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Traffic waves and delay differential equations

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"Though this be madness, yet there is method in 't"
(*Hamlet* 2.2.195)

Sommario

Questo elaborato si pone l'obiettivo di studiare il problema del traffico, concentrandosi su un modello semplificato in cui i veicoli sono confinati su una circonferenza e la cui velocità è determinata dal modello optimal velocity.

Il discorso si sviluppa su tre capitoli: nel primo viene presentato il modello optimal velocity per il flusso del traffico e si procede a uno studio della stabilità lineare attorno al punto di equilibrio stazionario. Nel secondo capitolo lo stesso modello viene studiato nel limite termodinamico per un numero infinito di veicoli. Si ricava una soluzione costituita da un'onda di traffico che si propaga in verso opposto al moto delle auto. Nel terzo e ultimo capitolo il modello viene studiato tramite teoria perturbativa nell'intorno del punto critico, introducendo un potenziale termodinamico e seguendo la teoria di Landau delle transizioni di fase. Vengono infine ricavate le medesime condizioni di stabilità del sistema trovate nel primo capitolo.

Abstract

The aim of this work is to study the problem of traffic focusing on a simplified model in which vehicles move on a loop with their velocity governed by the optimal velocity model.

It consists of three chapters: in the first, we present the optimal velocity model for traffic flow and we proceed in the study of linear stability near the stationary equilibrium point. In the second, we study the same model in the thermodynamic limit for an infinite number of vehicles and we find an analytic solution that describes a backward propagating traffic soliton wave. In the third and last chapter we study the model near the critical point, introducing a thermodynamic potential according to Landau's phase transition theory. We finally recover the same stability conditions obtained in the first chapter.

Contents

Introduction

Traffic models are a clear example of how physics of complex system can design mathematical descriptions of everyday phenomena. As a matter of fact, willingly or unwillingly, we all have to deal with traffic in our lives: we have to take it into consideration when we go to work, to university and especially when we go on vacation. The development of traffic science is therefore beneficial to the society: in fact, understanding the factors that cause traffic would mean to know how to prevent it and therefore enhance the living standards by building roads that are less prone to traffic jams.

Traffic science aims to discover the fundamental properties and laws in transportation systems [11]. Physicists have proposed simplified models in order to discover the essential factors affecting on the traffic behaviour and, despite the complexity of traffic and the complications of human behaviour, physical traffic theory is an example of a highly quantitative description of a living system that exhibits a rich variety of phenomena, such as dynamical jamming transition, metastability and nonlinear waves [11].

The scientific studies for traffic problems were started in 1935 [11]. Substantially, traffic can be viewed from two different point of views: macroscopically as a compressible fluid, microscopically as a collection of small entities. Several models were proposed for both of them: Lighthill and Whitham have presented the oldest and most popular macroscopic traffic model, in which traffic jam was studied as a shock wave [11],[8]. On the other hand, the most popular microscopic models are the car following models. Newell in 1961 introduced the *optimal velocity model* [11], that was later extended by Nagatani [10] with the *next-nearest-neighbour interaction.* Mason and Woods [9] introduced the presence of different vehicles, Nakayama *et. al* [12] the *backward-looking model.*

This work studies in detail an idealized situation of vehicles in motion on a loop, with their velocities obeying the optimal velocity model. The outline is as follows.

In the first chapter we briefly present the various traffic models, focusing on the optimal velocity model. We present our idealized model, in which N identical vehicles move on a loop with their velocities governed by the optimal velocity model. We study the linear stability near the stationary equilibrium state and we explicitly write the conditions for stable equilibrium in the presence of small delay.

In the second chapter, we develop a continuum analysis of our traffic model in the limit of an infinite number of vehicles. We rewrite the traffic equations in the continuum case and, following Hasebe [7], Bazzani [5], we find a backward-travelling traffic wave solution. We present the results of a numerical simulation of our traffic model.

In the third and last chapter we apply perturbation theory to our problem [5] and we analyze the stability and instability conditions near the critical equilibrium point using Landau's phase transition theory [13]. We introduce a thermodynamic potential and we study the stability and instability conditions according to the different values of the delay. We finally recover the same solutions we found in the first chapter.

Chapter 1

Modelling traffic

We present some models describing traffic flow on a highway, dividing them into two macro-categories: macroscopic and microscopic traffic models. We will later focus on the *optimal velocity* model, that will be the object of an analysis concerning the stability of its stationary equilibrium solution in presence of a small perturbation.

1.1 Traffic models

Among the various mathematical descriptions of traffic, we can identify two categories: macroscopic and microscopic models (see [11]) . The main idea at the foundation of the former is the parallelism with the motion of a compressible fluid. This description gives an overall view of the vehicles' motion without focusing on every single entity, but relying on average quantities. Microscopic models instead offer an opposite description of traffic, viewing it as made up of interacting individuals, the motion of each one taken into account simultaneously.

The typical example of microscopic traffic model is the *car-following* model [11], in which the j-th vehicle is only affected by the $(j + 1)$-th, called the *leading vehicle*. Non-integer car-following models are called follow-the-leader models. Among the car following models we can find the *optimal velocity model*, which we will discuss shortly, and the *next-nearest-neighbour model* [10], in which the j-th vehicle's dynamics is also influenced by the headway between the $(j + 1)$-th and the $(j + 2)$-th. Mason and Woods [9] have achieved a further generalization of the optimal velocity model, including vehicles of different types (cars and trucks) and introducing different delays depending on the type of vehicle. Another generalization comes from the work of Nakayama *et al* [12] in which the driver looks at the following vehicle as well as the preceding one and is, in fact, called *backward-looking model*. Macroscopic traffic models, on the other hand, treat traffic as a compressible one-dimensional fluid [11] whose state is described by the spatial vehicle density and the average velocity. The oldest continuum model, along with a continuity equation, was proposed by Lighthill and Whitham [8].

1.2 Optimal Velocity model

We focus on the *optimal velocity* model and we apply it to traffic on a circular lane. In 1961, Newell [11] proposed the delay differential equation that characterizes this model (more details on delay differential equations and the differences with ordinary differential equations can be found in App. A)

$$\dot{x}_k(t+\tau) = V_{\text{opt}}(\Delta x_k(t)) \qquad k \in [0; N] \tag{1.1}$$

where τ is the delay, $\Delta x_k(t)$ is the distance between the $(k+1)$-th and k-th vehicle at time t, N is the number of vehicles and $V_{\text{opt}}(\Delta x_k(t))$ is the optimal velocity function. In Sect. 1.5 we present all the properties that $V_{\text{opt}}(z)$ must satisfy so that our model adapts to our intuition of a real, even if simplified, traffic situation. All vehicles are ordered so that $x_{k+1} > x_k$ for all the duration of the motion. We will avoid situations of collisions and overtaking, since they require a more articulate discussion.

The idea that lead to this form of equation is that, while driving on a one-way lane without junctions, the driver focuses only on the preceding vehicle, adjusting its velocity according to the distance from it in order to avoid collisions. The driver's reaction time is mathematically described by the delay. The average brake reaction time for an unalerted driver in a surprise situation is 1 s [15].

Without any changes in our model, we can move the delay on the right hand side of Eq. (1.1) as following

$$\dot{x}_k(t) = V_{\text{opt}}(\Delta x_k(t-\tau)) \qquad k \in [0; N] \tag{1.2}$$

This is the form of the optimal velocity model that will be used in our future calculations.

1.3 Vehicles on a loop

We simplify our description of traffic assuming that the N vehicles are alike, each one measuring d_0 in length and travelling on a loop of length L without junctions. The absence of junctions implies the conservation of the number N of vehicles, since new vehicles can not enter the road nor already present vehicles can exit it. Thanks to this periodic boundary condition, the last vehicle (namely, the $(N-1)$-th) obeys the same dynamic as the others. The function $x_k(t)$ indicates the position of the *centre* of the k-th vehicle, so that the normalized distance between two consecutive vehicles can be written as:

$$\Delta x_k(t) = \frac{x_{k+1}(t) - x_k(t) - d_0}{d_0} \tag{1.3}$$

where d_0 is the vehicles' length. In our model, the optimal velocity equation takes the form:

$$\dot{x}_k(t) = V_{\text{opt}}\left(\frac{x_{k+1}(t-\tau) - x_k(t-\tau) - d_0}{d_0}\right) \tag{1.4}$$

We introduce a periodic boundary condition on the 0-th and N-th vehicle since we defined

$$x_0(t) = x_N(t) \tag{1.5}$$

In the case of a small delay, Eq. (1.2) can be approximated by

$$\dot{x}_k(t+\tau) \simeq \dot{x}_k(t) + \tau\ddot{x}_k(t) \qquad \tau \ll 1 \tag{1.6}$$

so that we have an expression for $\ddot{x}_k(t)$

$$\begin{aligned} \ddot{x}_k(t) &= \frac{1}{\tau}[V_{\text{opt}}(\Delta x_k(t)) - \dot{x}_k(t)] \\ &= \beta[V_{\text{opt}}(\Delta x_k(t)) - \dot{x}_k(t)] \end{aligned} \tag{1.7}$$

Where β is the inverse of delay time called *sensitivity*. We remark that Eq. (1.7) and Eq. (1.2) are not mathematically equivalent: the former is only an approximation for small values of τ of the latter and has lost the character of delay differential equation since it does not presents anymore a dependence from a delay.

Eq. (1.7) can be written as the system:

$$\begin{cases} \dot{x}_k(t) &= v_k(t) \\ \dot{v}_k(t) &= \beta[V_{\text{opt}}(\Delta x_k(t)) - \dot{x}_k(t)] \end{cases} \qquad k \in [0, N] \tag{1.8}$$

We can rescale it in order to get only one relevant parameter, namely β, leaving the others rid of the physical dimensions

$$x'_k \to x_k/d_0 \qquad v'_k \to v_k/v_\infty \qquad t' \to t v_\infty/d_0 \tag{1.9}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}x_k} = \frac{1}{d_0}\frac{\mathrm{d}}{\mathrm{d}x'_k} \qquad \frac{\mathrm{d}}{\mathrm{d}v} = \frac{1}{v_\infty}\frac{\mathrm{d}}{\mathrm{d}v'} \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{v_\infty}{d_0}\frac{\mathrm{d}}{\mathrm{d}t'} \tag{1.10}$$

and the rescaled system of Eq. (1.8) becomes

$$\begin{cases} \dot{x}'_k(t') &= v'_k(t') \\ \dot{v}'_k(t') &= \beta'\big(V'_{\text{opt}}(\Delta x'_k(t')) - \dot{x}'_k(t')\big) \end{cases} \quad k \in [0, N] \tag{1.11}$$

where $V'_{\text{opt}} = V_{\text{opt}}/v_\infty$ and the only relevant parameter is $\beta' = \beta \frac{d_0}{v_\infty}$ that includes the others.

1.4 Relevant parameters

We analyze a few relevant parameters and variables that will be frequently used in the following discussion:

- $w_k(t)$, the normalized distance between two consecutive vehicles
- C, the road's capacity
- ρ_0, the average vehicles' density

We start from the first one, $w_k(t)$. It is defined as

$$w_k(t) = \frac{x_{k+1}(t) - x_k(t) - d_0}{d_0} \qquad k \in [0, N-1] \tag{1.12}$$

summing over the index k we obtain:

$$\sum_{k=0}^{N-1} w_k(t) = \frac{L - Nd_0}{d_0} \tag{1.13}$$

that is, the normalized length of the free road, cleared from the space occupied by the vehicles. The sum in Eq. (1.13) is a useful indicator of the road's congestion level: for this reason, we introduce the parameter C that measures the road's capacity:

$$C = \sum_k w_k(t) = \frac{L}{d_0} - N \tag{1.14}$$

It is always $C \geq 0$, with the equality indicating a fully congested road. Lastly, we introduce the average vehicle density ρ_0

$$\rho_0 = \frac{Nd_0}{L} = \frac{N}{N_m} = 1 - C' \tag{1.15}$$

with $C' = Cd_0/L$ and where $N_m = L/d_0$ indicates the maximum number of vehicles that can be held by the lane. ρ_0 is a road's congestion indicator as well: this time we have $\rho_0 \leq 1$, with the equality always meaning the presence of a fully congested road: in fact, it is realized when $N = N_m$, that is, when there is no space between two consecutive vehicles.

