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Lie bialgebras and Etingof-Kazhdan quantization

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Relatore:
Chiar.mo Prof.
Luca Moci

Correlatore:
Chiar.mo Prof.
Andrea Appel

Presentata da:
Andrea Rivezzi

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Preface

In this thesis we present the solution given by Pavel Etingof and David Kazhdan in [EK96] and [EK98a] to the problem of quantization of Lie bialgebras, stated by Vladimir Drinfeld in [Dri92].

Historical introduction

Before 1980, Lie algebras were considered *rigid objects*, where the term rigid object was understood to mean that, given a Lie algebra $\mathfrak{g}$, its universal enveloping algebra $\mathcal{U}\mathfrak{g}$ cannot be deformed as algebras.

However, around 1982, motivated from physics, some mathematicians of the Leningrad school of mathematics discovered a deformation of the universal enveloping algebra $\mathcal{U}\mathfrak{sl}_2$ of the Lie algebra $\mathfrak{sl}_2$. This was a very unexpected result, because it went against the previous thoughts about Lie algebras, and also because the existence of this deformation was motivated by a further algebraic structure, that is the structure of Lie bialgebra.

In the articles [Dri85] and [Jim85], in 1985, Vladimir Drinfeld and Michio Jimbo generalized the deformation of $\mathfrak{sl}_2$ to the case of symmetrizable Kac–Moody algebras. This can be viewed as the birth of the concept of quantum group, that is a Hopf algebra that deforms the universal enveloping algebra of a Lie algebra. Hopf algebras turned out to be the best algebraic structure to formalize the concept of deformation of a Lie bialgebra; this connection was well explained in [Dri86], which is considered the manifesto of quantum groups.

Later, in [Dri92], Drinfeld published some unsolved problems in quantum group theory; in one of these problems he wondered if it was possible, given any Lie bialgebra, to find a Hopf algebra that would quantize it. The (positive) answer came some years later, with the series of articles of Etingof and Kazhdan [EK96], [EK98a], [EK98b], [EK00a], [EK00b] and [EK08]. Moreover, they found a quantization technique in a functorial way, further proving to be equivalent in the case of symmetrizable Kac–Moody algebras. Therefore, Drinfeld–Jimbo quantum groups became a special case of a bigger and more general construction.

Overview

Quantum $\mathfrak{sl}_2$

The first example of a quantization of a Lie bialgebra was made on the Lie algebra $\mathfrak{sl}_2$, and appeared in the context of the theory of quantum integrable systems. At first, this structure (which we call a quantum group) was not related to the concept of a Lie bialgebra, but manifested in the context of topological Hopf algebras. Indeed, this object appeared on the form of a sort of deformation of the universal enveloping algebra of $\mathfrak{sl}_2$.

More precisely, it was defined as the topologically free $\mathbb{C}[[\hbar]]$–algebra $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ generated from three elements X, Y, H and with relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}.$$

It is easy to see that, at $\hbar = 0$, $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ gives back the universal enveloping algebra $\mathcal{U}\mathfrak{sl}_2$, that is, there is an obvious isomorphism

$$\mathcal{U}_\hbar(\mathfrak{sl}_2)/(\hbar \cdot \mathcal{U}_\hbar(\mathfrak{sl}_2)) \simeq \mathcal{U}\mathfrak{sl}_2.$$

The interesting fact is that $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ is a deformation of $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ as a Hopf algebra rather than just an algebra. In particular, $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ is endowed with a coproduct

$$\Delta_\hbar : \mathcal{U}_\hbar(\mathfrak{sl}_2) \to \mathcal{U}_\hbar(\mathfrak{sl}_2) \otimes \mathcal{U}_\hbar(\mathfrak{sl}_2)$$

given on the generators by the assignments

$$\begin{aligned} X &\mapsto X \otimes e^{\hbar H} + 1 \otimes X \\ Y &\mapsto Y \otimes 1 + e^{-\hbar H} \otimes Y \\ H &\mapsto H \otimes 1 + 1 \otimes H. \end{aligned}$$

Moreover, this Hopf algebra structure is not cocommutative, but quasi–triangular. That is, $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ possesses an R–matrix

$$R_\hbar = e^{\frac{1}{2}\hbar(H \otimes H)} \sum_{n \geq 0} R_\hbar(n)(X^n \otimes Y^n)$$

satisfying the relations

$$R\Delta = \Delta^{op}R, \qquad (\Delta\otimes \mathsf{id})(R) = R_{13}R_{23}, \qquad (\mathsf{id}\otimes\Delta)(R) = R_{13}R_{12}.$$

As before, we obtain a Hopf algebra isomorphism

$$\mathcal{U}_\hbar(\mathfrak{sl}_2)/(\hbar\cdot\mathcal{U}_\hbar(\mathfrak{sl}_2)) \simeq \mathcal{U}\mathfrak{sl}_2,$$

and then we may interpret $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ as a deformation of the universal enveloping algebra of $\mathfrak{sl}_2$. Moreover, as opposed to $\mathcal{U}\mathfrak{sl}_2$ (that has a canonical cocommutative Hopf algebra structure), the Hopf algebra $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ is clearly non cocommutative. For any $x \in \mathfrak{sl}_2$, choose $\tilde{x} \in \mathcal{U}_\hbar(\mathfrak{sl}_2)$ such that $\tilde{x} = x \ mod\ \hbar$. Then we set

$$\delta(x) = \frac{\Delta(\tilde{x}) - \Delta^{op}(\tilde{x})}{\hbar}\ mod\ \hbar.$$

On the generators of $\mathfrak{sl}_2$, we get

$$\delta(e) = e\wedge h, \qquad \delta(f) = f\wedge h, \qquad \delta(h) = 0.$$

One then check that $\delta : \mathfrak{sl}_2 \to \mathfrak{sl}_2\wedge\mathfrak{sl}_2$ is a Lie cobracket and endows $\mathfrak{sl}_2$ with a Lie bialgebra structure. Therefore, $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ is a Hopf algebra that deforms the universal enveloping algebra of $\mathfrak{sl}_2$, and at the same describe the Lie bialgebra structure of $\mathfrak{sl}_2$. More concisely, we say that $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ is a quantization of $\mathfrak{sl}_2$.

Quantization of Lie bialgebras

Inspired by the case of $\mathfrak{sl}_2$, many mathematicians started to study how to generalize the construction of the quantum group of $\mathfrak{sl}_2$ to any Lie algebra. In 1985, Drinfeld and Jimbo defined a class of quantized universal enveloping algebras that generalizes the case of $\mathfrak{sl}_2$ to the family of symmetrizable Kac–Moody algebras. In full generality, as suggested by the case of $\mathfrak{sl}_2$, the right point of view is to focus on the Lie algebra structure. If H is a topological Hopf algebra and $(\mathfrak{g}, [\cdot,\cdot], \delta)$ is a Lie bialgebra, we say that H is a quantization of $\mathfrak{g}$ if

$$H/(\hbar\cdot H) \simeq \mathcal{U}\mathfrak{g}$$

as Hopf algebras, and if

$$\delta(x) = \frac{\Delta(\tilde{x}) - \Delta^{op}(\tilde{x})}{\hbar}\ mod\ \hbar,$$

where $\tilde{x}$ is a lifting of x in H. The Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \delta)$ is said to be the quasi–classical limit of H, while the Hopf algebra H is called a quantized universal enveloping algebra (QUE for short). As the reader can see in the fourth chapter, this gives rise to a functor *semi–classical limit*

$$SC : QUE \to LBA$$

that, to a quantized universal enveloping algebra H, assigns the Lie bialgebra $(Prim(H/(\hbar \cdot H)), [\cdot, \cdot] \ mod \ \hbar, \frac{\Delta - \Delta^{op}}{\hbar} \ mod \ \hbar)$, where $Prim(H/(\hbar \cdot H))$ denotes the set of primitive elements of $H/(\hbar \cdot H)$.

The question that mathematicians asked is: can we go back? In other words, can we construct a functor

$$Q : LBA \to QUE$$

that, given a Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot], \delta)$, gives us back a quantized enveloping algebra that deforms $\mathfrak{g}$? This is what we refer to as the quantization problem, which remained open until the second half of the 90's , and was solved by Etingof and Kazhdan.

The Etingof–Kazhdan strategy

The solution of the quantization problem given by Etingof and Kazhdan consists of several steps and is based on the resolution of various subproblems.

The basic idea of the Etingof–Kazhdan quantization is, given a Lie bialgebra $\mathfrak{g}$, to try to find a deformation of the category $\mathsf{Mod}(\mathcal{U}\mathfrak{g})$ instead of deforming directly the Hopf algebra $\mathcal{U}\mathfrak{g}$. In fact, these two different concepts of deformation are connected to each other by the Tannaka–Krein duality. In the case of bialgebras, we may summarize this theory by saying that, if A is an algebra, then to have a bialgebra structure on A it is necessary and sufficient to construct a tensor structure on $\mathsf{Mod}(A)$, together with a tensor structure on the forgetful functor

$$F : \mathsf{Mod}(A) \to \mathsf{Vect}$$

which assigns to any A–module its underlying vector space and to any morphism of A–modules its underlying linear map. Indeed, there is an isomorphism of algebras $\mathsf{End}(F) \simeq A$; while the tensor structure on F induces a bialgebra structure on $\mathsf{End}(F)$, and thus on A. Naturally, different tensor structures on $\mathsf{Mod}(A)$ and on F will give rise to different bialgebra structures on A.

For example, in the case of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$, we have that it has a standard bialgebra structure (that is also Hopf), and this can be interpreted by the Tannaka–Krein duality by considering the trivial tensor structure on $\mathsf{Mod}(\mathcal{U}\mathfrak{g})$, together with the trivial tensor structure on F.

Therefore, the first step of the Etingof–Kazhdan quantization is to find a non trivial tensor structure on $\mathsf{Mod}(\mathcal{U}\mathfrak{g})$, together with a non trivial tensor structure on the forgetful functor F. If we do, then we can apply the Tannakian reconstruction, obtaining a bialgebra that is a candidate to be a quantization of $\mathfrak{g}$.

The Drinfeld category

Given an algebra A, we have that finding a non trivial tensor structure on $\mathsf{Mod}(A)$ is not straightforward. However, the following theorem, due to Drinfeld, gives us a good start point.

Theorem (Drinfeld): Let $\mathfrak{g}$ be a Lie bialgebra and let $t \in \mathfrak{g}\otimes\mathfrak{g}$ be an invariant and symmetric tensor. Then there exists an associator $\Phi_{KZ} \in \mathcal{U}\mathfrak{g}^{\otimes 3}[[\hbar]]$. Moreover, the element $R = e^{\hbar t/2}$ defines a quasi–triangular structure on the (topological) quasi–bialgebra $(\mathcal{U}\mathfrak{g}[[\hbar]], \otimes, \mathbb{C}[[\hbar]], \Phi_{KZ})$.

In the case of simple Lie algebras, such an element it is given by the canonical element corresponding to the Killing form. For an arbitrary Lie bialgebra $\mathfrak{g}$, such element does not exists. However, we can embed $\mathfrak{g}$ into another Lie bialgebra, the Drinfeld double $\mathfrak{D}\mathfrak{g}$, which, as a vector space is $\mathfrak{D}\mathfrak{g} = \mathfrak{g} \otimes \mathfrak{g}^*$, and which is canonically endowed with an invariant and symmetric tensor. This fact, together with the Drinfeld theorem, suggests that a construction of a quantization of a Lie bialgebra $\mathfrak{g}$ should pass first through the quantization of its Drinfeld double $\mathfrak{D}\mathfrak{g}$. In fact, Etingof and Kazhdan first construct a quantization of $\mathfrak{D}\mathfrak{g}$, and then they isolate a Hopf subalgebra that quantizes $\mathfrak{g}$.

We first examine the case where $\mathfrak{g}$ is of finite dimension. Given a finite–dimensional Lie bialgebra $\mathfrak{g}$, we consider the category $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$, whose objects are the topologically free $\mathfrak{D}\mathfrak{g}$–modules, and whose class of morphisms is $\mathsf{Hom}_{\mathfrak{D}\mathfrak{g}[[\hbar]]}(V, W)$. The Drinfeld theorem applied to $\mathfrak{D}\mathfrak{g}$ defines a braided tensor structure on $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$; we refer to this braided tensor category as the Drinfeld category associated to $\mathfrak{g}$. In order to find a quantization of $\mathfrak{D}\mathfrak{g}$, we shall construct a tensor structure on the forgetful functor F.

The fiber functor and universal Verma modules

A crucial role in the Etingof–Kazhdan quantization is played by universal Verma modules, that are defined by

$$M_\pm := \mathsf{Ind}_{\mathfrak{g}_\pm[[\hbar]]}^{\mathfrak{D}\mathfrak{g}[[\hbar]]} c_\pm = \mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]] \otimes_{\mathcal{U}\mathfrak{g}_\pm[[\hbar]]} c_\pm,$$

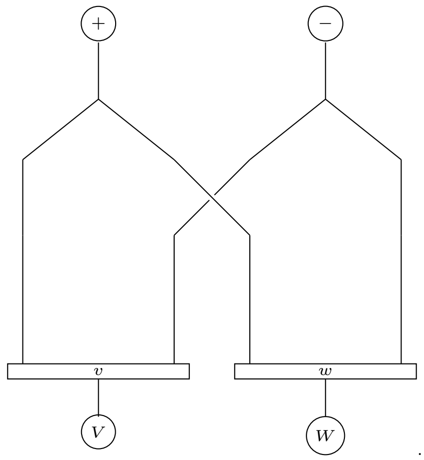
where $c_\pm$ is the trivial $\mathfrak{g}_\pm[[\hbar]]$–module of rank 1. We have that $M_\pm$ are coalgebra objects in $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$, and also we can represent the forgetful functor F with

$$F(V) = \mathsf{Hom}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-, V) \simeq V,$$

where the isomorphism sends f into $f(1_+ \otimes 1_-)$. Since $M_\pm$ are coalgebras, then so is the polarized representative $M_+ \otimes M_-$. Its non–cocommutative coalgebra structure induces a tensor structure on the forgetful functor F, that is a map

$$\begin{aligned} J_{V,W} : F(V) \otimes F(W) &\to F(V \otimes W) \\ v \otimes w &\mapsto J_{V,W}(v \otimes w), \end{aligned}$$

and is described by the picture



By Tannakian reconstruction, we obtain a Hopf algebra $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$, whose underlying vector space is $\mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]]$, and whose coproduct is given by

$$\Delta_\hbar := J^{-1}\Delta_0 J.$$

We have that $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ is a quantization of $\mathfrak{D}\mathfrak{g}$, and furthermore it carries a quasi–triangular structure, with R-matrix $R_\hbar := (J^{op})^{-1} e^{\hbar \cdot t/2} J$.

The solution of the problem

After defining a quantization of the Drinfeld double of a Lie bialgebra $\mathfrak{g}$, we want to define a Hopf subalgebra of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ that quantizes $\mathfrak{g}$. Unfortunately, the choice $\mathcal{U}\mathfrak{g}[[\hbar]] \subset \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^*[[\hbar]] \simeq \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ does not works, because in general it is not closed with respect to the coproduct; this fact makes the problem more complicated.

The idea of Etingof and Kazhdan to solve this problem is to use the isomorphism

$$\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g}) \simeq \mathsf{End}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-)$$

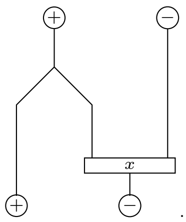
to define a quantization of $\mathfrak{g}_+ = \mathfrak{g}$ and of $\mathfrak{g}_- = \mathfrak{g}^*$, using the Verma modules $M_\pm$. Indeed, we consider the spaces

$$F(M_\pm) = \mathsf{Hom}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-, M_\pm),$$

and prove that they are naturally endowed with a Hopf algebra structure quantizing the Lie bialgebras $\mathfrak{g}_\mp$. To do this, we prove that $F(M_\pm)$ corresponds to Hopf subalgebras of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. In the case of $F(M_-)$, we define the embedding

$$m_+(x) : F(M_-) \to \mathsf{End}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-) = \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$$

that sends $x \in F(M_-)$ into the element of $\mathsf{End}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-)$ represented by the picture



Therefore, we define $\mathcal{U}_\hbar(\mathfrak{g})$ as the image of $F(M_-)$ through m_+. As a vector space, we have that $F(M_-) \simeq \mathcal{U}\mathfrak{g}[[\hbar]]$; furthermore, we can define a product $\mu_\hbar$ and a coproduct $\Delta_\hbar$ on $F(M_-)$ that respectively give the product and the coproduct of $\mathcal{U}\mathfrak{g}[[\hbar]]$ modulo $\hbar$. In order to prove that $F(M_-)$ is indeed a Hopf algebra, we first observe that, given $x, y \in F(M_-)$, we have

$$m_+(x) \circ m_+(y) = m_+(z)$$

for a certain $z \in F(M_-)$. This implies that $\mathcal{U}_\hbar(\mathfrak{g})$ is a subalgebra of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. Later, through a pictorial technique, we show that in particular $\mathcal{U}_\hbar(\mathfrak{g})$ is a Hopf subalgebra of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. Finally, we have that $F(M_-)$ is a Hopf algebra, and in particular a quantized universal enveloping algebra, whose quasi–classical limit is $\mathfrak{g}$.

Functoriality of the quantization

The previous construction is not functorial, since it involves the Drinfeld double of the Lie bialgebra. Furthermore, this technique is not valid in the infinite–dimensional setting. However, Etingof and Kazhdan solve both problems by considering a *new* setting, in that the Verma module M_+ does not appear.

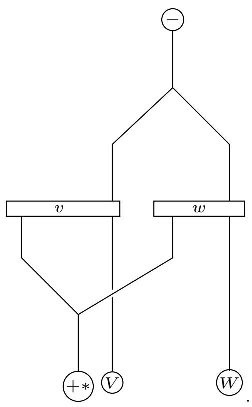
The new quantization technique is similar to the previous one, and passes again through the Tannaka–Krein duality. To avoid the functorial problems arising from the involvement of the Drinfeld double, the Drinfeld category of a Lie bialgebra is now given by the tensor category of all the Drinfeld-Yetter $\mathfrak{g}$–modules. Here, they define new universal Verma modules as

$$\tilde{M}_- := S(\mathfrak{g}) \qquad \text{and} \qquad \tilde{M}_+^* := \prod_{n\geq 0} S^n(\mathfrak{g}).$$

The latter can be thought of as the *dual* of M_+, which on the other hand is not a Drinfeld–Yetter module for infinite–dimensional $\mathfrak{g}$. At this point, the forgetful functor is now represented by

$$\tilde{F}(V) = \mathsf{Hom}_{\mathcal{DY}(\mathfrak{g})}(\tilde{M}_-, \tilde{M}_+^* \otimes V) \simeq V,$$

where now the isomorphism sends f into $< 1_+, f(1_-) >$. In this case, a tensor structure on the forgetful functor is given by



To define a quantization of $\mathfrak{g}$, we emulate the first construction: define a embedding

$$\tilde{m}_+ : \tilde{F}(\tilde{M}_-) \to \mathsf{End}(\tilde{F})$$

and define $\tilde{\mathcal{U}}_\hbar(\mathfrak{g})$ as the image of $\tilde{F}(\tilde{M}_-)$ through $\tilde{m}_+$. As before, we obtain a Hopf algebra that deforms $\mathcal{U}\mathfrak{g}$, and such that its quasi–classical limit is $\mathfrak{g}$.

Furthermore, this new construction is functorial, and it is also valid in the infinite–dimensional case. Moreover, it gives rise to a functor

$$Q^{EK} : LBA \to QUE,$$

which is adjoint to the quasi–classical limit functor $SC : QUE \to LBA$. The functoriality of the quantization is a crucial feature, which allows to obtain the following

Theorem. Let $(\mathfrak{g}, [\cdot,\cdot], \delta)$ be a symmetrizable Kac–Moody algebra. Then, the Etingof–Kazhdan quantization $Q^{EK}(\mathfrak{g})$ is isomorphic, as a Hopf algebra, to the Drinfeld–Jimbo quantum group $\mathcal{U}_\hbar^{DJ}(\mathfrak{g})$.

Outline

This thesis is structured in the following way.

In the first chapter we present the category theory tools necessary for the study of the representation theory of Hopf algebras. In particular, starting from the functors naturally defined in the category $\mathsf{Vect}(\mathbb{K})$, we define the concept of tensor product in a generic category, and the concepts of left unit constraint, right unit constraint, associativity constraint and of braiding. Which follows is the notion of tensor category first, and of braided tensor category then. The main result of this chapter is the Mac Lane's coherence theorem, that affirms that any tensor category is tensor equivalent to a strict one. In other words, if $\mathcal{C}$ is a tensor category, we may forgot the existence of brackets and consider tensors of the form $U \otimes V \otimes W$ in $\mathcal{C}$.

In the second chapter, following the point of view of [Kas12], we present the notions of algebra, coalgebra, bialgebra and Hopf algebra, with a particular attention to their representation theory. By analogy with the first chapter, we define the notion of quasi–triangular bialgebra, that is a bialgebra whose modules form a braided tensor category. In the last part of the chapter, we introduce the Drinfeld quantum double, that is, given a finite–dimensional Hopf algebra H, a quasi–triangular Hopf algebra whose underlying vector space is $H^* \otimes H$.

In the third chapter we first give a brief introduction on the theory of Lie algebras, with a particular attention to the semisimple case and to the case of symmetrizable Kac–Moody algebras. Secondly, we define the notions of Lie coalgebra and of Lie bialgebra. Since quantization theory can be interpreted as a bridge between Hopf algebras and Lie bialgebras, we present the notion of quasi–triangular Lie bialgebra and then we define the Drinfeld double of a Lie bialgebra $\mathfrak{g}$, that is a quasi–triangular Lie bialgebra whose underlying vector space is $\mathfrak{g} \oplus \mathfrak{g}^*$. Thus, as in the case of Hopf algebras, we may embed any Lie bialgebra into a quasi–triangular one. Finally, we present Manin triples, and we show that there is a one–one correspondence between finite–dimensional Manin triples $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ and Lie bialgebra structures on $\mathfrak{g}_+$. This correspondence allows us to define a Lie bialgebra structure on symmetrizable Kac–Moody algebras.

The aim of the fourth chapter is to define the concept of quantization and to study Drinfeld and Jimbo quantum groups. First, we introduce a new setting, that is through topologically free modules. This allows us to consider classical objects from a topological point of view, and also allows us to define the notion of a deformation of a Lie bialgebra. Indeed, in the central part of this chapter, we define the notion of quantization of a Lie algebra, and we prove that if $\mathfrak{g}$ is a Lie algebra and H is a deformation of $\mathcal{U}\mathfrak{g}$, then the Lie algebra $\mathfrak{g}$ has a natural structure of Lie bialgebra. After that, we present Drinfeld–Jimbo quantum groups, that is a class of Hopf algebras that deforms the universal enveloping algebra of Kac–Moody algebras. Furthermore, we see that the Drinfeld–Jimbo quantum group associated to a symmetrizable Kac–Moody algebra has a structure of a quasi–triangular Hopf algebra.

In the fifth chapter we present the universal construction of quantization of Lie bialgebras due to Pavel Etingof and David Kazhdan. We first introduce a non functorial quantization technique in the case of finite–dimensional Lie bialgebras, and in a second moment we make some changes to it, obtaining a functorial quantization, even valid in the infinite–dimensional case. The strategy of Etingof and Kazhdan is based on a Tannaka–Krein duality approach: they deform the category $\mathsf{Mod}(\mathcal{U}\mathfrak{g})$ instead of $\mathcal{U}\mathfrak{g}$, and then define a quantization $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ of the Drinfeld double of $\mathfrak{g}$ through Tannakian reconstruction. Later, they identify a Hopf subalgebra of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ and show that it gives a quantization of $\mathfrak{g}$.

Introduzione

In questa tesi viene presentata la soluzione data da Pavel Etingof e David Kazhdan in [EK96] e [EK98a] del problema della quantizzazione delle bialgebre di Lie, formulato da Vladimir Drinfeld in [Dri92].

Introduzione storica

Prima del 1980, le algebre di Lie venivano considerate "oggetti rigidi", nel senso che, data un'algebra di Lie $\mathfrak{g}$, la sua algebra inviluppante universale $\mathcal{U}\mathfrak{g}$ era considerata non adattabile al concetto di deformazione.

Tuttavia, intorno al 1982, alcuni matematici della Scuola di Leningrado scoprirono una deformazione dell'algebra inviluppante universale dell'algebra di Lie $\mathfrak{sl}_2$. Questo fu un risultato molto inaspettato, perché andava contro il pensiero precedente sulle algebre di Lie, e anche perché l'esistenza di questa deformazione era motivata da un'ulteriore struttura algebrica, ovvero la struttura di bialgebra di Lie.

Negli articoli [Dri85] e [Jim85] del 1985, Vladimir Drinfeld e Michio Jimbo generalizzarono la deformazione di $\mathfrak{sl}_2$ al caso delle algebre di Kac–Moody simmetrizzabili. Questo può essere visto come la nascita del concetto di quantum group, inteso come un'algebra di Hopf che deforma l'algebra inviluppante universale di un'algebra di Lie. Le algebre di Hopf si rivelarono infatti la struttura algebrica migliore per formalizzare il concetto di deformazione di una bialgebra di Lie; questa connessione fu ben spiegata in [Dri86], che è considerato il manifesto dei quantum groups.

Successivamente, in [Dri92], Drinfeld pubblicò alcuni problemi irrisolti nella teoria dei quantum groups; in uno di essi, Drinfeld si chiese se fosse possibile, data una qualsiasi bialgebra di Lie, trovare un'algebra di Hopf che la quantizzasse. La risposta positiva arrivò qualche anno dopo, con la serie articoli di Etingof e Kazhdan [EK96], [EK98a], [EK98b], [EK00a], [EK00b] e

[EK08]. Inoltre, Etingof e Kazhdan proposero una tecnica di quantizzazione funtoriale, dimostrando inoltre essere equivalente ai quantum group di Drinfeld e Jimbo nel caso di algebre di Kac–Moody simmetrizzabili. Pertanto, i quantum groups di Drinfeld e Jimbo divennero un caso particolare di una costruzione più generica e ampia.

Panoramica

Quantum $\mathfrak{sl}_2$

Il quantum group di $\mathfrak{sl}_2$ fu il primo esempio di quantizzazione di una bialgebra di Lie ed apparve nel contesto dei sistemi integrabili quantistici. All'inizio, questa struttura algebrica non era correlata al concetto di bialgebra di Lie, ma si inseriva invece nel contesto delle algebre di Hopf topologiche. Infatti, $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ venne presentato come una sorta di deformazione dell'algebra inviluppante universale di $\mathfrak{sl}_2$.

Più precisamente, si definisce $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ come la $\mathbb{C}[[\hbar]]$–algebra topologicamente libera generata da tre elementi X, Y, H e con le relazioni

$$[H,X] = 2X, \qquad [H,Y] = -2Y, \qquad [X,Y] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}.$$

È facile vedere che, per $\hbar = 0$, $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ restituisce l'algebra inviluppante universale di $\mathfrak{sl}(2)$, nel senso che esiste un isomorfismo banale

$$\mathcal{U}_\hbar(\mathfrak{sl}_2)/(\hbar \cdot \mathcal{U}_\hbar(\mathfrak{sl}_2)) \simeq \mathcal{U}\mathfrak{sl}_2.$$

Il fatto interessante è che $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ è una deformazione di $\mathcal{U}\mathfrak{sl}_2$ anche in quanto algebra di Hopf. In particolare, si ha che $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ è munita di un coprodotto

$$\Delta_\hbar : \mathcal{U}_\hbar(\mathfrak{sl}_2) \to \mathcal{U}_\hbar(\mathfrak{sl}_2) \otimes \mathcal{U}_\hbar(\mathfrak{sl}_2)$$

definito sui generatori da

$$\begin{aligned} X &\mapsto X \otimes e^{\hbar H} + 1 \otimes X \\ Y &\mapsto Y \otimes 1 + e^{-\hbar H} \otimes Y \\ H &\mapsto H \otimes 1 + 1 \otimes H. \end{aligned}$$

Inoltre, questa struttura di algebra di Hopf non è cocommutativa, ma è quasi triangolare. Infatti, si ha che esiste un elemento $R_\hbar \in \mathcal{U}_\hbar(\mathfrak{sl}_2) \otimes \mathcal{U}_\hbar(\mathfrak{sl}_2)$ che soddisfa le proprietà

$$R\Delta = \Delta^{op} R, \qquad (\Delta \otimes \mathsf{id})(R) = R_{13}R_{23}, \qquad (\mathsf{id} \otimes \Delta)(R) = R_{13}R_{12}.$$

Otteniamo dunque un isomorfismo di algebre di Hopf

$$\mathcal{U}_\hbar(\mathfrak{sl}_2)/(\hbar \cdot \mathcal{U}_\hbar(\mathfrak{sl}_2)) \simeq \mathcal{U}\mathfrak{sl}_2,$$

e quindi possiamo interpretare $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ come una deformazione dell'algebra inviluppante universale di $\mathfrak{sl}_2$. Inoltre, a differenza di $\mathcal{U}\mathfrak{sl}_2$ (che ha una struttura canonica di algebra di Hopf cocommutativa), l'algebra di Hopf $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ è non cocommutativa. Per ogni $x \in \mathfrak{sl}_2$, scegliamo $\tilde{x} \in \mathcal{U}_\hbar(\mathfrak{sl}_2)$ tale che $\tilde{x} = x \ mod \ \hbar$, e consideriamo la mappa

$$\delta(x) = \frac{\Delta(\tilde{x}) - \Delta^{op}(\tilde{x})}{\hbar} \ mod \ \hbar.$$

Sui generatori di $\mathfrak{sl}_2$, otteniamo

$$\delta(e) = e \wedge h, \qquad \delta(f) = f \wedge h, \qquad \delta(h) = 0.$$

Si può mostrare che $\delta : \mathfrak{sl}_2 \wedge \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$ è un cobracket di Lie, che fornisce a $\mathfrak{sl}_2$ una struttura di bialgebra di Lie. Pertanto, $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ è un'algebra di Hopf che deforma l'algebra inviluppante universale di $\mathfrak{sl}_2$, e allo stesso tempo induce una struttura di bialgebra di Lie su $\mathfrak{sl}_2$. Queste due proprietà rendono $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ una quantizzazione di $\mathfrak{sl}_2$.

Il problema della quantizzazione delle bialgebre di Lie

Ispirandosi al caso di $\mathfrak{sl}_2$, molti matematici iniziarono a studiare come generalizzare la costruzione del quantum group di $\mathfrak{sl}_2$ al caso di una generica algebra di Lie. Nel 1985, Drinfeld e Jimbo definirono una classe di algebre inviluppanti universali quantizzate che generalizzano il caso di $\mathfrak{sl}_2$ alla famiglia delle algebre di Kac–Moody simmetrizzabili. In questo contesto, come suggerito dal caso di $\mathfrak{sl}_2$, il punto di vista giusto è quello di coinvolgere la categoria LBA delle bialgebre di Lie. Se H è un'algebra di Hopf topologica e $(\mathfrak{g}, [\cdot,\cdot], \delta)$ è una bialgebra di Lie, diciamo che H è una quantizzazione di $\mathfrak{g}$ se esiste un isomorfismo

$$H/(\hbar \cdot H) \simeq \mathcal{U}\mathfrak{g}$$

di algebre di Hopf, e se

$$\delta(x) = \frac{\Delta(\tilde{x}) - \Delta^{op}(\tilde{x})}{\hbar} \ mod \ \hbar,$$

dove $\tilde{x}$ è un qualsiasi lifting di x in H. La bialgebra di Lie $(\mathfrak{g}, [\cdot,\cdot], \delta)$ è detta il limite quasi classico di H, mentre l'algebra di Hopf H è chiamata un'algebra inviluppante universale quantizzata. Come si vede nel quarto capitolo di questa tesi, esiste un funtore "limite semi classico"

$$SC : QUE \to LBA$$

che, ad ogni algebra inviluppante universale quantizzata H, assegna la bialgebra di Lie $(Prim(H/\hbar{\cdot}H), [\cdot,\cdot]\ mod\ \hbar, \frac{\Delta-\Delta^{op}}{\hbar}\ mod\ \hbar)$, in cui $Prim(H/\hbar{\cdot}H)$ denota l'insieme degli elementi primitivi di $H/\hbar{\cdot}H$. La domanda che si posero i matematici è la seguente: è possibile tornare indietro? In altre parole, è possibile costruire un funtore

$$Q : LBA \to QUE$$

che, data una bialgebra di Lie $(\mathfrak{g}, [\cdot,\cdot], \delta)$, restituisce un'algebra inviluppante quantizzata che deforma $\mathcal{U}\mathfrak{g}$? Questa domanda costituisce il problema di quantizzazione delle bialgebre di Lie, che rimase aperto fino alla seconda metà degli anni 90, e fu risolto da Etingof e Kazhdan.

La strategia risolutiva di Etingof e Kazhdan

La soluzione del problema di quantizzazione fornita da Etingof e Kazhdan consiste in diversi passaggi e si basa sulla risoluzione di diversi sottoproblemi.

L'idea di base della quantizzazione di Etingof–Kazhdan è la seguente: data una bialgebra di Lie $\mathfrak{g}$, si prova a trovare una deformazione della categoria $\mathsf{Mod}(\mathcal{U}\mathfrak{g})$, invece di deformare direttamente l'algebra di Hopf $\mathcal{U}\mathfrak{g}$. Infatti, questi due diversi concetti di deformazione sono collegati tra loro dalla teoria della dualità di Tannaka–Krein. Nel caso delle bialgebre, possiamo riassumere questa teoria dicendo che, se A è un'algebra, allora per avere una struttura di bialgebra su A è necessario e sufficiente costruire una struttura tensoriale su $\mathsf{Mod}(A)$, insieme ad una struttura tensoriale sul funtore fibra

$$F : \mathsf{Mod}(A) \to \mathsf{Vect}$$

che assegna ad ogni A–modulo il suo corrispondente spazio vettoriale e ad ogni morfismo di A–moduli la sua corrispondente mappa lineare. Infatti, si ha un isomorfismo di algebre $\mathsf{End}(F) \simeq A$; inoltre, si dimostra che ogni struttura tensoriale su F induce una struttura di bialgebra su $\mathsf{End}(F)$, e dunque su A. Naturalmente, differenti strutture tensoriali su $\mathsf{Mod}(A)$ e su F restituiscono differenti strutture di bialgebra su A.

Ad esempio, abbiamo che l'algebra inviluppante universale $\mathcal{U}\mathfrak{g}$ di un'algebra di Lie $\mathfrak{g}$ ha una struttura standard di bialgebra (la quale, in particolare è anche Hopf), e questa può essere interpretato dalla dualità di Tannaka–Krein considerando la struttura tensoriale banale su $\mathsf{Mod}(\mathcal{U}\mathfrak{g})$, insieme alla struttura tensoriale banale su F.

Pertanto, il primo passo della quantizzazione di Etingof e Kazhdan è quello di trovare una struttura tensoriale non banale su $\mathsf{Mod}(\mathcal{U}\mathfrak{g})$, insieme ad una struttura tensoriale non banale sul funtore fibra F, per poi applicare la ricostruzione Tannakiana, ottenendo una bialgebra candidata ad essere una quantizzazione di $\mathfrak{g}$.

La categoria di Drinfeld

Data un'algebra A, non è semplice trovare una struttura tensoriale non banale su $\mathsf{Mod}(A)$. Tuttavia, il seguente teorema, dimostrato da Drinfeld, ci fornisce un buon punto di partenza.

Teorema (Drinfeld): Sia $\mathfrak{g}$ una bialgebra di Lie e sia $t \in \mathfrak{g} \otimes \mathfrak{g}$ un tensore invariante e simmetrico. Allora esiste un associatore $\Phi_{KZ} \in \mathcal{U}\mathfrak{g}^{\otimes 3}[[\hbar]]$. Inoltre, il tensore $R = e^{\hbar t/2}$ definisce una struttura quasi triangolare sulla quasi bialgebra topologica $(\mathcal{U}\mathfrak{g}[[\hbar]], \otimes, \mathbb{C}[[\hbar]], \Phi_{KZ})$.

Nel caso delle algebre di Lie semplici, un elemento invariante e simmetrico è dato dall'elemento canonico corrispondente alla forma di Killing; nel caso di una arbitraria bialgebra di Lie $\mathfrak{g}$, invece, non è detto che esista. Tuttavia, possiamo immergere la bialgebra di Lie $\mathfrak{g}$ nel suo doppio di Drinfeld, che è una bialgebra di Lie il cui spazio vettoriale è $\mathfrak{D}\mathfrak{g} = \mathfrak{g} \otimes \mathfrak{g}^*$, ed ammette in modo canonico un tensore invariante e simmetrico. Questo fatto, insieme al teorema di Drinfeld, suggerisce che la costruzione di una quantizzazione di una bialgebra di Lie $\mathfrak{g}$ dovrebbe passare prima attraverso la quantizzazione del suo doppio di Drinfeld $\mathfrak{D}\mathfrak{g}$. Infatti, Etingof e Kazhdan costruiscono prima una quantizzazione di $\mathfrak{D}\mathfrak{g}$, per poi isolare una sottoalgebra di Hopf che quantizza $\mathfrak{g}$.

In primo luogo, esaminiamo il caso in cui $\mathfrak{g}$ è di dimensione finita. Data una bialgebra di Lie di dimensione finita $\mathfrak{g}$, consideriamo la categoria $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$, i cui oggetti sono i $\mathfrak{D}\mathfrak{g}$–moduli topologicamente liberi, e la cui classe di morfismi è $\mathsf{Hom}_{\mathfrak{D}\mathfrak{g}[[\hbar]]}(V, W)$. Applicando il teorema di Drinfeld a $\mathfrak{D}\mathfrak{g}$, possiamo definire una struttura di categoria tensoriale intrecciata su $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$, che chiamiamo categoria di Drinfeld. Per trovare una quantizzazione di $\mathfrak{D}\mathfrak{g}$, cerchiamo quindi di costruire una struttura tensoriale sul funtore fibra F.

Il funtore fibra e i moduli di Verma universali

I moduli di Verma universali svolgono un ruolo cruciale nella costruzione di quantizzazione di Etingof e Kazhdan. Essi sono definiti come

$$M_\pm := \mathsf{Ind}_{\mathfrak{g}_\pm[[\hbar]]}^{\mathfrak{D}\mathfrak{g}[[\hbar]]} c_\pm = \mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]] \otimes_{\mathcal{U}\mathfrak{g}_\pm[[\hbar]]} c_\pm,$$

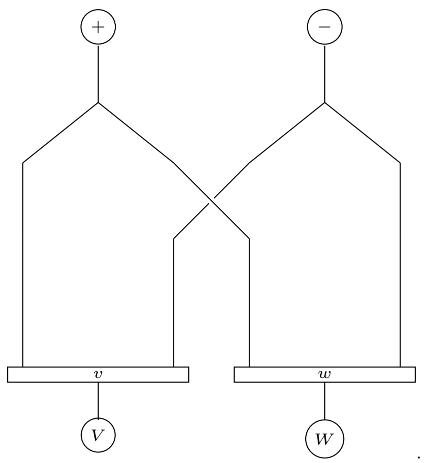
dove $c_\pm$ è il $\mathfrak{g}_\pm[[\hbar]]$–modulo banale di rango 1. Abbiamo che $M_\pm$ sono oggetti coalgebra in $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$; inoltre possiamo rappresentare il funtore fibra F tramite

$$F(V) = \mathsf{Hom}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-, V) \simeq V,$$

in cui l'isomorfismo manda f in $f(1_+ \otimes 1_-)$. Poiché $M_\pm$ sono coalgebre, lo è anche il loro prodotto tensoriale $M_+ \otimes M_-$. La sua struttura di coalgebra non cocommutativa induce una struttura tensoriale sul funtore fibra F, data dalla mappa

$$\begin{aligned} J_{V,W} : F(V) \otimes F(W) &\to F(V \otimes W) \\ v \otimes w &\mapsto J_{V,W}(v \otimes w) \end{aligned}$$

che rappresentiamo attraverso l'immagine



Tramite ricostruzione Tannakiana, otteniamo un'algebra di Hopf $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$, la cui struttura di spazio vettoriale è data da $\mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]]$, e il cui coprodotto è dato da

$$\Delta_\hbar := J^{-1}\Delta_0 J.$$

Abbiamo quindi che $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ è una quantizzazione di $\mathfrak{D}\mathfrak{g}$, e inoltre ha una struttura quasi triangolare, la cui R–matrice è $R_\hbar := (J^{op})^{-1}RJ$.

La soluzione del problema

Una volta definita una quantizzazione del doppio di Drinfeld di una bialgebra di Lie $\mathfrak{g}$, si cerca di trovare una sottoalgebra di Hopf di $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ che quantizzi $\mathfrak{g}$. Sfortunatamente, la scelta $\mathcal{U}\mathfrak{g}[[\hbar]] \subset \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^*[[\hbar]] \simeq \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ non funziona, in quanto in generale non è detto che sia chiusa rispetto al coprodotto; questo fatto rende il problema più complicato.

L'idea di Etingof e Kazhdan per risolvere questo problema è di usare l'isomorfismo

$$\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g}) \simeq \mathsf{End}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-)$$

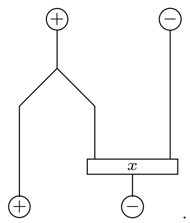
per definire una quantizzazione di $\mathfrak{g}_+ = \mathfrak{g}$ e di $\mathfrak{g}_- = \mathfrak{g}^*$, usando i moduli di Verma $M_\pm$. Infatti, consideriamo gli spazi

$$F(M_\pm) = \mathsf{Hom}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-, M_\pm)$$

e dimostriamo che sono naturalmente muniti di una struttura di algebra di Hopf, fornendo una quantizzazione di $\mathfrak{g}_\mp$. Per mostrarlo, proviamo che $F(M_\pm)$ corrisponde ad una sottoalgebra di Hopf di $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. Nel caso di $F(M_-)$, consideriamo l'embedding

$$m_+(x) : F(M_-) \to \mathsf{End}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-)$$

che manda $x \in F(M_-)$ nell'elemento di $\mathsf{End}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-)$ rappresentato dall'immagine



Pertanto, definiamo $\mathcal{U}_\hbar(\mathfrak{g})$ come l'immagine di $F(M_-)$ tramite m_+. Come spazio vettoriale, abbiamo che $F(M_-) \simeq \mathcal{U}\mathfrak{g}[[\hbar]]$; inoltre possiamo definire su $F(M_-)$ un prodotto $\mu_\hbar$ e un coprodotto $\Delta_\hbar$ che, modulo $\hbar$, restituiscono rispettivamente il prodotto e il coprodotto di $\mathcal{U}\mathfrak{g}[[\hbar]]$. Per dimostrare che $F(M_-)$ è effettivamente un'algebra di Hopf, osserviamo prima che, dati $x, y \in F(M_-)$, si ha

$$m_+(x) \circ m_+(y) = m_+(z)$$

per un opportuno $z \in F(M_-)$. Ciò implica che $\mathcal{U}_\hbar(\mathfrak{g})$ è una sottoalgebra di $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. Successivamente, attraverso una tecnica pittorica, mostriamo che in particolare $\mathcal{U}_\hbar(\mathfrak{g})$ è una sottoalgebra di Hopf di $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. Abbiamo dunque che $F(M_-)$ è un'algebra inviluppante universale quantizzata, il cui limite quasi classico è $\mathfrak{g}$.

