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Jarrow-Yildirim model for inflation:

theory and applications

Tesi di Laurea in Equazioni Differenziali Stocastiche

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Introduzione

Questa tesi ha come argomento l'inflazione, e in particolare il modello di Jarrow & Yildirim, che è quello attualmente utilizzato per la valutazione dei derivati su inflazione.
Nel Capitolo 1 vengono ricordati i principali risultati riguardanti i modelli short e forward per i tassi d'interesse; è inoltre analizzato come gestire i rapporti tra un mercato “domestico” ed uno “straniero”.
Nel Capitolo 2 è fornita una introduzione generale sull'inflazione; è poi introdotto il modello di Jarrow-Yildirim, e vengono derivate le dinamiche delle principali componenti del mercato.
Il Capitolo 3 presenta alcuni tra i più importanti derivati sull'inflazione: zero coupon swap, year-on-year, cap e floor. Servendosi di risultati probabilistici come il cambio di numeraire, viene calcolato il loro valore, mostrando il procedimento di valutazione passo passo.
Nel Capitolo 4 è introdotta la calibrazione: viene chiarito in cosa consiste in questo contesto e come può essere realizzata, spiegando quali dati sono effettivamente disponibili sul mercato. Sono illustrati due possibili metodi di ottimizzazione, uno dei quali non standard (di tipo euristico), e sono presentati alcuni esempi concreti di calibrazione realizzati in Matlab.
Il Capitolo 5 si occupa del metodo numerico Monte Carlo: dopo una spiegazione generale del suo funzionamento, viene approfondito il modo in cui esso può essere utilizzato per valutare derivati (sull'inflazione) complessi, per i quali non si dispone di una formula esplicita per il prezzo.
Infine, nel Capitolo 6 il modello viene arricchito con il rischio di credito, che

permette di tenere conto di un eventuale fallimento del garante per un contratto. In questo contesto è ricavato il metodo di valutazione di un derivato nel caso generale, tramite l'introduzione dei defaultable zero-coupon bond. E' poi approfondito in particolare un esempio di contratto, ed è spiegato come calcolarne il prezzo servendosi del Monte Carlo in due diversi modi.
Nelle Appendici sono richiamati alcuni interessanti risultati teorici utilizzati nella tesi: l' Appendice A si occupa delle misure martingale e presenta il Teorema di Girsanov e il cambio di numeraire. L' Appendice B tratta alcuni aspetti della teoria delle Equazioni Differenziali Stocastiche: in particolare, è fornita la soluzione per le EDS lineari, ed è enunciato il teorema di Feynman-Kač che lega le EDS alle Equazioni alle Derivate Parziali. Infine, nell' Appendice C vengono ricavati alcuni utili risultati riguardanti la distribuzione normale (integrale di un processo normale, attesa di una varibile lognormale, formula esplicita per $E[e^X(p\,e^Z - K)^+]$ con p e K costanti e X e Z normali).

Introduction

This thesis deals with inflation theory; in particular, we present the model of Jarrow & Yildirim, which is nowadays used when pricing inflation derivatives.

In Chapter 1 main results about short and forward interest rate models are recalled; moreover, it is shown how to deal with the relationships between a “domestic” market and a “foreign” one.

In Chapter 2 a general introduction about inflation is given; then the Jarrow-Yildirim model is introduced, and the dynamics of the main components of the market are derived.

Chapter 3 presents some of the most important inflation-indexed derivatives: zero coupon swap, year-on-year, cap and floor. Using probability results as the change of numeraire, their value is computed, explaining the pricing proceeding step by step. For the pricing of more complex derivatives, the Monte Carlo method is treated in detail.

Chapter 4 explains what calibration means in this context, and how it can be performed. Some remark about real market data is given, and concrete calibration examples in Matlab are presented, after having illustrated a common method and an heuristic and non standard one.

Finally, Chapter 6 enriches the model with credit risk, which allows to take into account the possibility of bankruptcy of the counterparty of a contract. In this context, the general method of pricing is derived, with the introduction of defaultable zero-coupon bonds. Then, a concrete example of contract is studied in detail, and its pricing is made in two different ways using Monte

Carlo.

In the appendixes there are some interesting theoretical results which are used in the thesis: Appendix A deals with martingale measures and presents Girsanov's theorem and the change of numeraire; Appendix B treats some aspects of the theory of Stochastic Differential Equations: in particular, the solution for linear EDSs is given, and we enunciate the Feynman-Kač theorem, which shows the connection between EDSs and Partial Differential Equations. Finally, in Appendix C some useful results about normal distribution are derived (integral of a normal process, expectation of a lognormal variable, explicit formula for $E[e^X(p\,e^Z - K)^+]$ with p and K constant and X and Z normally distributed).

Contents

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Chapter 1

Models for the market

In this chapter some important definitions are recalled, and we present some results about short and forward models that will be used later on; we will specifically deal with inflation from Chapter 2. For a general introduction on martingale measures and changes of numeraire see Appendix A.

1.1 T-bonds and interest rates

In this section we define two fundamental elements of the market: T-bonds and interest rates.

Definition 1.1 (**T-bond**). A *T-bond*, or *zero-coupon*[1] *bond with maturity* T, is a contract which guarantees its owner to cash a unit of currency (one dollar/ one euro...) at the date T. The price of a T-bond at time t is usually indicated with $P(t, T)$.

Note that the payoff of $P(t, T)$ is deterministic: $P(T, T)$=1 for every T. $P(t, T)$ thus represents the amount of currency to be owned at time t in order to be sure to have one unit at T. An analogous definition holds for the discount factor D (see def A.3 in Appendix A), but these two instruments

[1]The expression "zero-coupon" is commonly used for the contracts which do not provide for intermediate payments.

are different: once maturity T is fixed, $D(t,T)$ is an $\mathscr{F}_T^W$-measurable random variable, whose value is not known in instants t preceding T; $P(t,T)$ is instead $\mathscr{F}_t^W$-measurable, in fact it is available on the market at every instant t, because it represents the price of a contract (which can be sold, so its quotation is known). In particular, once a martingale measure Q is chosen, the following relation holds between P and D:

$$P(t,T) = E^Q[e^{-\int_t^T r(s)ds}P(T,T)|\mathscr{F}_t^W] = E^Q[D(t,T)|\mathscr{F}_t^W], \tag{1.1}$$

where r denotes the short interest rate.
In the following we will always assume that T-bonds exist on the market for every T. If an instant t is fixed, $P(t,T)$ as a function of T describes the so called *term structure*, or *bond price curve* at time t; we will also assume that for every fixed t that curve is differentiable, that is $P(t,T)$ is differentiable with respect to its second variable.

We now recall the definitions of the two fundamental "types" of interest rates.

Definition 1.2 (**Forward rate**)**.** The *instantaneous forward rate contracted in t and with maturity T* is the process

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}. \tag{1.2}$$

This definition connects bond prices with the forward rate. From this, it follows that

$$\int_s^T f(t,u)du = -\log P(t,T) + \log P(t,s) = \log\frac{P(t,s)}{P(t,T)}, \qquad \text{so}$$

$$e^{-\int_s^T f(t,u)du} = \frac{P(t,T)}{P(t,s)} \qquad \text{from which}$$

$$P(t,T) = P(t,s)\exp\Big\{-\int_s^T f(t,u)du\Big\} \quad \forall t \le s \le T$$

and if we choose s=t we obtain

$$P(t,T) = \exp\Big\{-\int_t^T f(t,u)du\Big\}. \tag{1.3}$$

Definition 1.3 (**Short rate**)**.** The *short instantaneous rate at time t* is the process

$$r(t) = f(t,t).$$

Thanks to the definitions and results of this section, it is possible to create a model for the market in different ways, mainly three:

1) giving the short rate dynamics

$$dr(t) = a(t)dt + b(t)dW(t) \tag{1.4}$$

and, using (1.1), deriving T-bond prices (short model);

2) giving the forward dynamics (for every maturity T)

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \tag{1.5}$$

and, using (1.3), deriving T-bond prices (forward model);

3) giving T-bond dynamics (for every maturity T)

$$dP(t,T) = P(t,T)m(t,T)dt + P(t,T)v(t,T)dW(t); \tag{1.6}$$

in each case W is a d-dimensional Brownian Motion, v and σ are d-dimensional row vectors, and every coefficient is a scalar adapted process (with respect to t). In this context we will always assume the regularity conditions, with respect to t and T, which are necessary for differentiating with continuity, deriving under the sign of integral and changing integration order. Under such hypothesis, the theorem in the following page gives some important relations among the coefficients[2]:

[2]We will indicate as a subscript the variable with respect to which we derive.

Theorem 1.1.1.

a) If $P(t,T)$ satisfies (1.6), the dynamics of f is of the type (1.5) with

$$\begin{cases} \alpha(t,T) = v_T(t,T) \cdot v(t,T) - m_T(t,T) \\ \alpha(t,T) = -v_T(t,T). \end{cases} \tag{1.7}$$

b) If $f(t,T)$ satisfies (1.5), the dynamics of r is of the type (1.4) with

$$\begin{cases} a(t) = f_T(t,T)|_{T=t} + \alpha(t,t) \\ b(t) = \sigma(t,t). \end{cases} \tag{1.8}$$

c) If $f(t,T)$ satisfies (1.5), the dynamics of P is of the type (1.6) with

$$\begin{cases} m(t,T) = r(t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}||v(t,T)||^2 \\ v(t,T) = -\int_t^T \sigma(t,s)ds. \end{cases} \tag{1.9}$$

In the two following chapters models of type 1) and 2) are analyzed more accurately.

1.2 Short models

Let us consider a model for the market without risky assets (N=0), that is in which the "bank account" B is the only security, with dynamics (A.1), and suppose the dynamics of the short rate r is assigned.
In this context, bonds are seen as derivatives on the interest rate, but that rate (the underlying) IS NOT a listed security.
In the search for a martingale measure, we note that every measure which is equivalent to P satisfies the requests, in fact the only asset is B, and if we discount it (with respect to the numeraire that is itself) we obtain a process which is identically equal to one, which is trivially a martingale under every measure; so, instead of giving the dynamics of r with respect to the measure P, it is expedient to assign it directly under the martingale measure Q (*martingale modeling*). Let thus assume that the Q-dynamics of r is

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t. \tag{1.10}$$

Note that the form of $P(t,T)$ in (1.1) corresponds to the one in the Feynman-Kač formula[3]. More precisely: if we indicate $P(t,T)$ with $F(t,r_t;T)$, for (1.1) we have that

$$F(t,r_t;T) = E^Q[e^{-\int_t^T r(s)ds}|\mathscr{F}_t^W]$$

so, thanks to the Feynman-Kač formula, $(t,r) \mapsto F(t,r;T)$ is solution of the system

$$\begin{cases} \mathscr{A}F - aF + \partial_t F = 0 & \text{for } t \in]0,T[,\ r_t \in \mathbb{R} \\ F(T,\cdot) = 1 \end{cases}$$

with $a(s,r) = r$ and $\mathscr{A}$ characteristic operator associated with (1.10), that is

$$\mathscr{A} = \frac{1}{2}\sigma^2(t,r)\partial_{rr} + \mu(t,r)\partial_r$$

which means $F(t,r_t)$ solves the *term structure equation*

$$\begin{cases} \frac{\sigma^2(t,r)}{2}\partial_{rr}F + \mu(t,r)\partial_r F - rF + \partial_t F = 0 & \text{for } t \in]0,T[,\ r_t \in \mathbb{R} \\ F(T,\cdot) = 1. \end{cases} \tag{1.11}$$

This equation can be easily solved in the case of the so called "affine models", which are the ones in which T-bond prices can be written in the form

$$P(t,T) = e^{A(t,T)-B(t,T)r(t)} = \tilde{A}(t,T)e^{-B(t,T)r(t)}. \tag{1.12}$$

In particular, among these models there are the ones in which μ and σ take the form

$$\mu(t,r) = \alpha(t)r + \beta(t), \quad \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)} \tag{1.13}$$

with α, β, γ and δ deterministic functions. In these cases, A and B turn out to be the ones which satisfy

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{1.14}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0. \end{cases} \tag{1.15}$$

[3]See Theorem B.2.1 in Appendix B.

Among the models of type (1.13) there is the one we will use in the following: it is the *Hull & White model*, in which the Q-dynamics of r is of the form

$$dr(t) = (\theta(t) - a(t)r(t))dt + \sigma(t)dW(t) \tag{1.16}$$

with a, σ and θ deterministic functions of time. In particular, if a and σ are constant, comparing (1.13) and (1.16) we see that (1.14) and (1.15) become

$$\begin{cases} B_t(t,T) - aB(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{1.17}$$

$$\begin{cases} A_t(t,T) = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) = 0. \end{cases} \tag{1.18}$$

so A and B are

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}) \tag{1.19}$$

$$A(t,T) = \int_t^T \Big(\frac{1}{2}\sigma^2 B^2(s,T) - \theta(s)B(s,T)\Big)ds. \tag{1.20}$$

Now we want that T-bond prices at time 0 $P(0,T)$ calculated with (1.12) coincide with real ones $P^*(0,T)$; actually, since we know forward rates and bond prices are in one-to-one correspondence, we perform the fitting between the theoretical forward rate curve $f(0,T)$, $T > 0$ and the observed one $f^*(0,T)$, $T > 0$ (where $f^*(t,T) = -\frac{\partial \log P^*(t,T)}{\partial T}$). Let us see this more in detail. From the definitions (1.2) and (1.12) of forward rates and affine models, we have

$$f(t,T) = -A_T(t,T) + B_T(t,T)r(t) \tag{1.21}$$

which for t=0 becomes

$$f(0,T) = -A_T(0,T) + B_T(0,T)r(0);$$

for (1.19), $B_T(t,T) = e^{-a(T-t)}$, and for (1.20)

$$\begin{aligned} A_T(t,T) &= \tfrac{1}{2}\sigma^2 B^2(T,T) - \theta(T)B(T,T) + \textstyle\int_t^T (\sigma^2 B(s,T)B_T(s,T) - \theta(s)B_T(s,T))ds \\ &= \tfrac{\sigma^2}{2a^2}(1 - e^{-a(T-t)})^2 - \textstyle\int_t^T \theta(s)B_T(s,T)ds \end{aligned}$$

so that (1.21) becomes

$$f(0,T) = e^{-aT}r(0) + \int_0^T \theta(s)B_T(s,T)ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2$$

and our problem becomes the search for a function θ which solves

$$f^*(0,T) = e^{-aT}r(0) + \int_0^T \theta(s)B_T(s,T)ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2.$$

If we now decompose $f^*(0,T)$ as $x(T) - g(T)$ with

$$\begin{cases} \dot{x}(t) = -ax(t) + \theta(t) \\ x(0) = r(0) \end{cases} \tag{1.22}$$

and $g(t) = \frac{\sigma^2}{2}B^2(0,t)$, θ must be

$$\theta(t) = \dot{x}(t) + ax(t) = \left.\frac{\partial f^*(0,T)}{\partial T}\right|_{T=t} + \dot{g}(t) + a(f^*(0,t) + g(t)). \tag{1.23}$$

Substituting (1.23) in (1.18), we have

$$\begin{aligned}
&A(t,T) = \\
&\tfrac{\sigma^2}{2a^2}\textstyle\int_t^T (1-e^{-a(T-s)})^2 ds - \int_t^T \Big(\left.\tfrac{\partial f^*(0,T)}{\partial T}\right|_{T=s} + \dot{g}(s) + a(f^*(0,s)+g(s))\Big)\tfrac{1}{a}(1-e^{-a(T-s)})ds \\
&= \tfrac{\sigma^2}{2a^2}\Big\{T - t + \tfrac{1}{2a} - \tfrac{e^{-2a(T-t)}}{2a} - \tfrac{2}{a} + \tfrac{2e^{-a(T-t)}}{a}\Big\} + \tfrac{1}{a}f^*(0,t)(1-e^{-a(T-t)}) \\
&- \textstyle\int_t^T f^*(0,s)e^{-a(T-s)}ds + \tfrac{1}{a}g(t)(1-e^{-a(T-t)}) - \int_t^T g(s)e^{-a(T-s)}ds - \log P^*(0,t)(1-e^{-a(T-t)}) \\
&+ \textstyle\int_t^T a\log P^*(0,s)e^{-a(T-s)}ds - \int_t^T g(s)ds + \int_t^T g(s)e^{-a(T-s)}ds \\
&= \tfrac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) + f^*(0,t)B(t,T) + \log\tfrac{P^*(0,T)}{P^*(0,t)}
\end{aligned}$$

and putting this expression and (1.18) in (1.12) we finally obtain the price of a T-bond in the Hull & White model:

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)}\exp\Big\{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r(t)\Big\}. \tag{1.24}$$

This is an explicit formula, but it cannot be easily used in practice, because the prices $P^*(0,T)$ are observable only for certain maturities T, while the forward rates f^* should be obtained through a derivative, which is thus impossible in concrete (unless we make an arbitrary interpolation of the bond prices). For this reason, we derive another expression for $P(t,T)$ in the Hull & White model. Let us consider a generic Hull & White interest rate process r:

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dW(t) \tag{1.25}$$

where θ is a deterministic function, and $a > 0$ and σ are constants. Thus r is the solution of a linear SDE so, according with (B.7) in Appendix B, its value is:

$$r(t) = e^{-at}\Big(r(0) + \int_0^t e^{as}\theta(s)ds + \int_0^t e^{as}\sigma dW(s)\Big) = x(t) + \phi(t) \tag{1.26}$$

where

$$x(t) = e^{-at}\sigma \int_0^t e^{as}dW(s) \tag{1.27}$$

is the stochastic part, and

$$\phi(t) = e^{-at}\Big(r(0) + \int_0^t e^{as}\theta(s)ds\Big)$$

is the deterministic one, which is still unknown explicitly (because it contains θ).

So, for $t \leq s$, the following holds:

$$r(s) = \phi(s) + e^{-a(s-t)}\Big(x(t) + \sigma \int_t^s e^{a(u-t)}dW(u)\Big) \tag{1.28}$$

from which, if we calculate the price of a bond:

$$\begin{aligned} P(t,T) &= E\big[e^{-\int_t^T r(s)ds}|\mathscr{F}_t\big] \\ &= E[e^{-\int_t^T \phi(s)ds - x(t)\int_t^T e^{-a(s-t)ds} - \sigma\int_t^T e^{-a(s-t)}\left(\int_t^s e^{a(u-t)}dW_u\right)ds}|\mathscr{F}_t] \\ &= E\big[e^{-\int_t^T \phi(s)ds - x(t)\int_t^T e^{-a(s-t)ds} - \sigma\int_t^T \left(\int_u^T e^{a(u-s)}ds\right)dW_u}|\mathscr{F}_t\big] \\ &= E\big[e^{H(T)}\big] \end{aligned} \tag{1.29}$$

where $H(T) = -\int_t^T \phi(s)ds - x(t)B(t,T) - \sigma\int_t^T B(u,T)dW(u)$, with $B(u,T) = \frac{1-e^{-a(T-u)}}{a}$. H is thus made up of a deterministic part plus a stochastic integral of a $\mathscr{C}^\infty$-function, so it is normally distributed, with mean and variance

$$\mu(t,T) = -\int_t^T \phi(s)ds - x(t)B(t,T), \quad V(t,T) = \sigma^2 \int_t^T (B(u,T))^2 du$$

respectively (thanks to the Itô formula). Thus, the price of the bond turns out to be the expectation of a lognormal variable, whose value is[4] $e^{\mu(t,T)+\frac{V(t,T)}{2}}$. If we now put ourselves in a concrete context, the prices of the bonds at time zero are known: if we call them $P^*(0,T)$, the following must hold:

$$P^*(0,T) = P(0,T) = e^{\mu(0,T)+\frac{V(0,T)}{2}} = e^{-\int_0^T \phi(s)ds + \frac{\sigma^2}{2}\int_0^T (B(u,T))^2 du}$$

$$= e^{-\int_0^t \phi(s)ds - \int_t^T \phi(s)ds + x(t)B(t,T) - x(t)B(t,T)}$$

$$\cdot e^{\frac{\sigma^2}{2}\left(\int_0^T B(u,T)^2du + \int_0^t B(u,t)^2du - \int_0^t B(u,t)^2du + \int_t^T B(u,T)^2du - \int_t^T B(u,T)^2du\right)}$$

$$= P(0,t)P(t,T)e^{x(t)B(t,T)+\frac{\sigma^2}{2}\left(\int_0^T B(u,T)^2du - \int_0^t B(u,t)^2du - \int_t^T B(u,T)^2du\right)}$$

$$= P(0,t)P(t,T)e^{x(t)B(t,T)+\frac{\sigma^2}{2}\int_0^t (B(u,T)^2 - B(u,t)^2)du}$$

so

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-x(t)B(t,T)+\frac{\sigma^2}{2}\int_0^t (B(u,t)^2 - B(u,T)^2)du} \tag{1.30}$$

in which only deterministic and observable addends appear.

Let us now determine the price of an option on a T-bond. We recall that a Call/Put option on a certain underlying asset Y is a derivative which ensures its holder the possibility (but not the obligation) of buying/selling Y at a future time T with a price K which is established at the issuing moment; its payoff is thus $(Y(T)-K)^+$, and the price is determined as usual with

$$Call(t,T) = E[e^{-\int_t^T r(s)ds}(Y(T)-K)^+ | \mathscr{F}_t].$$

If the underlying is an S-bond with $0 < T \leq S$, at time t=0 we have

$$\begin{aligned} Call(0,T) &= E[e^{-\int_0^T r(s)ds}(P(T,S)-K)^+] \\ &= E[e^{-\int_0^T \phi(s)ds - \int_0^T x(s)ds}\left(\tfrac{P(0,S)}{P(0,T)} e^{-x(T)B(T,S)+\frac{\sigma^2}{2}\int_0^T (B(u,T)^2 - B(u,S)^2)du} - K\right)^+] \\ &= \tfrac{P(0,S)}{P(0,T)} e^{-\int_0^T \phi(s)ds + \frac{\sigma^2}{2}\int_0^T (B(u,T)^2 - B(u,S)^2)du} E[e^{-\int_0^T x(s)ds}(e^{-x(T)B(T,S)} - \bar{K})^+] \end{aligned}$$

[4] See (C.1) in Appendix C.

where $\bar{K} = K\frac{P(0,T)}{P(0,S)}e^{\frac{\sigma^2}{2}\int_0^T(B(u,S)^2-B(u,T)^2)du}$.

Now, in order to obviate the fact that ϕ contains θ, which is not observable on the market, we can impose the equality between market prices of the bonds and model ones:

$$P^*(0,T) = P(0,T) = e^{\mu(0,T)+\frac{V(0,T)}{2}} = e^{-\int_0^T \phi(s)ds+\frac{V(0,T)}{2}} \Rightarrow$$

$$\int_0^T \phi(s)ds = \frac{V(0,T)}{2} - \log(P^*(0,T)). \tag{1.31}$$

Substituting this in the expression of the Call and taking in mind that all bond prices at time 0 must coincide with market ones, we obtain

$$Call(0,T) = P^*(0,S)e^{-\frac{\sigma^2}{2}\int_0^T B(u,S)^2du}E[e^{-\int_0^T x(s)ds}(e^{-x(T)B(T,S)} - \bar{K})^+]$$

in which we can see an expectation of the form $E[e^X(e^Z - M)^+]$, where (X,Z) have a bidimensional Gaussian as joint distribution: thanks to the computations in Section C.3 of Appendix C, we know the explicit expression of this type of integral. Let us calculate the moments of the distribution:

$$\begin{aligned}
E[X] &= E[-\textstyle\int_0^T x(s)ds] = E[-\int_0^T e^{-as}\sigma\int_0^s e^{au}dW(u)ds]\\
&= -\sigma E[\textstyle\int_0^T\int_u^T e^{a(u-s)}ds\, dW(u)] = 0;\\
E[Z] &= E[-x(T)\textstyle\int_T^S e^{-a(s-T)}ds] = E[-e^{-aT}\sigma\int_0^T e^{au}dW(u)\int_T^S e^{-a(s-T)}ds] = 0;\\
var[X] &= \sigma^2 E[(\textstyle\int_0^T\int_u^T e^{a(u-s)}ds\, dW(u))^2] = \sigma^2\int_0^T B(u,T)^2du;\\
var[Z] &= B(T,S)^2\, var[x(T)] = B(T,S)^2\tfrac{\sigma^2}{2}B(0,2T);\\
cov[X,Z] &= B(T,S)E[\textstyle\int_0^T \sigma e^{a(s-T)}dW(s)\int_0^T\int_0^s \sigma e^{a(u-s)}dW(u)ds]\\
&= \sigma^2 B(T,S)\textstyle\int_0^T e^{a(u-T)}(\int_u^T e^{a(u-s)}ds)du = \frac{\sigma^2}{2}B(T,S)B(0,T)^2.
\end{aligned}$$

Now we can apply formula (C.6) in Appendix C to compute the expectation, obtaining

$$\begin{aligned}
&Call(0,T) = P^*(0,S)e^{-\frac{\sigma^2}{2}\int_0^T(B(u,S))^2du}e^{E[X]+E[Z]+\frac{var[X]^2}{2}}\\
&\Big(e^{\frac{var[Z]^2}{2}+cov[X,Z]}\Phi\big(\tfrac{E[Z]-\log\bar{K}+cov[X,Z]+var[Z]^2}{\sqrt{var[Z]}}\big) - \bar{K}e^{-E[Z]}\Phi\big(\tfrac{E[Z]-\log\bar{K}+cov[X,Z]}{\sqrt{var[Z]}}\big)\Big).
\end{aligned}$$

We now note that, if we put $h = -\frac{\sigma^2}{2}\int_0^T (B(u,S)^2 - B(u,T)^2)\,du$, with a bit calculation we obtain that

$$\frac{h + cov[X,Z] + var[Z]}{\sqrt{var[Z]}} = \frac{\sqrt{var[Z]}}{2}$$

which also implies

$$h + cov[X,Z] + \tfrac{var[Z]}{2} = (h + cov[X,Z] + var[Z] - \tfrac{var[Z]}{2})\tfrac{\sqrt{var[Z]}}{\sqrt{var[Z]}}$$
$$= \tfrac{1}{2}\sqrt{var[Z]}\sqrt{var[Z]} - \tfrac{var[Z]}{2} = 0.$$

So our formula finally becomes

$$Call(0,T) = P^*(0,S)\Phi(d_1) - P^*(0,T)K\Phi(d_2), \quad \text{with}$$
$$d_1 = \tfrac{1}{\sigma_p}\log\left(\tfrac{P^*(0,S)}{KP^*(0,T)}\right) + \tfrac{1}{2}\sigma_p, \quad d_2 = \tfrac{1}{\sigma_p}\log\left(\tfrac{P^*(0,S)}{KP^*(0,T)}\right) - \tfrac{1}{2}\sigma_p,$$
$$\sigma_p = \sqrt{var[Z]} = \tfrac{1}{a}(1 - e^{-a(S-T)})\sqrt{\tfrac{\sigma^2}{2a}(1 - e^{-2aT})}$$

where Φ is the standard normal distribution function.
If we consider a Put, the computation is of the same type, so we obtain the more general formula:

Proposition 1.2.1 (Hull & White zero-coupon bond option). *Assuming the interest rate r follows the Hull & White dynamics in (1.16) with constant a and σ, the price of an option on a zero-coupon bond is*

$$ZBPC(0,T) = w\,P^*(0,S)\Phi(wd_1) - wP^*(0,T)K\Phi(wd_2), \quad \textit{with}$$
$$d_1 = \tfrac{1}{\sigma_p}\log\left(\tfrac{P^*(0,S)}{KP^*(0,T)}\right) + \tfrac{1}{2}\sigma_p, \quad d_2 = \tfrac{1}{\sigma_p}\log\left(\tfrac{P^*(0,S)}{KP^*(0,T)}\right) - \tfrac{1}{2}\sigma_p, \tag{1.32}$$
$$\sigma_p = \tfrac{1}{a}(1 - e^{-a(S-T)})\sqrt{\tfrac{\sigma^2}{2a}(1 - e^{-2aT})}$$

where w is 1 for a Call, -1 for a Put.

