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Finite-group Yang-Mills lattice gauge theories in the Hamiltonian formalism

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Sommario

Nuovi sviluppi nel campo nelle tecniche sperimentali potrebbero presto permettere la realizzazione di *simulatori quantistici*, ovvero di sistemi quantomeccanici realizzabili sperimentalmente che descrivano una specifica Hamiltoniana di nostra scelta. Una volta costruito il sistema, si possono effettuare esperimenti per studiare il comportamento della teoria descritta dall'Hamiltoniana scelta. Un'interessante applicazione riguarda le teorie di gauge non-Abeliane come la Cromodinamica Quantistica, per le quali si hanno un certo numero di problemi irrisolti, in particolare nella regione a potenziale chimico finito. La principale sfida teorica per la realizzazione di un simulatore quantistico è quella di rendere lo spazio di Hilbert della teoria di gauge finito-dimensionale. Infatti in un esperimento si possono controllare realisticamente solo alcuni gradi di libertà del sistema quantistico, e certamente solo un numero finito. Seguendo alcune linee già tracciate in letteratura, nel presente lavoro ottieniamo uno spazio di Hilbert finito-dimensionale sostituendo il gruppo di gauge - un gruppo di Lie - con un gruppo finito, ad esempio uno dei suoi sottogruppi. Dopo una rassegna della teoria di Yang-Mills nel continuo e su reticolo, ne diamo la formulazione Hamiltoniana enfatizzando l'introduzione del potenziale chimico. A seguire, introduciamo le teorie basate su un qualsiasi gruppo di gauge finito, e proponiamo una soluzione ad un problema irrisolto di tali teorie, cioè la determinazione degli autovalori della densità di energia elettrica. Effettuiamo inoltre alcuni calcoli analitici della tensione di stringa in teorie con gruppo di gauge finito, e risolveremo esattamente alcune di esse in un caso semplificato. A finire, studieremo il comportamento dello stato fondamentale di tali teorie tramite un metodo variazionale, e offriremo alcune considerazioni conclusive.

Abstract

Exciting new developments in experimental techniques may soon allow the realisation of *quantum simulators*. These are *ad hoc*, experimentally realisable quantum mechanical systems which describe a target Hamiltonian of choice. One can then perform experiments as a means of calculation, probing the behaviour of the theory described by the Hamiltonian. A particularly interesting application of quantum simulators is to non-Abelian gauge theories such as Quantum Chromodynamics. These have a number of unresolved issues, such as the inability to probe the phase diagram at finite chemical potential. The main theoretical challenge for the realisation of quantum simulators of Yang-Mills gauge theories is to find a suitable Hamiltonian formulation. In fact, one can realistically control experimentally only a few degrees of freedom, and in any case only a finite number of them. However, even when discretised on a lattice, quantum field theories have an infinite number of degrees of freedom. A prescription is then necessary for how to make the Hilbert space finite-dimensional. In this work, following developments in the literature, we investigate the idea of replacing the gauge group, a Lie group, with a finite group, perhaps one of its finite subgroups. This automatically makes the Hilbert space finite dimensional. We review Yang-Mills theory in the continuum and on the lattice, emphasising the Hamiltonian formulation and the introduction of a chemical potential. We then formulate Hamiltonians for lattice gauge theories with any finite gauge group, and propose a method for solving an outstanding issue, that of the determination of the eigenvalues of the electric energy density. We then perform some analytical calculations of the strong-coupling string tension for finite group gauge theories, and solve them exactly in a simplified situation. We then investigate the ground state with a variational method. Finally, we offer some conclusive remarks.

Contents

Introduction

A crucial part of our understanding of physical nature rests on the Standard Model of particle physics, which is described by a Yang-Mills gauge theory [1]. These are of great relevance in modern physics, as all the known forces in nature, except gravity, can be described by Yang-Mills gauge theories coupled to some matter fields. A gauge theory is defined by the requirement of local invariance with respect to a certain group of symmetries. Two prototypical examples of such theories are Quantum Electrodynamics, which is locally invariant under $U(1)$, and Quantum Chromodynamics, locally invariant under SU(3). Both are coupled to fermionic matter. While other interesting types of gauge theories, such as Chern-Simons theory, have attracted attention in the past, in this project we will be concerned with Yang-Mills gauge theories only.

The crucial difference between Quantum Electrodynamics and Chromodynamics is that the former is invariant under U(1), which is an Abelian group, while the latter is invariant under SU(3), a non-Abelian group. This fact turns out to have substantial implications. A basic feature of quantum field theories is that the coupling constant, which controls the strength of the interaction, is not actually a constant, but it depends on the energy scale of the process under consideration [1]. The behaviour of the coupling can be analysed via the renormalisation group. In the Abelian case of Quantum Electrodynamics, the coupling increases with the energy scale, and this fact admits a simple physical picture in terms of dielectric screening [1]. A striking property of non-Abelian gauge theories such as Quantum Chromodynamics is that the coupling decreases with increasing energy scale, a phenomenon known as *asymptotic freedom* [1]. As a consequence, the perturbative techniques that work in the case of Quantum Electrodynamics can only be applied to high-energy processes in Quantum Chromodynamics, and fail in the low-energy regime [1]. While high energy experiments confirm the predictions of Quantum Chromodynamics, a non-perturbative approach is needed to confirm the required properties in the low-energy regime [1, 2]. In particular, in order to explain why quarks are never observed in experiments one needs to establish the property of *confinement*, which means that the only states in the spectrum of QCD are SU(3) colour singlets [1, 2].

In order to extract predictions from non-Abelian gauge theories in the low-energy regime one needs a non-perturbative approach. Essentially the only such scheme is

"lattice gauge theory". The idea [2, 3] is to introduce a non-perturbative cut-off in the form of a lattice which discretises spacetime. The gauge theory is formulated in the Euclidean path-integral approach, which renders the integral strictly positive, so that it can be simulated in a computer via Monte Carlo algorithms [2, 3]. This program has achieved a number of successes, most importantly a demonstration of confinement [4], for which a simple test has been devised [3]. Moreover, the mass spectrum of various theories [5, 6] and the finite-temperature properties of QCD [7] have been studied. Nonetheless, lattice gauge theory presents some limitations:

1. When one includes fermions at finite chemical potential, the partition function becomes the sum of a number of highly oscillating terms, which makes numerical evaluation difficult [8]. This is known as the *sign problem*. A number of interesting phases have been predicted in this region, such as the quark-gluon plasma [7] and colour superconductivity [9].

2. Since the theory is Euclidean, it is hard to study its real-time dynamics. In other words, a number of interesting quantities, such as conductivity and viscosity, cannot be computed from the Euclidean path-integral [10]. Moreover, the details of the various stages of out-of-equilibrium phenomena such as heavy-ion collisions are also out of reach [11].

There is no known fully satisfactory solution to either of these problems [8, 11]. Researchers have then started looking at alternative ways to tackle these issues. A particularly interesting approach is analog quantum simulation [11, 12]. Feynman [13] first noted the difficulty of simulating quantum mechanical systems on a classical computer, and proposed the construction of a universal quantum simulator. The idea is that one can more efficiently simulate a quantum system via another quantum system, rather than via classical computation. In recent years, a number of advances in experimental techniques [14] has made possible the proposal and realisations of different setups to simulate physical theories via quantum mechanical systems [11]. These are all examples of special-purpose quantum simulators, as opposed to universal quantum simulators. In other words, one builds a *specific* system which is described by a specific, chosen Hamiltonian. Experiments can then investigate the properties of the chosen Hamiltonian, and as such, serve as a means of calculation. Some such proposals work with ultra-cold atoms in optical lattices [11, 12]. In this case, a lattice gauge theory is described by a physical lattice made of cold atoms trapped in a periodic potential by appropriately placed laser beams (i.e. an optical lattice). The interactions between the atoms can be tuned in a way to describe a surprising range of models. Other proposals involve superconducting qubits [11]. Quantum simulation is intrinsically free of the sign problem, and there is no issue in studying the real time dynamics of the theory [11]. The long-term hope is that experimental and theoretical advances may one day allow the quantum simulation

of some non-perturbative aspects of QCD, so that currently inaccessible regions of the phase diagram may be explored.

The realisation of quantum simulators poses important practical and technical challenges, but also a number of theoretical difficulties. In particular, one needs to find an appropriate formulation of Yang-Mills gauge theories. We see that:

1. The theory must be formulated in the Hamiltonian approach, rather than in the path-integral approach of usual lattice gauge theory, as condensed matter systems are typically described by Hamiltonians. In this case time is kept as continuous, while only space is discretised.

2. The theory must have a finite-dimensional Hilbert space. Even on a finite lattice, the Hilbert space of quantum field theories is infinite-dimensional. However, one can realistically control experimentally only a few degrees of freedom of each atom, and in any case only a finite number of them [11]. If there's to be any hope of simulating the gauge theory via a physical system, the theory's Hilbert space must be finite-dimensional.

The Hamiltonian formulation of lattice gauge theories was given in [15], and has been employed in a number of scenarios [16, 17, 18]. However, the Hilbert space is still infinite-dimensional and we need some prescription to make it finite-dimensional.

In order to see how this can be done, we consider the case of Quantum Electrodynamics. This allows us to describe the different prescriptions to make the Hilbert space finite dimensional without unnecessary complications. As we will see in section 1.3, the typical commutation relation for Electrodynamics in the *continuum* Hamiltonian formalism is schematically

$$[A, E] = i \tag{1}$$

for the gauge field A and the electric field E. This commutation relation cannot be implemented on a finite-dimensional Hilbert space: if that was the case, then one could take a trace of both sides, yielding 0 equal to the dimension of the Hilbert space. In an infinite-dimensional space, however, the above commutation relation can implemented by operators which are not trace-class [19]. This is impossible in a finite-dimensional space. Instead of the given commutation relation, in a finite-dimensional space one can only satisfy an exponentiated version of the above, for E and $U = \exp(iA)$ [19],

$$[E, U] = -U \tag{2}$$

This is the commutation relation in the Hamiltonian formalism for lattice gauge theory [15]. The commutation relation eq. (2) can be implemented in a finite-dimensional space, and implies its continuum counterpart eq. (1) when the Hilbert space is infinite-dimensional [19].

Simply replacing the commutation relation does not suffice to make the Hilbert space finite-dimensional, and some further prescription is needed. A common approach [20, 21], known as the "Quantum Link Model", makes the Hilbert finite-dimensional while preserving the commutation relation (2) exactly. This is achieved by replacing the A, U operators with appropriate spin operators. With the substitution

$$E \to S_z \qquad U \to S_- = S_x - iS_y,$$

we indeed have $[S_z, S_-] = -S_-$, which is eq. (2). Since this a purely formal substitution, one can choose any spin representation, and as such, the dimensionality of the resulting finite-dimensional Hilbert space. For example, one can improve accuracy, at the expense of complexity, by enlarging the Hilbert space. A similar substitution can be applied to the non-Abelian case [21].

The drawback of the above method is that the chosen operator S_- is not unitary, as $S_-^\dagger = S_+$. However, in lattice gauge theory U *is* a unitary operator. This corresponds in the continuum gauge theory to the so-called *minimal coupling* of gauge and matter fields, which is unitary, and one might want to preserve this important prescription.

In fact, the commutation relation eq. (2) cannot be satisfied on a finite-dimensional space with U unitary [19]. Therefore, one must choose whether to preserve the commutation relation or the unitarity of U. While the Quantum Link Model chooses to preserve the commutation relation, in the present work we consider the other possibility, that of keeping the unitarity of U. The basic idea was first proposed in [19] and further developed in [22, 23] and others. One replaces the gauge group, a Lie group such as U(1) or SU(N), with a finite group, for example one of its subgroups. This automatically makes the Hilbert space finite-dimensional. In [19, 22], the first papers dealing with this approach, $1+1$ dimensional $U(1)$ gauge theory was discretised to a $\mathbb{Z}_N$ gauge theory, where $\mathbb{Z}_N$ is the cyclic group with N elements. The Abelian case has also been treated in past Masters' theses in $1+1$ dimensions [24] and $2+1$ dimensions [25]. More generally, Hamiltonian Lattice Gauge Theory has been treated in the standard references [15, 16] and in the thesis [18].

As emphasised in this introduction, we are interested in finite-group versions of Yang-Mills gauge theories rather than other kinds of gauge theories. For example, Hamiltonian formulations of Quantum Double models with finite gauge groups have been considered in the past, such as in [26]. These are related, but different models than the ones we are considering here. In particular, they choose an entirely different form for the "electric" Hamiltonian (see section 2.2). Unlike in our models, the two parts of their Hamiltonian commute with each other, resulting in a much simpler structure. Another branch of physics in which finite-group gauge symmetries find applications is particle physics. In fact, it is believed that quantum gravity should break all non-gauge symmetries [27], and one might want to be able to preserve some useful discrete symmetries of the Standard Model. However, unlike in our case, these are continuum gauge theories (as opposed to

lattice gauge theories) and the finite group is achieved via spontaneous breaking of a continuous symmetry via a charged Higgs [27]. Finally, a differential calculus for finite groups has been developed, which allows the formulation of a Lie-group gauge theory with a finite group as its base space [28], but this is not what we're looking for in the present work.