Funtorialità della quantizzazione

La costruzione precedente risulta tuttavia non essere funtoriale, in quanto coinvolge il doppio di Drinfeld della bialgebra di Lie. Inoltre, questa tecnica non è valida nel caso di bialgebre di Lie di dimensione infinita. Etingof e Kazhdan risolvono entrambi i problemi considerando una nuova quantizzazione, in cui il modulo di Verma M_+ non appare.

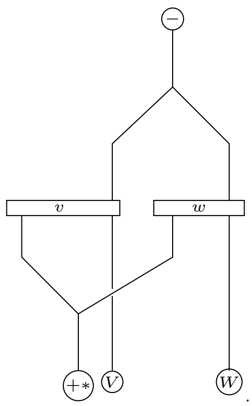
La nuova tecnica di quantizzazione è simile alla precedente, e passa di nuovo attraverso la dualità di Tannaka–Krein. Per evitare i problemi funtoriali derivanti dal coinvolgimento del doppio di Drinfeld, si utilizza come categoria di Drinfeld la categoria tensoriale dei $\mathfrak{g}$–moduli di Drinfeld–Yetter. Pertanto, vengono definiti i nuovi moduli di Verma come

$$\tilde{M}_- := S(\mathfrak{g}) \qquad \text{e} \qquad \tilde{M}_+^* := \prod_{n\geq 0} S^n(\mathfrak{g}),$$

in cui quest'ultimo viene considerato come *duale* di M_+. A questo punto, rappresentiamo il funtore fibra tramite

$$\tilde{F}(V) = \mathsf{Hom}_{\mathcal{DY}(\mathfrak{g})}(\tilde{M}_-, \tilde{M}_+^* \otimes V) \simeq V,$$

in cui adesso l'isomorfismo manda f in $< 1_+, f(1_-) >$. In questo caso, una struttura tensoriale sul funtore fibra è data da



Per definire una quantizzazione di $\mathfrak{g}$, emuliamo la prima costruzione: definiamo l'embedding

$$\tilde{m}_+ : \tilde{F}(\tilde{M}_-) \to \mathsf{End}(\tilde{F})$$

e definiamo $\tilde{\mathcal{U}}_\hbar(\mathfrak{g})$ come l'immagine di $\tilde{F}(\tilde{M}_-)$ tramite $\tilde{m}_+$. Come prima, otteniamo infatti un'algebra di Hopf che deforma $\mathcal{U}\mathfrak{g}$, e tale che il suo limite quasi classico è $\mathfrak{g}$. Inoltre, questa nuova costruzione è funtoriale, ed è valida anche nel caso di bialgebre di Lie di dimensione infinita. Infine, il corrispondente funtore

$$Q^{EK} : LBA \to QUE$$

è aggiunto al funtore limite semi classico $SC : QUE \to LBA$. Questo fatto ci consente di ottenere il seguente

Teorema. Sia $(\mathfrak{g}, [\cdot,\cdot], \delta)$ una algebra di Kac–Moody simmetrizabile. Allora la quantizzazione di Etingof e Kazhdan $Q^{EK}(\mathfrak{g})$ è isomorfa, come algebra di Hopf, al quantum group di Drinfeld e Jimbo $\mathcal{U}_\hbar^{DJ}(\mathfrak{g})$ associato a $\mathfrak{g}$.

Struttura della tesi

Questa tesi è strutturata nel modo seguente.

Nel primo capitolo presentiamo gli strumenti della teoria delle categorie necessari per lo studio della teoria delle rappresentazioni delle algebre di Hopf. In particolare, partendo dai funtori naturalmente definiti nella categoria $\mathsf{Vect}(\mathbb{K})$, definiamo il concetto di prodotto tensoriale in una generica categoria, e i concetti di vincolo unitario sinistro, vincolo unitario destro, vincolo di associatività e di intrecciamento. Ciò che segue è il concetto di categoria tensoriale prima, e di categoria tensoriale intrecciata poi. Il risultato principale di questo capitolo è il teorema di coerenza di Mac Lane, che afferma che ogni categoria tensoriale è tensorialmente equivalente ad una categoria rigorosa. In altre parole, se $\mathcal{C}$ è una categoria tensoriale, possiamo dimenticare l'esistenza delle parentesi e considerare tensori della forma $U \otimes V \otimes W$ in $\mathcal{C}$.

Nel secondo capitolo, seguendo il punto di vista di [Kas12], presentiamo le nozioni di algebra, coalgebra, bialgebra e di algebra di Hopf, con una particolare attenzione alla loro teoria delle rappresentazioni. Per analogia con il primo capitolo, definiamo la nozione di bialgebra quasi triangolare, che è una bialgebra i cui moduli formano una categoria tensoriale intrecciata. Nell'ultima parte del capitolo, introduciamo il quantum double di Drinfeld, il quale, data un'algebra di Hopf di dimensione finita H, è un'algebra di Hopf quasi triangolare il cui spazio vettoriale è $H^* \otimes H$.

Nel terzo capitolo viene data una breve introduzione sulla teoria delle algebre di Lie, con una particolare attenzione al caso semisemplice e al caso delle algebre di Kac–Moody simmetrizzabili. In secondo luogo, definiamo le nozioni di coalgebra di Lie e di bialgebra di Lie. Poiché la teoria della quantizzazione fa da ponte tra la teoria delle algebre di Hopf e la teoria delle bialgebre di Lie, presentiamo le nozioni di bialgebra di Lie quasi triangolare e di doppio di Drinfeld di una bialgebra di Lie $\mathfrak{g}$. Quest'ultimo, è in particolare una bialgebra di Lie quasi triangolare, il cui spazio vettoriale è $\mathfrak{g} \oplus \mathfrak{g}^*$; abbiamo dunque che, come nel caso delle algebre di Hopf, possiamo immergere qualsiasi bialgebra di Lie in una quasi triangolare. Infine, diamo la definizione di tripla di Manin, e mostriamo che esiste una corrispondenza biunivoca tra le triple di Manin di dimensione finita $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ e le strutture di bialgebra di Lie su $\mathfrak{g}_+$. Questa corrispondenza ci permette di definire una struttura di bialgebra di Lie sulle algebre di Kac–Moody simmetrizzabili.

Lo scopo del quarto capitolo è quello di definire il concetto di quantizzazione di una bialgebra di Lie e di presentare i quantum groups di Drinfeld e Jimbo. Per fare ciò, introduciamo la categoria dei moduli topologicamente liberi, che ci permette di considerare gli oggetti classici da un punto di vista topologico, e ci consente inoltre di dare senso al concetto di deformazione di una bialgebra di Lie. Nella parte centrale di questo capitolo, definiamo infatti la nozione di quantizzazione di un'algebra di Lie e dimostriamo che se $\mathfrak{g}$ è un'algebra di Lie e H è una deformazione di $\mathcal{U}\mathfrak{g}$, allora l'algebra di Lie $\mathfrak{g}$ ha una struttura naturale di bialgebra di Lie. Successivamente, presentiamo i quantum groups di Drinfeld e Jimbo, che sono una classe di algebre di Hopf quasi triangolari che deformano l'algebra inviluppante universale delle algebre di Kac–Moody.

Nel quinto capitolo presentiamo infine la costruzione universale della quantizzazione delle bialgebre di Lie dovuta a Pavel Etingof e David Kazhdan. In primo luogo, costruiamo una tecnica di quantizzazione non funtoriale, valida soltanto nel caso di bialgebre di Lie di dimensione finita; in un secondo momento, apportiamo alcune modifiche a tale costruzione, ottenendo una quantizzazione funtoriale e valida anche nel caso di bialgebre di Lie di dimensione infinita. La strategia di Etingof e Kazhdan si basa sulla dualità di Tannaka-Krein: l'idea è quella di deformare la categoria $\mathsf{Mod}(\mathcal{U}\mathfrak{g})$ invece di deformare $\mathcal{U}\mathfrak{g}$. Una volta fatto ciò, si definisce una quantizzazione $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ del doppio di Drinfeld di $\mathfrak{g}$ attraverso la ricostruzione Tannakiana. Successivamente, viene identificata una sottoalgebra Hopf di $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ e si mostra che essa fornisce una quantizzazione di $\mathfrak{g}$.

Contents

Chapter 1

Categories and functors

In this thesis $\mathbb{K}$ always denotes a field of characteristic zero.

1.1 Categories

Definition 1.1.1. *A category $\mathcal{C}$ consists:*

(1) of a class $\mathsf{Obj}(\mathcal{C})$ whose elements are called the objects of the category;

(2) of a class $\mathsf{Hom}(\mathcal{C})$ whose elements are called the morphisms (or the arrows) of the category;

(3) of maps

$$\begin{aligned} \textit{identity} \quad & id : \mathsf{Obj}(\mathcal{C}) \to \mathsf{Hom}(\mathcal{C}); \\ \textit{source} \quad & s : \mathsf{Hom}(\mathcal{C}) \to \mathsf{Obj}(\mathcal{C}); \\ \textit{target} \quad & t : \mathsf{Hom}(\mathcal{C}) \to \mathsf{Obj}(\mathcal{C}); \\ \textit{composition} \quad & \circ : \mathsf{Hom}(\mathcal{C}) \times_{\mathsf{Obj}(\mathcal{C})} \mathsf{Hom}(\mathcal{C}) \to \mathsf{Hom}(\mathcal{C}) \end{aligned}$$

such that:

(a) for any object V in $\mathsf{Obj}(\mathcal{C})$, we have

$$s(\mathsf{id}_V) = t(\mathsf{id}_V) = V;$$

(b) for any morphism f in $\mathsf{Hom}(\mathcal{C})$, we have

$$\mathsf{id}_{t(f)} \circ f = f \circ \mathsf{id}_{s(f)} = f;$$

(c) *for any morphisms* f, g, h *satisfying* $t(f) = s(g)$ *and* $t(g) = s(h)$, *we have*

$$(h \circ g) \circ f = h \circ (g \circ f),$$

where $\mathsf{Hom}(\mathcal{C}) \times_{\mathsf{Obj}(\mathcal{C})} \mathsf{Hom}(\mathcal{C})$ *denotes the class of all the composable morphisms, i.e. the class of all couples* (f, g) *of morphisms such that* $t(f) = s(g)$.

The object $s(f)$ is called the source of the morphism f, while the object $t(f)$ is called the target of the morphism f. We denote by:

- gf or $g \circ f$ the composition of two composable morphisms;
- $\mathsf{Hom}_{\mathcal{C}}(V, W)$ the class of all morphisms in $\mathsf{Hom}(\mathcal{C})$ whose source is the object V and whose target is the object W;
- $\mathsf{Hom}_{\mathcal{C}}(V, \cdot)$ the class of all morphisms in $\mathsf{Hom}(\mathcal{C})$ whose source is the object V;
- $\mathsf{Hom}_{\mathcal{C}}(\cdot, W)$ the class of all morphisms in $\mathsf{Hom}(\mathcal{C})$ whose target is the object W;
- $\mathsf{End}_{\mathcal{C}}(V)$ the class of all morphisms in $\mathsf{Hom}(\mathcal{C})$ whose source and target is the object V;
- $f : V \to W$ a morphism f in $\mathsf{Hom}_{\mathcal{C}}(V, W)$.

We say that a morphism f in $\mathsf{Hom}_{\mathcal{C}}(V, W)$ is an isomorphism if there exists a morphism f^{-1} in $\mathsf{Hom}_{\mathcal{C}}(W, V)$, called the inverse morphism of f, such that $f^{-1} \circ f = \mathsf{id}_V$ and $f \circ f^{-1} = \mathsf{id}_W$. If a morphism f admits an inverse morphism, this is unique and is an isomorphism whose inverse is f. Two morphisms f, g in a category $\mathcal{C}$ are said to be parallel if $s(f) = s(g)$ and $t(f) = t(g)$.

Example 1.1.2. *Let* $\mathbb{K}$ *be a field. We denote by* $\mathsf{Vect}(\mathbb{K})$ *the category whose objects are all the vector spaces over* $\mathbb{K}$ *and whose morphisms are all the* $\mathbb{K}$*–linear morphisms.*

Definition 1.1.3. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories. We define the product of* $\mathcal{C}$ *and* $\mathcal{D}$ *as the category* $\mathcal{C} \times \mathcal{D}$ *whose objects are all the pairs of objects* (V, W) *in* $\mathsf{Obj}(\mathcal{C}) \times \mathsf{Obj}(\mathcal{D})$ *and whose class of morphisms is*

$$\mathsf{Hom}_{\mathcal{C}\times\mathcal{D}}((V, W), (V', W')) = \mathsf{Hom}_{\mathcal{C}}(V, V') \times \mathsf{Hom}_{\mathcal{D}}(W, W').$$

Definition 1.1.4. *Let* $\mathcal{C}$ *be a category. A subcategory* $\mathcal{D}$ *of* $\mathcal{C}$ *consists of:*

- *a subclass* $\mathsf{Obj}(\mathcal{D})$ *of* $\mathsf{Obj}(\mathcal{C})$*;*

- *a subclass* $\mathsf{Hom}(\mathcal{D})$ *of* $\mathsf{Hom}(\mathcal{C})$

that are stable under the identity, source, target and composition maps of $\mathcal{C}$.

Example 1.1.5. *The category* $\mathsf{Vect}(\mathbb{K})_{<\infty}$ *of all finite–dimensional vector spaces over a field* $\mathbb{K}$ *is a subcategory of* $\mathsf{Vect}(\mathbb{K})$.

1.2 Functors

In this Section we define the concept of functor, that is the way of categories to communicate with each other ones.

Definition 1.2.1. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be two categories. A functor* $F : \mathcal{C} \to \mathcal{C}'$ *consists:*

- *of a map* $F : \mathsf{Obj}(\mathcal{C}) \to \mathsf{Obj}(\mathcal{C}')$;
- *of a map* $F : \mathsf{Hom}(\mathcal{C}) \to \mathsf{Hom}(\mathcal{C}')$

such that:

(F1) for any object V *in* $\mathsf{Obj}(\mathcal{C})$, *we have* $F(\mathsf{id}_V) = \mathsf{id}_{F(V)}$;

(F2) for any morphism f *in* $\mathsf{Hom}(\mathcal{C})$, *we have*

$$s(F(f)) = F(s(f)) \quad \text{and} \quad t(F(f)) = F(t(f));$$

(F3) if f, g *are composable morphisms in* $\mathsf{Hom}(\mathcal{C})$, *we have*

$$F(g \circ f) = F(g) \circ F(f).$$

We may think upon a functor as something that relates two categories in such a way the direction of the arrows is preserved.

Definition 1.2.2. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be two categories. A controvariant functor* $F : \mathcal{C} \to \mathcal{C}'$ *consists:*

- *of a map* $F : \mathsf{Obj}(\mathcal{C}) \to \mathsf{Obj}(\mathcal{C}')$;
- *of a map* $F : \mathsf{Hom}(\mathcal{C}) \to \mathsf{Hom}(\mathcal{C}')$

such that:

(F1) for any object V *in* $\mathsf{Obj}(\mathcal{C})$, *we have* $F(\mathsf{id}_V) = \mathsf{id}_{F(V)}$;

(F4) for any morphism f in $\mathsf{Hom}(\mathcal{C})$, we have

$$t(F(f)) = F(s(f)) \quad \textit{and} \quad s(F(f)) = F(t(f));$$

(F5) if f, g are composable morphisms in $\mathsf{Hom}(\mathcal{C})$, we have

$$F(g \circ f) = F(f) \circ F(g).$$

Contrary to the concept of a functor, we may think upon a controvariant functor as something that relates two categories, in such a way the direction of the arrows is reversed.

Definition 1.2.3. *Let F and G be functors between two categories $\mathcal{C}, \mathcal{C}'$.*

- *A natural transformation $\eta : F \to G$ is a family $\eta(V) : F(V) \to G(V)$ of morphisms in $\mathsf{Hom}(\mathcal{C}')$ indexed by the objects V in $\mathsf{Obj}(\mathcal{C})$ such that, for any morphism $f : V \to W$ in $\mathsf{Hom}(\mathcal{C})$, the square*

$$\begin{array}{ccc} F(V) & \xrightarrow{\eta(V)} & G(V) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(W) & \xrightarrow{\eta(W)} & G(W) \end{array}$$

  commutes.
  If furthermore $\eta(V)$ is an isomorphism for any V in $\mathsf{Obj}(\mathcal{C})$, we say that $\eta : F \to G$ is a natural isomorphism.

- *We say that (F, G) is a pair of adjoint functors, with right adjoint functor F and left adjoint functor G, if there exists a natural isomorphism*

$$\mathsf{Hom}_{\mathcal{C}'}(\cdot, F(\cdot)) \simeq \mathsf{Hom}_{\mathcal{C}}(G(\cdot), \cdot).$$

If $\eta : F \to G$ is a natural isomorphism, then the collection of all morphisms $\eta(V)^{-1}$ defines a natural isomorphism from G to F. Let $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ be three categories and let $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}''$ be two functors. Then we may define the composition $G \circ F : \mathcal{C} \to \mathcal{C}''$, that is is a functor too. If furthermore F and G are natural isomorphisms, then the composition $G \circ F$ is too. We denote by GF the composition of the functors F and G.

Definition 1.2.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.*

- *We say that F is faithful (resp. fully faithful) if, for any couple (V, V') of objects in $\mathcal{C}$, the map*

$$F : \mathsf{Hom}_{\mathcal{C}}(V, V') \to \mathsf{Hom}_{\mathcal{D}}(F(V), F(V'))$$

  is injective (resp. bijective).

- *We say that F is essentially surjective if, for any object W in $\mathcal{D}$, there exists an object V in $\mathcal{C}$ such that $F(V) \simeq W$.*

- *We say that F is an equivalence of categories if it is essentially surjective and fully faithful. This is equivalent to the existence of a functor $G : \mathcal{D} \to \mathcal{C}$ and of two natural isomorphisms*

$$\eta : \mathsf{id}_{\mathcal{D}} \to FG \qquad \text{and} \qquad \theta : GF \to \mathsf{id}_{\mathcal{C}}.$$

1.2.1 Abelian categories and exact functors

The aim of this Subsection is to give the definition of exact functor.

Definition 1.2.5. *Let $\mathcal{C}$ be a category.*

- *An object W of $\mathcal{C}$ is said to be terminal if to each object V in $\mathcal{C}$ there is exactly one arrow $f : V \to W$.*

- *An object V of $\mathcal{C}$ is said to be initial if to each object W in $\mathcal{C}$ there is exactly one arrow $f : V \to W$.*

- *An object 0 of $\mathcal{C}$ is said to be a null object if it is both initial and terminal.*

- *A monomorphism is a morphism $f : V \to W$ such that, for any object U and morphisms f_1, f_2 in $\mathsf{Hom}_{\mathcal{C}}(U, V)$ with $f \circ f_1 = f \circ f_2$, we have $f_1 = f_2$.*

- *An epimorphism is a morphism $g : V \to W$ such that, for any object U and morphisms g_1, g_2 in $\mathsf{Hom}_{\mathcal{C}}(W, U)$ with $g_1 \circ g = g \circ g_2$, we have $g_1 = g_2$.*

- *Given a morphism f in $\mathsf{Hom}_{\mathcal{C}}(V, W)$, we define an image of f as a couple (m, I), where I is an object of $\mathcal{C}$ and m is a monomorphism in $\mathsf{Hom}(I, W)$, such that there exists a morphism e in $\mathsf{Hom}_{\mathcal{C}}(V, I)$ with $f = m \circ e$, and such that, for any object I' with a morphism e' in $\mathsf{Hom}_{\mathcal{C}}(V, I')$ and a monomorphism m' in $\mathsf{Hom}_{\mathcal{C}}(I', W)$ with $f = m' \circ e'$, there exists a unique morphism v in $\mathsf{Hom}_{\mathcal{C}}(I, I')$ such that $m = m' \circ v$.*

- *A morphism f is called a left zero morphism if for any object V in $\mathcal{C}$ and any g, h in $\mathsf{Hom}_{\mathcal{C}}(V, \cdot)$ we have $f \circ g = f \circ h$. Similarly, a morphism f is called a right zero morphism if for any object W in $\mathcal{C}$ and any g, h in $\mathsf{Hom}_{\mathcal{C}}(\cdot, W)$ we have $g \circ f = h \circ f$. A zero morphism is one that is both a left zero morphism and a right zero morphism.*

- *Let V and W be two objects in $\mathcal{C}$ and let f and g be two morphisms in $\mathsf{Hom}_{\mathcal{C}}(V, W)$. An equalizer is a couple (E, eq), where E is an object and eq is a morphism in $\mathsf{Hom}_{\mathcal{C}}(E, V)$ satisfying $f \circ eq = g \circ eq$, and such that, given any object O and morphism m in $\mathsf{Hom}_{\mathcal{C}}(O, V)$ with $f \circ m = g \circ m$, there exists a unique morphism u in $\mathsf{Hom}_{\mathcal{C}}(O, E)$ such that $eq \circ u = m$.*

- *Suppose that $\mathcal{C}$ has a null object. Let V and W be two objects and let $f : V \to V$ be a morphism. a kernel of f is an equalizer of f and the zero morphism from V to W.*

- *Let V and W be two objects in $\mathcal{C}$ and let f and g be two morphisms in $\mathsf{Hom}_{\mathcal{C}}(V, W)$. A coequalizer is a couple (q, U), where U is an object and q is a morphism in $\mathsf{Hom}_{\mathcal{C}}(W, U)$ satisfying $q \circ f = q \circ g$, and such that, given any object O and morphism m in $\mathsf{Hom}_{\mathcal{C}}(W, O)$ with $m \circ f = m \circ g$, there exists a unique morphism u in $\mathsf{Hom}_{\mathcal{C}}(U, O)$ such that $u \circ q = m$.*

- *Suppose that $\mathcal{C}$ has a null object. Let V and W be two objects and let $f : V \to V$ be a morphism. a cokernel of f is a coequalizer of f and the zero morphism from V to W.*

Remark 1.2.6. *The notion of image, kernel and cokernel are universal. Therefore, given a morphism f and a image (resp. a kernel, resp. a cokernel) g of f, we may say that g is the image (resp. the kernel, resp. the cokernel) of f.*

Definition 1.2.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.*

- *We say that $\mathcal{C}$ is an Ab–category if, for any V and W in $\mathsf{Obj}(\mathcal{C})$, the class $\mathsf{Hom}_{\mathcal{C}}(V, W)$ has the structure of an abelian group, and composition of morphisms is bilinear, in the sense that composition of morphisms distributes over the group operation. In formulas, this means that, if f, g, h are in $\mathsf{Hom}(\mathcal{C})$, we have:*

$$f \circ (g + h) = (f \circ g) + (f \circ h) \quad \text{and} \quad (f + g) \circ h = (f \circ h) + (g \circ h),$$

where $+$ is the group operation.

- *A functor $T : \mathcal{C} \to \mathcal{D}$ between two Ab–categories is said to be additive if $T(f + f') = T(f) + T(f')$ for any pair (f, f') of parallel morphisms.*
- *We say that $\mathcal{C}$ is an abelian category if:*
  (1) $\mathcal{C}$ is an Ab–category;
  (2) $\mathcal{C}$ has a null object;
  (3) $\mathcal{C}$ has binary biproducts;
  (4) every morphism in $\mathcal{C}$ has a kernel and a cokernel;
  (5) every monomorphism in $\mathcal{C}$ is a kernel and any epimorphism in $\mathcal{C}$ is a cokernel.

Definition 1.2.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories.*

- *A diagram*

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

  is said to be a short exact sequence if f is a monomorphism, g is an epimorphism and the image of f is equal to the kernel of g.

- *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. We say that the functor F is exact if, whenever*

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

  is a short exact sequence in $\mathcal{C}$, then

$$0 \longrightarrow F(U) \xrightarrow{F(f)} F(V) \xrightarrow{F(g)} F(W) \longrightarrow 0$$

  is a short exact sequence in $\mathcal{D}$.

1.2.2 Representable functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The algebra of endomorphisms of F is the algebra whose elements are all the collections of maps

$$\varphi_V : F(V) \to F(V), \qquad V \text{ in } \mathsf{Obj}(\mathcal{C})$$

making commutative the following diagram:

$$\begin{array}{ccc} F(V) & \xrightarrow{\varphi_V} & F(V) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ F(W) & \xrightarrow{\varphi_W} & F(W) \end{array}$$

for any V and W in $\mathsf{Obj}(\mathcal{C})$ and f in $\mathsf{Hom}_{\mathcal{C}}(V, W)$. We denote this algebra by $\mathsf{End}(F)$.

Definition 1.2.9. *Let $\mathcal{C}$ be a category and let Set be the category of all sets.*

- *We say that $\mathcal{C}$ is a locally small category if, for any V and W in $\mathsf{Obj}(\mathcal{C})$, the class $\mathsf{Hom}_{\mathcal{C}}(V, W)$ is a set, called a homset.*
- *For each object V of a locally small category $\mathcal{C}$, we define the hom functor as the functor $\mathsf{Hom}_{\mathcal{C}}(V, \cdot)$ that maps an object W to the set $\mathsf{Hom}_{\mathcal{C}}(V, W)$.*
- *A functor $F : \mathcal{C} \to \mathsf{Set}$ is said to be representable if it is naturally isomorphic to $\mathsf{Hom}_{\mathcal{C}}(V, \cdot)$ for some object V of $\mathcal{C}$.*
- *If $F : \mathcal{C} \to \mathsf{Set}$ is representable, we define a representation of F as a pair (V, ϕ), where V is an object of $\mathcal{C}$ and $\phi : \mathsf{Hom}_{\mathcal{C}}(V, \cdot) \to F$ is a natural isomprhism.*

All the categories that we are interested on are locally small. Even if the definition of representable functor is made on the category Set, it is possible to adapt it to other categories (e.g. to the category $\mathsf{Vect}(\mathbb{K})$).

1.3 Tensor categories

Definition 1.3.1. *A tensor monoidal category or, simply, a tensor category is a sextuple $(\mathcal{C}, \otimes, I, a, l, r)$, where:*

- *$\mathcal{C}$ is a category;*
- *$\otimes$ is a functor*
$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$
called a tensor product on $\mathcal{C}$;
- *I is an object of $\mathcal{C}$, called the unit of $\mathcal{C}$;*
- *a is a natural isomorphism*
$$a : \otimes(\otimes \times \mathsf{id}) \to \otimes(\mathsf{id} \times \otimes),$$
called an associativity constraint. This means that for any triple U, V, W in $\mathsf{Obj}(C)$ there exists an isomorphism
$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

such that the square

$$\begin{array}{ccc} (U \otimes V) \otimes W & \xrightarrow{a_{U,V,W}} & U \otimes (V \otimes W) \\ {\scriptstyle (f\otimes g)\otimes h}\downarrow & & \downarrow{\scriptstyle f\otimes(g\otimes h)} \\ (U' \otimes V') \otimes W' & \xrightarrow{a_{U',V',W'}} & U' \otimes (V' \otimes W') \end{array}$$

commutes for any f *in* $\mathsf{Hom}_{\mathcal{C}}(U,\cdot)$, g *in* $\mathsf{Hom}_{\mathcal{C}}(V,\cdot)$ *and* h *in* $\mathsf{Hom}_{\mathcal{C}}(W,\cdot)$*;*

- l *is a natural isomorphism*

  $$l : \otimes(I \times \mathsf{id}) \to \mathsf{id},$$

  called a left unit constraint with respect to I*. This means that for any* V *in* $\mathsf{Obj}(\mathcal{C})$ *there exists an isomorphism*

  $$l_V : I \otimes V \to V$$

  such that the square

  $$\begin{array}{ccc} I \otimes V & \xrightarrow{l_V} & V \\ {\scriptstyle \mathsf{id}_I\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ I \otimes V' & \xrightarrow{l_{V'}} & V' \end{array}$$

  commutes for any f *in* $\mathsf{Hom}_{\mathcal{C}}(V,\cdot)$*;*

- r *is a natural isomorphism*

  $$r : \otimes(\mathsf{id} \times I) \to \mathsf{id},$$

  called a right unit constraint with respect to I*. This means that for any* V *in* $\mathsf{Obj}(\mathcal{C})$ *there exists an isomorphism*

  $$r_V : V \otimes I \to V$$

  such that the square

  $$\begin{array}{ccc} V \otimes I & \xrightarrow{r_V} & V \\ {\scriptstyle f\otimes\mathsf{id}_I}\downarrow & & \downarrow{\scriptstyle f} \\ V' \otimes I & \xrightarrow{r_{V'}} & V' \end{array}$$

  commutes for any f *in* $\mathsf{Hom}_{\mathcal{C}}(V,\cdot)$

such that the pentagonal diagram (P)

$$\begin{array}{ccc}
(U \otimes (V \otimes W)) \otimes X & \xleftarrow{a_{U,V,W} \otimes \mathsf{id}_X} & ((U \otimes V) \otimes W) \otimes X \\
 & & \downarrow{\scriptstyle a_{U \otimes V,W,X}} \\
\downarrow{\scriptstyle a_{U,V \otimes W,X}} & & (U \otimes V) \otimes (W \otimes X) \\
 & & \downarrow{\scriptstyle a_{U,V,W \otimes X}} \\
U \otimes ((V \otimes W) \otimes X) & \xrightarrow[\mathsf{id}_U \otimes a_{V,W,X}]{} & U \otimes (V \otimes (W \otimes X))
\end{array}$$

commutes for any U, V, W, X *in* $\mathsf{Obj}(\mathcal{C})$ *and the triangular diagram (T)*

$$\begin{array}{ccc}
(V \otimes I) \otimes W & \xrightarrow{a_{V,I,W}} & V \otimes (I \otimes W) \\
 & {\scriptstyle r_V \otimes \mathsf{id}_W} \searrow & \downarrow{\scriptstyle \mathsf{id}_V \otimes l_W} \\
 & & V \otimes W
\end{array}$$

commutes for any pair (V, W) *in* $\mathsf{Obj}(\mathcal{C}) \times \mathsf{Obj}(\mathcal{C})$.

The diagrams (P) and (T) are called respectively the pentagon axiom and the triangle axiom.

Remark 1.3.2. *Let* $\otimes$ *be a tensor product on* $\mathcal{C}$*. Then:*

- *we have an object* $V \otimes W$ *in* $\mathsf{Obj}(C)$ *associated to any pair* (V, W) *in* $\mathsf{Obj}(\mathcal{C}) \times \mathsf{Obj}(\mathcal{C})$*;*
- *we have a morphism* $f \otimes g$ *in* $\mathsf{Hom}(C)$ *associated to any pair* (f, g) *in* $\mathsf{Hom}(\mathcal{C}) \times \mathsf{Hom}(\mathcal{C})$*;*
- *from* $(F1)$ *we have* $\mathsf{id}_{V \otimes W} = \mathsf{id}_V \otimes \mathsf{id}_W$*;*
- *from* $(F2)$ *we have*

  $$s(f \otimes g) = s(f) \otimes s(g) \quad \textit{and} \quad t(f \otimes g) = t(f) \otimes t(g);$$

- *from* $(F3)$ *we have*

  $$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$$

  for any f, f', g, g' *in* $\mathsf{Hom}(\mathcal{C})$ *such that* $s(f') = t(f)$ *and* $s(g') = t(g)$*;*

- *using the previous equalities, we have*

$$f \otimes g = (f \otimes \mathsf{id}_{t(g)}) \circ (\mathsf{id}_{s(f)} \otimes g) = (\mathsf{id}_{t(f)} \otimes g) \circ (f \otimes \mathsf{id}_{s(g)}).$$

Remark 1.3.3. *Let* $\mathbb{K}$ *be a field. Then the classical tensor product on* $\mathsf{Vect}(\mathbb{K})$ *given by*

$$\begin{aligned} \otimes : \mathsf{Vect}(\mathbb{K}) \times \mathsf{Vect}(\mathbb{K}) &\to \mathsf{Vect}(\mathbb{K}) \\ (V, W) &\mapsto V \otimes W \end{aligned}$$

is a functor.

Proposition 1.3.4. *Let* U, V, W *be vector spaces in* $\mathsf{Vect}(\mathbb{K})$*. Then there are isomorphisms:*

$$(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$$

determined by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$*,*

$$\mathbb{K} \otimes V \simeq V \simeq V \otimes \mathbb{K}$$

determined by $\lambda \otimes v \mapsto \lambda v$ *and* $v \mapsto v \otimes 1$*, and*

$$V \otimes W \simeq W \otimes V$$

given by the flip $\tau_{V,W}$ *defined by* $\tau_{V,W}(v \otimes w) = w \otimes v$*.*

It follows from the previous Proposition that the category $\mathsf{Vect}(\mathbb{K})$ is a tensor category, where:

- the unit is the ground field $\mathbb{K}$;

- the associativity constraint is given by

$$a((u \otimes v) \otimes w) = u \otimes (v \otimes w);$$

- the unit constraints are

$$l(1 \otimes v) = v = r(v \otimes 1).$$

1.3.1 Properties of the unit

We now state some important properties of the unit of a tensor category. We refer the proofs of the results below to [Kas12].

Lemma 1.3.5. *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category. Then the triangles*

$$\begin{array}{ccc} (I \otimes V) \otimes W & \xrightarrow{a_{I,V,W}} & I \otimes (V \otimes W) \\ & \searrow^{l_V \otimes \mathrm{id}_W} & \downarrow l_{V \otimes W} \\ & & V \otimes W \end{array}$$

and

$$\begin{array}{ccc} (V \otimes W) \otimes I & \xrightarrow{a_{V,W,I}} & V \otimes (W \otimes I) \\ & \searrow^{r_{V \otimes W}} & \downarrow \mathrm{id}_V \otimes r_W \\ & & V \otimes W \end{array}$$

commute for any pair (V, W) of objects in $\mathcal{C}$.

Lemma 1.3.6. *Let I be a unit of a tensor category. For any object V we have*

- $l_{I \otimes V} = \mathrm{id}_I \otimes l_V$;
- $r_{V \otimes I} = r_V \otimes \mathrm{id}_I$;
- $l_I = r_I$.

Proposition 1.3.7. *The class $\mathrm{End}_{\mathcal{C}}(I)$ of all the endomorphisms of the unit object I is a commutative monoid for the composition. Moreover, for any pair (f, g) of endomorphisms of I, we have*

$$f \otimes g = g \otimes f = r_I^{-1} \circ (f \circ g) \circ r_I = r_I^{-1} \circ (g \circ f) \circ r_I.$$

1.3.2 Tensor functors and strictness

Definition 1.3.8. *A tensor category $(\mathcal{C}, \otimes, I, a, l, r)$ is said to be strict if the associativity and the unit constraints are all identities of the category.*

We now want to adapt the concepts of functor, natural transformation and equivalence to the case of tensor categories. So we have the following

Definition 1.3.9. *Let $(\mathcal{C}, \otimes, I, a, l, r)$ and $(\mathcal{D}, \otimes, I, a, l, r)$ be two tensor categories.*

- *A tensor functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$, where $F : \mathcal{C} \to \mathcal{D}$ is a functor, φ_0 is an isomorphism from I to $F(I)$, and*

$$\varphi_2(U, V) : F(U) \otimes F(V) \to F(U \otimes V)$$

is a family of natural isomorphisms indexed by all couples (U, V) of objects of $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc}
(F(U) \otimes F(V)) \otimes F(W) & \xrightarrow{a_{F(U),F(V),F(W)}} & F(U) \otimes (F(V) \otimes F(W)) \\
{\scriptstyle \varphi_2(U,V) \otimes \mathsf{id}_{F(W)}} \downarrow & & \downarrow {\scriptstyle \mathsf{id}_{F(U)} \otimes \varphi_2(V,W)} \\
F(U \otimes V) \otimes F(W) & & F(U) \otimes F(V \otimes W) \\
{\scriptstyle \varphi_2(U \otimes V, W)} \downarrow & & \downarrow {\scriptstyle \varphi_2(U, V \otimes W)} \\
F((U \otimes V) \otimes W) & \xrightarrow{F(a_{U,V,W})} & F(U \otimes (V \otimes W))
\end{array}$$

and the squares

$$\begin{array}{ccc}
I \otimes F(U) & \xrightarrow{l_{F(U)}} & F(U) \\
{\scriptstyle \varphi_0 \otimes \mathsf{id}_{F(U)}} \downarrow & & \uparrow {\scriptstyle F(l_U)} \\
F(I) \otimes F(U) & \xrightarrow{\varphi_2(I,U)} & F(I \otimes U)
\end{array}$$

and

$$\begin{array}{ccc}
F(U) \otimes I & \xrightarrow{r_{F(U)}} & F(U) \\
{\scriptstyle \mathsf{id}_{F(U)} \otimes \varphi_0} \downarrow & & \uparrow {\scriptstyle F(r_U)} \\
F(U) \otimes F(I) & \xrightarrow{\varphi_2(U,I)} & F(U \otimes I)
\end{array}$$

commute for all objects (U, V, W) in $\mathcal{C}$.

- *A tensor functor $(F, \varphi_0, \varphi_2)$ is said to be strict if the isomorphisms φ_0 and φ_2 are identities of $\mathcal{D}$.*

- *A natural tensor transformation $\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi_0,' \varphi_2')$ between tensor functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \to F'$ such that the diagrams*

$$\begin{array}{ccc}
I & \xrightarrow{\varphi_0} & F(I) \\
& {\scriptstyle \varphi_0'} \searrow & \downarrow {\scriptstyle \eta(I)} \\
& & F'(I)
\end{array}$$

and

$$\begin{array}{ccc} F(U)\otimes F(V) & \xrightarrow{\varphi_2(U,V)} & F(U\otimes V) \\ \eta(U)\otimes\eta(V)\downarrow & & \downarrow\eta(U\otimes V) \\ F'(U)\otimes F'(V) & \xrightarrow{\varphi_2'(U,V)} & F'(U\otimes V) \end{array}$$

commute for any couple (U,V) *of objects of* $\mathcal{C}$*.*

- *A natural tensor isomorphism is a natural tensor transformation that is also a natural isomorphism.*

- *A tensor equivalence between two tensor categories* $\mathcal{C}$ *and* $\mathcal{D}$ *is a tensor functor* $F:\mathcal{C}\to\mathcal{D}$ *such that there exist a tensor functor* $F':\mathcal{D}\to\mathcal{C}$ *and two natural tensor isomorphisms* $\eta:\mathsf{id}_{\mathcal{D}}\to FF'$ *and* $\theta:F'F\to\mathsf{id}_{\mathcal{C}}$*.*

- *In case there exists a tensor equivalence between* $\mathcal{C}$ *and* $\mathcal{D}$*, we say that* $\mathcal{C}$ *and* $\mathcal{D}$ *are tensor equivalent.*

Remark 1.3.10. *It is important to observe that the category* $\mathsf{Vect}(\mathbb{K})$ *is not strict, and so we have to use brackets accurately. However, the following result, due to Saunders Mac Lane, allows us to imagine that every tensor category is strict (and so also the category* $\mathsf{Vect}(\mathbb{K})$*). In order to lighten the notation, we will use this result very often, so that we can use less brackets in the discussion.*

Theorem 1.3.11 (**Mac Lane**). *Let* $\mathcal{C}=(\mathcal{C},\otimes,I,a,l,r)$ *be a tensor category. Then there exists a strict category* $\mathcal{C}^{str}$ *which is tensor equivalent to* $\mathcal{C}$*.*

A proof of this Theorem can be find in [ML13]. This result allows us to denote by $V^{\otimes n}$ the n–th tensor product of an object V of a tensor category.

1.4 Braided tensor categories

Definition 1.4.1. *A braided tensor category is a seventuple* $(\mathcal{C},\otimes,I,a,l,r,c)$*, where:*

- $(\mathcal{C},\otimes,I,a,l,r)$ *is a tensor category;*

- c *is a commutativity constraint, i.e. a natural isomorphism*

$$c:\otimes\to\otimes\tau.$$

This means that, for any couple (V, W) of objects in the category $\mathcal{C}$, we have an isomorphism

$$c_{V,W} : V \otimes W \to W \otimes V$$

such that the square

$$\begin{array}{ccc} V \otimes W & \xrightarrow{c_{V,W}} & W \otimes V \\ \downarrow{\scriptstyle f\otimes g} & & \downarrow{\scriptstyle g\otimes f} \\ V' \otimes W' & \xrightarrow{c_{V',W'}} & W' \otimes V' \end{array}$$

commutes for any f in $\mathsf{Hom}(V, \cdot)$ and g in $\mathsf{Hom}(W, \cdot)$;

such that the two hexagonal diagrams
$(E1)$

$$\begin{array}{ccccc} & U \otimes (V \otimes W) & \xrightarrow{c_{U,V\otimes W}} & (V \otimes W) \otimes U & \\ {\scriptstyle a_{U,V,W}} \nearrow & & & & \searrow {\scriptstyle a_{V,W,U}} \\ (U \otimes V) \otimes W & & & & V \otimes (W \otimes U) \\ {\scriptstyle c_{U,V}\otimes \mathsf{id}_W} \searrow & & & & \nearrow {\scriptstyle \mathsf{id}_V \otimes c_{U,W}} \\ & (V \otimes U) \otimes W & \xrightarrow[a_{V,U,W}]{} & V \otimes (U \otimes W) & \end{array}$$

and $(E2)$

$$\begin{array}{ccccc} & (U \otimes V) \otimes W & \xrightarrow{c_{U\otimes V,W}} & W \otimes (U \otimes V) & \\ {\scriptstyle a^{-1}_{U,V,W}} \nearrow & & & & \searrow {\scriptstyle a^{-1}_{W,U,V}} \\ U \otimes (V \otimes W) & & & & (W \otimes U) \otimes V \\ {\scriptstyle \mathsf{id}_U \otimes c_{V,W}} \searrow & & & & \nearrow {\scriptstyle c_{U,W} \otimes \mathsf{id}_V} \\ & U \otimes (W \otimes V) & \xrightarrow[a^{-1}_{U,W,V}]{} & (U \otimes W) \otimes V & \end{array}$$

commute for all objects U, V, W in $\mathcal{C}$. The functor c is called a braiding for the tensor category $(\mathcal{C}, \otimes, I, a, l, r)$.

Note that if c is a braiding for $\mathcal{C}$, then also the inverse c^{-1} it is. A braided tensor category $\mathcal{C}$ is said to be symmetric if its braiding c satisfies

$$c_{W,V} \circ c_{V,W} = \mathsf{id}_{V\otimes W}$$

for any V and W in $\mathsf{Obj}(\mathcal{C})$. When the tensor category $\mathcal{C}$ is strict, the commutativity of the diagrams $(E1)$ and $(E2)$ are respectively equivalent to the relations

- $c_{U,V\otimes W} = (\mathsf{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathsf{id}_W)$;
- $c_{U\otimes V,W} = (c_{U,W} \otimes \mathsf{id}_V)(\mathsf{id}_U \otimes c_{V,W})$.

Example 1.4.2. *Due to the standard isomorphisms between vector spaces, we have that the flip functor*

$$\tau_{V,W} : V \otimes W \to W \otimes V$$

is a braiding for the tensor category $\mathsf{Vect}(\mathbb{K})$*. Furthermore, the braiding* τ *satisfies* $\tau_{W,V} \circ \tau_{V,W} = \mathsf{id}_{V,W}$*, and so* $\mathsf{Vect}(\mathbb{K})$ *is a symmetric braided tensor category.*

Proposition 1.4.3. *For any object* V *in a braided tensor category* $\mathcal{C}$ *with unit* I*, we have*

- $l_V \circ c_{V,I} = r_V$*;*
- $r_V \circ c_{I,V} = l_V$*;*
- $c_{I,V} = c_{V,I}^{-1}$*.*

When the category is strict, these relations become

$$c_{I,V} = c_{V,I} = \mathsf{id}_V.$$

We refer the proof of this Proposition to [Kas12].

