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Thermodynamics and Scattering in 2D for 4D theories

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To all my supports, even those of null size

Abstract

Considering classical open string solutions with a null polygonal contour at the boundary of AdS_3, the geometric problem of minimal surfaces related to gluon scattering amplitudes at strong coupling is developed and solved in a set of TBA-like integral equations. Their form resembles that of a TBA system whose free energy yields the dynamic contribution to the area. The corresponding set of functional relations, the so called Y-system, is derived and extensively analyzed in the cases of octagon and decagon together with the set of integral equations. Useful rewrites of the latter establish the connection with the $\frac{SU(N)_k}{[U(1)]^{N-1}}$ HSG models and allow us to identify their Y-system (related to a universal TBA) with the one derived in AdS_3 when the algebra level k is set to 2. A useful insight that clarifies how to properly reduce to the A_n series is offered. Thanks to a well defined change of reference frame, whose mathematical structure has been studied in detail, the dynamic part of the remainder function coincides with the extremum of the Yang-Yang functional for the modified TBA equations. Explicit examples have been treated in terms of Y-functions and pseudo-energies, extrapolating useful links with several works concerning null WLs, Hitchin systems and integrable perturbations of CFTs corresponding to G_k-parafermions.

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Chapter 1

Introduction

The geometrical problem of computing minimal surfaces in the d-dimensional AdS space, which end on a null polygonal contour at the AdS boundary, it is a generalization of the problem of finding the shape of soap bubbles, or Plateau problem, to AdS_d. In the 19th century, the study of these surfaces was interesting for several reasons, including the size of atoms. For us, the study of these AdS minimal surfaces is interesting for what it could eventually teach us about their constituents, gluons, and their scattering amplitudes for all values of the coupling. While this is the broadest physical motivation, in this thesis we will limit ourselves to classical geometry (or strong coupling) inside AdS_3. In this way, we are dealing with scattering amplitudes at strong coupling which correspond to minimal surfaces that ends at the AdS boundary on a very peculiar null polygon contour [1]. When we consider a colour ordered amplitude involving n gluons with null momenta $k_1, k_2, ..., k_n$, we get a contour which is specified by its ordered vertices $x_1, x_2, ..., x_n$, with $x_i^\mu - x_{i-1}^\mu = k_i^\mu$. Conformal transformations act on this polygon and can change the positions of the vertices: one can define *conformal invariant cross ratios* which do not change under conformal transformations. Then, the problem becomes identical to the problem of computing a Wilson loop with this contour, i.e. a WL that consists of a sequence of light-like segments and depends on a finite number of parameters (the positions of the cusps x_i). In fact, one of the reasons why this simple subclass of Wilson loops is so interesting is that they are connected to scattering amplitudes in gauge theories [2, 3, 4]. Thanks to the gauge/string duality, at strong coupling they can be computed in terms of classical strings in AdS [1] and the classical equations of motion for the string tell us that the corresponding area should be extremized. This is the precise reason why we end up studying minimal surfaces in AdS space. It is possible to exploit the classical integrability of the problem to set a method for computing the area as a function of the shape of the boundary contour. This area is infinite due to the IR divergencies of the amplitudes or the UV divergencies of the WLs. It is possible to define a renormalized area as a function of the cross ratios, which is finite and conformal invariant [5, 6, 7, 8]. In the context of scattering amplitudes, this quantity is called *remainder function.* Thus, after a regularization, the amplitude can be seen as consisting of two pieces, a divergent part and a finite one. While the divergent part had a structure that is well understood (in the following we will indicate it with A_{cutoff}), only a piece of the finite part was known ($A_{BDS-like} - A_{BDS}$) [9]. From this setting, the remainder function represented the deviation from the conjectured BDS formula of the multi-loop amplitudes. The other piece of the finite part had not yet been calculated in general: only 2-loop perturbative computations were available in the literature [3, 10].

After [1], in which the minimal surface for the 4-point amplitude has been obtained by solving the Euler-Lagrange equation in the static gauge, the problem continued to be studied but it was very difficult to extend the 4-point solution to the general n-point amplitudes. In 2009, Luis F. Alday and Juan Maldacena, focusing on minimal surfaces in AdS_3, made remarkable progress and pointed out that this mathematical problem can be reduced to a certain generalized sinh-Gordon equation and to SU(2) Hitchin equations [5, 6]. The minimal area is obtained by finding the Stokes data of the associated linear problem, which is studied in detail by Gaiotto, Moore and Neitzke [11, 12] in the context of the moduli space of certain supersymmetric theories. This connection was specially useful to the authors of [6] because Gaiotto, Moore and Neitske had studied this mathematical problem, exploiting its integrability and obtaining exact solutions in some cases. Thanks to the explicit solutions found in [11], it was possible to device a method for computing the area of the surface as a function of the boundary contour, namely cross ratios made out from the positions of the cusps of the polygon, for the simplest non-trivial case, the null octagon.

Later, this work was generalized to AdS_4 and AdS_5 [7]. In the most relevant AdS_5 case, the minimal area problem is shown to be equivalent to solving the $SU(4)$ Hitchin system. Motivated by the connection between the solution of the associated linear problem and the thermodynamic Bethe ansatz (TBA) integral equations [11, App. E], Alday, Gaiotto and Maldacena found that the minimal area of the hexagon is evaluated by the free energy of the TBA equations of the A_3 integrable theory [7]. One might wonder if such a link with the A_n series of the ADE classification also exists for the AdS_3 case. In [13] the authors, focusing on minimal surfaces with a $2n$ sided polygonal boundary in AdS_3, determined the integral equations explicitly in the case of the decagon and the dodecagon, finding that they fit precisely in the general form proposed by Gaiotto, Moore and Neitzke in [11, App. E]; but, contrary to what happens in AdS_5, they identified the present integral equations with the TBA equations of the homogeneous sine-Gordon model [14, 15]. Following the path opened by them, we have studied in depth the parallelism between the TBA equations for AdS_3 surfaces and the integral equations for HSG models, stressing the appropriate reduction to the A_n series [20]. This means that the A_n series can be considered as immersed within general HSG models and we believe that this perspective could be useful to reveal who ODE/IM is for A_n. In particular, we found that our AdS_3 kernels actually correspond to the HSG component that does not lead to

A_n[1].

Subsequently, in [8] L. F. Alday et al. calculated the area of the surface as a function of the conformal cross ratios characterizing the general n sided polygon at the boundary, reducing the problem to a simple set of functional equations for the cross ratios as a functions of the spectral parameter, both for AdS_3 and AdS_5 cases. Then, they showed that the solution consists of a system of integral equations of the thermodynamic Bethe ansatz (TBA) form [36] and the area is given by the TBA free energy of the system. Exploiting this development, we will show that the AdS_3 Y-system found in [8] coincides with the HSG Y-system derived in [20], when the algebraic level is $k = 2$.

Finally in [37], deriving an OPE-like expansion for polygonal null WLs, Alday, Gaiotto and colleagues proposed a new version for the TBA integral equations that determine the strong coupling answer, in which the equations involve only the physical cross ratios (in contrast with the ones in [8] which involved some other auxiliary parameters Z_s). In addiction, the non-trivial part of the area, A_{free}, is given by the critical value of the Yang-Yang functional which reproduces, once extreme, the *modified TBA integral equations*. The computation of the regularized area or better, the remainder function, was performed following the basic steps of [8], but with a very general formalism inherited by general Hitchin systems. Here, we will rewrite the procedure with the more familiar formalism of [8], highlighting the change of reference frame that characterizes this new writing: if I do not switch to physical cross ratios, I do not obtain the correct Yang-Yang functional which reproduces the free energy in its saddle point. This means that we have better defined the physics behind universal cross ratios; in terms of the dimensionless parameters $Z_s, \bar{Z}_s$ the computation of the dynamic part of the remainder function brings out the free-energy, whilst considering the geometric properties of the WL, like universal cross ratios, the natural quantity that emerges is the Yang-Yang functional (to be seen as an action) and the area is related to its critical value.

Furthermore, this reformulation will allow us to derive the TBA integral equations in terms of Y-functions and pseudo-energies, allowing useful links with both [13] and [38, 39]. These latter connections will be very useful for future developments; indeed, from the knowledge of the AdS_3 kernels of the modified TBA equations it should be possible to construct a P form factor theory [40, 41, 42, 43, 44] for the HSG models that resembles the TBA outcomes, inspired by the AdS_5 case [38], with respect to which our attempt would merely constitute a three dimensional reduction. From the comparison with [38], we then see that our AdS_3 kernels coincide with the ones of the AdS_5 mesonic sector, up to simple factors depending on the indices s, s' which labels the Y-functions. This leads us to think that it will be possible to reproduce the calculation also in the AdS_3 case, paying attention to the different physical content we could have; since we

[1] In this regard, the paper [31] raises many questions regarding the derivation of TBA-like integral equations in the ADE universal form, as proposed by Zamolodchikov in [32], starting from the integral equations for general Hitchin systems [11, eq. (5.13)]. In fact, the ADE classification and the HSG models have TBA equations quite different in terms of kernels and mass scales.

do not have a previously defined geometric decomposition for the WL (squares and pentagons seem not to work for AdS_3), we will exploit more general properties regarding the polygonal transitions P (form factors) for the determination of their explicit form. Finally, based on the Hubbard-Stratonovich transform, we have defined the fundamental steps that allow to get back the TBA set-up from the re-summing of the BSV series corresponding to the decagon.

Chapter 2

The TBA equations for minimal surfaces in AdS_3

2.1 Preliminaries

To circumscribe and specify the appropriate context from which to start, it is advisable to summarize the fundamental steps of [6]. The starting point are the classical equations for a string world-sheet that ends on the null polygon at the boundary of the AdS_3 space. We consider only a special class of null polygons which can be embedded in a 2-dimensional subspace, such as an $\mathbb{R}^{1,1}$ subspace of the boundary of AdS[1]. When we consider the AdS_3 space as the following surface in $\mathbb{R}^{2,2}$,

$$\vec{Y}.\vec{Y} = -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1 \tag{2.1}$$

classical string in AdS_3 can be described by a reduced model in terms of the embedding coordinates $Y^\mu \in \mathbb{R}^{2,2}$, which takes into account both the equations of motion and the Virasoro constraints:

$$\partial\bar{\partial}\vec{Y} - \left(\partial\vec{Y}.\bar{\partial}\vec{Y}\right)\vec{Y} = 0 \qquad \partial\vec{Y}.\partial\vec{Y} = \bar{\partial}\vec{Y}.\bar{\partial}\vec{Y} = 0 \tag{2.2}$$

We parametrize the world-sheet in terms of the complex variables $z, \bar{z}$ and we are interested in space-like surfaces embedded in AdS. The area of the world-sheet depends on the positions of the cusps at the boundary, but conformal symmetry implies that it depends only on cross ratios of these positions, up to a simple term which arises due to the regulator. For null polygons living in $\mathbb{R}^{1,1}$ we have 6 conformal generators that move the positions of the cusps. We underline that, to have a closed null polygon, we need an

[1] For these loops the string world-sheet lives in an AdS_3 subspace of the full AdS_d space with $d \geq 3$. Since we are merely studying the classical equations, the solutions can be embedded in any string theory geometry which contains an AdS_3 factor.

even number of cusps, since each cusp joints a left moving with a right moving null line. If we consider $2n$ cusps, then we have $2(n-3)$ cross ratios. The positions of the cusps are determined by n coordinates x_i^+ and n coordinates x_i^-; so that, we have $n-3$ cross ratios made out of the x_i^+ and $n-3$ from the x_i^-. The first time we have a non-trivial dependence on the cross ratios is for $n=4$ which corresponds to a null octagon. We are interested in the area of the world-sheet that is a completely geometric problem. In order to analyze it, one can use a Pohlmeyer type reduction: this maps the problem of strings moving in AdS_3 to a problem involving a single field α, which obeys a generalized sinh-Gordon equation, and a holomorphic polynomial $p(z)$, whose degree is related to the number of cusps of the corresponding Wilson loop[2]. A polynomial of degree $n-2$ corresponds to $2n$ cusps. We define

$$e^{2\alpha(z,\bar{z})} = \frac{1}{2}\partial\vec{Y}.\bar{\partial}\vec{Y} \tag{2.3}$$

$$N_a = \frac{e^{-2\alpha}}{2}\epsilon_{abcd}Y^b\partial Y^c\bar{\partial}Y^d \tag{2.4}$$

$$p = -\frac{1}{2}\vec{N}.\partial^2\vec{Y} \qquad \bar{p} = \frac{1}{2}\vec{N}.\bar{\partial}^2\vec{Y} \tag{2.5}$$

where $\vec{N}$ is a purely imaginary vector whose imaginary part is a time-like vector orthogonal to the space-like surface we are dealing with,

$$\vec{N}.\vec{Y} = \vec{N}.\partial\vec{Y} = \vec{N}.\bar{\partial}\vec{Y} = 0 \qquad \vec{N}.\vec{N} = 1 \tag{2.6}$$

Then, as a consequence of (2.2), it can be shown [6] that $p = p(z)$ is a holomorphic function and $\alpha(z,\bar{z})$ satisfies the generalized sinh-Gordon equation

$$\partial\bar{\partial}\alpha(z,\bar{z}) - e^{2\alpha(z,\bar{z})} + |p(z)|^2 e^{-2\alpha(z,\bar{z})} = 0 \tag{2.7}$$

The area of the world-sheet is simply given by the conformal gauge action expressed in terms of the reduced fields

$$A = 4\int d^2z\, e^{2\alpha} \tag{2.8}$$

where, if $z = x+iy$, then $\int d^2z = \int dxdy$ and $\partial = \frac{1}{2}(\partial_x - i\partial_y)$. For solutions relevant to our problem this area is divergent and needs to be regularized. Note that (2.7) and (2.8) are invariant under conformal transformations, provided we transform α and p accordingly. This is the original conformal invariance of the theory which the homogenous Virasoro constraints have not broken. Given a solution of the generalized sinh-Gordon model,

[2]We can view this reduction as a sophisticated gauge choice that leaves us only the physical degrees of freedom.

one can reconstruct the classical string world-sheet in AdS_3 by solving 2 auxiliary linear problems involving the field α,

$$\partial\psi^L_\alpha + (B^L_z)_\alpha^\beta\,\psi^L_\beta = 0 \qquad \bar\partial\psi^L_\alpha + (B^L_{\bar z})_\alpha^\beta\,\psi^L_\beta = 0 \tag{2.9}$$

$$\partial\psi^R_{\dot\alpha} + (B^R_z)_{\dot\alpha}^{\dot\beta}\,\psi^R_{\dot\beta} = 0 \qquad \bar\partial\psi^R_{\dot\alpha} + (B^R_z)_{\dot\alpha}^{\dot\beta}\,\psi^R_{\dot\beta} = 0 \tag{2.10}$$

where the $SL(2)$ connections B^L and B^R are given by

$$B^L_z = \begin{pmatrix} \frac{1}{2}\partial\alpha & -e^{\alpha} \\ -e^{-\alpha}p(z) & -\frac{1}{2}\partial\alpha \end{pmatrix} \qquad B^L_{\bar z} = \begin{pmatrix} -\frac{1}{2}\bar\partial\alpha & -e^{-\alpha}\bar p(\bar z) \\ -e^{\alpha} & \frac{1}{2}\bar\partial\alpha \end{pmatrix} \tag{2.11}$$

$$B^R_z = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha}p(z) \\ -e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix} \qquad B^R_{\bar z} = \begin{pmatrix} \frac{1}{2}\bar\partial\alpha & -e^{\alpha} \\ e^{-\alpha}\bar p(\bar z) & -\frac{1}{2}\bar\partial\alpha \end{pmatrix} \tag{2.12}$$

which obey the following flatness conditions

$$\partial B^L_{\bar z} - \bar\partial B^L_z + [B^L_z, B^L_{\bar z}] = 0 \qquad \partial B^R_{\bar z} - \bar\partial B^R_z + [B^R_z, B^R_{\bar z}] = 0 \tag{2.13}$$

Internal $SL(2)_L \times SL(2)_R$ indices $\alpha, \dot\alpha$ denote rows and columns of these connections, while the indices $a, \dot a = 1, 2$ denote the two different linearly independent solutions of each linear problem, namely $\psi^L_{\alpha,a}$ and $\psi^R_{\dot\alpha,\dot a}$. The space-time isometry group $SO(2,2) = SL(2) \times SL(2)$ acts on these indices. These auxiliary linear problems display Stokes phenomena as $z \to \infty$, so that depending on the various angular sectors in z associated to each cusps, the two solutions takes different asymptotic forms [6, sec. 3]. Once a pair of solutions has been found, the explicit form of the space-time coordinates $Y_{a,\dot a}(z, \bar z)$ is given by a particular bilinear combination of the left and right solutions

$$Y_{a,\dot a} = \psi^L_{\alpha,a} M_1^{\alpha,\dot\beta} \psi^R_{\dot\beta,\dot a} \qquad M_1^{\alpha,\dot\beta} = \mathbb{I}_2 \tag{2.14}$$

It turns out that the left connection B^L can be promoted to a family of flat connections by introducing a spectral parameter ζ,

$$B_z = A_z + \Phi_z \quad \longrightarrow \quad B_z(\zeta) = A_z + \frac{1}{\zeta}\Phi_z \tag{2.15}$$

$$B_{\bar z} = A_{\bar z} + \Phi_{\bar z} \quad \longrightarrow \quad B_{\bar z}(\zeta) = A_{\bar z} + \zeta\Phi_{\bar z} \tag{2.16}$$

where we have decomposed the connection B_z into its diagonal part A_z, which has the interpretation of a gauge connection in two dimension, and off diagonal part Φ_z, which is a Higgs field. Actually, both left and right connections can be simply obtained (up to a constant gauge transformation) from $B(\zeta)$ by setting $\zeta = 1$ or $\zeta = i$ respectively,

$$B^L_z = B_z(1) \qquad B^R_z = U B_z(i) U^{-1} \qquad U = \begin{pmatrix} 0 & e^{i\frac{\pi}{4}} \\ e^{i\frac{3\pi}{4}} & 0 \end{pmatrix} \tag{2.17}$$

It is important to note that this decomposition agrees with the form used in [11, 12]. The zero curvature conditions (2.13) can be rephrased in the following $SU(2)$ Hitchin equations:

$$D_{\bar{z}}\Phi_z = D_z\Phi_{\bar{z}} = 0 \qquad F_{z\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] = 0 \tag{2.18}$$

where $D_\mu\Phi = \partial_\mu\Phi_z + [A_\mu, \Phi_z]$ is the covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the field strength. Furthermore, through (2.7), we can rewrite the area in terms of the Higgs fields of the Hitchin system,

$$Tr[\Phi_z\Phi_{\bar{z}}] = e^{2\alpha} + |p(z)|^2 e^{-2\alpha} = 2e^{2\alpha} - \partial\bar{\partial}\alpha \tag{2.19}$$

together with

$$p = \frac{1}{2}Tr\left[\Phi_z^2\right] \tag{2.20}$$

and then

$$A = 4\int d^2z\, e^{2\alpha} = 2\int d^2z\, Tr[\Phi_z\Phi_{\bar{z}}] + \text{total derivative} \tag{2.21}$$

The Hitchin equations can be considered for a generic gauge group. Here, we are in the particular case of the $SU(2)$ gauge group and we are interested in configurations which are invariant under a certain $\mathbb{Z}_2$-symmetry,

$$A \to \sigma_3 A \sigma_3 \qquad \Phi \to -\sigma_3\Phi\sigma_3 \tag{2.22}$$

where σ_3 is the Pauli matrix. We have projected onto the $\mathbb{Z}_2$ invariant subspace, so that the moduli space of our problem is a subspace of the full hyperKahler Hitchin space, sometimes called the *real section*[3].

Before recalling the results on the regularized area, we would like to make a comment that connects us to the work of Gaiotto, Moore and Neitske. Introducing a spectral parameter ζ, and thinking about the analytic structure of the gauge invariant information contained in the flat connection, is a standard tool for analyzing the solutions of integrable models. In our problem, the gauge invariant information is contained in the cross ratios, so that studying the cross ratios as a function of ζ is a way to solve the problem: this is precisely what we can found in [11, 12]. Exploiting the analytic structure of the cross ratios, the authors have written down a Riemann-Hilbert problem whose solution determines the metric in the moduli space, parametrized by the coefficients of the polynomial z_i,

$$p(z) = \prod_{i=1}^{n-2}(z - z_i) \qquad \sum_{i=1}^{n-2} z_i = 0 \tag{2.23}$$

[3]The full hyperKahler space can be represented as a torus fibration over this real section.

In [6, App. C] it is shown how to calculate the area from the expression of the known metric $g_{z_i\bar{z}_j}$,

$$A \sim \sum_{i=1}^{n-2}(z_i\partial_{z_i} + \bar{z}_i\partial_{\bar{z}_i})K \qquad \partial_{z_i}\partial_{\bar{z}_j}K = g_{z_i\bar{z}_j} \tag{2.24}$$

where K is the Kahler potential that leads to the metric $g_{z_i\bar{z}_j}$. In [11] this procedure is carried out explicitly for a quadratic polynomial, which is the first non-trivial case. It was found that the metric corresponds to that of a four dimensional $\mathcal{N}=2$ theory with a single hypermultiplet compactified on a circle.

It is possible to define formulas for the regularized area in terms of the solution of the generalized sinh-Gordon equation using a physical space-time regulator, which corresponds to placing a cut-off on the radial AdS_3 direction. Writing the AdS_3 metric through the Poincaré coordinates,

$$ds^2 = \frac{1}{r^2}(dr^2 + dx^+dx^-) \tag{2.25}$$

we put a cut-off that demands $r \geq \mu$ so that the regularized area is given by

$$A = 4\int_{r(z,\bar{z})\geq\mu} d^2z\, e^{2\alpha} \tag{2.26}$$

Since the problem is conformal invariant, one would have expected that the area would depend only on the conformal cross ratios. However, the introduction of the regulator spoils the conformal symmetry and, after removing the divergent piece, what remains is not conformal invariant. In order to extract the dependence on the regulator, it is convenient to rewrite

$$A = 4\int d^2z\,(e^{2\alpha} - \sqrt{p\bar{p}}) + 4\int_{r(z,\bar{z})\geq\mu} d^2z\,\sqrt{p\bar{p}} = A_{sinh} + 4\int_\Sigma d^2\omega \tag{2.27}$$

where the new variable $d\omega = \sqrt{p(z)}dz$ was introduced to simplify the generalized sinh-Gordon equation,

$$\partial_\omega\bar{\partial}_{\bar{\omega}}\hat{\alpha} - e^{2\hat{\alpha}} + e^{-2\hat{\alpha}} = 0 \qquad \hat{\alpha} \equiv \alpha - \frac{1}{4}\log(p\bar{p}) \tag{2.28}$$

A_{sinh} is the finite piece of the area and involve only the solution to the sinh-Gordon problem: it depends only on the coefficients of the polynomial $p(z)$, which in turn determine the space-time cross ratios. Thus A_{sinh} depends only on the space-time cross ratios. The second term in (2.27) involves an integral over a complicated region: since we have a Riemann surface, we have some structure of cuts which depend on the form of the polynomial and simplify at large $|\omega|$, see [6, sec. 5]. When n is odd, we can split this second term into two pieces,

$$4\int_\Sigma d^2\omega = A_{periods} + A_{cutoff} \tag{2.29}$$

one involves the region for finite values of ω that is sensitive to the branch cuts, $A_{periods}$, and the other one is the integral at very large values of ω, A_{cutoff}. While $A_{periods}$ is finite and can be expressed in a simple way[4] once the polynomial p is given, A_{cutoff} can be separated in

$$A_{cutoff} = A_{div} + A_{BDS-like} \tag{2.30}$$

where A_{div} contains all the μ dependence of the regularization scheme and $A_{BDS-like}$ depends on the distance between the cusps and not purely the cross ratios. Explicit expressions of these two pieces can be found in the computations of [6, App. B]. One common way to write the full answer is to introduce the *remainder function* $R(\chi)$ which is a finite function of the cross ratios and contains the non-trivial information,

$$A = A_{div} + A_{BDS} + R(\chi) \tag{2.31}$$

$$R = A_{BDS-like} - A_{BDS} + A_{periods} + A_{sinh} \tag{2.32}$$

where the difference $A_{BDS-like} - A_{BDS}$ is written explicitly in [6, App. E]. The complicated part of the problem is to compute A_{sinh}[5] and also to express the coefficients of the polynomial as a function of the space-time cross ratios. Note that the formula (2.24) is expected to give us $A_{sinh} + A_{periods}$ in a regularization scheme that introduces a cut-off in the ω-plane and throws away the divergent terms. The case with n even is a bit more complicated because the region at large $|\omega|$ is not simplified as in the odd case: there is a single branch cut that survives at infinity, which is reflected in the addition of a new term A_{extra}. In [6, sec. 6] we can find the full final answer for the octagon. After we compute this geometric area, we can relate these results to the vacuum expectation value of the Wilson loop at strong coupling (amplitude) in $\mathcal{N} = 4$ super Yang Mills as

$$\text{Amplitude} \sim \langle W \rangle \sim e^{-\frac{R^2}{2\pi\alpha'}A} = e^{-\frac{\sqrt{\lambda}}{2\pi}A} \qquad \frac{R^2}{\alpha'} = \sqrt{\lambda} = \sqrt{g^2 N} \tag{2.33}$$

with A is the geometrical area of the surface in units where the radius of AdS has been set to one.

As explained in [8], where the authors exploit the integrability of the classical AdS sigma model, the integrable coset theories (with Virasoro constraints) of the form G/H, where the Lie algebra $\mathcal{G}$ has a $\mathbb{Z}_2$-symmetry that ensure integrability, represents a more general way to formalize the above construction. Assuming that the Lie algebra admits the decomposition $\mathcal{G} = \mathcal{H} + \mathcal{K}$, in which $\mathcal{H}$ is the invariant component under the action of the $\mathbb{Z}_2$ generator while elements of $\mathcal{K}$ are sent to minus themselves, we can write the

[4]The structure of $A_{periods}$ is very similar to the structure of the Kahler potential in cases where the same Riemann surface appears in the description of the vector moduli space of $\mathcal{N} = 2$ supersymmetric theories in four dimension.

[5]After [7], the notation A_{sinh} will be replaced with A_{free} to highlight the structure of free-energy of the corresponding TBA system.

G invariant flats currents $J = g^{-1}dg = H + K$ with $dJ + J \wedge J = 0$. The equations of motion together with the flatness condition for J lead to

$$D_z K_{\bar{z}} = 0 = D_{\bar{z}} K_z \qquad [D_z, D_{\bar{z}}] + [K_z, K_{\bar{z}}] = 0 \qquad D_z X \equiv \partial X + [H_z, X] \tag{2.34}$$

which can be seen as equations for the connection. Once solved, we can find a coset by solving the flatness condition

$$(d + J)\psi = (d + H + K)\psi = 0 \tag{2.35}$$

for a set of independent vector solutions; then, we can orthonormalize and assemble them into g^{-1}, the representative coset called *flat section*. The equations (2.34) are identical to our $SU(2)$ Hitchin equations (2.18) after the identification $\hat{\Phi}_z = K_z$, $\hat{\Phi}_{\bar{z}} = K_{\bar{z}}$, $A = H$. If we introduce a spectral parameter ζ, they also correspond to the flatness of the one parameter family of connections,

$$d + \hat{\mathcal{A}}(\zeta) \qquad \hat{\mathcal{A}}(\zeta) = \frac{K_z dz}{\zeta^2} + H + \zeta^2 K_{\bar{z}} d\bar{z} \tag{2.36}$$

which, after a global gauge transformation, becomes [8, sec. 2]

$$\mathcal{A} = \frac{\Phi_z dz}{\zeta} + A + \zeta \Phi_{\bar{z}} d\bar{z} \tag{2.37}$$

Precisely for this reason, the problem of computing minimal surfaces in AdS_3 which in turn reduces to a $\mathbb{Z}_2$ projection of an $SU(2)$ Hitchin problem, can be rephrased in terms of sections of the flat connection which obey

$$\left(d + \frac{\Phi_z dz}{\zeta} + A + \Phi_{\bar{z}} d\bar{z}\, \zeta\right) \psi(\zeta) = 0 \tag{2.38}$$

The Z_2-symmetry relates solutions $\psi(\zeta)$ with different values of the spectral parameter. If $\psi(\zeta)$ is a flat section, then $\eta(\zeta) = U\psi(-\zeta)$ is another solution of the problem with $U = \sigma_3$. The small solutions change as we change ζ and we can track on it by looking them in the large z region. In a given Stokes sector, the small solution contains a factor behaving as

$$e^{\frac{-\int_z \sqrt{p(z')}dz'}{\zeta}} \sim e^{-\frac{z^{n/4}}{\zeta}} \tag{2.39}$$

where n (even) is determined by the degree of the polynomial and is equal to the number of cusps of the polygon. Thus, there are $\frac{n}{2}$ small solutions s_i in the corresponding $\frac{n}{2}$ Stokes sectors. Note that the solutions do not come back to themselves after a shift by $e^{2\pi i}\zeta$. If we start with the solution $s_i(\zeta)$, which is the small one in the i-th Stokes sector, then we find that $s_i(e^{2\pi i}\zeta) \propto s_{i+2}(\zeta)$. We can choose a solution s_1 in the first Stokes sector and define all others as

$$s_j(\zeta) = U^{j-1} s_1(e^{ji\pi}\zeta) \tag{2.40}$$

where $Us_i(e^{i\pi}\zeta) \propto s_{i+1}(\zeta)$. Then, as we go around, we have[6] $s_{\frac{n}{2}+1} = A(\zeta)s_1$.

[6]When $\frac{n}{2}$ is odd, $A(\zeta)$ can be set to one; otherwise it has a simple form.

The full connection with spectral parameter ζ is an $SL(2)$ flat connection and thus we can form an $SL(2)$ invariant product $\langle \psi, \psi' \rangle = -\langle \psi', \psi \rangle$ with two solutions of the linear problem (2.38), which satisfies

$$\langle s_i, s_j \rangle (e^{i\pi}\zeta) = \langle s_{i+1}, s_{j+1} \rangle (\zeta) \tag{2.41}$$

We can normalize s_1 so that $\langle s_1, s_2 \rangle = 1$, then the previous equation also implies that $\langle s_i, s_{i+1} \rangle = 1$. The invariant products just introduced allow us to define the cross ratios[7] by forming quantities like

$$\chi_{ijkl}(\zeta) = \frac{\langle s_i, s_j \rangle \langle s_k, s_l \rangle}{\langle s_i, s_k \rangle \langle s_j, s_l \rangle} \tag{2.42}$$

which can be related to the *(conformal invariant) space-time cross ratios*, formed from the positions of the cusps, by setting $\zeta = 1$ and $\zeta = i$. Namely,

$$\chi_{ijkl}(\zeta = 1) = \frac{x_{ij}^{+} x_{kl}^{+}}{x_{ik}^{+} x_{jl}^{+}} \tag{2.43}$$

$$\chi_{ijkl}(\zeta = i) = \frac{x_{ij}^{-} x_{kl}^{-}}{x_{ik}^{-} x_{jl}^{-}} \tag{2.44}$$

where $x_i^{\pm}$ are the space-time positions of the cusps for a polygon that is embedded in $\mathbb{R}^{1,1}$, the boundary of AdS_3. We just recall that these positions are given by a set of $\frac{n}{2}$ values x_i^{+} and an equivalent set of $\frac{n}{2}$ values x_i^{-}.

2.1.1 The AdS_3 Y-system

Starting from the Schouten identity for the inner products $\langle s_i, s_j \rangle(\zeta)$ made out of two small solutions of the linear problem,

$$\langle s_i, s_j \rangle \langle s_k, s_l \rangle + \langle s_i, s_l \rangle \langle s_j, s_k \rangle + \langle s_i, s_k \rangle \langle s_l, s_j \rangle = 0 \tag{2.45}$$

we can derive a set of functional equations if we start from a particular choice of small solutions where $i = k+1$, $j = -k$ and $l = -k-1$:

$$\langle s_{k+1}, s_{-k} \rangle \langle s_k, s_{-k-1} \rangle + \langle s_{k+1}, s_{-k-1} \rangle \langle s_{-k}, s_k \rangle + \langle s_{k+1}, s_k \rangle \langle s_{-k-1}, s_{-k} \rangle = 0 \tag{2.46}$$

In our normalization, the last term is equal to one. If we define

$$f^{\pm}(\zeta) \equiv f\left(e^{\pm i\frac{\pi}{2}}\zeta\right) \tag{2.47}$$

[7]The cross ratios do not depend on the arbitrary normalization of the s_i and, by construction, they are also invariant under the conformal symmetries of AdS_3.

together with

$$T_{2k} = \langle s_{-k-1}, s_k \rangle^+ \qquad T_{2k+1} = \langle s_{-k-1}, s_{k+1} \rangle \tag{2.48}$$

we can easily see that the following relations hold,

$$\langle s_{k+1}, s_{-k} \rangle = -T_{2k}^+ \tag{2.49}$$

$$\langle s_k, s_{-k-1} \rangle = -T_{2k}^- \tag{2.50}$$

$$\langle s_{k+1}, s_{-k-1} \rangle = -T_{2k+1} \tag{2.51}$$

$$\langle s_k, s_{-k} \rangle = -T_{2k-1} \tag{2.52}$$

Thus, our starting identity (2.46) becomes the *SU(2) Hirota equation*

$$T_s^+ T_s^- = T_{s+1} T_{s-1} + 1 \tag{2.53}$$

where $s = 2k$. T_s is non-zero for $s = 0, 1, 2, ..., \frac{n}{2} - 2$.

Finally, we introduce the *Y-function* as the product of two next-to-nearest T-functions,

$$Y_s \equiv T_{s-1} T_{s+1} \tag{2.54}$$

so that they are non-zero in a slightly smaller lattice parametrized by $s = 1, 2, ..., \frac{n}{2} - 3$. Note that the number of Y-functions coincides with the number of independent cross ratios. The Hirota equation (2.53) implies the AdS_3 *Y-system*

$$Y_s^+ Y_s^- = (1 + Y_{s+1})(1 + Y_{s-1}) \tag{2.55}$$

In order to fix the Y-functions and pick out the appropriate solutions to this set of equations, we have to consider the analytic properties of the Y's. Furthermore, to make these solutions useful, we must relate them to the actual expression for the area. However, before that, some general properties about the Hirota equations should be analyzed: the aim is to better understand the possible choices of normalization of the Y-functions and their implications.

The general form of the Hirota system, which generalizes the $SU(2)$ case derived above, is a set of functional equations for the functions $T_{a,s}(\zeta)$. The indices (a, s) take integer values and parametrize a two dimensional lattice, so that at each point of this lattice we have a function $T_{a,s}(\zeta)$. Then, for each site (a, s) we have an Hirota equation

$$T_{a,s}^+ T_{a,s}^- = T_{a,s-1} T_{a,s+1} + T_{a+1,s} T_{a-1,s} \tag{2.56}$$

which have a huge gauge symmetry

$$T_{a,s} \to \prod_{\alpha,\beta=\pm} g_{\alpha\beta}\left(e^{i\frac{\pi}{2}(\alpha a+\beta s)}\zeta\right) T_{a,s}(\zeta) \tag{2.57}$$

where $g_{\alpha\beta}$ are four arbitrary functions. By means of the latter transformation, we can define a set of gauge invariant quantities[8]

$$Y_{a,s} = \frac{T_{a,s-1}T_{a,s+1}}{T_{a+1,s}T_{a-1,\,s}} \tag{2.58}$$

for which hold the following general Y-system, derived from (2.56),

$$\frac{Y_{a,s}^{+}Y_{a,s}^{-}}{Y_{a+1,s}Y_{a-1,s}} = \frac{(1+Y_{a,s-1})\,(1+Y_{a,s+1})}{(1+Y_{a+1,s})\,(1+Y_{a-1,s})} \tag{2.59}$$

Different domain in (a,s), together with different boundary condition and analytic properties, describe different integrable models. In the previous treatment, we consider the case in which the T-functions live in a finite strip with three rows and $\frac{n}{2}-1$ columns, where n is the number of cusps (gluons). The functions (2.48) are the ones in the middle row, $T_s = T_{1,s}$, and similarly, $Y_s = Y_{1,s}$.

