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DIFFERENTIAL CALCULUS IN METRIC MEASURE SPACES

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To my family who always encouraged me to not give up in every situation:

"Are you frustrated to the fact that you are the weakest one?
If so, hold on to that feeling. That's proof that you haven't given up yet, on yourself.
Listen, don't give up saying that you're living within your boundaries.
Don't be a boring person like that.
If you have the guts to not give up, you can become anything you want to be.
We're living creatures that don't have wings, but still went to the moon."

- Chivalry of a Failed Knight

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Introduction

The aim of this thesis is the definition of the differential calculus' objects and the Laplace operator in metric measure spaces, following the presentation of [16]. It is well known that the classical weak definition of the condition $\Delta g = h$ is

$$\int \langle \nabla f, \nabla g \rangle \, \mathrm{d}x = \int f h \, \mathrm{d}x$$

for every test function f. Through the duality between tangent and cotangent space, the scalar product of the gradients can be expressed as the differential Df of f applied to the gradient ∇g of g. We will see that this definition can be extended to metric measure spaces, because it is expressed in terms of the properties of the measure and the metric of space.

In the first chapter we introduce the definitions and properties of metric measure spaces (in particular, $(X, \mathrm{d}, \mathfrak{m})$ is a metric measure space, (X, d) a Polish space and $\mathfrak{m}$ a non-negative Radon measure). In the second chapter we study the properties of Lipschitz functions and define the metric derivative for absolutely continuous curves.

In general, the norm does not derive from a scalar product and the Laplacian cannot be defined in the distributional sense as above. Therefore in the third chapter we define the concept of minimal p-weak upper gradient $|Dg|_w$ as in [4] and [15] and the Sobolev class S^p of the functions with finite minimal p-weak upper gradient.

In the fourth chapter we study the generalization of the concept of the differential of f applied to the gradient of g and this gives two objects:

$$D^+ f(\nabla g) := \inf_{\varepsilon > 0} \frac{|D(g + \varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon |Dg|_w^{p-2}}, \quad D^-(\nabla g) := \sup_{\varepsilon < 0} \frac{|D(g + \varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon |Dg|_w^{p-2}}$$

In general they are different, but if they agree the space will be called q-infinitely strictly convex because in the normed case it corresponds to the one in which the norm is strictly convex.

We will prove some chain rules for $D^\pm f(\nabla g)$, including the Leibniz rule but in a different way with respect to the Euclidian case: the tangent space is not available for the metric measure spaces (in the classical sense), so we'll study the duality between the space S^p and a suitable space of q-test plans, where q is the conjugate exponent of p. In particular, using the theory of optimal transport, with reference to [4] and [6], we'll be able to associate a transport plan to the gradient of a function in S^p, and the Leibniz rule will be a consequence of its validity on the real line.

Once $D^\pm f(\nabla g)$ has been defined, in the fifth chapter we define the p-Laplacian by saying that a function $g : X \to \mathbb{R}$ is in its domain, $g \in D(\boldsymbol{\Delta})$, if it belongs to the class S^p and if there is a Radon measure μ such that

$$-\int D^+ f(\nabla g) \mathrm{d}\mathfrak{m} \leq \int f d\mu \leq -\int D^- f(\nabla g) \mathrm{d}\mathfrak{m}$$

for every $L^1(X, \mathfrak{m})$ Lipschitz function f with support of finite $\mathfrak{m}$-measure. We then write $\mu \in \mathbf{\Delta} g$. The chain and Leibniz rules proved previously will be used to prove the analogous ones for $\mathbf{\Delta}$.

An important class of spaces are the so-called infinitesimally Hilbert spaces, i.e. those for which $W^{1,2}(X, d, \mathfrak{m})$ is a Hilbert space. In this case the Laplacian is single-valued and is linearly dependent on g. Furthermore the space is also 2-infinitesimally strictly convex, so $Df(\nabla g)$ is well defined and it can be proved that $Df(\nabla g) = Dg(\nabla f)$, i.e. an identification (duality) between differentials and gradients analogous to the one possible through the Riesz theorem.

In the last part of the fifth chapter we show an application of what we previously proved to the Heisenberg group, considering it as a metric measure space endowed with the Korany metric and the Lebesgue's measure. We show that on the whole space the metric Laplacian coincides with the sub-Laplacian that Hörmander in [17] proved to be an hypoelliptic operator. Then we consider the submanifold $\mathbb{X} = \{x = 0\}$: in this case with the differential approach we get the sub-Laplacian restricted to $\mathbb{X}$ and this is not an hypoelliptic operator. Hence we study what kind of operator we get if we apply the previous definitions to $\mathbb{X}$, using the Cheeger's energy functional as defined in [1].

Introduzione

L'obbiettivo di questa tesi è la definizione del calcolo differenziale e dell'operatore di Laplace in spazi metrici di misura, seguendo la presentazione di [16]. Come è ben noto la classica definizione debole della condizione $\Delta g = h$ è

$$\int \langle \nabla f, \nabla g \rangle \, \mathrm{d}x = \int f h \, \mathrm{d}x$$

per ogni funzione test f. Attraverso la dualità tra spazio tangente e cotangente, il prodotto scalare dei gradienti può essere reinterpretato come il differenziale Df di f applicato al gradiente ∇g di g. Vedremo che questa definizione si può estendere a spazi metrici di misura, perché è espressa in termine delle proprietà della misura e della metrica dello spazio.

Nel primo capitolo vengono introdotte le definizioni e proprietà principali degli spazi metrici di misura (in particolare, se $(X, \mathrm{d}, \mathfrak{m})$ è lo spazio metrico di misura, (X, d) è uno spazio polacco e $\mathfrak{m}$ una misura di Radon non negativa) mentre nel secondo quelle riguardanti le funzioni lipschitziane e la derivata metrica di curve assolutamente continue.

In generale la norma non deriva da un prodotto scalare e il laplaciano non si può definire in modo distribuzionale come sopra. Nel terzo capitolo quindi viene definito il concetto di p-supergradiente debole $|Dg|_w$ come in [4] e [15] attraverso l'utilizzo della derivata metrica e delle curve assolutamente continue e di conseguenza la classe di Sobolev S^p delle funzioni che hanno p-supergradente debole finito.

Nel quarto capitolo viene studiata la generalizzazione del concetto di differenziale di f applicato al gradiente di g che da luogo a due oggetti:

$$D^+ f(\nabla g) := \inf_{\varepsilon > 0} \frac{|D(g + \varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon |Dg|_w^{p-2}}, \quad D^-(\nabla g) := \sup_{\varepsilon < 0} \frac{|D(g + \varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon |Dg|_w^{p-2}}$$

In generale risultano diversi, ma se concincidono lo spazio verrà detto $q-$infinitesimamente strettamente convesso perché nel caso normato corrisponde al caso in cui la norma è strettamente convessa.

Vengono quindi dimostrate alcune regole della catena per $D^{\pm} f(\nabla g)$ fra cui anche la regola di Leibniz ma in modo diverso rispetto al caso normato: non avendo a disposizione lo spazio tangente per gli spazi metrici di misura, viene studiata la dualità fra lo spazio S^p e un opportuno spazio di misure dette q-piani test, dove q è l'esponente coniugato di p. In particolare mediante l'introduzione del funzionale energia di Cheeger, analogo a quello di Dirichlet nel caso euclideo, e lo studio del suo flusso-gradiente sarà possibile associare un piano di trasporto al gradiente di una funzione in S^p come in [4] e [6], e la regola di Leibniz sarà una conseguenza della sua validità sulla retta reale.

Una volta definito $D^{\pm} f(\nabla g)$ nel quinto capitolo viene definito il p-laplaciano affermando che una funzione $g : X \to \mathbb{R}$ è nel suo dominio, $g \in D(\mathbf{\Delta})$, se appartiene alla classe S^p ed esiste una misura di Radon μ tale che

$$-\int D^+f(\nabla g)\mathrm{d}\mathfrak{m} \leq \int f d\mu \leq -\int D^-f(\nabla g)\mathrm{d}\mathfrak{m}$$

per ogni funzione lipschitziana f sommabile con supporto di $\mathfrak{m}$-misura finita. Si scrive quindi $\mu \in \mathbf{\Delta} g$. Le regole della catena e di Leibniz provate precedentemente saranno usate per provare quelle per $\mathbf{\Delta}$.

Una classe importante di spazi sono quelli infitesimamente di Hilbert, cioè quelli per cui $W^{1,2}(X, d, \mathfrak{m})$ è uno spazio di Hilbert. In questo caso il laplaciano assume un solo valore e risulta linearmente dipendente da g. Inoltre lo spazio è anche 2-infinitesimamente strettamente convesso, per cui $Df(\nabla g)$ è ben definito e si dimostra che $Df(\nabla g) = Dg(\nabla f)$, cioè un'identificazione tra differenziali e gradienti analoga a quella possibile attraverso il teorema di Riesz.

Nell'ultima parte del quinto capitolo infine viene mostrata un'applicazione del calcolo differenziale in spazi metrici di misura al gruppo di Heisenberg, considerandolo uno spazio metrico di misura munito della metrica di Korany e la misura di Lebesgue. Nella prima parte si mostra che il laplaciano metrico coincide con quello subriemanniano che Hörmander in [17] ha mostrato essere un operatore ipoellittico. Viene poi considerata nella seconda parte la sottovarietà $\mathbb{X} = \{x = 0\}$: in questo caso attraverso l'approccio differenziale si ottiene il laplaciano subriemanniano ristretto a $\mathbb{X}$ e questo non è un operatore ipoellittico. Quindi viene applicata la teoria del calcolo in spazi metrici precedentemente sviluppata per studiare quale genere di operatore si ottiene, usando il funzionale energia di Cheeger definito come in [1].

Chapter 1

General measure theory and differentiation theorems

In this chapter we define and prove the essential tools from the Measure Theory we will use throughout the thesis, following [16]. We will define doubling spaces and state the Vitali's covering theorem, which we will use to introduce the Vitali spaces and to prove the main two results of the chapter: the Lebesgue's differentiation and the Radon-Nikodym's theorems. Finally, we will recall some notions from functional analysis regarding weak convergence of measures the duality between continuous functions and measures given by the Riesz theorem.

1.1 First definitions

We assume that $X \neq \emptyset$ and denote the power set of X as $\mathcal{P}(X)$.

Definition 1.1.1. A family of sets $\mathscr{S} \subseteq \mathcal{P}(X)$ is called *σ-algebra* over X if:

- $\emptyset, X \in \mathscr{S}$
- $A \in \mathscr{S} \Rightarrow A^c \in \mathscr{S}$
- $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{S} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathscr{S}$

Definition 1.1.2. A set function $\mathfrak{m} : \mathscr{S} \to [0, +\infty]$ is called *measure* if:

- $\mathfrak{m}(\emptyset) = 0$;
- $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{S}$ e $A_i \cap A_j = \emptyset, i \neq j \Rightarrow \mathfrak{m}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathfrak{m}(A_i)$ (*σ-additivity*).

If $\mathscr{S}$ isn't a σ-algebra $\mathfrak{m}$ is called *pre-measure*. La triple $(X, \mathscr{S}, \mu)$ is called *measure space*. If $\mathfrak{m}(X) = 1$ then X is called *probability space* and $\mathfrak{m}$ *probability measure*.

Remark 1.1.1. From the σ-additivity of $\mathfrak{m}$, if $A \subseteq B$ then using the De Morgan's relations we have

$$B = A \cup (A^c \cap B) \Rightarrow (A \cup B^c)^c \in \mathscr{S} \Rightarrow A^c \cap B \in \mathscr{S}$$

so $\mathfrak{m}(B) = \mathfrak{m}(A) + \mathfrak{m}(A^c \cap B) \Rightarrow \mathfrak{m}(B) \geq \mathfrak{m}(A)$. This property is called *monotonicity*.

Definition 1.1.3. Given any measure we can associate to it a σ-algebra whose sets are called *(Carathéodory-)measurable*: $A \in \mathscr{S}$ is said measurable if

$$\mathfrak{m}(T) = \mathfrak{m}(T \cap A) + \mathfrak{m}(T \setminus A), \qquad \forall T \in \mathscr{S}.$$

We say that A is $\mathfrak{m}$-*negligible* if $\mathfrak{m}(A) = 0$.

The collection of measurable sets $\mathscr{M}(\mathfrak{m})$ is a σ-algebra from definitions and the monotonicity property. Finally $\mathfrak{m}$ is *complete* over $\mathscr{M}(\mathfrak{m})$, i.e. all the $\mathfrak{m}$-negligible sets are $\mathfrak{m}$-measurable. The following theorem collects some properties of $\mathfrak{m}$, called *continuity of the measure* (see [7] or [2] for the proof):

Theorem 1.1.1. *Let $\{A_k\}_{k=1}^{+\infty}$ be a sequence of $\mathfrak{m}$-measurable sets. Then*

I) The sets $\bigcup_{k=1}^{+\infty} A_k, \bigcap_{k=1}^{+\infty} A_k$ are $\mathfrak{m}$-measurable.

II) If the A_k are pairwise disjoint then $\mathfrak{m}\Big(\bigcup_{k=1}^{+\infty} A_k\Big) = \sum_{k=1}^{+\infty} \mathfrak{m}(A_k)$

III) If $A_1 \subseteq \cdots \subseteq A_k \subseteq \dots$, then $\exists \lim_{k\to+\infty} \mathfrak{m}(A_k) = \mathfrak{m}\Big(\bigcup_{k=1}^{+\infty} A_k\Big)$

IV) If $A_1 \supseteq \cdots \supseteq A_k \supseteq \dots$ and $\mathfrak{m}(A_1) < +\infty$, then $\exists \lim_{k\to+\infty} \mathfrak{m}(A_k) = \mathfrak{m}\Big(\bigcap_{k=1}^{+\infty} A_k\Big)$

From now on if it is clear which measure we are referring to we call the $\mathfrak{m}$-measurable or $\mathfrak{m}$-negligible sets simply measurable or negligible. We consider also $\mathfrak{m}$ always restricted to $\mathscr{M}(\mathfrak{m})$.

Definition 1.1.4. We say that a function $f : X \to Y$ from a measure space $(X, \mathscr{S}, \mathfrak{m})$ to a measurable space $(Y, \mathscr{T})$ is *measurable* w.r.t. $\mathscr{M}(\mathfrak{m})$ if $f^{-1}(A) \in \mathscr{M}(\mathfrak{m})$ for any $A \in \mathscr{T}$.

Generally if X is a measurable space then the measurability is defined w.r.t. any σ-algebra $\mathscr{S}$ over X requiring just that $f^{-1}(A) \in \mathscr{S}$.

Proposition 1.1.2. *Let f, g be real measurable function over X. Then also $f + g$, fg, αf with $\alpha \in \mathbb{R}$ are measurable and $f \vee g := \max\{f, g\}$ e $f \wedge g := \min\{f, g\}$ are too. Moreover, composition of measurable function is measurable.*

Using the properties of $\mathfrak{m}$ we have also the following proposition.

Proposition 1.1.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of extended real-valued measurable functions. Then $\sup_{n\in\mathbb{N}} f_n$, $\inf_{n\in\mathbb{N}} f_n$, $\overline{\lim}_{n\in\mathbb{N}} f_n$ and $\underline{\lim}_{n\in\mathbb{N}} f_n$ are measurable.*

Proof. For example, $(\sup_{n\in\mathbb{N}} f_n)^{-1}((a, +\infty]) = \bigcup_{n\in\mathbb{N}} f^{-1}((a, +\infty]) \in \mathscr{M}(\mathfrak{m})$. □

1.2 Borel σ-algebra, Borel measures and restrictions

For every topological space X it can be defined a "natural" σ-algebra $\mathscr{B}$, called *Borel σ-algebra*, generated by open sets. More precisely, $\mathscr{B}$ is the smallest σ-algebra containing the open sets of X. In general, the smallest σ-algebra containing a fixed family of sets is the intersection of all the σ-algebras containing it. Notice that this intersection is surely not empty because $\mathcal{P}(X)$ contains it, so that the definition makes sense.

We consider now X as the base set of a metric space (X, d).

Definition 1.2.1. Let X be a non-empty set. A function $d : X \times X$ is called *distance* if

- $d(x,y) \geq 0 \; \forall x,y \in X$ e $d(x,y) = 0$ if and only if $x = y$;
- $d(x,y) = d(y,x) \; \forall x,y \in X$;
- $d(x,y) \leq d(x,z) + d(z,y) \; \forall x,y,z \in X$.

The couple (X,d) is called *metric space*.

Definition 1.2.2. A measure $\mathfrak{m}$ over X is said *Borel measure* if open sets are measurable. A Borel set is any set in a topological space that can be formed from open sets through operations of countable union, countable intersection and relative complement. We say that a measure is *regular* if for each measurable set A there exists a Borel set B such that $A \subset B$ and $\mathfrak{m}(A) = \mathfrak{m}(B)$. Finally, *Borel regular measure* is a Borel measure that is also regular.

We recall the following useful result.

Theorem 1.2.1 (Carathéodory's criterion)**.** *A measure $\mathfrak{m}$ over a metric space (X,d) is a Borel measure if and only if*

$$\mathfrak{m}(E_1 \cap E_2) = \mathfrak{m}(E_1) + \mathfrak{m}(E_2), \qquad \forall E_1, E_2 \in \mathscr{M}(\mathfrak{m}) \mid d(E_1.E_2) > 0.$$

Proof. Without loss of generality we can prove that all closed sets are measurable. Let C be a closed set and let us define

$$D_0 := \{x \in X \mid d(x,C) \geq 1\}, \quad D_n := \left\{x \in X \mid \frac{1}{2^n} \leq d(x,C) < \frac{1}{2^{n-1}}\right\}.$$

We want to prove that for every F

$$\mathfrak{m}(F) \geq \mathfrak{m}(F \cap C) + \mathfrak{m}(F \cap C^c),$$

so that we choose F such that $\mathfrak{m}(F) < +\infty$. We notice that D_i and D_j are disjoint if $i \neq j$, so that for an arbitrary index m, using the monotonicity property and the σ-additivity, we have

$$\mathfrak{m}(F) \geq \mathfrak{m}\left(F \cap \bigsqcup_{j=0}^{m} D_{2j}\right) = \sum_{j=0}^{m} \mathfrak{m}(F \cap D_{2j}).$$

The same inequality holds with $2j$ replaced by $2j+1$. Then the series in the right-hand side is convergent. From the fact that C and $\bigsqcup_{j=0}^{m} D_j$ are disjoint we have that $\forall m \geq 0$

$$\begin{aligned}\mathfrak{m}(F) &\geq \mathfrak{m}\left((F \cap C) \cup \left(F \cap \bigsqcup_{j=0}^{m} D_j\right)\right) = \mathfrak{m}(F \cap C) + \mathfrak{m}\left(F \cap \bigsqcup_{j=0}^{m} D_j\right) \\ &\geq \mathfrak{m}(F \cap C) + \mathfrak{m}(F \cap C^c) - \sum_{j>m} \mathfrak{m}(F \cap D_j),\end{aligned}$$

where the last inequality follows from the fact that $\mathfrak{m}(F \cap C^c) \leq \mathfrak{m}(F \cap \bigsqcup_{j=0}^{m} D_j) + \mathfrak{m}(F \cap \bigsqcup_{j>m} D_j)$. Taking the limit as $m \to +\infty$ we obtain the inequality, noticing that $\sum_j \mathfrak{m}(F \cap D_j) < +\infty$. $\square$

For every measure $\mathfrak{m}$ over a set X it can be defined another measure over any subset $Y \subset X$ simply restricting $\mathfrak{m}$ to Y. We denote it by $\mathfrak{m}_Y$. If $\mathfrak{m}$ is a Borel measure or Borel regular then $\mathfrak{m}_Y$ is too. In fact if $U \subset Y$ is open and $T \subset Y$ then $U = O \cap Y$ for some open set $O \subset X$. Hence

$$\begin{aligned}\mathfrak{m}_Y(T) = \mathfrak{m}(T) = \mathfrak{m}(T \cap O) + \mathfrak{m}(T \setminus O) &= \mathfrak{m}(T \cap U) + \mathfrak{m}(T \setminus U)\\ &= \mathfrak{m}_Y(T \cap U) + \mathfrak{m}_Y(T \setminus U),\end{aligned}$$

so that U is measurable. If $\mathfrak{m}$ is Borel regular let $E \subset Y$ and $B \subset X$ a Borel set containing E and such that $\mathfrak{m}(E) = \mathfrak{m}(B)$. Then $B \cap Y$ is a Borel set of Y, $E \subset B \cap Y$ and

$$\mathfrak{m}_Y(E) \leq \mathfrak{m}_Y(B \cap Y) \leq \mathfrak{m}(B) = \mathfrak{m}(E) = \mathfrak{m}_Y(E).$$

Another type of restriction can be defined: if $\mathfrak{m}$ is a measure over a set X and $Y \subset X$ then we define the $\mathfrak{m}$-*measure concentrated over* Y $\mathfrak{m} \llcorner Y$ in X as

$$\mathfrak{m} \llcorner Y(E) := \mathfrak{m}(E \cap Y), \qquad \forall E \subset X.$$

Lemma 1.2.2. *If $\mathfrak{m}$ is a Borel measure over a topological space X and $Y \subset X$ then $\mathfrak{m} \llcorner Y$ is a Borel measure over X. Moreover, if $\mathfrak{m}$ is Borel regular then also $\mathfrak{m} \llcorner Y$ is Borel regular if and only if Y admits a partition $Y = B_0 \cup N$ with B_0 a Borel set of X and N a negligible set.*

Proof. The first part follows proceeding exactly as before in the case of restrictions. Assume now that $\mathfrak{m}$ is Borel regular. If also $\mathfrak{m} \llcorner Y$ is Borel regular then there exists a Borel set $B \supset Z \setminus Y$ such that $\mathfrak{m} \llcorner Y(B) = \mathfrak{m} \llcorner Y(Z \setminus Y) = 0$ by definition. Putting $B_0 := Z \setminus Y$ and $N := Y \setminus B_0$ we obtain the direct implication. Vice versa, assuming that this partition exists, let $E \subset X$. Being $\mathfrak{m}$ Borel regular, we can pick the Borel sets $B \supset E \cap Y$ and $B' \supset N$ such that $\mathfrak{m}(B) = \mathfrak{m}(E \cap Y)$ and $\mathfrak{m}(B') = \mathfrak{m}(N) = 0$. Hence $B_1 := B' \cup B \cup (Z \setminus B_0)$ is a Borel set of X that contains E, so that

$$\begin{aligned}\mathfrak{m} \llcorner Y(E) \leq \mathfrak{m} \llcorner Y(B_1) &\leq \mathfrak{m} \llcorner Y(B') + \mathfrak{m} \llcorner Y(B) + \mathfrak{m} \llcorner Y(Z \setminus B_0)\\ &\leq \mathfrak{m}(B') + \mathfrak{m}(B) + \mathfrak{m}(N) = \mathfrak{m}(E \cap Y) = \mathfrak{m} \llcorner Y(E).\end{aligned}$$

□

Naturally the same conclusions hold if we consider extensions of measures: more precisely, if $\mathfrak{m}$ is a measure over $Y \subset X$ then it can be extended to a measure $\overline{\mathfrak{m}}$ over the whole X defining

$$\overline{\mathfrak{m}}(E) := \mathfrak{m}(E \cap Y) \qquad \forall E \subset X.$$

If $\mathfrak{m}$ is a measure over X then $\overline{\mathfrak{m}}_Y = \mathfrak{m} \llcorner Y$.

Lemma 1.2.3. *Let $\mathfrak{m}$ be a Borel measure over a subset Y of a topological space X. Then the extension $\overline{\mathfrak{m}}$ is a Borel measure over X. Moreover, if $\mathfrak{m}$ is Borel regular then $\overline{\mathfrak{m}}$ is too if and only if there exists a partition $Y = B_0 \cup N$ with B_0 a Borel set of X and N negligible.*

In the following we will use sometimes the following extension result for measurable functions:

Lemma 1.2.4. *If U is a measurable subset of X and $f : U \to [-\infty, \infty]$ is measurable then the extension of f given by $F : X \to [-\infty, \infty]$ with $F(x) = f(x)$ if $x \in U$ or 0 otherwise, is measurable.*

Proof. To prove the measurability of F it is sufficient to prove that for any $t \in \mathbb{R}$ the set $\{x \in X \mid F(x) > t\} \in \mathscr{M}(\mathfrak{m})$. Let $A \subset X$. Then if $E \subset U$ is $\mathfrak{m}_U-$measurable then we know that

$$\mathfrak{m}(A \cap U \cap E) + \mathfrak{m}((A \cap U) \setminus E) = \mathfrak{m}(A \cap U).$$

Hence, being $A \cap E = (A \cap U) \cap E$ and $A \setminus E = (A \setminus U) \cup ((A \cap U) \setminus E)$,

$$\begin{aligned} \mathfrak{m}(A) &\leq \mathfrak{m}(A \cap E) + \mathfrak{m}(A \setminus E) = \mathfrak{m}(A \cap U \cap E) + \mathfrak{m}((A \setminus U) \cup ((A \cap U) \setminus E)) \\ &\leq \mathfrak{m}(A \cap U \cap E) + \mathfrak{m}(A \setminus U) + \mathfrak{m}((A \cap U) \setminus E) = \mathfrak{m}(A \cap U) + \mathfrak{m}(A \setminus U) = \mathfrak{m}(A), \end{aligned}$$

so that E is $\mathfrak{m}-$measurable. This concludes the proof because if $t \geq 0$, then the set $\{x \in X \mid F(x) > t\}$ is equal to $\{x \in U \mid f(x) > t\}$ and if $t < 0$ is equal to $\{x \in U \mid f(x) > t\} \cap U^c$. Thus to prove that F is measurable it is enough to prove that if $E \subset U$ is $\mathfrak{m}_U-$measurable then it is also $\mathfrak{m}-$measurable. $\square$

Definition 1.2.3. We say that a function $f : Y \to Z$, with Y, Z topological spaces, is a *Borel function* if the preimage of every open set is a Borel subset of Y.

From this definition it follows that the preimage of every Borel set is a Borel set as well. Actually a more general result holds:

Theorem 1.2.5. *We have the following properties*

- *If $f : Y \to Z$ is a Borel function then the family $\{B \subset Z \mid f^{-1}(B) \in \mathscr{B}(Y)\}$ is a σ-algebra.*
- *Continuous functions and the composition of Borel functions are Borel functions.*

A measurable function f between to sets Y and Z and a measure $\mathfrak{m}$ over Y naturally induces a measure over Z:

Definition 1.2.4. We call *pushforward-measure of* $\mathfrak{m}$ *through* f the measure $f_\sharp\mathfrak{m}$ over Z defined by

$$f_\sharp\mathfrak{m}(E) := \mathfrak{m}(f^{-1}(E)) \qquad \forall E \subset Z.$$

For example, the extension $\overline{\mathfrak{m}}$ is the pushforward of $\mathfrak{m}$ through the inclusion map.

Proposition 1.2.6. *Let Y, Z topological spaces, $\mathfrak{m}$ a Borel measure over Y and $f : Y \to Z$ a Borel function. Then $f_\sharp\mathfrak{m}$ is a Borel measure over Z. If $f_\sharp\mathfrak{m}$ is also Borel regular then Y admits a partition $Y = B_0 \cup N$ with $f(B_0)$ a Borel set of Z and N $\mathfrak{m}-$negligible.*
Moreover, if $\mathfrak{m}$ is also Borel regular, f is a bijection between Y and its image and Y admits a partition $Y = B_0 \cup N$ as before, then $f_\sharp\mathfrak{m}$ is Borel regular.

Proof. By definition $f_\sharp\mathfrak{m}$ is a measure over Z. To prove that it is Borel regular if also $\mathfrak{m}$ is Borel regular, let $T \subset Z$ and $U \subset Z$ be an open set. Then

$$\begin{aligned} f_\sharp\mathfrak{m}(T) &\triangleq \mathfrak{m}(f^{-1}(T)) = \mathfrak{m}(f^{-1}(T) \cap f^{-1}(U)) + \mathfrak{m}(f^{-1}(T) \setminus f^{-1}(U)) \\ &= \mathfrak{m}(f^{-1}(T \cap U)) + \mathfrak{m}(f^{-1}(T \setminus U)) \triangleq f_\sharp\mathfrak{m}(T \cap U) + f_\sharp\mathfrak{m}(T \setminus U). \end{aligned}$$

Now we assume that $f_\sharp \mathfrak{m}$ is Borel regular. Then there exists a Borel set B of Z containing $Z \setminus f(Y)$ such that

$$\mathfrak{m}(f^{-1}(B)) \triangleq f_\sharp \mathfrak{m}(B) = f_\sharp \mu(Z \setminus f(Y)) = 0.$$

Choosing $B_0 := f^{-1}(Z \setminus B)$ and $N := f^{-1}(B)$ we get the thesis.
Finally we assume that $\mathfrak{m}$ is Borel regular, that $f : Y \to f(Y)$ is a bijection and that Y admits a partition $Y = B_0 \cup N$ as above. Let $E \subset Z$. Being $\mathfrak{m}$ Borel regular, then it exists a Borel set $B' \subset Y$ containing $f^{-1}(E)$ such that $\mathfrak{m}(B') = \mathfrak{m}(f^{-1}(E))$. But f is a bijection then $f(B') = B'' \cap f(Y)$ for some Borel set $B'' \subset Z$. Putting $B := (Z \setminus f(B_0)) \cup B''$ then B is a Borel set of Z, $E \subset B$ and

$$\begin{aligned} f_\sharp \mathfrak{m}(E) &\leq f_\sharp \mathfrak{m}(B) = \mathfrak{m}(f^{-1}(B)) \\ &\leq \mathfrak{m}(N) + \mathfrak{m}(B') = \mathfrak{m}(f^{-1}(E)) \triangleq f_\sharp \mathfrak{m}(E). \end{aligned}$$

□

Definition 1.2.5. We define *diameter* of a subset A of a metric space (X, d) the real number

$$\mathrm{diam}(A) := \sup\{d(x, y) \mid x, y \in A\}$$

We now state without proof the Vitali's covering theorem (for the proof we refer to [16]). It is a purely metric result that does not involve any measure. We recall that a topological space X is separable if it contains a dense and countable subset.

Theorem 1.2.7 (Vitali's covering theorem)**.** *Let (X, d) be a metric space. Then every family $\mathscr{F}$ of balls of X with uniformly bounded diameter has a subfamily $\mathscr{G}$ of pairwise disjoint balls such that for any $B \in \mathscr{F}$ exists $B' \in \mathscr{G}$ with $B \cap B' \neq \emptyset$ and $\mathrm{diam}(B) < 2\,\mathrm{diam}(B')$. Moreover, we have that*

$$\bigcup_{B \in \mathscr{F}} B \subset \bigcup_{B \in \mathscr{G}} 5B. \tag{1.1}$$

If X is also separable then $\mathscr{G}$ is necessarily countable.

Finally we define a property of topological spaces that will be fundamental for the following chapters.

Definition 1.2.6. We say that a topological space X has the *Lindelöf property* if every open cover admits a countable subcover. A separable and complete metric space is called *Polish space*.

Corollary 1.2.8. *Every separable metric space has the Lindelöf property.*

Proof. Let $\{U\} =: \mathscr{U}$ be an open cover of X. Fixing $k \in \mathbb{N}$, for every $x \in X$ we consider an open set $U \in \mathscr{U}$ and a ball $B_x := B(x, r_x)$ such that $r_x \leq k$ e $5B_x \subset U$. Then by the Vitali's covering theorem from the family $\mathscr{F} := \{B_x \mid x \in X\}$ we can extract a subcover $\mathscr{G} := \{B_i\}$ such that

$$X = \bigcup_{B_x \in \mathscr{F}} B_x \subset \bigcup_{B_i \in \mathscr{G}} 5B_i.$$

If for every i we choose an open set $U_i \in \mathscr{U}$ such that $5B_i \subset U_i$ then we obtain the subcover $\mathscr{U}$ we are looking for, taking the countable union for k. □

1.3 Integration in measure spaces

In this section we define integrals in measure spaces, i.e. integration w.r.t. any type of measure. Being classical results, we state them without proof.

Definition 1.3.1. Let $(X, \mathscr{S})$ be a measurable space. A measurable function $f : X \to \mathbb{R}$ is said *simple* if it has finite image, i.e. if there exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $A_1, \dots, A_n \in \mathscr{S}$ disjoint such that

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where $\chi_{A_i}(x) = 1$ if $x \in A_i$ and 0 otherwise (*characteristic function of* A_i). We will denote the family of real-valued simple functions with $\mathscr{S}_{\mathbb{R}}$.

Remark 1.3.1. $\mathscr{S}_{\mathbb{R}}$ is a $\mathbb{R}$-vector space.

The following lemma is the classical starting point for the integration in measure spaces.

Lemma 1.3.1. *Let $f : X \to \mathbb{R}$ a positive measurable function. Then there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathscr{S}_{\mathbb{R}^+}$ such that $\phi_n \nearrow f$ for $n \to \infty$.*

Definition 1.3.2. Let $\phi \in \mathscr{S}_{\mathbb{R}^+}$, $\phi = \sum_{k=1}^{n} a_k \chi_{A_k}$. We define

$$J_{\mathfrak{m}}(\phi) := \sum_{k=1}^{n} a_k \mathfrak{m}(A_k).$$

If f is positive and measurable, thanks to the preceding lemma we define the integral

$$\int f \, \mathrm{d}\mathfrak{m} := \sup\{J_{\mathfrak{m}}(\phi) \mid \phi \in \mathscr{S}_{\mathbb{R}^+}, \phi \to f\}.$$

The two integrals defined above are linear and satisfy the monotonicity property. We recall the following fundamental theorems:

Theorem 1.3.2 (Beppo Levi). *If $\{f_n\}_{n\in\mathbb{N}}$ is an increasing sequence of positive measurable functions and $f := \sup_{n\in\mathbb{N}} f_n$ then*

$$\int f \, \mathrm{d}\mathfrak{m} = \sup_{n\in\mathbb{N}} \int f_n \, \mathrm{d}\mathfrak{m}$$

Lemma 1.3.3 (Fatou). *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative measurable functions then*

$$\int \varliminf_{n\in\mathbb{N}} f_n \, \mathrm{d}\mathfrak{m} \le \varliminf_{n\in\mathbb{N}} \int f_n \, \mathrm{d}\mathfrak{m}.$$

Definition 1.3.3. A measurable function f is called *integrable* if $\int f_+ \, \mathrm{d}\mathfrak{m} < \infty$ and $\int f_- \, \mathrm{d}\mathfrak{m} < \infty$, with $f_+ := f \vee 0$ and $f_- := f \wedge 0$, and we write $f \in L^1(X, \mathfrak{m})$. We say that $f \in L^p(X, \mathfrak{m})$ if $|f|^p$ is integrable. The integral of f over a subset A of X is by definition the integral of $\chi_A f$.

Remark 1.3.2. $L^1(X, \mathfrak{m})$ is a $\mathbb{R}-$vector space.

Definition 1.3.4. We say that a property $\mathcal{P}$ holds $\mathfrak{m}$*−almost everywhere* ($\mathfrak{m}$*−a.e.*) if $\mathcal{N} := \{x \in X \mid \mathcal{P} \text{ is false}\}$ is negligible.

Directly from definitions, the following statements hold.

Proposition 1.3.4. *Let* $(X, \mathscr{S}, \mathfrak{m})$ *be a measure space and* $u, v \in L^1(X, \mathfrak{m})$*. Then* $u \wedge v$*,* $u \vee v \in L^1(X, \mathfrak{m})$*. Moreover, if* $u \leq v$ *then* $\int u \,\mathrm{d}\mathfrak{m} \leq \int v \,\mathrm{d}\mathfrak{m}$*. It always holds that* $\left|\int u \,\mathrm{d}\mathfrak{m}\right| \leq \int |u| \,\mathrm{d}\mathfrak{m}$*.*

Proposition 1.3.5. *If* $f \in L^1(X, \mathfrak{m})$ *and* $\int_A f \,\mathrm{d}\mathfrak{m} = 0$ *for every* $A \in \mathscr{S}$ *then* $f = 0$ $\mathfrak{m}$*−a.e..*

The last classical result is the following theorem.

Theorem 1.3.6 (Lebesgue's dominated convergence theorem)**.** *If* $\{f_n\}_{n \in \mathbb{N}}$ *is a sequence of measurable functions such that there exists* $g \in L^p(X, \mathfrak{m})$ *such that* $|f_n| \leq g$ *for every* $n \in \mathbb{N}$*,* $f_n \to f$ $\mathfrak{m}$*−a.e. as* $n \to \infty$*, then*

$$\int |f - f_n|^p \,\mathrm{d}\mathfrak{m} \overset{n\to\infty}{\longrightarrow} 0.$$

Theorem 1.3.7. *If* $f : Y \to Z$ *and* $h : Z \to \overline{\mathbb{R}}^+$ *are measurable and* $\mathfrak{m}$ *is a measure over* Y *then*

$$\int h \,\mathrm{d} f_\sharp \mathfrak{m} = \int h \circ f \,\mathrm{d}\mathfrak{m}.$$

We can also define another measure associated to any non-negative integrable function g.

Definition 1.3.5. We define $g_\mathfrak{m}$ with the following formula:

$$g_\mathfrak{m}(A) := \int_A g \,\mathrm{d}\mathfrak{m}.$$

We say that g is a *density* for $g_\mathfrak{m}$ w.r.t. $\mathfrak{m}$.

Proposition 1.3.8. *If* $f : X \to \mathbb{R}$ *is measurable then*

$$\int f \,\mathrm{d} g_\mathfrak{m} = \int f\, g \,\mathrm{d}\mathfrak{m}.$$

Hence f *is integrable w.r.t.* $g_\mathfrak{m}$ *if and only if* $f\, g$ *is integrable w.r.t.* $\mathfrak{m}$*.*

1.4 Metric measure spaces

Definition 1.4.1. A measure $\mathfrak{m}$ over a non-empty set X is said

- *σ-finite* if X can be decomposed in a countable union of measurable sets each with finite measure;
- *locally finite* if for every $x \in X$ there exists a neighbourhood of x with finite measure.

Definition 1.4.2. We call *metric measure space* the triple $(X, d, \mathfrak{m})$, where we assume that (X, d) is a separable metric space and $\mathfrak{m}$ is a locally finite Borel regular measure over X. Sometimes we'll denote by X the triple $(X, d, \mathfrak{m})$.

Being (X, d) separable then it has the Lindelöf property. Then the following two lemmas follow

Lemma 1.4.1. *Every metric measure space can be decomposed in a countable union of balls each with finite measure.*

Lemma 1.4.2. *Let E be a subset of a metric measure space X. If for every $x \in E$ there exists a neighbourhood U_x of x such that $\mathfrak{m}(E \cap U_x) = 0$ then $\mathfrak{m}(E) = 0$.*

Notice that the thesis of the second lemma is false if (X, d) is not separable: in fact, if X is not countable and not separable and endowed with the discrete distance function $d(x, y) = 1$ if $x \neq y$, then if we define $\mathfrak{m}$ over E as 0 if E is countable and ∞ if E is uncountable then locally $\mathfrak{m}$ is 0 but $\mathfrak{m}(E) = \infty$.

Definition 1.4.3. In a metric measure space $(X, d, \mathfrak{m})$ we define the *support* of $\mathfrak{m}$ as

$$\operatorname{supp}(\mathfrak{m}) := X \setminus \bigcup\{O \mid O \subset X \text{ open and } \mathfrak{m}(O) = 0\}.$$

Thanks to the Lemma 1.4.2 we have

$$\mathfrak{m}(X \setminus \operatorname{supp}(\mathfrak{m})) = 0.$$

A fundamental property of Borel regular measures is that they can be approximated using open and closed sets.

Proposition 1.4.3. *Let $(X, d, \mathfrak{m})$ be a metric measure space. Then*

$$\mathfrak{m}(A) = \sup\{\mathfrak{m}(C) \mid C \subset A, C \subset X \textit{ closed}\}, \tag{1.2}$$

$$\mathfrak{m}(E) = \inf\{\mathfrak{m}(O) \mid E \subset O, O \subset X \textit{ open}\} \tag{1.3}$$

for every $A, E \in \mathscr{M}(\mathfrak{m})$.