The goal of this project is to extend the results of [19], which formulated lattice gauge theories with finite Abelian groups, to the non-Abelian case. In particular, we aim to find a formulation of Hamiltonian lattice gauge theory with any finite gauge group. In the $U(1)$ case, the group can be approximated by one of its $\mathbb{Z}_N$ subgroups, and one can improve the approximation by increasing the order of the group. This is essentially only possible in the Abelian case [29, 30]. It should be emphasised that the non-trivial aspect of our aim is formulating Yang-Mills lattice gauge theories with finite gauge groups in the Hamiltonian formalism. As we will see in section 2.1, in the path-integral approach finite gauge groups can be immediately considered. In fact, even if one cannot approximate non-Abelian gauge groups by finite subgroups, for computing many quantities of interest one can achieve a good approximation to the Lie group gauge theory if the order of the finite subgroup is large enough [31, 32].

In chapter 1 we'll review some aspects of Yang-Mills gauge theories in the continuum, emphasising the Wick rotation to a Euclidean theory and the Hamiltonian formalism, which will be useful in later chapters. We'll also see in more detail the origin of the sign problem, and construct explicitly QCD in $0 + 1$ dimensions, which showcases this formulation. In chapter 2 we review lattice gauge theory in both the path-integral and Hamiltonian approaches, offering a full derivation of the Hamiltonian for Lie group gauge theories in a modern language. In chapter 3 we formulate lattice gauge theories with finite gauge groups, and we propose a way to resolve an outstanding issue, the determination of the exact form of the electric term in the non-Abelian case. In chapter 4 we perform various calculations with finite-group gauge theories. First we compute the string tension analytically in the strong-coupling regime. We then solve the theory exactly in a highly simplified case, that of a single plaquette. Finally, we study the phase structure of some finite group gauge theories of interest via a variational method. We then offer some conclusive remarks and possible directions to extend the present work.

Chapter 1

Continuum Yang-Mills Gauge Theory

In the continuum, a gauge theory is a quantum field theory defined on Minkowski space $\mathbb{R}^{d,1}$, with the property of local invariance with respect to a certain group of transformations. A local transformation is one which is given by a different group element at each point in spacetime. In the present project, we're interested in a specific kind of gauge theory known as Yang-Mills gauge theory. In the following sections, we'll review Yang-Mills theory in Minkowski spacetime and then perform a Wick rotation to Euclidean spacetime. We will then consider the Hamiltonian formulation of the theory, and we will give our formulation of SU(N) Yang-Mills theory in $0+1$ dimensions. Finally, we discuss the introduction of a chemical potential, the QCD phase diagram, and related issues such as the *sign problem*.

1.1 Yang-Mills with fermions

A Yang-Mills gauge theory is defined by a choice of gauge group, typically $U(1)$ or SU(N) and by a choice of representation for the matter fields in the theory. If there are no matter fields present, the theory is said to be "pure". Otherwise, one obtains a different theory depending on the choice of matter fields. These are typically Dirac spinors transforming in the defining (fundamental) representation [1, 33]. For example, Quantum Chromodynamics is an SU(3) gauge theory with Dirac spinors in the fundamental representation. One can choose other gauge representations for the matter fields (for example, the adjoint representation) and also different Poincaré representations (for example, they could be scalars) [33].
In what follows, we will keep in mind the example of QCD, i.e. Yang-Mills theory with fermions in the fundamental representation, even though we will work with general SU(N) rather than specialise ourselves to the QCD case $N = 3$. The theory is described

by the classical Lagrangian [1, 33]

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{1.1}$$

where fermions are taken in the fundamental representation of $\mathrm{SU}(N)$ and the covariant derivative is $D_\mu = \partial_\mu - iA_\mu$. We choose the convention where the Lie algebra generators T^a are Hermitian and $[T^a, T^b] = if^{abc}T^c$ with real structure constants f^{abc}. Moreover, the convention for the Killing form is such that $\mathrm{tr}\,(T^aT^b) = \frac{1}{2}\delta^{ab}$. The curvature tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

and transforms in the adjoint representation of $\mathrm{SU}(N)$. Both the gauge field A_μ and the curvature tensor $F_{\mu\nu}$ live in the Lie algebra $\mathfrak{su}(N)$. Under a gauge transformation given by a group-valued function $g(x) \in \mathrm{SU}(N)$, such that $\psi(x) \to g(x)\psi(x)$, then

$$A_\mu(x) \to g(x)A_\mu(x)g(x)^{-1} + ig(x)\partial_\mu g(x)^{-1} \tag{1.2}$$

so that $D_\mu\psi(x) \to g(x)D_\mu\psi(x)$, while

$$F_{\mu\nu} \to g(x)F_{\mu\nu}g(x)^{-1}$$

leaving the action invariant. Note that any choice of representation of $\mathrm{SU}(N)$ is equally valid, albeit with the appropriate modifications. In fact, we've been a bit sloppy as here $g(x)$ is not quite a group element, but rather a matrix in the defining representation.
Sometimes a different convention is chosen, whereby one replaces $A_\mu \to gA_\mu$ in all of the above formulas, and moreover $F_{\mu\nu} \to gF_{\mu\nu}$, which has the effect of changing the normalisation of the kinetic term, $-\frac{1}{2g^2} \to -\frac{1}{2}$. In any case, the action of the theory is then given by

$$S[A, \psi, \bar{\psi}] = \int d^{d+1}x\,\mathcal{L}$$

in $d+1$ spacetime dimensions. The path integral is given in natural units

$$Z = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS}$$

We will not consider in detail the issue of the quantisation of Yang-Mills theories, which is rather complicated [34]. The problem arises because due to gauge-invariance the integral over the gauge field A greatly overcounts the possible field configurations, which results in the partition function being infinite [1]. This problem does not occur on the lattice [2, 33], and as such, we will not consider this issue here.
It will turn out to be important to note that one can formulate the theory in terms of

the curvature tensor $F_{\mu\nu}$ instead of the gauge field A_μ [35]. With appropriate gauge-fixing, A_μ can be expressed in terms of $F_{\mu\nu}$. In the path-integral approach this change of variables can be formulated as

$$\int \mathcal{D}A\, e^{iS[A]} = \int \mathcal{D}F\, \delta(I(F))\, e^{iS[F]}$$

where $I(F)$ is the Bianchi identity, which in $3+1$ dimensions reads $I(F) = \epsilon_{\mu\nu\lambda\sigma} D_\nu F_{\lambda\sigma}$. In other words, one can use F instead of A as an integration variable, as long as one only integrates over those F which come from some A. This is especially useful in those cases where the Bianchi identity is trivial, and we'll exploit this fact in the following chapters.

1.2 Euclidean Yang-Mills theory

In order to formulate the lattice gauge theory in the path-integral approach, we need to first perform a Wick rotation to a Euclidean field theory [2]. This has the effect of changing the path-integral integrand from e^{iS}, which is oscillatory, to e^{-S}, which is positive, and as such can be interpreted as a probability distribution for Monte Carlo simulations.
The Wick rotation is standard but can be quite tricky. The Euclidean path integral is given by

$$Z = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S}$$

so that the Minkowski action and the Euclidean action satisfy $iS_M = -S_E$. The respective Lagrangians are given by

$$S_M = \int d^{d+1}x_M\, \mathcal{L}_M \qquad S_E = \int d^{d+1}x_E\, \mathcal{L}_E$$

where the Euclidean and Minkowski measures differ in the time component. For example in four dimensions,

$$d^4x_M = dx_0 d^3\mathbf{x} \qquad d^4x_E = dx_4 d^3\mathbf{x}$$

The relationship between x_4 and x_0 is achieved by the usual Wick rotation [1], where one sets $x_4 = ix_0$ or $x_0 = -ix_4$. This leads to $\mathcal{L}_E = -\mathcal{L}_M$. Now consider the gauge kinetic term in the Minkowski Lagrangian. We have

$$-\frac{1}{2g^2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{2g^2}\,\mathrm{tr}\left(2F_{0i}F^{0i} + 2F_{ij}F^{ij}\right)$$

where indices are raised and lowered using the Minkowski metric, with the convention that $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. Therefore

$$F^{0i} = -F_{0i} \qquad F^{ij} = F_{ij}$$

so that we get

$$-\frac{1}{2g^2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{2g^2}\,\mathrm{tr}\left(-2F_{0i}F_{0i} + 2F_{ij}F^{ij}\right)$$

In order to go to Euclidean signature, where we have a sum over all indices with the metric $\delta_{\mu\nu}$, we need to turn the minus sign in F_{0i} into a + sign. In other words, we need to set $F_{0i} = iF_{4i}$. We have that

$$F_{0i} = \partial_0 A_i - \partial_i A_0 - i[A_0, A_i]$$

Since $x_0 = -ix_4$, we see that $\partial_0 = i\partial_4$ and therefore we need to define $A_0 = iA_4$ in order to get $F_{0i} = iF_{4i}$. After this substitution, the kinetic term for the gauge field in the Minkowski Lagrangian written in terms of Euclidean quantities is

$$-\frac{1}{2g^2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})$$

where the sum is simply a Euclidean sum, and now we can raise/lower indices without introducing minus signs. In four dimensions, we would have $\mu = 1, 2, 3, 4$ and similarly in higher/lower dimensions.
Given this, we can consider the fermionic part of the Lagrangian. Here we'll simply perform the Wick rotation of the Dirac operator. The Minkowski Lagrangian is

$$\bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma^\mu_M \partial_\mu + \gamma^\mu_M A_\mu - m)\psi$$

where we have introduced a subscript in the gamma matrices to indicate whether they are Euclidean or Minkowskian. These satisfy

$$\{\gamma^\mu_M, \gamma^\nu_M\} = 2\eta^{\mu\nu} \qquad \{\gamma^\mu_E, \gamma^\nu_E\} = 2\delta^{\mu\nu}$$

Consider first the time term $i\gamma^0_M\partial_0$. Using $\partial_0 = i\partial_4$ this becomes $-\gamma^0_M\partial_4$, so that we must have $\gamma^0_M = \pm\gamma^4_E$. Choosing the positive sign, $i\gamma^0_M\partial_0 = -\gamma^4_E\partial_4$. Then the other time term, $\gamma^0_M A_0$ becomes $\gamma^4_E iA_4$. In order to get a Euclidean sum, we then need to choose $i\gamma^i_M = \gamma^i_E$, which satisfy the correct Clifford algebra. Also note that with this convention, the Euclidean gamma matrices are all Hermitian. In the end we find

$$\bar{\psi}(i\gamma^\mu_M \partial_\mu + \gamma^\mu_M A_\mu - m)\psi = -\bar{\psi}(\gamma^\mu_E \partial_\mu + i\gamma^\mu_E A_\mu + m)\psi$$

Since $\mathcal{L}_E = -\mathcal{L}_M$, we finally arrive at

$$\mathcal{L}_E = \frac{1}{2g^2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(\gamma^\mu D_\mu + m)\psi$$

where the indices are all Euclidean, sums are understood, and now $D_\mu = \partial_\mu + iA_\mu$. The gamma matrices are Hermitian and satisfy the Euclidean Clifford algebra.