Now, since the T-functions (2.48) are inner products of small solutions, they are sensitive to their normalization. In the derivation of (2.53), we have used the normalization $\langle s_i, s_{i+1}\rangle = 1$ which correspond to the following gauge choice:

$$\begin{aligned} T_{0,2k} &\equiv \langle s_{-k-1}, s_{-k}\rangle = 1 \\ T_{2,2k} &\equiv \langle s_{k}, s_{k+1}\rangle = 1 \\ T_{0,2k+1} &\equiv \langle s_{-k-2}, s_{-k-1}\rangle^{+} = 1 \\ T_{2,2k+1} &\equiv \langle s_{k}, s_{k+1}\rangle^{+} = 1 \end{aligned} \tag{2.60}$$

We could of course choose not to fix a normalization for the T-functions, but then we should use the gauge invariant combination (2.58) when defining the Y-functions,

$$Y_{2k} = \frac{T_{1,2k-1}T_{1,2k+1}}{T_{0,2k}T_{2,2k}} = \frac{\langle s_{-k}, s_k\rangle\langle s_{-k-1}, s_{k+1}\rangle}{\langle s_{-k-1}, s_{-k}\rangle\langle s_k, s_{k+1}\rangle} \tag{2.61}$$

$$Y_{2k+1} = \frac{T_{1,2k}T_{1,2k+2}}{T_{0,2k+1}T_{2,2k+1}} = \left[\frac{\langle s_{-k-1}, s_k\rangle\langle s_{-k-2}, s_{k+1}\rangle}{\langle s_{-k-2}, s_{-k-1}\rangle\langle s_k, s_{k+1}\rangle}\right]^{+} \tag{2.62}$$

which are now manifestly independent of the choice of the normalization of the small solutions and reproduce the physical cross ratios (2.43) and (2.44) when computed for $\zeta = 1$ or $\zeta = i$. In the following, we will use normalized definitions and, in order to conform the arguments, we will do a little shift.

[8] We can see the Dynkin diagram structure of the T's and the Y's in [8].

2.1.2 Analytic properties of the Y-functions

For finite values of ζ other than zero, it is clear from (2.48) that the T's are analytic functions of ζ. While, in general, the Y's will be meromorphic functions, in our case we see that they have no poles and are thus analytic away from $\zeta = 0, \infty$ due to the precise choice of normalization that set the denominators to one. For $\zeta \to 0$ and $\zeta \to \infty$, they will have essential singularities and in this section we will briefly analyze the behaviour in these two regions.

When $\zeta \to 0$, we can solve the equations for the flat sections by making a WKB approximation, where ζ plays the role of $\hbar$. This is explain in great detail in [12] and here we apply it to our case. We are considering the equation

$$\left(d + \frac{\Phi_z dz}{\zeta} + A + \zeta \Phi_{\bar{z}} d\bar{z}\right) s = 0 \tag{2.63}$$

in which it is convenient to make similarity transformation that diagonalize $\Phi_z \to \sqrt{p}\,\mathrm{diag}(1,-1)$. The solutions in this approximation go like $\exp\left(\pm\frac{1}{\zeta}\int\sqrt{p}dz\right)$ times constant vectors. The WKB is a good approximation if we are following the line of steepest descent,

$$\Im\left\{\frac{\sqrt{p}\dot{z}}{\zeta}\right\} = 0 \tag{2.64}$$

but fails at the zeros of p, the turning points. From each Stokes sector we have WKB lines that emanates from it: these lines can end in another Sokes sector or, for very special lines, on the zeros of p. If a line connects two Stokes sectors, say i and j, then we can use it to approximate the inner product $\langle s_i, s_j\rangle$. This estimate is good in a sector of width π in the phase of ζ, centered on the value of ζ where the line exists. As we change the phase of ζ or the polynomial p, the pattern of flows lines changes. For $\zeta = 1$ or $\zeta = i$ only some inner products can be evaluated. Alternatively, we can set $\zeta = e^{i\frac{\pi}{4}}$ and evaluate them all at once.

Using these types of flow patterns it is a simple matter to evaluate various inner products. It turns out that the inner products in the definitions of the Y-functions (2.61) and (2.62) combine to give a contour integral around a certain cycles [8, sec. 3]. Thus, each Y_s, is estimated by the integral

$$Z_s = -\oint_{\gamma^s}\sqrt{p}dz \tag{2.65}$$

and the corresponding Y-function have the small ζ behaviour

$$\log Y_{2k} \sim \frac{Z_{2k}}{\zeta} + \ldots \qquad \log Y_{2k+1} \sim \frac{Z_{2k+1}}{i\zeta} + \ldots \tag{2.66}$$

For what follow, it is convenient to define the parameters[9] m_s in this way

$$m_{2k} = -2Z_{2k} \qquad m_{2k+1} = -\frac{2Z_{2k+1}}{i} \tag{2.67}$$

A similar computation for $\zeta \to \infty$ gives a similar result where now $\log Y_{2k} \sim \zeta \bar{Z}_{2k}$ and $\bar{m}_{2k} = -2\bar{Z}_{2k}$. The barred parameters are introduced only to differentiate the two regions in ζ. Thus, we have showed that all the Y-functions have the asymptotic behaviour

$$\log Y_s(\theta) = -m_s \cosh\theta + ... \tag{2.68}$$

for large θ where $\zeta = e^{\theta}$. It is important to note that this behaviour is good over a range of $(-\pi, \pi)$ in the imaginary part of θ. In fact, for each Y_s, the region $\Im\{\theta\} = 0$ corresponds to the center of the region where the WKB lines exists. Thus, the corresponding WKB lines exists for a sector of angular size π around this line. In addition, we have mentioned that the WKB approximation continues to be good for a further sector of $\frac{\pi}{2}$ on each side.

2.2 The TBA-like integral equations

Starting from the AdS_3 Y-system [8], i.e. a set of $\frac{n}{2} - 3$ functional equations,

$$Y_s^+ Y_s^- = (1 + Y_{s+1})(1 + Y_{s-1}) \qquad s = 1, 2, ..., \frac{n}{2} - 3 \tag{2.69}$$

we can determine uniquely the Y-functions by studying their analytic properties. We recall that the number of Y-functions coincides with the number of independent cross ratios. For a polygon that is embedded in $R^{1,1}$ (the boundary of AdS_3), the positions of its cusps are given by a set of $\frac{n}{2}$ values x_i^+ and an equivalent set of x_i^-, where n is the number of cusps. The resulting cross ratios are then $\frac{n}{2} - 3$. For practical purposes, it is useful to have an equivalent formulation of these functional relations in terms of *TBA-like integral equations*. To derive them, we follow the usual procedure:

i. We introduce the quantity

$$l_s(\theta) \equiv \log\left(\frac{Y_s(\theta)}{e^{-m_s \cosh\theta}}\right) = \log Y_s(\theta) + m_s \cosh\theta \tag{2.70}$$

that is analytic in the strip $|\Im\{\theta\}| \le \frac{\pi}{2}$ and vanishes for $\theta \to \pm\infty$ by virtue of the Y's properties;

[9] For our choice of polynomial, the m_s are all real and positive.

a) Small ζ behavior ($\theta \to -\infty$)

$$m_s = -2Z_s \in \mathbb{R}_+ \tag{2.71}$$

$$\begin{cases} \cosh\theta \sim \frac{e^{-\theta}}{2} \\ \log Y_s(\theta) \sim \frac{Z_s}{\zeta} = e^{-\theta}\left(-\frac{m_s}{2}\right) \end{cases} \tag{2.72}$$

$$l_s(\theta) = \log Y_s(\theta) + m_s \cosh\theta \sim e^{-\theta}\left(-\frac{m_s}{2}\right) + m_s \frac{e^{-\theta}}{2} = 0 \tag{2.73}$$

b) Large ζ behavior ($\theta \to +\infty$)

$$\bar{m}_s = -2\bar{Z}_s \tag{2.74}$$

$$\begin{cases} \cosh\theta \sim \frac{e^{\theta}}{2} \\ \log Y_s(\theta) \sim \zeta \bar{Z}_s = e^{\theta}\left(-\frac{\bar{m}_s}{2}\right) \end{cases} \tag{2.75}$$

$$l_s(\theta) = \log Y_s(\theta) + \bar{m}_s \cosh\theta \sim e^{\theta}\left(-\frac{\bar{m}_s}{2}\right) + \bar{m}_s \frac{e^{\theta}}{2} = 0 \tag{2.76}$$

ii. We take the logarithm of the Y-system equations (2.69),

$$\log Y_s^+(\theta) + \log Y_s^-(\theta) = \log\left[\left(1 + Y_{s+1}(\theta)\right)\left(1 + Y_{s-1}(\theta)\right)\right] \tag{2.77}$$

and we add a null contribution in the l.h.s,

$$m_s \cosh\left(\theta + i\frac{\pi}{2}\right) + m_s \cosh\left(\theta - i\frac{\pi}{2}\right) \tag{2.78}$$

so as to rewrite it in this way

$$\log\left(\frac{Y_s^+(\theta)}{e^{-m_s \cosh(\theta + i\frac{\pi}{2})}}\right) + \log\left(\frac{Y_s^-(\theta)}{e^{-m_s \cosh(\theta - i\frac{\pi}{2})}}\right) = l_s^+(\theta) + l_s^-(\theta) \tag{2.79}$$

iii. Then the equations become

$$l_s^+(\theta) + l_s^-(\theta) = \log\left[\left(1 + Y_{s+1}(\theta)\right)\left(1 + Y_{s-1}(\theta)\right)\right] \tag{2.80}$$

and we convolute them with the kernel $\mathcal{K}(\theta) = \frac{1}{2\pi \cosh\theta}$:

$$\begin{aligned} \mathcal{K}(\theta) * (l_s^+ + l_s^-) &\equiv \int_{-\infty}^{+\infty} d\theta' \, \mathcal{K}(\theta - \theta')(l_s^+ + l_s^-)(\theta') = \\ &= \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \frac{l_s(\theta' + i\frac{\pi}{2}) + l_s(\theta' - i\frac{\pi}{2})}{\cosh(\theta - \theta')} = \\ &= \mathcal{K}(\theta) * \log\left[\left(1 + Y_{s+1}\right)\left(1 + Y_{s-1}\right)\right] \end{aligned} \tag{2.81}$$

iv. We separate the convolution product into two addenda and, on each of them, we perform one of the following substitutions, that move the integration to the edges of the analytic strip:

$$\begin{cases} \theta' + i\frac{\pi}{2} \to \theta' & \text{in the first integral} \\ \theta' - i\frac{\pi}{2} \to \theta' & \text{in the second integral} \end{cases}$$

$$\mathcal{K}(\theta) * (l_s^+ + l_s^-) =$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \frac{l_s(\theta' + i\frac{\pi}{2})}{\cosh(\theta - \theta')} + \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \frac{l_s(\theta' - i\frac{\pi}{2})}{\cosh(\theta - \theta')} = \\
&= \int_{\mathbb{R}+i\frac{\pi}{2}} \frac{d\theta'}{2\pi} \frac{l_s(\theta')}{\cosh(\theta - \theta' + i\frac{\pi}{2})} + \int_{\mathbb{R}-i\frac{\pi}{2}} \frac{d\theta'}{2\pi} \frac{l_s(\theta')}{\cosh(\theta - \theta' - i\frac{\pi}{2})} = \\
&= \int_{\mathbb{R}+i\frac{\pi}{2}} \frac{d\theta'}{2\pi} \frac{l_s(\theta')}{i\sinh(\theta - \theta')} - \int_{\mathbb{R}-i\frac{\pi}{2}} \frac{d\theta'}{2\pi} \frac{l_s(\theta')}{i\sinh(\theta - \theta')} = \\
&= \oint_\gamma \frac{d\theta'}{2\pi i} \frac{l_s(\theta')}{\sinh(\theta' - \theta)} = l_s(\theta)
\end{aligned} \tag{2.82}$$

where γ is the (rectangular) integration contour consisting of the edges of the physical strip together with two vertical segments at $\Re\{\theta\} \to \pm\infty$. In order to be able to add these extra segments it was important to use the quantity l_s instead of $\log Y_s$: this is why we have introduced it. Furthermore, in the last step we used the fact that l_s has no singularities inside the physical strip[10].

Therefore, by rewriting the last equation in terms of $Y_s(\theta)$, we get the integral form of the Y-system (2.69):

$$l_s(\theta) = \log Y_s(\theta) + m_s \cosh\theta = \mathcal{K}(\theta) * \log\left[(1 + Y_{s+1})(1 + Y_{s-1})\right] \tag{2.83}$$

$$\log Y_s(\theta) = -m_s \cosh\theta + \mathcal{K}(\theta) * \log\left[(1 + Y_{s+1})(1 + Y_{s-1})\right] \tag{2.84}$$

$$\log Y_s(\theta) = -m_s \cosh\theta + \sum_{s'=s-1}^{s+1} \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta')} \log\left[1 + Y_{s'}(\theta')\right] \tag{2.85}$$

For a given choice of masses m_s (auxiliary parameters), the solution is unique and a basis of cross ratios can be read from evaluating the $Y_s(\theta)$'s at $\theta = 0$. These equations are the desired TBA-like integral equations that determine the result at strong coupling in the case that the polygon can be embedded in $R^{1,1}$ (and the surface in AdS_3), in the particular case in which the zeros of the polynomial are along the real axis.

[10]This is an important input on the analytic properties of the Y-functions.

2.2.1 General complex case

When we move the zeroes away from the real axis, the functional Y-system equations continue to be true because they do not depend on the polynomial; contrarily, the asymptotic boundary conditions change: now, the quantities Z_s and m_s are more general complex numbers and the asymptotic behaviour is

$$\begin{cases} m_s = -2Z_s, & \log Y_s \sim -\frac{m_s}{2\zeta} \quad \text{for } \zeta \to 0 \\ \bar{m}_s = -2\bar{Z}_s, & \log Y_s \sim -\frac{\bar{m}_s}{2}\zeta \quad \text{for } \zeta \to +\infty \end{cases} \tag{2.86}$$

Furthermore, when we derive the integral equations, it is convenient to shift the line where the Y-functions are integrated to be along the direction where $\frac{m_s}{\zeta}$ is real and positive, which also makes $\bar{m}_s\zeta$ real and positive. We have[11],

$$m_s = |m_s|e^{i\varphi_s} \tag{2.87}$$

$$\frac{m_s}{\zeta} = \frac{|m_s|e^{i\varphi_s}}{e^{\Re\{\theta\}+i\Im\{\theta\}}} \in \mathbb{R}_+ \tag{2.88}$$

$$\sin\left(\varphi_s - \Im\{\theta\}\right) = 0 \quad \longrightarrow \quad \varphi_s = \Im\{\theta\} + k\pi \qquad k = 0, 1, 2, ...$$

$$\cos\left(\varphi_s - \Im\{\theta\}\right) > 0 \quad \longrightarrow \quad -\frac{\pi}{2} < \varphi_s - \Im\{\theta\} < \frac{\pi}{2}$$

It will be useful to define

$$\tilde{Y}_s(\theta) = Y_s(\theta + i\varphi_s) \tag{2.89}$$

where here θ is real. Like before, we start from equations (2.69), but we introduce the quantity l_s in a slightly different way,

$$l_s(\theta + i\varphi_s) = \tilde{l}_s(\theta) \equiv \log\left(\frac{Y_s(\theta + i\varphi_s)}{e^{-|m_s|\cosh\theta}}\right) = \log\tilde{Y}_s(\theta) + |m_s|\cosh\theta \tag{2.90}$$

Consequently, when we compute the behaviour for large θ, we have:

a) Small ζ behavior: $\quad l_s \sim -\frac{m_s}{2}e^{-\theta - i\varphi_s} + |m_s|\frac{e^{-\theta}}{2} = 0$

b) Large ζ behavior: $\quad l_s \sim -\frac{\bar{m}_s}{2}e^{\theta + i\varphi_s} + |\bar{m}_s|\frac{e^{\theta}}{2} = 0$

Now we take the logarithm,

$$\log\tilde{Y}_s^+(\theta) + \log\tilde{Y}_s^-(\theta) = \log\left[\left(1 + \tilde{Y}_{s+1}(\theta)\right)\left(1 + \tilde{Y}_{s-1}(\theta)\right)\right] \tag{2.91}$$

[11] First, let's assume $\zeta = e^\theta$ with θ generically complex. This allow us to understand that $\Re\{\theta\}$ must be translate by the argument of m_s. Then we can replace $\Re\{\theta\} \to \theta$.

and we add a null contribution to the l.h.s so that we can write

$$\log \tilde{Y}_s^+(\theta) + \log \tilde{Y}_s^-(\theta) + |m_s| \cosh\left(\theta + i\frac{\pi}{2}\right) + |m_s| \cosh\left(\theta - i\frac{\pi}{2}\right) =$$

$$= \log\left(\frac{\tilde{Y}_s^+(\theta)}{e^{-|m_s|\cosh(\theta+i\frac{\pi}{2})}}\right) + \log\left(\frac{\tilde{Y}_s^-(\theta)}{e^{-|m_s|\cosh(\theta-i\frac{\pi}{2})}}\right) =$$

$$= \tilde{l}_s^+(\theta) + \tilde{l}_s^-(\theta) = \log\left[\left(1+\tilde{Y}_{s+1}(\theta)\right)\left(1+\tilde{Y}_{s-1}(\theta)\right)\right] \tag{2.92}$$

The next step is to convolute with the kernel $\mathcal{K}(\theta + i\varphi_s) = \frac{1}{2\pi\cosh(\theta+i\varphi_s)}$:

$$\begin{aligned}
\mathcal{K}(\theta+i\varphi_s) * (l_s^+ + l_s^-) &\equiv \int_{\mathbb{R}+i\alpha_s} d\theta' \, \mathcal{K}(\theta + i\varphi_s - \theta')(l_s^+ + l_s^-)(\theta') = \\
&= \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{l_s(\theta' + i\alpha_s + i\frac{\pi}{2}) + l_s(\theta' + i\alpha_s - i\frac{\pi}{2})}{\cosh(\theta + i\varphi_s - \theta' - i\alpha_s)} = \\
&= \oint_\gamma \frac{d\theta'}{2\pi i} \frac{l_s(\theta' + i\alpha_s)}{\sinh(\theta' + i\alpha_s - \theta - i\varphi_s)} = \\
&= \tilde{l}_s(\theta) = \log \tilde{Y}_s(\theta) + |m_s|\cosh\theta
\end{aligned} \tag{2.93}$$

where γ is the (rectangular) integration contour created by two different shifts in the variable θ' together with two vertical segments at $\Re\{\theta\} \to \pm\infty$. Note that in order to write the second line, it must be valid that $|\varphi_s - \alpha_s| < \frac{\pi}{2}$. Thus, we can rewrite the above convolution in the following form

$$\begin{aligned}
\log \tilde{Y}_s(\theta) + |m_s|\cosh\theta &= \mathcal{K}(\theta + i\varphi_s) * \log\left[(1+Y_{s+1})(1+Y_{s-1})\right] = \\
&= \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{\log\left[1 + Y_{s'}(\theta')\right]}{\cosh(\theta + i\varphi_s - \theta')}
\end{aligned} \tag{2.94}$$

or equivalently,

$$\ln \tilde{Y}_s(\theta) = -|m_s|\cosh\theta + \sum_{s'=s-1}^{s+1} \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \frac{\log\left[1 + Y_{s'}(\theta' + i\varphi_{s'})\right]}{\cosh(\theta - \theta' + i\varphi_s - i\varphi_{s'})} \tag{2.95}$$

Finally, we find the TBA-like integral equations for the general complex case[12]:

$$\log \tilde{Y}_s(\theta) = -|m_s|\cosh\theta + \sum_{s'=s-1}^{s+1} \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \frac{\log\left[1 + \tilde{Y}_{s'}(\theta')\right]}{\cosh(\theta - \theta' + i\varphi_s - i\varphi_{s'})} \tag{2.96}$$

[12]These equations coincide with [8, eq. (26)] and make contact with those of [11]. See comments at the end of [8, sec 3.5] and subsequent sections of this document.

As long as $|\varphi_s - \varphi_{s'}| < \frac{\pi}{2}$ the integral equations conserve the form that we have derived. If we deform the phases beyond this regime, we will have to change the form of the equations [8, App. B] by picking the appropriate pole contributions from the kernels, which become singular for $|\varphi_s - \varphi_{s'}| = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. Of course, the integral equations change but the Y's (and therefore the area) are continuous thanks to the analytic extension that takes into account the pole[13]. This is the *wall crossing phenomenon* discussed in [11, 12]. These integral equations are a special case of the general one discussed in [11], where the equations are true for an arbitrary $\mathcal{N} = 2$ theory (our Hitchin problem is just a special case). Due to the $\mathbb{Z}_2$ projection, we can easily map the kernel in [11] to the one found here; we report the calculation in appendix A and section 2.7.

2.3 Rewriting of the TBA

The integral equations (2.96) can be rewritten in a slightly different form if we consider the complex redefinition of the variable θ and the antisymmetric intersection form $\theta^{ss'}$ of the cycles involved to determine the asymptotic behaviour of Y_s[14]. The latter follows from the general theory in [11], but it can be easily checked in this particular case by examining the integral equations (2.96) for different values of s. We refer to subsequent sections for an exhaustive check. As a consequence of the redefinition, the integration path moves along a straight line, parallel to the real axis and with an imaginary part equals to $\varphi_{s'}$. If we set,

$$\begin{cases} \theta + i\varphi_s & \longrightarrow \quad \theta \\ \theta' + i\varphi_{s'} & \longrightarrow \quad \theta' \end{cases} \tag{2.97}$$

then

$$\log Y_s(\theta) = -|m_s| \cosh(\theta - i\varphi_s) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + Y_{s'}(\theta')\right]}{\cosh(\theta - \theta')} \tag{2.98}$$

According to the redefinition, it continues to be true that $|\varphi_s - \varphi_{s'}| < \frac{\pi}{2}$. It should be noted that the previous rewrite recovers the original Y-functions and represents the 3-dimensional analogue of equations (3.1)-(3.3) of [39]. In fact, we can make another small change in the notation by introducing the kernel

$$\mathcal{K}_{ss'}(\theta, \theta') = \frac{(-1)^{s+1}\,\theta^{ss'}}{2\pi \cosh(\theta - \theta')} \tag{2.99}$$

[13] Actually, there are two types of analytic extension: the one that takes into account the pole, called in the literature "going smoothly", and the one that doesn't take it into account

[14] The intersection form $\theta^{ss'}$ for all the cycles associated to the Y-functions is computed in [8]: considering s even, if an arrow points from $\hat{Y}_s$ to $\hat{Y}_{s'}$, then we have $\theta^{ss'} = \langle \gamma_s, \gamma_{s'} \rangle = -1$, otherwise the intersection vanishes. Note that this convention is opposite to that depicted in [8].

and the quantity

$$\mathcal{L}_s(\theta) = \log\left[1 + Y_s(\theta)\right] \tag{2.100}$$

finally coming to write

$$\log Y_s(\theta) = -|m_s|\cosh(\theta - i\varphi_s) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} d\theta'\, \mathcal{K}_{ss'}(\theta,\theta')\, \mathcal{L}_{s'}(\theta') \tag{2.101}$$

At this point we can follow two equivalent ways, each of which leads, through different notations, to useful developments:

***route a*)** we can introduce the *pseudo-energies* $\varepsilon(\theta - i\varphi_s)$ and recast the TBA equations as function of them, separating the forcing term into two exponentials multiplied by the Gaiotto's Z. In doing so, we move the integration at $\hat{\varphi}_{s'} = \mathbb{R} + \varphi_{s'} + i\frac{\pi}{2}$ and we point out a clear link with the $\frac{SU(N)_2}{[U(1)]^{N-1}}$ HSG models [13];

***route b*)** alternatively, we can define the *hatted Y-functions* $\hat{Y}_s(\theta)$ and, in a second time, change the integration line according to *route a*. In doing so, the forcing term remains a cosh and the relationships between $\hat{Y}$ and ε guarantee the equivalence of the two paths. We will follow this way especially in the last part of this chapter when, starting from section 2.4, we are going to calculate the regularized area.

2.3.1 *Route a*: the pseudo-energies

The pseudo-energies (with real argument) are defined by the relation

$$-\ln Y_s(\theta) = \varepsilon(\theta - i\varphi_s) \tag{2.102}$$

which allows us to rewrite (2.98) in the following form

$$\varepsilon(\theta - i\varphi_s) = |m_s|\cosh(\theta - i\varphi_s) - \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + e^{-\varepsilon(\theta' - i\varphi_{s'})}\right]}{\cosh(\theta - \theta')} \tag{2.103}$$

Now, we can modify the arguments of the pseudo-energies by following three steps:

i) $\varepsilon(\theta' - i\varphi_{s'}) = \varepsilon(\theta' + i\frac{\pi}{2} - i\hat{\varphi}_{s'})$ where $\hat{\varphi}_{s'} = \varphi_{s'} + i\frac{\pi}{2}$

ii) $\theta' + i\frac{\pi}{2} \longrightarrow \theta'_+$ that implies

$$\varepsilon(\theta' + i\frac{\pi}{2} - i\hat{\varphi}_{s'}) \longrightarrow \varepsilon(\theta'_+ - i\hat{\varphi}_{s'})$$

$$\mathbb{R} + i\varphi_{s'} \longrightarrow \mathbb{R} + i\hat{\varphi}_{s'}$$

$$\cosh(\theta - \theta') \longrightarrow \cosh(\theta - \theta'_+ + i\frac{\pi}{2})$$

iii) $\theta + i\frac{\pi}{2} \longrightarrow \theta_+$ that implies

$$\cosh(\theta - \theta'_+ + i\frac{\pi}{2}) \longrightarrow \cosh(\theta_+ - \theta'_+)$$

$$\cosh(\theta - i\varphi_s) \longrightarrow \cosh(\theta_+ - i\hat{\varphi}_s)$$

$$\varepsilon(\theta - i\varphi_s) \longrightarrow \varepsilon(\theta_+ - i\hat{\varphi}_s)$$

In the end, taking into account the equality $|m_s| \equiv 2|Z_s|$ and omitting the subscript $+$ to lighten the notation, we get

$$\varepsilon(\theta - i\hat{\varphi}_s) = 2|Z_s|\cosh(\theta - i\hat{\varphi}_s) - \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\hat{\varphi}_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + e^{-\varepsilon(\theta' - i\hat{\varphi}_{s'})}\right]}{\cosh(\theta - \theta')} \tag{2.104}$$

Even without the subscript $+$, one can easily deduce who the complex variable θ is by noting that the argument of the pseudo-energies must be real. The previous shifts in the rapidity variables allow us to decompose the forcing term into two addenda

$$2|Z_s|\cosh(\theta - i\hat{\varphi}_s) = -\frac{Z_s}{e^{\theta - i\frac{\pi}{2}b_s}} - \bar{Z}_s e^{\theta - i\frac{\pi}{2}b_s} \tag{2.105}$$

where

$$Z_s = -|Z_s|e^{i\alpha_s} \quad \text{and} \quad \alpha_s = \begin{cases} \varphi_s & \text{for } s \text{ even} \\ \varphi_s + \frac{\pi}{2} & \text{for } s \text{ odd} \end{cases} \tag{2.106}$$

allow us to make contact with the equations of [13][15]. Starting from (2.98), *route a* leads then to the following equation

$$\varepsilon(\theta - i\hat{\varphi}_s) = -\frac{Z_s}{e^{\theta - i\frac{\pi}{2}b_s}} - \bar{Z}_s\, e^{\theta - i\frac{\pi}{2}b_s} - \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\hat{\varphi}_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + e^{-\varepsilon(\theta' - i\hat{\varphi}_{s'})}\right]}{\cosh(\theta - \theta')} \tag{2.107}$$

where the variable $\theta \equiv \theta_+$ is defined by means of the previous shifts. It is important to note that the above equation is the 3-dimensional reduction of equations (F.10)-(F.11) of [37, app. F], which represent the starting point for the identification procedure $A_{free} = YY_{cr}$ in the AdS_5 case[16]. This means that, starting from (2.107), we can reproduce the procedure developed in [37, app. F] for the 3-dimensional case. Moreover, it also means that what was done in [37, app. E], where we started from $\sinh^{-1}$-type kernels, can be reproduce right here on $\cosh^{-1}$-type kernels: there is a connection between (2.107) and [37, eq. (E.2)], which we will point out soon in section 2.7.

[15] See section 2.7 for explicit details.

[16] In both [37, app. F] and [39], we see that the equations for the hexagon coincide with the ones in [7], which correspond to the A_3 series. We will explore the link with the A_n series just for this reason in section 3.4. also for the AdS_3 case

2.3.2 *Route b*: the hatted Y-functions

The equations (2.98) depends on the two sets of constants $|m_s|$ and φ_s, which eventually will be related to the $\frac{n}{2}-3$ conformal ratios of a polygonal Wilson loop with n null edges. This is directly related to the fact that the number of Y-functions is equal to the number of cross ratios. For the continuation, it turns out convenient to introduce the *hatted Y-functions* $\hat{Y}_s(\theta)$,

$$\begin{cases} \hat{Y}_{2k}(\theta) = Y_{2k}(\theta) \\ \hat{Y}_{2k+1}(\theta) = Y_{2k+1}\left(\theta - i\frac{\pi}{2}\right) \end{cases} \tag{2.108}$$

which, thanks to the quantity

$$b_s = \frac{1+(-1)^s}{2}$$

can be defined in a more compact way

$$\hat{Y}_s(\theta) = Y_s\left(\theta - i\frac{\pi}{2}b_{s+1}\right) \tag{2.109}$$

As well as in [8], [37, app. F] or [39], the physical cross ratios can be calculated directly from the $\hat{Y}_s(\theta)$'s by placing $\theta = 0$. We define the pseudo-energies like in (2.102),

$$\varepsilon(\theta - i\varphi_s) = -\ln Y_s(\theta) \tag{2.110}$$

although it will turn out useful to express the ε-functions also in terms of the hatted-Y's,

$$\begin{aligned} \varepsilon(\theta - i\hat{\varphi}_s) &= -\ln Y_s(\theta - i\frac{\pi}{2}) = \\ &= \begin{cases} -\ln \hat{Y}_s(\theta - i\frac{\pi}{2}) & \text{if } s \text{ even} \\ -\ln \hat{Y}_s(\theta) & \text{if } s \text{ odd} \end{cases} = \\ &= -\ln \hat{Y}_s(\theta - i\frac{\pi}{2}b_s) \end{aligned} \tag{2.111}$$

Now, we would like to rewrite (2.98) using these new $\hat{Y}$'s. Thus, our starting point is:

$$\log Y_s(\theta) = -|m_s|\cosh(\theta - i\varphi_s) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}\log\left[1+Y_{s'}(\theta')\right]}{\cosh(\theta-\theta')} \tag{2.112}$$

We must consider that, if $\theta_{s+1} \equiv \theta + i\frac{\pi}{2}b_{s+1}$,

$$\begin{aligned} \log Y_s(\theta) = \log Y_s(\theta_{s+1} - i\frac{\pi}{2}b_{s+1}) = \log \hat{Y}_s(\theta_{s+1}) = \\ = \begin{cases} \log \hat{Y}_s(\theta) & \text{if } s \text{ even} \\ \log \hat{Y}_s(\theta + i\frac{\pi}{2}) & \text{if } s \text{ odd} \end{cases} \end{aligned} \tag{2.113}$$

so that we can also write

$$Y_{s'}(\theta') = \hat{Y}_{s'}(\theta'_{s'+1}) \quad \text{with} \quad \theta'_{s'+1} \equiv \theta' + i\frac{\pi}{2}b_{s'+1} = \theta' + i\frac{\pi}{2}b_s \equiv \theta'_s$$

because s and s' have opposite parity. Considering that the integration path change in $\mathbb{R} + i\bar{\varphi}_{s'} := \mathbb{R} + i\varphi_{s'} + i\frac{\pi}{2}b_{s'+1}$, we can recast the TBA equations (2.98) in the particular form

$$\log \hat{Y}_s(\theta_{s+1}) = -|m_s|\cosh(\theta_{s+1} - i\bar{\varphi}_s) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\bar{\varphi}_{s'}} \frac{d\theta'_s}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + \hat{Y}_{s'}(\theta'_s)\right]}{\cosh(\theta_{s+1} - i\frac{\pi}{2}b_{s+1} - \theta'_s + i\frac{\pi}{2}b_s)} \tag{2.114}$$

where

$$i\bar{\varphi}_{s'} = \begin{cases} i\varphi_{s'} + i\frac{\pi}{2} = i\hat{\varphi}_{s'} & \text{if } s' \text{ odd} \\ i\varphi_{s'} & \text{if } s' \text{ even} \end{cases} \tag{2.115}$$

The previous equation can be rewritten in a more compact form

$$\log \hat{Y}_s(\theta + i\frac{\pi}{2}b_{s+1}) = -|m_s|\cosh(\theta - i\varphi_s) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + \hat{Y}_{s'}(\theta' + i\frac{\pi}{2}b_{s'+1})\right]}{\cosh(\theta - \theta')} \tag{2.116}$$

As anticipated, the above equation coincides with (2.107) and relations (2.111) can be used to prove it.

2.4 The remainder function

As we have seen in section 2.1, in order to compute scattering amplitudes at strong coupling, we need to compute the area of minimal surfaces in AdS given by

$$A = 2\int d^2z\, Tr\left[\Phi_z \Phi_{\bar{z}}\right] \tag{2.117}$$

Since for solutions relevant to scattering amplitudes the relation (2.20) implies

$$Tr\left[\Phi_z \Phi_{\bar{z}}\right] \sim \sqrt{p\bar{p}} \tag{2.118}$$

this area diverges and needs to be regularized. This can be done by dividing it in different contributions [6, 7]. For the simpler odd case (n = odd for a general $2n$-gon), we have

$$A = A_{reg} + A_{periods} + A_{cutoff} \tag{2.119}$$

where

$$A_{reg} = 2\int d^2z \left(Tr\left[\Phi_z \Phi_{\bar{z}}\right] - 2\sqrt{p\bar{p}}\right) \tag{2.120}$$

$$A_{periods} = 4\int d^2z\,\sqrt{p\bar{p}} - 4\int_{\Sigma_0, z_{AdS}>\epsilon} d^2\omega = 4\int_{\Sigma} d^2\omega - A_{cutoff} \tag{2.121}$$

where we have defined the ω-plane by $d\omega = \sqrt{p(z)}dz$ and Σ denotes the region in this plane related to the regulator. As we will see in this section, A_{reg} is the non-trivial dynamic function that is computed by the free-energy of the TBA system. Therefore, the interesting part of the area is given by the integral equivalent to the two pieces $A_{reg} + A_{free}$ [8]

$$A = 2\int d^2z\, Tr\left[\Phi_z \Phi_{\bar{z}}\right] \tag{2.122}$$

which, by definition, is independent of ζ. Improving the WKB approximation we talked about in section 2.1.2, we can find other terms that describe the behaviour of the Y functions in the two regimes considered by expanding the expressions for the inner products. We will take complex masses but with small enough phases so that the WKB approximations that we find in [8] continue to be valid, with the same cycles. In general, the cross ratios that have a simple WKB approximation will change as we change the phase of the masses beyond certain point, see [12]. As in [8], for our purposes it is enough to do the derivation for some range of masses and then consider the analytic continuation as mentioned before.