1.3 Forward models

In these models forward rate dynamics is assigned for every maturity T:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW_t \\ f(0,T) &= f^*(0,T) \end{aligned} \tag{1.33}$$

where W is a d-dimensional Brownian Motion, and where we have chosen to put ourselves directly under the martingale measure Q; such models were introduced by Heath Jarrow and Morton, so that they are referred to as *HJM*.
In order to guarantee that (1.1) and (1.3) give the same result, the following condition must hold:

Theorem 1.3.1 (HJM drift condition). *Let us assume f is described by (1.33) under the martingale measure. Then*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds \quad \forall t \leq T \tag{1.34}$$

where the apex indicates transposition.

Proof. From c) of Theorem 1.1.1, we have

$$\frac{dP(t,T)}{P(t,T)} = \left[r(t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}\Big|\Big| \int_t^T \sigma_r(s)ds \Big|\Big|^2 \right] dt - \int_t^T \sigma_r(s)ds\, dW(t).$$

Under a martingale measure we also know that a T-bond must have the same value of the bank account on the average, that is

$$r(t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}\Big|\Big| \int_t^T \sigma_r(s)ds \Big|\Big|^2 = r(t)$$

so that

$$-\int_t^T \alpha(t,s)ds + \frac{1}{2}\Big|\Big| \int_t^T \sigma_r(s)ds \Big|\Big|^2 = 0. \tag{1.35}$$

Deriving (1.35) with respect to T we obtain the desired relation. □

Condition (1.34) establishes that, once we fix the volatility, the drift coefficient of the forward rate is automatically defined.

In this context, the Hull & White model takes the form

$$df(t,T) = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW_t. \tag{1.36}$$

1.4 Foreign market

Let us now consider a domestic market and a foreign one with risk-neutral probability measures Q and Q^f and money market accounts B and B^f respectively, and an exchange rate $\mathcal{Q}$, i.e. at time t one unit of foreign currency corresponds to $\mathcal{Q}(t)$ units of domestic currency.
If X^f is a derivative in the foreign market, the no-arbitrage pricing formula gives us

$$X^f(t) = B^f(t)E^f\Big[\frac{X^f(T)}{B^f(T)}\Big|\mathcal{F}_t\Big]$$

whose domestic value is

$$\mathcal{Q}(t)B^f(t)E^f\Big[\frac{X^f(T)}{B^f(T)}\Big|\mathcal{F}_t\Big]. \tag{1.37}$$

For a domestic investor, X^f is equivalent to the derivative $X^f\mathcal{Q}$, so (1.37) must be equal to

$$B(t)E\Big[\frac{X^f(T)\mathcal{Q}(T)}{B(T)}\Big|\mathcal{F}_t\Big].$$

This equality at time $t=0$ implies

$$E^f\Big[\frac{X^f(T)\mathcal{Q}(0)}{B^f(T)}\Big] = E\Big[\frac{X^f(T)\mathcal{Q}(T)}{B(T)}\Big] = E\Big[\frac{dQ^f}{dQ}\frac{X^f(T)\mathcal{Q}(0)}{B^f(T)}\Big]$$

with

$$\frac{dQ^f}{dQ} = \frac{\mathcal{Q}(T)B^f(T)}{\mathcal{Q}(0)B(T)}. \tag{1.38}$$

As a consequence, switching from the the measure Q^f to Q corresponds to changing the numeraire from B^f to $B/\mathcal{Q}$, in fact the change-of-numeraire formula (A.4) tells that if we want to pass from numeraire B^f to U we must use

$$\frac{dQ^U}{dQ^f} = \frac{U(T)B^f(0)}{U(0)B^f(T)} = \frac{U(T)}{U(0)B^f(T)} \tag{1.39}$$

so if we take $Q^U = Q$ and compare (1.39) with (1.38) we obtain

$$\frac{dQ}{dQ^f} = \frac{\mathcal{Q}(0)B(T)}{\mathcal{Q}(T)B^f(T)} = \frac{U(T)}{U(0)B^f(T)}$$

which says that it is enough to take

$$U(t) = \frac{B(t)}{\mathcal{Q}(t)}.$$

Chapter 2

Inflation: definition and model

2.1 Introduction

Inflation represents the general rise of the prices of goods in an economy (in case of decrease, it is called *deflation*). Its dynamics is complex; an increase in the money in circulation can cause its depreciation and thus prices escalate. Broadly, inflation tends to rise when the demand for goods and services exceeds the effective possibility of the economy of furnishing them. Moreover, its expectation should be bound to the setting of costs and wages, since an increase in prices reduces the purchasing power; effectively most legislations give the employees the right of a periodical salary increase as an inflation compensation.
Seasonal variations of inflation have been observed: for example, it tends to rise during Christmas period and to drop in January and July due to sales. We note that this seasonal effects change from one country to another, and they can be very definite such as in UK or almost non influential as in Italy. Governments and central banks generally shoot for an annual inflation around 2-3%, which is advantageous for the economy beacause it stimulates consumers to buy goods and services (since prices tend to rise, postponing the buying would lead to pay more). Moreover, in periods of low inflation interest rates are usually low, so people are stimulated in asking for loans. On

the contrary, deflation is very negative for an economy because consumers often wait for purchasing since prices tend to decrease.
Inflation is measured as the percentage increase of an index which represents the price of a certain basket of goods and whose value is determined by statistical institutes. Several indexes exist: the main ones are the European HICPxT (Harmonized Index of Consumer Prices excluding Tobacco, calculated by Eurostat considering all the countries in the monetary union, weighted by their consumption level), the French FRCPI (given by INSEE), the English RPI (Retail Price Index, from National Statistics) and the US CPI (Consumer Price Index, from BLS).
In Italy, Istat produces three indices:
-NIC (Intera Comunità Nazionale), which is the reference parameter for economic policy;
-FOI (Famiglie di Operai e Impiegati), which takes into account the families which depend on a (not agricultural) employee and which is used to periodically adapt the monetary values;
-IPCA (Indice dei Prezzi al Consumo Armonizzato: it is the italian version of HICP) which allows a Europe-wide comparison of inflation measure: the economies of the different members of the European Union must converge, and this index is used to check the possibility of permanence in the monetary union.
NIC and FOI are based on the same basket but give different weights to the goods; IPCA instead follows the comunitary agreement to exclude lotto, lotteries, forecast competitions and life-insurance services, and refers to the prices that are really payed by the consumers (for example, it takes sales into account).
It is common to generically indicate the inflation index with CPI, and we will do this too.
Finally, we note that the inflation index is computed at regular intervals, typically monthly, but its present value is never known as its elaboration requires a certain time; for instance, HICPxT index is published around the

15^{th} of the following month. More precisely, that publication corresponds to the so-called "unrevised index", which is effectively used, but which can be revised if, on the basis of new data, Eurostat decides it is inaccurate.

2.2 Nominal and real rates and bonds

When we usually speak about interest rates we refer to nominal rates, which are bound to the variation of the amount of money but not to its effective purchasing power; real rates instead reckon with the adjustment due to inflation. For example, in case of a loan, if the agreed interest rate turns out to be equal to the inflation rate, the real rate received by the lending of the loan is zero.

Notation 1. In the following we will indicate with the subscripts n or r what is nominal or real respectively.

The relation between the two rates is *Fisher equation*

$$(1 + r_r)(1 + i) = (1 + r_n)$$

where i is the inflation rate. Such relation is commonly approximated by

$$r_r = r_n - i. \tag{2.1}$$

Actually, we can consider a foreign-currency analogy, where nominal rates are the interest rates in the domestic economy, real rates are the ones in the foreign (real) economy, and the exchange rate is given by the CPI; our model will follow this interpretation.

A nominal T-bond in T corresponds to a unit of currency, without taking care of inflation, while the value of a real T-bond is one unit of CPI at maturity; in line with Notation 1, we indicate with $P_n(t, T)$ and $P_r(t, T)$ the prices at time t of a nominal/real T-bond respectively. We point out that $P_r(t, T)$ is expressed in CPI units: it means that, in order to obtain the nominal price in t of a real T-bond, we have to multiply $P_r(t, T)$ for the value of the CPI at time t (and we indicate with P_{TIPS} the quantity we get in such a way).

Notation 2. We indicate with $I(t)$ the value of the inflation index at time t.

As we told in advance, the index is the price of a basket (which is expressed in a certain currency, as euros/dollars). In order to refer to inflation, instead of working with percentage increments of the index, the direct use of the value of the index is common, and we will follow this line: $I(t)$ will thus be the CPI value at time t, that is the price in t of the reference basket. With the new notation we thus have

$$P_{TIPS}(t,T) = I(t)P_r(t,T).$$

Really, since a certain time is necessary for the determination of the inflation index, $I(t)$ turns out to be not the CPI in the instant t, but in $t-L$, where L denotes the time lag[1].

2.3 Jarrow-Yildirim model

The Jarrow-Yildirim model is an HJM model, in which forward rate dynamics are assigned (nominal and real, for every maturity T); let us assume the probability space associated to the effective economy is $(\Omega, \mathscr{F}, P)$, but as like as in Section 1.3 we will describe the processes directly under the martingale measure Q. The dynamics of the CPI I is assigned, too. Thus, the basis of the model are:

$$df_n(t,T) = \alpha_n(t,T)dt + \sigma_n(t,T)dW_n(t), \quad f_n(0,T) = f_n^*(0,T) \tag{2.2}$$

$$df_r(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dW_r(t), \quad f_r(0,T) = f_r^*(0,T) \tag{2.3}$$

$$dI(t) = I(t)\mu_I(t)dt + I(t)\sigma_I(t)dW_I(t), \quad I(0) = I_0 > 0 \tag{2.4}$$

with W_n, W_r and W_I Brownian Motions with correlations $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$, and with σ_n, σ_r and σ_I deterministic functions in L^2. For (2.2) the HJM condition (1.34) must hold. Moreover, for Q being a martingale measure,

[1]See also the introduction of Chapter 3.

it is necessary that not only the discounted P_ns, but also the discounted P_{TIPS}s and discounted $I(t)B_r$ are martingales with respect to Q; this implies analogous conditions for (2.3) and (2.4). In fact: let us set $\xi(t,T) = \frac{I(t)P_r(t,T)}{B_n(t,T)}$. So, according to the Itô formula we have

$$d\xi(t,T) = -\frac{1}{B_n^2(t)} I(t)P_r(t,T)dB_n(t) + \frac{1}{B_n(t)} d(I(t)P_r(t,T)) =$$

$$= -\frac{1}{B_n(t)} I(t)P_r(t,T)r_n(t)dt + \frac{1}{B_n(t)}\{I(t)dP_r(t,T) + P_r(t,T)dI(t)\} =$$

using (2.4) and c) of Theorem 1.1.1

$$\xi(t)\Big[-r_n(t) + r_r(t) - \textstyle\int_t^T \alpha(t,s)ds + \frac{1}{2}\Big|\Big|\int_t^T \sigma_r(t,s)ds\Big|\Big|^2 - \sigma_I(t)\rho_{r,I}\int_t^T \sigma_r(t,s)ds + \mu_I(t)\Big]dt$$

$$+\xi(t,T)[-\textstyle\int_t^T \sigma_r(t,s)ds\, dW_r(t) + \sigma_I(t)dW_I(t)].$$

In order to be a martingale, ξ must have null drift, so if we derive the drift with respect to T we must obtain zero:

$$-\alpha_r(t,T) + \sigma_r(t,T)\int_t^T \sigma_r(t,s)ds - \sigma_r(t,T)\sigma_I(t)\rho_{rI} = 0$$

from which we gain the desired condition for the drift of f_r:

$$\alpha_r(t,T) = \sigma_r(t,T)\Big[\int_t^T \sigma_r(t,s)ds - \sigma_I(t)\rho_{rI}\Big]. \tag{2.5}$$

In an analogous way, setting $\zeta(t,T) = \frac{I(t)B_r(t,T)}{B_n(t,T)}$, thanks to the Itô formula we have

$$\begin{aligned} d\zeta(t,T) &= -\tfrac{1}{B_n^2(t)} I(t)B_r(t,T)dB_n(t) + \tfrac{1}{B_n(t)} d(I(t)B_r(t,T)) \\ &= -\tfrac{1}{B_n(t)} I(t)B_r(t,T)r_n(t)dt + \tfrac{1}{B_n(t)}(I(t)dB_r(t,T) + B_r(t,T)dI(t)) \\ &= \zeta(t,T)[-r_n(t) + r_r(t) + \mu_I(t)]dt + \zeta(t,T)\sigma_I(t)dW_I(t). \end{aligned}$$

In order to be a martingale, ζ must have null drift, so deriving the drift with respect to T we must obtain zero; thus, the desired condition on the drift of I is

$$\mu_I = r_n(t) - r_r(t), \tag{2.6}$$

and it corresponds to Fisher equation (2.1).

Using the relations we have found, let us now derive the dynamics of the different T-bond prices.
For what concerns P_n, joining c) of Theorem 1.1.1 and (1.35) of the proof of HJM condition, we immediately obtain

$$\frac{dP_n(t,T)}{P_n(t,T)} = r_n(t)dt - \int_t^T \sigma_n(s)ds\, dW_n(t). \tag{2.7}$$

For P_r, using c) of Theorem 1.1.1 again, and (2.5), we obtain

$$\frac{dP_r(t,T)}{P_r(t,T)} = \left[r_r(t) - \int_t^T \alpha_r(t,s)ds + \frac{1}{2}\Big(\int_t^T \sigma_r(t,s)ds \Big)^2 \right] dt - \int_t^T \sigma_r(t,s)ds\, dW_r(t)$$

with

$$\int_t^T \alpha_r(t,s)ds = \int_t^T \Big(\sigma_r(t,s) \int_t^s \sigma_r(t,u)du \Big) ds - \int_t^T \sigma_r(t,s)\sigma_I(t)\rho_{rI} ds.$$

Let us call A the first addend, and let us integrate by parts:

$$\begin{aligned} A &= \left[\Big(\int_t^s \sigma_r(t,u)du \Big)^2 \right]_{s=t}^{s=T} - \int_t^T \Big(\int_t^s \sigma_r(t,u)du\, \sigma_r(t,s) \Big) ds \\ &= \Big(\int_t^T \sigma_r(t,u)du \Big)^2 - A \end{aligned}$$

so

$$2A = \Big(\int_t^T \sigma_r(t,u)du \Big)^2.$$

Thus

$$\begin{aligned} \frac{dP_r(t,T)}{P_r(t,T)} &= r_r(t)dt - \tfrac{1}{2}\Big(\textstyle\int_t^T \sigma_r(t,u)du \Big)^2 dt + \int_t^T \sigma_r(t,s)\sigma_I(t)\rho_{rI} ds\, dt \\ &\quad + \tfrac{1}{2}\Big(\textstyle\int_t^T \sigma_r(t,u)du \Big)^2 dt - \int_t^T \sigma_r(s)ds\, dW_r(t) \\ &= \Big[r_r(t) + \rho_{rI}\sigma_I(t) \textstyle\int_t^T \sigma_r(t,s)ds \Big] dt - \int_t^T \sigma_r(s)ds\, dW_r(t). \end{aligned} \tag{2.8}$$

Finally, keeping in mind that $P_{TIPS}(t,T) = I(t)P_r(t,T)$, we have

$$dP_{TIPS}(t,T) = I(t)dP_r(t,T) + P_r(t,T)dI(t) + d < I(t), P_r(t,T) >=$$

using (2.8) and (2.6)

$$
\begin{aligned}
&= P_{TIPS}(t,T)\Big[\Big(r_r(t)+\sigma_I(t)\rho_{rI}\textstyle\int_t^T\sigma_r(t,s)ds\Big)dt-\int_t^T\sigma_r(t,s)ds\,dW_r(t)\\
&\quad +(r_n(t)-r_r(t))dt+\sigma_I(t)dW_I(t)-\sigma_I(t)\rho_{rI}\textstyle\int_t^T\sigma_r(t,s)ds\,dt\Big]\\
&= P_{TIPS}(t,T)r_n(t)dt-P_{TIPS}(t,T)\textstyle\int_t^T\sigma_r(t,s)ds\,dW_r(t)+P_{TIPS}(t,T)\sigma_I(t)dW_I(t).
\end{aligned}
$$

For sake of clarity, let us sum up the dynamics we obtained up to now.

Proposition 2.3.1. *Under the martingale measure Q, we have:*

$$
\begin{aligned}
df_n(t) &= \sigma_n(t,T)\textstyle\int_t^T\sigma_n(t,s)ds\,dt+\sigma_n(t,T)dW_n(t)\\
df_r(t) &= \sigma_r(t,T)\Big[\textstyle\int_t^T\sigma_r(t,s)ds-\rho_{rI}\sigma_I(t)\Big]dt+\sigma_r(t,T)dW_r(t)\\
\tfrac{dI(t)}{I(t)} &= [r_n(t)-r_r(t)]dt+\sigma_I(t)dW_I(t)\\
\tfrac{dP_n(t,T)}{P_n(t,T)} &= r_n(t)dt-\textstyle\int_t^T\sigma_n(t,s)ds\,dW_n(t) \qquad (2.9)\\
\tfrac{dP_r(t,T)}{P_r(t,T)} &= \Big[r_r(t)dt+\sigma_I(t)\rho_{rI}\textstyle\int_t^T\sigma_r(t,s)ds\Big]dt\\
&\quad -\textstyle\int_t^T\sigma_r(t,s)ds\,dW_r(t)\\
\tfrac{dP_{TIPS}(t,T)}{P_{TIPS}(t,T)} &= r_n(t)dt+\sigma_I(t)dW_I(t)-\textstyle\int_t^T\sigma_r(t,s)ds\,dW_r(t).
\end{aligned}
$$

From now on we assume nominal and real volatility to be of the form $\sigma_k(t,T)=\sigma_k e^{-a_k(T-t)}$, $k\in n,r$, with σ_k constant, as in (1.36) (Hull & White); moreover we assume $\sigma_I(t)$ is constant (independent from t).
We thus have

$$
\begin{aligned}
df_n(t,T) &= \Big(\sigma_n e^{-a_n(T-t)}\textstyle\int_t^T\sigma_n e^{-a_n(s-T)}ds\Big)dt+\sigma_n e^{-a_n(T-t)}dW_n(t)\\
&= \tfrac{\sigma_n^2}{a_n}(-e^{-2a_n(T-t)}+e^{-a_n(T-t)})dt+\sigma_n e^{-a_n(T-t)}dW_n(t)
\end{aligned}
$$

from which, transforming into integral form and imposing the term structure:

$$f_n(t,T)=f_n^*(0,T)+\frac{\sigma_n^2}{a_n}\int_0^t\Big(-e^{-2a_n(T-s)}+e^{-a_n(T-s)}\Big)ds+\sigma_n\int_0^t e^{-a_n(T-s)}dW_n(s).$$

So, recalling (1.3):

$$\begin{aligned} dr_n(t) &= \frac{\partial f_n(0,T)}{\partial T}\Big|_{T=t} dt + \frac{\sigma_n^2}{a_n}\int_0^t \Big(-e^{-2a_n(t-s)}(-2a_n) + e^{-a_n(t-s)}(-a_n)\Big) ds\, dt \\ &\quad +\sigma_n dW_n(t) + \sigma_n \int_0^t e^{-a_n(t-s)}(-a_n) dW_n(s)\, dt \\ &= \Big\{ \frac{\partial f_n(0,T)}{\partial T}\Big|_{T=t} - a_n r_n(t) - a_n\Big(-f_n(0,t) - \frac{\sigma_n^2}{a_n}\int_0^t e^{-2a_n(t-s)} ds\Big)\Big\} dt \\ &\quad +\sigma_n dW_n(t) \end{aligned}$$

and using (1.23), we obtain the nominal short rate dynamics

$$dr_n(t) = [\theta_n(t) - a_n r_n(t)]dt + \sigma_n dW_n(t). \tag{2.10}$$

For real short rate calculation is analogous:

$$df_r(t,T) = \frac{\sigma_r^2}{a_r}(-e^{-2a_r(T-t)} + e^{-a_r(T-t)})dt - \sigma_r e^{-a_r(T-t)}\rho_{rI}\sigma_I dt + \sigma_r e^{-a_r(T-t)} dW_n(t)$$

from which, transforming into integral form and imposing the term structure:

$$\begin{aligned} f_r(t,T) &= f_r^*(0,T) + \frac{\sigma_r^2}{a_r}\int_0^t \Big(-e^{-2a_r(T-s)} + e^{-a_r(T-s)}\Big) ds \\ &\quad -\frac{\sigma_r\sigma_I\rho_{rI}}{a_r}(e^{-a_r(T-t)} - e^{-a_r T}) + \sigma_r \int_0^t e^{-a_r(T-s)} dW_r(s). \end{aligned}$$

So

$$\begin{aligned} dr_r(t) &= \frac{\partial f_r(0,T)}{\partial T}\Big|_{T=t} dt + \frac{\sigma_r^2}{a_r}\int_0^t \Big(-e^{-2a_r(t-s)}(-2a_r) + e^{-a_r(t-s)}(-a_r)\Big) ds\, dt \\ &\quad -\sigma_r\sigma_I\rho_{rI} e^{-a_r t} dt + \sigma_r dW_r(t) + \sigma_r \int_0^t e^{-a_r(t-s)}(-a_r) dW_r(s)\, dt \\ &= \frac{\partial f_r(0,T)}{\partial T}\Big|_{T=t} dt - a_r r_r(t) dt - a_r\Big(-f_r(0,t) - \frac{\sigma_r^2}{a_r}\int_0^t e^{-2a_r(t-s)} ds\Big) dt \\ &\quad -\sigma_r\sigma_I\rho_{rI} dt + \sigma_r dW_r(t) \end{aligned}$$

from which we deduce the real short rate dynamics

$$dr_r(t) = [\theta_r(t) - a_r r_r(t) - \sigma_r\sigma_I\rho_{rI}]dt + \sigma_r dW_r(t). \tag{2.11}$$

Let us sum up the results we found about short rates, adding their extended expression (which is obtained solving the linear SDEs (2.10) and (2.11))[2]:

[2]See (B.7) in Appendix B.

Proposition 2.3.2. *Under the martingale measure Q, we have:*

$$\begin{aligned}
dr_n(t) &= [\theta_n(t) - a_n r_n(t)]dt + \sigma_n dW_n(t) \\
r_n(t) &= r_n(s)e^{-a_n(t-s)} + \int_s^t e^{a_n(u-t)}\theta_n(u)du + \int_s^t e^{a_n(u-t)}\sigma_n dW_n(u) \\
dr_r(t) &= [\theta_r(t) - a_r r_r(t) - \sigma_r\sigma_I\rho_{rI}]dt + \sigma_r dW_r(t) \\
r_r(t) &= r_r(s)e^{-a_r(t-s)} + \int_s^t e^{a_r(u-t)}(\theta_r(u) - \rho_{rI}\sigma_r\sigma_I)du + \int_s^t e^{a_r(u-t)}\sigma_r dW_r(u).
\end{aligned} \tag{2.12}$$

Let us now analyze the consequences of the foreign-currency analogy. By Section 1.4, moving from the real measure Q^r to the nominal one Q^n corresponds to a change of numeraire from B^r to B^n/I; it means that moving back from Q^n to Q^r corresponds to a change of numeraire from B^n to $B^r I$. From (2.3.1) we obtain

$$I(T) = I(t)e^{\int_t^T (r_n(s)-r_r(s))ds - \frac{1}{2}\sigma_I^2(T-t) + \sigma_I(W_I(T)-W_I(t))} \tag{2.13}$$

so, recalling

$$dB^k(t) = r^k(t)B^k(t)dt \Rightarrow B^k(t) = e^{\int_0^t r_k(s)ds}, \quad k = n, r$$

we have

$$B^r(t)I(t) = I(0)e^{\int_0^t r_n(s)ds - \frac{1}{2}\sigma_I^2 t + \sigma_I W_I(t)} = B^r(0)I(0)e^{\int_0^t r_n(s)ds - \frac{1}{2}\sigma_I^2 t + \sigma_I W_I(t)}.$$

Moreover

$$\begin{aligned}
&d(B^r I)(t) = I(t)dB^r(t) + B^r(t)dI(t) = \\
&I(t)(r_r(t)B^r(t)dt) + B^r(t)I(t)((r_n(t) - r_r(t))dt + \sigma_I dW_I(t)) = \\
&r_n(t)(B^r I)(t)dt + \sigma_I(B^r I)(t)dW_I(t).
\end{aligned}$$

Using the change-of-numeraire formula (A.2.4), we obtain

$$dW_r^{B^r I}(t) = dW_r^{B^n}(t) - \rho_{rI}\Big(\frac{\sigma_I(B^r I)(t)}{(B^r I)(t)} - \frac{0}{B^n(t)}\Big)dt$$

from which

$$dW_r^{B^n}(t) = dW_r^{B^r I}(t) + \rho_{rI}\sigma_I dt;$$

so, the dynamics for r_r under Q^r is

$$dr_r(t) = [\theta_r(t) - a_r r_r(t)]dt + \sigma_r dW_r(t) \tag{2.14}$$

which is exactly of the same form as the one of r_n under Q (2.10).

Chapter 3

Inflation-linked derivatives

The market for inflation-linked derivatives has known a strong growth in the last ten years: in 2001 it almost did not exist, then inflation market quickly developed in Europe first, and now the US market is very active for what concerns inflation, too. Among the causes of this rapid development there probably was the desire for new kinds of structured products, since the traditional fixed-income ones only allowed small returns, and part of the investors were reluctant to the idea of taking too many risks.
Inflation-indexed derivatives have the purpose of transferring the inflation risk, and to deal with real returns instead of nominal ones. As a consequence of the lag in the index, the covering of the inflation risk is not perfect, but shifted backwards (generally of three months): it means that instead of referring to present inflation, derivatives deal with a three-months back inflation, so that they cover a period which starts three months before the issuing date and ends three months before maturity.
Inflation-indexed derivatives are numerous and diversified in order to satisfy the client's needs; in this chapter we present some of the most well known derivatives on the inflation rate and we derive their price.