1.5 Optimal Velocity function

We analyze the properties of the key feature of the optimal velocity model: the optimal velocity function. In its most general expression $V_{\text{opt}}(w)$ is a sigmoid function of the normalized distance between two consecutive vehicles. We require it to fulfil the physical conditions:

- $\lim_{w \to 0} V_{\text{opt}}(w) = 0$
- $\lim_{w \to \infty} V_{\text{opt}} = v_\infty$
- V_{opt} must be monotonically increasing

The first two properties only reflect the behaviour of the vehicles in two extreme cases: in a congested road, as the distance between vehicles tends to zero, so does their velocity; while in a free road the velocity of each vehicle tends to the speed limit, here indicated as v_∞. Mathematically, and these properties will be extensively exploited in Ch. 3, we need our optimal velocity function to present a flex point at a distance $w = w_C^0$, that is the critical distance between two consecutive vehicles (see Sect. 1.7). We will also assume that $V_{\text{opt}}(w) \in C^3(\mathbb{R})$, so that $V'_{\text{opt}}(w)$ presents a maximum at $w = w_C^0$ and $V''_{\text{opt}}(w_C^0) = 0$ along with $V'''_{\text{opt}}(w_C^0) < 0$.

The analytical form of the optimal velocity function should be deduced from empirical data and probably does not exist a universal function that can be used for all models. For our discussion, we will assume that the optimal velocity has the expression

$$V_{\text{opt}}(t) = \frac{v_\infty}{2}\Big(\tanh\big(aw - w_C^0\big) + \tanh\big(w_C^0\big)\Big) \tag{1.16}$$

where $w_C^0 \gg 1$ so that $\lim_{w \to \infty} V_{\text{opt}} = v_\infty$ is satisfied. Fig. 1.1 shows the optimal velocity function with the following choice of parameters: $v_\infty = 2, a = 1, w_C^0 = 6$. In Fig. 1.2 is plotted its first derivative.

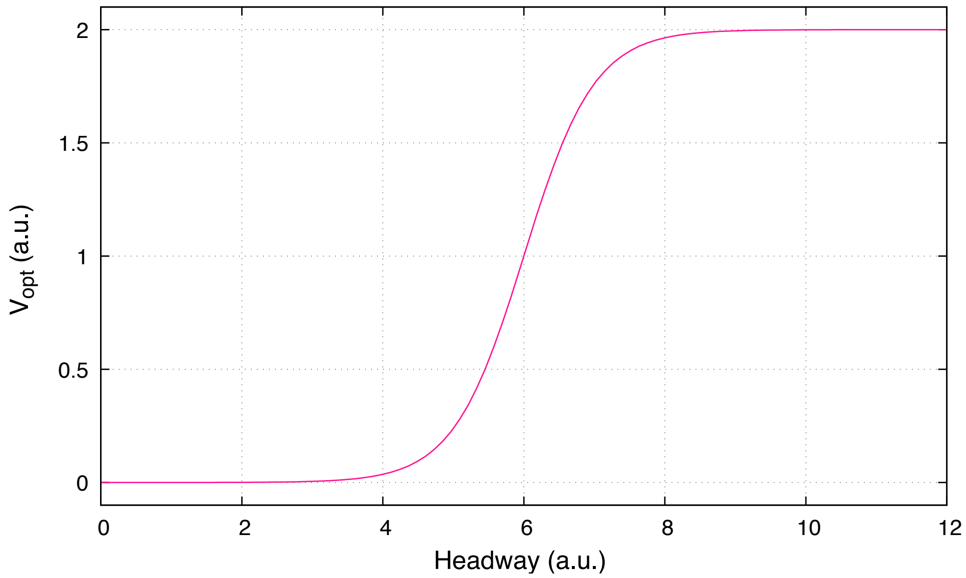


Figure 1.1: Plot of $V_{\text{opt}}(w)$ (1.16) for $v_\infty = 2, a = 1, w_C^0 = 6$

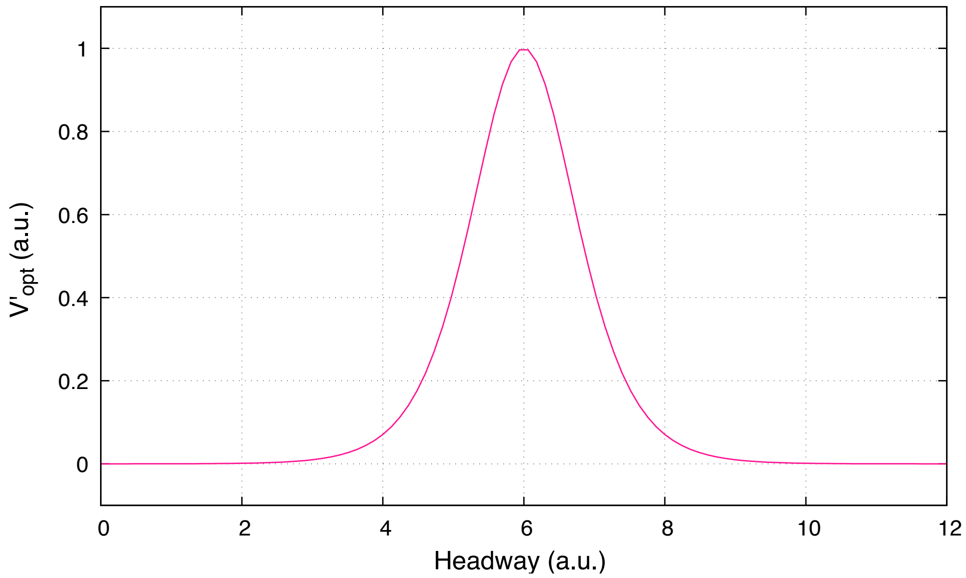


Figure 1.2: Plot of $V'_{\text{opt}}(w)$ (1.16) for $v_\infty = 2, a = 1, w_C^0 = 6$

1.6 Linear stability

Following Bazzani [5], we now proceed to study the stability near an equilibrium point using linear stability theory, that is one of the most common and fruitful approach to the study of any dynamical system, since it provides satisfying approximate solutions to non-linear systems that allow to understand their main features without the need of analitically solving them.

1.6.1 Equilibrium points

A dynamical system described by an equation of the form:

$$\dot{x}(t) = f(x)$$

has an equilibrium point when the condition $\dot{x}(t) = 0$ is satisfied. In our problem x is also a function of the index k, that identifies the vehicle's number, and $f(x)$ is given by the optimal velocity function. The condition $\dot{x}(t) = 0$ causes the optimal velocity function to vanish, implying the stillness of all vehicles: it is not the kind of equilibrium we are looking for, since it does not allow the system to evolve in time.

We therefore look for an equilibrium solution in which all vehicles, independently from time and vehicle number, move with the same velocity, that remains constant during motion. Looking at our problem, it is easy to see that if we put the reciprocal distance between vehicles equals to $w^0 = C/N$, we obtain the stationary solution we are looking for. That is easily explained using the definition of w_k and C found in Eqs. (1.12) and (1.14)

$$\begin{aligned} w^0 &= N\frac{x^0_{k+1} - x^0_k - d_0}{d_0} \\ C &= \frac{L}{d_0} - N \end{aligned} \tag{1.17}$$

inserting the definition of C in the expression for w^0

$$\frac{x^0_{k+1} - x^0_k - d_0}{d_0} = \frac{L}{d_0 N} - 1 \tag{1.18}$$

and isolating the distance between the centres of two consecutive vehicles, we obtain

$$x^0_{k+1} - x^0_k = \frac{L}{N} \tag{1.19}$$

that is, all vehicles are placed at the reciprocal distance obtained by dividing the total length of the lane by the number of vehicles. The optimal velocity for the trivial equilibrium condition is written as: $V_{\text{opt}}(w^0)$.

1.6.2 Perturbation of the stationary solution

We now proceed to study the conditions for stability in presence of a small oscillating perturbation, that we can write as

$$w_k(t) = w^0(1 + \epsilon \exp(\lambda t + 2\pi i n k/N)) \qquad k \in [0, N-1] \tag{1.20}$$

where n is the oscillation number and λ is a complex parameter. The perturbation is periodic of period N in the spatial variable. Our approach for finding the stability conditions consists in studying under which conditions the real part of the parameter λ is non-positive: in fact, when this happens, the solution does not diverge, but it oscillates for $\operatorname{Re}\lambda = 0$ and it decays for $\operatorname{Re}\lambda < 0$ respectively.

We rewrite Eq. (1.2) as the time derivative of the distance between two consecutive vehicles by subtracting term by term two of its successive instances

$$\begin{aligned} \dot{x}_{k+1}(t) &= V_{\text{opt}}(w_{k+1}(t-\tau)) \quad - \\ \dot{x}_k(t) &= V_{\text{opt}}(w_k(t-\tau)) \end{aligned} \tag{1.21}$$

so that we obtain, setting $d_0 = 1$

$$\dot{w}_k(t) = [V_{\text{opt}}(w_{k+1}(t-\tau)) - V_{\text{opt}}(w_k(t-\tau))] \tag{1.22}$$

We will show in App. C how this relation can be used to describe the dynamics of nodes in a network.

We expand up to the first order in ϵ, inserting the perturbed expression for $w_k(t)$ in Eq. (1.22). On the left hand side we obtain

$$\dot{w}_k(t) = w^0 \lambda \epsilon \exp(\lambda t + 2\pi i n k/N) \tag{1.23}$$

while on the right hand side we have

$$\begin{aligned} &V_{\text{opt}}(w_{k+1}(t-\tau)) - V_{\text{opt}}(w_k(t-\tau)) = \\ &V_{\text{opt}}(w^0) + V'_{\text{opt}}(w^0) w^0 \epsilon \exp(\lambda(t-\tau) + 2\pi i n(k+1)/N) \\ &- V_{\text{opt}}(w^0) - V'_{\text{opt}}(w^0) w^0 \epsilon \exp(\lambda(t-\tau) + 2\pi i n k/N) = \\ &V'_{\text{opt}}(w^0) w^0 \epsilon \exp(\lambda(t-\tau) + 2\pi i n k/N)[\exp(2\pi i n/N) - 1] \end{aligned} \tag{1.24}$$

Eq. (1.22) then becomes

$$\begin{aligned} &w^0 \lambda \epsilon \exp(\lambda t + 2\pi i n k/N) = \\ &V'_{\text{opt}}(w^0) w^0 \epsilon \exp(\lambda(t-\tau) + 2\pi i n k/N)[\exp(2\pi i n/N) - 1] \end{aligned} \tag{1.25}$$

simplifying,

$$\lambda e^{\lambda \tau} = V'_{\text{opt}}(w^0)[\exp(2\pi i n/N) - 1] \tag{1.26}$$

using Euler's identity we get

$$\lambda e^{\lambda\tau} = V'_{\text{opt}}(w^0)[\cos(2\pi n/N) - 1 + i\sin(2\pi n/N)] \tag{1.27}$$

We first see what happens if the delay is absent, $\tau = 0$. We have that Eq. (1.27) becomes

$$\lambda = V'_{\text{opt}}(w^0)[\cos(2\pi n/N) - 1 + i\sin(2\pi n/N)] \tag{1.28}$$

Writing $\lambda = \lambda_R + i\lambda_I$ and dividing the real from the imaginary part

$$\begin{aligned} \lambda_R &= V'_{\text{opt}}(w^0)[\cos(2\pi n/N) - 1] \\ \lambda_I &= V'_{\text{opt}}(w^0)\sin(2\pi n/N) \end{aligned} \tag{1.29}$$

We notice that

$$\lambda_R = V'_{\text{opt}}(w^0)[\cos(2\pi n/N) - 1] \leq 0 \qquad \forall n \in \mathbb{N} \tag{1.30}$$

We can conclude that, when the delay is absent, the initial perturbation evolves into a situation of linear stability for all ns: the distance between two vehicles does not grow uncontrollably, instead, it oscillates decaying exponentially following the parameter λ_R and therefore resulting in a damped oscillating motion. For $\operatorname{Re}\lambda = 0$ the distance between vehicles oscillates without damping.

1.6.3 Finite delay

We now study the evolution of the system for nonzero values of τ.