2.4.1 Small ζ regime

Thanks to the WKB approximation described in [8] and using slightly different functions defined by (here $\zeta = e^{\theta}$)

$$\hat{Y}_{2k}(\zeta) = Y_{2k}(\zeta) \qquad \hat{Y}_{2k+1}(\zeta) = Y_{2k+1}(\zeta e^{-i\frac{\pi}{2}})$$

we find that for $\theta \to -\infty$ holds

$$\log \hat{Y}_k(\zeta = e^{\theta + i\frac{\pi}{2}b_{k+1}}) \sim -\left[\frac{1}{\zeta}\oint_{\gamma_k}\lambda + \oint_{\gamma_k}\alpha + \zeta\oint_{\gamma_k} u + ...\right] \tag{2.123}$$

where $\lambda = \sqrt{p(z)}dz$ and u is an exact one form on the Riemann surface with component u_z and $u_{\bar{z}}$. For our purposes, it will only be important to compute the diagonal component $u^i_{\bar{z}} = \Phi^{ii}_{\bar{z}}$. Here α is given by the diagonal components of the connection A and, due to the Z_2 projection valid in our case, is equal to zero. In the basis where Φ_z is diagonal, we can rewrite (2.122) as

$$A = i\int \lambda \wedge u = -i\sum_{r,s}\omega_{rs}\oint_{\gamma^r}\lambda\oint_{\gamma^s} u \tag{2.124}$$

where γ^r are a basis of cycles and ω_{rs} is the inverse of the intersection form of the cycles [8]. By expanding the TBA-like integral equations (2.116), we can compute the small

ζ-behavior of $\hat{Y}_k$. Let us proceed step by step: first we rewrite our starting point,

$$\log \hat{Y}_s(\theta+i\frac{\pi}{2}b_{s+1}) = -|m_s|\cosh(\theta-i\varphi_s)+\sum_{s'=s-1}^{s+1}\int_{\mathbb{R}+i\varphi_{s'}}\frac{d\theta'}{2\pi}\frac{(-1)^{b_s}\theta^{ss'}\log\left[1+\hat{Y}_{s'}(\theta'+i\frac{\pi}{2}b_{s'+1})\right]}{\cosh(\theta-\theta')} \tag{2.125}$$

inside which we can find the conserved charges of $\log\hat{Y}_s$,

$$\log \hat{Y}_s(\theta+i\frac{\pi}{2}b_{s+1}) = \log\hat{Y}_s(\theta_{s+1}) \sim \sum_{n=-1}^{+\infty}\tilde{c}_{n,s}\,e^{n\theta_{s+1}} = \tilde{c}_{-1,s}\,e^{-\theta_{s+1}}+\tilde{c}_{1,s}\,e^{\theta_{s+1}}+\ldots =$$

$$= \frac{\tilde{c}_{-1,s}}{e^{\theta+i\frac{\pi}{2}b_{s+1}}}+\tilde{c}_{1,s}\,e^{\theta+i\frac{\pi}{2}b_{s+1}}+\ldots \tag{2.126}$$

We find out that

$$\tilde{c}_{-1,s} = -\frac{m_s}{2}(i)^{b_{s+1}} = Z_s \tag{2.127}$$

$$\tilde{c}_{1,s} = -\frac{\bar{m}_s}{2}(-i)^{b_{s+1}}+\sum_{s'}\int\frac{d\theta'}{\pi}\frac{e^{-i\frac{\pi}{2}b_{s+1}}(-1)^{b_s}\theta^{ss'}\log\left[1+\hat{Y}_{s'}\left(\theta'+i\frac{\pi}{2}b_{s'+1}\right)\right]}{e^{\theta'}} =$$

$$= \bar{Z}_s+\frac{(-i)^{b_{s+1}}}{\pi}\sum_{s'}(-1)^{b_s}\int\frac{d\zeta'}{\zeta'}\frac{\theta^{ss'}}{\zeta'}\log\left[1+\hat{Y}_{s'}\left(\theta'+i\frac{\pi}{2}b_{s'+1}\right)\right] \tag{2.128}$$

and therefore

$$\log\hat{Y}_s(\theta+i\frac{\pi}{2}b_{s+1}) \sim \frac{Z_s}{e^{\theta+i\frac{\pi}{2}b_{s+1}}}+$$

$$+\,e^{\theta+i\frac{\pi}{2}b_{s+1}}\left\{\bar{Z}_s+\frac{(-i)^{b_{s+1}}}{\pi}\sum_{s'}(-1)^{b_s}\int\frac{d\zeta'}{\zeta'}\frac{\theta^{ss'}}{\zeta'}\log\left[1+\hat{Y}_{s'}\left(\theta'+i\frac{\pi}{2}b_{s'+1}\right)\right]\right\} \tag{2.129}$$

where

$$\bar{Z}_s = -\frac{\bar{m}_s}{2}(-i)^{b_{s+1}} = \begin{cases} -\frac{\bar{m}_s}{2} & \text{if } s \text{ even} \\ i\frac{\bar{m}_s}{2} & \text{if } s \text{ odd}\end{cases}$$

and the integration is carried out on $\mathbb{R}+i\varphi_{s'}$. Comparing expansions (2.123) and (2.126), we can find the formula for the contributions $A_{periods}+A_{free}$ in the small ζ-regime ($s=k$),

$$\begin{cases} -\oint_{\gamma_s}\lambda &= \tilde{c}_{-1,s} \\ -\oint_{\gamma_s}u &= \tilde{c}_{1,s}\end{cases} \tag{2.130}$$

$$A = -i\sum_{r,s}\omega_{rs}\oint_{\gamma^r}\lambda\oint_{\gamma^s}u = -i\sum_{r,s}\omega_{rs}\,\tilde{c}_{-1,r}\,\tilde{c}_{1,s} \tag{2.131}$$

Instead, comparing expansions (2.123) and (2.129), we can separately rewrite the contributions in a convenient form to highlight the free energy structure of A_{free},

$$\left\{ \begin{array}{lcl} -\oint_{\gamma^s} \lambda & = & Z^s \\ -\oint_{\gamma^s} u & = & \bar{Z}^s + \frac{(-i)^{b_{s+1}}}{\pi} \sum_{s'} (-1)^{b_s} \int \frac{d\zeta'}{\zeta'} \frac{\theta^{ss'}}{\zeta'} \log\left[1 + \hat{Y}_{s'}\left(\theta' + i\frac{\pi}{2} b_{s'+1}\right)\right] \end{array} \right. \tag{2.132}$$

$$A = A_{periods} + A^{small}_{free} =$$

$$= -i \sum_{r,s} \omega_{rs}\, Z^r \left\{ \bar{Z}_s + \frac{(-i)^{b_{s+1}}}{\pi} \sum_{s'} (-1)^{b_s} \int \frac{d\zeta'}{\zeta'} \frac{\theta^{ss'}}{\zeta'} \log\left[1 + \hat{Y}_{s'}\left(\theta' + i\frac{\pi}{2} b_{s'+1}\right)\right] \right\} =$$

$$= -i\, \omega_{rs}\, Z^r\, \bar{Z}^s + \frac{(-i)^{b_{s+1}+1}(-1)^{b_s}}{\pi} \sum_r \int \frac{d\zeta'}{\zeta'} \frac{Z^r}{\zeta'} \log\left[1 + \hat{Y}_r\left(\theta' + i\frac{\pi}{2} b_{r+1}\right)\right] \tag{2.133}$$

Thus, we obtain

$$A_{periods} = -i\, \omega_{rs}\, Z^r\, \bar{Z}^s \tag{2.134}$$

$$A^{small}_{free} = \frac{(-i)^{b_{s+1}+1}(-1)^{b_s}}{\pi} \sum_r \int \frac{d\zeta'}{\zeta'} \frac{Z^r}{\zeta'} \log\left[1 + \hat{Y}_r(\theta' + i\frac{\pi}{2} b_{r+1})\right] \tag{2.135}$$

To reach the structure of free energy, we have to average the result from (2.135) with the result we obtain from the large ζ-expansion.

2.4.2 Large ζ regime

We can retrace the steps (2.123)-(2.135) for $\theta \to +\infty$,

$$\log \hat{Y}_k \;\sim\; -\left[\zeta \oint_{\gamma_k} \lambda + \oint_{\gamma_k} \alpha + \frac{1}{\zeta} \oint_{\gamma_k} u + \ldots\right] \quad \text{with} \quad \zeta = e^{\theta + i\frac{\pi}{2} b_{k+1}} \tag{2.136}$$

$$A = i \sum_{r,s} \omega_{rs} \oint_{\gamma^r} \lambda \oint_{\gamma^s} u = i \sum_{r,s} \omega_{rs}\, c_{-1,r}\, c_{1,s} \tag{2.137}$$

$$\log \hat{Y}_s(\theta + i\frac{\pi}{2} b_{s+1}) = \log \hat{Y}_s(\theta_{s+1}) \sim \sum_{n=-1}^{+\infty} c_{n,s}\, e^{-n\theta_{s+1}} \tag{2.138}$$

$$c_{-1,s} = -\frac{\bar{m}_s}{2}(-i)^{b_{s+1}} = \bar{Z}_s \tag{2.139}$$

$$c_{1,s} = Z_s + \frac{(i)^{b_{s+1}}}{\pi} \sum_{s'} (-1)^{b_s} \int \frac{d\zeta'}{\zeta'} \frac{\theta^{ss'}}{(\zeta')^{-1}} \log\left[1 + \hat{Y}_{s'}\left(\theta' + i\frac{\pi}{2} b_{s'+1}\right)\right] \tag{2.140}$$

$$\log \hat{Y}_s(\theta + i\frac{\pi}{2} b_{s+1}) \sim e^{\theta + i\frac{\pi}{2} b_{s+1}}\, \bar{Z}^s +$$

$$+\frac{1}{e^{\theta+i\frac{\pi}{2}b_{s+1}}}\left[Z^s+\frac{(i)^{b_{s+1}}}{\pi}\sum_{s'}(-1)^{b_s}\int\frac{d\zeta'}{\zeta'}\,\theta^{ss'}\zeta'\log\left[1+\hat{Y}_{s'}(\theta'+i\frac{\pi}{2}b_{s'+1})\right]\right] \tag{2.141}$$

$$A=A_{periods}+A^{large}_{free}=$$

$$=-i\,\omega_{rs}\,Z^r\,\bar{Z}^s+\frac{(i)^{b_{s+1}+1}(-1)^{b_s}}{\pi}\sum_r\int\frac{d\zeta'}{\zeta'}\,\bar{Z}^r\zeta'\log\left[1+\hat{Y}_r(\theta'+i\frac{\pi}{2}b_{r+1})\right] \tag{2.142}$$

2.4.3 Averaged results: the free energy structure

If we set,

$$Z_r=-\frac{m_r}{2}(i)^{b_{r+1}}\qquad \bar{Z}_r=-\frac{\bar{m}_r}{2}(-i)^{b_{r+1}}$$

$$m_r\equiv|m_r|e^{i\varphi_r}\qquad \bar{m}_r\equiv|m_r|e^{-i\varphi_r}$$

$$\tilde{Y}_r(\theta)=Y_r(\theta+i\varphi_r)$$

we can rewrite A^{small}_{free} and A^{large}_{free} in the form

$$A^{small}_{free}=\sum_r\int_{\mathbb{R}}\frac{d\theta'}{2\pi}|m_r|e^{-\theta'}\log\left[1+Y_r(\theta'+i\varphi_r)\right] \tag{2.143}$$

$$A^{large}_{free}=\sum_r\int_{\mathbb{R}}\frac{d\theta'}{2\pi}|m_r|e^{\theta'}\log\left[1+Y_r(\theta'+i\varphi_r)\right] \tag{2.144}$$

through the following steps: let's replace Z_r first,

$$A^{small}_{free}=\frac{(-i)^{b_{s+1}+1}(-1)^{b_s}}{\pi}\sum_r\int_{\mathbb{R}+i\varphi_r}d\theta'\left(-\frac{m_r}{2}(i)^{b_{r+1}}\right)e^{-\theta'}\log\left[1+\hat{Y}_r(\theta'+i\frac{\pi}{2}b_{r+1})\right]=$$

$$=\frac{(-1)^{b_{s+1}+2+b_s}(i)^{b_{s+1}+1+b_{r+1}}}{\pi}\sum_r\int_{\mathbb{R}}d\theta'\frac{|m_r|}{2}e^{-\theta'}\log\left[1+\hat{Y}_r(\theta'+i\varphi_r+i\frac{\pi}{2}b_{r+1})\right] \tag{2.145}$$

and, considering that r and s have opposite parity, we get

$$(-1)^{b_{s+1}+2+b_s}(i)^{b_{s+1}+1+b_{r+1}}=1$$

$$A^{small}_{free}=\sum_r\int_{\mathbb{R}}\frac{d\theta'}{2\pi}|m_r|e^{-\theta'}\log\left[1+Y_r(\theta'+i\varphi_r)\right]$$

If we repeat very similar steps for A^{large}_{free}, we will finally get

$$A_{free}=\frac{A^{small}_{free}+A^{large}_{free}}{2}=\sum_r\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}|m_r|\cosh\theta'\log\left[1+\tilde{Y}_r(\theta')\right] \tag{2.146}$$

This derivation requires that $\frac{n}{2}$ is odd because we have considered θ^{rs} invertible. If $\frac{n}{2}$ is even, like in the octagon case, we start from $\frac{n}{2}+1$ and we take away one zero of the polynomial (soft collinear limit). Then, the result contains two pieces: one is A_{free} discussed above and the other contains an extra piece called precisely A_{extra} [6]. Finally, considering that it is possible to obtain an explicit writing of $A_{periods}$ in terms of the masses, each contribution to the regularized area is expressed in terms of the complex masses m_s that appear in the integral equations (2.98).

2.5 The octagon

Here we derived some of the results of [6] from our point of view, highlighting what important differences occur in cases where $\frac{n}{2}$ is even. For the octagon we have $n=8$ and only one Y-function, because s takes values up to $\frac{n}{2}-3=1$; so that, the Y-system (2.69) is very simple

$$Y_s^+ Y_s^- = 1 \tag{2.147}$$

We introduce the quantity l_s in the usual way,

$$l_s(\theta) \equiv \log\left(\frac{Y_s(\theta)}{e^{-|m_s|\cosh\theta}}\right) = \log Y_s(\theta) + |m_s|\cosh\theta \tag{2.148}$$

and, after the introduction of a null contribution, we can write,

$$\log\left(\frac{\tilde{Y}_s^+(\theta)}{e^{-|m_s|\cosh(\theta+i\frac{\pi}{2})}}\right) + \log\left(\frac{\tilde{Y}_s^-(\theta)}{e^{-|m_s|\cosh(\theta-i\frac{\pi}{2})}}\right) = \tilde{l}_s^+(\theta) + \tilde{l}_s^-(\theta) = 0 \tag{2.149}$$

The subsequent convolution with the kernel $\mathcal{K}(\theta)$ produces

$$\mathcal{K}(\theta) * (\tilde{l}_s^+ + \tilde{l}_s^-) =$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\left[\tilde{l}_s^+(\theta') + \tilde{l}_s^-(\theta')\right] = \\
&= \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi}\frac{\tilde{l}_s(\theta'+i\frac{\pi}{2}) + \tilde{l}_s(\theta'-i\frac{\pi}{2})}{\cosh(\theta-\theta')} = \\
&= \oint_\gamma \frac{d\theta'}{2\pi i}\frac{\tilde{l}_s(\theta')}{\sinh(\theta'-\theta)} = \\
&= \tilde{l}_s(\theta) = \log\tilde{Y}_s(\theta) + |m_s|\cosh\theta
\end{aligned} \tag{2.150}$$

Finally, we reach the following integral equation for the Y-system

$$\log\tilde{Y}_s(\theta) + |m_s|\cosh\theta = 0 \tag{2.151}$$

which solution is

$$\begin{aligned}\log Y_s(\theta) &= -\frac{|m_s|}{2}e^{-i\varphi_s+\theta} - \frac{|m_s|}{2}e^{i\varphi_s-\theta} = \\ &= -\frac{\bar{m}_s}{2}e^{\theta} - \frac{m_s}{2}e^{-\theta} = \\ &= \frac{Z_s}{\zeta} + \zeta\bar{Z}_s \quad \longrightarrow \quad Y_s(\theta) = e^{\frac{Z_s}{\zeta}+\zeta\bar{Z}_s}\end{aligned} \tag{2.152}$$

In order to reach the free energy structure, we have to derive the conserved charges from a slightly different integral equations, even if the integral term in our case ($s = 1$) vanishes. Furthermore, since $\frac{n}{2}$ is even, the inverse of the intersection form cannot be defined.

$$\frac{n}{2} = \text{even} \quad \rightarrow \quad \nexists\omega_{rs} \quad \rightarrow \quad \text{Soft Collinear Limit}: \left(\frac{n}{2}+1\right) \quad \rightarrow \quad s = 1, 2$$

Our starting point becomes then (2.116),

$$\log Y_s(\theta) = -|m_s|\cosh(\theta - i\varphi_s) + \sum_{s'=s-1}^{s+1}\int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi}\frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta-\theta')}\log\left[1 + Y_{s'}(\theta')\right] \tag{2.153}$$

We will compute the conserved charges as if we were in the case of the decagon, reserving for ourself the possibility of eliminating a zero of the polynomial and making the limit for $s = 1, 2 \rightarrow s = 1$ (soft collinear limit). We start with $s = 1$,

$$\log \hat{Y}_1(\theta + i\frac{\pi}{2}) = -|m_1|\cosh(\theta - i\varphi_1) + \int_{\mathbb{R}+i\varphi_2} \frac{d\theta'}{2\pi}\frac{(-1)^2\,\theta^{12}}{\cosh(\theta-\theta')}\log\left[1 + \hat{Y}_2(\theta')\right] \tag{2.154}$$

and we obtain the following charges in the two regimes:

Small ζ regime

$$\tilde{c}_{-1,1} = -i\frac{m_1}{2} = Z_1 \tag{2.155}$$

$$\tilde{c}_{1,1} = \bar{Z}_1 + \int_{\mathbb{R}+i\varphi_2}\frac{d\theta'}{\pi}\frac{(-i)}{e^{\theta'}}\log\left[1 + \hat{Y}_2(\theta')\right] \tag{2.156}$$

Large ζ regime

$$c_{-1,1} = \bar{Z}_1 \tag{2.157}$$

$$c_{1,1} = Z_1 + \frac{i}{\pi}\int_{\mathbb{R}+i\varphi_2} d\theta' e^{\theta'}\log\left[1 + \hat{Y}_2(\theta')\right] \tag{2.158}$$

In the case $s = 2$, we obtain very similar results starting from

$$\log \hat{Y}_2(\theta) = -|m_2|\cosh(\theta - i\varphi_2) + \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\frac{(-1)^3\,\theta^{21}}{\cosh(\theta-\theta')}\log\left[1 + \hat{Y}_1(\theta' + i\frac{\pi}{2})\right] \tag{2.159}$$

Small ζ regime

$$\tilde{c}_{-1,2} = -\frac{m_2}{2} = Z_2 \tag{2.160}$$

$$\tilde{c}_{1,2} = \bar{Z}_2 + \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{\pi} \frac{1}{e^{\theta'}} \log\left[1 + \hat{Y}_1(\theta' + i\frac{\pi}{2})\right] \tag{2.161}$$

Large ζ regime

$$c_{-1,2} = \bar{Z}_2 \tag{2.162}$$

$$c_{1,2} = Z_2 + \frac{1}{\pi} \int_{\mathbb{R}+i\varphi_1} d\theta' e^{\theta'} \log\left[1 + \hat{Y}_1(\theta' + i\frac{\pi}{2})\right] \tag{2.163}$$

When $s = 1, 2$ it is possible to define the inverse of the incidence matrix because starting with an extra zero $\left(\frac{n}{2} + 1\right)$ is like starting with $n/2$ no longer even. So, in principle, we could use the usual formulas for calculating the area in the small ζ regime:

$$\begin{aligned}
A = -i \sum_{r,s} \omega_{rs} \oint_{\gamma^r} \lambda \oint_{\gamma^s} u = \\
-i\omega_{21} \oint_{\gamma^2} \lambda \oint_{\gamma^1} u - i\omega_{12} \oint_{\gamma^1} \lambda \oint_{\gamma^2} u = \\
= -i\,\omega_{21}\,\tilde{c}_{-1,2}\,\tilde{c}_{1,1} - i\,\omega_{12}\,\tilde{c}_{-1,1}\,\tilde{c}_{1,2}
\end{aligned} \tag{2.164}$$

However, as mentioned before, in order to recover the case of our interest (the octagon), we must execute the limit for $s = 1, 2 \to s = 1$:

$$\begin{aligned}
\left(\zeta = e^{\theta}\right)\ 1 \cdot \tilde{c}_{-1,2} &\longrightarrow \tilde{c}_{-1,1} \cdot e^{-i\frac{\pi}{2}}\ \left(\zeta = e^{\theta + i\frac{\pi}{2}}\right) \\
Z_2 = -\tfrac{m_2}{2} \cdot 1 &\longrightarrow -\tfrac{m_1}{2} i \cdot (-i) = Z_1 \cdot (-i) \\
\log\left[1 + \hat{Y}_2(\theta')\right] &\longrightarrow \log\left[1 + \hat{Y}_1(\theta' + i\tfrac{\pi}{2})\right]
\end{aligned} \tag{2.165}$$

Similar substitution are valid for the large ζ regime; consequently, to compute the contribution of the area we use the following modified formulas:

$$A^{small} = -ie^{-i\frac{\pi}{2}} \tilde{c}_{-1,1}\tilde{c}_{1,1} = -\tilde{c}_{-1,1}\tilde{c}_{1,1} \tag{2.166}$$

$$A^{large} = ie^{i\frac{\pi}{2}} c_{-1,1}c_{1,1} = -c_{-1,1}c_{1,1} \tag{2.167}$$

which produce these contributions,

$$A^{small} = -Z_1\bar{Z}_1 - \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{\pi} \frac{(-i)}{e^{\theta'}} Z_1 \log\left[1 + Y_1(\theta')\right] = A_{periods} + A^{small}_{free} \tag{2.168}$$

$$\begin{aligned} A^{small}_{free} &= -\int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{\pi}\frac{(-i)}{e^{\theta'}} Z_1 \log\left[1+Y_1(\theta')\right] = \\ &= -\int_{-\infty}^{+\infty} \frac{d\theta'}{\pi}(-i)e^{-\theta'-i\varphi_1}\left(-i\frac{|m_1|e^{i\varphi_1}}{2}\right)\log\left[1+Y_1(\theta'+i\varphi_1)\right] = \\ &= \int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}|m_1|e^{-\theta'}\log\left[1+\tilde{Y}_1(\theta')\right] \end{aligned} \tag{2.169}$$

$$A^{large} = -\bar{Z}_1 Z_1 - \frac{i}{\pi}\int_{\mathbb{R}+i\varphi_1} d\theta' e^{\theta'}\bar{Z}_1 \log\left[1+Y_1(\theta')\right] = A_{periods} + A^{large}_{free} \tag{2.170}$$

$$\begin{aligned} A^{large}_{free} &= -\frac{i}{\pi}\int_{\mathbb{R}+i\varphi_1} d\theta' e^{\theta'}\bar{Z}_1 \log\left[1+Y_1(\theta')\right] = \\ &= -\frac{i}{\pi}\int_{-\infty}^{+\infty} d\theta'\left(i\frac{|m_1|e^{-i\varphi_1}}{2}\right)e^{\theta'+i\varphi_1}\log\left[1+Y_1(\theta'+i\varphi_1)\right] = \\ &= \int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}|m_1|e^{\theta'}\log\left[1+\tilde{Y}_1(\theta')\right] \end{aligned} \tag{2.171}$$

Then we obtain

$$\begin{aligned} A_{free} &= \frac{A^{small}_{free}+A^{large}_{free}}{2} = \\ &= \int_{\mathbb{R}}\frac{d\theta'}{2\pi}|m_1|\cosh\theta'\log\left[1+\tilde{Y}_1(\theta')\right] = \\ &= \int_{\mathbb{R}}\frac{d\theta'}{2\pi}|m_1|\cosh\theta'\log\left[1+e^{-|m_1|\cosh\theta'}\right] \end{aligned} \tag{2.172}$$

which agrees with what was called A_{sinh} in [6].

2.6 The decagon

In this case we have $n = 10$ and the number of Y-functions is $s = 1, 2$. Thus, the Y-system reads

$$Y_s^+ Y_s^- = (1+Y_{s+1})(1+Y_{s-1}) \tag{2.173}$$

corresponding to these two integral equations

$$\log\tilde{Y}_1(\theta) = -|m_1|\cosh\theta + \int_{\mathbb{R}}\frac{d\theta'}{2\pi}\frac{\log\left[1+\tilde{Y}_2(\theta')\right]}{\cosh(\theta-\theta'+i\varphi_1-i\varphi_2)} \tag{2.174}$$

$$\log\tilde{Y}_2(\theta) = -|m_2|\cosh\theta + \int_{\mathbb{R}}\frac{d\theta'}{2\pi}\frac{\log\left[1+\tilde{Y}_1(\theta')\right]}{\cosh(\theta-\theta'+i\varphi_2-i\varphi_1)} \tag{2.175}$$

which derive from the following compact form

$$\log \tilde{Y}_s(\theta) = -|m_s| \cosh\theta + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta-\theta'+i\varphi_s - i\varphi_{s'})} \log\left[1+\tilde{Y}_{s'}(\theta')\right] \tag{2.176}$$

Now, we repeat the calculation developed in section 2.5 for both equations:

(s=1) We start from

$$\log Y_1(\theta) = -|m_1| \cosh(\theta - i\varphi_1) + \int_{\mathbb{R}+i\varphi_2} \frac{d\theta'}{2\pi} \frac{\log\left[1+Y_2(\theta')\right]}{\cosh(\theta-\theta')} \tag{2.177}$$

where

$$\begin{cases} Y_1(\theta_+ - i\frac{\pi}{2}) = \hat{Y}_1(\theta_+) \quad \text{with} \quad \theta_+ = \theta + i\frac{\pi}{2} \\ Y_2(\theta') = \hat{Y}_2(\theta') \end{cases} \tag{2.178}$$

and, omitting[17] the subscript $+$, we finally get

$$\log \hat{Y}_1(\theta) = -|m_1| \cosh(\theta - i\hat{\varphi}_1) + \int_{\mathbb{R}+i\hat{\varphi}_2} \frac{d\theta'}{2\pi} \frac{\log\left[1+\hat{Y}_2(\theta'-i\frac{\pi}{2})\right]}{\cosh(\theta-\theta')} \tag{2.179}$$

The WKB approximation, which is ζ-independent, for the first equation in the small ζ-regime produces

$$\log \hat{Y}_k \sim -\left[\frac{1}{\zeta_+}\oint_{\gamma_k} \lambda + \zeta_+ \oint_{\gamma_k} u + ...\right] \qquad \text{with} \quad \zeta_+ = e^{\theta + i\frac{\pi}{2}} \tag{2.180}$$

The conserved charges are hidden inside the log-expansion,

$$\log \hat{Y}_1(\theta_+) \sim \sum_{n=-1}^{+\infty} \tilde{c}_{n,1}\, e^{n\theta_+} \tag{2.181}$$

and reads

$$\tilde{c}_{-1,1} = -i\frac{m_1}{2} = Z_1 \tag{2.182}$$

$$\tilde{c}_{1,1} = \bar{Z}_1 + \frac{\theta^{12}}{\pi}\int \frac{d\zeta'_+}{\zeta_+}\frac{1}{\zeta'_+} \log\left[1+\hat{Y}_2\left(\zeta'_+ - i\frac{\pi}{2}\right)\right] \tag{2.183}$$

Comparing the previous expressions we can deduce:

$$\begin{cases} -\oint_{\gamma^1}\lambda &= \tilde{c}_{-1,1} &= Z_1 \\ -\oint_{\gamma^1} u &= \tilde{c}_{1,1} &= \bar{Z}_1 + \frac{\theta^{12}}{\pi}\int \frac{d\zeta'_+}{\zeta'_+}\frac{1}{\zeta'_+}\log\left[1+\hat{Y}_2\left(\zeta'_+ - i\frac{\pi}{2}\right)\right] \end{cases} \tag{2.184}$$

[17]Here we rename the variable θ_+ as simply θ, remembering that its complex definition can be read from the argument of the forcing which must be real.

(s=2) We start from

$$\log Y_2(\theta) = -|m_2|\cosh(\theta - i\varphi_2) + \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi} \frac{\log\left[1 + Y_1(\theta')\right]}{\cosh(\theta - \theta')} \tag{2.185}$$

where

$$\begin{cases} Y_1(\theta_+ - i\frac{\pi}{2}) = \hat{Y}_1(\theta_+) \quad \text{with} \quad \theta_+ = \theta + i\frac{\pi}{2} \\ Y_2(\theta') = \hat{Y}_2(\theta') \end{cases} \tag{2.186}$$

and, omitting the subscript $+$ like before, we finally get

$$\log \hat{Y}_2(\theta - i\frac{\pi}{2}) = -|m_2|\cosh(\theta - i\hat{\varphi}_2) + \int_{\mathbb{R}+i\hat{\varphi}_1} \frac{d\theta'}{2\pi} \frac{\log\left[1 + \hat{Y}_1(\theta')\right]}{\cosh(\theta - \theta')} \tag{2.187}$$

The WKB approximation for the second equation in the small ζ-regime produces,

$$\log \hat{Y}_k \sim -\left[\frac{1}{\zeta}\oint_{\gamma_k} \lambda + \zeta \oint_{\gamma_k} u + ...\right] \qquad \text{with} \quad \zeta = e^{\theta} \tag{2.188}$$

The conserved charges are hidden inside the log-expansion,

$$\log \hat{Y}_2(\theta_+ - i\frac{\pi}{2}) = \log \hat{Y}_2(\theta) \sim \sum_{n=-1}^{+\infty} \tilde{c}_{n,2}\, e^{n\theta} \tag{2.189}$$

and reads

$$\tilde{c}_{-1,2} = -\frac{m_2}{2} = Z_2 \tag{2.190}$$

$$\tilde{c}_{1,2} = \bar{Z}_2 - \frac{i\,\theta^{21}}{\pi} \int \frac{d\zeta'_+}{\zeta_+} \frac{1}{\zeta'_+} \log\left[1 + \hat{Y}_1(\zeta'_+)\right] \tag{2.191}$$

Comparing the previous expressions we can deduce:

$$\begin{cases} -\oint_{\gamma^2} \lambda &= & \tilde{c}_{-1,2} &= & Z_2 \\ -\oint_{\gamma^2} u &= & \tilde{c}_{1,2} &= & \bar{Z}_2 - \frac{i\,\theta^{21}}{\pi} \int \frac{d\zeta'_+}{\zeta'_+} \frac{1}{\zeta'_+} \log\left[1 + \hat{Y}_1(\zeta'_+)\right] \end{cases} \tag{2.192}$$

Now, we can calculate the contributions to the area in the small ζ-regime through the usual formula,

$$A = A_{periods} + A^{small}_{free} = -i \sum_{r,s} \omega_{rs} \oint_{\gamma^r} \lambda \oint_{\gamma^s} u =$$

$$= -i\,\omega_{21} \oint_{\gamma^2} \lambda \oint_{\gamma^1} u - i\,\omega_{12} \oint_{\gamma^1} \lambda \oint_{\gamma^2} u =$$

$$
\begin{aligned}
&= -i\omega_{21} Z^2 \bar{Z}^1 - \frac{i}{\pi}\int \frac{d\zeta'_+}{\zeta'_+}\frac{1}{\zeta'_+} Z^2 \log\left[1+\hat{Y}_2\left(\zeta'_+ - i\frac{\pi}{2}\right)\right] - \\
&-i\omega_{12} Z^1 \bar{Z}^2 - \frac{1}{\pi}\int \frac{d\zeta'_+}{\zeta'_+}\frac{1}{\zeta'_+} Z^1 \log\left[1+\hat{Y}_1\left(\zeta'_+\right)\right] = \\
&= -i\omega_{12}\left(Z^1\bar{Z}^2 - Z^2\bar{Z}^1\right) - \\
&-\frac{i}{\pi}\int_{\mathbb{R}+i\hat{\varphi}_2} d\theta' e^{-\theta'}\left(-\frac{|m_2|}{2}e^{i\varphi_2}\right)\log\left[1+\hat{Y}_2\left(\theta' - i\frac{\pi}{2}\right)\right] - \\
&-\frac{1}{\pi}\int_{\mathbb{R}+i\hat{\varphi}_1} d\theta' e^{-\theta'}\left(-i\frac{|m_1|}{2}e^{i\varphi_1}\right)\log\left[1+\hat{Y}_1\left(\theta'\right)\right] = \\
&= A_{periods} + \\
&+\frac{i}{\pi}\int_{\mathbb{R}+i\varphi_2} d\theta' e^{-\theta'-i\frac{\pi}{2}}\frac{|m_2|}{2}e^{i\varphi_2}\log\left[1+\hat{Y}_2(\theta')\right] + \\
&+\frac{i}{\pi}\int_{\mathbb{R}+i\varphi_1} d\theta' e^{-\theta'-i\frac{\pi}{2}}\frac{|m_1|}{2}e^{i\varphi_1}\log\left[1+\hat{Y}_1(\theta' + i\frac{\pi}{2})\right] = \\
&= A_{periods} + \sum_{r=1}^{2}\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}e^{-\theta'}|m_r|\log\left[1+\tilde{Y}_r(\theta')\right]
\end{aligned} \tag{2.193}
$$

If we repeat the same calculus in the large ζ-regime, we obtain the desired form of A^{large}_{free} to reach the free energy structure,

$$
A^{large}_{free} = \sum_{r=1}^{2}\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}e^{\theta'}|m_r|\log\left[1+\tilde{Y}_r(\theta')\right] \tag{2.194}
$$

that implies

$$
A_{free} = \frac{A^{small}_{free} + A^{large}_{free}}{2} = \sum_{r=1}^{2}\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}|m_r|\cosh\theta'\log\left[1+\tilde{Y}_r(\theta')\right] \tag{2.195}
$$

which correspond to (2.146) for the case $r = 1, 2$.

2.7 Equivalent TBA equations in the context of Hitchin systems

In this chapter, we would like to emphasize the link existing between our integral equations (2.98) and those we find in [37, App. E], [13] and [11, App. E]. From the former, we can make contact with these three important papers concerning general Hitchin's systems, in which the authors argue without using the Y-system (and hence the Y-functions) from which we started instead.

2.7.1 Integral equations with X-functions

First of all, we show how to connect with [37, eq. (E.2)] starting from our TBA-like integral equation for the general complex case (2.98), which we report for convenience,

$$\log Y_s(\theta) = -|m_s| \cosh(\theta - i\varphi_s) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + Y_{s'}(\theta')\right]}{\cosh(\theta - \theta')} \tag{2.196}$$

Considering that the quantities Z_s have different definitions depending on whether the subscript s is even or odd,

$$\begin{aligned} Z_s &= \begin{cases} -\frac{m_s}{2} = -\frac{|m_s|}{2} e^{i\varphi_s} & \text{for } s \text{ even} \\ -i\frac{m_s}{2} = -\frac{|m_s|}{2} e^{i\varphi_s + i\frac{\pi}{2}} & \text{for } s \text{ odd} \end{cases} = \\ &= -\frac{|m_s|}{2} e^{i\varphi_s + i\frac{\pi}{2} b_{s+1}} \end{aligned} \tag{2.197}$$

in order to respect the convention on the integration line[18], we have to consider the following integration variable, which changes from the even to the odd case:

$$\begin{aligned} \theta &= \begin{cases} \Re\{\theta\} + i\varphi_s & \text{for } s \text{ even} \\ \Re\{\theta\} + i\varphi_s + i\frac{\pi}{2} & \text{for } s \text{ odd} \end{cases} = \\ &= \Re\{\theta\} + i\varphi_s + i\frac{\pi}{2} b_{s+1} = \\ &= \Re\{\theta\} + i\bar{\varphi}_s \end{aligned} \tag{2.198}$$

In doing so, we can rewrite (2.196) in the following general form

$$\begin{aligned} \log Y_s(\theta) = \frac{Z_s}{e^{\theta + i\frac{\pi}{2} b_{s+1}}} + \bar{Z}_s e^{\theta + i\frac{\pi}{2} b_{s+1}} + \\ + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\bar{\varphi}_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta - \theta' + i\frac{\pi}{2} b_{s'+1})} \log\left[1 + Y_{s'}\left(\theta' - i\frac{\pi}{2} b_{s'+1}\right)\right] \end{aligned} \tag{2.199}$$

After rewriting the integral in terms of sinh, a step that differs from the even case to the odd one, we can consider the relation between Y and X-functions, described in [37, App. E]: these functions coincide on the integration line up to an appropriate shift in θ by the argument of Z_s. Thanks to the previous choice of the integration lines, we can say in general that

$$Y_s(\theta) \equiv X_s\left(\theta + i\frac{\pi}{2} b_{s+1}\right) \tag{2.200}$$

[18]The integration line is set in such a way that $\frac{Z_s}{\zeta} \in \mathbb{R}_-$, with $\zeta = e^\theta$, in agreement with [37, App. E].