3.1 Zero-coupon swap

A *zero-coupon swap* (ZCIIS, which stands for *zero-coupon inflation-indexed swap*) is a contract in which one of the two parties accepts to pay the inflation rate at maturity and to receive from the other party a fixed rate K which is established at the beginning, where both the rates are computed referring to the nominal value N. If we indicate with M the length of the contract, the inflation rate is computed as percentage increase of the index between the dates $T_0 := 0$ and T_M of the start and the end of the contract respectively; note that $I(0)$ is known at the moment of the stipulation of the contract, while $I(T_M)$ is not. The *fixed leg* is the leg connected with the fixed rate K, and we will indicate it with the subscript "fix"; the *floating leg* is the leg connected with inflation, and we will indicate it just with the subscript "f". Their values at maturity are

$$N[(1+K)^M - 1] \quad \text{and} \quad N\left[\frac{I(T_M)}{I(0)} - 1\right]$$

respectively. Let us now focus on the floating leg. Under the martingale measure Q with numeraire B_n its arbitrage price at time t is given by

$$ZCIIS_f(t, T_M, I(0), N) = N\, E\left[e^{-\int_t^{T_M} r_n(u)du}\left[\frac{I(T_M)}{I(0)} - 1\right]\middle|\mathscr{F}_t\right]. \tag{3.1}$$

Remembering that P_{TIPS} corresponds to the nominal value of a real bond, we have:

$$I(t)P_r(t,T) = E[e^{-\int_t^{T_M} r_n(u)du} I(T)|\mathscr{F}_t] \tag{3.2}$$

so

$$ZCIIS_f(t, T_M, I(0), N) = N\left[\frac{I(t)}{I(0)}P_r(t, T_M) - P_n(t, T_M)\right] \tag{3.3}$$

and, in particular, when $t = 0$,

$$ZCIIS_f(0, T_M, I(0), N) = N[P_r(0, T_M) - P_n(0, T_M)]. \tag{3.4}$$

The price of the fix leg is simply

$$\begin{aligned} ZCIIS_{fix}(0,T_M,I(0),N) &= N\,E\left[e^{-\int_0^{T_M} r_n(u)du}((1+K)^M-1)\middle|\mathscr{F}_0\right] \\ &= NP_n(0,T_M)((1+K)^M-1) \end{aligned}$$

so, if we are the party who pays the fix leg, we finally obtain

$$\begin{aligned} ZCIIS(0,T_M,I(0),N) &= N[P_r(0,T_M)-P_n(0,T_M)]-NP_n(0,T_M)[(1+K)^M-1] \\ &= N[P_r(0,T_M)-P_n(0,T_M)(1+K)^M]. \end{aligned}$$

It is interesting to note that this price is independent from the model.
If we look for the value of the fix rate K which makes the swap null at time 0, we obtain

$$K = \left(\frac{P_r(0,T_M)}{P_n(0,T_M)}\right)^{\frac{1}{M}} - 1; \tag{3.5}$$

these values are known on the market for some maturities T_M, so that it is possible to derive the P_r prices:

$$P_r(0,T_M) = P_n(0,T_M)(1+K(T_M))^M. \tag{3.6}$$

3.2 Year-on-year swap

An *year-on-year swap* (YYIIS, which stands for *year-on-year inflation-indexed swap*) is a contract for which a set of dates $T_0 = 0, T_1, \ldots, T_M$ is given, and on each subinterval one of the two parties pays the inflation rate and receives from the other party the fixed rate K, where both the rates are computed referring to the nominal value N and multiplied for the year fraction φ_i for the interval $[T_{i-1}, T_i]$. This means that the value of the fixed leg and of the floating one at time T_i are

$$N\varphi_i K \quad \text{and} \quad N\varphi_i\left[\frac{I(T_i)}{I(T_{i-1})}-1\right]$$

respectively; so, indicating with the subscript "f" the inflation-indexed leg, we have

$$YYIIS_f(t, T_{i-1}, T_i, \varphi_i, N) = N\varphi_i E\left[e^{-\int_t^{T_i} r_n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]\middle|\mathscr{F}_t\right]. \quad (3.7)$$

If $t \geq T_{i-1}$, (3.7) is completely similar to the floating leg of a ZCIIS, so for its price see (3.3); if instead $t < T_{i-1}$, we can write

$$YYIIS_f(t, T_{i-1}, T_i, \varphi_i, N) = N\varphi_i E\left[e^{-\int_t^{T_{i-1}} r_n(u)du} E\left[e^{-\int_{T_{i-1}}^{T_i} r_n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]\middle|\mathscr{F}_{T_{i-1}}\right]\middle|\mathscr{F}_t\right]$$

where the inner expectation is $ZCIIS_f(T_{i-1}, T_i, I(T_{i-1}), 1)$ (see (3.1)), so

$$\begin{aligned} YYIIS_f(t, T_{i-1}, T_i, \varphi_i, N) &= N\varphi_i E\left[e^{-\int_t^{T_{i-1}} r_n(u)du}[P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)]\middle|\mathscr{F}_t\right] \\ &= N\varphi_i E\left[e^{-\int_t^{T_{i-1}} r_n(u)du} P_r(T_{i-1}, T_i)\middle|\mathscr{F}_t\right] - N\varphi_i P_n(t, T_i) \end{aligned} \quad (3.8)$$

(in the last passage we used the martingality of discounted $P_n(\cdot, T_i)$).
Now, in order to calculate the expectation in (3.8), we change the numeraire[1] from $B_n(\cdot)$ to $P_n(\cdot, T_{i-1})$, so that

$$\begin{aligned} YYIIS_f(t, T_{i-1}, T_i, \varphi_i, N) &= N\varphi_i E_n^{T_{i-1}}\left[\tfrac{P_n(t,T_{i-1})}{P_n(T_{i-1},T_{i-1})} P_r(T_{i-1}, T_i)\middle|\mathscr{F}_t\right] - N\varphi_i P_n(t, T_i) \\ &= N\varphi_i P_n(t, T_{i-1}) E_n^{T_{i-1}}[P_r(T_{i-1}, T_i)|\mathscr{F}_t] - N\varphi_i P_n(t, T_i) \end{aligned} \quad (3.9)$$

where $E_n^{T_{i-1}}$ indicates the expectation under the forward nominal martingale measure $Q_n^{T_{i-1}}$. In order to calculate this, we recall the Hull & White dynamics of the real short rate (2.11) under the martingale measure with numeraire B_n

$$dr_r(t) = [\theta_r(t) - \rho_{rI}\sigma_r\sigma_I - a_r r_r(t)]dt + \sigma_r(t)dW_r(t)$$

where W_r is a correlated Brownian Motion with correlation ρ_{nr}. From (1.12), we remember that $P_n(t, T) = \tilde{A}_n(t, T)e^{-B_n(t,T)r}$ where $\tilde{A}_n$ and B_n are the

[1] See (A.3) in Appendix A

deterministic functions given by the exponential of (1.20) and by (1.19), so

$$\begin{aligned} dP_n(t,T) &= (d\tilde{A}_n(t,T))e^{-B_n(t,T)r_n(t)} - \tilde{A}_n(t,T)e^{-B_n(t,T)r_n(t)}d(B_n(t,T)r_n(t)) \\ &= P_n(t,T)[(\ldots)dt - B_n(t,T)\sigma_n dW_n(t)] \end{aligned}$$

where the dots $(\ldots)$ are allowed as we are only interested in the diffusion coefficient in order to apply the change of numeraire[2]. We thus obtain

$$dW_n(t) = dW_n^{T_{i-1}}(t) - B_n(t,T_{i-1})\sigma_n dt \tag{3.10}$$

from which

$$\begin{aligned} dW_r(t) &= \rho_{nr} dW_n(t) = \rho_{nr} dW_n^{T_{i-1}}(t) - \rho_{nr}\sigma_n B_n(t,T_{i-1})dt \\ &= dW_r^{T_{i-1}}(t) - \rho_{nr}\sigma_n B_n(t,T_{i-1})dt \end{aligned}$$

and finally

$$dr_r(t) = [\theta_r(t) - \rho_{rI}\sigma_r\sigma_I - a_r r_r(t) - \rho_{nr}\sigma_n\sigma_r B_n(t,T_{i-1})]dt + \sigma_r dW_r^{T_{i-1}}(t). \tag{3.11}$$

Equation (3.11) is a linear EDS, and analogously to (2.3.2) we have

$$\begin{aligned} r_r(t) = e^{-a_r(t-t_0)}\{ \quad & r_r(t_0) + \int_{t_0}^t e^{a_r(s-t_0)}(\theta_r(s) - \rho_{rI}\sigma_r\sigma_I - \rho_{nr}\sigma_n\sigma_r B_n(s,T_{i-1}))ds \\ & + \int_{t_0}^t e^{a_r(s-t_0)}\sigma_r dW_r^{T_{i-1}}(s)\}; \end{aligned}$$

taking $t = T_{i-1}$ and simply t instead of t_0, we obtain

$$\begin{aligned} r_r(T_{i-1}) = {} & e^{-a_r(T_{i-1}-t)}r_r(t) + e^{-a_r T_{i-1}}\int_t^{T_{i-1}} e^{a_r s}(\theta_r(s) - \rho_{rI}\sigma_r\sigma_I - \rho_{nr}\sigma_n\sigma_r B_n(s,T_{i-1}))ds \\ & + e^{-a_r T_{i-1}}\int_t^{T_{i-1}} e^{a_r s}\sigma_r dW_r^{T_{i-1}}(s) \end{aligned} \tag{3.12}$$

[2]See theorem A.2.4 in Appendix A.

from which we deduce that $r(T_{i-1})$ has a normal distribution, with expectation (under the martingale measure $Q^{T_{i-1}}$):

$$E^{T_{i-1}}[r(T_{i-1})|\mathscr{F}_t] = e^{-a_r(T_{i-1}-t)}r_r(t) + e^{-a_rT_{i-1}}\int_t^{T_{i-1}} e^{a_rs}\theta_r(s)ds$$
$$-\rho_{rI}\sigma_r\sigma_I e^{-a_rT_{i-1}}\tfrac{e^{a_rT_{i-1}}-e^{a_rt}}{a_r} - \rho_{nr}\sigma_n\sigma_r e^{-a_rT_{i-1}}\int_t^{T_{i-1}} e^{a_rs}B_n(s,T_{i-1})ds$$
$$= e^{-a_r(T_{i-1}-t)}r_r(t) - \rho_{rI}\sigma_r\sigma_I B_r(t,T_{i-1}) + e^{-a_rT_{i-1}}\int_t^{T_{i-1}} e^{a_rs}\Big(\left.\tfrac{\partial f^*r(0,T)}{\partial T}\right|_{T=s}$$
$$+a_rf_r^*(0,s) + \tfrac{\sigma_r^2}{2a_r}(1-e^{-2a_rs})\Big)ds - \rho_{nr}\sigma_n\sigma_r e^{-a_rT_{i-1}}\int_t^{T_{i-1}} e^{a_rs}\tfrac{1-e^{-a_n(T_{i-1}-s)}}{a_n}ds.$$

Let us now focus on the last two addends: they are equal to

$$e^{-a_rT_{i-1}}\{[f_r^*(0,s)e^{a_rs}]_{s=t}^{s=T_{i-1}} - \int_t^{T_{i-1}} f_r^*(0,s)a_re^{a_rs}ds + \int_t^{T_{i-1}} f_r^*(0,s)a_re^{a_rs}ds$$
$$+\tfrac{\sigma_r^2}{2a_r}\int_t^{T_{i-1}}(e^{a_rs}-e^{-a_rs})ds\} - \rho_{nr}\sigma_n\sigma_r e^{-a_rT_{i-1}}\tfrac{1}{a_n}\int_t^{T_{i-1}}(e^{a_rs}-e^{-a_n(T_{i-1}-s)+a_rs})ds$$
$$= f_r^*(0,T_{i-1}) - e^{-a_r(T_{i-1}-t)}f_r^*(0,t) + e^{-a_rT_{i-1}}\tfrac{\sigma_r^2}{2a_r}(e^{a_rT_{i-1}} - e^{a_rt} + e^{-a_rT_{i-1}} - e^{-a_rt})$$
$$-\rho_{nr}\sigma_n\sigma_r\tfrac{1}{a_n}\Big\{\tfrac{1}{a_r} - \tfrac{e^{-a_r(T_{i-1}-t)}}{a_r} - \tfrac{1}{a_n+a_r} + \tfrac{e^{-a_n(T_{i-1}-t)-a_r(T_{i-1}-t)}}{a_n+a_r}\Big\}$$
$$= f_r^*(0,T_{i-1}) + \tfrac{\sigma_r^2}{2a_r^2}(1+e^{-2a_rT_{i-1}} - 2e^{-a_rT_{i-1}}) - e^{-a_r(T_{i-1}-t)}f_r^*(0,t)$$
$$-\tfrac{\sigma_r^2}{2a_r^2}e^{-a_r(T_{i-1}-t)}(1+e^{-2a_rt}-2e^{-a_rt})$$
$$-\rho_{nr}\sigma_n\sigma_r\tfrac{1}{a_n+a_r}\Big\{\tfrac{1-e^{-a_r(T_{i-1}-t)}}{a_r} + \tfrac{e^{-a_n(T_{i-1}-t)-a_r(T_{i-1}-t)}}{a_n} - \tfrac{e^{-a_r(T_{i-1}-t)}}{a_n}\Big\}$$

so

$$\begin{aligned}E^{T_{i-1}}[r(T_{i-1})|\mathscr{F}_t] &= e^{-a_r(T_{i-1}-t)}r_r(t) - \rho_{rI}\sigma_r\sigma_I B_r(t,T_{i-1}) + f_r^*(0,T_{i-1})\\ &+\tfrac{\sigma_r^2}{2a_r^2}(1-e^{-a_rT_{i-1}})^2 - e^{-a_r(T_{i-1}-t)}\big\{f_r^*(0,t) + \tfrac{\sigma_r^2}{2a_r^2}(1-e^{-a_rt})^2\big\} \\ &-\tfrac{\rho_{nr}\sigma_n\sigma_r}{a_n+a_r}[B_r(t,T_{i-1}) + a_rB_n(t,T_{i-1})B_r(t,T_{i-1}) - B_n(t,T_{i-1})].\end{aligned} \tag{3.13}$$

From (3.12) we can also derive the variance of $r(T_{i-1})$:

$$\mathrm{var}^{T_{i-1}}[r(T_{i-1})|\mathscr{F}_t] = e^{-2a_rT_{i-1}}\int_t^{T_{i-1}} e^{2a_rs}\sigma_r^2ds = \frac{\sigma_r^2}{2a_r}(1-e^{-2a_r(T_{i-1}-t)}). \tag{3.14}$$

Keeping in mind that $P_r(t,T) = \tilde{A}_r(t,T)e^{-B_r(t,T)r(t)}$ (1.12) with $\tilde{A}_r$ and B_r deterministic functions, we obtain that $P_r(T_{i-1},T_i)$ is lognormally distributed, so we are able to calculate its expectation under the martingale measure $Q^{T_{i-1}}$, which was left unexpanded in (3.9):

$$E^{T_{i-1}}[P_r(T_{i-1},T_i)|\mathscr{F}_t] = \int_{-\infty}^{+\infty} \tilde{A}_r(T_{i-1},T_i)e^{-B_r(T_{i-1},T_i)x}p(x)dx$$

where p is the density of $r_{T_{i-1}}$, and we know that $r_{T_{i-1}} \sim \mathcal{N}(m,V)$, with $m = E^{T_{i-1}}[r_{T_{i-1}}|\mathscr{F}_t]$ and $V = E^{T_{i-1}}[r_{T_{i-1}}|\mathscr{F}_t]$, so that (omitting the pedices and arguments of $\tilde{A}$ and B in order to simplify the notation)

$$\begin{aligned} E^{T_{i-1}}[P_r(T_{i-1},T_i)|\mathscr{F}_t] &= \tilde{A}\int_{-\infty}^{+\infty} e^{-Bx}\tfrac{1}{\sqrt{2\pi V}}e^{-\frac{(x-m)^2}{2V}}dx \\ &= \tilde{A}e^{\frac{B^2V}{2}-mB}\tfrac{1}{\sqrt{2\pi V}}\int_{-\infty}^{+\infty} e^{-\frac{(x-(m-BV))^2}{2V}}dx. \end{aligned} \tag{3.15}$$

Now, remembering that the integral of the normal density is 1, using the expressions in (3.13) and (3.14) for m and V, and substituting the explicit form of $\tilde{A}$ derived from (1.24), we obtain that (3.15) is equal to

$$\begin{aligned} &\tfrac{P_r^*(0,T_i)}{P_r^*(0,T_{i-1})}\exp\Big\{B_r(T_{i-1},T_i)f_r^*(0,T_{i-1}) - \tfrac{\sigma_r^2}{4a_r}(1-e^{-2a_rT_{i-1}})B_r(T_{i-1},T_i)^2 \\ &+\tfrac{\sigma_r^2}{4a_r}(1-e^{-2aT_{i-1}-t})B_r(T_{i-1},T_i)^2 - B_r(T_{i-1},T_i)\big[r_r(t)e^{-a_r(T_{i-1}-t)} + f_r^*(0,T_{i-1}) \\ &+\tfrac{\sigma_r^2}{2a_r^2}(1-e^{-a_rT_{i-1}})^2 - e^{-a_r(T_{i-1}-t)}\big(f_r^*(0,t)+\tfrac{\sigma_r^2}{2a_r^2}(1-e^{-a_rt})^2\big) - \rho_{rI}\sigma_r\sigma_I B_r(t,T_{i-1}) \\ &-\tfrac{\rho_{nr}\sigma_n\sigma_r}{a_n+a_r}[B_r(t,T_{i-1})+a_rB_n(t,T_{i-1})B_r(t,T_{i-1})-B_n(t,T_{i-1})]\big]\Big\} = \end{aligned}$$

$$\begin{aligned} &= \tfrac{P_r^*(0,T_i)}{P_r^*(0,T_{i-1})}\exp\Big\{f_r^*(0,t)\tfrac{1}{a_r}(e^{-a_r(T_{i-1}-t)} - e^{-a_r(T_i-t)}) + \rho_{rI}\sigma_r\sigma_I B_r(T_{i-1},T_i)B_r(t,T_{i-1}) \\ &+\tfrac{\rho_{nr}\sigma_n\sigma_r}{a_n+a_r}B_r(T_{i-1},T_i)[B_r(t,T_{i-1})+a_rB_n(t,T_{i-1})B_r(t,T_{i-1})-B_n(t,T_{i-1})] \\ &+\tfrac{\sigma_r^2}{4a_r}\Big[\tfrac{1}{a_r^2}(1+e^{-2a_r(T_i-T_{i-1})}-2e^{-a_r(T_i-T_{i-1})})(e^{-2a_rT_{i-1}}-e^{-2a_r(T_{i-1}-t)}) \\ &+\tfrac{2}{a_r^2}(1-e^{-a_r(T_i-T_{i-1})})e^{-a_r(T_{i-1}-t)}(1+e^{-2a_rt}-2e^{-a_rt}) \\ &-\tfrac{2}{a_r^2}(1-e^{-a_r(T_i-T_{i-1})})(1+e^{-2a_rT_{i-1}}-2e^{-a_rT_{i-1}})\Big] - \tfrac{r_r(t)}{a_r}(1-e^{-a_r(T_i-T_{i-1})})e^{-a_r(T_{i-1}-t)}\Big\}. \end{aligned} \tag{3.16}$$

Now we notice that

$$\frac{P_r(t,T_i)}{P_r(t,T_{i-1})} = \frac{\tilde{A}_r(t,T_i)}{\tilde{A}_r(t,T_{i-1})}e^{-B_r(t,T_i)r_r(t)}e^{B_r(t,T_{i-1})r_r(t)} \tag{3.17}$$

with

$$\begin{aligned} \tfrac{\tilde{A}_r(t,T_i)}{\tilde{A}_r(t,T_{i-1})} &= \tfrac{P_r^*(0,T_i)}{P_r^*(0,t)}\exp\Big\{B_r(t,T_i)f_r^*(0,t) - \tfrac{\sigma_r^2}{4a_r}(1-e^{-2a_rt})B_r(t,T_i)^2\Big\} \\ \cdot\tfrac{P_r^*(0,t)}{P_r^*(0,T_{i-1})}&\exp\Big\{-B_r(t,T_{i-1})f_r^*(0,t) + \tfrac{\sigma_r^2}{4a_r}(1-e^{-2a_rt})B_r(t,T_{i-1})^2\Big\} \end{aligned} \tag{3.18}$$

$$\begin{aligned} &= \tfrac{P_r^*(0,T_i)}{P_r^*(0,T_{i-1})} \exp\big\{ f_r^*(0,t) \tfrac{1}{a_r}(-e^{-a_r(T_i-t)} + e^{-a_r(T_{i-1}-t)}) \\ &+ \tfrac{\sigma_r^2}{4a_r}(1-e^{-2a_r t})\tfrac{1}{a_r^2}(e^{-2a_r(T_{i-1}-t)} - 2e^{-a_r(T_{i-1}-t)} - e^{-2a_r(T_i-t)} + 2e^{-a_r(T_i-t)})\big\} \end{aligned} \tag{3.19}$$

so from (3.16), (3.17) and (3.19) with some calculation we finally obtain

$$E^{T_{i-1}}[P_r(T_{i-1},T_i)|\mathscr{F}_t] = \frac{P_r(t,T_i)}{P_r(t,T_{i-1})} e^{C(t,T_{i-1},T_i)} \tag{3.20}$$

where

$$\begin{aligned} C(t,T_{i-1},T_i) = \;& \sigma_r B_r(T_{i-1},T_i)\big\{ B_r(t,T_{i-1})(\rho_{rI}\sigma_I - \tfrac{1}{2}\sigma_r B_r(t,T_{i-1}) \\ & + \tfrac{\rho_{nr}\sigma_n}{a_n+a_r}(1+a_r B_n(t,T_{i-1})) - \tfrac{\rho_{nr}\sigma_n}{a_n+a_r} B_n(t,T_{i-1})\big\}. \end{aligned} \tag{3.21}$$

Thus we are able to write explicitly the price at time t, for $t < T_{i-1}$, of the floating part of a YYIIS, in fact substituting (3.20) in (3.9) we obtain

$$YYIIS_f(t,T_{i-1},T_i,\varphi_i,N) = N\varphi_i P_n(t,T_{i-1}) \frac{P_r(t,T_i)}{P_r(t,T_{i-1})} e^{C(t,T_{i-1},T_i)} - N\varphi_i P_n(t,T_i). \tag{3.22}$$

In this formula, $\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}$ is the forward price of a real bond, that is the price at time t of a real bond which starts in T_{i-1} with maturity T_i, and it is multiplied by the "correction factor" e^C which disappears if $\sigma_r = 0$. Summing up, and remembering that for $t \geq T_{i-1}$ (3.3) holds, the value at time t of the floating part of a YYIIS is

$$\begin{aligned} YYIIS_f(t,\mathcal{T},\phi,N) = \;& N\varphi_{i(t)}\big[\tfrac{I(t)}{I(T_{i(t)-1})} P_r(t,T_{i(t)}) - P_n(t,T_{i(t)})\big] \\ & + N\textstyle\sum_{i=i(t)+1}^{M} \varphi_i \big[P_n(t,T_{i-1}) \tfrac{P_r(t,T_i)}{P_r(t,T_{i-1})} e^{C(t,T_{i-1},T_i)} - P_n(t,T_i)\big] \end{aligned} \tag{3.23}$$

where $\mathcal{T} = \{T_1,\dots,T_M\}$, $\phi = \varphi_1,\dots,\varphi_M$ and $i(t) = min\{i : T_i > t\}$;
for $t = 0$ (3.23) reduces to

$$\begin{aligned} & YYIIS_f(0,\mathcal{T},\phi,N) = \\ & N\varphi_1[P_r(0,T_1) - P_n(0,T_1)] + N\textstyle\sum_{i=2}^{M} \varphi_i \big[P_n(0,T_{i-1}) \tfrac{P_r(0,T_i)}{P_r(0,T_{i-1})} e^{C(0,T_{i-1},T_i)} - P_n(0,T_i)\big]. \end{aligned} \tag{3.24}$$

If we consider both legs of the YYIIS, we finally obtain

$$\begin{aligned} YYIIS(0,\mathcal{T},\phi,N) = \;& N\varphi_1 P_r(0,T_1) + N\textstyle\sum_{i=2}^{M} \varphi_i \big[P_n(0,T_{i-1}) \tfrac{P_r(0,T_i)}{P_r(0,T_{i-1})} e^{C(0,T_{i-1},T_i)}\big] \\ & - N(1+K)\textstyle\sum_{i=1}^{M} \varphi_i P_n(0,T_i). \end{aligned} \tag{3.25}$$

To conclude, we give an alternative formulation for (3.24), using the forward rates: we recall the definition

Definition 3.1 (**Forward rate**). The simply-compounded forward interest rate prevailing at time t for the expiry $T > t$ and maturity $S > T$ is

$$F(t; T, S) = \frac{P(t,T) - P(t,S)}{\tau_i P(t,S)}.$$

Substituting this in (3.24) we thus obtain

$$YYIIS_f(0, \mathcal{T}, \phi, N) = N \sum_{i=1}^{M} \varphi_i P_n(0, T_i) \Big[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)} e^{C(0, T_{i-1}, T_i)} - 1 \Big].$$

3.3 IICap/IIFloor

An *Inflation-Indexed Caplet/Floorlet* (IIC/IIF) is a Call/Put option on the inflation rate: if the time interval is $[T_{i-1}, T_i]$, φ_i the correspondent year fraction, N the nominal value of the contract and k the strike, the payoff is

$$N\varphi_i \Big[w\Big(\frac{I(T_i)}{I(T_{i-1})} - 1 - k \Big) \Big]^+ \tag{3.26}$$

where w is 1 for a caplet, -1 for a floorlet. So, setting $K = 1 + k$, the price at time $t < T_{i-1}$ is

$$\begin{aligned} IICpltFlt(t, T_{i-1}, T_i, \varphi_i, K, N, w) &= N\varphi_i E_n \Big[e^{-\int_t^{T_i} r_n(s)ds} \Big[w\Big(\tfrac{I(T_i)}{I(T_{i-1})} - K \Big) \Big]^+ \Big| \mathscr{F}_t \Big] \\ &= N\varphi_i P_n(t, T_i) E_n^{T_i} \Big[\Big[w\Big(\tfrac{I(T_i)}{I(T_{i-1})} - K \Big) \Big]^+ \Big| \mathscr{F}_t \Big]. \end{aligned} \tag{3.27}$$