1.4.1 The Yang–Baxter equation

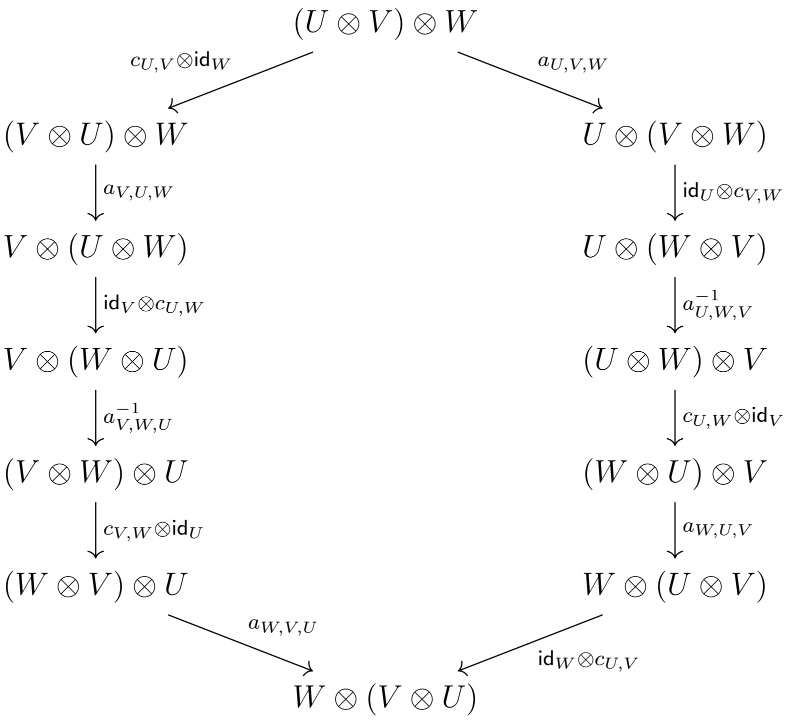
Definition 1.4.4. *Let* V *be a vector space over a field* $\mathbb{K}$*. A linear automorphism* c *of* $V \otimes V$ *is said to be a* R*–matrix if it is a solution of the Yang–Baxter equation*

$$(c \otimes \mathsf{id}_V)(\mathsf{id}_V \otimes c)(c \otimes \mathsf{id}_V) = (\mathsf{id}_V \otimes c)(c \otimes \mathsf{id}_V)(\mathsf{id}_V \otimes c)$$

that holds in the automorphism group of $V^{\otimes 3}$*. We abbreviate the Yang–Baxter equation with* YBE*.*

The following result can be considered the categorical version of the Yang–Baxter equation and is one of the main properties of braided tensor categories.

Theorem 1.4.5. *Let U, V, W be three objects of a braided tensor category. Then the dodecagon*



commutes. If the category $\mathcal{C}$ is strict, then the commutativity of the dodecagon is equivalent to the relation

$$(c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}),$$

that implies that the natural isomorphism $c_{V,V}$ is a solution of the Yang–Baxter equation for any object V in $\mathcal{C}$.

We refer the proof of this proposition to [Kas12].

1.4.2 Braided tensor functors

We now want to adapt the definitions seen in the Subsection 1.3.2 to the braided case.

Definition 1.4.6. *Let $(\mathcal{C}, \otimes, I, a, l, r, c)$ and $(\mathcal{D}, \otimes, I, a, l, r, c)$ be two braided tensor categories.*

- *A tensor functor $(F, \varphi_0, \varphi_2)$ from $\mathcal{C}$ to $\mathcal{D}$ is said to be braided if, for any pair (V, V') of objects in $\mathcal{C}$, the square*

$$\begin{array}{ccc} F(V) \otimes F(V') & \xrightarrow{\varphi_2(V,V')} & F(V \otimes V') \\ {\scriptstyle c_{F(V),F(V')}}\downarrow & & \downarrow{\scriptstyle F(c_{V,V'})} \\ F(V') \otimes F(V) & \xrightarrow{\varphi_2(V',V)} & F(V' \otimes V) \end{array}$$

  commutes.

- *A braided tensor functor $(F, \varphi_0, \varphi_2)$ is said to be strict if the isomorphisms φ_0 and φ_2 are identities of $\mathcal{D}$.*

- *A natural tensor braided isomorphism is a natural tensor braided transformation that is also a natural isomorphism.*

1.5 Limits and inverse systems

From the fourth chapter onwards we will need the concept of inverse system. For more precise details on inverse systems we refer to the eighth chapter of [ES15].

1.5.1 Limits in categories

Definition 1.5.1. *Let $\mathcal{C}$ and $\mathcal{J}$ be two categories.*

- *A diagram of shape $\mathcal{J}$ in $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$, where the category $\mathcal{J}$ is thought of as an index category, and the functor F is thought of as indexing a collection of objects and morphisms in $\mathcal{C}$ patterned on $\mathcal{J}$. The category $\mathcal{J}$ is called the index category or the scheme of the diagram F.*

- *Given a diagram F of shape $\mathcal{J}$ in $\mathcal{C}$, we define a cone to F as a couple (N, ψ), where N is an object of $\mathcal{C}$ and ψ is a family*

$$\psi_X : N \to F(X)$$

  of morphisms in $\mathsf{Hom}_{\mathcal{C}}(N, \cdot)$ indexed by the objects X of $\mathcal{J}$, such that for every morphism $f : X \to Y$ in $\mathsf{Hom}(\mathcal{J})$, we have

$$F(f) \circ \psi_X = \psi_Y.$$

We now have the tools to define the concept of limit in the context of categories.

Definition 1.5.2. *Let $\mathcal{C}$ and $\mathcal{J}$ be two categories and let $F : \mathcal{J} \to \mathcal{C}$ be a diagram of shape $\mathcal{J}$ in $\mathcal{C}$. A limit of the diagram $F : \mathcal{J} \to \mathcal{C}$ is a cone (L, ϕ) to F such that for any other cone (N, ψ) to F there exists a unique morphism*

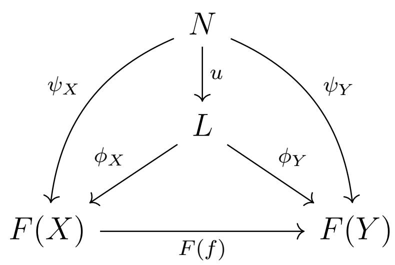
$$u : N \to L$$

such that

$$\phi_X \circ u = \psi_X$$

for all X in $\mathsf{Obj}(J)$.

We can represent this definiton with the following diagram



1.5.2 Inverse systems and inverse limits

Inverse limits are a special case of the concept of a limit in category theory. They can be defined in an arbitrary category through a universal construction, that is the following.

Definition 1.5.3. *A directed set is a couple $(I, \leq)$, where I is a nonempty set and $\leq$ is a reflexive and transitive binary relation, with the additional property that every pair of elements has an upper bound. In other words, for any couple i, j in I there exists k in I such that $i \leq k$ and $j \leq k$.*

Example 1.5.4. *The set of natural numbers $\mathbb{N}$ is a directed set.*

Definition 1.5.5. *Let $\mathcal{C}$ be a category. An inverse system in $\mathcal{C}$ consists:*

- *of a directed set $(I, \leq)$;*
- *of a collection $\{X_i\}_{i \in I}$ in $\mathsf{Obj}(\mathcal{C})$;*
- *of a collection of morphisms $f_{ji} : X_j \to X_i$ for any $i \leq j$*

such that:

(i) $f_{ii} = \mathsf{id}_{X_i}$ *for all* $i \in I$*;*

(ii) $f_{ji} \circ f_{kj} = f_{ki}$ *whenever* $i \le j \le k$*.*

We denote an inverse system by a triple $(I, \{X_i\}, \{f_{ji}\})$*.*

Definition 1.5.6. *Let* $\mathcal{C}$ *be a category and let* $(I, \{X_i\}, \{f_{ji}\})$ *be an inverse system in* $\mathcal{C}$*. An inverse limit for the inverse system* $(I, \{X_i\}, \{f_{ji}\})$ *is a couple* $(X, \{\pi_i\})$*, where:*

- X *is in* $\mathsf{Obj}(\mathcal{C})$*;*
- π_i *is in* $\mathsf{Hom}_{\mathcal{C}}(X, X_i)$ *for any* $i \in I$

such that:

(i) For any $i \le j$ *the following diagram commutes:*

$$\begin{array}{ccc} X & & \\ {\scriptstyle \pi_j}\downarrow & \searrow^{\pi_i} & \\ X_j & \xrightarrow[f_{ji}]{} & X_i \end{array} \quad ;$$

(ii) Given any Y *in* $\mathsf{Obj}(\mathcal{C})$ *and any collection of morphisms* ϕ_i *in* $\mathsf{Hom}(Y, X_i)$ *such that the diagram*

$$\begin{array}{ccc} Y & & \\ {\scriptstyle \phi_j}\downarrow & \searrow^{\phi_i} & \\ X_j & \xrightarrow[f_{ji}]{} & X_i \end{array}$$

commutes for any $i \le j$*, there exists a unique morphism* ϕ *in* $\mathsf{Hom}(Y, X)$ *such that the following diagram commutes for all* $i \in I$*:*

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & X \\ & \searrow_{\phi_i} & \downarrow{\scriptstyle \pi_i} \\ & & X_i \end{array} \quad .$$

If an inverse limit exists, then it is unique up to $\mathcal{C}$*–isomorphism and is denoted by* $\varprojlim_{i \in I} X_i$*.*

The inverse limit of an inverse system does not always exists. However, in the cases of our interest they exist and admit the following description:

$$\varprojlim_{i \in I} X_i = \left\{ x \in \prod_{i \in I} X_i \mid f_{ji}(x_j) = x_i \text{ for all } i \le j \right\}.$$

1.5.3 The inverse limit topology

We will consider inverse systems whose directed set is $\mathbb{N}$ and whose inverse limit is of the form described above.
The inverse limit of an inverse system $(\mathbb{N}, \{X_n\}, \{f_{mn}\})$ possesses a natural topology, called the inverse limit topology. It is obtained as follows. Consider the inverse limit $(X, \{\pi_n\})$ of the inverse system $(\mathbb{N}, \{X_n\}, \{f_{mn}\})$ and consider the discrete topology on each X_n.

Definition 1.5.7. *The inverse limit topology of* $\varprojlim_{i\in I} X_i$ *is the restriction of the direct product topology of* $\prod_{n\in\mathbb{N}} X_n$ *to* $\varprojlim_{i\in I} X_i$.

Therefore, a basis of open sets of the inverse limit topology is given by the family of all subsets $\pi_n^{-1}(U_n)$, where n runs in $\mathbb{N}$ and U_n is any subset of X_n. By definition of this topology, we observe that:

- the map $\pi_n : \varprojlim_{n\in\mathbb{N}} X_n \to X_n$ is continuous for any $n \in \mathbb{N}$;
- if Y is a topological space and $f : Y \to \varprojlim_{n\in\mathbb{N}} X_i$ is a map, then f is continuous if and only if the map

$$\pi_n \circ f : Y \to X_n$$

  is continuous for any $n \in \mathbb{N}$.

Chapter 2

The language of Hopf algebras

2.1 Algebras

Definition 2.1.1. *Let $\mathbb{K}$ be a field. A $\mathbb{K}$–algebra or, simply, an algebra is a triple (A, μ, η), where:*

(i) A is a vector space over $\mathbb{K}$;

(ii) $\mu : A \otimes A \to A$ is a $\mathbb{K}$–linear map, called the multiplication;

(iii) $\eta : \mathbb{K} \to A$ is a $\mathbb{K}$–linear map, called the unit

such that:

(A1) (associativity axiom): the square

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \mathsf{id}} & A \otimes A \\ {\scriptstyle \mathsf{id} \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

commutes;

(A2) (unit axiom): the diagram

$$\begin{array}{ccccc} \mathbb{K} \otimes A & \xrightarrow{\eta \otimes \mathsf{id}} & A \otimes A & \xleftarrow{\mathsf{id} \otimes \eta} & A \otimes \mathbb{K} \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \mu} & {}^{\simeq}\swarrow & \\ & & A & & \end{array}$$

commutes.

If moreover the triangle

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & {\scriptstyle\mu}\searrow & \downarrow{\scriptstyle\mu} \\ & & A \end{array}$$

commutes, we say that the algebra (A, μ, η) *is commutative.*

Definition 2.1.2. *Let* (A, μ, η) *and* (A', μ', η') *be two* $\mathbb{K}$*–algebras. A morphism of algebras is a* $\mathbb{K}$*–linear map* $f : A \to A'$ *such that:*

(MA1) $\mu' \circ (f \otimes f) = f \circ \mu$*;*

(MA2) $f \circ \eta = \eta'$*.*

A morphism is said to be an isomorphism if it is invertible. If A, A' *and* A'' *are three algebras and* $f : A \to A'$ *and* $g : A' \to A''$ *are morphisms of algebras, then the composition* $g \circ f : A \to A''$ *is a morphism of algebras. Composition of isomorphisms is still an isomorphism.*

Example 2.1.3. *We now give some examples of algebras.*

- *Any field* $\mathbb{K}$ *is a commutative* $\mathbb{K}$*–algebra.*

- *Let* V *be a vector space over a field* $\mathbb{K}$*. The set* $\mathsf{End}(V)$ *of all the linear endomorphisms of* V *is an algebra, with multiplication given by the composition and unit given by the identity map* id_V *of* V*.*

- *For any algebra* (A, μ, η)*, set*

$$\mu^{op} := \mu \circ \tau_{A,A}.$$

  Then the triple (A, μ^{op}, η) *is an algebra, which we call the opposite algebra of* A *and denote by* A^{op}*.*

- *Let* (A, μ, η) *and* (A', μ', η') *be two algebras. Then the triple*

$$(A \otimes A', (\mu \otimes \mu')(\mathsf{id}_A \otimes \tau_{A,A'} \otimes \mathsf{id}_{A'}), \eta \otimes \eta')$$

  is an algebra.

2.1.1 Modules

Definition 2.1.4. *Let (A, μ, η) be an algebra. A left A–module or, simply, an A–module is a couple (M, μ_M), where:*

- *M is a vector space;*
- *$\mu_M : A \otimes M \to M$ is a linear map, called the action of A on M, such that*

  (i) the square

$$\begin{array}{ccc} (A \otimes A) \otimes M & \xrightarrow{\mu \otimes \mathsf{id}} & A \otimes M \\ {\scriptstyle \mathsf{id} \otimes \mu_M} \downarrow & & \downarrow {\scriptstyle \mu_M} \\ A \otimes M & \xrightarrow{\mu_M} & M \end{array}$$

  commutes;

  (ii) The triangle

$$\begin{array}{ccc} \mathbb{K} \otimes M & \xrightarrow{\eta \otimes \mathsf{id}} & A \otimes M \\ & {\scriptstyle \simeq} \searrow & \downarrow {\scriptstyle \mu_M} \\ & & M \end{array}$$

  commutes.

We denote by $a \cdot v$ the action of $a \in A$ on the vector $v \in M$.

Example 2.1.5. *Let (A, μ, η) be an algebra. Then:*

- *(A, μ) is an A–module, with action given by the left multiplication;*
- *given (M, μ_M) and $(M', \mu_{M'})$ two A–modules, we have that*

$$(M \otimes M', \mu_M \otimes \mu_{M'})$$

  is an $A \otimes A$–module, whose action is given by

$$(a \otimes a') \cdot (v \otimes v') \coloneqq a \cdot v \otimes a' \cdot v'.$$

Definition 2.1.6. *Let (A, μ, η) be an algebra and let M, M' be two A–modules. A linear map $f : M \to M'$ is said to be a morphism of A–modules if*

$$f(a \cdot v) = a \cdot f(v)$$

for all $a \in A$ and $v \in M$. Morphisms of A–modules are usually called A–linear maps.

A morphism is said to be an isomorphism if it is invertible. If M, M' and M'' are three A–modules and $f : M \to M'$ and $g : M' \to M''$ are morphisms of A–modules, then the composition $g \circ f : M \to M''$ is a morphism of A–modules. Composition of isomorphisms is still an isomorphism.

Definition 2.1.7. *Let (M, μ_M) be an A–module. The algebra morphism*

$$\begin{aligned} \rho : A &\to \mathsf{End}(M) \\ a &\mapsto \mu_M(a, \cdot) \end{aligned}$$

is called a representation of A on M.

Given an algebra (A, μ, η), we have that the class of all A–modules with the class of all morphisms of A–modules form a category, that we denote by $\mathsf{Mod}(A)$.

2.2 Coalgebras

Definition 2.2.1. *Let $\mathbb{K}$ be a field. A $\mathbb{K}$–coalgebra or, simply, a coalgebra is a triple (C, Δ, ε), where:*

(i) C is a vector space over $\mathbb{K}$;

(ii) $\Delta : C \to C \otimes C$ is a $\mathbb{K}$–linear map, called the comultiplication;

(iii) $\varepsilon : C \to \mathbb{K}$ is a $\mathbb{K}$–linear map, called the counit

such that:

(C1) (coassociativity axiom): the square

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathsf{id}\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes\mathsf{id}} & C \otimes C \otimes C \end{array}$$

commutes;

(C2) (counit axiom): the diagram

$$\begin{array}{ccccc} \mathbb{K} \otimes C & \xleftarrow{\varepsilon\otimes\mathsf{id}} & C \otimes C & \xrightarrow{\mathsf{id}\otimes\varepsilon} & C \otimes \mathbb{K} \\ & {\scriptstyle \simeq}\nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle \simeq} & \\ & & C & & \end{array}$$

commutes.

If moreover the triangle

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ & \searrow^{\Delta} & \downarrow \tau \\ & & C \otimes C \end{array}$$

commutes, we say that the coalgebra (C, Δ, ε) *is cocommutative.*

Definition 2.2.2. *Let* (C, Δ, ε) *and* $(C', \Delta', \varepsilon')$ *be two* $\mathbb{K}$*–coalgebras. A morphism of coalgebras is a* $\mathbb{K}$*–linear map* $f : C \to C'$ *such that:*

(MC1) $(f \otimes f) \circ \Delta = \Delta' \circ f$;

(MC2) $\varepsilon = \varepsilon' \circ f$.

A morphism is said to be an isomorphism if it is invertible. If C, C' and C'' are three coalgebras and $f : C \to C'$ and $g : C' \to C''$ are morphisms of coalgebras, then the composition $g \circ f : C \to C''$ is a morphism of coalgebras. Composition of isomorphisms is still an isomorphism.

Example 2.2.3. *We now give some examples of coalgebras.*

- *Any field* $\mathbb{K}$ *is a* $\mathbb{K}$*–coalgebra with* $\Delta(1) = 1 \otimes 1$ *and* $\varepsilon(1) = 1$*. Moreover, for any coalgebra* (C, Δ, ε)*, the map* $\varepsilon : C \to \mathbb{K}$ *is a morphism of coalgebras.*
- *For any coalgebra* (C, Δ, ε)*, set*

$$\Delta^{op} := \tau_{C,C} \circ \Delta.$$

  Then the triple $(C, \Delta^{op}, \varepsilon)$ *is a coalgebra, which we call the opposite coalgebra of* C *and denote by* C^{op}.
- *Let* (C, Δ, ε) *and* $(C', \Delta', \varepsilon')$ *be two coalgebras. Then the triple*

$$(C \otimes C', (\mathsf{id}_C \otimes \tau_{C,C'} \otimes \mathsf{id}_{C'}) \circ (\Delta \otimes \Delta'), \varepsilon \otimes \varepsilon')$$

  is a coalgebra.

Convention. We now introduce the Sweedler's sigma notation. If x is an element of a coalgebra (C, Δ, ε), we denote the element $\Delta(x) \in C \otimes C$ by

$$\Delta(x) = \sum_{(x)} x' \otimes x''.$$

Using this notation, we may express the coassociativity constraint of a map Δ by the following equality:

$$\sum_{(x,x')} (x')' \otimes (x')'' \otimes x'' = \sum_{(x,x'')} x' \otimes (x'')' \otimes (x'')''.$$

2.2.1 Comodules

Definition 2.2.4. *Let (C, Δ, ε) be a coalgebra. A left C–comodule or, simply, a C–comodule is a pair (N, Δ_N), where:*

- *N is a $\mathbb{K}$–vector space;*
- *$\Delta_N : N \to C \otimes N$ is a $\mathbb{K}$–linear map, called the coaction of C on N, such that:*

  (i) the square

$$\begin{array}{ccc} N & \xrightarrow{\Delta_N} & C \otimes N \\ {\scriptstyle \Delta_N}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta_N} \\ C \otimes N & \xrightarrow{\Delta\otimes\mathrm{id}} & (C \otimes C) \otimes N \end{array}$$

  commutes;

  (ii) the triangle

$$\begin{array}{ccc} \mathbb{K} \otimes N & \xrightarrow{\varepsilon\otimes\mathrm{id}} & C \otimes N \\ & \nwarrow{\scriptstyle \simeq} & \uparrow{\scriptstyle \Delta} \\ & & N \end{array}$$

  commutes.

Definition 2.2.5. *Let (C, Δ, ε) be a coalgebra and let (N, Δ_N) and $(N', \Delta_{N'})$ be two C–comodules. A linear map $f : N \to N'$ is said to be a morphism of C–comodules if*

$$(\mathrm{id}_C \otimes f) \circ \Delta_N = \Delta_{N'} \circ f.$$

A morphism is said to be an isomorphism if it is invertible. If N, N' and N'' are three C–comodules and $f : N \to N'$ and $g : N' \to N''$ are morphisms of C–comodules, then the composition $g \circ f : N \to N''$ is a morphism of C–comodules. Composition of isomorphisms is still an isomorphism.

2.2.2 Duality between algebras and coalgebras

We now want to see the link between the notion of algebra and the notion of coalgebra. Let (C, Δ, ε) be a coalgebra and consider the map

$$\lambda : C^* \otimes C^* \to (C \otimes C)^*$$

defined by

$$\lambda(f \otimes g)(a \otimes b) = f(a) \otimes g(b).$$

Set $\bar{\lambda} = \lambda \circ \tau_{C,C}$ and consider the triple (A, μ, η), where:

- $A = C^*$;
- $\mu = \Delta^* \circ \bar{\lambda}$;
- $\eta = \varepsilon^*$;

where the symbol $*$ indicate the transpose of a linear map. We have that the triple (A, μ, η) is an algebra, and we so we have that the dual vector space of a coalgebra has a natural structure of algebra. Conversely, let (A, μ, η) be a finite–dimensional algebra. Then the map $\bar{\lambda}$ defined above is an isomorphism from $A^* \otimes A^*$ to $(A \otimes A)^*$, which allows us to consider the triple (C, Δ, ε), where:

- $C = A^*$;
- $\Delta = \bar{\lambda}^{-1} \circ \mu^*$;
- $\varepsilon = \eta^*$.

We have that the triple (C, Δ, ε) is a coalgebra, and so the dual vector space of a finite–dimensional algebra has a natural structure of coalgebra.
In general, the dual vector space of an algebra does not carry a natural coalgebra structure. However, in the infinite–dimensional case, we can use the following trick. Let (A, μ, η) be an infinite–dimensional algebra and let us consider the set

$$A^o := \{f \in A^* | f(I) = 0 \text{ for some ideal } I \text{ of } A,\ \dim A/I < +\infty\}.$$

The set A^o is called the finite dual of A, and it has always a coalgebra structure. A proof of this fact can be find in the sixth chapter of [Swe69].

2.3 Bialgebras

Let H be a vector space with an algebra structure (H, μ, η) and a coalgebra structure (H, Δ, ε). How we have seen in the previous sections, the vector space $H \otimes H$ is naturally endowed with an algebra and a coalgebra structure. We now want to see when these two structures on H *work well* together.

Theorem 2.3.1. *The following statements are equivalent:*

(i) the maps μ and η are morphisms of coalgebras;

(ii) the maps Δ and ε are morphisms of algebras.

Proof. The proof consists in writing down the commutativive diagrams expressing both statements. The fact that μ is a morphism of coalgebras is equivalent, using (MC1) and (MC2), to the commutativity of the two squares

$$\begin{CD} H\otimes H @>{\mu}>> H \\ @V{(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\Delta\otimes\Delta)}VV @VV{\Delta}V \\ (H\otimes H)\otimes(H\otimes H) @>{\mu\otimes\mu}>> H\otimes H \end{CD} \qquad \begin{CD} H\otimes H @>{\varepsilon\otimes\varepsilon}>> \mathbb{K}\otimes\mathbb{K} \\ @V{\mu}VV @VV{\mathrm{id}}V \\ H @>{\varepsilon}>> \mathbb{K} \end{CD}$$

whereas the fact that η is a morphism of coalgebras is expressed, using (MC1) and (MC2), by the commutativity of the two diagrams

$$\begin{CD} \mathbb{K} @>{\eta}>> H \\ @V{\mathrm{id}}VV @VV{\Delta}V \\ \mathbb{K}\otimes\mathbb{K} @>{\eta\otimes\eta}>> H\otimes H \end{CD} \qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\eta} & H \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle\varepsilon} \\ & & \mathbb{K} \end{array} .$$

Otherwise, the fact that Δ is a morphism of algebras is equivalent, using (MA1) and (MA2), to the commutativity of the two squares

$$\begin{CD} H\otimes H @>{\Delta\otimes\Delta}>> (H\otimes H)\otimes(H\otimes H) \\ @V{\mu}VV @VV{(\mu\otimes\mu)(\mathrm{id}\otimes\tau\otimes\mathrm{id})}V \\ H @>{\Delta}>> H\otimes H \end{CD} \qquad \begin{CD} \mathbb{K} @>{\eta}>> H \\ @V{\mathrm{id}}VV @VV{\Delta}V \\ \mathbb{K}\otimes\mathbb{K} @>{\eta\otimes\eta}>> H\otimes H \end{CD}$$

whereas the fact that ε is a morphism of algebras is expressed, using (MA1) and (MA2), by the commutativity of the two diagrams

$$\begin{CD} H\otimes H @>{\varepsilon\otimes\varepsilon}>> \mathbb{K}\otimes\mathbb{K} \\ @V{\mu}VV @VV{\mathrm{id}}V \\ H @>{\varepsilon}>> \mathbb{K} \end{CD} \qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\eta} & H \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle\varepsilon} \\ & & \mathbb{K} \end{array} .$$

The first four diagrams are the same of the second ones, and so the claim is proved. □

The Theorem above allows us to give the following

Definition 2.3.2. *A bialgebra is a quintuple* $(H, \mu, \eta, \Delta, \varepsilon)$*, where:*

- *the triple* (H, μ, η) *is an algebra;*
- *the triple* (H, Δ, ε) *is a coalgebra;*
- *the quintuple* $(H, \mu, \eta, \Delta, \varepsilon)$ *satisfies one of the two equivalent statements of the Theorem 2.3.1.*

Definition 2.3.3. *Let* $(H, \mu, \eta, \Delta, \varepsilon)$ *and* $(H', \mu', \eta', \Delta', \varepsilon')$ *be two bialgebras. A linear map* $f : H \to H'$ *is a morphism of bialgebras if:*

- f *is a morphism of algebras;*
- f *is a morphism of coalgebras.*

Proposition 2.3.4. *Let* $(H, \mu, \eta, \Delta, \varepsilon)$ *be a bialgebra. Then:*

(1) $H^{op} = (H, \mu^{op}, \eta, \Delta, \varepsilon)$*,*

(2) $H^{cop} = (H, \mu, \eta, \Delta^{op}, \varepsilon)$*, and*

(3) $H^{opcop} = (H, \mu^{op}, \eta, \Delta^{op}, \varepsilon)$

are bialgebras.

We refer the proofs of the results above to [Kas12].

2.4 Representations

In this section we want to study the category $\mathsf{Mod}(A)$ of all modules of a bialgebra A. In particular, we want to have the axioms of braided tensor category on $\mathsf{Mod}(A)$, and so we have to investigate under which conditions they exist. We start the reasoning taking a cue from the braided tensor category we already know: $\mathsf{Vect}(\mathbb{K})$.
Let (A, μ, η) be an algebra and let $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{K}$ be morphisms between algebras. We have seen in the Subsection 3.1.1 that if V, W are A–modules, then $V \otimes W$ is too. Furthermore, the map Δ allows to pull back this $A \otimes A$–module structure into an A–module structure given by

$$a \cdot (u \otimes v) = \Delta(a) \cdot (u \otimes v).$$

Moreover, the map ε endows $\mathbb{K}$ with an A–module structure given by

$$a \cdot \lambda = \varepsilon(a)\lambda.$$

This means that the tensor product of $\mathsf{Vect}(\mathbb{K})$ restricts to a functor

$$\otimes : \mathsf{Mod}(A) \times \mathsf{Mod}(A) \to \mathsf{Mod}(A)$$

for which $\mathbb{K}$ is a unit.

Lemma 2.4.1. *Let* $(A, \mu, \eta, \Delta, \varepsilon)$ *be a bialgebra and let* U, V, W *be three* A*–modules. Then:*

(i) $(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$ *as A–modules;*

(ii) $\mathbb{K} \otimes V \simeq V \simeq V \otimes \mathbb{K}$ *as A–modules.*

If furthermore A is cocommutative, then the flip

$$\tau_{V,W} : V \otimes W \to W \otimes V$$

is an isomorphism of A–modules.

Proof. It suffices to consider the canonical isomorphisms of the category $\mathsf{Vect}(\mathbb{K})$. □

The next result gives us an interpretation of bialgebras in terms of their modules.

Theorem 2.4.2. *Let* (A, μ, η) *be an algebra and let* $\Delta : A \to A \otimes A$ *and* $\varepsilon : A \to \mathbb{K}$ *be morphisms of algebras. Then the quintuple* $(A, \mu, \eta, \Delta, \varepsilon)$ *is a bialgebra if and only if the category* $\mathsf{Mod}(A)$ *equipped with the tensor product described above and the constraints* a, l, r *of* $\mathsf{Vect}(\mathbb{K})$ *is a tensor category.*

Proof. Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. It follows from the Lemma 2.4.1 that $(\mathsf{Mod}(A), \otimes, \mathbb{K}, a, l, r)$ is a tensor category. Conversely, let (A, μ, η) be an algebra together with two morphisms of algebras

$$\Delta : A \to A \otimes A$$

and

$$\varepsilon : A \to \mathbb{K}$$

and suppose that $(\mathsf{Mod}(A), \otimes, \mathbb{K}, a, l, r)$ is a tensor category. We have to show that the triple (A, Δ, ε) satisfies the properties (C1) and (C2).

(C1) : Consider the associativity constraint $a_{A,A,A}$. By hypothesis, it is A–linear, which means that for $a, u, v, w \in A$ we have

$$a_{A,A,A}(a((u \otimes v) \otimes w)) = a \cdot a_{A,A,A}((u \otimes v) \otimes w).$$

By definition of the associativity constraint, this can be expressed by

$$(\Delta \otimes \mathsf{id})(\Delta(a))(u \otimes (v \otimes w)) = (\mathsf{id} \otimes \Delta)(\Delta(a))(u \otimes (v \otimes w)).$$

Setting $u = v = w = 1$ we get

$$(\Delta \otimes \mathsf{id})(\Delta(a)) = (\mathsf{id} \otimes \Delta)(\Delta(a)).$$

(C2) : By hypothesis, the left unit constraint l_A and the right unit constraint r_A are A–linear, and then similarly to the previous case we obtain that

$$(\varepsilon \otimes \mathsf{id})(\Delta(a)) = a$$
$$(\mathsf{id} \otimes \varepsilon)(\Delta(a)) = a.$$

for all $a \in A$.

□

We now want to construct a braiding for the category $\mathsf{Mod}(A)$, and so we have to define a functor that satisfies the hexagon axioms. We proceed by cases. How we have seen in the previous chapter, in the case of $\mathsf{Vect}(\mathbb{K})$, we have that the flip functor τ is a braiding. Let us see if the flip functor does the same in the category $\mathsf{Mod}(A)$. Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. We want that, for any V, W in $\mathsf{Mod}(A)$, the map

$$\tau_{V,W} : V \otimes W \to W \otimes V$$
$$v \otimes w \mapsto w \otimes v$$

is an isomorphism of A–modules. In other words, we want that the following relation holds:

$$\tau_{V,W}(a \cdot (v \otimes w)) = a \cdot \tau_{V,W}(v \otimes w)$$

for any $a \in A$, $v \in V$ and $w \in W$. Let $a \in A$, $v \in V$ and $w \in W$. We have:

$$\begin{aligned}
\tau_{V,W}(a \cdot (v \otimes w)) &= \tau_{V,W}(\Delta(a) \cdot (v \otimes w)) \\
&= \tau_{V,W}\left(\sum_{(a)} a' \cdot v \otimes a'' \otimes w\right) \\
&= \sum_{(a)} a'' \cdot w \otimes a' \cdot v \\
&= \Delta^{op}(a) \cdot (w \otimes v),
\end{aligned}$$

while

$$a \cdot \tau_{V,W}(v \otimes w) = \Delta(a) \cdot (w \otimes v).$$

So we proved the following

Proposition 2.4.3. *Let $A = (A, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra and let V, W be A–modules. Then the flip $\tau_{V,W}$ is an isomorphism of A–modules if and only if $\Delta = \Delta^{op}$.*

In other words, we have that the flip functor is a good candidate to be a braiding for the category $\mathsf{Mod}(A)$ if and only if the bialgebra A is cocommutative. However, as we will see in the next chapters, there exist non–cocommutative bialgebras, and so the flip functor is not enough. We can solve this problem introducing the concept of universal R–matrix, that allows us to define a special class of bialgebras, in which the cocommutative ones are included.

Definition 2.4.4. *Let $A = (A, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. We say that A is quasi–cocommutative if there exists an invertible element $R \in A \otimes A$ such that, for all $a \in A$, we have*

$$\Delta^{op}(a) = R\Delta(a)R^{-1}.$$

An element R satisfying this condition is called a universal R–matrix for A. We denote a quasi–cocommutative bialgebra by a sextuple $(A, \mu, \eta, \Delta, \varepsilon, R)$.

We can look upon a quasi–cocommutative bialgebra as a bialgebra whose non–cocommutativity is controlled by its universal R–matrix. In fact, we have that:

Remark 2.4.5. *If $A = (A, \mu, \eta, \Delta, \varepsilon)$ is a cocommutative bialgebra, then $1 \otimes 1$ is a universal R–matrix for A. This means that if A is cocommutative, then it is also quasi–cocommutative.*

The following result gives us a good candidate to be a braiding for the category $\mathsf{Mod}(A)$. Moreover this result generalizes the case of cocommutative bialgebras.

Proposition 2.4.6. *Let $(A, \mu, \eta, \Delta, \varepsilon, R)$ be a quasi–cocommutative bialgebra, and let V, W be two A–modules. Then the map*

$$\begin{aligned} \beta_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto \tau_{V,W}(R \cdot (v \otimes w)) \end{aligned}$$

is an isomorphism of A–modules.

Proof. We have to prove that $\beta_{V,W}$ is A–linear. Let $a \in A$, $v \in V$ and $w \in W$. Then:

$$\begin{aligned} \beta_{V,W}(a \cdot (v \otimes w)) &= \tau_{V,W}(R \cdot \Delta(a) \cdot (v \otimes w)) \\ &= \tau_{V,W}(\Delta^{op}(a) \cdot R \cdot (v \otimes w)) \\ &= \Delta(a) \cdot \tau_{V,W}(R \cdot (v \otimes w)) \\ &= a \cdot \beta_{V,W}(v \otimes w). \end{aligned}$$

□

However, this is not enough, because in the definition of a braided tensor category we have the hexagonal axioms, and so the next step is to investigate under which conditions they hold. Before continuing the discussion we have to introduce a useful notation:
Convention. Let A be an algebra and let

$$X = \sum_i x_i^{(1)} \otimes \ldots \otimes x_i^{(p)} \in A^{\otimes p} \quad (p > 1).$$

For any p–tuple $(k_1, \ldots, k_p)$ of distinct elements of $\{1, \ldots, n\}$ $(n \geq p)$, we denote by $X_{k_1 \ldots k_p}$ the element of $A^{\otimes n}$ given by

$$X_{k_1 \ldots k_p} = \sum_i y_i^{(1)} \otimes \ldots \otimes y_i^{(n)}$$

where $y_i^{(k)} = x_i^{(j)}$ if $k = k_j$ for $j \leq p$ and $y_i^{(k)} = 1$ otherwise.

Definition 2.4.7. *A quasi–cocommutative bialgebra $A = (A, \mu, \eta, \Delta, \varepsilon, R)$ is said to be braided, or quasi–triangular, if its universal R–matrix R satisfies the relations*

$$(\Delta \otimes \mathsf{id}_A)(R) = R_{13}R_{23} \qquad \textit{and} \qquad (\mathsf{id}_A \otimes \Delta)(R) = R_{13}R_{12}.$$

If moreover we have $RR_{12} = 1$ we say that A is a quasi–cocommutative triangular bialgebra or, simply, a triangular bialgebra. If $A = (A, \mu, \eta, \Delta, \varepsilon, R)$ is a quasi–triangular (resp. triangular) bialgebra, we say that its universal R–matrix R is a quasi–triangular (resp. triangular) structure on A.

Let us see which conditions are required so that the functor β satisfies the hegaxon axioms. We have to check under which conditions are satisfied the following two relations:

(H1) $a_{V,W,U} \circ \beta_{U,V\otimes W} \circ a_{U,V,W} = \mathsf{id}_V \otimes \beta_{U,W} \circ a_{V,U,W} \circ \beta_{U,V} \otimes \mathsf{id}_W;$

(H2) $a^{-1}_{W,U,V} \circ \beta_{U\otimes V,W} \circ a^{-1}_{U,V,W} = \beta_{U,W} \otimes \mathsf{id}_V \circ a^{-1}_{U,W,V} \circ \mathsf{id}_U \otimes \beta_{V,W}$

for all U, V, W in $\mathsf{Mod}(A)$. Let $u \in U$, $v \in V$ and $w \in W$ and let

$$R = \sum_i s_i \otimes t_i.$$

Then

$$\begin{aligned}
&(a_{V,W,U} \circ \beta_{U,V\otimes W} \circ a_{U,V,W})((u \otimes v) \otimes w) \\
=&(a_{V,W,U} \circ \beta_{U,V\otimes W})(u \otimes (v \otimes w)) \\
=&a_{V,W,U}(\tau(R \cdot (u \otimes (v \otimes w)))) \\
=&a_{V,W,U}\left(\tau\left(\sum_i s_i \cdot u \otimes t_i \cdot (v \otimes w)\right)\right) \\
=&a_{V,W,U}\left(\tau\left(\sum_i s_i \cdot u \otimes \Delta(t_i) \cdot (v \otimes w)\right)\right) \\
=&a_{V,W,U}\left(\sum_i \Delta(t_i) \cdot (v \otimes w) \otimes s_i \cdot u\right) \\
=&\sum_{i,(t_i)} t_i' \cdot v \otimes (t_i'' \cdot w \otimes s_i \cdot u) \\
=&(\Delta \otimes \mathsf{id})(R^{op}) \cdot (v \otimes (w \otimes u)).
\end{aligned}$$

On the other side, we have

$$\begin{aligned}
&(\mathsf{id}_V \otimes \beta_{U,W} \circ a_{V,U,W} \circ \beta_{U,V} \otimes \mathsf{id}_W)((u \otimes v) \otimes w) \\
=&(\mathsf{id}_V \otimes \beta_{U,W} \circ a_{V,U,W})(\tau(R \cdot (u \otimes v) \otimes w)) \\
=&(\mathsf{id}_V \otimes \beta_{U,W} \circ a_{V,U,W})\left(\left(\sum_i t_i \cdot v \otimes s_i \cdot u\right) \otimes w\right) \\
=&(\mathsf{id}_V \otimes \beta_{U,W})\left(\sum_i t_i \cdot v \otimes (s_i \cdot u \otimes w)\right) \\
=&\sum_i t_i \cdot v \otimes \beta(s_i \cdot u \otimes w) \\
=&\sum_i t_i \cdot v \otimes \tau(R \cdot (s_i \cdot u \otimes w)) \\
=&\sum_{i,j} t_i \cdot v \otimes (t_j \cdot w \otimes s_j \cdot s_i \cdot u) \\
=&R_{31}.R_{32} \cdot (v \otimes (w \otimes u)).
\end{aligned}$$

Then the first Hexagon (H1) commutes if and only if

$$(\Delta \otimes \mathsf{id})(R^{op}) = R_{31}R_{32}.$$

Applying the permutation (123) of the variables in $A \otimes A \otimes A$, the condition above is equivalent to

$$(123)(\Delta \otimes \mathsf{id})(R^{op}) = (123)(R_{31}R_{32}),$$

that gives us the following condition:

$$(\Delta \otimes \mathsf{id})(R) = R_{13}R_{23}.$$

The second hexagon gives us the following equality:

$$\begin{aligned}
&(a^{-1}_{W,U,V} \circ \beta_{U\otimes V,W} \circ a^{-1}_{U,V,W})(u \otimes (v \otimes w)) \\
=&(a^{-1}_{W,U,V} \circ \beta_{U\otimes V,W})((u \otimes v) \otimes w) \\
=&a^{-1}_{W,U,V}(\tau(R \cdot (u \otimes v) \otimes w)) \\
=&a^{-1}_{W,U,V}\left(\tau\left(\sum_i s_i \cdot (u \otimes v) \otimes t_i \cdot w\right)\right) \\
=&a^{-1}_{W,U,V}\left(\tau\left(\sum_{i,(s_i)} (s'_i \cdot u \otimes s''_i \cdot v) \otimes t_i \cdot w\right)\right) \\
=&a^{-1}_{W,U,V}\left(\sum_{i,(s_i)} t_i \cdot w \otimes (s'_i \cdot u \otimes s''_i \cdot v)\right) \\
=&\sum_{i,(s_i)} (t_i \cdot w \otimes s'_i \cdot u) \otimes s''_I \cdot v \\
=&(\mathsf{id} \otimes \Delta)(R^{op}) \cdot ((w \otimes u) \otimes v),
\end{aligned}$$

while on the other side, we have

$$\begin{aligned}
&(\beta_{U,W} \otimes \mathsf{id}_V \circ a^{-1}_{U,W,V} \circ \mathsf{id}_U \otimes \beta_{V,W})(u \otimes (v \otimes w)) \\
=&(\beta_{U,W} \otimes \mathsf{id}_V \circ a^{-1}_{U,W,V})(u \otimes \tau(R \cdot (v \otimes w))) \\
=&(\beta_{U,W} \otimes \mathsf{id}_V \circ a^{-1}_{U,W,V})\left(u \otimes \sum_i t_i \cdot w \otimes s_i \cdot v\right) \\
=&(\beta_{U,W} \otimes \mathsf{id}_V)\left(\sum_i (u \otimes t_i \cdot w) \otimes s_i \cdot v\right) \\
=&\sum_{i,j} (t_j \cdot t_i \cdot w \otimes s_j \cdot u) \otimes s_i \cdot v \\
=&R_{31}R_{21} \cdot ((w \otimes u) \otimes v).
\end{aligned}$$

Then the second Hexagon (H2) commutes if and only if

$$(\mathsf{id} \otimes \Delta)(R^{op}) = R_{31}R_{21}.$$

Applying the permutation (123) of the variables in $A \otimes A \otimes A$, the condition above is equivalent to

$$(123)(\mathsf{id} \otimes \Delta)(R^{op}) = (123)(R_{31}R_{21}),$$

that gives us the following condition:

$$(\mathsf{id} \otimes \Delta)(R) = R_{13}R_{12}.$$

Hence, we have the following

Theorem 2.4.8. *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Then the tensor category $\mathsf{Mod}(H)$ is braided if and only if H is braided. Moreover, the category $\mathsf{Mod}(H)$ is a symmetric braided tensor category if and only if H is triangular.*

Proof. We prove the first part of the statement. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a braided bialgebra with universal R–matrix R. It follows from the discussion above and from Proposition 2.4.6 that β is a braiding in the category $\mathsf{Mod}(H)$. Conversely, let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra and suppose that there exists a braiding c for the tensor category $\mathsf{Mod}(H)$. Define an invertible element $R \in H \otimes H$ by

$$R = \tau_{H,H}(c_{H,H}(1 \otimes 1)).$$

Let us show that R is a universal R–matrix for H. Let V, W be in $\mathsf{Mod}(H)$, $v \in V$, $w \in W$. By the naturality of the braiding, we have that the square

$$\begin{array}{ccc} H \otimes H & \xrightarrow{c_{H,H}} & H \otimes H \\ \downarrow{\scriptstyle \bar{v}\otimes\bar{w}} & & \downarrow{\scriptstyle \bar{w}\otimes\bar{v}} \\ V \otimes W & \xrightarrow{c_{V,W}} & W \otimes V \end{array}$$

commutes, where $\bar{v}$ and $\bar{w}$ are the H–linear maps defined by

$$\begin{aligned} \bar{v} : H &\to V \\ 1 &\mapsto v \end{aligned}$$

and

$$\begin{aligned} \bar{w} : H &\to W \\ 1 &\mapsto w. \end{aligned}$$

This implies that

$$\begin{aligned} c_{V,W}(v \otimes w) &= (\bar{w} \otimes \bar{v})(c_{H,H}(1 \otimes 1)) \\ &= \tau_{V,W}((\bar{v} \otimes \bar{w})(R)) \\ &= \tau_{V,W}(R \cdot (v \otimes w)) \\ &= \beta_{V,W}(v \otimes w). \end{aligned}$$

The H–linearity of $c_{H,H}$ means that for any $a \in H$ we have

$$a \cdot c_{H,H}(1 \otimes 1) = c_{H,H}(a \cdot (1 \otimes 1)),$$

that means that

$$\Delta(a) \cdot \tau_{H,H}(R) = \tau_{H,H}(R \cdot \Delta(a)),$$

and this is equivalent to

$$\Delta^{op}(a)R = R\Delta(a).$$

This proves that R is a universal R–matrix for H. Finally, the commutativity of the hexagons (H1) and (H2) implies the relations

$$(\mathsf{id} \otimes \Delta)(R) = R_{13}R_{12} \quad \text{and} \quad (\Delta \otimes \mathsf{id})(R) = R_{13}R_{23},$$

and so the claim is proved. □

2.5 Quasi–bialgebras

In this section we introduce the notion of quasi–bialgebra, that generalizes the concept of bialgebra. The aim of this discussion is to define an equivalence relation between the categories of all modules of two different quasi–bialgebras. For the proofs of all the results presented in this section we refer to [Kas12].