For simplicity, let us set s even and the previous equation simplify in

$$\log Y_s(\theta) = \frac{Z_s}{e^\theta} + \bar{Z}_s e^\theta + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\hat{\varphi}_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)\,\theta^{ss'}}{\cosh(\theta-\theta'+i\frac{\pi}{2})} \log\left[1+Y_{s'}\left(\theta'-i\frac{\pi}{2}\right)\right] \tag{2.201}$$

where $\hat{\varphi}_{s'} = \varphi_{s'} + \frac{\pi}{2}$. Finally, we can write

$$\log X_s(\theta) = \frac{Z_s}{e^\theta} + \bar{Z}_s e^\theta + \sum_{s'=1}^{\frac{n}{2}-3} \int_{\mathbb{R}+i\hat{\varphi}_{s'}} \frac{d\theta'}{2\pi i} \frac{\theta^{ss'}}{\sinh(\theta'-\theta)} \log\left[1+X_{s'}\left(\theta'\right)\right] \tag{2.202}$$

which coincide with [37, eq. (E.2)] by means of the following identifications:

$$\begin{aligned} n-gon &\longleftrightarrow 2N-gon \\ s,s' = 1,2,...,\tfrac{n}{2}-3 &\longleftrightarrow \gamma,\gamma' \in \Gamma^+ = 1,2,...,N-3 \\ \mathbb{R}+i\bar{\varphi}_{s'} &\longleftrightarrow l_{\gamma'} \\ \theta^{ss'} &\longleftrightarrow \langle\gamma,\gamma'\rangle \\ 1 &\longleftrightarrow \Omega(\gamma') \end{aligned} \tag{2.203}$$

Further details about the equivalence just shown can be found in the derivation of this last equation in the context of general Hitchin's systems (appendix A) or in the analysis of some particular case, such as the decagon, already studied in section 2.6 and with which we want to confront for a check.

We will now derive the integral equations for the decagon from the X-functions formalism: the polygon has $2N = 10$ sides and the index that labels the equations takes values $s = 1,2,...,N-3$. Therefore, we have to consider only the two values $s = 1,2$. For the value $s = 1$, eq. (E.2) of [37] becomes

$$\log X_1(\theta) = \frac{Z_1}{e^\theta} + \bar{Z}_1 e^\theta + \int_{l_2} \frac{d\theta'}{2\pi i} \frac{\langle 1,2\rangle}{\sinh(\theta'-\theta)} \log\left[1+X_2(\theta')\right] \tag{2.204}$$

where the analysis of the integration paths leads us to fix

$$\begin{cases} l_1 : \frac{Z_1}{e^\theta} \in \mathbb{R}_- \longleftrightarrow \theta = \Re\{\theta\} + i\varphi_1 + i\frac{\pi}{2} \equiv \theta_1 + i\frac{\pi}{2} \\ l_2 : \frac{Z_2}{e^{\theta'}} \in \mathbb{R}_- \longleftrightarrow \theta' = \Re\{\theta'\} + i\varphi_2 \equiv \theta'_2 \end{cases} \tag{2.205}$$

Thus, we can rewrite in the following form

$$\log X_1(\theta) = \frac{Z_1}{e^\theta} + \bar{Z}_1 e^\theta + \int_{\mathbb{R}+i\varphi_2} \frac{d\theta'}{2\pi i} \frac{(-1)\langle 1,2\rangle}{\sinh(\theta_1+i\frac{\pi}{2}-\theta')} \log\left[1+X_2(\theta')\right] \tag{2.206}$$

finally getting

$$\log Y_1(\theta + i\varphi_1) = -|m_1| \cosh \theta + \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{\theta^{12} \log\left[1 + Y_2(\theta' + i\varphi_2)\right]}{\cosh(\theta - \theta' + i\varphi_1 - i\varphi_2)} \tag{2.207}$$

where in the last step we have renamed $\Re\{\theta\} \to \theta$, so that $\theta_s \to \theta + i\varphi_s$. In a completely analogous way, we can find the corresponding equation for the value $s = 2$,

$$\log Y_2(\theta + i\varphi_2) = -|m_2| \cosh \theta + \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{(-1)\theta^{21} \log\left[1 + Y_1(\theta' + i\varphi_1)\right]}{\cosh(\theta - \theta' + i\varphi_2 - i\varphi_1)} \tag{2.208}$$

These equations for the decagon, derived working with X-functions, are in agreement with the general compact form (2.176) obtained in section 2.6.

2.7.2 Integral equations with ε-functions

To show the equivalence between (2.98) and [13, eq. (3.38)], we can proceed in two equivalent ways, directly from the Y-functions formalism of chapter 2 or, considering the previous subsection, from [37, eq. (E.2)] paying attention to the conventions used in the two contexts. In order to guarantee a logical thread to the exposition, we choose the former. As before, we rewrite the starting point for convenience,

$$\log Y_s(\theta) = -|m_s| \cosh(\theta - i\varphi_s) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + Y_{s'}(\theta')\right]}{\cosh(\theta - \theta')} \tag{2.209}$$

Considering the relation between Y-functions and pseudo-energies,

$$Y_s(\theta) = e^{-\varepsilon(\theta - i\varphi_s)} \tag{2.210}$$

together with

$$Z_s = |Z_s| e^{i\alpha_s} \equiv \frac{|m_s|}{2} e^{i\alpha_s} \qquad \alpha_s = \varphi_s + \frac{\pi}{2} b_{s+1} \tag{2.211}$$

we can rewrite (2.209) in the following form

$$\varepsilon(\theta - i\varphi_s) = |m_s| \cosh(\theta - i\varphi_s) - \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta - \theta')} \log\left[1 + e^{-\varepsilon(\theta' - i\varphi_{s'})}\right] \tag{2.212}$$

With the following shift, we underline the real argument of the ϵ-functions,

$$\begin{cases} \theta - i\varphi_s \longrightarrow \theta \\ \varepsilon(\theta - i\varphi_s) \longrightarrow \varepsilon_s(\theta) \end{cases} \tag{2.213}$$

and we get

$$\varepsilon_s(\theta) = 2|Z_s|\cosh\theta - \sum_{s'=s-1}^{s+1}\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}\frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta-\theta'+i\varphi_s-i\varphi_{s'})}\log\left[1+e^{-\varepsilon_{s'}(\theta')}\right] \quad (2.214)$$

Now, with the aim of bringing up the quantities α_s and $\alpha_{s'}$ in the argument of cosh, we add and subtract terms of the type $i\frac{\pi}{2}b_{s+1}$, so that we obtain in the denominator

$$\begin{aligned}\cosh\left[\theta-\theta'+i\alpha_s-i\alpha_{s'}+i\frac{\pi}{2}(b_{s'+1}-b_{s+1})\right] &= \\ = \sinh(\theta-\theta'+i\alpha_s-i\alpha_{s'})\sinh\left[i\frac{\pi}{2}(b_{s'+1}-b_{s+1})\right] &= \\ = (-1)^s i\sinh(\theta-\theta'+i\alpha_s-i\alpha_{s'}) & \end{aligned} \quad (2.215)$$

Finally, we can write

$$\varepsilon_s(\theta) = 2|Z_s|\cosh\theta - \sum_{s'=s-1}^{s+1}\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi i}\frac{(-1)\,\theta^{ss'}}{\sinh(\theta-\theta'+i\alpha_s-i\alpha_{s'})}\log\left[1+e^{-\varepsilon_{s'}(\theta')}\right] \quad (2.216)$$

which coincide with [13, eq. (3.38)] by means of the following identifications:

$$\begin{aligned} n-gon &\longleftrightarrow 2n-gon \\ s,s' = 1,2,...,\tfrac{n}{2}-3 &\longleftrightarrow k,l = 1,2,...,n-3 \\ (-1)\theta^{ss'} &\longleftrightarrow \langle\gamma_k,\gamma_l\rangle \\ 1 &\longleftrightarrow \Omega(\gamma') \end{aligned} \quad (2.217)$$

Like before, further details about the equivalence just shown can be found in the analysis of the decagon within the ε-functions formalism.

We will now derive the integral equations for the decagon from the ε-functions formalism: the polygon has $2n = 10$ sides and the index that labels the equations takes values $s = 1,2,...,n-3$. Therefore, we have to consider only the two values $s = 1,2$. For the value $s = 1$, eq. (3.38) of [13] becomes

$$\varepsilon_1(\theta) = 2|Z_1|\cosh\theta - \int_{\mathbb{R}}\frac{d\theta'}{2\pi}\frac{(-i)\langle\gamma_1,\gamma_2\rangle}{\sinh(\theta-\theta'+i\alpha_1-i\alpha_2)}\log\left[1+e^{-\varepsilon_2(\theta')}\right] \quad (2.218)$$

Considering that $\langle\gamma_1,\gamma_2\rangle \equiv \theta^{12} = -1$, we can recast the previous equation,

$$\varepsilon_1(\theta) = 2|Z_1|\cosh\theta - \int_{\mathbb{R}}\frac{d\theta'}{2\pi i}\frac{(-1)}{\sinh(\theta-\theta'+i\alpha_1-i\alpha_2)}\log\left[1+e^{-\varepsilon_2(\theta')}\right] \quad (2.219)$$

and then

$$\varepsilon_1(\theta) = 2|Z_1|\cosh\theta - \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' - i\tilde{\alpha})} \log\left[1 + e^{-\varepsilon_2(\theta')}\right] \tag{2.220}$$

where, from (2.211), we have

$$\begin{cases} -i\tilde{\alpha} = i\alpha_1 - i\alpha_2 - i\frac{\pi}{2} = i\varphi_1 - i\varphi_2 \\ \alpha_1 = \varphi_1 + i\frac{\pi}{2} \\ \alpha_2 = \varphi_2 \end{cases} \tag{2.221}$$

Now, for a generic value of k, holds

$$-\,\varepsilon_k(\theta) = \ln X_{\gamma_k}(\theta + i\alpha_k) \tag{2.222}$$

so that we can write

$$\begin{aligned} -\varepsilon_1(\theta) &= \ln X_1(\theta + i\alpha_1) = \\ &= \ln X_1(\theta + i\varphi_1 + i\frac{\pi}{2}) = \\ &= \ln Y_1(\theta + i\varphi_1) = \\ &= \ln Y_1(\theta_1) \end{aligned} \tag{2.223}$$

while

$$\begin{aligned} -\varepsilon_2(\theta') &= \ln X_2(\theta' + i\alpha_2) = \\ &= \ln X_2(\theta' + i\varphi_2) = \\ &= \ln Y_2(\theta' + i\varphi_2) = \\ &= \ln Y_2(\theta'_2) \end{aligned} \tag{2.224}$$

in the end, we can rewrite in the following form

$$\log Y_1(\theta + i\varphi_1) = -|m_1|\cosh\theta + \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{\theta^{12}\log\left[1 + Y_2(\theta' + i\varphi_2)\right]}{\cosh(\theta - \theta' + i\varphi_1 - i\varphi_2)} \tag{2.225}$$

In a completely analogous way, we can find the corresponding equation for the value $s = 2$,

$$\log Y_2(\theta + i\varphi_2) = -|m_2|\cosh\theta + \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{(-1)\theta^{21}\log\left[1 + Y_1(\theta' + i\varphi_1)\right]}{\cosh(\theta - \theta' + i\varphi_2 - i\varphi_1)} \tag{2.226}$$

These equations for the decagon, derived working with ε-functions, are in agreement with the general compact form (2.176) obtained in section 2.6.

Chapter 3

The relevant HSG models for minimal surfaces in AdS_3

In this chapter, we would like to exploit the rewriting (2.107) to make contact with the homogenous sine-Gordon (HSG) models [14, 15], which are a class of 2-dimensional integrable models generalizing the sine-Gordon model. They are obtained by integrable perturbations of CFTs corresponding to G_k-parafermions, or $\frac{G_k}{[U(1)]^{r_g}}$ cosets [16, 17, 18], where G is a simple compact Lie group with Lie algebra g and r_g is the rank of g. The S-matrices describing the HSG models for simply laced G groups are proposed in [19].

As we learn from [13, 7], concerning both the AdS_3 and AdS_5 cases, in order to derive the integral equations characterizing minimal surfaces with a null polygonal boundary in AdS, it is necessary to write down a Riemann-Hilbert problem from the data of the polynomial $p(z)$ and other inherent constraints. Until the work of Y. Hatsuda and colleagues [13] for AdS_3, it was known that these integral equations possess the structure of TBA equations [11, App. E], but it was not at all clear what models are described by them in practice: they found that the TBA integral equations for the AdS_3 case are identified with those of the $\frac{SU(N)_2}{[U(1)]^{N-1}}$ HSG models, which is discussed in detail in [20, 21]. From our rewriting (2.107), it is now possible to derive this result in a similar way linking the integral equations (2.96) found in the previous chapter with those of [13], through very similar steps of section 2.7.

Furthermore, in the last part of the chapter, we will exploit this link to show that our AdS_3 Y-system (2.69) actually coincides with the very general one of [20] in a particular configuration. The case of the decagon, studied in section 2.6, is offered as an example to check this statement. Also the subsequent restriction to the $su(k)$-minimal ATFTs, namely the A_n series of the ADE classification [33], is inserted as a particular limit on the resonance parameter [20, 21].

3.1 The $\frac{SU(N)_2}{[U(1)]^{N-1}}$ HSG models

For convenience, we rewrite the starting point (2.107), which will allow us to connect with the HSG models in a similar way to what we find in [13]:

$$\varepsilon(\theta - i\hat{\varphi}_s) = -\frac{Z_s}{e^{\theta - i\frac{\pi}{2}b_s}} - \bar{Z}_s\, e^{\theta - i\frac{\pi}{2}b_s} - \sum_{s'} \int_{\mathbb{R}+i\hat{\varphi}_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + e^{-\varepsilon(\theta' - i\hat{\varphi}_{s'})}\right]}{\cosh(\theta - \theta')} \tag{3.1}$$

Recalling the current definition[1] of the complex variable θ set by the shifts of subsection 2.3.1, we can recast the forcing term in this fashion,

$$-\frac{Z_s}{e^{\theta - i\frac{\pi}{2}b_s}} - \bar{Z}_s\, e^{\theta - i\frac{\pi}{2}b_s} = 2|Z_s| \cosh\left[\Re\left\{\theta\right\}\right] \tag{3.2}$$

while the integrating function, once the integration has been moved to the real axis, becomes

$$\frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + e^{-\varepsilon(\theta' - i\hat{\varphi}_{s'})}\right]}{\cosh(\theta - \theta')} = \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + e^{-\varepsilon_{s'}(\Re\{\theta'\})}\right]}{\cosh\left[\Re\left\{\theta\right\} - \Re\left\{\theta'\right\} + i\varphi_s - i\varphi_{s'}\right]} \tag{3.3}$$

Finally, recasting in terms of real variables, we get the useful writing to connect with the HSG models,

$$\varepsilon_s(\theta) = 2|Z_s| \cosh\theta - \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'} \log\left[1 + e^{-\varepsilon_{s'}(\theta')}\right]}{\cosh(\theta - \theta' + i\varphi_s - i\varphi_{s'})} \tag{3.4}$$

To see the connection, we need to compare the previous equation with [13, eq. (3.40)], which are the TBA equations for the $\frac{SU(N)_2}{[U(1)]^{N-1}}$ homogeneous sine-Gordon models: we rewrite them to clarify,

$$\varepsilon_a(\theta) = m_a R \cosh\theta - \sum_{b=1}^{N-1} \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{i I_{ab}}{\sinh(\theta - \theta' + \sigma_{ab} + i\frac{\pi}{2})} \log\left[1 + e^{-\varepsilon_b(\theta')}\right] \tag{3.5}$$

where $a = 1, 2, ..., N-1$ labels the particles with mass m_a, I_{ab} is the incidence matrix, R is the inverse temperature and $\sigma_{ab} = -\sigma_{ba}$ are some parameters. It is important to note that these latter equations have been derived in [13] following the usual procedure [36, 33] from the S-matrix

$$S_{ab}(\theta) = (-1)^{\delta_{ab}} \left\{ c_a \tanh\left[\frac{1}{2}\left(\theta + \sigma_{ab} - i\frac{\pi}{2}\right)\right]\right\}^{I_{ab}} \tag{3.6}$$

[1] Just look at the ε-function's argument and remember that it must be real.

where $c_a = \pm 1$ [21]. Now it is clear that the link holds under the following identifications:

$$\begin{array}{rcl} s, s' = 1, ..., \frac{n}{2} - 3 & \longleftrightarrow & a, b = 1, ..., N-1 \\ & N = \frac{n}{2} - 2 & \\ 2|Z_s| & \longleftrightarrow & m_a R \\ (-1)^{s+1}\theta^{ss'} & \longleftrightarrow & \epsilon_{ab} I_{ab} \\ & \epsilon_{ab} = -\epsilon_{ba} = \pm 1 & \\ i(\varphi_s - \varphi_{s'}) & \longleftrightarrow & \sigma_{ab} \end{array} \tag{3.7}$$

We therefore connected with a very specific class of HSG coset-models, which we will continue to study later in this chapter.

3.2 The universal TBA equations for the HSG models

For our purposes, it is important to say something about the rich particle structure [20, 21, 22] of the deformations around the CFT point of the HSG models, such as the description of scattering matrices [19] and their properties related to mass scales, resonance parameter and parity breaking [20].

In [19], a new family of S-matrices with resonance poles is conjectured to correspond to the HSG theories associated with simply laced compact Lie groups. These theories have been constructed as integrable perturbations of the WZNW-coset [16, 17, 18] of the form $\frac{G_k}{H}$, where G is a compact simple Lie group, $H \subset G$ is a maximal abelian torus and $k > 1$ is an integer called the "level". A characteristic feature of these suggested S-matrices is that some elements are not parity invariant and contain resonance shifts which allow for the formation of unstable bound states. Only the specific choice of the groups ensures that these theories possess a mass gap [15]. The defining action of the HSG models thus constructed reads

$$S_{HSG}[g] = S_{CFT}[g] + \frac{m^2}{\pi\beta^2} \int d^2x \, \langle \Lambda_+, g(x)^{-1} \Lambda_- g(x) \rangle \tag{3.8}$$

where S_{CFT} is the coset action, $\langle \, , \rangle$ is a Killing form of G and $g(x)$ is a group valued bosonic scalar field. $\Lambda_\pm$ are semi-simple elements of the Cartan subalgebra associated with H and play the role of continuous vector coupling constants. They determine the mass ratios of the particle spectrum as well as the behaviour of the model under a parity transformation. The parameters m and β^2 are the mass scale and the coupling constant.

The proposed scattering matrices consist *only partially* of l copies of minimal $su(k)$-ATFTs [23], whose mass scales are free parameters. The scattering between solitons belonging to different copies is described by an S-matrix which violates parity [19] and this type of matrix possesses resonance poles where the related resonance parameters characterize the formation of unstable bound states. In comparison with the models studied until [20], the HSG models which we will cover in this chapter are distinguished in two aspects: first, they break parity invariance and second some of the resonance poles can be associated directly to unstable particles[2]. One of the main outcomes of [20] is that their TBA-analysis gives strong support to the scattering matrix proposed in [19]. Now, let us more specific briefly recalling the main features of the proposed HSG scattering matrix in a form most suitable for our discussion. The solitons are labelled by two quantum numbers (a,i): we refer to a as the main quantum number, which runs within $1 \leq a \leq k-1$, while i is the colour and runs within $1 \leq i \leq l$, where l denotes the rank of G and h its Coxeter number. Therefore, the 2-p scattering matrix between the soliton (a,i) and the soliton (b,j) will have a general form $S_{ab}^{ij}(\theta)$, where θ is the rapidity difference.

In [19] it was proposed to describe the scattering of solitons which possess the same colour by the S-matrix of the $\mathbb{Z}_k$-Ising model or, equivalently, the minimal $su(k)$-ATFT [23]:

$$
\begin{aligned}
S_{ab}^{ii}(\theta) &= (a+b)_\theta(|a-b|)_\theta \prod_{n=1}^{min(a,b)-1} (a+b-2n)_\theta^2 & (3.9)\\
&= \exp\left\{ \int \frac{dt}{t}\, 2\cosh\left(\frac{\pi t}{k}\right)\left[2\cosh\left(\frac{\pi t}{k}-I\right)\right]_{ab}^{-1} e^{-it\theta} \right\} & (3.10)
\end{aligned}
$$

where we have introduced the abbreviation

$$
(x)_\theta = \frac{\sinh\left[\frac{1}{2}\left(\theta + i\frac{\pi x}{k}\right)\right]}{\sinh\left[\frac{1}{2}\left(\theta - i\frac{\pi x}{k}\right)\right]} \tag{3.11}
$$

for the general building blocks. The second line (3.10) is the integral representation of the block form (3.9) and I denotes the incidence matrix of the $su(k)$-Dynkin diagram. Instead, the scattering of solitons with different colour quantum numbers was proposed to be described by

$$
\begin{aligned}
S_{ab}^{ij}(\theta) &= (\eta_{ij})^{ab} \prod_{n=0}^{min(a,b)-1} (-|a-b|-1-2n)_{\theta+\sigma_{ij}} & (3.12)\\
&= (\eta_{ij})^{ab} \exp\left\{ -\int \frac{dt}{t}\left[2\cosh\left(\frac{\pi t}{k}\right)-I\right]_{ab}^{-1} e^{-it(\theta+\sigma_{ij})} \right\} & (3.13)
\end{aligned}
$$

with K^g denoting the Cartan matrix of the simply laced Lie algebra g and $K_{ij}^g \neq 0, 2$. Here the $\eta_{ij} = \eta_{ji}^*$ are arbitrary k-th roots of -1 and the shifts in the rapidity variables

[2]The resonance parameter characterizes the mass scale of the unstable particles.

are functions of the vector couplings σ_{ij}, which are anti-symmetric in the colour values $\sigma_{ij} = -\sigma_{ji}$. An explicit expressions for the classical resonance shifts is

$$\sigma_{ij} = \ln \sqrt{\frac{(\alpha_i \cdot \Lambda_+)\,(\alpha_j \cdot \Lambda_-)}{(\alpha_i \cdot \Lambda_-)\,(\alpha_j \cdot \Lambda_+)}} \tag{3.14}$$

where the α_i are simple roots. Due to the fact that these shifts are real, the function $S_{ab}^{ij}(\theta)$ for $i \neq j$ will have poles beyond the imaginary axis: the parameters σ_{ij} characterize resonance poles. An important feature is that (3.12) is not parity invariant, where parity is broken by the phase factors η as well as the shifts σ. As a consequence, while the parity invariant objects (3.9) satisfy the usual relations

$$S_{ab}^{ii}(\theta) = S_{ba}^{ii}(\theta) = \left[S_{ab}^{ii}(-\theta^*)\right]^* \quad \text{and} \quad S_{ab}^{ii}(\theta) S_{ab}^{ii}(-\theta) = 1 \tag{3.15}$$

the matrix elements for the scattering between solitons with different colour satisfy

$$S_{ab}^{ij}(\theta) = \left[S_{ba}^{ji}(-\theta^*)\right]^* \quad \text{and} \quad S_{ab}^{ij}(\theta) S_{ba}^{ji}(-\theta) = 1 \tag{3.16}$$

Analyzing the above S-matrices, we can say the following statement about the formation of bound states: two solitons with the same colour may form a bound state of the same colour, whilst solitons of different colours, say (a, i) and (b, j), may only form an unstable state (c, k), whose lifetime and energy scale are characterized by the parameter σ through the Breit-Wigner formulas [24]

$$\left(M_c^k\right)^2 - \frac{\left(\Gamma_c^k\right)^2}{4} = \left(M_a^i\right)^2 + \left(M_b^j\right)^2 + 2 M_a^i M_b^j \cosh\sigma \cos\Theta \tag{3.17}$$

$$M_c^k \Gamma_c^k = 2 M_a^i M_b^j \sinh|\sigma| \sin\Theta \tag{3.18}$$

where the resonance pole in $S_{ab}^{ij}(\theta)$ is situated at $\theta_R = \sigma - i\Theta$ and the Γ_c^k denote the decay width. For what will be developed later, it is important to note that, in the limit $\sigma \to 0$, the relation (3.18) shows us that the unstable particles become stable, but they are not yet like the other asymptotically stable particles. They become virtual states characterized by poles on the imaginary axis beyond the physical sheet.

The HSG models are characterized by l different mass scales[3] $m_1, m_2, ..., m_l$ and the explicit expression for the mass ratios is

$$\frac{m_i}{m_j} = \frac{M_a^i}{M_a^j} = \sqrt{\frac{(\alpha_i \cdot \Lambda_+)\,(\alpha_i \cdot \Lambda_-)}{(\alpha_j \cdot \Lambda_-)\,(\alpha_j \cdot \Lambda_+)}} \tag{3.19}$$

[3] In addition there are $l-1$ independent parameters in form of the possible phase shifts $\sigma_{ij} = -\sigma_{ji}$ for each i, j such that $K^g \neq 0, 2$. This means that we have $2l - 1$ independent parameters in total.

In [25] the semi-classical mass for the soliton (a, i) was found to be

$$M_a^i = \frac{m_i}{\pi \beta^2} \sin\left(\frac{\pi a}{k}\right) \tag{3.20}$$

where β is a coupling constant and the m_i are the l different mass scales.

The two values of the resonance parameter 0 and ∞ are special: in the former case parity is restored on the classical as well as on the TBA-level, whereas in the latter case the l copies of the minimal ATFT are decoupled and unstable bound states may not be produced [20, 21]. We here report the full set of TBA integral equations in the general parity-violation case for completeness [20, sec. 3]. From the two sets[4] of BA equations, which are inherent in the party-violation case (L denote the length of the compactified space direction),

$$\begin{cases} e^{iLM_A \sinh\theta_A} \prod_{B\neq A} S_{AB}(\theta_A - \theta_B) = 1 \\ e^{-iLM_A \sinh\theta_A} \prod_{B\neq A} S_{BA}(\theta_B - \theta_A) = 1 \end{cases} \tag{3.21}$$

we may carry out the usual procedure [36] and obtain the following sets of NLIEs,

$$\begin{cases} \epsilon_A^+(\theta) + \sum_B \Phi_{AB} * L_B^+(\theta) = rM_A \cosh\theta \\ \epsilon_A^-(\theta) + \sum_B \Phi_{BA} * L_B^-(\theta) = rM_A \cosh\theta \end{cases} \tag{3.22}$$

where, as usual, the symbol $*$ denote the rapidity convolution defined by

$$(f * g)(\theta) := \int \frac{d\theta'}{2\pi} f(\theta - \theta') g(\theta') \tag{3.23}$$

and $r = m_1 T^{-1}$ is the inverse temperature times the overall mass scale m_1 of the lightest particle. We have also redefined the masses by $M_a^i \to M_a^i/m_1$. As very common in these context, we have introduced the so-called pseudo-energies $\epsilon_A^+(\theta) = \epsilon_A^-(-\theta)$ and the related functions

$$L_A^{\pm}(\theta) = \ln(1 + e^{-\epsilon_A^{\pm}(\theta)}) \tag{3.24}$$

The kernels in the integrals are given by

$$\Phi_{AB}(\theta) = \Phi_{BA}(-\theta) = -i\frac{d}{d\theta} \ln S_{AB}(\theta) \tag{3.25}$$

Notice that the second equation in (3.22) may be obtained from the first one simply by the parity transformation $\theta \to -\theta$ and the first equality in (3.25). The main difference of these equations in comparison with the parity invariant case is that we have lost the usual symmetry of the pseudo-energies as a function of the rapidities, such that now in general it holds $\epsilon_A^+(\theta) \neq \epsilon_A^-(\theta)$: this symmetry may be recovered by restoring parity.

[4]These two sets of equations are of course not entirely independent and may be obtained from each other by complex conjugation with the help of relation (3.16).

To proceed further, it is convenient to consider the following separation of the kernel (3.25) into two parts:

$$\phi_{ab}(\theta) = \phi^{ii}_{ab}(\theta) = -i\frac{d}{d\theta}\ln S^{ii}_{ab}(\theta) + \int dt\,\delta_{ab}\,e^{-it\theta} \tag{3.26}$$

$$= \int dt \left\{ \delta_{ab} - \frac{2\cosh\left(\frac{\pi t}{k}\right)}{\left[2\cosh\left(\frac{\pi t}{k}\right) - I\right]_{ab}} \right\} e^{-it\theta} \tag{3.27}$$

$$\psi_{ab}(\theta) = \phi^{ij}_{ab}(\theta + \sigma_{ji}) = -i\frac{d}{d\theta}\ln S^{ij}_{ab}(\theta) \tag{3.28}$$

$$= \int dt\,\frac{e^{-it\theta}}{\left[2\cosh\left(\frac{\pi t}{k}\right) - I\right]_{ab}} \tag{3.29}$$

where $\phi_{ab}(\theta)$ is just the TBA kernel of the $su(k)$-minimal ATFT and in the remaining kernels $\psi_{ab}(\theta)$ we have removed the resonance shift. The integral representations for these new kernels are obtained easily from the expressions (3.10) and (3.13). They are generically valid for all values of the level k. Then, the convolution term in the first of equations (3.22) can be recast as

$$\sum_{j=1}^{l}\sum_{b=1}^{k-1}\phi^{ij}_{ab} * L^{j}_{b}(\theta) = \sum_{b=1}^{k-1}\phi_{ab} * L^{i}_{b}(\theta) + \sum_{j\neq i,j=1}^{l}\sum_{b=1}^{k-1}\psi_{ab} * L^{j}_{b}(\theta - \sigma_{ji}) \tag{3.30}$$

This rewriting shows that, whenever we are in a regime in which the second term is negligible ($\sigma \to \infty$), we are left with l non-interacting copies of the $su(k)$-minimal ATFT. We will return to this limit in section 3.4, but now let us focus on the special limit $\sigma \to 0$, which correspond to the parity invariant case: for the classical theory it was pointed out in [14] that only then the equations of motion and the TBA-equations are parity invariant. However, even in the absence of the resonance shifts, the S-matrix still violates parity through the phase factors η. Then, from (3.22), we remove the split $(+,-)$ and we are left with l copies of the system

$$\epsilon^{i}_{a}(\theta) + \sum_{b=1}^{k-1}(\phi_{ab} + \psi_{ab}) * L^{i}_{b}(\theta) = rM^{i}_{a}\cosh\theta \tag{3.31}$$

Now that everything necessary has been properly introduced and defined, we are ready to derive the universal form for the TBA integral equations of the HSG coset models, particularly advantageous when one wants to discuss properties of the model leaving the level k generic. By means of the convolution theorem and the Fourier transforms of the TBA kernels ϕ_{ab} and ψ_{ab}, which can be read off directly from (3.27) and (3.29), in the following we illustrate the calculus in detail. Once we rename $\nu_a^i(\theta) = rM_a^i \cosh\theta$, we can rewrite the first equation in (3.22) splitting the convolution term,

$$\epsilon_a^i(\theta) + \sum_{b=1}^{k-1} \left(\phi_{ab} * L_b^i\right)(\theta) + \sum_{j\neq i, j=1}^{l} \sum_{b=1}^{k-1} \left(\psi_{ab} * L_b^j\right)(\theta - \sigma_{ji}) = \nu_a^i(\theta) \tag{3.32}$$

After reformulating in momentum space through the usual definition

$$\tilde{f}_a^i(k) = \int_{-\infty}^{+\infty} d\theta\, f_a^i(\theta)\, e^{ik\theta} \tag{3.33}$$

we multiply and divide for $e^{ik\theta'}$ inside the convolution terms in order to get

$$\tilde{\epsilon}_a^i(k) + \sum_{b=1}^{k-1} \int_{\mathbb{R}} \frac{d\theta'}{2\pi}\, e^{ik\theta'} L_b^i(\theta') \int_{-\infty}^{+\infty} d\theta\, e^{ik(\theta-\theta')} \phi_{ab}(\theta-\theta') +$$

$$+ \sum_{j=1}^{l} \sum_{b=1}^{k-1} \int_{\mathbb{R}} \frac{d\theta'}{2\pi}\, e^{ik\theta'} L_b^j(\theta' - \sigma_{ji}) \int_{-\infty}^{+\infty} d\theta\, e^{ik(\theta-\theta')} \psi_{ab}(\theta-\theta') = \tilde{\nu}_a^i(k) \tag{3.34}$$

where we have shifted the rapidity variable $\theta' + \sigma_{ji} \to \theta'$ so that to make the arguments of the kernels homogeneous. At this point, we must return to expressions (3.27) and (3.29) to identify their Fourier transform:

$$\phi_{ab}(\theta) = \int_{-\infty}^{+\infty} dt\, \tilde{\phi}_{ab}(t)\, e^{-it\theta} \quad \longrightarrow \quad \tilde{\phi}_{ab}(t) = \delta_{ab} - \frac{2\cosh\left(\frac{\pi t}{k}\right)}{\left[2\cosh\left(\frac{\pi t}{k}\right) - I\right]_{ab}} \tag{3.35}$$

for which holds the Zamolodchikov's identity[5] [35]

$$\left(\delta_{ab} - \frac{1}{2\pi}\tilde{\phi}_{ab}\right)^{-1} = \delta_{ab} - \frac{I_{ab}}{2\cosh\left(\frac{\pi t}{k}\right)} = \delta_{ab} - \tilde{R}(t) I_{ab} \tag{3.36}$$

or equivalently

$$\tilde{\phi}_{ab}(k) = \sum_c I_{bc}\tilde{\phi}_{ac}\tilde{R} - 2\pi I_{ab}\tilde{R} \tag{3.37}$$

with I_{ab} is the incidence matrix of the $su(k)$-Dynkin diagram. For the other kernel ψ_{ab} we can say that

$$\psi_{ab}(\theta) = \int_{-\infty}^{+\infty} dt\, \tilde{\psi}_{ab}(t)\, e^{-it\theta} \quad \longrightarrow \quad \tilde{\psi}_{ab}(t) = \frac{1}{\left[2\cosh\left(\frac{\pi t}{k}\right) - I\right]_{ab}} \tag{3.38}$$

[5] Here k plays the role of h in [35, 32], that is the Coxeter number of the simple compact Lie group G.

for which we have a slightly different identity

$$\tilde{\psi}_{ab}(k) = \sum_c I_{bc}\tilde{\psi}_{ac}\tilde{R} + 2\pi\delta_{ab}\tilde{R} \tag{3.39}$$

Now we can continue, renaming $b \to c$ and multiplying both side for $(\delta_{ab} - \tilde{R}I_{ab})$, following the same procedure illustrated in [35]; terms like these appear for the integral contributions:

$$\frac{1}{2\pi}\sum_{c=1}^{k-1}\int_{\mathbb{R}} d\theta'\, e^{ik\theta'} L_c^i(\theta')\tilde{\phi}_{bc}(k) \tag{3.40}$$

$$-\frac{1}{2\pi}\sum_{c=1}^{k-1}\int_{\mathbb{R}} d\theta'\, e^{ik\theta'} \tilde{R}I_{ab}L_c^i(\theta')\tilde{\phi}_{ac}(k) \tag{3.41}$$

and

$$\frac{1}{2\pi}\sum_{j=1}^{l}\sum_{c=1}^{k-1}\int_{\mathbb{R}} d\theta'\, e^{ik\theta'} L_c^j(\theta' - \sigma_{ji})\tilde{\psi}_{bc}(k) \tag{3.42}$$

$$-\frac{1}{2\pi}\sum_{j=1}^{l}\sum_{c=1}^{k-1}\int_{\mathbb{R}} d\theta'\, e^{ik\theta'} \tilde{R}I_{ab}L_c^j(\theta' - \sigma_{ji})\tilde{\psi}_{ac}(k) \tag{3.43}$$