Writing $\lambda = \lambda_R + i\lambda_I$, Eq. (1.27) becomes

$$\begin{aligned} &[\lambda_R + i\lambda_I]e^{\lambda_R\tau}[\cos(\lambda_I\tau) + i\sin(\lambda_I\tau)] = \\ &\qquad V'_{\text{opt}}(w^0)[\cos(2\pi n/N) - 1 + i\sin(2\pi n/N)] \end{aligned} \tag{1.31}$$

separating real and imaginary parts, we obtain the system

$$\begin{cases} e^{\lambda_R\tau}[\lambda_R\cos(\lambda_I\tau) - \lambda_I\sin(\lambda_I\tau)] = V'_{\text{opt}}(w^0)[\cos(2\pi n/N) - 1] \\ e^{\lambda_R\tau}[\lambda_I\cos(\lambda_I\tau) + \lambda_R\sin(\lambda_I\tau)] = V'_{\text{opt}}(w^0)\sin(2\pi n/N) \end{cases} \tag{1.32}$$

We are interested in the conditions that keep $\lambda_R \leq 0$. We first impose the condition $\lambda_R = 0$ in the system of Eq. (1.32) in order to obtain the critical value of τ that demarcates instability from stability. We get the following

$$\begin{cases} \lambda_I\sin(\lambda_I\tau) = -V'_{\text{opt}}(w^0)[\cos(2\pi n/N) - 1] = 2V'_{\text{opt}}(w^0)\sin^2(\pi n/N) \\ \lambda_I\cos(\lambda_I\tau) = V'_{\text{opt}}(w^0)\sin(2\pi n/N) = 2V'_{\text{opt}}(w^0)\sin(\pi n/N)\cos(\pi n/N) \end{cases} \tag{1.33}$$

we have a solution in terms of τ and λ_I

$$\lambda_I \tau = \frac{\pi n}{N} \qquad \lambda_I = 2V'_{\text{opt}}(w^0)\sin(\pi n/N) \tag{1.34}$$

so that we can define the *critical value* of τ, τ_C: the value in correspondence of which the parameter λ_R goes to zero and the solution is stable and oscillating

$$\tau_c = \frac{\pi n}{2NV'_{\text{opt}}(w^0)\sin(\pi n/N)} \tag{1.35}$$

in the limit $n/N \ll 1$ we can write it as

$$\tau_c \simeq [2V'_{\text{opt}}(w^0)]^{-1} \qquad n/N \ll 1 \tag{1.36}$$

taking advantage of the known limit $\lim_{x\to 0} \sin(x)/x$ with $x = \pi n/N$.

In the case of small τ it is possible to get an explicit estimate for $\lambda_R < 0$. We follow the work of Bazzani [4]. Starting from two consecutive instances of Eq. (1.8)

$$\begin{cases} \dot{x}_{k+1}(t) = v_{k+1}(t) \\ \dot{v}_{k+1}(t) = \dfrac{1}{\tau}[V_{\text{opt}}(\Delta x_{k+1}(t)) - v_{k+1}(t)] \end{cases}$$
$$\begin{cases} \dot{x}_{k}(t) = v_{k}(t) \\ \dot{v}_{k}(t) = \dfrac{1}{\tau}[V_{\text{opt}}(\Delta x_{k}(t)) - v_{k}(t)] \end{cases} \tag{1.37}$$

subtracting memberwise every equation in the system, we move on the frame of reference in which the variables are the relative distances between the vehicles

$$\begin{cases} \dot{w}_k(t) &= u_k(t) \\ \dot{u}_k(t) &= \dfrac{1}{\tau}[V_{\text{opt}}(w_{k+1}) - V_{\text{opt}}(w_k) - u_k(t)] \end{cases} \tag{1.38}$$

where we wrote $u_k(t) = v_{k+1}(t) - v_k(t)$. We perturb $w_k(t)$ as we have already done before, obtaining

$$\dot{w}_k = u_k = \epsilon w^0 \lambda \exp\Big(2\pi i k \frac{n}{N} + \lambda t\Big)$$

$$\dot{u}_k = \epsilon w^0 \lambda^2 \exp\Big(2\pi i k \frac{n}{N} + \lambda t\Big) \tag{1.39}$$

$$V_{\text{opt}}(w^0 + \Delta w) = V_{\text{opt}}(w^0) + V'_{\text{opt}}(w^0)\Delta w$$

$$= V_{\text{opt}}(w^0) + V'_{\text{opt}}(w^0)\epsilon w^0 \exp\Big(2\pi i k \frac{n}{N} + \lambda t\Big)$$

So that the second equation in the system (1.38) looks like

$$\begin{aligned} \epsilon w^0 \exp\Big(2\pi i k \frac{n}{N} + \lambda t\Big)\lambda^2 = \\ \frac{1}{\tau}\epsilon w^0 \exp\Big(2\pi i k \frac{n}{N} + \lambda t\Big)\Big[-\lambda + V_{\text{opt}}(w^0)'\Big(\exp\Big(2\pi i \frac{n}{N}\Big) - 1\Big)\Big] \end{aligned} \tag{1.40}$$

reducing, and writing $V_{\text{opt}}(w_0)' = K$, we get

$$\lambda^2 = \frac{1}{\tau}\Big[-\lambda + K\Big(\exp\Big(2\pi i \frac{n}{N}\Big) - 1\Big)\Big] \tag{1.41}$$

which is a second order equation in the variable λ. Solving it, we obtain

$$\lambda_\pm = -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} + \frac{K}{\tau}\Big[\exp\Big(2\pi i \frac{n}{N}\Big) - 1\Big]} \tag{1.42}$$

that can be written in the form

$$\lambda_\pm = \frac{1}{2\tau}\left[-1 \pm \sqrt{1 + 4\tau K\left[\cos\left(\frac{2\pi n}{N}\right) - 1 + i\sin\left(\frac{2\pi n}{N}\right)\right]}\right] \tag{1.43}$$

For future convenience, we introduce $\theta = 2\pi n/N$ and $A = 4K\tau$. The solution λ_- is always stable, since its real part will clearly be negative. We focus on λ_+ and we look for the instability conditions, resulting from $\operatorname{Re}\lambda_+ > 0$, that can be written as

$$\operatorname{Re}\sqrt{1 - A + A\exp(i\theta)} > 1 \tag{1.44}$$

We can compute the square modulus of the square root argument

$$\begin{aligned} |1 - A + A\exp(i\theta)|^2 = |1 - A + A\cos\theta + iA\sin\theta|^2 = \\ (1-A)^2 + 2A\cos\theta(1-A) + A^2 \end{aligned} \tag{1.45}$$

thus we can explicitly write its phase ϕ

$$\cos\phi = \frac{1 - A(1-\cos\theta)}{\sqrt{(1-A)^2 + 2A(1-A)\cos\theta + A^2}} \tag{1.46}$$

using the equality

$$\begin{aligned} &\left[\mathrm{Re}\sqrt{1 - A + A\exp(i\theta)}\right]^2 = \\ &\sqrt{(1-A)^2 + 2A(1-A)\cos\theta + A^2}\cos^2\frac{\phi}{2} = \\ &\sqrt{(1-A)^2 + 2A(1-A)\cos\theta + A^2}\,\frac{\cos\phi + 1}{2} \end{aligned} \tag{1.47}$$

We can write the instability condition as follows

$$1 - A(1-\cos\theta) + \sqrt{(1-A)^2 + 2A(1-A)\cos\theta + A^2} > 2 \tag{1.48}$$

We get to

$$(1-A)^2 + A^2 + 2A(1-A)\cos\theta > [1 + A(1-\cos\theta)]^2 \tag{1.49}$$

where

$$[1 + A(1-\cos\theta)]^2 = (1+A)^2 - 2A(1+A)\cos\theta + A^2\cos^2\theta \tag{1.50}$$

Developing the calculations a little bit further we get to the final condition

$$\begin{aligned} &4A - 4A\cos\theta + A^2\cos^2\theta - A^2 < 0 \\ &\Rightarrow A(1-\cos\theta)[4 - A(1+\cos\theta)] < 0 \end{aligned} \tag{1.51}$$

In order for it to be satisfied, we then must have

$$\cos\theta > \frac{4}{A} - 1 = \frac{1}{\tau K} - 1 \tag{1.52}$$

that is the condition for *linear instability*. When the sensitivity satisfies $\beta' < 2K$ and the value of N or n makes θ decrease to zero (and therefore $\cos\theta \to 1$) we have an unstable solution, since the conditions for linear instability are satisfied

$$\cos\frac{2n\pi}{N} > \frac{\beta'}{K} - 1 \tag{1.53}$$

For example, we can say that perturbations characterized by long wavelengths, namely with small n, and with $\beta' < 2K$ are linearly unstable. On the contrary, the condition for *linear stability* reads as

$$\frac{1}{\tau K} - 1 > 1$$

that is

$$\begin{aligned}\frac{1}{\tau} \geq 2K = 2V'_{\text{opt}}(w^0) = \frac{1}{\tau_C} \\ \Rightarrow \tau \leq \frac{1}{2V'_{\text{opt}}(w^0)} \\ \Rightarrow \tau \leq \tau_C\end{aligned} \tag{1.54}$$

1.7 Conclusions

We summarize the results obtained with the previous analysis:

- $\tau > \tau_C$: the traffic flow is unstable, small perturbations tend to amplify and the flow changes to a different dynamical state.
- $\tau < \tau_C$: the traffic flow remains stable and perturbations decay with time.
- $\tau = \tau_C$ is a condition of neutral stability.

From Eq. (1.35) we can write the condition on τ as a condition on N: there is a critical value of N, N_C, beyond which the traffic flow becomes unstable. Usually, when one has to deal with a finite number of vehicles, fixes τ and writes Eq. (1.35) as a condition on N. When there are a few vehicles, w^0 increases (see Eqs. (1.19), (1.17)) and $V'_{\text{opt}} \ll 1$, so the condition $\tau < [2V'_{\text{opt}}(w^0)]^{-1}$ is satisfied and the system is stable. This means that any perturbation caused, for example, by a vehicle that moves slightly out of the uniform flow, will be reabsorbed in the system, that continues in its uniform evolution.

When the number of vehicles increases, we assist to a decrease of w^0 and to an increase in $V'_{\text{opt}}(w^0)$: there are critical values N_C and w^0_C reached when $V_{\text{opt}}(w^0)$ is in the flex point and $V'_{\text{opt}}(w^0)$ is in the maximum point (see Fig. 1.2). For these values, the equilibrium solution becomes unstable

$$\tau \simeq \left[2V'_{\text{opt}}(w^0_C)\right]^{-1} \tag{1.55}$$

Any variation from the uniform motion will cause the entire system to change its dynamical state: a backward-moving traffic soliton will appear and the time evolution of the distance between two consecutive vehicles will be controlled by this soliton wave. This phenomenon will be the object of study of the successive chapters of this work. When $N > N_C$, V'_{opt} decreases (see Fig. 1.2), so a stable solution is recovered.

Chapter 2

Continuum limit

In this chapter we present the continuum version of the optimal velocity model, we perform an analogous study on the linear stability near the equilibrium point and we look for a travelling wave solution called *traffic soliton*. In Sect. 2.5 we present the results of a numerical simulation carried out in order to show the pattern of oscillating congested traffic on a circular lane.

2.1 Thermodynamic limit

In order to obtain the continuum version of the traffic model (see [5]), we will employ a frequently used approach derived from statistical mechanics called *thermodynamic limit* that will allows us to focus on average quantities without considering fluctuations about their mean values.

We take the limit for an infinite number of vehicles, $N \to \infty$, with the constraint of keeping the stationary equilibrium distance between consecutive vehicles constant $w^0 = C/N$: this means that C must go to infinity as well. Otherwise, the vehicles' distances and velocities would reduce to zero and we would get to a still state that does not evolve in time.

Our thermodynamic limit takes the form:

$$N \to \infty, C \to \infty \quad | \quad w^0 = C/N = \text{constant} \tag{2.1}$$

2.2 Traffic equations in the continuum limit

We now introduce three parameters frequently used in the following discussion:

- n, the oscillation number
- $\delta = n/C$, the wave number

- $z = k\frac{C}{N} = kw^0$ a real index used, as in the discrete case, to indicate the equilibrium position of the k-th vehicle

In analogy with $w_k(t)$, the distance between consecutive vehicles in the discrete case, we introduce the quantity $g(t, z)$, that is necessary to write the traffic equations in the continuum limit. The dependence of w from k is realized in g with the parameter z: to maintain an appropriate notation, $w_{k+a}(t)$ $(a \in N)$ will be indicated as $g(t, z + a\Delta z)$ with $\Delta z = w^0$.