In the Jarrow-Yildirim model the nominal and real rates are normally distributed (remember the dynamics (2.10) and (2.11)), and in the previous section we have seen that this continues to hold if we move to a forward nominal measure. Moreover, we recall from (2.13) that

$$I(T) = I(t) e^{\int_t^T (r_n(s) - r_r(s))ds - \frac{1}{2}\sigma_I^2 (T-t) + \sigma_I (W_I(T) - W_I(t))}$$

so the CPI index follows a lognormal distribution[3].
Taking $P_n(\cdot, T_i)$ as numeraire (with the same proceeding which lead us to (3.10)), we obtain

$$dW_n(t) = dW_n^{T_i}(t) - B_n(t,T_i)\sigma_n dt \tag{3.28}$$

from which

$$\begin{aligned} dW_I(t) &= \rho_{nI} dW_n(t) = \rho_{nI} dW_n^{T_i}(t) - \rho_{nI}\sigma_n B_n(t,T_i)dt \\ &= dW_I^{T_i}(t) - \rho_{nI}\sigma_n B_n(t,T_i)dt \end{aligned}$$

so that

$$dI(t) = I(t)[r_n(t) - r_r(t) - \rho_{nI}\sigma_n\sigma_I B_n(t,T_i)]dt + \sigma_I dW_I^{T_i}(t). \tag{3.29}$$

Taking T_{i-1} as "starting point" and calculating at time T_i, we can finally write

$$\frac{I(T_i)}{I(T_{i-1})} = e^{\int_{T_{i-1}}^{T_i}(r_n(s)-r_r(s))ds - \frac{1}{2}\sigma_I^2(T-t) - \rho_{nI}\sigma_n\sigma_I\int_{T_{i-1}}^{T_i} B_n(s,T_i)ds + \sigma_I(W_I^{T_i}(T_i) - W_I^{T_i}(T_{i-1}))}.$$

The expectation in the second line of (3.27) is thus of the form $E[(w(e^Z - K))^+]$ with $Z = \log\frac{I(T_i)}{I(T_{i-1})}$ normal variable, so if we knew the expectation and variance of Z, we could apply formula (C.6) in Appendix C.
With this aim, we recall (3.20):

$$E_n^{T_{i-1}}[P_r(T_{i-1},T_i)|\mathscr{F}_t] = \frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)} \tag{3.30}$$

and we note that a change of numeraire on the first member gives

$$= E_n^{T_i}[P_r(T_{i-1},T_i)\frac{P_n(T_{i-1},T_{i-1})}{P_n(t,T_{i-1})}\frac{P_n(t,T_i)}{P_n(T_{i-1},T_i)}|\mathscr{F}_t] = \frac{P_n(t,T_i)}{P_n(t,T_{i-1})}E_n^{T_i}[\frac{P_r(T_{i-1},T_i)}{P_n(T_{i-1},T_i)}|\mathscr{F}_t] =$$

using (3.2)

$$= \frac{P_n(t,T_i)}{P_n(t,T_{i-1})}E_n^{T_i}\Big[\frac{I(T_i)}{I(T_{i-1})}\Big|\mathscr{F}_t\Big] \tag{3.31}$$

[3]Remember that the integral of a Gaussian process is Gaussian: see Section C.1 in Appendix C.

and equating the second members of (3.30) and (3.31) we obtain

$$E_n^{T_i}\Big[e^Z|\mathscr{F}_t\Big] = E_n^{T_i}\Big[\frac{I(T_i)}{I(T_{i-1})}|\mathscr{F}_t\Big] = \frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)} \quad (3.32)$$

with C defined in (3.21).
If we now put $E[Z] = \mu$ and $\mathrm{var}[Z] = v^2$, according to (C.1) in Appendix C, we have

$$m = E[e^Z] = e^{\mu+\frac{v^2}{2}} \Rightarrow \mu = \log m - \frac{v^2}{2}. \quad (3.33)$$

We need to calculate v^2; for this aim, we use the dynamics under the martingale measure Q, since the change of numeraire only determines a deterministic addend which does not affect the variance.

$$v^2 = \mathrm{var}_n^{T_i}\Big[\log\frac{I(T_i)}{I(T_{i-1})}|\mathscr{F}_t\Big] = \mathrm{var}_n\Big[\int_{T_{i-1}}^{T_i}(r_n(s)-r_r(s))ds+\sigma_I(W_I(T_i)-W_I(T_{i-1}))|\mathscr{F}_t\Big].$$

Since $\mathrm{var}[A+B] = \mathrm{var}[A] + \mathrm{var}[B] + 2\mathrm{Cov}[A,B]$, let us calculate the single addends. Using the expression of r_n in (2.3.2)

$$\mathrm{var}_n\Big[\int_{T_{i-1}}^{T_i} r_n(s)ds|\mathscr{F}_t\Big] = \mathrm{var}_n\Big[\sigma_n\int_{T_{i-1}}^{T_i}\int_t^s e^{-a_n(s-u)}dW_n(u)ds|\mathscr{F}_t\Big].$$

We now observe that $\int_{T_{i-1}}^{T_i}\big(\int_t^s dW_u\big)ds = \int_t^{T_{i-1}}\big(\int_{T_{i-1}}^{T_i} ds\big)dW_u + \int_{T_{i-1}}^{T_i}\big(\int_u^{T_i} ds\big)dW_u$, so we treat the two parts separately, calling them a) and b) (keeping in mind that the increment of Brownian Motions over a certain interval $[T_{i-1}, T_i]$ is independent from $\mathscr{F}_{T_{i-1}}$, so the two parts are independent).
a):

$$\begin{aligned}\mathrm{var}_n\Big[\sigma_n\int_t^{T_{i-1}}\big(\textstyle\int_{T_{i-1}}^{T_i} e^{-a_n(s-u)}ds\big)dW_n(u)|\mathscr{F}_t\Big] &= \sigma_n^2\int_t^{T_{i-1}}\big(\textstyle\int_{T_{i-1}}^{T_i} e^{-a_n(s-u)}ds\big)^2du \\ &= \tfrac{\sigma_n^2}{a_n^2}\big(1-e^{-a_n(T_i-T_{i-1})}\big)^2\int_t^{T_{i-1}} e^{-2a_n(T_i-u)}du \qquad (3.34)\\ &= \tfrac{\sigma_n^2}{2a_n^3}\big(1-e^{-a_n(T_i-T_{i-1})}\big)^2\big(1-e^{-2a_n(T_{i-1}-t)}\big).\end{aligned}$$

b):

$$\begin{aligned}\mathrm{var}_n\Big[\sigma_n\int_{T_{i-1}}^{T_i}\big(\textstyle\int_u^{T_i} e^{-a_n(s-u)}ds\big)dW_n(u)|\mathscr{F}_t\Big] &= \sigma_n^2\int_{T_{i-1}}^{T_i}\big(\textstyle\int_u^{T_i} e^{-a_n(s-u)}ds\big)^2du \\ &= \tfrac{\sigma_n^2}{a_n^2}\int_{Ti-1}^{T_i}(1+e^{-2a_n(T_i-u)}-2e^{-a_n(T_i-u)})du \qquad (3.35)\\ &= \tfrac{\sigma_n^2}{a_n^2}\Big\{T_i-T_{i-1}-\tfrac{e^{-2a_n(T_i-T_{i-1})}}{2a_n}+2\tfrac{e^{-a_n(T_i-T_{i-1})}}{a_n}-\tfrac{3}{2a_n}\Big\}.\end{aligned}$$

Since the dynamics of r_r differs from the one of r_n only for a deterministic factor (see (2.3.2)), the calculus for the variance of its integral is exactly the same, and gives:

$$\begin{aligned}\mathrm{var}_n\Big[\int_{T_{i-1}}^{T_i} r_r(s)ds|\mathscr{F}_t\Big] = & \tfrac{\sigma_r^2}{2a_r^3}\big(1-e^{-a_r(T_i-T_{i-1})}\big)^2\big(1-e^{-2a_r(T_{i-1}-t)}\big) \\ & +\tfrac{\sigma_r^2}{a_r^2}\big\{T_i-T_{i-1}-\tfrac{e^{-2a_r(T_i-T_{i-1})}}{2a_r}+2\tfrac{e^{-a_r(T_i-T_{i-1})}}{a_r}-\tfrac{3}{2a_r}\big\}.\end{aligned} \tag{3.36}$$

The third (and last) variance we need is

$$\mathrm{var}_n\Big[\sigma_I(W_I(T_i)-W_I(T_{i-1}))|\mathscr{F}_t\Big] = \sigma_I^2(T_i-T_{i-1}).$$

Let us now calculate covariances:

$$\begin{aligned}&\mathrm{Cov}_n\Big[\int_{T_{i-1}}^{T_i} r_n(s)ds, \int_{T_{i-1}}^{T_i} r_r(s)ds\Big] = \\ &\sigma_n\sigma_r E\Big[\Big(\int_{T_{i-1}}^{T_i}\big(\int_t^s e^{-a_n(s-u)}dW_n(u)\big)ds\Big)\Big(\int_{T_{i-1}}^{T_i}\big(\int_t^s e^{-a_r(s-u)}dW_r(u)\big)ds\Big)\Big].\end{aligned}$$

Applying again the decomposition in a) and b) of each integral, we have: the covariance of part a) of each integral

$$\begin{aligned}&\sigma_n\sigma_r E\Big[\Big(\int_t^{T_{i-1}}\big(\int_{T_{i-1}}^{T_i} e^{-a_n(s-u)}ds\big)dW_n(u)\Big)\Big(\int_t^{T_{i-1}}\big(\int_{T_{i-1}}^{T_i} e^{-a_r(s-u)}ds\big)dW_r(u)\Big)\Big] \\ &= \rho_{nr}\sigma_n\sigma_r\int_t^{T_{i-1}}\big(\int_{T_{i-1}}^{T_i} e^{-a_n(s-u)}ds\int_{T_{i-1}}^{T_i} e^{-a_r(s-u)}ds\big)du \\ &= \tfrac{\rho_{nr}\sigma_n\sigma_r}{a_na_r(a_n+a_r)}\big(1-e^{-a_n(T_i-T_{i-1})}\big)\big(1-e^{-a_r(T_i-T_{i-1})}\big)\big(1-e^{-(a_n+a_r)(T_{i-1}-t)}\big);\end{aligned}$$

the covariance of part b) of each integral

$$\begin{aligned}&\sigma_n\sigma_r E\Big[\Big(\int_{T_{i-1}}^{T_i}\big(\int_u^{T_i} e^{-a_n(s-u)}ds\big)dW_n(u)\Big)\Big(\int_{T_{i-1}}^{T_i}\big(\int_u^{T_i} e^{-a_r(s-u)}ds\big)dW_r(u)\Big)\Big] \\ &= \rho_{nr}\sigma_n\sigma_r\int_{T_{i-1}}^{T_i}\big(\int_u^{T_i} e^{-a_n(s-u)}ds\int_u^{T_i} e^{-a_r(s-u)}ds\big)du \\ &= \tfrac{\rho_{nr}\sigma_n\sigma_r}{a_na_r}\int_{T_{i-1}}^{T_i}\big(1-e^{-a_n(T_i-u)}\big)\big(1-e^{-a_r(T_i-u)}\big)du \\ &= \tfrac{\rho_{nr}\sigma_n\sigma_r}{a_na_r}\big\{T_i-T_{i-1}-\tfrac{1-e^{-a_n(T_i-T_{i-1})}}{a_n}-\tfrac{1-e^{-a_r(T_i-T_{i-1})}}{a_r}+\tfrac{1-e^{-(a_n+a_r)(T_i-T_{i-1})}}{a_n+a_r}\big\};\end{aligned}$$

the cross covariances of parts a) and b) of the two integrals, which is zero; the covariances between part a) of the integral of r and the inflation addend, which is zero for both rates; the covariances between part b) of the integral of r and the inflation addend

$$\text{Cov}_n\Big[\sigma_n \int_{T_{i-1}}^{T_i} \big(\int_u^{T_i} e^{-a_n(s-u)} ds\big) dW_n(u), \sigma_I \int_{T_{i-1}}^{T_i} dW_I(u)\Big]$$
$$= \rho_{nI}\sigma_n\sigma_I \int_{T_{i-1}}^{T_i} \big(\int_u^{T_i} e^{-a_n(s-u)} ds\big) du = \frac{\rho_{nI}\sigma_n\sigma_I}{a_n}\big\{T_i - T_{i-1} - \frac{1-e^{-a_n(T_i-T_{i-1})}}{a_n}\big\}$$

and the same for r_r just changing the pedices from n to r. Summing up:

$$\begin{aligned}
&v^2 = v^2(t, T_{i-1}, T_i) = \\
&\frac{\sigma_n^2}{2a_n^3}\big(1 - e^{-a_n(T_i-T_{i-1})}\big)^2\big(1 - e^{-2a_n(T_{i-1}-t)}\big) + \frac{\sigma_r^2}{2a_r^3}\big(1 - e^{-a_r(T_i-T_{i-1})}\big)^2\big(1 - e^{-2a_r(T_{i-1}-t)}\big) \\
&+\frac{\sigma_n^2}{a_n^2}\big\{T_i - T_{i-1} - \frac{e^{-2a_n(T_i-T_{i-1})}}{2a_n} + 2\frac{e^{-a_n(T_i-T_{i-1})}}{a_n} - \frac{3}{2a_n}\big\} \\
&+\frac{\sigma_r^2}{a_r^2}\big\{T_i - T_{i-1} - \frac{e^{-2a_r(T_i-T_{i-1})}}{2a_r} + 2\frac{e^{-a_r(T_i-T_{i-1})}}{a_r} - \frac{3}{2a_r}\big\} \\
&+\sigma_I^2(T_i - T_{i-1}) \\
&-2\frac{\rho_{nr}\sigma_n\sigma_r}{a_n a_r(a_n+a_r)}\big(1 - e^{-a_n(T_i-T_{i-1})}\big)\big(1 - e^{-a_r(T_i-T_{i-1})}\big)\big(1 - e^{-(a_n+a_r)(T_{i-1}-t)}\big) \\
&-2\frac{\rho_{nr}\sigma_n\sigma_r}{a_n a_r}\big\{T_i - T_{i-1} - \frac{1-e^{-a_n(T_i-T_{i-1})}}{a_n} - \frac{1-e^{-a_r(T_i-T_{i-1})}}{a_r} + \frac{1-e^{-(a_n+a_r)(T_i-T_{i-1})}}{a_n+a_r}\big\} \\
&+2\frac{\rho_{nI}\sigma_n\sigma_I}{a_n}\big\{T_i - T_{i-1} - \frac{1-e^{-a_n(T_i-T_{i-1})}}{a_n}\big\} \\
&-2\frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\big\{T_i - T_{i-1} - \frac{1-e^{-a_r(T_i-T_{i-1})}}{a_r}\big\}.
\end{aligned} \tag{3.37}$$

Now, since we have the expressions (3.33) and (3.37), we can finally apply (C.6) obtaining

$$\begin{aligned}
&IICpltFlt(t, T_{i-1}, T_i, \varphi_i, K, N, w) \\
&= wN\varphi_i P_n(t, T_i)e^{\mu}\Big(e^{\frac{v^2}{2}}\Phi\big(w\frac{\mu - \log K + v^2}{\sqrt{v^2}}\big) - Ke^{-\mu}\Phi\big(w\frac{\mu - \log K}{\sqrt{v^2}}\big)\Big) \\
&= wN\varphi_i P_n(t, T_i)me^{-\frac{v^2}{2}}\Big(e^{\frac{v^2}{2}}\Phi\big(w\frac{\log m - \log K + \frac{v^2}{2}}{\sqrt{v^2}}\big) - \frac{K}{m}e^{\frac{v^2}{2}}\Phi\big(w\frac{\log m - \log K - \frac{v^2}{2}}{\sqrt{v^2}}\big)\Big) \\
&= wN\varphi_i P_n(t, T_i)\Big[\frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)}\Phi\Big(w\frac{\log\frac{P_n(t,T_{i-1})P_r(t,T_i)}{KP_n(t,T_i)P_r(t,T_{i-1})} + C(t,T_{i-1},T_i) + \frac{1}{2}v^2(t,T_{i-1},T_i)}{v(t,T_{i-1},T_i)}\Big) \\
&-K\Phi\Big(w\frac{\log\frac{P_n(t,T_{i-1})P_r(t,T_i)}{KP_n(t,T_i)P_r(t,T_{i-1})} + C(t,T_{i-1},T_i) - \frac{1}{2}v^2(t,T_{i-1},T_i)}{v(t,T_{i-1},T_i)}\Big)\Big]
\end{aligned} \tag{3.38}$$

where Φ is the standard normal distribution function.

Given a set of dates $\mathcal{T} = \{T_0, T_1, \dots, T_M\}$, an *Inflation-Indexed Cap/Floor* (IICap/IIFloor) is a sequence of inflation-indexed caplets/floorlets which are set on each subinterval $[T_{i-1}, T_i]$: it means that its price at time t is the sum of the discounted prices of the caplets/floorlets, and in particular at $t = 0$:

$$\begin{aligned} IICapFloor(0, \mathcal{T}, \phi, K, N, w) &= N \sum_{i=1}^{M} \varphi_i E_n \left[e^{-\int_0^{T_i} r_n(u)du} \left[w \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \middle| \mathscr{F}_t \right] \\ &= N \sum_{i=1}^{M} \varphi_i P_n(0, T_i) E_n^{T_i} \left[\left[w \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \middle| \mathscr{F}_t \right]. \end{aligned} \tag{3.39}$$

Chapter 4

Calibration

The Jarrow-Yildirim model is defined through the eight parameters a_n, a_r, σ_n, σ_r, σ_I, ρ_{nr}, ρ_{nI}, ρ_{rI}; until now we have not taken care about their values, but when dealing with concrete problems as derivative pricing it is necessary to know them. *Calibration* is the determination of reasonable values for the parameters of a model, where "reasonable" means that the prices obtained with the model fit the real ones of the market as much as possible. Thus, calibration reduces to a problem of minimization of the distance between market prices and model ones, where the distance should be defined in line with one's aims: the most used is the sum of the squares of the differences (least square problem), but it can be modified in order to give more importance to the precision on some prices than on others, for example multiplying the differences in the sum for opportune weights.
Calibration presents many difficulties. First of all, the minimization must be done with all the parameters as variables: in our model, it implies that the problem is set in $\mathbb{R}^8$. Moreover, the function to be minimized has not got properties which guarantee the existence of only one point of minimum (it is not convex), so usual numerical methods may fail. Finally, in the inflation context the available market data are not so many, and this affects the precision of the results.
In this chapter we highlight the fundamental steps for calibration for the

Jarrow-Yildirim model. Firstly we show what kind of data is really available and how to use it; then we compare two different methods for the minimization (Matlab *lsqnonlin* and the heuristic method *differential evolution*), showing the results of some experiments.

4.1 Data and implementation

Calibration for the Jarrow-Yildirim model is usually done using market data both about interest rates and concerning inflation-linked derivatives. For example, calibration can be done on interest rates caps and floors and on inflation-indexed caps and floors, and we will focus on this choice.
Interest rates caps and floors are contracts in which at any T_i of a set of dates $\mathcal{T} = \{T_1, \ldots, T_M\}$ a Call/Put on the simply-compounded spot interest rate L [1] takes place: its discounted payoff is

$$\sum_{i=1}^{M} D(t,T_i) N \varphi_i (w(L(T_{i-1},T_i) - K))^+$$

where K is the strike, N the notional and w is 1 for a cap, -1 for a floor. For these contracts there exists the pricing formula (Black's formula)

$$CapFloor(0,\mathcal{T},\phi,K,N,\sigma_{1,M},w) = N \sum_{i=1}^{M} P_n(0,T_i)\varphi_i Bl(K,F(0;T_{i-1},T_i),v_i,w) \tag{4.1}$$

where F is the one in definition (3.1), $v_i = \sigma_{1,M}\sqrt{T_{i-1}}$ and

$$\begin{aligned} &Bl(K,F,v,w) = Fw\Phi(wd_1(K,F,v)) - Kw\Phi(wd_2(K,F,v)), \\ &d_1(K,F,v) = \frac{\log(\frac{F}{K}) + \frac{v^2}{2}}{v}, \qquad d_2(K,F,v) = \frac{\log(\frac{F}{K}) - \frac{v^2}{2}}{v}. \end{aligned} \tag{4.2}$$

$\sigma_{1,M}$ is the volatility parameter on the chosen time interval, and it is available on the market for ATM caps/floors, where "ATM" stands for "at the money"

[1] We remember that $L(t,T) = \frac{1-P(t,T)}{\varphi(t,T)P(t,T)}$, where $\varphi(t,T)$ denotes the year fraction between t and T.

and means that the strike K is the one that makes the "corresponding" interest rate swap fair at time 0:

$$K = S(0)_{1,M} = \frac{P_n(0,T_1) - P_n(0,T_M)}{\sum_{i=2}^{M} \varphi_i P_n(0,T_i)}. \tag{4.3}$$

With ATM strikes the price of a cap equals the one of the corresponding floor, so calibration is made on caps only. Summing up: in order to determine the market price of interest rate caps (where the plural refers to the fact that we deal with different maturities) we have to calculate the par strikes K according to (4.3), and we need to know the market volatilities $\sigma_{1,M}$, so that we can apply formula (4.1). Of course, instead of doing this, we may directly use the market prices of interest rate caps, if they are available.
For what concerns the model prices we can note that a caplet (one addend of the sum in cap's form) can be seen as a put on a bond, so that we can use the pricing formula for that derivative in the Hull & White model. More precisely:

$$\begin{aligned}
&Cplt(t, T_{i-1}, T_i, \varphi_i, K, N) = E\big[e^{-\int_t^{T_i} r_n(s)ds} N\varphi_i (L(T_{i-1}, T_i) - K)^+ | \mathscr{F}_t\big] \\
&= NE\big[E\big[e^{-\int_t^{T_{i-1}} r_n(s)ds} e^{-\int_{T_{i-1}}^{T_i} r_n(s)ds} \varphi_i (L(T_{i-1}, T_i) - K)^+ | \mathscr{F}_{T_{i-1}}\big] | \mathscr{F}_t\big] \\
&= NE\big[e^{-\int_t^{T_{i-1}} r_n(s)ds} P_n(T_{i-1}, T_i) \varphi_i (L(T_{i-1}, T_i) - K)^+ | \mathscr{F}_t\big] \\
&= NE\big[e^{-\int_t^{T_{i-1}} r_n(s)ds} P_n(T_{i-1}, T_i) \big(\tfrac{1}{P(T_{i-1}, T_i)} - 1 - K\varphi_i\big)^+ | \mathscr{F}_t\big] \\
&= NE\big[e^{-\int_t^{T_{i-1}} r_n(s)ds} (1 - (1 + K\varphi_i) P_n(T_{i-1}, T_i))^+ | \mathscr{F}_t\big] \\
&= N(1 + K\varphi_i) E\big[e^{-\int_t^{T_{i-1}} r_n(s)ds} (\tfrac{1}{(1+K\varphi_i)} - P_n(T_{i-1}, T_i))^+ | \mathscr{F}_t\big]
\end{aligned} \tag{4.4}$$

We now note that the last expression in (4.4) is the price of a put with maturity T_{i-1} on a T_i-zero-coupon bond with nominal value $N' = N(1+K\varphi_i)$ and strike $K' = \frac{1}{(1+K\varphi_i)}$:

$$Cplt(t, T_{i-1}, T_i, \varphi_i, K, N) = N' ZBP(t, T_{i-1}, T_i, K'). \tag{4.5}$$

Thanks to this, the pricing of a caplet/floorlet reduces to the calculus of a zero-coupon bond put, and for what concerns interest rates our model is an Hull & White (see (2.3.2)), so (1.32) holds. Joining that formula with (4.5) we finally obtain the pricing formula for an interest-rate caplet in our model:

$$
\begin{aligned}
&Cplt(t, T_{i-1}, T_i, \varphi_i, K, N) = -N' P_n(t, T_i)\Phi(-d_1') + N' P_n(t, T_{i-1}) K' \Phi(-d_2')),\\
&\text{with} \quad N' = N(1 + K\varphi_i), \qquad K' = \tfrac{1}{(1+K\varphi_i)},\\
&d_1' = \tfrac{1}{\sigma_p} \log\left(\tfrac{P_n(t,T_i)}{K' P_n(t,T_{i-1})}\right) + \tfrac{1}{2}\sigma_p, \quad d_2' = \tfrac{1}{\sigma_p} \log\left(\tfrac{P_n(t,T_i)}{K' P_n(t,T_{i-1})}\right) - \tfrac{1}{2}\sigma_p,\\
&\sigma_p = \tfrac{1}{a_n}\left(1 - e^{-a_n(T_i - T_{i-1})}\right)\sqrt{\tfrac{\sigma_n^2}{2a_n}\left(1 - e^{-2a_n(T_{i-1}-t)}\right)}.
\end{aligned}
\tag{4.6}
$$

To conclude, the price of a cap is obviously obtained adding its caplets, which are two each year.
Common tenors for interest rate caps are 3, 4, 5, 6, 7, 8, 9, 10, 15, 20 years. We point out that nominal bond prices $P_n(0,T)$ are available on the market but not for every T, so it could be necessary to interpolate them to the set of dates of interest.

Let us now move to inflation-indexed caps and floors. Since market prices for certain tenors are usually available, we only have to apply formula (3.38) in order to calculate caplet/floorlet prices in our model, and then to sum over the caplets, which in this case are one each year.
Take care that market prices often do not include the first caplet/floorlet, since its value is known at the issuing instant; in that case, also model prices must avoid to sum them. Moreover, since cap/floors with tenor=1 coincide with first caplets/floorlets, they are not traded either, so in that circumstance we should not calibrate on them.
Common tenors for inflation-indexed caps/floors are (1, 2,) 3, 5, 7, 10, 12, 15, 20, 30 years, and common strikes are 0.01, 0.015, 0.02, 0.025, $\ldots$, 0.03.
Remember that $P_r(0,T)$ values are obtained from the par strikes of zero-coupon swaps through formula (3.6), and that an interpolation is often necessary in order to have their values for every T in the set of caplet maturities.

To conclude this section we would like to point out that two types of price are available on the market, that is "BID" and "ASK"[2]; a way to deal with this duality is to consider both prices making a linear combination between the two (for example with weights $(\frac{1}{2}, \frac{1}{2})$ if we have no reason to give predominance to one more than to the other).