Proof. Notice that for every measurable set A of finite measure there exists two Borel sets such that $\mathfrak{m}(B') = \mathfrak{m}(A) = \mathfrak{m}(B)$. The existence of B follows from the Borel regularity of $\mathfrak{m}$. For the existence of B', there exists $B'' \subset X$ containing $B \setminus A$ such that $\mathfrak{m}(B'') = \mathfrak{m}(B \setminus A) = 0$, so it is enough to consider $B' := B \setminus B''$. Now we assume that $\mathfrak{m}(X) < \infty$ and the general case will follow from Lemma 1.4.1.
Assume that A is a Borel set of X and we consider the family $\mathscr{F}$ of over subsets of X for which (1.2) holds. It contains all open sets but also the closed ones because from the separability of X they can be decomposed into countable unions of open sets. Being $\mathfrak{m}(X) < \infty$ then the measure of these unions can be calculated using those open sets. Using the fact that $\mathscr{F}$ is $\cap-$closed and $\cup-$closed then the family

$$\mathscr{G} := \{A \in \mathscr{F} \mid X \setminus A \in \mathscr{F}\}$$

is a σ-algebra that contains all closed sets, so it must contain also all the Borel sets.
To prove (1.3), from the Borel regularity of $\mathfrak{m}$ we obtain a set $E_0 \supset E$ such that $\mathfrak{m}(E) = \mathfrak{m}(E_0)$. As before we assume that X has finite measure and fix $\varepsilon > 0$. Thanks to (1.2) we have a closed set $C \subset X \setminus E_0$ such that $\mathfrak{m}(C) > \mathfrak{m}(X \setminus E_0) - \varepsilon$. Hence $O = X \setminus C$ is open, contains E and satisfies $\mathfrak{m}(O) < \mathfrak{m}(E_0) + \varepsilon = \mathfrak{m}(E) + \varepsilon$. $\square$

Definition 1.4.4. A Borel measure $\mathfrak{m}$ over a metric space X is called *Radon measure* if $\mathfrak{m}(K) < \infty$ for every compact set $K \subset X$, (1.3) holds and

$$\mathfrak{m}(O) = \sup\{\mathfrak{m}(K) \mid K \subset O, K \subset X \text{ compact}\} \qquad \forall O \subset X \text{ open}. \tag{1.4}$$

Notice that a Radon measure is also Borel regular.

Proposition 1.4.4. *Let $(X, d, \mathfrak{m})$ be a metric measure space and $\mathfrak{m}$ a Radon measure. Then*

$$\mathfrak{m}(A) = \sup\{\mathfrak{m}(K) \mid K \subset A \textit{ compact}\} \qquad \forall A \subset X \textit{ measurable}. \tag{1.5}$$

Proof. We can assume, thanks again to Lemma 1.4.1, that $\mathfrak{m}(X) < \infty$. Let $A \subset X$ be a measurable set and fix $\varepsilon > 0$. Being $\mathfrak{m}$ Borel regular there exists a closed set $C \subset A$ and an open set $O \supset A$ such that $\mathfrak{m}(O \setminus C) < \varepsilon$ then thanks to (1.2). By definition of Radon measure, we can find a compact set $K \subset O$ such that $\mathfrak{m}(K) > \mathfrak{m}(O) - \varepsilon$, so that the compact set $K \cap C \subset A$ satisfies

$$\mathfrak{m}(A) \geq \mathfrak{m}(K \cap C) = \mathfrak{m}(K) - \mathfrak{m}(K \backslash C) > \mathfrak{m}(O) - 2\varepsilon \geq \mathfrak{m}(A) - 2\varepsilon.$$

□

If (X, d) is a Polish space then it has quite interesting properties.

Proposition 1.4.5. *Let $(X, d, \mathfrak{m})$ be a metric measure space with (X, d) a Polish space. Then $\mathfrak{m}$ is a Radon measure. Moreover, X can be decomposed in a countable union of compact sets plus a negligible set.*

Proof. We first observe that being $\mathfrak{m}$ locally finite then it is possible to cover every compact set with a finite number of balls each of finite measure, hence every compact set of X has finite measure.

Let now $A \subset X$ be a closed set such that $\mathfrak{m}(A) < \infty$ and $\varepsilon > 0$. Being X separable for every $n \in \mathbb{N}$ we can find a countable family of closed balls $\overline{B}_{n1}, \overline{B}_{n2}, \dots$ with centers in A and radii $\frac{1}{n}$ such that

$$A \subset \bigcup_{i=1}^{\infty} \overline{B}_{ni}.$$

We now choose i_n such that $\mathfrak{m}(A \cap C_n) > \mathfrak{m}(A) - \frac{\varepsilon}{2^n}$, with $C_n := \overline{B}_{n_1} \cup \cdots \cup \overline{B}_{ni_n}$. Denoting

$$K := \bigcap_{n=1}^{\infty} C_n,$$

and choosing $T \subset K$ we fix $\delta > 0$ such that $d(x, y) > \delta$ for every $x, y \in T$. Being $T \subset C_n$ for every n, if n satisfies $\frac{2}{n} < \delta$ then we can't find two different points of T both in a single $\overline{B}_{ni}$ which union is C_n. Hence T is finite and K is totally bounded and, being closed, it is also compact. Moreover, thanks to the continuity of $\mathfrak{m}$,

$$\mathfrak{m}(A \cap K) = \lim_{m \to \infty} \mathfrak{m}(A \cap C_1 \cap \cdots \cap C_m).$$

Hence

$$\begin{aligned}
\mathfrak{m}(A) &\leq \mathfrak{m}(A \cap C_1 \cap \cdots \cap C_m) + \sum_{n=1}^{m} \mathfrak{m}(A \setminus C_n) \\
&\leq \mathfrak{m}(A \cap C_1 \cap \cdots \cap C_m) + \varepsilon,
\end{aligned}$$

so $\mathfrak{m}(A \cap K) > \mathfrak{m}(A) + \varepsilon$.
Finally, we can cover X with a countable family of B_i each one of finite measure. Being $\mathfrak{m}$ a Radon measure, for every $i \in \mathbb{N}$ we can find a compact set $K_{i,j} \subset B_i$ such that $\mathfrak{m}(K_{i,j}) \geq \mathfrak{m}(B_i) - \frac{1}{j}$. Hence $\mathfrak{m}(B_i \setminus \bigcup_j K_{i,j}) = 0$ and we obtain the desired decomposition of X. □

1.5 Differentiation: Lebesgue's and Radon-Nikodym's theorems

Definition 1.5.1. A covering $\mathscr{B}$ of closed balls of a subset $A \subset X$ is called *fine* if

$$\inf\{r \mid r > 0, \overline{B}(x,r) \in \mathscr{B}\} = 0, \qquad \forall x \in A. \tag{1.6}$$

When $\mathfrak{m}$ is the Lebesgue's measure, a corollary of the Vitali's covering theorem ensures that from every fine covering $\mathscr{B}$ of a subset $A \subset \mathbb{R}^n$ made of closed balls it is possible to extract a subcover $\mathscr{C} \subset \mathscr{B}$ such that the measure of $A \setminus \bigcup_{B \in \mathscr{C}} B$ is 0 (see for instance [12]). We want to mimic this corollary to define a new class of metric measure spaces.

Definition 1.5.2. We say that a metric measure space $(X, d, \mathfrak{m})$ is a *Vitali space* if for every subset A of X and every covering $\mathscr{B}$ of A made of closed balls, for every $x \in A$ there exists a subcover $\mathscr{C} \subset \mathscr{B}$ of pairwise disjoint balls such that

$$\mathfrak{m}\left(A \setminus \bigcup_{B \in \mathscr{C}} B\right) = 0. \tag{1.7}$$

Theorem 1.5.1. *Let $(X, d, \mathfrak{m})$ be a metric measure space such that*

$$\mathcal{D}(x) := \varlimsup_{r \to 0} \frac{\mathfrak{m}(B(x,2r))}{\mathfrak{m}(B(x,r))} < \infty \tag{1.8}$$

for almost all $x \in X$. Then X is a Vitali space.

Proof. Let $A \subset X$ and let $\mathscr{B}$ be a fine subcover of A made of closed balls. We assume that the balls $\mathscr{B}$ has uniformly bounded radius, so we can apply the Vitali covering theorem repeatedly. Thanks to Lemma 1.4.1 we can decompose X as

$$X = \bigcup_{k=1}^{\infty} D_k, \quad D_k \text{ open}, \quad D_k \subset D_{k+1}, \quad \mathfrak{m}(D_k) < \infty \quad \forall k \in \mathbb{N}.$$

Put $A_k := \{x \in A \mid \mathcal{D}(x) < 2^k\} \cap D_k$. Clearly $A_k \subset A_{k+1}$ and $A = \bigcup_{k=1}^{\infty} A_k$. Using induction, we'll build a finite number of families of balls $\mathscr{C}_l \subset \mathscr{B}$ such that $\mathscr{C}_l \subset \mathscr{C}_{l+1}$ and that

$$\mathfrak{m}\left(A_k \setminus \bigcup_{B \in \mathscr{C}_l} B\right) \leq 2^{-l}\mathfrak{m}(A_k) \tag{1.9}$$

every time $1 \leq k \leq l$. Then the family

$$\mathscr{C} = \bigcup_{l=1}^{\infty} \mathscr{C}_l = \{B \mid B \in \mathscr{C}_l, l \in \mathbb{N}\}$$

will satisfy the Vitali condition (1.7).

To prove it, let $\mathscr{B}_1$ be the family of balls $\overline{B}(x,r) \in \mathscr{B}$ with $r \leq 1$ such that $x \in A, \overline{B}(x,r) \subset D_1$ and that

$$\mathfrak{m}(\overline{B}(x,5r)) \leq 2^3\,\mathfrak{m}(\overline{B}(x,r)). \tag{1.10}$$

We have that $\mathscr{B}_1$ is a fine covering of A_1 made of balls with uniformly bounded radius. Then using the Vitali's covering theorem we can extract from $\mathscr{B}_1$ a subcover $\mathscr{C}_1'$ made of pairwise disjoint balls such that if $B \in \mathscr{B}_1$ then it exists a ball $B' \in \mathscr{C}_1'$ such that $B \cap B' \neq \emptyset$ and that $\operatorname{diam}(B) < 2\operatorname{diam}(B')$. Enumerate the elements of $\mathscr{C}_1' = \{B_1^1, B_2^1, \dots\}$. Being the balls B_i^1 closed and the covering fine there exist a positive integer N such that for every $x \in A_1 \setminus (B_1^1 \cup \dots \cup B_N^1)$ there exists a ball $B \in \mathscr{B}_1$ with x as center that does not intersect $B_1^1 \cup \dots \cup B_N^1$. Hence this ball intersects another one $B_i^1 \in \mathscr{C}_1'$ for some $i > N$ such that $\operatorname{diam}(B) < 2\operatorname{diam}(B_i^1)$. In particular we have that

$$A_1 \setminus (B_1^1 \cup \dots \cup B_N^1) \subset \bigcup_{i \geq N+1} 5B_i^1.$$

From this and from (1.10) we have that

$$\mathfrak{m}(A_1 \setminus (B_1^1 \cup \dots \cup B_N^1)) \leq \sum_{i \geq N+1} \mathfrak{m}(5B_i^1) \leq 2^3 \sum_{i \geq N+1} \mathfrak{m}(B_i^1)$$

When $N \to \infty$, the right-hand side of this inequality tends to 0 being $\mathscr{C}_1'$ made of disjoint balls of a fixed subset of X, i.e. D_1, of finite measure. We pick N_1 such that

$$\mathfrak{m}(A_1 \setminus (B_1^1 \cup \dots \cup B_{N_1}^1) \leq \frac{1}{2}\mathfrak{m}(A_1)$$

and we set $\mathscr{C}_1 := \{B_1^1, \dots, B_{N_1}^1\}$. For the induction step, we assume that all the families $\mathscr{C}_1 \subset \dots \subset \mathscr{C}_l$ satisfy (1.9). Let $\mathscr{B}_{l+1}$ be a family of closed balls $\overline{B}(x,r)$ such that $x \in A_{l+1}, \overline{B}(x,r) \subset D_{l+1} \setminus \bigcup_{B \in \mathscr{C}_l} B$ and that

$$\mathfrak{m}(\overline{B}(x,5r)) \leq 2^{3(l+1)}\,\mathfrak{m}(\overline{B}(x,r)).$$

Consequently the family $\mathscr{B}_{l+1}$ is a fine covering of $A_{l+1} \setminus \bigcup_{B \in \mathscr{C}_l} B$ made of closed balls with uniformly bounded radius. By the Vitali's covering theorem we can extract a subcover $\mathscr{C}_{l+1}'$ of pairwise disjoint balls of $\mathscr{B}_{l+1}$ such that if $B \in \mathscr{B}_{l+1}$ and $B' \in \mathscr{C}_{l+1}'$ then $B \cap B' \neq \emptyset$ and such that $\operatorname{diam}(B) < 2\operatorname{diam}(B')$. Enumerate as before $\mathscr{C}_{l+1}' = \{B_1^{l+1}, B_2^{l+1}, \dots\}$ and reasoning in the same way we have that

$$A_{l+1} \Big\backslash \Bigg[\bigcup_{B \in \mathscr{C}_l} B \cup (B_1^{l+1} \cup B_2^{l+1} \cup \dots \cup B_N^{l+1}) \Bigg] \subset \bigcup_{i \geq N+1} 5B_i^{l+1}.$$

Hence

$$\mathfrak{m}\Bigg(A_{l+1} \Big\backslash \Bigg[\bigcup_{B \in \mathscr{C}_l} B \cup (B_1^{l+1} \cup B_2^{l+1} \cup \dots \cup B_N^{l+1}) \Bigg] \Bigg) \leq 2^{3(l+1)} \sum_{i \geq N+1} \mathfrak{m}(B_i^{l+1}). \tag{1.11}$$

As before the right-hand side of this inequality tends to 0 if $N \to \infty$, being $\mathscr{C}_{l+1}'$ made of pairwise disjoint balls of D_{l+1}. Now let $N_{l+1} \in \mathbb{N}$ such that when $N = N_{l+1}$ the expression in (1.11) is less or equal to $2^{-(l+1)}\mathfrak{m}(A_k)$ for every $1 \le k \le l+1$ for which $\mathfrak{m}(A_k) > 0$, i.e.

$$\sum_{i \ge N_{l+1}+1} \mathfrak{m}(B_i^{l+1}) \le 2^{-4(l+1)} \min\{\mathfrak{m}(A_k) \mid 1 \le k \le l+1, \mathfrak{m}(A_k) > 0\}.$$

Then the family

$$\mathscr{C}_{l+1} := \mathscr{C}_l \cup \{B_1^{l+1}, \ldots, B_{N_{l+1}}^{l+1}\}$$

completes the induction step of the proof. □

We now introduce a fundamental property for the two main theorems of this chapter.

Definition 1.5.3. A Borel regular measure over a metric space (X, d) is called *doubling measure* if every ball in X has positive finite measure and there exists $C \ge 1$ such that

$$\mathfrak{m}(B(x, 2r)) \le C\,\mathfrak{m}(B(x, r)) \tag{1.12}$$

for every $x \in X$ and $r > 0$.

The *doubling constant* $C_{\mathfrak{m}}$ is the minimal constant in (1.12). We notice that if we iterate this inequality then

$$\mathfrak{m}(B(x, \lambda r)) \le \lambda^{\log_2 C_{\mathfrak{m}}}\mathfrak{m}(B(x, r)),$$

where $\lambda \ge 1$ and the number $\log_2 C_{\mathfrak{m}}$ is sometimes called "dimension" of the metric measure space $(X, d, \mathfrak{m})$ if $\mathfrak{m}$ is a doubling measure.

The following theorem is the first fundamental one of this chapter:

Theorem 1.5.2 (Lebesgue's differentiation theorem)**.** *Let $(X, d, \mathfrak{m})$ be a Vitali space and $f : X \to \mathbb{R}$ be a locally integrable function. Then almost every $x \in X$ is a Lebesgue point, i.e.*

$$\lim_{r \to 0} \frac{1}{\mathfrak{m}(\overline{B}(x, r))} \int_{\overline{B}(x,r)} |f(y) - f(x)| \,\mathrm{d}\mathfrak{m}(y) = 0 \tag{1.13}$$

for almost every $x \in X$. Moreover,

$$\lim_{r \to 0} \frac{1}{\mathfrak{m}(\overline{B}(x, r))} \int_{\overline{B}(x,r)} f(y) \,\mathrm{d}\mathfrak{m}(y) = f(x) \tag{1.14}$$

for almost every $x \in X$.

Proof. We assume that f is non-negative, that $\mathfrak{m}(X) < \infty$ and that $\mathfrak{m}(B(x, r)) > 0$ for every $x \in X$ e $r > 0$. Let $c > 0$ and define

$$F_c := \left\{ x \in X \mid \overline{\lim_{r \to 0}} \frac{1}{\mathfrak{m}(\overline{B}(x, r))} \int_{\overline{B}(x,r)} f \,\mathrm{d}\mathfrak{m} > c \right\}.$$

We fix c and let O be an open set containing F_c. The family

$$\mathscr{B} := \left\{ \overline{B}(x, r) \subset O \mid x \in F_c, \frac{1}{\mathfrak{m}(\overline{B}(x, r))} \int_{\overline{B}(x,r)} f \,\mathrm{d}\mathfrak{m} > c \right\}$$

is a fine covering of F_c. By the Vitali's theorem we can extract a subcover $\mathscr{C}$ of balls of $\mathscr{B}$ that covers F_c almost everywhere. It follows that

$$c\,\mathfrak{m}(F_c) \le c \sum_{B \in \mathscr{C}} \mathfrak{m}(B) \le \sum_{B \in \mathscr{C}} \int_B f \, \mathrm{d}\mathfrak{m} \le \int_O f \, \mathrm{d}\mathfrak{m} \le \int_X f \, \mathrm{d}\mathfrak{m} < \infty,$$

so $\overline{\lim}_{r \to 0} \frac{1}{\mathfrak{m}(\overline{B}(x,r))} \int_{\overline{B}(x,r)} f \, \mathrm{d}\mathfrak{m} < \infty$.
Now, for $c > 0$ we define

$$E_c := \left\{ x \in X \mid \underline{\lim}_{r \to 0} \frac{1}{\mathfrak{m}(\overline{B}(x,r))} \int_{\overline{B}(x,r)} f \, \mathrm{d}\mathfrak{m} < c \right\}.$$

The set of points of X for which the limit (1.14) does not exist is contained in a countable union of sets $G_{s,t} := E_s \cap F_t$, with $s < t$ and $s, t \in \mathbb{Q}$. Being $\mathfrak{m}(G_{s,t}) \le \mathfrak{m}(X) < \infty$ we have that $\mathfrak{m}(G_{s,t}) = 0$. We fix $G_{s,t}$ and a Borel set A containing it such that $\mathfrak{m}(G_{s,t}) = \mathfrak{m}(A)$. For every open set O containing A, reasoning in the same way as before, we have that

$$t\mathfrak{m}(G_{s,t}) \le \int_O f \, \mathrm{d}\mathfrak{m} \le \int_A f \, \mathrm{d}\mathfrak{m} + \int_{O \setminus A} f \, \mathrm{d}\mathfrak{m}.$$

Taking the infimum over the open sets O, using the Borel regularity of $\mathfrak{m}$ and the continuity of the integral we obtain

$$t\mathfrak{m}(G_{s,t}) \le \int_A f \, \mathrm{d}\mathfrak{m}.$$

We fix then $\varepsilon > 0$ and choose $0 < \delta < \varepsilon$ such that $\int_H f \, \mathrm{d}\mathfrak{m} < \varepsilon$ for every measurable $H \subset X$ with $\mathfrak{m}(H) < \delta$. Thanks to the Borel regularity of $\mathfrak{m}$, we can find an open set $O \supset A$ such that $\mathfrak{m}(O) \le \mathfrak{m}(G_{s,t}) + \delta$. Using again the Vitali property of $\mathfrak{m}$ we can find a covering $\mathscr{C}$ made of pairwise disjoint balls in O such that

$$\int_B f \, \mathrm{d}\mathfrak{m} < s\,\mathfrak{m}(B),$$

for every $B \in \mathscr{C}$ and $\mathfrak{m}(G_{s,t} \setminus \bigcup_{B \in \mathscr{C}} B) = 0$. In particular we have that

$$\mathfrak{m}(G_{s,t}) \le \mathfrak{m}\Big(G_{s,t} \setminus \bigcup_{B \in \mathscr{C}} B \Big) + \mathfrak{m}\Big(\bigcup_{B \in \mathscr{C}} B \Big) = \mathfrak{m}\Big(\bigcup_{B \in \mathscr{C}} B \Big).$$

Hence

$$\mathfrak{m}\Big(A \setminus \bigcup_{B \in \mathscr{C}} B \Big) \le \mathfrak{m}(O) - \mathfrak{m}\Big(\bigcup_{B \in \mathscr{C}} B \Big) \le \mathfrak{m}(G_{s,t}) + \delta - \mathfrak{m}\Big(\bigcup_{B \in \mathscr{C}} B \Big) \le \delta,$$

so that

$$\int_A f \, \mathrm{d}\mathfrak{m} \le \int_{A \setminus \bigcup_{B \in \mathscr{C}} B} f \, \mathrm{d}\mathfrak{m} + \sum_{B \in \mathscr{C}} \int_B f \, \mathrm{d}\mathfrak{m} \le \varepsilon + s\mathfrak{m}(O) \le \varepsilon + s\mathfrak{m}(G_{s,t}) + s\delta.$$

Then if $\varepsilon \to 0$ we have that

$$\int_A f \, \mathrm{d}\mathfrak{m} \le s\mathfrak{m}(G_{s,t})$$

from which we get

$$t\mathfrak{m}(G_{s,t}) \leq s\mathfrak{m}(G_{s,t}). \tag{1.15}$$

Thus the limits in the thesis is finite $x-$a.e..

We want to prove now that the function in the left-hand side of (1.14) defined by

$$g(x) := \lim_{r\to 0} \frac{1}{\mathfrak{m}(\overline{B}(x,r))} \int_{\overline{B}(x,r)} f(y)\,\mathrm{d}\mathfrak{m}(y) \tag{1.16}$$

is measurable. We can rewrite this function as the pointwise limit of the sequence g_n defined by

$$g_n(x) := \frac{1}{\mathfrak{m}(\overline{B}(x,r))} \int_{\overline{B}(x,\frac{1}{n})} f\,\mathrm{d}\mathfrak{m}$$

so it is sufficient to prove that for fixed $\delta > 0$

$$u(x) := \mathfrak{m}(\overline{B}(x,\delta)), \qquad v(x) := \int_{\overline{B}(x,\delta)} f\,\mathrm{d}\mathfrak{m}$$

are measurable over $U_\delta = \{x \mid B(x,2\delta) \subset B\}$. Fix $x \in U_\delta$ and let $\{x_i\}_{i\in\mathbb{N}} \subset X$ be a sequence converging to x. Fix an open set O containing $\overline{B}(x,\delta)$. The balls $\overline{B}(x_i,\delta) \subset O$ for i large enough, so

$$\overline{\lim_{i\to\infty}}\, u(x_i) \leq \mathfrak{m}(O), \qquad \overline{\lim_{i\to\infty}}\, v(x_i) \leq \int_O f\,\mathrm{d}\mathfrak{m}.$$

Taking the infimum over the open sets O, we get $\overline{\lim}_{i\to\infty} u(x_i) \leq u(x)$ and $\overline{\lim}_{i\to\infty} v(x_i) \leq v(x)$. It follows that both u and v are upper semicontinuous, so that also g is measurable.

The last step consists in proving that $g = f$ $x-$a.e.. To this end, let $A \subset X$ be a measurable set and fix $t > 1$. A can be expressed, modulo a negligible set, as a disjoint union of the following measurable sets, for $n \in \mathbb{N}$:

$$A_n := A \cap \{x \in X \mid t^n \leq g(x) < t^{n+1}\}, \tag{1.17}$$

$$A_{-(n+1)} := A \cap \{x \in X \mid t^{-(n+1)} \leq g(x) < t^{-n}\}, \tag{1.18}$$

$$A_\infty := A \cap \{x \in X \mid g(x) = 0\}. \tag{1.19}$$

We notice that

$$\int_{A_\infty} f\,\mathrm{d}\mathfrak{m} = 0 = \int_{A_\infty} g\,\mathrm{d}\mathfrak{m}. \tag{1.20}$$

The reasoning made before can be used for every A of F_c or of E_c, hence observing that $A_n \subset \{g > s^n\}$ for every $s < t$ and letting $s \to t$ we have that

$$t^n\mathfrak{m}(A_n) \leq \int_{A_n} f\,\mathrm{d}\mathfrak{m}, \qquad \int_{A_n} f\,\mathrm{d}\mathfrak{m} \leq t^{n+1}\mathfrak{m}(A_n). \tag{1.21}$$

Hence

$$\frac{1}{t}\int_{A_n} f\,\mathrm{d}\mathfrak{m} \leq t^n\mathfrak{m}(A_n) \leq \int_{A_n} g\,\mathrm{d}\mathfrak{m} \leq t^{n+1}\mathfrak{m}(A_n) \leq t\int_{A_n} f\,\mathrm{d}\mathfrak{m}.$$

$$\frac{1}{t}\int_{A_{-(n+1)}} f\,\mathrm{d}\mathfrak{m} \leq \int_{A_{-(n+1)}} g\,\mathrm{d}\mathfrak{m} \leq t\int_{A_{-(n+1)}} f\,\mathrm{d}\mathfrak{m}.$$

Summing over n and using (1.20) together with (1.21) we get

$$\frac{1}{t}\int_A f\,\mathrm{d}\mathfrak{m} \le \int_A g\,\mathrm{d}\mathfrak{m} \le t\int_A f\,\mathrm{d}\mathfrak{m}.$$

So if $t \to 1$ we obtain

$$\int_A f\,\mathrm{d}\mathfrak{m} = \int_A g\,\mathrm{d}\mathfrak{m}.$$

Being A chosen arbitrarily, it follows that $f = g$ $x-$a.e.. □

We now introduce the definition of a derivative of a measure w.r.t. another one.

Definition 1.5.4. Let $(X, d, \mathfrak{m})$ be a metric measure space and $\mathfrak{h}$ a Borel regular locally finite measure over X. The *derivative of* $\mathfrak{h}$ *w.r.t.* $\mathfrak{m}$ *at a point* $x \in X$ is the limit, if it exists and it is finite,

$$\lim_{r\to 0}\frac{\mathfrak{h}(\overline{B}(x,r))}{\mathfrak{m}(\overline{B}(x,r))} =: \frac{\mathrm{d}\mathfrak{h}}{\mathrm{d}\mathfrak{m}}(x) \tag{1.22}$$

Remark 1.5.1. Being $\mathfrak{m}$ locally finite, we have that $\mathfrak{m}(\overline{B}(x,r)) > 0$ for every $x \in \mathrm{supp}(\mathfrak{m})$ and for any $r > 0$ so the limit in the definition exists $x-$a.e..

Theorem 1.5.3 (Lebesgue-Radon-Nikodym's theorem)**.** *Let* $(X, d, \mathfrak{m})$ *be a Vitali space and* $\mathfrak{h}$ *be a Borel regular locally finite measure over* X*. Then there exist and are univocally determined two Borel regular locally finite measures* $\mathfrak{h}^s$ *and* $\mathfrak{h}^a$ *over* X *such that*

$$\mathfrak{h}(A) = \mathfrak{h}^s(A) + \mathfrak{h}^a(A) \tag{1.23}$$

for every Borel subset $A \subset X$ *and that there exists* $D \subset X$ *such that* $\mathfrak{h}^s(D) = 0, \mathfrak{h}^a(D^c) = 0$ *and that* $\mathfrak{h}^a = \mathfrak{h} \llcorner D$*. Moreover, the derivatives of* $\mathfrak{h}$ *and* $\mathfrak{h}^a$ *w.r.t.* $\mathfrak{m}$ *exist almost everywhere over* X*, are* $\mathfrak{m}-$*measurable and locally integrable over* X*, with*

$$\mathfrak{h}^a(A) = \int_A \frac{\mathrm{d}\mathfrak{h}}{\mathrm{d}\mathfrak{m}}(x)\,\mathrm{d}\mathfrak{m}(x) = \int_A \frac{\mathrm{d}\mathfrak{h}^a}{\mathrm{d}\mathfrak{m}}(x)\,\mathrm{d}\mathfrak{m}(x) \tag{1.24}$$

for every Borel subset $A \subset X$*. In particular,*

$$\frac{\mathrm{d}\mathfrak{h}}{\mathrm{d}\mathfrak{m}} = \frac{\mathrm{d}\mathfrak{h}^a}{\mathrm{d}\mathfrak{m}} \qquad \mathfrak{m}\text{-}a.e.\ in\ X. \tag{1.25}$$

Proof. We first prove how to obtain the *Lebesgue's decomposition* (1.23). Let $E \subset X$ and we define

$$\mathfrak{h}^a(E) := \inf \mathfrak{h}(B),$$

where the infimum is taken over the Borel subsets $B \subset X$ such that $\mathfrak{m}(E \setminus B) = 0$. $\mathfrak{h}^a$ is a measure and $\mathfrak{h}^a(E) \le \mathfrak{h}(E)$ for every E thanks to the regularity of $\mathfrak{h}$ and $\mathfrak{h}^a(N) = 0$ for every N such that $\mathfrak{m}(N) = 0$. it is also locally finite. We have to prove that it is Borel regular: let $E_1, E_2 \subset X$ such that $d(E_1, E_2) > 0$. Let $B \subset X$ a Borel set such that $\mathfrak{m}((E_1 \cup E_2) \setminus B) = 0$ and let O_1, O_2 be two open set containing E_1, E_2 respectively such that $d(O_1, O_2) > 0$ and we put $B_1 := B \cap O_1$ e $B_2 := B \cap O_2$. Then $\mathfrak{m}(E_1 \setminus B_1) = \mathfrak{m}(E_2 \setminus B_2) = 0$ and so

$$\mathfrak{h}(B) \ge \mathfrak{h}(B_1 \cup B_2) = \mathfrak{h}(B_1) + \mathfrak{h}(B_2) \ge \mathfrak{h}^a(E_1) + \mathfrak{h}^a(E_2).$$

Taking the infimum over all the Borel sets B as before we can conclude, by Carathéodory's criterion, that $\mathfrak{h}^a$ is a Borel measure.
To prove that it is also regular, let $E \subset X$ and we assume that $\mathfrak{h}^a(E) < \infty$. Taking a decreasing sequence of Borel sets $B_1 \supset B_2 \supset \cdots$ such that $\mathfrak{m}(E \setminus B_j) = 0$ for every $j \in \mathbb{N}$ and that $\lim_{j\to\infty} \mathfrak{h}(B_j) = \mathfrak{h}^a(E)$. By the continuity of $\mathfrak{m}$ this sequence exists because if $\mathfrak{m}(E \setminus B') = \mathfrak{m}(E \setminus B'') = 0$ then $\mathfrak{m}(E \setminus (B' \cap B'')) = 0$. For the same reason there exists also a Borel set $B_0 \supset \bigcup_{j\in\mathbb{N}}(E\setminus B_j)$ such that $\mathfrak{m}(B_0) = 0$. Finally we put $B := (\bigcap_{j\in\mathbb{N}} B_j) \cup B_0$. Then $B \supset E$, is a Borel set and

$$\mathfrak{h}^a(E) \le \mathfrak{h}^a(B) \le \mathfrak{h}^a\left(\bigcap_{j\in\mathbb{N}} B_j\right) \le \mathfrak{h}^a(B_k) \le \mathfrak{h}(B_k) \qquad \forall k \in \mathbb{N}.$$

Hence $\mathfrak{h}^a(E) = \mathfrak{h}^a(B)$ and this proves the regularity of $\mathfrak{h}^a$. Moreover, setting $D = \bigcap_{j\in\mathbb{N}} B_j$, we have that D is a Borel set and $\mathfrak{m}(E\setminus D) = 0$. Hence $\mathfrak{h}^a(E) \le \mathfrak{h}^a(D) \le \lim_{j\in\mathbb{N}} \mathfrak{h}(B_j) = \lim_{j\in\mathbb{N}} \mathfrak{h}^a(B_j) = \mathfrak{h}^a(E)$, i.e. $\mathfrak{h}^a(E) = \mathfrak{h}(D)$.
Now, if E is a Borel set we set $D' := D \cap E$ where D is the same as before. Then also D' is a Borel set such that $\mathfrak{m}(E \setminus D') = \mathfrak{m}(E \setminus D) = 0$ and so $\mathfrak{h}^a(E) \le \mathfrak{h}(D') \le \mathfrak{h}(D) = \mathfrak{h}^a(E)$, i.e. $\mathfrak{h}^a(E) = \mathfrak{h}(D')$. Now we want to prove that

$$\mathfrak{h}^a(B) = \mathfrak{h}(B \cap D') \qquad \forall B \subset E \text{ Borel set.} \tag{1.26}$$

In fact, if $B \subset D$ is a Borel set then

$$\mathfrak{h}^a(B) + \mathfrak{h}^a(E\setminus B) = \mathfrak{h}^a(D) = \mathfrak{h}(D') = \mathfrak{h}(B \cap D') + \mathfrak{h}(D'\setminus B).$$

But being $\mathfrak{m}(B \setminus D') = 0$ and $\mathfrak{m}((E \setminus B) \setminus (D' \setminus B)) \le \mathfrak{m}(E \setminus D')) = 0$, then $\mathfrak{h}^a(B) \le \mathfrak{h}(B \cap D')$ and $\mathfrak{h}^a(D \setminus B) \le \mathfrak{h}(D' \setminus B)$, so we have (1.26).
Consider now the decomposition of X in a countable family of pairwise disjoint Borel sets $\{D_i\}_{i\in\mathbb{N}}$, possible thanks to Lemma (1.4.1), such that $\mathfrak{m}(D_i)$ and $\mathfrak{h}(D_i)$ are finite for every $i \in \mathbb{N}$. Applying the preceding construction we obtain the Borel sets $D_i' \subset D_i$ such that $\mathfrak{m}(D_i') = \mathfrak{m}(D_i)$ and $\mathfrak{h}^a(B) = \mathfrak{h}^a(B \cap D_i')$ for every $B \subset D_i$ Borel set. We set $D := \bigcup_{i\in\mathbb{N}} D_i'$, so $\mathfrak{m}(X \setminus D) = 0$ and fix $E \subset X$ and $B \supset E$ such that $\mathfrak{h}^a(E) = \mathfrak{h}^a(B)$. Then

$$\mathfrak{h}^a(E) = \sum_{i\in\mathbb{N}} \mathfrak{h}^a(B \cap D_i) = \sum_{i\in\mathbb{N}} \mathfrak{h}(B \cap D_i') = \mathfrak{h}(B \cap D) \ge \mathfrak{h}(E \cap D)$$

and $\mathfrak{h}(B \setminus D) = 0$. Hence $\mathfrak{h}^a(E \setminus D) \le \mathfrak{h}^a(B \setminus D) = 0$ and we get

$$\mathfrak{h}^a(E) \le \mathfrak{h}^a(E \cap D) + \mathfrak{h}^a(E \setminus D) = \mathfrak{h}^a(E \cap D) \le \mathfrak{h}(E \cap D).$$

So $\mathfrak{h}^a(E) = \mathfrak{h}(E \cap D)$ for every $E \subset X$ and then $\mathfrak{h}^a = \mathfrak{h} \llcorner D$.
As last step for the decomposition we set $\mathfrak{h}^s := \mathfrak{h} - \mathfrak{h}^a$. Then the preceding reasoning gives us $\mathfrak{h}^s(D) = 0$ and that $\mathfrak{h}^s$ is a Borel regular measure being so $\mathfrak{h}$ and $\mathfrak{h}^a$.
Assume now that $\mathfrak{m}(X)$ and $\mathfrak{h}(X)$ are finite and that $\mathfrak{m}(\overline{B}(x,r)) > 0$ for every $x \in X$ and for every $r > 0$. Let $c > 0$ and we define

$$E_c := \left\{x \in X \mid \varliminf_{r\to 0} \frac{\mathfrak{h}(\overline{B}(x,r))}{\mathfrak{m}(\overline{B}(x,r))} < c\right\}, \qquad F_c := \left\{x \in X \mid \varlimsup_{r\to 0} \frac{\mathfrak{h}(\overline{B}(x,r))}{\mathfrak{m}(\overline{B}(x,r))} < c\right\}.$$

We fix $c > 0$ and let $E_c' \subset E_c$ be any of his subsets. Then we fix $\varepsilon > 0$ and pick an open set $O \supset E_c'$ such that $\mathfrak{m}(O) \le \mathfrak{m}(E_c') + \varepsilon$. We can use a fine covering E_c' made of closed balls $\overline{B}(x,r) \subset O, x \in E'_c$, such that $\mathfrak{h}(\overline{B}(x,r)) < c\,\mathfrak{m}(\overline{B}(x,r))$ and the hypothesis of $\mathfrak{m}$ having the Vitali property permits us to conclude that there exists a countable family of pairwise disjoint balls $\mathscr{C} := \{B\}$ in O such that $\mathfrak{h}(B) < c\,\mathfrak{m}(B)$ for every $B \in \mathscr{C}$ and $\mathfrak{m}(E'_c \setminus \bigcup_{B\in\mathscr{C}} B) = 0$. Hence

$$\mathfrak{h}^a(E'_c) \leq \mathfrak{h}\Bigg(\bigcup_{B\in\mathscr{C}} B\Bigg) \leq c \sum_{B\in\mathscr{C}} \mathfrak{m}(B) \leq c\,\mathfrak{m}(O) \leq c\,\mathfrak{m}(E'_c) + c\,\varepsilon,$$

and letting $\varepsilon \to 0$ we conclude that

$$\mathfrak{h}^a(E'_c) \leq c\,\mathfrak{m}(E'_c). \tag{1.27}$$

Now we want to prove the opposite inequality, with F'_c in place of E'_c. To this aim, for every fixed $c > 0$, let $F'_c \subset F_c$ be any of its subsets and let $\varepsilon > 0$. We choose an open set $O \supset F'_c \cap D$ such that $\mathfrak{h}(O) \leq \mathfrak{h}(F'_c \cap D) + \varepsilon$, with D such that $\mathfrak{h}^a = \mathfrak{h} \llcorner D$, $D \subset X$ and $\mathfrak{m}(D^c) = 0$. Exactly as before, using the Vitali property and a fine covering we have that

$$c\,\mathfrak{m}(F'_c) = c\,\mathfrak{m}(F'_c \cap D) \leq \mathfrak{h}(O) \leq \mathfrak{h}(F'_c \cap D) + \varepsilon = \mathfrak{h}^a(F'_c) + \varepsilon.$$

Letting as before $\varepsilon \to 0$ we obtain

$$c\,\mathfrak{m}(F'_c) \leq \mathfrak{h}^a(F'_c). \tag{1.28}$$

Moreover,

$$c\,\mathfrak{m}(F_c) \leq \mathfrak{h}^a(F_c) \leq \mathfrak{h}(X) < \infty$$

for every $c > 0$ hence

$$\overline{\lim_{r\to 0}} \frac{\mathfrak{h}(\overline{B}(x,r))}{\mathfrak{m}(\overline{B}(x,r))} < \infty \qquad x\text{-a.e.}.$$

Arguing in the same way as in the Lebesgue's differentiation theorem, the set of points $x \in X$ such that (1.22) does not hold is contained in countable union of sets in the form $G_{s,t} := E_c \cap F_t$, where $s < t, s, t \in \mathbb{Q}$. So from (1.27), (1.28) and the assumption that $\mathfrak{m}(G_{s,t}) \leq \mathfrak{m}(X) < \infty$, we get that $\mathfrak{m}(G_{s,t}) = 0$ and the function $g(x) := \frac{\mathrm{d}\mathfrak{h}(x)}{\mathrm{d}\mathfrak{m}(x)}$ exists and is finite $\mathfrak{m}-$a.e.. Let then $A \subset X$ be a Borel set and fix $t > 1$. We define the sets A_n as in (1.17). Then, modulo a $\mathfrak{m}-$negligible set, A can expressed as the union of the A_ns. Using (1.27) and (1.28) in place of (1.15) and (1.21) we conclude that $\mathfrak{h}^a(A_\infty) = 0$ and that

$$\frac{1}{t}\,\mathfrak{h}^a(A) \leq \int_A g\,\mathrm{d}\mathfrak{m} \leq t\,\mathfrak{h}^a(A).$$

Letting $t \to 1$ we get the thesis.
The second equality in (1.24) follows from the first and from the Lebesgue's differentiation theorem. (1.25) follows directly from the preceding ones. □

Definition 1.5.5. We say that a measure $\mathfrak{h}$ over X is *absolutely continuous w.r.t.* $\mathfrak{m}$, and we write $\mathfrak{h} \ll \mathfrak{m}$, if for every measurable set $A \subset X$ such that $\mathfrak{m}(A) = 0$ we have that $\mathfrak{h}(A) = 0$. (1.23) is called *Lebesgue's decomposition* of $\mathfrak{h}$ in its *singular part* $\mathfrak{h}^s$ and *absolutely continuous part* $\mathfrak{h}^a$ w.r.t. $\mathfrak{m}$.

So now it is justified the name "absolutely continuous" for $\mathfrak{h}^a$. The function $\frac{\mathrm{d}\mathfrak{h}}{\mathrm{d}\mathfrak{m}}$ often is called *Radon-Nikodym's derivative* or *density of* $\mathfrak{h}$ *w.r.t.* $\mathfrak{m}$ and has the same properties of an ordinary derivative, for example the linearity and the chain rule. Historically the Radon-Nikodym's theorem and the Lebesgue's decomposition were two different results (in fact the Radon-Nikodym's theorem is usually stated for two measures one absolutely continuous w.r.t. the other), but they are closely related as the theorem shows.