We finally rewrite Eq. (1.22) using continuous variables

$$\frac{\partial g(t,z)}{\partial t} = [V_{\text{opt}}(g(t-\tau, z+\Delta z)) - V_{\text{opt}}(g(t-\tau, z))] \tag{2.2}$$

The equation depends on the parameters (w^0, τ). For a finite number of vehicles we have a constraint on $w_k(t)$, (1.14), translating in the continuum case, the sum becomes an integral extended from 0 to C, since we are integrating on the spatial variable z in order to obtain the total length of the free road

$$\sum_{k=0}^{N-1} w_k(t) = C \Rightarrow \lim_{\substack{C\to\infty \\ N\to\infty}} \frac{1}{w^0}\int_0^C \mathrm{d}z\, g(t,z) = C \tag{2.3}$$

since $\mathrm{d}z = w^0 \mathrm{d}k$. We can write it as

$$\frac{1}{C}\sum_{k=0}^{N-1} w_k(t) = 1 \Rightarrow \lim_{\substack{C\to\infty \\ N\to\infty}} \frac{1}{C}\int_0^C \mathrm{d}z\, g(t,z) = w^0 \tag{2.4}$$

2.2.1 Consistency with the discrete case

We can now demonstrate the consistency between the two models. The stationary solution, in which all vehicles move keeping the same reciprocal distance, has obviously no time dependence: we can write it as $g_0 = g_0(z)$. Since vehicles move with non-zero velocity, it also must be periodic of period w^0:

$$g_0(z) = g_0(z + \Delta z) \tag{2.5}$$

where $\Delta z = w^0$ is the gap between vehicles in the stationary solution. The stationary continuum case has also a physical solution that is $g_0(z) = \text{const}$, where the distance between consecutive vehicles is everywhere the same. If we insert the physical solution in the constraint in Eq. (2.4), we finally have $g_0(z) = w^0$: this means that the equilibrium solution is the same as the discrete case and our continuum model is consistent with the discrete one.

2.3 Linear stability in the continuum limit

We now introduce a perturbation in the stationary solution as we previously did in the discrete case (Sect. 1.6.2)

$$g(t,z) = \lim_{N\to\infty} w_k(t) = w^0(1+\epsilon \exp(\lambda t + 2\pi i \delta z)) \tag{2.6}$$

where the second addend in the exponential term was obtained by multiplying and dividing by C and using the definitions of z and δ. We proceed with the same linear stability study already carried out in Sect. 1.6.2 for a discrete number of vehicles, expanding Eq. (2.2) up to the first order in ϵ. For the left side we obtain

$$\frac{\partial g(t,z)}{\partial t} = w^0 \lambda \epsilon \exp(\lambda t + 2\pi i \delta z) \tag{2.7}$$

while the right side gives

$$\begin{aligned} &V_{\text{opt}}(g(t-\tau, z+\Delta z)) - V_{\text{opt}}(g(t-\tau, z)) = \\ &V_{\text{opt}}(g_0) + V'_{\text{opt}}(g_0) w^0 \epsilon \exp(\lambda(t-\tau) + 2\pi i \delta(z+\Delta z)) \\ &- V_{\text{opt}}(g_0) - V'_{\text{opt}}(g_0) w^0 \epsilon \exp(\lambda(t-\tau) + 2\pi i \delta z) = \\ &V'_{\text{opt}}(g_0) w^0 \epsilon \exp(\lambda(t-\tau) + 2\pi i \delta z)[\exp(2\pi i \delta \Delta z) - 1] \end{aligned} \tag{2.8}$$

So that Eq. (2.2) becomes

$$\begin{aligned} &w^0 \lambda \epsilon \exp(\lambda t + 2\pi i \delta z) = \\ &V'_{\text{opt}}(g_0) w^0 \epsilon \exp(\lambda(t-\tau) + 2\pi i \delta z)[\exp(2\pi i \delta \Delta z) - 1] \end{aligned} \tag{2.9}$$

simplifying

$$\lambda e^{\lambda \tau} = V'_{\text{opt}}(g_0)[\exp(2\pi i \delta \Delta z) - 1] \tag{2.10}$$

As we did before, focusing on the case in which the delay is absent, and writing $\lambda = \lambda_R + i\lambda_I$, we obtain the following expressions for its real and imaginary part

$$\begin{cases} \lambda_R &= V'_{\text{opt}}(g_0)[\cos(2\pi\delta\Delta z) - 1] \\ \lambda_I &= V'_{\text{opt}}(g_0)\sin(2\pi\delta\Delta z) \end{cases} \tag{2.11}$$

This result is clearly analogous to the one obtained in the Sect. 1.6.2, Eq. (1.30)

$$\lambda_R = V'_{\text{opt}}(g_0)[\cos(2\pi\delta\Delta z) - 1] \leq 0 \qquad \forall \delta \in \mathbb{R} \tag{2.12}$$

We can conclude that, in the absence of delay, the equilibrium is always stable, as in the discrete case.

Even when the delay τ is finite, we get the same result obtained in Sect. 1.6.3. We repeat the most relevant passages, firstly dividing the real and imaginary part obtained by substituting $\lambda = \lambda_R + i\lambda_I$ in Eq. (2.10)

$$\begin{cases} e^{\lambda_R \tau}[\lambda_R \cos(\lambda_I \tau) - \lambda_I \sin(\lambda_I \tau)] = V'_{\text{opt}}(g_0)\big[\cos\big(2\pi\delta w^0\big) - 1\big] \\ e^{\lambda_R \tau}[\lambda_I \cos(\lambda_I \tau) + \lambda_R \sin(\lambda_I \tau)] = V'_{\text{opt}}(g_0)\sin\big(2\pi\delta w^0\big) \end{cases} \tag{2.13}$$

Setting $\lambda_R = 0$

$$\begin{cases} \lambda_I \sin(\lambda_I \tau) = -V'_{\text{opt}}(g_0)\big[\cos\big(2\pi\delta w^0\big) - 1\big] = 2V'_{\text{opt}}(g_0)\sin^2(\pi\delta w^0) \\ \lambda_I \cos(\lambda_I \tau) = V'_{\text{opt}}(g_0)\sin\big(2\pi\delta w^0\big) = 2V'_{\text{opt}}(g_0)\sin\big(\pi\delta w^0\big)\cos\big(\pi\delta w^0\big) \end{cases} \tag{2.14}$$

The result still being

$$\lambda_I \tau = \pi\delta w^0 \qquad \lambda_I = 2V'_{\text{opt}}(g_0)\sin\big(\pi\delta w^0\big) \tag{2.15}$$

with the same critical value of τ

$$\tau_C = \frac{\pi\delta w^0}{2V'_{\text{opt}}(g_0)\sin(\pi\delta w^0)} \tag{2.16}$$

along with the same stable equilibrium condition for small delays

$$\tau < \frac{1}{2V'_{\text{opt}}(g_0)} = \tau_C \tag{2.17}$$

2.4 Analytic solution

We look for a travelling wave solution of Eq. (2.2), in which we will use the expression for $V_{\text{opt}}(w)$ found in Sect. 1.5 that we remind here for clarity's sake

$$V_{\text{opt}}(w) = \frac{v_\infty}{2}\big[\tanh\big(aw - w_C^0\big) + \tanh\big(w_C^0\big)\big] \qquad w \geq 0$$

with $w^0 \gg 1$ so that $\lim_{w\to\infty} V_{\text{opt}} = v_\infty$. Following Bazzani [5], Hasebe [7] the solution we are looking for should have the travelling wave form

$$g(z,t) = H(z + \nu t) \tag{2.18}$$

With ν being the wave's velocity. We set $u = z + \nu t$ for the sake of convenience. Inserting the wave form in Eq. (2.2), we obtain

$$\frac{\partial H(z + \nu t)}{\partial t} = \big[V_{\text{opt}}\big(H\big(z + w^0 + \nu(t-\tau)\big)\big) - V_{\text{opt}}(H(z + \nu(t-\tau)))\big] \tag{2.19}$$

In terms of the variable u, the previous equation can be written as

$$\nu \frac{\mathrm{d}H(u)}{\mathrm{d}u} = [V_{\text{opt}}\big(H(u + w^0 - \nu\tau)\big) - V_{\text{opt}}(H(u - \nu\tau))] \tag{2.20}$$

We proceed with a rescalation of the variable u, $u' \to u/w^0$, to fix the spatial scale and we set the travelling wave's velocity $\nu = w^0/2\tau$. We can write Eq. (2.20) as

$$\frac{\nu}{w^0}\frac{\mathrm{d}H(u')}{\mathrm{d}u'} = V_{\text{opt}}\left(H\left(u' + 1 - \frac{1}{2\tau}\tau\right)\right) - V_{\text{opt}}\left(H\left(u' - \frac{1}{2\tau}\tau\right)\right) \tag{2.21}$$

that becomes

$$\frac{\nu}{w^0}\frac{\mathrm{d}H(u')}{\mathrm{d}u'} = V_{\text{opt}}(H(u' + 1/2)) - V_{\text{opt}}(H(u' - 1/2)) \tag{2.22}$$

Expliciting the optimal velocity function, the right hand side takes the form

$$\begin{aligned} V_{\text{opt}}(H(u' + 1/2)) - V_{\text{opt}}(H(u' - 1/2)) = \\ \tanh(H(u' + 1/2)) - \tanh(H(u' - 1/2)) \end{aligned} \tag{2.23}$$

Defining $G(u') = \tanh(H(u'))$, we obtain

$$\frac{\mathrm{d}G(u')}{\mathrm{d}u'} = \left[1 - \tanh^2 H(u')\right]\frac{\mathrm{d}H(u')}{\mathrm{d}u'} = \left[1 - G^2(u')\right]\frac{\mathrm{d}H(u')}{\mathrm{d}u'} \tag{2.24}$$

so that Eq.(2.22) becomes

$$\frac{\nu}{w^0}\frac{dG(u')/du'}{1 - G^2(u')} = G\left(u' + \frac{1}{2}\right) - G\left(u' - \frac{1}{2}\right) \tag{2.25}$$

2.4.1 Jacobi elliptic functions

We now give a brief insight on the definition and some properties of *Jacobi elliptic functions* and we will show that they constitute a solution of Eq. (2.25). Jacobi elliptic functions belong to the class of elliptic functions, that appear in common mathematical and physical problems: as trigonometric functions appear in the problem of the parametrization of the arc-length of the circle, elliptic functions emerge in the parametrization of the arc-length of an ellipse so that they are often referred to as a generalization of trigonometric functions. In physics, we can find them in the calculations of the period of a pendulum abandoning the approximation of small angles.

We can introduce Jacobi elliptic functions from integrals of the form

$$u = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k\sin^2\theta}} \qquad k \in [0; 1] \tag{2.26}$$

where we call the angle φ *amplitude*: $\varphi = \operatorname{am} u$. There are in total twelve Jacobi elliptic functions, we will focus on the most used ones that are [1]

$$\begin{aligned} \operatorname{sn}(u) &= \sin(\varphi) \\ \operatorname{cn}(u) &= \cos(\varphi) \\ \operatorname{dn}(u) &= (1 - k\sin^2(\varphi))^{1/2} = \Delta(\varphi) \end{aligned} \tag{2.27}$$

We now demonstrate that, using the properties of Jacobi elliptic functions, we can obtain an equation analogous to Eq. (2.25).

Focusing on the addition property for $\operatorname{sn}(u)$ [1]

$$\operatorname{sn}(x+y) = \frac{\operatorname{sn}(x)\operatorname{cn}(y)\operatorname{dn}(y) + \operatorname{sn}(y)\operatorname{cn}(x)\operatorname{dn}(x)}{1 - k\operatorname{sn}^2(x)\operatorname{sn}^2(y)} \tag{2.28}$$

with $x + y = u' \pm 1/2$

$$\begin{aligned} &\operatorname{sn}\left(u' + \frac{1}{2}, k\right) - \operatorname{sn}\left(u' - \frac{1}{2}\right) = \\ &\frac{\operatorname{sn}(u')\operatorname{cn}(\frac{1}{2})\operatorname{dn}(\frac{1}{2}) + \operatorname{sn}\left(\frac{1}{2}\right)\operatorname{cn}(u')\operatorname{dn}(u')}{1 - k\operatorname{sn}^2(u')\operatorname{sn}^2\left(\frac{1}{2}\right)} \\ &- \frac{\operatorname{sn}(u')\operatorname{cn}\left(-\frac{1}{2}\right)\operatorname{dn}\left(-\frac{1}{2}\right) + \operatorname{sn}\left(-\frac{1}{2}\right)\operatorname{cn}(u')\operatorname{dn}(u')}{1 - k\operatorname{sn}^2(u')\operatorname{sn}^2\left(-\frac{1}{2}\right)} \end{aligned} \tag{2.29}$$

and, knowing that $\operatorname{sn}(x) = -\operatorname{sn}(-x)$, $\operatorname{cn}(x) = \operatorname{cn}(-x)$ and $\operatorname{dn}(x) = \operatorname{dn}(-x)$, we obtain

$$\operatorname{sn}\left(u' + \frac{1}{2}\right) - \operatorname{sn}\left(u' - \frac{1}{2}\right) = 2\frac{\operatorname{cn}(u')\operatorname{dn}(u')\operatorname{sn}\left(\frac{1}{2}\right)}{1 - k\operatorname{sn}^2\left(\frac{1}{2}\right)\operatorname{sn}^2(u')} \tag{2.30}$$

The derivative of $\operatorname{sn}(u')$ takes the form

$$\frac{\mathrm{d}\operatorname{sn}(u')}{\mathrm{d}u'} = \operatorname{cn}(u')\operatorname{dn}(u') \tag{2.31}$$

so that we can write the previous equation as

$$2\operatorname{sn}\left(\frac{1}{2}\right)\frac{\mathrm{d}\operatorname{sn}(u')/\mathrm{d}u'}{1 - k\operatorname{sn}^2\left(\frac{1}{2}\right)\operatorname{sn}^2(u')} = \operatorname{sn}\left(u' + \frac{1}{2}\right) - \operatorname{sn}\left(u' - \frac{1}{2}\right) \tag{2.32}$$

We finally obtained the same form of Eq. (2.25).