4.2 Minimization

As already said, calibration consists in a minimization of the distance between market and model prices. A first problem is that the parameters must satisfy some constraints, first of all the fact that the covariance matrix should be positive definite; in order to solve this problem, a change of variable is suggested. For example, it is possible to make a sort of Cholesky decomposition of the correlation matrix which works even if the constraint is not satisfied, using Q-R decomposition[3] of the product of the square root of the eigenvalues for the eigenvectors matrix, and then taking the upper triangular part of R. The result of this proceeding can be not real, so it is necessary to get rid of the imaginary part, for example taking the absolute value of the result (or simply taking the real part, but perhaps it is a too rough method; another possibility is to take the absolute value but preserving the sign of the real part). Then, minimization must be done on the new parameters (without constraints), and finally it will be necessary to come back to the first ones, taking the transpose of the covariance matrix we have obtained and multiplying it for the covariance matrix itself.
We now show two possible ways of performing the minimization.

[2]BID and ASK denote the best price of buying or selling respectively, which means that for someone who wants to buy the ASK price is the most convenient, while for a seller the BID one is the best.

[3]The Q-R decomposition of a matrix A gives Q orthogonal and R upper triangular such that A=QR.

4.2.1 Matlab lsqnonlin

The first way is to use function “lsqnonlin” of Matlab optimization toolbox. It asks as input the vectorial function to be minimized (it provides to sum the squares of the differences by itself) and the starting point for the search. Options can be added in order to choose the algorithm and the tolerance.
This method meets quite difficulties in dealing with non-convex functions, since it presents an high risk of finding a local minimum instead of the absolute one. On the other hand, it is easy and quick, so it is possible to run it many times with different starting points and to choose the best result.

4.2.2 Differential evolution

“Differential evolution” is a method which deals with populations of n_p solutions: it means that at each iteration it carries n_p vectors whose length is the number of variables on which we are minimizing (eight in our case). The next population is obtained through random linear combinations: more precisely, for each new solution three solution indices l_1, l_2, l_3 are generated (from a uniform distribution), and the corresponding vectors of the previous population are combined as $v_{l_1} + F v_{l_2} - F v_{l_3}$, where F is a weight parameter. Then another random number is generated from the uniform distribution in [0,1], and only if it is less than a fixed “crossover” parameter CR the new solution substitutes the previous one. At this point, if the function to be minimized turns out to assume a smaller value with the new set of parameters, the new solution is kept, otherwise the old one remains.
In this method, the numerosity of each population and the number of iterations (that is, the number of populations) are to be decided at the beginning. At the end, the best solution of the last population is chosen.
Such method has the quality of “jumping” randomly, so permitting to explore different regions and reducing the risk of focussing on a local minimum. But even if it can quickly identify promising areas of the search space, then

its convergence is only linear. To reduce this drawback, differential evolution is frequently coupled with local searches involving a single solution. In this case, firstly it is to decide when to make the local search; it is possible to perform it not for every population but only periodically. Then the starting solutions have to be chosen: for example, we can take the best three ones in the current population, or the best one and other two ones which are determined randomly (but also the number three is just an example). At this point, from each of these solutions a local search is started: a typical function which is suitable for this aim is Matlab “minsearch”, which is based on Nelder and Mead algorithm.

4.2.3 Examples and results

In this section we present the results of some calibrations performed on a set of data with the two different methods; in both cases we minimize the percentage differences between market and model prices of interest rate caps and inflation-indexed caps and floors (including the ones with one year as tenor): if X_k, $k \in \{mk, m\}$, is the price at time 0 of a derivative on the market ($k = mk$) or in the model ($k = m$), the vector we want to minimize is $\frac{X_m - X_{mk}}{X_{mk}}$.
First of all let us sum up the market data.
These are the prices of nominal T-bonds at time 0 for $T = 1, 2, \ldots, 29$:

$$\begin{aligned} P_n = (1;\quad & 0,999862; 0,999329; 0,998383; 0,997332; 0,996128; \\ & 0,994764; 0,993209; 0,983567; 0,972189; 0,959461; \\ & 0,931639; 0,901994; 0,870120; 0,836433; 0,803523; \\ & 0,771517; 0,740070; 0,709119; 0,678481; 0,648315; \\ & 0,568554; 0,462336; 0,391743; 0,342419; 0,302396; \\ & 0,264317; 0,229523; 0,198594). \end{aligned}$$

The swap rates we use to calculate the real T-bonds for tenors $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 30$ are

$$K = (\quad 0,02285; 0,02070; 0,02090; 0,02120; 0,02160; 0,02180; 0,02190; \\ 0,02210; 0,02235; 0,02250; 0,02297; 0,02334; 0,02372; 0,02489).$$

Market volatilities for interest rate caps with tenors $(3, 4, 5, 6, 7, 8, 9, 10, 15, 20)$ are

$$\sigma = (0,3610; 0,3481; 0,3313; 0,3147; 0,2997; 0,2874; 0,2766; 0,2670; 0,2321; 0,2200).$$

Finally, market prices for inflation-indexed caps and floors are the following (T stands for tenor, S for strike):

T \ S	0.01	0.015	0.02	0.025	0.03
1	0,012825	0,008632	0,005057	0,002448	0,000964
2	0,025500	0,017930	0,011516	0,006690	0,004009
3	0,040197	0,029368	0,020118	0,012947	0,008042
5	0,073879	0,056779	0,041926	0,029948	0,021108
7	0,109447	0,086630	0,066608	0,050092	0,03740
10	0,161572	0,130978	0,103890	0,081139	0,06312
12	0,193569	0,158216	0,126807	0,100263	0,079038
15	0,237421	0,195428	0,157970	0,126126	0,100475
20	0,298752	0,247006	0,200662	0,161126	0,129244
30	0,399418	0,331416	0,269965	0,217097	0,17437

Table 4.1: Inflation-Indexed Caps

T \ S	0.01	0.015	0.02	0.025	0.03
1	0,000838	0,001557	0,002895	0,005200	0,008628
2	0,004103	0,006232	0,010360	0,0143900	0,021013
3	0,008726	0,011000	0,017330	0,0245000	0,033937
5	0,021154	0,027206	0,035505	0,0466790	0,060989
7	0,034749	0,043229	0,054504	0,069285	0,087891
10	0,056108	0,067824	0,083047	0,102606	0,126898
12	0,069196	0,082725	0,100199	0,122537	0,150194
15	0,086369	0,102058	0,122281	0,148118	0,180148
20	0,110190	0,128503	0,152219	0,182743	0,220920
30	0,144429	0,165697	0,193516	0,229918	0,276462

Table 4.2: Inflation-Indexed Floors

If we use Matlab lsqnonlin with a random starting point, mainly two things can happen: it can find the true solution, or the maximum number of iterations is reached (we set it to 501). The positive aspects of this method is that it is quick, so we can launch it several times and keep the best solution. Through lsqnonlin options we have chosen levenberg-marquardt method. With our data and a tolerance of 10^{-9} some of the possible results are the following ones (we report the set of parameters, the error in terms of the mean of the absolute values of the percentage differences, and the number of iterations):

solution: $(a_n, a_r, \sigma_n, \sigma_r, \sigma_I, \rho_{nr}, \rho_{nI}, \rho_{rI})$	error	iter
(0.0562, 0.1551 , 0.0108, 0.0056, 0.0163, -1.0000, -1.0000, 1.0000)	0.0528	60
(0.0562, 0.1551 , 0.0108, 0.0056, 0.0163, -1.0000, -1.0000, 1.0000)	0.0528	66
(0.1344, 0.1192, 0.0119, 0.0286, 0.0167 , 1.0000 , 1.0000 , 1.0000)	0.0980	501
(0.1367, 0.1047 , 0.0118 , 0.0272 , 0.0164 , 1.0000, 1.0000 , 1.0000)	0.1022	501
(0.0561 , 0.1550 , 0.0108, 0.0056 , 0.0163, -1.0000, -1.0000 , 1.0000)	0.0528	57
(0.0562, 0.1550 , 0.0108, 0.0056, 0.0163, -1.0000, -1.0000, 1.0000)	0.0528	51

Table 4.3: Calibration results: lsqnonlin

We note that when 501 iterations are reached the error is big; in the other cases instead the result is good, in fact it is around 0.05; the tolerance we have chosen is already very small, so that reducing its value does not give better solutions.

Differential evolution is a bit less quick than lsqnonlin, but anyway it gives the result in a few minutes; its greatest quality is that it always reaches a good solution. Here are some examples of results, obtained with populations of 150 individuals, 50 iterations, $F = 0.6$, $CR = 0.9$, and performing a local search with minsearch every five iterations:

In the last column we report the maximum number of function evaluations: we see that it does not affect the validity of the results, which are in fact all satisfactory, with an error around 0.05 also in this case.

In this context we reported just a few examples, but we can conclude that

solution: $(a_n, a_r, \sigma_n, \sigma_r, \sigma_I, \rho_{nr}, \rho_{nI}, \rho_{rI})$	error	max eval
(0.0498, 0.2324, 0.0108, 0.0080, 0.0142, -0.8669, -0.9775, 0.8869)	0.0505	200
(0.0521, 0.2357, 0.0109, 0.0070, 0.0850, -0.9964, -0.1515, 0.0675)	0.0506	200
(0.0523, 0.2619, 0.0109, 0.0074, 0.3439, -0.9943, -0.0048, -0.1014)	0.0516	200
(0.0521, 0.2357 , 0.0109, 0.0070, 0.0850, -0.9964, -0.1515, 0.0675)	0.0506	200
(0.0498, 0.2553, 0.0108, 0.0082, 0.0425, -0.8851, -0.1039, -0.3684)	0.0513	200
(0.0499 , 0.2458 , 0.0108, 0.0072, 0.0130, -0.9987, -0.8422, 0.8336)	0.0507	200
(0.0498 , 0.2362, 0.0108, 0.0077, 0.0373, -0.9150, -0.4036, 0.0001)	0.0504	300
(0.0498, 0.2414, 0.0108, 0.0088, 9.2718, -0.8131, -0.0015, -0.5809)	0.0505	500
(0.0498, 0.2273, 0.0108, 0.0078, 0.0339, -0.8755, -0.3861, 0.7799)	0.0506	500

Table 4.4: Calibration results: differential evolution

both methods work well; the second one is more reliable and gives errors which are a bit lower, while the first one is quicker.

Chapter 5

Monte Carlo method

Not every contract can be priced with an explicit formula as the ones we obtained in the previous sections; derivatives in fact can have complicated payoffs so that the expectation of their discounted values cannot be easily calculated. In those cases a good way to determine the price of the contract can be to perform Monte Carlo simulation, that is to generate the discounted payoff, according to its distribution, a high number of times, and then to make a mean of the obtained values. Of course this method is not exact, but when the number of simulations goes to infinity it converges to the true result.
In the following paragraph we briefly give some more explanations about this method.

5.1 Theory

The Monte Carlo method is a numerical method which allows to calculate the expected value of a random variable whose distribution is known. It is based on probabilistic results, in particular on the Law of Large Numbers:

Theorem 5.1.1 (**Law on Large Numbers (strong version)**). *Let (X_n) be a sequence of i.i.d. random variables with $E[X_1] < +\infty$. If we define*

$M_n = \frac{\sum_{i=1}^{n} X_i}{n}$, *then*

$$M_n \xrightarrow[n\to+\infty]{} E[X_1] \quad a.s. \tag{5.1}$$

This theorem tells that if we are able to generate many realizations X_1, X_2, ..., X_n of the random variable X in an independent way, then we can almost surely use their mean M_n as an approximation of E[X]; in order to estimate the error we make using this procedure, we recall the following result:

Proposition 5.1.2 (Markov inequality). *Let X be a real random variable, $\lambda \in \mathbb{R}$, $1 \le p < +\infty$. Then*

$$P(|X - E[X]| \ge \lambda) \le \frac{var[X]}{\lambda^2}. \tag{5.2}$$

If we apply this proposition with $X = M_n$ (noting that $E[M_n] = \mu$) we obtain

$$P(|M_n - \mu]| \ge \varepsilon) \le \frac{\mathrm{var}[M_n]}{\varepsilon^2} = \frac{1}{n^2\varepsilon^2}\sum_{i=1}^{n}\mathrm{var}[X_i] = \frac{n\mathrm{var}[X_1]}{n^2\varepsilon^2} = \frac{\mathrm{var}[X_1]}{n\varepsilon^2} \tag{5.3}$$

where we have used the independence of the generations X_i, $i = 1, \ldots, n$ (thus the variance of their sum is the sum of their variances), and the fact that they are identically distributed (so the variances are all equal to $\mathrm{var}[X_1]$). Note that the result of the Monte Carlo method are not numbers but random variables, as the proceeding is based on random generations. Anyway, (5.3) gives an objective estimation of the probability of finding a result whose distance from the true one is greater then a certain ε, showing that it depends on the number of realizations n, on the approximation error ε we have chosen and on the variance of the generated variable. In line with Theorem 5.1.1, with n tending to infinity the probability of obtaining a bad result (where the "badness" is fixed by ε) tends to zero; what is interesting to note is that if X has a big variance, the upper bound for the probability is greater (this fact is evident, as a big variance allows the generations to distribute more widely around the mean).

An important property of the Monte Carlo method is the independence of the error from the dimension of the problem. To explain this, we recall another important asymptotic result:

Theorem 5.1.3 (Central limit theorem). *Let (X_n) be a sequence of i.i.d. random variables with $\sigma^2 = var[X_1] < +\infty$. If we define $\mu = E[X_1]$, $M_n = \frac{\sum_{i=1}^{n} X_i}{n}$ and $Y_n = \sqrt{n}(\frac{M_n - \mu}{\sigma})$, then*

$$Y_n \xrightarrow[n\to+\infty]{d} Z, \quad Z \sim \mathcal{N}(0,1) \tag{5.4}$$

where the letter "d" over the arrow of limit indicates convergence in distribution[1]*; in particular*

$$P(Y_n \leq x) \xrightarrow[n\to+\infty]{} \Phi(x) \quad \forall x \in \mathbb{R} \tag{5.5}$$

where Φ is the standard normal distribution function[2]*.*

Since in our case the hypothesis of Theorem 5.1.3 hold, from (5.5) we derive that (for $n \to +\infty$) we asymptotically have

$$Q\Big(\sqrt{n}\Big(\frac{M_n - \mu}{\sigma}\Big) \leq x\Big) \approx \Phi(x) \quad \forall x \in \mathbb{R}$$

so

$$Q\Big(M_n \in \Big[\mu - \frac{\sigma x}{\sqrt{n}}, \mu + \frac{\sigma x}{\sqrt{n}}\Big]\Big) \approx p, \quad p = 2\Phi(x) - 1. \tag{5.6}$$

Let us now fix $p \in]0,1[$, which means we establish that we want a probabilistic error inferior to $1-p$; then, which is the error of the solution obtained with Monte Carlo? If we call ε the distance between M_n, which is our Monte Carlo result, and the mean μ, which is the true value, from (5.6) we have

$$Q(M_n \in [\mu - \varepsilon, \mu + \varepsilon]) \approx p, \quad p = 2\Phi\Big(\frac{\sqrt{n}\varepsilon}{\sigma}\Big) - 1$$

[1] A sequence of random variables (X_n) converges "in distribution" (or "in law") to a random variable X if the sequence of the corresponding distributions weakly converges to the distribution of X.

[2] $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx$.

from which

$$\varepsilon = \frac{\sigma}{\sqrt{n}}\Phi^{-1}\Big(\frac{p+1}{2}\Big). \tag{5.7}$$

This result shows that the error decreases with $\frac{1}{\sqrt{n}}$, and it is independent from the dimension of the problem.
We can interpret (5.7) in the following way: if we fix a probability p, with that degree of certainty the true result will stay in the interval

$$\Big[M_n - z\frac{\sigma}{\sqrt{n}}, M_n + z\frac{\sigma}{\sqrt{n}}\Big] \quad \text{with } z = \Phi^{-1}\Big(\frac{p+1}{2}\Big).$$

The estimate of z is possible thanks to the tables which give the values of Φ^{-1} for certain arguments. Here we report the values of z correspondent to some of the most used confidence levels:

$p = 2\Phi(z) - 1$	z
99%	2.58
98%	2.33
95%	1.96
90%	1.65

Table 5.1: Confidence levels

For example, if we fix $p = 99\%$, the true result will stay in the window $\Big[M_n - 2.58\frac{\sigma}{\sqrt{n}}, M_n + 2.58\frac{\sigma}{\sqrt{n}}\Big]$ with probability $p = 99\%$.
Most of times the standard deviation σ of the variables we are generating is unknown; in those cases, in order to estimate Monte Carlo window, we can approximate σ with the sample standard deviation

$$\hat{\sigma}_n = \sqrt{\frac{\sum_{i=1}^n X(i)^2}{n} - \Big(\frac{\sum_{i=1}^n X(i)}{n}\Big)^2}$$

which is known, as the $X(i)$s are our generations, and the second term under the square root is M_n^2. Thus, we can estimate Monte Carlo window with

$$\Big[M_n - z\frac{\hat{\sigma}_n}{\sqrt{n}}, M_n + z\frac{\hat{\sigma}_n}{\sqrt{n}}\Big] \quad \text{with} \quad z = \Phi^{-1}\Big(\frac{p+1}{2}\Big).$$

One of the few drawbacks of the Monte Carlo method is that it is almost impossible to have completely independent generations of the random variable: since we need a great number of realizations, we must use a calculator, which can only work following algorithms, thus giving results which are only pseudo-random. The error due to the non-independence of the generations cannot be easily estimated, but if the generator works well the method can be applied and gives good results anyway.

5.2 Example: IICap

The Monte Carlo method is useful when we want to calculate the value of a derivative for which an explicit pricing formula is not available. Here instead we give a short explanation about how Monte Carlo can be implemented for the pricing on an inflation-indexed cap: since we have already derived the formula for it (in Section 3.3), this chapter has the aim of showing a possible way to apply the method, and the results can be compared with the exact ones to check the validity of the procedure. Moreover, some of the calculations we will make have general interest and can be used for the pricing of other derivatives, too. For a “real” application of the Monte Carlo method, see the Example in Section 6.3.
As a cap is simply the sum of caplets, we focus on a single caplet. According to the second line of (3.27), we have to calculate

$$N\varphi_i\Big[w\Big(\frac{I(T_i)}{I(T_{i-1})}-1-k\Big)\Big]^+, \tag{5.8}$$

under the forward measure with numeraire $P_n(\cdot, T_i)$. From Section 3.3, we know that

$$\frac{I(T_i)}{I(T_{i-1})} = e^{\int_{T_{i-1}}^{T_i}(r_n(s)-r_r(s))ds-\frac{1}{2}\sigma_I^2(T-t)+\sigma_I(W_I(T_i)-W_I(T_{i-1}))}. \tag{5.9}$$

To perform Monte Carlo, we can thus simulate (5.8) many times and finally take the mean. To do this, the processes we have to generate are three:

$\int_{T_{i-1}}^{T_i} r_n(s)ds$, $\int_{T_{i-1}}^{T_i} r_r(s)ds$ and $(W_I(T_i) - W_I(T_{i-1}))$. According to Section 3.3, their joint distribution is a tridimensional normal. In order to calculate the moments of this distribution, let us remember that if we have a generic Hull & White interest rate process r

$$dr(t) = (h(t) - ar(t))dt + \sigma dW(t)$$

where h is a deterministic function, and $a > 0$ and σ are constants, recalling the computations made in Section 1.2 and in particular equation (1.31), we know that $r(t) = x(t) + \phi(t)$ with

$$\int_0^T \phi(s)ds = \frac{V(T)}{2} - \log(P^*(0,T)) \tag{5.10}$$

with $V(T) = \sigma^2 \int_0^T (B(u,T))^2 du$. We note that the explicit dynamics of the processes are always written with respect to a starting point in an initial time s (see (2.3.2)); in this context we always take $s = 0$, because we are calculating the price of the caplet at time 0, so we are conditioned to $\mathscr{F}_0$. This allows to make only one generation of the processes, directly in the interval of interest (other choices for s would require us to know the value of the process at time s, often forcing us to make a first generation from 0 to s, and then another from s to the instant of interest: see Example in Section 6.3). Now, if we consider the dynamics of r_n and r_r in (2.3.2), we see that under the martingale measure Q both rates follow Hull & White dynamics of the type (1.25), so for $k = n, r$ we have

$$\int_{T_{i-1}}^{T_i} r_k(s)ds = \int_{T_{i-1}}^{T_i} (x_k(s) + \phi_k(s))ds. \tag{5.11}$$

Let us now consider the two addends separately. The first one is

$$\begin{aligned}
&\textstyle\int_{T_{i-1}}^{T_i} x_k(s)ds = \int_{T_{i-1}}^{T_i} e^{-a_k s}\sigma_k \int_0^s e^{a_k u} dW_k(u)ds = \\
&\textstyle\sigma_k \int_0^{T_{i-1}} e^{a_k u} dW_k(u) \int_{T_{i-1}}^{T_i} e^{-a_k s} ds + \sigma_k \int_{T_{i-1}}^{T_i} e^{-a_k s} \int_{T_{i-1}}^{s} e^{a_k u} dW_k(u) ds = \\
&\textstyle\sigma_k \int_0^{T_{i-1}} e^{a_k u} dW_k(u) \int_{T_{i-1}}^{T_i} e^{-a_k s} ds + \sigma_k \int_{T_{i-1}}^{T_i} e^{a_k u} \int_u^{T_i} e^{-a_k s} ds\, dW_k(u),
\end{aligned}$$

which has normal distribution with zero mean and variance equal to the variance of the whole $\int_{T_{i-1}}^{T_i} r_k(s)ds$, which we have already computed in Section 3.3 and is given in (3.34), (3.35) and (3.36).
The second addend of (5.11) is

$$\int_{T_{i-1}}^{T_i} \phi_k(s)ds = \int_0^{T_i} \phi_k(s)ds - \int_0^{T_{i-1}} \phi_k(s)ds.$$

Accordingly to (5.10) and keeping in mind the definition of B in (1.19) and adding to it the parameter a as a subscript, for $k = n$ we have

$$\begin{aligned} &\int_{T_{i-1}}^{T_i} \phi_n(s)ds = \tfrac{V_n(T_i)}{2} - \log(P_n^*(0,T_i)) - \tfrac{V_n(T_{i-1})}{2} + \log(P_n^*(0,T_{i-1})) = \\ &\log \tfrac{P_n^*(0,T_{i-1})}{P_n^*(0,T_i)} + \tfrac{\sigma_n^2}{2}\Big(\int_0^{T_i} B(u,T_i)^2 du - \int_0^{T_{i-1}} B(u,T_{i-1})^2 du \Big) = \\ &\log \tfrac{P_n^*(0,T_{i-1})}{P_n^*(0,T_i)} + \tfrac{\sigma_n^2}{2a_n^2}(T_i - 2B_{a_n}(0,T_i) + B_{2a_n}(0,T_i) - T_{i-1} + 2B_{a_n}(0,T_{i-1}) - B_{2a_n}(0,T_{i-1})). \end{aligned} \tag{5.12}$$

For $k = r$ it is enough to remember Proposition 2.3.2 and note that, taking $s = 0$, $r_r(t)$ has the form as $r_n(t)$ (with all the subscripts r instead of n) plus the addend $-\rho_{rI}\sigma_r\sigma_I \int_0^t e^{a_r(u-t)}du$; so $\int_{T_{i-1}}^{T_i} \phi_r(s)ds$ will be of the same form of $\int_{T_{i-1}}^{T_i} \phi_r(s)ds$ (with all the subscripts r instead of n), plus the addend

$$\begin{aligned} &-\rho_{rI}\sigma_r\sigma_I \int_{T_{i-1}}^{T_i} \int_0^s e^{a_r(u-s)}du\,ds = -\tfrac{\rho_{rI}\sigma_r\sigma_I}{a_r} \int_{T_{i-1}}^{T_i} (1 - e^{-a_r s})ds \\ &= -\tfrac{\rho_{rI}\sigma_r\sigma_I}{a_r}\big(T_i - T_{i-1} - \tfrac{e^{-a_r T_{i-1}}}{a_r}(1 - e^{-a_r(T_i - T_{i-1})})\big) \\ &= -\tfrac{\rho_{rI}\sigma_r\sigma_I}{a_r}\big(T_i - T_{i-1} - e^{-a_r T_{i-1}} B_{a_r}(T_{i-1},T_i)\big). \end{aligned}$$

Summing up[3]:

$$\begin{aligned} &\int_{T_{i-1}}^{T_i} \phi_r(s)ds = \log \tfrac{P_r^*(0,T_{i-1})}{P_r^*(0,T_i)} - \tfrac{\rho_{rI}\sigma_r\sigma_I}{a_r}\big(T_i - T_{i-1} - e^{-a_r T_{i-1}} B_{a_r}(T_{i-1},T_i)\big) \\ &+\tfrac{\sigma_r^2}{2a_r^2}(T_i - 2B_{a_r}(0,T_i) + B_{2a_r}(0,T_i) - T_{i-1} + 2B_{a_r}(0,T_{i-1}) - B_{2a_r}(0,T_{i-1})). \end{aligned} \tag{5.13}$$

Let us finally note that

$$\begin{aligned} &E[\textstyle\int_{T_{i-1}}^{T_i} r_k(s)ds] = \int_{T_{i-1}}^{T_i} \phi_k(s)ds, \quad k = n, r; \\ &E[\sigma_I(W_I(T_i) - W_I(T_{i-1}))] = 0. \end{aligned}$$

[3] Note that $P_r(0,T_{i-1})$ and $P_r(0,T_i)$ do not bring any correction term since $P_r(t,T) = E^r\big[e^{-\int_t^T r_r(s)ds}\big|\mathscr{F}_t\big]$ and for the foreign currency analogy the dynamics of r_r under Q^r are of the same form of the ones of r_n under Q (see (2.14)).