Remark 1.5.2. In the proof we used coverings made of closed balls because the Lebesgue's differentiation theorem was stated for closed balls since it uses the Vitali property of the base space X. it is possible to use also open balls: if $(X, d, \mathfrak{m})$ is a metric measure space and $f : X \to \mathbb{R}$ is a locally integrable function then

$$\frac{1}{\mathfrak{m}(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|\,\mathrm{d}\mathfrak{m}(y) \le \frac{\mathfrak{m}(B(x,2r))}{\mathfrak{m}(B(x,r))}\frac{1}{\mathfrak{m}(\overline{B}(x,\frac{3r}{2}))}\int_{\overline{B}(x,\frac{3r}{2})} |f(y)-f(x)|\,\mathrm{d}\mathfrak{m}(y)$$

$x-$a.e. and for every $r > 0$ small enough. In particular, if $\mathfrak{m}$ is a doubling measure then we get

$$\lim_{r\to 0}\frac{1}{\mathfrak{m}(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|\,\mathrm{d}\mathfrak{m}(y) = 0 \tag{1.29}$$

$x-$a.e., hence

$$\lim_{r\to 0}\frac{1}{\mathfrak{m}(B(x,r))}\int_{B(x,r)} f(y)\,\mathrm{d}\mathfrak{m}(y) = f(x) \tag{1.30}$$

$x-$a.e.. Now using (1.25) and (1.30) we obtain

$$\frac{\mathrm{d}\mathfrak{h}}{\mathrm{d}\mathfrak{m}}(x) = \lim_{r\to 0}\frac{\mathfrak{h}(B(x,r))}{\mathfrak{m}(B(x,r))}, \tag{1.31}$$

$\mathfrak{m}-$a.e. $x \in X$ if $\mathfrak{m}$ is a doubling measure and if $\mathfrak{h}$ is a Borel regular locally finite measure over X. The same holds if $(X, \mathfrak{m})$ is a doubling space.

For future usage, we collect here some properties and definitions about signed and vector measures.

Definition 1.5.6. Let X be a set and $\mathscr{M} \subset \mathcal{P}(X)$ be a $\sigma-$algebra. A function $\mathfrak{m} : \mathscr{M} \to \mathbb{R}^n$ is called a *vector-valued measure* if $\mathfrak{m}$ is countably additive, in the sense that

$$\mathfrak{m}\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty}\mathfrak{m}(A_i), \quad A_i \cap A_j = \emptyset \quad \text{if} \quad i \neq j \quad \text{and} \quad A_i \in \mathscr{M}. \tag{1.32}$$

Moreover, given $\mathfrak{m}$ as above we define the function $|\mathfrak{m}| : \mathscr{M} \to [0, \infty)$ as

$$|\mathfrak{m}|(A) := \sup\left\{\sum_{i=1}^{\infty}\|\mathfrak{m}(A_i)\| : A = \bigsqcup_{i=1}^{\infty} A_i \quad A_i \in \mathscr{M}\right\}. \tag{1.33}$$

The function $|\mathfrak{m}|$ is called *variation of* $\mathfrak{m}$ and the quantity $|\mathfrak{m}|(X)$ is called *total variation of* $\mathfrak{m}$.

Proposition 1.5.4. *Let* X, $\mathscr{M}$ *and* $\mathfrak{m}$ *as before. Then the following hold:*

I) *Every infinite sum in* (1.32) *is absolutely convergent.*

II) *The variation* $|\mathfrak{m}|$ *is countably on* $\mathscr{M}$ *and therefore it is a measure itself.*

III) *The quantity* $|\mathfrak{m}|(X)$ *is finite, hence* $|\mathfrak{m}|$ *is a finite measure.*

1.6 Weak convergence of measures

Definition 1.6.1. Given a normed space V we say that a sequence $\{v_n\} \subset V$ *weakly converges* to $v \in V$ and we write $v_n \rightharpoonup v$ if for all $f \in V^*$ we have $f(v_n) \overset{n\to\infty}{\longrightarrow} f(v)$.

We distinguish this type of convergence from the usual one on V simply calling the latter *strong convergence*. The first properties of the weak convergence are the following. Denote with $J : X \to X^{**}$ the classic isometry from V to its bidual V^{**}, namely $J_V(v)(f) := f(v)$ for all $v \in V, f \in V^*$.

Theorem 1.6.1. *If $v_n \rightharpoonup v$ in the normed space V then $\{\|v_n\|_V\}_{n\in\mathbb{N}}$ is bounded and $\underline{\lim}_{n\to\infty} \|v_n\|_V \geq \|v\|_V$.*

Proof. Thanks to the Hahn-Banach theorem there exists $f \in V^*$ such that $\|f\|_{V^*} = 1$ and that $f(v) = \|v\|_V$. But being $v_n \rightharpoonup v$ in V we have that $f(v_n) \to f(v)$. So we can conclude that

$$\|v\|_V = f(v) = \lim_{n\to\infty} f(v_n) = \underline{\lim}_{n\to\infty} f(v_n) \leq \underline{\lim}_{n\to\infty} \|f\|_{V^*}\|v_n\|_V = \underline{\lim}_{n\to\infty} \|v_n\|_V.$$

Being $v_n \rightharpoonup v$ in V then $J_V(v_n)(f) \to J_V(v)(f)$ for every $f \in V^*$. Thus $\{J_V(v_n)\}_{n\in\mathbb{N}}$ converges pointwise to $J_V(v)$ in V^*. Hence

$$\sup_{n\in\mathbb{N}} |J_V(v_n)(f)| < \infty \qquad \forall\, f \in V^*.$$

Finally, using the uniform boundedness principle we can conclude:

$$\sup_{n\in\mathbb{N}} \|v_n\|_V = \sup_{n\in\mathbb{N}} \|J_V(v_n)\|_{V^{**}} < \infty.$$

□

A tool that sometimes we will use is the following lemma. Recall that the *convex hull* of a subset A of V is the intersection of all convex sets of V containing A.

Lemma 1.6.2 (Mazur). *Let $\{v_n\}_{n\in\mathbb{N}} \subset V$ be a normed space and $v_n \rightharpoonup v \in V$. Then v belongs to the convex hull of the sequence $\{v_n\}_{n\in\mathbb{N}}$.*

Proof. Denote with H the convex hull of $\{v_n\}_{n\in\mathbb{N}}$. We can replace $\{v_n\}$ with $\{v_n - h\}$, for $h \in H$ if $0 \notin H$, so we assume that this is not the case. Suppose that there exists $\varepsilon > 0$ such that

$$\|v - w\|_V > 2\varepsilon \qquad \forall\, w \in H.$$

Thus $v \neq 0$. If $\|a - a'\|_V < \varepsilon$ and $\|b - b'\|_V < \varepsilon$ then we would have that $\|(ta + (1 - t)b) - (ta' + (1-t)b'\|_V < \varepsilon$ if $a, a', b, b' \in V$ and $t \in [0, 1]$, so

$$H_\varepsilon := \{w \in V \mid (w, H) < \varepsilon\} \subset H$$

is convex and is an open set containing $0 \in V$. So we define the functional (called *Minkowski functional*)

$$|w|_\varepsilon := \inf\{\lambda > 0 \mid \lambda^{-1}w \in H_\varepsilon\} \qquad w \in V.$$

Applying the Hahn-Banach theorem to the linear map $tv \mapsto t|v|_\varepsilon$ there exists $\overline{v}^* : V \to \mathbb{R}$ such that $\overline{v}^*(v) = |v|_\varepsilon$ e $\overline{v}^*(w) \leq |w|_\varepsilon$ for all $w \in V$. So we get

$$1 < |v|_\varepsilon = \lim_{n\to\infty} \overline{v}^*(v_n) \leq \underline{\lim}_{n\to\infty} |v_n|_\varepsilon \leq 1$$

that is a contradiction. □

Thanks to the Mazur Lemma, if $v_n \rightharpoonup v$ in V then there exists a sequence $\{\overline{v}_k\}$ of convex combinations

$$\overline{v}_k = \sum_{n=1}^{m_k} \lambda_{n,k} v_n, \qquad \lambda_{n,k} \geq 0, \qquad \lambda_{k,k} + \cdots + \lambda_{m_k,k} = 1$$

strongly convergent to v in V (slightly more informally, we can extract from a weakly convergent sequence a strongly convergent one).

Proposition 1.6.3. *Let $(X, \mathfrak{m})$ be a measure space with $\mathfrak{m}$ a σ-finite measure and $p \in [1, \infty)$. If a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L^p(X, \mathfrak{m})$ weakly converges to $f \in L^p(X, \mathfrak{m})$ and*

$$\lim_{n\to\infty} f_n(x) = g(x) \qquad x - a.e.$$

then $g = f$ almost everywhere.

Proof. From the Mazur Lemma there exist a (sub-)sequence $\{\overline{f}_k\}$ made of convex combinations of f_n convergent to f in $L^p(X, \mathfrak{m})$. We can assume that this convergence is pointwise almost everywhere X. Being $\overline{f}_k \to g$ almost everywhere then by the assumptions $f = g$ almost everywhere. □

If X is a topological space, then $C_c(X, \mathbb{R}^n)$ denotes the space of all vector-valued functions with compact support in X, normed with

$$\|f\|_\infty := \sup\{\|f(x)\|_{\mathbb{R}^n} : x \in X\}.$$

The completition of $C_c(X, \mathbb{R}^n)$ w.r.t. the above norm is the Banach space $C_0(X, \mathbb{R}^n)$ of all continuous functions vanishing at infinity, i.e. of all continuous f such that for every $\varepsilon > 0$ there exists a compact set K such that $\|f(x)\| < \varepsilon$ whenever $x \in X \setminus K$.

The following two theorems provide an useful link between linear functionals on $C_0(X, \mathbb{R}^n)$ and measures. For the proof see [2] and [7].

Theorem 1.6.4 (Riesz). *Suppose that X is a locally compact Hausdorff topological space and let L be a positive linear functional on $C_0(X, \mathbb{R})$. Then there exists a unique Borel measure $\mathfrak{m} : \mathscr{B}(X) \to [0, \infty]$, finite on compact sets, such that*

$$L(f) = \int_X f \, \mathrm{d}\mathfrak{m} \qquad \forall f \in C_0(X, \mathbb{R}).$$

If L is a bounded linear functional on $C_0(X, \mathbb{R}^n)$ then there exists a vector-valued measure $\mathfrak{m} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_n) : \mathscr{B}(X) \to \mathbb{R}^n$ such that

$$L(f) = \int_X f \, \mathrm{d}\mathfrak{m} := \sum_{i=1}^n \int_X f_i \, \mathrm{d}\mathfrak{m}_i, \qquad \forall f = (f_1, \ldots, f_n) \in C_0(X, \mathbb{R}^n).$$

Moreover, there holds

$$\|L\|_{C_0(X,\mathbb{R}^n)} = |\mathfrak{m}|(X).$$

Corollary 1.6.5. *The vector space $\mathcal{M}(X, n)$ of all vector-valued measures $\mathfrak{m} : \mathscr{B}(X) \to \mathbb{R}^n$, endowed with norm $\|\mathfrak{m}\| := |\mathfrak{m}|(X)$, is a Banach space.*

Definition 1.6.2. Given V a normed space we say that a sequence of functionals $\{f_n\} \subset V^*$ *converges weakly** to $f \in V^*$, writing $f_n \overset{*}{\rightharpoonup} f$, if for every $v \in V$ we have $f_n(v) \overset{n\to\infty}{\longrightarrow} f(v)$.

In V^* weak convergence implies the weakly* one by definition but in general the converse does not hold.

Applying the definition on the space $\mathcal{M}(X,n)$ we can define the weak* convergence in this space, induced by $C_0(X,\mathbb{R}^n)$. Given a sequence $\{\mathfrak{m}_k\}$ in $\mathcal{M}(X,n)$ we have

$$\mathfrak{m}_k \overset{*}{\rightharpoonup} \mathfrak{m} \qquad \Leftrightarrow \qquad \lim_{k\to\infty} \int_X f \, \mathrm{d}\mathfrak{m}_k = \int_X f \, \mathrm{d}\mathfrak{m} \quad \forall f \in C_0(X,\mathbb{R}^n).$$

Remark 1.6.1. We remark that if the space X is separable then the space $C_0(X,\mathbb{R}^n)$ is itself separable, hence the weak* topology restricted to bounded sets of $\mathcal{M}(X,n)$ can be proven to be metrizable. In particular, from any sequence $\{\mathfrak{m}_k\} \subset \mathcal{M}(X,n)$ with equibounded total variations one can extract a subsequence $\{\mathfrak{m}_{j_k}\}$ such that $\mathfrak{m}_{j_k} \overset{*}{\rightharpoonup} \mathfrak{m}$ for some $\mathfrak{m} \in \mathcal{M}(X,n)$ (Banach-Alaoglu theorem).

Now we'll show some properties concerning the weak* convergence of measures, and remand to the Definition 2.1.4 for lower/upper semicontinuous functions and to Proposition 2.7 for the approximation from below by Lipschitz function of a lower semicontinuous function.

Proposition 1.6.6. *If $\{\mathfrak{m}_h\}$ is a sequence of Radon measures on the locally compact, separable metric space X, such that $\mathfrak{m}_h \overset{*}{\to} \mathfrak{m}$, then*

I) If the measures $\mathfrak{m}_h$ are positive, then for every lower semicontinuous function $u : X \to [0,\infty]$

$$\varliminf_{h\to\infty} \int_X u \, \mathrm{d}\mathfrak{m}_h \geq \int_X u \, \mathrm{d}\mathfrak{m}$$

and for every upper semicontinuous function $v : X \to [0,\infty)$ with compact support

$$\varlimsup_{h\to\infty} \int_X v \, \mathrm{d}\mathfrak{m}_h \leq \int_X v \, \mathrm{d}\mathfrak{m}.$$

II) If $|\mathfrak{m}_h|$ locally weakly converges to $\mathfrak{h}$, then $\mathfrak{h} \geq |\mathfrak{m}|$. Moreover, if X is relatively compact and $\mathfrak{h}(\partial X) = 0$ then $\mathfrak{m}_h(X) \to \mathfrak{m}(X)$ as $h \to \infty$. More generally,*

$$\int_X u \, \mathrm{d}\mathfrak{m} = \lim_{h\to\infty} \int_X u \, \mathrm{d}\mathfrak{m}_h$$

for any bounded Borel function $u : X \to \mathbb{R}$ with compact support such that the set of its discontinuity points is $\mathfrak{h}-$negligible.

Chapter 2

Lipschitz functions and curves in metric spaces

We prove in this chapter the density property of Lipschitz functions in the L^p spaces, studying their properties and the connection with lower semicontinuous functions. Another tool that we introduce are the Lipschitz partitions of unity, for which the properties of doubling spaces studied Chapter 1 will be useful. Finally, we will study the metric derivative of asbolutely continuous curves in a metric measure space, that is the key tool we will use to introduce the weak upper gradients in the following chapter.

2.1 Lipschitz functions

Definition 2.1.1. Let(X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is called *L-Lipschitz* if there exists a constant $L \geq 0$ such that

$$d_Y(f(a), f(b)) \leq L\, d_X(a, b) \qquad \forall a, b \in X. \tag{2.1}$$

The smallest L for which (2.1) holds is called *Lipschitz constant*.

Lipschitz functions will be the substitute of test functions in the Euclidian spaces (smooth with compact support). The first result on Lipschitz functions is the following extension theorem:

Theorem 2.1.1 (Whitney-Mc. Shane extension theorem)**.** *Let (X, d) be a metric space, $A \subset X$ and $f : A \to \mathbb{R}$ a $L-$Lipschitz function. Then there exists a $L-$Lipschitz function $F : X \to \mathbb{R}$ such that $F|_A = f$.*

Proof. Assume $A \neq \emptyset$. Given $x \in X$ we define the function

$$F(x) = \inf\{f(a) + L\, d(a, x) \mid a \in A\}. \tag{2.2}$$

Fixing a point $a_0 \in A$ we have that

$$f(a) + L\, d(a, x) \geq f(a) + L\, d(a, a_0) - L\, d(a_0, x) \geq f(a_0) - L\, d(a_0, x), \tag{2.3}$$

so $F(x) > -\infty$ for all $x \in X$. The function $x \mapsto f(a) + L\, d(a, x)$ is L-Lipschitz for fixed $a \in A$, then F is the pointwise infimum of a family of L-Lipschitz functions, hence also F is. In fact, if $x, y \in X$, fixing $\varepsilon > 0$ we can find $a^y_\varepsilon \in A$ such that $F(y) \geq f(a^y_\varepsilon) + L\, d(a^y_\varepsilon, y) - \varepsilon$. By the definition of F, we have that $F(x) \leq f(a^y_\varepsilon) + L\, d(a^y_\varepsilon, y)$, so

$$F(x) - F(y) \leq L\, d(a^y_\varepsilon, y) - L\, d(a^y_\varepsilon, x) + \varepsilon \leq L\, d(x, y) + \varepsilon.$$

Letting $\varepsilon \to 0$ we have $F(x) - F(y) \leq L\, d(x, y)$. By symmetry, also $F(y) - F(x) \leq L\, d(x, y)$: F is $L-$Lipschitz. Finally, from (2.3) we have that $F(a) = f(a)$ if $a \in A$. □

Remark 2.1.1. Formula (2.2) gives the largest extension of f that is $L-$Lipschitz, in the sense that if $G : X \to \mathbb{R}$ is $L-$Lipschitz and such that $G|_A = f$ then $G \leq F$. Similarly,

$$F(x) = \sup\{f(a) - L\, d(a, x) \mid a \in A\} \tag{2.4}$$

defines the smallest extension of f that is $L-$Lipschitz.

Remark 2.1.2. The Kirszbraum theorem states that the conclusion of the preceding theorem still holds if $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$, with $n, m \geq 1$.

Applying the preceding theorem to the coordinate functions of a vector function with values in $\mathbb{R}^n$ we get the following corollary.

Corollary 2.1.2. *Let (X, d) be a metric space, $A \subset X$ and $f : X \to \mathbb{R}^n$ a $L-$Lipschitz function. Then there exists a $L\sqrt{n}-$Lipschitz function $F : X \to \mathbb{R}^n$ such that $F|_A = f$.*

A tool that we will use sometimes in the following chapters will be Lipschitz partition of unity, i.e. partitions of unity made by Lipschitz functions. To this aim, we need to introduce a new class of measure spaces.

Definition 2.1.2. A metric space (X, d) is called *ε-separable* if every two distinct points of the space have distance at least ε. The space X is called *doubling space with constant* N, $N \geq 1$ an integer, if for every ball $B(x, r)$ every subset of $B(x, r)$ that is $\frac{r}{2}-$separable contains at most N points.

Remark 2.1.3. If the metric measure space $(X, d, \mathfrak{m})$ is a doubling space then also (X, d) is a doubling space. In fact, if a subset of $B(x, r)$ $\frac{r}{2}-$separable contains k points that we denote by $x_1, \ldots, x_k$ then for the doubling properties of $\mathfrak{m}$ and being the $B(x_i, \frac{r}{4})$ pairwise disjoint for every $i = 1, \ldots, k$

$$\frac{k}{C^4_{\mathfrak{m}}}\mathfrak{m}(B(x, 2r)) \leq \sum_{i=1}^{k} \frac{1}{C_{\mathfrak{m}}}\mathfrak{m}\left(B\left(x_i, \frac{r}{2}\right)\right) \leq \sum_{i=1}^{k} \mathfrak{m}\left(B\left(x_i, \frac{r}{4}\right)\right) \leq \mathfrak{m}(B(x, 2r)).$$

Hence, being the balls of finite measure, $k \leq C^4_{\mathfrak{m}}$. The same reasoning shows that if X is endowed with a locally finite doubling measure then X is separable.

Open subsets of doubling spaces can be covered by balls that constitute a covering akin to the classical Whitney decomposition of open subset of $\mathbb{R}^n$:

Theorem 2.1.3 (Whitney decomposition)**.** *Let (X, d) be a doubling metric space with doubling constant N and let Ω be an open subset of X sich that $\Omega^c \neq \emptyset$. There exists a countable family of balls in Ω $\mathscr{W}_\Omega := \{B(x_i, r_i)\}_{i \in \mathbb{N}}$ such that*

$$\Omega = \bigcup_{i \in \mathbb{N}} B(x_i, r_i)$$

$$\sum_{i \in \mathbb{N}} \chi_{B(x_i, 2r_i)} \leq 2N^5, \qquad \textit{where} \qquad r_i = \frac{1}{8} d(x_i, X \setminus \Omega).$$

Proof. For $x \in \Omega$, denote $d(x) := d(x, X \setminus \Omega)$. Notice that if we use $\partial\Omega$ in place of $X \setminus \Omega$ it can happen that $B(x, d(x, \partial\Omega)) \cap X \setminus \Omega \neq \emptyset$. For $k \in \mathbb{Z}$ set

$$\mathscr{F}_k := \left\{ B\left(x, \frac{d(x)}{40}\right) \mid x \in \Omega, 2^{k-1} < d(x) \leq 2^k \right\}.$$

By the Vitali theorem, we can consider a subfamily $\mathscr{G}_k \subset \mathscr{F}_k$ of disjoint balls such that

$$\bigcup_{B \in \mathscr{F}_k} B \subset \bigcup_{B \in \mathscr{G}_k} 5B.$$

We want to prove that we can consider for the thesis the family

$$\mathscr{W}_\Omega := \bigcup_{k \in \mathbb{N}} \{5B \mid B \in \mathscr{G}_k\}.$$

By construction we just have to prove $\sum_{i \in \mathbb{N}} \chi_{B(x_i, 2r_i)} \leq 2N^5$. Suppose that there is a point in Ω that belongs to M balls of the form $2B, B \in \mathscr{W}_\Omega$ and label them as $B(x_j, \frac{1}{4}d(x_j))$, with $d(x_1) \geq d(x_i), j = 1, \dots, M$. Using the triangle inequality

$$d(x_i) \geq \frac{3}{5}d(x_1), \qquad B\left(x_i, \frac{1}{4}d(x_i)\right) \subset B\left(x_1, \frac{3}{4}d(x_1)\right), \qquad i = 1, \dots, M. \tag{2.5}$$

If x_i and x_j are the centers of balls of the same family $\mathscr{F}_k$ then

$$d(x_i, x_j) \geq \frac{1}{20} \min\{d(x_i), d(x_j)\} \geq \frac{1}{40} d(x_1)$$

if $i \neq j$.

Remark 2.1.4. Being X a doubling space with doubling constant N, every ball with radius $r > 0$ can be covered by N balls of radius $\frac{r}{2}$. If the vice versa holds, then X is a doubling space with doubling constant N^2. So every set that is $\frac{r}{2^k}$−separable in a ball $B(x, r)$ has at most N^k points.

By this remark, in our case at most N^5 of these balls can have centers in $\mathscr{F}_k$ for fixed k. Suppose now that $x_1 \in \mathscr{F}_{k_1}$, so $d(x_1) \geq d(x_i) \geq \frac{3}{5}d(x_1), i = 2, \dots, M$. Hence all the centers must be contained in $\mathscr{F}_{k_1-1} \cap \mathscr{F}_{k_1}$. □

Let X be a doubling metric space with doubling constant N. If Ω is an open subset of X such that $X \setminus \Omega \neq \emptyset$ and let $\mathscr{W}_\Omega$ be the family constructed with the Whitney extension theorem. Given a ball $B(x_i, r_i) \in \mathscr{W}_\Omega$ we define

$$\psi_i(x) := \min\left\{ \frac{1}{r_i} d(x, X \setminus B(x_i, 2r_i)), 1 \right\}.$$

By definition, ψ_i is $\frac{1}{r_i}$−Lipschitz. Moreover,

$$1 \leq \sum_{i \in \mathbb{N}} \psi_i(x) \leq 2N^5.$$

Set

$$\phi_i(x) := \frac{\psi_i(x)}{\sum_k \psi_k(x)}. \tag{2.6}$$

The functions ϕ_i satisfy the following properties for some constant $C \geq 1$ that depends only by N:

- $\phi_i(x) = 0$ for $x \notin B(x_i, 2r_i)$ and for every $x \in \Omega$ we have that $\phi_i(x) \neq 0$ for at most C indices i;
- $0 \le \phi_i \le 1$ and $\phi|_{B(x_i,r_i)} \ge \frac{1}{C}$;
- ϕ_i is $\frac{C}{r_i}-$Lipschitz;
- $\sum_i \phi_i(x) = 1$ for every $x \in \Omega$.

Definition 2.1.3. We call a family of function $\{\phi_i\}$ as above a *Lipschitz partition of unity*.

Using these partitions it can be shown that a $L-$Lipschitz function over doubling metric space can be extended with a $CL-$Lipschitz function, where $C \ge 1$ depends only on the doubling constant of X.

We then want to prove an essential density property concerning Lipschitz functions, but before we need some definitions.

Definition 2.1.4. Let (X, d) be a metric space. A function $f : X \to (-\infty, \infty]$ is said to be *lower semicontinuous* if the set $\{x \in X : f(x) > a\} =: \{f > a\}$ is open for each $a \in \mathbb{R}$. it is said *upper semicontinuous* if $-f$ is lower semicontinuous.

By definition it follows that f is lower semicontinuous if and only if

$$\varliminf_{y \to x} f(y) \ge f(x) \qquad \forall x \in X. \tag{2.7}$$

Thus, if f is lower semicontinuous and $f(x) = \infty$ for a point $x \in X$ then f is continuous at x in the extended sense.

If f and g are lower semicontinuous and if $x \ge 0$ then both $cf + g$ and $\min\{f, g\}$ are lower semicontinuous. Moreover, the pointwise supremum of an arbitrary family of lower semicontinuous functions if lower semicontinuous.

Proposition 2.1.4. *Let (X, d) be a metric space, $c \in \mathbb{R}$ and $f : X \to [c, \infty]$ be lower semicontinuous. Then there exists a sequence $\{f_i\}$ of Lipschitz functions on X such that $c \le f_i \le f_{i+1} \le f$ and $\lim_{i\to\infty} f_i(x) = f(x)$ for each $x \in X$.*

Proof. Define, for each $i = 1, 2, \dots$ a function f_i on X by

$$f_i(x) := \inf\{f(y) + id(x, y) : y \in X\}.$$

Following the argument in the proof of the Whitney-Mc. Shane extension theorem we have that each f_i is $i-$Lipschitz with $c \le f_i(x) \le f_{i+1}(x) \le f(x)$ for each $X \in X$. Fix $x \in X$. Assume first that $f(x) = \infty$. Let $M > 0$ and choose $\varepsilon > 0$ such that $f > M$ on the ball $B(x, \varepsilon)$. Therefore $f_i(x)$ is at least the minimum of the numbers M and $c + i\varepsilon$. For every i large enough that $c + i\varepsilon > M$ we have that $f_i(x) \ge M$, which implies $\lim_{i\to\infty} f_i(x) = f(x) = \infty$.
Next, assume that $f(x) < \infty$. Let $M < f(x)$, and choose $\varepsilon > 0$ such that $f > M$ on the same ball as before. As above, we find $f_i(x) \ge M$ for all large i and hence that $\lim_{i\to\infty} f_i(x) = f(x)$ in this case as well. □

Combining the dominated convergence theorem and Proposition 2.1.4 we get the following corollary.

Corollary 2.1.5. *Let $X = (X, d, \mathfrak{m})$ be a metric measure space, $p \in [1, \infty)$ and let $f : X \to [0, \infty]$ be a $p-$integrable lower semicontinuous function. Then there exists a sequence $\{f_i\}$ of Lipschitz functions on X such that $0 \le f_i \le f_{i+1} \le f$ and $f_i \to f$ both pointwise and in $L^p(X, \mathfrak{m})$ as $i \to \infty$.*

Quite similarly, in every metric measure space non-negative $p-$integrable functions can be approximated in L^p by a pointwise decreasing sequence of lower semicontinuous functions.

Theorem 2.1.6 (Vitali-Carathéodory). *Let $(X, d, \mathfrak{m})$ be a metric measure space and let $p \in [1, \infty)$. For every $p-$integrable function $f : X \to [0, \infty]$ there exists a pointwise decreasing sequence $\{g_i\}$ of lower semicontinuous functions on X such that $f \le g_{i+1} \le g_i$ and $g_i \to f$ in $L^p(X, \mathfrak{m})$.*

Proof. Let $f : X \to [0, \infty]$ be a $p-$integrable function on X. Pick an increasing sequence $\{\phi_i\}$ of non-negative simple functions converging pointwise to f. By using the representation

$$f = \phi_1 + \sum_{i=2}^{\infty} (\phi_i - \phi_{i-1})$$

we find that f admits the following expression:

$$f = \sum_{j=0}^{\infty} a_j \chi_{E_j}.$$

Here $a_0 = \infty, a_j \in (0, \infty)$ for $j \ge 1$ and $E_j \subset X$ is a measurable set for all $j = 0, 1, \ldots$ and note that $\mathfrak{m}(E_0) = 0$.

Fix then $\varepsilon > 0$. By (1.3) we can choose for each $g \ge 1$ an open set $U_j \supset E_j$ such that

$$\mathfrak{m}(U_j) \le \mathfrak{m}(E_j) + \varepsilon^p \, 2^{-jp} \, a_j^{-p}.$$

Moreover, we can choose a sequence of open sets $V_j \supset E_0$ such that

$$\mathfrak{m}(V_j) \le \varepsilon^p \, 2^{-jp}$$

for $j = 1, 2, \ldots$. Then for the lower semicontinuous function

$$g := \sum_{j=1}^{\infty} a_j \chi_{U_j} + \sum_{j=1}^{\infty} \chi_{V_j}$$

we have that both $f \le g$ on X and

$$\|g - f\|_{L^p(X,\mathfrak{m})} \le \sum_{j=1}^{\infty} a_j \mathfrak{m}(U_j \setminus E_j)^{\frac{1}{p}} + \sum_{j=1}^{\infty} \mathfrak{m}(V_j)^{\frac{1}{p}} \le 2\varepsilon.$$

Being ε arbitrary, and being the minimum of two lower semicontinuous functions lower semicontinuous the theorem follows. □

Combining the Vitali-Carathéodory theorem with Corollary 2.1.5 we get the main result.

Theorem 2.1.7. *Let $X = (X, d, \mathfrak{m})$ be a metric measure space and $p \in [1, \infty)$. Then Lipschitz functions are dense in $L^p(X, \mathfrak{m})$. If in addition (X, d) is locally compact then Lipschitz functions with compact support are dense in $L^p(X, \mathfrak{m})$.*

2.2 Absolute continuity and curves in metric spaces

To introduce the concept of (weak) upper gradients and Sobolev classes we need some definitions and properties of absolute continuous curves.

Definition 2.2.1. Let (X, d) be a complete metric space and let $\gamma : [0, 1] \to X$ be a curve. We say that γ belongs to $AC([0, 1], \mathbb{R})$ if there exists $f \in L^1([0, 1])$ such that

$$d(\gamma(s), \gamma(t)) \leq \int_s^t f(r)\,\mathrm{d}r \qquad \forall t, s \in (a, b) \tag{2.8}$$

For $p = 1$ γ will be called *absolutely continuous curve*. If $f \in L^q([0, 1]), q \in [1, \infty]$ then the curve will be called *q-absolutely continuous* and set of such curves denoted by $AC^q([0, 1], X)$.

With a little abuse of the notation we will use often the term "curve" both for the map γ and its image $\gamma([0, 1])$.

Remark 2.2.1. In general the domain of γ is an open interval (a, b), but observe that in this case the limit for $t \downarrow a$ and $t \uparrow b$ of γ exist being X complete by assumptions, even if a or b are ∞. Recall also that a curve in $AC^q([0, 1], X)$ is uniformly continuous.

Among all the possible choices of f in (2.8) there exists a minimal one, which is provided by the following theorem.

Theorem 2.2.1. *Let $p \in [1, +\infty]$. Then for any curve $\gamma \in AC^q([0, 1], X)$ the limit*

$$|\dot{\gamma}_t| := \lim_{s \to t} \frac{d(\gamma_s, \gamma_t)}{|s - t|} \tag{2.9}$$

exists for $\mathcal{L}^1-$a.e. $t \in [0, 1]$. Moreover, the function $t \mapsto |\dot{\gamma}_t| \in L^p([0, 1])$ is an admissible integrand for the right-hand side of (2.8)*, and it is minimal in the following sense:*

$$|\dot{\gamma}_t| \leq f(t) \quad \mathcal{L}^1 - a.e.\ t \in [0, 1] \quad \forall f \text{ satisfying } (2.8). \tag{2.10}$$

Proof. Let $\{y_n\}_{n \in \mathbb{N}}$ be a dense subset of $\gamma([0, 1]) \subset X$ and define

$$d_n(t) := d(y_n, \gamma_t), \quad n \in \mathbb{N}.$$

By continuity of the distance function all the d_n's are absolutely continuous on $[0, 1]$. Hence the function

$$d(t) := \sup_{n \in \mathbb{N}} |\dot{d}_n(t)|$$

is well defined $\mathcal{L}^1-$a.e. in $[0, 1]$. If $t \in [0, 1]$ is a point where all the functions d_n are differentiable notice that by the reverse triangle inequality

$$\varliminf_{h \to 0} \frac{d(\gamma_{t+h}, \gamma_t)}{|h|} \geq \sup_{n \in \mathbb{N}} \varliminf_{h \to 0} \frac{|d_n(t + h) - d_n(t)|}{|h|} \triangleq d(t).$$

From the Definition 2.2.5 we have $d \leq f$ $\mathcal{L}^1-$a.e., therefore $d \in L^p([0, 1], \mathcal{L}^1)$ being so f by definition. But from the definition of d we have that

$$d(\gamma_{t+h}, \gamma_t) = \sup_{n \in \mathbb{N}} |d_n(t + h) - d_n(t)| \leq \int_t^{t+h} d(r)\mathrm{d}r$$

and so

$$\varlimsup_{h \to 0} \frac{d(\gamma_{t+h}, \gamma_t)}{|h|} \leq d(t),$$

at any Lebesgue point t of d. $\square$

Definition 2.2.2. Given a curve $\gamma : [0,1] \to X$ we define its *length* $\mathscr{L}(\gamma) \in \overline{\mathbb{R}}$ as the supremum of the numbers

$$\sum_{i=1}^{k} d(\gamma_{t_i}, \gamma_{t_{i-1}}) \tag{2.11}$$

over all the scompositions of $[0,1]$ in $0 = t_0 < t_1 < \cdots < t_k = 1$. If $\mathscr{L}(\gamma) < \infty$ we say that γ is *rectifiable*.

Given a rectifiable curve $\gamma : [0,1] \to X$, if $f : X \to Y$ is a $L-$Lipschitz map between metric spaces then $f \circ \gamma$ is rectifiable and

$$\mathscr{L}(f \circ \gamma) \leq L\,\mathscr{L}(\gamma). \tag{2.12}$$

Moreover, all Lipschitz curves are locally rectifiable.

Definition 2.2.3. If $\gamma : [0,1] \to X$ is rectifiable then we define its *length function* $s_\gamma : [0,1] \to [0, \mathscr{L}(\gamma)]$ as

$$s_\gamma(t) := \mathscr{L}(\gamma|_{[0,t]})$$

By definition, it follows that

$$d(\gamma_{t_2}, \gamma_{t_1}) \leq \mathscr{L}(\gamma|_{[t_1,t_2]}) = s_\gamma(t_2) - s_\gamma(t_1) \tag{2.13}$$

Lemma 2.2.2. *The function s_γ associated to a rectifiable curve $\gamma : [0,1] \to X$ is increasing and continuous.*

Proof. s_γ is increasing by (2.11). To prove that it is also continuous, fix $0 \leq t_0 \leq 1$. Since s_γ is increasing, both the right and left sided limit $s_\gamma^-(t_0)$ and $s_\gamma^+(t_0)$ exist. Suppose first that $s_\gamma(t_0) - s_\gamma^-(t_0) > \delta > 0$. Then $t_0 > 0$ and let $0 < t_1 < 1$. Since we have

$$\mathscr{L}(\gamma|_{[t_1,t_0]}) = s_\gamma(t_0) - s_\gamma(t_1) = s_\gamma|_{[t_1,t_0]}(t_0) > \delta,$$

by continuity of γ there exist $t_1 =: a_0 < \cdots < a_k < t_0$ such that

$$\sum_{j=1}^{k} d(\gamma_{a_j}, \gamma_{a_{j-1}}) > \delta.$$

Define $t_2 := a_k$. Then $\mathscr{L}(\gamma|_{[t_1,t_2]}) > \delta$ and

$$\mathscr{L}(\gamma|_{[t_2,t_0]}) = s_\gamma(t_0) - s_\gamma(t_2) > \delta.$$

By induction we build a sequence $\{t_i\}$ of times $t_1 < t_2 < \cdots < t_i < \cdots < t_0$ such that $\mathscr{L}(\gamma|_{[t_i,t_{i+1}]}) > \delta$. This means that

$$\mathscr{L}(\gamma|_{[t_1,t_0]}) \geq \mathscr{L}(\gamma|_{[t_i,t_{i+1}]}) > (i-1)\delta \qquad \forall i = 2,3,\ldots$$

and this contradicts the rectifiability of γ. We conclude that $s_\gamma^-(t_0) = s_\gamma(t_0)$.
Similarly $s_\gamma^+(t_0) = s_\gamma(t_0)$ and the thesis follows. $\square$

Proposition 2.2.3. *A rectifiable curve $\gamma : [0,1] \to X$ is absolutely continuous if and only if its associated length function s_γ is absolutely continuous.*

Proof. The absolute continuity of γ follows from the one of s_γ thanks to formula (2.13). Suppose that γ absolutely continuous. Let $\varepsilon > 0$ and $\delta > 0$ and we assume to have a family of k disjoint intervals $[a_i, b_i]$, $i = 1, \dots, k$ of $[0,1]$ such that $\sum_{i=1}^k b_i - a_i < \delta$. Then, being

$$s_\gamma(b_i) - s_\gamma(a_i) = \mathscr{L}(\gamma|_{[a_i,b_i]}) < \infty,$$

we can decompose each interval $[a_i, b_i]$ in k_i subintervals $[a_i^j, b_i^j]$ such that

$$\sum_{j=1}^{k_i} d(\gamma_{b_i^j}, \gamma_{a_i^j}) > s_\gamma(b_i) - s_\gamma(a_i) - \frac{\varepsilon}{k}.$$

Hence $\sum_{i=1}^k \sum_{j=1}^{k_i} b_i^j - a_i^j = \sum_{i=1}^k b_i - a_i < \delta$ and

$$\sum_{i=1}^k |s_\gamma(b_i) - s_\gamma(a_i)| \leq 2\varepsilon,$$

that is the definition of absolute continuous function. □

Definition 2.2.4. The *arc-length reparametrization* γ_s of a rectifiable curve $\gamma : [0,1] \to X$ is the function $\gamma_s : [0, \mathscr{L}(\gamma)] \to X$ defined by

$$\gamma_s(t) := \gamma(s_\gamma^{-1}(t)),$$

where $s_\gamma^{-1}(t) := \sup\{s \mid s_\gamma(s) = t\} = \max\{s \mid s_\gamma(s) = t\}$ exists from the continuity of s_γ.

Notice that s_γ^{-1} is continuous from the right, i.e. $\lim_{t \to t_0^+} s_\gamma^{-1}(t) = s_\gamma^{-1}(t_0)$. If $\lim_{t \to t_0^-} s_\gamma^{-1}(t) = s_0 < s_\gamma^{-1}(t_0)$ then γ is constant on $[s_0, s_\gamma^{-1}(t_0)]$. Hence $\gamma_s : [0, \mathscr{L}(\gamma)] \to X$ is the unique curve satisfying

$$\gamma(t) = \gamma_s(s_\gamma(t)) \qquad \forall\, t \in [0,1].$$

By definition it follows that

$$\mathscr{L}(\gamma_s|_{[t,t+h]}) = h \qquad t \in [0, \mathscr{L}(\gamma)].$$

and so we have the following proposition

Proposition 2.2.4. *The arc-length reparametrization γ_s of a compact rectifiable curve γ is $1-$Lipschitz, hence absolutely continuous and $|\dot\gamma_s(t)| = 1$ $t-$a.e. in $[0, \mathscr{L}(\gamma)]$.*

Definition 2.2.5. We define the integral of a Borel function $f : X \to [0, \infty]$ along a rectifiable curve $\gamma : [a,b] \to X$ as

$$\int_\gamma f \,\mathrm{d}s := \int_0^{\mathscr{L}(\gamma)} f(\gamma_s(t))\,\mathrm{d}t \tag{2.14}$$

Notice that $f \circ \gamma_s$ is a non-negative Borel function over $[0, \mathscr{L}(\gamma)]$ so that the integral in (2.14) exists. Moreover, line integrals are always defined over curves that are locally rectifiable.

Proposition 2.2.5. *Let* $\gamma : [0, L] \to X$ *be a* $1-$*Lipschitz curve. Then*

$$\int_\gamma f \,\mathrm{d}s \leq \int_0^L f(\gamma_t)\,\mathrm{d}t$$

for every Borel function $f : X \to [0, \infty]$.