Definition 2.5.1. *Let A be an algebra and let $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{K}$ be morphisms of algebras. We say that A is a quasi–bialgebra if the category* $\mathsf{Mod}(A)$ *equipped with the tensor product of* $\mathsf{Vect}(\mathbb{K})$ *is a tensor category.*

In other words, A is a quasi–bialgebra if there exist an associativity constraint a, a left unit constraint l, and a right unit constraint r satisfying the pentagon axiom and the triangle axiom. Note that, as seen in the previous section, if these constraints are the same of $\mathsf{Vect}(\mathbb{K})$, then A is a bialgebra. The following result, due to Vladimir Drinfeld, gives us a characterization of quasi–bialgebras. In the following we assume that the unit costraints are the trivial ones.

Proposition 2.5.2. *Let A be an algebra and suppose that $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{K}$ are morphisms of algebras. Then A is a quasi–bialgebra if and only if there exists an invertible element $\Phi \in A \otimes A \otimes A$ such that:*

(i) $(\mathsf{id} \otimes \Delta)(\Delta(a)) = \Phi\big((\Delta \otimes \mathsf{id})(\Delta(a))\big)\Phi^{-1}$;

(ii) $(\varepsilon \otimes \mathsf{id}) \circ \Delta = \mathsf{id} = (\mathsf{id} \otimes \varepsilon) \circ \Delta$;

(iii) $(\mathsf{id} \otimes \mathsf{id} \otimes \Delta)(\Phi)(\Delta \otimes \mathsf{id} \otimes \mathsf{id})(\Phi) = \Phi_{234}(\mathsf{id} \otimes \Delta \otimes \mathsf{id})(\Phi_{123})$;

(iv) $(\mathsf{id} \otimes \varepsilon \otimes \mathsf{id})(\Phi) = 1.$

for all $a \in A$*, here* $\Phi_{123} = \Phi \otimes 1$ *and* $\Phi_{234} = 1 \otimes \Phi$*.*

We denote a quasi–bialgebra by $(A, \Delta, \varepsilon, \Phi)$. Since Φ generates an associativity constraint, we call it an associator of A. A morphism of quasi–bialgebras

$$\alpha : (A, \Delta, \varepsilon, \Phi) \to (A', \Delta', \varepsilon', \Phi')$$

is a morphism of algebras between the underlying algebras such that

$$(\alpha \otimes \alpha) \circ \Delta = \Delta' \circ \alpha \quad \text{and} \quad (\alpha \otimes \alpha \otimes \alpha)(\Phi) = \Phi'.$$

It is an isomorphism of quasi–bialgebra if, in addition, it is invertible.

2.5.1 Gauge transformations

The purpose of this subsection is to introduce an equivalence relation on quasi–bialgebras such that the categories of modules of two equivalent quasi–bialgebras are tensor equivalent.

Definition 2.5.3. *Let* $A = (A, \Delta, \varepsilon, \Phi)$ *be a quasi–bialgebra. A gauge transformation on* A *is an invertible element* F *of* $A \otimes A$ *such that*

$$(\varepsilon \otimes \mathsf{id}_A)(F) = (\mathsf{id}_A \otimes \varepsilon)(F) = 1.$$

Starting from a quasi–bialgebra $(A, \Delta, \varepsilon, \Phi)$, we can obtain another quasi–bialgebra $(A, \Delta_F, \varepsilon, \Phi_F)$ using a gauge transformation as follows. We define an algebra morphism

$$\begin{aligned} \Delta_F : A &\to A \otimes A \\ a &\mapsto F\Delta(a)F^{-1} \end{aligned}$$

and a new associator Φ_F by

$$\Phi_F := F_{23}(\mathsf{id}_A \otimes \Delta)(F)\Phi(\Delta \otimes \mathsf{id}_A)(F^{-1})F_{12}^{-1} \in A \otimes A \otimes A.$$

Proposition 2.5.4. *Let* $(A, \Delta_F, \varepsilon, \Phi_F)$ *be a quasi–bialgebra and let* $F \in A \otimes A$ *be a gauge transformation on* A*. Then* $(A, \Delta_F, \varepsilon, \Phi_F)$ *is a quasi–bialgebra.*

Remark 2.5.5. *We have to do some important remarks:*

- *In general, if* A *is a bialgebra and* F *is a gauge transformation of* A*, then* A_F *is not a bialgebra.*

- *If A is a quasi–bialgebra and F is a gauge transformation on A, then F^{-1} is too and we have*

$$(A_F)_{F^{-1}} = A = (A_{F^{-1}})_F.$$

- *If A is a quasi–bialgebra and F, F' are gauge transformations, then*

$$(A_{F'})_F = A_{FF'}.$$

Definition 2.5.6. *Two quasi–bialgebras $(A, \Delta_F, \varepsilon, \Phi_F)$ and $(A', \Delta'_F, \varepsilon', \Phi'_F)$ are said to be equivalent if there exist a gauge transformation F on A' and an isomorphism $\alpha : A \to A'_F$ of quasi–bialgebras.*

The remarks above imply that this is an equivalence relation. However, our purpose is to have that equivalent quasi–bialgebras have equivalent tensor categories of modules. To have this, we first introduce the following

Lemma 2.5.7. *Let $(A, \Delta_F, \varepsilon, \Phi_F)$ be a quasi–bialgebra, V, W be two A–modules and F be a gauge transformation on A. Define*

$$\varphi_2^F(V, W)(v \otimes w) = F^{-1}(v \otimes w),$$

where v and w belong to V and W respectively. Then the triple $(\mathsf{id}, \mathsf{id}, \varphi_2^F)$ is a tensor functor from the tensor category $\mathsf{Mod}(A)$ to the tensor category $\mathsf{Mod}(A_F)$.

Let A and A' be equivalent quasi–bialgebras with a gauge transformation F on A' and an isomorphism $\alpha : A \to A'_F$ of quasi–bialgebras. The map α induces a strict tensor functor $(\alpha^*, \mathsf{id}, \mathsf{id})$ from $\mathsf{Mod}(A'_F)$ to $\mathsf{Mod}(A)$ as follows: if V is a A'_F–module and $v \in V$, define the action of A on v given by

$$a \cdot v = f(a) \cdot v.$$

Theorem 2.5.8. *The tensor functor $(\alpha^*, \mathsf{id}, \varphi_2^F)$ is a tensor equivalence between $\mathsf{Mod}(A')$ and $\mathsf{Mod}(A)$.*

2.5.2 Braided quasi–bialgebras

Definition 2.5.9. *A quasi–bialgebra $(A, \Delta, \varepsilon, \Phi)$ is said to be braided if $\mathsf{Mod}(A)$ is a braided tensor category.*

Braided quasi–bialgebras are also called quasi–triangular quasi–bialgebras, and they are characterizated by the following

Proposition 2.5.10. *A quasi–bialgebra* $(A, \Delta, \varepsilon, \Phi)$ *is braided if and only if there exists an inverible element* $R \in A \otimes A$*, called the universal* R*–matrix of* A*, such that for any* $a \in A$ *we have:*

- $\Delta^{op}(a) = R\Delta(a)R^{-1}$;
- $(\mathsf{id} \otimes \Delta)(R) = (\Phi_{231})^{-1}R_{13}\Phi_{213}R_{12}(\Phi_{123})^{-1}$;
- $(\Delta \otimes \mathsf{id})(R) = \Phi_{312}R_{13}(\Phi_{132})^{-1}R_{23}\Phi_{123}$.

A braided quasi–bialgebra is denoted by a quintuple $(A, \Delta, \varepsilon, \Phi, R)$. A morphism of braided quasi–bialgebras

$$\alpha : (A, \Delta, \varepsilon, \Phi, R) \to (A', \Delta', \varepsilon', \Phi', R')$$

is a morphism of the underlying quasi–bialgebras such that

$$(\alpha \otimes \alpha)(R) = R'.$$

We now extend the notion of gauge transformation to the case of braided quasi–bialgebras. Given a braided quasi–bialgebra $A = (A, \Delta, \varepsilon, \Phi, R)$ and a gauge transformation F on A, define

$$R_F = F_{21}RF^{-1}.$$

Proposition 2.5.11. *Under these hypotesis, we have that:*

- *The quintuple* $(A, \Delta_F, \varepsilon, \Phi_F, R_F)$ *is a braided quasi–bialgebra;*
- *the tensor functor* $(\mathsf{id}, \mathsf{id}, \varphi_2^F)$ *is a braided tensor functor from* $\mathsf{Mod}(A)$ *to* $\mathsf{Mod}(A_F)$*.*

We now adapt the definition of equivalence between tensor quasi–bialgebras to the case of the braided quasi–bialgebras.

Definition 2.5.12. *We say that two braided quasi–bialgebras* $(A, \Delta, \varepsilon, \Phi, R)$ *and* $(A', \Delta', \varepsilon', \Phi', R')$ *are equivalent if there exist a gauge transformation* F *on* A' *and an isomorphism* $\alpha : A \to A'_F$ *of braided quasi–bialgebras.*

As a consequence of the previuous results, we have the following

Theorem 2.5.13. *The tensor functor* $(\alpha^*, \mathsf{id}, \varphi_2^F)$ *is a braided tensor equivalence between the braided tensor categories* $\mathsf{Mod}(A')$ *and* $\mathsf{Mod}(A)$*.*

2.6 Hopf algebras

Definition 2.6.1. *Let* (A, μ, η) *be an algebra and* (C, Δ, ε) *be a coalgebra. We define the convolution map in* $\mathsf{Hom}(C, A)$ *by*

$$\begin{aligned} \star : \mathsf{Hom}(C, A) \times \mathsf{Hom}(C, A) &\to \mathsf{Hom}(C, A) \\ (f, g) &\mapsto f \star g \coloneqq \mu \circ (f \otimes g) \circ \Delta. \end{aligned}$$

Proposition 2.6.2. *The triple* $(\mathsf{Hom}(C, A), \star, \eta \circ \varepsilon)$ *is an algebra.*

Proof. We have to prove that $(\mathsf{Hom}(C, A), \star, \eta \circ \varepsilon)$ satisfies the axioms (A1),(A2).

- The associativity of $\star$ is given by the associativity of μ and the coassociativity of Δ, so (A1) is satisfied.

- By the properties of η and ε, we have

$$\begin{aligned} ((\eta \circ \varepsilon) \star f)(x) &= (\mu \circ ((\eta \circ \varepsilon) \otimes f) \circ \Delta)(x) \\ &= (\mu \circ ((\eta \circ \varepsilon) \otimes f)) \Big(\sum_{(x)} x' \otimes x'' \Big) \\ &= \mu \Big(\sum_{(x)} (\eta \circ \varepsilon)(x') \otimes f(x'') \Big) \\ &= \sum_{(x)} (\eta \circ \varepsilon)(x') f(x'') \\ &= \sum_{(x)} \varepsilon(x') f(x'') \\ &= f \Big(\sum_{(x)} \varepsilon(x') x'' \Big) = f(x) \\ &= f \Big(\sum_{(x)} x' \varepsilon(x'') \Big) \\ &= \sum_{(x)} f(x') \varepsilon(x'') \\ &= \sum_{(x)} f(x') (\eta \circ \varepsilon)(x'') \\ &= (\mu \circ (f \otimes (\eta \circ \varepsilon))) \Big(\sum_{(x)} x' \otimes x'' \Big) \\ &= (\mu \circ (f \otimes (\mu \circ \varepsilon)) \circ \Delta)(x) \\ &= (f \star (\eta \circ \varepsilon))(x), \end{aligned}$$

and so also (A2) is satisfied.

□

When $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra, we may consider the case $A = C = H$ and thus define the convolution map on the vector space $\mathsf{End}(H)$.

Definition 2.6.3. *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra.*

- *An endomorphism $S \in \mathsf{End}(H)$ is called an antipode for H if*

$$S \star \mathsf{id}_H = \mathsf{id}_H \star S = \eta \circ \varepsilon.$$

- *A Hopf algebra is a bialgebra with an antipode.*
- *A morphism of Hopf algebras is a morphism between the underlying bialgebras commuting with the antipodes.*
- *A quasi–cocommutative Hopf algebra is a Hopf algebra whose bialgebra structure is quasi–cocommutative.*
- *A quasi–triangular (resp. triangular) Hopf algebra is a Hopf algebra whose bialgebra structure has a quasi–triangular (resp. triangular) strucutre on it.*
- *A quasi–Hopf algebra is a quasi–bialgebra A equipped with an antihomomorphism S, called the antipode, and with two elements $\alpha, \beta \in A$ such that:*

  (1) $\mu(S \otimes \alpha)(\Delta(x)) = \varepsilon(x)\alpha$;
  (2) $\mu(\mathsf{id}_A \otimes \beta S)(\Delta(x)) = \varepsilon(x)\beta$;
  (3) $\sum_i X_i \beta S(Y_i) \alpha Z_i = \mathsf{id}_A$;
  (4) $\sum_i S(\bar{X}_i) \alpha \bar{Y}_i \beta S(\bar{Z}_i) = \mathsf{id}_A$,

  where $\Phi = X_i \otimes Y_i \otimes Z_i$ and $\Phi^{-1} = \bar{X}_i \otimes \bar{Y}_i \otimes \bar{Z}_i$.

- *A quasi–triangular quasi–Hopf algebra is a quasi–Hopf algebra whose underlying quasi–bialgebra structure is quasi–triangular.*

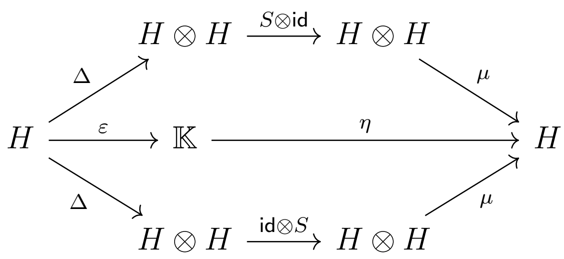
We denote a Hopf algebra by $(H, \mu, \eta, \Delta, \varepsilon, S)$.

Remark 2.6.4. *Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a Hopf algebra and let S, S' be two antipodes. Then*

$$S = S \star (\eta \circ \varepsilon) = S \star (\mathsf{id}_H \star S') = (S \star \mathsf{id}_H) \star S' = (\eta \circ \varepsilon) \star S' = S'.$$

Then if a bialgebra has an antipode, it is unique.

We can represent a Hopf algebra with the following diagram:



Also, we can think upon the antipode of a Hopf algebra as the inverse of the identity map for the convolution product.

Proposition 2.6.5. *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. Then:*

- *S is a bialgebra morphism from H to H^{cop}.*
- *The following statements are equivalent:*
  - *(i) $S^2 = \mathsf{id}$;*
  - *(ii) for all $x \in H$ we have $\sum_{(x)} S(x'')x' = \varepsilon(x)1$;*
  - *(iii) for all $x \in H$ we have $\sum_{(x)} x''S(x') = \varepsilon(x)1$.*
- *If H is commutative or cocommutative, then $S^2 = \mathsf{id}_H$.*
- *$H^{opcop} = (H, \mu^{op}, \eta, \Delta^{op}, \varepsilon, S)$ is a Hopf algebra.*
- *$S : H \to H^{opcop}$ is a morphism of Hopf algebras.*
- *if S is an isomorphism, then also $H^{op} = (H, \mu^{op}, \eta, \Delta, \varepsilon, S^{-1})$ and $H^{cop} = (H, \mu, \eta, \Delta^{op}, \varepsilon, S^{-1})$ are Hopf algebras.*

We refer the proofs of the results above to [Kas12].

Definition 2.6.6. *Let H be a Hopf algebra. The quantum Yang–Baxter equation is the following equation for an element $R \in H \otimes H$:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

We abbreviate the quantum Yang–Baxter equation with $QYBE$.

Proposition 2.6.7. *Let H be a Hopf algebra and let R be a quasi–triangular structure on H. Then R is a solution of the $QYBE$.*

Proof. We have

$$\begin{aligned}
R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes \mathsf{id})(R) \\
&= (\Delta^{op} \otimes \mathsf{id})(R)R_{12} \\
&= (\tau \otimes \mathsf{id})(\Delta \otimes \mathsf{id})(R)R_{12} \\
&= (\tau \otimes \mathsf{id})(R_{13}R_{23})R_{12} \\
&= R_{23}R_{13}R_{12}.
\end{aligned}$$

□

Note that if H is a quasi–triangular Hopf algebra with universal R–matrix R and V is a H–module, then the image of R in $\mathsf{End}(V) \otimes \mathsf{End}(V)$ through the associated representation is a solution of the YBE in $\mathsf{End}(V)^{\otimes 3}$.

2.7 The Drinfeld quantum double

In this Section we present the construction of the Drinfeld quantum double. All the proofs of the results below can be find in [Kas12].

2.7.1 Matched bialgebras

Definition 2.7.1. *Let A be an algebra, C be a coalgebra and H, X be Hopf algebras.*

- *We say that A is a module–algebra over H if A is a H–module and the multiplication and the unit of A are morphisms of H–modules.*

- *We say that C is a module–coalgebra over H if there exists a morphism of coalgebras $H \otimes C \to C$ inducing a H–module structure on C.*

- *We say that the pair (X, H) of bialgebras is matched if there exist linear maps $\alpha : H \otimes X \to X$ and $\beta : H \otimes X \to X$ turning X into a module–coalgebra over A, and turning A into a right module–coalgebra X, such that, setting*

$$\alpha(h \otimes x) = h \cdot x \quad \textit{and} \quad \beta(h \otimes x) = h^x,$$

the following conditions are satisfied:

$$(i) \quad h\cdot(xy) = \sum_{(h,x)} (h'\cdot x')(h''^{x''}\cdot y),$$

$$(ii) \quad h\cdot 1 = \varepsilon(h)1,$$

$$(iii) \quad (hk)^x = \sum_{(k,x)} h^{k'\cdot x'} k''^{x''},$$

$$(iv) \quad 1^x = \varepsilon(x)1,$$

$$(v) \quad \sum_{(h,x)} h'^{x'} \otimes h''\cdot x'' = \sum_{(h,x)} h''^{x''} \otimes h'\cdot x'$$

for all $h, k \in H$ and $x, y \in X$.

Theorem 2.7.2. *Let (X, H) be a matched pair of bialgebras. There exists a unique bialgebra structure on the vector space $X \otimes H$, with unit equal to $1 \otimes 1$, such that its product is given by*

$$(x\otimes h)(y\otimes k) = \sum_{(h,y)} = x(h'\cdot y')\otimes h''^{y''}k),$$

its coproduct by

$$\Delta(x\otimes h) = \sum_{(h,x)} (x'\otimes h')\otimes(x''\otimes h''),$$

and its counit by

$$\varepsilon(x\otimes h) = \varepsilon(x)\varepsilon(h)$$

for all $x, y \in X$ and $h, k \in H$. We denote this bialgebra by $X \bowtie H$. If moreover the bialgebras X and H have antipodes, respectively denoted by S_X and S_H, then the bicrossed product is a Hopf algebra with antipode given by

$$S(x\otimes h) = \sum_{(x,a)} = S_H(h'')\cdot S_X(x'')\otimes S_A(a')^{S_X(x')}.$$

We now give the definition of the Drinfeld's Quantum Double. First we need the following

Theorem 2.7.3. *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a finite–dimensional Hopf algebra with invertible antipode and consider the Hopf algebra*

$$X = (H^{op})^* = (H^*, \Delta^*, \varepsilon^*, (\mu^{op})^*, \eta^*, (S^{-1})^*).$$

Let $\alpha : H \otimes X \to X$ and $\beta : H \otimes X \to H$ be the linear maps given by

$$\alpha(h \otimes x) = h \cdot x = \sum_{(h)} x(S^{-1}(h'')?a')$$

and

$$\beta(h \otimes x) = h^x = \sum_{(h)} f(S^{-1}(h''')h')h''$$

where $h \in H$ and $x \in X$ and the question mark serves as a mute variable. Then the pair (X, H) of Hopf algebras is matched.

2.7.2 The quantum double and its universal R–matrix

Definition 2.7.4. *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a finite–dimensional Hopf algebra with invertible antipode. The quantum double $D(H)$ of the Hopf algebra H is the bicrossed product of H and $X = (H^{op})^*$:*

$$D(H) \coloneqq X \bowtie H = (H^{op})^* \bowtie H.$$

Consider now the injective maps

$$\begin{aligned} i_X : X &\to D(H) \\ x &\mapsto x \otimes 1 \end{aligned}$$

and

$$\begin{aligned} i_H : H &\to D(H) \\ h &\mapsto 1 \otimes h. \end{aligned}$$

Choose a basis $\{e_i\}_{i\in I}$ of the vector space H and consider its dual basis $\{e^i\}_{i\in I}$ of the vector space X. Set

$$\rho = \sum_{i\in I} e_i \otimes e^i$$

and

$$R = (i_X \otimes i_H)(\rho) = \sum_{i\in I} (1 \otimes e_i) \otimes (e^i \otimes 1) \in D(H) \otimes D(H).$$

Then we have the following

Theorem 2.7.5. *The tensor R defined above equips $D(H)$ with a quasi–triangular Hopf algebra structure.*

We may interpret this Theorem by saying that every Hopf algebra H can be embedded into a quasi–triangular one, that is its Drinfeld double.

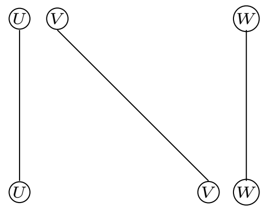
2.8 Pictorial representation

The aim of this Section is to present a pictorial approach for equalities in braided tensor categories. The advantage of this notation is that we do not have to evaluate associators, that sometimes are very difficult to explicit.

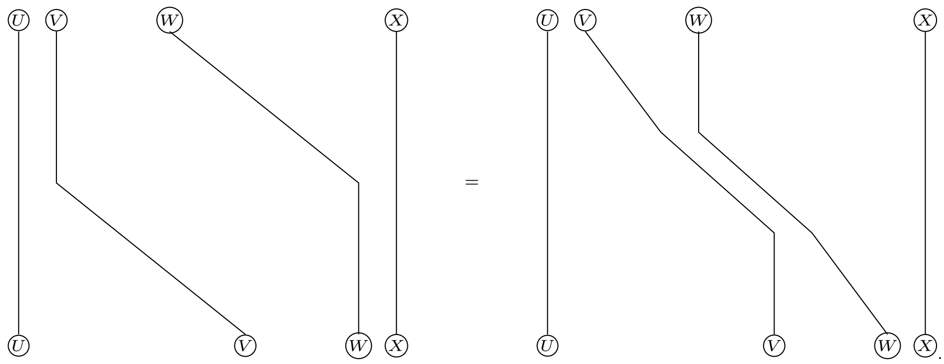
Let $(\mathcal{C}, \otimes, I, a, l, r, c)$ be a braided tensor category and let U, V, W and X be objects of $\mathcal{C}$. Then we represent the isomorphism

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

with the picture



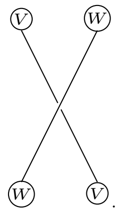
where we interpret the tensor product of two objects as two close circles. Using this notation, we express the pentagon axiom by



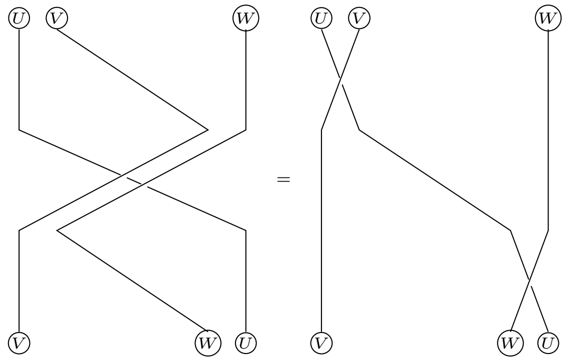
Similarly, we represent the isomorphism

$$c_{V,W} : V \otimes W \to W \otimes V$$

with the picture

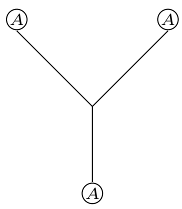


Then the first hexagonal diagram is represented with

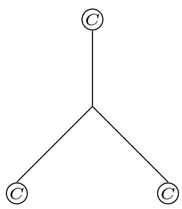


and the same can be done for the second one.

In this pictorial context, if there is a crossing (i.e. a braiding), it is important to distinguish the object that passes under with the object that passes over. By the Mac Lane's coherence Theorem, we may forgot the existence of the associativity constraint during a pictorial representation, so that we do not have to pay attention to the distance of the circles. However, it is important to remember that associators are a relevant part of the discussion, and so we may ignore them only during a pictorial representation. In fact, we will use a pictorial approach to prove only equalities which are pretty complicated to evaluate algebrically. Furthermore, we will denote products and coproducts respectively by



and



and other functions with a squared box.

Chapter 3

Lie bialgebras

3.1 Lie algebras

Definition 3.1.1. *Let $\mathbb{K}$ be a field.*

- *A Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ *is a vector space* $\mathfrak{g}$ *endowed with a bilinear map* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, called the Lie bracket, satisfying the following two conditions for all* $x, y, z \in \mathfrak{g}$ *:*

  (i) (antisymmetry):
  $$[x, y] = -[y, x];$$

  (ii) (Jacoby identity):
  $$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

- *A Lie subalgebra* $\mathfrak{g}'$ *of a Lie algebra* $\mathfrak{g}$ *is a subspace* $\mathfrak{g}'$ *of* $\mathfrak{g}$ *that is stable under the Lie bracket.*

- *An ideal* $\mathfrak{i}$ *of a Lie algebra* $\mathfrak{g}$ *is a subspace* $\mathfrak{i}$ *of* $\mathfrak{g}$ *such that* $[x, y] \in \mathfrak{i}$ *for any* $x \in \mathfrak{i}$ *and* $y \in \mathfrak{g}$*.*

- *A morphism of Lie algebras is a linear map* $f : \mathfrak{g} \to \mathfrak{g}'$ *such that* $f([x, y]) = [f(x), f(y)]$*.*

Example 3.1.2. *We now give some examples of Lie algebras.*

- *If* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are two Lie algebras, we may equip the direct sum* $\mathfrak{g} \oplus \mathfrak{g}'$ *with a Lie bracket given by:*
  $$[(x, x'), (y, y')] := ([x, y], [x', y']).$$

- *Given a Lie algebra $\mathfrak{g}$, we define the opposite Lie algebra $\mathfrak{g}^{op}$ of $\mathfrak{g}$ as the vector space $\mathfrak{g}$ with Lie bracket given by*
$$[x, y]^{op} := [y, x].$$
- *Let $\mathfrak{i}$ be an ideal of a Lie algebra $\mathfrak{g}$. There exists a unique Lie algebra structure on the vector space $\mathfrak{g}/\mathfrak{i}$ such that the canonical projection from $\mathfrak{g}$ into $\mathfrak{g}/\mathfrak{i}$ is a morphism of Lie algebras.*
- *Let $f : \mathfrak{g} \to \mathfrak{g}'$ be a morphism of Lie algebras. Then $\ker f$ is an ideal of $\mathfrak{g}$ and $\mathsf{Im} f$ is a subalgebra of $\mathfrak{g}'$. Furthermore, the map $\mathfrak{g}/\ker f \to \mathsf{Im} f$ is an isomorphism of Lie algebras.*
- *Let A be an associative algebra. Set $[a, b] := ab - ba$ for $a, b \in A$. Then $(A, [\cdot, \cdot])$ is a Lie algebra, which we denote by $\mathcal{L}(A)$.*
- *For any $\mathbb{K}$–vector space V, we denote by $\mathfrak{gl}(V)$ the Lie algebra $\mathcal{L}(\mathsf{End}(V))$ of all endomorphisms of V. When V is of finite dimension n, then $\mathfrak{gl}(V)$ is isomorphic to the Lie algebra $\mathfrak{gl}_n = \mathcal{L}(M_{n,n}(\mathbb{K}))$ of all $n \times n$ matrices with entries in $\mathbb{K}$.*
- *The vector space $\mathfrak{sl}_n$ of all traceless $n \times n$ matrices with entries in $\mathbb{K}$ is a Lie subalgebra of $\mathfrak{gl}_n$.*

Note that if $(\mathfrak{g}, [,])$ is a Lie algebra, we may rewrite the antisymmetry axiom and the Jacobi identity respectively by

(i) $[\cdot, \cdot] = -([\cdot, \cdot] \circ \tau)$;

(ii) $[\cdot, \cdot] \circ (\mathsf{id}_{\mathfrak{g}} \otimes [\cdot, \cdot]) \circ (\mathsf{id}_{\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}} + \sigma + \sigma^2) = 0$,

where $[\cdot, \cdot]$ is thought of as a linear map from $\mathfrak{g} \otimes \mathfrak{g}$ to $\mathfrak{g}$, and σ denotes the map which cyclically permutes the cordinates in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. In the following we will sometimes use the language of permutations for permutation maps in $\mathfrak{g}^{\otimes 3}$. For example, the map σ will be denoted by (123).

3.1.1 Lie modules

Definition 3.1.3. *Let $\mathfrak{g}$ be a Lie algebra.*

- *A Lie $\mathfrak{g}$–module is a pair (V, μ), where V is a vector space and μ is a linear map*
$$\mu : \mathfrak{g} \times V \to V$$
$$(x, v) \mapsto x \cdot v$$
such that $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for any $x, y \in \mathfrak{g}$ and $v \in V$.

- *Let (V, μ) be a Lie $\mathfrak{g}$–module. A Lie $\mathfrak{g}$–submodule of V is a subspace W of V such that $x \cdot w \in W$ for any $x \in \mathfrak{g}$ and $w \in W$.*
- *A Lie $\mathfrak{g}$–module (V, μ) is said to be irreducible if no admits non trivial submodules (V and $\{0_V\}$ are always Lie $\mathfrak{g}$–submodules of V).*
- *Let (V, μ) be a Lie $\mathfrak{g}$–module. The representation associated to (V, μ) is the Lie algebra morphism given by*

$$\begin{aligned} \rho : \mathfrak{g} &\to \mathfrak{gl}(V) \\ x &\mapsto \mu(x, \cdot). \end{aligned}$$

- *Let (V_1, μ_1) and (V_2, μ_2) be two Lie $\mathfrak{g}$–modules. A morphism of Lie $\mathfrak{g}$–modules is a linear map $f : V_1 \to V_2$ such that the diagram*

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle \rho_1(x)}\downarrow & & \downarrow{\scriptstyle \rho_2(x)} \\ V_1 & \xrightarrow[f]{} & V_2 \end{array}$$

  commutes for any $x \in \mathfrak{g}$.
- *Let (V, μ) be a finite–dimensional Lie $\mathfrak{g}$–module and let V^* be its dual vector space. We define a structure of Lie $\mathfrak{g}$–module on V^* defining the action of $x \in \mathfrak{g}$ on a function $f \in V^*$ by $(x \cdot f)(v) := -f(x \cdot v)$.*

Example 3.1.4. *Let $\mathfrak{g}$ be a finite–dimensional Lie algebra. The adjoint representation is*

$$\begin{aligned} ad : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto [x, \cdot]. \end{aligned}$$

So every Lie algebra $\mathfrak{g}$ is a Lie module on itself. If $\mathfrak{g}$ is finite–dimensional, we may consider its dual vector space $\mathfrak{g}^*$ and the dual of the adjoint representation: for $x \in \mathfrak{g}$ we define

$$ad_x^* = -(ad_x)^*.$$

Thus, ad_x^* is an element of $\mathfrak{gl}(\mathfrak{g}^*)$ satisfying

$$(y, ad_x(x')) = -(ad_x^*(y), x')$$

for all $x' \in \mathfrak{g}$ and $y \in \mathfrak{g}^*$, where $(\cdot, \cdot)$ denotes the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The representation

$$\begin{aligned} ad^* : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}^*) \\ x &\mapsto ad_x^* \end{aligned}$$

is called the coadjoint representation of $\mathfrak{g}$ in $\mathfrak{g}^*$.

3.1.2 Semisimple Lie algebras

In this Subsection we present a fast discussion on the main results regarding complex semisimple Lie algebras. This is a very classical discussion, and so the literature is very ample; for example, more technical insights on the results below can be consulted in [Hum12].

Definition 3.1.5. *A Lie algebra is said to be simple if no admits non trivial ideals. A Lie algebra is said to be semisimple if it is the direct sum of simple Lie algebras.*

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then $\mathfrak{g}$ admits the following decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where:

- $\mathfrak{h}$ is the maximal subalgebra of $\mathfrak{g}$ whose elements are ad–diagonalizable;
- $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} | x \neq 0, [h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}\}$ are called the root spaces of $\mathfrak{g}$.

The decompisition above is called the Cartan decomposition of the semisimple Lie algebra $\mathfrak{g}$. The subalgebra $\mathfrak{h}$ is called the Cartan subalgebra of $\mathfrak{g}$, while the set $\Phi := \{\alpha \in \mathfrak{h}^* | \mathfrak{g}_\alpha \neq \emptyset\}$ is called the root system of $\mathfrak{g}$ and its elements are called roots.

Definition 3.1.6. *Let $\mathfrak{g}$ be a Lie algebra. The Killing form of $\mathfrak{g}$ is the bilinear form $(\cdot, \cdot)$ given by*

$$(x, y) := tr(ad_x ad_y).$$

We have that $(\cdot, \cdot)$ is invariant, and in the case of semisimple Lie algebras is nondegenerate. In particular, $(\cdot, \cdot)$ is also nondegenerate if restricted to the Cartan subalgebra $\mathfrak{h}$.

We now give some properties of the root system of a complex semisimple Lie algebra. Most of them are consequences of the non degeneration of the Killing form on the Cartan subalgebra $\mathfrak{h}$.

- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \Phi$.
- If $\alpha \in \Phi$, then also $-\alpha \in \Phi$.
- $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$.

- for all $\alpha, \beta \in \Phi$ we have that $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer number and we have $\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Phi$, where now $(\cdot,\cdot)$ denotes the dual of the Killing form on the dual vector space $\mathfrak{h}^*$ of $\mathfrak{h}$.

- There exists a partition $\Delta^+ \sqcup \Delta^- = \Phi$ of the root system such that $\Delta^+ = -\Delta^-$. Using this fact we can express the Cartan decomposition by
$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$
where
$$\mathfrak{n}_\pm := \bigoplus_{\alpha\in\Delta^\pm} \mathfrak{g}_\alpha.$$

3.1.3 Kac–Moody algebras

Kac–Moody algebras are the infinite–dimensional analogue of semisimple Lie algebras, and then they share many of their properties. The theory of Kac–Moody algebras is explained in depth in [Kac90].

Definition 3.1.7. *The construction of a Kac–Moody algebra is the following:*

- *Let n be a positive integer number and let A be a $n \times n$ matrix with integers entries. We say that A is a generalized Cartan matrix if*

  (i) $a_{ii} = 2$*;*

  (ii) $a_{ij} \leq 0 \ \forall i \neq j$*;*

  (iii) $a_{ij} = 0$ *if and only if* $a_{ji} = 0$.

- *A generalized Cartan matrix is said to be symmetrizable if there exist* $d_i \in \mathbb{Z}$, $d_i \neq 0$ *such that* $d_i a_{ij} = d_j a_{ji} =: b_{ij}$.

- *The Kac–Moody algebra associated to a symmetrizable generalized Cartan matrix A is the Lie algebra $\mathfrak{g}(A)$ generated from e_i, f_i, h_i for $i = 1, \ldots, n$, and whose bracket is defined by the following relations:*

  (1) $[e_i, f_i] = h_i$*;*

  (2) $[e_i, f_j] = 0$ *for* $i \neq j$*;*

  (3) $[h_i, e_j] = a_{ij}e_j$*;*

  (4) $[h_i, f_j] = -a_{ij}f_j$*;*

  (5) $[h_i, h_j] = 0$*;*

  (6) $(ad_{e_i})^{1-a_{ij}}(e_j) = 0$*;*

(7) $(ad_{f_i})^{1-a_{ij}}(f_j) = 0.$

The subalgebra generated by $h_1, \dots, h_n$ is called the Cartan subalgebra of $\mathfrak{g}(A)$, while the subalgebras $\mathfrak{n}_+ := \mathsf{Span}(e_1, \dots, e_n)$ and $\mathfrak{n}_- := \mathsf{Span}(f_1, \dots, f_n)$ are called respectively the positive and the negative nilpotent subalgebras of $\mathfrak{g}(A)$. As in the finite–dimensional semisimple case, the algebra $\mathfrak{g}(A)$ decomposes into a sum of root spaces

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

In this case the root spaces are not one–dimensional (but are still finite–dimensional). The set of roots Φ admits a split $\Phi = \Delta^+ \sqcup \Delta^-$ such that $\Delta^+ = -\Delta^-$, corresponding to the decomposition $\mathfrak{g}(A) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.
For reasons that will be clearer later, we need the infinite–dimensional analogue of the Killing form on Kac–Moody algebras. Given a symmetrizable generalized Cartan matrix A and the associated Kac–Moody algebra $\mathfrak{g}(A)$, we define a symmetric bilinear form on the Cartan subalgebra $\mathfrak{h}$ by

$$(h_i, h_j) := d_i^{-1} d_j^{-1} b_{ij}.$$

This form may be degenerate; let $\mathfrak{i}$ be its kernel and let $\mathfrak{i}'$ be the complement to $\mathfrak{i}$ in $\mathfrak{h}$. We extend $(,)$ to $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathfrak{i}^*$ by $(h, x) = x(h)$ and $(\mathfrak{i}^*, \mathfrak{i}^*) = (\mathfrak{i}^*, \mathfrak{i}') = 0$. Therefore, $(\cdot, \cdot)$ is non degenerate on $\tilde{\mathfrak{h}}$. Furthermore, we extend the simple roots α_i on $\tilde{\mathfrak{h}}^*$ by $\alpha_i(z) = d_i z(h_i)$ for $z \in \mathfrak{i}^*$. The extendend Kac–Moody algebra is

$$\tilde{\mathfrak{g}}(A) = \mathfrak{n}_+ \oplus \tilde{\mathfrak{h}} \oplus \mathfrak{n}_-$$

as a vector space, and is generated by $e_i, f_i, \tilde{\mathfrak{h}}$ with the following relations:

- $[e_i, f_i] = h_i$;
- $[e_i, f_j] = 0$ for $i \neq j$;
- $[h_i, e_j] = \alpha_i(h) e_j$;
- $[h_i, f_i] = -\alpha_i(h) f_j$;
- $[\tilde{\mathfrak{h}}, \tilde{\mathfrak{h}}] = 0$;
- $(ad_{e_i})^{1-a_{ij}}(e_j) = 0$;
- $(ad_{f_i})^{1-a_{ij}}(f_j) = 0$.

The next result is due to Kac and we refer the proof to [Kac90].

Proposition 3.1.8. *There is a unique non degenerate invariant bilinear form on $\tilde{\mathfrak{g}}(A)$ which extends the form $(\cdot, \cdot)$ on $\tilde{\mathfrak{h}}$.*

3.2 Lie coalgebras

Definition 3.2.1. *Let $\mathbb{K}$ be a field.*

- *A Lie coalgebra $(\mathfrak{c}, \delta)$ is a vector space $\mathfrak{c}$ with a linear map $\delta : \mathfrak{c} \to \mathfrak{c} \otimes \mathfrak{c}$, called the Lie cobracket, such that:*

  (i) (antisymmetry):

  $$\delta = -(\tau \circ \delta);$$

  (ii) (coJacoby identity):

  $$(\mathsf{id}_{\mathfrak{c}\otimes\mathfrak{c}\otimes\mathfrak{c}} + \sigma + \sigma^2) \circ (\mathsf{id}_{\mathfrak{c}} \otimes \delta) \circ \delta = 0.$$

- *A Lie subcoalgebra $\mathfrak{c}'$ of a Lie coalgebra $\mathfrak{c}$ is a subspace $\mathfrak{c}'$ of $\mathfrak{c}$ such that $\delta(\mathfrak{c}') \subseteq \mathfrak{c}' \otimes \mathfrak{c}'$.*

- *A coideal $\mathfrak{j}$ of a Lie coalgebra $\mathfrak{c}$ is a subspace $\mathfrak{j}$ of $\mathfrak{c}$ such that $\delta(\mathfrak{j}) \subseteq \mathfrak{j} \otimes \mathfrak{c} + \mathfrak{c} \otimes \mathfrak{j}$.*

- *A morphism between two Lie coalgebras $(\mathfrak{c}, \delta)$ and $(\mathfrak{c}', \delta')$ is a linear map $f : \mathfrak{c} \to \mathfrak{c}'$ such that $\delta' \circ f = (f \otimes f) \circ \delta$.*

Example 3.2.2. *We now give some examples of Lie coalgebras.*

- *Given a Lie coalgebra $\mathfrak{c}$, we define its opposite Lie coalgebra $\mathfrak{c}^{op}$ as the vector space $\mathfrak{c}$ with Lie cobracket given by*

  $$\delta^{op} := \tau \circ \delta.$$

- *Let (C, Δ, ε) be a coalgebra and set*

  $$\delta := \Delta - \Delta^{op}.$$

  Then (C, δ) is a Lie coalgebra, which we denote by $\mathcal{L}^c(C)$.