1.3 Hamiltonian formulation

To obtain the Hamiltonian formulation of Yang-Mills theory with fermions, one proceeds in the usual way from the Lagrangian by computing the conjugate momenta [33, 36]. Treating fermions as Grassmann variables, the quantisation of the fermionic part of the action is straightforward and does not require further analysis than the free Dirac case [1]. On the other hand, the gauge field action requires more analysis. The main issue is that the A_0 component of the gauge field does not have a conjugate momentum:

$$\frac{\delta \mathcal{L}}{\delta \dot{A}_0} = 0$$

As such, the transformation from "velocities" to "momenta" is not invertible, and it would appear that the Hamiltonian does not exist. This situation can be remedied by imposing the gauge condition $A_0 = 0$, which has been referred to as "canonical gauge" [19], "temporal gauge" [36], "Hamiltonian" gauge [37] or "Weyl gauge" [38]. With this choice of gauge, the kinetic part of the Lagrangian reduces to

$$\mathcal{L} = -\frac{1}{2g^2} \operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right) = \frac{1}{g^2} \operatorname{tr}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = \frac{1}{2g^2}\left(E_i^a E_i^a - B_i^a B_i^a\right)$$

where the "chromoelectric" and "chromomagnetic" fields $\mathbf{E}$ and $\mathbf{B}$ are defined implicitly by the components $F_{\mu\nu}$. In the temporal gauge $\mathbf{E} = \dot{\mathbf{A}}$, and as such $\mathbf{E}/g^2$ is the conjugated momentum to the gauge field $\mathbf{A}$. The chromomagnetic field $\mathbf{B}$ is defined by the spatial components of the curvature tensor, and as such it does not contain time derivatives. The Legendre transformation then gives

$$\mathcal{H} = \frac{1}{g^2} E_i^a \dot{A}_i^a - \frac{1}{2g^2}\left(E_i^a E_i^a - B_i^a B_i^a\right) = \frac{1}{2g^2}\left(E_i^a E_i^a + B_i^a B_i^a\right)$$

In other words, the Hamiltonian is given in d spatial dimensions by

$$H = \int d^d x \, \frac{1}{g^2} \operatorname{tr}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$$

akin to the case of electromagnetism. In the Hamiltonian formulation, the fields $\mathbf{A}$ and $\mathbf{E}$ are now operators, satisfying the commutation relations

$$\begin{aligned}
[A_i^a(\mathbf{x}), E_j^b(\mathbf{y})] &= i g^2 \delta_{ij} \delta_{ab} \delta(\mathbf{x} - \mathbf{y}), \\
[E_i^a(\mathbf{x}), E_j^b(\mathbf{y})] &= [A_i^a(\mathbf{x}), A_j^b(\mathbf{y})] = 0
\end{aligned}$$

The $\mathbf{B}$ operator is defined from $\mathbf{A}$. The presence of the couplings in the commutation relation is rather awkward. To avoid it, one can choose the different convention for the gauge fields outlined in section 1.1, or redefine the components of $\mathbf{A}$, $\mathbf{E}$ (but not $\mathbf{A}$, $\mathbf{E}$

themselves) to include the coupling g. A simpler choice is to work with the canonical momentum $\boldsymbol{\pi} = \mathbf{E}/g^2$, in which case the Hamiltonian becomes

$$H = \int d^d x \ \mathrm{tr} \left(g^2 \boldsymbol{\pi}^2 + \frac{1}{g^2} \mathbf{B}^2 \right)$$

and the commutation relations are canonical. It will turn out to be important to note that the conjugate momenta $\boldsymbol{\pi}$ are the infinitesimal generators of translations of $\mathbf{A}$. In fact, their commutation relations are analogous to $[x_i, p_j] = i\delta_{ij}$, and these imply that p is the generator of x translations. The analogy can be made precise if we pick as our Hilbert space the space of wavefunctionals of field configurations. An element of this Hilbert space is a functional $\psi[\mathbf{A}]$, and the action of the gauge field $\mathbf{A}$ is given by simple multiplication, in a manner analogous to the action of $\hat{x}$ in the position Hilbert space of ordinary quantum mechanics. The conjugate momenta can then be represented as

$$\pi_i^a = -i \frac{\delta}{\delta A_i^a}$$

analogously to the momenta $\hat{p}$ of ordinary quantum mechanics. This perspective will turn out to be useful in 2.2 when placing gauge fields on the lattice in the Hamiltonian formalism.

The choice of the temporal gauge $A_0 = 0$ does not come for free. In order to impose this gauge condition, one must make sure that the equation of motion for A_0 is satisfied. Intuitively, A_0 can be thought of as a Lagrange multiplier enforcing a constraint. While the analysis is not simple [36, 39], the end result is that one must have

$$D_i E_i = 0$$

where the D_i are the spatial components of the covariant derivative and the E_i are the spatial components of the chromoelectric field. This equation is inconsistent with the commutation relations, and as such it cannot be implemented as an operator equation [33, 36]. The solution is to impose it on states, by requiring that the allowed, "physical" states satisfy [33, 36]

$$D_i E_i \left| \psi \right\rangle = 0$$

for each colour component of $D_i E_i$. These constraints select a subspace of the overall Hilbert space, which we will call the *physical* Hilbert space. In the case of a $U(1)$ gauge theory, the constraint reduces to $\nabla \cdot \mathbf{E} = 0$, which is Gauss' Law. As such, the constraint for a general gauge group, and in the presence of matter fields, are sometimes also called Gauss' law. Note that in the case of a non-Abelian group the gauge field carries colour charge, unlike the photon which has no $U(1)$ charge. This is reflected in the Gauss' law constraint, which in that case can be written as $\nabla \cdot E = \rho_C$ where ρ_C is colour charge. In the presence of matter fields, the constraint is modified in order to include the matter colour charge.

1.3.1 QCD in 0+1 Dimensions

In this subsection we consider the simplified model of an SU(N) gauge theory in $0+1$ dimensions, with fermions in the defining representation. This allows us to describe the above ideas in a simpler setting and it showcases Gauss' law for fermions and the introduction of a chemical potential term. Quantum Chromodynamics corresponds to the case $N = 3$. Such a theory was considered in the path-integral approach in [40], as a case study for the introduction of the chemical potential on the lattice. Here we develop the Hamiltonian formulation of the theory, which requires continuum time while space is reduced to a single point. While the calculations are not hard, to our knowledge they haven't appeared previously in the literature.

In $0+1$ dimensions, the tensor $F_{\mu\nu}$ only has one component, which is forced to be zero by antisymmetry. Therefore the kinetic term for the gauge field is exactly zero and there is no gauge field dynamics. The only other term in the Lagrangian eq. (1.1) is the spinor term coupled to the gauge field. In the $0+1$ dimensional Minkowski theory, spinors have $2^0 = 1$ components. Therefore the ψ vector will have only one spinor component and N colour components in the fundamental representation of SU(N). There is only one gamma matrix, γ^0, and it satisfies the Clifford algebra $(\gamma^0)^2 = 1$. Since spinors only have one component, γ^0 is simply a number and we can take $\gamma^0 = 1$. It follows that $\bar{\psi} = \psi^\dagger$. The Lagrangian is therefore

$$\mathcal{L} = \bar{\psi}(iD_0 - m)\psi = \bar{\psi}(i\partial_0 + A_0 - m)\psi$$

For $A_0 = 0$ and Grassmannian ψ this is the Lagrangian of N decoupled fermionic harmonic oscillators with frequency m in appropriate units. This is because in $0+1$ dimensions, which corresponds to ordinary quantum mechanics, a classical (bosonic) harmonic oscillator is given by the Lagrangian

$$L = p\dot{x} - \frac{1}{2}\left(p^2 + \omega^2 x^2\right)$$

which is simply a Legendre transformation from the usual Hamiltonian. The Lagrangian can be reformulated in terms of "creation/annihilation" variables

$$a = \frac{1}{\sqrt{2\omega}}(\omega x + ip) \qquad \bar{a} = \frac{1}{\sqrt{2\omega}}(\omega x - ip)$$

as

$$L = i\bar{a}\dot{a} - \omega\bar{a}a$$

which upon canonical quantisation leads to the commutation relations $[a, a^\dagger] = 1$, where $a, \bar{a} \to a, a^\dagger$ are now operators. In our case, as we saw, the Lagrangian is

$$\mathcal{L} = i\bar{\psi}\dot{\psi} + \bar{\psi}(A_0 - m)\psi$$

or rather, making explicit the colour components,

$$\mathcal{L} = i\bar{\psi}_i\dot{\psi}_i + \bar{\psi}_i\left[(A_0)^a(T^a)_{ij} - m\delta_{ij}\right]\psi_j$$

where the T^a are the Hermitian generators of SU(N) and the sums over i and over a are understood. The canonical momenta conjugate to ψ are

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = -i\bar{\psi}$$

where π is again an $N-$component vector, leading to the classical Hamiltonian

$$\mathcal{H} = \dot{\psi}\pi - \mathcal{L} = \bar{\psi}(m - A_0)\psi$$

The Poisson structure is given by $\{\pi_i, \psi_j\}_{PB} = -\delta_{ij}$, or $\{\psi_i, \bar{\psi}_j\}_{PB} = -i\delta_{ij}$ which leads to the quantum anticommutators

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij} \qquad \{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0$$

giving a two-dimensional Fock space for each component of ψ, for a total 2^N- dimensional Fock space. The quantum Hamiltonian suffers from ordering ambiguities. The simplest quantisation choice leads to

$$\hat{\mathcal{H}} = \psi^\dagger(m - A_0)\psi$$

Adopting a "Wigner-Weyl" symmetric prescription instead, one first rewrites the classical hamiltonian as

$$\mathcal{H} = \frac{m}{2}\left(\bar{\psi}\psi - \psi\bar{\psi}\right) - \frac{1}{2}\left(\bar{\psi}A_0\psi - \psi A_0^*\bar{\psi}\right)$$

where we have used the convention that the $\mathfrak{su}(N)$ generators are Hermitian, and that the components of A_0 are real. In general, note that only for $\mathfrak{su}(2)$ can all the generators of the Lie algebra be chosen to be real. This leads to the quantum Hamiltonian

$$\hat{\mathcal{H}} = \frac{m}{2}\left(\psi^\dagger\psi - \psi\psi^\dagger\right) - \frac{1}{2}\left(\psi^\dagger A_0\psi - \psi A_0^*\psi^\dagger\right)$$

Using the anticommutation relations, the Hamiltonian can be rewritten as

$$\hat{\mathcal{H}} = m\left(\psi^\dagger\psi - \frac{N}{2}\right) - \psi^\dagger A_0\psi$$

where we have used the fact that the generators of $\mathfrak{su}(N)$ are traceless. This is the Hamiltonian of N fermionic harmonic oscillators coupled via the classical gauge field A_0, which may possibly be time dependent. One can add a chemical potential term $-\mu\psi^\dagger\psi$ which may be simply absorbed into the mass, with the caveat that now the mass can be negative.

If instead we want to treat A_0 as dynamical, we can compute its conjugate momentum from the Lagrangian. As usual, we find that it is identically zero and we can treat A_0 as the multiplier of a constraint. The equation of motion for A_0 is imposed at the level of the states, while using "gauge-invariance" we set $A_0 = 0$. The Hamiltonian then becomes simply the Hamiltonian of N free fermionic harmonic oscillators:

$$\hat{\mathcal{H}} = m\left(\psi^\dagger\psi - \frac{N}{2}\right)$$

with the constraint that the physical Hilbert space is given by those states $|\psi\rangle$ which satisfy

$$G^a\,|\psi\rangle = 0$$

where the constraint is given by

$$G^a = \psi_i^\dagger (T^a)_{ij}\psi_j$$

This is nothing but the non-Abelian Gauss' law, where we impose the constraint that states in the physical Hilbert space carry no colour charge.

1.4 Chemical potential and the sign problem

The QCD phase diagram shows the different phases of QCD in the $\mu - T$ plane, where T is temperature and μ is the chemical potential [8]. A rough outline of the phase diagram is given in figure 1.1. While a number of interesting phases have been predicted such as the quark-gluon plasma [7] and colour superconductivity [9], detailed quantitative investigation has been limited to the $\mu = 0$ region [8]. This is due in the first place to the difficulty of studying QCD in the low energy regime, as already outlined in the Introduction. Moreover, lattice gauge theory, the most common non-perturbative approach for quantitative study of Yang-Mills theory, is not applicable in the finite μ region due to the *sign problem* which will be outlined in the present section.

In the Hamiltonian formulation, the chemical potential potential can be easily introduced as usual in statistical mechanics [41]. If $\hat{H}$ is the Hamiltonian and $\hat{N}$ is the number operator, then one may simply replace $\hat{H}$ with $\hat{H} - \mu\hat{N}$ where μ is the chemical potential. In QCD the number operator is related to the fermions, and is given by $\hat{N} = \psi^\dagger\psi$, the fermion number, which is given by the number of particles minus the number of antiparticles.