Proof. Being γ $1-$Lipschitz also the associated length function $s_\gamma : [0, L] \to [0, \mathscr{L}(\gamma)]$ is $1-$Lipschitz and absolutely continuous. Hence

$$\int_\gamma f\,\mathrm{d}s = \int_0^{\mathscr{L}(\gamma)} f(\gamma_s(t))\,\mathrm{d}t = \int_0^L f(\gamma_s(s_\gamma(t)))s'_\gamma(t)\,\mathrm{d}t \leq \int_0^L f(\gamma_s(s_\gamma(t)))\,\mathrm{d}t = \int_0^L f(\gamma_t)\,\mathrm{d}t.$$

□

Chapter 3

Sobolev classes and $q-$test plans

With the notion of absolutely continuous curve and metric derivative we can now define the analogous of a norm of a gradient. Recall that given a function $f \in C^1(\mathbb{R}^n, \mathbb{R})$ and a curve $\gamma \in C^1([0,1], \mathbb{R}^n)$ then by the chain rule, the fundamental theorem of calculus and the Cauchy-Schwarz inequality, we have

$$|f(\gamma_1) - f(\gamma_0)| = \left| \int_0^1 \langle \nabla f(\gamma_s), \dot{\gamma_s} \rangle \, \mathrm{d}s \right| \leq \int_0^1 \|\nabla f(\gamma_s)\| \, \|\dot{\gamma_s}\| \, \mathrm{d}s.$$

Then we define the Sobolev classes and the properties of Sobolev functions, proving three fundamental inequalities (Proposition 3.2.4). Finally we introduce the Cheeger's energy functional and from the study of its gradient flow we will get a density result of Lipschitz functions in Sobolev spaces.

3.1 Upper gradients and Sobolev classes

Definition 3.1.1. Given a Borel function $f : X \to \mathbb{R}$ we say that a Borel function $G : X \to [0, \infty]$ is an *upper gradient* of f if

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t) |\dot{\gamma}_t| \, \mathrm{d}t, \qquad \forall \gamma \in AC([0,1], X).$$

The following definition is the most simple example of an upper gradient.

Definition 3.1.2. Given $f : X \to \mathbb{R}$ we define the *local Lipschitz constant* $\mathrm{lip}(f) : X \to [0, \infty]$ as

$$\mathrm{lip}(f)(x) := \varlimsup_{y \to x} \frac{|f(y) - f(x)|}{d(x,y)}$$

if x is not an isolated point, 0 otherwise. If in the numerator we consider the positive part (respectively the negative part) of f we get the definition of $\mathrm{lip}^+(f)$ and respectively of $\mathrm{lip}^-(f)$.

Remark 3.1.1. If f is locally Lipschitz then $\mathrm{lip}^{\pm}(f)$ and $\mathrm{lip}(f)$ are upper gradients of f by definition.

What we are going to do now is to bring all these metric definitions on the space $\mathscr{P}(X)$ of probability measures over X because this will allow us to get a duality for particular measures and to get a gradient in a metric measure space.

Definition 3.1.3. Let $\mathscr{P}(X)$ be the set of probability measures over X. We define the *Wasserstein space* as

$$\mathscr{P}_q(X) := \left\{ \mu \in \mathscr{P}(X) : \int_X d(x, x_0) \, \mathrm{d}\mathfrak{m} < \infty \quad \forall x_0 \in X \right\}$$

We endow it with the $q-$*Wasserstein distance* defined by

$$W_q(\mu, \nu) := \inf_{\gamma} \left(\int_X d^q(x, y) \, \mathrm{d}\boldsymbol{\gamma}(x, y) \right)^{\frac{1}{q}},$$

where the infimum is taken over all the measures $\boldsymbol{\gamma} \in \mathscr{P}(X \times X)$ with marginals μ and ν ($\boldsymbol{\gamma}$'s are called *transport plans* or *couplings*).

Remark 3.1.2. it is possible to prove that if X is a Polish space then there exists always a transport plan between two probability measures.

Proposition 3.1.1. *W_q is a distance.*

Proof. The symmetry of W_q follows from the symmetry of d.
For the triangle inequality (called also *gluing lemma*) let $\mu_1, \mu_2, \mu_3 \in \mathscr{P}_q(X)$, $\boldsymbol{\gamma}_{ij}$ be transport plans from μ_i to μ_j, for $i, j \in \{1, 2, 3\}$, and $\mu \in \mathscr{P}_q(X \times X \times X)$ with marginals $\boldsymbol{\gamma}_{12}$ and $\boldsymbol{\gamma}_{23}$. Then

$$\begin{aligned}
W_q(\mu_1, \mu_3) &\leq \left(\int_{X \times X} d^q(x, z) \, \mathrm{d}\boldsymbol{\gamma}_{13}(x, z) \right)^{\frac{1}{q}} = \left(\int_{X \times X \times X} d^q(x, z) \, \mathrm{d}\mu(x, y, z) \right)^{\frac{1}{q}} \\
&\leq \left(\int_{X \times X \times X} d^q(x, y) \, \mathrm{d}\mu(x, y, z) \right)^{\frac{1}{q}} + \left(\int_{X \times X \times X} d^q(y, z) \, \mathrm{d}\mu(x, y, z) \right)^{\frac{1}{q}} \\
&= \left(\int_{X \times X} d^q(x, y) \, \mathrm{d}\boldsymbol{\gamma}_{12}(x, y) \right)^{\frac{1}{q}} + \left(\int_{X \times X} d^q(y, z) \, \mathrm{d}\boldsymbol{\gamma}_{23}(y, z) \right)^{\frac{1}{q}} \\
&\triangleq W_q(\mu_1, \mu_2) + W_q(\mu_2, \mu_3).
\end{aligned}$$

For the homogeneity of W_q consider $f : X \to \Delta \subset X \times X$ defined by $f(x) = (x, x)$ and let

$$\boldsymbol{\nu} := f_\sharp \mu,$$

with $\mu \in \mathscr{P}_q(X)$. Then we have that $\boldsymbol{\nu} \in \mathscr{P}_q(\Delta)$ and if we define $\boldsymbol{\pi} \in \mathscr{P}_q(X \times X)$ as

$$\boldsymbol{\pi}(A) := \boldsymbol{\nu}(A \cap \Delta) \quad \forall A \in \mathscr{B}(X \times X)$$

then $\boldsymbol{\pi}$ is a transport plan from μ to μ (not optimal in general) and by its definition $\boldsymbol{\pi}(\Delta^c) = 0$ $(*)$. Hence by the homogeneity of d $(**)$

$$\begin{aligned}
W_q^q(\mu, \mu) &\leq \int_{X \times X} d^q(x, y) \, \mathrm{d}\boldsymbol{\pi}(x, y) \\
&= \int_\Delta d^q(x, y) \, \mathrm{d}\boldsymbol{\pi}(x, y) + \int_{\Delta^c} d^q(x, y) \, \mathrm{d}\boldsymbol{\pi}(x, y) \\
&\overset{(*)}{=} \int_\Delta d^q(x, x) \, \mathrm{d}\boldsymbol{\pi}(x, x) + 0 \overset{(**)}{=} 0 \Rightarrow W_q(\mu, \mu) = 0.
\end{aligned}$$

□

Definition 3.1.4. We define the *evaluation map* $e_t : C([0,1],X) \to X$ and the *restriction map* $\mathrm{restr}_t^s : C([0,1],X) \to C([0,1],X)$ for $s,t \in [0,1]$ as

$$e_t(\gamma) := \gamma_t, \qquad (\mathrm{restr}_t^s(\gamma))_r := \gamma_{t+r(s-t)}.$$

Notice that for $t > s$ we have a change of orientation. The following theorem is the first metric result in $(\mathscr{P}_q(X), W_q)$, called also *superposition principle*, and we will use it often in the nexe chapter:

Theorem 3.1.2 (Lisini). *Let (X,d) be a Polish space, $q \in (1,\infty)$ and $\mu_t : [0,1] \to \mathscr{P}_q(X)$ a $q-$absolutely continuous curve w.r.t. W_q. Then there exists a measure $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ concentrated on $AC^q([0,1],X)$ such that*

$$(e_t)_\sharp \boldsymbol{\pi} = \mu_t \qquad \forall t \in [0,1],$$
$$\int |\dot\gamma_t|^q \,\mathrm{d}\boldsymbol{\pi}(\gamma) = |\dot\mu_t|^q \qquad t-a.e.$$

For the proof we refer to [3].

Remark 3.1.3. For $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ such that $\iint_0^1 |\dot\gamma_t|^q \mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty$ and $(e_t)_\sharp \boldsymbol{\pi} = \mu_t$ for all t, the following inequality always holds

$$\int |\dot\gamma_t|^q \mathrm{d}\boldsymbol{\pi}(\gamma) \geq |\dot\mu_t|^q \qquad t-a.e.$$

Proof. Firstly we have to prove that $|\dot\gamma_t|$ exists $\boldsymbol{\pi}-$a.e. for any $\gamma \in C([0,1],X)$.
The set

$$\Lambda := \{(t,\gamma) \in [0,1] \times C([0,1],X) \mid |\dot\gamma_t| \text{ does not exist}\}$$

is a Borel subset of $[0,1] \times C([0,1],X)$ being the map $(t,\gamma) \mapsto \frac{d(\gamma_{t+h},\gamma_t)}{|h|}$ continuous for all $h \neq 0$. Since $\boldsymbol{\pi}$ is concentrated on $AC^q([0,1],X)$ we have that if $\gamma \in C([0,1],X)$ then $\mathcal{L}^1(\{t \in [0,1] \mid (t,\gamma) \in \Lambda\}) = 0$. Hence by Fubini's theorem, $t-$a.e. and $\boldsymbol{\pi}-$a.e. we have that

$$\boldsymbol{\pi}(\{\gamma \in C([0,1],X) \mid (t,\gamma) \in \Lambda\}) = 0.$$

We now prove that $\mu_t = (e_t)_\sharp \boldsymbol{\pi} \in \mathscr{P}_q(X)$ for all $t \in [0,1]$. Fixed $\overline{x} \in X$ we have that

$$\begin{aligned}
\int_X d^q(x,\overline{x})\,\mathrm{d}\mu_t(x) &= \int d^q(\gamma_t,\overline{x})\,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq 2^{q-1} \int (d^q(\gamma_0,\overline{x}) + d^q(\gamma_0,\gamma_t))\,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&\leq 2^{q-1} \int \left(d^q(\gamma_0,\overline{x}) + \left(\int_0^t |\dot\gamma_t|(r)\,\mathrm{d}r \right)^q \right) \mathrm{d}\boldsymbol{\pi}(\gamma) \\
&\leq 2^{q-1} \int \left(d^q(\gamma_0,\overline{x}) + \int_0^T |\dot\gamma_t|(r)\,\mathrm{d}r \right) \mathrm{d}\boldsymbol{\pi}(\gamma)
\end{aligned}$$

and it is finite by our assumptions.
Pick now $t,s \in [0,1], s < t$ and let $\boldsymbol{\gamma}_{s,t} := (e_s,e_t)_\sharp \boldsymbol{\pi}$ be the pushforward measure of $\boldsymbol{\pi}$ through (e_s,e_t) with marginals μ_s,μ_t. Then by hypotheses and Hölder's inequality we have

$$\begin{aligned}
W_q^q(\mu_s,\mu_t) &\leq \int_{X\times X} d^q(x,y)\,\mathrm{d}\boldsymbol{\gamma}_{s,t}(x,y) = \int d^q(e_s(\gamma),e_t(\gamma))\,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&\leq \int \left(\int_s^t |\dot\gamma_t|(r)\,\mathrm{d}r \right)^q \mathrm{d}\boldsymbol{\pi}(\gamma) \leq \int |s-t|^{q-1} \int_s^t |\dot\gamma_t|(r)\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= |s-t|^{q-1} \int_s^t \int |\dot\gamma_t|(r)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}r,
\end{aligned}$$

i.e. the absolute continuity of μ_t w.r.t. W_q. Then the thesis follows applying the Lebesgue's differentiation theorem. $\square$

Assume from now on that

$$\begin{gathered}(X,d) \text{ is a Polish space,}\\ \mathfrak{m} \text{ is a non-negative Radon measure on } X.\end{gathered} \tag{3.1}$$

From the fact that $\mathfrak{m}$ is locally finite and (X,d) is separable, by the Lindelöf property there exists a Borel probability measure $\tilde{\mathfrak{m}} \in \mathscr{P}(X)$ such that

$$\begin{aligned}&\mathfrak{m} \ll \tilde{\mathfrak{m}} \leq C\mathfrak{m}, \qquad C>0,\\ &\text{with } \tfrac{\mathrm{d}\tilde{\mathfrak{m}}}{\mathrm{d}\mathfrak{m}} \text{ locally bounded from below by a positive constant}\end{aligned} \tag{3.2}$$

where locally bounded from below means that for any $x \in X$ there exists a neighbourhood U_x and a constant $c_x > 0$ such that $\mathfrak{m}-$a.e. on U_x it holds $\frac{\mathrm{d}\tilde{\mathfrak{m}}}{\mathrm{d}\mathfrak{m}} \geq c_x$. It also can be proven that $\tilde{\mathfrak{m}}$ can be chosen so that

$$\int_X d^q(x,x_0)\,\mathrm{d}\tilde{\mathfrak{m}} < \infty \qquad \forall q \geq 1, \text{ for some, and thus for any, } x_0 \in X. \tag{3.3}$$

Now we want to define the Sobolev classes on X with a different approach from the classic one used in $\mathbb{R}^n$, although it can be shown that they are equivalent.

Definition 3.1.5. Let $(X,d,\mathfrak{m})$ be a metric measure space as in (3.1) and consider $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$. We say that $\boldsymbol{\pi}$ has *bounded compression* if there exists $C>0$ such that

$$(e_t)_\sharp \boldsymbol{\pi} \leq C\mathfrak{m} \qquad \forall t \in [0,1].$$

For $q \in (1,\infty)$ we call $\boldsymbol{\pi}$ a $q-$*test plan* if it has bounded compression, is concentrated on $AC^q([0,1],X)$ and if

$$\iint_0^1 |\dot{\gamma}_t|^q \,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty.$$

Definition 3.1.6. Let $(X,d,\mathfrak{m})$ be a metric measure space as in (3.1), $p,q \in (1,\infty)$ conjugate exponents. We say that a Borel function $f: X \to \mathbb{R}$ belongs to the *Sobolev class* $\mathrm{S}^p(X,d,\mathfrak{m})$ (respectively $\mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m})$) if there exists a function $G \in L^p(X,\mathfrak{m})$ (respectively in $L^p_{\mathrm{loc}}(X,d,\mathfrak{m})$) such that

$$\int |f(\gamma_1)-f(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_s)|\dot{\gamma}_s|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma), \qquad \text{for every } q-\text{test plan } \boldsymbol{\pi}.$$

In this case G is called $p-$*weak upper gradient* of f.

Remark 3.1.4. The class of $q-$test plans contains the one of $q'-$test plans if $q \leq q'$, for the inclusions among L^p-spaces. Hence if $\mathrm{S}^p(X,d,\mathfrak{m}) \subset \mathrm{S}^{p'}(X,d,\mathfrak{m})$ for $p \geq p'$ and if $f \in \mathrm{S}^p(X,d,\mathfrak{m})$ and G is a $p-$weak upper gradient then G is also a $p'-$weak upper gradient of f.

As G in the preceding definition we want the minimal one, if it exists. Hence first of all we need to prove that $f \circ \gamma$ admits an absolutely continuous representative-

Proof. If $\boldsymbol{\pi}$ is a $q-$test plan, then also $(\mathrm{restr}_t^s)_\sharp\boldsymbol{\pi}$ is a $q-$test plan. Hence if G is a $p-$weak upper gradient of f such that $\int_\gamma g < \infty$ a.e., then for every $t, s \in [0,1]$ we have that

$$|f(\gamma_s) - f(\gamma_t)| \leq \int_t^s G(\gamma_r)|\dot{\gamma}_r|\,\mathrm{d}r.$$

We then apply the Fubini theorem to the product measure $\mathcal{L}^2 \otimes \boldsymbol{\pi}$ over the set $(0,1)^2 \times C([0,1], X)$. So the function f satisfies

$$|f(\gamma_s) - f(\gamma_t)| \leq \left|\int_t^s G(\gamma_r)|\dot{\gamma}_r|\,\mathrm{d}r\right| \quad (t,s) - \text{a.e.}$$

Similarly,

$$\begin{cases} |f(\gamma_s) - f(\gamma_0)| \leq \int_0^s g(\gamma_r)|\dot{\gamma}_r|\,\mathrm{d}r \\ |f(\gamma_1) - f(\gamma_s)| \leq \int_s^1 g(\gamma_r)|\dot{\gamma}_r|\,\mathrm{d}r \end{cases} \qquad s - \text{a.e.}$$

Lemma 3.1.3. *Let $f : (0,1) \to \mathbb{R}, q \in [1,\infty]$ and $g \in L^q((0,1))$ be a non-negative function such that*

$$|f(s) - f(t)| \leq \left|\int_s^t g(r)\,\mathrm{d}r\right| \qquad (t,s) - a.e..$$

Then $f \in W^{1,q}\big((0,1)\big)$ and $|f'| \leq g$ a.e. in $(0,1)$.

Proof. Let $\mathcal{N} \subset (0,1)^2$ be a $\mathcal{L}^2-$negligible set for which the thesis is false. By the Fubini's theorem we can choose $s \in (0,1)$ such that $(s,t) \notin \mathcal{N}$ $t-$a.e. so that $f \in L^\infty(0,1)$. Since the set $\{(t,h) \in (0,1)^2 \mid (t, t+h) \in \mathcal{N} \cap (0,1)^2\}$ is $\mathcal{L}^2-$negligible, we can use again the Fubini's theorem to claim that $h-$a.e. $(t,t+h) \notin \mathcal{N}$ $t-$a.e.. Hence if $\{h_i\}_{i\in\mathbb{N}}$ is a sequence of points with this property and such that $h_i \downarrow 0$, using the identity

$$\int_0^1 f(t)\frac{\phi(t+h) - \phi(t)}{h}\,\mathrm{d}t = -\int_0^1 \frac{f(t+h) - f(t)}{-h}\phi(t)\,\mathrm{d}t$$

with $\phi \in C_0^1(0,1)$ and $h = h_i$ small enough we obtain

$$\left|\int_0^1 f(t)\phi'(t)\,\mathrm{d}t\right| \leq \int_0^1 g(t)|\phi(t)|\,\mathrm{d}t.$$

Hence we can interpret the distributional derivative of f as a signed measure η with finite total variation satisfying

$$-\int_0^1 f\phi'\,\mathrm{d}t = \int_0^1 \phi\,\mathrm{d}\eta, \qquad \left|\int_0^1 \phi\,\mathrm{d}\eta\right| \leq \int_0^1 g|\phi|\,\mathrm{d}t \quad \text{for all } \phi \in C_0^1(0,1).$$

So $\eta \ll \mathcal{L}^1$ with $|\eta| \leq g\mathcal{L}^1$. Hence $|f'| \leq g$ a.e. in $(0,1)$. The case $q > 1$ follows using the same argument with $g \in L^q(0,1)$. □

Being $g\circ\gamma|\dot{\gamma}| \in L^1(0,1)$ $\boldsymbol{\pi}-$a.e., thanks to the lemma we have that $f\circ\gamma \in W^{1,1}(0,1)$ $\boldsymbol{\pi}-$a.e. and that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(f\circ\gamma)\right| \leq g\circ\gamma|\dot{\gamma}| \qquad \text{a.e. in } (0,1) \text{ and } \boldsymbol{\pi} - \text{a.e..} \tag{3.4}$$

But from the arbitrariness of $\boldsymbol{\pi}$ we have that $f\circ\gamma \in W^{1,1}(0,1)$ $\gamma-$a.e. so it admits an absolutely continuous representative f_γ. □

With the same argument $\gamma-$a.e. if G_1, G_2 are two $p-$weak upper gradients of f then $\min\{G_1, G_2\}$ is a $p-$weak upper gradient of f too. Hence there exists a minimal function $G \geq 0$ $\mathfrak{m}-$a.e. in $L^p(X, \mathfrak{m})$ such that the preceding definition holds.

Definition 3.1.7. We call this function *minimal $p-$weak upper gradient* of f and we denote it by $|Df|_w$.

Remark 3.1.5. With this notation however it is not explicit the dependence on p of the definition (or potential). Thanks to the Cheeger's results in [10], if the measure $\mathfrak{m}$ is doubling and the space support a local and weak version of the Poincaré inequality, then for $f \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}), p \geq p'$ we have that $|Df|_{w,p'} = |Df|_{w,p}$ $\mathfrak{m}-$a.e.. Hence in this case the dependence on the Sobolev exponent p can be omitted.

3.2 Properties of functions in $\mathrm{S}^p(X, d, \mathfrak{m})$

Using definitions and the triangle inequality we get those first two properties:

- $\mathrm{S}^p(X, d, \mathfrak{m})$ and $\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ are $\mathbb{R}-$vector spaces and for $\alpha, \beta \in \mathbb{R}$ we have that $\mathfrak{m}-$a.e.
$$|D(\alpha f + \beta g)|_w \leq |\alpha||Df|_w + |\beta||Dg|_w.$$
- The spaces $\mathrm{S}^p(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$ and $\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}) \cap L^\infty_{\mathrm{loc}}(X, \mathfrak{m})$ are algebras for which $\mathfrak{m}-$a.e.
$$|D(fg)|_w \leq |f||Dg|_w + |g||Df|_w.$$

We can now prove also other two fundamental properties that we will use for several proofs in the following chapters.

- (Locality principle): for every $f \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ we have that $|Df|_w = 0$ $\mathfrak{m}-$a.e. over $f^{-1}(\mathcal{N})$, $\forall \mathcal{N} \subset \mathbb{R}$ $\mathcal{L}^1-$negligible. Moreover, $\forall f, g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$, $\mathfrak{m}-$a.e. on $\{f = g\}$ it holds that
$$|Df|_w = |Dg|_w.$$

  Proof. Denoting with
$$G(x) := \begin{cases} |Df|_w(x) & f(x) \in \mathbb{R} \setminus \mathcal{N}, \\ 0 & f(x) \in \mathcal{N}, \end{cases}$$
  G is a $p-$weak upper gradient (considering the case of $\mathbb{R}-$valued absolutely continuous functions) and so the thesis follows from the minimality of $|Df|_w$. □

- (Chain rule for $f \in \mathrm{S}^p(X, d, \mathfrak{m})$): if $\phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then $\phi \circ f \in \mathrm{S}^p(X, d, \mathfrak{m})$ and $\mathfrak{m}-$a.e.
$$|D(\phi \circ f)|_w = |\phi' \circ f||Df|_w. \tag{3.5}$$

  Proof. As in (3.4) we can prove that $|D(\phi \circ f)|_w \leq |\phi' \circ f||Df|_w$. To get the equality, assume $\phi \in C^1$ (the Rademacher's theorem ensures that the right-hand side makes sense even in the Lipschitz case) and $0 \leq \phi' \leq 1$. By definition,

$(1 - \phi' \circ f)|Df|_w$ and $(\phi' \circ f)|Df|_w$ are upper gradients of $f - \phi \circ f$ and f respectively. Hence

$$|Df|_w \le |D(f - \phi \circ f)|_w + |D(\phi \circ f)|_w \le \big((1 - \phi' \circ f) + \phi' \circ f)\big)|Df|_w = |Df|_w.$$

and so all the inequalities are equalities and we have the thesis. □

Proposition 3.2.1. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1)*, $p \in (1, \infty)$ and $\Omega \subset X$ an open set. Then the following hold:*

I) *For $f \in \mathrm{S}^p(X, d, \mathfrak{m})$ (resp. $f \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$, $f|_{\overline{\Omega}} \in \mathrm{S}^p(X, d, \mathfrak{m})$ (resp. $f \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ and $\mathfrak{m}-$a.e.*

$$(|Df|_w)_X = (|Df|_w)_{\overline{\Omega},} \tag{3.6}$$

where $(|Df|_w)_X$ (resp. $(|Df|_w)_{\overline{\Omega}}$) denotes the minimal $p-$weak upper gradient of f in $(X, d, \mathfrak{m})$ (resp. of $f\ |_{\overline{\Omega}}$ in $(\overline{\Omega}, d, \mathfrak{m})$).

II) *Conversely, if $f \in \mathrm{S}^p(\overline{\Omega}, d, \mathfrak{m})$ (resp. $f \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$) with $\mathrm{supp}(f) \subset \Omega$ such that $d(\mathrm{supp}(f), X \setminus \Omega) > 0$ then extending f over all X setting it equal to 0 on $X \setminus \Omega$ we have have that $f \in \mathrm{S}^p(X, d, \mathfrak{m})$ and* (3.6) *holds.*

Proof. I) Since the class of curves to test if f is Sobolev in Ω is smaller w.r.t. the X's one then $f|_{\overline{\Omega}} \in \mathrm{S}^p(X, d, \mathfrak{m})$. Moreover, we have that $(|Df|_w)_{\overline{\Omega}} \le (|Df\|_w)_X$ $\mathfrak{m}-$a.e. in $\overline{\Omega}$ so it is sufficient to prove the converse inequality. Define the function $G : X \to [0, \infty]$ as

$$G(x) := \begin{cases} (|Df|_w)_{\overline{\Omega}}(x) & x \in \Omega, \\ +\infty & \text{otherwise}. \end{cases}$$

By definition G is a $p-$weak upper gradient of f in X. But $(|Df|_w)_X$ is the minimal $p-$weak upper gradient so $(|Df|_w)_X \le G$ $\mathfrak{m}-$a.e. in X.

II) We denote with $C := \mathrm{supp}(f)$. Then we need the following coarea lemma, for which we refer to [13]:

Lemma 3.2.2 (Eilenberg inequality)**.** *Let $f : X \to \mathbb{R}$ be a map over a Polish space X and denote by $N(f, y) := \sharp f^{-1}\{y\}$ the cardinality of $f^{-1}\{y\}$ for $y \in \mathbb{R}$. Then if f is Lipschitz, for $m \in [0, \infty]$ and A Borel subset of X the following inequality holds, with $\mathcal{H}^m$ the m-dimensional Hausdorff measure on X:*

$$(\mathrm{Lip}(f))^m \cdot \mathcal{H}^m(A) \ge \int N(f|_A, y)\, \mathrm{d}\mathcal{H}^m(y).$$

For any absolutely continuous curve γ the set $L_r := \{t \in [0, 1] \mid d(\gamma_t, C) = r\}$ is finite $r-$a.e. thanks to the Eilenberg inequality with $m = 0$, noticing also that the left-hand side is finite by assumptions. Setting $R := d(C, \partial\Omega) > 0$ and choosing $r \in (0, R)$ such that L_r is finite we can use L_r a set of times to decompose γ in a finite number of curves, some in Ω and others that does not intersect C. Hence the inequality (3.6) follows by locality and (I).

□

We endow $\mathrm{S}^p(X,d,\mathfrak{m})$ with the seminorm

$$\|f\|_{\mathrm{S}^p(X,d,\mathfrak{m})} := \||Df|_w\|_{L^p(X,\mathfrak{m})}$$

and define the Sobolev space $W^{1,p}(X,d,\mathfrak{m}) := \mathrm{S}^p(X,d,\mathfrak{m}) \cap L^p(X,\mathfrak{m})$ endowed with the norm, mimicking the Euclidian case,

$$\|f\|^p_{W^{1,p}(X,d,\mathfrak{m})} := \|f\|^p_{L^p(X,\mathfrak{m})} + \|f\|_{\mathrm{S}^p(X,d,\mathfrak{m})}.$$

it is not known if $\mathrm{S}^p(X,d,\mathfrak{m})$ is complete w.r.t. its seminorm. However, $W^{1,p}(X,d,\mathfrak{m})$ is always a Banach space thanks to the following proposition, that follows from definitions and the Mazur's lemma:

Proposition 3.2.3. *Let $(X,d,\mathfrak{m})$ be a metric measure space as in* (3.1), $p \in (1,\infty)$, $\{f_n\} \subset \mathrm{S}^p(X,d,\mathfrak{m})$ *and* $\{G_n\} \subset L^p(X,\mathfrak{m})$. *Assuming that G_n is a $p-$weak upper gradient for f_n $\forall n \in \mathbb{N}$, that $f_n \to f$ pointwise $\mathfrak{m}-$a.e. and that $G_n \rightharpoonup G$ in $L^p(X,\mathfrak{m})$ then $f \in \mathrm{S}^p(X,d,\mathfrak{m})$ and G is a $p-$weak upper gradient of f.*

We now prove two results concerning the duality between $q-$test plans and functions $\mathrm{S}^p(X,d,\mathfrak{m})$ that will be useful for the next chapter.

Definition 3.2.1. We define the *$q-$energy* $E_{q,t} : C([0,1],X) \to [0,\infty]$ of a curve $\gamma \in C([0,1],X)$, with $q \in (1,\infty)$, as

$$E_{q,t}(\gamma) := \begin{cases} t\sqrt[q]{\frac{1}{t}\int_0^t |\dot\gamma_s|^q \,\mathrm{d}s}, & \text{if } \mathrm{restr}_0^t(\gamma) \in AC^q([0,1],X), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.7}$$

Proposition 3.2.4. *Let $(X,d,\mathfrak{m})$ be a metric measure space as in* (3.1), *$p \in (1,\infty)$ and q its conjugate, $f \in S^p(X,d,\mathfrak{m})$ and $\boldsymbol{\pi}$ a $q-$test plan. Then the following inequalities hold:*

$$\left|\frac{f(\gamma_t)-f(\gamma_0)}{E_{q,t}}\right|^p \le \frac{1}{t}\int_0^t |Df|^p_w(\gamma_s)\,\mathrm{d}s \qquad \boldsymbol{\pi}-a.e., \forall\gamma \in [0,1], \tag{3.8}$$

$$\varlimsup_{t\downarrow 0}\int \left|\frac{f(\gamma_t)-f(\gamma_0)}{E_{q,t}}\right|^p \mathrm{d}\boldsymbol{\pi}(\gamma) \le \int |Df|^p_w(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma), \tag{3.9}$$

$$\varlimsup_{t\downarrow 0}\int \frac{f(\gamma_t)-f(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \frac{1}{p}\int |Df|^p_w(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma) + \varlimsup_{t\downarrow 0}\frac{1}{qt}\iint_0^t |\dot\gamma_s|^q\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma). \tag{3.10}$$

Proof. From the definition of minimal $p-$weak upper gradient we immediately get (3.8):

$$|f(\gamma_t)-f(\gamma_0)| \le \int_0^t |Df|_w(\gamma_s)|\dot\gamma_s\,\mathrm{d}s \overset{\text{Hölder}}{\le} \sqrt[p]{\int_0^t |Df|^p_w(\gamma_s)\,\mathrm{d}s}\sqrt[q]{\int_0^t |\dot{\gamma_s}^q|\,\mathrm{d}s}.$$

Diving both sides by $\sqrt[q]{\int_0^t |\dot{\gamma_s}^q|\,\mathrm{d}s}$ and using the definition of $E_{q,t}$ we obtain the thesis. For (3.9) if ρ_s is the density of $(e_s)_\sharp\boldsymbol{\pi}$ w.r.t. $\mathfrak{m}$, integrating (3.8) we have that

$$\int\left|\frac{f(\gamma_t)-f(\gamma_0)}{E_{q,t}}\right|\mathrm{d}\boldsymbol{\pi}(\gamma) \le \frac{1}{t}\int_0^t\int |Df|^p_w\,\mathrm{d}(e_s)_\sharp\boldsymbol{\pi}\,\mathrm{d}s = \int |Df|^p_w\left(\frac{1}{t}\int_0^t \rho_s\,\mathrm{d}s\right)\mathrm{d}\mathfrak{m}.$$

By definition, $|Df|_w^p \in L^1(X,\mathfrak{m})$ and $\rho_s\mathfrak{m} \overset{*}{\rightharpoonup} \rho_0\mathfrak{m}$ hence $\rho_s \to \rho_0$ as $s \to 0$. Hence

$$\lim_{t\downarrow 0}\int |Df|_w^p\left(\frac{1}{t}\int_0^t \rho_s\,\mathrm{d}s\right)\mathrm{d}\mathfrak{m} = \int |Df|_w^p(\gamma_0)\,\mathrm{d}\boldsymbol{\pi}(\gamma).$$

Finally, for(3.10), using Young's inequality we get

$$\frac{|f(\gamma_t)-f(\gamma_0)|}{t} = \frac{|f(\gamma_t)-f(\gamma_0)|}{E_{q,t}(\gamma)}\frac{E_{q,t}(\gamma)}{t} \overset{\text{Young}}{\leq} \frac{1}{p}\left|\frac{f(\gamma_t)-f(\gamma_0)}{E_{q,t}(\gamma)}\right|^p + \frac{1}{q}\left|\frac{E_{q,t}(\gamma)}{t}\right|^q. \tag{3.11}$$

The thesis follows integrating w.r.t. $\boldsymbol{\pi}$ and using (3.9) in passing to the limit as $t \downarrow 0$. $\square$

Proposition 3.2.5. *Let $(X,d,\mathfrak{m})$ be a metric measure space as in* (3.1), *$p \in (1,\infty)$ and q its conjugate, $f \in \mathrm{S}^p(X,d,\mathfrak{m})$ and $\boldsymbol{\pi}$ a $q-$test plan. Then*

$$\lim_{t\downarrow 0}\sqrt[p]{\frac{1}{t}\int_0^t |Df|_w^p \circ e_s\,\mathrm{d}s} = |Df|_w \circ e_0 \qquad \textit{in } L^p(\boldsymbol{\pi}). \tag{3.12}$$

Furthermore, if the family of functions $\frac{E_{q,t}}{t}$ is dominated in $L^q(\boldsymbol{\pi})$ then the family of functions $\frac{f(\gamma_t)-f(\gamma_0)}{t}$ is dominated in $L^1(\boldsymbol{\pi})$.

Proof. it is sufficient to prove that

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^t |Df|_w^p \circ e_s\,\mathrm{d}s = |Df|_w^p \circ e_0 \qquad \text{in } L^1(\boldsymbol{\pi}). \tag{3.13}$$

Let $h \in L^1(X,\mathfrak{m})$ and define $H_t \in L^1(\boldsymbol{\pi})$, for $t \in [0,1]$, as

$$H_t := \frac{1}{t}\int_0^t h \circ e_s\,\mathrm{d}s, \qquad H_0 := h \circ e_0.$$

Notice that if h is $L-$Lipschitz then

$$|H_t(\gamma) - H_0(\gamma)| \leq L\,\frac{1}{t}\int_0^t d(\gamma_s,\gamma_0)\,\mathrm{d}s \leq L\int_0^t |\dot{\gamma}_s|\mathrm{d}s$$

so that $\|H_t - H_0\|_{L^1(\boldsymbol{\pi})} \overset{t\downarrow 0}{\to} 0$. Using the density of Lipschitz functions in $L^1(X,\mathfrak{m})$ and by definition there exists a constant $C>0$ such that $(e_t)_\sharp\boldsymbol{\pi} \leq C\mathfrak{m}$ $\forall t \in [0,1]$, we have

$$\int \frac{1}{t}\left|\int_0^t h(\gamma_s)\,\mathrm{d}s\right|\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \frac{1}{t}\iint_0^t |h(\gamma_s)|\,\mathrm{d}s\,\mathrm{d}\pi(\gamma) \leq C\|h\|_{L^1(X,\mathfrak{m})},$$
$$\int |h(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq C\|h\|_{L^1(X,\mathfrak{m})}.$$

We get the thesis choosing $h = |Df|_w^p$. For the second part, from (3.11) and (3.8) we have that

$$\frac{|f(\gamma_t)-f(\gamma_0)|}{t} \leq \frac{1}{pt}\int_0^t |Df|_w^p(\gamma_s)\,\mathrm{d}s + \frac{1}{q}\left|\frac{E_{q,t}(\gamma)}{t}\right|^q,$$

and from (3.13) the family of functions $\frac{1}{pt}\int_0^t |Df|_w^p(\gamma_s)\,\mathrm{d}s$ is dominated in $L^1(\boldsymbol{\pi})$ and we obtain also the second thesis. $\square$

3.3 The Cheeger energy

In this section we define the Cheeger's energy functional that will be useful in several proof later. It will be useful also to establish some connections between gradient flows in L^2 and the Wasserstein geometry.

This first result, proven in section 8.2 of [4], concerns the dependence of $\mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m})$ on the reference measure $\mathfrak{m}$ of the metric measure space X:

Theorem 3.3.1. *Let (X,d) be a Polish space and $\mathfrak{m},\mathfrak{m}'$ two non-negative Radon measures on X. Assume that*

$$\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m} \quad \textit{and that} \quad \frac{\mathrm{d}\mathfrak{m}'}{\mathrm{d}\mathfrak{m}} \textit{ is locally bounded from below by a positive constant.}$$

Then for every $p \in (1,\infty)$ the spaces $\mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m})$ and $\mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m}')$ coincide and for $f \in \mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m}) = \mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m}')$ a function G is a $p-$weak upper gradient of f in $\mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m})$ if and only if it is so in $\mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m}')$.

In summary, the notion of being a Sobolev function is unchanged if we replace the reference measure $\mathfrak{m}$ with an equivalent one $\mathfrak{m}'$ such that $\ln(\frac{\mathrm{d}\mathfrak{m}'}{\mathrm{d}\mathfrak{m}}) \in L^\infty(X,\mathfrak{m})$.

Remark 3.3.1. Let $(X,d,\mathfrak{m})$ be a metric measure space as in (3.1) and $\tilde{\mathfrak{m}} \in \mathscr{P}(X)$ as in (3.2) and (3.3). Since $\tilde{\mathfrak{m}} \le C\mathfrak{m}$ for some $C>0$ and from Proposition 3.3.1 we have that

$$\mathrm{S}^p(X,d,\mathfrak{m}) \subset \mathrm{S}^p(X,d,\tilde{\mathfrak{m}})$$

Definition 3.3.1. Let $p \in (1,\infty)$. We define the *Cheeger energy functional* $\mathrm{Ch}_p : L^2(X,\mathfrak{m}) \to [0,\infty]$ by

$$\mathrm{Ch}_p := \begin{cases} \frac{1}{p}\int |Df|_w^p \,\mathrm{d}\mathfrak{m} & f \in \mathrm{S}^p(X,d,\mathfrak{m}) \cap L^2(X,\mathfrak{m}), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.14}$$

Similarly, we define $\tilde{\mathrm{Ch}}_p : L^2(X,\mathfrak{m}) \to [0,\infty]$ in the same way using $\tilde{\mathfrak{m}}$ instead of $\mathfrak{m}$.

Proposition 3.3.2. *Ch_p is convex, lower semicontinuous and has dense domain, hence for $f \in L^2(X,\mathfrak{m})$ there exists a unique gradient-flow of Ch_p starting from f.*

Proof. The energy is convex by definition. Let $\{f_n\}$ be convergent to f in $L^2(X,\mathfrak{m})$ and assume, after possibly extracting a subsequence and with no loss of generality, that $\mathrm{Ch}_p(f_n)$ converges to a finite limit.