We can simplify the notation in this way

$$\begin{aligned}&\tilde{\epsilon}_b^i(k) - \tilde{R}I_{ab}\tilde{\epsilon}_a^i(k) + \frac{1}{2\pi}\sum_c \mathcal{F}\left[L_c^i\right]\tilde{\phi}_{bc}(k) - \frac{1}{2\pi}\sum_c \mathcal{F}\left[L_c^i\right]\tilde{R}I_{ab}\tilde{\phi}_{ac}(k) + \\ &+\frac{1}{2\pi}\sum_{j,c}\mathcal{F}\left[L_c^j\right]\tilde{\psi}_{bc}(k) - \frac{1}{2\pi}\sum_{j,c}\mathcal{F}\left[L_c^j\right]\tilde{R}I_{ab}\tilde{\psi}_{ac}(k) = \tilde{\nu}_b^i(k) - \tilde{R}I_{ab}\tilde{\nu}_a^i(k)\end{aligned} \tag{3.44}$$

Now, using (3.37) and (3.39), we can rewrite in the following form

$$\tilde{\epsilon}_a^i(k) + \tilde{R}I_{ab}\tilde{\nu}_b^i(k) + \tilde{R}I_{ab}\tilde{\nu}_b^i(k) - \sum_b \mathcal{F}\left[L_b^i\right] I_{ab}\tilde{R} + \sum_{j,c}\mathcal{F}\left[L_c^j\right]\delta_{cb}\tilde{R} = \tilde{\nu}_a^i(k)$$

Considering the Fourier anti-transform of $\tilde{R}(t)$

$$\varphi_k(\alpha) = \int_{-\infty}^{+\infty} dt\, \tilde{R}(t)e^{-it\alpha} = \frac{k/2}{\cosh\left(\frac{k\alpha}{2}\right)} = 2\pi\bar{\mathcal{F}}[\tilde{R}] \tag{3.45}$$

and merging appropriately the terms, we can recast as

$$\tilde{\epsilon}_a^i(k) + \tilde{R} I_{ab} \tilde{\epsilon}_b^i(k) - \sum_b \int_{-\infty}^{+\infty} \frac{d\alpha}{2\pi} e^{ik\alpha} \varphi_k(\alpha) I_{ab} \int_{\mathbb{R}} d\theta' e^{ik\theta'} \left[\epsilon_b^i(\theta') + L_b^i(\theta')\right] +$$

$$+ \sum_{j,b} \int_{-\infty}^{+\infty} \frac{d\alpha}{2\pi} e^{ik\alpha} \varphi_k(\alpha) \delta_{ab} \int_{\mathbb{R}} d\theta' e^{ik\theta'} L_b^j(\theta' - \sigma_{ji}) = \tilde{\nu}_a^i(k) \tag{3.46}$$

Finally, setting $\alpha = \theta - \theta'$, we get

$$\tilde{\epsilon}_a^i(k) + \sum_{j,b} \delta_{ab} \int_{-\infty}^{+\infty} d\theta e^{ik\theta} \left(\varphi_k * L_b^j\right)(\theta - \sigma_{ji}) = \sum_b I_{ab} \int_{-\infty}^{+\infty} d\theta e^{ik\theta} \left[\varphi_k * \left(\epsilon_b^i + L_b^i\right)\right](\theta) \tag{3.47}$$

where we have dropped out

$$\sum_b \tilde{R} I_{ab} \tilde{\nu}_b^i(k) - \tilde{\nu}_a^i(k) \tag{3.48}$$

because of the crucial property of the mass spectrum inherited from ATFT [28]

$$\sum_b I_{ab} M_b^i = 2\cos\left(\frac{\pi}{k}\right) M_a^i \tag{3.49}$$

which implies [29, eq. (80)]

$$\frac{r}{2\pi} \sum_b I_{ab} M_b^i \int d\theta' \frac{k/2}{\cosh\left[\frac{k}{2}(\theta - \theta')\right]} \cosh\theta' = r M_a^i \cosh\theta' \tag{3.50}$$

Coming back to the rapidity space, we obtain the desired *universal TBA equations* for the $\frac{SU(N)_k}{[U(1)]^{N-1}}$ HSG model

$$\epsilon_a^i(\theta) + \Omega_k * L_a^j(\theta - \sigma_{ji}) = \sum_{b=1}^{k-1} I_{ab}\, \Omega_k * \left(\epsilon_b^i + L_b^i\right)(\theta) \tag{3.51}$$

where I_{ab} denotes the incidence matrix of the $su(k)$ algebras and the kernel Ω_k is found to be

$$\Omega_k(\theta) = \frac{k}{2\cosh\left(\frac{k}{2}\theta\right)} \tag{3.52}$$

Closely related to the TBA equations in the form (3.51) are the following functional relations (also called *Y-system*). Using complex continuation [29] and defining the quantity

$$Y_a^i(\theta) = e^{-\epsilon_a^i(\theta)} \tag{3.53}$$

the integral equations are replaced by the HSG Y-system

$$Y_a^i\left(\theta+i\frac{\pi}{k}\right)Y_a^i\left(\theta-i\frac{\pi}{k}\right)=\left[1+Y_a^j(\theta-\sigma_{ji})\right]\prod_{b=1}^{k-1}\left[1+Y_b^i(\theta)^{-1}\right]^{-I_{ab}} \tag{3.54}$$

These systems are useful in many aspects, for instance they may be exploited in order to establish periodicities in the Y-functions, which in turn can be used to provide approximate analytical solutions of the TBA-equations.

3.3 The algebra level $k=2$

In section 3.1 we have connected our TBA-like integral equations (2.98), which represent the solution to the problem of minimal surfaces in AdS_3, with the NLIEs for the $\frac{SU(N)_2}{[U(1)]^{N-1}}$ HSG models coming from the following S-matrix (here we write only the non-trivial part) [13]

$$S_{ab}(\theta)=(-1)^{\delta_{ab}}\left\{c_a\tanh\left[\frac{1}{2}\left(\theta+\sigma_{ab}-i\frac{\pi}{2}\right)\right]\right\}^{I_{ab}} \tag{3.55}$$

Precisely for this reason, one might think that there is a deeper connection with our AdS_3 case, such as with our Y-system (2.69), perhaps hidden in the universal form (3.51) and in the corresponding set of functional relations (3.54). This would also imply that the S-matrix considered by [13] coincides, for $k=2$, with those previously proposed to describe the HSG models [19, 20]. In this section, we will highlight this link exploiting [21] and test it for the case of the decagon in AdS_3: useful correspondences for the relevant indices, inherent to the two different contexts, will be specified.

Therefore, let us focus on the $k=2$ case and see how we can rewrite the scattering matrices (3.9) and (3.12) in a more compact form, by considering very general properties. The theory has a fairly rich particle content: $N-1$ asymptotically stable particles characterized by a mass scale m_i and $N-2$ unstable particles whose energy scale is characterized by the resonance parameters σ_{ij} $(1\le i,j\le N-1)$. The stable particles are in a 1-1 fashion to the vertices of the $SU(N)$-Dynkin diagram and we associate to the link between vertex i and j the $N-2$ linearly independent resonance parameters σ_{ij}. Once an unstable particle becomes extremely heavy, the original coset decouples into a direct product of two cosets different from the original one [21]

$$\lim_{\sigma_{i,i+1}\to\infty}=\frac{SU(N)_2}{[U(1)]^{N-1}}\equiv\frac{SU(i+1)_2}{[U(1)]^{i}}\otimes\frac{SU(N-i)_2}{[U(1)]^{N-i-1}} \tag{3.56}$$

This is equivalent to cutting the related Dynkin diagram at the largest resonance parameter at some energy scale, such as $\sigma_{i,i+1}$ in the previous relation.

In the general case $\frac{SU(N)_2}{[U(1)]^{N-1}}$ where only the level is fixed, the scattering of two stable particles of type $(1,i)$ and $(1,j)$, with $1 \leq i, j \leq N-1$, is described by the 2-p S-matrices (3.9) and (3.12), which can be rewritten in a slightly different way [21]

$$S_{ij}(\theta) = (-1)^{\delta_{ij}} \left\{ c_i \tanh \left[\frac{1}{2} \left(\theta + \sigma_{ij} - i\frac{\pi}{2} \right) \right] \right\}^{I_{ij}} \tag{3.57}$$

Already at this level we can identify correspondences in the indices if we compare with (3.55). The incidence matrix of the $SU(N)$-Dynkin diagram is denoted by I. The parity breaking which is characteristic for the HSG models and manifests itself by the fact that $S_{ij} \neq S_{ji}$, takes place through the resonance parameters $\sigma_{ij} = -\sigma_{ji}$ and the colour value c_i. The latter quantity arises from a partition of the Dynkin diagram into two disjoint sets, which we refer to as "+" and "-". We then associate the values $c_i = +1$ to the vertices i of the Dynkin diagram of SU(N), in such a way that no two vertices related to the same set are linked together. The resonance poles in $S_{ij}(\theta)$ at $(\theta_R)_{ij} = -\sigma_{ij} - i\frac{\pi}{2}$ are associated in the usual Breit-Wigner fashion to the $N-2$ unstable particles. It is important to recall that the mass of the unstable particle M_c, formed in the scattering between the stable particles i and j, behaves as $M_c \sim e^{|\sigma_{ij}|/2}$. There are no poles present on the imaginary axis, which indicates that no stable bound states may be formed. It is clear from the expression of the scattering matrix (3.57), that whenever a resonance parameter σ_{ij} with $I_{ij} \neq 0$ goes to infinity, we may view the whole system as consisting out of two sets of particles which only interact freely amongst each other. The unstable particle, which was created in interaction process between these two theories before taking the limit, becomes so heavy that it can not be formed anymore at any energy scale.

Considering (3.9), we can easily see that $(1 \leq a, b \leq 1)$

$$\begin{aligned} S_{11}^{ii}(\theta) &= (2)_\theta (0)_\theta \prod_{n=1}^{min(1,1)-1} (2-2n)_\theta^2 = \\ &= \frac{\sinh\left[\frac{1}{2}\left(\theta + \frac{2\pi i}{2}\right)\right]}{\sinh\left[\frac{1}{2}\left(\theta - \frac{2\pi i}{2}\right)\right]} = -1 \end{aligned} \tag{3.58}$$

Instead, from (3.12), we obtain

$$\begin{aligned} S_{11}^{ij}(\theta) &= \eta_{ij} \prod_{n=0}^{min(1,1)-1} \frac{\sinh\left[\frac{1}{2}\left(\theta + \sigma_{ij} + \frac{i\pi}{k}(-1-2n)\right)\right]}{\sinh\left[\frac{1}{2}\left(\theta + \sigma_{ij} - \frac{i\pi}{k}(-1-2n)\right)\right]} = \\ &= \eta_{ij} \frac{\sinh\left[\frac{1}{2}\left(\theta + \sigma_{ij} + \frac{i\pi}{2}(-1)\right)\right]}{\sinh\left[\frac{1}{2}\left(\theta + \sigma_{ij} - \frac{i\pi}{2}(-1)\right)\right]} = \\ &= \tanh\left[\frac{1}{2}\left(\theta + \sigma_{ij} - i\frac{\pi}{2}\right)\right] \end{aligned} \tag{3.59}$$

provided that an appropriate value is chosen for η_{ij}. So that, we can reformulate the S-matrix for this particular situation in the compact form (3.57). For $N = 3$ we can check this statement and get [20] (there are only two self-conjugated solitons)

$$S_{11}^{11} = S_{11}^{22} = -1 \qquad \text{and} \qquad S_{11}^{12}(\theta - \sigma) = -S_{11}^{21}(\theta + \sigma) = \tanh\left[\frac{1}{2}\left(\theta - i\frac{\pi}{2}\right)\right] \tag{3.60}$$

where $\eta_{12} = -\eta_{21} = i$ and $\sigma := \sigma_{12} = -\sigma_{21}$. For the case considered, $I_{12} = I_{21} = +1$ is the incidence matrix of the $SU(3)$-Dynkin diagram.

Now we are ready to derive from (3.54) a set of functional relations equivalent to our AdS_3 Y-system (2.69). Setting $k = 2$ and $a = b = 1$ we get

$$Y_1^i\left(\theta + i\frac{\pi}{2}\right) Y_1^i\left(\theta - i\frac{\pi}{2}\right) = \left[1 + Y_1^j(\theta - \sigma_{ji})\right] \tag{3.61}$$

where $1 \leq i, j \leq 2$, $i \neq j$ and $I_{11} = 0$ allow us to drop out the factor with $\prod$. To check the equivalence[6], we must only explicit the functional relations and compare them with the some explicit example, such as the decagon studied in section 2.6. Then, we consider (2.173) and, explaining the system of functional relations, we find

$$Y_1^+ Y_1^- = Y_1\left(\theta + i\frac{\pi}{2}\right) Y_1\left(\theta - i\frac{\pi}{2}\right) = [1 + Y_2(\theta)] \tag{3.62}$$

and

$$Y_2^+ Y_2^- = Y_2\left(\theta + i\frac{\pi}{2}\right) Y_2\left(\theta - i\frac{\pi}{2}\right) = [1 + Y_1(\theta)] \tag{3.63}$$

paying attention to the fact that θ differs from (2.173) to (3.54) for a complex definition: see the definition (3.53) where θ is real. Therefore, once we consider a more general complex definition of the Y-functions implemented in (3.54),

$$Y_a^i(\theta + i\varphi_s) = e^{-\varepsilon_a^i(\theta)} \tag{3.64}$$

we can relate the quantity σ_{ij} to the phases φ_s of the complex rapidity used in (2.173):

$$\sigma_{ij} = i(\varphi_i - \varphi_j) \tag{3.65}$$

As a last note, if we consider higher values of N, in order to reproduce similar results for polygons with more sides, the systems of functional equations no longer exactly match, so that we assume that showing the equivalence means going back to the original definition of the Y-functions used for the AdS_3 case, see section 2.1 or [8], and considering their properties and periodicity.

As we can see from the explicit calculation of the previous matrices S_{11}^{ii} and S_{11}^{ij}, the non-trivial dynamical informations comes only from the scattering of two solitons with

[6] Another check could be the derivation of the integral equations coming from (3.61) following the usual procedure depicted in section 2.2.1, which would be coincide with the ones found for the decagon.

different colours, in general of type (a,i) and (b,j) with $i \neq j$, and therefore from the kernel

$$\psi_{ab}(\theta) = -i\frac{d}{d\theta}\ln S^{ij}_{ab}(\theta) \tag{3.66}$$

obtained through the partition of the full HSG kernel $\Phi_{AB}(\theta)$, as we said previously. These kernels correspond exactly with those of the TBA integral equations of our AdS_3 configuration (2.96), providing another check to the aforementioned link.

$$\begin{aligned}
\psi_{ab}(\theta) &= -i\frac{d}{d\theta}\ln\left\{(\eta_{ij})^{ab}\prod_{n=0}^{min(a,b)-1}\frac{\sinh\left[\frac{1}{2}\left(\theta+\sigma_{ij}+\frac{i\pi}{k}(-|a-b|-1-2n)\right)\right]}{\sinh\left[\frac{1}{2}\left(\theta+\sigma_{ij}-\frac{i\pi}{k}(-|a-b|-1-2n)\right)\right]}\right\} = \\
&= -i\sum_{n=0}^{min(a,b)-1}\frac{d}{d\theta}\ln\left\{\frac{\sinh\left[\frac{1}{2}\left(\theta+\sigma_{ij}+\frac{i\pi}{k}(-|a-b|-1-2n)\right)\right]}{\sinh\left[\frac{1}{2}\left(\theta+\sigma_{ij}-\frac{i\pi}{k}(-|a-b|-1-2n)\right)\right]}\right\} = \\
&= \frac{i}{2}\sum_{n=0}^{min(a,b)-1}\left\{\coth\left[\frac{1}{2}(\theta+\sigma_{ij}-\frac{i\pi}{k}\alpha)\right]-\coth\left[\frac{1}{2}(\theta+\sigma_{ij}+\frac{i\pi}{k}\alpha)\right]\right\} = \\
&= \sum_{n=0}^{min(a,b)-1}\frac{\sin\left[\frac{\pi}{k}(|a-b|+1+2n)\right]}{\cosh(\theta+\sigma_{ij})-\cos\left[\frac{\pi}{k}(|a-b|+1+2n)\right]}
\end{aligned} \tag{3.67}$$

In the $\frac{SU(N)_2}{[U(1)]^{N-1}}$ case,

$$k=2 \qquad 1\leq a,b\leq 1 \qquad 1\leq i,j\leq N-1=l \tag{3.68}$$

so that

$$\psi_{11}(\theta) = \frac{1}{\cosh(\theta+\sigma_{ij})} \tag{3.69}$$

in agreement with our AdS_3 kernels (2.96) through (3.65).

3.4 Narrowing down to the A_n series

In this section we would like to explain the reduction that leads to the Y-system of the $su(k)$-minimal ATFTs, namely the A_n series of the ADE classification, derived originally in [32]. This reduction is consistent with the limit $\sigma \to \infty$, noting that the asymptotic behaviour of the Y-functions (3.53) is

$$\lim_{\theta\to\infty} Y^i_a(\theta) \sim e^{-rM^i_a\cosh\theta} \tag{3.70}$$

from the asymptotic condition [20]

$$\lim_{\theta\to\infty}\epsilon^i_a(\theta) = rM^i_a e^{\pm\theta} \tag{3.71}$$

In this case, the Y-functions have a period related to the dimension of the perturbing operator as conjectured in [20],

$$Y_a^i(\theta + \frac{i\pi}{1-\Delta} + \sigma_{ji}) = Y_{\bar{a}}^j(\theta) \tag{3.72}$$

where, for vanishing resonance parameters $\sigma_{ii} = 0$ and the choice $g = su(2)$, this behaviour coincides with the one obtained in [32] for the A_n series:

$$A_n: \quad h = n+1 \quad \text{and} \quad \Delta(A_n) = \frac{2}{n+3} = \frac{2}{h+2} \tag{3.73}$$

from which

$$1 - \Delta = \frac{n+1}{n+3} = \frac{h}{h+2} \tag{3.74}$$

so that

$$Y_a^i\left(\theta + i\pi\frac{h+2}{2}\right) = Y_{n-a+1}^i(\theta) \tag{3.75}$$

with $\bar{a} = n - a + 1$ [33].

When σ tends to infinity, the mass ratio $\frac{m_1}{m_2}$ is not fixed: it may be chosen to be very large or very small! This is due to the fact that sending σ to infinity is equivalent to decouple the TBA systems for solitons with different colours by shifting one system to the infrared with respect to the scale parameter r. In other words, looking at relation (3.19), it is as if the two masses were no longer comparable and one of the two, say M_a^j, tended to infinity. The special value $\sigma = \infty$ corresponds in the classical theory to a choice of the vector couplings in (3.19) orthogonal to a simple root of G, so that we agree that the speech is fully equivalent to saying that one of the two masses becomes too large compared to the other.

Therefore, in the limit that we are dealing with, the second term in (3.30) is negligible and we are left with l non-interacting copies of the $su(k)$-minimal ATFT [21]. Later, we will see that this fact allow us to reproduce the TBA equations in the universal form proposed by Zamolodchikov in [32]. But, let us try to derive the ADE Y-system from (3.54) by performing the limit $\sigma \to \infty$:

$$\begin{aligned}
\lim_{\sigma\to\infty} Y_a^i\left(\theta + \frac{i\pi}{k}\right) Y_a^i\left(\theta - \frac{i\pi}{k}\right) &= \lim_{\sigma\to\infty} \left[1 + Y_a^j(\theta - \sigma_{ji})\right] \prod_{b=1}^{k-1} \left[1 + Y_b^i(\theta)^{-1}\right]^{-I_{ab}} \\
&= \prod_{b=1}^{k-1} \left[1 + Y_b^i(\theta)^{-1}\right]^{-I_{ab}} \qquad (3.76)
\end{aligned}$$

due to (3.70) and the above discussion on the mass ratio $\frac{m_i}{m_j}$. Then, whereas our definition of Y-functions (3.53) is exactly the reciprocal of that of Zamolodchikov [32] here indicated by $Y^i_{a,Zam}(\theta)$, we get

$$\left[Y^i_{a,Zam}(\theta+\frac{i\pi}{k})\right]^{-1}\left[Y^i_{a,Zam}(\theta-\frac{i\pi}{k})\right]^{-1}=\prod_{b=1}^{k-1}\left[1+Y^i_{b,Zam}(\theta)\right]^{-I_{ab}} \tag{3.77}$$

and therefore

$$\frac{1}{Y^i_{a,Zam}(\theta+\frac{i\pi}{k})}\frac{1}{Y^i_{a,Zam}(\theta-\frac{i\pi}{k})}=\prod_{b=1}^{k-1}\frac{1}{\left[1+Y^i_{b,Zam}(\theta)\right]^{I_{ab}}} \tag{3.78}$$

which exactly matches with the system of functional relations found in [32]. With the aim of establishing useful correspondences in the indices, in the following we will compare the notations used here with those used in [32] and [13]. From (3.78), we see that our k plays the role of h, the Coxeter number of the algebra considered in [32], so that the index b takes values between 1 and $h-1$, which in [32] corresponds to the number of particles N (the rank of the algebra). For the A_n series, $h=n-1$ where n indicates the number of particles [33]. Definitely, $h-1=n$ is satisfied only for the A_n series. Another way to see this correspondence is to consider the rank of the algebras $W(A_n)$, which from [34] is known to be n. When we deal with a general $su(k)$-ATFT algebra, the rank is simply $k-1$ so that, in our correspondence $k\leftrightarrow h$, this implies $h-1$. Then, only for the A_n series holds $h-1=n$ which then finally coincides with the number of particles. Summing up, we can establish a relation between the level k of the simple compact Lie group G of our coset $\frac{G_k}{H}$ and the index n which labels the A_n series: for $k=2$ we found the A_1 series, for $k=3$ the A_2 series, and so on following the rule $k=h=n+1$.

Wanting instead to make a comparison with [13], we have to note that the coset considered there is of the form $\frac{SU(n-2)_2}{[U(1)]^{n-3}}$, where $2n$ is the number of sides of the polygon (gluons). In this context, $n-3$ corresponds to the rank l of our simple compact Lie group G and the level k has been set to 2. Finally, since in [13] the index a, which labels the particles corresponding to each simple roots, runs until $n-3$, we can say that it corresponds to the index i, which in [20] label the colours and runs from 1 to l.

Now we would like to show the agreement existing with the series A_n taking into account the scattering matrices (block form) in the limit $\sigma\to\infty$, where only the component $S^{ii}_{ab}(\theta)$ describing the scattering between solitons with the same colour contributes (see (3.30)). From [33], we know the block form of the S-matrix for the A_n series,

$$S_{ab}(\theta)=f_{\frac{|a-b|}{n+1}}(\theta)f_{\frac{a+b}{n+1}}(\theta)\left[\prod_{k=1}^{min(a,b)-1}f_{\frac{|a-b|+2k}{n+1}}\right]^2 \tag{3.79}$$

with

$$f_\alpha(\theta)=\frac{\sinh\left(\frac{\theta+i\alpha\pi}{2}\right)}{\sinh\left(\frac{\theta-i\alpha\pi}{2}\right)} \tag{3.80}$$

It is clear that the k used in (3.9) corresponds to the $n+1$ used here in $S_{ab}(\theta)$, strengthening the relationship $k = h = n+1$. Furthermore, a direct computation of the building blocks for different values of k, taking in mind that $1 \leq a, b \leq k-1$, explicit the equivalence of the two matrices.

What we have discussed so far, allows us to understand how in the limit $\sigma \to \infty$ only the kernels $\varphi_{ab}(\theta)$,

$$\varphi_{ab}(\theta) = \phi_{ab}^{ii}(\theta) = -i\frac{d}{d\theta} \ln S_{ab}^{ii}(\theta) + \int dt \, \delta_{ab} \, e^{-it\theta} \tag{3.81}$$

participate at the dynamics in a non-trivial way, a situation somehow opposite to what we saw in the previous section 3.3. Thus, we expect that from the universal TBA equations (3.51) it is possible to reproduce the universal form of Zamolodchikov [32]. So, let us start from (3.46)

$$\tilde{\epsilon}_a^i(k) + \tilde{R} I_{ab} \tilde{\nu}_b^i(k) - \tilde{\nu}_a^i(k) - \sum_b \int_{-\infty}^{+\infty} \frac{d\alpha}{2\pi} e^{ik\alpha} \varphi_k(\alpha) I_{ab} \int_{\mathbb{R}} d\theta' e^{ik\theta'} \left[\epsilon_b^i(\theta') + L_b^i(\theta')\right] = 0 \tag{3.82}$$

Then, through passages very similar to section 3.2, we can write down

$$\epsilon_a^i(\theta) + \sum_b I_{ab} (\varphi_k * \nu_b^i)(\theta) - \nu_a^i(\theta) = \sum_b I_{ab} \left[\varphi_k * \left(\epsilon_b^i + L_b^i\right)\right](\theta) \tag{3.83}$$

and since

$$\epsilon_b^i(\theta) + L_b^i(\theta) = \log\left(1 + e^{\epsilon_b^i(\theta)}\right) \tag{3.84}$$

we obtain

$$\epsilon_a^i(\theta) - \nu_a^i(\theta) = \sum_b I_{ab} \left\{\varphi_k * \left[-\nu_b^i + \log\left(1 + e^{\epsilon_b^i}\right)\right]\right\}(\theta) \tag{3.85}$$

which is in agreement with [32, eq. (7)].

Chapter 4

The remainder function in the cross ratios frame: the modified TBA equations

4.1 The Yang-Yang functional for symmetric TBA equations

The general TBA equations with a symmetric kernel,

$$\ln Y_a(u) = L_a(u) + \frac{1}{2\pi}\sum_b \int_{l_b} du'\, \mathcal{K}_{ab}(u,u') \log[1+Y_b(u')] \tag{4.1}$$

can be written as *extreme conditions* for the following Yang-Yang functional [37],

$$\begin{aligned} YY = &\frac{1}{2\pi}\sum_a \int_{l_a} du \left[\rho_a(u)\phi_a(u) - Li_2\left(-e^{L_a(u)-\phi_a(u)}\right)\right] + \\ &+ \frac{1}{8\pi^2}\sum_{a,b}\int_{l_a} du \int_{l_b} du'\, \mathcal{K}_{ab}(u,u')\rho_a(u)\rho_b(u') \end{aligned} \tag{4.2}$$

To show this fact, we consider the variation with respect to $\rho_a(u)$,

$$\delta_\rho(YY) = \frac{1}{2\pi}\sum_a \int_{l_a} du\, \delta\rho_a(u) \left[\phi_a(u) + \frac{1}{2\pi}\sum_b \int_{l_b} du'\, \mathcal{K}_{ab}(u,u')\rho_b(u')\right] \tag{4.3}$$

$$\frac{\delta(YY)}{\delta\rho_a(u)} = 0 \quad \Longleftrightarrow \quad \phi_a(u) + \frac{1}{2\pi}\sum_b \int_{l_b} du'\, \mathcal{K}_{ab}(u,u')\rho_b(u') = 0 \tag{4.4}$$

and the variation with respect to $\phi_a(u)$,

$$\delta_\phi(YY) = \frac{1}{2\pi}\sum_a \int_{l_a} du\, \delta\phi_a(u) \left[\rho_a(u) - \log\left(1 + e^{L_a(u)-\phi_a(u)}\right)\right] \tag{4.5}$$

$$\frac{\delta(YY)}{\delta\phi_a(u)} = 0 \quad \Longleftrightarrow \quad \rho_a(u) - \log\left(1 + e^{L_a(u)-\phi_a(u)}\right) = 0 \tag{4.6}$$

To write the last condition we have used the following result, starting from the definition of the Rogers dilogarithm function:

$$Li_2\left(-e^{L_a(u)-\phi_a(u)}\right) \stackrel{def}{=} \sum_{n=1}^{+\infty} \frac{\left(-e^{L_a(u)-\phi_a(u)}\right)^n}{n^2} \tag{4.7}$$

$$\begin{aligned} \delta(Li_2) &= \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}\, \delta[e^{n(L_a-\phi_a)}] = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} e^{n(L_a-\phi_a)}(-n)\, \delta\phi_a = \\ &= \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(e^{L_a-\phi_a})^n}{n}\, \delta\phi_a = \ln(1 + e^{L_a-\phi_a})\, \delta\phi_a \end{aligned} \tag{4.8}$$

If we define $\ln Y_a(u) \equiv L_a(u) - \phi_a(u)$, from (4.6) we obtain

$$\rho_a(u) = \log[1 + Y_a(u)] \tag{4.9}$$

which inserted in (4.4) reproduces the general TBA equations (4.1).

4.1.1 The critical value YY_{cr}

Relations (4.4), (4.6) and (4.9), being the conditions that extreme the Yang-Yang functional, can be used to find its critical value YY_{cr} after a simple substitution:

$$\begin{aligned} YY_{cr} &= \frac{1}{2\pi}\sum_a \int_{l_a} du\, \log\left[1 + Y_a(u)\right] \frac{(-1)}{2\pi} \sum_b \int_{l_b} du'\, \mathcal{K}_{ab}(u,u') \log\left[1 + Y_b(u')\right] - \\ &\quad - \frac{1}{2\pi}\sum_a \int_{l_a} du\, Li_2\left[-Y_a(u)\right] + \\ &\quad + \frac{1}{8\pi^2}\sum_{a,b} \int_{l_a} du \int_{l_b} du'\, \mathcal{K}_{ab}(u,u') \log\left[1 + Y_a(u)\right] \log\left[1 + Y_b(u')\right] = \\ &= -\frac{1}{2\pi}\sum_a \int_{l_a} du\, Li_2\left[-Y_a(u)\right] - \\ &\quad - \frac{1}{8\pi^2}\sum_{a,b} \int_{l_a} du \int_{l_b} du'\, \mathcal{K}_{ab}(u,u') \log\left[1 + Y_a(u)\right] \log\left[1 + Y_b(u')\right] \end{aligned} \tag{4.10}$$

At the end of this chapter we will see that the interesting part of the regularized area (*the remainder function*) of the minimal surfaces in AdS_3 coincides with the extreme of the Yang-Yang functional for the *(symmetric) modified TBA equations.* Accordingly, the area will turn out to be the extreme of an action functional with fixed boundary conditions given by the choice of physical cross ratios.

4.2 The modified TBA equations

Our starting point is the complex TBA-like integral equations (2.98),

$$\log Y_s(\theta) = -|m_s|\cosh(\theta - i\varphi_s) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta-\theta')} \log\left[1 + Y_{s'}(\theta')\right] \quad (4.11)$$

which, after a simple manipulation of the forcing, can be recast in this fashion

$$\log Y_s(\theta) = \frac{Z_s}{e^{\theta + i\frac{\pi}{2}b_{s+1}}} + \bar{Z}_s e^{\theta + i\frac{\pi}{2}b_{s+1}} + I_0^b \quad (4.12)$$

where

$$I_0^b \equiv \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta-\theta')} \log\left[1 + Y_{s'}(\theta')\right] \quad (4.13)$$

The previous rewrite holds thanks to the setting relationships about the variable Z_s (and its complex conjugate $\bar{Z}_s$),

$$Z_s = \begin{cases} -\frac{|m_s|}{2} e^{i\varphi_s} & \text{for } s \text{ even} \\ -\frac{|m_s|}{2} e^{i\varphi_s + i\frac{\pi}{2}} & \text{for } s \text{ odd} \end{cases} \quad (4.14)$$

and the choice of the integration lines such that $\frac{Z_s}{\zeta} \in \mathbb{R}_-$, in agreement with [8], [37] and [13]. To derive the modified TBA equations (in short MTBA), we have to eliminate the variables Z_s and $\bar{Z}_s$ in favour of the *physical cross ratios* $\ln y_s^{\pm}$, which are obtained by setting[1] $\zeta = 1$ and $\zeta = i$ inside (4.12). As we can easily see for the even case, θ has to take the values 0 and $i\frac{\pi}{2}$ to recover the corresponding values $\zeta = 1$ and $\zeta = i$; conversely, in the odd case the respective values are $-i\frac{\pi}{2}$ and 0. Thus, we can say that $\theta = -i\frac{\pi}{2}b_{s+1}$ corresponds to $\zeta = 1$ whilst $\theta = i\frac{\pi}{2}b_s$ matches with $\zeta = i$, coming to define the above mentioned physical cross ratios:

$$\begin{cases} \log Y_s(\theta = -i\frac{\pi}{2}b_{s+1}) \equiv \ln y_s^+ & \text{for} \quad \zeta = 1 \\ \log Y_s(\theta = i\frac{\pi}{2}b_s) \equiv \ln y_s^- & \text{for} \quad \zeta = i \end{cases} \quad (4.15)$$

[1]Here $\zeta = e^{\theta + i\frac{\pi}{2}b_{s+1}}$ depends on the quantity b_s, which plays the role of a parity qualifier in a purely arithmetic sense. In other words, b_s discriminates between even and odd indices.

We can now bring out the cross ratios $\ln y_s^+$ replacing $\zeta = 1$ inside (4.12),

$$\ln y_s^+ = Z_s + \bar{Z}_s + I_2^b \tag{4.16}$$

where

$$I_2^b \equiv \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh\left[-i\frac{\pi}{2}b_{s+1} - \theta'\right]} \mathcal{L}_{s'}(\theta') \tag{4.17}$$

and $\mathcal{L}_{s'}(\theta') = \log\left[1 + Y_{s'}(\theta')\right]$. Then we can extract the quantity $\bar{Z}_s$,

$$\bar{Z}_s = \ln y_s^+ - Z_s - I_2^b \tag{4.18}$$

An equivalent substitution for $\zeta = i$ leads to the cross ratios $\ln y_s^-$,

$$\ln y_s^- = -iZ_s + i\bar{Z}_s + I_1^b \tag{4.19}$$

where

$$I_1^b \equiv \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh\left[i\frac{\pi}{2}b_s - \theta'\right]} \mathcal{L}_{s'}(\theta') \tag{4.20}$$

After some algebra we get

$$Z_s = \frac{1}{2}\ln y_s^+ + \frac{i}{2}\ln y_s^- - \frac{i}{2}I_1^b - \frac{1}{2}I_2^b \tag{4.21}$$

together with

$$\bar{Z}_s = \frac{1}{2}\ln y_s^+ - \frac{i}{2}\ln y_s^- + \frac{i}{2}I_1^b - \frac{1}{2}I_2^b \tag{4.22}$$

Now we are ready to replace Z_s and $\bar{Z}_s$ inside (4.12) to obtain the *(not yet symmetric) MTBA equations*,

$$\log Y_s(\theta) = L_s(\theta) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^s\theta^{ss'}\sinh(2\theta)}{\sinh(2\theta')\cosh(\theta-\theta')} \mathcal{L}_{s'}(\theta') \tag{4.23}$$

where $L_s(\theta)$ indicates the forcing term,

$$L_s(\theta) \equiv \ln y_s^+ \cosh\left(\theta + i\frac{\pi}{2}b_{s+1}\right) - i\ln y_s^- \sinh\left(\theta + i\frac{\pi}{2}b_{s+1}\right) \tag{4.24}$$

If we introduce the new rapidity variable $u = \coth(2\theta)$, our previous MTBA equations (4.23) become symmetric and this feature is fundamental to reach their description through the Yang-Yang functional (4.2). Thus, we have

$$u = \frac{\cosh(2\theta)}{\sinh(2\theta)} \quad \longrightarrow \quad d\theta = -\frac{1}{2}\,du\,\sinh^2(2\theta) \tag{4.25}$$

$$\log Y_s(\theta) = L_s(\theta) + \frac{1}{2\pi} \sum_{s'=s-1}^{s+1} \int_D du' \, \mathcal{K}_{ss'}(\theta,\theta') \, \mathcal{L}_{s'}(\theta') \tag{4.26}$$

where we set

$$\mathcal{K}_{ss'}(\theta,\theta') = -\frac{(-1)^s \theta^{ss'} \sinh(2\theta') \sinh(2\theta)}{2\cosh(\theta-\theta')} \tag{4.27}$$

while the integration over u-rapidity is intended on a section of a straight line on the real axis [38, sec. 11], namely $D = (-1, -\infty) \cup (+\infty, +1)$.