2.4.2 Traffic soliton

We showed that the solution of Eq. (2.25) must have the form $G(u') = \beta\operatorname{sn}(u', k)$, where we highlighted the dependence from the parameter k. Using it to rewrite the left side of Eq. (2.25)

$$\frac{\nu}{w^0}\frac{dG(u')/du'}{1-G^2(u')} = \frac{\nu\beta}{w^0}\frac{\mathrm{sn}'(u',k)}{1-\beta^2\,\mathrm{sn}^2(u',k)} \tag{2.33}$$

for the right hand side we obtain

$$G\left(u'+\frac{1}{2}\right) - G\left(u'-\frac{1}{2}\right) = \beta\,\mathrm{sn}\left(u'+\frac{1}{2}\right) - \beta\,\mathrm{sn}\left(u'-\frac{1}{2}\right) \tag{2.34}$$

So that Eq. (2.25) becomes

$$\frac{\nu}{w^0}\frac{\mathrm{sn}'(u',k)}{1-\beta^2\,\mathrm{sn}^2(u',k)} = \mathrm{sn}\left(u'+\frac{1}{2}\right) - \mathrm{sn}\left(u'-\frac{1}{2}\right) \tag{2.35}$$

The similarities between Eqs. (2.32) and (2.35) lead us to set some constraints in order to have a solution of the chosen form

$$\begin{aligned} \frac{\nu}{w^0} &= 2\,\mathrm{sn}\left(\frac{1}{2},k\right) \\ \beta &= \pm\sqrt{k}\,\mathrm{sn}\left(\frac{1}{2},k\right) \end{aligned} \tag{2.36}$$

Recalling that $\nu = w^0/2\tau$, we obtain for the first constraint

$$\mathrm{sn}\left(\frac{1}{2},k\right) = \frac{1}{4\tau} \tag{2.37}$$

an equation that determines $k < 1$, that is related to a spatial scale. The parameter β is determined in the second constraint using the result that we have just obtained

$$\beta = \pm\frac{\sqrt{k}}{4\tau} \tag{2.38}$$

Therefore, we have found that the fundamental parameter is in fact τ, since alone it determines the soliton velocity, ν, and the wave form via k and β. The soliton velocity also helds a dependence from the number of vehicles, since it contains the parameter w^0.

At the critical density, we have:

$$V'_{\mathrm{opt}}(w^0_C) = 1/2\tau_C \tag{2.39}$$

so that the critical wave velocity can be expressed as a function of $V'_{\mathrm{opt}}(w^0_C)$:

$$\nu_C = w^0_C V'_{\mathrm{opt}}(w^0_C) \tag{2.40}$$

We can conclude that when the number of vehicles reaches the critical value, any small perturbation leads to the formation of a localized moving cluster of traffic with velocity ν_C as the result of a dynamical phase transition. The traffic cluster is backward propagating with velocity $\nu_C = 1/2\tau_C$.

We now write the expression of the traffic soliton wave $H(u)$ by first inverting the relation

$$G(u) = \tanh(H(u)) \tag{2.41}$$

that can be solved with

$$H(u) = \operatorname{arctanh}(G(u)) = \frac{1}{2}\ln\frac{1+G(u)}{1-G(u)} \tag{2.42}$$

and we have an analytic expression of the traffic soliton wave

$$H\left(z + \frac{w^0}{2\tau}t\right) = \frac{1}{2}\ln\frac{1+k/(4\tau)\,\mathrm{sn}(z+w^0/(2\tau)t,k)}{1-k/(4\tau)\,\mathrm{sn}(z+w^0/(2\tau)t,k)} \tag{2.43}$$

In App. B are reported the defining properties of solitons along with a brief historical introduction.

2.5 Numerical simulation

We performed a numerical simulation in order to concretely view the development of the oscillating congested traffic, that shows the backward moving traffic waves. We employed a C++ code to obtain the data. The algorithm works following this outline: firstly, the equilibrium positions between vehicles are calculated, using the initial parameters provided by the user. Then, the program adds a sinusoidal perturbation in the form

$$x_k(0) = w^0(1 + \alpha\sin\big((2\pi w^0 k25)/L\big)) + x_{k-1}(0) \tag{2.44}$$

with $\alpha = 0.1$. It then memorizes the distances between consecutive vehicles at the present time, at the delayed time and at all the time steps in between the two (a parameter is used in order to choose the minimum time step that discretizes the time interval τ). It then calculates the next time step distance between two vehicles as the sum of the distance they had in the present with the product of the time step for the difference in the velocities of the two consecutive vehicles. Finally, it prints on a text file positions and distances between vehicles.

Figs. 2.1, 2.5 and 2.5 show the results of a simulation performed with the following parameters: length $L = 5000\,\mathrm{m}$, time step $0.1\,\mathrm{s}$, delay $\tau = 1\,\mathrm{s}$, vehicles' length $d_0 = 5\,\mathrm{m}$, $v_\infty = 7\mathrm{m/s}$. As a result, we have an inhomogeneous traffic flow, with the coexistence of both free and jammed traffic states. The former is represented by the higher values of the headway, while the latter is represented by its the lower value. In Figs. 2.5, 2.5 we can appreciate the backward propagation of the traffic waves.

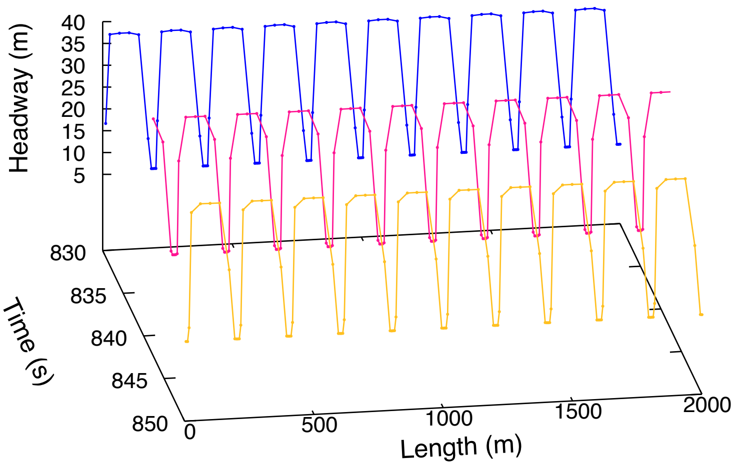
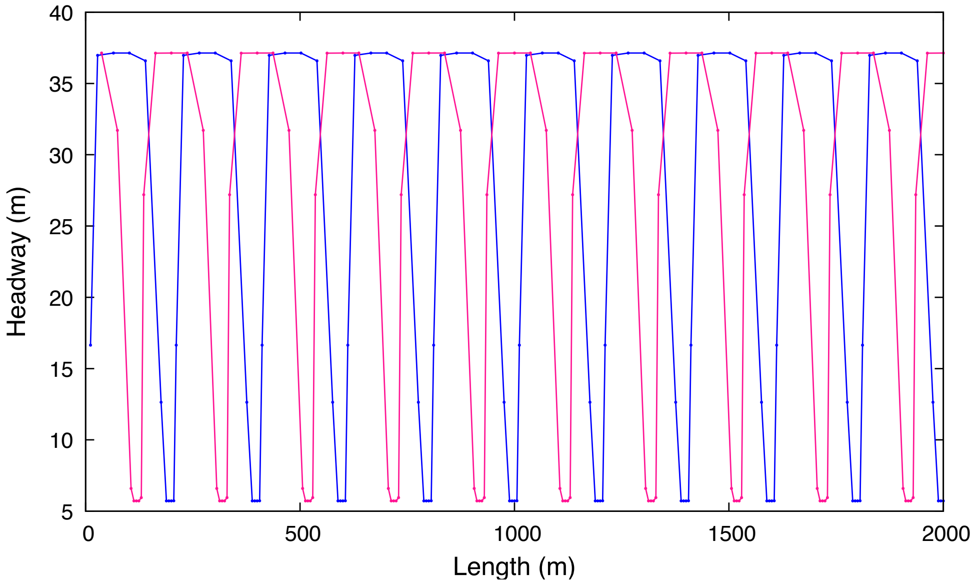


Figure 2.1: Oscillating congested traffic pattern for $N = 250$ vehicles on a circumference. We plotted three headway profiles every 10 s



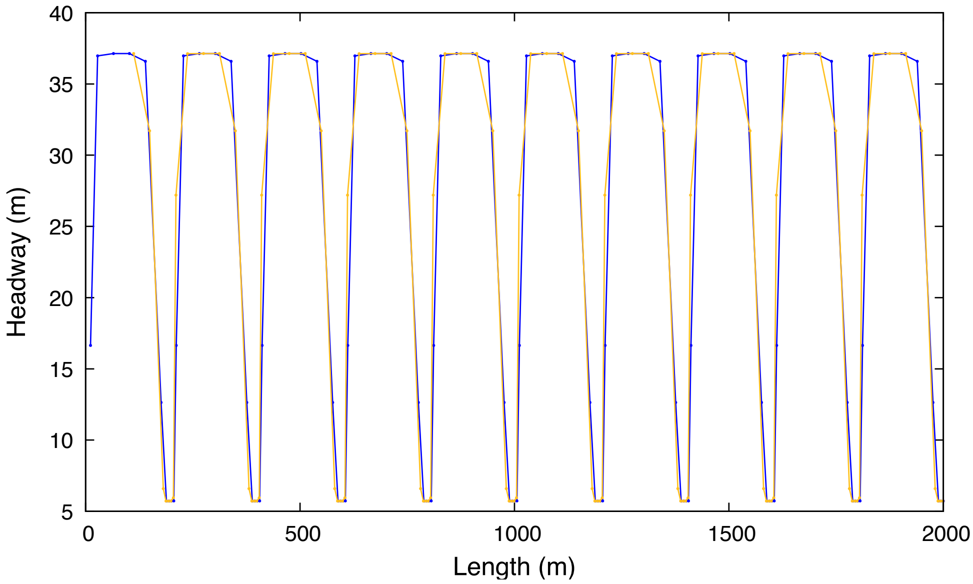


Figure 2.2: The first figure shows the headway profile at consecutive time frames, $t = 820\,\mathrm{s}$ (blue), $t = 830\,\mathrm{s}$ (red), we can appreciate how the traffic waves propagate backwards. The second figure presents the headway profile at $t = 820\,\mathrm{s}$ (blue) and the headway profile at $t = 830\,\mathrm{s}$ (yellow) obtained by translating the positions ahead of the space travelled backward by the traffic wave in the time interval between the two plots (10 s) with a wave velocity $v_0 = 7.5\mathrm{m/s}$ calculated using the simulation parameters.

Chapter 3

Perturbative approach

This chapter is dedicated to solving our traffic equations applying perturbation theory. We remark that this perturbative approach analyzes the dynamics near the critical state transition point $w = w_C^0$ and it is not an approximation of the soliton solution obtained in Ch. 2 for a generic stationary equilibrium state.