Convention: We will also use the Sweedler's sigma notation defined in 2.2 for the Lie cobracket of Lie coalgebras.

3.2.1 Duality between Lie algebras and Lie coalgebras

We now want to see the link between the notion of Lie algebra and the notion of Lie coalgebra. This is similar to the discussion made in the Subsection 2.2.2.
Let $(\mathfrak{c}, \delta)$ be a Lie coalgebra and consider the linear map

$$\lambda : \mathfrak{c}^* \otimes \mathfrak{c}^* \to (\mathfrak{c} \otimes \mathfrak{c})^*$$

defined as in the Subsection 2.2.2. Then the dual vector space $\mathfrak{c}^*$ of $\mathfrak{c}$ has a natural structure of Lie algebra, whose bracket is

$$\delta^* \circ \lambda : \mathfrak{c}^* \otimes \mathfrak{c}^* \to \mathfrak{c}^*,$$

where δ^* denotes the transpose of the linear map δ. Conversely, let $(\mathfrak{g}, [\cdot, \cdot])$ be a finite–dimensional Lie algebra. Then, similarly to the case of the dual of a finite–dimensional algebra, we can invert the map λ and consider the linear map given by

$$\lambda^{-1} \circ [\cdot, \cdot]^* : \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*.$$

This linear map endows the vector space $\mathfrak{g}^*$ with a structure of Lie coalgebra, and so the dual vector space of a finite–dimensional Lie algebra has a natural structure of a Lie coalgebra. As in the case of algebras, one can define a Lie coalgebra structure on the dual vector space of a infinite–dimensional Lie algebra. However, this association is not made in a natural way (because it passes through a controvariant functor). For more details on this construction we refer to [Mic80].

3.3 The universal enveloping algebra of a Lie algebra

The purpose of this Section is to assign to any Lie algebra $\mathfrak{g}$ an algebra $\mathcal{U}\mathfrak{g}$ together with a morphism of Lie algebras $i_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{L}(\mathcal{U}\mathfrak{g})$.

Definition 3.3.1. *Let V be a vector space over a field $\mathbb{K}$. We define*

- *the tensor algebra of V as the vector space*

$$T(V) := \bigoplus_{n \in \mathbb{N}} V^{\otimes n},$$

where we set $V^{\otimes 0} := \mathbb{K}$;

- *the symmetric algebra of V as the vector space*

$$S(V) \coloneqq T(V)/I(V) \ ,$$

where $I(V)$ is the two sided ideal generated by all the elements of the form $v \otimes w - w \otimes v$;

- *the exterior algebra of V as the vector space*

$$\Lambda(V) \coloneqq T(V)/J(V),$$

where $J(V)$ is the two sided ideal generated by all the elements of the form $v \otimes v$.

The grading in $T(V)$ induces a grading in both $S(V)$ and $T(V)$, and so we write

$$S(V) = \bigoplus_{n\in\mathbb{N}} S^n(V)$$

and

$$\Lambda(V) = \bigoplus_{n\in\mathbb{N}} \Lambda^n(V).$$

Definition 3.3.2. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{I}(\mathfrak{g})$ be the two sided ideal of $T(\mathfrak{g})$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$. The enveloping algebra of $\mathfrak{g}$ is*

$$\mathcal{U}\mathfrak{g} \coloneqq T(\mathfrak{g})/\mathcal{I}(\mathfrak{g}).$$

Note that the generators of $\mathcal{I}(\mathfrak{g})$ are not homogeneus for the grading of $T(\mathfrak{g})$; therefore there is no grading on the enveloping algebra compatible with the grading of the tensor algebra.

Theorem 3.3.3 (Poincaré–Birkhoff–Witt). *Let $\mathfrak{g}$ be a Lie algebra and let $(x_i)_{i\in I}$ be a basis for $\mathfrak{g}$, where I is a ordered index set. Then the set*

$$\{x_{i_1}^{e_1} \cdot \ldots \cdot x_{i_n}^{e_n} | i_1 < \ldots < i_n, e_i \in \mathbb{Z}^+\}$$

is a basis for $\mathcal{U}\mathfrak{g}$.

A proof of the Poincaré–Birkhoff–Witt Theorem can be consulted in [Hum12]. Let $i : \mathfrak{g} \to T(\mathfrak{g})$ be the canonical injection of $\mathfrak{g}$ into $T(\mathfrak{g})$ and let

$$\pi : T(\mathfrak{g}) \to \mathcal{U}\mathfrak{g}$$

be the canonical surjection of the tensor algebra onto the enveloping algebra. Consider

$$i_{\mathfrak{g}} = \pi \circ i : \mathfrak{g} \to \mathcal{U}\mathfrak{g}.$$

Since $i_{\mathfrak{g}}([x, y]) = xy - yx$, we have that $i_{\mathfrak{g}}$ is a Lie algebra morphism from $\mathfrak{g}$ into $\mathcal{L}(\mathcal{U}\mathfrak{g})$. We now state the universal property of the enveloping algebra.

Theorem 3.3.4. *Let $\mathfrak{g}$ be a Lie algebra. Given any algebra A and any morphism of Lie algebras $f : \mathfrak{g} \to \mathcal{L}(A)$, there exists a unique morphism of algebras $\varphi : \mathcal{U}\mathfrak{g} \to A$ such that $\varphi \circ i_{\mathfrak{g}} = f$.*

We refer the proof of the last proposition to [Kas12].

3.3.1 Restricted and induced representations

Note that the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a Lie $\mathfrak{g}$–module, with action given by left multiplication. This fact allows us to define the concept of induced representation.

Definition 3.3.5. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$, (V, μ_V) be a Lie $\mathfrak{g}$–module and (W, μ_W) be a Lie $\mathfrak{h}$–module.*

- *The restriction of the Lie $\mathfrak{g}$–module (V, μ_V) on the subalgebra $\mathfrak{h}$ is the Lie $\mathfrak{h}$–module $(V, \tilde{\mu})$, where $\tilde{\mu}$ is defined by*

$$\begin{aligned} \tilde{\mu} : \mathfrak{h} \times V &\to V \\ (a, v) &\mapsto \mu_V(a, v). \end{aligned}$$

  We denote by $\mathsf{Res}_{\mathfrak{h}}^{\mathfrak{g}} V$ this Lie $\mathfrak{h}$–module structure.

- *The couple $(\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{h}} W, \bar{\mu})$, where*

$$\begin{aligned} \bar{\mu} : \mathfrak{g} \times \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{h}} W &\to \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{h}} W \\ (a, u \otimes w) &\mapsto (a \cdot u) \otimes w \end{aligned}$$

  is Lie g–module, called the Lie $\mathfrak{g}$–module induced by the Lie $\mathfrak{h}$–module W. We denote by $\mathsf{Ind}_{\mathfrak{h}}^{\mathfrak{g}} W$ this Lie $\mathfrak{g}$–module structure.

The following theorem gives us a bridge between the notion of induced representation and of restricted representation.

Theorem 3.3.6 (Frobenius). *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$, (V, μ_V) be a Lie $\mathfrak{g}$–module and (W, μ_W) be a Lie $\mathfrak{h}$–module. Then*

$$\mathsf{Hom}_{\mathfrak{h}}(\mathsf{Res}_{\mathfrak{h}}^{\mathfrak{g}} V, W) \simeq \mathsf{Hom}_{\mathfrak{g}}(\mathsf{Ind}_{\mathfrak{h}}^{\mathfrak{g}} W, V).$$

3.3.2 A Hopf algebra structure on the universal enveloping algebra

Proposition 3.3.7. *Let $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}''$ be three Lie algebras. Then:*

(i) *for any morphism of Lie algebras* $f : \mathfrak{g} \to \mathfrak{g}'$, *there exists a unique morphism of algebras* $U(f) : \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}'$ *such that* $U(f) \circ i_{\mathfrak{g}} = i_{\mathfrak{g}} \circ f$.

(ii) $U(\mathsf{id}_{\mathfrak{g}}) = \mathsf{id}_{\mathcal{U}\mathfrak{g}}$.

(iii) *If* $f : \mathfrak{g} \to \mathfrak{g}'$ *and* $f' : \mathfrak{g}' \to \mathfrak{g}''$ *are composable morphisms of Lie algebras, then* $U(f' \circ f) = U(f') \circ U(f)$.

(iv) $\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}') \simeq \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}'$.

A proof of this Proposition can be consulted in [Kas12]. This result allows us to define a Hopf algebra structure on the enveloping algebra $\mathcal{U}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$. More specifically, we have that:

- The comultiplication Δ on $\mathcal{U}\mathfrak{g}$ is defined by $\Delta = \varphi \circ U(\delta)$, where δ is the diagonal map

$$\begin{aligned} \delta : \mathcal{U}\mathfrak{g} &\to \mathcal{U}\mathfrak{g} \oplus \mathcal{U}\mathfrak{g} \\ x &\mapsto (x, x) \end{aligned}$$

  and φ is the isomorphism $\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) \to \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$.

- The counit is given by $\varepsilon = U(0)$, where 0 denotes the zero morphism.

- The antipode is defined by $S = U(\tau)$, where τ is the flip map $\tau : \mathfrak{g} \to \mathfrak{g}^{op}$.

In particular, if $\mathfrak{g}$ is a Lie algebra and $\mathcal{U}\mathfrak{g}$ is its universal enveloping algebra, the coproduct of $\mathcal{U}\mathfrak{g}$ is given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

3.3.3 Primitive elements

Definition 3.3.8. *Let* H *be a bialgebra. An element* x *of* H *is said to be primitive if*

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

We denote by $Prim(H)$ the set of all the primitive elements of a bialgebra H.

Lemma 3.3.9. *If* x *and* y *are primitive elements of a bialgebra, then the commutator* $xy - yx$ *is a primitive element.*

Proof. We have

$$\Delta(xy) = (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) = 1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1,$$

and so

$$\Delta(xy - yx) = 1 \otimes (xy - yx) + (xy - yx) \otimes 1.$$

□

Remark 3.3.10. *If $\mathfrak{g}$ is a Lie algebra, then $Prim(\mathcal{U}\mathfrak{g}) = \mathfrak{g}$.*

3.4 Lie bialgebras

Definition 3.4.1. *Let $\mathbb{K}$ be a field.*

- *A Lie bialgebra is a triple $(\mathfrak{g}, [\cdot,\cdot], \delta)$, where:*

  (i) $(\mathfrak{g}, [\cdot,\cdot])$ *is a Lie algebra;*

  (ii) $(\mathfrak{g}, \delta)$ *is a Lie coalgebra;*

  (iii) the following relation, called the cocycle condition, is satisfied

  $$\begin{aligned}\delta([x,y]) &= \sum_{(x)} (x' \otimes [x'', y]) + ([x', y] \otimes x'') + \sum_{(y)} (y' \otimes [x, y'']) + ([x, y'] \otimes y'') \\ &=: [\delta(x), 1 \otimes y + y \otimes 1] + [1 \otimes x + x \otimes 1, \delta(y)]\end{aligned}$$

  for all $x, y \in \mathfrak{g}$.

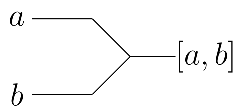
- *A Lie subbialgebra $\mathfrak{h}$ of a Lie bialgebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ that is both a Lie subalgebra and a Lie subcoalgebra.*

- *A Lie biideal $\mathfrak{i}$ of a Lie bialgebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ that is both a Lie ideal and a Lie coideal.*

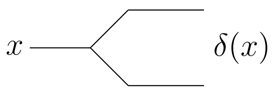
- *A morphism f of Lie bialgebras $\mathfrak{g}, \mathfrak{g}'$ is a linear map $f : \mathfrak{g} \to \mathfrak{g}'$ that is both a morphism of Lie algebras and a morphism of Lie coalgebras.*

Proposition 3.4.2. *Let $(\mathfrak{g}, [\cdot,\cdot], \delta)$ be a finite–dimensional Lie bialgebra. Then $(\mathfrak{g}^*, \delta^*, [\cdot,\cdot]^*)$ is a Lie bialgebra.*

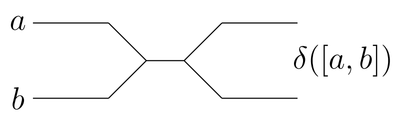
Proof. As seen in the Subsection 3.2.1, we have that the pair $(\mathfrak{g}^*, [\cdot,\cdot]^*)$ is a Lie coalgebera, while the pair $(\mathfrak{g}^*, \delta^*)$ is a Lie algebra. It remains to prove the cocycle condition for $(\mathfrak{g}^*, \delta^*, [\cdot,\cdot]^*)$. To do this, we use a pictorial approach: we represent the Lie bracket $[\cdot,\cdot]$ by the picture



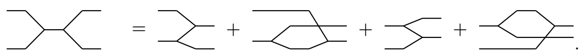
and the Lie cobracket δ by the picture



In this representation, we denote the composition of maps by adjoining diagrams from left to right. For example, $\delta([a,b])$ corresponds to the diagram



Furthermore, the operation of taking the dual corresponds to interchanging left and right. In particular, the cocycle condition of $\mathfrak{g}$ can be written as:



We have that this picture is self dual, and so we conclude that the cocycle condition still holds in $(\mathfrak{g}^*, \delta^*, [\cdot,\cdot]^*)$. $\square$

Let $(\mathfrak{g}, [\cdot,\cdot], \delta)$ be a Lie bialgebra and let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{g}$. The structure's constants of $\mathfrak{g}$ with respect to the basis $\{e_1, \dots, e_n\}$ are $\{\alpha_{i,j}^k, \beta_k^{i,j}\}$, where

$$[e_i, e_j] = \sum_{k=1}^{n} \alpha_{i,j}^k e_k$$

and

$$\delta(e_k) = \sum_{i,j} \beta_k^{i,j} e_i \otimes e_j$$

for $1 \leq i, j, k \leq n$. We now want to write the cocycle condition in terms of the structure's constants. We have

$$\delta([e_i, e_j]) = \sum_k \alpha_{i,j}^k \delta(e_k) = \sum_{k,u,v} \alpha_{i,j}^k \beta_k^{u,v} e_u \otimes e_v,$$

while

$$\begin{aligned}[1 \otimes e_i + e_i \otimes 1, \delta(e_j)] &= \sum_{s,t} \beta_j^{s,t}[1 \otimes e_i + e_i \otimes 1, e_s \otimes e_t] \\ &= \sum_{s,t} \beta_j^{s,t} e_s \otimes [e_i, e_t] + [e_i, e_s] \otimes e_t \\ &= \sum_{s,t,u} \beta_j^{s,t} \alpha_{i,t}^u e_s \otimes e_u + \sum_{s,t,v} \beta_j^{s,t} \alpha_{i,s}^v e_v \otimes e_t,\end{aligned}$$

and

$$\begin{aligned}[\delta(e_i), 1 \otimes e_j + e_j \otimes 1] &= -[1 \otimes e_j + e_j \otimes 1, \delta(e_i)] \\ &= -\sum_{p,q} \beta_i^{p,q}[1 \otimes e_j + e_j \otimes 1, e_p \otimes e_q] \\ &= -\sum_{p,q} \beta_i^{p,q} e_p \otimes [e_j, e_q] + [e_j, e_p] \otimes e_q \\ &= -\sum_{p,q,l} \beta_i^{p,q} \alpha_{j,q}^l e_p \otimes e_l - \sum_{p,q,h} \beta_i^{p,q} \alpha_{j,p}^h e_h \otimes e_q.\end{aligned}$$

So the cocycle condition is satisfied if and only if

$$\sum_k \alpha_{r,s}^k \beta_k^{i,j} = \sum_p \left(\alpha_{p,r}^i \beta_s^{j,p} + \alpha_{p,s}^j \beta_r^{i,p} - \alpha_{p,r}^j \beta_s^{i,p} - \alpha_{p,s}^i \beta_r^{j,p} \right)$$

for any $i, j, r, s \in \{1, \dots n\}$.

3.4.1 Coboundary Lie bialgebras

We now introduce some concepts of Lie algebras cohomology. Let $\mathfrak{g}$ be a Lie algebra and V be a Lie $\mathfrak{g}$–module. For any $n \in \mathbb{N}$, we denote by $C^n(\mathfrak{g}, V)$ the set of all linear maps $f : \mathfrak{g}^{\otimes n} \to V$ such that

$$f(x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}) = sgn(\sigma) f(x_1 \otimes \dots \otimes x_n)$$

for all permutations $\sigma \in S_n$. Define the n–th differential map

$$\begin{aligned}\partial_n : C^n(\mathfrak{g}, V) &\to C^{n+1}(\mathfrak{g}, V) \\ f &\mapsto \partial_n f\end{aligned}$$

by

$$\begin{aligned}\partial_n f(x_1 \otimes \dots \otimes x_{n+1}) &\coloneqq \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot f(x_1 \otimes \dots \otimes \hat{x_i} \otimes \dots \otimes x_{n+1}) \\ &+ \sum_{1 \le i < j \le n+1} (-1)^{i+j} f([x_i, x_j] \otimes x_1 \otimes \dots \otimes \hat{x_i} \otimes \dots \otimes \hat{x_j} \otimes \dots \otimes x_{n+1}).\end{aligned}$$

We have that $\partial_n \circ \partial_{n+1} = 0$ for any $n \in \mathbb{N}$, and so we may define the following groups:

- the n–th cocycles group of $\mathfrak{g}$ with cofficients in V is the group
$$Z^n(\mathfrak{g}, V) \coloneqq \ker \partial_n;$$

- the n–th coboundaries group of $\mathfrak{g}$ with cofficients in V is the group
$$B^n(\mathfrak{g}, V) \coloneqq \mathsf{Im}\partial_{n-1};$$

- the n–th cohomology group of $\mathfrak{g}$ with cofficients in V is the group
$$H^n(\mathfrak{g}, V) \coloneqq Z_n/B_n.$$

Remark 3.4.3. *Let* $(\mathfrak{g}, [\cdot,\cdot], \delta)$ *be a Lie bialgebra. Then:*

- *if* V *and* W *are Lie* $\mathfrak{g}$*–modules, then* $V \otimes W$ *is too, with action given by*
$$x \cdot (v \otimes w) \coloneqq (x \cdot v) \otimes w + v \otimes (x \cdot w).$$
Therefore, the vector space $\mathfrak{g}^{\otimes n}$ *is a Lie* $\mathfrak{g}$*–module for any* $n \in \mathbb{N}$ *and we can consider the cohomology of* $\mathfrak{g}$ *with cofficients in* $\mathfrak{g}^{\otimes n}$*.*

- *The cobracket* δ *is an element of* $C^1(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$*. Furthermore, the cocycle condition means that* $\delta \in \ker \partial_1$*, so* δ *is a* 1*–cocycle and so we may define a Lie bialgebra as a Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ *equipped with a* 1*–cocycle* δ *with values in* $\mathfrak{g} \otimes \mathfrak{g}$*, satisfying the coJacoby identity.*

- *Let* $r \in \Lambda^2(\mathfrak{g}) = C^0(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$*. Then*
$$\begin{aligned} \partial_0 r(x) = x \cdot r &= (\mathsf{id}_{\mathfrak{g}} \otimes ad_x + ad_x \otimes \mathsf{id}_{\mathfrak{g}})(r) \\ &=: [1 \otimes x + x \otimes 1, r]. \end{aligned}$$

Definition 3.4.4. *An element* $r \in \Lambda^2(\mathfrak{g})$ *is said to be a coboundary structure for a Lie bialgebra* $(\mathfrak{g}, [\cdot,\cdot], \delta)$ *if* $\delta = \partial_0 r$*, that means that*
$$\delta(x) = \partial_0 r(x) = [1 \otimes x + x \otimes 1, r]$$
for all $x \in \mathfrak{g}$*. In this case we say that the triple* $(\mathfrak{g}, [\cdot,\cdot], r)$ *is a coboundary Lie bialgebra.*

3.4.2 Triangular Lie bialgebras and the classical Yang–Baxter equation

Not any cocycle defined by some $r \in \Lambda^2(\mathfrak{g})$ will give rise to a Lie bialgebra structure on $\mathfrak{g}$, as the coJacoby identity may not be satisfied. So we now want to investigate when an element $r \in \Lambda^2(\mathfrak{g})$ turns a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ into a Lie bialgebra structure.

Definition 3.4.5. *The classical Yang–Baxter map is the map*

$$\begin{aligned} CYB : \mathfrak{g}^{\otimes 2} &\to \mathfrak{g}^{\otimes 3} \\ r &\mapsto [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \end{aligned}$$

The equation $CYB(r) = 0$ is called the classical Yang–Baxter equation and is denoted by $CYBE$. Any solution of the $CYBE$ is called a r–matrix.

We have that the $CYBE$ restricts to a map $\Lambda^2\mathfrak{g} \to \Lambda^3\mathfrak{g}$. The next result, due to Drinfeld, gives us a characterization for coboundaries structures on a Lie algebra $\mathfrak{g}$.

Theorem 3.4.6. *Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra and let $r \in \Lambda^2(\mathfrak{g})$. Then $(\mathfrak{g}, [\cdot,\cdot], \partial_0 r)$ is a Lie bialgebra if and only if $CYB(r)$ is $\mathfrak{g}$–invariant.*

We refer the proof of this result to [ES02].

Definition 3.4.7. *A coboundary Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], r)$ is said to be triangular if $CYB(r) = 0$. Likewise, a triangular structure on a Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \delta)$ is an element $r \in \Lambda^2(\mathfrak{g})$ such that $\partial_0 r = \delta$ and $CYB(r) = 0$.*

Thus, if $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra, there is a one–one correspondence between triangular Lie bialgebra structures $(\mathfrak{g}, [\cdot,\cdot], r)$ and the solutions of the $CYBE$ in $\Lambda^2(\mathfrak{g})$. Note that, if $(\mathfrak{g}, [\cdot,\cdot], \delta)$ and $(\mathfrak{g}', [\cdot,\cdot]', \delta')$ are Lie bialgebras, $f : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism and $r \in \Lambda^2(\mathfrak{g})$, then

$$(f \otimes f \otimes f)(CYB(r)) = CYB((f \otimes f)(r)) \in \Lambda^3(\mathfrak{g}').$$

This means that if r is a triangular structure on $\mathfrak{g}$, then $(f \otimes f)(r)$ is a triangular structure on $\mathfrak{g}'$; this good property is not satisfied by a general coboundary structure, and so this makes triangular Lie bialgebras better to study than the coboundary ones.

3.5 Manin triples and the Drinfeld double construction

In this Subsection we consider only finite–dimensional Lie algebras.

3.5.1 Finite–dimensional Manin triples

Definition 3.5.1. *A finite–dimensional Manin triple is a triple* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$*, where:*

(i) $\mathfrak{g}$ *is a finite–dimensional Lie algebra equipped with a non degenerate and invariant bilinear form* $< \cdot, \cdot >$*, that means that*

$$< [x, y], z >=< x, [y, z] >$$

for all $x, y, z \in \mathfrak{g}$*;*

(ii) $\mathfrak{g}_+$ *and* $\mathfrak{g}_-$ *are Lie subalgebras of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *as vector spaces;*

(iii) $\mathfrak{g}_+$ *and* $\mathfrak{g}_-$ *are isotropic subspaces of* $\mathfrak{g}$ *with respect to* $< \cdot, \cdot >$*, that means that* $< \cdot, \cdot >\restriction_{\mathfrak{g}_+,\mathfrak{g}_+}= 0$ *and* $< \cdot, \cdot >\restriction_{\mathfrak{g}_-,\mathfrak{g}_-}= 0$*.*

Warning: in a Manin triple we have $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space, but not as a Lie algebra. In other words, the Lie bracket of $\mathfrak{g}$ is not the Lie bracket of the direct sum of the Lie algebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ defined in 3.1.2.
Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a finite–dimensional Manin triple and consider the map

$$\begin{aligned} \Phi_+ : \mathfrak{g}_+ &\to \mathsf{Hom}(\mathfrak{g}_-, \mathbb{K}) = \mathfrak{g}_-^* \\ x &\mapsto < x, \cdot > . \end{aligned}$$

Since $< \cdot, \cdot >$ is non degenerate in $\mathfrak{g}$, it follows that $\ker \Phi_+ = \{0\}$. This implies that

$$\dim \mathfrak{g}_+ \leq \dim \mathfrak{g}_-^* = \dim \mathfrak{g}_-.$$

Similarly, we can consider the map

$$\begin{aligned} \Phi_- : \mathfrak{g}_- &\to \mathsf{Hom}(\mathfrak{g}_+, \mathbb{K}) = \mathfrak{g}_+^* \\ y &\mapsto < \cdot, y > . \end{aligned}$$

Since $\ker \Phi_- = \{0\}$, we obtain

$$\dim \mathfrak{g}_- \leq \dim \mathfrak{g}_+^* = \dim \mathfrak{g}_+.$$

This means that if $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple, we have $\dim \mathfrak{g}_+ = \dim \mathfrak{g}_-$. In particular, we have

$$\mathfrak{g}_+^* \simeq \mathfrak{g}_-,$$

and so

$$\mathfrak{g} \simeq \mathfrak{g}_+ \oplus \mathfrak{g}_+^*.$$

By duality, since $\mathfrak{g}_+^*$ has a Lie algebra structure, then the dual $\mathfrak{g}_+$ has a Lie coalgebra structure.

Proposition 3.5.2. *Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a finite–dimensional Manin triple and let δ be the Lie coalgebra structure on $\mathfrak{g}_+$ defined above. Then $(\mathfrak{g}_+, [\cdot,\cdot], \delta)$ is a Lie bialgebra.*

Proof. We have to check the cocycle condition. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{g}_+$ and let $\{e_1^*, \dots, e_n^*\}$ be the dual basis of $\{e_1, \dots, e_n\}$ in $\mathfrak{g}_-$. Let $(\cdot,\cdot)$ be the natural pairing between $\mathfrak{g}_+$ and $\mathfrak{g}_- \simeq \mathfrak{g}_+^*$. Then we have

$$
\begin{aligned}
(e_r^* \otimes e_s^*, \delta([e_k, e_l])) &= \sum_c \alpha_{k,l}^c (e_r^* \otimes e_s^*, \delta(e_c)) \\
&= \sum_{c,d,f} \alpha_{k,l}^c \beta_c^{d,f} (e_r^* \otimes e_s^*, e_d \otimes e_f) \\
&= \sum_{c,d,f} \alpha_{k,l}^c \beta_c^{d,f} \delta_{r,d} \delta_{s,f} \\
&= \sum_c \alpha_{k,l}^c \beta_c^{r,s},
\end{aligned}
$$

while

$$
\begin{aligned}
&(e_r^* \otimes e_s^*, [1 \otimes e_k + e_k \otimes 1, \delta(e_l)]) = \\
&= \sum_{i,j} \beta_l^{i,j} (e_r^* \otimes e_s^*, e_i \otimes [e_k, e_j] + [e_k, e_i] \otimes e_j) \\
&= \sum_{i,j,t} \beta_l^{i,j} \alpha_{k,j}^t (e_r^* \otimes e_s^*, e_i \otimes e_t) + \sum_{i,j,h} \beta_l^{i,j} \alpha_{k,i}^h (e_r^* \otimes e_s^*, e_h \otimes e_j) \\
&= \sum_{i,j,t} \beta_l^{i,j} \alpha_{k,j}^t \delta_{r,i} \delta_{s,t} + + \sum_{i,j,h} \beta_l^{i,j} \alpha_{k,i}^h \delta_{r,h} \delta_{s,j} \\
&= \sum_j \beta_l^{r,j} \alpha_{k,j}^s + \sum_j \beta_l^{i,s} \alpha_{k,i}^r,
\end{aligned}
$$

and

$$
\begin{aligned}
&-(e_r^* \otimes e_s^*, [1 \otimes e_l + e_l \otimes 1, \delta(e_k)]) = \\
&= -\sum_{p,q} \beta_k^{p,q} (e_r^* \otimes e_s^*, e_p \otimes [e_l, e_q] + [e_l, e_p] \otimes e_q) \\
&= -\sum_{p,q,a} \beta_k^{p,q} \alpha_{l,q}^a (e_r^* \otimes e_s^*, e_p \otimes e_a) - \sum_{p,q,b} \beta_k^{p,q} \alpha_{l,p}^b (e_r^* \otimes e_s^*, e_b \otimes e_q) \\
&= -\sum_{p,q,a} \beta_k^{p,q} \alpha_{l,q}^a \delta_{r,p} \delta_{s,a} - \sum_{p,q,b} \beta_k^{p,q} \alpha_{l,p}^b \delta_{r,b} \delta_{s,q} \\
&= -\sum_q \beta_k^{r,q} \alpha_{l,q}^s - \sum_q \beta_k^{p,s} \alpha_{l,p}^r.
\end{aligned}
$$

Using the antisymmetry of the structure's constants and some smart changes of variables, we obtain the cocycle identity.

$\square$

Note that $\mathfrak{g}_-$ is the dual Lie bialgebra of $\mathfrak{g}_+$.

Proposition 3.5.3. *Let* $(\mathfrak{g}, [\cdot,\cdot], \delta)$ *be a Lie bialgebra. Then* $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ *is a Manin triple.*

Proof. We have to construct a non degenerate and invariant bilinear form that satisfies the definition of a finite–dimensional Manin triple. Also, we have to define the mixed bracket $[x, y]$ for $x \in \mathfrak{g}$ and $y \in \mathfrak{g}^*$. Consider the bilinear form given by

$$< x + y, x' + y' > := y(x') + y'(x)$$

for $x, x' \in \mathfrak{g}$ and $y, y' \in \mathfrak{g}^*$. It is clear that $< \cdot, \cdot > \upharpoonright_{\mathfrak{g},\mathfrak{g}} = 0$ and $< \cdot, \cdot > \upharpoonright_{\mathfrak{g}^*,\mathfrak{g}^*} = 0$. We now have to define the mixed bracket $[x, y]$ for $x \in \mathfrak{g}$, $y \in \mathfrak{g}^*$ in such a way that $< \cdot, \cdot >$ is invariant. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{g}$. Then we have

$$\begin{aligned} < [e_i^*, e_j], e_k^* > &= - < e_j, [e_i^*, e_k^*] > \\ &= - \sum_t \beta_t^{i,k} < e_j, e_t^* > \\ &= -\beta_j^{i,k} \end{aligned}$$

and

$$\begin{aligned} < [e_i^*, e_j], e_k > &= < e_i^*, [e_j, e_k] > \\ &= \sum_s \alpha_{j,k}^s < e_i^*.e_s > \\ &= \alpha_{j,k}^i. \end{aligned}$$

Then we obtain the following mixed bracket:

$$[e_i^*, e_j] = \sum_{k=1}^{n} \alpha_{j,k}^i e_k^* - \beta_j^{i,k} e_k.$$

We now have to prove that this bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$ satisfies the Jacoby identity. It is clear that the identity is satisfied in the case of $x, x', x'' \in \mathfrak{g}$ and in the case of $y, y', y'' \in \mathfrak{g}^*$, and so we have to prove it in the two mixed cases. Let $e_i \in \mathfrak{g}$ and $e_j^*, e_k^* \in \mathfrak{g}^*$. Then we have:

$$\begin{aligned}[e_i,[e_j^*,e_k^*]] &= \sum_t \beta_t^{j,k}[e_i,e_t^*] \\ &= -\sum_{t,p}\beta_t^{j,k}\alpha_{t,p}^i e_p^* + \sum_{t,p}\beta_t^{j,k}\beta_t^{i,p}e_p,\end{aligned}$$

$$\begin{aligned}[e_k^*,[e_i,e_j^*]] &= \sum_s \beta_i^{j,s}[e_k^*,e_s] - \sum_s \alpha_{i,s}^j[e_k^*,e_s^*] \\ &= \sum_{s,m}\beta_i^{j,s}\alpha_{s,m}^k e_m^* - \sum_{s,m}\beta_i^{j,s}\beta_s^{k,m}e_m - \sum_{s,n}\alpha_{i,s}^j\beta_n^{k,s}e_n^*,\end{aligned}$$

and

$$\begin{aligned}[e_j^*,[e_k^*,e_i]] &= \sum_q \alpha_{i,q}^k[e_j^*,e_q^*] - \sum_q \beta_i^{k,q}[e_j^*,e_q] \\ &= \sum_{q,r}\alpha_{i,q}^k\beta_r^{j,q}e_r^* - \sum_{q,v}\beta_i^{k,q}\alpha_{q,v}^j e_v^* + \sum_{q,v}\beta_i^{k,q}\beta_q^{j,v}e_v.\end{aligned}$$

The sum of the terms whose constants are mixed products of α's and β's, using some smart change of varialbes and of signs, gives us the cocycle condition, while the sum of the terms whose constants are products of β's gives us the Jacoby identity of $\mathfrak{g}^*$, and so we have

$$[e_i,[e_j^*,e_k^*]] + [e_k^*,[e_i,e_j^*]] + [e_j^*,[e_k^*,e_i]] = 0.$$

In a similar way, if we take $e_j, e_k \in \mathfrak{g}$ and $e_i^* \in \mathfrak{g}^*$ and we compute the sum

$$[e_i^*,[e_j,e_k]] + [e_k,[e_i^*,e_j]] + [e_j,[e_k,e_i^*]],$$

we obtain the sum of the cocycle condition with the Jacoby identity of $\mathfrak{g}$, and so we have

$$[e_i^*,[e_j,e_k]] + [e_k^{,}[e_i^*,e_j]] + [e_j,[e_k,e_i^*]] = 0,$$

that proves the Jacoby identity for $\mathfrak{g} \oplus \mathfrak{g}^*$. □

Remark 3.5.4. *If $(\mathfrak{g}, [\cdot,\cdot], \delta)$ is a Lie bialgebra and $\mathfrak{g} \oplus \mathfrak{g}^*$ is the associated Manin triple, we may write the mixed bracket in a coordinate–free way by*

$$[x,y] = ad_x^*(y) - ad_y^*(x)$$

for any $x \in \mathfrak{g}$ and $y \in \mathfrak{g}^*$.

3.5.2 Quasi–triangular Lie bialgebras and the Drinfeld double

The preceding construction shows that the notions of finite–dimensional Manin triple and of finite–dimensional Lie bialgebra are equivalent, and so it comes naturally to introduce the next

Definition 3.5.5. *Let* $(\mathfrak{g}, [\cdot,\cdot], \delta)$ *be a finite–dimensional Lie bialgebra and let* $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ *be the finite–dimensional Manin triple associated to* $\mathfrak{g}$*. The Drinfeld double of* $\mathfrak{g}$ *is the finite–dimensional Lie algebra* $\mathfrak{g} \oplus \mathfrak{g}^*$*, and we denote it by* $\mathfrak{D}\mathfrak{g}$*.*

We now want to define a Lie bialgebra structure on the Drinfeld double $\mathfrak{D}\mathfrak{g}$ of a finite–dimensional Lie bialgebra $\mathfrak{g}$. Before that, we introduce the Definition of quasi–triangular Lie bialgebra.

Definition 3.5.6. *A quasi–triangular Lie bialgebra is a triple* $(\mathfrak{g}, [\cdot,\cdot], \tilde{r})$ *such that:*

(i) $(\mathfrak{g}, [\cdot,\cdot], \tilde{r})$ *is a coboundary Lie bialgebra;*

(ii) $CYB(\tilde{r}) = 0$*;*

(iii) $\tilde{r} + \tau(\tilde{r}) \in \mathfrak{g} \otimes \mathfrak{g}$ *is* $\mathfrak{g}$*–invariant.*

Theorem 3.5.7. *Let* $\mathfrak{g}$ *be a finite–dimensional Lie bialgebra. Then* $\mathfrak{D}\mathfrak{g}$ *is a quasi–triangular Lie bialgebra.*

Proof. We have to check the three conditions above.

(i) Set $\delta \coloneqq \delta_{\mathfrak{g}} \oplus -\delta_{\mathfrak{g}^*}$. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{g}$ and let

$$\tilde{r} = \sum_{j=1}^{n} e_j \otimes e_j^*.$$

Let $e_i \in \mathfrak{g}$ be an element of the basis. Then

$$\begin{aligned}
\partial\tilde{r}(e_i) &= [e_i \otimes 1 + 1 \otimes e_i, \tilde{r}] \\
&= [e_i \otimes 1 + 1 \otimes e_i, \sum_j^n e_j \otimes e_j^*] \\
&= \sum_j^n \big([e_i, e_j] \otimes e_j^* + e_j \otimes [e_i, e_j^*]\big) \\
&= \sum_j^n \big([e_i, e_j] \otimes e_j^* - e_j \otimes [e_j^*, e_i]\big) \\
&= \sum_j^n [e_i, e_j] \otimes e_j^* + \sum_{j,k} \beta_i^{j,k} e_j \otimes e_k - \sum_{j,k} \alpha_{i,k}^j e_j \otimes e_k^* \\
&= \sum_j^n [e_i, e_j] \otimes e_j^* + \sum_{j,k} \beta_i^{j,k} e_j \otimes e_k - \sum_k [e_i, e_k] \otimes e_k^* \\
&= \sum_{j,k} \beta_i^{j,k} e_j \otimes e_k \\
&= \delta_{\mathfrak{g}}(e_i).
\end{aligned}$$

Let $e_i^* \in \mathfrak{g}^*$ be an element of the dual basis. Then

$$\begin{aligned}
\partial\tilde{r}(e_i^*) &= [e_i^* \otimes 1 + 1 \otimes e_i^*, \tilde{r}] \\
&= [e_i^* \otimes 1 + 1 \otimes e_i^*, \sum_j e_j \otimes e_j^*] \\
&= \sum_j \big([e_i^*, e_j] \otimes e_j^* + e_j \otimes [e_i^*, e_j^*]\big) \\
&= \sum_{j,k} \alpha_{j,k}^i e_k^* \otimes e_j^* - \sum_{j,k} \beta_j^{i,k} e_k \otimes e_j^* + \sum_j e_j \otimes [e_i^*, e_j^*] \\
&= -\sum_{j,k} \alpha_{k,j}^i e_k^* \otimes e_j^* - \sum_k e_k \otimes [e_i^*, e_k^*] + \sum_j e_j \otimes [e_i^*, e_j^*] \\
&= -\sum_{j,k} \alpha_{k,j}^i e_k^* \otimes e_j^* \\
&= -\delta_{\mathfrak{g}^*}(e_i^*).
\end{aligned}$$

Therefore $\delta = \partial_0\tilde{r}$ and so $\mathfrak{D}\mathfrak{g}$ is a coboundary Lie bialgebra.

(ii)

$$\begin{aligned}
CYB(\tilde{r}) &= [\tilde{r}_{12}, \tilde{r}_{13}] + [\tilde{r}_{12,\tilde{r}_{23}}] + [\tilde{r}_{13}, \tilde{r}_{2,3}] \\
&= \sum_{i,j} [e_i, e_j] \otimes e_i^* \otimes e_j^* + e_i \otimes [e_i^*, e_j] \otimes e_j^* + e_i \otimes e_j \otimes [e_i^*, e_j^*] \\
&= \sum_{i,j,k} \alpha_{i,j}^k e_k \otimes e_i^* \otimes e_j^* + \sum_{i,j,k} \alpha_{j,k}^i e_i \otimes e_k^* \otimes e_j^* + \\
&- \sum_{i,j,k} \beta_j^{i,k} e_i \otimes e_k \otimes e_j^* + \sum_{i,j,k} \beta_k^{i,j} e_i \otimes e_j \otimes e_k \\
&= \sum_{i,j,k} \alpha_{i,j}^k e_k \otimes e_i^* \otimes e_j^* - \sum_{i,j,k} \alpha_{k,j}^i e_i \otimes e_k^* \otimes e_j^* + \\
&- \sum_{i,j,k} \beta_j^{i,k} e_i \otimes e_k \otimes e_j^* + \sum_{i,j,k} \beta_k^{i,j} e_i \otimes e_j \otimes e_k \\
&= 0.
\end{aligned}$$

(iii) We have that $\tilde{r} + \tau(\tilde{r})$ is the Casimir element corresponding to the nondegenerate bilinear form $< \cdot, \cdot >$ on $\mathfrak{D}\mathfrak{g}$, and thus it is $\mathfrak{D}\mathfrak{g}$–invariant.

□

Let $\mathfrak{g}$ be a finite–dimensional Lie bialgebra and let $\mathfrak{D}\mathfrak{g}$ be its Drinfeld double. Since $\delta = \delta_{\mathfrak{g}} \oplus -\delta_{\mathfrak{g}^*}$, we have that the embeddings

$$i_{\mathfrak{g}} : \mathfrak{g} \hookrightarrow \mathfrak{D}\mathfrak{g}$$

and

$$i_{\mathfrak{g}^*} : \mathfrak{g}^* \hookrightarrow \mathfrak{D}\mathfrak{g}$$

are Lie bialgebra maps. In particular, this tells us that for any finite dimensional Lie bialgebra $\mathfrak{g}$, there exists a canonical embedding of $\mathfrak{g}$ into a quasi–triangular one, in which the r–matrix is explicitly described. This fact is explained in more precise terms by the following

Proposition 3.5.8. *Let $(\mathfrak{g}, [\cdot,\cdot], r)$ be a finite–dimensional quasi–triangular Lie bialgebra and consider the Lie subalgebras of $\mathfrak{g}$ given by*

$$\begin{aligned}
\mathfrak{g}_+ &:= \mathsf{Span}\{(\mathsf{id} \otimes f)(r) | f \in \mathfrak{g}^*\} \\
\mathfrak{g}_- &:= \mathsf{Span}\{(f \otimes \mathsf{id})(r) | f \in \mathfrak{g}^*\}.
\end{aligned}$$

Suppose that r is such that $\mathfrak{g}_+ + \mathfrak{g}_- = \mathfrak{g}$. Then $\mathfrak{g}$ is isomorphic, as a quasi–triangular Lie bialgebra, to a quotient of $\mathfrak{D}\mathfrak{g}_+$.

A proof of this result can be find in [ES02]. The Proposition above tells us that if $\mathfrak{g}$ is a finite–dimensional Lie algebra endowed with a r–matrix, then there exist a finite–dimensional Lie bialgebra $(\mathfrak{g}_+, [\cdot,\cdot], \delta)$ and a Lie algebra map $\phi : \mathfrak{D}\mathfrak{g}_+ \to \mathfrak{g}$ such that $\phi_{|\mathfrak{g}_+}$ is injective and with $(\phi \otimes \phi)(r) = \tilde{r}$, where $\tilde{r}$ is the usual quasi–triangular structure on $\mathfrak{D}\mathfrak{g}_+$. As a direct consequence of this fact, we can reduce the study of Lie bialgebras to the case of quasi–triangular ones.