If we first assume that all the f_n have $p-$weak upper gradient then $|Df_n|_w$ is uniformly bounded in $L^p(X,\mathfrak{m})$. Let $f_{n(k)}$ be a subsequence such that $|Df_{n(k)}|_w$ converges weakly to g in $L^p(X,\mathfrak{m})$. Then g is $p-$weak upper gradient of f and

$$\mathrm{Ch}_p(f) \le \frac{1}{q}\int_X |g|^q \,\mathrm{d}\mathfrak{m} \le \varliminf_{k\to\infty} \frac{1}{q}\int_X |Df_{n(k)}|_w^p \,\mathrm{d}\mathfrak{m} = \varliminf_{n\to\infty} \mathrm{Ch}_p(f_n).$$

Denote now by $f^N := \max\{-N, \min\{f,N\}\}$ and set

$$\mathscr{C} := \{f : X \to \mathbb{R} : f^N \text{ has a } p-\text{ weak upper gradient for all } N \in \mathbb{N}\}.$$

In the general case when $f_n \in \mathscr{C}$, we consider the functions f_n^N and to conclude is sufficient to notice that from $|Df_n^N|_w \leq |Df_n|_w$ we have that f_N^N has $p-$weak upper gradient for any $N \in \mathbb{N}$ and

$$\int_X |Df^N|_w^p \,\mathrm{d}\mathfrak{m} \leq \underline{\lim}\, n \to \infty \int_X |Df_n^N|_w^p \,\mathrm{d}\mathfrak{m} \leq \varliminf_{n\to\infty} \int_X |Df_n|_w^p \,\mathrm{d}\mathfrak{m}.$$

Passing to the limit as $N \to \infty$ the conclusion follows by monotone convergence. Since the finiteness domain of Ch_p is dense in $L^2(X,\mathfrak{m})$, the Hilbertian theory of gradient flows (see, for instance, [9]) can be applied to Cheeger's energy functional to provide, for all $f_0 \in L^2(X,\mathfrak{m})$ a locally absolutely continuous map $t \mapsto f_t$ from $(0,\infty)$ to $L^2(X,\mathfrak{m})$ with $f_t \to f_0$ as $t \downarrow 0$, whose derivatives satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t \in -\partial^- \mathrm{Ch}_p(f_t) \quad (0\infty) \ni t - \text{a.e.}.$$

□

We then collect here some properties of the gradient-flow of the Cheeger energy, for the proof see [3] and [4]:

Theorem 3.3.3. *Let $(X,d,\mathfrak{m})$ be a metric measure space as in* (3.1) *and $\tilde{\mathfrak{m}} \in \mathscr{P}(X)$ as in* (3.2) *and* (3.3). *Moreover, let $p \in (1,\infty)$, $f \in L^2(X,\tilde{\mathfrak{m}})$ and $(f_t) \subset L^2(X,\tilde{\mathfrak{m}})$ be the gradient-flow of $\tilde{\mathrm{Ch}}_p$ starting from f. Then the following hold:*

I) Mass preservation*:*

$$\int f_t \,\mathrm{d}\tilde{\mathfrak{m}} = \int f \,\mathrm{d}\tilde{\mathfrak{m}} \quad \forall t \geq 0.$$

II) Maximum principle*: if $f \leq C$ (resp. $f \geq c$) $\tilde{\mathfrak{m}}-$a.e. then $f_t \leq C$ (rrsp. $f_t \geq c$) $\tilde{\mathfrak{m}}-$a.e. for any $t \geq 0$.*

III) Entropy dissipation*: if $c \leq f \leq C$ $\tilde{\mathfrak{m}}-$a.e. and $u : [c,C] \to \mathbb{R}$ is a C^2-function then $t \mapsto \int u(f_t)\,\mathrm{d}\mathfrak{m}$ is a $C^1(0,\infty)-$function and the following equality holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int u(f_t)\,\mathrm{d}\tilde{\mathfrak{m}} = -\int u''(f_t)|Df_t|_w^p \,\mathrm{d}\tilde{\mathfrak{m}} \quad \forall t > 0. \tag{3.15}$$

IV) Dissipation at $t = 0$*: if with the same assumptions as in (III) we also assume that $f \in \mathrm{S}^p(X,d,\mathfrak{m})$ then* (3.15) *holds also at $t = 0$ and it holds that*

$$\lim_{t\downarrow 0} \frac{\mathrm{d}}{\mathrm{d}t} \int u(f_t)\,\mathrm{d}\tilde{\mathfrak{m}} = -\int u''(f)|Df|_w^p \,\mathrm{d}\tilde{\mathfrak{m}}. \tag{3.16}$$

V) Kuwada's lemma*: assume that $c \leq f \leq C$ with $c, C > 0$ and that $\int f\,\mathrm{d}\tilde{\mathfrak{m}} = 1$. Then the curve $t \mapsto \mu_t := f_t\tilde{\mathfrak{m}}$ is $q-$absolutely continuous w.r.t. W_q, with q the conjugate exponent of p, and for its metric derivative $|\dot{\mu}_t|$ we have that*

$$|\dot{\mu}_t|^q \leq \int \frac{|Df_t|_w^p}{f_t^{q-1}} \,\mathrm{d}\tilde{\mathfrak{m}} \quad t - a.e.. \tag{3.17}$$

The first application of this theorem is to prove the following ones as in [1], [4] and [6] and they can be regarded as a metric version of the Meyers-Serrin theorem.

Theorem 3.3.4. *Let* $(X, d, \mathfrak{m})$ *be a metric measure space as in* (3.1), $p \in (1, \infty)$ *and assume that* $\mathfrak{m}$ *is finite on bounded sets. Then, for every* $f \in W^{1,p}(X, d, \mathfrak{m})$ *there exists a sequence* $\{f_n\} \subset W^{1,p}(X, d, \mathfrak{m})$ *of Lipschitz functions such that*

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(X,\mathfrak{m})} = 0,$$
$$\lim_{n\to\infty} \|f_n\|_{\mathrm{S}^p(X,d,\mathfrak{m})} = \lim_{n\to} \||Df_n|\|_{L^p(X,\mathfrak{m})} = \lim_{n\to} \|\overline{|Df_n|}\|_{L^p(X,\mathfrak{m})} = \|f\|_{\mathrm{S}^p(X,d,\mathfrak{m})},$$

where given $h : X \to \mathbb{R}$, $\overline{|Dh|} : X \to [0, \infty]$ *is set by definition to* 0 *at isolated points and*

$$\overline{|Dh|}(x) := \inf_{r>0} \sup_{y_1 \neq y_2 \in B_r(x)} \frac{|h(y_1) - h(y_2)|}{d(y_1, y_2)}.$$

Corollary 3.3.5. *With the same definition as before, assuming also that* $W^{1,p}(X, d, \mathfrak{m})$ *is uniformly convex then* $\mathrm{LIP}(X) \cap W^{1,p}(X, d, \mathfrak{m})$ *is dense in* $W^{1,p}(X, d, \mathfrak{m})$.

Chapter 4

Differentials and gradients

We now want to study how the differential of a function f operates on the gradient of another function g. To this aim, we first define the two functions $D^{\pm}f(\nabla g)$ and define for the following the infinitesimally strictly convex spaces, where the two functions agree. Next we will prove the duality between test plans and gradients which will allow us to get the gradient ∇g, justifying the expression $D^{\pm}f(\nabla g)$ because $|Dg|_w$ in the smooth setting is the norm of the gradient. Finally we prove the chain and Leibniz rules for $D^{\pm}f(\nabla g)$ and how they turns into equalities in infinitesimally strictly convex spaces.

4.1 Basic definitions and first properties

Considering $a, b \in \mathbb{R}^+$ we have

$$a^n - b^n = (a-b)(b^{n-1} + ab^{n-2} + \cdots + a^{n-2}b + a^{n-1}).$$

With a convex function $\phi : \mathbb{R} \to \mathbb{R}^+$ and using the preceding equality, for $p \in (1, \infty)$ we get

$$\frac{\phi(\varepsilon)^p - \phi(0)^p}{p\varepsilon\phi(0)^{p-2}} = \frac{\phi(\varepsilon) - \phi(0)}{p\varepsilon\phi(0)^{p-2}}\left(\phi(\varepsilon)^{p-1} + \phi(\varepsilon)^{p-2}\phi(0) + \cdots + \phi(0)^{p-1}\right) \xrightarrow{\varepsilon \to 0^+} \phi'(0^+)\phi(0).$$

Hence the two quantities

$$\inf_{\varepsilon > 0} \frac{\phi(\varepsilon)^p - \phi(0)^p}{p\varepsilon\phi(0)^{p-2}}, \qquad \sup_{\varepsilon < 0} \frac{\phi(\varepsilon)^p - \phi(0)^p}{p\varepsilon\phi(0)^{p-2}}$$

are independent of the value of p and are equal to $\phi(0)\phi'(0^+)$ and $\phi(0)\phi'(0^-)$ respectively if $\phi(0) \neq 0$. Moreover, the convexity of ϕ allows us to replace the $\sup$ and $\inf$ with the right and left limit respectively for $\varepsilon \downarrow 0$ and $\varepsilon \uparrow 0$.

So now we fix $p \in (1, \infty)$ and let $f, g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$. Thanks to the upper gradients' properties we have that the map $\varepsilon \mapsto |D(g + \varepsilon f)|_w$ is convex w.r.t. ε, i.e.

$$|D(g + ((1-\lambda)\varepsilon_0 + \lambda\varepsilon_1)f|_w \leq (1-\lambda)|D(g + \varepsilon_0 f)|_w + \lambda|D(g + \varepsilon_1 f)|_w, \quad \mathfrak{m} - \text{a.e.},$$

for every $\lambda \in [0, 1], \varepsilon_0, \varepsilon_1 \in \mathbb{R}$ so mimicking the preceding argument we have the following definition:

Definition 4.1.1. Let $(X, d, \mathfrak{m})$ be a metric measure space as in (3.1), $p \in (1, \infty)$ and $f, g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$. The two functions $D^\pm f(\nabla g) : X \to \mathbb{R}$ are well defined $\mathfrak{m}-$a.e. by the formulas

$$D^+ f(\nabla g) := \inf_{\varepsilon>0} \frac{|D(g+\varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon |Dg|_w^{p-2}}, \tag{4.1}$$

$$D^- f(\nabla g) := \sup_{\varepsilon<0} \frac{|D(g+\varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon |Dg|_w^{p-2}}, \tag{4.2}$$

on $\{x \in X : |Dg|_w(x) \neq 0\}$ and are equal to 0 by definition on the complement.

If we apply this definition in the Euclidian case we get the $p-$Laplacian. To see it more clearly, recall that by Riesz theorem we have

$$Df(\nabla g) = \langle \nabla f, \nabla g \rangle, \qquad |Dg|_w = \|\nabla g\|_{\mathbb{R}^n},$$

and the formulas (4.1) and (4.3) agree. For example, if $p = 2$ once integrated we just get the first variation of the Dirichlet energy functional (also, we remark that the metric equivalent of the Dirichlet energy is the Cheeger's one).

Remark 4.1.1. In our notations we omit the explicit dependence on the Sobolev exponent p in writing $D^\pm(\nabla g)$. By Remark 3.1.5, if the space X is doubling and supports a $(1-p')-$Poincaré inequality, $D^\pm f(\nabla g)$ is unambiguously defined for $f, g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$, where $p \geq p'$.

Remark 4.1.2. From now on, the expressions $D^\pm f(\nabla g)|Dg|_w^{p-2}$ on the set $\{x \in X : |Dg|_w(x) = 0\}$ will always be taken 0 by definition. In this way we obtain $\mathfrak{m}-$a.e. on X

$$D^+ f(\nabla g)|Dg|_w^{p-2} = \inf_{\varepsilon>0} \frac{|D(g+\varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon},$$

$$D^- f(\nabla g)|Dg|_w^{p-2} = \sup_{\varepsilon<0} \frac{|D(g+\varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon}.$$

Thanks to the inequality $|D(g+\varepsilon f)|_w \leq |Dg|_w + |\varepsilon||Df|_w$ we have, setting $|Dg|_w = a$ and $|Df|_w = b$, that

$$|D^\pm f(\nabla g)| \leq \frac{(a+\varepsilon b)^p - a^p}{p\varepsilon\, a^{p-2}} = \frac{a^p + \varepsilon\, a^{p-1} b + o(\varepsilon) - a^p}{p\varepsilon\, a^{p-2}} \overset{\varepsilon \to 0^\pm}{\longrightarrow} ab$$

so that

$$|D^\pm f(\nabla g)| \leq |Df|_w |Dg|_w \qquad \mathfrak{m} - \text{a.e.} \tag{4.3}$$

In particular, we have also that $D^\pm f(\nabla g)|Dg|_w^{p-2} \in L^1(X, \mathfrak{m})$ for all $f, g \in \mathrm{S}^p(X, d, \mathfrak{m})$ (respectively in $L^1_{\mathrm{loc}}(X, \mathfrak{m})$ for all $f, g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$).

Setting $g_\varepsilon := g + \varepsilon f$, the convexity of $f \mapsto |D(g+\varepsilon f)|_w^p$ yields

$$D^- f(\nabla g) \leq D^+ f(\nabla g), \qquad \mathfrak{m} - \text{a.e.} \tag{4.4}$$

and

$$D^+ f(\nabla g)|Dg|_w^{p-2} = \inf_{\varepsilon>0} D^- f(\nabla g)|Dg|_w^{p-2} = \inf_{\varepsilon>0} D^+ f(\nabla g)|Dg|_w^{p-2}, \qquad \mathfrak{m} - \text{a.e.}, \tag{4.5}$$

$$D^- f(\nabla g)|Dg|_w^{p-2} = \sup_{\varepsilon<0} D^+ f(\nabla g)|Dg|_w^{p-2} = \sup_{\varepsilon<0} D^- f(\nabla g)|Dg|_w^{p-2}, \qquad \mathfrak{m} - \text{a.e.}. \tag{4.6}$$

Moreover, from the definition it directly follows that

$$D^+(-f)(\nabla g) = -D^- f(\nabla g) = D^+ f(\nabla(-g)), \qquad \mathfrak{m}\text{-a.e..} \tag{4.7}$$

Setting $f = g$ we get

$$\frac{|D(g+\varepsilon g)|_w^p - |Dg|_w^p}{p\varepsilon |Dg|_w^{p-2}} = \frac{(|1+\varepsilon|^p - 1)|Dg|_w^p}{p\varepsilon |Dg|_w^{p-2}} = \frac{1}{p\varepsilon}|Dg|_w^2 \sum_{i=0}^{p} \binom{p}{i} \varepsilon^i \overset{\varepsilon \to 0^\pm}{\longrightarrow} |Dg|_w^2,$$

therefore

$$D^\pm(\nabla g) = |Dg|_w^2, \qquad \mathfrak{m} - \text{a.e..} \tag{4.8}$$

As a consequence of the locality properties we also have

$$D^\pm f(\nabla g) = 0, \quad \mathfrak{m} - \text{a.e. on} \quad f^{-1}(\mathcal{N}) \cup g^{-1}(\mathcal{N}),\ \forall \mathcal{N} \subset \mathbb{R} : \mathcal{L}^1(\mathcal{N}) = 0, \tag{4.9}$$

$$D^\pm(\nabla g) = D^\pm f(\nabla \tilde{g}), \qquad \mathfrak{m} - \text{a.e. on} \qquad \{f = \tilde{f}\} \cap \{g = \tilde{g}\}. \tag{4.10}$$

Some further properties of $D^\pm f(\nabla g)$ are collected in the following proposition

Proposition 4.1.1. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1), *$p \in (1, \infty)$ and $g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$.Then the function*

$$\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}) \ni f \mapsto D^+ f(\nabla g)$$

is positively $1-$homogeneous, convex $\mathfrak{m}-$a.e. in the sense that

$$D^+((1-\lambda)f_0 + \lambda f_1)(\nabla g) \leq (1-\lambda)D^+ f_1(\nabla g) + \lambda D^+ f_2(\nabla g), \qquad \mathfrak{m} - \text{a.e.},$$

for all $f_0, f_1 \in \mathrm{S}^+_{\mathrm{loc}}(X, d, \mathfrak{m})$ and $\lambda \in [0,1]$, and $1-$Lipschitz in the following sense:

$$|D^+ f_1(\nabla g) - D^+ f_2(\nabla g)| \leq |D(f_1 - f_2)|_w |Dg|_w, \qquad \mathfrak{m} - \text{a.e.}\ \forall f_1, f_2 \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}). \tag{4.11}$$

Similarly, the function

$$\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}) \ni f \mapsto D^- f(\nabla g)$$

is positively $1-$homogeneous, concave and $1-$Lipschitz.
Conversely, for all $f \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ it holds

- *$\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}) \ni g \mapsto D^+ f(\nabla g)$ is positively $1-$homogeneous and upper semicontinuous,*
- *$\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}) \ni g \mapsto D^- f(\nabla g)$ is positively $1-$homogeneous and lower semicontinuous,*

where the upper semicontinuity is intended as follows: if $g_n, g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}), n \in \mathbb{N}$, and for some Borel set $E \subset X$ we have that $\int_E |Df|_w^p \,\mathrm{d}\mathfrak{m} < \infty$, $\sup_{n \in \mathbb{N}} \int_E |Dg_n|_w^p \,\mathrm{d}\mathfrak{m} < \infty$ and $\int_E |D(g_n - g)|_w^p \,\mathrm{d}\mathfrak{m} \to 0$, then

$$\varlimsup_{n\to\infty} \int_{E'} D^+ f(\nabla g_n)|Dg_n|_w^{p-2} \,\mathrm{d}\mathfrak{m} \leq \int_{E'} D^+ f(\nabla g)|Dg|_w^{p-2} \,\mathrm{d}\mathfrak{m}, \qquad \forall \text{ Borel set } E' \subset E.$$

Similarly for lower semicontinuity.

Proof. The positive $1-$ homogeneity in f, g follows from definitions.

For convexity (respectively concavity) in f, we have that

$$|Dg + \varepsilon((1-\lambda)f_0 + \lambda f_1)|_w - |Dg|_w \leq (1-\lambda)\Big(|D(g+\varepsilon f_0)_w - |Dg|_w\Big) \\ +\lambda\Big(|D(g+\varepsilon f_1)|_w |Dg|_w\Big),$$

so dividing by $\varepsilon > 0$ (resp. $\varepsilon < 0$) and letting $\varepsilon \downarrow 0$ (resp. $\varepsilon \uparrow 0$) we get the thesis.

For the Lipschitz continuity, just notice that using convexity and adding and subtracting $\frac{|Dg|_w}{\varepsilon}$ we get

$$\left|\frac{|D(g+\varepsilon f)|_w - |Dg|_w}{\varepsilon} - \frac{|D(g+\varepsilon \tilde{f})|_w - |Dg|_w}{\varepsilon}\right| \leq |D(f-\tilde{f})|_w, \qquad \forall\, \varepsilon \neq 0.$$

Finally let $\varepsilon \to 0$.

For the semicontinuity in g, let $E \subset X$ be a Borel set as in the assumptions and let

$$V := \Big\{ g \in \mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m}) : \int_E |Dg|_w^p \,\mathrm{d}\mathfrak{m} < \infty \Big\}.$$

Endow V with the seminorm $\|g\|_{\mathrm{S}^p,E} := \sqrt[p]{\int_E |Dg|_w^p \,\mathrm{d}\mathfrak{m}}$ and notice that $\forall\, \varepsilon \neq 0$ the real valued map

$$V \ni g \quad \mapsto \quad \int_{E'} \frac{|D(g+\varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon}\,\mathrm{d}\mathfrak{m}$$

is continuous. Hence by the properties of the weak convergence the map

$$V \ni g \quad \mapsto \quad \int_{E'} D^+f(\nabla g)|Dg|_w^{p-2}\,\mathrm{d}\mathfrak{m} = \inf_{\varepsilon>0}\int_{E'} \frac{|D(g+\varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon}\,\mathrm{d}\mathfrak{m}$$

is lower semicontinuous. Similarly for $D^-f(\nabla g)$. □

In general $D^+f(\nabla g) \neq D^-f(\nabla g)$, but for example in strictly convex normed spaces then they agree $\mathfrak{m}-$a.e.. So we have the following definition.

Definition 4.1.2. Let $(X,d,\mathfrak{m})$ be a metric measure space as in (3.1), $p \in (1,\infty)$ and q its conjugate exponent. We say that $(X,d,\mathfrak{m})$ is *$q-$infinitesimally stricly convex* (shortly $q-$i.s.c.) if

$$\int D^+f(\nabla g)|Dg|_w^{p-2}\,\mathrm{d}\mathfrak{m} = \int D^-f(\nabla g)|Dg|_w^{p-2}\,\mathrm{d}\mathfrak{m}, \qquad \forall\, f,g \in \mathrm{S}^p(X,d,\mathfrak{m}). \tag{4.12}$$

Remark 4.1.3. In the case X normed space the condition (4.12), whatever q is, is equivalent to the strict convexity of the norm, as analyzed in the introduction.

From inequality (4.4) we get that the integral equality (4.12) is equivalent to the pointwise one:

$$D^+f(\nabla g) = D^-f(\nabla g) \qquad \mathfrak{m}-\text{a.e.} \qquad \forall\, f,g \in \mathrm{S}^p(X,d,\mathfrak{m}). \tag{4.13}$$

Then thanks to the locality properties and using a cut-off argument we have that (4.13) is true also for $f,g \in \mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m})$. Furthermore, from Remark 3.1.5 we know that if $\mathfrak{m}$

is a doubling measure and the space supports a p'−weak Poincaré inequality, if X is q'−i.s.c. then is also q−i.s.c. for every $q \in (1, q')$, with q' conjugate exponent of p'.

If X is q−i.s.c.then se denote

$$D^+ f(\nabla g) = D^- f(\nabla g) =: Df(\nabla g) \qquad f, g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}).$$

Directly from Proposition 4.1.1 we deduce the following corollary.

Corollary 4.1.2. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1)*, $p, q \in (1, \infty)$ conjugate exponents. Assume that X is q−i.s.c.. Then*

- *for any $g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ the map*

$$\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}) \ni f \quad \mapsto \quad Df(\nabla g)$$

is linear $\mathfrak{m}$−a.e., i.e.

$$D(\alpha_1 f_1 + \alpha_2 f_2)(\nabla g) = \alpha_1 Df_1(\nabla g) + \alpha_2 Df_2(\nabla g) \quad \mathfrak{m} - \textit{i.e.}$$

for any $f_1, f_2 \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}), \alpha_1, \alpha_2 \in \mathbb{R}$.

- *For any $f \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ the map*

$$\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}) \ni g \quad \mapsto \quad Df(\nabla g)$$

is 1−homogeneous and continuous: if $g_n, g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}), n \in \mathbb{N}$ and for some Borel set $E \subset X$ it holds $\sup_{n \in \mathbb{N}} \int_E |Dg|^p_w \,\mathrm{d}\mathfrak{m} < \infty$ and $\int_E |D(g_n - g)|^p_w \,\mathrm{d}\mathfrak{m} \to 0$ then

$$\lim_{n\to\infty} \int_{E'} Df(\nabla g_n)|Dg_n|^{p-2}_w \,\mathrm{d}\mathfrak{m} = \int_{E'} Df(\nabla g)|Dg|^{p-2}_w \,\mathrm{d}\mathfrak{m}, \qquad \forall\, \textit{Borel set } E' \subset E.$$

Remark 4.1.4. From Proposition 4.1.1 we get the weak lower semicontinuity of f in $S^p(X, d, \mathfrak{m})$: for $p \in (1, \infty)$ and $g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ the map

$$\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m}) \ni f \quad \mapsto \quad \int D^+ f(\nabla g)|Dg|^{p-2}_w \,\mathrm{d}\mathfrak{m}$$

is weakly lower semicontinuous, i.e. if $L(f_n) \overset{n\to\infty}{\longrightarrow} L(f)$ for every $L \in \left(\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})\right)^*$ then

$$\varliminf_{n\to\infty} \int D^+ f_n(\nabla g)|Dg|^{p-2}_w \,\mathrm{d}\mathfrak{m} \geq \int D^+ f(\nabla g)|Dg|^{p-2}_w \,\mathrm{d}\mathfrak{m}.$$

In fact just notice that that $f \mapsto \int D^+ f(\nabla g)|Dg|^{p-2}_w \,\mathrm{d}\mathfrak{m}$ is convex and continuous and $f \mapsto \int D^- f(\nabla g)|Dg|^{p-2}_w \,\mathrm{d}\mathfrak{m}$ is weakly upper semicontinuous in $\mathrm{S}^p(X, d, \mathfrak{m})$.

4.2 Duality between test plans and gradients

In the preceding section we defined how the differential of f operates on the gradient of g, for $f, g \in \mathrm{S}^p(X, d, \mathfrak{m})$. But recall that $|Dg|_w$ is the equivalent of the *norm* of the gradient of g, so we need to define a *real* gradient, justifying the expression $D^{\pm}(\nabla g)$.

On a flat normed space, for any two smooth functions f, g it can be proven that the following equalities hold

$$\inf_{\varepsilon>0} \frac{\|D(g+\varepsilon f)(x)\|_*^2 - \|Dg(x)\|_*^2}{2\varepsilon} = \max_{v\in\nabla g(x)} Df(v),$$
$$\sup_{\varepsilon<0} \frac{\|D(g+\varepsilon f)(x)\|_*^2 - \|Dg(x)\|_*^2}{2\varepsilon} = \min_{v\in\nabla g(x)} Df(v).$$

We study in this section the validity of this statement in our metric setting. More precisely, we will prove that

$$\inf_{\varepsilon>0} \frac{\|D(g+\varepsilon f)(x)\|_*^2 - \|Dg(x)\|_*^2}{2\varepsilon} \geq Df(v) \geq \sup_{\varepsilon<0} \frac{\|D(g+\varepsilon f)(x)\|_*^2 - \|Dg(x)\|_*^2}{2\varepsilon} \tag{4.14}$$

for all $v \in \nabla g(x)$.

The two functions $D^+f(\nabla g)$ e $D^-f(\nabla g)$ will replace the leftmost and rightmost sides in (4.14).

Definition 4.2.1. Let $q \in (1,\infty)$ and $\boldsymbol{\pi} \in \mathscr{P}\big(C([0,1],X)\big)$. We define the $q-$*norm* $\|\boldsymbol{\pi}\|_q \in [0,\infty]$ of $\boldsymbol{\pi}$ by

$$\|\boldsymbol{\pi}\|_q := \sqrt[q]{\varlimsup_{t\downarrow 0} \int \left(\frac{E_{q,t}}{t}\right)^q \mathrm{d}\boldsymbol{\pi}} \triangleq \sqrt[q]{\varlimsup_{t\downarrow 0} \frac{1}{t} \iint_0^t |\dot\gamma_s|^q \,\mathrm{d}s \,\mathrm{d}\boldsymbol{\pi}(\gamma)} \tag{4.15}$$

if $(\mathrm{restr}_0^T)_\sharp \boldsymbol{\pi}$ is concentrated on absolutely continuous curves for some $T \in (0,1]$ and $+\infty$ otherwise.

Remark 4.2.1. By calling $\|\cdot\|_q$ a "norm" we are abusing the notation because $\mathscr{P}\big(C([0,1],X)\big)$ is not a vector space.

Remark 4.2.2. If $\boldsymbol{\pi}$ has bounded compression and $\|\boldsymbol{\pi}\|_q < \infty$ then not necessarily $\boldsymbol{\pi}$ is a $q-$test plan because it might happen that $\iint_0^1 |\dot\gamma_t|^q \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma) = \infty$. However, for $T > 0$ small enough $(\mathrm{restr}_0^T)_\sharp \boldsymbol{\pi}$ is $q-$test plan.

Hence, if $p, q \in (1,\infty)$ are conjugate exponents and $\boldsymbol{\pi}$ has bounded compression with $\|\boldsymbol{\pi}\|_q < \infty$ and $g \in \mathrm{S}^p(X,d,\mathfrak{m})$ from (3.10) we have that

$$\varlimsup_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \,\mathrm{d}\boldsymbol{\pi} \leq \frac{\||Dg|_w\|^q_{L^p(X,(e_0)_\sharp\boldsymbol{\pi}}}{p} + \frac{\|\boldsymbol{\pi}\|_q^q}{q}. \tag{4.16}$$

Notice that the left-hand side of this inequality reminds a derivative along absolutely continuous curves. Hence we have the following definition:

Definition 4.2.2. Let $(X,d,\mathfrak{m})$ be a metric measure space as in (3.1), $p, q \in (1,\infty)$ conjugate exponents and $g \in \mathrm{S}^p(X,d,\mathfrak{m})$. We say that $\boldsymbol{\pi} \in \mathscr{P}\big(C([0,1],X)\big)$ $q-$*represents* ∇g if $\boldsymbol{\pi}$ has bounded compression, $\|\boldsymbol{\pi}\|_q < \infty$ and the following inequality holds (converse of (4.16)):

$$\varliminf_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \,\mathrm{d}\boldsymbol{\pi} \geq \frac{\||Dg|_w\|^q_{L^p(X,(e_0)_\sharp\boldsymbol{\pi}}}{p} + \frac{\|\boldsymbol{\pi}\|_q^q}{q}. \tag{4.17}$$

Remark 4.2.3. Given the disintegration $\{\boldsymbol{\pi}_x\}_{x\in X}$ of $\boldsymbol{\pi}$ w.r.t. e_0 we get a Borel map $X \ni x \mapsto \boldsymbol{\pi}_x \in \mathscr{P}\big(C([0,1],X)\big)$ that associates to $(e_0)_\sharp\boldsymbol{\pi}-$a.e. x a set of curves that is a gradient-flow at time $t=0$ of g starting from x. In this way for every measure μ such that $\mu \le C(e_0)_\sharp\boldsymbol{\pi}$ for some $C>0$ the plan

$$\boldsymbol{\pi}_\mu := \int \boldsymbol{\pi}_x \,\mathrm{d}\mu(x)$$

also $q-$represents ∇g, as we will prove later.

Remark 4.2.4. Assume that $\boldsymbol{\pi}$ $q-$represents ∇g. By Lisini Theorem, the curve $t\mapsto \mu_t := (e_t)_\sharp\boldsymbol{\pi}$ is $q-$absolutely continuous w.r.t. W_q in a neighbourhood of 0. Now consider the plan $\tilde{\boldsymbol{\pi}} \in \mathscr{P}\big(C([0,1],X)\big)$ associated to (μ_t) by Theorem 3.1.2. We have that

$$\int \frac{g(\gamma_t)-g(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}(\gamma) = \frac{1}{t}\bigg(\int g\,\mathrm{d}\mu_t - \int g\,\mathrm{d}\mu_0\bigg) = \int \frac{g(\gamma_t)-g(\gamma_0)}{t}\,\mathrm{d}\tilde{\boldsymbol{\pi}}(\gamma),$$

$\||Dg|_w\|_{L^p(X,(e_0)_\sharp\boldsymbol{\pi})} = \||Dg|_w\|_{L^p(X,\mu_0)} = \||Dg|_w\|_{L^p(X,(e_0)_\sharp\tilde{\boldsymbol{\pi}})}$ and by Remark 3.1.3 and by Lisini Theorem we get

$$\varlimsup_{t\downarrow 0}\frac{1}{t}\iint_0^t |\dot\gamma_t|^q\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) \ge \varlimsup_{t\downarrow 0}\frac{1}{t}\int_0^t |\dot\mu_t|^q\,\mathrm{d}t = \varlimsup_{t\downarrow 0}\frac{1}{t}\iint_0^t |\dot\gamma_t|^q\,\mathrm{d}t\,\mathrm{d}\tilde{\boldsymbol{\pi}}(\gamma). \tag{4.18}$$

Consequently $\tilde{\boldsymbol{\pi}}$ $q-$represents ∇g. Moreover, the inequalities in (4.18) have to be be equalities otherwise (4.16) would fail for $\tilde{\boldsymbol{\pi}}$.

We defined the two objects $D^\pm f(\nabla g)$ using the quantity $\frac{|D(g+\varepsilon f)|_w^p - |Dg|_w^p}{p\varepsilon|Dg|_w^p}$ and $q-$test plans representing ∇g by (4.16) and (4.17). We want to compare those two definitions and this is the key technical point that will allow us to get the desired duality.

Theorem 4.2.1. *Let* $(X,d,\mathfrak{m})$ *be a metric measure space as in* (3.1), $p,q\in(1,\infty)$ *conjugate exponents and* $f,g\in \mathrm{S}^p(X,d,\mathfrak{m})$. *Then for every plan* $\boldsymbol{\pi}\in\mathscr{P}\big(C([0,1],X)\big)$ *which* $q-$*represents* ∇g *then the following inequalities hold*

$$\begin{aligned}\int D^+f(\nabla g)|Dg|^{p-2}\,\mathrm{d}(e_0)_\sharp\boldsymbol{\pi} &\ge \varlimsup_{t\downarrow 0}\int \frac{f(\gamma_t)-f(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &\ge \varliminf_{t\downarrow 0}\int \frac{f(\gamma_t)-f(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}(\gamma) \ge \int D^-f(\nabla g)|Dg|^{p-2}\,\mathrm{d}(e_0)_\sharp\boldsymbol{\pi}.\end{aligned}$$

Proof. From (4.16) applied to $g+\varepsilon f$ we get

$$\varlimsup_{t\downarrow 0}\int \frac{(g+\varepsilon f)(\gamma_t)-(g+\varepsilon f)(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \frac{\||D(g+\varepsilon f)|_w\|^p_{L^p(X,(e_0)_\sharp\boldsymbol{\pi})}}{p} + \frac{\|\boldsymbol{\pi}\|_q^q}{q}.$$

We know that $\boldsymbol{\pi}$ $q-$represents ∇g, hence

$$\varliminf_{t\downarrow 0}\int \frac{g(\gamma_t)-g(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}(\gamma) \ge \frac{\||Dg|_w\|^p_{L^p(X,(e_0)_\sharp\boldsymbol{\pi})}}{p} + \frac{\|\boldsymbol{\pi}\|_q^q}{q}.$$

Subtracting the second inequality from the first we get

$$\varlimsup_{t\downarrow 0}\varepsilon\int \frac{f(\gamma_t)-f(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \int \frac{|D(g+\varepsilon f)|_w^p - |Dg|_w^p}{p}\,\mathrm{d}(e_0)_\sharp\boldsymbol{\pi}$$

and dividing for $\varepsilon>0$ (respectively for $\varepsilon<0$) and using the dominated converge theorem we get the first (respectively the third) inequality of the thesis. □

it is also possible to characterize the plans $q-$representing a gradient as follows:

Theorem 4.2.2. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in (3.1), $p, q \in (1, \infty)$ conjugate exponents, $g \in \mathrm{S}^p(X, d, \mathfrak{m})$ and $\boldsymbol{\pi} \in \mathscr{P}\big(C([0,1], X)\big)$ a $q-$test plan with bounded compression. Then $\boldsymbol{\pi}$ $q-$represents ∇g if and only if*

$$\lim_{t\downarrow 0} \frac{g\circ e_t - g\circ e_0}{E_{q,t}} = \lim_{t\downarrow 0}\left(\frac{E_{q,t}}{t}\right)^{\frac{q}{p}} = |Dg|_w \circ e_0 \qquad \textit{in} \quad L^p\big(C([0,1],X), \boldsymbol{\pi}\big). \tag{4.19}$$

Proof. If the thesis is true then $\boldsymbol{\pi}$ $q-$represents ∇g by definition.
Suppose that $\boldsymbol{\pi}$ $q-$represents ∇g. Then by definition, (4.16) the Young's inequality, we have

$$\varlimsup_{t\downarrow 0} \int \frac{g\circ e_t - g\circ e_0}{t}\,\mathrm{d}\boldsymbol{\pi} = \varliminf_{t\downarrow 0} \int \frac{g\circ e_t - g\circ e_0}{t}\,\mathrm{d}\boldsymbol{\pi} = \|\boldsymbol{\pi}\|_q^q = \||Dg|_w\|^p_{L^p(X,(e_0)_\sharp\boldsymbol{\pi})} =: L \tag{4.20}$$

Now define the three functions $A_t, B_t, C_t : C([0,1], X) \to \mathbb{R}\cup\{\pm\infty\}$ as follows

$$A_t := \frac{g\circ e_t - g\circ e_0}{E_{q,t}}, \qquad B_t := \frac{E_{q,t}}{t}, \qquad C_t := \sqrt[p]{\frac{1}{t}\int_0^t |Dg|_w^p \circ e_s\,\mathrm{d}s}.$$

From (3.8) we get

$$|A_t| \le C_t \qquad \boldsymbol{\pi} - \text{a.e.} \tag{4.21}$$

and from (3.12)

$$C_t \to |Dg|_w \circ e_0 \qquad \text{in } L^p(\boldsymbol{\pi}). \tag{4.22}$$

Using (4.20), (4.21) and (4.22) we conclude that

$$\begin{aligned} L &= \lim_{t\downarrow 0}\int \frac{g\circ e_t - g\circ e_0}{t}\,\mathrm{d}\boldsymbol{\pi} = \lim_{t\downarrow 0}\int A_t\,B_t\,\mathrm{d}\boldsymbol{\pi} \le \varliminf_{t\downarrow 0}\int |A_t| B_t \\ &\le \varliminf_{t\downarrow 0}\left(\frac{\|A_t\|^p_{L^p(\boldsymbol{\pi})}}{p} + \frac{\|B_t\|^p_{L^q(\boldsymbol{\pi})}}{q}\right) \le \varlimsup_{t\downarrow 0}\frac{\|A_t\|^p_{L^p(\boldsymbol{\pi})}}{p} + \varlimsup_{t\downarrow 0}\frac{\|B_t\|^p_{L^q(\boldsymbol{\pi})}}{q} \\ &\le \varlimsup \frac{\|C_t\|^p_{L^p(\boldsymbol{\pi})}}{p} + \frac{\|\boldsymbol{\pi}\|_q^q}{q} = L. \end{aligned} \tag{4.23}$$

But we have also the equality

$$\varliminf_{t\downarrow 0}\left(\frac{\|A_t\|^p_{L^p(\boldsymbol{\pi})}}{p} + \frac{\|B_t\|^p_{L^q(\boldsymbol{\pi})}}{q}\right) = \varlimsup_{t\downarrow 0}\frac{\|A_t\|^p_{L^p(\boldsymbol{\pi})}}{p} + \varlimsup_{t\downarrow 0}\frac{\|B_t\|^p_{L^q(\boldsymbol{\pi})}}{q}$$

that implies that the limits of $\|A_t\|^p_{L^p(\boldsymbol{\pi})}$ and $\|B_t\|^q_{L^q(\boldsymbol{\pi})}$ exist as $t \downarrow 0$. From here we deduce that $\lim_{t\downarrow 0}\|A_t\|^p_{L^p(\boldsymbol{\pi})} = \lim_{t\downarrow 0}\|C_t\|^p_{L^p(\boldsymbol{\pi})}$ which together with (4.21) and (4.22) guarantees that $|A_t| \to |Dg|_w \circ e_0$ as $t \downarrow 0$ in $L^p(\boldsymbol{\pi})$. Noticing that also the first inequality in (4.23) is an equality then

$$A_t \xrightarrow{t\downarrow 0} |Dg|_w \circ e_0 \qquad \text{in } L^p(\boldsymbol{\pi}).$$

Using again (4.23) we have that $\lim_{t\downarrow 0}\|B_t\|^q_{L^q(\boldsymbol{\pi})} = L$. Let $B \in L^q(\boldsymbol{\pi})$ be any weak limit of B_t as $t \downarrow 0$ in $L^q(\boldsymbol{\pi})$. Since

$$L = \lim_{t\downarrow 0} \int A_t B_t \,\mathrm{d}\boldsymbol{\pi} = \int |Dg| \circ e_0\, B \,\mathrm{d}\boldsymbol{\pi} \leq \frac{\||Dg|_w \circ e_0\|^p_{L^p(\boldsymbol{\pi})}}{p} + \frac{\|B\|^q_{L^q(\boldsymbol{\pi})}}{q} \leq L, \tag{4.24}$$

then $\|B\|^q_{L^q(\boldsymbol{\pi})}$ so that $B_t \to B$ (i.e. the weak convergence is actually strong because we have the weak convergence of B_t and the convergence of the norms). But in (4.24) the first inequality is an equality, so $B^q = |Dg|^p_w \circ e_0$ $\boldsymbol{\pi}-$a.e.. □

Thanks to this characterization of $q-$test plans, we have that if $g \in \mathrm{S}^p(X, d, \mathfrak{m})$ then

- if $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ are $q-$test plans that $q-$represent ∇g then

$$\boldsymbol{\pi} := \lambda\boldsymbol{\pi}_1 + (1-\lambda)\boldsymbol{\pi}_2 \qquad q - \text{represents}\, \nabla g, \quad \lambda \in [0,1]; \tag{4.25}$$

- if $\boldsymbol{\pi}$ is a plan that $q-$represent ∇g and $F : C([0,1], X) \to \mathbb{R}$ is a non-negative bounded Borel function such that $\int F \,\mathrm{d}\boldsymbol{\pi} = 1$, then

$$\tilde{\boldsymbol{\pi}} := F\boldsymbol{\pi} \qquad q - \text{represents}\, \nabla g. \tag{4.26}$$

Now we have only to prove the existence of such plans. To this aim we will take advantage of the measure $\tilde{\mathfrak{m}} \in \mathscr{P}(X)$ and of the Cheeger's energy functional $\tilde{\mathrm{Ch}}_p : L^2(X, \mathfrak{m}) \to [0, \infty]$ defined before.

Lemma 4.2.3. *Let* $(X, d, \mathfrak{m})$ *be a metric measure space as in* (3.1), $\tilde{\mathfrak{m}}$ *as in* (3.2) *and* (3.3), $p, q \in (1, \infty)$ *conjugate exponents and* $g \in \mathrm{S}^p(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$. *Then there exists a plan* $\boldsymbol{\pi}$ *which* $q-$*represents* ∇g *and such that*

$$c\tilde{\mathfrak{m}} \leq (e_0)_\sharp \boldsymbol{\pi} \leq C\tilde{\mathfrak{m}},$$

with $c, C > 0$.