We start again from Eq.(1.22)

$$\dot{w}_k(t) = V_{\text{opt}}(w_{k+1}(t-\tau)) - V_{\text{opt}}(w_k(t-\tau)) \tag{3.1}$$

Following Bazzani [5], we look for a wave solution in the form

$$w_k(t) = w_C^0 + H(k\Delta z + \nu\Delta z t, t) = w_C^0 + H(X, t) \tag{3.2}$$

where $H(X,t)$ is the undulatory term that moves with velocity $\nu\Delta z$. Δz denotes the gap between two consecutive vehicles in the stationary state, here it is used as an index; later we will set $\Delta z = 1$ therefore we will simply indicate the wave velocity with ν. Our study addresses the analysis of the system near the critical transition point, namely the distance $w = w_C^0$ in correspondence of which the optimal velocity function presents a flex point. For this reason, we write ν as

$$\nu = V'_{\text{opt}} + \Delta\nu \tag{3.3}$$

where

$$V'_{\text{opt}} = \left.\frac{\mathrm{d}V_{\text{opt}}}{\mathrm{d}z}\right|_{w=w_C^0} = 1 \qquad \Delta\nu = \epsilon V'_{\text{opt}} \tag{3.4}$$

In our perturbative expansion we will consider $\partial H/\partial t \ll 1$ since the perturbation is about the stationary solution and therefore variations with respect to time are very slow. For this reason, in our future calculations we will not proceed beyond the first order when expanding our function with respect to the variable t. For clarity's sake, we remark a few properties of the optimal velocity function along with a few results we obtained in the previous chapters:

- V_{opt} is a sigmoidal function with a flex point at $w = w_C^0$, therefore $V'_{\text{opt}}(w_C^0)$ is a maximum point, $V''_{\text{opt}}(w_C^0) = 0$ and $V'''_{\text{opt}}(w_C^0) < 0$. In the development of our calculations, we will set $w_C^0 = \Delta z = 1$, $V'_{\text{opt}}(w_C^0) \simeq 1$, $V'''_{\text{opt}}(w_C^0) = -2$ (see Figs. 1.2, 3.1).

- We have a critical value of τ, $\tau_C = 1/\big(2V'_{\text{opt}}(w_C^0)\big)$. If $\tau < \tau_C$ the equilibrium solution is always stable, while if $\tau = \tau_C(1+\epsilon)$, the dynamical state changes and a traffic soliton wave develops.

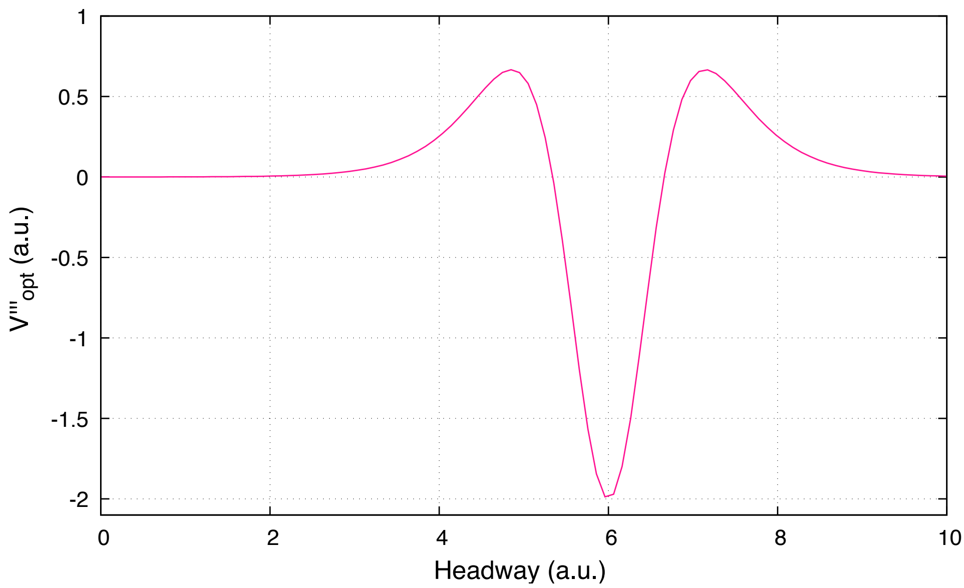


Figure 3.1: Plot of $V'''_{\text{opt}}(w)$ (1.16) for $v_\infty = 2, a = 1, w_C^0 = 6$

3.1 Development

Firstly, we remark that here and in the following sections we will write V_{opt}, V'_{opt} and V'''_{opt} intending the values of these quantities evaluated at the critical point w_C^0.

We insert our ansatz in Eq. (3.1) and we expand up to terms of order $\mathcal{O}(\tau^3)$ in the X variable, while in the t variable we consider only a first order expansion, since $H(X,t)$ has a slow dependence on t, as previously said. For the left side we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} w_k(t+\tau)) = \frac{\mathrm{d}}{\mathrm{d}t} H(X, t+\tau) = \\
&\partial_t H(X, t+\tau) + \partial_X H(X, t+\tau) \frac{\mathrm{d}X}{\mathrm{d}t} = \\
&\partial_t H(X, t+\tau) + \partial_X H(X, t+\tau) \Delta z \nu
\end{aligned} \tag{3.5}$$

Now expanding up to the third order in τ we get

$$\begin{aligned}
&\partial_t H(X, t+\tau) + \partial_X H(X, t+\tau) \Delta z \nu = \\
&\partial_t H + \nu \Delta z \partial_X H + \nu \Delta z \tau \partial_X \partial_t H + \\
&(\nu \Delta z)^2 \tau \partial_X^2 H + (\nu \Delta z)^3 \frac{\tau^2}{2} \partial_X^3 H + (\nu \Delta z)^4 \frac{\tau^3}{6} \partial_X^4 H
\end{aligned} \tag{3.6}$$

We expand $V_{\text{opt}}(w_k)$ considering H as a perturbation

$$\begin{aligned}
&V_{\text{opt}}(w_k(t)) = V_{\text{opt}}(w_C^0 + H(X,t)) = \\
&V_{\text{opt}} + V'_{\text{opt}} H(k,t) + \frac{V'''_{\text{opt}}}{6} H^3(k,t)
\end{aligned} \tag{3.7}$$

where we used the fact that $V''_{\text{opt}} = 0$ since V'_{opt} has a maximum at $w = w_C^0$. Therefore for the right side of Eq. (3.1) we obtain the expansion

$$\begin{aligned}
&V_{\text{opt}}(w_{k+1}(t)) - V_{\text{opt}}(w_k(t)) = \\
&V'_{\text{opt}}(H(k+1,t) - H(k,t)) + \frac{V'''_{\text{opt}}}{6}\big(H^3(k+1,t) - H^3(k,t)\big) = \\
&V'_{\text{opt}}\bigg(H(k,t) + \Delta z \partial_X H(k,t) + \frac{\Delta z^2}{2}\partial_X^2 H + \frac{\Delta z^3}{6}\partial_X^3 H + \frac{\Delta z^4}{24}\partial_X^4 H - H(k,t)\bigg) \\
&+ \frac{V'''_{\text{opt}}}{6}\bigg(H^3(k,t) + \Delta z \partial_X H^3 + \frac{\Delta z^2}{2}\partial_X^2 H^3 - H^3(k,t)\bigg) = \\
&V'_{\text{opt}}\Delta z\bigg[\partial_X H + \frac{\Delta z}{2}\partial_X^2 H + \frac{\Delta z^2}{6}\partial_X^3 H + \frac{\Delta z^3}{24}\partial_X^4 H\bigg] \\
&+ \frac{V'''_{\text{opt}}}{6}\Delta z\bigg[\partial_X H^3 + \frac{\Delta z}{2}\partial_X^2 H^3\bigg]
\end{aligned} \tag{3.8}$$

Equating Eqs. (3.6) and (3.8) and letting $\Delta z = 1$, we get

$$\begin{aligned}
&\partial_t H + \nu \partial_X H + \nu\tau\partial_X\partial_t H + \nu^2\tau\partial_X^2 H + \nu^3\frac{\tau^2}{2}\partial_X^3 H + \\
&\nu^4\frac{\tau^3}{6}\partial_X^4 H = V'_{\text{opt}}\bigg[\partial_X H + \frac{1}{2}\partial_X^2 H + \frac{1}{6}\partial_X^3 H + \frac{1}{24}\partial_X^4 H\bigg] \\
&+ \frac{V'''_{\text{opt}}}{6}\bigg[\partial_X H^3 + \frac{1}{2}\partial_X^2 H^3\bigg]
\end{aligned} \tag{3.9}$$

that we can write as

$$\begin{aligned}
&\big(V'_{\text{opt}} - \nu\big)\partial_X H + \frac{1}{2}\big(V'_{\text{opt}} - 2\tau\nu^2\big)\partial_X^2 H + \\
&\frac{1}{6}\big(V'_{\text{opt}} - 3\tau^2\nu^3\big)\partial_X^3 H + \frac{1}{6}\bigg(\frac{V'_{\text{opt}}}{4} - \tau^3\nu^4\bigg)\partial_X^4 H + \\
&\frac{V'''_{\text{opt}}}{6}\bigg[\partial_X H^3 + \frac{1}{2}\partial_X^2 H^3\bigg] = \\
&\partial_t H + \nu\tau\partial_X\partial_t H
\end{aligned} \tag{3.10}$$

We now write $\nu = 1/(2\tau)$ and we also know that $\nu = V'_{\text{opt}} + \epsilon V'_{\text{opt}}$ according to Eq. (3.3),that is, we are considering the dynamics near the transition point (i.e. the flex point of V_{opt}). We simplify the terms between parentheses keeping only the leading terms

$$
\begin{aligned}
&V'_{\text{opt}} - \nu = V'_{\text{opt}} - V'_{\text{opt}} - \epsilon V'_{\text{opt}} = -\epsilon V'_{\text{opt}} \\
&\frac{1}{2}\left(V'_{\text{opt}} - 2\tau\nu^2\right) = \frac{1}{2}\left(V'_{\text{opt}} - 2\tau\frac{\nu}{2\tau}\right) = -\frac{\epsilon V'_{\text{opt}}}{2} \\
&\frac{1}{6}\left(V'_{\text{opt}} - 3\tau^2\nu^3\right) = \frac{1}{6}\left(V'_{\text{opt}} - 3\tau^2\frac{\nu}{4\tau^2}\right) = \frac{1}{24}V'_{\text{opt}} \\
&\frac{1}{6}\left(\frac{V'_{\text{opt}}}{4} - \tau^3\nu^4\right) = \frac{1}{6}\left(\frac{V'_{\text{opt}}}{4} - \tau^3\frac{\nu}{8\tau^3}\right) = \frac{1}{48}V'_{\text{opt}}
\end{aligned}
$$

So that Eq. (3.10) becomes

$$
\begin{aligned}
&-\epsilon V'_{\text{opt}}\partial_X H - \frac{\epsilon V'_{\text{opt}}}{2}\partial_X^2 H + \frac{V'_{\text{opt}}}{24}\partial_X^3 H + \frac{V'_{\text{opt}}}{48}\partial_X^4 H \\
&+ \frac{V'''_{\text{opt}}}{6}\left[\partial_X H^3 + \frac{1}{2}\partial_X^2 H^3\right] = \left[1 + \frac{1}{2}\partial_X\right]\partial_t H
\end{aligned}
\tag{3.11}
$$

carrying everything on the left side

$$
\left[1 + \frac{\partial_X}{2}\right]\left[\partial_X\left(-\epsilon V'_{\text{opt}}H + \frac{V'_{\text{opt}}}{24}\partial_X^2 H - \frac{\left|V'''_{\text{opt}}\right|}{6}H^3\right) - \partial_t H\right] = 0 \tag{3.12}
$$

That leads us to the wave equation

$$
\partial_t H = \partial_X\left(-\epsilon V'_{\text{opt}}H + \frac{V'_{\text{opt}}}{24}\partial_X^2 H - \frac{\left|V'''_{\text{opt}}\right|}{6}H^3\right) \tag{3.13}
$$

We can now introduce the thermodynamical potential

$$
\phi(H) = \epsilon V'_{\text{opt}}\frac{H^2}{2} + \frac{\left|V'''_{\text{opt}}\right|}{24}H^4 \tag{3.14}
$$

and the functional

$$
\Phi(H) = \int \mathrm{d}X\left[\frac{V'_{\text{opt}}}{48}(\partial_X H)^2 + \phi(H)\right] \tag{3.15}
$$

so that we can write Eq. (3.13) as

$$
\partial_t H + \partial_X\frac{\delta\Phi}{\delta H} = 0 \tag{3.16}
$$

where we varied the functional as it is customary to do in classical mechanics.