3.5.3 The standard structure

We now present the standard Lie bialgebra structure of simple Lie algebras. By the discussion above, this Lie bialgebra structure can be presented as a quotient of a double of a Lie bialgebra. When not specified, the ground field is the set of complex numbers $\mathbb{C}$.
Let $\mathfrak{g}$ be a finite–dimensional simple Lie algebra and let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ its Cartan decomposition. Consider the Lie algebra given by

$$\tilde{\mathfrak{g}} = \mathfrak{n}_+ \oplus \mathfrak{h}^{(1)} \oplus \mathfrak{h}^{(2)} \oplus \mathfrak{n}_-$$

as a vector space, where $\mathfrak{h}^{(1)} \simeq \mathfrak{h}^{(2)} \simeq \mathfrak{h}$ and with the following relations:

- $[\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)}] = 0$;
- $[h^{(i)}, e_\alpha] = h(\alpha) e_\alpha$;
- $[h^{(i)}, f_\alpha] = -h(\alpha) f_\alpha$;
- $[e_\alpha, f_\alpha] = \frac{1}{2}(h_\alpha^{(1)} + h_\alpha^{(2)})$.

Consider now the Lie algebra map $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ that is the identity on $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ and such that $\pi(h_\alpha^{(1)}) = \pi(h_\alpha^{(2)}) = h_\alpha$. This map allows us to construct a non degenerate bilinear form $(\cdot,\cdot)_{\tilde{\mathfrak{g}}}$ on $\tilde{\mathfrak{g}}$ using the bilinear form $(\cdot,\cdot)$ of $\mathfrak{g}$ in the following way:

$$(x+h^{(1)}+h^{(2)}, x'+h'^{(1)}+h'^{(2)})_{\tilde{\mathfrak{g}}} = 2\big((\pi(h^{(1)}), \pi(h'^{(2)}))+(\pi(h^{(2)}), \pi(h'^{(1)}))\big)+(x, x').$$

Proposition 3.5.9. *The triple of Lie algebras $(\tilde{\mathfrak{g}}, \mathfrak{n}_+ \oplus \mathfrak{h}^{(1)}, \mathfrak{n}_- \oplus \mathfrak{h}^{(2)})$ endowed with the bilinear form above is a Manin triple.*

Thus, $\tilde{\mathfrak{g}}$ has a quasi–triangular Lie bialgebra structure, with r–matrix given by

$$\tilde{r} = \sum_{\alpha \in \Delta^+} e_\alpha \otimes f_\alpha + \frac{1}{2} \sum_i k_i^{(1)} \otimes k_i^{(2)},$$

where (k_i) is a orthonormal basis of $\mathfrak{h}$ for $(\cdot,\cdot)$. The Lie algebra map π defined above endows $\mathfrak{g}$ with a quasi–triangular Lie bialgebra structure, whose r–matrix is

$$r = \sum_{\alpha\in\Delta^+} e_\alpha \otimes f_\alpha + \frac{1}{2}\sum_i k_i \otimes k_i.$$

This structure is called the standard structure. Moreover, the Lie subalgebras

$$\mathfrak{b}_\pm \coloneqq \mathfrak{n}_\pm \oplus \mathfrak{h}$$

are Lie subbialgebras, and the above construction shows that $\mathfrak{g}$ is *almost* the double of $\mathfrak{b}_+$ (or $\mathfrak{b}_-$).
Let us compute the cobracket of the standard structure. For the cocycle condition, it is sufficient to evaluate it on the Cartan subalgebra $\mathfrak{h}$ and on simple roots. For Cartan elements we have

$$\begin{aligned}
\delta(k_j) &= [1\otimes k_j + k_j \otimes 1, r]\\
&= \sum_{\alpha\in\Delta^+}[1\otimes k_j + k_j\otimes 1, e_\alpha\otimes f_\alpha] + \frac{1}{2}\sum_i [1\otimes k_j + k_j\otimes 1, k_i\otimes k_i]\\
&= \sum_{\alpha\in\Delta^+}\big(e_\alpha\otimes[k_j,f_\alpha] + [k_j,e_\alpha]\otimes f_\alpha\big) + \frac{1}{2}\sum_i\big(k_i\otimes[k_j,k_i] + [k_j,k_i]\otimes k_i\big)\\
&= \sum_{\alpha\in\Delta^+}\big(-\alpha(k_j)(e_\alpha\otimes f_\alpha) + \alpha(k_j)(e_\alpha\otimes f_\alpha)\big)\\
&= 0,
\end{aligned}$$

while for a simple root β we obtain

$$\begin{aligned}
\delta(e_\beta) &= [1\otimes e_\beta + e_\beta\otimes 1, r]\\
&= \sum_{\alpha\in\Delta^+}\big(e_\alpha\otimes[e_\beta,f_\alpha] + [e_\beta,e_\alpha]\otimes f_\alpha\big) + \frac{1}{2}\sum_i\big(k_i\otimes[e_\beta,k_i] + [e_\beta\otimes k_i]\otimes k_i\big)\\
&= e_\beta\otimes h_\beta - \frac{1}{2}(e_\beta\otimes h_\beta + h_\beta\otimes e_\beta)\\
&= e_\beta\wedge h_\beta,
\end{aligned}$$

and

$$
\begin{aligned}
\delta(f_\beta) &= [1 \otimes f_\beta + f_\beta \otimes 1, r] \\
&= \sum_{\alpha \in \Delta^+} \big(e_\alpha \otimes [f_\beta, f_\alpha] + [f_\beta, e_\alpha] \otimes f_\alpha\big) + \frac{1}{2} \sum_i \big(k_i \otimes [f_\beta, k_i] + [f_\beta \otimes k_i] \otimes k_i\big) \\
&= -h_\beta \otimes f_\beta + \frac{1}{2}(f_\beta \otimes h_\beta + h_\beta \otimes f_\beta) \\
&= f_\beta \wedge h_\beta.
\end{aligned}
$$

3.5.4 Infinite Manin triples and the standard structure for Kac–Moody algebras

We now want to define a standard Lie bialgebra structure on Kac–Moody algebras. We want to emulate the previous discussion, and so we need some definitions that extend the finite–dimensional case to the infinite–dimensional one, since Kac–Moody algebras are often infinite–dimensional. For simplicity, we consider only Lie algebras of countable dimension.

Definition 3.5.10. *A Manin triple is a triple* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$*, where:*

(i) $\mathfrak{g}$ *is a Lie algebra equipped with a non degenerate and invariant bilinear form* $(\cdot, \cdot)$*;*

(ii) $\mathfrak{g}_+$ *and* $\mathfrak{g}_-$ *are Lie subalgebras of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *as vector spaces;*

(iii) $\mathfrak{g}_+$ *and* $\mathfrak{g}_-$ *are isotropic subspaces of* $\mathfrak{g}$ *with respect to* $(\cdot, \cdot)$*;*

(iv) The bilinear form $(\cdot, \cdot)$ *induces an isomorphism* $\mathfrak{g}_- \to \mathfrak{g}_+^*$*.*

With the above Definition, the notions of a Manin triple and a Lie bialgebra are equivalent. Note that, contrary to the finite–dimensional case, in the infinite–dimensional case the notion of Manin triple is not symmetric. In fact if $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is an infinite Manin triple, then the triple $(\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+)$ is not an infinite Manin triple, since in general $\mathfrak{g}_-^*$ is not isomorphic to $\mathfrak{g}_+$.
As in the finite–dimensional case, we define the standard Lie bialgebra structure on a Kac–Moody algebra as follows: let A be a $n \times n$ symmetrizable Cartan matrix, and let $\tilde{\mathfrak{g}}(A)$ be the associated extended Kac–Moody algebra. Let (k_j) be a orthonormal basis for $\tilde{\mathfrak{h}}$, and let

$$
r = \sum_{\alpha \in \Delta^+} \sum_i e^i_\alpha \otimes f^i_\alpha + \frac{1}{2} \sum_j k_j \otimes k_j.
$$

The cobracket of $\tilde{\mathfrak{g}}(A)$ is defined by $\delta(x) = [x \otimes 1 + 1 \otimes x, r]$. In particular, as in the semisimple case, we have

$$\delta(h_i) = 0, \qquad \delta(e_i) = d_i e_i \wedge h_i, \qquad \delta(f_i) = d_i f_i \wedge h_i.$$

Chapter 4

Quantized universal enveloping algebras

4.1 Topologically free modules

4.1.1 Topological vector spaces

Definition 4.1.1. *A topological vector space is a $\mathbb{C}$–vector space V endowed with a Haussdorf topology such that the maps*

$$\begin{aligned} V \times V &\to V \\ (x, y) &\mapsto x + y \end{aligned}$$

and

$$\begin{aligned} \mathbb{C} \times V &\to V \\ (\lambda, x) &\mapsto \lambda x \end{aligned}$$

are continuous.

Let V be a topological vector space, $x_0 \in V$, and $\{U_i\}_{i \in I}$ be a base of neighbourhoods for 0_V, then:

- the family $\{x_0 + U_i\}_{i \in I}$ is a base of neighbourhoods for x_0;
- the set αU_i is a neighbourhood of 0_V for all $\alpha \in \mathbb{C}$ and for all $i \in I$.

Therefore, to have a vector topology on a vector space V it suffices to assign a base of neighborhoods for 0_V.

Example 4.1.2. *Let V be a vector space together with a norm $|| \cdot ||$. Then V is a topological vector space. In fact we may consider the metric d induced by $|| \cdot ||$ and the topology induced by d, obtaining that:*

- *the sum map $V \times V \to V$ is continuous; this fact follows from the triangular inequality of the norm.*
- *the product map $\mathbb{C} \times V \to V$ is continuous; this fact follows from the triangular inequality and homogeneity of the norm.*

It is clear that this is a Haussdorf topology.

4.1.2 The algebra of complex formal series

The goal of this chapter is to define the concept of quantization of a Lie bialgebra. First, we have to introduce the algebra of the complex formal series, which plays a crucial role in this context. For more details on the theory of the complex formal series we refer to [Car95].

Definition 4.1.3. *The algebra of the complex formal series in one variable $\hbar$ is the set*

$$\mathbb{C}[[\hbar]] = \left\{ \sum_{n \geq 0} a_n \hbar^n \right\}$$

where $\{a_n\}_{n \geq 0}$ is a family of complex numbers indexed by $\mathbb{N}$.

Note that the ring $\mathbb{C}[\hbar]$ is a subset of $\mathbb{C}[[\hbar]]$ (more precisely, it is a subring). In fact, let $f = \sum_{n \geq 0} a_n \hbar^n$ and $f' = \sum_{n \geq 0} a'_n \hbar^n$ be two elements of $\mathbb{C}[[\hbar]]$. We define the sum of f and f' by

$$f + f' := \sum_{n \geq 0} (a_n + a'_n) \hbar^n$$

and the product of f and f' by

$$f f' := \sum_{n \geq 0} \left(\sum_{p+q=n} a_p a'_q \right) \hbar^n \ .$$

These two operations endow $\mathbb{C}[[\hbar]]$ with the structure of a ring, with unit for the sum given by the constant polynomial 0 and unit for the product given by the constant polynomial 1.

Proposition 4.1.4. *A formal series $f = \sum_{n \geq 0} a_n \hbar^n$ is invertible in $\mathbb{C}[[\hbar]]$ if and only if a_0 is invertible in $\mathbb{C}$.*

We refer the proof of this result to [Car95]. This Proposition may be interpreted as saying that the ring $\mathbb{C}[[\hbar]]$ is a local ring, that is a ring with a unique maximal ideal. The maximal ideal of $\mathbb{C}[[\hbar]]$ is the ideal $(\hbar)$ generated by $\hbar$. For any $n \geq 0$, consider the algebra $\mathbb{C}[\hbar]/(\hbar^n)$. Then the map

$$\pi_n : \mathbb{C}[[\hbar]] \to \mathbb{C}[\hbar]/(\hbar^n)$$
$$\sum_{n\geq 0} a_n \hbar^n \mapsto \sum_{k=0}^{n-1} a_k \hbar^k \quad mod\ (\hbar^n)$$

is surjective and furthermore its kernel is the ideal $(\hbar^n)$. Then π_n induces a isomorphism of algebras

$$\mathbb{C}[[\hbar]]/(\hbar^n) \simeq \mathbb{C}[\hbar]/(\hbar^n).$$

For any $n \geq 0$, there is also a surjective morphism of algebras

$$p_n : \mathbb{C}[\hbar]/(\hbar^n) \to \mathbb{C}[\hbar]/(\hbar^{n-1})$$

induced by the inclusion of ideals $(\hbar^n) \subset (\hbar^{n-1})$. Consider the inverse system of algebras $(\mathbb{N}, \{\mathbb{C}[\hbar]/(\hbar^n)\}, p_{mn})$. We have that $p_n \circ \pi_n = \pi_{n-1}$, and then, for the universal property of the inverse limit, there exists a unique morphism of algebras

$$\pi : \mathbb{C}[[\hbar]] \to \varprojlim_{n\in\mathbb{N}} \mathbb{C}[\hbar]/(\hbar^n)$$

such that $p_n \circ \pi = \pi_n$.

Proposition 4.1.5. *The map*

$$\pi : \mathbb{C}[[\hbar]] \to \varprojlim_{n\in\mathbb{N}} \mathbb{C}[\hbar]/(\hbar^n)$$

is an isomorphism of algebras.

The Proposition above allows us to equip $\mathbb{C}[[\hbar]]$ with the inverse limit topology. This topology is called the $\hbar$–adic topology. Since $\{0\}$ is a family of open neighbourhoods of 0 in the discrete topology of $\mathbb{C}[\hbar]/(\hbar^n)$ for any $n \geq 0$, then the family $\{\pi_n^{-1}(0)\}_{n\geq 0} = \{(\hbar^n)\}_{n\geq 0}$ is a family of open neighbourhoods of 0 in $\mathbb{C}[[\hbar]]$. Also, we have that the sum and the multiplication by scalars in $\mathbb{C}[[\hbar]]$ are continuous, and so $\mathbb{C}[[\hbar]]$ is a topological vector space, whose topology is generated by the ideals $\{(\hbar^n)\}_{n\geq 0}$.

4.1.3 Topologically free modules

Let M be a $\mathbb{C}[[\hbar]]$–module and consider the family of submodules given by $(\hbar^n \cdot M)_{n>0}$ with the canonical morphisms of modules

$$p_n : M_n := \frac{M}{\hbar^n \cdot M} \to M_{n-1} := \frac{M}{\hbar^{n-1} \cdot M}$$

They form an inverse system of $\mathbb{C}[[\hbar]]$–modules, and so we may consider its inverse limit

$$\tilde{M} \coloneqq \varprojlim_{n\in\mathbb{N}} M_n,$$

which has a natural structure of $\mathbb{C}[[\hbar]]$–module. The module $\tilde{M}$ is called the $\hbar$–adic completion of M.

Definition 4.1.6. *Let $V \in \mathsf{Vect}(\mathbb{C})$. The topologically free module $V[[\hbar]]$ associated to V is the set of all formal series*

$$\sum_{n\geq 0} v_n \hbar^n$$

where $(v_0, v_1, \ldots)$ is an infinite family of elements of V indexed by $\mathbb{N}$. The structure of vector space in $V[[\hbar]]$ is defined by

$$\sum_{n\geq 0} v_n\hbar^n + \sum_{n\geq 0} v'_n\hbar^n \coloneqq \sum_{n\geq 0}(v_n + v'_n)\hbar^n$$

and

$$\lambda \sum_{n\geq 0} v_n\hbar^n \coloneqq \sum_{n\geq 0}(\lambda v_n)\hbar^n,$$

while the structure of $\mathbb{C}[[\hbar]]$–module is given by

$$\sum_{n\geq 0} a_n\hbar^n \cdot \sum_{n\geq 0} v_n\hbar^n \coloneqq \sum_{n\geq 0}\left(\sum_{p+q=n} a_p \cdot v_q\right)\hbar^n.$$

Let M and N be two $\mathbb{C}[[\hbar]]$–modules and consider the $\mathbb{C}[[\hbar]]$–module given by

$$M \otimes_{\mathbb{C}[[\hbar]]} N \coloneqq M \otimes N/(f \cdot m \otimes n - m \otimes f \cdot n).$$

Then $\dfrac{M \otimes_{\mathbb{C}[[\hbar]]} N}{\hbar^n \cdot (M \otimes_{\mathbb{C}[[\hbar]]} N)}$ is an inverse system of $\mathbb{C}[[\hbar]]$–modules.

Definition 4.1.7. *The topological tensor product $M\tilde{\otimes}N$ of M and N is the $\hbar$–adic completion of $M \otimes_{\mathbb{C}[[\hbar]]} N$:*

$$M\tilde{\otimes}N \coloneqq \varprojlim_{n\in\mathbb{N}} \frac{M \otimes_{\mathbb{C}[[\hbar]]} N}{\hbar^n \cdot (M \otimes_{\mathbb{C}[[\hbar]]} N)}.$$

Moreover, the usual associativity and commutativity constraints of $\mathsf{Vect}(\mathbb{C})$ induce the following $\mathbb{C}[[\hbar]]$–isomorphisms:

- $(M\tilde{\otimes}N)\tilde{\otimes}P \simeq M\tilde{\otimes}(N\tilde{\otimes}P)$;
- $M\tilde{\otimes}N \simeq N\tilde{\otimes}M$;
- $\mathbb{C}[[\hbar]]\tilde{\otimes}M \simeq M \simeq M\tilde{\otimes}\mathbb{C}[\hbar]]$.

Furthermore, if M, M', N, N' are four $\mathbb{C}[[\hbar]]$–modules and $f : M \to M'$ and $g : N \to N'$ are $\mathbb{C}[[\hbar]]$–linear maps, then there exists a $\mathbb{C}[[\hbar]]$–linear map

$$f\tilde{\otimes}g : M\tilde{\otimes}N \to M'\tilde{\otimes}N'$$

satisfying the formal properties of the classical tensor product.

Proposition 4.1.8. *If M and N are topologically free modules, then $M\tilde{\otimes}N$ is too, and also we have*

$$V[[\hbar]]\tilde{\otimes}W[[\hbar]] \simeq (V \otimes W)[[\hbar]].$$

The Proposition above allows us to extend some definitions given in the second chapter to the setting of $\mathbb{C}[[\hbar]]$–modules. More precisely, we define:

- a topological algebra as a triple (A, μ, η), where:
  - (ii) A is a $\mathbb{C}[[\hbar]]$–module;
  - (ii) $\mu : A\tilde{\otimes}A \to A$ and $\eta : \mathbb{C}[[\hbar]] \to A$ are morphisms of $\mathbb{C}[[\hbar]]$–modules such that

    $$\mu \circ (\mu\tilde{\otimes}\mathsf{id}_A) = \mu \circ (\mathsf{id}_A\tilde{\otimes}\mu)$$

    and

    $$\mu \circ (\eta\tilde{\otimes}\mathsf{id}_A) = \mathsf{id}_A = \mu \circ (\mathsf{id}_A\tilde{\otimes}\eta);$$

- a morphism $f : (A, \mu, \eta) \to (A', \mu', \eta')$ of topological algebras as a morphism of $\mathbb{C}[[\hbar]]$–modules such that

  $$f \circ \mu = \mu' \circ (f\tilde{\otimes}f)$$

  and

  $$f \circ \eta = \eta';$$

- a topological Hopf algebra as a $\mathbb{C}[[\hbar]]$–module H with a coproduct, product, counit, unit and antipode which satisfies the axioms of a Hopf algebra with respect to the tensor product $\tilde{\otimes}$.

- For any other algebraic structure defined in the previous chapters, we may define its topological counterpart as in the previous examples.

In the following, in order to lighten the notation, we omit the symbol $\sim$ over tensor products and we treat topological objects as classical objects. We now introduce the notion of a topological algebra presented by generators and relations.

Definition 4.1.9. *Let X be a set.*

- *The topologically free algebra generated by X is the algebra of the formal series over the free complex algebra generated by the set X and we denote it by $\mathsf{Span}_{\mathbb{C}}(X)[[\hbar]]$. We equip $\mathsf{Span}_{\mathbb{C}}(X)[[\hbar]]$ with the $\hbar$–adic topology.*
- *Let R be a subset of $\mathsf{Span}_{\mathbb{C}}(X)[[\hbar]]$. A topological algebra A is said to be the $\mathbb{C}[[\hbar]]$–algebra topologically generated by the set X of generators and the set R of relations if A is isomorphic to the quotient of $\mathsf{Span}_{\mathbb{C}}(X)[[\hbar]]$ by the closure (for the $\hbar$–adic topology) of the two–sided ideal generated by R.*

4.2 Quantized universal enveloping algebras

Definition 4.2.1. *Let H be a topological Hopf algebra.*

- *We say that H is a deformation of a topological Hopf algebra H_0 if*

$$H/(\hbar \cdot H) \simeq H_0$$

  as topological Hopf algebras.

- *A deformation H of a topological Hopf algebra is called a quantized universal enveloping algebra if*

$$H/(\hbar \cdot H) \simeq \mathcal{U}\mathfrak{g}$$

  for some Lie algebra $\mathfrak{g}$.

In the following we will sometimes drop the word *universal* for brevity. Note that, since in $\mathcal{U}\mathfrak{g}$ the coproduct is

$$\Delta(x) = 1 \otimes x + x \otimes 1,$$

then $\mathcal{U}\mathfrak{g}$ is cocommutative and so also $H/(\hbar \cdot H)$ it is. For this reason we say that H is cocommutative up to the first order (while H may be not cocommutative).

Proposition 4.2.2. *Let H be a quantized enveloping algebra with $H/(\hbar{\cdot}H) \simeq \mathcal{U}\mathfrak{g}$. Then the Lie algebra $\mathfrak{g}$ is naturally equipped with a bialgebra structure defined by*

$$\delta(x) = \frac{\Delta(\tilde{x}) - \Delta^{op}(\tilde{x})}{\hbar} \quad mod\ \hbar$$

where $\tilde{x}$ is any lifting of x to H, i.e. any element of H such that $\tilde{x}$ mod $\hbar \mapsto x \in \mathfrak{g} \subset \mathcal{U}\mathfrak{g}$.

Proof. We have to check severals facts:

(i) The formula above makes sense: from $H/(\hbar{\cdot}H) \simeq \mathcal{U}\mathfrak{g}$ and from the fact that $\mathcal{U}\mathfrak{g}$ is cocommutative, we have that $H/(\hbar \cdot H)$ is cocommutative, and so $\Delta(x) - \Delta^{op}(x)$ is a multiple of $\hbar$ for all $x \in \mathfrak{g}$.

(ii) δ is well–defined: let $x \in \mathfrak{g}$ and let x' and $x'' \in H$ be two liftings of x in H. Then there exists $u \in H$ such that $x'' = x' + \hbar u$. We have:

$$\begin{aligned}
&\frac{\Delta(x'') - \Delta^{op}(x'')}{\hbar} \quad mod\ \hbar \\
&= \frac{\Delta(x' + \hbar u) - \Delta^{op}(x' + \hbar u)}{\hbar} \quad mod\ \hbar \\
&= \frac{\Delta(x') + \Delta(\hbar u) - \Delta^{op}(x') - \Delta^{op}(\hbar u)}{\hbar} \quad mod\ \hbar \\
&= \frac{\Delta(x') + \hbar\Delta(u) - \Delta^{op}(x') - \hbar\Delta^{op}(u)}{\hbar} \quad mod\ \hbar \\
&= \frac{\Delta(x') - \Delta^{op}(x')}{\hbar} + \frac{\hbar\Delta(u) - \hbar\Delta^{op}(u)}{\hbar} \quad mod\ \hbar \\
&= \frac{\Delta(x') - \Delta^{op}(x')}{\hbar} + \Delta(u) - \Delta^{op}(u) \quad mod\ \hbar \\
&= \frac{\Delta(x') - \Delta^{op}(x')}{\hbar} \quad mod\ \hbar,
\end{aligned}$$

and so δ is well–defined.

(iii) δ is skew–symmetric: let $x \in \mathfrak{g}$ and let $\tilde{x}$ be a lifting of x in H. We have:

$$\begin{aligned}
\delta(x) &= \frac{\Delta(\tilde{x}) - \Delta^{op}(\tilde{x})}{\hbar} \quad mod\ \hbar \\
&= -\frac{\Delta^{op}(\tilde{x}) - \Delta(x)}{\hbar} \quad mod\ \hbar \\
&= -\delta^{op}(x).
\end{aligned}$$

(iv) δ satisfies the coJacobi rule: let $x \in \mathfrak{g}$ and let $a \in H$ be a lifting of x. We have to prove that $((\mathsf{id} + \sigma + \sigma^2) \circ (\mathsf{id} \otimes \delta) \circ \delta)(x) = 0$. We have

$$((\mathsf{id} \otimes \delta) \circ \delta)(x) = (\mathsf{id} \otimes \delta)\left(\frac{\Delta(a) - \Delta^{op}(a)}{\hbar} \quad mod\ \hbar\right) =$$
$$= \frac{(\mathsf{id} \otimes \delta)(\Delta(a)) - (\mathsf{id} \otimes \delta)(\Delta^{op}(a))}{\hbar} \quad mod\ \hbar$$
$$= \frac{1}{\hbar^2}\big((\mathsf{id} \otimes \Delta)(\Delta(a)) - (\mathsf{id} \otimes \Delta^{op})(\Delta(a)) - (\mathsf{id} \otimes \Delta)(\Delta^{op}(a)) + (\mathsf{id} \otimes \Delta^{op})(\Delta^{op}(a))\big)$$
$$\mod\ \hbar.$$

The evaluation of the map $\mathsf{id} + \sigma + \sigma^2$ on the four terms in the round brackets gives us a sum of twelve elements. More precisely, using the Sweedler's notation and denoting σ by (123), we obtain:

(1)
$$(\mathsf{id} \otimes \Delta)\left(\sum_{(a)} a' \otimes a''\right) = \sum_{(a,a'')} a' \otimes (a'')' \otimes (a'')'';$$

(2)
$$-(\mathsf{id} \otimes \Delta^{op})\left(\sum_{(a)} a' \otimes a''\right) = -\sum_{(a,a'')} a' \otimes (a'')'' \otimes (a'')'$$
$$= -((23) \circ (\mathsf{id} \otimes \Delta) \circ \Delta)(a);$$

(3)
$$-(\mathsf{id} \otimes \Delta)\left(\sum_{(a)} a'' \otimes a'\right) = -\sum_{(a,a')} a'' \otimes (a')' \otimes (a')''$$
$$= -((13) \circ (12) \circ (\Delta \otimes \mathsf{id}) \circ \Delta)(a)$$
$$= -((123) \circ (\Delta \otimes \mathsf{id}) \circ \Delta)(a);$$

(4)
$$(\mathsf{id} \otimes \Delta^{op})\left(\sum_{(a)} a'' \otimes a'\right) = \sum_{(a,a')} a'' \otimes (a')'' \otimes (a')';$$

(5)
$$(123)\left(\sum_{(a,a'')} a' \otimes (a'')' \otimes (a'')''\right) = \sum_{(a,a'')} (a'')'' \otimes a' \otimes (a'')'$$
$$= ((12) \circ (23) \circ (\Delta \otimes \mathsf{id}) \circ \Delta)(a)$$
$$= ((123) \circ (\Delta \otimes \mathsf{id}) \circ \Delta)(a);$$

(6)

$$(123)\Big(-\sum_{(a,a'')} a'\otimes(a'')''\otimes(a'')'\Big) = -\sum_{(a,a'')}(a'')'\otimes a'\otimes(a'')''$$
$$= -((\tau\otimes\mathsf{id})\circ(\mathsf{id}\otimes\Delta)\circ\Delta)(a)$$
$$= -((\tau\otimes\mathsf{id})\circ(\Delta\otimes\mathsf{id})\circ\Delta)(a)$$
$$= -((\Delta^{op}\otimes\mathsf{id})\circ\Delta)(a);$$

(7)

$$(123)\Big(-\sum_{(a,a')} a''\otimes(a')'\otimes(a')''\Big) = -\sum_{(a,a')}(a')''\otimes a''\otimes(a')'$$
$$= -((\tau\otimes\mathsf{id})\circ(\mathsf{id}\otimes\Delta^{op})\circ\Delta^{op})(a)$$
$$= -((\tau\otimes\mathsf{id})\circ(\Delta^{op}\otimes\mathsf{id})\circ\Delta^{op})(a)$$
$$= -((\Delta\otimes\mathsf{id})\circ\Delta^{op})(a);$$

(8)

$$(123)\Big(\sum_{(a,a')} a''\otimes(a')''\otimes(a')'\Big) = \sum_{(a,a')}(a')'\otimes a''\otimes(a')''$$
$$= ((12)\circ(13)\circ(12)\circ(\mathsf{id}\otimes\Delta)\circ\Delta)(a)$$
$$= ((23)\circ(\mathsf{id}\otimes\Delta)\circ\Delta)(a);$$

(9)

$$(123)\Big(\sum_{(a,a'')}(a'')''\otimes a'\otimes(a'')'\Big) = \sum_{(a,a'')}(a'')'\otimes(a'')''\otimes a'$$
$$= ((\Delta\otimes\mathsf{id})\circ\Delta^{op})(a);$$

(10)

$$(123)\Big(-\sum_{(a,a'')}(a'')'\otimes a'\otimes(a'')''\Big) = -\sum_{(a,a'')}(a'')''\otimes(a'')'\otimes a';$$

(11)

$$(123)\Big(-\sum_{(a,a')}(a')''\otimes a''\otimes(a')'\Big) = -\sum_{(a,a')}(a')'\otimes(a')''\otimes a'';$$

(12)

$$(123)\left(\sum_{(a,a')}(a')'\otimes a''\otimes(a')''\right)=\sum_{(a,a')}(a')''\otimes(a')'\otimes a''$$
$$=((\Delta^{op}\otimes \mathsf{id})\circ\Delta)(a).$$

It is clear that $(2)+(8)=0$, $(7)+(9)=0$ and $(6)+(12)=0$. Furthermore, from the coassociativity of Δ we obtain that $(1)+(11)=0$ and $(3)+(5)=0$, while from the coassociativity of Δ^{op} we obtain that $(4)+(10)=0$, and so the CoJacobi rule is satisfied.

(v) $\delta(x)\in\mathfrak{g}\otimes\mathfrak{g}$: we refer the proof of this fact to [ES02].

(vi) $(\mathfrak{g},[\cdot,\cdot],\delta)$ satisfies the cocycle condition: Let $a,b\in\mathfrak{g}$ and a',b' be liftings respectively of a and b. Then we have that $a'b'-b'a'$ is a lifting of $[a,b]$, and so we obtain:

$$\begin{aligned}\delta([a,b]) &= \frac{\Delta(a'b'-b'a')-\Delta^{op}(a'b'-b'a')}{\hbar} \quad mod\ \hbar\\ &= \frac{\Delta(a'b')-\Delta(b'a')-\Delta^{op}(a'b')+\Delta^{op}(b'a')}{\hbar} \quad mod\ \hbar\\ &= \frac{\Delta(a')\Delta(b')-\Delta(b')\Delta(a')-\Delta^{op}(a')\Delta^{op}(b')+\Delta^{op}(b')\Delta^{op}(a')}{\hbar} \quad mod\ \hbar\\ &= \frac{\Delta(a')\Delta(b')-\Delta^{op}(a')\Delta^{op}(b')}{\hbar}+\frac{\Delta^{op}(b')\Delta^{op}(a')-\Delta(b')\Delta(a')}{\hbar} \quad mod\ \hbar\\ &= [1\otimes a+a\otimes 1,\delta(b)]-[1\otimes b+b\otimes 1,\delta(a)].\end{aligned}$$

□

Definition 4.2.3. *The quasi–classical limit of a quantized enveloping algebra H is the Lie bialgebra $\mathfrak{g}=Prim(H/(\hbar\cdot H))$, where the cobracket is defined by the proposition above. Conversely, we say that H is a quantization of the Lie bialgebra $\mathfrak{g}$.*

In particular, one can show that there is a functor from the category QUE of topological quantized universal enveloping algebras to the category LBA of Lie bialgebras

$$SC: QUE\to LBA$$

that assigns to a quantized enveloping algebra H its quasi–classical limit $\mathfrak{g}=Prim(H/\hbar\cdot H)$. The functor SC is commonly called the quasi–classical limit functor, or the semiclassical limit functor.

4.3 Quantum groups

4.3.1 Quantum $\mathfrak{sl}_2$

We now present an example of a Hopf algebra that is neither commutative nor cocommutative. We recall that $\mathfrak{sl}_2$ is the simple Lie algebra of 2×2 traceless matrices with entries in $\mathbb{C}$. In the following, if q is a complex number, we assume that it is not a root of the unity. A discussion in the case of q is a root of the unity can be find in [Lus90].

Definition 4.3.1. *Let $q \in \mathbb{C}$. We define $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ as the algebra generated by the four variables E, F, K, K^{-1} with the relations:*

- $KK^{-1} = K^{-1}K = 1$;
- $KEK^{-1} = q^2 E$;
- $KFK^{-1} = q^{-2} F$;
- $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$.

The algebra $\mathcal{U}_q$ was the first well–defined object that appeared in the literature with the name of a quantum group. Is clear from the relations above that $\mathcal{U}_q$ is not commutative.

We now present the non–cocommutative Hopf algebra structure on $\mathcal{U}_q$.

Proposition 4.3.2. *The maps*

$$\begin{aligned} \Delta : \mathcal{U}_q &\to \mathcal{U}_q \otimes \mathcal{U}_q \\ E &\mapsto 1 \otimes E + E \otimes K \\ F &\mapsto K^{-1} \otimes F + F \otimes 1 \\ K &\mapsto K \otimes K \\ K^{-1} &\mapsto K^{-1} \otimes K^{-1} \end{aligned}$$

and

$$\begin{aligned} \varepsilon : \mathcal{U}_q &\to \mathbb{C} \\ E, F &\mapsto 0 \\ K, K^{-1} &\mapsto 1 \end{aligned}$$

define a Hopf structure on $\mathcal{U}_q$, with antipode given by

$$\begin{aligned} S : \mathcal{U}_q &\to \mathcal{U}_q \\ E &\mapsto -EK^{-1} \\ F &\mapsto -KF \\ K &\mapsto K^{-1} \\ K^{-1} &\mapsto K. \end{aligned}$$

Proof. We have to check several facts:

(i) Δ defines a morphism of algebras: it suffices to check it on the four generators. We have:

$$\Delta(K)\Delta(K^{-1}) = \Delta(K^{-1})\Delta(K) = 1;$$

$$\begin{aligned}\Delta(K)\Delta(E)\Delta(K^{-1}) &= (K \otimes K)(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1}) \\ &= 1 \otimes KEK^{-1} + KEK^{-1} \otimes K \\ &= q^2(1 \otimes E + E \otimes K) \\ &= q^2\Delta(E);\end{aligned}$$

$$\begin{aligned}\Delta(K)\Delta(F)\Delta(K^{-1}) &= (K \otimes K)(K^{-1} \otimes F + F \otimes 1)(K^{-1} \otimes K^{-1}) \\ &= K^{-1} \otimes KFK^{-1} + KFK^{-1} \otimes 1 \\ &= q^{-2}(K^{-1} \otimes F + F \otimes 1) \\ &= q^{-2}\Delta(F);\end{aligned}$$

$$\begin{aligned}\Delta(E)\Delta(F) - \Delta(F)\Delta(E) &= (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) \\ &- (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K) \\ &= K^{-1} \otimes EF + F \otimes E + EK^{-1} \otimes KF + EF \otimes K \\ &- K^{-1} \otimes FE - K^{-1}E \otimes FK - F \otimes E - FE \otimes K \\ &= K^{-1} \otimes (EF - FE) + (EF - FE) \otimes K \\ &= \frac{K^{-1} \otimes (K - K^{-1}) + (K - K^{-1}) \otimes K}{q - q^{-1}} \\ &= \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}.\end{aligned}$$

(ii) Δ is coassociative: also in this case it suffices to check it on the four generators. We have:

$$(\Delta \otimes \mathsf{id})(\Delta(E)) = (\Delta \otimes \mathsf{id})(1 \otimes E + E \otimes K) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K;$$

$$(\mathsf{id} \otimes \Delta)(\Delta(E)) = (\mathsf{id} \otimes \Delta)(1 \otimes E + E \otimes K) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K;$$

$$(\Delta \otimes \mathsf{id})(\Delta(F)) = (\Delta \otimes \mathsf{id})(K^{-1} \otimes F + F \otimes 1) = K^{-1} \otimes K^{-1} \otimes F + K^{-1} \otimes F \otimes 1 + F \otimes 1 \otimes 1;$$

$$(\mathsf{id} \otimes \Delta)(\Delta(F)) = (\mathsf{id} \otimes \Delta)(K^{-1} \otimes F + F \otimes 1) = K^{-1} \otimes K^{-1} \otimes F + K^{-1} \otimes F \otimes 1 + F \otimes 1 \otimes 1;$$

$$(\Delta \otimes \mathsf{id})(\Delta(K)) = K \otimes K \otimes K = (\mathsf{id} \otimes \Delta)(\Delta(K));$$

$$(\Delta \otimes \mathsf{id})(\Delta(K^{-1})) = K^{-1} \otimes K^{-1} \otimes K^{-1} = (\mathsf{id} \otimes \Delta)(\Delta(K^{-1})).$$

(iii) ε is a morphism of algebras:

$$\varepsilon(K)\varepsilon(K^{-1}) = 1 = \varepsilon(K^{-1})\varepsilon(K);$$
$$\varepsilon(K)\varepsilon(E)\varepsilon(K^{-1}) = 0 = \varepsilon(q^2E);$$
$$\varepsilon(K)\varepsilon(F)\varepsilon(K^{-1}) = 0 = \varepsilon(q^{-2}F);$$
$$\varepsilon(E)\varepsilon(F) - \varepsilon(F)\varepsilon(E) = 0 = \varepsilon\left(\frac{K-K^{-1}}{q-q^{-1}}\right).$$

(iv) ε satisfies the counit axiom:

$$(\varepsilon \otimes \mathsf{id})(\Delta(K)) = (\varepsilon \otimes \mathsf{id})(K \otimes K) = 1 \otimes K;$$
$$(\mathsf{id} \otimes \varepsilon)(\Delta(K)) = (\mathsf{id} \otimes \varepsilon)(K \otimes K) = K \otimes 1;$$
$$(\varepsilon \otimes \mathsf{id})(\Delta(K^{-1})) = (\varepsilon \otimes \mathsf{id})(K^{-1} \otimes K^{-1}) = 1 \otimes K^{-1};$$
$$(\mathsf{id} \otimes \varepsilon)(\Delta(K^{-1})) = (\mathsf{id} \otimes \varepsilon)(K^{-1} \otimes K^{-1}) = K^{-1} \otimes 1;$$
$$(\varepsilon \otimes \mathsf{id})(\Delta(E)) = (\varepsilon \otimes \mathsf{id})(1 \otimes E + E \otimes K) = 1 \otimes E;$$
$$(\mathsf{id} \otimes \varepsilon)(\Delta(E)) = (\mathsf{id} \otimes \varepsilon)(1 \otimes E + E \otimes K) = E \otimes 1;$$
$$(\varepsilon \otimes \mathsf{id})(\Delta(F)) = (\varepsilon \otimes \mathsf{id})(K^{-1} \otimes F + F \otimes 1) = 1 \otimes F;$$
$$(\mathsf{id} \otimes \varepsilon)(\Delta(F)) = (\mathsf{id} \otimes \varepsilon)(K^{-1} \otimes F + F \otimes 1) = F \otimes 1.$$

(v) S is an antipode: we refer this fact to [Kas12].

□

One expect to recover $\mathcal{U}\mathfrak{sl}_2$ from $\mathcal{U}_q$ by setting $q = 1$. Unfortunately, this is impossible with the current definition of $\mathcal{U}_q$, since when $q = 1$ we have that $\mathcal{U}_q$ is not defined. However, we can give another presentation for $\mathcal{U}_q$, that is the following.

Proposition 4.3.3. *The algebra $\mathcal{U}_q$ is isomorphic to the algebra $\mathcal{U}'_q$ generated by the five variables E, F, K, K^{-1}, L and with the relations:*

- $KK^{-1} = K^{-1}K = 1$;
- $KEK^{-1} = q^2E$;
- $KFK^{-1} = q^{-2}F$;
- $EF - FE = L$;
- $(q - q^{-1})L = K - K^{-1}$;

- $LE - EL = q(EK + K^{-1}E)$;
- $LF - FL = -q^{-1}(FK + K^{-1})F$.

Observe that, contrary to $\mathcal{U}_q$, the algebra $\mathcal{U}'_q$ is well–defined for all values of the parameter q, in particular when $q = 1$. However, the presentation of $\mathcal{U}_q$ is pretty simpler, and so most mathematicians prefers to use it instead of $\mathcal{U}'_q$. The next Proposition tells us that we can obtain $\mathcal{U}\mathfrak{sl}_2$ from $\mathcal{U}'_q$ by setting $q = 1$.

Proposition 4.3.4. *Denoting the enveloping algebra of $\mathfrak{sl}_2$ with $\mathcal{U}$, we have that*

$$\mathcal{U}'_1 \simeq \mathcal{U}[K]/(K^2 - 1) \quad \text{and} \quad \mathcal{U} \simeq \mathcal{U}'_1/(K - 1).$$

We refer to [Kas12] the proof of the last two propositions.

4.3.2 Drinfeld–Jimbo quantum groups

We need some notations to present Drinfeld–Jimbo quantum groups. Let q be a complex number.

- The q–integers are

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \ldots + q^{-n+1};$$

- the q–factorial is defined by

$$[n]_q! := \prod_{k=1}^{n} [k]_q;$$

- the q–binomial cofficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

We now define Drinfeld–Jimbo quantum groups, that are topological Hopf algebras that quantize the enveloping algebra of a Kac–Moody algebra. We start the discussion analyzing the case of $\mathfrak{sl}_2$.

Definition 4.3.5. *We define $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ as the $\mathbb{C}[[\hbar]]$–algebra topologically generated by the three variables X, Y, H and by the relations*

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = \frac{sinh(\hbar H)}{sinh(\hbar)} = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}},$$

where e^x is the formal series

$$e^x := \sum_{n\geq 0} \frac{x^n}{n!}$$

and sinh is the formal series

$$sinh(x) := \frac{e^x - e^{-x}}{2} = \sum_{n\geq 0} e(n)\frac{x^n}{n!},$$

where $e(n) = 0$ if n is even and $e(n) = 1$ if n is odd.

Remark 4.3.6. *We have that, although $sinh(\hbar)$ is not invertible, it is the product of $\hbar$ with a unique invertible element, so that $sinh(\hbar H)/sinh(\hbar)$ is a well–defined element of $\mathsf{Span}_{\mathbb{C}}(X, Y, H)$. Furthermore we have*

$$\frac{sinh(\hbar H)}{sinh(\hbar)} \equiv H \ mod \ \hbar.$$

Theorem 4.3.7. *Let $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ as above. Then we have that:*

(i) $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ is a topological Hopf algebra;

(ii) $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ is a quantized enveloping algebra, whose quasi–classical limit is $\mathfrak{sl}_2$;

(iii) $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ has a R–matrix.