Proof. Define the function $u_q : [0, \infty) \to \mathbb{R}$ by

$$\begin{aligned} u_q(z) &:= \frac{z^{3-q} - (3-q)}{(3-q)(2-q)} \qquad \text{if } q \neq 2, 3, \\ u_2(z) &:= z \ln z - z, \\ u_3(z) &:= z - \ln z. \end{aligned}$$

The computation of the derivatives of u_q shows that this function is convex for every q. If $q = 2$ we put $\rho_0 := \tilde{c}e^{-g}$ otherwise we choose $a \in \mathbb{R}$ such that $1 + (a-g)(2-q) > b > 0$ $\mathfrak{m}-$a.e. and we define

$$\rho_0 := \tilde{c}\big(1 + (a-g)(2-q)\big)^{\frac{1}{2-q}}$$

where in both cases $\tilde{c}$ is chosen so that $\int \rho_0 \,\mathrm{d}\mathfrak{m} = 1$ (i.e. ρ_0 is a density for $(e_0)_\sharp\boldsymbol{\pi}$). By construction, $c \leq \rho_0 \leq C$ $\mathfrak{m}-$a.e., with $c, C > 0$, $u_q'(\rho_0) = -g + \text{cost.}$ and, by chain rule $\rho_0 \in \mathrm{S}^p(X, d, \mathfrak{m}) \subset \mathrm{S}^p(X, d, \tilde{\mathfrak{m}})$.

Now we consider the gradient flow (ρ_t) of the Cheeger energy $\tilde{\mathrm{Ch}}_p$ in $L^2(X, \tilde{\mathfrak{m}})$ starting from ρ_0. (II) of Theorem 3.3.3 ensures that the densities ρ_t are non-negative and uniformly bounded in $L^\infty(X, \tilde{\mathfrak{m}})$, while (I) of the same theorem grants $\int \rho_t \,\mathrm{d}\tilde{\mathfrak{m}} = 1$

for every $t \geq 0$. Hence if we define $\mu_t := \rho_t \tilde{\mathfrak{m}} \in \mathscr{P}(X)$ then $\mu_t \in \mathscr{P}_q(X)$ for every $t \geq 0$. Moreover, by (V) of Theorem 3.3.3 we have that the curve $t \mapsto \mu_t$ is $q-$absolutely continuous w.r.t. W_q. Hence we can use Lisini Theorem to associate to (μ_t) a plan $\boldsymbol{\pi} \in \mathscr{P}\big(C([0,1],X)\big)$ concentrated on $AC^q([0,1],X)$.

Since the ρ_ts are uniformly bounded and $\tilde{\mathfrak{m}} \leq \tilde{C}\mathfrak{m}$ for some $\tilde{C} > 0$, $\boldsymbol{\pi}$ has bounded compression. Using again (V) of the Theorem 3.3.3 we have that

$$\iint_0^t |\dot\gamma_s|^q \,\mathrm{d}s \,\mathrm{d}\boldsymbol{\pi}(\gamma) = \int_0^t |\dot\mu_s|^q \,\mathrm{d}s \leq \iint_0^t \frac{|D\rho_s|_w^p}{\rho_s^{q-1}} \,\mathrm{d}s \,\mathrm{d}\mathfrak{m} = \iint_0^t u_q''(\rho_s)|D\rho_s|_w^p \,\mathrm{d}s \,\mathrm{d}\mathfrak{m}.$$

Since $\rho_0 \in \mathrm{S}^p(X,d,\mathfrak{m})$, then applying (3.16) we get

$$\|\boldsymbol{\pi}\|_q^q = \varlimsup_{t\downarrow 0} \frac{1}{t} \iint_0^t |\dot\gamma_s|^q \,\mathrm{d}s \leq \varlimsup_{t\downarrow 0} \frac{1}{t} \iint_0^t u_q''(\rho_s)|D\rho_s|_w^p \,\mathrm{d}s \,\mathrm{d}\tilde{\mathfrak{m}} \overset{(3.16)}{=} \int u_q''(\rho_0)|D\rho_0|_w^p \,\mathrm{d}\tilde{\mathfrak{m}}. \tag{4.27}$$

But the u_qs are convex so that

$$\begin{aligned}\int \frac{g(\gamma_t) - g(\gamma_0)}{t} \,\mathrm{d}\boldsymbol{\pi}(\gamma) &= \int \frac{u_q'\big(\rho_0(\gamma_0)\big) - u_q'\big(\rho_0(\gamma_t)\big)}{t} \,\mathrm{d}\boldsymbol{\pi}(\gamma) = \frac{1}{t}\int u_q'(\rho_0)(\rho_0 - \rho_t)\,\mathrm{d}\tilde{\mathfrak{m}} \\ &\geq \frac{1}{t}\int \big(u_q(\rho_0) - u_q(\rho_t)\big)\,\mathrm{d}\tilde{\mathfrak{m}}.\end{aligned}$$

Now using (III) and (IV) of Theorem 3.3.3 we have

$$\varliminf_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \,\mathrm{d}\boldsymbol{\pi}(\gamma) \geq \int u_q''(\rho_0)|D\rho_0|_w^p \,\mathrm{d}\tilde{\mathfrak{m}}. \tag{4.28}$$

But $u_q''(\rho_0)|D\rho_0|_w^p = \rho_0|D\rho_0|_w^p$ $\mathfrak{m}-$a.e., so that

$$\int u_q''(\rho_0)|D\rho_0|_w^p \,\mathrm{d}\tilde{\mathfrak{m}} = \||Dg|_e\|_{L^p((e_0)_\sharp\boldsymbol{\pi})}^p$$

and the thesis follows from (4.27) and (4.28). □

Now we are ready to prove the existence of plans representing gradients:

Theorem 4.2.4. *Let $(X,d,\mathfrak{m})$ be a metric measure space as in* (3.1), $\tilde{\mathfrak{m}}$ *as in* (3.2) *and* (3.3), *$p,q \in (1,\infty)$ conjugate exponents, $g \in \mathrm{S}^p(X,d,\mathfrak{m})$ and $\mu \in \mathscr{P}(X)$ a probability measure such that $\mu \leq C\tilde{\mathfrak{m}}$ for some $C > 0$.*

Then there exists $\boldsymbol{\pi} \in \mathscr{P}\big(C([0,1],X)\big)$ which $q-$represents ∇g and such that $(e_0)_\sharp\boldsymbol{\pi} = \mu$.

Proof. Let $\phi : \mathbb{R} \to [0,1]$ be defined by

$$\phi(x) := \begin{cases} x - 2n, & \text{if } x \in [2n, 2n+1), \text{ for some } n \in \mathbb{Z}, \\ 2n - x, & \text{if } x \in [2n-1, 2n), \text{ for some } n \in \mathbb{Z}, \end{cases}$$

and $\psi := 1 - \phi$. Being linear both ϕ and ψ are $1-$Lipschitz. Applying the preceding lemma to $\phi \circ g$ and to $\psi \circ g$ we get the plans $\boldsymbol{\pi}^1$ and $\boldsymbol{\pi}^2$ which $q-$represent respectively

$\nabla(\phi \circ g)$ and $\nabla(\psi \circ g)$, with $c\tilde{\mathfrak{m}} \leq (e_0)_\sharp \boldsymbol{\pi}^i \leq C\tilde{\mathfrak{m}}$, $i = 1, 2$, for some $c, C > 0$. We put $A := \{g^{-1}(\bigcup_{n\in\mathbb{Z}}[2n, 2n+1))\}$ and define the two functions $F^1, F^2 : C([0,1], X) \to \mathbb{R}$ by

$$F^1(\gamma) := \chi_A(\gamma_0)\frac{\mathrm{d}\mu}{\mathrm{d}(e_0)_\sharp \boldsymbol{\pi}^1}(\gamma_0),$$
$$F^2(\gamma) := \chi_{X\setminus A}(\gamma_0)\frac{\mathrm{d}\mu}{\mathrm{d}(e_0)_\sharp \boldsymbol{\pi}^2}(\gamma_0).$$

By the assumptions the $\frac{\mathrm{d}(e_0)_\sharp \boldsymbol{\pi}^i}{\mathrm{d}\tilde{\mathfrak{m}}}$s are bounded from below, $i = 1, 2$, hence F^1 and F^2 are bounded. Moreover, by locality, $\tilde{\mathfrak{m}}-$a.e. in $g^{-1}(\mathbb{Z})$ it holds $|Dg|_w = |D(-g)|_w = 0$, being $\mathbb{Z}$ negligible because is countable. Now if

$$\boldsymbol{\pi} := F^1\boldsymbol{\pi}^1 + F^2\boldsymbol{\pi}^2,$$

by (4.25) and (4.26) $\boldsymbol{\pi}$ $q-$represents ∇g and by construction $(e_0)_\sharp\boldsymbol{\pi} = \mu$. □

4.3 Differential calculus for $D^{\pm} f(\nabla g)$

In this section we prove the chain and Leibniz rules for $D^{\pm} f(\nabla g)$.

Theorem 4.3.1. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1), *$p \in (1, \infty)$ and $f, g \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$. Also, let $\phi : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that for every $x \in X$ there exists a neighbourhood $U_x \subset X$ of x and an open interval $I_x \subset \mathbb{R}$ such that $\mathfrak{m}\big((U_x \setminus f^{-1}(I_x)\big) = 0$ and $\phi|_{I_x}$ is Lipschitz.*

Then $\phi \circ f \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ and the following relation holds:

$$D^{\pm}(\phi \circ f)(\nabla g) = (\phi' \circ f)\, D^{\pm \mathrm{sgn}(\phi'\circ f)} f(\nabla g) \qquad \mathfrak{m} - a.e. \tag{4.29}$$

where at points x where ϕ is not differentiable at $f(x)$ the value $\phi' \circ f$ is taken arbitrarily.

Similarly, if $\phi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, for any $x \in X$ there exists a neighbourhood $x \in U_x \subset X$ and an open interval $I_x \subset \mathbb{R}$ such that $\mathfrak{m}\big(U_x \setminus g^{-1}(I_x)\big) = 0$ and $\phi|_{I_x}$ is Lipschitz then $\phi \circ g\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ and the following relation holds:

$$D^{\pm} f\big(\nabla(\phi \circ g)\big) = (\phi' \circ g)\, D^{\pm \mathrm{sgn}(\phi'\circ g)} f(\nabla g) \qquad \mathfrak{m} - a.e. \tag{4.30}$$

where at points x where ϕ is not differentiable at $g(x)$ the value $\phi' \circ g$ is taken arbitrarily.

Proof. From assumptions and formula (3.5) both $\phi \circ f$ and $\phi \circ g$ belong to the space $\mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$. Without loss of generality we can assume that $f, g \in \mathrm{S}^p(X, d, \mathfrak{m})$ and that $\phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz.

We start with (4.29). Let $\mathcal{N} \subset \mathbb{R}$ be the $\mathcal{L}^1-$ negligible set of non-differentiability points of ϕ (negligible by Rademacher's Theorem). Then thanks to locality $\phi(\mathcal{N})$ is $\mathcal{L}^1-$negligible so the thesis holds $\mathfrak{m}-$a.e. on $f^{-1}(\mathcal{N})$ being both sides of (4.29) equal to 0. If ϕ is affine the thesis is true and arguing as before it holds also if ϕ is piecewise affine. Now take a general ϕ. Let $\{\phi_n\}$ be a sequence of piecewise affine functions such that $\phi'_n \to \phi'$ $\mathcal{L}^1-$a.e.. Let $\mathcal{N}'$ be the $\mathcal{L}^1-$negligible set of points z such that either ϕ or ϕ_n is not differentiable at z or $\phi'_n(z)$ does not converge to $\phi'(z)$. Arguing as before, we still get the thesis $\mathfrak{m}-$a.e. on $f^{-1}(\mathcal{N}')$ being both sides of (4.29) equal to 0. On the set $X \setminus f^{-1}(\mathcal{N}')$ we use the continuity property (4.11) and the chain rule (3.5) to obtain

$$\begin{aligned}|D^{\pm}(\phi \circ f)(\nabla g) - D^{\pm}(\phi_n \circ f)(\nabla g)| &\leq |D\big((\phi - \phi_n) \circ f\big)|_w\, |Dg|_w \\ &= \big(|\phi' - \phi'_n| \circ f\big)\, |Df|_w\, |Dg|_w.\end{aligned}$$

By construction the right-hand side tends 0 $\mathfrak{m}-$a.e. on $X \setminus f^{-1}(\mathcal{N}')$ as $n \to \infty$.

Now we prove (4.30). Using (4.7) we only need to prove that

$$D^+ f\big(\nabla(\phi \circ g)\big) = (\phi' \circ g)\, D^{\mathrm{sgn}(\phi'\circ g)} f(\nabla g).$$

Notice that the identity

$$\frac{|D(ag+\varepsilon f)|_w^p - |D(ag)|_w^p}{p\varepsilon |D(ag)|_w^{p-2}} = a \frac{|D(g+\frac{\varepsilon}{a} f)|_w^p - |Dg|_w^p}{p\frac{\varepsilon}{a}|Dg|_w^{p-2}}$$

holds $\mathfrak{m}-$a.e. for every $\varepsilon, a \neq 0$ and thanks to (4.29) it implies that the thesis is true for a linear ϕ. Hence (4.30) holds also for an affine ϕ, and by locality also for a piecewise affine ϕ, arguing as in (4.29).

Let $\tilde{J} \subset J$ an interval where ϕ is Lipschitz and let $\{\phi_n\}$ be a sequence of uniformly Lipschitz and piecewise affine functions such that $\phi_n' \to \phi'$ $\mathcal{L}^1-$a.e. on $\tilde{J}$. By construction, for every $E \subset g^{-1}(\tilde{J})$ the sequence of functions $(\phi_n' \circ g)\, D^{\mathrm{sgn}(\phi_n'\circ g)} f(\nabla g)|D(\phi_n \circ g)|_w^{p-2}$ is dominated in $L^1(E, \mathfrak{m}|_E)$ and pointwisely $\mathfrak{m}-$a.e. converges to the function $(\phi' \circ g)\, D^{\mathrm{sgn}(\phi'\circ g)} f(\nabla g)|D(\phi_n \circ g)|_w^{p-2}$. Since $\int_E |D(\phi_n \circ g - \phi \circ g)|_w^p \,\mathrm{d}\mathfrak{m} \overset{n\to\infty}{\longrightarrow} 0$, from the semicontinuity statement of Proposition 4.3 we deduce that

$$\begin{aligned}
\int_E D^+ f\big(\nabla(\phi\circ g)\big)\,|D(\phi\circ g)|_w^{p-2}\,\mathrm{d}\mathfrak{m} &\geq \varlimsup_{n\to\infty} \int_E D^+ f\big(\nabla(\phi_n\circ g)\big)\,|D(\phi_n\circ g)|_w^{p-2}\,\mathrm{d}\mathfrak{m} \\
&= \varlimsup_{n\to\infty} \int_E (\phi_n'\circ g)\, D^{\mathrm{sgn}(\phi_n'\circ g)} f(\nabla g)\,|D(\phi_n\circ g)|_w^{p-2}\,\mathrm{d}\mathfrak{m} \\
&= \int_E (\phi'\circ g)\, D^{\mathrm{sgn}(\phi'\circ g)} f(\nabla g)\,|D(\phi\circ g)|_w^{p-2}\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

By the arbitrariness of $\tilde{J}$ and E we get

$$D^+ f\big(\nabla(\phi\circ g)\big) \geq (\phi'\circ g)\, D^{\mathrm{sgn}(\phi'\circ g)} f(\nabla g) \qquad \mathfrak{m} - \text{a.e.}. \tag{4.31}$$

To conclude it is sufficient to apply this inequality with $\phi \circ g$ replacing g and ϕ^{-1} replacing ϕ. To make this rigorous, assume that $\phi \in C^1_{\mathrm{loc}}$. Notice that (4.31) holds $\mathfrak{m}-$a.e. on $g^{-1}(\{\phi' = 0\})$. Pick z such that $\phi'(z) \neq 0$ (we can assume that $\phi'(z) > 0$ since the the proof is similar if $\phi'(z) < 0$). Set $\tilde{g} := \min\{\max\{g, a\}, b\}$ and notice that ϕ is invertible on $\tilde{g}(X)$ with $C^1_{\mathrm{loc}}-$inverse. Hence using (4.31) we have that

$$\begin{aligned}
D^+ f(\nabla \tilde{g}) = D^+ f\big(\nabla(\phi^{-1}\circ(\phi\circ\tilde{g}))\big) &\geq (\phi^{-1})'\circ(\phi\circ\tilde{g})\, D^+ f\big(\nabla(\phi\circ\tilde{g})\big) \\
&= \frac{1}{\phi'\circ\tilde{g}}\, D^+ f\big(\nabla(\phi\circ\tilde{g})\big) \qquad \mathfrak{m} - \text{a.e.}.
\end{aligned}$$

Thanks to locality and to the arbitrariness of z we can conclude that (4.30) holds for any function $\phi \in C^1_{\mathrm{loc}}$. The general case follows by approximating ϕ with a sequence $\{\phi_n\} \subset C^1_{\mathrm{loc}}$ such that $\mathcal{L}^1(\{\phi_n \neq \phi\} \cup \{\phi_n' \neq \phi'\}) \overset{n\to\infty}{\longrightarrow} 0$ and using again the locality principle. $\square$

To prove the Leibniz's rule, we will use the following lemma:

Lemma 4.3.2. *Let* $(X, d, \mathfrak{m})$ *be a metric measure space as in* (3.1), $p, q \in (1, \infty)$ *conjugate exponents,* $f_1, f_2 \in \mathrm{S}^p(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$, $g \in \mathrm{S}^p(X, d, \mathfrak{m})$ *and* $\boldsymbol{\pi} \in \mathscr{P}\big(C([0,1], X)\big)$ *a plan* $q-$*representing* ∇g.

Then

$$\lim_{t\downarrow 0} \int \left| \frac{\big(f_1(\gamma_t) - f_1(\gamma_0)\big)\big(f_2(\gamma_t) - f_2(\gamma_0)\big)}{t} \right| \mathrm{d}\boldsymbol{\pi}(\gamma) = 0.$$

Proof. Let $F_t^i(\gamma) := f_i(\gamma_t) - f_i(\gamma_0), i = 1,2, t \in (0,1]$. From (4.19) the family of functions $\frac{E_{q,t}}{t}$ is dominated in $L^q(\boldsymbol{\pi})$, so that from the second part of Proposition 3.2.5 the family $\frac{F_t^1}{t}$ is dominated in $L^1(\boldsymbol{\pi})$. By definition

$$\|F_t^2\|_{L^\infty(\boldsymbol{\pi})} \leq 2\|f_2\|_{L^\infty(\mathfrak{m})} \qquad t \in (0,1].$$

Hence to conclude is sufficient to notice that $\boldsymbol{\pi}-$a.e. $F_t^2 \xrightarrow{t\downarrow 0} 0$ being also $\frac{F_t^2}{t}$ dominated in $L^1(\boldsymbol{\pi})$. □

Theorem 4.3.3. *Let* $(X,d,\mathfrak{m})$ *be a metric measure space as in* (3.1)*,* $p \in (1,\infty)$*,* $f_1, f_2 \in \mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m}) \cap L^\infty(X,\mathfrak{m})$ *and* $g \in \mathrm{S}^p_{\mathrm{loc}}(X,d,\mathfrak{m})$*.*

Then $\mathfrak{m}-$*a.e. the following inequalities hold:*

$$\begin{aligned} D^+(f_1f_2)(\nabla g) &\leq f_1\, D^{\mathrm{sgn} f_1} f_2(\nabla g) + f_2\, D^{\mathrm{sgn} f_2} f_1(\nabla g),\\ D^-(f_1f_2)(\nabla g) &\geq f_1\, D^{-\mathrm{sgn} f_1} f_2(\nabla g) + f_2\, D^{-\mathrm{sgn} f_2} f_1(\nabla g). \end{aligned}$$

Proof. Thanks to the locality property (4.9) and using a cut-off argument we can assume that $f_1, f_2 \in \mathrm{S}^p(X,d,\mathfrak{m}) \cap L^\infty(X,\mathfrak{m})$ and that $g \in \mathrm{S}^p(X,d,\mathfrak{m})$. Moreover, replacing f_1 and f_2 with $|f_1|$ e $|f_2|$ and using the chain rule we can reduce ourselves to consider the case $f_1, f_2 \geq 0$.

With these assumptions, we want to prove that for every measure $\mu \in \mathscr{P}(X)$ with $\mu \leq C\tilde{\mathfrak{m}}$ for some $C > 0$ ($\tilde{\mathfrak{m}}$ as in (3.2) and (3.3)) the following assertion holds

$$\int D^+(f_1f_2)(\nabla g)|Dg|_w^{p-2}\,\mathrm{d}\mu \leq \int \big(f_1 D^+ f_2(\nabla g) + f_2 D^+ f_1(\nabla g)\big)|Dg|_w^{p-2}\,\mathrm{d}\mu \tag{4.32}$$

and from the arbitrariness of μ and from the fact that $\mathfrak{m} \ll \tilde{\mathfrak{m}}$ we will get the thesis. The second inequality will follow using $-g$ instead of g and from (4.7).

Fix μ and notice that if $f_1f_2 = 0$ $\mu-$a.e. then the thesis is obvious, hence we can assume that $\int f_i\,\mathrm{d}\mu > 0, i = 1,2$.

We fix $\varepsilon > 0$ and set $g_\varepsilon := g + \varepsilon f_1 f_2 \in \mathrm{S}^p(X,d,\mathfrak{m})$. Let $\boldsymbol{\pi}^\varepsilon$ be a $q-$test plan $q-$representing ∇g such that $(e_0)_\sharp \boldsymbol{\pi}^\varepsilon = \mu$. We know that it exists from Theorem 4.2.4 (with q conjugate exponent of p). Using (4.5) and Theorem 4.2.1 we have that

$$\begin{aligned} \int D^+(f_1f_2)(\nabla g)|Dg|_w^{p-2}\,\mathrm{d}\mu &\leq \int D^-(f_1f_2)(\nabla g_\varepsilon)|Dg_\varepsilon|_w^{p-2}\,\mathrm{d}\mu\\ &\leq \varliminf_{t\downarrow 0} \int \frac{f_1(\gamma_t)f_2(\gamma_t) - f_1(\gamma_0)f_2(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma)\\ &\leq \varlimsup_{t\downarrow 0} \int \frac{f_1(\gamma_t)f_2(\gamma_t) - f_1(\gamma_0)f_2(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma). \end{aligned}$$

Thanks to Lemma 4.3.2 we have that

$$\begin{aligned} &\varlimsup_{t\downarrow 0} \int \frac{f_1(\gamma_t)f_2(\gamma_t) - f_1(\gamma_0)f_2(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma)\\ &= \varlimsup_{t\downarrow 0} \int \left(f_1(\gamma_0)\frac{f_2(\gamma_t) - f_2(\gamma_0)}{t} + f_2(\gamma_0)\frac{f_1(\gamma_t) - f_1(\gamma_0)}{t}\right)\mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma)\\ &= \varlimsup_{t\downarrow 0} \int f_1(\gamma_0)\frac{f_2(\gamma_t) - f_2(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma) + \varlimsup_{t\downarrow 0} \int f_2(\gamma_0)\frac{f_1(\gamma_t) - f_1(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma). \end{aligned}$$

From (4.26) we know that the plans $\boldsymbol{\pi}_i^\varepsilon := \frac{f_i \circ e_0}{\int f_i \,\mathrm{d}\mu}\boldsymbol{\pi}^\varepsilon, i = 1,2$ $q-$represent ∇g_ε with $(e_0)_\sharp \boldsymbol{\pi}_i^\varepsilon = \frac{f_i \circ e_0}{\int f_i \,\mathrm{d}\mu}\mu$. Hence using again 4.2.1 we have that

$$\overline{\lim_{t\downarrow 0}} \int f_1(\gamma_0)\frac{f_2(\gamma_t)-f_2(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma) \le \int f_1 D^+ f_2(\nabla g_\varepsilon)|Dg_\varepsilon|_w^{p-2}\,\mathrm{d}\mu,$$
$$\overline{\lim_{t\downarrow 0}} \int f_2(\gamma_0)\frac{f_1(\gamma_t)-f_1(\gamma_0)}{t}\,\mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma) \le \int f_2 D^+ f_1(\nabla g_\varepsilon)|Dg_\varepsilon|_w^{p-2}\,\mathrm{d}\mu.$$

Now we have proved that for every $\varepsilon > 0$ it holds

$$\int D^+(f_1 f_2)(\nabla g)|Dg|_w^{p-2}\,\mathrm{d}\mu \le \int \big(f_1 D^+ f_2(\nabla g_\varepsilon)|Dg_\varepsilon|_w^{p-2} + f_2 D^+ f_1(\nabla g_\varepsilon)|Dg_\varepsilon|_w^{p-2}\big)\,\mathrm{d}\mu.$$

But the right-hand side of the preceding inequality id dominated in $L^1(X,\mathfrak{m})$ by the assumptions, so that using (4.5) the limit for $\varepsilon \downarrow 0$ allows us to conclude. □

Remark 4.3.1. If $(X,d,\mathfrak{m})$ is $q-$i.s.c. with q conjugate exponent of p, then with the same assumptions of the preceding two theorems, the following equalities hold $\mathfrak{m}-$a.e.:

$$D(\phi\circ f)(\nabla g) = (\phi'\circ f)\,Df(\nabla g), \tag{4.33}$$
$$Df\big(\nabla(\phi\circ g)\big) = (\phi'\circ g)\,Df(\nabla g), \tag{4.34}$$
$$D(f_1 f_2)(\nabla g) = f_1 Df_2(\nabla g) + f_2 Df_1(\nabla g). \tag{4.35}$$

Remark 4.3.2. A natural question is whether a Leibniz's rule of the form

$$Df\big(\nabla(g_1 g_2)\big) = g_1\,Df(\nabla g_2) + g_2\,Df(\nabla g_1) \qquad \forall\, f, g_1, g_1 \text{ "smooth"} \tag{4.36}$$

is valid, possibly with equality replaced by an inequality and with appropriate sign choices in $D^\pm$.

In general this is false: for example, in a flat normed space the preceding equality would be true if the norm comes from a scalar product. Indeed, recalling that $\nabla g = \mathrm{Dual}^{-1}(Dg)$ then (4.36) holds for any f, g_1, g_2 if and only if

$$\mathrm{Dual}^{-1}\big(D(g_1 g_2)\big) = g_1\,\mathrm{Dual}^{-1}(Dg_2) + g_2\,\mathrm{Dual}^{-1}(Dg_1).$$

Thanks to the Leibniz rule for differentials we know that the left-hand side is equal to

$$\mathrm{Dual}^{-1}(g_1 Dg_2 + g_2 Dg_1).$$

Hence, since Dual^{-1} is always $1-$homogeneous, then (4.36) holds if and only if Dual^{-1} is linear: i.e. if the norm comes from a scalar product. We will analyze this problem in the next chapter introducing the infinitesimally Hilbertian spaces.

Remark 4.3.3. In Theorem 4.2.1 that links the derivatives $D^\pm f(\nabla g)$ and $\underline{\lim}_{t\downarrow 0}\int \frac{f\circ e_t - f\circ e_0}{t}\,\mathrm{d}\boldsymbol{\pi}$, $\overline{\lim}_{t\downarrow 0}\int \frac{f\circ e_t - f\circ e_0}{t}\,\mathrm{d}\boldsymbol{\pi}$, the first ones are called *vertical derivatives* because obtained perturbating the dependent variable while the second ones *horizontal derivatives* because obtained perturbing the independent variable instead.

We have got the chain rule as a consequence of the vertical derivatives and the Leibniz rule as a consequence of the horizontal ones. Actually we can get the Leibniz rule via the "vertical" approach only: as in the proof of Theorem 4.3.3 we can reduce ourselves to prove

$$D^+(f_1 f_2)(\nabla g) \le f_1\,D^+ f_2(\nabla g) + f_2\,D^+ f_1(\nabla g) \qquad \mathfrak{m}-\text{a.e.},$$

considering $f_1, f_2 \in \mathrm{S}^p(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$ positive and $g \in \mathrm{S}^p(X, d, \mathfrak{m})$. Using the chain rule (4.29) and the convexity and 1-homogeneity of the map $f \mapsto D^+ f(\nabla g)$ we get

$$\begin{aligned} D^+(f_1 f_2)(\nabla g) &= f_1 f_2\, D^+\big(\ln(f_1 f_2)\big)(\nabla g) \leq f_1 f_2 \big(D^+(\ln(f_1))(\nabla g) + D^+(\ln(f_2))(\nabla g)\big) \\ &= f_1 D^+ f_2(\nabla g) + f_2 D^+ f_1(\nabla g) \qquad \mathfrak{m} - \text{a.e..} \end{aligned}$$

Chapter 5

Laplacian

In this final chapter we define the Laplacian and prove the Leibniz and chain rule for it, checking also its stability under convergence and proving the locality property. We then focus our attention to the linear case by introducing the infinitesimally Hilbertian spaces and prove the duality property between differentials and gradients, possible essentially because in this spaces $D^+f(\nabla g)$ and $D^-f(\nabla g)$ agree. Recalling that in the smooth setting the Laplacian can be defined via the Dirichlet's energy, in this metric setting we give a different definition of the Laplacian based on the Cheeger's energy and study the compatibility of this definition with the first one we give. In the last part we will apply the construction of the Laplacian to the Heisenberg group, observing how different can be the differential approach and the Cheeger's one.

5.1 Definition and first properties

Thanks to the results of the preceding section we are now ready to define the Laplacian operator. Notice first that Proposition 3.2.1 allows us to define the Sobolev class $\mathrm{S}^p_{\mathrm{loc}}(\Omega)$ in this way:

Definition 5.1.1. Let $(X, d, \mathfrak{m})$ be a metric measure space as in (3.1), $\Omega \subset X$ an open subset and $p \in (1, \infty)$. Define the space

$$\mathrm{S}^p_{\mathrm{loc}}(\Omega) := \{g \text{ Borel functions} \mid g\chi \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})\ \forall\, \chi : X \to [0,1] \text{ Lipschitz function with } d(\mathrm{supp}(\chi), X \setminus \Omega) > 0\}$$

Thanks to the locality property if $g \in \mathrm{S}^p_{\mathrm{loc}}(\Omega)$ then $|Dg|_w \in L^p_{\mathrm{loc}}(\Omega, \mathfrak{m}|_\Omega)$ is well defined as

$$|Dg|_w := |D(g\chi)|_w \qquad \mathfrak{m}-\text{a.e. on } \{\chi = 1\}, \tag{5.1}$$

with $\chi : X \to [0,1]$ any Lipschitz function such that $d(\mathrm{supp}(\chi), X \setminus \Omega) > 0$. Again by locality the functions $D^{\pm}f(\nabla g)$ are well defined $\mathfrak{m}-$a.e. on Ω and $\mathfrak{m}-$a.e. the chain rules (4.29) and (4.30) and the Leibniz rule hold for every couple of functions $f, g \in \mathrm{S}^p_{\mathrm{loc}}(\Omega)$, $p \in (1, \infty)$.

We need some form of integrability $|Dg|_w$ in order to define the Laplacian.

Definition 5.1.2. Let $(X, d, \mathfrak{m})$ be a metric measure space as in (3.1) and $\Omega \subset X$ an open subset. We define the class

$$\mathrm{Int}(\Omega) := \{\Omega' \subset \Omega \mid \Omega' \text{ is bounded }, d(\Omega', X \setminus \Omega) > 0 \text{ and } \mathfrak{m}(\Omega') < \infty\}.$$

If $p \in (1, \infty)$ we define the space $\mathrm{S}^p_{int}(\Omega)$ of *functions Sobolev internally in* Ω as

$$\mathrm{S}^p_{\mathrm{int}}(\Omega) := \left\{ g \in \mathrm{S}^p_{\mathrm{loc}}(\Omega) : \int_{\Omega'} |Dg|_w^p \, \mathrm{d}\mathfrak{m} < \infty, \forall \Omega' \Subset \mathrm{Int}(\Omega) \right\}.$$

We need also a space of test functions: as said many times before, maximum regularity is achieved by Lipschitz functions so we have the following definition.

Definition 5.1.3. Let $(X, d, \mathfrak{m})$ be a metric measure space as in (3.1) and $\Omega \subset X$ an open subset. We define the space

$$\mathrm{Test}(\Omega) := \left\{ f \in \mathrm{Lip}(X) : \mathrm{supp}(f) \subset \Omega', \Omega' \Subset \mathrm{Int}(\Omega) \right\}.$$

Remark 5.1.1. Recall that if $f : X \to \mathbb{R}$ is Lipschitz then $f \in \mathrm{S}^p_{\mathrm{loc}}(X, d, \mathfrak{m})$ $\forall p \in (1, \infty)$ and its minimal $p-$weak upper gradient $|Df|_w$ is uniformly bounded by $\mathrm{Lip}(f)$.

Naturally, if $g \in \mathrm{S}^p_{\mathrm{int}}(\Omega)$ and $f \in \mathrm{Test}(\Omega)$ we have that $g + \varepsilon f \in \mathrm{S}^p_{\mathrm{int}}(\Omega)$ for every $\varepsilon \in \mathbb{R}$. Thus, if $g \in \mathrm{S}^p_{\mathrm{int}}(\Omega)$ and $f \in \mathrm{Test}(\Omega)$ then thanks to (5.1) the functions $D^{\pm} f(\nabla g)$ are defined $\mathfrak{m}-$a.e. on Ω as

$$D^{\pm} f(\nabla g) := D^{\pm} f(\nabla(g\chi)) \qquad \text{on } \{\chi = 1\} \tag{5.2}$$

for every$\chi : X \to [0, 1]$ Lipschitz function with that $\mathrm{supp}(\chi) \subset \Omega'$ for some Ω' and from the fact that $\mathfrak{m}(\mathrm{supp}(f)) < \infty$ we have that $D^{\pm} f(\nabla g) \in L^1(\Omega, \mathfrak{m}|_{\Omega})$.

Now we are ready to define the distributional Laplacian.

Definition 5.1.4. Let $(X, d, \mathfrak{m})$ be a metric measure space as in (3.1), $\Omega \subset X$ an open subset and $g : \Omega \to \mathbb{R}$ a Borel function. We say that g *is in the domain of the Laplacian* and write $g \in D(\mathbf{\Delta}, \Omega)$ if $g \in \mathrm{S}^p_{\mathrm{int}}(\Omega)$ for some $p > 1$ and there exists a Radon measure μ on Ω such that for every $f \in \mathrm{Test}(\Omega) \cap L^1(\Omega, |\mu|)$ it holds

$$-\int_{\Omega} D^+ f(\nabla g) \, \mathrm{d}\mathfrak{m} \leq \int_{\Omega} f \, \mathrm{d}\mu \leq -\int_{\Omega} D^- f(\nabla g) \, \mathrm{d}\mathfrak{m}. \tag{5.3}$$

In this case we write $\mu \in \mathbf{\Delta} g|_{\Omega}$. If $\Omega = X$ then we write $g \in D(\mathbf{\Delta})$ and $\mu \in \mathbf{\Delta} g$.

Notice that $\mathbf{\Delta}$ is a set of measures that can be multi-valued.

Remark 5.1.2. As for $|Df|_w$ and $D^{\pm} f(\nabla g)$ the choice of p as Sobolev exponent may affect the definition of the Laplacian. But in the definition we required that $g \in \mathrm{S}^p_{\mathrm{int}}(\Omega)$ for some $p > 1$ to ensure to compute $|D(g + \varepsilon f)|_w$ and hence $D^{\pm} f(\nabla g)$.

Remark 5.1.3. We have already observed that $|Df|_w \in L^{\infty}$. In the Euclidian case (i.e. $X = \mathbb{R}^n$ and $\mathfrak{m} = \mathcal{L}^n$) to write $\Delta g = \mu$ in the distributional sense it is sufficient to require $g \in L^1_{\mathrm{loc}}$. Thanks to the Sobolev embedding theorems, if $\Delta g = \mu$ in the distributional sense then its distributional gradient $\nabla g \in L^p_{\mathrm{loc}}$ for every $p \in [1, 1 + \frac{1}{n})$. Thus, in the Euclidian case we may not know n but we are still sure that $\Delta g = \mu$ implies $\nabla g \in L^p_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$ for $p > 1$.

The Laplacian defined in (5.3) can be multi-valued as the following example shows: consider the metric measure space $X = (X, d, \mathfrak{m}) = (\mathbb{R}^2, d, \mathcal{L}^2)$ with d the distance induced by the 1−norm (i.e. if $(x_1, y_1), (x_2, y_2) \in X$ then $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$). As said before, now $|Dg|_w = \|\nabla g\|_1 = |\partial_x f| + |\partial_y f|$. If $p = 2$ then

$$\begin{aligned}\frac{\|\nabla(g+\varepsilon f)\|_1^2 - \|\nabla g\|_1^2}{2\varepsilon} &= \frac{\big(|\partial_x(g+\varepsilon f)| + |\partial_y(g+\varepsilon f)|\big)^2 - \big(|\partial_x g| + |\partial_y g|\big)^2}{2\varepsilon} \\ &= \frac{2\varepsilon\big((\partial_x g)(\partial_x f) + (\partial_y g)(\partial_y f)\big) + 2|(\partial_x g + \varepsilon\partial_x f)(\partial_y g + \varepsilon\partial_y f)| - 2|\partial_x g||\partial_y g| + o(\varepsilon)}{2\varepsilon} \\ &= \frac{\varepsilon\big((\partial_x g)(\partial_x f) + (\partial_y g)(\partial_y f)\big) + |(\partial_x g)(\partial_y g) + \varepsilon\big((\partial_x g)(\partial_y f) + (\partial_x f)(\partial_y g)\big) + o(\varepsilon)|}{\varepsilon} - \\ &- \frac{|\partial_x g||\partial_y g| + o(\varepsilon)}{\varepsilon}.\end{aligned}$$

Now using the triangle inequality first and the converse triangle inequality then we get

$$\begin{aligned}&\le \big((\partial_x g)(\partial_x f) + (\partial_y g)(\partial_y f)\big) + |(\partial_x g)(\partial_y f)| + |(\partial_x f)(\partial_y g)| + o(1), \\ &\ge \big((\partial_x g)(\partial_x f) + (\partial_y g)(\partial_y f)\big) + |(\partial_x g)(\partial_y f) + (\partial_x f)(\partial_y g)| + o(1).\end{aligned}$$

Therefore both $D \pm f(\nabla g)$ are bounded by

$$\begin{aligned}\big((\partial_x g)(\partial_x f) + (\partial_y g)(\partial_y f)\big) + |(\partial_x g)(\partial_y f) + (\partial_x f)(\partial_y g)| \le D^{\pm} f(\nabla g) \le \\ \le \big((\partial_x g)(\partial_x f) + (\partial_y g)(\partial_y f)\big) + |(\partial_x g)(\partial_y f)| + |(\partial_x f)(\partial_y g)|\end{aligned}$$

and so the Laplacian is multivalued.

Nevertheless we have some immediate properties:

- homogeneity:

$$g \in D(\mathbf{\Delta}, \Omega),\ \mu \in \mathbf{\Delta} g|_\Omega \quad \Rightarrow \quad \lambda g \in D(\mathbf{\Delta}, \Omega),\ \lambda\mu \in \mathbf{\Delta}(\lambda g)|_\Omega, \quad \forall \lambda \in \mathbb{R}; \tag{5.4}$$

- translations invariance:

$$g \in D(\mathbf{\Delta}, \Omega),\ \mu \in \mathbf{\Delta} g|_\Omega \quad \Rightarrow \quad c + g \in D(\mathbf{\Delta}, \Omega),\ \mu \in \mathbf{\Delta}(c+g)|_\Omega, \quad \forall c \in \mathbb{R};$$

- $\mathbf{\Delta} g|_\Omega$ is convex and weakly closed, i.e. if $\{\mu_n\} \subset \mathbf{\Delta} g|_\Omega$ and μ is a Radon measure over Ω such that $f \in \mathrm{Test}(\Omega) \cap L^1(X, |\mu|)$ then $f \in L^1(\Omega, |\mu_n|)$ for n large enough and

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu,$$

  then $\mu \in \mathbf{\Delta} g|_\Omega$;

- locality:

$$\tilde{\Omega} \subset \Omega \text{ open sets, } g \in D(\mathbf{\Delta}, \Omega),\ \mu \in \mathbf{\Delta} g|_\Omega \quad \Rightarrow \quad g \in D(\mathbf{\Delta}, \tilde{\Omega}),\ \mu|_{\tilde{\Omega}} \in \mathbf{\Delta} g|_{\tilde{\Omega}}; \tag{5.5}$$

- The measures in $\mathbf{\Delta} g|$ are concentrated on $\mathrm{supp}(\mathfrak{m})$.

Theorem 5.1.1. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1)*, $\Omega \subset X$ an open subset, $p \in (1, \infty)$ and $g \in \mathrm{S}_{\mathrm{int}}(\Omega) \cap D(\boldsymbol{\Delta}, \Omega)$. Suppose that $(X, d, \mathfrak{m})$ is $q-$i.s.c. with q conjugate exponent of p.*

Then $\boldsymbol{\Delta} g|_{\Omega}$ contains only one measure.

Proof. From (5.2) and (4.13), for all $f \in \mathrm{Test}(\Omega)$ we have that $D^+ f(\nabla g) = D^- f(\nabla g)$ $\mathfrak{m}-$a.e. on Ω. □

Now we give a different definition of the Laplacian based on the $\mathrm{Ch}_2 : L^2(X, \mathfrak{m}) \to [0, \infty]$ (analogous in the Euclidian case to defining the 2-Laplacian weakly through the Dirichlet energy) and recall that it is convex and lower semicontinuous.