In the path-integral formalism, fermions are taken to be Grassmann variables. In order to avoid working with such cumbersome anticommuting variables, one integrates them out [1, 8],

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\int d^{d+1}x\,\bar{\psi}K\psi} = \det K$$

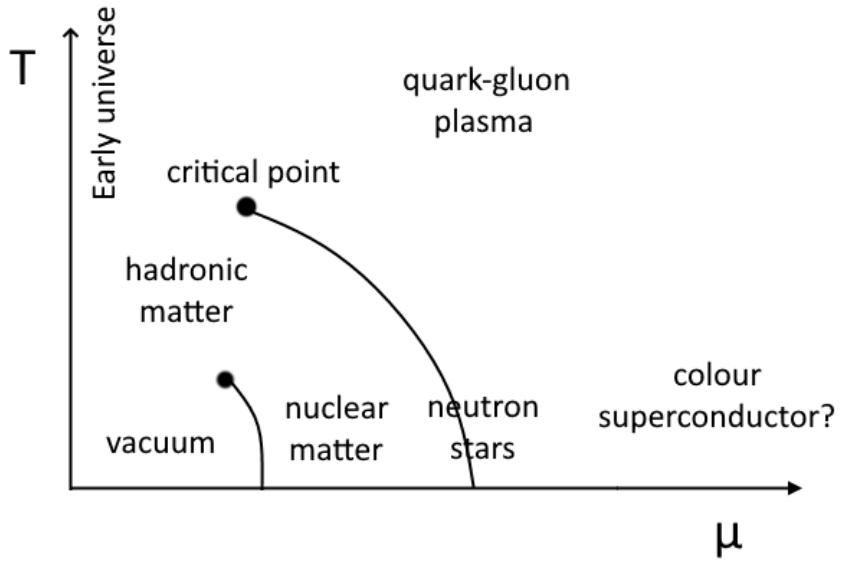


Figure 1.1: The QCD phase diagram

In a gauge theory the fermions will be coupled to the gauge fields, so that K has some complicated dependence on A_μ. In many interesting cases, one would like to include a chemical potential term $\mu\psi^\dagger\psi$ in the Lagrangian. However, as shown in [8], the fermion determinant for the resulting operator K turns out to be complex, with a non-trivial phase factor. As a result, the integrand of the path-integral is not positive anymore, and it cannot be interpreted as a probability distribution. This is called the "sign problem", and poses a severe limitation on the applicability of lattice Monte Carlo simulations in the finite μ region. A quantum simulator, as outlined in the Introduction, works with a Hamiltonian and does not need to perform Monte Carlo averages. It is thus free of the sign problem, at least in principle, and could be used to probe the QCD phase diagram in the $\mu \neq 0$ region.

Chapter 2

Lattice Gauge Theory

In the previous chapter we considered Yang-Mills theory in the continuum. Since perturbative calculations break down in the low-energy regime of the theory, a non-perturbative approach is needed. In this chapter, we'll describe the most successful such non-perturbative scheme, lattice gauge theory. First, we'll review pure gauge theory on the lattice in the path-integral approach, together with the main results achieved via this scheme. Afterwards, the Hamiltonian formulation of pure gauge theory is described in a modern approach, and we give our own full derivation of the Hamiltonian. A further section outlines the well-known lattice formulation of fermions, in both the path-integral and Hamiltonian approaches. The issue of "fermion doubling" is considered, and a well-known method to mitigate it, the so-called *staggered fermions*, is described. A final section considers the specific issues relating to the chemical potential on the lattice. We perform calculations to show that the Hamiltonian approach does not suffer from the known issues with the naive chemical potential term in the path-integral approach.

2.1 Pure gauge theory in the path-integral approach

The most common approach to lattice gauge theory is based on the lattice discretisation of a continuum gauge theory in the Euclidean path integral approach. We need to perform two separate steps:

1. We move from a theory defined on Minkowski spacetime to a Euclidean theory (i.e. we perform a Wick rotation).

2. We replace the continuum spacetime with a discrete lattice.

The first step is necessary in order to obtain a positive integrand in the path integral, so that afterwards a Monte Carlo simulation may be carried out.

We saw how to perform the Wick rotation in section 1.2 and we can now perform the second step, i.e. discretising spacetime. We replace continuous $d+1$ dimensional

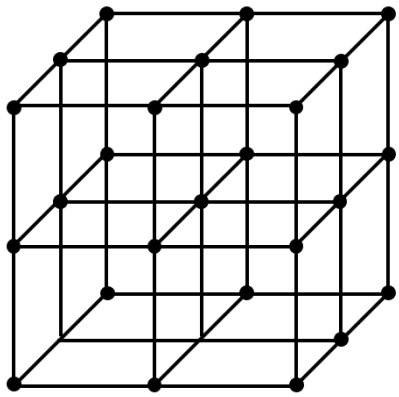


Figure 2.1: A three-dimensional cubic lattice

Euclidean spacetime with a $d+1$ dimensional hypercubic lattice, with lattice spacing a, equal in all directions, as shown in figure 2.1.

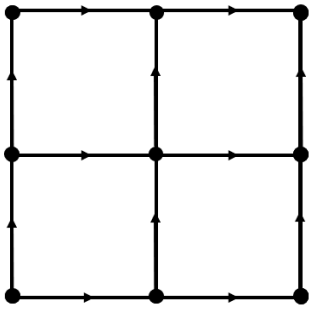
The choice of a hypercubic lattice is the simplest and the only one used in practice. In principle, one could formulate the theory on any lattice, or more generally on any graph [42]. However, the hypercubic lattice is the only one used in practice [2]. Formally, in four dimension the lattice is defined as

$$\Gamma = \left\{ x \in \mathbb{R}^4 \text{ s. t. } x = \sum_{\mu=1}^{4} n_\mu a\hat{\mu} \quad n_\mu \in \mathbb{Z} \right\}$$

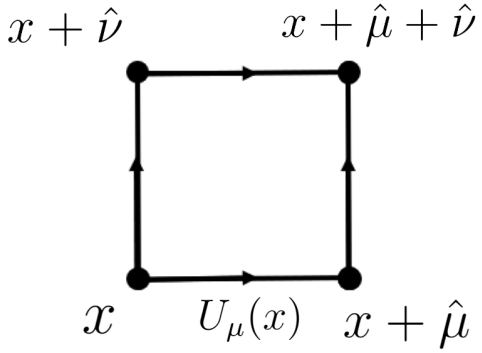
where $\mu = 1, 2, 3, 4$ and $\hat{\mu}$ is the unit vector in direction μ. The extension to a different number of dimensions is obvious.

On the lattice, vertices host matter fields (usually fermions), while the gauge fields A_μ live on the links between different vertices. Intuitively, $\psi(x)$ is a fermion defined on lattice site x, while $A_\mu(x)$ is a gauge field defined on the link between lattice sites x and $x + \hat{\mu}$.

Let's now construct the theory with care. We start from the hypercubic lattice Γ. We'll choose a preferred orientation for each direction in the lattice, i.e. we pick $+\hat{\mu}$ rather than $-\hat{\mu}$ as a lattice vector. For example in two dimensions we choose the positive x direction and the positive y direction, rather than say, the negative x direction and positive y direction or any other choice. It doesn't matter what orientation is chosen as long as one is chosen (see for example figure 2.2a). We identify each link on the lattice by a lattice site and a direction. So for example the link $(x, \hat{\mu})$ is the link between lattice sites x and $x + \hat{\mu}$. Choosing the lattice vectors to be in the direction of positive orientation, then the link $(x, \hat{\mu})$ is traversed in the positive direction. Note that the link $(x + a\hat{\mu}, -\hat{\mu})$ is the same link as $(x, \hat{\mu})$, but traversed in the negative sense, that is opposite the chosen orientation [2].



(a) A possible orientation on a two-dimensional lattice



(b) The gauge fields on the lattice live on the links, while the fermions live on the sites

Figure 2.2: Lattice orientation and gauge fields

To each link $(x, \hat{\mu})$ traversed in the positive direction we associate a group element $U_\mu(x) \in \mathrm{SU}(N)$ as in figure 2.2b. On the other hand, if the same link is traversed in the negative direction, then we associate to it the Hermitian conjugate of the same matrix, $U_\mu(x)^\dagger = U_\mu(x)^{-1} \in \mathrm{SU}(N)$ [2]. We recover the continuous case by writing

$$U_\mu(x) = \exp\left(iaA_\mu(x)\right) \tag{2.1}$$

Note that some authors include the coupling g in the exponent, and this different choice depends on the original convention chosen for the gauge field Lagrangian in section 1.1. Of course the μ index in U_μ cannot be a Lorentz index, but this is of no concern as the lattice explicitly breaks the Lorentz symmetry [2].

As in the pure gauge theory we don't have fermions, the only thing left to do is to write down a lattice action. In order to do so, we need to understand what requirements we want to impose on a suitable action. Two reasonable things to require are the following:

1. The lattice action should reduce to the continuum action in the continuum limit, $a \to 0$.

2. The lattice action should respect the gauge symmetry.

These are the two fundamental conditions that we would like our action to satisfy. In fact, they are quite generic and it's not surprising to discover that one can find many gauge-invariant lattice actions with the correct continuum limit [2, 43]. This is an important remark to which we'll come back several times. Note that we would really like to state

condition 2 as "the lattice action should respect all the symmetries of the continuum action". However, apart from gauge invariance, the other fundamental symmetry of the continuum action is Poincaré symmetry, which is necessarily broken by the lattice. We do expect however that it is restored in the continuum limit [2].

Before we continue, we need to specify what we mean by gauge invariance on the lattice. Suppose that we perform a local gauge transformation on the fermions (even though for now we don't have them). If the gauge transformation is described by a group-valued function $g(x)$ in the appropriate representation, then the fermions at each site transform as $\psi(x) \to g(x)\psi(x)$. Typically, one chooses the fundamental representation of SU(N), which coincides with its defining representation. The gauge transformation is local because it is given by a different matrix at each lattice site. The transformation on the links, where we have group-valued lattice variables $U_\mu(x)$, is then given by

$$U_\mu(x) \to g(x)U_\mu(x)g(x+a\hat{\mu})^\dagger \tag{2.2}$$

This is the same transformation law of the group-valued Wilson lines in the continuum theory [1] and basically the only way to match the two transformations. In the continuum limit, eq. (2.2) reduces to the correct trasformation law for the gauge field A_μ, eq. (1.2). To shows this, we perform a Taylor expansion of the last term, remembering that we need to keep $g(x+a\hat{\mu})^{-1}g(x+a\hat{\mu}) = 1$, which implies

$$g(x+a\hat{\mu})^{-1} = g(x) - a\partial_\mu g(x)^{-1} + \mathcal{O}(a^2)$$

Then expanding U_μ using eq. (2.1), we see that eq. (2.2) reduces to

$$A_\mu(x) \to g(x)A_\mu(x)g(x)^{-1} + ig(x)\partial_\mu g(x)^{-1} + \mathcal{O}(a)$$

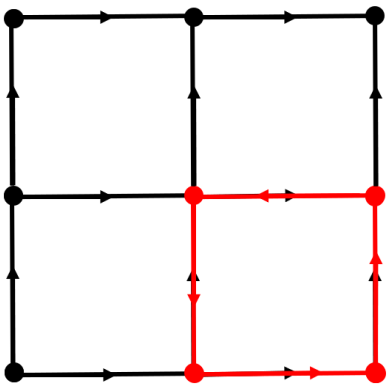
which is eq. 1.2, the correct transformation law, in the $a \to 0$ limit.

In order to construct a lattice action, we need to find a gauge-invariant function of the variables U_μ. The most important such function is the so-called *Wilson loop* [2]. Let C be a closed path on the lattice, that is a path that starts and ends at the same lattice site, while traversing only lattice links. Then we take the product of all the matrices on each successive link, remembering to take Hermitian conjugates if the link is traversed in the negative direction. We can finally take a trace of the resulting matrix and we will have obtained a gauge invariant object, the Wilson loop $\operatorname{tr} W_C$.

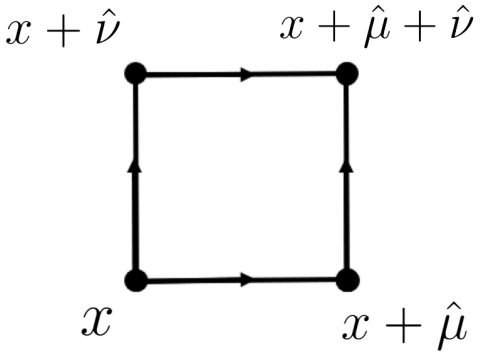
The construction is illustrated in the simplest example of Wilson loop, where the path C is taken to be the smallest possible, i.e. a *plaquette*. This a square path with all sides having minimum length, a (see figure 2.3a). The corresponding Wilson loop is

$$\operatorname{tr} W_\square = \operatorname{tr}\left[U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu(x+a\hat{\nu})^\dagger U_\nu(x)^\dagger\right]$$

as illustrated in figure 2.3b. Note that there is no sum over μ, ν. Instead, these denote the two directions of the plaquette under consideration. The conventions to define Wilson loops can vary: some prefer to include the trace in the definition, some do not [2, 42].



(a) A plaquette on a two dimensional lattice (red). The lattice orientation is in black, while the direction in which the plaquette traverses the links is in red.



(b) The four links in a plaquette as in the calculation of a Wilson loop in the two directions μ, ν.

Figure 2.3: Details of plaquettes

The useful property of the plaquette Wilson loop is its continuum limit [2]:

$$\operatorname{tr} W_\square \approx N - \frac{a^4}{2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(a^6) \tag{2.3}$$

where again there's no summation over μ, ν and $N = \operatorname{tr} 1$ is the size of the U matrices.