4.2.1 The Yang-Yang functional for the $\frac{SU(N)_2}{[U(1)]^{N-1}}$ HSG models

As we explained in section 4.1, the symmetric MTBA equations (4.26) can be derived as the extreme conditions of the Yang-Yang functional (4.2), which we rewrite for convenience in our case of interest,

$$\begin{aligned} YY = & \frac{1}{2\pi} \sum_s \int_D du \left[\rho_s(u)\phi_s(u) - Li_2\left(-e^{L_s(u)-\phi_s(u)}\right)\right] + \\ & + \frac{1}{8\pi^2} \sum_{s,s'} \int_D du \int_{D'} du' \mathcal{K}_{ss'}(u,u')\rho_{s'}(u')\rho_s(u) \end{aligned} \tag{4.28}$$

Variations with respect to ρ_s and ϕ_s produce

$$\frac{\delta(YY)}{\delta\rho_s(u)} = 0 \quad \Longleftrightarrow \quad \phi_s(u) + \frac{1}{2\pi} \sum_{s'} \int_{D'} du' \, \mathcal{K}_{ss'}(u,u')\rho_{s'}(u') = 0 \tag{4.29}$$

together with

$$\frac{\delta(YY)}{\delta\phi_s(u)} = 0 \quad \Longleftrightarrow \quad \rho_s(u) = \log\left[1 + e^{L_s(u)-\phi_s(u)}\right] \tag{4.30}$$

If we set

$$L_s(u) - \phi_s(u) \equiv \log Y_s(u) \tag{4.31}$$

from (4.30) we obtain

$$\rho_s(u) = \log\left[1 + Y_s(u)\right] \tag{4.32}$$

and, consequently, replacing these last two relations inside (4.29) we can reproduce the symmetric MTBA equations (4.26). To find the critical value of the Yang-Yang functional, we have to replace its extreme conditions inside (4.28) and then we will get

$$\begin{aligned} YY_{cr} &= -\frac{1}{2\pi} \sum_s \int_D du \, Li_2\left[-Y_s(\theta)\right] - \\ & \quad -\frac{1}{8\pi^2} \sum_{s,s'} \int_D du \int_{D'} du' \, \mathcal{K}_{ss'}(\theta,\theta') \log\left[1 + Y_{s'}(\theta')\right] \log\left[1 + Y_s(\theta)\right] = \\ &= YY_{cr}^{(1)} + YY_{cr}^{(2)} \end{aligned} \tag{4.33}$$

This critical value YY_{cr} will be very useful at the end of this chapter when, after calculating the two contributions of the remainder functions $A = A_{periods} + A_{free}$, we will try to identify it with A_{free}. But now, we would like to underline an important link started in chapter 3 where, starting from (2.98) and taking advantage of *route b*, we had connected with a particular class of HSG models thanks to equation (2.107). As a matter of fact, we now want to specify that the Yang-Yang functional (4.28) that we have built before (and also its critical value (4.33) just computed) turns out to be the functional describing the $\frac{SU(N)_2}{[U(1)]^{N-1}}$ HSG models that we have considered. This important link is valid by virtue of the equation from which we started to derive the MTBA equations, that is (2.98), the same equation that, following *route b*, leads to the HSG models as described in section 3.1.

4.3 The remainder function

In this section we will compute the interesting part of the regularized area, the so called remainder function, exploiting the procedure outlined in section 2.4. For this purpose, we need to bring out the conserved charges hidden in the large θ-expansion of $\log Y_s(\theta)$. Each of the two consequent regimes will produce three kinds of terms with none, one and two integrals: but a most useful rewriting of that results is given by their average, which reveals vast simplifications on the two integrals terms. As we have already seen, only in the end we will split the result of this latter average in two contributions, which we identify with $A_{periods}$, the term without integrals, and A_{free}, the term with only one integral. So far there is no difference from what has been done in chapter 2; but the next step is crucial within our new framework, where we want to express the remainder function only in terms of physical cross ratios. This particular step consists of a very precise replacement within A_{temp}, the average of the one integral terms. The latter substitution will allow us to identify A_{temp} with YY_{cr}, after appropriate simplifications and the well known change of the rapidity variable that guarantees the symmetry.

4.3.1 Small ζ regime

We expand (4.23) for $\theta \to -\infty$,

$$\log Y_s(\theta) \sim \sum_{n=-1}^{+\infty} \tilde{c}_{n,s}\, e^{n\left(\theta + i\frac{\pi}{2}b_{s+1}\right)} \tag{4.34}$$

and we extract the following charges:

$$\tilde{c}_{-1,s} = \frac{1}{2}(\ln y_s^+ + i\ln y_s^-) - \sum_{s'} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^s \theta^{ss'} e^{-\theta' + i\frac{\pi}{2}b_{s+1}}}{\sinh(2\theta')} \mathcal{L}_{s'}(\theta') \tag{4.35}$$

$$\tilde{c}_{1,s} = \frac{1}{2}(\ln y_s^+ - i \ln y_s^-) + \sum_{s'} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^s \theta^{ss'} e^{-3\theta' - i\frac{\pi}{2}b_{s+1}}}{\sinh(2\theta')} \mathcal{L}_{s'}(\theta') \tag{4.36}$$

Now, using (2.131), which is valid for the area when $\theta \to -\infty$, we get three main contributions with none, one and two integrals:

$$A = -i \sum_{s,s'} \omega_{ss'} \tilde{c}_{-1,s} \, \tilde{c}_{1,s'} = A_0^- + A_{temp}^{(1)} + A_2^- \tag{4.37}$$

In the previous expression, A_0^- is the term with none integral,

$$\begin{aligned} A_0^- &= -\frac{i}{4} \sum_{s,s'} \omega_{ss'} \left(-i \ln y_s^+ \ln y_{s'}^- + i \ln y_s^- \ln y_{s'}^+ \right) = \\ &= -\frac{1}{2} \sum_{s,s'} \omega_{ss'} \ln y_s^+ \ln y_{s'}^- \end{aligned} \tag{4.38}$$

$A_{temp}^{(1)}$ is the term with only one integral,

$$\begin{aligned} A_{temp}^{(1)} = -\frac{i}{2} \sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} \frac{(-1)^{s+1}}{\sinh(2\theta)} \mathcal{L}_s(\theta) \times \\ \times \left[\left(e^{-3\theta - i\frac{\pi}{2}b_s} + e^{-\theta + i\frac{\pi}{2}b_s} \right) \ln y_s^+ - i \left(e^{-\theta + i\frac{\pi}{2}b_s} - e^{-3\theta - i\frac{\pi}{2}b_s} \right) \ln y_s^- \right] \end{aligned} \tag{4.39}$$

while A_2^- is the last term with two integrals,

$$A_2^- = -\frac{i}{4\pi^2} \sum_{s',s} (-1)^{s'+s+2} \theta^{s's} \int_{\mathbb{R}+i\varphi_{s'}} d\theta' \frac{e^{-\theta' + i\frac{\pi}{2}b_{s'}}}{\sinh(2\theta')} \mathcal{L}_{s'}(\theta') \int_{\mathbb{R}+i\varphi_s} d\theta \frac{e^{-3\theta - i\frac{\pi}{2}b_s}}{\sinh(2\theta)} \mathcal{L}_s(\theta) \tag{4.40}$$

4.3.2 Large ζ regime

We expand (4.23) also for $\theta \to +\infty$,

$$\log Y_s(\theta) \sim \sum_{n=-1}^{+\infty} c_{n,s} \, e^{-n\left(\theta + i\frac{\pi}{2}b_{s+1}\right)} \tag{4.41}$$

and we get another two conserved charges:

$$c_{-1,s} = \frac{1}{2}(\ln y_s^+ - i \ln y_s^-) + \sum_{s'} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^s \theta^{ss'} e^{\theta' - i\frac{\pi}{2}b_{s+1}}}{\sinh(2\theta')} \mathcal{L}_{s'}(\theta') \tag{4.42}$$

$$c_{1,s} = \frac{1}{2}(\ln y_s^+ + i \ln y_s^-) - \sum_{s'} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^s \theta^{ss'} e^{3\theta' + i\frac{\pi}{2}b_{s+1}}}{\sinh(2\theta')} \mathcal{L}_{s'}(\theta') \tag{4.43}$$

Now, using (2.137), which is valid for the area when $\theta \to +\infty$, we get three main contributions with none, one and two integrals:

$$A = i\sum_{s,s'} \omega_{ss'} c_{-1,s}\, c_{1,s'} = A_0^+ + A_{temp}^{(2)} + A_2^+ \tag{4.44}$$

In the previous expression, A_0^+ is the term with none integral equivalent to A_0^-,

$$A_0^+ = -\frac{1}{2}\sum_{s,s'} \omega_{ss'} \ln y_s^+ \ln y_{s'}^- \tag{4.45}$$

$A_{temp}^{(2)}$ is the term with only one integral,

$$A_{temp}^{(2)} = -\frac{i}{2}\sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} \frac{(-1)^{s+1}}{\sinh(2\theta)} \mathcal{L}_s(\theta)\times$$
$$\times \left[\left(e^{3\theta+i\frac{\pi}{2}b_s} + e^{\theta-i\frac{\pi}{2}b_s}\right)\ln y_s^+ - i\left(e^{3\theta+i\frac{\pi}{2}b_s} - e^{\theta-i\frac{\pi}{2}b_s}\right)\ln y_s^-\right] \tag{4.46}$$

while A_2^+ is the last term with two integrals,

$$A_2^+ = \frac{i}{4\pi^2}\sum_{s',s}(-1)^{s'+s+2}\theta^{s's} \int_{\mathbb{R}+i\varphi_{s'}} d\theta' \frac{e^{\theta'-i\frac{\pi}{2}b_{s'}}}{\sinh(2\theta')}\mathcal{L}_{s'}(\theta') \int_{\mathbb{R}+i\varphi_s} d\theta \frac{e^{3\theta+i\frac{\pi}{2}b_s}}{\sinh(2\theta)}\mathcal{L}_s(\theta) \tag{4.47}$$

4.3.3 Averaged results

In order to find a simple expression that describes the remainder function, we have to average the previous results of the area. We will denote with A_0, A_{temp} and A_2 their respective averages. A_0 is very simple:

$$A_0 = \frac{A_0^- + A_0^+}{2} = -\frac{1}{2}\sum_{s,s'} \omega_{ss'} \ln y_s^+ \ln y_{s'}^- \tag{4.48}$$

A_{temp} requires a greater effort, also because inside it hides the rigid mathematical structure without which identification with YY_{cr} would be impossible. Let us see in what follows the complete steps that led to the critical value of the Yang-Yang functional. The average of the one integral terms reads[2]

$$A_{temp} = \frac{A_{temp}^{(1)} + A_{temp}^{(2)}}{2} =$$
$$= -\sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} \frac{\cosh(2\theta)}{\sinh(2\theta)} \mathcal{L}_s(\theta)\times$$

[2]The desired mathematical structure lies inside the square brackets.

$$\times\left[(-1)^{s+1} i \ln y_s^+ \cosh(\theta + i\frac{\pi}{2} b_s) - i(-1)^{s+1} i \ln y_s^- \sinh(\theta + i\frac{\pi}{2} b_s)\right] \tag{4.49}$$

At this point it is important to note that the term in parenthesis is the derivative of the forcing term of the (not yet symmetric) MTBA equation (4.23),

$$A_{temp} = -\sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} \frac{\cosh(2\theta)}{\sinh(2\theta)} \mathcal{L}_s(\theta)\times$$

$$\times\left[\ln y_s^+ \sinh(\theta + i\frac{\pi}{2} b_{s+1}) - i \ln y_s^- \cosh(\theta + i\frac{\pi}{2} b_{s+1})\right] =$$

$$= -\sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} \frac{\cosh(2\theta)}{\sinh(2\theta)} \mathcal{L}_s(\theta)\times$$

$$\times\partial_\theta\left[\ln y_s^+ \cosh(\theta + i\frac{\pi}{2} b_{s+1}) - i \ln y_s^- \sinh(\theta + i\frac{\pi}{2} b_{s+1})\right] =$$

$$= -\sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} (u)\, \partial_\theta\left[L_s(\theta)\right] \mathcal{L}_s(\theta) \tag{4.50}$$

Replacing $L_s(\theta)$ inside (4.50) we get

$$A_{temp} = -\sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} (u)\, \partial_\theta\left[\ln Y_s(\theta)\right] \mathcal{L}_s(\theta)+$$

$$+\sum_{s,s'} \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} (u) \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^s \theta^{ss'}}{\sinh(2\theta')} \partial_\theta\left[\frac{\sinh(2\theta)}{\cosh(\theta-\theta')}\right] \mathcal{L}_{s'}(\theta')\mathcal{L}_s(\theta) \tag{4.51}$$

Using the property

$$\partial_\theta\left[\ln Y_s(\theta)\right] \log\left[1+Y_s(\theta)\right] = \partial_\theta\left[-Li_2\left(-Y_s\right)\right] \tag{4.52}$$

we can recast A_{temp} in this fashion

$$A_{temp} = -\sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} (u)\, \partial_\theta\left[-Li_2\left(-Y_s(\theta)\right)\right]+$$

$$+\frac{1}{4\pi^2}\sum_{s,s'} \int_{\mathbb{R}+i\varphi_s} \frac{d\theta \mathcal{L}_s(\theta)}{\sinh(2\theta)} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta' \mathcal{L}_{s'}(\theta')}{\sinh(2\theta')} (-1)^s \theta^{ss'}\left[\cosh(2\theta)\partial_\theta\left(\frac{\sinh(2\theta)}{\cosh(\theta-\theta')}\right)\right] \tag{4.53}$$

which, after an integration by parts, becomes

$$A_{temp} = b.t. - \sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} \partial_\theta(u)\left[Li_2\left(-Y_s(\theta)\right)\right]+$$

$$+\frac{1}{4\pi^2}\sum_{s,s'}\int_{\mathbb{R}+i\varphi_s}\frac{d\theta\mathcal{L}_s(\theta)}{\sinh(2\theta)}\int_{\mathbb{R}+i\varphi_{s'}}\frac{d\theta'\mathcal{L}_{s'}(\theta')}{\sinh(2\theta')}(-1)^s\theta^{ss'}\left[\cosh(2\theta)\partial_\theta\left(\frac{\sinh(2\theta)}{\cosh(\theta-\theta')}\right)\right] \tag{4.54}$$

Let's focus on the integral terms: the first is none other than $YY_{cr}^{(1)}$, that is the first addendum in (4.33),

$$-\frac{1}{2\pi}\sum_s\int_D du\, Li_2\left[-Y_s(\theta)\right]=YY_{cr}^{(1)} \tag{4.55}$$

while the second one, after an appropriate symmetry, becomes

$$\frac{1}{16\pi^2}\sum_{s,s'}\int_D du\int_{D'}du'(-1)^s\theta^{ss'}\sinh(2\theta)\sinh(2\theta')\left[\cosh(2\theta)\partial_\theta\left(\frac{\sinh(2\theta)}{\cosh(\theta-\theta')}\right)\right]\mathcal{L}_{s'}(\theta')\mathcal{L}_s(\theta) \tag{4.56}$$

Now, in order to show that $YY_{cr}^{(2)}$ emerges from this second integral term, we have to rewrite the square brackets in a simpler form; immediately after, we will see that the latter simpler form cancels out with the remaining two-integrals terms, once they have been averaged. Let's proceed to simplify the square brackets first:

$$\cosh(2\theta)\partial_\theta\left(\frac{\sinh(2\theta)}{\cosh(\theta'-\theta)}\right)=$$

$$=\frac{2\cosh^2(2\theta)}{\cosh(\theta'-\theta)}+\frac{\cosh(2\theta)\sinh(2\theta)\sinh(\theta'-\theta)}{\cosh^2(\theta'-\theta)}=$$

$$=\frac{1}{\cosh(\theta'-\theta)}+\frac{3\cosh(3\theta+\theta')+\cosh(5\theta-\theta')}{4\cosh^2(\theta'-\theta)}=$$

$$=\frac{1}{\cosh(\theta'-\theta)}+\frac{\sinh(2\theta+2\theta')\sinh(\theta-\theta')[3+\cosh(2\theta-2\theta')]}{4\cosh^2(\theta'-\theta)}+$$

$$+\frac{\sinh(2\theta-2\theta')\cosh(\theta-\theta')\sinh(2\theta+2\theta')}{4\cosh^2(\theta'-\theta)}+$$

$$+\frac{\cosh(2\theta+2\theta')\cosh(\theta-\theta')[3+\cosh(2\theta-2\theta')]}{4\cosh^2(\theta'-\theta)}+$$

$$+\frac{\sinh(2\theta-2\theta')\sinh(\theta-\theta')\cosh(2\theta+2\theta')}{4\cosh^2(\theta'-\theta)}=$$

$$=\frac{1}{\cosh(\theta'-\theta)}+\sinh(2\theta+2\theta')\sinh(\theta-\theta')+$$

$$+\cosh(2\theta+2\theta')\cosh(\theta-\theta')+\frac{\sinh(2\theta+2\theta')\sinh(\theta-\theta')}{2\cosh^2(\theta'-\theta)}$$

These four addenda are exactly what we need to reproduce $YY_{cr}^{(2)}$ and simplify the remaining two-integrals terms. In particular, we see that the first term, once replaced inside (4.56), produces

$$\frac{1}{16\pi^2}\sum_{s,s'}\int_D du\int_{D'}du'(-1)^s\theta^{ss'}\frac{\sinh(2\theta)\sinh(2\theta')}{\cosh(\theta-\theta')}\mathcal{L}_{s'}(\theta')\mathcal{L}_s(\theta)=$$

$$=-\frac{1}{8\pi^2}\sum_{s,s'}\int_D du\int_{D'}du'\left[-\frac{(-1)^s\theta^{ss'}\sinh(2\theta)\sinh(2\theta')}{2\cosh(\theta-\theta')}\right]\mathcal{L}_{s'}(\theta')\mathcal{L}_s(\theta)=YY_{cr}^{(2)} \quad (4.57)$$

which corresponds with the second addendum in (4.33). Instead, the second and third term cancel out with the average of the two-integrals terms which we will calculate in the following. What about the last term? This particular addendum does not contribute to the integral thanks to the antisymmetric property of both the intersection form $\theta^{ss'}$ and the resulting kernel. Now, we focus on A_2, the average of the remaining two-integrals terms,

$$A_2=\frac{A_2^-+A_2^+}{2}=$$

$$=\frac{1}{4\pi^2}\sum_{s,s'}(-1)^{s+s'+1}\theta^{ss'}\int_{\mathbb{R}+i\varphi_{s'}}\frac{d\theta'\mathcal{L}_{s'}(\theta')}{\sinh(2\theta')}\int_{\mathbb{R}+i\varphi_s}\frac{d\theta\mathcal{L}_s(\theta)}{\sinh(2\theta)}\times$$

$$\times\left[\frac{-e^{-i\frac{\pi}{2}(b_s-b_{s'}+1)-\theta'-3\theta}+e^{i\frac{\pi}{2}(b_s-b_{s'}+1)+\theta'+3\theta}}{2}\right]=$$

$$=\frac{1}{4\pi^2}\sum_{s,s'}(-1)^{s+s'+1}\theta^{ss'}\int_{\mathbb{R}+i\varphi_{s'}}\frac{d\theta'\mathcal{L}_{s'}(\theta')}{\sinh(2\theta')}\int_{\mathbb{R}+i\varphi_s}\frac{d\theta\mathcal{L}_s(\theta)}{\sinh(2\theta)}\cosh\left[i\frac{\pi}{2}(b_s-b_{s'}+1)+3\theta+\theta'\right]=$$

$$=\frac{1}{4\pi^2}\sum_{s,s'}(-1)^{s+2s'+1}\theta^{ss'}\int_{\mathbb{R}+i\varphi_{s'}}\frac{d\theta'\mathcal{L}_{s'}(\theta')}{\sinh(2\theta')}\int_{\mathbb{R}+i\varphi_s}\frac{d\theta\mathcal{L}_s(\theta)}{\sinh(2\theta)}\cosh(3\theta+\theta')=$$

$$=\frac{1}{4\pi^2}\sum_{s,s'}(-1)^{s+1}\theta^{ss'}\int_{\mathbb{R}+i\varphi_{s'}}\frac{d\theta'\mathcal{L}_{s'}(\theta')}{\sinh(2\theta')}\int_{\mathbb{R}+i\varphi_s}\frac{d\theta\mathcal{L}_s(\theta)}{\sinh(2\theta)}\cosh(3\theta+\theta')=$$

$$=\frac{1}{16\pi^2}\sum_{s,s'}\int_{D'}du'\int_D du\,(-1)^{s+1}\theta^{ss'}\sinh(2\theta')\sinh(2\theta)\cosh(3\theta+\theta')\mathcal{L}_s(\theta)\mathcal{L}_{s'}(\theta')$$

where

$$\cosh(3\theta+\theta')=\cosh(2\theta+2\theta')\cosh(\theta-\theta')+\sinh(2\theta+2\theta')\sinh(\theta-\theta') \quad (4.58)$$

Summing up, we just showed that the interesting part of the regularized area, once averaged, can be written in terms of A_0 (4.48) and the critical value of the Yang-Yang functional (4.28) from which we derive the MTBA equations depending on the physical cross ratios only; in particular, we can finally say that

$$A_{periods} + A_{free} = A_0 + YY_{cr} \tag{4.59}$$

where $YY_{cr} = YY_{cr}^{(1)} + YY_{cr}^{(2)}$ is the sum of expressions (4.55) and (4.57).

As a final comment to this section, it is important to note that the previous identification $A_{free} = YY_{cr}$ is no longer true if we do not eliminate the quantities Z_s and $\bar{Z}_s$ in favour of the physical cross ratios $y_s^{\pm}$. This can be easily demonstrated by repeating the exact same calculation just illustrated and replacing inside (4.49) and (4.48) the quantities (4.16) and (4.19), that is returning to the system of parameters that includes the quantities $Z_s, \bar{Z}_s$. The integral terms combine to produce a free energy structure (like in chapter 2) but other terms appear, so that the critical value of the YY functional is no longer just the free energy of the corresponding TBA system. The calculation we have performed in this section therefore shows that the modified TBA equations, as a function of the physical cross ratios, correspond to a *change of reference frame* that allows to identify, through a saddle point evaluation, the extreme of the Yang-Yang functional with the free-energy of the thermodynamic system described by them.

4.4 MTBA equations and YY_{cr} for the octagon

In the case of the octagon ($n = 8$), the degree of the polynomial is $\frac{n}{2} - 2 = 2$ and $s = 1, ..., \frac{n}{2} - 3 = 1$. When $\frac{n}{2}$ is even, we can use the *soft collinear limit* [8, 37], which consists in considering a polynomial with one more zero. We then start from $\left(\frac{n}{2} + 1\right) - 2 = 3$ zeroes and $s = 1, 2$ possible values. As we will see in the next section when we will analyse the case of the decagon ($n = 10$), many aspects of these two cases are very similar and this is a direct consequence of the soft collinear limit. Thus, our starting point is the equation (2.98),

$$\log Y_s(\theta) = -|m_s| \cosh(\theta - i\varphi_s) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta - \theta')} \log\left[1 + Y_{s'}(\theta')\right] \tag{4.60}$$

The forcing term becomes

$$\frac{Z_s}{e^{\theta + i\frac{\pi}{2}b_{s+1}}} + \bar{Z}_s e^{\theta + i\frac{\pi}{2}b_{s+1}}$$

so that the equation can be rewritten as

$$\log Y_s(\theta) = \frac{Z_s}{e^{\theta + i\frac{\pi}{2}b_{s+1}}} + \bar{Z}_s e^{\theta + i\frac{\pi}{2}b_{s+1}} + I_0^b \quad \text{with} \quad s = 1, 2 \tag{4.61}$$

where

$$I_0^b = \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta - \theta')} \log\left[1 + Y_{s'}(\theta')\right]$$

As in the previous section, we can find the (non-symmetric) MTBA equations that reads

$$\log Y_s(\theta) = L_s(\theta) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^s \theta^{ss'} \sinh(2\theta)}{\sinh(2\theta') \cosh(\theta-\theta')} \mathcal{L}_{s'}(\theta') \tag{4.62}$$

where

$$L_s(\theta) = \ln y_s^+ \cosh\left(\theta + i\frac{\pi}{2} b_{s+1}\right) - i \ln y_s^- \sinh\left(\theta + i\frac{\pi}{2} b_{s+1}\right) \tag{4.63}$$

When we perform the soft collinear limit ($s = 1, 2 \to s = 1$), the MTBA equation simplify in the following form

$$\log Y_s(\theta) = L_s(\theta) \tag{4.64}$$

Since we found the structure of the modified TBA equations, the next step is to build the YY functional; for the octagon, we easily see that definition (4.2) reduces to the first term only:

$$YY = \frac{1}{2\pi} \sum_s \int_D du \left[\rho_s(u)\phi_s(u) - Li_2\left(-e^{L_s(u)-\phi_s(u)}\right)\right] \tag{4.65}$$

where the domain is the same of (4.26). The variation with respect to ρ_s and ϕ_s produce respectively

$$\phi_s(u) = 0 \tag{4.66}$$

and

$$\rho_s(u) = \log\left[1 + e^{L_s(u)-\phi_s(u)}\right] \tag{4.67}$$

If we set

$$L_s(u) - \phi_s(u) = \log Y_s(\theta)$$

then (4.66), replaced inside the previous equation, let us to get the desired MTBA equation with $s = 1$. Now we are ready to calculate its critical value YY_{cr},

$$YY_{cr} = -\frac{1}{2\pi} \sum_s \int_D du\, Li_2\left[-Y_s(\theta)\right] \tag{4.68}$$

In order to show that $YY_{cr} = A_{free}$, we have to perform the calculation of the octagon's area following the steps depicted in the previous section. We start with the value $s = 1$ and we find this two conserved charges (recall that $\theta^{12} = 1$)

$$\tilde{c}_{-1,1} = \frac{1}{2}\left(\ln y_1^+ + i \ln y_1^-\right) + \int_{\mathbb{R}+i\varphi_2} \frac{d\theta'}{2\pi} \frac{e^{-\theta'+i\frac{\pi}{2}}}{\sinh(2\theta')} \mathcal{L}_2(\theta')$$

$$\tilde{c}_{1,1} = \frac{1}{2}\left(\ln y_1^+ - i \ln y_1^-\right) - \int_{\mathbb{R}+i\varphi_2} \frac{d\theta'}{2\pi} \frac{e^{-3\theta'-i\frac{\pi}{2}}}{\sinh(2\theta')} \mathcal{L}_2(\theta')$$

Instead, starting from the equation for $s = 2$, we obtain the following conserved charges

$$\tilde{c}_{-1,2} = \frac{1}{2}\left(\ln y_2^+ + i\ln y_2^-\right) + \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{e^{-\theta'}}{\sinh(2\theta')}\,\mathcal{L}_1(\theta')$$

$$\tilde{c}_{1,2} = \frac{1}{2}\left(\ln y_2^+ - i\ln y_2^-\right) - \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{e^{-3\theta'}}{\sinh(2\theta')}\,\mathcal{L}_1(\theta')$$

To calculate the area we use the modified formula [37, App. F] which we have seen to be useful in the case $\frac{n}{2} = $ even[3],

$$A = -i\,\omega_{21}\,\tilde{c}_{-1,2}\,\tilde{c}_{1,1} \quad\longrightarrow\quad A = -i\,e^{-i\frac{\pi}{2}}\,\tilde{c}_{-1,1}\,\tilde{c}_{1,1} = -\tilde{c}_{-1,1}\,\tilde{c}_{1,1} \tag{4.69}$$

so that

$$\begin{aligned}
A = &-\frac{1}{4}\left(\ln y_1^+ + i\ln y_1^-\right)\left(\ln y_1^+ - i\ln y_1^-\right) + \\
&+\frac{1}{2}\left(\ln y_1^+ + i\ln y_1^-\right)\int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{(-i)\,e^{-3\theta'}}{\sinh(2\theta')}\,\mathcal{L}_1(\theta') - \\
&-\frac{1}{2}\left(\ln y_1^+ - i\ln y_1^-\right)\int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{(i)\,e^{-\theta'}}{\sinh(2\theta')}\,\mathcal{L}_1(\theta') + \\
&+\int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{e^{-\theta'}}{\sinh(2\theta')}\mathcal{L}_1(\theta')\int_{\mathbb{R}+i\varphi_1} \frac{d\theta''}{2\pi}\,\frac{e^{-3\theta''}}{\sinh(2\theta'')}\mathcal{L}_1(\theta'')
\end{aligned} \tag{4.70}$$

where the second and third lines define $A^{(1)}_{temp}$,

$$\begin{aligned}
A^{(1)}_{temp} &= \frac{1}{2}\int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{\mathcal{L}_1(\theta')}{\sinh(2\theta')}\left[\left(-ie^{-3\theta'} - ie^{-\theta'}\right)\ln y_1^+ - i\left(ie^{-3\theta'} - ie^{-\theta'}\right)\ln y_1^-\right] = \\
&= -\frac{i}{2}\int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{\mathcal{L}_1(\theta')}{\sinh(2\theta')}\left[\left(e^{-3\theta'} + e^{-\theta'}\right)\ln y_1^+ - i\left(e^{-\theta'} - e^{-3\theta'}\right)\ln y_1^-\right]
\end{aligned} \tag{4.71}$$

In a similar way, we can find $A^{(2)}_{temp}$,

$$A^{(2)}_{temp} = -\frac{i}{2}\int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{\mathcal{L}_1(\theta')}{\sinh(2\theta')}\left[\left(e^{3\theta'} + e^{\theta'}\right)\ln y_1^+ - i\left(e^{3\theta'} - e^{\theta'}\right)\ln y_1^-\right] \tag{4.72}$$

Now, we are ready to find the contribution to the regularized area with only one integral for the octagon,

$$A^{octagon}_{temp} = \frac{A^{(1)}_{temp} + A^{(2)}_{temp}}{2} =$$

[3]When we consider the soft collinear limit we eliminate the extra zero of the polynomial and we note that the integrating function changes as a consequence.

$$= -i\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}\frac{\cosh(2\theta)}{\sinh(2\theta)}\left[\ln y_1^+\cosh\theta - i\ln y_1^-\sinh\theta\right]\mathcal{L}_1(\theta) =$$

$$= -\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}(u)\left[i\ln y_1^+\cosh\theta - i(i)\ln y_1^-\sinh\theta\right]\mathcal{L}_1(\theta) =$$

$$= -\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}(u)\left[\ln y_1^+\sinh\left(\theta+i\frac{\pi}{2}\right) - i\cosh\left(\theta+i\frac{\pi}{2}\right)\ln y_1^-\right]\mathcal{L}_1(\theta) =$$

$$= -\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}(u)\,\partial_\theta\left[\ln y_1^+\cosh(\theta+i\frac{\pi}{2}) - i\sinh\left(\theta+i\frac{\pi}{2}\right)\ln y_1^-\right]\mathcal{L}_1(\theta) \tag{4.73}$$

We note that the term in parenthesis nothing else is that the forcing of the non-symmetric MTBA equations,

$$L_1(\theta) = \ln y_1^+\cosh\left(\theta+i\frac{\pi}{2}\right) - i\ln y_1^-\sinh\left(\theta+i\frac{\pi}{2}\right) \tag{4.74}$$

so that we can rewrite

$$A^{octagon}_{temp} = -\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}(u)\,\partial_\theta\left[L_1(\theta)\right]\mathcal{L}_1(\theta) =$$

$$= -\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}(u)\,\partial_\theta\left[\log Y_1(\theta)\right]\mathcal{L}_1(\theta) =$$

$$-\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}(u)\,\partial_\theta\left[-Li_2\left(-Y_1(\theta)\right)\right] \stackrel{P}{=}$$

$$= b.t. - \int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}\partial_\theta(u)\,Li_2\left[-Y_1(\theta)\right] =$$

$$= b.t. - \frac{1}{2\pi}\int_D du\,Li_2\left[-Y_1(\theta)\right] = YY_{cr} \tag{4.75}$$

Here we have focused only on the terms with one integral because the treatment of the other types is pretty much the same as in the general case.