Definition 5.1.5. Let $(X, d, \mathfrak{m})$ be a metric measure space as in (3.1) and $g \in L^2(X, \mathfrak{m})$. We say that g is in the domain of the Laplacian if $\mathrm{Ch}_2(g) < \infty$ and the subdifferential $\partial^- \mathrm{Ch}_2(g) \neq \emptyset$. In this case the Laplacian of g is defined as the element with minimal $L^2(X, \mathfrak{m})-$norm in $-\partial^- \mathrm{Ch}_2(g)$.

Theorem 5.1.2. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1)*, $g \in L^2(X, \mathfrak{m})$ and $h \in L^2(X, \mathfrak{m})$. Assume that $\mathrm{Ch}_2(g) < \infty$ and $-h \in \partial^- \mathrm{Ch}_2(g)$. Then $g \in D(\boldsymbol{\Delta})$ and $h\mathfrak{m} \in \boldsymbol{\Delta} g$.*

Proof. Fix $f \in L^2(X, \mathfrak{m})$ with $\mathrm{Ch}_2(f) < \infty$ and notice that applying the definition of subdifferential to Ch_2 we have that

$$\mathrm{Ch}_2(g) - \varepsilon \int f h \, \mathrm{d}\mathfrak{m} \leq \mathrm{Ch}_2(g + \varepsilon f) \qquad \forall \varepsilon \in \mathbb{R}.$$

So for $\varepsilon > 0$ we have that

$$- \int f h \, \mathrm{d}\mathfrak{m} \leq \int \frac{|D(g + \varepsilon f)|_w^2 - |Dg|_w^2}{2\varepsilon} \, \mathrm{d}\mathfrak{m}.$$

Letting $\varepsilon \downarrow 0$ and using the dominated convergence on the right-hand side of this inequality we have

$$- \int f h \, \mathrm{d}\mathfrak{m} \leq \int D^+ f(\nabla g) \, \mathrm{d}\mathfrak{m}$$

and this holds for $f \in \mathrm{Test}(X) \subset \{\mathrm{Ch}_2 < \infty\} \cap L^1(X, |h|\mathfrak{m})$. Replacing f with $-f$ we get the other inequality of (5.3) and so the thesis. □

Remark 5.1.4. Notice that the choice of representing the Laplacian with the element of minimal norm has been done just to identify it in a unique way but w.r.t the definition 5.1.4 and thanks to the preceding theorem every element in $-\partial^- \mathrm{Ch}_2(g)$ is admissible for representing the Laplacian.

Remark 5.1.5. it is natural to ask if the converse of theorem 5.1.2 holds: suppose that $g \in D(\mathrm{Ch}_2) \cap D(\boldsymbol{\Delta})$ and for some $\mu \in \boldsymbol{\Delta} g$ we have that $\mu \ll \mathfrak{m}$ with density $h \in L^2(X, \mathfrak{m})$. Can we say that $-h \in \partial^- \mathrm{Ch}_2(g)$? We want to understand that if from

$$\mathrm{Ch}_2(g) - \int f h \, \mathrm{d}\mathfrak{m} \leq \mathrm{Ch}_2(g + f) \qquad \forall f \in \mathrm{Test}(X)$$

we can deduce that the same inequality holds or not for all $f \in L^2(X, \mathfrak{m})$. The answer lies in the density of Lipschitz functions in $W^{1,2}(X, d, \mathfrak{m})$ and it can be proven to be affirmative if $W^{1,2}(X, d, \mathfrak{m})$ is uniformly convex and $\mathfrak{m}$ is finite on finite sets.

5.2 Calculus rules with the Laplacian

In this section we collect the basic calculus rules of the Laplacian proving also that they are very similar to the ones in the Euclidian setting.

Theorem 5.2.1. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1), *$\Omega \subset X$ an open set and $g \in D(\mathbf{\Delta}, \Omega)$. Suppose that g is Lipschitz on Ω' for any $\Omega' \in \mathrm{Int}(\Omega)$ and let $\phi : g(\Omega) \to \mathbb{R}$ a $C^{1,1}_{\mathrm{loc}}$ map. Then $\phi \circ g \in D(\mathbf{\Delta}, \Omega)$ and for any $\mu \in \mathbf{\Delta} g|_\Omega$ we have that*

$$\mathbf{\Delta}(\phi \circ g)|_\Omega \ni \tilde{\mu} := (\phi' \circ g)\,\mu + (\phi'' \circ g)\,|Dg|_w^2 \mathfrak{m}. \tag{5.6}$$

Proof. Being g continuous $(\phi' \circ g)\mu$ makes sense and defines a locally finite measure. Similarly for $(\phi'' \circ g)|Dg|_w^2 \in L^\infty_{\mathrm{loc}}(\Omega, \mathfrak{m}|_\Omega)$.

Moreover $\phi' \circ g$ is Lipschitz on Ω' and $(\phi'' \circ g)|Dg|_w^2$ is bounded on Ω' for any $\Omega' \in \mathrm{Int}(\Omega)$. Therefore if $f \in \mathrm{Test}(\Omega) \cap L^1(\Omega, |\tilde{\mu}|)$ then $f(\phi' \circ g) \in \mathrm{Test} \cap L^1(|\mu|)$. Now fix such f and using the chain rules (4.29) and (4.30) and the Leibniz rules and (4.8) we have that $\mathfrak{m}-$a.e. on Ω

$$\begin{aligned} D^+ f \nabla(\phi \circ g) &= (\phi' \circ g) D^{\mathrm{sgn}(\phi' \circ g)} f(\nabla g) \geq D^+\big(f(\phi' \circ g)\big)(\nabla g) - f D^{\mathrm{sgn} f}(\phi' \circ g)(\nabla g) \\ &= D^+\big(f(\phi' \circ g)\big)(\nabla g) - f(\phi'' \circ g) D^{\mathrm{sgn}(f(\phi'' \circ g))} g(\nabla g) \\ &= D^+\big(f(\phi' \circ g)\big) - f(\phi'' \circ g)|Dg|_w^2. \end{aligned}$$

Integrating we obtain

$$\int D^+ f\big(\nabla(\phi \circ g)\big)\,\mathrm{d}\mathfrak{m} \geq -\int f\,\mathrm{d}\tilde{\mu}.$$

Replacing f with $-f$ we conclude. □

Remark 5.2.1. The Lipschitz-continuity assumption on Ω' on g was needed to ensure that $f \in \mathrm{Test}(\Omega)$ implies $f(\phi' \circ g) \in \mathrm{Test}(\Omega)$, so that from the assumption one could deduce

$$\int D^+\big(f(\phi' \circ g)\big)(\nabla g)\,\mathrm{d}\mathfrak{m} \geq -\int f\,\mathrm{d}\mu.$$

it is well known, apart technicalities, that in the Euclidian setting a non-negative distribution can be seen as a non-negative measure. The next theorem proves a similar statement for the metric Laplacian providing a sufficient condition on g that ensures that it is in the domain of the Laplacian, giving also a bound on elements of $\mathbf{\Delta} g|_\Omega$.

Theorem 5.2.2. *Let $(X, d, \mathfrak{m})$ a metric measure space as in* (3.1) *and assume also that (X, d) is a proper space. Let $\Omega \subset X$ be an open set, $g \in \mathrm{S}^p_{\mathrm{int}}(\Omega)$, $p \in (1, \infty)$ and $\tilde{\mu}$ a Radon measure on Ω. Assume that for any non-negative $f \in \mathrm{Test}(\Omega) \cap L^1(\Omega, |\tilde{\mu}|)$ it holds*

$$-\int_\Omega D^- f(\nabla g)\,\mathrm{d}\mathfrak{m} \leq \int_\Omega f\,\mathrm{d}\tilde{\mu}. \tag{5.7}$$

Then $g \in D(\mathbf{\Delta}, \Omega)$ and for any $\mu \in \mathbf{\Delta} g|_\Omega$ it holds $\mu \leq \tilde{\mu}$.

Proof. Consider the $\mathbb{R}-$valued map

$$\mathrm{Test}(\Omega) \ni f \;\mapsto\; T(f) := -\int_\Omega D^- f(\nabla g)\,\mathrm{d}\mathfrak{m}.$$

From Proposition 4.1.1 T is a sublinear map so by the Hahn-Banach Theorem there exists a linear map $L : \mathrm{Test} \to \mathbb{R}$ such that $L(f) \le T(f)\ \forall f \in \mathrm{Test}(\Omega)$. By (4.7) we then have

$$-\int_\Omega D^+ f(\nabla g)\,\mathrm{d}\mathfrak{m} \le L(f) \le -\int_\Omega D^- f(\nabla g)\,\mathrm{d}\mathfrak{m} \qquad \forall f \in \mathrm{Test}(\Omega).$$

By (5.7) it holds

$$\int_\Omega f\,\mathrm{d}\tilde{\mu} - L(f) \ge 0 \qquad \forall f \in \mathrm{Test}(\Omega), f \ge 0.$$

Fix a compact set $K \subset \Omega$ and a function $\chi_K \in \mathrm{Test}(\Omega)$ such that $0 \le \chi_K \le 1$ everywhere and $\chi_K \equiv 1$ on K. Let $V_K \subset \mathrm{Test}(\Omega)$ be the set of those χ_K with support contained in K. For any non-negative $f \in V_K$ the fact that $(\max f)\chi_K - f \in \mathrm{Test}(\Omega)$ and non-negative yields

$$\begin{aligned} L(f) = -L\big((\max f)\chi_K - g\big) + L\big((\max f)\chi_K\big) &\ge -\int \big((\max f)\chi_K - f\big)\,\mathrm{d}\tilde{\mu} + L\big((\max f)\chi_K\big) \\ &\ge -(\max f)\big(\tilde{\mu}(\mathrm{supp}(\chi_K)) + L(\chi_K)\big). \end{aligned}$$

Thus for a generic $f \in V_K$ it holds

$$\begin{aligned} L(f) = L(f^+ - f^-) = L(f^+) - L(f^-) &\le \int_\Omega f^+\,\mathrm{d}\tilde{\mu} + (\max f^-)\big(\tilde{\mu}(\mathrm{supp}(\chi_K)) + L(\chi_K)\big) \\ &\le (\max |f|)\big(\tilde{\mu}(K) + \tilde{\mu}(\mathrm{supp}(\chi_K)) + L(\chi_K)\big), \end{aligned}$$

so $L : V_K \to \mathbb{R}$ is continuous w.r.t the supremum norm. Hence it can be extended to a linear bounded functional on the set $C_K \subset C(\Omega)$ of continuous functions with support contained in K and this extension is unique by the density of Lipschitz functions in the uniform norm. Being K arbitrary, by the Riesz's Theorem there exists a Radon measure μ such that

$$L(f) = \int f\,\mathrm{d}\mu \qquad \forall f \in \mathrm{Test}(\Omega).$$

Thus $g \in D(\boldsymbol{\Delta}, \Omega)$ and $\mu \in \boldsymbol{\Delta} g|_\Omega$.

By (5.7) it is immediate to get that for any $\mu' \in \boldsymbol{\Delta} g|_\Omega$ it holds $\mu' \le \tilde{\mu}$. □

Remark 5.2.2. If $(X, d, \mathfrak{m})$ is $q-$i.s.c. and $g \in \mathrm{S}^p_{\mathrm{int}}(\Omega)$, with p and q conjugate exponents, the map T is linear.

The next one is a convergence result for the Laplacian (*stability* under convergence).

Theorem 5.2.3. *Let $(X, d, \mathfrak{m})$ a metric measure space as in* (3.1)*, $\Omega \subset X$ an open set and $p \in (1, \infty)$. Let $\{g_n\} \subset \mathrm{S}^p_{\mathrm{int}}(\Omega)$ be a sequence and $g \in \mathrm{S}^p_{\mathrm{int}}(\Omega)$ be such that $\int_{\Omega'} |D(g_n - g)|_w\,\mathrm{d}\mathfrak{m} \to 0$ for every $\Omega' \in \mathrm{Int}(\Omega)$. Assume also that $g_n \in D(\boldsymbol{\Delta}, \Omega)$ for every $n \in \mathbb{N}$, let $\mu_n \in \boldsymbol{\Delta} g_n|_\Omega$ and suppose that for some locally finite measure μ on Ω it holds*

$$f \in \mathrm{Test}(\Omega) \cap L^1(\Omega, |\mu|) \quad \Rightarrow \quad f \in L^1(\Omega, |\mu_n|) \text{ for } n \text{ large enough and } \mu_n \overset{*}{\rightharpoonup} \mu. \tag{5.8}$$

Then $g \in D(\boldsymbol{\Delta}, \Omega)$ and $\mu \in \boldsymbol{\Delta} g|_\Omega$.

Proof. For any $f \in \mathrm{Test}(\Omega)$ and $g \in \mathrm{S}^p_{\mathrm{int}}(\Omega)$ it holds

$$\int D^+ f(\nabla g)\,\mathrm{d}\mathfrak{m} = \inf_{\varepsilon>0} \int |Dg|_w \frac{|D(g+\varepsilon f)|_w - |Dg|_w}{\varepsilon}\,\mathrm{d}\mathfrak{m},$$
$$\int D^- f(\nabla g)\,\mathrm{d}\mathfrak{m} = \sup_{\varepsilon<0} \int |Dg|_w \frac{|D(g+\varepsilon f)|_w - |Dg|_w}{\varepsilon}\,\mathrm{d}\mathfrak{m}.$$

By our assumptions for any $f \in \mathrm{Test}(\Omega)$ it holds

$$\int_{\mathrm{supp}(f)} |D(g_n - g)|_w\,\mathrm{d}\mathfrak{m} \to 0,$$

the sequence $\frac{|D(g_n+\varepsilon f)|_w - |Dg_n|_w}{\varepsilon}$ is uniformly bounded in $L^\infty(\Omega, \mathfrak{m}|_\Omega)$ by $\mathrm{Lip}(f)$ and converges to $\frac{|D(g+\varepsilon f)|_w - |Dg|_w}{\varepsilon}$ in $L^1(\Omega, \mathfrak{m}|_\Omega)$ for any $\varepsilon \neq 0$. Hence

$$\overline{\lim_{n\to\infty}} \int_\Omega D^+ f(\nabla g_n)\,\mathrm{d}\mathfrak{m} \leq \int_\Omega D^+ f(\nabla g)\,\mathrm{d}\mathfrak{m}, \tag{5.9}$$
$$\underline{\lim_{n\to\infty}} \int_\Omega D^- f(\nabla g_n)\,\mathrm{d}\mathfrak{m} \geq \int_\Omega D^- f(\nabla g)\,\mathrm{d}\mathfrak{m}. \tag{5.10}$$

If furthermore $f \in L^1(\Omega, |\mu|)$ (5.8) ensures that $f \in L^1(\Omega, |\mu_n|)$ for n sufficiently large. So from $\mu_n \in \mathbf{\Delta} g_n|_\Omega$ we have

$$-\int D^+ f(\nabla g_n)\,\mathrm{d}\mathfrak{m} \leq \int f\,\mathrm{d}\mu_n \leq -\int D^- f(\nabla g_n)\,\mathrm{d}\mathfrak{m} \qquad \forall n \gg 0.$$

Using (5.9) and (5.10) we can pass to the limit in these inequalities and get the thesis. □

The next statement shows that the definition we gave for the Laplacian is compatible with the Cheeger's one.

Proposition 5.2.4. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1)*, $\Omega \subset X$ an open set and $g \in \mathrm{S}^2_{\mathrm{int}}(\Omega)$ with $\int_\Omega |Dg|_w^2\,\mathrm{d}\mathfrak{m} < \infty$. If we assume that*

$$\int_\Omega |Dg|_w^2\,\mathrm{d}\mathfrak{m} \leq \int_\Omega |D(g+h)|_w^2\,\mathrm{d}\mathfrak{m} \qquad \forall h \in \mathrm{S}^2(X, d, \mathfrak{m}) : \mathrm{supp}(h) \subset \Omega,$$

then $g \in D(\mathbf{\Delta}, \Omega)$ and $0 \in \mathbf{\Delta} g|_\Omega$.

Proof. First of all, if $f \in \mathrm{Test}(\Omega)$ we certainly have $\mathrm{supp}(f) \subset \Omega$ and $f \in \mathrm{S}^2(X, d, \mathfrak{m})$. Thus for $f \in \mathrm{Test}(\Omega)$ and $\varepsilon \in \mathbb{R}$ the additional assumption made yields

$$\int_\Omega |D(g+\varepsilon f)|_w^2\,\mathrm{d}\mathfrak{m} \geq \int_\Omega |Dg|_w^2\,\mathrm{d}\mathfrak{m}.$$

Therefore

$$\int_\Omega \frac{|D(g+\varepsilon f)|_w^2 - |Dg|_w^2}{2\varepsilon}\,\mathrm{d}\mathfrak{m} \lesseqgtr 0 \qquad \text{if} \quad \varepsilon \lessgtr 0.$$

By letting ε tend to 0 we get the thesis. □

The Laplacian also has a *local-to-global* property, explained in the next theorem:

Theorem 5.2.5. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1) *and suppose that (X, d) is a proper space. Let $p \in (1\infty)$ and q its conjugate exponent and assume $(X, d, \mathfrak{m})$ q–i.s.c.. Let $\Omega \subset X$ be an open, $\{\Omega_i\}_{i\in I}$ a family of open sets such that $\Omega = \bigcup_{i\in I} \Omega_i$, $g \in \mathrm{S}^2_{\mathrm{int}}(\Omega)$ with $g \in D(\mathbf{\Delta}, \Omega_i)$ $\forall i \in I$ and μ_i the only element of $\mathbf{\Delta} g|_{\Omega_i}$. Then*

$$\mu_i|_{\Omega_i\cap\Omega_j} = \mu_j|_{\Omega_i\cap\Omega_j} \qquad \forall i, j \in I, \tag{5.11}$$

$g \in D(\mathbf{\Delta}, \Omega)$ and the measure μ on Ω defined by

$$\mu|_{\Omega_i} := \mu_i \qquad \forall i \in I \tag{5.12}$$

is the only element of $\mathbf{\Delta} g|_\Omega$.

Proof. Being (X, d) proper, for any $\overline{\Omega} \subset X$ open and for any Radon measure ν on $\overline{\Omega}$ and any $f \in \mathrm{Test}(\overline{\Omega})$ it holds $f \in L^1(\overline{\Omega}, |\nu|)$ because the support of f is compact.

Let $i, j \in I$, $f \in \mathrm{Test}(\Omega_i \cap \Omega_j)$. By definition we have that

$$-\int_{\Omega_i\cap\Omega_j} f \,\mathrm{d}\mu_i = \int_{\Omega_i\cap\Omega_j} Df(\nabla g)\,\mathrm{d}\mathfrak{m} = -\int_{\Omega_i\cap\Omega_j} f\,\mathrm{d}\mu_j,$$

which yields (5.11). In particular the measure μ is well defined by (5.12).

Fix now $f \in \mathrm{Test}(\Omega)$. Since the support of f is compact there exists a finite set $I_f \subset I$ of indices such that $\mathrm{supp}(f) \subset \bigcup_{i\in I_f} \Omega_i$. From the fact that (X, d) is proper we can build Lipschitz partition of unity $\{\chi_i\}_{i\in I_f}$. Hence $f\chi_i \in \mathrm{Test}(\Omega_i)$ for any $i \in I_f$ and by the linearity of the differential expressed by the Corollary 4.1.2 we have

$$\begin{aligned}\int Df(\nabla g)\,\mathrm{d}\mathfrak{m} &= \int D\Big(\sum_{i\in I_f} \chi_i f\Big)(\nabla g)\,\mathrm{d}\mathfrak{m} = \sum_{i\in I_f}\int D(f\chi_i)(\nabla g)\,\mathrm{d}\mathfrak{m}\\ &= -\sum_{i\in I_f}\int f\chi_i\,\mathrm{d}\mu_i = -\int f\,\mathrm{d}\Big(\sum_{i\in I_f}\chi_i\mu_i\Big).\end{aligned}$$

□

We conclude this section showing the effect of a change of the reference measure $\mathfrak{m}$. As for the previous proposition we need the infinitesimally strict convexity to express the formula.

Theorem 5.2.6. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1) *and $V : X \to \mathbb{R}$ a locally Lipschitz function, which is Lipschitz when restricted to bounded sets. Defining the new measure*

$$\mathfrak{m}' := e^{-V}\mathfrak{m}$$

let $\mathbf{\Delta}'$ be the Laplacian in $(X, d, \mathfrak{m}')$. Let $\Omega \subset X$ be an open set, $g \in D(\mathbf{\Delta}) \cap \mathrm{S}^p_{\mathrm{int}}(\Omega)$ for some $p \in (1, \infty)$ and assume that $(X, d, \mathfrak{m})$ is q–i.s.c. with q the conjugate exponent of p. Then $g \in D(\mathbf{\Delta}', \Omega)$ and the measure

$$\mu' := e^{-V}\mu - DV(\nabla g)e^{-V}\mathfrak{m}$$

is the only element in $\mathbf{\Delta}' g|_\Omega$, where μ is the only element of $\mathbf{\Delta} g|_\Omega$.

Proof. Since e^{-V} and $|DV|_w$ are locally bounded then μ' is a locally finite measure so the statement makes sense. For $f \in \mathrm{Test}'(\Omega) \cap L^1(\Omega, |\mu'|)$, where $\mathrm{Test}'(\Omega)$ is the set $\mathrm{Test}(\Omega)$ with $\mathfrak{m}$ replaced by $\mathfrak{m}'$, the fact that $\mathrm{supp}(f)$ is bounded yields that $V|_{\mathrm{supp}(f)}$ is Lipschitz and bounded. It follows that $fe^{-V} \in \mathrm{Test}(\Omega) \cap L^1(\Omega, |\mu|)$ so from the chain rule (4.33) and the Leibniz's one (4.35) we get

$$\begin{aligned}\int f \, \mathrm{d}\mu' &\triangleq \int fe^{-V} \, \mathrm{d}\mu - \int fDV(\nabla g)e^{-V} \, \mathrm{d}\mathfrak{m} \\ &= -\int \big(D(fe^{-V})(\nabla g) - fD(e^{-V})(\nabla g)\big) \, \mathrm{d}\mathfrak{m} = -\int e^{-V} Df(\nabla g) \, \mathrm{d}\mathfrak{m}.\end{aligned}$$

□

5.3 The linear case

In this section we introduce a sufficient condition in order for the Laplacian to be linear.

Definition 5.3.1. Let $(X, d, \mathfrak{m})$ be a metric measure space as in (3.1). We say that $(X, d, \mathfrak{m})$ is *infinitesimally Hilbertian* if the seminorm $\| \cdot \|_{\mathrm{S}^2(X,d,\mathfrak{m})}$ on $\mathrm{S}^2(X, d, \mathfrak{m})$ satisfies the parallelogram rule.

Remark 5.3.1. Recall that if $(A, \| \cdot \|)$ is a normed space then the parallelogram rule (called also *polarization identity*) reads as

$$\|x\|^2 + \|y\|^2 = \frac{\|x+y\|^2}{2} + \frac{\|x-y\|^2}{2} \qquad \forall x, y \in A$$

and $(A, \| \cdot \|)$ is a Hilbert space if and only if the parallelogram rule is satisfied.

Being the infinitesimally Hilbertian spaces $2-$i.s.c. then $\forall f, g \in \mathrm{S}^2_{\mathrm{loc}}(X, d, \mathfrak{m})$ the function $Df(\nabla g)$ is well defined $\mathfrak{m}-$a.e. and for $g \in D(\boldsymbol{\Delta}) \cap \mathrm{S}^2(X, d, \mathfrak{m})$ the set $\boldsymbol{\Delta} g$ contains only one element which we will denote again by $\boldsymbol{\Delta} g$ with a little abuse of notation.

The most beautiful property of this kind of spaces is that we can prove a duality between differential and gradients, analogous to the one possible via Riesz Theorem.

Theorem 5.3.1. *Let $(X, d, \mathfrak{m})$ be a metric measure space as in* (3.1)*. Then is it infinitesimally Hilbertian if and only if it is $2-$i.s.c. and $\forall f, g \in \mathrm{S}^2_{\mathrm{loc}}(X, d, \mathfrak{m})$ it holds*

$$Df(\nabla g) = Dg(\nabla f) \qquad \mathfrak{m} - a.e.. \tag{5.13}$$

Proof. Assume first that the space is $2-$i.s.c. and that (5.13) holds. Fix $f, g, \in \mathrm{S}^2(X, d, \mathfrak{m})$ and notice that

$$\begin{aligned}\|g + f\|^2_{\mathrm{S}^2(X,d,\mathfrak{m})} - \|g\|^2_{\mathrm{S}^2(X,d,\mathfrak{m})} &\triangleq \||D(g+f)|_w\|^2_{L^2(X,\mathfrak{m})} - \||Dg|_w\|^2_{L^2(X,\mathfrak{m})} \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \||D(g+tf)|_w\|^2_{L^2(X,\mathfrak{m})} \, \mathrm{d}t \\ &= 2\iint_0^1 Df\big(\nabla(g+tf)\big) \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} = 2\iint_0^1 D(g+tf)(\nabla f) \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} \\ &= 2\int Dg(\nabla f) \, \mathrm{d}\mathfrak{m} + \|f\|^2_{\mathrm{S}^2(X,d,\mathfrak{m})}.\end{aligned}$$

Then replacing f with $-f$ and adding up we can conclude.

For the converse, with a cut-off argument we can reduce to the case $f, g \in \mathrm{S}^2(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$. Now we're assuming that $\|\cdot\|_{\mathrm{S}^2(X,d,\mathfrak{m})}$ satisfies the parallelogram rule so we have that

$$\frac{\|g + \varepsilon f\|^2_{\mathrm{S}^2(X,d,\mathfrak{m})} - \|g\|^2_{\mathrm{S}^2(X,d,\mathfrak{m})}}{\varepsilon} = \frac{\|f + \varepsilon g\|^2_{\mathrm{S}^2(X,d,\mathfrak{m})} - \|f\|^2_{\mathrm{S}^2(X,d,\mathfrak{m})}}{\varepsilon} + O(\varepsilon) \quad \forall f, g \in \mathrm{S}^2(X, d, \mathfrak{m}).$$

So by definition of $Df(\nabla g)$ and using (4.8) we get

$$\int Df(\nabla g)\,\mathrm{d}\mathfrak{m} = \int Dg(\nabla f)\,\mathrm{d}\mathfrak{m} \qquad \forall f, g \in \mathrm{S}^2(X, d, \mathfrak{m}). \tag{5.14}$$

Our goal now is to pass from this integral equality to the pointwise statement (5.13). Fix $h \in \mathrm{S}^2(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$ and using (4.8), the Leibniz rule (4.35), the chain rule and (5.14) we have

$$\begin{aligned}
\int h|Df|^2_w\,\mathrm{d}\mathfrak{m} = \int h\,Df(\nabla f)\,\mathrm{d}\mathfrak{m} &= \int \big(D(hf)(\nabla f) - f\,Dh(\nabla f)\big)\,\mathrm{d}\mathfrak{m} \\
&= \int D(hf)(\nabla f)\,\mathrm{d}\mathfrak{m} - \frac{1}{2}\int Dh\big(\nabla(f^2)\big)\,\mathrm{d}\mathfrak{m} \\
&= \int D(hf)(\nabla f)\,\mathrm{d}\mathfrak{m} - \frac{1}{2}\int D(f^2)(\nabla h)\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

From Corollary 4.1.2 the map $\mathrm{S}^2(X, d, \mathfrak{m}) \ni g \mapsto \int Dg(\nabla h)\,\mathrm{d}\mathfrak{m}$ is linear, hence the map

$$\mathrm{S}^2(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m}) \ni f \mapsto \int D(f^2)(\nabla h)\,\mathrm{d}\mathfrak{m}$$

is a quadratic form and similarly, being $\mathrm{S}^2(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m}) \ni g \mapsto \int D(hg)(\nabla f)\,\mathrm{d}\mathfrak{m}$ and $\mathrm{S}^2(X, d, \mathfrak{m}) \ni g \mapsto \int D(hf)(\nabla g)\,\mathrm{d}\mathfrak{m}$ linear then

$$\mathrm{S}^2(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m}) \ni f \mapsto \int D(hf)(\nabla f)\,\mathrm{d}\mathfrak{m}$$

is a quadratic form.
So the map

$$\mathrm{S}^2(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m}) \ni f \mapsto \int h|Df|^2_w\,\mathrm{d}\mathfrak{m}$$

is a quadratic form for any $h \in \mathrm{S}^2(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$ and this gives the thesis. □

On infinitesimally Hilbertian spaces we'll denote $Df(\nabla g)$ by $\nabla f \cdot \nabla g$ in order to highlight its symmetry. A first consequence of (5.13) and the linearity of the differential (Corollary 4.1.2) is the bilinearity of $\nabla f \cdot \nabla g$, i.e.

$$\nabla(\alpha_1 f_1 + \alpha_2 f_2) \cdot \nabla g = \alpha_1 \nabla f_1 \cdot \nabla g + \alpha_2 \nabla f_2 \cdot \nabla g, \qquad \forall f_1, f_2, g \in \mathrm{S}^2_{\mathrm{loc}}(X, d, \mathfrak{m}),\ \alpha_1, \alpha_2 \in \mathbb{R}, \tag{5.15}$$

$$\nabla f \cdot \nabla(\beta_1 g_1 + \beta_2 g_2) = \beta_1 \nabla f \cdot \nabla g_1 + \beta_2 \nabla f \cdot \nabla g_2 \qquad \forall f, g_1, g_2 \in \mathrm{S}^2_{\mathrm{loc}}(X, d, \mathfrak{m}),\ \beta_1, \beta_2 \in \mathbb{R}, \tag{5.16}$$

and from (4.35) we also get $\mathfrak{m}-$a.e.

$$\nabla(f_1 f_2) \cdot \nabla g = f_1 \nabla f_2 \cdot \nabla g + f_2 \nabla f_2 \cdot \nabla g, \qquad \forall f_1, f_2 \in \mathrm{S}^2_{\mathrm{loc}}(X, d, \mathfrak{m}) \cap L^\infty_{\mathrm{loc}}(X, \mathfrak{m}),\ g \in \mathrm{S}^2_{\mathrm{loc}}(X, d, \mathfrak{m}) \tag{5.17}$$

$$\nabla f \cdot \nabla(g_1 g_2) = g_1 \nabla f \cdot \nabla g_2 + g_2 \nabla f \cdot \nabla g_1 \qquad \forall f \in \mathrm{S}^2_{\mathrm{loc}}(X, d, \mathfrak{m}),\ g_1, g_2 \in \mathrm{S}^2_{\mathrm{loc}}(X, d, \mathfrak{m}) \cap L^\infty_{\mathrm{loc}}(X, \mathfrak{m}). \tag{5.18}$$

Theorem 5.3.2. *Let* $(X, d, \mathfrak{m})$ *be a metric measure space as in* (3.1). *Then the following are equivalent:*

I) $(X, d, \mathfrak{m})$ *is infinitesimally Hilbertian.*

II) For any $\Omega \subset X$ *open with* $\mathfrak{m}(\partial\Omega) = 0$ *the space* $(\overline{\Omega}, d, \mathfrak{m}|_\Omega)$ *is infinitesimally Hilbertian.*

III) $W^{1,2}(X, d, \mathfrak{m})$ *is an Hilbert space.*

IV) The Cheeger's energy $\mathrm{Ch}_2 : L^2(X, \mathfrak{m}) \to [0, \infty]$ *is a quadratic form, i.e.*

$$\mathrm{Ch}_2(f + g) + \mathrm{Ch}_2(f - g) = 2\big(\mathrm{Ch}_2(f) + \mathrm{Ch}_2(g)\big) \qquad \forall f, g \in L^2(X, \mathfrak{m}). \tag{5.19}$$

Proof. We have that:

I)⇒II) Follows from Proposition 3.2.1 and the equivalence stated in Proposition 5.3.1.

II)⇒I) Just take $\Omega := X$.

I)⇒III) It follows from the definition of the $W^{1,2}(X, d, \mathfrak{m})-$norm.

III)⇒I) We already know that $W^{1,2}(X, d, \mathfrak{m}) \ni f \mapsto \||Df|_w\|^2_{L^2(X,\mathfrak{m})}$ is a quadratic form. Proceeding exactly as in the proof of Theorem 5.3.1 above we have that

$$W^{1,2}(X, d, \mathfrak{m}) \ni f \qquad \mapsto \qquad |Df|_w^2 \in L^2(X, \mathfrak{m}) \tag{5.20}$$

is a quadratic form. Using the Lindelöf property of (X, d) and the local finiteness of $\mathfrak{m}$ we can build an increasing sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact sets such that $\mathfrak{m}(X \bigcup_{n\in\mathbb{N}} K_n) = 0$ and an increasing sequence $\{\chi_n\}_{n\in\mathbb{N}}$ of Lipschitz bounded functions such that $\mathfrak{m}(\mathrm{supp}(\chi_n)) < \infty$ and $\chi_n \equiv 1$ on K_n for every $n \in \mathbb{N}$. Then for every $f \in \mathrm{S}^2(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$ and $n \in \mathbb{N}$ it holds $f\chi_n \in W^{1,2}(X, d, \mathfrak{m})$. Since $|Df\|_w = |D(f\chi_n)|_w$ $\mathfrak{m}-$a.e. on K_n, from (5.20) and letting $n \to \infty$ we get that

$$\mathrm{S}^2(X, d, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m}) \ni f \qquad \mapsto \qquad |Df|_w^2 \in L^1(X, \mathfrak{m})$$

is a quadratic form as well. With a truncation argument we get also that $\mathrm{S}^2(X, d, \mathfrak{m}) \ni f \mapsto |Df|_w^2 \in L^1$ is a quadratic form. Integrating we can conclude.

III)⇔IV) The formula $\|f\|_{W^{1,2}(X,d,\mathfrak{m})} = \|f\|^2_{L^2(X,\mathfrak{m})} + 2\mathrm{Ch}_2(f)$ shows that Ch_2 satisfies the parallelogram rule if and only if it so does the $W^{1,2}(X, d, \mathfrak{m})-$norm.

□

The importance of infinitesimally Hilbertian spaces is that the Laplacian is linear, as explained by the following theorem. We will denote by

$$D_{\mathrm{ifm}}(\mathbf{\Delta}, \Omega) \subset D(\mathbf{\Delta}, \Omega)$$

the set of those g's whose Laplacian has finite internal mass, i.e. for any $\mu \in \mathbf{\Delta} g|_\Omega$ it holds $|\mu|(\Omega') < \infty$ for any $\Omega' \Subset \mathrm{Int}(\Omega)$.

Remark 5.3.2. If (X, d) is proper then $D_{\mathrm{ifm}}(\mathbf{\Delta}, \Omega) = D(\mathbf{\Delta}, \Omega)$.

Theorem 5.3.3. *Let $(X, d, \mathfrak{m})$ an infinitesimally Hilbertian space. Then for any $\Omega \subset X$ open the set $D_{\mathrm{ifm}}(\boldsymbol{\Delta}, \Omega) \cap \mathrm{S}^2_{\mathrm{int}}(X, d, \mathfrak{m})$ is a vector space and for $g \in D_{\mathrm{ifm}}(\boldsymbol{\Delta}, \Omega) \cap \mathrm{S}^2_{\mathrm{int}}(X, d, \mathfrak{m})$ $\boldsymbol{\Delta} g|_\Omega$ is single-valued and linearly depends on g.*

Proof. Fix $\Omega \subset X$ open. We already know from Theorem 5.1.1 that $\boldsymbol{\Delta} g|_\Omega$ is single valued for any $g \in \mathrm{S}^2_{\mathrm{int}}(\Omega) \cap D(\boldsymbol{\Delta}, \Omega)$ being $(X, d, \mathfrak{m})$ $2-$i.s.c.. With a cut-off argument we deduce that (5.15) and (5.16) are satisfied also for functions in $\mathrm{S}^2_{\mathrm{int}}(\Omega)$. Pick $g_1, g_2 \in D_{\mathrm{ifm}}(\boldsymbol{\Delta}, \Omega) \cap \mathrm{S}^2_{\mathrm{int}}(\Omega)$ and $\beta_1, \beta_2 \in \mathbb{R}$. We have that $|\boldsymbol{\Delta} g_i|\big(\mathrm{supp}(f)\big) < \infty$ for any $f \in \mathrm{Test}(\Omega)$ so that in particular $\mathrm{Test}(\Omega) \subset L^1(\Omega, |\boldsymbol{\Delta} g|_\Omega|)$, $i = 1, 2$. Also, $\mathrm{Test}(\Omega) \subset \mathrm{S}^2(X, d, \mathfrak{m})$. Let $f \in \mathrm{Test}(\Omega)$ and conclude with

$$\begin{aligned}
\int_\Omega f \, \mathrm{d}(\beta_1 \boldsymbol{\Delta} g_1|_\Omega + \beta_2 \boldsymbol{\Delta} g_2|_\Omega) &= \beta_1 \int f \, \mathrm{d}\boldsymbol{\Delta} g_1|_\Omega + \beta_2 \int_\Omega f \, \mathrm{d}\boldsymbol{\Delta} g_2|_\Omega \\
&\triangleq -\beta_1 \int_\Omega \nabla f \cdot \nabla g_1 \, \mathrm{d}\mathfrak{m} - \beta_2 \int_\Omega \nabla f \cdot \nabla g_2 \, \mathrm{d}\mathfrak{m} \\
&= -\int_\Omega \nabla f \cdot \nabla(\beta_1 g_1 + \beta_2 g_2) \, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

□

Now on infinitesimally Hilbertian spaces from (5.19) we deduce that the formula

$$\mathscr{E}(f, g) := \mathrm{Ch}_2(f + g) - \mathrm{Ch}_2(f) - \mathrm{Ch}_2(g)$$

defines a symmetric bilinear form on $W^{1,2}(X, d, \mathfrak{m})$. Now we will take from [14] some definitions regarding the theory of bilinear forms on Hilbert spaces.

Definition 5.3.2. Let $(H, \langle \cdot, \cdot \rangle)$ be an Hilbert space. Given a symmetric bilinear form

$$\mathscr{E} : \mathcal{D}(\mathscr{E}) \times \mathcal{D}(\mathscr{E}) \to \mathbb{R}$$

with $\mathcal{D}(\mathscr{E})$ a dense linear subspace of H we say that $\mathscr{E}$ is *closed* if $\mathcal{D}(\mathscr{E})$ is complete w.r.t. the metric induced by $\mathscr{E}$, i.e.

$$\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(\mathscr{E}),\ \mathscr{E}_1(u_n - u_m, u_n - u_m) \overset{n,m \to \infty}{\longrightarrow} 0 \Rightarrow \exists u \in \mathcal{D}(\mathscr{E}) : \mathscr{E}_1(u_n - u, u_n - u) \overset{n \to \infty}{\longrightarrow} 0$$

where $\mathscr{E}_1(u, v) := \mathscr{E}(u, v) + \langle u, v \rangle$.

Definition 5.3.3. Let $(X, \mathfrak{m})$ be a $\sigma-$finite measure space and consider the Hilbert space $H := L^2(X, \mathfrak{m})$. A symmetric form $\mathscr{E}$ on H is called *Markovian symmetric* if the following property holds: for any $\varepsilon > 0$ there exists a real function ϕ_ε such that

$$\phi_\varepsilon(t) = t \quad \forall t \in [0, 1], \quad -\varepsilon \le \phi_\varepsilon(t) \le 1 + \varepsilon \quad t \in \mathbb{R} \text{ and } 0 \le \phi_\varepsilon(t') - \phi_\varepsilon(t) \le t' - t, t < t',$$

$$u \in \mathcal{D}(\mathscr{E}) \Rightarrow \phi_\varepsilon(u) \in \mathcal{D}(\mathscr{E}),\ \mathscr{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \le \mathscr{E}(u, u).$$

A *Dirichlet* form is by definition a symmetric form on H that is not only Markovian but also closed.

Taking as $\mathscr{E}$ the one previously defined then the chain rule (3.5) ensures that it is Markovian and the semicontinuity of Ch_2 means that it is closed. Hence $\mathscr{E}$ is a Dirichlet form on $L^2(X,\mathfrak{m})$. The generator of this form will be denoted by Δ so that

$$\Delta : D(\Delta) \subset L^2(X,\mathfrak{m}) \to L^2(X,\mathfrak{m})$$

and Δ and its domain $D(\Delta)$ are defined by

$$g \in D(\Delta), h = \Delta g \quad \Leftrightarrow \quad \mathscr{E}(f,g) = -\int fh \,\mathrm{d}\mathfrak{m} \quad \forall f \in D(\mathscr{E}). \tag{5.21}$$

Remark 5.3.3. Notice that $D(\mathscr{E}) = D(\mathrm{Ch}_2) = W^{1,2}(X,d,\mathfrak{m})$.