Proof. (i): We define the coproduct, the counit and the antipode by:

$$\begin{aligned} \Delta_\hbar : \mathcal{U}_\hbar(\mathfrak{sl}_2) &\to \mathcal{U}_\hbar(\mathfrak{sl}_2) \otimes \mathcal{U}_\hbar(\mathfrak{sl}_2) \\ X &\mapsto X \otimes e^{\hbar H} + 1 \otimes X \\ Y &\mapsto Y \otimes 1 + e^{-\hbar H} \otimes Y \\ H &\mapsto H \otimes 1 + 1 \otimes H \end{aligned}$$

$$\begin{aligned} \varepsilon_\hbar : \mathcal{U}_\hbar(\mathfrak{sl}_2) &\to \mathbb{C}[[\hbar]] \\ X, Y, H &\mapsto 0 \end{aligned}$$

$$\begin{aligned} S_\hbar : \mathcal{U}_\hbar(\mathfrak{sl}_2) &\to \mathcal{U}_\hbar(\mathfrak{sl}_2) \\ X &\mapsto -Xe^{-\hbar H} \\ Y &\mapsto -e^{\hbar H}Y \\ H &\mapsto -H. \end{aligned}$$

We have that $\Delta_\hbar$ and $\varepsilon_\hbar$ are morphisms of algebras, while $S_\hbar$ is an anti–morphism of algebras; these facts can be consulted in more details in [CP+95].
(ii): Let us recall that $\mathcal{U}\mathfrak{sl}_2$ is the free algebra generated by the elements e, f, h with relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h$$

and it has a Hopf algebra structure as described in the Subsection 3.3.2. Then, due to (i), to have a Hopf algebras isomorphism it is enough to consider the map

$$\begin{aligned} \varphi : \mathcal{U}_\hbar(\mathfrak{sl}_2)/\hbar \cdot \mathcal{U}_\hbar(\mathfrak{sl}_2) &\to \mathcal{U}\mathfrak{sl}_2 \\ [X] &\mapsto e \\ [Y] &\mapsto f \\ [H] &\mapsto h. \end{aligned}$$

We also observe that, from 4.2.2, we obtain a cobracket on $\mathfrak{sl}_2$ given by

$$\delta(e) = e \wedge h, \quad \delta(f) = f \wedge h, \quad \delta(h) = 0,$$

that is the standard Lie bialgebra structure of $\mathfrak{sl}_2$ described in the Subsection 3.5.3.
(iii): Consider the tensor

$$R_\hbar = e^{\frac{1}{2}\hbar(H\otimes H)} \sum_{n\geq 0} R_\hbar(n)(X^n \otimes Y^n),$$

where

$$R_\hbar(n) = \frac{q^{\frac{1}{2}n(n+1)}(1-q^{-2})^n}{[n]_q!} \quad \text{and} \quad q = e^\hbar.$$

We have to prove that:

(1) $(\Delta_\hbar \otimes \mathsf{id})(R_\hbar) = (R_\hbar)_{13}(R_\hbar)_{23}$,

(2) $(\mathsf{id} \otimes \Delta_\hbar)(R_\hbar) = (R_\hbar)_{13}(R_\hbar)_{12}$, and

(3) $\Delta_\hbar^{op} = R_\hbar \Delta_\hbar R_\hbar^{-1}$.

To prove (1), we first compute $(R_\hbar)_{13}(R_\hbar)_{23}$, obtaining

$$(R_\hbar)_{13}(R_\hbar)_{23} = \sum_{n,m\geq 0} R_\hbar(n)R_\hbar(m)e^{\frac{1}{2}\hbar(H\otimes 1\otimes H)}(X^n\otimes 1\otimes Y^n)e^{\frac{1}{2}\hbar(1\otimes H\otimes H)}(1\otimes X^m\otimes Y^m).$$

Since

$$e^{-\frac{1}{2}\hbar(1\otimes H\otimes H)}(X^n \otimes 1 \otimes Y^n)e^{\frac{1}{2}\hbar(1\otimes H\otimes H)} = (1 \otimes e^{n\hbar H} \otimes 1)(X^n \otimes 1 \otimes Y^n),$$

we have that

$$(X^n \otimes 1 \otimes Y^n) e^{\frac{1}{2}\hbar(1\otimes H\otimes H)} = e^{\frac{1}{2}\hbar(1\otimes H\otimes H)}(1 \otimes e^{n\hbar H} \otimes 1)(X^n \otimes 1 \otimes Y^n),$$

and so

$$(R_\hbar)_{13}(R_\hbar)_{23} = \sum_{n,m\geq 0} R_\hbar(n)R_\hbar(m) e^{\frac{1}{2}\hbar(H\otimes 1\otimes H + 1\otimes H\otimes H)}(1\otimes e^{n\hbar H}\otimes 1)(X^n\otimes X^m\otimes Y^{n+m}).$$

On the other side, we have that

$$(\Delta_\hbar \otimes \mathsf{id})(R_\hbar) = \sum_{n,m\geq 0} a_{nm}(X^n \otimes X^m \otimes Y^{n+m}),$$

where

$$a_{nm} = q^{-nm}\frac{[n+m]_q!}{[n]_q![m]_q!} R_\hbar(n+m)\{(\Delta_\hbar \otimes \mathsf{id})(e^{\frac{1}{2}\hbar(H\otimes H)})\}(1 \otimes e^{n\hbar H} \otimes 1).$$

Therefore, to prove (1) it suffices to observe that

$$q^{-nm}\frac{[n+m]_q!}{[n]_q![m]_q!} R_\hbar(n+m) = R_\hbar(n)R_\hbar(m).$$

The proof of (2) is similar. To prove (3), first observe that both $\Delta_\hbar^{op} R_\hbar$ and $R_\hbar \Delta_\hbar$ are algebra morphisms from $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ to $\mathcal{U}_\hbar(\mathfrak{sl}_2) \otimes \mathcal{U}_\hbar(\mathfrak{sl}_2)$, and so it is enough to prove that

$$\Delta_\hbar^{op}(a) R_\hbar = R_\hbar \Delta_\hbar(a)$$

for $a = X, Y, H$. For $a = X$ we have

$$\begin{aligned}
\Delta_\hbar^{op}(X) R_\hbar &= (\tau \circ \Delta_\hbar)(X) R_\hbar \\
&= \tau(X \otimes e^{\hbar H} + 1 \otimes X) R_\hbar \\
&= (e^{\hbar H} \otimes X + X \otimes 1) R_\hbar \\
&= (e^{\hbar H} \otimes X + X \otimes 1)\left(e^{\frac{1}{2}\hbar(H\otimes H)} \sum_{n\geq 0} R_\hbar(n)(X^n \otimes Y^n)\right),
\end{aligned}$$

while

$$R_\hbar \Delta_\hbar(X) = \left(e^{\frac{1}{2}\hbar(H\otimes H)} \sum_{n\geq 0} R_\hbar(n)(X^n \otimes Y^n)\right)(X \otimes e^{\hbar H} + 1 \otimes X).$$

Then we have

$$\begin{aligned}
\Delta_\hbar^{op}(X) R_\hbar - R_\hbar \Delta_\hbar(X) = \sum_{n\geq 0} e^{\frac{1}{2}\hbar(H\otimes H)}\Bigg(& R_\hbar(n)(1 \otimes e^{-\hbar H} - q^{2n} e^{\hbar H}) \\
& + R_\hbar(n+1)[n+1]_q!\left(1 \otimes \frac{q^n e^{\hbar H} - q^{-n} e^{-\hbar H}}{q - q^{-1}}\right)(X^{n+1} \otimes Y^n)\Bigg),
\end{aligned}$$

and this is zero because

$$R_\hbar(n+1)[n+1]_q! = q^n(q-q^{-1})R_\hbar(n).$$

The cases of $a = Y$ and $a = H$ are similar to the previous one. $\square$

The following result relates the Hopf algebra $\mathcal{U}_q$ of the previous subsection with $\mathcal{U}_\hbar(\mathfrak{sl}_2)$. We have to assume that the ground field of $\mathcal{U}_q$ is the field of the fractions of the algebra $\mathbb{C}[[\hbar]]$.

Proposition 4.3.8. *There exists an injective map of Hopf algebras*

$$i : \mathcal{U}_q \to \mathcal{U}_\hbar(\mathfrak{sl}_2)$$

with $i(q) = e^\hbar$*, and so we can identify* $\mathcal{U}_q$ *with a subalgebra of* $\mathcal{U}_\hbar(\mathfrak{sl}_2)$.

We now generalize the construction above from $\mathfrak{sl}_2$ to any symmetrizable Kac–Moody algebra.

Definition 4.3.9. *Let* $A = (a_{ij})$ *be a* $n \times n$ *symmetrizable generalized Cartan matrix and let* $\mathfrak{g}$ *be the symmetrizable Kac–Moody algebra associated to* A*. We define the Drinfeld–Jimbo quantum group of* $\mathfrak{g}$ *as the* $\mathbb{C}[[\hbar]]$*–algebra* $\mathcal{U}_\hbar^{DJ}(\mathfrak{g})$ *topologically generated by the set of generators* $\{X_i, Y_i, H_i\}_{1\le i\le n}$ *and the relations:*

(1) $[H_i, H_j] = 0$;

(2) $[X_i, Y_j] = \delta_{ij}\frac{sinh(d_i\hbar H_i)}{sinh(d_i\hbar)}$;

(3) $[H_i, X_j] = a_{ij}X_j$;

(4) $[H_i, Y_j] = -a_{ij}Y_j$;

and if $i \neq j$

$$(5) \qquad \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} X_i^k X_j X^{1-a_{ij}-k} = 0$$

and

$$(6) \qquad \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} Y_i^k Y_j Y^{1-a_{ij}-k} = 0,$$

where we set $q_i = e^{d_i\hbar}$.

Remark 4.3.10. *As in the case of* $\mathfrak{sl}_2$ *we have that, although* $sinh(d_i\hbar)$ *is not invertible, it is the product of* $\hbar$ *with a unique invertible element, so that* $sinh(d_i\hbar H_i)/sinh(d_i\hbar)$ *is a well–defined element of* $\mathsf{Span}_{\mathbb{C}}(\{X_i, Y_i, H_i\}_{1\leq i\leq n})$*. Furthermore we have*

$$\frac{sinh(d_i\hbar H_i)}{sinh(d_i\hbar)} \equiv H_i \ mod \ \hbar.$$

The next result generalizes the quantization from $\mathfrak{sl}_2$ to any symmetrizable Kac–Moody algebra.

Theorem 4.3.11. *Let* $A = (a_{ij})$ *be a* $n \times n$ *symmetrizable generalized Cartan matrix and let* $\mathfrak{g}$ *be the symmetrizable Kac–Moody algebra associated to* A*. Let* $\mathcal{U}_\hbar^{DJ}(\mathfrak{g})$ *be the associated Drinfeld–Jimbo quantum group associated to* $\mathfrak{g}$*. Then we have that:*

(i) $\mathcal{U}_\hbar^{DJ}(\mathfrak{g})$ *is a topological Hopf algebra;*

(ii) $\mathcal{U}_\hbar^{DJ}(\mathfrak{g})$ *is a quantized enveloping algebra, whose quasi–classical limit is* $\mathfrak{g}$*;*

(iii) $\mathcal{U}_\hbar^{DJ}(\mathfrak{g})$ *has a R–matrix.*

Proof. (i): As in the case of $\mathfrak{sl}_2$, we define the coproduct, the counit and the antipode by:

$$\begin{aligned}\Delta_\hbar : \mathcal{U}_\hbar^{DJ}(\mathfrak{g}) &\to \mathcal{U}_\hbar^{DJ}(\mathfrak{g}) \otimes \mathcal{U}_\hbar^{DJ}(\mathfrak{g})\\ X_i &\mapsto X_i \otimes e^{d_i\hbar H_i} + 1 \otimes X_i\\ Y_i &\mapsto Y_i \otimes 1 + e^{-d_i\hbar H_i} \otimes Y_i\\ H_i &\mapsto H_i \otimes 1 + 1 \otimes H_i\end{aligned}$$

$$\begin{aligned}\varepsilon_\hbar : \mathcal{U}_\hbar^{DJ}(\mathfrak{g}) &\to \mathbb{C}[[\hbar]]\\ X_i, Y_i, H_i &\mapsto 0\end{aligned}$$

$$\begin{aligned}S_\hbar : \mathcal{U}_\hbar^{DJ}(\mathfrak{g}) &\to \mathcal{U}_\hbar^{DJ}(\mathfrak{g})\\ X_i &\mapsto -X_i e^{-d_i\hbar H_i}\\ Y_i &\mapsto -e^{d_i\hbar H_i} Y_i\\ H_i &\mapsto -H_i\end{aligned}$$

We have that $\Delta_\hbar$ and $\varepsilon_\hbar$ are morphisms of algebras, while $S_\hbar$ is an anti–morphism of algebras; these facts can be consulted in more details in [CP+95].

(ii): To prove this it is enough to emulate its corresponding of the Theorem 4.3.7.
(iii): Let $\Phi = \Delta^+ \sqcup \Delta^-$ be a Cartan decomposition of $\mathfrak{g}$ and consider the tensor

$$R_\hbar = e^{\hbar \sum_{ij} (B^{-1})_{ij} (H_i \otimes H_j)} \prod_{\alpha \in \Delta^+} exp_{q_\alpha}\big((1 - q_\alpha^{-2})(e_\alpha \otimes f_\alpha)\big),$$

where B is the matrix $(B)_{ij} = d_j^{-1} a_{ij}$, $q_\alpha = e^{d_\alpha \hbar}$ and

$$exp_{q_\alpha}(x) = \sum_{k \geq 0} q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!}.$$

For the rest of computations we refer to [CP+95]. □

Remark 4.3.12. *If A is the 1×1 matrix (2), we have that $\mathfrak{g}(A) = \mathfrak{sl}_2$, and also we obtain that*

$$\mathcal{U}_\hbar^{DJ}(\mathfrak{g}(A)) = \mathcal{U}_\hbar(\mathfrak{sl}_2).$$

Chapter 5

Quantization of Lie bialgebras

In this chapter we present a universal quantization technique for Lie bialgebras. The idea is, given a Lie bialgebra $\mathfrak{g}$, to quantize first its Drinfeld double $\mathfrak{D}\mathfrak{g}$, and then look for the quantization of $\mathfrak{g}$ inside that of $\mathfrak{D}\mathfrak{g}$. This quantization technique is due to Pavel Etingof and David Kazhdan and we will follow their reasoning as in [EK96] and [ES02].

5.1 The Tannaka–Krein duality

The idea of the Tannaka–Krein duality is to reconstruct an algebraic object starting from the category of its representations through a fiber functor, that is a functor that *forgets* certain properties of a category. The Tannaka–Krein duality can be applied in several contexts; we now present the case of bialgebras.

Definition 5.1.1. *Let $(\mathcal{C}, \otimes, I, a, l, r, c)$ be a symmetric braided tensor category. A braided tensor functor*

$$F : \mathcal{C} \to \mathsf{Vect}$$

is called a fiber functor if it is exact and faithful.

Example 5.1.2. *Let A be a $\mathbb{K}$–algebra. Then there is a natural forgetful functor*

$$F : \mathsf{Mod}(A) \to \mathsf{Vect}(\mathbb{K})$$

which assigns to any A–module its underlying $\mathbb{K}$–vector space and to any morphism of A–modules the underlying $\mathbb{K}$–linear map. We have that F is a fiber functor, that we call the forgetful functor.

The next Proposition is the base of the idea of the Tannaka–Krein duality.

Proposition 5.1.3. *Let A be an algebra and let F be the forgetful functor as above. Then we have that:*

(i) F is representable;

(ii) there exists an algebra isomorphism $\theta : A \to \mathsf{End}(F)$.

Proof. (i): We recall that any bialgebra A is an A–module with action given by the left multiplication. Then a representation of F is given by (A, ϕ), where

$$\phi : F \to \mathsf{Hom}_{\mathsf{Mod}(A)}(A, \cdot)$$

assigns to any A–module V its action in $\mathsf{Hom}_{\mathsf{Mod}(A)}(A, V)$.
(ii): Consider

$$\begin{aligned} \theta : A &\to \mathsf{End}(F) \\ a &\mapsto \theta(a) \end{aligned}$$

defined by $\theta(a)_V = a_V$, where a_V denotes the action of $a \in A$ on the A–module V. Since $a_V = a'_V$ implies $a = a'$, we have that θ is injective. The fact that θ is surjective arises from the fact that, given an algebra A, then every A–module is a quotient of a free A–module. For more details on this fact we refer to [ES02]. □

The Proposition above tells us that the knowledge of an algebra A is equivalent to the knowledge of the algebra of endomorphisms of the forgetful functor associated to A. We now want to generalize this discussion to the case of bialgebras, and so the question is: what information do we need to add to the forgetful functor to have a bialgebra structure on A? The next result gives us an answer to this question and also gives us a new interpretation of bialgebras in terms of fiber functors.

Theorem 5.1.4. *Let A be an algebra and let F be the forgetful functor. Then any structure of tensor category on $\mathsf{Mod}(A)$ together with a structure of tensor functor on F equips A with a bialgebra structure.*

Proof. We have to define a coproduct ad a counit on A. Since $A \simeq \mathsf{End}(F)$, it is sufficient to define the operations on $\mathsf{End}(F)$. Consider the functor

$$\begin{aligned} F^2 : \mathsf{Mod}(A) \times \mathsf{Mod}(A) &\to \mathsf{Mod}(A) \\ (V, W) &\mapsto F(V) \otimes F(W). \end{aligned}$$

We have that $\mathsf{End}(F^2) = \mathsf{End}(F) \otimes \mathsf{End}(F)$; an explanation of this fact can be consulted in [ES02]. Let V, W be objects in $\mathsf{Mod}(A)$ and let

$$J_{V,W} : F(V) \otimes F(W) \to F(V \otimes W)$$

be the family of natural isomorphisms that equips F with a tensor structure. We define $\Delta : \mathsf{End}(F) \to \mathsf{End}(F^2) = \mathsf{End}(F) \otimes \mathsf{End}(F)$ in the following way. Given $a \in A$, we take the element $a_{(\cdot)}$ of $\mathsf{End}(F)$ via the isomorphism θ of the previous proposition. To $a_{(\cdot)}$ we associate the element of $\mathsf{End}(V) \otimes \mathsf{End}(V)$ given by the composition

$$F(V) \otimes F(W) \xrightarrow{J_{V,W}} F(V \otimes W) \xrightarrow{a_{V\otimes W}} F(V \otimes W) \xrightarrow{J^{-1}_{V,W}} F(V) \otimes F(W).$$

Furthermore, since F has a tensor structure, we have that $F(\mathbb{K}) \simeq \mathbb{K}$, and so we define the counit map by

$$\begin{aligned} \varepsilon : \mathsf{End}(F) &\to \mathsf{End}(F(\mathbb{K})) \\ a_{(\cdot)} &\mapsto a_{F(\mathbb{K})}. \end{aligned}$$

If we look at the definition of a tensor functor, we observe that the commutativity of the hexagonal diagram implies that Δ is a morphism of algebras, while the commutativity of the two squared diagrams implies that ε is a morphism of algebras, and so the claim is proved. □

As a consequence of this Theorem, we can think upon to a bialgebra as an algebra together with a tensor structure on $\mathsf{Mod}(A)$ and a tensor functor $F : \mathsf{Mod}(A) \to \mathsf{Vect}(\mathbb{K})$ that is exact and faithful. Moreover, the preceding Theorem admits a braided version, that is the following.

Theorem 5.1.5. *Let A be an algebra and let F be the forgetful functor. Then any structure of braided tensor category on* $\mathsf{Mod}(A)$ *together with a structure of braided tensor functor on F equips A with a quasi–triangular bialgebra structure.*

5.2 Quantization of the Drinfeld double

The first step to construct a quantization of a finite–dimensional Lie bialgebra is to find a quantization of its Drinfeld double. To do this, we use the Tannaka–Krein approach: we construct a tensor category, a fiber functor and a tensor structure on it.

5.2.1 The Drinfeld category

Let $\mathfrak{g}$ be a finite–dimensional Lie bialgebra and let $\mathfrak{D}\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ be its Drinfeld double.

Definition 5.2.1. *The Drinfeld category associated to $\mathfrak{D}\mathfrak{g}$ is $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$, where:*

- *the objects are all the topologically free $\mathfrak{D}\mathfrak{g}$–modules;*
- $\mathsf{Hom}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(V, W) = \mathsf{Hom}_{\mathfrak{D}\mathfrak{g}[[\hbar]]}(V, W)$.

We now have to define a tensor structure on the Drinfeld category. As seen in the Section 2.5, this is equivalent to define a quasi–bialgebra structure in $\mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]]$, and by the Proposition 2.5.2, this is equivalent to have an associator $\Phi \in \mathcal{U}(\mathfrak{D}\mathfrak{g})^{\otimes 3}[[\hbar]]$. In general it is not easy to find an associator; however, the following Theorem assures us the existence in the case of the Drinfeld double of a Lie bialgebra.

Theorem 5.2.2 (Drinfeld). *Let $\mathfrak{g}$ be a Lie bialgebra and let $t \in \mathfrak{g} \otimes \mathfrak{g}$ be an invariant and symmetric tensor. Then there exists*

$$\Phi_{KZ} := \Phi_{KZ}(\hbar \cdot t_{12}, \hbar \cdot t_{23}) \in \mathcal{U}\mathfrak{g}^{\otimes 3}[[\hbar]]$$

such that Φ_{KZ} is an associator. Moreover, the element $R = e^{\hbar t/2}$ defines a quasi–triangular structure on the quasi–bialgebra $(\mathcal{U}\mathfrak{g}[[\hbar]], \otimes, \mathbb{C}[[\hbar]], \Phi_{KZ})$.

The proof of this Theorem can be find in [Dri90]. In particular, the element Φ_{KZ} is defined by the equality

$$F_0 = F_1 \Phi_{KZ},$$

where F_0 and F_1 are the unique solutions of the differential equation

$$\frac{dF}{dz} = \frac{1}{2i\pi}\left(\frac{\hbar \cdot t_{12}}{z} + \frac{\hbar \cdot t_{23}}{z-1}\right)F,$$

and t_{ij} is defined by the action of t on the i–th and j–th components of $\mathcal{U}\mathfrak{g}^{\otimes n}[[\hbar]]$. Moreover, the Drinfeld associator satisfies $\Phi \equiv 1 \; mod \; \hbar^2$; this will be crucial in our discussion.

Remark 5.2.3. *Let $\mathfrak{g}$ be a finite–dimensional Lie bialgebra and let $(\mathfrak{D}\mathfrak{g}, \mathfrak{g}, \mathfrak{g}^*)$ be the Manin triple associated to $\mathfrak{g}$. How we have seen in the third chapter, we have that, if r is the canonical r–matrix of $\mathfrak{D}\mathfrak{g}$, then $\Omega = r + \tau(r)$ is an invariant and symmetric element of $\mathfrak{D}\mathfrak{g} \otimes \mathfrak{D}\mathfrak{g}$. Therefore we may apply the Drinfeld theorem and obtain an associator Φ.*

As a consequence of the Drinfeld Theorem, we have a braided tensor structure on the Drinfeld category.

5.2.2 The fiber functor

Henceforth we denote $\mathfrak{g}$ by $\mathfrak{g}_+$ and $\mathfrak{g}^*$ by $\mathfrak{g}_-$.
We now have to construct a fiber functor together with a tensor structure on it. While the choice of the functor is trivial, the same cannot be said for its tensor structure. In fact, denoting the tensor category of all the topologically free modules over $\mathbb{K}$ by $\mathcal{A}$, we take as fiber functor the forgetful functor

$$F : \mathcal{M}_{\mathfrak{D}\mathfrak{g}} \to \mathcal{A}$$

assigning to any object of $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$ the topologically free module associated to its underlying vector space. To construct a tensor structure on this functor, we need the following two objects of $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$:

Definition 5.2.4. *The universal Verma modules associated to* $\mathfrak{D}\mathfrak{g}$ *are*

$$M_\pm := \mathsf{Ind}_{\mathfrak{g}_\pm[[\hbar]]}^{\mathfrak{D}\mathfrak{g}[[\hbar]]} c_\pm = \mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]] \otimes_{\mathcal{U}\mathfrak{g}_\pm[[\hbar]]} c_\pm,$$

where $c_\pm$ *is the trivial* $\mathfrak{g}_\pm[[\hbar]]$*–module of rank* 1.

By the Poincaré–Birkhoff–Witt Theorem, we have that the multiplication maps

$$\mathcal{U}\mathfrak{g}_\pm \otimes \mathcal{U}\mathfrak{g}_\mp \to \mathcal{U}\mathfrak{D}\mathfrak{g}$$

are linear isomorphisms. This allows us, as in the case of highest vector modules theory, to identify $M_\pm$ with $\mathcal{U}\mathfrak{g}_\mp \cdot 1_\mp$, where 1_+ and 1_- are vectors such that $\mathfrak{g}_\pm \cdot 1_\pm = 0$. This identification allows us to transport the coprodut and counit maps of $\mathcal{U}\mathfrak{g}_\pm$ on $M_\mp$. In fact we have the $\mathfrak{D}\mathfrak{g}[[\hbar]]$–morphisms

$$i_\pm : M_\pm \to M_\pm \otimes M_\pm$$

and

$$\varepsilon_\pm : M_\pm \to \mathbb{C}[[\hbar]],$$

with $i_\pm(1_\pm) = 1_\pm \otimes 1_\pm$ and $\varepsilon_\pm(u_\mp 1_\pm) = \varepsilon(u_\mp)$ for $u_\mp \in \mathcal{U}\mathfrak{g}_\mp$.

Lemma 5.2.5. *Under the previous hypothesis, we have that:*

(i) The maps $i_\pm$ *are coassociative, i.e.*

$$(i_\pm \otimes \mathsf{id}) \circ i_\pm = (\mathsf{id} \otimes i_\pm) \circ i_\pm;$$

(ii) the maps $\varepsilon_\pm$ *satisfies the counit axiom, i.e.*

$$(\varepsilon_\pm \otimes \mathsf{id}) \circ i_\pm = \mathsf{id} = (\mathsf{id} \otimes \varepsilon_\pm) \circ i_\pm;$$

(iii) The assignment $1 \to 1_+ \otimes 1_-$ *extends to an isomorphism of* $\mathfrak{D}\mathfrak{g}$*–modules*

$$\phi : \mathcal{U}\mathfrak{D}\mathfrak{g} \to M_+ \otimes M_-.$$

Proof. (i): We have to show that the following diagram is commutative:

$$\begin{array}{ccc} & M_\pm & \\ {}^{i_\pm}\swarrow & & \searrow^{i_\pm} \\ M_\pm & & M_\pm \\ \Big\downarrow {\scriptstyle \mathsf{id}\otimes i_\pm} & & \Big\downarrow {\scriptstyle i_\pm \otimes \mathsf{id}} \\ M_\pm \otimes (M_\pm \otimes M_\pm) & \xrightarrow[\Phi^{-1}]{} & (M_\pm \otimes M_\pm) \otimes M_\pm \end{array} .$$

Since $i_\pm$ are identified with the usual coproduct maps, we have that

$$(i_\pm \otimes \mathsf{id}) \circ i_\pm = (\mathsf{id} \otimes i_\pm) \circ i_\pm.$$

Furthermore, by definition of Φ, we have that $\Phi \equiv \mathsf{id} \otimes \mathsf{id} \otimes \mathsf{id}$ on the image of $(i_\pm \otimes \mathsf{id}) \circ i_\pm$, and so the diagram commutes.
(ii): The counit axiom of $\varepsilon_\pm$ follows by the usual counit relation

$$(\varepsilon \otimes \mathsf{id}) \circ \Delta = (\mathsf{id} \otimes \varepsilon) \circ \Delta$$

via the identification of $M_\pm$ with $\mathcal{U}\mathfrak{g}_\mp$.
(iii): We have that ϕ preserves the standard grading (by the identification of $M_\pm$ with $\mathcal{U}\mathfrak{g}_\mp$). Therefore, ϕ defines a map on the associated graded objects

$$S(\mathfrak{D}\mathfrak{g}) \to S(\mathfrak{g}_+) \otimes S(\mathfrak{g}_-),$$

which is the isomorphism induced by the identification $\mathfrak{D}\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. This implies that ϕ is an isomorphism. □

5.2.3 The tensor structure on the forgetful functor

From the previous Lemma we obtain that the forgetful functor F is represented by $M_+ \otimes M_-$, and so we have

$$F(V) = \mathsf{Hom}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-, V).$$

Proposition 5.2.6. *For* V *and* W *in* $\mathsf{Obj}(\mathcal{M}_{\mathfrak{D}\mathfrak{g}})$*, consider the map*

$$\begin{aligned} J_{V,W} : F(V) \otimes F(W) &\to F(V \otimes W) \\ v \otimes w &\mapsto J_{V,W}(v \otimes w) \end{aligned}$$

defined by

$$\begin{array}{rcl} M_+ \otimes M_- & \xrightarrow{i_+ \otimes i_-} & (M_+ \otimes M_+) \otimes (M_- \otimes M_-) \\ & \swarrow^{a} & \\ (M_+ \otimes (M_+ \otimes M_-)) \otimes M_- & \xrightarrow{(\mathsf{id} \otimes \beta) \otimes \mathsf{id}} & (M_+ \otimes (M_- \otimes M_+)) \otimes M_- \\ & \swarrow^{a^{-1}} & \\ (M_+ \otimes M_-) \otimes (M_+ \otimes M_-) & \xrightarrow{v \otimes w} & V \otimes W, \end{array}$$

where β denotes the braiding and a denotes a natural change of bracketing

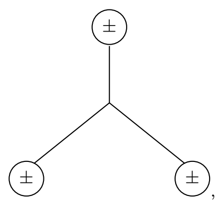
$$a : (\bullet\bullet)(\bullet\bullet) \to (\bullet(\bullet\bullet))\bullet .$$

Then the collection of maps $(J_{V,W})_{V,W \in \mathcal{M}_{\mathfrak{D}\mathfrak{g}}}$ is a tensor structure on F.

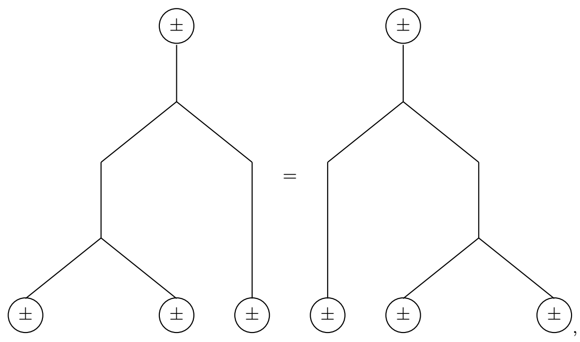
Proof. We have to check that the collection $(J_{V,W})_{V,W \in \mathcal{M}_{\mathfrak{D}\mathfrak{g}}}$ satisfies the definition of natural tensor isomorphism given in the Subsection 1.3.2. By definition of Φ, we have that $J_{V,W} \equiv \mathsf{id}$ *mod* $\hbar$, and so $J_{V,W}$ is an isomorphism for any V and W. Furthermore, we have that $J_{V,\mathbb{C}} = J_{\mathbb{C},V} = \mathsf{id}_V$ for any V in $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$. It remains to check the relation

$$J_{U \otimes V,W} \circ (J_{U,V} \otimes \mathsf{id}_W) = J_{U,V \otimes W} \circ (\mathsf{id}_U \otimes J_{V,W}).$$

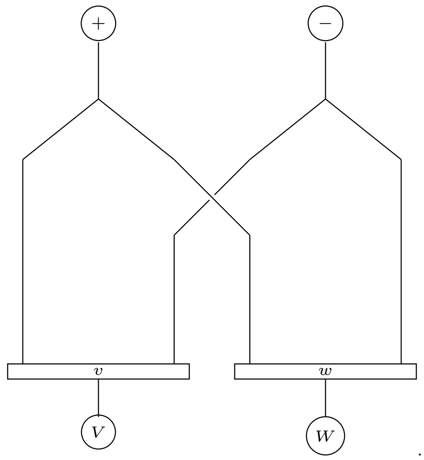
Since the constraints that we are considering may be very complicated, we give a pictorial proof of this fact. According with the notations given in the Section 2.8, we represent the morphisms $i_\pm$ by



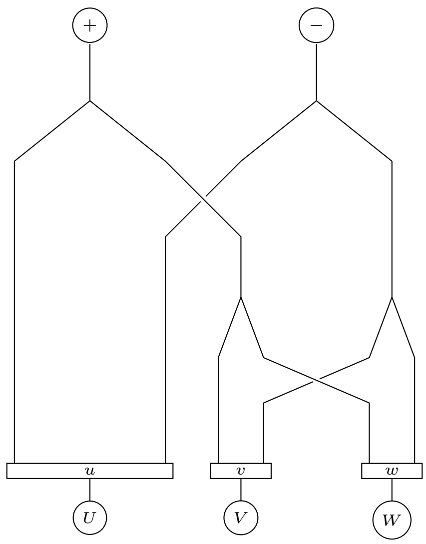
where we denote $M_\pm$ with $\pm$. The coassociativity of $i_\pm$ is represented by the equality



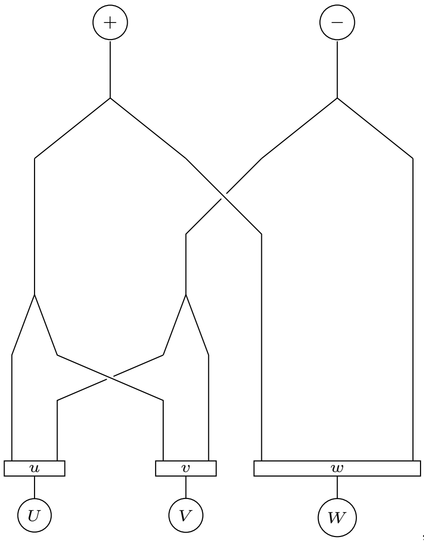
while the map $J_{V,W}$ is represented by



The terms $J_{U,V\otimes W} \circ (\mathsf{id}_U \otimes J_{V,W})$ and $J_{U\otimes V,W} \circ (J_{U,V} \otimes \mathsf{id}_W)$ are represented respectively by

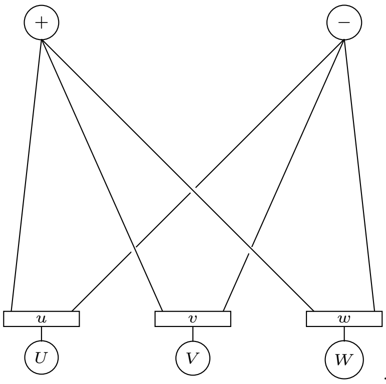


and



,

and we have that both are equivalent to the picture



.

□

We now give an explicit formula for J. Let V and W be objects in $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$ and let $v \in V$ and $w \in W$. Let us apply $J_{V\otimes W}(v \otimes w)$ to the vector $1_+ \otimes 1_-$. Making explicit the form of the associativity constraint, we have to apply to $1_+ \otimes 1_-$ the following composition of maps:

$$M_+ \otimes M_- \xrightarrow{i_+\otimes i_-} (M_+ \otimes M_+) \otimes (M_- \otimes M_-) \xrightarrow{\Phi_{1,2,34}} M_+ \otimes (M_+ \otimes (M_- \otimes M_-))$$

$$\xrightarrow{\mathsf{id}\otimes\Phi^{-1}_{2,3,4}} M_+ \otimes ((M_+ \otimes M_-) \otimes M_-) \xrightarrow{(23)e^{\hbar\Omega_{23}/2}} M_+ \otimes ((M_- \otimes M_+) \otimes M_-)$$

$$\xrightarrow{\mathsf{id}\otimes\Phi_{2,3,4}} M_+ \otimes (M_- \otimes (M_+ \otimes M_-)) \xrightarrow{\Phi^{-1}_{1,2,34}} (M_+ \otimes M_-) \otimes (M_+ \otimes M_-)$$

$$\xrightarrow{\qquad v\otimes w \qquad} V \otimes W.$$

Therefore, we obtain

$$\begin{aligned}
&J_{V,W}(v \otimes w)(1_- \otimes 1_-) \\
&= (v \otimes w)(\Phi^{-1}_{1,2,34}(\mathsf{id} \otimes \Phi_{2,3,4})(23)e^{\hbar\Omega_{23}/2}(\mathsf{id} \otimes \Phi^{-1}_{2,3,4})\Phi_{1,2,34}(1_+ \otimes 1_+ \otimes 1_- \otimes 1_-)) \\
&= (v \otimes w)(\Phi^{-1}_{1,2,34}(\mathsf{id} \otimes \Phi_{2,3,4})e^{\hbar\Omega_{23}/2}(\mathsf{id} \otimes \Phi^{-1}_{3,2,,4})\Phi_{1,3,24}(1_+ \otimes 1_- \otimes 1_+ \otimes 1_-)).
\end{aligned}$$

Using the map ϕ defined in the Lemma 5.2.5, we obtain that

$$J = (\phi^{-1} \otimes \phi^{-1})(\Phi^{-1}_{1,2,34}(\mathsf{id} \otimes \Phi_{2,3,4})e^{\hbar\Omega_{23}/2}(\mathsf{id} \otimes \Phi^{-1}_{3,2,,4})\Phi_{1,3,24}(1_+ \otimes 1_- \otimes 1_+ \otimes 1_-)).$$

5.2.4 Quantization of the Drinfeld double

We now have all the ingredients to apply the Tannaka–Krein duality. In fact, the tensor structure J on the forgetful functor F equips $\mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]]$ with a bialgebra structure, whose operations are

$$\Delta_\hbar = J^{-1}\Delta_0 J \quad \text{and} \quad \varepsilon_\hbar = \varepsilon_0,$$

where Δ_0 and ε_0 are the usual coproduct and counit. Since $J \equiv 1 \; mod \; \hbar$, we have that this bialgebra is a deformation of the Hopf algebra $\mathcal{U}\mathfrak{D}\mathfrak{g}$.

Proposition 5.2.7. *Let A be a bialgebra which is a deformation of a Hopf algebra A_0. Then A has a unique Hopf algebra structure compatible with its bialgebra structure and is a deformation of A_0 as a Hopf algebra.*

Proof. Let us consider the map

$$\begin{aligned}
\chi : \mathsf{End}(A) &\to \mathsf{End}(A) \\
T &\mapsto \mu \circ (T \otimes \mathsf{id}) \circ \Delta
\end{aligned}$$

where μ and Δ are respectively the product and the coproduct of A. Consider also the analogue of χ for A_0

$$\begin{aligned}
\chi_0 : \mathsf{End}(A_0) &\to \mathsf{End}(A_0) \\
T &\mapsto \mu_0 \circ (T \otimes \mathsf{id}) \circ \Delta_0.
\end{aligned}$$

We have that χ_0 is invertible; in particular its inverse is given by

$$\begin{aligned}
\chi_0^{-1} : \mathsf{End}(A_0) &\to \mathsf{End}(A_0) \\
T &\mapsto \mu_0 \circ (T \otimes S_0) \circ \Delta_0,
\end{aligned}$$

where S_0 is the antipode of A_0. Furthermore, we have that

$$\chi \equiv \chi_0 \; mod \; \hbar,$$

and so also χ is invertible. To define a Hopf structure on the bialgebra A, it suffices to consider as an antipode the map given by

$$S = \chi^{-1}(\eta \circ \varepsilon).$$

□

As a consequence of this Proposition, we have that $\mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]]$ has a unique Hopf algebra structure that deforms $\mathcal{U}\mathfrak{D}\mathfrak{g}$; we denote this Hopf algebra by $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$.

Lemma 5.2.8. *We have $J \equiv 1 + \frac{\hbar r}{2}$ mod $\hbar^2$.*

Proof. Since $\Phi \equiv 1$ *mod* $\hbar^2$, we have

$$\begin{aligned} J &\equiv (\phi^{-1} \otimes \phi^{-1})(e^{\hbar\Omega_{23}/2}(1_+ \otimes 1_- \otimes 1_+ \otimes 1_-)) \ mod\ \hbar^2 \\ &\equiv (\phi^{-1} \otimes \phi^{-1})((1 + \frac{\hbar\Omega_{23}}{2})(1_+ \otimes 1_- \otimes 1_+ \otimes 1_-)) \ mod\ \hbar^2 \\ &\equiv 1 + \frac{\hbar}{2}(\phi^{-1} \otimes \phi^{-1})((r_{23} + r_{32})(1_+ \otimes 1_- \otimes 1_+ \otimes 1_-)) \ mod\ \hbar^2. \end{aligned}$$

We have that the tensor r is such that $\tau(r)(1_- \otimes 1_+) = 0$. Furthermore, by definition we have $r(1_- \otimes 1_+) = \sum_i x_i 1_- \otimes x^i 1_-$, where $\{x_i\}$ is a basis of $\mathfrak{g}_+$ and $\{x^i\}$ is the dual basis. Thus we obtain

$$J \equiv 1 + \frac{\hbar r}{2} \ mod\ \hbar^2.$$

□

Theorem 5.2.9. *Under the previous hypothesis, we have that:*

(i) $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ is a quantization of $\mathfrak{D}\mathfrak{g}$;

(ii) $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ has a quasi–triangular structure.