In order to get the appropriate lattice action we need to sum over all plaquettes,

$$S = -\frac{1}{g^2} \sum_\square (\operatorname{tr} W_\square + \operatorname{tr} W_\square^\dagger) = a^{3-d} \int d^{d+1}x \left[\frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(a^2) \right] \tag{2.4}$$

where now we have a sum over μ, ν, which is appropriate for a Euclidean theory, and we have ignored the constant term. The factor of $1/2$ comes from the antisymmetry of $F_{\mu\nu}$, which causes an overcounting of terms in the sum. This is known as the *Wilson action*, as it was first derived by Wilson [3]. Some authors prefer to keep the constant term so that action is close to zero on plaquette configurations close to the identity [2].

A point that needs to be remembered is that the normalisation for the above lattice action only holds for SU(N) gauge theories, as we are using the convention $\operatorname{tr}\left(T^a T^b\right) = \frac{1}{2}\delta_{ab}$ which leads to

$$\frac{1}{2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

In the U(1) case, the only generator is $T^a = 1$. There is no need in this case to trace over the Wilson loop, so we need to add a factor of $1/2$ to the lattice action in order to obtain the correct $1/4$ normalisation.

As we noted previously, the lattice action is not unique. The Wilson action is in some sense the simplest possible, but there are definitely other lattice actions which satisfy the two requirements we've set forth [2]. For example, one could include larger or non-square Wilson loops. These all have the same leading order continuum limit, however the higher order terms may differ [18, 44]. As we have seen, the Wilson action gives the continuum action up to terms $\mathcal{O}(a^2)$. With appropriately chosen additional terms one can push the error to $\mathcal{O}(a^4)$ or higher, and this can be useful in numerical calculations approaching the continuum limit [44]. However, we won't consider here any action other than the Wilson action.

Finally, we can formulate the lattice path integral,

$$Z = \int \prod_{(x,\hat{\mu})} dU_\mu(x) e^{-S}$$

where the integration measure on the group is understood to be the Haar measure. The compactness of the group ensures that this is well-defined and gives a finite value. There is no need in the lattice formulation for gauge-fixing in order to avoid overcounting configurations, which leads to infinities in the continuum theory. This is avoided by integrating over the compact group rather than over the Lie algebra [33].

An important point is that while in the continuum gauge theory the gauge field A_μ and the field strength $F_{\mu\nu}$ take values in the Lie algebra of the gauge group, lattice gauge theory in the path integral approach can be formulated in terms of group-valued variables only.

2.1.1 Methods and main results

In the path-integral approach, one mainly computes averages of observables via a Monte Carlo simulation. The average of an observable O is given in the lattice theory by [2]

$$\langle O \rangle = \frac{1}{Z} \int \prod_{(x,\hat{\mu})} dU_\mu(x)\, O\, e^{-S}$$

Using an appropriate Monte Carlo algorithm, like the Metropolis or heat-bath algorithms [2], one can sample configurations with the appropriate probability distribution and then compute averages.

One of the most important averages is that of Wilson loops for generic closed paths C, without self-intersections. In this case, confinement has been shown to be equivalent to the "area-law" behaviour of Wilson loops [3], that is

$$\langle W(C) \rangle \sim \exp\left(-\sigma A(C)\right)$$

as the Wilson loops get larger, where $A(C)$ is the area inside the closed path C and σ is the *string tension*, that is the coefficient of the linear potential between two quarks. On the other hand, on a theory without confinement one expects "perimeter-law" behaviour [3],

$$\langle W(C) \rangle \sim \exp\left(-kp(C)\right)$$

where $p(C)$ is the perimeter of the path C and k is some constant. In this sense, the string tension σ can be seen as an order parameter of confinement [2]: it is finite in a confined phase, and zero in a deconfined phase.

The above confinement criteria only hold for the pure gauge theory without fermions. In fact, with the inclusion of fermionic matter the Wilson loops lose the area-law behaviour [2] and one needs different confinement criteria [45]. However, one should note that confinement is a property of the *pure* gauge theory. There is a useful analogy with superconductivity [33]. Imagine placing two magnetic monopoles inside a superconductor. In the normal phase, the magnetic flux lines can spread out as usual, giving the Coulomb behaviour of the potential between the charges. In the superconducting phase, on the contrary, the flux lines are prevented from spreading out due to the Meissner effect. The result is that the flux lines are now forced into a collimated flux tube between the two monopoles, giving a much stronger linear potential. In this analogy, confinement is the statement of the Meissner effect, i.e. the expulsion of magnetic fields, and as such, it is a property of the gauge fields, rather than of the specifics of the matter.

We now mention some of the main results obtained in the path-integral approach. With some effort, one can use ensemble averages to compute the energy of excited states of lattice gauge theories [6], but we will not consider this in detail. We are mainly interested in the phases of lattice gauge theories as a function of the coupling g. While finite temperature gauge theory is quite interesting [7], in the present work we limit ourselves to zero temperature. In the $U(1)$ case, one finds that four is a critical spacetime dimension for the phase structure [43]: in lower spacetime dimensions the theory shows a single, confined phase, while in four or higher dimensions it shows a first-order transition between a confined phase at strong coupling (g large) and a deconfined, massless phase at weak coupling (g small). The situation in the non-Abelian case is more complicated. However, in four dimensions both SU(2) [46] and SU(3) [43] show a single, confined phase. The $SU(2)$ lattice gauge theory shows a single confined phase in a lower number of dimensions [17, 47], while it shows a deconfinement transition in five dimensions [47].

As noted previously, lattice gauge theories in the path-integral approach can be entirely formulated in terms of group-valued quantities only. Therefore lattice gauge theories based on finite gauge groups can be formulated and studied. In [48] the $\mathbb{Z}_2$ lattice gauge theory was introduced, while [32] computed the energy of excitations of SU(2) lattice gauge theory by approximating SU(2) with one of its finite subgroups. Finally, in [31] various $\mathbb{Z}_N$ and non-Abelian gauge theories were studied in four spacetime dimensions. The author found that all the non-Abelian theories show a two-phase structure

with a first-order deconfinement transition. The same is observed in the $\mathbb{Z}_N$ theories for $N \leq 4$, while for $N > 4$ an additional intermediate "spin-wave" non-confining phase is observed (also see [43]).

Finally, it should be noted that the phase diagram of a theory is not a property of the gauge group or the dimensionality, but it depends crucially on the precise discretisation of the action. In fact, different lattice actions with the same continuum limit may have different phase diagrams [43].

2.2 Hamiltonian lattice gauge theory

As we've explained in the Introduction, we need to formulate the lattice gauge theory in the Hamiltonian formalism. As we have seen in section 1.3, in the temporal gauge $A_0 = 0$ we can write the continuum Hamiltonian as

$$H = \int d^d x \ \mathrm{tr} \left(g^2 \boldsymbol{\pi}^2 + \frac{1}{g^2} \mathbf{B}^2 \right)$$

where $\boldsymbol{\pi} = \mathbf{E}/g^2$ is the conjugate momentum to the gauge field $\mathbf{A}$. We choose to work with $\boldsymbol{\pi}$ instead of $\mathbf{E}$ to avoid the cumbersome presence of the coupling in the commutation relations. As the price to pay for choosing the temporal gauge, we need to impose the constraint $D_i E_i = 0$ on physical states. We will limit ourselves here to pure gauge theory without fermions. We will consider fermions on the lattice in section 2.3.

In the Hamiltonian formulation time is kept continuous, while space is discretised in the same manner as in the Euclidean path-integral approach. We see that the gauge choice $A_0 = 0$ is especially convenient: we know that the gauge field A_μ (or, equivalently, its group-valued version U_μ) lives on the links of the lattice. However, if time is kept continuous, we have no links in the continuous time direction, and it's unclear how to treat the time component of the gauge field: the choice $A_0 = 0$ solves this problem immediately.

The classic formulation of Hamiltonian lattice gauge theory is due to Kogut and Susskind [15]. The Kogut-Susskind Hamiltonian, which we'll derive below, is to be thought of as the Hamiltonian corresponding to the Wilson action. In fact, the former can be obtained from the latter by the transfer matrix procedure [2]: one assigns two different lattice spacings to time and spatial dimensions, and then the limit where the time spacing goes to zero is taken. A simpler derivation, through a *lattice* Legendre transform, is also possible [49]. Here we we follow the setup of a more modern approach [42], but we provide full justification of the various results and a derivation of the Hamiltonian.

We will construct separately the magnetic and electric part of the Hamiltonian, in correspondance with the continuum Hamiltonian:

$$H = H_E + H_B$$

The requirements that we place on the Hamiltonian are the same that we set forth in the path-integral approach: it needs to be gauge-invariant, and it must have the correct continuum limit.

From the path-integral approach we remember (eq. (2.3)) that in the continuum limit the sum over plaquettes Wilson loops gives a Euclidean sum over the square of the curvature tensor (see also eq. (2.4)). Since $\mathbf{B}^2 = \frac{1}{2}F_{ij}F^{ij}$, which is a Euclidean sum over the spatial components, the magnetic part of the Hamiltonian will again be given by the same construction using Wilson loops of plaquettes, only that now we only take plaquettes in the spatial directions. As such, the magnetic term does not give rise to particular issues. The electric field term is more complicated, as $\mathbf{E}^2 = F_{0i}F^{0i}$ but now time is continuous, so we can't construct plaquettes extending in the time direction, and therefore we can't use Wilson loops to construct the electric field term. Moreover, in contrast with the path-integral approach, now both the electric field and the plaquette term are operators, rather than simple variables: as such, we first need to define an appropriate Hilbert space.

Classically, we assign a matrix $U \in \mathrm{SU}(N)$ to each link, and therefore the configuration space on each link is precisely $\mathrm{SU}(N)$. Since different links are independent, the overall configuration space is the product of one copy of $\mathrm{SU}(N)$ on each link, [42]

$$\prod_{\text{links}} \mathrm{SU}(N) = \underbrace{\mathrm{SU}(N) \times \cdots \times \mathrm{SU}(N)}_{\#\,\text{links}}$$

The system can be quantised immediately by considering wavefunctions from the configuration space to the complex numbers, i.e.

$$\psi : \prod_{\text{links}} \mathrm{SU}(N) \to \mathbb{C}$$

This means that states in the Hilbert space of each link are given by

$$|\psi\rangle = \int dU\, \psi(U)\, |U\rangle$$

where $\{|U\rangle\}$ is the group element orthonormal basis, which can be though of as a "position basis" on the group. The $\psi(U)$ is a wavefunction, and as such it must be square-integrable. The Hilbert space on each link can then be identified with $L^2(\mathrm{SU}(N))$, i.e. the space of square-integrable functions on $\mathrm{SU}(N)$ [42]. The overall Hilbert space is then given by the tensor product

$$\mathcal{H} = \bigotimes_{\text{links}} L^2(\mathrm{SU}(N))$$

One can then define *position observables* $\hat{u}_{mn}$ on the single-link Hilbert space via

$$\hat{u}_{mn}\, |U\rangle = U_{mn}\, |U\rangle$$

where U_{mn} is the mn element of the matrix U taken in the fundamental representation of SU(N). If one wishes to include fermions in the theory, this position operator needs to be modified in a manner to output the matrix elements in the appropriate representation, and the rest of the discussion carries on unchanged. One can then define a matrix of operators $\hat{u}$, whose elements are precisely the operators $\hat{u}_{mn}$. In the defining representation of SU(N) it is an $N \times N$ matrix, but this depends on the chosen representation. Note that while $\hat{u}$ is unitary as long as the chosen representation is unitary (which can always be done for a compact Lie group, see appendix A), its elements $\hat{u}_{mn}$ are not unitary, and are not even necessarily invertible. In fact, a short calculation using the above definition shows that $(\hat{u}_{mn})^\dagger = \left(\hat{u}^\dagger\right)_{nm}$.

We can now construct the magnetic part of the Hamiltonian. As we noted before, since $\mathbf{B}^2 = \frac{1}{2}F_{ij}F^{ij}$, the magnetic part of the Hamiltonian is again given by the same construction using Wilson loops of plaquettes, only that now we only take plaquettes in the spatial directions. We can then construct spatial Wilson loops in our Hilbert space. Consider a path $C = \langle e_1, \ldots, e_k \rangle$ where e_i are links. Then define

$$\hat{u}_{mn}(e) = \begin{cases} \hat{u}_{mn} & \text{if } C \text{ traverses } e \text{ in the direction of } e \\ \left(\hat{u}^\dagger\right)_{mn} & \text{if } C \text{ traverses } e \text{ in the direction opposite to } e \end{cases}$$

This is the same definition of as in the path-integral approach, where we assigned a matrix and its Hermitian conjugate to a link traversed in the positive and negative orientations respectively, albeit in terms of operators. In particular, with this definition, if the path in the Wilson line traverses a link back and forth consecutively, then the corresponding terms cancel each other. We then define the *Wilson line observable* $\hat{W}_{mn}(C)$ as follows [42]:

$$\hat{W}_{mn}(C) \equiv \sum_{j_1, j_2, \ldots j_{k-1}} \hat{u}_{mj_1}(e_1)\hat{u}_{j_1 j_2}(e_2) \cdots \hat{u}_{j_{k-1} n}(e_k)$$

In other words, we take the matrix product of $\hat{u}$ matrix operators on successive links, leaving the first and last matrix index explicit. Note that the case where the path self-intersects is allowed. The Wilson loop $\mathrm{tr}(\hat{W}_C)$ then corresponds to the case when C is a closed path, where, moreover, we take a trace over the two free indices,

$$\mathrm{tr}(\hat{W}_C) \equiv \sum_m \hat{W}_{mm}(C)$$

As already noted, the conventions to define Wilson loops can vary: some prefer to include the trace in the definition, some do not. We denote the plaquette Wilson loop with $\mathrm{tr}\, \hat{W}_\square$.