4.5 MTBA equations and YY_{cr} for the decagon

The decagon case ($n = 10$) is characterized by $s = 1, ..., \frac{n}{2} - 3 = 1, 2$ and we will adopt the more usual procedure valid for the cases $\frac{n}{2} = $ odd. As for the octagon, we start from (2.98) and, after an appropriate identification of the quantity Z_s and $\bar{Z}_s$ for odd or even values of s, we get again the following starting equation:

$$\log Y_s(\theta) = \frac{Z_s}{e^{\theta+i\frac{\pi}{2}b_{s+1}}} + \bar{Z}_s e^{\theta+i\frac{\pi}{2}b_{s+1}} + \sum_{s'=s-1}^{s+1}\int_{\mathbb{R}+i\varphi_{s'}}\frac{d\theta'}{2\pi}\frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta-\theta')}\log\left[1+Y_{s'}(\theta')\right] \tag{4.76}$$

Through the usual procedure illustrated in the previous sections, we can find the (non-symmetric) MTBA equations

$$\log Y_s(\theta) = L_s(\theta) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^s \, \theta^{ss'} \sinh(2\theta)}{\sinh(2\theta') \cosh(\theta-\theta')} \, \mathcal{L}_{s'}(\theta') \tag{4.77}$$

where

$$L_s(\theta) = \ln y_s^+ \cosh\left(\theta + i\frac{\pi}{2} b_{s+1}\right) - i \ln y_s^- \sinh\left(\theta + i\frac{\pi}{2} b_{s+1}\right) \tag{4.78}$$

and

$$\mathcal{L}_{s'}(\theta') = \log\left[1 + Y_{s'}(\theta')\right] \tag{4.79}$$

Using the new rapidity variable $u = \coth(2\theta)$, we obtain the useful (symmetric) MTBA equations, which can be deduced from the extreme conditions of the associated Yang-Yang functional,

$$\log Y_s(\theta) = L_s(\theta) + \frac{1}{2\pi} \sum_{s'=s-1}^{s+1} \int_D du' \, \mathcal{K}_{ss'}(\theta, \theta') \, \mathcal{L}_{s'}(\theta') \tag{4.80}$$

with

$$\mathcal{K}_{ss'}(\theta, \theta') = -\frac{(-1)^s \, \theta^{ss'} \sinh(2\theta') \sinh(2\theta)}{2\cosh(\theta-\theta')} \tag{4.81}$$

The associated Yang-Yang functional is, by definition,

$$\begin{aligned} YY &= \frac{1}{2\pi} \sum_s \int_D du \left[\rho_s(u)\phi_s(u) - Li_2\left(-e^{L_s(u)-\phi_s(u)}\right)\right] + \\ &+ \frac{1}{8\pi^2} \sum_{s,s'} \int_D du \int_{D'} du' \, \mathcal{K}_{ss'}(\theta, \theta') \, \rho_s(u)\rho_{s'}(u') \end{aligned} \tag{4.82}$$

and its extreme conditions are written:

$$\phi_s(u) + \frac{1}{2\pi} \sum_{s'} \int_D du' \, \mathcal{K}_{ss'}(\theta, \theta') \, \rho_{s'}(u') = 0 \tag{4.83}$$

together with

$$\rho_s(u) - \log\left[1 + e^{L_s(u)-\phi_s(u)}\right] = 0 \tag{4.84}$$

If we set, as usual, $\ln Y_s(u) \equiv L_s(u) - \phi_s(u)$ we find

$$\rho_s(u) = \log\left[1 + Y_s(u)\right] = \mathcal{L}_s(u) \tag{4.85}$$

and,replacing the previous equation inside (4.83), we obtain the desired MTBA equations (4.80). Relations (4.83) and (4.84), once replaced into the definition of the Yang-Yang functional, allow us to get its critical value

$$\begin{aligned} YY_{cr} &= -\frac{1}{2\pi}\sum_s \int_D du\, Li_2\left[-Y_s(\theta)\right] - \\ &\quad -\frac{1}{8\pi^2}\sum_{s,s'}\int_D du \int_{D'} du'\, \mathcal{K}_{ss'}(\theta,\theta')\, \mathcal{L}_{s'}(u')\mathcal{L}_s(u) = \\ &= YY_{cr}^{(1)} + YY_{cr}^{(2)} \end{aligned} \tag{4.86}$$

From equation (4.77), valid for $s = 1$ and $s = 2$, we can deduce the conserved charges hidden in the large θ expansion: for $\theta \to -\infty$ we have

$$\tilde{c}_{-1,1} = \frac{1}{2}\left(\ln y_1^+ + i \ln y_1^-\right) + \int_{\mathbb{R}+i\varphi_2} \frac{d\theta'}{2\pi}\,\frac{\theta^{12}(i)e^{-\theta'}}{\sinh(2\theta')}\,\mathcal{L}_2(\theta') \tag{4.87}$$

$$\tilde{c}_{1,1} = \frac{1}{2}\left(\ln y_1^+ - i \ln y_1^-\right) - \int_{\mathbb{R}+i\varphi_2} \frac{d\theta'}{2\pi}\,\frac{\theta^{12}(-i)e^{-3\theta'}}{\sinh(2\theta')}\,\mathcal{L}_2(\theta') \tag{4.88}$$

$$\tilde{c}_{-1,2} = \frac{1}{2}\left(\ln y_2^+ + i \ln y_2^-\right) - \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{\theta^{21}\, e^{-\theta'}}{\sinh(2\theta')}\,\mathcal{L}_1(\theta') \tag{4.89}$$

$$\tilde{c}_{1,2} = \frac{1}{2}\left(\ln y_2^+ - i \ln y_2^-\right) + \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{\theta^{21}\, e^{-3\theta'}}{\sinh(2\theta')}\,\mathcal{L}_1(\theta') \tag{4.90}$$

while for $\theta \to +\infty$ we have

$$c_{-1,1} = \frac{1}{2}\left(\ln y_1^+ - i \ln y_1^-\right) - \int_{\mathbb{R}+i\varphi_2} \frac{d\theta'}{2\pi}\,\frac{\theta^{12}(-i)e^{\theta'}}{\sinh(2\theta')}\,\mathcal{L}_2(\theta') \tag{4.91}$$

$$c_{1,1} = \frac{1}{2}\left(\ln y_1^+ + i \ln y_1^-\right) + \int_{\mathbb{R}+i\varphi_2} \frac{d\theta'}{2\pi}\,\frac{\theta^{12}(i)e^{3\theta'}}{\sinh(2\theta')}\,\mathcal{L}_2(\theta') \tag{4.92}$$

$$c_{-1,2} = \frac{1}{2}\left(\ln y_2^+ - i \ln y_2^-\right) + \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{\theta^{21}\, e^{\theta'}}{\sinh(2\theta')}\,\mathcal{L}_1(\theta') \tag{4.93}$$

$$c_{1,2} = \frac{1}{2}\left(\ln y_2^+ + i \ln y_2^-\right) - \int_{\mathbb{R}+i\varphi_1} \frac{d\theta'}{2\pi}\,\frac{\theta^{21}\, e^{3\theta'}}{\sinh(2\theta')}\,\mathcal{L}_1(\theta') \tag{4.94}$$

When $\theta \to -\infty$, the area is computed from the following formula:

$$\begin{aligned} A &= -i\sum_{s,s'} \omega_{ss'}\, \tilde{c}_{-1,s}\, \tilde{c}_{1,s'} = \\ &= -i\,\omega_{21}\,\tilde{c}_{-1,2}\,\tilde{c}_{1,1} - i\,\omega_{12}\,\tilde{c}_{-1,1}\,\tilde{c}_{1,2} = \\ &= A_0^- + A_{temp}^{(1)} + A_2^- \end{aligned} \tag{4.95}$$

where

$$A^{(1)}_{temp} = -\frac{i}{2}\int_{\mathbb{R}+i\varphi_1}\frac{d\theta'}{2\pi}\frac{\mathcal{L}_1(\theta')}{\sinh(2\theta')}\left[\left(e^{-3\theta'}+e^{-\theta'}\right)\ln y_1^+ - i\left(e^{-\theta'}-e^{-3\theta'}\right)\ln y_1^-\right] -$$

$$-\frac{i}{2}\int_{\mathbb{R}+i\varphi_2}\frac{d\theta'}{2\pi}\frac{(-1)\mathcal{L}_2(\theta')}{\sinh(2\theta')}\left[\left(e^{-3\theta'-i\frac{\pi}{2}}+e^{-\theta'+i\frac{\pi}{2}}\right)\ln y_2^+ - i\left(e^{-\theta'+i\frac{\pi}{2}}-e^{-3\theta'-i\frac{\pi}{2}}\right)\ln y_2^-\right] \tag{4.96}$$

which agrees with the general result (4.39). Instead, when $\theta \to +\infty$, the area is calculated from the following formula:

$$\begin{aligned} A &= i\sum_{s,s'}\omega_{ss'}\,c_{-1,s}\,c_{1,s'} = \\ &= i\,\omega_{21}c_{-1,2}\,c_{1,1} + i\,\omega_{12}c_{-1,1}\,c_{1,2} = \\ &= A_0^+ + A^{(2)}_{temp} + A_2^+ \end{aligned} \tag{4.97}$$

where

$$A^{(2)}_{temp} = -\frac{i}{2}\int_{\mathbb{R}+i\varphi_1}\frac{d\theta'}{2\pi}\frac{\mathcal{L}_1(\theta')}{\sinh(2\theta')}\left[\left(e^{3\theta'}+e^{\theta'}\right)\ln y_1^+ - i\left(e^{3\theta'}-e^{\theta'}\right)\ln y_1^-\right] -$$

$$-\frac{i}{2}\int_{\mathbb{R}+i\varphi_2}\frac{d\theta'}{2\pi}\frac{(-1)\mathcal{L}_2(\theta')}{\sinh(2\theta')}\left[\left(e^{3\theta'+i\frac{\pi}{2}}+e^{\theta'-i\frac{\pi}{2}}\right)\ln y_2^+ - i\left(e^{3\theta'+i\frac{\pi}{2}}-e^{\theta'-i\frac{\pi}{2}}\right)\ln y_2^-\right] \tag{4.98}$$

which agrees again with the general result (4.46). Then, the contribution to the area with only one integral term is obtained through the average,

$$\begin{aligned} A^{decagon}_{temp} &= \frac{A^{(1)}_{temp}+A^{(2)}_{temp}}{2} = \\ &= -\frac{i}{4}\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}\frac{\mathcal{L}_1(\theta)}{\sinh(2\theta)}\times \\ &\times\left\{\left[\left(e^{-3\theta}+e^{-\theta}\right)+\left(e^{3\theta}+e^{\theta}\right)\right]\ln y_1^+ - i\left[\left(e^{-\theta}-e^{-3\theta}\right)+\left(e^{3\theta}-e^{\theta}\right)\right]\ln y_1^-\right\} - \\ &-\frac{i}{4}\int_{\mathbb{R}+i\varphi_2}\frac{d\theta}{2\pi}\frac{(-1)\mathcal{L}_2(\theta)}{\sinh(2\theta)}\times \\ &\times\left\{\left[\left(e^{-3\theta-i\frac{\pi}{2}}+e^{-\theta+i\frac{\pi}{2}}\right)+\left(e^{3\theta+i\frac{\pi}{2}}+e^{\theta-i\frac{\pi}{2}}\right)\right]\ln y_2^+ - i\left[\left(e^{-\theta+i\frac{\pi}{2}}-e^{-3\theta-i\frac{\pi}{2}}\right)+\left(e^{3\theta+i\frac{\pi}{2}}-e^{\theta-i\frac{\pi}{2}}\right)\right]\ln y_2^-\right\} \end{aligned}$$

After a little algebra, we get an useful final writing, which agrees with the general result (4.49),

$$\begin{aligned} A_{temp}^{decagon} &= -i\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}\,\frac{\mathcal{L}_1(\theta)\cosh(2\theta)}{\sinh(2\theta)}\left[\cosh\theta\ln y_1^+ - i\sinh\theta\ln y_1^-\right] - \qquad (4.99)\\ &- i\int_{\mathbb{R}+i\varphi_2}\frac{d\theta}{2\pi}\,\frac{(-1)\mathcal{L}_2(\theta)\cosh(2\theta)}{\sinh(2\theta)}\left[\cosh\left(\theta+i\frac{\pi}{2}\right)\ln y_2^+ - i\sinh\left(\theta+i\frac{\pi}{2}\right)\ln y_2^-\right] =\\ &= -\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}\,\frac{\mathcal{L}_1(\theta)\cosh(2\theta)}{\sinh(2\theta)}\left[\sinh\left(\theta+i\frac{\pi}{2}\right)\ln y_1^+ - i\cosh\left(\theta+i\frac{\pi}{2}\right)\ln y_1^-\right] -\\ &- \int_{\mathbb{R}+i\varphi_2}\frac{d\theta}{2\pi}\,\frac{\mathcal{L}_2(\theta)\cosh(2\theta)}{\sinh(2\theta)}\left[\sinh\theta\ln y_2^+ - i\cosh\theta\ln y_2^-\right] =\\ &= -\sum_{s=1}^{2}\int_{\mathbb{R}+i\varphi_s}\frac{d\theta}{2\pi}\,\frac{\cosh(2\theta)}{\sinh(2\theta)}\,\times\\ &\times \left[\ln y_s^+\sinh\left(\theta+i\frac{\pi}{2}b_{s+1}\right) - i\ln y_s^-\cosh\left(\theta+i\frac{\pi}{2}b_{s+1}\right)\right]\mathcal{L}_s(\theta) \qquad (4.100)\end{aligned}$$

We note that the terms in parenthesis coincide with the derivative of the forcing of the (non-symmetric) MTBA equations (4.77), in particular:

$$\begin{aligned} A_{temp}^{decagon} &= -\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}\,\frac{\cosh(2\theta)}{\sinh(2\theta)}\,\partial_\theta\left[L_1(\theta)\right]\mathcal{L}_1(\theta) -\\ &- \int_{\mathbb{R}+i\varphi_2}\frac{d\theta}{2\pi}\,\frac{\cosh(2\theta)}{\sinh(2\theta)}\,\partial_\theta\left[L_2(\theta)\right]\mathcal{L}_2(\theta) =\\ &= -\sum_{s=1}^{2}\int_{\mathbb{R}+i\varphi_s}\frac{d\theta}{2\pi}\,(u)\,\partial_\theta L_s(\theta)\,\mathcal{L}_s(\theta) \qquad (4.101)\end{aligned}$$

Using (4.77), valued for $s=1$ and $s=2$ separately, we can substitute inside $A_{temp}^{decagon}$ to obtain

$$\begin{aligned} A_{temp}^{decagon} = &-\int_{\mathbb{R}+i\varphi_1}\frac{d\theta}{2\pi}\,(u)\,\partial_\theta\left[-Li_2\left(-Y_1\right)\right] - \int_{\mathbb{R}+i\varphi_2}\frac{d\theta}{2\pi}\,(u)\,\partial_\theta\left[-Li_2\left(-Y_2\right)\right] +\\ &+\frac{1}{4\pi^2}\int_{\mathbb{R}+i\varphi_1}d\theta(u)\int_{\mathbb{R}+i\varphi_2}d\theta'\,\frac{(-1)\,\theta^{12}}{\sinh(2\theta')}\partial_\theta\left[\frac{\sinh(2\theta)}{\cosh(\theta-\theta')}\right]\mathcal{L}_2(\theta')\mathcal{L}_1(\theta) +\\ &+\frac{1}{4\pi^2}\int_{\mathbb{R}+i\varphi_2}d\theta\,(u)\int_{\mathbb{R}+i\varphi_1}d\theta'\,\frac{(-1)^2\,\theta^{21}}{\sinh(2\theta')}\partial_\theta\left[\frac{\sinh(2\theta)}{\cosh(\theta-\theta')}\right]\mathcal{L}_1(\theta')\mathcal{L}_2(\theta) \stackrel{P}{=}\end{aligned}$$

$$= b.t. - \int_{\mathbb{R}+i\varphi_1} \frac{d\theta}{2\pi} \partial_\theta (u)\, Li_2 [-Y_1] - \int_{\mathbb{R}+i\varphi_2} \frac{d\theta}{2\pi} \partial_\theta(u)\, Li_2 [-Y_2] +$$

$$+ \frac{1}{16\pi^2} \sum_{s,s'=1}^{2} \int_{D_s} du \int_{D_{s'}} du' \left[(-1)^s \theta^{ss'} \sinh(2\theta) \sinh(2\theta') \right] \times$$

$$\times \left[\cosh(2\theta) \partial_\theta \left(\frac{\sinh(2\theta)}{\cosh(\theta - \theta')} \right) \right] \mathcal{L}_{s'}(\theta') \mathcal{L}_s(\theta) =$$

where the second and third integral term coincide with $YY_{cr}^{(1)}$ of (4.86) while the last addendum can be recast as

$$-\frac{1}{8\pi^2} \sum_{s,s'=1}^{2} \int_{D_s} du \int_{D_{s'}} du' \left[-\frac{(-1)^s \theta^{ss'} \sinh(2\theta) \sinh(2\theta')}{2} \right] \left[\frac{1}{\cosh(\theta - \theta')} + ... \right] \mathcal{L}_{s'}(\theta') \mathcal{L}_s(\theta) \tag{4.102}$$

where the dots indicate terms that are simplified with the remaining two-integral contributions to the area, like in the previous general case; we will calculate these in a moment, but let us underline that the last term, after the above simplifications, becomes exactly $YY_{cr}^{(2)}$ of (4.86).

We end this section by calculating the other contributions of the area left behind. From (4.95) we have

$$A_0^- = \frac{\omega_{12}}{2} \ln y_2^+ \ln y_1^- - \frac{\omega_{12}}{2} \ln y_2^- \ln y_1^+ \tag{4.103}$$

and

$$A_2^- = \theta^{12} \int \frac{d\theta}{2\pi} \frac{e^{-\theta} \mathcal{L}_1(\theta)}{\sinh(2\theta)} \int \frac{d\theta'}{2\pi} \frac{e^{-3\theta'} \mathcal{L}_2(\theta')}{\sinh(2\theta')} -$$

$$- \theta^{21} \int \frac{d\theta}{2\pi} \frac{e^{-\theta} \mathcal{L}_2(\theta)}{\sinh(2\theta)} \int \frac{d\theta'}{2\pi} \frac{e^{-3\theta'} \mathcal{L}_1(\theta')}{\sinh(2\theta')} \tag{4.104}$$

while from (4.97) we obtain

$$A_0^+ = -\frac{\omega_{12}}{2} \ln y_1^+ \ln y_2^- + \frac{\omega_{12}}{2} \ln y_1^- \ln y_2^+ \tag{4.105}$$

and

$$A_2^+ = \theta^{12} \int \frac{d\theta}{2\pi} \frac{e^{\theta} \mathcal{L}_1(\theta)}{\sinh(2\theta)} \int \frac{d\theta'}{2\pi} \frac{e^{3\theta'} \mathcal{L}_2(\theta')}{\sinh(2\theta')} -$$

$$- \theta^{21} \int \frac{d\theta}{2\pi} \frac{e^{\theta} \mathcal{L}_2(\theta)}{\sinh(2\theta)} \int \frac{d\theta'}{2\pi} \frac{e^{3\theta'} \mathcal{L}_1(\theta')}{\sinh(2\theta')} \tag{4.106}$$

All these contributions are in agreement with the general case. Then, the average produce

$$A_0 = \frac{A_0^- + A_0^+}{2} = \frac{\omega_{12}}{2} (\ln y_1^- \ln y_2^+ - \ln y_1^+ \ln y_2^-) \tag{4.107}$$

together with

$$A_2 = \frac{A_2^- + A_2^+}{2} =$$

$$= -\frac{1}{8\pi^2} \sum_{s,s'=1}^{2} \int_{\mathbb{R}+i\varphi_s} du \int_{\mathbb{R}+i\varphi_{s'}} du' \left[-\frac{(-1)^{s'} \theta^{ss'} \sinh(2\theta) \sinh(2\theta')}{2} \right] \cosh(\theta+3\theta') \mathcal{L}_{s'}(\theta') \mathcal{L}_s(\theta) \tag{4.108}$$

Note that the simplification with the two integral terms present within $A_{temp}^{decagon}$ is guaranteed by (4.58) and the fact that s and s' have opposite parity. In this way, the general identification $A_{free} = YY_{cr}$ is also exactly satisfied in the case of the decagon.

Chapter 5

Future perspectives on the BSV polygonal transitions for AdS_3 WLs

In this last part, we would like to exploit some results of the previous chapter to properly introduce the main reason that prompted us from the beginning to analyze planar scattering amplitudes (dual to null polygonal WLs) in AdS_3: to attempt a three-dimensional reduction for the strong coupling re-summation of the BSV series, as illustrated in [38] for the hexagon. Contextually, this reduction would allow us to derive the P form factors that describe our decagon (in the same limit) in terms of an all-terms extended OPE series (BSV) [40, 41, 42, 43, 44]. Thanks to chapter 3, these intermediate transition amplitudes P would then be related to the HSG models there considered, obtaining a form factors description for the $\frac{SU(N)_2}{[U(1)]^{N-1}}$ HSG models.

5.1 A comparison with the AdS_5 case

In [38], after finding the complete set of ABA equations and the S-matrix for excitations over the GKP vacuum, the knowledge of all the necessary scattering data allowed to build the pentagonal amplitudes P in the perturbative regime contextually to the series of papers by Basso, Sever and Vieira (BSV)[1]. The authors' purpose was to investigate the multi-particle contributions (flux tube) to the MHV gluon scattering amplitudes, the so called BSV series, and re-sum them for the hexagon case so to reproduce the TBA set-up of [7, 8, 37]. From this point of view, light-like polygonal Wilson loops can be seen as a infinite sum over more fundamental polygons (squares and pentagons in the AdS_5 case) whose knowledge relies on the GKP scattering factors. In the strong coupling limit, this superposition of squares and pentagons leads to the classical string regime,

[1]Their work allow to extend to all terms the operator product expansion of [37], thus proposing a non-perturbative approach to 4D null polygonal WLs in $\mathcal{N} = 4$ SYM, which in turn depend on the 2D scattering factors [40, 41, 42, 43, 44].

that is the minimization of the supersymmetric string action [1]. From what has been discussed so far, we know that this is a complicated problem of minimal area solved by a set of non-linear coupled integral equations. Besides, we know that their form resembles that of a TBA system whose free-energy corresponds to the only dynamic part of the area [8]. This same TBA set-up can be obtained by re-summing the infinite BSV series and performing a saddle point evaluation, as has been properly demonstrated in [38]. The purpose of this chapter is to lay the foundations to be able to fully reproduce in the future the same analysis even on AdS_3. So let us briefly summarize the key steps and results of [38], which we will use as a guideline.

Considering the AdS_5 framework at generic finite coupling, the BSV series [40] is a sum over the intermediate multi-particle states, where the particles can be scalars, fermions, gluons and bound states of thereof, as analyzed in [38]. The simplest example is given by the hexagonal Wilson loop

$$W_{hex} = \sum_{N=0}^{+\infty} \frac{1}{N!} \sum_{a_1} \dots \sum_{a_N} \int \prod_{k=1}^{N} \left[\frac{du_k}{2\pi} \mu_{a_k}(u_k)\, e^{-\tau E_{a_k}(u_k) + i\sigma p_{a_k}(u_k) + i m_{a_k}\phi} \right] \times$$
$$\times P_{a_1 \dots a_N}(0|u_1 \dots u_N) P_{a_1 \dots a_n}(-u_1 \dots -u_N|0) \tag{5.1}$$

where the measures $\mu_{a_i}(u_i)$ correspond to quadrangular amplitudes and the pentagonal amplitude $P_{a_1 \dots a_N}(0|u_1 \dots u_N)$ represents the transition from the vacuum to an intermediate state of N particles of any kind a_i listed above. By virtue of the notations of [40], the general n-sided null WL is decomposed into a sequence of simpler fundamental building blocks, namely $n-3$ null squares and $n-4$ null pentagons. Any two adjacent squares form a pentagon so that we remain with $n-5$ null middle squares, each of which has three symmetries parametrized by a GKP time τ_i, space σ_i, and angle ϕ_i for rotations in the two dimensional space transverse to this middle square [37, 40]. The set $\{\tau_i, \sigma_i, \phi_i\}_{i=1}^{n-5}$ parametrizes the $3(n-5)$ independent conformal cross ratios of a n-edge null polygon [45]. Every middle square in the decomposition shares two of its opposite cusps with the big polygon; the positions of the other two cusps are fixed by the condition that they are null separated from their neighbours. For the case of the hexagon we are considering we have only 2 pentagons and 1 middle squares. This describes the kinematics of the decomposition. For what concerns the dynamic, we can say that we start with the GKP vacuum in the bottom and evolve it all the way to the top where it is overlapped with the vacuum again. In between, we decompose the flux tube state in the i-th middle square over a basis of GKP eigenstates ψ_i. Each eigenstate ψ_i propagates trivially in the corresponding square for a time τ_i. It then undergoes a pentagon transition P to the consecutive square where it is decomposed again and so on. The N-particle eigenstates ψ_i have definite energies E_i, $U(1)$ charges m_i and momenta p_i, with $N = 0, 1, 2, \dots$. The charges $\{E_i, m_i, p_i\}$ are the sum of the charges of the N individual excitations. We can parametrize each state with a set of rapidities $u = \{u_1, \dots, u_N\}$ and a set of indices $a = \{a_1, ..., a_N\}$ which labels the kind of the j-th excitation [46] belonging to the i-th eigenstate ψ_i. This explains the writing (5.1) and more details about the measures $\mu_{a_i}(u_i)$ can be found in [40].

When we go to the strong coupling limit, we have to disentangle the integrations over internal rapidities; this procedure means that we have to add different contributions: the leading contributions in the perturbative regime are due to gluons, mesons and their bound states [38]. To construct the pentagonal amplitudes P it is necessary to solve a series of axioms depending on the GKP S-matrix entries [40, 41]. For the gluonic sector we have

$$P(-u|-v) = P(v|u) \tag{5.2}$$

$$P(u|v) = S(u|v)P(v|u) \tag{5.3}$$

$$P(u^{-\gamma}|v) = \bar{P}(v|u) \tag{5.4}$$

where $P(u|v) = P_{FF}(u|v)$ and $\bar{P}(u|v) = P_{F\bar{F}}(u|v)$ denote the pentagonal amplitudes of two gluonic excitations $F, \bar{F}$. Thanks to the parametrization $u = \sqrt{2}g\tanh(2\theta)$, inherent in the gluonic sector, the axioms can be rewritten in terms of the rapidities θ, θ' and solved for generic bound states [38]

$$\alpha_{ml}P^{(gg)}_{ml}(\theta,\theta') = 1 + \frac{iml}{2\sqrt{2}g}\frac{\cosh(2\theta)\cosh(2\theta')}{\sinh(2\theta-2\theta')}[1+\cosh(\theta-\theta') - i\sinh(\theta-\theta')] + O(1/g^2) \tag{5.5}$$

where m and l identify the mesons constituting the bonded states before and after the transition respectively. We recover the transition between single gluons for $m = l = 1$, see result (10.7) in [38]. A very similar procedure, but with a new parametrization $u = \sqrt{2}g\coth(2\theta)$, allows to obtain for the mesonic sector the following pentagonal transition about bound states,

$$\beta_{ml}P^{(MM)}_{ml}(\theta,\theta') = 1 - \frac{iml}{\sqrt{2}g}\frac{\sinh(2\theta)\sinh(2\theta')}{\sinh(\theta-\theta')}\sqrt{2}\cosh\left(\theta-\theta'-i\frac{\pi}{4}\right) + O(1/g^2) \tag{5.6}$$

At this point, it is possible to compute (at strong coupling) any contributions of the BSV series which describes the hexagon Wilson loop. To do this we must take into account important properties[2] concerning the pentagonal P factors:

$$P_{a_1,...,a_N|0}(u_1,...,u_N|0) = \prod_{i<j} P_{a_i,a_j|0}(u_i,u_j|0) \tag{5.7}$$

together with

$$P_{a,b|0}(u,v|0)P_{-a,b|0}(u^{2\gamma},v|0) = P_{a,b|0}(u,v|0)P_{b|a}(v|u) = P_{b|-a}(v|u^{-2\gamma})P_{b|a}(v|u) = 1 \tag{5.8}$$

[2]These properties derive from the axioms, among which we mention the Watson, the monodromy and the residue condition.

which lead, when all the particles are in, to

$$P_{a_1,...,a_N|0}(u_1,...,u_N|0) = \prod_{i>j} \frac{1}{P_{a_i|a_j}(u_i|u_j)} \tag{5.9}$$

Instead, when all the particles are out, a similar computation produces

$$P_{0|a_1,...,a_N}(0|u_1,...,u_N) = \prod_{i<j} \frac{1}{P_{a_i|a_j}(u_i|u_j)} \tag{5.10}$$

In this way, it is possible to show that the following simple manipulation holds,

$$P_{a_1,...,a_N}(0|u_1,...,u_N)P_{a_1,...,a_N}(-u_1,...,-u_N|0) = \prod_{i<j}^{N} \frac{1}{P_{a_i|a_j}(u_i|u_j)P_{a_j|a_i}(u_j|u_i)} \tag{5.11}$$

In [38, sec. 11], we can find the explicit results for both one-particle and two-particle contributions, but even more interesting is the coincidence of the kernels of our modified TBA equations for the decagon (4.77) with the kernels $K^{(MM)}_{sym}(\theta_1,\theta_2)$ of the mesonic sector alone (see kernels (11.23)). Being more specific, we can recast the modified TBA equations for the decagon in the form[3]

$$\varepsilon(\theta - i\varphi_s) = -L_s(\theta) + \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi \sinh^2(2\theta')} \mathcal{K}^{ss'}_{sym}(\theta,\theta')\, \mathcal{L}_{s'}(\theta') \tag{5.12}$$

where

$$\mathcal{K}^{ss'}_{sym}(\theta,\theta') = -\frac{(-1)^s \theta^{ss'} \sinh(2\theta)\sinh(2\theta')}{\cosh(\theta-\theta')} \tag{5.13}$$

together with

$$-L_s(\theta) = \varepsilon^+_s \cosh\left(\theta + i\frac{\pi}{2} b_{s+1}\right) - i\,\varepsilon^-_s \sinh\left(\theta + i\frac{\pi}{2} b_{s+1}\right) \tag{5.14}$$

and

$$\mathcal{L}_{s'}(\theta') = \log\left[1 + e^{-\varepsilon_{s'}(\theta' - i\varphi_{s'})}\right] \tag{5.15}$$

highlighting that the measures also coincide if we consider the mesonic sector of the hexagon in AdS_5. From the comparison, we can assume that having a single equation with a single kernel type, the AdS_3 decagon corresponds to a single type of excitation on the GKP vacuum. Regarding the nature of this excitation, we limit ourselves to saying that in [42] it has been shown that, at strong coupling, there are three AdS massive string modes: the first two correspond to the two gluonic excitation $F, \bar{F}$ which

[3]The rewriting makes use of the notations and results reported in the appendix B.

behave as relativistic particles of mass $\sqrt{2}$ belonging to the two AdS_5 directions that are orthogonal to the AdS_3 subspace in which the classical string is moving; the third mode instead correspond to a particle of mass 2 associated to fluctuations inside the AdS_3 subspace and, since there are no fundamental excitations with this mass, in [42] it is explained how this particle emerges in the OPE context as a bound state of a fermion and of a anti-fermion (see also [38]).

Returning to [38, sec. 11], we can quickly verify (up to two-particle contributions) the equivalence announced in [40],

$$W_{hex} = \exp\left\{-\frac{\sqrt{\lambda}}{2\pi} YY_{cr}\right\} \tag{5.16}$$

where YY_{cr} corresponds to the dynamic part A_{free} of the full renormalized area, as we know from what has been said so far. This agreement is not restricted to one and two-particle contributions, but instead it does also extend to any number of particles. The BSV series for the hexagon can be fully re-summed by exploiting some standard technique, the Hubbard-Stratonovich transform [47]. In the next section, we will try to reproduce these technique in our AdS_3 case: in this way, we hope to set the future development of a BSV series for our AdS_3 decagon in which the polygonal amplitudes P (form factors) are defined starting from a new set of axioms and, maybe, a new physical background. In fact, considering the change of geometry $AdS_5 \to AdS_3$, for the moment we cannot say that the geometric decomposition of the WL will take place through squares and pentagons. For the same reason, we can only fix a very general set of axioms that hold for a new fundamental n-gon transition.

5.2 The path integral trick for the re-summation

Very often, to compute physical quantities, we have to sum over states: this is true, for example, when we compute partition functions in statistical physics or when we study correlation functions in a CFT. Actually, this sum consists in the subsequent application of the operator product expansion (OPE) to write the general n-point function as multiple sums over the states generated by the fusion of the local operators. A similar strategy can be applied for computing the vacuum expectation values of null polygonal Wilson loops in conformal gauge theories [37]. It entails, however, summing over a rather different class of states, namely the complete set of excitations ψ_i of the flux tube supported by two null Wilson lines [48]. For a generic polygon we have to perform this sum as many times as needed to fully decompose the evolution of the flux-tube state along the loop. Contrary to what happens in the AdS_5 case, where the decomposition is based on $n-5$ intermediate squares and $n-4$ pentagons, in our current AdS_3 case we do not have previous results that inform us on which decomposition to implement and which fundamental polygons to consider. In [40], this idea is presented in terms of pentagons, but for our case of interest we can organize these multiple sums in the following sequence

$$W = \sum_{\psi_i} e^{\sum_j [-E_j \tau_j + i p_j \sigma_j]} P(0|\psi_1) P(\psi_1|\psi_2) \ldots P(\psi_n|0) \tag{5.17}$$

where $P(\psi_i|\psi_j)$ is the generic polygonal transition between the states ψ_i and ψ_j that represents the elementary building block of the (unknown) geometric decomposition. A significant generalization of the previous decomposition is based on the following sequence of transitions and propagations

$$W = \langle \mathrm{vac}| \hat{\mathcal{P}}\, e^{-\tau_n \hat{H} + i\sigma_n \hat{P}}\, \hat{\mathcal{P}} \ldots e^{-\tau_1 \hat{H} + i\sigma_1 \hat{P}} \hat{\mathcal{P}} |\mathrm{vac}\rangle \tag{5.18}$$

where $\hat{H}$ and $\hat{P}$ are the generators of the two conformal symmetries inherent in the AdS_3 case [45]. This representation can be connected with the previous tessellation of the n-gon WL and the operator $\hat{\mathcal{P}}$ represents the transition between two fundamental polygons. To make contact with the decomposition (5.17) we have to insert a resolution of the identity at any propagation and consider that the states ψ_i are eigenstates of the flux tube hamiltonian $\hat{H}$ of the i-th intermediate polygon.

Inspired by [38], we would like to find a way to reproduce the manipulation (5.11) that allows to rewrite the product of polygonal transitions of many-particle states as the reciprocal of the product of single-particle P factors. This manipulation is based on the specific properties of the P factors or, equivalently, of the fundamental pentagon in the AdS_5 case. This precise step is crucial for the continuation (the path-integral trick we want to discuss) and we must generalize it by means of more general properties about P factors, which are listed below. Through these new properties, which will lead to a generalization of (5.8) valid for the pentagon, we hope to deduce a manipulation useful for our purposes.

Considering a general object of the form $P_{a_1,\ldots,a_k|b_1,\ldots,b_h}(u_1,\ldots,u_k|v_1,\ldots,v_h)$, the following axioms must be satisfied:

Watson relation

$$P_{a_1,\ldots,a_i,a_{i+1},\ldots,a_k|0}(\ldots,u_i,u_{i+1},\ldots|0) = S_{a_i,a_{i+1}}(u_i,u_{i+1}) P_{a_1,\ldots,a_{i+1},a_i,\ldots,a_k|0}(\ldots,u_{i+1},u_i,\ldots|0) \tag{5.19}$$

where the S-matrix satisfies unitarity $S_{a,b}(u,v)S_{b,a}(v,u) = 1$, crossing symmetry $S_{-a,b}(u^{2\gamma},v)S_{a,b}(u,v) = 1$ and mirror symmetry $S_{a,b}(u^{\gamma},v^{\gamma}) = S_{a,b}(u,v)$;

Monodromy relation

$$P_{a,a_1,\ldots,a_k|0}(u^{4\gamma n},u_1,\ldots,u_k|0) = P_{a_1,\ldots,a_k,-a|0}(u_1,\ldots,u_k,u|0) \tag{5.20}$$

where $4n$ is the number of sides of the polygon;

Residue condition

$$i\text{Res}_{\bar{u}=u}P_{\bar{a},a,a_1,...,a_k|0}(\bar{u}^{2\gamma},u,u_1,...,u_k|0)=\delta_{\bar{a},-a}P_{a_1,...,a_k|0}(u_1,...,u_k|0) \tag{5.21}$$

Reflection

$$P_{a_1,...,a_k,a|0}(u_1,...,u_k,u|0)=P_{0|a,a_k,...,a_1}(0|u,u_k,...,u_1) \tag{5.22}$$

Crossing transformation

$$\begin{aligned} P_{0|-a,b_k,...,b_1}(0|u,v_k,...,v_1) &= P_{a|b_k,...,b_1}(u^{2\gamma}|v_k,...,v_1)= \\ &= P_{-b_1,...,-b_k,a|0}(v_1^{2\gamma},...,v_k^{2\gamma},u^{2\gamma}|0) \end{aligned} \tag{5.23}$$

Helicity flip

$$P_{a_1,...,a_k|b_1,...,b_h}(u_1,...,u_k|v_1,...,v_h)=P_{-a_1,...,-a_k|-b_1,...,-b_h}(u_1,...,u_k|v_1,...,v_h) \tag{5.24}$$

Analyticity As a function of the rapidities, P is meromorphic in each variable within a physical region and exhibits only simple poles. For relativistic theories, when P is written in terms of relative hyperbolic rapidities θ_{ij}, the physical region is the strip $0\leq \text{Im}(\theta)\leq 2\pi n$.

A quick way to test these axioms is to consider the case of the pentagon $n=\frac{5}{4}$. Then, the fundamental property (5.8) is obtained by applying the crossing transformation twice. We hope to be able to reproduce this crucial manipulation also for fundamental polygons with more sides in a short time. Once this is done, we can follow the procedure (the Hubbard-Stratonovich transform) illustrated in [38] to fully re-sum the BSV series describing the decagon in AdS_3 and reproduce the entire TBA set-up. Below, we will show only the main steps of this technique which allows us to find, by means of a saddle point evaluation, our modified TBA equation for the decagon (4.77) and the corresponding Yang-Yang functional.