3.2 Landau phase transition theory

Landau proposed a theory of phase transitions that allows us to study how our thermodynamical potential varies according to the variations of the parameter τ. Landau's phase transition theory requires that for a non-equilibrium state of the system we can write a thermodynamical potential function of an *order parameter* whose average over the whole system is set. As described by Zacharovič and Leonidovič [13], Landau's potential can be developed in the proximity of the critical point as

$$F(\phi) = F_0 + V\left(a\frac{\phi^2}{2} + b\frac{\phi^4}{4} + ...\right) \tag{3.17}$$

where V indicates the system's volume. In our case, we have a density of potential, since the volume is not explicit, and our order parameter is H

$$\phi(H) = \epsilon V'_{\text{opt}}\frac{H^2}{2} + \frac{\left|V'''_{\text{opt}}\right|}{24}H^4 \tag{3.18}$$

We want to express our density of potential in terms of the parameter τ. We start from the equation for ϵ as a function of τ, namely

$$\frac{1}{2\tau} = V'_{\text{opt}}(1+\epsilon) \tag{3.19}$$

that we have deduced from Eq. (3.13). We can write the expressions for τ and ϵ

$$\tau = \frac{1}{2V'_{\text{opt}}(1+\epsilon)} \tag{3.20}$$

$$\epsilon = \frac{1}{2V'_{\text{opt}}\tau} - 1 \tag{3.21}$$

We remark that, as we previously found with our linear stability analysis in Ch. 1, our system's equilibrium solution is stable if $\epsilon > 0$, meaning $2V'\tau < 1$, while is unstable for $-1 < \epsilon < 0$, that would mean $2V'\tau > 1$. For $\epsilon < -1$, we would have $\tau < 0$ that is not a physical value, since it represents a time interval and therefore must be non-negative. In Fig. 3.2 we have plotted the functional dependence of τ from ϵ (3.20) for physically acceptable values of τ.

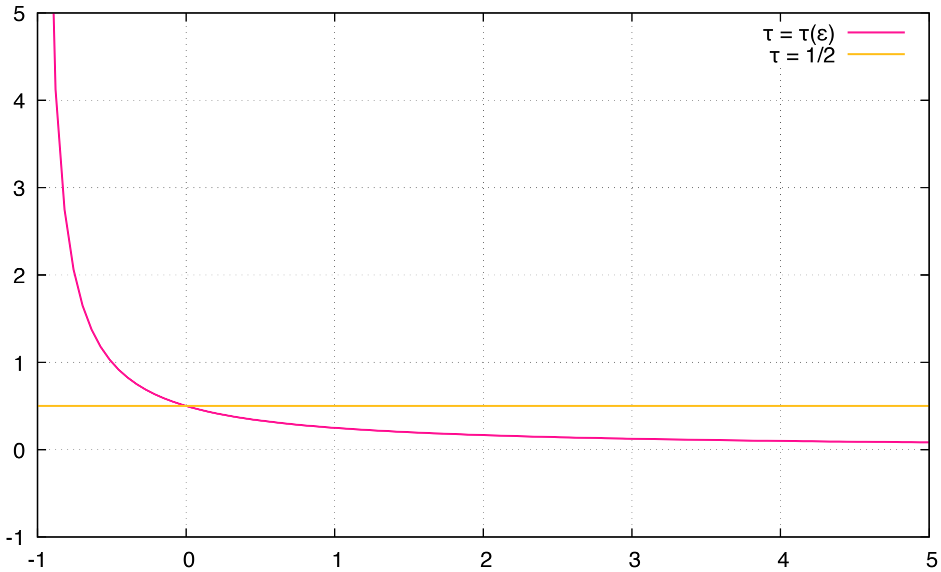


Figure 3.2: Plot of $\tau = \tau(\epsilon)$ and $\tau = 1/2$ for physical values of τ. We can see that for $-1 < \epsilon < 0$ τ is greater than its critical value, leading the system to instability; while it's smaller than its critical value for $\epsilon > 0$.

We can write our density of thermodynamic potential expliciting ϵ as a function of V'_{opt}, τ

$$\phi(H) = \left(\frac{1}{2V'_{\text{opt}}\tau} - 1 \right) \frac{H^2}{2} + \frac{\left| V'''_{\text{opt}} \right|}{24} H^4 \tag{3.22}$$

Now writing $V'_{\text{opt}}(w_C^0) = 1$, $V'''_{\text{opt}}(w_C^0) = -2$, we get to

$$\phi(H) = \left(\frac{1}{2\tau} - 1 \right) \frac{H^2}{2} + \frac{H^4}{12} \tag{3.23}$$

In Fig. 3.3 we finally study $\phi(H)$ as a function of τ. For $\tau \leq 1/2$ we have a stable dynamics that presents one minimum, while for $\tau > 1/2$ our potential presents two wells symmetric with respect to the origin, that therefore is an unstable point. Its dynamics in fact forces the system to "choose" one of the two wells, causing a break in the symmetry.

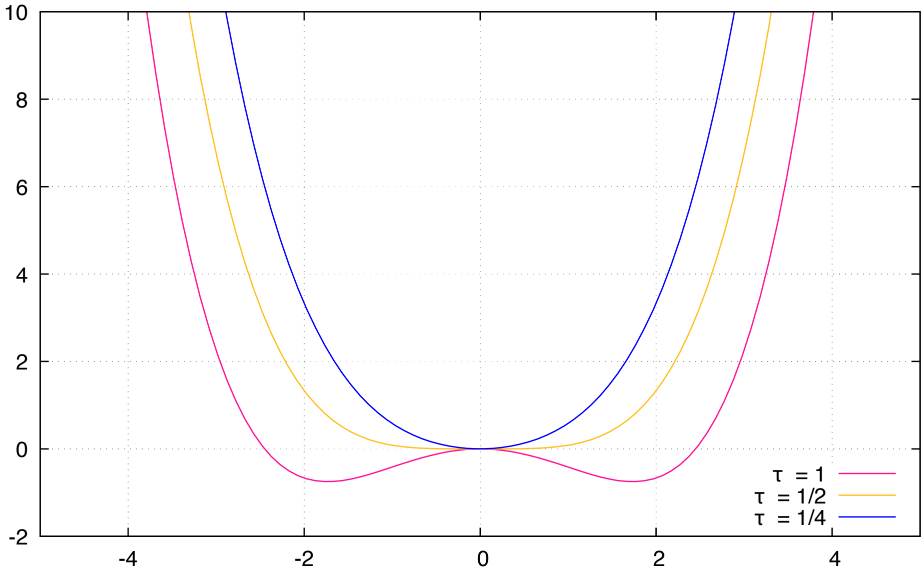


Figure 3.3: Plot of $\phi(\tau)$ (3.23) for various values of τ: $\tau = 1/2$ is the critical value and along with $\tau = 1/4 < 1/2$ constitutes the stable case, $\tau = 1 > 1/2$ presents a structure with two wells and the dynamics at the origin is therefore unstable.

The functional $\Phi(H)$ is an integral of motion for Eq. (3.13) since

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}X \left[\frac{V'_{\mathrm{opt}}}{48}(\partial_X H)^2 + \phi(H)\right] \\ &= \int \mathrm{d}X \left[\frac{1}{24}\partial_X H \partial_X \partial_t H + \frac{\mathrm{d}\phi}{\mathrm{d}H}\partial_t H\right]\end{aligned} \tag{3.24}$$

integrating by parts the first term we get to

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \int \mathrm{d}X \left[-\frac{V'_{\mathrm{opt}}}{24}\partial_X^2 H + \frac{\mathrm{d}\phi}{\mathrm{d}H}\right]\partial_t H \\ &= -\int \mathrm{d}X \left[\frac{V'_{\mathrm{opt}}}{24}\partial_X^2 H - \frac{\mathrm{d}\phi}{\mathrm{d}H}\right]\partial_X \left[\frac{V'_{\mathrm{opt}}}{24}\partial_X^2 H - \frac{\mathrm{d}\phi}{\mathrm{d}H}\right] \\ &= -\frac{1}{2}\int \mathrm{d}X \,\partial_X \left(\frac{V'_{\mathrm{opt}}}{24}\partial_X^2 H - \frac{\mathrm{d}\phi}{\mathrm{d}H}\right)^2 = 0\end{aligned} \tag{3.25}$$

where, in the second line, we used Eq. (3.16). The equality to zero is due to the boundary conditions.

3.3 Minimal action principle

We can also introduce a Lagrangian function

$$\mathcal{L} = \frac{V'_{\mathrm{opt}}}{48}(\partial_X H)^2 + \phi(H) \tag{3.26}$$

since the right hand side can be seen as a sum of a kinetic term and a potential term given by $-\phi(H)$. Therefore, we can write the functional $\Phi(H)$ in the form of an action

$$\Phi(H) = \int \mathrm{d}X \,\mathcal{L} \tag{3.27}$$

We now use the minimal action principle, firstly writing the variation of the action (3.27)

$$\begin{aligned}\frac{\delta\Phi(H)}{\delta H} &= \int \mathrm{d}X \left[\frac{\partial \mathcal{L}}{\partial H} - \frac{\mathrm{d}}{\mathrm{d}X}\frac{\partial \mathcal{L}}{\partial \dot{H}}\right] = \\ &\int \mathrm{d}X \left[-\frac{V'_{\mathrm{opt}}}{24}\partial_X^2 H + \epsilon V'_{\mathrm{opt}} H + \frac{\left|V'''_{\mathrm{opt}}\right|}{6} H^3\right]\delta H\end{aligned} \tag{3.28}$$

where $\dot{H} = \partial_X H$. We equate to zero

$$\int \mathrm{d}X \left[-\frac{V'_{\text{opt}}}{24} \partial_X^2 H + \epsilon V'_{\text{opt}} H + \frac{|V'''_{\text{opt}}|}{6} H^3 \right] = 0 \tag{3.29}$$

That is true for any choice of δH, therefore the term between parentheses must be equal to zero

$$\frac{V'_{\text{opt}}}{24} \partial_X^2 H = \epsilon V'_{\text{opt}} H + \frac{|V'''_{\text{opt}}|}{6} H^3 = \frac{\mathrm{d}\phi}{\mathrm{d}H} \tag{3.30}$$

We recovered Newton's equation for a potential of the form $-\phi(H)$.

We can find the equilibrium states studying the potential $-\phi(H)$, as we did before in Sect. 3.2 (see also Fig. 3.3). For $\epsilon > 0$ the only equilibrium state is for $H = 0$, while for $\epsilon < 0$ we have three equilibrium points

$$H_s = \pm \sqrt{\frac{6|\epsilon| V'_{\text{opt}}}{|V'''_{\text{opt}}|}} \quad H_i = 0 \tag{3.31}$$

where the subscript i indicates that the point is of unstable equilibrium.

When $\epsilon < 0$ (i.e. $2V'\tau > 1$) the solution with zero energy is a separatrix that can be analytically computed in the form $H(X) = \beta \tanh(\alpha X)$. At the separatrix we have

$$\frac{V'_{\text{opt}}}{2|V'''_{\text{opt}}|} (\partial_X H)^2 - \left(H^2 - \frac{6|\epsilon| V'_{\text{opt}}}{|V'''_{\text{opt}}|} \right)^2 = 0 \tag{3.32}$$

so that

$$\alpha^2 \beta^2 \frac{V'_{\text{opt}}}{2|V'''_{\text{opt}}|} \left(1 - \tanh^2(\alpha X)\right)^2 - \beta^4 \left(\tanh^2(\alpha X) - \frac{6|\epsilon| V'_{\text{opt}}}{|V'''_{\text{opt}}| \beta^2} \right)^2 = 0 \tag{3.33}$$

It follows that

$$\beta = \pm \sqrt{\frac{6|\epsilon| V'_{\text{opt}}}{|V'''_{\text{opt}}|}} \qquad \alpha = \sqrt{12|\epsilon|} \tag{3.34}$$

Recalling our expression for $X = z + \nu t$ we get the soliton wave

$$w_k(t) = w_C^0 \pm \sqrt{\frac{6|\epsilon| V'_{\text{opt}}}{|V'''_{\text{opt}}|}} \tanh\left(\sqrt{12|\epsilon|} (k\Delta z + \nu \Delta z t) \right) \tag{3.35}$$

where we have two asymptotic values

$$w_{as}(t) = w_C^0 \pm \sqrt{\frac{6|\epsilon| V'_{\rm opt}}{\left|V'''_{\rm opt}\right|}} \tag{3.36}$$

corresponding to the stopping and go traffic phases respectively.

Appendix A

Delay differential equations

We give a brief introduction to delay differential equations and their main properties, along with the differences with ordinary differential equations.