Remembering the result for the continuum limit of the Wilson loop, eq.s (2.3) and (2.4), the magnetic part of the Hamiltonian with the correct continuum limit is then given by

$$H_B = -\frac{1}{g^2 a^{4-d}} \sum_\square \left(\mathrm{tr}(\hat{W}_\square) + \mathrm{tr}(\hat{W}_\square)^\dagger \right)$$

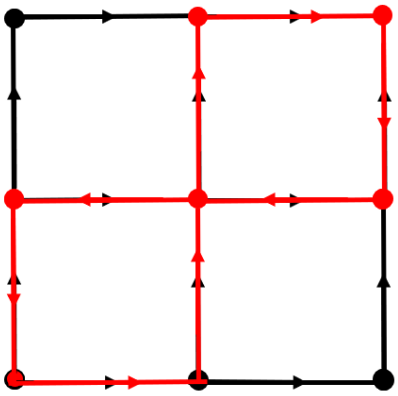


Figure 2.4: A closed loop on the lattice. The black arrows show the lattice orientation, while the red arrows show the direction in which links are traversed. Self-intersections are allowed.

where the sum is taken over spatial plaquettes $\square$ and d is the number of spatial dimensions. Some authors prefer to add a constant term to each plaquette, so that the Hamiltonian vanishes on configurations whose plaquettes are equal to the identity [2, 17]. As in section 2.1, we need an extra factor of $1/2$ in front for U(1), in order to get the correct continuum limit.

The electric field term is more complicated, as $\mathbf{E}^2 = F_{0i}F_{0i}$ but now time is continuous, so we can't construct plaquettes extending in the time direction, and therefore we can't use Wilson loops to construct the electric field term. However, we remember from the continuum formulation in section 1.3 that in the $A_0 = 0$ gauge, the conjugate momentum $\boldsymbol{\pi}$ is the infinitesimal generator of translations in gauge field space. On the lattice, we can then look at the generator of infinitesimal translations in U space.

Since the gauge group is non-Abelian two different translation operators can be defined on the group, i.e. left translation and right translation,

$$L_U |V\rangle = |UV\rangle \qquad R_U |V\rangle = |VU^{-1}\rangle$$

where L_U and R_U are, respectively, left and right "translations" in the analogy where $\{|U\rangle\}$ is a "position basis" for the group. Note that the inverse in the definition of R_U is necessary to ensure that the family of operators R_U satisfies the group law $R_U R_V = R_{UV}$, like the L do. In fact, the L_U and R_U commute with each other, and they are unitary operators. We'll check this for L_U, and it suffices to do so on a basis. Given two states $|V_1\rangle$ and $|V_2\rangle$, under L_U we have

$$|V_1\rangle \mapsto L_U |V_1\rangle = |UV_1\rangle$$

and similarly $|V_2\rangle \mapsto |UV_2\rangle$ so that their inner product

$$\langle V_1|V_2\rangle \mapsto \langle UV_1|UV_2\rangle = \langle V_1|V_2\rangle$$

where we have used the fact that the position basis is orthonormal, and that $V_1 = V_2$ if and only if $UV_1 = UV_2$. This shows that $L_U^\dagger L_U = 1$, and a similar calculation shows that $R_U^\dagger R_U = 1$. From the group transformation law it is not hard to see that $(L_U)^{-1} = L_{U^{-1}}$ so that we also have

$$L_U^\dagger = (L_U)^{-1} = L_{U^{-1}} = L_{U^\dagger}$$

Therefore L_U and also R_U (or rather $U \mapsto L_U$ as a map from $\mathrm{SU}(N)$ to operators on $L^2(\mathrm{SU}(N))$) are infinite-dimensional unitary representations of $\mathrm{SU}(N)$. In the mathematical community, they are known as the *left* and *right regular* representations [50, 51, 52]. As they are infinite-dimensional and $\mathrm{SU}(N)$ is compact, they must be reducible (see appendix A). These representations can be defined for any group, and in the Abelian case they are the same representation, $L_U = R_U^{-1}$. The properties of the regular representations are very special, and at the heart of the representation theory of compact Lie groups (see appendix A).

The action of say, L_U, is akin to the action of the translation operator in ordinary quantum mechanics, $e^{ix\hat{p}}$, which operates a translation by x in position space. One can then try to define "momentum" operators on the group based on this analogy, which are the infinitesimal generators of translations that we're looking for. This corresponds to finding a Lie algebra representation corresponding to the regular representation of the Lie group. As usual [52], for a Lie algebra element X, this is defined as the linear map $X \to \hat{\ell}(X)$ from the Lie algebra to operators on $L^2(\mathrm{SU}(N))$ such that

$$L_{e^{iX}} = \exp\left(i\hat{\ell}_L(X)\right) \qquad R_{e^{iX}} = \exp\left(i\hat{\ell}_R(X)\right)$$

As we have left and right regular Lie group representations, we'll also have left and right Lie algebra representations. As we will see later, it doesn't matter which one we use, so we choose to work with the left regular representations. All the calculations can be carried out in a similar manner with the right regular representations. By virtue of the L_U being unitary, the $\hat{\ell}_L(X)$ are Hermitian. Moreover, since L_U is a Lie group representation, it follows by construction that $\hat{\ell}_L(X)$ is a Lie algebra representation [52]. If the Lie algebra has Hermitian generators $\{T^a\}$, satisfying the commutation relation

$$[T^a, T^b] = if^{abc}T^c$$

then we can define the corresponding momentum operators for the various directions,

$$\hat{\ell}_L^a \equiv \hat{\ell}_L(T^a)$$

and these, by virtue of being a Lie algebra representation, satisfy

$$[\hat{\ell}_L^a, \hat{\ell}_L^b] = if^{abc}\hat{\ell}_L^c$$

It is sometimes useful to retrieve the $\hat{\ell}^a_L$ from the L_U by differentiation. In this case, a translation in the direction of a specific generator T^a is defined as L_U where $U = e^{i\epsilon T^a}$ for some ϵ. Then the momentum operators can be explicitly obtained by differentiation:

$$\hat{\ell}^a_L \equiv -i\frac{d}{d\epsilon}L_{e^{i\epsilon T^a}}\bigg|_{\epsilon=0}$$

The operators $\hat{\ell}^a_L$ will play the role of the lattice $\boldsymbol{\pi}$ operators, as both are infinitesimal generators of translations in gauge field space. Both $\hat{\ell}^a$ and $\boldsymbol{\pi}$ carry a colour index, while $\boldsymbol{\pi}$ carries a spatial index, unlike $\hat{\ell}^a$. In fact, the discussion about translations has been based on the single-link Hilbert space, so the spatial direction in the lattice is indicated simply by the link at which the $\hat{\ell}^a$ operators are defined. In particular, $\hat{\ell}^a_L$ corresponds to the link traversed in the positive orientation, while $\hat{\ell}^a_R$ corresponds to the link traversed in the negative orientation.

We can then obtain the commutation relations satisfied by the lattice chromoelectric field $\hat{\ell}^a_L$ with the other operator that we have defined, i.e. $\hat{u}$. We find that

$$[\hat{\ell}^a_L, \hat{u}_{ij}] = -[T^a\hat{u}]_{ij} \tag{2.5}$$

or equivalently,

$$[\hat{\ell}^a_L, \hat{u}] = -T^a\hat{u} \tag{2.6}$$

This can be shown by starting from the translation operator and noting that on a basis,

$$\begin{aligned}
[L_U, \hat{u}_{ij}]\,|V\rangle &= (L_U\hat{u}_{ij} - \hat{u}_{ij}L_U)\,|V\rangle = \\
&= L_U V_{ij}\,|V\rangle - \hat{u}_{ij}\,|UV\rangle = \\
&= V_{ij}\,|UV\rangle - (UV)_{ij}\,|UV\rangle = \\
&= (V_{ij} - (UV)_{ij})L_U\,|V\rangle
\end{aligned}$$

Now setting $U = \exp(i\epsilon T^a)$ and differentiating, we obtain

$$[\hat{\ell}^a_L, \hat{u}_{ij}]\,|V\rangle = -[T^aV]_{ij}\,|V\rangle$$

Using the $\hat{u}$ operator, we can rewrite the right hand side without reference to $|V\rangle$, and we obtain the above commutation relation on any $|V\rangle$. Since it is satisfied on a basis, it is valid as an operator equation.

We can now consider the continuum limit of the lattice chromoelectric field operator. This is necessary in order to make sure that the electric part of the Hamiltonian is normalised correctly. From the above commutation relation, on the space of wavefunctions

$$\hat{\ell}^a = -[T^aU]_{ij}\frac{\partial}{\partial U_{ij}}$$

We can then note that

$$\frac{\partial}{\partial A^a} U = \frac{\partial}{\partial A^a} \exp\left(iaA^b T^b\right) = iaT^a U$$

It follows that we can write

$$\hat{\ell}^a = -\frac{1}{ia} \frac{\partial U_{ij}}{\partial A^a} \frac{\partial}{\partial U_{ij}} = -\frac{1}{ia} \frac{\partial}{\partial A^a}$$

We now need to relate the partial derivative with the functional derivative that appeared in the definition of the conjugate momentum $\boldsymbol{\pi}$ in the continuum. This follows from considering the respective actions. On the lattice, space is simply a discrete index, so that

$$\frac{\partial}{\partial A^a(x)} A^b(y) = \delta_{ab}\delta_{xy}$$

while in the continuum we have a proper delta function on the right hand side:

$$\frac{\delta}{\delta A^a(x)} A^b(y) = \delta_{ab}\delta(x-y)$$

The relation between the continuum and discrete delta is, in the continuum limit

$$\delta_{xy} \approx a^d \delta(x-y)$$

This gives the correct dimensions; intuitively, the two formulas agree for $x \neq 0$ as they both give zero. For $x = y$, the discrete delta is equal to 1, while the continuum delta is infinite, and in the continuum limit the vanishing term in front cancels the infinity. This leads to

$$\frac{\partial}{\partial A^a(x)} \approx a^d \frac{\delta}{\delta A^a(x)}$$

The continuum limit of the lattice chromoelectric field is then

$$\hat{\ell}^a_L \to -a^{d-1}\pi^a \left[1 + \mathcal{O}(a)\right]$$

This is also dimensionally correct, as $\hat{\ell}$ is dimensionless by construction. We'll now show that this reproduces the correct continuum commutation relations. At the same link and spacetime point,

$$[\hat{\ell}^a_L, \hat{u}] = -T^a \hat{u}$$

implies

$$[a^{d-1}\pi^a(1 + \mathcal{O}(a)), 1 + iaA^b T^b + \mathcal{O}(a^2)] = -T^a(1 + \mathcal{O}(a))$$

Then we can carry on the calculation to get

$$-[\pi^a, iA^b]T^b + \mathcal{O}(a) = -T^a a^{-d}(1 + \mathcal{O}(a))$$

We recognise a^{-d} as the dimensions of the d-dimensional spatial delta function present in the continuum commutation relation. Then restoring the spatial indices and spatial points, the commutation relation becomes

$$[\pi_i^a(\mathbf{x}), A_j^b(\mathbf{y})]T^b = -iT^a\delta_{ij}\delta(\mathbf{x}-\mathbf{y})$$

where the sum over b is implied. Since the T^a are a basis of the Lie algebra,

$$[A_i^a(\mathbf{x}), \pi_j^b(\mathbf{y})] = i\delta_{ij}\delta_{ab}\delta(\mathbf{x}-\mathbf{y})$$

where we also exchanged A and π in the commutator. This is the same as the one we found in section 1.3.

Then one can define the "Laplacian" Δ_e at link e, given by

$$\Delta_e = \sum_a \left(\hat{\ell}_L^a\right)^2 = \sum_a \left(\hat{\ell}_R^a\right)^2$$

The reason why these two are equal is that they have the same action on $L^2(\mathrm{SU}(N))$, but this will be shown later, in section 3.1. The name "Laplacian" is chosen in analogy with ordinary quantum mechanics, where the square of the momentum operator, the generator of translations, is indeed the ordinary Laplacian.