We thus have the means of the three processes we want to generate, when they are expressed under the martingale measure Q.
Now, in the same way we did in Section 3.2, the change of numeraire from B_n to $P_n(\cdot, T_i)$ gives

$$\begin{aligned} dW_n(t) &= dW_n^{T_i}(t) - \sigma_n B_n(t,T_i)dt \\ dW_r(t) &= dW_r^{T_i}(t) - \rho_{nr}\sigma_n B_n(t,T_i)dt \\ dW_I(t) &= dW_I^{T_i}(t) - \rho_{nI}\sigma_n B_n(t,T_i)dt \end{aligned} \tag{5.14}$$

so that the dynamics of r_n, r_r and $W_I(T_i) - W_I(T_{i-1})$ must be modified with opportune additive terms in the drift, which lead to additive terms in the means of our three processes. Precisely:
∘ $r_n(t)$ has the additional addend $-\sigma_n^2 \int_0^t e^{-a_n(t-u)} B_n(u,T_i)\,du$, so $\int_{T_{i-1}}^{T_i} r_n(s)\,ds$ has the additional addend

$$\begin{aligned} &-\sigma_n^2 \int_{T_{i-1}}^{T_i} \Big(\int_0^t e^{-a_n(t-u)} B_n(u,T_i) du \Big) dt \\ &= -\tfrac{\sigma_n^2}{a_n} \int_{T_{i-1}}^{T_i} \Big(\int_0^t (e^{-a_n(t-u)} - e^{-a_n(T_i+t-2u)}) du \Big) dt \\ &= -\tfrac{\sigma_n^2}{a_n} \int_{T_{i-1}}^{T_i} \Big(\tfrac{1}{a_n} - \tfrac{e^{-a_n(T_i-t)}}{2a_n} - \tfrac{e^{-a_n t}}{a_n} + \tfrac{e^{-a_n(T_i+t)}}{2a_n} \Big) dt \\ &= -\tfrac{\sigma_n^2}{a_n^2} \Big(T_i - T_{i-1} - \tfrac{1}{2} B_n(T_{i-1},T_i) - e^{-a_n T_{i-1}} B_n(T_{i-1},T_i) - \tfrac{e^{-2a_n T_i}}{2} B_n(T_i, T_{i-1}) \Big). \end{aligned} \tag{5.15}$$

∘ $r_r(t)$ has the additional addend $-\sigma_n\sigma_r\rho_{nr} \int_0^t e^{-a_r(t-u)} B_n(u,T_i) du$, so $\int_{T_{i-1}}^{T_i} r_r(s) ds$ has the additional addend

$$\begin{aligned} &-\sigma_n\sigma_r\rho_{nr} \int_{T_{i-1}}^{T_i} \Big(\int_0^t e^{-a_r(t-u)} B_n(u,T_i) du \Big) dt \\ &= -\tfrac{\sigma_n\sigma_r\rho_{nr}}{a_n} \int_{T_{i-1}}^{T_i} \Big(\int_0^t (e^{-a_r(t-u)} - e^{-a_r(t-u) - a_n(T_i-u)}) du \Big) dt \\ &= -\tfrac{\sigma_n\sigma_r\rho_{nr}}{a_n} \int_{T_{i-1}}^{T_i} \Big(\tfrac{1}{a_r} - \tfrac{e^{-a_n(T_i-t)}}{a_n+a_r} - \tfrac{e^{-a_r t}}{a_r} + \tfrac{e^{-a_r t - a_n T_i)}}{a_n+a_r} \Big) dt \\ &= -\tfrac{\sigma_n\sigma_r\rho_{nr}}{a_n} \Bigg(\tfrac{T_i - T_{i-1}}{a_r} - \tfrac{1}{a_n(a_n+a_r)} + \tfrac{e^{-a_n(T_i-T_{i-1})}}{a_n(a_n+a_r)} + \tfrac{e^{-a_r T_i}}{a_r^2} - \tfrac{e^{-a_r T_{i-1}}}{a_r^2} \\ &\qquad - \tfrac{e^{-a_r T_i - a_n T_i}}{a_r(a_n+a_r)} + \tfrac{e^{-a_r T_{i-1} - a_n T_i}}{a_r(a_n+a_r)} \Bigg) \\ &= -\tfrac{\sigma_n\sigma_r\rho_{nr}}{a_n} \Bigg(\tfrac{T_i - T_{i-1}}{a_r} - \tfrac{1}{a_n+a_r} B_n(T_{i-1},T_i) + e^{-a_r T_{i-1}} B_r(T_{i-1},T_i) \big(\tfrac{e^{-a_n T_i}}{a_n+a_r} - \tfrac{1}{a_r} \big) \Bigg). \end{aligned} \tag{5.16}$$

$\circ$ $W_I(T_i) - W_I(T_{i-1})$ has the additional addend

$$-\sigma_n \sigma_I \rho_{nI} \int_{T_{i-1}}^{T_i} B_n(t, T_i) dt = -\frac{\sigma_n \sigma_I \rho_{nI}}{a_n}(T_i - T_{i-1} - B_n(T_{i-1}, T_i)). \quad (5.17)$$

These three deterministic addends only affect the means of the three processes so they can be simply added after the generation. This way of working gives us, with only one generation, the processes both under risk-neutral and under forward measure (in this context we are interested only in the second ones).

Let us now consider the variances and covariances of the three processes we want to generate.
First of all,

$$\mathrm{var}[\sigma_I(W_I(T_i) - W_I(T_{i-1}))] = \sigma_I^2(T_i - T_{i-1}).$$

The variances of the two integrals and the covariances between the three processes have already been calculated in Section 3.3[4] and are contained respectively in the first four lines of formula (3.37):

$$\begin{aligned} \mathrm{var}\Big[\textstyle\int_{T_{i-1}}^{T_i} r_k(s)ds\Big] = & \tfrac{\sigma_k^2}{a_k^2}\Big\{T_i - T_{i-1} - \tfrac{e^{-2a_k(T_i - T_{i-1})}}{2a_k} + 2\tfrac{e^{-a_k(T_i - T_{i-1})}}{a_k} - \tfrac{3}{2a_k}\Big\} \\ & + \tfrac{\sigma_k^2}{2a_k^3}\big(1 - e^{-a_k(T_i - T_{i-1})}\big)^2\big(1 - e^{-2a_k(T_{i-1} - t)}\big); \end{aligned} \quad (5.18)$$

and in the last four ones:

$$\begin{aligned} & \mathrm{cov}\Big[\textstyle\int_{T_{i-1}}^{T_i} r_n(s)ds, \int_{T_{i-1}}^{T_i} r_r(s)ds\Big] = \\ & \tfrac{\rho_{nr}\sigma_n\sigma_r}{a_n a_r(a_n + a_r)}\big(1 - e^{-a_n(T_i - T_{i-1})}\big)\big(1 - e^{-a_r(T_i - T_{i-1})}\big)\big(1 - e^{-(a_n + a_r)(T_{i-1} - t)}\big) \\ & + \tfrac{\rho_{nr}\sigma_n\sigma_r}{a_n a_r}\Big\{T_i - T_{i-1} - \tfrac{1 - e^{-a_n(T_i - T_{i-1})}}{a_n} - \tfrac{1 - e^{-a_r(T_i - T_{i-1})}}{a_r} + \tfrac{1 - e^{-(a_n + a_r)(T_i - T_{i-1})}}{a_n + a_r}\Big\}; \end{aligned} \quad (5.19)$$

$$\mathrm{cov}\Big[\int_{T_{i-1}}^{T_i} r_k(s)ds, \sigma_I(W_I(T_i) - W_I(T_{i-1}))\Big] = \frac{\rho_{kI}\sigma_k\sigma_I}{a_k}\Big\{T_i - T_{i-1} - \frac{1 - e^{-a_k(T_i - T_{i-1})}}{a_k}\Big\}. \quad (5.20)$$

[4]Remember that a change of measure does only affect the drifts, so it has no influence on variances and covariances.

It is thus possible to generate the three processes, calculate (5.9) and consequently (5.8).

Under the forward measure the discount factor is the nominal T-bond; precisely, in this case we need $P_n(0,T_i)$, which is known from market data. Summing up: we can generate the processes $\int_{T_{i-1}}^{T_i} r_n(s)ds$, $\int_{T_{i-1}}^{T_i} r_r(s)ds$ and $W_I(T_i)-W_I(T_{i-1})$ under the risk-neutral measure as a tridimensional normal with mean

$$\begin{aligned}
&[\log \tfrac{P_n^*(0,T_{i-1})}{P_n^*(0,T_i)} + \tfrac{\sigma_n^2}{2a_n^2}(T_i - 2B_{a_n}(0,T_i) + B_{2a_n}(0,T_i) - T_{i-1} + 2B_{a_n}(0,T_{i-1}) - B_{2a_n}(0,T_{i-1});\\
&\log \tfrac{P_r^*(0,T_{i-1})}{P_r^*(0,T_i)} - \tfrac{\rho_{rI}\sigma_r\sigma_I}{a_r}\big(T_i - T_{i-1} - e^{-a_r T_{i-1}} B_{a_r}(T_{i-1},T_i)\big)\\
&\tfrac{\sigma_r^2}{2a_r^2}(T_i - 2B_{a_r}(0,T_i) + B_{2a_r}(0,T_i) - T_{i-1} + 2B_{a_r}(0,T_{i-1}) - B_{2a_r}(0,T_{i-1}); \quad 0]
\end{aligned}$$

and covariance matrix

$$\begin{pmatrix}
\mathrm{var}\Big[\int_{T_{i-1}}^{T_i} r_n(s)ds\Big] & \mathrm{cov}\Big[\int_{T_{i-1}}^{T_i} r_n(s)ds,\ \int_{T_{i-1}}^{T_i} r_r(s)ds\Big] & \mathrm{cov}\Big[\int_{T_{i-1}}^{T_i} r_n(s)ds,\ W_I(T_i)-W_I(T_{i-1})\Big]\\
\mathrm{cov}\Big[\int_{T_{i-1}}^{T_i} r_n(s)ds,\ \int_{T_{i-1}}^{T_i} r_r(s)ds] & \mathrm{var}\Big[\int_{T_{i-1}}^{T_i} r_r(s)ds\Big] & \mathrm{cov}\Big[\int_{T_{i-1}}^{T_i} r_r(s)ds,\ W_I(T_i)-W_I(T_{i-1})\Big]\\
\mathrm{cov}\Big[\int_{T_{i-1}}^{T_i} r_n(s)ds,\ W_I(T_i)-W_I(T_{i-1})\Big] & \mathrm{cov}\Big[\int_{T_{i-1}}^{T_i} r_r(s)ds,\ W_I(T_i)-W_I(T_{i-1})\Big] & \mathrm{var}\Big[W_I(T_i)-W_I(T_{i-1})\Big]
\end{pmatrix}$$

whose elements are defined in (5.18), (5.19) and (5.20); then, in order to obtain the values of the processes under the measure with numeraire $P_n(\cdot,T_i)$, we must add to the three processes the terms in (5.15), (5.16) and (5.17) respectively. Thus, we can assembly $\frac{I(T_i)}{I(T_{i-1})}$ using its expression in (5.9), and calculate the payoff in (5.8). Finally, we discount from time T_i to time 0 with $P_n(0,T_i)$, obtaining one price of the caplet; the mean of the results obtained performing this procedure a great number of times (i.e: making a lot of generations of the three processes) is Monte Carlo result.

Chapter 6

Credit risk

A very important matter, especially in recent days, is credit risk, that is the risk associated with the possibility of bankruptcy. More precisely: if a derivative provides for a payment at a certain time T but before that time the counterparty defaults, at maturity the payment cannot be effectively performed, so the owner of the contract loses it entirely, or a part of it (actually, a recovery is often given). It means that the payoff of the derivative, and consequently its price, depends on the risk of bankruptcy of the counterparty.

6.1 Default

The standard way to model the counterparty risk of bankruptcy is to introduce the *default time* τ: it is a stopping time[1] which represents the instant in which the counterparty goes bankrupt.
Default time is introduced because it gives information which is not contained in the usual filtration $(\mathscr{F}_t)$ (which only deals with the behaviour of the underlyings). It means that if we want a filtration which provides for the whole flow of information, we should introduce

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\{\tau < u\}, u \leq t) \tag{6.1}$$

[1]A random variable $\tau : \Omega \to [0, +\infty]$ is a stopping time with respect to the filtration $(\mathscr{F}_t)$ if $\{\tau \leq t\} \in \mathscr{F}_t \quad \forall t \geq 0$.

which describes the default-free market variables up to t (filtration $\mathscr{F}_t$) and tells whether default occurred before t and, in that case, when exactly (σ-algebra $(\sigma(\{\tau \leq u\}, u \leq t))$.
Thus, taking care of the credit risk of the counterparty, the price at time t of a derivative with payoff X and maturity T is given by

$$E[\mathbb{1}_{\{\tau>T\}}X|\mathscr{G}_t]. \tag{6.2}$$

In order to calculate this, it is necessary to know how to deal with default time and the new filtration. The following proposition shows how to express (6.2) in terms of $(\mathscr{F}_t)$ instead of $(\mathscr{G}_t)$:

Proposition 6.1.1 (Filtration switching formula). *Let X be a $\mathscr{G}_\infty$-measurable payoff, and $\mathscr{G}_t$ the filtration in (6.1). So the following holds:*

$$E[\mathbb{1}_{\{\tau>T\}}X|\mathscr{G}_t] = \frac{\mathbb{1}_{\{\tau>t\}}}{Q[\tau > t|\mathscr{F}_t]}E[\mathbb{1}_{\{\tau>T\}}X|\mathscr{F}_t]. \tag{6.3}$$

Proof. Obviously

$$\begin{aligned} &E[\mathbb{1}_{\{\tau>T\}}X|\mathscr{G}_t] = E[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>T\}}X|\mathscr{G}_t] = \\ &\mathbb{1}_{\{\tau>t\}}E[\mathbb{1}_{\{\tau>T\}}X|\mathscr{G}_t] = \mathbb{1}_{\{\tau>t\}}E[\mathbb{1}_{\{\tau>T\}}X|\mathscr{F}_t \vee \sigma(\{\tau \leq u\}, u \leq t)]. \end{aligned} \tag{6.4}$$

Now, $\mathbb{1}_{\{\tau>T\}}X$ gives zero if $\tau < t$, so the only useful information about default contained in $\sigma(\{\tau \leq u\}, u \leq t)$ is whether $\tau \geq t$; thus, (6.4) is equal to

$$\mathbb{1}_{\{\tau>t\}}E[\mathbb{1}_{\{\tau>T\}}X|\mathscr{F}_t \vee \tau \geq t]. \tag{6.5}$$

Moreover, we recall that for any expectation E, random variable Y and event A

$$E[Y|A] = \frac{E[Y\mathbb{1}_A]}{P[A]},$$

so that, taking $E[\cdot] = E[\cdot|\mathscr{F}_t]$, (6.5) becomes

$$\frac{\mathbb{1}_{\{\tau>t\}}}{Q[\tau > t|\mathscr{F}_t]}E[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>T\}}X|\mathscr{F}_t] = \frac{\mathbb{1}_{\{\tau>t\}}}{Q[\tau > t|\mathscr{F}_t]}E[\mathbb{1}_{\{\tau>T\}}X|\mathscr{F}_t].$$

□

A common hypothesis is the independence of τ from all the other components of the market, and we will assume it too.

6.2 Defaultable zero-coupon bonds

Definition 6.1 (**Defaultable zero-coupon bonds**)**.** A *defaultable zero-coupon bond* P^d is a contract which gives 1 at maturity T if the issuing company does not default before T, zero in the other case.

Indicating with τ the default time variable, we give sense to $P^d(t,T)$ only in the case $\tau > t$. So, the following holds:

$$P^d(t,T) = E[e^{-\int_t^T r_n(s)ds}\mathbb{1}_{\{\tau>T\}}|\mathscr{G}_t]$$

which for $t = 0$ becomes

$$P^d(0,T) = E[e^{-\int_0^T r_n(s)ds}\mathbb{1}_{\{\tau>T\}}]$$

and thanks to the assumption of independence of τ from $(\mathscr{F}_t)$

$$= E[e^{-\int_0^T r_n(s)ds}]E[\mathbb{1}_{\{\tau>T\}}] = P_n(0,T)Q(\tau > T).$$

Summing up:

$$P^d(0,T) = P_n(0,T)Q(\tau > T). \tag{6.6}$$

In practice, a defaultable zero-coupon bond is a nominal bond multiplied by a factor, less or equal then one, which takes care of the risk of default of the issuer.
If we now want the price at time 0 of a defaultable derivative X^d with maturity T and default-free payoff X_T, we have

$$E[e^{-\int_0^T r_n(s)ds}X_T\mathbb{1}_{\{\tau>T\}}]$$

and thanks to the independence of τ from X, r and $(\mathscr{F}_t)$

$$= E[e^{-\int_0^T r_n(s)ds}X_T]Q(\tau > T); \tag{6.7}$$

using (6.6) and indicatig with X_0 the price at time 0 of the derivative without considering the default risk, (6.7) becomes

$$= X_0\frac{P^d(0,T)}{P_n(0,T)}.$$

Summing up:

$$X_0^d = X_0 \frac{P^d(0,T)}{P_n(0,T)}. \tag{6.8}$$

More generally, if we want the price at time t, for Proposition 6.1.1 we have

$$E[e^{-\int_0^T r_n(s)ds} X_T \mathbb{1}_{\{\tau>T\}} | \mathscr{G}_t] = \frac{\mathbb{1}_{\{\tau>t\}}}{Q[\tau > t|\mathscr{F}_t]} E[e^{-\int_0^T r_n(s)ds} \mathbb{1}_{\{\tau>T\}} X_T | \mathscr{F}_t]$$

which, thanks to the independence of τ from X, r and $(\mathscr{F}_t)$, is equal to

$$= \frac{\mathbb{1}_{\{\tau>t\}}}{Q[\tau > t]} E[\mathbb{1}_{\{\tau>T\}}] E[e^{-\int_0^T r_n(s)ds} X_T | \mathscr{F}_t] = \frac{\mathbb{1}_{\{\tau>t\}}}{Q[\tau > t]} Q[\tau > T] X_t =$$

using (6.6)

$$\mathbb{1}_{\{\tau>t\}} \frac{P_n(0,t)}{P^d(0,t)} \frac{P^d(0,T)}{P_n(0,T)} X_t.$$

Summing up:

$$X_t^d = \mathbb{1}_{\{\tau>t\}} X_t \frac{P_n(0,t) P^d(0,T)}{P_n(0,T) P^d(0,t)}. \tag{6.9}$$

Note that, in order to obtain these results, we have not assumed a specific distribution for τ.

If the values of the Spanish T-bonds $P^d(0,T)$ are not available, in order to compute them it is common to use the *Z-spread* zs, which is the value to be added to the yield rate Y for the calculus of the bonds[2] and is available on the market; more precisely, the following holds:

$$P^d(0,T)(1 + Y + zs)^T = 1.$$

[2] $P(0,T)(1+Y)^T = 1$.

6.3 Example

Let us consider a defaultable inflation-linked derivative issued on T_0 with maturity T_1 unless an extension event occurs, in which case the maturity is T_2. The counterparty with risk of default is Spain. The underlying is the Inflation index, with which the "inflation participation" is defined:

$$IP(T_1, T_2) = \left(\frac{I(T_2) - I(T_1)}{I(T_1)}\right)^+ = \left(\frac{I(T_2)}{I(T_1)} - 1\right)^+.$$

The contract pays the minimum between:
◦ 1 in T_1 if Spain has not defaulted before T_1 (and zero in case of early default);
◦ $1 + IP(T_1, T_2)$ in T_2 if Spain has not defaulted before T_2 (and zero in case of early default).
In order to price it, in T_1 a comparison should be made between 1 and the value of $1+IP(T_1, T_2)$ discounted from T_2 to T_1 and taking care of the default risk between T_1 and T_2: if 1 is the minimum between these two quantities, the contract ends in T_1 and corresponds to a Spanish T_1-bond, i.e. a T_1-bond with the risk of Spanish default; in the other case, the extension takes place, the contract ends in T_2 and its price is the one of $1 + IP(T_1, T_2)$, calculated in T_0 and considering the risk of Spanish default between T_0 and T_2.
The present value of the Spanish Z-spread is 0.022.
We will consider a notional value $N = 1$ (if $N \neq 1$, it will be enough to multiply the price for N).
Let us price this contract in two different ways.

6.3.1 Under risk neutral measure

Working under the risk neutral measure, it is possible to generate many times all the processes of interest (including the discount factors) at time T_1 and T_2; then, for any generation we can check the extension condition in T_1, and find the price of the derivative opportunely discounting the payoff from T_1 or T_2 to T_0 and taking a mean over the generations (Monte Carlo).

The things it is important to take in mind are that:
-since the contract provides for the check of the extension condition being at time T_1, the generation of the processes in T_2 should be made taking T_1 as a starting point;
-starting from each of the m_1 generations of the processes in T_1, m_2 generations of the processes in T_2 should be made, and their mean is taken: in this way, Monte Carlo is performed two times, and the scheme of the generations appears as a tree with m_1 principal branches and $m_1^{m_2}$ secondary ones.
For simplicity, performing a shift in time we will take $T_0 = 0$.
Let us now see the dynamics of the processes we must generate. We need to know the nominal discount factors from T_1 to T_0 and from T_2 to T_1, and the ratio of the inflation indices

$$I(T_2)/I(T_1) = e^{\int_{T_1}^{T_2}(r_n(s)-r_r(s))ds-\frac{1}{2}\sigma_I^2(T_2-T_1)+\sigma_I(W_I(T_2)-W_I(T_1))} \tag{6.10}$$

(see (2.13)).
Splitting r in α (deterministic) and x (stochastic) as (1.26), for $t < s$ we have (see (1.27))

$$x(s) = e^{-a(s-t)}x(t) + \sigma \int_t^s e^{-a(s-u)}dW(u), \quad x(0) = 0 \tag{6.11}$$

which means that $x(s)$ is normally distributed with mean $e^{-a(s-t)}x(t)$ and variance

$$var[x(s)] = \sigma^2 \int_s^t e^{-2a(t-u)}du = \sigma^2 B(2a, t-s)$$

and this holds both for the nominal and the real case. Working as we did in Section 1.2 to obtain (1.29) and (1.30), we have that the price at time t_1 of a nominal t_2-bond knowing the information up to $t \leq t_1 \leq t_2$ is:

$$P_n(t, t_1, t_2) = E[e^{-\int_{t_1}^{t_2} r_n(s)ds}|\mathscr{F}_t] =$$

thanks to (1.28)

$$E\left[e^{-\int_{t_1}^{t_2}\phi_n(s)ds - x_n(t)\int_{t_1}^{t_2} e^{-a_n(s-t)}ds - \sigma_n \int_{t_1}^{t_2} e^{-a_n(s-t)}(\int_t^s e^{a_n(u-t)}dW_n(u))ds}|\mathscr{F}_t\right]$$

where, keeping in mind the definition of B in (1.19), the last double integral in the exponent is equal to

$$\begin{aligned}&\int_{t_1}^{t_2}(\int_t^s e^{a_n(u-s)}dW_n(u))ds = \int_t^{t_1}(\int_{t_1}^{t_2} e^{a_n(u-s)}ds)dW_n(u) + \int_{t_1}^{t_2}(\int_{t_1}^s e^{a_n(u-s)}dW_n(u))ds \\ &= \int_t^{t_1} e^{a(u-t_1)}B_{a_n}(t_1,t_2)dW_n(u) + \int_{t_1}^{t_2} B_{a_n}(u,t_2)dW_n(u)\end{aligned} \tag{6.12}$$

so that $P_n(t,t_1,t_2) = E[e^{H(t,t_1,t_2)}|\mathscr{F}_t]$ with $H(t,t_1,t_2)$ which is normally distributed with mean

$$\mu(t,t_1,t_2) = -\int_{t_1}^{t_2}\phi_n(s)ds - x_n(t)\int_{t_1}^{t_2} e^{-a_n(s-t)}ds \tag{6.13}$$

and variance $V(t,t_1,t_2) = V_1(t,t_1,t_2) + V_2(t,t_1,t_2)$ where

$$\begin{aligned}&V_1(t,t_1,t_2) = var\Big[-\sigma_n\int_t^{t_1} e^{a(u-t_1)}B_{a_n}(t_1,t_2)dW_n(u)\Big] = \\ &\sigma_n^2\int_t^{t_1} e^{2a(u-t_1)}B_{a_n}(t_1,t_2)^2du = \tfrac{\sigma_n^2}{a_n^2}\int_t^{t_1} e^{2a(u-t_1)}(1-e^{-2a_n(t_2-t_1)}-2e^{-a_n(t_2-t_1)})du \\ &= \tfrac{\sigma_n^2}{a_n^2}(B_{2a_n}(t_1-t) + e^{-2a_n(t_2-t_1)}B_{2a_n}(t_1-t) - 2e^{-a_n(t_2-t_1)}B_{2a_n}(t_1-t)) \\ &= \tfrac{\sigma_n^2}{a_n^2}(1-e^{-a_n(t_2-t_1)})^2 B_{2a_n}(t_1-t)\end{aligned} \tag{6.14}$$

and

$$\begin{aligned}&V_2(t,t_1,t_2) = var\Big[-\sigma_n\int_{t_1}^{t_2} B_n(u,t_2)dW_n(u)\Big] = \sigma_n^2\int_{t_1}^{t_2} B_n(u,t_2)^2du \\ &= \tfrac{\sigma_n^2}{a_n^2}(t_2-t_1+B_n(2a_n,t_2-t_1)-2B_n(a_n,t_2-t_1))\end{aligned} \tag{6.15}$$

(note that the covariance between the two stochastic integrals in (6.12) is zero). So, according to (C.1) we have

$$P_n(t,t_1,t_2) = e^{\mu(t,t_1,t_2)+\frac{V(t,t_1,t_2)}{2}}. \tag{6.16}$$

In particular,

$$\begin{aligned}&P_n(t,t,t_2) = e^{\mu(t,t,t_2)+\frac{V(t,t,t_2)}{2}} = e^{-\int_t^{t_2}\phi_n(s)ds - x_n(t)\int_t^{t_2}e^{-a_n(s-t)}ds + \frac{V_2(t,t,t_2)}{2}} = \\ &e^{-\int_t^{t_1}\phi_n(s)ds - \int_{t_1}^{t_2}\phi_n(s)ds - x_n(t)\int_t^{t_1}e^{-a_n(s-t)}ds - x_n(t)\int_{t_1}^{t_2}e^{-a_n(s-t)}ds} \\ &\cdot e^{+\frac{V_2(t,t,t_2)}{2}+\frac{V_2(t,t,t_1)}{2}-\frac{V_2(t,t,t_1)}{2}+\frac{V_1(t,t,t_1)}{2}-\frac{V_1(t,t,t_1)}{2}+\frac{V_2(t,t_1,t_2)}{2}-\frac{V_2(t,t_1,t_2)}{2}} = \\ &P_n(t,t,t_1)P_n(t,t_1,t_2)e^{\frac{V_2(t,t,t_2)}{2}-\frac{V_2(t,t,t_1)}{2}-\frac{V_1(t,t,t_1)}{2}-\frac{V_2(t,t_1,t_2)}{2}}\end{aligned}$$

from which

$$P_n(t,t_1,t_2) = \frac{P_n(t,t,t_2)}{P_n(t,t,t_1)} e^{-\frac{1}{2}(V_2(t,t,t_2)-V_2(t,t,t_1)-V_1(t,t,t_1)-V_2(t,t_1,t_2))}. \tag{6.17}$$

Now, if we compare (6.16) and (6.17) and solve with respect to μ, we obtain

$$\mu(t,t_1,t_2) = \log \frac{P_n(t,t,t_2)}{P_n(t,t,t_1)} - \frac{1}{2}(V_2(t,t,t_2) - V_2(t,t,t_1)). \tag{6.18}$$

Let us now derive another expression for $P(t,t_1,t_2)$ which only depends on $x(t_1)$ (which we are able to generate: remember (6.11) and the following lines of explanation), so that we can know its value and put it in (6.18). Inserting in (6.13) the expression for the integral of ϕ in (5.12), substituting in (6.16) we obtain

$$P_n(t,t_1,t_2) = \frac{P_n(0,t_2)}{P_n(0,t_1)} e^{-\frac{\sigma_n^2}{2a_n^2}(t_2-2B_{a_n}(0,t_2)+B_{2a_n}(0,t_2)-t_1+2B_{a_n}(0,t_1)-B_{2a_n}(0,t_1))}$$
$$\cdot e^{-x_n(t)\int_{t_1}^{t_2} e^{-a_n(s-t)}ds + \frac{\sigma_n^2}{2a_n^2}(1-e^{-a_n(t_2-t_1)})^2 B_{2a_n}(t_1-t) + \frac{\sigma_n^2}{2a_n^2}(t_2-t_1+B_n(2a_n,t_2-t_1)-2B_n(a_n,t_2-t_1))} =$$

setting $\tilde{V}_{a,\sigma}(t) = \frac{\sigma^2}{a^2}(t - 2B_a(0,t) + B_{2a}(0,t))$

$$\frac{P_n(0,t_2)}{P_n(0,t_1)} e^{-x_n(t)e^{-a_n(t_1-t)}B_{a_n}(t_1,t_2)\frac{\sigma_n^2}{2a_n^2}(\tilde{V}_{a_n,\sigma_n}(t_1)-\tilde{V}_{a_n,\sigma_n}(t_2)+\tilde{V}_{a_n,\sigma_n}(t_2-t_1)+(1-e^{-a_n(t_2-t_1)})^2 B_{2a_n}(t_1-t))}.$$

In particular, if $t = t_1$ we obtain

$$P_n(t_1,t_1,t_2) = \frac{P_n(0,t_2)}{P_n(0,t_1)} e^{-x_n(t)B_{a_n}(t_1,t_2)+\frac{\sigma_n^2}{2a_n^2}(\tilde{V}_{a_n,\sigma_n}(t_1)-\tilde{V}_{a_n,\sigma_n}(t_2)+\tilde{V}_{a_n,\sigma_n}(t_2-t_1))} \tag{6.19}$$

which only requires the generation of x_n.
With our results, the nominal discount factor $e^{-\int_{t_1}^{t_2} r_n(s)ds}$, knowing the information up to t, is $e^{H(t,t_1,t_2)}$, so if we want to generate $\int_{t_1}^{t_2} r_n(s)ds$ it is enough to generate $-H(t,t_1,t_2)$ which is normally distributed with mean $\mu(t,t_1,t_2)$ in (6.18) (with P in (6.19)) and variance $V(t,t_1,t_2)$.