Proof. (i): We first note that $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ is a deformation of $\mathcal{U}\mathfrak{D}\mathfrak{g}$. In particular, from the construction of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ it follows that the product is undeformed, and so $\mu_\hbar = \mu_0$, where $\mu_\hbar$ is the product of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ and μ_0 is the product of $\mathcal{U}\mathfrak{D}\mathfrak{g}$ (and the same can be said for the unit and for the counit). Otherwise, the coproduct $\Delta_\hbar = J^{-1}\Delta_0 J$ is deformed, and from 5.2.8 we obtain that $\Delta_\hbar \equiv \Delta_0$ *mod* $\hbar$.
It remains to show that $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ gives back the correct quasi–classical limit. Let $x \in \mathfrak{D}\mathfrak{g} \subset \mathcal{U}\mathfrak{D}\mathfrak{g}$ and let $\tilde{x} \in \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ be a lifting of x. Set

$$\delta(x) = \frac{\Delta_\hbar(\tilde{x}) - \Delta_\hbar^{op}(\tilde{x})}{\hbar} \quad mod\ \hbar.$$

We have to prove that $\delta(a) = \delta_{\mathfrak{D}\mathfrak{g}}(a) = \delta_{\mathfrak{g}}(a) \oplus -\delta_{\mathfrak{g}^*}(a)$. Let r be the canonical quasi–triangular structure of $\mathfrak{D}\mathfrak{g}$. Using the previous lemma and the fact that $r + \tau(r)$ is $\mathfrak{D}\mathfrak{g}$– invariant, we have that

$$\begin{aligned}
\frac{\Delta_\hbar(\tilde{x}) - \Delta_\hbar^{op}(\tilde{x})}{\hbar} &\equiv \frac{J^{-1}\Delta_0(\tilde{x})J - (J^{op})^{-1}\Delta_0^{op}(\tilde{x})J^{op}}{\hbar} \ mod\ \hbar \\
&= \frac{J^{-1}\Delta_0(\tilde{x})J - (J^{op})^{-1}\Delta_0(\tilde{x})J^{op}}{\hbar} \ mod\ \hbar \\
&\equiv \frac{1}{2}\big([\Delta_0(\tilde{x}), r] - [\Delta_0(\tilde{x}), \tau(r)]\big) \ mod\ \hbar \\
&\equiv [\Delta_0(\tilde{x}), r] \ mod\ \hbar \\
&= \partial_0 r(x) \\
&= \delta_{\mathfrak{D}\mathfrak{g}}(x).
\end{aligned}$$

(ii): Let r be the usual quasi–triangular structure on $\mathfrak{D}\mathfrak{g}$ and let $\Omega = r + \tau(r)$. Set

$$R = (J^{op})^{-1} e^{\hbar\Omega/2} J.$$

We want to prove that R is a quasi–triangular structure for $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. Let $x \in \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. Then we have

$$\begin{aligned}
R\Delta_\hbar(x) &= (J^{op})^{-1} e^{\hbar\Omega/2} J J^{-1} \Delta_0(x) J \\
&= (J^{op})^{-1} e^{\hbar\Omega/2} \Delta_0(x) J \\
&= (J^{op})^{-1} \Delta_0(x) e^{\hbar\Omega/2} J \\
&= (J^{op})^{-1} \Delta_0(x) J^{op} (J^{op})^{-1} e^{\hbar\Omega/2} J \\
&= (J^{-1}\Delta_0(x)J)^{op} R \\
&= \Delta_\hbar^{op}(x) R.
\end{aligned}$$

To prove the first hexagon axiom, we have to check that $(\Delta_\hbar \otimes \mathsf{id})(R)$ and $R_{13}R_{23}$ act in the same way on a tensor $v \otimes w \otimes u$, where U, V, W are objects in $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$ and v, w, u belongs to $F(V), F(W), F(U)$ respectively. By the explicit formula of J, we have

$$R \cdot (v \otimes w) = (12) J_{W,V}^{-1} F(\beta) J_{V,W}.$$

Hence

$$\begin{aligned}
(\Delta_\hbar \otimes \mathsf{id})(R)(v \otimes w \otimes u) &= (J_{W,V}^{-1}) R (J_{V,W} \otimes \mathsf{id})(v \otimes w \otimes u) \\
&= (132)(\mathsf{id} \otimes J_{V,W}^{-1}) J_{U,V\otimes W}^{-1} F(\beta_{V\otimes W,U}) J_{V\otimes W,U} (J_{V,W} \otimes \mathsf{id})(v \otimes w \otimes u).
\end{aligned}$$

The functoriality of the braiding β implies that

$$\beta_{V\otimes W,U} = (\beta_{V,U} \otimes \mathsf{id})(\mathsf{id} \otimes \beta_{W,U}),$$

while the associativity of J implies that

$$J^{-1}_{U\otimes V,W}(F(\beta_{V,U}) \otimes \mathsf{id})J_{V\otimes U,W} = F(\beta_{V,U}) \otimes \mathsf{id}$$

and

$$J^{-1}_{V,U\otimes W}(\mathsf{id} \otimes F(\beta_{W,U}))J_{V,W\otimes U} = \mathsf{id} \otimes F(\beta_{W,U}).$$

Therefore, we obtain

$$\begin{aligned}(\Delta_\hbar \otimes \mathsf{id})(R)(v \otimes w \otimes u) &= (13)J^{-1}_{U,V}F(\beta_{V,U})J_{V,U}(23)J^{-1}_{U,W}F(\beta_{W,U})J_{W,U} \\ &= R_{13}R_{23}.\end{aligned}$$

The proof of the second hexagonal identity can be made in the same way. □

In particular, this construction is a preferred quantization, i.e. a quantization that only deforms the coproduct and the antipode. More specifically, the antipode of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ is given by

$$S_\hbar = QS_0Q^{-1},$$

where S_0 is the usual antipode of $\mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]]$ and

$$Q = \mu_0(S_0 \otimes \mathsf{id})(J).$$

Remark 5.2.10. *The construction of a quantization of the Drinfeld double depends on a choice of an associator* Φ*.*

5.3 Quantization of finite–dimensional Lie bialgebras

We now want to construct a quantization of any finite–dimensional Lie bialgebra $\mathfrak{g} = \mathfrak{g}_+$, starting from the quantization of its Drinfeld double $\mathfrak{D}\mathfrak{g}$. Since we have the algebra isomorphism

$$\mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]] \simeq \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g}),$$

and also we have that $\mathcal{U}\mathfrak{g}[[\hbar]]$ is a Hopf subalgebra of $\mathcal{U}\mathfrak{D}\mathfrak{g}[[\hbar]]$, we expect to use its corresponding in $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ to construct a quantization for $\mathfrak{g}$. Unfortunately, this choice does not works, because this subalgebra is in general not closed under the coproduct. Since the forgetful functor is represented by $M_+ \otimes M_-$, the idea of Etingof and Kazhdan is to consider

$$F(M_\pm) = \mathsf{Hom}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-, M_\pm)$$

as quantizations of $\mathfrak{g}_\mp$. In the following we present the case of $F(M_-)$ and we show that it is a quantization of $\mathfrak{g}_+ = \mathfrak{g}$. In fact, we have that $F(M_-)$ has a bialgebra structure, with the following operations.

- Product: given $x, y \in F(M_-) = \mathsf{Hom}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-, M_-)$, we define $\mu_\hbar(x, y)$ as the element of $F(M_-)$ given by the composition

$$M_+ \otimes M_- \xrightarrow{i_+ \otimes \mathsf{id}} (M_+ \otimes M_+) \otimes M_- \xrightarrow{\Phi} M_+ \otimes (M_+ \otimes M_-) \xrightarrow{\mathsf{id} \otimes y} M_+ \otimes M_- \xrightarrow{x} M_-.$$

- Unit: since in $F(M_-)$ we have a unique map such that $1_+ \otimes 1_- \to 1_-$, then we define the unit by

$$\eta_\hbar : \mathbb{C}[[\hbar]] \to F(M_-)$$

  that assigns to any $a \in \mathbb{C}[[\hbar]]$ the morphism

$$1_+ \otimes 1_- \to a \cdot 1_-.$$

- Coproduct: it suffices to consider

$$\Delta_\hbar = J^{-1}_{M_-, M_-} \circ F(i_-) : F(M_-) \to F(M_-) \otimes F(M_-).$$

- Counit: since F is a tensor functor, we have an isomorphism

$$\varphi_0 : F(\mathbb{C}[[\hbar]]) \to \mathbb{C}[[\hbar]].$$

  Then we define the counit by

$$\varepsilon_\hbar = F(\varepsilon_-) \circ \varphi_0 : F(M_-) \to \mathbb{C}[[\hbar]].$$

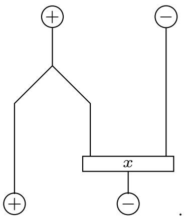
Hence, our aim is to show the following

Theorem 5.3.1. *Let $(\mathfrak{g}, [\cdot, \cdot], \delta)$ be a finite–dimensional Lie bialgebra. Then the quintuple $(F(M_-), \mu_\hbar, \eta_\hbar, \Delta_\hbar, \varepsilon_\hbar)$ as above is a quantization of $(\mathfrak{g}, [\cdot, \cdot], \delta)$.*

In fact, we have $F(M_-) \simeq \mathcal{U}\mathfrak{g}[[\hbar]]$ and, since $\Phi \equiv 1 \; mod \; \hbar^2$, we have also $\mu_\hbar \equiv \mu_0 \; mod \; \hbar$ and $\Delta_\hbar \equiv \Delta_0 \; mod \; \hbar$. We remain to show that $F(M_-)$ is indeed a Hopf algebra. However, in order to prove this, we will show that we can identify $F(M_-)$ as a Hopf subalgebra of $\mathsf{End}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-) \simeq \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. In fact, we introduce the following embedding. Given x in $F(M_-)$, we define $m_+(x)$ as the element of $\mathsf{End}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-)$ given by the composition

$$m_+(x) := (\mathsf{id} \otimes x) \circ \Phi \circ (i_+ \otimes \mathsf{id}),$$

that we represent with the following picture:



We define $\mathcal{U}_\hbar(\mathfrak{g}_+)$ as the subspace of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g}) \simeq \mathsf{End}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_+ \otimes M_-)$ given by the image of $F(M_-)$ through m_+. By the definition of $\mathcal{U}_\hbar(\mathfrak{g}_+)$, we observe that, if we prove that it is a Hopf algebra, then $\mathcal{U}_\hbar(\mathfrak{g}_+)/(\hbar \cdot \mathcal{U}_\hbar(\mathfrak{g}_+))$ is isomorphic to $\mathcal{U}\mathfrak{g}_+$ as a Hopf algebra, and so we have that $\mathcal{U}_\hbar(\mathfrak{g}_+)$ is a quantized enveloping algebra. So we need to prove that $\mathcal{U}_\hbar(\mathfrak{g}_+)$ is a Hopf algebra.

Proposition 5.3.2. *$\mathcal{U}_\hbar(\mathfrak{g}_+)$ is a subalgebra of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$.*

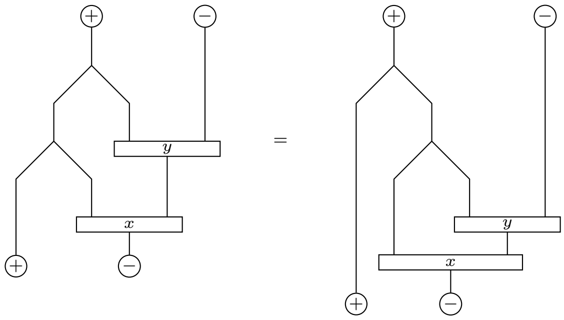
Proof. We have to prove that $\mathcal{U}_\hbar(\mathfrak{g}_+)$ is closed under the product. Let x and y be in $F(M_-)$. Then we have

$$\begin{aligned} m_+(x) \circ m_+(y) &= (\mathsf{id} \otimes x) \circ \Phi \circ (i_+ \otimes \mathsf{id}) \circ (\mathsf{id} \otimes y) \circ \Phi \circ (i_+ \otimes \mathsf{id}) \\ &= (\mathsf{id} \otimes x) \circ \Phi \circ ((\mathsf{id} \otimes \mathsf{id}) \otimes y) \circ \Phi_{12,3,4} \circ ((i_+ \otimes \mathsf{id}) \otimes \mathsf{id}) \circ (i_+ \otimes \mathsf{id}) \\ &= (\mathsf{id} \otimes x) \circ \Phi \circ ((\mathsf{id} \otimes \mathsf{id}) \otimes y) \circ \Phi_{12,3,4} \circ (\Phi^{-1} \otimes \mathsf{id}) \circ ((\mathsf{id} \otimes i_+) \otimes \mathsf{id}) \circ (i_+ \otimes \mathsf{id}) \\ &= (\mathsf{id} \otimes x) \circ (\mathsf{id} \otimes (\mathsf{id} \otimes y)) \circ \Phi_{1,2,34} \circ \Phi_{12,3,4} \circ (\Phi^{-1} \otimes \mathsf{id}) \circ ((\mathsf{id} \otimes i_+) \otimes \mathsf{id}) \circ (i_+ \otimes \mathsf{id}) \\ &= (\mathsf{id} \otimes x) \circ (\mathsf{id} \otimes (\mathsf{id} \otimes y)) \circ (\mathsf{id} \otimes \Phi) \circ \Phi_{1,23,4} \circ ((\mathsf{id} \otimes i_+) \otimes \mathsf{id}) \circ (i_+ \otimes \mathsf{id}), \end{aligned}$$

where we used the coassociativity of i_+ and the pentagon axiom. To prove the claim it suffices to consider $z = x \circ (\mathsf{id} \otimes y) \circ \Phi \circ (i_+ \otimes \mathsf{id})$, obtaining that

$$m_+(x) \circ m_+(y) = (z \otimes \mathsf{id}) \circ \Phi \circ (i_+ \otimes \mathsf{id}) = m_+(z).$$

We can represent the algebraic proof above with the following equality of pictures



□

It remains to show that $\mathcal{U}_\hbar(\mathfrak{g}_+)$ is a Hopf subalgebra of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$, and so we need a coproduct map on $\mathcal{U}_\hbar(\mathfrak{g}_+)$ that maps $\mathcal{U}_\hbar(\mathfrak{g}_+)$ into $\mathcal{U}_\hbar(\mathfrak{g}_+) \otimes \mathcal{U}_\hbar(\mathfrak{g}_+)$. Since we have a coproduct map $\Delta_\hbar$ on $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$, we have to prove that

$$\Delta_\hbar(m_+(x)) \in \mathcal{U}_\hbar(\mathfrak{g}_+) \otimes \mathcal{U}_\hbar(\mathfrak{g}_+)$$

for any $x \in F(M_-)$. Note that, for any $y \in \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$, the element $\Delta_\hbar(y) \in \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g}) \otimes \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ is determined by the commutativity of the diagram

$$\begin{array}{ccc} F(V) \otimes F(W) & \xrightarrow{J_{V,W}} & F(V \otimes W) \\ {\scriptstyle \Delta_\hbar(y)}\big\downarrow & & \big\downarrow{\scriptstyle y} \\ F(V) \otimes F(W) & \xrightarrow{J_{V,W}} & F(V \otimes W) \end{array}$$

that, for $y = m_+(x)$, gives us

$$\Delta_\hbar(m_+(x))_{v,w} = J_{V,W}^{-1}(m_+(x)_{v\otimes w})J_{V,W}.$$

Proposition 5.3.3. *$\mathcal{U}_\hbar(\mathfrak{g}_+)$ is a Hopf subalgebra of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$.*

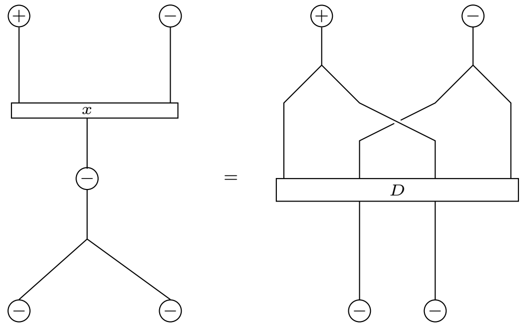
Proof. Let $x \in F(M_-)$ and let $\Delta_\hbar$ be the coproduct of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. We want to prove that $\Delta_\hbar(m_+(x))$ belongs to $\mathcal{U}_\hbar(\mathfrak{g}_+) \otimes \mathcal{U}_\hbar(\mathfrak{g}_+)$. More precisely, we want to prove that

$$\Delta_\hbar(m_+(x)) = (m_+ \otimes m_+)(J_{M_-,M_-}^{-1}(i_- \circ x)).$$

Let $D = J_{M_-,M_-}^{-1}(i_- \circ x)$. Then we have that

$$i_- \circ x = J_{M_-,M_-}D$$

and we represent this fact by the following equality of pictures



(♣)

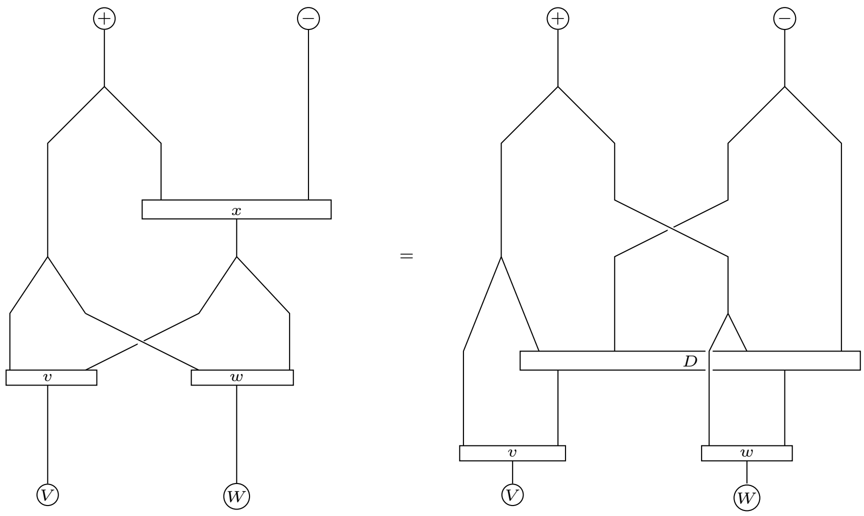
Our goal is to prove that

$$(m_+ \otimes m_+)(D) = \Delta_\hbar(m_+(x))_{v,w} = J_{V,W}^{-1}(m_+(x)_{v\otimes w})J_{V,W},$$

that is equivalent to

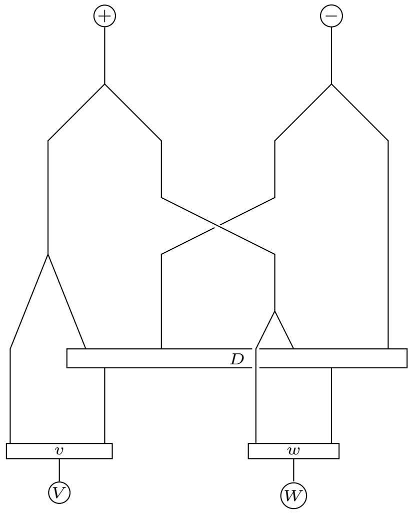
$$J_{V,W}\Delta_\hbar(m_+(x))_{v,w} = (m_+(x)_{v\otimes w})J_{V,W}.$$

In terms of pictures, we have to prove the following equality

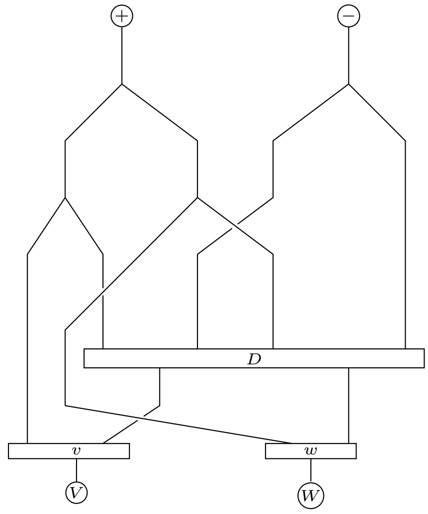


(♠)

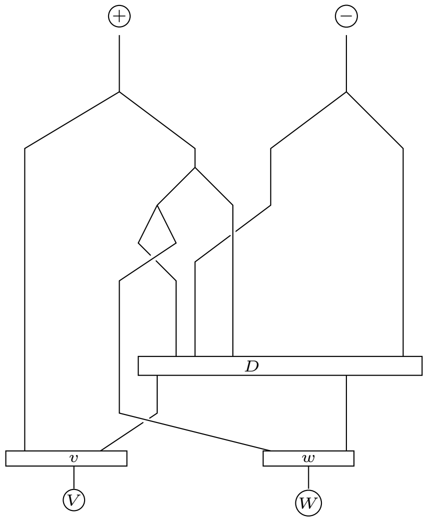
for any $v \in F(V)$ and $w \in F(W)$. Using the properties of i_+, we obtain the following sequence of equalities



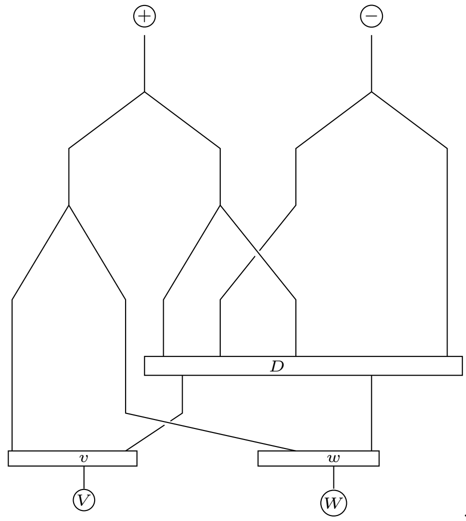
$=$



$=$



$=$



.

Substituting the left hand side of (♣) on the last picture, we obtain the left hand side of (♠), and so the claim is proved. □

From the result above it follows that $F(M_-)$ is a quantization of $\mathfrak{g}$, and so the Theorem 5.3.1 is proved. Note that, contrary to the case of the Drinfeld double, this is not a preferred quantization, since the product is deformed.

Remark 5.3.4. *We can make a similar discussion to show that $F(M_+)$ is a quantization of $\mathfrak{g}_-$.*

5.4 Quantization of infinite-dimensional Lie bialgebras and functoriality

The aim of this Section is to generalize the previous quantization technique to the infinite–dimensional case, in such a way to have a functor

$$Q : LBA \to QUE$$

from the category LBA of topological Lie bialgebras to the category QUE of quantized universal enveloping algebras over $\mathbb{C}[[\hbar]]$, such that $Q((\mathfrak{g}, [\cdot,\cdot], \delta))$ is a quantization of $(\mathfrak{g}, [\cdot,\cdot], \delta)$.

5.4.1 The limits of the previous construction

Basically, we shall extend the construction seen in the previous Sections; however, if we look at the finite–dimensional construction, we note two main problems in the passage at the infinite–dimensional setting:

(1): if $\mathfrak{g}$ is an infinite–dimensional Lie bialgebra and $(\mathfrak{D}\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is the associated infinite Manin triple, we have that the Casimir element of $\mathfrak{D}\mathfrak{g}$

$$\Omega = \sum_i (x_i^+ \otimes x_i^-) + (x_i^- \otimes x_i^+)$$

is a series that in general does not converge by acting on a $\mathfrak{D}\mathfrak{g} \otimes \mathfrak{D}\mathfrak{g}$–module. In particular, Ω does not converge if applied to $M_+ \otimes M_-$, that is a relevant object of the finite–dimensional setting.

(2): The operation

$$\mathfrak{g} \mapsto \mathfrak{D}\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$$

of taking the Drinfeld double of a Lie bialgebra is not functorial, because mixes the functor $\mathfrak{g} \mapsto \mathfrak{g}$ with the controvariant functor $\mathfrak{g} \mapsto \mathfrak{g}^*$.

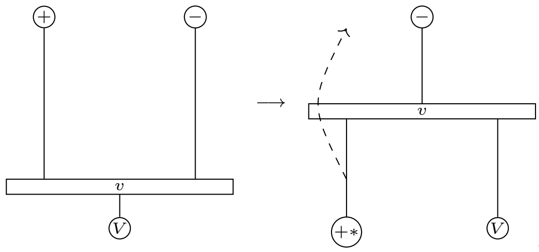
Both problems suggest that we should modify the previous quantization technique, by considering a new quantization that involves only one of the two Verma modules $M_\pm$, and possibly that it is independent from $\mathfrak{g}^*$. In fact, since there is a vector spaces isomorphism $M_+ \simeq S(\mathfrak{g}^*)[[\hbar]]$, Etingof and Kazhdan reformulate the previous quantization in a free–M_+ setting.

5.4.2 Quantization of finite–dimensional Lie bialgebras revisited

The first step to define the new quantization consists of modify the construction of the previous section, in such a way M_+ does not appear. Again, the approach of this technique is through the Tannaka–Krein duality, and so we have to define a tensor structure on the forgetful functor $F : \mathcal{M}_{\mathfrak{Dg}} \to \mathcal{A}$, without using the module M_+ (we keep the Drinfeld associator as in the first construction). For this, we set

$$\tilde{F}(V) = \mathsf{Hom}_{\mathcal{M}_{\mathfrak{Dg}}}(M_-, M_+^* \otimes V),$$

where we are considering the transformation of F into $\tilde{F}$ represented by the picture



,

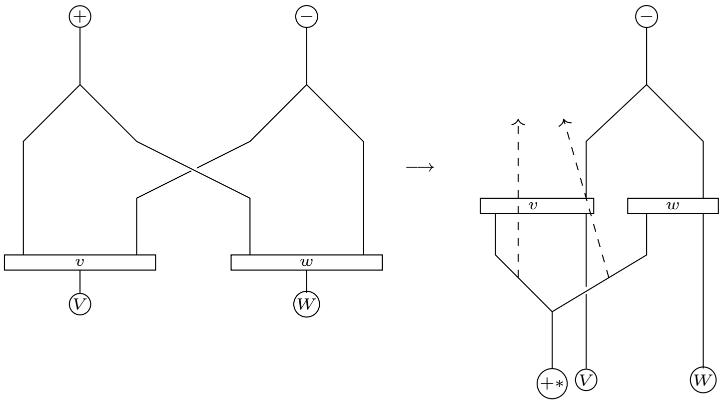
where M_+^* is the dual of M_+. We have that these two pictures are isotopic: in fact, the M_+–part is pulled around into the M_+^*–part. The tensor structure on $\tilde{F}$ is defined in the following way: given V, W in $\mathsf{Obj}(\mathcal{M}_{\mathfrak{Dg}})$, $v \in \tilde{F}(V)$, and $w \in \tilde{F}(W)$, we define $\tilde{J}_{V,W}(v \otimes w)$ as the composition

$$\begin{array}{l} M_- \xrightarrow{\;i_-\;} M_- \otimes M_- \xrightarrow{v \otimes w} (M_+^* \otimes V) \otimes (M_+^* \otimes W) \\ \xrightarrow{\;a\;} (M_+^* \otimes (V \otimes M_+^*)) \otimes W \xrightarrow{(\mathsf{id} \otimes \beta) \otimes \mathsf{id}} (M_+^* \otimes (M_+^* \otimes V)) \otimes W \\ \xrightarrow{\;a'\;} ((M_+^* \otimes M_+^*) \otimes V) \otimes W \xrightarrow{(i_+^* \otimes \mathsf{id}) \otimes \mathsf{id}} (M_+^* \otimes V) \otimes W \xrightarrow{\;\Phi\;} M_+^* \otimes (V \otimes W), \end{array}$$

where β denotes the braiding and a and a' denotes natural changes of bracketing

$$a : (\bullet\bullet)(\bullet\bullet) \to (\bullet(\bullet\bullet))\bullet$$
$$a' : (\bullet(\bullet\bullet))\bullet \to ((\bullet\bullet)\bullet)\bullet\,.$$

In terms of pictures, we are making the following transformation on the tensor structure defined in the Section 5.2:



Again, we have that these two pictures are isotopic, and so this new construction is formally equivalent to the first one. Indeed, the fact that $\tilde{J}$ defines a tensor structure on $\tilde{F}$ can be proved with the same pictorial technique that we used in 5.2.6, overturning the M_+–part into the M_+^*–part. As in the first construction, the identification $\tilde{\mathcal{U}}_\hbar(\mathfrak{Dg}) = \mathsf{End}_{\mathcal{M}_{\mathfrak{Dg}}}(\tilde{F})$ gives us a quantization of $\mathfrak{Dg}$. Furthermore, we have the analogue of the Theorem 5.3.1 in this new setting, that is:

Theorem 5.4.1. *The algebra $\tilde{F}(M_-)$ is a quantized universal enveloping algebra, whose quasi–classical limit is $\mathfrak{g}$.*

In fact, we transport the operations of $F(M_-)$ to $\tilde{F}(M_-)$ through the identifications $F \mapsto \tilde{F}$ and $J \mapsto \tilde{J}$. In order to prove that $\tilde{F}(M_-)$ is indeed a Hopf algebra, we shall the embedding m_+ to this new setting. In fact, we define the embedding of $\tilde{F}(M_-)$ into $\mathsf{End}_{\mathcal{M}_{\mathfrak{Dg}}}(\tilde{F})$ by

$$\tilde{m}_+ : \tilde{F}(M_-) \to \mathsf{End}_{\mathcal{M}_{\mathfrak{Dg}}}(\tilde{F})$$

that, given V in $\mathsf{Obj}(\mathcal{M}_{\mathfrak{Dg}})$, assigns to $x \in \tilde{F}(M_-)$ the endomorphism given by

$$\begin{aligned} \tilde{m}_+(x) : \tilde{F}(V) &\to \tilde{F}(V) \\ v &\mapsto (i_+^* \otimes \mathsf{id}) \circ \Phi \circ (\mathsf{id} \otimes v) \circ x. \end{aligned}$$

We define $\tilde{\mathcal{U}}_\hbar(\mathfrak{g})$ as the image of $\tilde{F}(M_-)$ through $\tilde{m}_+$. By analogy with the first construction, we have that:

- $\tilde{\mathcal{U}}_\hbar(\mathfrak{g})$ is a subalgebra of $\tilde{\mathcal{U}}_\hbar(\mathfrak{Dg})$;

- $\tilde{\mathcal{U}}_\hbar(\mathfrak{g})$ is a Hopf subalgebra of $\tilde{\mathcal{U}}_\hbar(\mathfrak{D}\mathfrak{g})$. In particular, we have

$$\Delta_\hbar(m(x)) = (m \otimes m)(\tilde{J}^{-1}_{M_-,M_-}(\mathsf{id} \otimes i_-) \circ x).$$

The next Theorem tells us that the result above is coherent with the construction of $\mathcal{U}_\hbar(\mathfrak{g})$.

Theorem 5.4.2. *The quantized universal enveloping algebras $\mathcal{U}_\hbar(\mathfrak{g})$ and $\tilde{\mathcal{U}}_\hbar(\mathfrak{g})$ are non canonically isomorphic.*

In fact, the correspondences $F \mapsto \tilde{F}$ and $J \mapsto \tilde{J}$ lead to a non canonical isomorphism $\mathcal{U}_\hbar(\mathfrak{g}) \simeq \tilde{\mathcal{U}}_\hbar(\mathfrak{g})$. However, the proof of this result is quite technical, and so we refer to [EK96] for more details.
This new construction solves the problem (2) of 5.4.1, since it not involves the universal Verma module M_+. However, the same cannot be said for the problem (1) of 5.4.1, because we have that M_+ is still an object of $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$. Therefore, in order to work with infinite–dimensional Lie bialgebras, we have to define a new Drinfeld category.

5.4.3 Drinfeld–Yetter modules

We now define the Drinfeld–Yetter category, which will take the place of the Drinfeld category in the new quantization construction.

Definition 5.4.3. *Let $(\mathfrak{g}, \delta)$ be a Lie coalgebra. We say that a vector space V is a Lie $\mathfrak{g}$–comodule if it is endowed with a linear map*

$$\pi^* : V \to \mathfrak{g} \otimes V$$

such that

$$(23) \circ (\mathsf{id} \otimes \pi^*) \circ \pi^* - (\mathsf{id} \otimes \pi^*) \circ \pi^* = (\delta \otimes \mathsf{id}) \circ \pi^*.$$

We have seen in 3.4 that if $\mathfrak{g}$ is a vector space with both the structures of a Lie algebra and of a Lie coalgebra, there is a compatibility relation (the cocycle condition) that controls if these two structures are compatible to the each other. Therefore, given a Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \delta)$ with a vector space that is both a Lie $\mathfrak{g}$–module and a Lie $\mathfrak{g}$–comodule, we expect that there is a further compatibility condition between action and coaction. In fact, we have the following

Definition 5.4.4. *Let $(\mathfrak{g}, [\cdot,\cdot], \delta)$ be a Lie bialgebra. A Drinfeld–Yetter $\mathfrak{g}$–module is a triple (V, π, π^*), where:*

- *the couple* (V, π) *is a Lie* $\mathfrak{g}$*–module;*
- *the couple* (V, π^*) *is a Lie* $\mathfrak{g}$*–comodule;*
- *the action and the coaction satisfy the following consistency condition:*

$$\pi^* \circ \pi - (\pi \otimes \mathsf{id}) \circ (12) \circ (\mathsf{id} \otimes \pi^*) = (\phi \otimes \mathsf{id}) \circ (\mathsf{id} \otimes \pi^*) - (\mathsf{id} \otimes \pi) \circ (\delta \otimes \mathsf{id}).$$

Moreover, the Drinfeld–Yetter category has a structure of tensor category. In fact, let $(\mathfrak{g}, [\cdot,\cdot], \delta)$ be a Lie bialgebra and let V, W be two Drinfeld–Yetter $\mathfrak{g}$–modules with actions given by

$$\pi_V : \mathfrak{g} \otimes V \to V \qquad \text{and} \qquad \pi_W : \mathfrak{g} \otimes W \to W,$$

and coactions given by

$$\pi_V^* : V \to \mathfrak{g} \otimes V \qquad \text{and} \qquad \pi_W^* : W \to \mathfrak{g} \otimes W.$$

Then we have that $V \otimes W$ is a Drinfeld–Yetter $\mathfrak{g}$–module, with action given by

$$\begin{aligned} \pi_{V\otimes W} : \mathfrak{g} \otimes (V \otimes W) &\to V \otimes W \\ x \otimes (v \otimes w) &\mapsto \pi_V(x \otimes v) \otimes w + v \otimes \pi_W(x \otimes w) \end{aligned}$$

and with coaction given by

$$\begin{aligned} \pi_{V\otimes W}^* : V \otimes W &\to \mathfrak{g} \otimes (V \otimes W) \\ v \otimes w &\mapsto \pi_V^*(v) \otimes w + (23)(\pi_W^*(w) \otimes v). \end{aligned}$$

Also, we may define the notions above in a topological setting, by considering topological Lie bialgebras $(\mathfrak{g}[[\hbar]], [\cdot,\cdot], \delta)$ and topologically free modules $V[[\hbar]]$. Henceforth, for any Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \delta)$, we denote by $\mathcal{DY}(\mathfrak{g})$ the category of Drinfeld–Yetter modules over the topologically free Lie bialgebra $(\mathfrak{g}[[\hbar]], [\cdot,\cdot], \hbar\delta)$. In order to define an associator for $\mathcal{DY}(\mathfrak{g})$, we first need to introduce a new Casimir operator. Before that, we do an important remark.

Remark 5.4.5. *Let us consider a Lie bialgebra* $(\mathfrak{g}, [\cdot,\cdot], \delta)$*.*

- *If* $(\mathfrak{g}, [\cdot,\cdot], \delta)$ *is of finite dimension, then the notion of a Drinfeld–Yetter* $\mathfrak{g}$*–module corresponds to the usual notion of a Lie* $\mathfrak{D}\mathfrak{g}$*–module, and so there is an equivalence of categories*

$$\mathcal{DY}(\mathfrak{g}) \simeq \mathcal{M}_{\mathfrak{D}\mathfrak{g}}.$$

- *If* $(\mathfrak{g}, [\cdot,\cdot], \delta)$ *is of infinite dimension, we have that the notion of a Drinfeld–Yetter* $\mathfrak{g}$*–module corresponds to the notion of equicontinuous* $\mathfrak{Dg}$*–module, in the sense of [EK96]. Therefore, there exists an equivalence of categories*
$$\mathcal{DY}(\mathfrak{g}) \simeq \mathcal{M}^e_{\mathfrak{Dg}},$$
where $\mathcal{M}^e_{\mathfrak{Dg}}$ *is the subcategory of* $\mathcal{M}_{\mathfrak{Dg}}$ *of all equicontinuous* $\mathfrak{Dg}$*–modules. In particular, whenever* $\dim \mathfrak{g} = \infty$, $\mathcal{DY}(\mathfrak{g})$ *identifies with a* proper *subcategory of* $\mathcal{M}_{\mathfrak{Dg}}$.

The Drinfeld–Yetter category allows us to define the following Casimir operator $\tilde{\Omega}$ in an arbitrary Lie bialgebra.

Definition 5.4.6. *Let* $(\mathfrak{g}, [\cdot,\cdot], \delta)$ *be a Lie bialgebra,* V *and* W *be two objects of* $\mathcal{DY}(\mathfrak{g})$*, and set*

$$r_{V,W} := (\pi_V \otimes \mathsf{id}_W) \circ (12) \circ (\mathsf{id}_V \otimes \pi^*_W).$$

We define the Casimir operator associated to V *and* W *as the element*

$$\tilde{\Omega}_{V,W} := r_{V,W} + \tau(r_{V,W}) \in \mathsf{End}(V \otimes W),$$

where $\tau(r_{V,W})$ *is the composition*

$$(12) \circ r_{W,V} \circ (12).$$

It is clear that $\tilde{\Omega}$ is a symmetric operator, i.e. that $\tilde{\Omega}_{V,W} = \tau(\tilde{\Omega}_{V,W})$. Moreover, it is easy to show that it is a morphism in $\mathsf{End}_{\mathcal{DY}(g)}(V \otimes W)$. Furthermore, the Casimir operator $\tilde{\Omega}$ allows us to extend the Drinfeld Theorem 5.2.2 to the case of the Drinfeld–Yetter category:

Theorem 5.4.7. *Let* $(\mathfrak{g}, [\cdot,\cdot], \delta)$ *be a Lie bialgebra. Then the Drinfeld–Yetter category* $\mathcal{DY}(\mathfrak{g})$ *is a braided tensor category, with associator given by*

$$\Phi_{KZ} = \Phi_{KZ}(\hbar \cdot \tilde{\Omega}_{12}, \hbar \cdot \tilde{\Omega}_{23}),$$

and whose R–matrix is given by

$$R = e^{\hbar \cdot \tilde{\Omega}/2}.$$

5.4.4 Verma modules and fiber functors for $\mathcal{DY}(\mathfrak{g})$

In order to extend the quantization to this new setting, we have to show that the Verma modules belong to $\mathcal{DY}(\mathfrak{g})$.

- We define $\tilde{M}_-$ as the object of $\mathcal{DY}(\mathfrak{g})$ whose vector space structure is given by
$$\tilde{M}_- := S(\mathfrak{g})[[\hbar]] = \bigoplus_{n\geq 0} S^n(\mathfrak{g})[[\hbar]],$$
whose Lie $\mathfrak{g}$–module structure is the usual action of $\mathfrak{g}$ on $S(\mathfrak{g})$ (as topologically free module), and whose Lie $\mathfrak{g}$–comodule structure is trivial on $S^0(\mathfrak{g}) = \mathbb{C}[[\hbar]]$. In fact, one can show that the assignment $\pi^*(1) = 0$ extends in a unique way the coaction on all the elements of $S(\mathfrak{g})$, by using the consinstency condition.

- We define $\tilde{M}_+^*$ as the object of $\mathcal{DY}(\mathfrak{g})$ whose vector space structure is given by
$$\tilde{M}_+^* := \prod_{n\geq 0} S^n(\mathfrak{g})[[\hbar]],$$
and whose Lie $\mathfrak{g}$–module structure and Lie $\mathfrak{g}$–comodule structure are respectively given by the previous coaction and action of $\tilde{M}_-$, by interchanging bracket and cobracket, reversing the order of composition and changing the signs.

One can show that the action and the coaction of $\tilde{M}_-$ and $\tilde{M}_+^*$ are well–defined, and also satisfy the consistency condition. Therefore, we have that $\tilde{M}_-$ and $\tilde{M}_+^*$ are two objects of $\mathcal{DY}(\mathfrak{g})$. Moreover, it is easy to show that, if $\mathfrak{g}$ is a finite–dimensional Lie bialgebra, the identification $\mathcal{DY}(\mathfrak{g}) \simeq \mathcal{M}^e_{\mathfrak{D}\mathfrak{g}}$ provide the identifications
$$\tilde{M}_- \mapsto M_- \qquad \text{and} \qquad \tilde{M}_+^* \mapsto M_+^*.$$
It is then easy to show that the results from Sections 5.1, 5.2 and 5.3 naturally extend to this case. We therefore have the following

Theorem 5.4.8. *Let $\tilde{F} : \mathcal{DY}(\mathfrak{g}) \to \mathcal{A}$ be the functor given by the assignment $\tilde{F}(V) := \mathsf{Hom}_{\mathcal{DY}(\mathfrak{g})}(\tilde{M}_-, \tilde{M}_+^* \otimes V)$ for any $V \in \mathcal{DY}(\mathfrak{g})$. Then, $\tilde{F}$ is a fiber functor with tensor structure given by the map $J_{VW} : \tilde{F}(V) \otimes \tilde{F}(W) \to \tilde{F}(V \otimes W)$*
$$J_{VW}(v \otimes w) := i_+^* \otimes \mathsf{id} \otimes \mathsf{id} \circ a^{-1} \circ \beta_{23} \circ a \circ v \otimes w \circ i_-$$
where as usual a denotes the change of bracketing $(\bullet\bullet)(\bullet\bullet) \to (\bullet(\bullet\bullet))\bullet$.

5.4.5 Functorial quantization of Lie bialgebras

Let us briefly recall the quantization of finite–dimensional Lie bialgebras as explained in Sections 5.1, 5.2 and 5.3. The procedure goes as follows. One first defines

- the category $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$;
- the Casimir operator Ω;
- the Drinfeld associator Φ;

These ingredients allows to define the Drinfeld category deforming $\mathcal{M}_{\mathfrak{D}\mathfrak{g}}$. Then, one introduces the universal Verma modules M_- and M_+^* and the forgetful functor

$$F(V) \coloneqq \mathsf{Hom}_{\mathcal{M}_{\mathfrak{D}\mathfrak{g}}}(M_-, M_+^* \otimes V)$$

The coalgebra structure i_- on M_- and the algebra structure i_+^* on M_+^* are then used to define the tensor structure $J_{VW} : F(V) \otimes F(W) \to F(V \otimes W)$. Finally, one shows that the object $F(M_-)$ is naturally endowed with a bialgebra structure with product $\mu : F(M_-) \otimes F(M_-) \to F(M_-)$

$$\mu(v \otimes v') \coloneqq i_+^* \otimes \mathsf{id} \circ \Phi^{-1} \circ \mathsf{id} \otimes v \circ v'$$

and coproduct $\Delta : F(M_-) \to F(M_-) \otimes F(M_-)$

$$\Delta(v) = J^{-1}_{M_-,M_-}(\mathsf{id} \otimes i_- \circ v)$$

Finally, one shows easily that it provides a quantization of the Lie bialgebra $\mathfrak{g}$. In Sections 5.4.3 and 5.4.4, we explained precisely how to obtain the same fundamental ingredients in the case of infinite–dimensional Lie bialgebras. First of all, we replace the category of representations by considering the Drinfeld–Yetter $\mathfrak{g}$–modules $\mathcal{DY}(\mathfrak{g})$. Here, we have a well–defined Casimir operator $\tilde{\Omega}$, which generalizes Ω and allows to obtain the analogue of the Drinfeld category, depending upon the choice of a universal associator Φ, as usual. This category is similarly endowed with two universal Verma modules $\tilde{M}_-, \tilde{M}_+^*$ and a forgetful functor

$$\tilde{F}(V) \coloneqq \mathsf{Hom}_{\mathcal{DY}(\mathfrak{g})}(\tilde{M}_-, \tilde{M}_+^* \otimes V)$$

Using the same formulas as before, we obtain a bialgebra

$$\tilde{\mathcal{U}_\hbar}(\mathfrak{g}) \coloneqq (\tilde{F}(\tilde{M}_-), \tilde{\mu}, \tilde{\eta}, \tilde{\Delta}, \tilde{\varepsilon})$$

which quantizes the arbitrary Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \delta)$.
Moreover, one can prove that the structure of Drinfeld–Yetter modules on

$\tilde{M}_-$ and $\tilde{M}_+^*$ are given by formulas which involve only the bracket $[\cdot,\cdot]$ and the (rescaled) cobracket $\hbar\delta$. This means that these formulas are *universal*, that is, they do not depend on the Lie bialgebra and are therefore functorial in the following sense.

Theorem 5.4.9 (Etingof–Kazhdan). *Let LBA and QUE denotes the categories of Lie bialgebras and quantum universal enveloping algebras. The assignment*

$$Q^{EK}(\mathfrak{g},[\cdot,\cdot],\delta) \coloneqq (\tilde{F}(\tilde{M}_-),\tilde{\mu},\tilde{\eta},\tilde{\Delta},\tilde{\varepsilon})$$

gives rise to a functor $Q^{EK} : LBA \to QUE$, which is adjoint to the quasi–classical limit functor $SC : QUE \to LBA$.

Remark 5.4.10. *As we pointed out in Section 5.3, it is clear that the functor Q^{EK} defined above (as the quantization $\tilde{\mathcal{U}}_\hbar(\mathfrak{g})$) depends upon the choice of an associator Φ.*

The functoriality of the quantization is a crucial feature, which allows to obtain the following

Theorem 5.4.11. *Let $(\mathfrak{g},[\cdot,\cdot],\delta)$ be a symmetrizable Kac–Moody algebra. Then, the Etingof–Kazhdan quantization $Q^{EK}(\mathfrak{g})$ is isomorphic, as a Hopf algebra, to the Drinfeld–Jimbo quantum group $\mathcal{U}_\hbar^{DJ}(\mathfrak{g})$.*

Very roughly, the proof goes as follows. One first prove that $Q^{EK}(\mathfrak{sl}_2) \simeq \mathcal{U}_\hbar^{DJ}(\mathfrak{sl}_2)$. Then, one relies on functoriality to show that $Q^{EK}(\mathfrak{g})$ is built out of copies of $\mathfrak{sl}_2$, as in the case of $\mathcal{U}_\hbar^{DJ}(\mathfrak{g})$.

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