In many physical problems it is quite natural to introduce a delay in the system's response time that causes the system's evolution at time t to depend on the state at time $t-\tau$. This description of a system is in many case more realistic than the ones that consider its evolution a function of only the present's state. This is where the necessity of studying delayed equations arises.

We can define a delay differential equation (DDE) as following

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), x(t-\tau)) \tag{A.1}$$

with $\tau > 0$ being a constant not surprisingly called "delay".

Following Smith [14], we can first distinguish between discrete and distributed delay: in the former the delay is contained in terms of the type $x(t-\tau)$, while the latter contains a weighted average of delays $x(t-\tau)$ expressed as

$$\int_{t-\tau}^{t} \mathrm{d}s\, k(t-s)x(s) = \int_{0}^{\tau} \mathrm{d}z\, k(z)x(t-z) \tag{A.2}$$

with $\tau \in [0; \infty]$. They are considered more realistic, but are harder to solve and it is difficult to estimate the expression of the kernel k from data. Discrete delay differential equations can contain more than one delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), x(t-\tau_1), ..., x(t-\tau_n)) \tag{A.3}$$

Another classification arises distinguishing the cases in which the kernel $k(u)$ is identically zero for all $u > u_{MAX}$; in this case the distributed delay represented by

$$\int_{-\infty}^{t} \mathrm{d}s\, x(s)k(t-s) = \int_{0}^{\infty} \mathrm{d}u\, x(t-u)k(u) \tag{A.4}$$

is classified (Smith [14]) as bounded delay because the integral considers values of $x(t)$ only for a bounded set of past times $[t - u_{MAX}, t]$; otherwise, it is called unbounded delay or infinite delay. In our later discussion we will deal with discrete delay differential equations that are obviously bounded delays.

If $x(t)$ satisfies a discrete delay differential equation of the form (A.1) on the interval $[a, b]$, then $x(t - \tau)$ must at least be defined on $[a - \tau, b]$ although it does not need to be differentiable on $[a - \tau, a]$. This motivates the following definition of solution of a discrete delay differential equation [14].

Definition A.0.1. Solution of a discrete differential equation
$x(t) : [a - \tau, b] \to \mathbb{R}$ is a solution of a delay differential equation of the form (A.1) if it is continuous, differentiable on $[a, b)$ and satisfies the Eq. (A.1) for $a \leq t < b$.

Another difference between ordinary and delayed differential equations lies in the form of the general initial value problem: in order to obtain the unique solution of a n-th order ordinary differential equations it is necessary to give n initial conditions, namely n values of the unknown function and its derivatives; on the contrary, for a first order delayed differential equation we need to specify a function on an interval $[-\tau, 0]$, where τ is the delay, as initial condition(Dads [2]). DDE therefore are systems with infinite degrees of freedom.

Following Bazzani [3] we study an interesting case of a linear delay differential equation

$$\dot{\vec{x}} = A\vec{x}(t - \tau) \tag{A.5}$$

in which the eigenvalues of A are negatives so that the equilibrium at $\vec{x} = 0$ is stable. Diagonalizing A via a linear change of variables $\vec{x} = T\vec{y}$, we obtain this type of equations

$$\dot{y} = -\gamma y(t - \tau) \tag{A.6}$$

in which γ represents an eigenvalue of A. As it is customary to do with ordinary differential equations, we look for solutions in the form $y(t) = y_i e^{\lambda t}$ but this time with the initial condition $y_i(s) = y_0 e^{\lambda s}$ with $s \in [-\tau, 0]$. We obtain the characteristic equation:

$$\lambda = -\gamma e^{-\lambda\tau} \tag{A.7}$$

Writing $\lambda = \eta + i\omega$, we get

$$\eta + i\omega = -\gamma e^{-\eta\tau}[\cos(\omega\tau) - i\sin(\omega\tau)] \tag{A.8}$$

that we can divide in its real and imaginary part as

$$\begin{aligned} \eta &= -\gamma e^{-\eta\tau}\cos(\omega\tau) \\ \omega &= \gamma e^{-\eta\tau}\sin(\omega\tau) \end{aligned} \tag{A.9}$$

When $\tau = 0$ we have a solution with $\eta = -\gamma$ and $\omega = 0$. We are interested in following with continuity this solution until the real part vanishes. So, assuming $\omega = 0$, we write Eq. (A.7) as

$$x = -\gamma\tau e^{-x} \qquad \text{with} \quad x = \eta\tau \tag{A.10}$$

That admits two solutions for $x < 0$ as long as $\gamma\tau \ll 1$.

When $\gamma\tau = -1$ and $\gamma\tau = -e$ we have a double solution, thereafter there are not anymore real solutions.

Varying ω we get

$$\eta^2 + \omega^2 = \gamma^2 e^{-2\eta\tau} \tag{A.11}$$

from which we get the constraint

$$\omega = \pm\sqrt{\gamma^2 e^{-2\eta\tau} - \eta^2} \tag{A.12}$$

We have then found an equation for η, namely

$$\eta = -\gamma e^{-\eta\tau} \cos\left(\sqrt{\gamma^2 e^{-2\eta\tau} - \eta^2}\,\tau\right) \tag{A.13}$$

We look for a pure imaginary solution, $\eta = 0$. In that case we must have $\omega\tau = \pm\pi/2 + k\pi$, therefore the following must hold

$$\tau_k = \gamma^{-1}\left(\frac{\pi}{2} + k\pi\right) \tag{A.14}$$

that leads $\omega = \gamma$.

If τ exceeds this value of a quantity $\Delta\tau$

$$\tau = \frac{\pi}{2\gamma} + \Delta\tau \tag{A.15}$$

let $\phi = \omega\tau = \pi/2 + \Delta\phi$ and Taylor expanding up to the first order we get

$$\cos(\phi) \simeq -\Delta\phi \tag{A.16}$$

Substituting in Eq. (A.13) we get

$$\eta \simeq \gamma e^{-\eta\gamma}\Delta\phi \tag{A.17}$$

that means $\eta > 0$ and the solution is unstable. At the first order follows that

$$\eta \simeq \gamma\Delta\phi \quad \Rightarrow \quad \Delta\phi \simeq \frac{\eta}{\gamma} \tag{A.18}$$

It follows that

$$\omega = \gamma + \Delta\omega \simeq \gamma\left(1 - \eta\frac{\pi}{2\gamma}\right) \tag{A.19}$$

that means

$$\Delta\omega \simeq -\eta\frac{\pi}{2} \tag{A.20}$$

Putting altogether the conditions, we get

$$\Delta\phi = \frac{\pi}{2\gamma}\Delta\omega + \gamma\Delta\tau \simeq \frac{\eta}{\gamma} \tag{A.21}$$

and we finally obtain the relation that connects η and $\Delta\tau$

$$\gamma\Delta\tau \simeq \frac{\eta}{\gamma}\left(1 + \left(\frac{\pi}{2}\right)^2\right) \tag{A.22}$$

rewriting

$$\eta = \frac{\gamma^2\Delta\tau}{1 + (\pi/2)^2} \tag{A.23}$$

We now report an example from Smith [14] to concretely describe the differences arising when we solve one of the most simple delayed equations

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -u(t-\tau) \qquad \tau > 0 \tag{A.24}$$

that describes a system governed by a negative delayed feedback indicated by the minus sign on the right hand side.

The associated ordinary differential equation is found setting $\tau = 0$

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -u(t) \tag{A.25}$$

that can be easily solved for $u(t) = u_0 e^{-t}$.

On the contrary, the solution of Eq. A.24 is not so straightforward. To have a more simple solution, we firstly prescribe the initial condition for $u(t)$:

$$u(t) = 1 \qquad -\tau \leq t \leq 0 \tag{A.26}$$

Then, we can find the expression for $u(t)$ for the time interval $0 \leq t \leq \tau$ by simply substituting the initial condition and integrating. We iterate this method for the successive time interval: we use the function we have just found as the initial condition for finding the expression of $u(t)$ in the time interval $\tau \leq t \leq 2\tau$. Iterating n times we can find the expression for $u(t)$ in the time interval $[(n-1)\tau, \tau]$

as shown by Smith [14]

$$u(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t-(k-1)\tau]^k}{k!} \quad \tau \in [(n-1)\tau, n\tau] \tag{A.27}$$

this procedure, called "the method of steps", is frequently used to find the solution of delay differential equations and clearly show that even for resolving the simplest DDE one has to use more elaborate methods than for ODE.

We conclude this brief introduction by remarking that a delay differential equation is completely different from an ordinary differential equation, but, for very small delays, the solutions of delay and ordinary equations have the same conditions for stability. This fact will be exploited in Sect. 1.6.3 where we find the conditions for linear stability in the case of a small delay.

Appendix B

Solitons: history and defining properties

On a historical note [6], the first description of a soliton was provided by John Scott Russell in 1834 when he observed the phenomenon of a solitary wave produced by a boat that kept moving maintaining its shape even when the boat stopped, in the Union Canal in Scotland. He wrote down what he had experienced to the British Association in his "Report on Waves":

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

He later reproduced the phenomenon in a wave tank and called it "wave of translation" and was able to deduce a formula for the speed of the solitary wave. His contemporaries, between the others is appropriate to mention George Airy and George Stokes, were appalled by his experimental observations as they were not explainable with the current theories on hydrodynamics. Finally, in 1870, Joseph

Boussinesq and Lord Rayleigh recognized his work and the discovery of a new phenomenon and were able to deduce, from the equations of motion of an inviscid compressible fluid, Russel's formula and the functional form of the wave profile. This initial acceptance of Russel's observations was then followed by the publication of the Kurteweg-de Vries equation for the solitary wave's profile by Diederik Korteweg and Gustav de Vries. Further discoveries were made by Zabusky and Kruskal many years later (1965) in a completely different context, studying the solution of an instance of the Kurteweg-de Vries equation. They gave these non-linear waves the name *soliton* in order to emphasize their particle-like property: they could, in fact, interact strongly with each other and then continue with their motion almost as if there had not been any interaction at all.

We now present a summary of the solitons' properties to clarify which phenomenon we have encountered with our traffic equations.

The term soliton comprehend a wide variety of solutions of non-linear differential equations that have in common the following defining properties [6]:

- constitute a wave of permanent form
- are localised, meaning that they decay or approach a constant at infinity
- interact strongly with each other preserving their identity unaffected

Appendix C

Network view

Following Bazzani [5] we can re-interpret Eq. 1.22 in a different framework: suppose we have a network of N nodes connected in a circle structure with $w_k(t)$ being the state of the k-th node. Its evolution is a function of the states of the connected nodes $(k-1)$ and $(k+1)$ and can decay to zero due to dissipative effects. Our system is for all intents a micro-canonical ensemble, that is, a collection of identical copies of a single entity that maintain constant its total energy and the number of particles in it. The average activity of the network is fixed by the micro-canonical condition

$$\sum_k w_k = C \tag{C.1}$$

We introduce the *activation function* $U(w)$ in analogy with the optimal velocity function as a positive sigmoid function, that satisfies

$$\begin{cases} U(0) = 0 \\ \lim\limits_{w \to \infty} U(w) = U_\infty \end{cases} \tag{C.2}$$

In our network model, $U(w)$ synchronizes the internal activity of a node with the activity of the others.

The dynamic of the k-th node is governed by

$$\dot{w}_k(t) = U\left(\sum_{hk} \pi_{kh} w_h(t-\tau)\right) - U(w_k(t-\tau)) \tag{C.3}$$

with π_{kh} being the interaction structure related to the network. Eq. C.3 is a re-reading of Eq. 1.22 in terms of our network model. The delay is necessary because signals are not instantaneous, but propagate with finite velocity.

Without external signal, the delay differential equation becomes

$$\dot{w}_k(t) = -U(w(t-\tau)) \tag{C.4}$$

and has a stable fixed point at $w_k(t) = 0$ since $U(0) = 0$. We show the stability:

$$\dot{w}_k(t) = -\frac{\mathrm{d}U(0)}{\mathrm{d}w} w(t-\tau) \tag{C.5}$$

$\frac{\mathrm{d}U(0)}{\mathrm{d}w} > 0$ since we assumed our activation function to be positive.

Taking $w \sim \exp(\lambda t)$, gives

$$e^{\lambda t}\lambda = -\frac{\mathrm{d}U(0)}{\mathrm{d}w} \tag{C.6}$$

that forces $\lambda < 0$ in any case, since $U'(0) \geq 0$.

The network structure and activation function define when a global synchronization is possible and when there exist synchronization soliton waves. Then, one should consider the stability problem for the soliton wave solutions.

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