By Proposition 2.3.2, for $t < s$ $r_r(s)$ contains the extra addend $-\rho_{rI}\sigma_r\sigma_I \int_t^s e^{a_r(u-s)}du$, so $-\int_{t_1}^{t_2} r_r(s)ds$ contains the extra addend

$$\rho_{rI}\sigma_r\sigma_I \int_{t_1}^{t_2}\int_t^s e^{a_r(u-s)}du\,ds = \rho_{rI}\sigma_r\sigma_I \int_{t_1}^{t_2} B_{a_r}(t,s)ds =$$
$$\frac{\rho_{rI}\sigma_r\sigma_I}{a_r}(t_2 - t_1 - e^{a(t_1-t)}B_{a_r}(t_1,t_2)). \tag{6.20}$$

Now we recall that

$$P_r(t,t_1,t_2) = E^r[e^{-\int_{t_1}^{t_2} r_r(s)ds}|\mathscr{F}_t]$$

where the superscript r denotes the "real" measure, i.e. the one with numeraire B_rI, and exploiting (2.14) and the calculations which lead to (6.20), this is equal to

$$= E[e^{-\int_{t_1}^{t_2} r_r(s)ds}|\mathscr{F}_t]\, e^{-\frac{\rho_{rI}\sigma_r\sigma_I}{a_r}(t_2-t_1-e^{a(t_1-t)}B_{a_r}(t_1,t_2))} \tag{6.21}$$

from which we obtain the analogous to formula (6.18) in the real case: the mean of $-\int_{t_1}^{t_2} r_r(s)ds$ knowing the information up to t is

$$\log\frac{P_r(t,t,t_2)}{P_r(t,t,t_1)} - \frac{1}{2}(V_2^r(t,t,t_2) - V_2^r(t,t,t_1)) + \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}(t_2-t_1-e^{a(t_1-t)}B_{a_r}(t_1,t_2)) \tag{6.22}$$

(where V_2^r has the same form of V_2 in (6.15) but with all the subscripts r instead of n).

Now if we apply the formula for the expectation of a lognormal variable to (6.21), we obtain

$$P_r(t,t_1,t_2) = e^{\mu_r(t,t_1,t_2)+\frac{V_r}{2}}\, e^{-\frac{\rho_{rI}\sigma_r\sigma_I}{a_r}(t_2-t_1-e^{a(t_1-t)}B_{a_r}(t_1,t_2))} \tag{6.23}$$

where, as in the nominal case, $\mu_r(t,t_1,t_2) = -\int_{t1}^{t2}\phi_n(s)ds - x_n(t)\int_{t_1}^{t_2} e^{-a_n(s-t)}ds$. In Chapter 5 we have already calculated $-\int_{t1}^{t2}\phi_n(s)ds$: substituting in (6.23) the expression in formula (5.13) we obtain

$$P_r(t,t_1,t_2) = \frac{P_r(0,t_2)}{P_r(0,t_1)} e^{\frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(t_2-t_1-e^{-a_r t_1}B_{a_r}(t_1,t_2)\right)}$$
$$\cdot e^{-\frac{\sigma_r^2}{2a_r^2}(t_2-2B_{a_r}(0,t_2)+B_{2a_r}(0,t_2)-t_1+2B_{a_r}(0,t_1)-B_{2a_r}(0,t_1)) - \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(t_2-t_1-e^{a(t_1-t)}B_{a_r}(t_1,t_2)\right)}$$

which for $t = t_1$ gives

$$\begin{aligned} P_r(t_1,t_1,t_2) &= \frac{P_r(0,t_2)}{P_r(0,t_1)} \cdot e^{-\frac{\sigma_r^2}{2a_r^2}(t_2-2B_{a_r}(0,t_2)+B_{2a_r}(0,t_2)-t_1+2B_{a_r}(0,t_1)-B_{2a_r}(0,t_1))} \\ &\cdot e^{\frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(t_2-t_1-e^{-a_r t_1}B_{a_r}(t_1,t_2)-t_2+t_1+B_{a_r}(t_1,t_2)\right)} = \\ P_r(t_1,t_1,t_2) &= \frac{P_r(0,t_2)}{P_r(0,t_1)} \cdot e^{-\frac{\sigma_r^2}{2a_r^2}(t_2-2B_{a_r}(0,t_2)+B_{2a_r}(0,t_2)-t_1+2B_{a_r}(0,t_1)-B_{2a_r}(0,t_1))} \\ &\cdot e^{\rho_{rI}\sigma_r\sigma_I B_{a_r}(t_1,t_2)B_{a_r}(0,t_1)}. \end{aligned} \tag{6.24}$$

So, in order to compute the mean of $-\int_{T_1}^{T_2} r_r(s)ds$ we use expression (6.24) for calculating $P_r(T_1, T_1, T_2)$ and insert it in (6.22); since the differences between r_n and r_r only lay on the drift, the variance of $-\int_{T_1}^{T_2} r_r(s)ds$ is simply

$$V_r(t, t_1, t_2) = \frac{\sigma_r^2}{a_r^2}(1-e^{-a_r(t_2-t_1)})^2 B_{2a_r}(t_1-t)+\frac{\sigma_r^2}{a_r^2}(t_2-t_1+B_r(2a_r, t_2-t_1)-2B_r(a_r, t_2-t_1)).$$

Finally, we have to generate the Brownian increment $W_I(t_2) - W_I(t_1)$, which is normally distributed with mean zero and variance $t_2 - t_1$.
Let us now analyze the covariances between the five processes $x_n(t_2)$, $x_r(t_2)$, $\int_{t_1}^{t_2} x_n(s)ds$, $\int_{t_1}^{t_2} x_r(s)ds$ and $W_I(t_2)-W_I(t_1)$ (we are interested in $(t_1, t_2) = (0, T_1)$ and $(t_1, t_2) = (T_1, T_2)$), knowing the information up to t_1. For this aim, recalling (6.11) and (6.12), we have

$$\begin{aligned}
&Cov\Big[x_k(t_2),\, \textstyle\int_{t_1}^{t_2} x_k(u)du\Big] \\
&= Cov\Big[\sigma_k e^{-a(t_2-t_1)}x(t_1) + \sigma \textstyle\int_{t_1}^{t_2} e^{-a(t_2-u)}dW_k(u),\, \sigma_n \int_{t_1}^{t_2} B_{a_k}(u, t_2)dW_k(u)\Big] \\
&= \sigma_k^2 \textstyle\int_{t_1}^{t_2} \frac{e^{-a_k(t_2-u)}-e^{-2a_k(t_2-u)}}{a_k}du = \frac{\sigma_k^2}{a_k}\Big\{\frac{1-e^{-a_k(t_2-t_1)}}{a_k} - \frac{1-e^{-2a_k(t_2-t_1)}}{2a_k}\Big\} \\
&= \tfrac{\sigma_k^2}{2}B_{a_k}(t_1, t_2)^2
\end{aligned}$$

$$\begin{aligned}
&Cov\Big[x_n(t_2),\, \textstyle\int_{t_1}^{t_2} x_r(u)du\Big] \\
&= \rho_{nr}\sigma_n\sigma_r \textstyle\int_{t_1}^{t_2} e^{-a_n(t_2-u)}\frac{1-e^{-a_r(t_2-u)}}{a_r}du = \frac{\rho_{nr}\sigma_n\sigma_r}{a_r}(B_{a_n}(t_1, t_2) - B_{a_n+a_r}(t_1, t_2))
\end{aligned}$$

$$Cov\Big[x_r(t_2),\, \textstyle\int_{t_1}^{t_2} x_n(u)du\Big] = \frac{\rho_{nr}\sigma_n\sigma_r}{a_n}(B_{a_r}(t_1, t_2) - B_{a_n+a_r}(t_1, t_2))$$

$$\begin{aligned}
&Cov\Big[\textstyle\int_{t_1}^{t_2} x_n(u)du,\, \int_{t_1}^{t_2} x_r(u)du\Big] \\
&= \tfrac{\rho_{nr}\sigma_n\sigma_r}{a_n a_r}(t_2 - t_1 - B_{a_n}(t_1, t_2) - B_{a_r}(t_1, t_2) + B_{a_n+a_r}(t_1, t_2))
\end{aligned}$$

$$Cov\Big[x_n(t_2),\, x_r(t_2)\Big] = \rho_{nr}\sigma_n\sigma_r B_{a_n+a_r}(t_1,t_2)$$

$$Cov[x_k(t_2),\, W_I(t_2) - W_I(t_1)] = \rho_{kI}\sigma_k\sigma_I B_{a_k}(t_1,t_2), \quad k = n, r$$

$$Cov\Big[\textstyle\int_{t_1}^{t_2} x_k(u)du, W_I(t_2) - W_I(t_1)\Big] = \frac{\rho_{kI}\sigma_k\sigma_I}{a_k}(t_2 - t_1 - B_{a_k}(t_1,t_2)).$$

Let us sum up. In order to price the derivative, we must:
-generate m_1 times the five processes $x_n(T_1)$, $x_r(T_1)$, $\int_0^{T_1} x_n(s)ds$, $\int_0^{T_1} x_r(s)ds$ and $W_I(T_1)$ as a five-dimensional normal (with the mean vector and covariance matrix calculated in this section);
-compute $P_n(T_1,T_2)$ and $P_r(T_1,T_2)$ as in (6.19) and (6.24) (m_1 values each, since we use the generations of $x_n(T_1)$ and $x_r(T_1)$);
-from each of the m_1 previous generations, generate m_2 times the five processes $x_n(T_2)$, $x_r(T_2)$, $\int_{T_1}^{T_2} x_n(s)ds$, $\int_{T_1}^{T_2} x_r(s)ds$ and $W_I(T_2) - W_I(T_1)$ as a five-dimensional normal (with the mean vector and covariance matrix calculated in this section) and, from these, $I(T_2)/I(T_1)$ (with formula (6.10)) and the value in T_1 of the payoff in case of extension, i.e.

$$\frac{P^{sp}(0,T_2)P_n(0,T_1)}{P^{sp}(0,T_1)P_n(0,T_2)} e^{-\int_{T_1}^{T_2} r_n(s)ds}\Big(1 + \Big(0, \frac{I(T_2)}{I(T_1)} - 1\Big)^+\Big)$$

(note that we have also considered the Spanish risk of default between T_1 and T_2, according to (6.9));
-make a mean of such discounted payoffs over the m_2 generations (obtaining m_1 values);
-implement the extension condition: the payoff in T_1 of the contract (on each of the m_1 generations) is the minimum between 1 and the mean in the previous step;
-calculate the m_1 prices of the contract discounting from T_1 to 0 the payoff obained in the previous step (and taking into account the Spanish risk of

default in this interval); i.e.: we multiply the T_1-payoff for

$$\frac{P^{sp}(0,T_1)}{P_n(0,T_1)}e^{-\int_0^{T_1} r_n(s)ds};$$

-make a mean over the m_1 obtained result (Monte Carlo).

6.3.2 Under forward measures: Monte Carlo and analytic inflation-indexed caplet formula

In this section we will find an expression for the price of the contract which requires the generation of the processes only at time T_1, moving under two different forward measures and exploiting the explicit formula for inflation-indexed caplets.
First of all, let us analyze the extension condition: the contract is estinguished at time T_1 if (remember (6.9))

$$\begin{aligned}
&\mathbb{1}_{\{\tau>T_1\}} \leq \mathbb{1}_{\{\tau>T_1\}}\tfrac{P^{sp}(0,T_2)P_n(0,T_1)}{P^{sp}(0,T_1)P_n(0,T_2)}E\Big[e^{-\int_{T_1}^{T_2} r_n(s)ds}\Big(1+\Big(\tfrac{I(T_2)}{I(T_1)}-1\Big)^+\Big)|\mathcal{F}_{T_1}\Big]\\
\Leftrightarrow\quad &\mathbb{1}_{\{\tau>T_1\}} \leq \mathbb{1}_{\{\tau>T_1\}}\tfrac{P^{sp}(0,T_2)P_n(0,T_1)}{P^{sp}(0,T_1)P_n(0,T_2)}(P_n(T_1,T_2)+\tfrac{IICplt(T_1,T_1,T_2,\varphi=T_2-T_1,K=1)}{T_2-T_1}).
\end{aligned}$$

So, indicating with E^T the forward measure with numeraire $P_n(\cdot,T)$, the price in 0 of the contract is

$$\begin{aligned}
&E\Big[e^{-\int_0^{T_1} r_n(s)ds}\mathbb{1}_{\{\tau>T_1\}}\mathbb{1}_{\mathbb{1}_{\{\tau>T_1\}}\leq\mathbb{1}_{\{\tau>T_1\}}\frac{P^{sp}(0,T_2)P_n(0,T_1)}{P^{sp}(0,T_1)P_n(0,T_2)}\left(P_n(T_1,T_2)+\frac{IICplt(T_1,T_1,T_2,\varphi=T_2-T_1,K=1)}{T_2-T_1}\right)}\Big]\\
&+E\Big[e^{-\int_0^{T_2} r_n(s)ds}\mathbb{1}_{\{\tau>T_2\}}\mathbb{1}_{\mathbb{1}_{\{\tau>T_1\}}>\mathbb{1}_{\{\tau>T_1\}}\frac{P^{sp}(0,T_2)P_n(0,T_1)}{P^{sp}(0,T_1)P_n(0,T_2)}\left(P_n(T_1,T_2)+\frac{IICplt(T_1,T_1,T_2,\varphi=T_2-T_1,K=1)}{T_2-T_1}\right)}\\
&\qquad\Big(1+\Big(\tfrac{I(T_2)}{I(T_1)}-1\Big)^+\Big)\Big]\\
&=P_n(0,T_1)\tfrac{P^{sp}(0,T_1)}{P_n(0,T_1)}E^{T_1}\Big[\mathbb{1}_{1\leq\frac{P^{sp}(0,T_2)P_n(0,T_1)}{P^{sp}(0,T_1)P_n(0,T_2)}\left(P_n(T_1,T_2)+\frac{IICplt(T_1,T_1,T_2,\varphi=T_2-T_1,K=1)}{T_2-T_1}\right)}\Big]\\
&+P_n(0,T_2)\tfrac{P^{sp}(0,T_2)}{P_n(0,T_2)}E^{T_2}\Big[\mathbb{1}_{1>\frac{P^{sp}(0,T_2)P_n(0,T_1)}{P^{sp}(0,T_1)P_n(0,T_2)}\left(P_n(T_1,T_2)+\frac{IICplt(T_1,T_1,T_2,\varphi=T_2-T_1,K=1)}{T_2-T_1}\right)}\\
&\qquad\qquad E^{T_2}\Big[\Big(1+\Big(\tfrac{I(T_2)}{I(T_1)}-1\Big)^+\Big)|\mathcal{F}_{T_1}\Big]\Big]
\end{aligned}$$

$$
\begin{aligned}
&= P^{sp}(0,T_1)E^{T_1}\left[\mathbb{1}_{1\le\frac{P^{sp}(0,T_2)P_n(0,T_1)}{P^{sp}(0,T_1)P_n(0,T_2)}}\left(P_n(T_1,T_2)+\frac{IICplt(T_1,T_1,T_2,\varphi=T_2-T_1,K=1)}{T_2-T_1}\right)\right]\\
&+P^{sp}(0,T_2)E^{T_2}\left[\mathbb{1}_{1>\frac{P^{sp}(0,T_2)P_n(0,T_1)}{P^{sp}(0,T_1)P_n(0,T_2)}}\left(P_n(T_1,T_2)+\frac{IICplt(T_1,T_1,T_2,\varphi=T_2-T_1,K=1)}{T_2-T_1}\right)\right.\\
&\qquad\left.\left(\frac{IICplt(T_1,T_1,T_2,\varphi=T_2-T_1,K=1)}{(T_2-T_1)P_n(T_1,T_2)}+1\right)\right]
\end{aligned}
\tag{6.25}
$$

where the expression for $IICplt$ is the one in (3.38) with C in (3.21). So the only things we have to generate are $P_n(T_1,T_2)$ and $P_n(T_1,T_2)$ under the forward measures with numeraire $P_n(\cdot,T_1)$ and $P_n(\cdot,T_1)$; for this aim, we can use (6.19) and (6.24) respectively, which only require x_n and x_r, but we must take care that their generations must be made under the two forward measures. From (6.11), under the risk-neutral measure (starting from $t=0$)

$$x_k(T_1)=\sigma\int_0^{T_1}e^{-a(T_1-u)}dW_k(u),\quad k=n,r. \tag{6.26}$$

In order to move under the forward measure with numeraire $P_n(\cdot,T)$, we recall (see Section 5.2) we must add to $x_n(T_1)$ the term

$$-\sigma_n^2\int_0^{T_1}e^{-a_n(T_1-u)}B_n(u,T)\,du=-\frac{\sigma_n^2}{a_n}(B_{a_n}(0,T_1)-e^{-a_n(T-T_1)}B_{2a_n}(0,T_1))$$

which gives

$$
\begin{aligned}
&-\tfrac{\sigma_n^2}{a_n}(B_{a_n}(0,T_1)-B_{2a_n}(0,T_1)) && \text{for } T=T_1\\
&-\tfrac{\sigma_n^2}{a_n}(B_{a_n}(0,T_1)-e^{-a_n(T_2-T_1)}B_{2a_n}(0,T_1)) && \text{for } T=T_2.
\end{aligned}
$$

For $x_r(T_1)$, the additional term is

$$-\sigma_n\sigma_r\rho_{nr}\int_0^{T_1}e^{-a_r(T_1-u)}B_n(u,T)du=-\frac{\sigma_n\sigma_r\rho_{nr}}{a_n}\left(B_{a_r}(T_1,T_2)-\frac{e^{-a_n(T-T_1)}}{a_n+a_r}+\frac{e^{-a_nT-a_rT_1}}{a_n+a_r}\right)$$

which gives

$$
\begin{aligned}
&-\tfrac{\sigma_n\sigma_r\rho_{nr}}{a_n}(B_{a_r}(T_1,T_2)-B_{a_n+a_r}(0,T_1)) && \text{for } T=T_1\\
&-\tfrac{\sigma_n\sigma_r\rho_{nr}}{a_n}(B_{a_r}(T_1,T_2)-e^{-a_n(T_2-T_1)}B_{a_n+a_r}(0,T_1)) && \text{for } T=T_2.
\end{aligned}
$$

Variances and covariances are not affected by the change of measure, so they are the same as in the previous section; we thus know mean and covariance matrix of the multinormal variable we have to generate in order to gain x_n and x_r; from these, we can calculate $P_n(T_1, T_2)$, $P_r(T_1, T_2)$ and the whole expression of the price in (6.25).

6.3.3 Results

In this section we show some examples of results of the two pricing methods. In line with the terms of the real contract we refer to, we set $T_1 = 7$ and $T_2 = 30$.
If we use the data corresponding to 29^{th} April 2011, a possible set of parameters obtained through calibration (with differential evolution) is

$$a_n = 0.0485,\ a_r = 0.1778,\ \sigma_n = 0.0101,\ \sigma_r = -0.0030,\ \sigma_I = -0.0046$$
$$\rho_{nr} = -0.0021,\ \rho_{nI} = 0.0001,\ \rho_{rI} = -0.0054$$

while the values of the bonds we need are

$$P_n(0,7) = 0,803736,\ P_n(0,30) = 0,338797$$
$$P_r(0,7) = 0,940484,\ P_r(0,30) = 0,709818$$

and the Spanish Z-spread is 0.022. With these data, applying the two methods described in the previous section, we see that results agree on a price which is around 33% of the notional. Since both proceedings use the Monte Carlo method, they give a confidence interval as result; for the first method, which uses simulation two times, we calculated the interval according to the outer Monte Carlo.
Some possible results are reported in the next tables. Since the first method requires a double application of Monte Carlo, if we choose a great number of internal and external iterations it turns out to be quite slow; keeping that number not enormous (we mean something like 20000 internal and 50000 external iterations, as in the first following examples) allows to have the result in few minutes. The speed of the second method instead allows to run it with a great number of iterations (like 1000000) obtaining the result immediately.
In the tables we also report the number of extensions occurred among Monte

Carlo simulations. We recall that in the second method the extension condition is checked under two different forward measures, so there are two different number of extensions.

MC iterations (Int,Ext)	99%-confidence interval	"solution"	extensions
(20000,50000)	[3.317550e-001, 3.328019e-001]	3.322785e-001	50000
(20000,50000)	[3.319285e-001, 3.329779e-001]	3.324532e-001	49999
(50000,50000)	[3.318032e-001, 3.328532e-001]	3.323282e-001	50000
(20000,100000)	[3.320010e-001, 3.327388e-001]	3.323699e-001	100000
(20000,100000)	[3.320397e-001, 3.327745e-001]	3.324071e-001	100000
(20000,100000)	[3.320417e-001, 3.327868e-001]	3.324143e-001	99998

Table 6.1: 29-04-2011, Price of the contract: risk neutral method

MC iterations	99%-confidence interval	"solution"	extensions
1000000	[3.322078e-001, 3.327093e-001]	3.324585e-001	(999993, 999981)
1000000	[3.321884e-001, 3.326897e-001]	3.324390e-001	(999991, 999970)
1000000	[3.320696e-001, 3.325705e-001]	3.323201e-001	(999998, 999980)
1000000	[3.322949e-001, 3.327960e-001]	3.325455e-001	(999999, 999980)
1000000	[3.321244e-001, 3.326250e-001]	3.323747e-001	(999995, 999979)

Table 6.2: 29-04-2011, Price of the contract: forward measure method

We can see that the confidence interval of the second method is contained in the ones obtained with the first method, which is exactly what we hoped, since for the first method we chose a lower number of iterations, so the interval should be larger; there just could be some negligible imprecision in some extremes (the supremum of the fourth result with the second method does not stay in some of the intervals obtained with the first method, but this does not affect the validity of the result). As we have already anticipated, the price of the contract is around 33.2% of the notional (and this value turns out to be consistent with market requests). Moreover, we note that the extension event occurs almost surely (the frequency of early ends is less then 1 per 50000), and this result is concordant with market expectations, too. Finally, we observe that the number of extensions under the forward measure with numeraire $P(\cdot, T_1)$ is greater then the one under the forward measure with numeraire $P(\cdot, T_2)$; the difference between the two numbers could appear strange, but we must remember that extensions under different measures can have different weights (under one measure there could be less extensions, but each of them can be more incisive on the result).

If we use the data corresponding to another date, the result can obviously be different, preserving the consistence between the two methods. For example, if we use consider the data corresponding to 22^{nd} February 2011, a possible set of parameters obtained through calibration (with differential evolution) is

$$a_n = 0,0583,\ a_r = 0,1467,\ \sigma_n = 0,0112,\ \sigma_r = 0,0055,\ \sigma_I = 0,0255$$
$$\rho_{nr} = -0,8926,\ \rho_{nI} = -0,6302,\ \rho_{rI} = 0,9124$$

while the values of the bonds we need are

$$P_n(0,7) = 0,803852,\ P_n(0,30) = 0.342517$$
$$P_r(0,7) = 0,935480,\ P_r(0,30) = 0,716144$$

and for the Spanish Z-spread we take 0.022 again.
The tables in the next page report some possible results obtained with these data.

MC iterations (Int,Ext)	99%-confidence interval	"solution"	extensions
(20000,50000)	[3.406044e-001, 3.415387e-001]	3.410716e-001	50000
(20000,50000)	[3.405215e-001, 3.414547e-001]	3.409881e-001	50000
(50000,50000)	[3.409114e-001, 3.418549e-001]	3.413831e-001	49999
(20000,100000)	[3.409090e-001, 3.415738e-001]	3.412414e-001	99999
(20000,100000)	[3.407231e-001, 3.413901e-001]	3.410566e-001	100000

Table 6.3: 22-02-2011, Price of the contract: risk neutral method

MC iterations	99%-confidence interval	"solution"	extensions
1000000	[3.408049e-001, 3.413915e-001]	3.410982e-001	(999995, 999979)
1000000	[3.408841e-001, 3.414703e-001]	3.411772e-001	(999997, 999983)
1000000	[3.408582e-001, 3.414438e-001]	3.411510e-001	(999993, 999985)
1000000	[3.408325e-001, 3.414175e-001]	3.411250e-001	(999996, 999986)
1000000	[3.408260e-001, 3.414117e-001]	3.411189e-001	(999998, 999987)

Table 6.4: 22-02-2011, Price of the contract: forward measure method

We can see that also in this case the two methods give consistent results, and that the price of the contract this time is around 34.1%.

Appendix A

Preliminary results

In this Appendix some important definitions and theoretical results are recalled.