Theorem 5.3.4. *Let $(X,d,\mathfrak{m})$ be an infinitesimally Hilbertian space, $g \in W^{1,2}(X,d,\mathfrak{m})$ and $h \in L^2(X,\mathfrak{m})$. Consider the following:*

I) $g \in D(\boldsymbol{\Delta})$ and $\boldsymbol{\Delta} g = h\mathfrak{m}$,

II) $h \in -\partial^-\mathrm{Ch}_2(g)$,

III) $g \in D(\Delta)$ and $\Delta g = h$.

Then

$$\textit{(II)} \quad \Leftrightarrow \quad \textit{(III)} \quad \Rightarrow \quad \textit{(I)},$$

and if $\mathfrak{m}$ is finite on bounded sets it also holds

$$\textit{(I)} \quad \Rightarrow \quad \textit{(II), (III)}.$$

Proof. We proceed by steps:

(III)$\Rightarrow$(II) We need to prove that for any $f \in L^2(X,\mathfrak{m})$ it holds

$$\mathrm{Ch}_2(g) - \int (f-g)h \,\mathrm{d}\mathfrak{m} \le \mathrm{Ch}_2(f). \tag{5.22}$$

If $f \notin D(\mathrm{Ch}_2) = D(\mathscr{E})$ there is nothing to prove. Otherwise $f - g \in D(\mathscr{E})$ and by definition of Δ we have that

$$-\int (f-g)h \,\mathrm{d}\mathfrak{m} = \mathscr{E}(f-g,g) = \mathscr{E}(f,g) - \mathscr{E}(g,g) \le \frac{1}{2}\mathscr{E}(f,f) - \frac{1}{2}\mathscr{E}(g,g).$$

(II)$\Rightarrow$(III) We pick $f \in D(\mathscr{E})$ and notice that by definition of $\partial^-\mathrm{Ch}_2(g)$ it holds

$$\mathrm{Ch}_2(g) - \int \varepsilon f h \,\mathrm{d}\mathfrak{m} \le \mathrm{Ch}_2(g+\varepsilon f), \qquad \forall \varepsilon \in \mathbb{R}.$$

We conclude by observing that $\mathrm{Ch}_2(g+\varepsilon f) = \mathrm{Ch}_2(g) + \varepsilon^2\mathrm{Ch}_2(f) - \varepsilon\mathscr{E}(f,g)$, diving by ε and letting it to tend to 0.

(II)$\Rightarrow$(I) This is a particular case of Theorem 5.1.2.

(I)⇒(III) We now assume that $\mathfrak{m}$ is finite on bounded sets. We know that

$$-\int \nabla f \cdot \nabla g \,\mathrm{d}\mathfrak{m} = \int fh \,\mathrm{d}\mathfrak{m} \qquad \forall f \in \mathrm{Test}(X) \cap L^1(X, |h|\mathfrak{m})$$

and we want to conclude that the same is true for any $f \in W^{1,2}(X, d, \mathfrak{m})$. Pick $f \in W^{1,2}(X, d, \mathfrak{m})$ and assume for the moment that $\mathrm{supp}(f)$ is bounded. Let χ be a Lipschitz bounded function with bounded support and identically equal to 1 on $\mathrm{supp}(f)$. Also, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Lipschitz functions converging to f in $W^{1,2}(X, d, \mathfrak{m})$ (Corollary 3.3.5). Then $f_n\chi \to f$ in $W^{1,2}(X, d, \mathfrak{m})$ and $f_n\chi \in \mathrm{Test} \cap L^1(X, |h|\mathfrak{m})$ for any $n \in \mathbb{N}$. Thus passing to the limit in

$$-\int \nabla(f_n\chi) \cdot \nabla g \,\mathrm{d}\mathfrak{m} = \int f_n\chi h \,\mathrm{d}\mathfrak{m}$$

we get

$$-\int \nabla f \cdot \nabla g \,\mathrm{d}\mathfrak{m} = \int fh \,\mathrm{d}\mathfrak{m}$$

for any $f \in W^{1,2}(X, d, \mathfrak{m})$ with bounded support.
To achieve the general case let $\{\chi_n\}_{n\in\mathbb{N}}$ be an increasing sequence of non-negative $1-$Lipschitz functions with bounded support and such that $\chi_n \equiv 1$ on $B_n(x_0)$, $x_0 \in X$ fixed. Fixing $f \in W^{1,2}(X, d, \mathfrak{m})$ we have that $f\chi_n \in W^{1,2}(X, d, \mathfrak{m})$ and has bounded support. Using the dominate convergence theorem we get that

$$\|f - f\chi_n\|_{L^2(X,\mathfrak{m})} \overset{n\to\infty}{\longrightarrow} 0.$$

Also, we have that

$$\begin{aligned} |D(f - f\chi_n)|_w &\overset{n\to\infty}{\longrightarrow} 0 \quad \mathfrak{m}-\text{a.e.} \\ |D(f - f\chi_n)|_w &\leq |Df|_w|1-\chi_n| + |f| \quad \mathfrak{m}-\text{a.e.} \end{aligned}$$

so by dominated convergence we have

$$|D(f - f\chi_n)|_w \overset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(X, \mathfrak{m})$$

So we can conclude by letting $n \to \infty$ in

$$-\int \nabla(f\chi_n) \cdot \nabla g \,\mathrm{d}\mathfrak{m} = \int f\chi_n h \,\mathrm{d}\mathfrak{m}.$$

□

Recall that a metric space (X, d) is said *proper* is every closed ball is compact.

Lemma 5.3.5. *Let $(X, d, \mathfrak{m})$ be a proper infinitesimally Hilbertian space, $\Omega \subset X$ an open set and $g \in D(\mathbf{\Delta}, \Omega) \cap \mathrm{S}^2_{\mathrm{int}}(\Omega)$. Then for every $\psi \in \mathrm{S}^2(X, d, \mathfrak{m}) \cap C_c(X)$ with support contained in Ω it holds*

$$-\int_\Omega \nabla\psi \cdot \nabla g \,\mathrm{d}\mathfrak{m} = \int_\Omega \psi \,\mathrm{d}\mathbf{\Delta} g|_\Omega.$$

Proof. Since (X, d) is proper $\mathrm{supp}(\psi)$ is compact and therefore has $\mathbf{\Delta} g|_\Omega-$finite measure. Similarly $\mathfrak{m}(\mathrm{supp}(\psi)) < \infty$ and thus $\psi \in L^2(X, \mathfrak{m})$. Hence $\psi \in W^{1,2}(X, d, \mathfrak{m})$ and from Corollary 3.3.5 we know there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset W^{1,2}(X, d, \mathfrak{m})$ of Lipschitz functions converging to ψ in $W^{1,2}(X, d, \mathfrak{m})$. Define $\psi^{t,+}, \psi^{t,-} : X \to \mathbb{R}$ by

$$\psi^{t,+}(x) := \inf_{y\in X} \left\{\psi(y) + \frac{d^2(x, y)}{2t}\right\}, \qquad \psi^{t,-}(x) := \sup_{y\in X} \left\{\psi(y) + \frac{d^2(x, y)}{2t}\right\}.$$

It can be proven that these two functions are Lipschitz, equibounded and it holds $\psi^{t,+}(x) \uparrow \psi(x)$, $\psi^{t,-}(x) \downarrow \psi(x)$ as $t \downarrow 0$ for any $x \in X$. Putting

$$\psi_{n,t} := \min\{\max\{\psi_n, \psi^{t,+}\}, \psi^{t,-}\}$$

we observe that $\psi_{n,t}$ is Lipschitz for any $n \in \mathbb{N}$ and $t > 0$. Let $\chi \in \mathrm{Test}(\Omega)$ be identically 1 on $\mathrm{supp}(\psi)$ and consider the functions $\chi\psi_{n,t} \in \mathrm{Test}(\Omega)$. Since $\psi^{t,+}, \psi^{t,-}$ and χ are Lipschitz we can use the dominate convergence theorem to ensure that for any $t > 0$ the sequence $\{\psi_{n,t}\}_{n\in\mathbb{N}}$ converges to ψ in $W^{1,2}(X, d, \mathfrak{m})-$energy as $n \to \infty$. Thus this convergence is also w.r.t. the $W^{1,2}-$norm and from the fact that $\mathrm{supp}(\chi\psi_{n,t}) \subset \mathrm{supp}(\chi)$ and $\mathfrak{m}(\mathrm{supp}(\chi)) < \infty$ we get

$$\lim_{n\to\infty} \int_\Omega \nabla(\chi\psi_{n,t}) \cdot \nabla g \,\mathrm{d}\mathfrak{m} = \int_\Omega \nabla\psi \cdot \nabla g \,\mathrm{d}\mathfrak{m} \qquad \forall t > 0. \tag{5.23}$$

By construction, $\{\psi_{n,t}\}_{n\in\mathbb{N}}$ is bounded in both n and t and pointwise converges to ψ as $t \to 0$ uniformly w.r.t. n. Taking into account that $\mathrm{supp}(\chi\psi_{n,t}) \subset \mathrm{supp}(\chi)$, $|\mathbf{\Delta} g|_\Omega|(\mathrm{supp}(\chi)) < \infty$ as said before and the dominated convergence theorem we get

$$\lim_{t\downarrow 0} \int_\Omega \chi\psi_{n,t} \,\mathrm{d}\mathbf{\Delta} g|_\Omega = \int_\Omega \psi \,\mathrm{d}\mathbf{\Delta} g|_\Omega \qquad \text{uniformly on } n. \tag{5.24}$$

Since $\chi\psi_{n,t} \in \mathrm{Test}(\Omega)$ then (5.23) and (5.24) together with a diagonalization argument give the thesis. □

Lemma 5.3.6. *Let $(X, d, \mathfrak{m})$ be a proper infinitesimally Hilbertian space, $\Omega \subset X$ an open set and $g \in D(\mathbf{\Delta}, \Omega) \cap \mathrm{S}^2_{\mathrm{int}}(\Omega)$ with $\mathbf{\Delta} g|_\Omega \ll \mathfrak{m}$ with density $h \in L^2_{\mathrm{loc}}(\Omega, \mathfrak{m}|_\Omega)$. Then for every $\psi \in W^{1,2}(X, d, \mathfrak{m})$ with bounded support contained in Ω it holds*

$$-\int_\Omega \nabla\psi \cdot \nabla g \,\mathrm{d}\mathfrak{m} = \int_\Omega \psi h \,\mathrm{d}\mathfrak{m}.$$

Proof. By Corollary 3.3.5 we know there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset W^{1,2}(X, d, \mathfrak{m})$ of Lipschitz functions converging to ψ in $W^{1,2}(X, d, \mathfrak{m})$. Let $\chi \in \mathrm{Test}(\Omega)$ be identically 1 on $\mathrm{supp}(\psi)$ and notice that $\chi\psi_n \in \mathrm{Test}(\Omega) \cap L^1(\Omega, |h|\mathfrak{m})$ and $\chi\psi_n \to \psi$ in $W^{1,2}(X, d, \mathfrak{m})$. Hence we can pass to the limit in

$$-\int_\Omega \nabla(\chi\psi_n) \cdot \nabla g \,\mathrm{d}\mathfrak{m} = \int_\Omega \chi\psi_n h \,\mathrm{d}\mathfrak{m}$$

getting the thesis. □

Now we can establish the validity of the chain rule for the Laplacian.

Theorem 5.3.7. *Let $(X, d, \mathfrak{m})$ be an infinitesimally Hilbertian metric measure space, $\Omega \subset X$ an open set and $g \in D(\boldsymbol{\Delta}, \Omega) \cap \mathrm{S}^2_{\mathrm{int}}(\Omega)$, $I \subset \mathbb{R}$ an open interval such that $\mathfrak{m}(g^{-1}(\mathbb{R} \setminus I)) = 0$ and $\phi \in C^{1,1}_{\mathrm{loc}}(I)$. Then the following holds:*

I) Assume that $g|_{\Omega'}$ is Lipschitz for every $\Omega' \Subset \mathrm{Int}(\Omega)$. Then

$$\phi \circ g \in D(\boldsymbol{\Delta}, \Omega) \qquad \textit{and} \qquad \boldsymbol{\Delta}(\phi \circ g)|_\Omega = \phi' \circ g \boldsymbol{\Delta} g|_\Omega + \phi'' \circ g |Dg|^2_w \mathfrak{m}|_\Omega. \tag{5.25}$$

II) Assume that (X, d) is proper and $g \in C(\Omega)$. Then (5.25) *holds.*

III) Assume that (X, d) is proper, $g \in L^2_{\mathrm{loc}}(X, \mathfrak{m})$ and $\boldsymbol{\Delta} g|_\Omega \ll \mathfrak{m}$ with Radon-Nikodym derivative in $L^2_{\mathrm{loc}}(X, \mathfrak{m}|_\Omega)$. Then (5.25) *holds.*

Proof. The first is just a particular case of Theorem 5.2.1.

II) Let $\Omega' \Subset \mathrm{Int}(\Omega)$ and observe that since $\overline{\Omega}'$ is compact then its image under g is compact as well and thus ϕ'' is bounded on $g(\Omega')$. It easily follows that the formula

$$\tilde{\mu} := (\phi' \circ g)\boldsymbol{\Delta} g|_\Omega + (\phi'' \circ g)|Dg|^2_w \mathfrak{m}|_\Omega$$

defines a locally finite measure on Ω, so that the statement makes sense.
Being $\mathrm{Test}(\Omega) \subset L^1(\Omega, |\tilde{\mu}|)$ pick $f \in \mathrm{Test}(\Omega)$ and use (5.17) and (5.18) to get

$$\begin{aligned}\nabla f \cdot \nabla(\phi \circ g) = (\phi' \circ g)\nabla f \cdot \nabla g &= \nabla(f\phi' \circ g) \cdot \nabla g - f\nabla(\phi' \circ g) \cdot \nabla g \\ &= \nabla(f\phi' \circ g) \cdot \nabla g - f(\phi'' \circ g)|Dg|^2_w \qquad \mathfrak{m} - \text{a.e..}\end{aligned}$$

Integrating we obtain

$$-\int \nabla f \cdot \nabla(\phi \circ g)\, \mathrm{d}\mathfrak{m} = -\int \nabla(f\phi' \circ g) \cdot \nabla g\, \mathrm{d}\mathfrak{m} + \int f(\phi'' \circ g)|Dg|^2_w\, \mathrm{d}\mathfrak{m}. \tag{5.26}$$

Hence to conclude is sufficient to show that

$$-\int \nabla(f\phi' \circ g) \cdot \nabla g\, \mathrm{d}\mathfrak{m} = \int f\phi' \circ g\, \mathrm{d}\boldsymbol{\Delta} g|_\Omega \tag{5.27}$$

but this is a consequence of Lemma 5.3.5 applied to $\psi := f\phi' \circ g$ which by our assumptions belongs to $\mathrm{S}^2(X, d, \mathfrak{m}) \cap C_c(\Omega)$.

III) By hypotheses we know that $\phi' \circ g \in L^2_{\mathrm{loc}}(\Omega, \mathfrak{m}|_\Omega)$ and $\phi'' \circ g \in L^\infty_{\mathrm{loc}}(\Omega, \mathfrak{m}|_\Omega)$. Therefore, since $\boldsymbol{\Delta} g|_\Omega \ll \mathfrak{m}$ with L^2_{loc} density w.r.t.$\mathfrak{m}$ the same formula as before for $\tilde{\mu}$ defines a locally finite measure on Ω and the statement makes sense. As before, we have $\mathrm{Test}(\Omega) \subset L^1(\Omega, |\tilde{\mu}|)$. With the same computations we get (5.26) as well so we reduce to show that (5.27) holds also in this case. But this is a consequence of Lemma 5.3.6 applied to $\psi :) f\phi' \circ g$ which belongs to $W^{1,2}(X, d, \mathfrak{m})$ and has bounded support contained in Ω.

□

The last thing we want to prove is that also the Leibniz rule holds for the Laplacian (analogous to the Euclidian case) in this setting, so it is not available in the general non-linear setting.

Theorem 5.3.8. *Let $(X, d, \mathfrak{m})$ be an infinitesimally Hilbertian metric measure space, $\Omega \subset X$ an open set and $g_1, g_2 \in D(\mathbf{\Delta}, \Omega) \cap \mathrm{S}^2_{\mathrm{int}}(\Omega)$. Then the following holds.*

I) If g_1, g_2 are Lipschitz on Ω' for every $\Omega' \Subset \mathrm{Int}(\Omega)$ and $g_1, g_2 \in D_{\mathrm{ifm}}(\mathbf{\Delta}, \Omega)$, then $g_1, g_2 \in D(\mathbf{\Delta}, \Omega)$ and

$$\mathbf{\Delta}(g_1 g_2)|_\Omega = g_1 \mathbf{\Delta} g_2|_\Omega + g_2 \mathbf{\Delta} g_1|_\Omega + 2\nabla g_1 \cdot \nabla g_2 \mathfrak{m}. \tag{5.28}$$

II) If (X, d) is proper and $g_1, g_2 \in C(\Omega)$ then $g_1 g_2 \in D(\mathbf{\Delta}, \Omega)$ and (5.28) holds.

III) if (X, d) is proper, $g_1, g_2 \in L^2_{\mathrm{loc}}(\Omega, \mathfrak{m}|_\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega, \mathfrak{m}|_\Omega)$ and $\mathbf{\Delta} g_i|\Omega \ll \mathfrak{m}$ with $L^2_{\mathrm{loc}}(\Omega, \mathfrak{m}|\Omega)-$density, $i = 1, 2$, then $g_1 g_2 \in D(\mathbf{\Delta}, \Omega)$ and (5.28) holds.

Proof. We proceed case by case.

I) Being Lipschitz, g_1 and g_2 are bounded on Ω' for any $\Omega' \Subset \mathrm{Int}(\Omega)$, hence $g_1 g_2 \in \mathrm{S}^2_{\mathrm{int}}(\Omega)$. It is also clear that the right hand side of (5.28) defines a locally finite measure μ on Ω, so the statement makes sense. The fact that $|\mathbf{\Delta} g_i|_\Omega|(\Omega')$ is finite for every $\Omega' \Subset \mathrm{Int}(\Omega)$, $i = 1, 2$, grants that $\mathrm{Test}(\Omega) \subset L^1(\Omega, |\mathbf{\Delta} g_i|_\Omega|)$, $i = 1, 2$, and $\mathrm{Test}(\Omega) \subset L^1(\Omega, |\mu|)$. To conclude, pick $f \in \mathrm{Test}(\Omega)$ and notice that $fg_1, fg_2 \in \mathrm{Test}(\Omega)$ and take the Leibniz's rule (5.17) and (5.18) into account to get

$$\nabla f \cdot \nabla(g_1 g_2) = g_1 \nabla f \cdot \nabla g_2 + g_2 \nabla f \cdot \nabla g_1 = \nabla(fg_1) \cdot \nabla g_2 + \nabla(fg_2) \cdot \nabla g_1 - 2f \nabla g_1 \cdot \nabla g_2, \tag{5.29}$$

which integrated gives the thesis.

II) As before, the right hand side of (5.28) defines a locally finite measure μ and as before $g_1 g_2 \in \mathrm{S}^2_{\mathrm{int}}(\Omega)$, $\mathrm{Test}(\Omega) \subset L^1(\Omega, |\mathbf{\Delta} g_i|_\Omega|)$, $i = 1, 2$, and $\mathrm{Test}(\Omega) \subset L^1(\Omega, |\mu|)$. Pick $f \in \mathrm{Test}(\Omega)$ and notice that with the same computations done in (5.29) the thesis follows if we show that

$$\int_\Omega \nabla(fg_1) \cdot \nabla g_2 \, \mathrm{d}\mathfrak{m} = -\int_\Omega fg_1 \, \mathrm{d}\mathbf{\Delta} g_2|_\Omega, \tag{5.30}$$

$$\int_\Omega \nabla(fg_2) \cdot \nabla g_1 \, \mathrm{d}\mathfrak{m} = -\int_\Omega fg_2 \, \mathrm{d}\mathbf{\Delta} g_1|_\Omega. \tag{5.31}$$

These are a consequence of Lemma 5.3.5 applied to $\psi := fg_1$, $g := g_2$ and $\psi := fg_2$ and $g := g_1$ respectively.

III) Same as in (II) but using Lemma 5.3.6 in place of Lemma 5.3.5 to justify (5.30) and (5.31).

□

5.4 An example: the Heisenberg group $\mathbb{H}$

The first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, \circ)$, where $\circ$ is its standard group law, owns a sub-Riemannian structure given by the horizontal distribution $\mathscr{S}$ generated by the 2 left invariant vector fields $X = \partial_x - (y/2)\partial_t$ and $Y = \partial_y + (x/2)\partial_t$, whose commutator is $[X, Y] = \partial_t$. Since these vectors X and Y and their commutator generate all the tangent space of $\mathbb{R}^3$, L. Hormander in [17] was able to show that the sub-Laplacian, defined by $\Delta_{\mathbb{H}^1} = X^2 + Y^2$, is an hypoelliptic operator. However when we consider a surface $\mathcal{S}$ immersed in $\mathbb{H}^1$ the intersection between the tangent space $T_p\mathcal{S}$ and the distribution $\mathscr{S}_p$ is given by a one dimensional space. Therefore from the differential point of view the sub-Laplacian restricted to $\mathcal{S}$ is given by the second derivative in the horizontal direction tangent to $\mathcal{S}$ and this clearly is not an hypoellitic operator. In order to understand this problem we can consider as immersed surface the plane $\mathbb{X} := \{x = 0\}$. Since we have developed a general theory for the Laplacian in metric spaces we want to understand if this metric Lapacian coincide with the differential one or give us a different hypoellitic operator.

5.4.1 The sub-Laplacian

Consider

$$\mathbb{R}^3 \quad \text{with coordinates } x, y \text{ and } t$$

and the three vector fields

$$X := \partial_x - \frac{y}{2}\partial_t, \qquad Y := \partial_y + \frac{x}{2}\partial_t \quad \text{and} \quad T := [X, Y] = \partial_t.$$

We define the *degree* of a vector field as the number of commutators $+1$, done on the basis vector fields, required to obtain it. In our case, X and Y are vector fields of degree 1 while T of degree 2. Notice also that X, Y are Hörmander vector fields because X, Y and T generate the tangent space of $\mathbb{R}^3$. Hence if we consider the distribution

$$\mathscr{S} := \operatorname{span}(X, Y),$$

the triple $(\mathbb{R}^3, \mathscr{S}, g)$, with g the Riemannian metric w.r.t. which X, Y and T are orthonormal, is a sub-Riemannian manifold called the *Heisenberg group* $\mathbb{H}^1$.

To obtain a metric measure space, we need to compute the distance function d induced by X, Y and T. Since $\mathscr{S}$ satisfies the Hörmander condition $\forall \xi, \xi_0 \in \mathbb{H}^1$ there exists an integral curve γ with values in $\mathbb{H}^1$ and endpoints ξ, ξ_0. Therefore we can define d between ξ, ξ_0 as the minimum length of the integral curves connecting ξ and ξ_0. A first estimate of d is given by this theorem:

Theorem 5.4.1 (Nagel, Stein, Wainger). *If $(\mathcal{M}, \mathscr{S}, g)$ is a sub-Riemannian manifold and $X_1, \ldots, X_n$ form a base of the tangent space adapted to a distribution $\mathscr{S}$ satisfying the Hörmander condition then $\forall \xi_0 \in \mathcal{M}$ there exists a neighbourhood U of ξ_0 such that we can represent any $\xi \in U$ as*

$$\xi := \exp\{t_1 X_1 + \cdots + t_n X_n\}(\xi_0) \tag{5.32}$$

where $(t_1, \ldots, t_n)$ are called canonical coordinates. Moreover, for all $K \subset U$ compact subset there exist two positive constants C_1, C_2 such that

$$C_1 \sum_{i=1}^{n} |t_i|^{\frac{1}{\deg(X_i)}} \leq d(\xi, \xi_0) \leq C_2 \sum_{i=1}^{n} |t_i|^{\frac{1}{\deg(X_i)}} \tag{5.33}$$

In our case with X, Y and T as vector fields we have that

$$d(\xi, \xi_0) \sim |t_X| + |t_Y| + \sqrt{|t_T|}$$

Putting $\gamma(s) := (x(s), y(s), t(s))$, with $s \in [0,1]$, by definition of integral curve we need to solve the following system of differential equations

$$\begin{cases} \gamma'(s) = (t_X X + t_Y Y + t_T T) I\big(\gamma(s)\big), \\ \gamma(0) = \xi_0, \ \gamma(1) = \xi \end{cases}$$

Integrating the system we get the expressions for t_X, t_Y and t_T:

$$\begin{cases} t_X = \xi_x - \xi_{0_x}, \\ t_Y = \xi_y - \xi_{0_y}, \\ t_T = (\xi_t - \xi_{0_t}) - \frac{\xi_x \xi_{0_y} + \xi_y \xi_{0_x}}{2}. \end{cases}$$

If we consider $\xi_0 = (0,0,0)$ we get

$$d(\xi, 0) \sim |\xi_x| + |\xi_y| + \sqrt{|\xi_t|},$$

which tells us that this distance is similar to the Euclidian one in all directions except for the vertical one t. It can be defined a norm on $\mathbb{H}$ which defines an equivalent distance, called *Korany metric*:

$$\|\xi\|_K := \sqrt[4]{(x^2+y^2)^2 + t^2}, \quad \xi = (x, y, t)$$

We can compute d also as the Carnot-Carathéodory distance d_{CC} by writing the geodesic equations: as in [8] we write $\lambda := \xi dx + \eta dy + \theta dt$ for any element λ in the cotangent space. Being X and Y orthonormal w.r.t. g the Hamiltonian function is, with $q \in \mathbb{R}^3$,

$$H(q, \lambda) := \frac{1}{2}\left(\left(\xi - \frac{1}{2}\theta y\right)^2 + \left(\eta + \frac{1}{2}\theta x\right)^2\right).$$

The Hamiltonian system and the initial conditions are

$$\begin{cases} \dot{x} &= \xi - \frac{1}{2}\theta y, \\ \dot{y} &= \eta + \frac{1}{2}\theta x, \\ \dot{t} &= \frac{1}{2}(\eta x - \xi y) + \frac{1}{4}\theta(x^2 + y^2) \\ \dot{\xi} &= -\frac{1}{2}\eta\theta - \frac{1}{4}\theta^2 x \\ \dot{\eta} &= -\frac{1}{2}\xi\theta - \frac{1}{4}\theta^2 y, \\ \dot{\theta} &= 0, \end{cases} \qquad \begin{cases} x(0) = y(0) = t(0) = 0, \\ \xi(0) = \xi_0, \eta(0) = \eta_0, \theta(0) = \theta_0. \end{cases}$$

Integrating the system and reducing it to a system containing only the variables (x, y, t) we get that the solution $\gamma(s) = (x(s), y(s), t(s))$ is

$$\begin{aligned} x(s) &= \frac{\xi_0}{|\theta_0|}\sin(|\theta_0|s) - \frac{\eta_0}{|\theta_0|}(\cos(|\theta_0|s) - 1), \\ y(s) &= -\frac{\xi_0}{|\theta_0|}(\cos(|\theta_0|s) - 1) - \frac{\eta_0}{|\theta_0|}\sin(|\theta_0|s), \qquad \text{if} \quad \theta_0 \neq 0 \\ t(s) &= \frac{\xi_0^2 + \eta_0^2}{2|\theta_0|^2}(|\theta_0|s - \sin(|\theta_0|s)), \end{aligned}$$

and

$$x(s) = \xi_0 s, \quad y(s) = \eta_0 s, \quad t(s) = 0, \qquad \text{if} \quad \theta_0 = 0.$$

Computing the length of the geodesics now we get the d_{CC} distance:

$$\begin{cases} d_{CC}(0, (x, y, t)) = C(|t| + x^2 + y^2), & C > 0, \\ d_{CC}((x, y, t), (x, y, t')) = \sqrt{\pi|t - t'|}. \end{cases}$$

As reference measure $\mathfrak{m}$ we consider the $3-$dimensional Lebesgue's measure $\mathcal{L}^3$, so that we can consider the metric measure space

$$\mathcal{H} := (\mathbb{H}^1, d_K, \mathcal{L}^3).$$

We want to compute the Laplacian over all $\mathcal{H}$, considering $|Dg|_w = \|\nabla_{\mathcal{H}} g\|_K$ with $\nabla_{\mathcal{H}} g := (Xg, Yg)$ (actually this can be proven).

Remark 5.4.1. We already know that the space is 2-i.s.c., so the two functions $D^{\pm} f(\nabla g)$ agree. Moreover, we can compute $Df(\nabla g)$ as $\langle \nabla f, \nabla g \rangle_{\mathbb{H}^1}$, so we expect that the metric Laplacian coincides with the sub-Riemannian one defined by $X^2 + Y^2$.

By definitions we have that

$$\begin{aligned} \|\nabla_{\mathcal{H}} g\|_K^2 &\triangleq \left\{ \left[\left(\partial_x g - \frac{y}{2} \partial_t g \right)^2 + \left(\partial_y g + \frac{x}{2} \partial_t g \right)^2 \right]^{\frac{4}{2} \cdot \frac{1}{4}} \right\}^2, \\ &= (\partial_x g)^2 + (\partial_y g)^2 + \frac{x^2 + y^2}{4} (\partial_t g)^2 - (\partial_t g) \big[x(\partial_y g) - y(\partial_x g) \big] \\ \|\nabla_{\mathcal{H}}(g + \varepsilon f)\|_K^2 &= \|\nabla_{\mathcal{H}} g\|_K^2 + \varepsilon \Big[2(\partial_x g)(\partial_x f) + 2(\partial_y g)(\partial_y f) + \frac{x^2 + y^2}{2} (\partial_t g)(\partial_t f) \\ &\quad - x(\partial_t g)(\partial_y f) + y(\partial_t g)(\partial_x f) - x(\partial_t f)(\partial_y f) + y(\partial_t f)(\partial_x f) \Big] + o(\varepsilon). \end{aligned}$$

Now inserting those two quantities in the definition $Df(\nabla g)$ we obtain

$$\begin{aligned} Df(\nabla g) := \inf_{\varepsilon > 0} \frac{|D(g + \varepsilon f)|_w^2 - |Dg|_w^2}{2\varepsilon} &= \frac{1}{2} \Big[2(\partial_x g)(\partial_x f) + 2(\partial_y g)(\partial_y f) + \frac{x^2 + y^2}{2} (\partial_t g)(\partial_t f) \\ - x(\partial_t g)(\partial_y f) + y(\partial_t g)(\partial_x f) - x(\partial_t f)(\partial_y f) + y(\partial_t f)(\partial_x f) \Big] \end{aligned}$$

and integrating by parts

$$\int_{\mathcal{H}} Df(\nabla g) \, \mathrm{d}\mathcal{L}^3 = \int_{\mathcal{H}} \Big[\Delta_{\mathbb{R}^2} g + \frac{x^2 + y^2}{4} (\partial_{tt}^2 g) - x(\partial_{yt}^2 g) + y(\partial_{xt}^2 g) \Big] f \, \mathrm{d}\mathcal{L}^3$$

As expected, we got the sub-Laplacian.

5.4.2 The submanifold $\{x = 0\}$

We now consider the submanifold $\mathbb{X} := \{x = 0\}$ of $\mathbb{H}$ with the distance function d_K restricted to $\mathbb{X}$ and as reference measure $\mathfrak{m}$ the $2-$dimensional Lebesgue measure $\mathcal{L}^2$, so that we have the metric measure space

$$\mathcal{X} := (\mathbb{X}, d_K|_{\mathbb{X}}, \mathcal{L}^2).$$

The gradient of the function $f(x,y,t)=x$ is a base for the normal subspace to $\mathbb{X}$ and is given by $(1,0,0)$. So the unique vector field orthogonal to ∇f that is both in $\mathscr{S}$ and in the tangent space of $\mathbb{X}$ is $(0,1,0)$, i.e. ∂_y. Hence the distance function now is the Euclidian one in direction y but is equal to $+\infty$ in direction t. With just one derivation available we do not have a subriemannian structure so we have to compute the Laplacian through the tools developed in this thesis.

First we use the two functions $D^{\pm}f(\nabla g)$. We now have that

$$|Dg|_w = \|\nabla_{\mathbb{X}} f\|_K = |\partial_y f|$$

and if we do the same computations as before we get

$$\frac{\big(\partial_y(g+\varepsilon f)\big)^2 - (\partial_y g)^2}{2\varepsilon} \xrightarrow{\varepsilon\to 0} (\partial_y g)(\partial_y f)$$

which integrated by parts w.r.t $\mathcal{L}^2$ gives us the degenerate second order operator

$$\Delta_{\mathbb{H}^1}|_{\mathcal{X}} = \partial^2_{yy}.$$

Therefore from the differential point of view the sub-Laplacian restricted to $\mathbb{X}$ is given by the second derivative in the horizontal direction tangent to $\mathbb{X}$, and clearly this is not an hypoelliptic operator.

Now we want to use the Cheeger's energy to define the Laplacian, using the definition of Ch_2 given in [1]:

$$\mathrm{Ch}_2(f) := \inf\left\{ \varliminf_{i\to\infty} \int_{\mathbb{X}} |\nabla f_i|^p \,\mathrm{d}\mathcal{L}^2 : f_i \xrightarrow{L^p} f,\ f_i \in \mathrm{Lip}_b(\mathbb{X}, d_K) \right\}$$

and the *slope* $|\nabla f_i|$ of f_i is the function defined by

$$|\nabla f_i|(x) := \varlimsup_{y\to x} \frac{|f(y)-f(x)|}{d_K(y,x)}.$$

Being on $\mathbb{X}$ we can consider the points $z_0=(y_0,t_0)$ and $z_1=(y_1,t_1)$ as x and y in the above definition. The distance d_K between z_0 and z_1 is

$$d(z_0,z_1) = |y_1-y_0| + \sqrt{|t_1-t_0|}.$$

We plug this in the slope's definition and approximate each f_i with a product function $g_i h_i$, with $g_i=g_i(y)$ and $h_i=h_i(t)$ for every $i\in\mathbb{N}$, being the product functions dense in L^2, the domain of Ch_2. Hence we have that

$$\begin{aligned}
|\nabla f_i|(z_0) &\triangleq \varlimsup_{z_1\to z_0} \frac{|g_i(y_1)h_i(t_1) - g_i(y_0)h_i(t_0)|}{|y_1-y_0|+\sqrt{|t_1-t_0|}} \\
&= \varlimsup_{z_1\to z_0} \frac{|g_i(y_1)\big(h_i(t_1)-h_i(t_0)\big) + h_i(t_0)\big(g_i(y_1)-g_i(y_0)\big)|}{|y_1-y_0|+\sqrt{|t_1-t_0|}} \\
&\le \varlimsup_{z_1\to z_0} \left| g_i(y_1)\frac{h_i(t_1)-h_i(t_0)}{|y_1-y_0|+\sqrt{|t_1-t_0|}} + h_i(t_0)\frac{g_i(y_1)-g_i(y_0)}{|y_1-y_0|+\sqrt{|t_1-t_0|}} \right| \\
&\le \varlimsup_{t_1\to t_0} |g_i(y_0)| \frac{|h_i(t_1)-h_i(t_0)|}{\sqrt{|t_1-t_0|}} + \varlimsup_{y_1\to y_0} |h_i(t_0)| \frac{|g_i(y_1)-g_i(y_0)|}{|y_1-y_0|}.
\end{aligned}$$

If g_i and h_i are Lipschitz, then

$$\exists L_{g_i} > 0 : |g_i(y_1) - g_i(y_0)| \le L_{g_i}|y_1 - y_0|,$$
$$\exists L_{h_i} > 0 : |h_i(t_1) - h_i(t_0)| \le L_{h_i}|t_1 - t_0|,$$

hence we can continue the computations as:

$$\begin{aligned}&\le \varlimsup_{t_1\to t_0} L_{h_i}|g_i(y_0)|\frac{|t_1-t_0|}{\sqrt{|t_1-t_0|}} + \varlimsup_{y_1\to y_0} L_{g_i}|h_i(t_0)|\frac{|y_1-y_0|}{|y_1-y_0|}\\ &= L_{g_i}|h_i(t_0)|,\end{aligned}$$

so that

$$|\nabla f_i|(z_0) \le L_{g_i}|h_i(t_0)|. \tag{5.34}$$

Remark 5.4.2. Consider $X = \mathbb{R}$ and $d(t_1,t_0) = \sqrt{|t_1-t_0|}$. In this case we have that

$$|\nabla f_i|(t_0) = \varlimsup_{t_1\to t_0}\frac{|f_i(t_1)-f_i(t_0)|}{\sqrt{|t_1-t_0|}} \overset{\text{as before}}{\le} L_{f_i}\varlimsup_{t_1\to t_0}\frac{|t_1-t_0|}{\sqrt{|t_1-t_0|}} = 0.$$

Hence $\mathrm{Ch}_2(f) \equiv 0$ so that $\partial\mathrm{Ch}_2(f) = \{0\}$ and the metric Laplacian is the null operator.

We assume that the opposite inequality in (5.34) holds. By definition of $D^{\pm}f(\nabla g)$ and denoting $f = \phi_f\psi_f$ and $g = \phi_g\psi_g$ with ϕ_f, ϕ_g functions in y and ψ_f, ψ_g functions in t, we have that

$$\begin{aligned}|\nabla(g+\varepsilon f)|^2 - |\nabla g|^2 &= \big||\psi_g|L_{\phi_g} + \varepsilon|\psi_f|L_{\phi_f}\big|^2 - |\psi_g|^2L_{\phi_g}^2\\ &= 2\varepsilon|\psi_f||\psi_g|L_{\phi_f}L_{\phi_g} + o(\varepsilon)\end{aligned}$$

so that

$$D^+f(\nabla g) = D^-f(\nabla g) = |\psi_f||\psi_g|L_{\phi_f}L_{\phi_g}.$$

Integrating we aim to find a measure μ that satisfies

$$-\int_{\mathbb{X}}|\psi_f||\psi_g|L_{\phi_f}L_{\phi_g}\,\mathrm{d}\mathcal{L}^2 = \int_{\mathbb{X}} f\,\mathrm{d}\mu$$

If we formally integrate by parts, recalling that $f = \phi_f\psi_f$ and interpreting L_{ϕ_f} as ϕ_f' and L_{ϕ_g} as ϕ_g', we get

$$\mu = \phi_g''\psi_g\,\mathcal{L}^2,$$

which tells us that the metric Laplacian μ can "see" the vertical direction t (expressed by the presence of $\psi_g = \psi_g(t)$). With the differential approach we only obtained ∂^2_{yy}, which is not hypoelliptic nor can "detect" the t direction.

Remark 5.4.3. This phenomenon does not occur in case of submanifolds immersed the Riemannian manifolds (that inherits the Riemannian structure by the ambient space) and in case of hypersurfaces immersed in the higher dimensional Heisenberg group $\mathbb{H}^n$, with $n > 1$, where the new distribution given by the intersection of the ambient distribution and the tangent space to the hypersurface still induces a sub-Riemannian structure on the hypersurface.

Bibliography

[1] Ambrosio L., *Calculus, heat flow and curvature dimension bounds in metric measure spaces*, Scuola Normale Superiore, Pisa, Italy, November 29th 2017

[2] Ambrosio L., Da Prato G., Mennucci A., *Introduction to Measure Theory and Integration*, Edizioni della Normale, Pisa, Italy, 2007

[3] Ambrosio L., Gigli N., Savaré G., *Calculus and heat flows in metric measure spaces with Ricci curvature bounded from below*, arXiv:1106.2090, 2011

[4] Ambrosio L., Gigli N., Savaré G., *Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces*, arXiv:1111.3730, 2011

[5] Ambrosio L., Gigli N., Savaré G., *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2005

[6] Ambrosio L., Gigli N., Savaré G., *Metric measure spaces with Riemannian Ricci curvature bounded from below*, arXiv:1109.0222, 2011

[7] Ambrosio L, Tilli P., *Selected topics on "Analysis in Metric Spaces"*, Scuola Normale Superiore, Pisa, 2000

[8] Bellaïche A., *The tangent space in sub-Riemannian geometry*, 1996.

[9] Brezis H., *Operateurs Maximaux Monotones*, North Holland, January 15th 1973

[10] Cheeger J., *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Func. Anal., 9, 1999

[11] Evans C. L., *Partial Differential Equations*, American Mathematical Society, India Universities Press, Second Indian edition, 2014

[12] Evans C. L., Gariepy R. F., *Measure Theory and Fine Properties of Functions*, CRC Press, Revised edition, 2015

[13] Federer H., *Geometric Measure Theory*, Springer, Berlin, 1996

[14] Fukushima M., Oshima Y., Takeda M., *Dirichlet forms and symmetric Markov processes*, Second edition, 2011

[15] Gigli N., *On the differential structure of metric measure spaces and applications*, June 5th 2018.

[16] Heinonen J., Koskela P., Shanmugalingam N., T. Tyson J., *Sobolev Spaces on Metric Measure Spaces, An Approach Based on Upper Gradients*, new mathematical monographs: 27, Cambridge University Press, 2015

[17] Hörmander L., *Hypoelliptic second order differential equations*, Acta Math., 119:147–171, 1967.

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