Once we have found the appropriate manipulation for the product of generic polygonal transitions involving N arbitrary excitations, we hope to be able to rewrite our WL (5.17) in the following form

$$W_{dec}=\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{a_1}...\sum_{a_N}\int\prod_{k=1}^{N}\left[\frac{du_k}{2\pi}\mu_{a_k}(u_k)\,e^{-\tau E_{a_k}(u_k)+i\sigma p_{a_k}(u_k)}\right]\prod_{i<j}^{N}e^{\langle X_{(a_i)}(u_i)X_{(a_j)}(u_j)\rangle} \tag{5.25}$$

where the field $X_{(a)}$ satisfies the well know identity about functional gaussian integration in the presence of a linear source term [38],

$$\prod_{i<j}^{N}e^{\langle X_{(a_i)}(u_i)X_{(a_j)}(u_j)\rangle}=\left\langle e^{X_{(a_1)}(u_1)\,...\,X_{(a_N)}(u_N)}\right\rangle \tag{5.26}$$

This step is clearly based on the appropriate manipulation of the polygonal amplitudes P, but also on the possibility of connecting them with the correlators of the gaussian fields. To clarify what we mean, let us consider what it was made for the AdS_5 hexagon:

$$\begin{aligned} P_{a_1,\ldots,a_N}(0|u_1,\ldots,u_N)P_{a_1,\ldots,a_N}(-u_1,\ldots,-u_N|0) &= \prod_{i<j}^{N}\frac{1}{P_{a_i|a_j}(u_i|u_j)P_{a_j|a_i}(u_j|u_i)} = \\ &= \prod_{i<j}^{N} e^{\langle X_{(a_i)}(u_i)X_{(a_j)}(u_j)\rangle} \end{aligned} \tag{5.27}$$

Now, thanks to this substitution, it is possible to interpret (5.25) as a Kac-Feynman path integral (partition function) for any value of the coupling,

$$W_{dec} = \langle \exp\left\{\int \frac{du}{2\pi}\sum_a \left[\mu_a(u)\, e^{f_a(u)+X_{(a)}(u)}\right]\right\}\rangle \tag{5.28}$$

with $f_a(u) = -\tau E_a(u) + i\sigma p_a(u)$; however, only at strong coupling important simplifications ensure that it can be added up. At this point, the previous analogy with the meson sector leads us to say that the following form is desirable,

$$W_{dec} = \langle \exp\left\{-\int \frac{du}{2\pi}\,\mu(u)\, Li_2\left[-e^{f(u)+X(u)}\right]\right\}\rangle \tag{5.29}$$

where only the right definition of the measure $\mu_a(u)$ allows to bring out the dilogarithm function $Li_2(x)$. Also the physical properties of the states (including bound states if any) will be fundamental to obtain the previous writing. Now, it is possible to read (5.29) as a quantum mechanics partition function for the field $X(u)$

$$W_{dec} = Z[X] = \int \mathcal{D}X\, e^{-\mathcal{S}[X]} \tag{5.30}$$

thanks to the definition of the action[4]

$$\mathcal{S}[X] = \frac{1}{2}\sum_{a,b}\int d\theta d\theta' X_a(\theta)T_{ab}(\theta,\theta')X_b(\theta') + \sum_a \int \frac{d\theta}{2\pi}\,\mu_a(\theta)\, Li_2\left[-e^{L_a(\theta)+X_a(\theta)}\right] \tag{5.31}$$

which, under extremisation, gives the equation of motion

$$X_a(\theta) - \sum_b \int \frac{d\theta'}{2\pi}\,\mathcal{G}_{ab}(\theta,\theta')\,\mu_b(\theta')\log\left[1+e^{L_b(\theta')+X_b(\theta')}\right] \tag{5.32}$$

[4]This action should be related to our Yang-Yang functional (4.2) while the function L_a is given by our modified TBA equations (5.12).

where the subscript takes the values $a = 1, 2$ and the Green function has been introduced as

$$\int d\theta' \, \mathcal{G}(\theta, \theta') \, T(\theta', \theta'') = \delta(\theta - \theta'') \quad \text{or} \quad \mathcal{G}(\theta, \theta') = \langle X(\theta) X(\theta') \rangle \tag{5.33}$$

It is interesting to note that the Green function, at strong coupling, can be related only to the symmetric part of the transition amplitude[5],

$$\mathcal{G}(\theta, \theta') = -\frac{2\pi}{\sqrt{\lambda}} \mathcal{K}^{ab}_{sym}(\theta, \theta') + O\left(\frac{1}{\lambda}\right) \tag{5.34}$$

Finally, if we introduce the function $\varepsilon(\theta - i\varphi_a) = -L_a(\theta) - X_a(\theta)$, we obtain the modified TBA equations for the decagon case (5.12) after having exploited once again the analogy with the mesonic sector for the correct definition of the measure $\mu_b(\theta')$. As a last remark, we note that, from (5.33) and (5.34), the action (5.31) possesses the divergent prefactor $\sqrt{\lambda}$, so that the decagonal WL (5.30) is dominated by the classical configuration, achieved by imposing the equations of motion on the fields. This saddle point evaluation gives us back the critical value of the Yang-Yang functional, which should be related to that computed in section 4.5.

[5]Equivalently, we could say that the anti-symmetric part of the P-factors is fixed in the strong coupling limit.

Chapter 6

Conclusions

We started from the purely geometric problem of computing minimal area surfaces with null polygonal contour at the boundary of AdS_3, related to gluon scattering amplitude at strong coupling by virtue of the gauge/string duality in the context of 4D supersymmetric gauge theories. The problem becomes identical to the problem of computing a Wilson loop with the same contour and it is solved by a set of non-linear coupled integral equations which form resembles that of a relativistic TBA system [8]. The area can be computed as function of the conformal cross ratios characterizing the null polygon at the boundary: it has a divergent part, which can be regularized in a well known fashion, and a finite part that consists in two contributions, only one of which is dynamic depending on the pseudo-energies. We denote this latter finite piece with A_{free}, the interesting part of the so called remainder function. In chapter 2, we have showed how to calculate these finite contributions through a procedure that simultaneously uses an appropriate WKB approximation and the conserved charges hidden in the log-expansion. This then allowed us to expand the integral equations in the two regimes of interest as a function of the spectral parameter, determining both contributions to the remainder function. The octagon and decagon cases are analyzed in detail, underlying the importance of the soft collinear limit in the cases where $\frac{n}{2}$ is even.

The TBA integral equations that determine the strong coupling answer have been rewritten in two equivalent ways, one with the pseudo-energies and the other with the hatted Y-functions. These two equivalent formulation, obtained through appropriate shifts in the rapidity variables, allow us to connect with [11], [13] and [37, App. E], where one works in the fully equivalent context of general Hitchin systems. Furthermore, the connection with [13] is more important for us because the study there reported merges the TBA equations for the cases of decagon and dodecagon to those describing a particular class of integrable perturbations of CFTs corresponding to G_k-parafermions, the $\frac{SU(N)_2}{[U(1)]^{N-1}}$ HSG models. After studying the general features of this class of models, such as mass scales, S-matrices, resonance poles and kernels, we have derived the universal form of the TBA equations by means of a fundamental decomposition of the full HSG kernel [20].

The analysis of the resonance parameter in the two special regimes $\sigma = 0, \infty$ is crucial for the understanding of subsequent developments. We have derived the complete set of functional relations (Y-system) closely related to the universal form of the integral equations. In chapter 3, we show that it reproduces our AdS_3 Y-system, previously introduced, for the particular algebra level $k = 2$. The decagon case is proposed as an useful check. Concerning the relevant regime $\sigma = \infty$, we have seen that only one part of the full HSG kernel contributes to the dynamics: this is exactly the part that does not attend to the dynamics of the HSG models above mentioned [21]. In this limit, by means of the Fourier transform of the only kernel involved, it is possible to get both the universal TBA equation and the Y-system of the A_n series (ADE).

In [37] was proposed a new version for the TBA integral equation derived in [8], replacing the auxiliary parameters $Z, \bar{Z}$ in favour of the physical cross ratios. These latter equations are known as modified TBA equation (MTBA) and they can be related to the extremization of an appropriate Yang-Yang functional, once the due symmetrization has been implemented. By means of a saddle point evaluation, it is possible to relate its critical value YY_{cr} to A_{free}, the dynamic part of the remainder function. In chapter 4, we pointed out that this relation depends on a rigid mathematical structure, which can be though as a precise change of reference frame: this result make manifest an important property of the area, namely it is the extremum of an action functional with fixed boundary condition given by the choice of physical cross ratios.

The last, but not least, reason that led us to study strongly coupled scattering amplitudes in AdS_3 (dual to null polygonal WLs) is to develop some kind of three-dimensional reduction of [38], where it was shown that from the re-summation of the BSV series corresponding to the null WL considered, it is possible to get back the aforementioned TBA set-up. Although it is possible to reproduce the main steps for our case of interest, many questions await answers: we do not have previous results that inform us on the geometric decomposition (squares and pentagons do not seem to fit AdS_3) to be applied to the WL as well as on the interpretative physical background (the complete set of excitations over the GKP vacuum). For this reason, the last chapter is dedicated to illustrating some possible future developments that for reasons of time and space have not been included in this thesis. We hope to be able to exploit more general properties valid for new polygonal transitions P (form factors) with the aim of manipulating the product of P-factors entering in the decomposition of the WL and corresponding to transitions of an arbitrary number N of particles. We are looking for properties that agree with the standard technique used for re-summing, the Hubbard-Stratonovich transform. Finally, (hopefully useful) comments were included on what was noted by the comparison with the AdS_5 case.

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Appendix A

The TBA-like integral equation from general Hitchin systems

The specific geometric problem of computing minimal surfaces that end on null polygons in AdS_3 is a special case of the general theory about Hitchin systems on Riemann surfaces [11, 12]. From these works, it results that the integral equations given for general Hitchin systems and the integral equations for polygons in AdS_3, derived in [8] as a simpler warm up problem than the full AdS_5 space, actually coincide in the simplest kinematic region. There is a large amount of freedom in setting up the Riemann-Hilbert problem and, as a consequence, we have many different ways of writing the integral equations. The purpose of this appendix is to present these integral equations in a very general form, which allows us to connect, once the appropriate conventions have been established, with different equivalent contexts, such as [8] and [13]. The *TBA-like integral equations* coming from general Hitchin systems can be written as

$$\ln X_\gamma(\zeta) = \frac{Z_\gamma}{\zeta} + i\theta_\gamma + \bar{Z}_\gamma \zeta - \frac{1}{4\pi i} \sum_{\gamma' \in \Gamma} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{l'_\gamma} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 + X_{\gamma'}(\zeta')\right] \quad \text{(A.1)}$$

Some discrete data, which are the final product of a WKB analysis of some differential equations on the surface, appear inside this equation: for what follows, we only need to know some basic properties about them. The labels γ, γ' run within the set of possible values Γ and come in pairs, say γ and $-\gamma$. From [11, App. E], we know that the cardinality of this set is equal to the number of BPS rays. The $\Omega(\gamma')$ are certain integer numbers and the $\langle \gamma, \gamma' \rangle$ is an antisymmetric pairing. The quantities Z_γ are auxiliary complex numbers while the θ_γ are angles, which will be set to zero[1]. Concerning the lines of integration $l_{\gamma'}$, straight rays from $\zeta' = 0$ to $\zeta' = \infty$, there is a canonical choice to set them: $\frac{Z'_\gamma}{\zeta'} \in \mathbb{R}_-$.

[1]In the language of Hitchin systems, we are restricting to a real section.

Once the angles θ_γ are set to zero, a simplification occurs:

$$\begin{aligned} X_{-\gamma}(-\zeta) &= X_\gamma(\zeta) \\ \Omega(-\gamma) &= \Omega(\gamma) \\ \langle -\gamma, \gamma' \rangle = \langle \gamma, -\gamma' \rangle &= -\langle \gamma, \gamma' \rangle \\ Z_{-\gamma} &= -Z_\gamma \end{aligned} \tag{A.2}$$

This is due to the $\mathbb{Z}_2$-symmetry, which is inherent in the present AdS_3 case. Now, splitting the pair of labels $(\gamma', -\gamma')$, we can write

$$\begin{aligned} \ln X_\gamma(\zeta) &= \frac{Z_\gamma}{\zeta} + \bar{Z}_\gamma \zeta - \\ &-\frac{1}{4\pi i} \sum_{\gamma' \in \Gamma^+} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{l_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 + X_{\gamma'}(\zeta')\right] - \\ &- \frac{1}{4\pi i} \sum_{-\gamma' \in \Gamma^-} \Omega(-\gamma') \langle \gamma, -\gamma' \rangle \int_{l_{-\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 + X_{-\gamma'}(\zeta')\right] \end{aligned} \tag{A.3}$$

Using the above properties, we can recast in this form

$$\begin{aligned} \ln X_\gamma(\zeta) &= \frac{Z_\gamma}{\zeta} + \bar{Z}_\gamma \zeta - \\ &-\frac{1}{4\pi i} \sum_{\gamma' \in \Gamma^+} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{l_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 + X_{\gamma'}(\zeta')\right] + \\ &+ \frac{1}{4\pi i} \sum_{-\gamma' \in \Gamma^-} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{l_{-\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 + X_{-\gamma'}(\zeta')\right] \end{aligned} \tag{A.4}$$

and performing $\zeta' \to -\zeta'$ in the second integral, which also implies $l_{-\gamma'} \to l_{\gamma'}$, we get

$$\begin{aligned} \ln X_\gamma(\zeta) &= \frac{Z_\gamma}{\zeta} + \bar{Z}_\gamma \zeta - \\ &- \frac{1}{4\pi i} \sum_{\gamma' \in \Gamma^+} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{l_{\gamma'}} \frac{d\zeta'}{\zeta'} \left[\frac{\zeta' + \zeta}{\zeta' - \zeta} + \frac{\zeta - \zeta'}{\zeta' + \zeta} \right] \log\left[1 + X_{\gamma'}(\zeta')\right] \end{aligned} \tag{A.5}$$

The definition of $\zeta = e^\theta$, allow us to reach the following equation

$$\ln X_\gamma(\theta) = \frac{Z_\gamma}{e^\theta} + \bar{Z}_\gamma e^\theta + \\ + \frac{1}{2\pi i} \sum_{\gamma' \in \Gamma+} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{l_{\gamma'}} \frac{d\theta'}{\sinh(\theta - \theta')} \log\left[1 + X_{\gamma'}(\theta')\right] \tag{A.6}$$

which coincide with [37, eq. (E.2)] if we multiply the incidence matrix $\langle \gamma, \gamma' \rangle$ by minus 1. We must pay attention to this last step and to the definition of the incidence matrix in establishing the link with [8, 13].

Appendix B

Equivalent computation of A_{free} in terms of ε-functions

In this appendix we would perform the computation of the main contribution A_{temp} to the regularized area in terms of the pseudo-energies, so as to have a more convenient formalism for any future connections with [38, 39]. Starting from (2.107) is completely equivalent to consider (2.98) and replace (2.102): in both cases, we get the following equation as our useful starting point

$$\epsilon(\theta - i\varphi_s) = -\frac{Z_s}{e^{\theta_{s+1}}} - \bar{Z}_s e^{\theta_{s+1}} - \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta-\theta')} \mathcal{L}_{s'}(\theta') \tag{B.1}$$

where $\theta_{s+1} \equiv \theta + i\frac{\pi}{2}b_{s+1}$ and $\mathcal{L}_{s'}(\theta') = \log\left[1 + e^{-\epsilon(\theta' - i\varphi_{s'})}\right]$. It will soon be very useful to simplify this equation further and write

$$\epsilon(\theta - i\varphi_s) = -\frac{Z_s}{e^{\theta + i\frac{\pi}{2}b_{s+1}}} - \bar{Z}_s e^{\theta + i\frac{\pi}{2}b_{s+1}} - I_0^b \tag{B.2}$$

where

$$I_0^b \equiv \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh(\theta-\theta')} \log\left[1 + e^{-\epsilon(\theta' - i\varphi_{s'})}\right] \tag{B.3}$$

The next step is to define the physical cross ratios:

$$\begin{cases} \epsilon(-i\frac{\pi}{2}b_{s+1} - i\varphi_s) \equiv \epsilon_s^+ & \xleftrightarrow{\zeta=1} \quad -\ln Y_s(-i\frac{\pi}{2}b_{s+1}) \equiv -\ln y_s^+ \\ \epsilon(i\frac{\pi}{2}b_s - i\varphi_s) \equiv \epsilon_s^- & \xleftrightarrow{\zeta=i} \quad -\ln Y_s(i\frac{\pi}{2}b_s) \equiv -\ln y_s^- \end{cases} \tag{B.4}$$

We are therefore ready to rewrite the integral equations in terms of physical cross ratios only, eliminating the quantities Z_s and $\bar{Z}_s$ through an adequate substitution. For $\zeta = 1$ we can write,

$$\epsilon_s^+ = -Z_s - \bar{Z}_s - I_2^b \tag{B.5}$$

where

$$I_2^b \equiv \sum_{s'} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh\left(-i\frac{\pi}{2}b_{s+1}-\theta'\right)} \mathcal{L}_{s'}(\theta') \tag{B.6}$$

so that

$$\bar{Z}_s = -\epsilon_s^+ - Z_s - I_2^b \tag{B.7}$$

Instead, for $\zeta = i$ we have

$$\epsilon_s^- = iZ_s - i\bar{Z}_s - I_1^b = 2iZ_s + i\epsilon_s^+ + iI_2^b - I_1^b \tag{B.8}$$

where

$$I_1^b \equiv \sum_{s'} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi} \frac{(-1)^{s+1}\theta^{ss'}}{\cosh\left(i\frac{\pi}{2}b_s-\theta'\right)} \mathcal{L}_{s'}(\theta') \tag{B.9}$$

so that

$$Z_s = -\frac{1}{2}\epsilon_s^+ - \frac{i}{2}\epsilon_s^- - \frac{i}{2}I_1^b - \frac{1}{2}I_2^b \tag{B.10}$$

$$\bar{Z}_s = -\frac{1}{2}\epsilon_s^+ + \frac{i}{2}\epsilon_s^- + \frac{i}{2}I_1^b - \frac{1}{2}I_2^b \tag{B.11}$$

finally getting

$$\begin{aligned} \epsilon(\theta - i\varphi_s) &= -e^{-\theta-i\frac{\pi}{2}b_{s+1}}\left[-\frac{1}{2}\epsilon_s^+ - \frac{i}{2}\epsilon_s^- - \frac{i}{2}I_1^b - \frac{1}{2}I_2^b\right] - \\ &- e^{\theta+i\frac{\pi}{2}b_{s+1}}\left[-\frac{1}{2}\epsilon_s^+ + \frac{i}{2}\epsilon_s^- + \frac{i}{2}I_1^b - \frac{1}{2}I_2^b\right] - I_0^b \end{aligned} \tag{B.12}$$

A simple calculation produces the following forcing term, which we will call $-L_s(\theta)$ to underline the analogy with the forcing obtained by working with Y-functions,

$$-L_s(\theta) = \epsilon_s^+ \cosh\left(\theta + i\frac{\pi}{2}b_{s+1}\right) - i\,\epsilon_s^- \sinh\left(\theta + i\frac{\pi}{2}b_{s+1}\right) \tag{B.13}$$

Performing the calculation also for the integral terms, we arrive at the *modified (non-symmetric) TBA equations*,

$$\epsilon(\theta - i\varphi_s) = -L_s(\theta) - \sum_{s'=s-1}^{s+1} \int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi}\left[\frac{(-1)^s\theta^{ss'}\sinh(2\theta)}{\sinh(2\theta')\cosh(\theta-\theta')}\right]\mathcal{L}_{s'}(\theta') \tag{B.14}$$

B.1 The Yang-Yang functional and its critical value

Thanks to the new rapidity variable $u = \frac{\cosh(2\theta)}{\sinh(2\theta)}$, we can symmetrize the MTBA equations (B.14) and show that exist a Yang-Yang functional which, once extreme, reproduce them exactly. To achieve this, it is important to change the sign of the equations so that the argument of the exponential in the logarithm of the r.h.s. appears precisely as in the l.h.s.,

$$\begin{aligned}
-\epsilon(\theta - i\varphi_s) &= L_s(\theta) + \frac{1}{2\pi}\sum_{s'}\int_D du' \left[-\frac{(-1)^s \theta^{ss'} \sinh(2\theta')\sinh(2\theta)}{2\cosh(\theta-\theta')} \right] \mathcal{L}_{s'}(\theta') \\
&= L_s(\theta) + \frac{1}{2\pi}\sum_{s'}\int_D du'\, \mathcal{K}_{ss'}(\theta,\theta') \log\left[1 + e^{-\epsilon(\theta' - i\varphi_{s'})}\right] \qquad \text{(B.15)}
\end{aligned}$$

From the definition of the Yang-Yang functional (4.2), the variations with respect to ρ_s and ϕ_s produce respectively

$$\phi_s(u) + \frac{1}{2\pi}\sum_{s'}\int_D du'\, \mathcal{K}_{ss'}(\theta,\theta')\, \rho_{s'}(u') = 0 \qquad \text{(B.16)}$$

and

$$\rho_s(u) = \log\left[1 + e^{L_s(u) - \phi_s(u)}\right] \qquad \text{(B.17)}$$

Now we set

$$L_s(u) - \phi_s(u) = -\epsilon(\theta - i\varphi_s) \qquad \text{(B.18)}$$

so that

$$\rho_s(u) = \log\left[1 + e^{-\epsilon(\theta - i\varphi_s)}\right] \qquad \text{(B.19)}$$

and (B.16) reproduce the desired MTBA equations (B.15). Finally, replacing (B.16) and (B.17) inside (4.2), we extract the critical value YY_{cr},

$$\begin{aligned}
YY_{cr} = YY_{cr}^{(1)} + YY_{cr}^{(2)} = -\frac{1}{2\pi}\sum_s \int_D du\, Li_2\left[-e^{-\epsilon(\theta - i\varphi_s)}\right] - \\
-\frac{1}{8\pi^2}\sum_{s,s'}\int_D du \int_{D'} du'\, \mathcal{K}_{ss'}(\theta,\theta')\, \log\left[1 + e^{-\epsilon(\theta' - i\varphi_{s'})}\right] \log\left[1 + e^{-\epsilon(\theta - i\varphi_s)}\right] \qquad \text{(B.20)}
\end{aligned}$$

B.2 The main contributions to the regularized area

Going back to the (non-symmetric) MTBA equations (B.14), we can derive the main contributions to the regularized area through the conserved charges hidden in the large θ regime. As in section 4.3, we will report first the charges coming from the small ζ-regime and then those from the large ζ-regime. Consequently, we will calculate the interesting part of the regularized area ($A_{periods} + A_{free}$) separately in the previous regimes and, only at the end, we will average the results. Thus, we consider this equation

$$\begin{aligned}
\epsilon(\theta - i\varphi_s) &= \epsilon_s^+ \cosh\left(\theta + i\frac{\pi}{2}b_{s+1}\right) - i\,\epsilon_s^- \sinh\left(\theta + i\frac{\pi}{2}b_{s+1}\right) - \\
&- \sum_{s'}\int_{\mathbb{R}+i\varphi_{s'}} \frac{d\theta'}{2\pi}\, \frac{(-1)^s\, \theta^{ss'} \sinh(2\theta)}{\sinh(2\theta')\cosh(\theta-\theta')}\, \mathcal{L}_{s'}(\theta') \qquad \text{(B.21)}
\end{aligned}$$

From $\theta \to -\infty$ we get

$$\tilde{c}_{-1,s} = \frac{1}{2}\left(\epsilon_s^+ + i\epsilon_s^-\right) + \sum_{s'} \int \frac{d\theta'}{2\pi} \frac{(-1)^s \theta^{ss'}}{\sinh(2\theta')}\, e^{-\theta' + i\frac{\pi}{2}b_{s+1}}\, \mathcal{L}_{s'}(\theta') \tag{B.22}$$

$$\tilde{c}_{1,s} = \frac{1}{2}\left(\epsilon_s^+ - i\epsilon_s^-\right) - \sum_{s'} \int \frac{d\theta'}{2\pi} \frac{(-1)^s \theta^{ss'}}{\sinh(2\theta')}\, e^{-3\theta' - i\frac{\pi}{2}b_{s+1}}\, \mathcal{L}_{s'}(\theta') \tag{B.23}$$

$$A^{small} = -i \sum_{s,s'} \omega_{ss'}\, \tilde{c}_{-1,s}\, \tilde{c}_{1,s'} = A_0^{small} + A_{temp}^{(1)} + A_2^{small} \tag{B.24}$$

$$\begin{aligned} A_{temp}^{(1)} &= -\frac{i}{2} \sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} \frac{(-1)^{s+1}\mathcal{L}_s(\theta)}{\sinh(2\theta)} \times \\ \times &\left[-\epsilon_s^+ \left(e^{-3\theta - i\frac{\pi}{2}b_s} + e^{-\theta + i\frac{\pi}{2}b_s}\right) + i\,\epsilon_s^- \left(e^{-\theta + i\frac{\pi}{2}b_s - e^{-3\theta - i\frac{\pi}{2}b_s}}\right)\right] \end{aligned} \tag{B.25}$$

while from $\theta \to +\infty$ we obtain

$$A^{large} = i \sum_{s,s'} \omega_{ss'}\, c_{-1,s}\, c_{1,s'} = A_0^{large} + A_{temp}^{(2)} + A_2^{large} \tag{B.26}$$

$$\begin{aligned} A_{temp}^{(2)} &= -\frac{i}{2} \sum_s \int_{\mathbb{R}+i\varphi_s} \frac{d\theta}{2\pi} \frac{(-1)^{s+1}\mathcal{L}_s(\theta)}{\sinh(2\theta)} \times \\ \times &\left[-\epsilon_s^+ \left(e^{3\theta + i\frac{\pi}{2}b_s} + e^{\theta - i\frac{\pi}{2}b_s}\right) + i\epsilon_s^- \left(e^{3\theta + i\,\frac{\pi}{2}b_s - e^{\theta - i\frac{\pi}{2}b_s}}\right)\right] \end{aligned} \tag{B.27}$$

The subsequent averaging process

$$A_{temp} = \frac{A_{temp}^{(1)} + A_{temp}^{(2)}}{2} \tag{B.28}$$

produces two contributions, one for ϵ_s^+ and one for ϵ_s^-:

$$\left\{ \begin{array}{lccc} \text{for} & -\epsilon_s^+ & \to & \cosh(2\theta)\cosh\left(\theta + i\frac{\pi}{2}b_s\right) \\ \text{for} & -\epsilon_s^- & \to & \cosh(2\theta)\sinh\left(\theta + i\frac{\pi}{2}b_s\right) \end{array} \right. \tag{B.29}$$

so that

$$\begin{aligned} A_{temp} &= -i \sum_s \int \frac{d\theta}{2\pi} \frac{(-1)^{s+1}\cosh(2\theta)}{\sinh(2\theta)} \mathcal{L}_s(\theta) \times \\ &\times \left[-\epsilon_s^+ \cosh\left(\theta + i\frac{\pi}{2}b_s\right) - i(-\epsilon_s^-)\sinh\left(\theta + i\frac{\pi}{2}b_s\right)\right] \end{aligned} \tag{B.30}$$

We see that the following relations hold,

$$\left\{ \begin{array}{l} (-1)^{s+1} i \cosh\left(\theta + i\frac{\pi}{2}b_s\right) = \sinh\left(\theta + i\frac{\pi}{2}b_{s+1}\right) \\ (-1)^{s+1} i \sinh\left(\theta + i\frac{\pi}{2}b_s\right) = \cosh\left(\theta + i\frac{\pi}{2}b_{s+1}\right) \end{array} \right. \tag{B.31}$$

and then we can rewrite A_{temp} in such a way that the term in square brackets highlights the *derivative of the forcing of the symmetric MTBA equations* (B.15),

$$A_{temp} = -\sum_s \int \frac{d\theta}{2\pi} \begin{pmatrix} \cosh(2\theta) \\ \sinh(2\theta) \end{pmatrix} \mathcal{L}_s(\theta) \times$$

$$\times \partial_\theta \left[-\epsilon_s^+ \cosh\left(\theta + i\frac{\pi}{2} b_{s+1}\right) - i(-\epsilon_s^-) \sinh\left(\theta + i\frac{\pi}{2} b_{s+1}\right)\right] =$$

$$= -\sum_s \int \frac{d\theta}{2\pi} (u)\, \partial_\theta \left[L_s(\theta)\right] \mathcal{L}_s(\theta) \tag{B.32}$$

As we have seen on several occasions during this discussion, we have again arrived at the rigid mathematical structure that allows the identification $A_{free} = YY_{cr}$, by means of a change of reference frame ($Z_s, \bar{Z}_s \to \ln y_s^{\pm}$) and consequent vast simplifications among the integral contributions to the regularized area.

To connect easily with [38], in analogy to [37, App. F] for the AdS_5 case, we should compute the explicit expressions of YY_{cr} as a function of the pseudo-energies. We can take these results from the general case treated in section 4.2.1 and appropriately replace the Y-functions with (2.102).

Bibliography

[1] L. F. Alday and J. Maldacena, *Gluon scattering amplitudes at strong coupling*, arXiv:0705.0303 [hep-th].

[2] J. M. Drummond, G. P. Korchemsky and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, arXiv:0707.0243 [hep-th].
A. Brandhuber, P. Heslop and G. Travaglini, *MHV amplitudes in N=4 super Yang-Mills and Wilson loops*, arXiv:0707.1153 [hep-th].

[3] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, *The two-loop six-gluon MHV amplitude in maximally supersymmetric Yang-Mills theory*, arXiv:0803.1465 [hep-th].
J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *Hexagon Wilson loop = six-gluon MHV amplitude*, arXiv:0803.1466 [hep-th].

[4] L. F. Alday and R. Roiban, *Scattering amplitudes, Wilson loops and the string/gauge theory correspondence*, arXiv:0807.1889 [hep-th].

[5] L. F. Alday and J. Maldacena, *Minimal surfaces in AdS and the eight-gluon scattering amplitude at strong coupling*, arXiv:0903.4707 [hep-th].

[6] L. F. Alday and J. Maldacena, *Null polygonal Wilson loops and minimal surfaces in Anti-de-Sitter space*, arXiv:0904.0663 [hep-th].

[7] L. F. Alday, D. Gaiotto and J. Maldacena, *Thermodynamic Bubble Ansatz*, arXiv:0911.4708 [hep-th].

[8] L. F. Alday, J. Maldacena, A. Sever and P. Vieira, *Y-system for scattering amplitudes*, arXiv:1002.2459 [hep-th].

[9] Z. Bern, L. J. Dixon and V. A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, arXiv:0505205 [hep-th].
J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes*, arXiv:0712.1223 [hep-th].

[10] C. Anastasiou, A. Brandhuber, P. Heslop, V. V. Khoze, B. Spence and G. Travaglini, *Two-loop polygon Wilson loops in N=4 SYM*, arXiv: 0902.2245 [hep-th].
V. Del Duca, C. Duhr and V. A. Smirnov, *An analytic result for the two-loop hexagon Wilson loop in N=4 SYM*, arXiv:0911.5332 [hep-th].

[11] D. Gaiotto, G. Moore and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, arXiv:0807.4723 [hep-th].

[12] D. Gaiotto, G. Moore and A. Neitzke, *Wall-crossing, Hitchin systems and the WKB approximation*, arXiv:0907.3987 [hep-th].

[13] Y. Hatsuda. K. Ito, K. Sakai and Y. Satoh, *Thermodynamic Bethe ansatz equations for minimal surfaces in AdS_3*, arXiv:1002.2941 [hep-th].

[14] C. R. Fernandez-Pousa, M. V. Gallas, T. J. Hollowood and J. L. Miramontes, *The symmetric space and homogeneous sine-Gordon theories*, arXiv:9606032 [hep-th].

[15] C. R. Fernandez-Pousa, M. V. Gallas, T. J. Hollowood and J. L. Miramontes, *Solitonic integrable perturbations of parafermionic theories*, arXiv:9701109 [hep-th].

[16] D. Gepner, *New conformal field theories associated with Lie algebras and their partition functions*, Nucl. Phys. B 290 (1987) 10.

[17] D. Gepner and Z. Qiu, *Modular invariant partition functions for parafermionic field theories*, Nucl. Phys. B 285 (1987) 423.

[18] V. A. Fateev and A. B. Zamolodchikov, *Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in Z_N-symmetric statistical systems*, Sov. Phys. JETP 62 (1985) 215 [Zh. Eksp. Teor. Fiz. 89 (1985) 380].

[19] J. L. Miramontes and C. R. Fernandez-Pousa, *Integrable quantum field theories with unstable particles*, arXiv:9910218 [hep-th].

[20] O. A. Castro-Alvaredo, A. Fring, C. Korff and J. L. Miramontes, *Thermodynamic Bethe ansatz of the homogeneous sine-Gordon models*, arXiv:9912196 [hep-th].

[21] O. A. Castro-Alvaredo and A. Fring, *Decoupling the $SU(N)_2$-homogeneous sine-Gordon model*, arXiv:0010262 [hep-th].

[22] P. Dorey and J. L. Miramontes, *Mass scales and crossover phenomena in the homogeneous sine-Gordon models*, arXiv:0405275 [hep-th].

[23] R. Köberle and J. A. Swieca, *Factorizable Z(N) models*, Phys. Lett. 86B (1979) 209.

[24] G. Breit and E. P. Wigner, *Capture of slow neutrons*, Phys. Rev. 49 (1936) 519.

[25] C. R. Fernandez-Pousa and L. J. Miramontes, *Semi-classical spectrum of the homogeneous sine-Gordon theories*, arXiv:9706203 [hep-th].

[26] O. A. Castro-Alvaredo, A. Fring and C. Korff, *Form factors of the homogeneous sine-Gordon models*, arXiv:0004089 [hep-th].

[27] O. A. Castro-Alvaredo and A. Fring, *Identifying the operator content, the homogeneous sine-Gordon models*, arXiv:0008044 [hep-th].

[28] A. Fring, H. C. Liao and D. I. Olive, *The mass spectrum and coupling in affine Toda theories*, Phys. Lett. B266 (1991) 82.

[29] A. Fring, C. Korff and B. J. Schulz, *The ultraviolet behaviour of integrable quantum field theories, affine Toda field theory*, arXiv:9902011 [hep-th].

[30] P. Dorey and J. L. Miramontes, *Aspects of the homogeneous sine-Gordon models*, arXiv:0211174 [hep-th].

[31] S. Cecotti and M. Del Zotto, *Y systems, Q systems and 4D N=2 supersymmetric QFT*, arXiv:1403.7613 [hep-th].

[32] Al. B. Zamolodchikov, *On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories*, Phy. Lett. B253 (1991) 391.

[33] T. R. Klassen and E. Melzer, *Purely elastic scattering theories and their ultraviolet limits*, Nucl. Phys. B338 (1990) 485 [preprint EFI 89-58, UMTG-153 (1989)].

[34] V. A. Fateev and S. L. Lukyanov, *The model of two-dimensional conformal quantum field theory with Z_n-symmetry*, Intern. J. Mod. Phys. A3 (1988) 507.

[35] F. Ravanini, R. Tateo and A. Valleriani, *Dynkin TBA's*, arXiv:9207040 [hep-th].

[36] C. N. Yang and C. P. Yang, *Thermodynamics of one-dimensional systems of bosons with repulsive delta function interaction*, J. Math Phys. 10 (1969) 1115
Al. B. Zamolodchikov, *Thermodynamic Bethe ansatz in relativistic models: scaling 3-state Potts and Lee-Yang models*, Nucl. Phys. B342 (1990) 695.

[37] L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *An operator product expansion for polygonal null Wilson loops*, arXiv:1006.2788 [hep-th].

[38] D. Fioravanti, S. Piscaglia and M. Rossi, *Asymptotic Bethe Ansatz on the GKP vacuum as a defect spin chain: scattering, particles and minimal area Wilson loops*, arXiv:1503.08795 [hep-th].

[39] A. Bonini, D. Fioravanti, S. Piscaglia and M. Rossi, *Strong Wilson polygons from the lodge of free and bound mesons*, arXiv:1511.05851 [hep-th].

[40] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and flux-tube S-matrix at finite coupling*, arXiv:1303.1396 [hep-th].

[41] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and flux-tube S-matrix II. Extracting and matching data*, arXiv:1306.2058 [hep-th].

[42] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and flux-tube S-matrix III. The two-particle contributions*, arXiv:1402.3307 [hep-th].

[43] B. Basso, A. Sever and P. Vieira, *On the collinear limit of scattering amplitudes at strong coupling*, arXiv:1405.6350 [hep-th].

[44] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and flux-tube S-matrix IV. Gluons and fusion*, arXiv:1407.1736 [hep-th].

[45] A. Sever and P. Vieira, *Multichannel conformal blocks for polygon Wilson loops*, arXiv:1105.5748 [hep-th].

[46] B. Basso, *Exciting the GKP string at any coupling*, arXiv:1010.5237 [hep-th].

[47] M. Henkel, *Conformal invariance and critical phenomena*, Springer 1999.

[48] L. F. Alday and J. Maldacena, *Comments on operators with large spin*, arXiv:0708.0672 [hep-th].