A.1 Martingale measures

Let's recall the following definition:

Definition A.1 (**Exponential martingale**)**.** Given a d-dimensional Brownian Motion $(W_t)_{t\in[0,T]}$ on the probability space $(\Omega, \mathscr{F}, P, (\mathscr{F}_t))$, and given a d-dimensional process $\lambda \in \mathbb{L}^2_{loc}$[1], the *exponential martingale associated to* λ is the process

$$Z_t^\lambda = \exp\Big(- \int_0^t \lambda_s \cdot dW_s - \frac{1}{2} \int_0^t |\lambda_s|^2 ds \Big), \quad t \in [0,T].$$

If we set $X_t = -\int_0^t \lambda_s \cdot dW_s - \frac{1}{2}\int_0^t |\lambda_s|^2 ds$, we have $Z_t^\lambda = e^{X_t}$, so the application of the Itô formula gives

$$dZ_t^\lambda = e^{X_t} dX_t + \frac{1}{2} e^{X_t} d < X >_t = e^{X_t}(-\lambda_t \cdot dW_t - \frac{1}{2}|\lambda_t|^2 dt + \frac{1}{2}|\lambda_t|^2 dt) =$$

$$= -\lambda_t Z_t^\lambda \cdot dW_t.$$

[1] A stochastic process u_t is in $\mathbb{L}^2_{loc}[0,T]$ if it is progressively measurable with respect to the filtration $(\mathscr{F})$ and such that $\int_0^T |u_t|^2 dt < +\infty$ a.s.

Note 1. An exponential martingale is a local martingale, and, since it is positive, it is a supermartingale; so it is a martingale if and only if $E[Z_T^\lambda] = 1$. A sufficient condition for Z^λ to be a martingale is the existence of a constant C such that

$$\int_0^T |\lambda_t|^2 dt < C \ a.s.$$

The following theorems provide for the instruments which are necessary for switching from a probability measure to another, modifying the corresponding Brownian Motions.

Theorem A.1.1 (**Girsanov's theorem**). *Let Z^λ be the exponential martingale associated to the process $\lambda \in \mathbb{L}^2_{loc}$, and let it be a P-martingale*[2]*. Let us define the measure Q*

$$\frac{dQ}{dP} = Z_T^\lambda.$$

So the process

$$W_t^\lambda = W_t + \int_0^t \lambda_s ds, \quad t \in [0, T]$$

is a Brownian Motion on $(\Omega, \mathscr{F}, Q, (\mathscr{F}_t))$.

Theorem A.1.2 (**Change of drift**). *Let Q be a probability measure equivalent to P. Thus*

$$\left.\frac{dQ}{dP}\right|_{\mathscr{F}_t^W} = Z_T^\lambda, \quad dZ_t^\lambda = -Z_t^\lambda \lambda_t \cdot dW_t, \quad \lambda \in \mathbb{L}^2_{loc}.$$

Moreover, the process W^λ defined by

$$dW_t = dW_t^\lambda - \lambda_t dt$$

is a Brownian Motion on $(\Omega, \mathscr{F}, Q, (\mathscr{F}_t))$.

[2] A typical condition which ensures Z^λ is a strict martingale (under the hypothesis $\lambda \in \mathbb{L}^2_{loc}$) is the so-called *Novikov condition*:

$$E\Big[\exp\Big(\frac{1}{2}\int_0^T |\lambda_s|^2 ds\Big)\Big] < \infty$$

.

Note 2. Under the hypothesis of Theorem A.1.2, if X is an Itô process of the form

$$dX_t = \mu_t dt + \sigma_t dW_t$$

under the measure P, its dynamics under the measure Q becomes

$$dX_t = \mu_t dt + \sigma_t (dW_t^{\lambda} - \lambda_t dt) = (\mu_t - \sigma_t \lambda_t) dt + \sigma_t dW_t^{\lambda}.$$

We thus observe that the change of measure modifies only the drift of X, while its diffusion coefficient remains the same.

Definition A.2 (Correlated Brownian Motion). Given a probability space $(\Omega, \mathscr{F}, P)$, a d-dimensional *correlated Brownian Motion* is a process of the form

$$W_t = A\bar{W}_t$$

with $\bar{W}$ standard d-dimensional Brownian Motion and A non singular $d \times d$ matrix such that, setting $\rho = AA^*$, that matrix (which is called *correlation matrix*) has diagonal elements equal to one:

$$\rho^{ii} = \sum_{j=1}^{d} (A^{ij})^2 = 1 \quad \forall i = 1, \dots, d.$$

In this case, each component of W is a one-dimensional Brownian Motion, and

$$d < W^i, W^j >_t = \rho^{ij} dt \quad i, j = 1, \dots, d.$$

Theorem A.1.3 (Change of drift with correlation). *Let Q be a probability measure equivalent to P. Thus*

$$\left.\frac{dQ}{dP}\right|_{\mathscr{F}_t^W} = Z_T, \quad dZ_t = -Z_t \lambda_t \cdot dW_t, \quad \lambda \in \mathbb{L}^2_{loc}.$$

Moreover, the process W^{λ} defined by

$$dW_t = dW_t^{\lambda} - \rho \lambda_t dt$$

is a Brownian Motion on $(\Omega, \mathscr{F}, Q, (\mathscr{F}_t))$ with correlation matrix ρ.

Note 3. Under the hypothesis of Theorem A.1.3, if X is an Itô process of the form

$$dX_t = \mu_t dt + \sigma_t dW_t$$

under the measure P, its dynamics under the measure Q becomes

$$dX_t = \mu_t dt + \sigma_t(dW_t^\lambda - \rho\lambda_t dt) = (\mu_t - \sigma_t\rho\lambda_t)dt + \sigma_t dW_t^\lambda.$$

We thus observe that also in this case the change of measure entails a correction only in the drift of X, and not in the diffusion coefficient.

A.2 Numeraire

Given the probability space $(\Omega, \mathscr{F}, P)$, let us consider a model for the market in which there are N risky assets $(S^1, \dots, S^N)$ and the "bank account" B with dynamics

$$\begin{aligned} dB_t &= r_t B_t dt, \qquad \text{i.e.} \\ B_t &= \exp\{\textstyle\int_0^t r_s ds\}, \end{aligned} \tag{A.1}$$

where r indicates the interest rate and it is a progressively measurable process.

Definition A.3 (Discount factor). The *discount factor* is the stochastic process

$$D(t,T) = \frac{B_t}{B_T} = e^{-\int_t^T r_s ds},$$

which is unknown at time $t < T$ and represents the amount of money to be possessed at time t in order to obtain one unit at time T.

We now recall the following:

Definition A.4 (Discounting). Given an asset S, it discounted price is

$$\tilde{S}_t = \frac{S_t}{B_t} = e^{-\int_0^t r_s ds} S_t.$$

Let us now define a class of processes which possess the fundamental features of prices:

Definition A.5 (**Q-price processes**)**.** Given the probability space $(\Omega, \mathscr{F}, P)$ and the equivalent martingale measure Q, a *Q-price process* is a stochastic process U which is positive and such that the discounted prices $\tilde{U}_t = \frac{U_t}{B_t}$ are Q-martingales $\forall t < T$.

As the name suggests, among these processes there are the ones of the prices of the risky assets $S^i,\ i = 1, \ldots, N$ (for the definition of martingale measures); processes of this type can be chosen as numeraire as an alternative to B:

Definition A.6 (**Equivalent martingale measure with numeraire U**)**.** Let U be a Q-price process. We thus define *equivalent martingale measure with numeraire* U on $(\Omega, \mathscr{F})$ a probability measure Q^U equivalent to P such that the processes of the prices of the assets discounted with respect to U are Q^U-martingales, that is:

$$\begin{aligned} E^{Q^U}\left[\tfrac{B_T}{U_T}|\mathscr{F}_t^W\right] &= \tfrac{B_t}{U_t} \quad \forall t < T \\ E^{Q^U}\left[\tfrac{S_T^i}{U_T}|\mathscr{F}_t^W\right] &= \tfrac{S_t^i}{U_t} \quad \forall i = 1, \ldots, N, \quad \forall t < T. \end{aligned} \tag{A.2}$$

In practice, as we already told, the process U is used as numeraire instead of B. Thus, analogously to Definition A.3, it is possible to define the new discount factor

$$D^U(t, T) = \frac{U_t}{U_T}.$$

With this definition and from (A.2) the following risk-neutral pricing formulas immediately come out:

$$B_t = E^{Q^U}[D^U(t, T)B_T|\mathscr{F}_t^W] \quad t \in [0, T]$$

$$S_t = E^{Q^U}[D^U(t, T)S_T|\mathscr{F}_t^W] \quad t \in [0, T].$$

Next theorems show how we can switch from a numeraire to another and how the dynamics of the assets are modified as a consequence of this change.

Theorem A.2.1. *Let Q be a martingale measure with numeraire B, and let U be a Q-price process. We define the probability measure Q^U on $(\Omega, \mathscr{F})$ as*

$$\frac{dQ^U}{dQ} = \frac{U_T B_0}{B_T U_0}.$$

So, for every $X \in L^1(\Omega, Q)$ the following holds:

$$E^Q[D(t,T)X|\mathscr{F}_t^W] = E^{Q^U}[D^U(t,T)X|\mathscr{F}_t^W] \quad t \in [0,T]. \tag{A.3}$$

Thus, Q^U is an equivalent martingale measure with numeraire U, and the Q-risk-neutral prices of a European derivative X are equivalently given by the first or the second member of (A.3).

Corollary A.2.2. *If U and V are Q-price processes, then*

$$\left.\frac{dQ^V}{dQ^U}\right|_{\mathscr{F}_t^W} = \frac{V_t U_0}{U_t V_0}. \tag{A.4}$$

It can be useful to know how to deal with the change from a numeraire to another in the case they are Itô processes. For this purpose, we present the following lemma which calculates the quotient between two Itô processes: since for the change of numeraire only the diffusion coefficient is necessary, we focus on this, and use the dots $(\ldots)$ in place of the drifts.

Lemma A.2.3. *Let W be a d-dimensional correlated Brownian Motion, and let σ^U and σ^V be two d-dimensional processes in $\mathbb{L}^2_{loc}$; thus, given the Itô processes U e V*

$$dU_t = (\ldots)dt + \sigma^U \cdot dW_t \tag{A.5}$$

$$dV_t = (\ldots)dt + \sigma^V \cdot dW_t, \tag{A.6}$$

$\frac{V}{U}$ turns out to be an Itô process of the form

$$d\frac{V_t}{U_t} = (\ldots)dt + \frac{V_t}{U_t}\Big(\frac{\sigma_t^V}{V_t} - \frac{\sigma_t^U}{U_t}\Big) \cdot dW_t.$$

Proof. For Itô's formula

$$d\frac{V_t}{U_t} = \frac{1}{U_t}dV_t - \frac{V_t}{U_t^2}dU_t + \frac{1}{2}\frac{2}{U_t^3}V_t d\langle U,U\rangle_t - 2\frac{1}{2}\frac{1}{U_t^2}d\langle U,V\rangle_t$$

and the last two addends only give drift contribution. $\square$

The following theorem sums up the rule for the change of numeraire we were looking for:

Theorem A.2.4 (**Change of numeraire**). *Let U and V be two Itô processes with the dynamics (A.5) and (A.6), and let ρ be the correlation matrix of W. Thus*

$$dW_t^U = dW_t^V + \rho\Big(\frac{\sigma_t^V}{V_t} - \frac{\sigma_t^U}{U_t}\Big)dt.$$

Appendix B

Stochastic differential equations

A *stochastic differential equation* (SDE) is an equation of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{B.1}$$

with $b : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ and $\sigma : [0,T]\times\mathbb{R}^N \to \mathbb{R}^{N\times d}$ deterministic functions named drift and diffusion coefficient respectively, and with W d-dimensional Brownian Motion, with d≤N, on the space $(\Omega, \mathscr{F}, P, (\mathscr{F}_t))$.

B.1 Linear SDE

A particular type of SDE are the so called *linear* ones, in which the coefficients of (B.1) are linear functions of X_t; it means they are the ones of the form

$$dX_t = (b(t) + B(t))dt + (\sigma(t) + \Sigma(t)X_t)dW_t. \tag{B.2}$$

Among these, the most common ones are the ones in which Σ=0, that is of the form

$$dX_t = (b(t) + B(t))dt + \sigma(t)dW_t, \tag{B.3}$$

coupled with an initial condition

$$X_{t_0} = x, \tag{B.4}$$

with $x \in \mathbb{R}^N$, b, B and $\sigma \in L^\infty_{loc}[t_0, +\infty[$ and with values in $\mathbb{R}^N$, $\mathbb{R}^{N\times N}$ and $\mathbb{R}^{N\times d}$ respectively.
For this kind of SDEs the form of the solution is explicitly known:

Theorem B.1.1. *The solution of the SDE (B.3) with initial condition (B.4) is of the form*

$$X_t = \Phi_{t_0}(t)\Big(x + \int_{t_0}^t \Phi_{t_0}^{-1}(s)b(s)ds + \int_{t_0}^t \Phi_{t_0}^{-1}(s)\sigma(s)dW_s\Big), \tag{B.5}$$

where Φ_{t_0} is the solution of the Cauchy problem

$$\begin{cases} \Phi'(t) = B(t)\phi(t) \\ \Phi(t_0) = I_N \end{cases}$$

with I_N N-dimensional identity matrix.

If B is constant (independent from time), the solution of the Cauchy problem turns out to be

$$\Phi_{t_0}(t) := e^{B\,(t-t_0)}, \tag{B.6}$$

taking in mind that, by definition, the exponential of a matrix $A \in \mathbb{R}^{N\times N}$ is

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \quad t \in \mathbb{R}.$$

Thus, if B is constant (B.5) becomes

$$X_t = e^{B\,(t-t_0)}x + \int_{t_0}^t e^{-B\,(s-t)}b(s)ds + \int_{t_0}^t e^{-B\,(s-t)}\sigma(s)dW_s. \tag{B.7}$$

B.2 SDE and PDE

Definition B.1 (Characteristic operator). Given the SDE (B.1), the *characteristic operator associated with* X is the operator L defined by

$$L_t f(x) = \frac{1}{2}\sum_{i\,j=1}^{N} C_{ij}(t,x)\partial_{x_i x_j} f(x) + \sum_{i=1}^{N} b_i(t,x)\partial_{x_i} f(x), \tag{B.8}$$

where $C = \sigma\sigma^*$.

The following theorem establishes an important link between SDEs and partial differential equations:

Theorem B.2.1 (**Feynman-Kač formula).** *Let* S_T *be the strip* $]0, T[\times \mathbb{R}^N$, *and let* $a \in \mathscr{C}(S_T)$, $a \geqslant a_0$, $a_0 \in \mathbb{R}$, b, C *and* $f \in L^\infty(\overline{S_T}) \cap \mathscr{C}^\alpha(S_T)$. *Moreover, let* $u \in \mathscr{C}^2(S_T) \cap \mathscr{C}(\overline{S_T})$ *be the solution to the Cauchy problem*

$$\begin{cases} \mathscr{A}u - au + \partial_t u = f & \text{in } S_T \\ u(T, \cdot) = \phi \end{cases}$$

with $\mathscr{A}$ *characteristic operator associated to (B.1). Finally, let us assume there exist two positive constants* M *and* β *such that*

$$|u(t,x)| + |f(t,x)| \leq Me^{\beta|x|^2}, \quad (t,x) \in \overline{S_T}.$$

Thus

$$u(t,x) = E\Big[e^{-\int_t^T a(s,X_s)ds}\phi(X_T) - \int_t^T e^{-\int_t^s a(r,X_r)dr} f(s,X_s)ds\Big], \qquad \text{(B.9)}$$

where $X = X^{t,x}$ *is the solution of the SDE (B.1) with initial value* x *at the instant* t.

Note that if $f = 0$ the solution (B.9) reduces to

$$u(x,t) = E\Big[e^{-\int_t^T a(s,X_s)ds}\phi(X_T)\Big].$$

Appendix C

Normal distribution

Normal distributions frequently recur when dealing with the Jarrow-Yildirim model; for this reason, it seems useful to point out some results about them.

C.1 Integral of a normal process

In this section we prove that the integral of a process with Hull & White dynamics is still normal.
Let X_t be a stochastic process of the form

$$dX_t = (b(t) + BX_t)dt + \sigma(t)dW_t, \qquad X_{t_0} = x.$$

According to (B.7), this linear EDS has solution

$$X_t = e^{B(t-t_0)}x + \int_{t_0}^{t} e^{-B(s-t)}b(s)ds + \int_{t_0}^{t} e^{-B(s-t)}\sigma(s)dW_s = \int_{t_0}^{t} f(s)ds + \int_{t_0}^{t} g(s)dW_s$$

with f and g deterministic functions, which implies that both X_t and $Y_t := \int_{t_0}^{t} g(s)dW_s$ are stochastic processes (in t) with normal distribution. By Itô formula $d(tY_t) = tdY_t + Y_t dt$, which can be written in the integral form

$$tY_t = t_0 Y_{t_0} + \int_{t_0}^{t} sg(s)dW_s + \int_{t_0}^{t} Y_s ds \quad \Rightarrow \quad \int_{t_0}^{t} Y_s ds = -t_0 Y_{t_0} + \int_{t_0}^{t} (t-s)g(s)dW_s$$

which implies that $\int_{t_0}^t Y_s ds$ is normally distributed. Finally

$$\int_{t_0}^t X_s ds = \int_{t_0}^t \Big(\int_{t_0}^s f(x)dx \Big) ds + \int_{t_0}^t Y_s ds$$

which thus has normal distribution.

C.2 Lognormal distribution

A random variable X is said to be lognormally distributed if it is of the form $X = e^Z$ with $Z \sim \mathcal{N}(\mu, \sigma^2)$; in this case, we can split Z in $\mu + \sigma U$ where $U \sim \mathcal{N}(0,1)$, so that

$$E[X] = E[e^Z] = E[e^{\mu+\sigma U}] = e^{\mu} E[e^{\sigma U}]$$

with

$$E[e^{\sigma U}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\sigma x - \frac{x^2}{2}} dx = e^{\frac{\sigma^2}{2}}$$

and finally obtain

$$E[X] = e^{\mu + \frac{\sigma^2}{2}}. \tag{C.1}$$

For completeness let us calculate the variance too:

$$E[X^2] = E[e^{2Z}] = E[e^{2\mu + 2\sigma U}] = e^{2\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{2\sigma x - \frac{x^2}{2}} dx = e^{2\mu + 2\sigma^2}$$

so that

$$\mathrm{var}[X] = E[X^2] - E[X]^2 = e^{2(\mu + \frac{\sigma^2}{2})} - e^{2\mu + 2\sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1). \tag{C.2}$$

C.3 Expectation formula

First of all, we recall the form of the density of a multinormal distribution:

Definition C.1. A random variable $X : \Omega \to \mathbb{R}^N$ has multinormal distribution with mean $\mu \in \mathbb{R}^N$ and covariance matrix $C \in \mathbb{R}^{N \times N}$ symmetric and positive definite if its density has the form

$$f_{\mu,C}(x) = \frac{1}{\sqrt{(2\pi)^N \det C}} \exp\Big(-\frac{1}{2} \langle C^{-1}(x-\mu), (x-\mu) \rangle \Big), \quad x \in \mathbb{R}^N. \tag{C.3}$$

In particular, for $N = 2$, (C.3) becomes

$$f_{\mu,C}(x) = \frac{1}{2\pi\sqrt{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2}} \exp\Big(-\frac{\sigma_{22}^2(x_1-\mu_1)^2 - 2\sigma_{12}^2(x_1-\mu_1)(x_2-\mu_2) + \sigma_{11}^2(x_2-\mu_2)^2}{2(\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2)}\Big)$$

for $x = (x_1, x_2) \in \mathbb{R}^2$, and if $\mu = 0$ it reduces to

$$f_{(0,0),C}(x_1, x_2) = \frac{1}{2\pi\sqrt{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2}} \exp\Big(-\frac{\sigma_{22}^2x_1^2 - 2\sigma_{12}^2x_1x_2 + \sigma_{11}^2x_2^2}{2(\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2)}\Big).$$

In finance, when pricing derivatives it is frequent to meet expectations of the form

$$E[e^X(p\,e^Z - K)^+] \tag{C.4}$$

where $p \in \mathbb{R}$ and (X, Z) is a bidimensional random variable with mean and covariance matrix

$$\mu = (\mu_X, \mu_Z), \qquad C = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12} \\ \sigma_{12} & \sigma_{22}^2 \end{pmatrix}.$$

In order to calculate (C.4) it is useful to re-express it in the form

$$E[e^{X-\mu_X}e^{\mu_X}(p\,e^{Z-\mu_Z}e^{\mu_Z} - K)^+] = e^{\mu_X+\mu_Z}p\,E[e^{X-\mu_X}(p\,e^{Z-\mu_Z} - \tfrac{K}{p}e^{-\mu_Z})^+]$$
$$= e^{\mu_X+\mu_Z}p\,E[e^{X_{bis}}(p\,e^{Z_{bis}} - K_{bis})^+]$$

where $X_{bis} = X - \mu_X$, $Z_{bis} = Z - \mu_Z$ and $K_{bis} = \frac{K}{p}e^{-\mu_Z}$; in this way, (X_{bis}, Z_{bis}) has a bidimensional normal distribution with mean $\mu = (0, 0)$ and covariance matrix still C, so that (C.4) can be calculated as

$$e^{\mu_X+\mu_Z}p\int_{-\infty}^{+\infty}\int_{\log K_{bis}}^{+\infty} e^x(e^z - K_{bis})f_{(0,0),C}(x,z)\,dx\,dz$$
$$= e^{\mu_X+\mu_Z}p\int_{\log K_{bis}}^{+\infty}(e^z - K_{bis})\Big(\int_{-\infty}^{+\infty} e^x f_{(0,0),C}(x,z)\,dx\Big)dz.$$

Let us calculate the internal integral. For brevity we set $S = \sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2$:

$$\int_{-\infty}^{+\infty} e^x f_{(0,0),C}(x,z)\,dx =$$
$$\frac{1}{2\pi\sqrt{S}}\int_{-\infty}^{+\infty} e^{x-\frac{\sigma_{22}^2x^2 - 2\sigma_{12}^2xz + \sigma_{11}^2z^2}{2S}}dx = \frac{1}{2\pi\sqrt{S}}e^{-\frac{\sigma_{11}^2z^2}{2S}}\int_{-\infty}^{+\infty} e^{-\frac{\sigma_{22}^2(x^2+2x(-\sigma_{11}^2+\frac{\sigma_{12}^2}{\sigma_{22}^2}-\frac{\sigma_{12}}{\sigma_{22}^2}z))}{2S}}dx$$
$$= \frac{1}{\sqrt{2\pi\sigma_{22}^2}}e^{-\frac{\sigma_{11}^2z^2}{2S}}e^{\frac{(S+\sigma_{12}z)^2}{2S}} = \frac{1}{\sqrt{2\pi\sigma_{22}^2}}e^{-\frac{(z-\sigma_{12})^2}{2\sigma_{22}^2}+\frac{\sigma_{11}^2}{2}}.$$

So the double integral is

$$\frac{1}{\sqrt{2\pi\sigma_{22}^2}}\int_{\log K_{bis}}^{+\infty}(e^z - K_{bis})e^{-\frac{(z-\sigma_{12})^2}{2\sigma_{22}^2}+\frac{\sigma_{11}^2}{2}}dz =$$
$$\frac{1}{\sqrt{2\pi\sigma_{22}^2}}e^{\frac{\sigma_{11}^2}{2}-\frac{\sigma_{12}^2}{2\sigma_{22}^2}}\Big(e^{\frac{(\frac{\sigma_{12}}{\sigma_{22}}+\sigma_{22})^2}{2}}\int_{\log K_{bis}}^{+\infty}e^{-\frac{(z-\sigma_{12}-\sigma_{22}^2)^2}{2\sigma_{22}^2}}dz - K_{bis}e^{\frac{\sigma_{12}^2}{2\sigma_{22}^2}}\int_{\log K_{bis}}^{+\infty}e^{-\frac{(z-\sigma_{12})^2}{2\sigma_{22}^2}}dz\Big).$$

Let us now make the changes of variable $\xi = \frac{z-\sigma_{12}-\sigma_{22}^2}{\sigma_{22}}$ in the first integral and $\nu = \frac{z-\sigma_{12}}{\sigma_{22}}$ in the second one:

$$e^{\frac{\sigma_{11}^2}{2}+\frac{\sigma_{22}^2}{2}+\sigma_{12}}\frac{1}{\sqrt{2\pi}}\int_{\frac{\log K_{bis}-\sigma_{12}-\sigma_{22}^2}{\sigma_{22}}}^{+\infty}e^{-\frac{\xi^2}{2}}d\xi - K_{bis}e^{\frac{\sigma_{11}^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{\frac{\log K_{bis}-\sigma_{12}}{\sigma_{22}}}^{+\infty}e^{-\frac{\nu^2}{2}}d\nu$$
$$= e^{\frac{\sigma_{11}^2}{2}+\frac{\sigma_{22}^2}{2}+\sigma_{12}}\Phi\Big(-\frac{\log K_{bis}-\sigma_{12}-\sigma_{22}^2}{\sigma_{22}}\Big) - K_{bis}e^{\frac{\sigma_{11}^2}{2}}\Phi\Big(-\frac{\log K_{bis}-\sigma_{12}}{\sigma_{22}}\Big)$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{x^2}{2}}dx$ is the standard normal distribution function. We have thus obtained the result:

$$E[e^X(p\,e^Z - K)^+] =$$
$$e^{\mu_X+\mu_Z+\frac{\sigma_{11}^2}{2}}p\Big(e^{\frac{\sigma_{22}^2}{2}+\sigma_{12}}\Phi\Big(\frac{\mu_Z+\log p-\log K+\sigma_{12}+\sigma_{22}^2}{\sigma_{22}}\Big) - \frac{Ke^{-\mu_Z}}{p}\Phi\Big(\frac{\mu_Z+\log p-\log K+\sigma_{12}}{\sigma_{22}}\Big)\Big). \tag{C.5}$$

Sometimes instead of (C.4) we need to calculate

$$E[e^X(K - p\,e^Z)^+].$$

The proceeding is exactly the same, and allows us to obtain the following formula which generalizes (C.5):

$$E[e^X(w(p\,e^Z - K))^+] =$$
$$we^{\mu_X+\mu_Z+\frac{\sigma_{11}^2}{2}}p\Big(e^{\frac{\sigma_{22}^2}{2}+\sigma_{12}}\Phi\Big(w\frac{\mu_Z+\log p-\log K+\sigma_{12}+\sigma_{22}^2}{\sigma_{22}}\Big) - \frac{Ke^{-\mu_Z}}{p}\Phi\Big(w\frac{\mu_Z+\log p-\log K+\sigma_{12}}{\sigma_{22}}\Big)\Big) \tag{C.6}$$

where w can be ± 1.

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