With the scaling we found, the correct form for the electric Hamiltonian is

$$H_E = \frac{g^2}{2a^{d-2}} \sum_{e\in\text{links}} \Delta_e$$

where the sum is taken over links.

Therefore, overall, the pure gauge Kogut-Susskind Hamiltonian is given by

$$H = \frac{g^2}{2a^{d-2}} \sum_{\text{links}} \sum_a \left(\hat{\ell}_L^a\right)^2 - \frac{1}{g^2 a^{4-d}} \sum_{\square} \left(\mathrm{tr}(\hat{W}_\square) + \mathrm{tr}(\hat{W}_\square)^\dagger\right) \tag{2.7}$$

with the commutation relations

$$[\hat{\ell}_L^a, \hat{\ell}_L^b] = if^{abc}\hat{\ell}_L^c$$

$$[\hat{\ell}_L^a, \hat{u}] = -T^a\hat{u}$$

which are the same as those given by other authors [2, 15, 42], as long as the different definitions are taken into account. As noted previously, sometimes an additional constant term is added to the magnetic Hamiltonian. As different definitions are sometimes used for the lattice chromoelectric field and for the plaquette term, the commutation relations and the normalisations of the various terms may differ in different texts.

Now that we have found the Hamiltonian, it only remains to write down the gauge transformations. A local gauge transformation in this formalism is given as before by a choice of matrix $U_v \in \mathrm{SU}(N)$ at every site v of the lattice [42]. Then the appropriate transformation on each vertex v, which gives the same transformation as the one previously seen, is given by

$$G_v = \bigotimes_{e_-=v} L_U(e) \bigotimes_{e_+=v} R_U(e)$$

where e_- denotes the source vertex to link e and e_+ denotes the target vertex to link e. In order words, we apply L_U to those matrices which live in the directed links which start from v and we apply R_U to those matrices which live in the directed links which end at v. This clearly gives the same prescription as we've previously seen.

Once this law is applied at all vertices, we will have obtained the overall gauge transformation $\bigotimes_v G_v$. This obviously commutes with the magnetic Hamiltonian. It also commutes with the electric Hamiltonian, as we will see in section 3.1. Therefore the Kogut-Susskind Hamiltonian is gauge-invariant. The physical Hilbert space is then selected from the overall Hilbert space by requiring that states in the physical Hilbert space be gauge-invariant, i.e. $|\psi\rangle$ is in the physical Hilbert space if

$$\bigotimes_v G_v |\psi\rangle = |\psi\rangle$$

This requirement is also called "Gauss' law", as in the continuum limit it reduces to the constraint on physical states that we saw in the continuum theory.

If one were to add fermions, the overall Hilbert space would be enlarged to the tensor product of the pure gauge overall Hilbert space with the fermions' Hilbert space [23]. Then the requirement of gauge-invariance will involve both spaces at the same time, and a condition akin to the previous one will select the physical Hilbert space. The condition would then translate in the continuum to Gauss' law on physical states, where now one has a non-zero charge density carried by the fermions.

The gauge-invariant subspace

We conclude this section with a description of the gauge-invariant subspace. A basis for the gauge-invariant subspace in the pure gauge case is given by all Wilson loops corresponding to any closed path on the lattice, where we allow self-intersections [15]. If we include fermions, then we also allow open Wilson lines, as long as they begin with a fermion and terminate with another fermion, in a way that the gauge transformation matrices all cancel.

The issue with this basis of states is that it is highly overcomplete [53]. This is particularly striking in the finite-group case, where the Hilbert space is finite dimensional, and

even on a finite lattice we have infinitely many Wilson loops [36]. There is no known general way of eliminating the redundant Wilson loops, although the redundancy conditions can be formulated for Lie groups as non-linear trace identites known as "Mandelstam constraints" [53]. In some highly simplified situations, for example in section 4.3, we will be able to obtain a complete description of the physical Hilbert space.

2.2.1 Methods and main results

While most numerical calculations have been performed in the path-integral approach, the Hamiltonian formulation of lattice gauge theories has also attracted some attention. It has mainly been used for the analytical calculation of the string tension [54] and for the determination of the energies of excited states in the confined phase [17, 18, 37, 55].

It is important to note that the results obtained from the Hamiltonian formulation and from the path-integral formulation may not be directly compared. For example, the coupling constants in the Hamiltonian and in the action are not in principle the same. The results obtained with the Euclidean and Hamiltonian methods may however be compared via the renormalisation group [56]. The methods required for the comparison are not however applicable to the case of finite groups, and it is such unclear how to compare results obtained with the Hamiltonian and path-integral approaches in the finite-group case.

2.3 Fermions on the lattice

In order to describe a theory such as Quantum Chromodynamics on the lattice, one needs to discretise the fermionic action. However, this turns out to be non-trivial, due to the issue known as *fermion doubling*. This essentially consists in the appearance of unphysical degrees of freedom with the naive discretisation of the action [33, 57]. In order to illustrate the problem, we will compute the momentum-space propagator on the lattice for the naive discretisation of the free fermionic action.

In the continuum Euclidean formulation, the free Dirac action is

$$S = \int d^{d+1}x\, \bar{\psi}\left(\gamma^\mu \partial_\mu + m\right)\psi$$

Then the integral will be approximated by a sum and the derivative by a finite difference,

$$\int d^{d+1}x \to a^{d+1} \sum_x$$

$$\bar{\psi}\gamma^\mu \partial_\mu \psi \to \sum_\mu \bar{\psi}\gamma^\mu \frac{\psi(\mathbf{x} + a\hat{\mu}) - \psi(\mathbf{x} - a\hat{\mu})}{2a}$$

So that overall the "naive" lattice fermionic action is given by

$$S = a^{d+1} \sum_x \left[m\bar{\psi}(x)\psi(x) + \frac{1}{2a} \sum_\mu \left(\bar{\psi}(x)\gamma^\mu \psi(x + a\hat{\mu}) - \bar{\psi}(x + a\hat{\mu})\gamma^\mu \psi(x) \right) \right] \tag{2.8}$$

where we have redefined $x - a\hat{\mu} \to x$ in the last term to make everything symmetric. Now we can go to momentum space,

$$\psi(x) = \int_{BZ} \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{ikx} c_k$$

where the integral is taken over the Brillouin zone, $[-\pi/a, \pi/a]^{d+1}$ [41]. A short computation shows that

$$\sum_x \bar{\psi}(x)\psi(x) = \int_{BZ} \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{c}_k c_k$$

and similarly

$$\sum_x \bar{\psi}(x)\gamma^\mu \psi(x + a\hat{\mu}) = \int_{BZ} \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{ik_\mu a} \bar{c}_k \gamma^\mu c_k$$

so that the momentum space action is

$$S = a^{d+1} \int_{BZ} \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{c}_k D(k) c_k$$

where

$$D(k) = m + i \sum_\mu \gamma^\mu \frac{\sin(k_\mu a)}{a}$$

The propagator is then the inverse of $D(k)$, giving

$$D(k)^{-1} = \frac{m - i\sum_\mu \gamma^\mu \sin(k_\mu a)/a}{m^2 + \sum_\mu \sin(k_\mu a)^2/a^2}$$

We see that naively, we recover the usual fermion propagator in the limit $a \to 0$. However, consider for a moment the case $m = 0$. We know that physical particles correspond to poles in the propagator [1]. However, with the above lattice propagator, these occur not only near $k_\mu = 0$, which corresponds to the physical particle spectrum, but also at the edges of the Brillouin zone, at momentum vectors

$$k_\mu = (\pm\pi/a, 0, 0, \ldots), (0, \pm\pi/a, 0, \ldots), \ldots$$

which correspond to two unphysical particle excitations per dimension, giving a total of 2^{d+1} excitations in $d + 1$ dimensions. This problem is known as *fermion doubling*

[33, 57]. It definitely occurs also in the case $m \neq 0$, only that since we're in a Euclidean theory now the propagator does not have poles. One could nonetheless imagine moving to Minkowski time, and now we would again have unphysical massive particles showing up as poles in the propagator.

In other words, not only do we have excitations with the correct continuum limit near $k = 0$, but also near $k = \pm\pi/a$. In fact, mathematically, the continuum limit doesn't hold everywhere in the Brillouin zone. This is easier to see in $0 + 1$ dimensions, where

$$D(k)^{-1} = \frac{m - i \sin(ka)/a}{m^2 + \sin(ka)^2/a^2}$$

Then at some definite points, for example $k = \pm\pi/(2a)$ or at the edges, $k = \pm\pi/a$, the given propagator does not reduce to the usual fermion propagator. The correct continuum limit is achieved at most in the central region $(-\frac{\pi}{2a}, \frac{\pi}{2a})$ of the Brillouin zone. In an attempt to investigate what happens at the edges, we can compute the real space propagator

$$G(x) = \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} D(k)^{-1} e^{ikx}$$

Due to the above issues, we will see that the naive continuum limit does not give the correct continuum propagator. Instead, one can split the integration into a central region, $[-\frac{\pi}{2a}, \frac{\pi}{2a}]$, and two edge regions, $[-\frac{\pi}{a}, -\frac{\pi}{2a}]$ and $[\frac{\pi}{2a}, \frac{\pi}{a}]$. The integral over the first edge region will be given by

$$\int_{-\pi/a}^{-\pi/(2a)} \frac{dk}{2\pi} D(k)^{-1} e^{ikx}$$

Under the substitution $k \to -k - \pi/a$ the momentum-space propagator is unchanged, and one gets

$$\int_{-\pi/(2a)}^{0} \frac{dk}{2\pi} D(k)^{-1} e^{ikx}$$

The substitution $k \to -k + \pi/a$ achieves a similar result for the other edge, giving

$$\int_{0}^{\pi/(2a)} \frac{dk}{2\pi} D(k)^{-1} e^{ikx}$$

so that the two edge pieces together give the same contribution as the central region. The overall result is that

$$G(x) = 2 \int_{-\pi/(2a)}^{\pi/(2a)} \frac{dk}{2\pi} D(k)^{-1} e^{ikx}$$

In this range, the sine function is well approximated by the linear function, and now in the continuum limit the lattice propagator $G(x)$ reduces to twice the continuum propagator. This is a manifestation of the presence of the doublers.

It turns out that the presence of doublers is generic, and not simply due to a bad choice of discretisation. If one wants to preserve locality and chiral symmetry when $m = 0$, then doublers necessarily appear [33, 57], a result known as the Nielsen-Ninomiya theorem [33].

Note that the doublers also appear in the Hamiltonian formalism, only that here we will have only 2^d doublers in $d + 1$ dimensions, rather than 2^{d+1} doublers in the action formulation, because time is continuous and not discrete [2].

On the other hand, there is no issue with placing other matter fields on the lattice, for example scalar fields [33]. In this case, some short calculations akin to the ones we've performed before lead to the momentum-space lattice propagator in $0 + 1$ dimensions,

$$D(k)^{-1} = \left[m^2 + \frac{\sin\left(ka/2\right)^2}{(a/2)^2} \right]^{-1}$$

In this case, we see that as k varies over the Brillouin zone, the argument of the sine ranges over $[-\frac{\pi}{2}, \frac{\pi}{2}]$, where it is well approximated by the linear function. In this case, we have no doublers.

There are many ways to avoid or mitigate doublers on the lattice, but any prescription will have to break some of the assumptions of the Nielsen-Ninomiya theorem [57]. For example, Wilson fermions [3] break the chiral symmetry on the lattice by adding a term in the action which vanishes in the continuum limit. Here we will follow a different approach, called *staggered fermions*, or *Kogut-Susskind fermions* [15, 33].

The idea is to mildly break locality by spreading the fermionic degrees of freedom over multiple lattice sites [33]. For example, in two dimensions, if instead of placing a fermion on each lattice site, we place particles on "even" sites and antiparticles on "odd" sites, we have effectively doubled the size of the lattice for each kind of fermion, leading to a dispersion relation similar to that of scalar fields, so that the sine is well-approximated by a linear function on the whole Brillouin zone.

To obtain the staggered fermions [33], we can define a new fermionic species $\chi(x)$ such that

$$\psi(x) = \prod_\mu (\gamma^\mu)^{x_\mu} \chi(x)$$

By $(\gamma^\mu)^{x_\mu}$ we mean here $(\gamma^\mu)^{n_\mu}$ where $x_\mu = a n_\mu$, so that n_μ is an integer. Since in both Minkowski and Euclidean space $(\gamma^\mu)^2 = \pm 1$, each gamma matrix is contained in the above prefactor above either once or not at all, alternating between neighbouring sites, possibly with some extra signs.

Since the prefactor contains all the gamma matrices, the action is diagonalised in spinor space. In fact we see that, for example in Euclidean spacetime,

$$\bar{\psi}(x)\psi(x) = \chi^\dagger \left(\prod_\mu (\gamma^\mu)^{x_\mu} \right)^\dagger \gamma^4 \prod_\mu (\gamma^\mu)^{x_\mu} \chi(x) = (-1)^{\eta_1(x)} \bar{\chi}(x)\chi(x)$$

where $\eta_1(x)$ is determined by the commutation of the various gamma matrices. The other term in the action can be similarly treated, as

$$\bar{\psi}(x)\gamma^\mu\psi(x+a\hat{\mu}) = (-1)^{\eta_2(x,\mu)}\bar{\chi}(x)\chi(x+a\hat{\mu})$$

where the $x+a\hat{\mu}$ causes an extra γ^μ in the prefactor, which cancels the γ^μ in front. The sign $\eta_2(x,\mu)$ is again determined by the commutation of the various gamma matrices among themselves. The action is now given by

$$S = a^{d+1}\sum_x \Bigg[m(-1)^{\eta_1(x)}\bar{\chi}(x)\chi(x) + \\ + \frac{1}{2a}\sum_\mu (-1)^{\eta_2(x,\mu)}\left(\bar{\chi}(x)\chi(x+a\hat{\mu}) + \bar{\chi}(x+a\hat{\mu})\chi(x)\right)\Bigg]$$

which is diagonal in spinor space. We can then discard the multiple spin degrees of freedom on each site [33]. The end result is that we still have all the doublers, but now each doubler contributes only one spin degree of freedom, and the different doublers play the role of spin components of the staggered fermions [33].

Once the issue of fermion doubling has been addressed, it remains to show how to couple the fermions to the gauge fields. The mass term in the action is unchanged. Instead, the derivative term will have to be modified in a manner so as to reduce to the covariant derivative in the continuum limit. Since the gauge fields live in the lattice links, and in the lattice theory we have a matrix $U_\mu(x)$ on each link, there is essentially only one way of adding these matrices to the above action [2, 33], that is

$$S = a^{d+1}\sum_x \Bigg[m(-1)^{\eta_1(x)}\bar{\chi}(x)\chi(x) + \\ + \frac{1}{2a}\sum_\mu (-1)^{\eta_2(x,\mu)}\left(\bar{\chi}(x)U_\mu(x)\chi(x+a\hat{\mu}) + \bar{\chi}(x+a\hat{\mu})U_\mu^\dagger(x)\chi(x)\right)\Bigg]$$

This lattice action has the correct continuum limit, and it's easy to see that it has precisely the correct form to ensure gauge invariance [2, 33]. Moreover, note that the staggering procedure doesn't impact the coupling to the gauge fields.

The generalisation to the Hamiltonian formulation is straightforward. Suppose we consider the Minkowski-space continuum Dirac theory in $d+1$ dimensions. Then the quantum Hamiltonian is given by

$$H = \int d^d\mathbf{x}\ \left[\bar{\psi}(-i\nabla\!\!\!/ + m)\psi\right]$$

where $\{\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{y})\} = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y})$ and all other anticommutators vanish. The naive fermion Hamiltonian is obtained via the discretisation,

$$\int d^d\mathbf{x} \to a^d \sum_{\mathbf{x}}$$

$$\bar{\psi}(\mathbf{x})\psi(\mathbf{x}) \to \psi^\dagger(\mathbf{x})\gamma^0\psi(\mathbf{x})$$

$$\bar{\psi}\not{\nabla}\psi \to \sum_i \psi^\dagger\gamma^0\gamma^i \frac{\psi(\mathbf{x}+a\hat{i}) - \psi(\mathbf{x}-a\hat{i})}{2a}$$

where $\mathbf{x} \in a\mathbb{Z}^d$ indexes lattice positions and $i = 1, 2, \ldots, d$ indexes lattice directions. Therefore the naive quantum lattice Hamiltonian is given by

$$H = a^d \sum_{\mathbf{x}} \Bigg[m\psi^\dagger(\mathbf{x})\gamma^0\psi(\mathbf{x}) + \\ - \frac{1}{2a}\sum_i \Big(\psi^\dagger(\mathbf{x})i\gamma^0\gamma^i\psi(\mathbf{x}+a\hat{i}) - \psi^\dagger(\mathbf{x}+a\hat{i})i\gamma^0\gamma^i\psi(\mathbf{x})\Big)\Bigg] \quad (2.9)$$

where in the last term we've sent $\mathbf{x} - a\hat{i} \to \mathbf{x}$ to ensure a more symmetric form of the Hamiltonian. The staggering procedure can be carried out in the same manner as in the action formulation, with the difference that now we only consider spatial gamma matrices [15]. Moreover, an extra matrix factor can be included to get rid of the γ^0 matrix, leading to the staggered Hamiltonian

$$H = a^d \sum_{\mathbf{x}} \Bigg[m(-1)^{\eta_1(\mathbf{x})}\chi^\dagger(\mathbf{x})\chi(\mathbf{x}) + \\ -\frac{1}{2a}\sum_i (-1)^{\eta_1(\mathbf{x})}\Big(\chi^\dagger(\mathbf{x})\chi(\mathbf{x}+a\hat{i}) + \chi^\dagger(\mathbf{x}+a\hat{i})\chi(\mathbf{x})\Big)\Bigg]$$

for some signs $\eta_1(\mathbf{x})$ and $\eta_2(\mathbf{x}, i)$, determined by the commutation of the various gamma matrices. The coupling to the gauge field is achieved in a similar manner to what we've previously seen, i.e.

$$H = a^d \sum_{\mathbf{x}} \Bigg[m(-1)^{\eta_1(\mathbf{x})}\chi^\dagger(\mathbf{x})\chi(\mathbf{x}) + \\ -\frac{1}{2a}\sum_i (-1)^{\eta_1(\mathbf{x})}\Big(\chi^\dagger(\mathbf{x})U_i(\mathbf{x})\chi(\mathbf{x}+a\hat{i}) + \chi^\dagger(\mathbf{x}+a\hat{i})U_i^\dagger(\mathbf{x})\chi(\mathbf{x})\Big)\Bigg]$$

as in [15]. This concludes the construction of the fermionic Hamiltonian on the lattice.

2.4 Chemical potential on the lattice

As reviewed in section 1.4, a number of interesting phenomena are described by QCD at finite chemical potential, but investigation of this region is prevented by the *sign problem*. No fully satisfactory solution to the sign problem is known. As described in the Introduction, one may approach the problem in a different way, by means of a *quantum simulation* as opposed to a computer simulation. This requires studying the introduction of a finite chemical potential on the lattice in the Hamiltonian formalism. In the action formulation, it has been shown [58] that the naive way to include a finite chemical potential (simply adding $\mu\psi^\dagger\psi$ to the action) leads to unphysical divergences, as reviewed below. In order to be able to perform quantum simulations at finite chemical potential, we need to consider whether this problem also arises in the Hamiltonian formulation. Our calculations show that the Hamiltonian does not suffer from these unphysical divergences, and we offer some reasons why this may be so.

In the Hamiltonian formulation, a finite chemical potential term of the form $-\mu\psi^\dagger\psi$, where $\psi^\dagger\psi$ is the fermion number, may added to the Hamiltonian in a manner similar to that of usual statistical mechanics [8, 41]. Therefore, one might expect that the corresponding prescription in the action formulation would be to add a term $+\mu\psi^\dagger\psi$ to the action. Employing the naive Euclidean action in 3+1 dimensions eq. (2.8), we would get

$$S = a^4 \sum_x \Bigg[m\bar{\psi}(x)\psi(x) + \mu\bar{\psi}(x)\gamma^0\psi(x) + \\ + \frac{1}{2a}\sum_\mu \left(\bar{\psi}(x)\gamma^\mu\psi(x+a\hat{\mu}) - \bar{\psi}(x+a\hat{\mu})\gamma^\mu\psi(x)\right)\Bigg]$$

It has been shown [58] that, even after removing the vacuum contribution and taking the zero-temperature limit, the energy density of the free relativistic fermion gas with the above lattice action is divergent in the continuum limit for $\mu \neq 0$.

The authors of [58] show that this is due to an improper way of placing the chemical potential on the lattice. In the continuum theory, the action with a chemical potential is given by

$$S = \int d^4x\, \bar{\psi}\left(\not{\partial} + m + \gamma^0\mu\right)\psi$$

In other words, the chemical potential formally enters the action as the "time" component of an imaginary gauge field [8]. As such, the proper lattice regularisation of the above

action would be akin to adding an imaginary gauge field in the time direction [58, 8]:

$$S = a^4 \sum_x m\bar{\psi}(x)\psi(x) + \frac{1}{2a}\left(e^{\mu a}\bar{\psi}(x)\gamma^0\psi(x+a\hat{\mu}) - e^{-\mu a}\bar{\psi}(x+a\hat{\mu})\gamma^0\psi(x)\right) + \\ + \frac{1}{2a}\sum_i \left(\bar{\psi}(x)\gamma^i\psi(x+a\hat{i}) - \bar{\psi}(x+a\hat{i})\gamma^i\psi(x)\right)$$

where the second sum is taken only over the "spatial" components. It has been shown that this action leads to no divergences [8, 58].

The question is now whether the naive addition of a chemical potential term leads to divergences in the Hamiltonian formulation. We'll perform an explicit calculation and show that there are no divergences. As in section 2.2, the quantum Hamiltonian is given by

$$H = \int d^3\mathbf{x}\ \left[\bar{\psi}(-i\not{\nabla} + m)\psi\right]$$

where $\{\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{y})\} = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y})$ and all other anticommutators vanish. Going to the lattice, we obtain the same quantum Hamiltonian as in the previous section, eq. (2.9), with the addition of a chemical potential term:

$$H = a^3 \sum_{\mathbf{x}} \Bigg[m\psi^\dagger(\mathbf{x})\gamma^0\psi(\mathbf{x}) - \mu\psi^\dagger(\mathbf{x})\psi(\mathbf{x}) + \\ - \frac{1}{2a}\sum_i \left(\psi^\dagger(\mathbf{x})i\gamma^0\gamma^i\psi(\mathbf{x}+a\hat{i}) - \psi^\dagger(\mathbf{x}+a\hat{i})i\gamma^0\gamma^i\psi(\mathbf{x})\right)\Bigg]$$

We can then expand the field in momentum space,

$$\psi(\mathbf{x}) = \int_{BZ} \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} c_{\mathbf{k}}$$

where the Brillouin zone is $BZ = \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]^3$ and the $c_{\mathbf{k}}$ are Dirac spinors satisfying analogous anticommutation relations to those of the ψ field. In continuum QFT the expansion is slightly different but equivalent. Expanding the various terms we obtain after a simple calculation

$$\sum_{\mathbf{x}} \psi^\dagger(\mathbf{x})\gamma^0\psi(\mathbf{x}) = \int_{BZ} \frac{d^3\mathbf{k}}{(2\pi)^3} c^\dagger_{\mathbf{k}}\gamma^0 c_{\mathbf{k}}$$

The other terms can be dealt with similarly:

$$\sum_{\mathbf{x},i} \psi^\dagger(\mathbf{x})i\gamma^0\gamma^i\psi(\mathbf{x}+a\hat{i}) = \sum_i \int_{BZ} \frac{d^3\mathbf{k}}{(2\pi)^3} e^{ik_i a} c^\dagger_{\mathbf{k}} i\gamma^0\gamma^i c_{\mathbf{k}}$$

Putting everything together we obtain the momentum space Hamiltonian

$$H = a^3 \int_{BZ} \frac{d^3\mathbf{k}}{(2\pi)^3} c^\dagger_{\mathbf{k}} D(\mathbf{k}) c_{\mathbf{k}}$$

where

$$D(\mathbf{k}) = m\gamma^0 - \mu + \sum_i \frac{\sin(k_i a)}{a} \gamma^0\gamma^i$$

Now we can choose a specific representation for the gamma matrices, the Dirac representation:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

so that

$$\gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

and therefore

$$D(\mathbf{k}) = \begin{pmatrix} -(\mu - m)\mathbb{1} & \sum_i \frac{\sin(k_i a)}{a}\sigma^i \\ \sum_i \frac{\sin(k_i a)}{a}\sigma^i & -(\mu + m)\mathbb{1} \end{pmatrix}$$

This can be diagonalised leading to two distinct double eigenvalues

$$\lambda_\pm = -\mu \pm \sqrt{m^2 + \sum_i \frac{\sin^2(k_i a)}{a^2}} \equiv -\mu \pm \omega(\mathbf{k})$$

Now since the fermion number operators at different sites commute with each other, the overall partition function will be the infinite product of the single-momentum partition functions [59]:

$$Z = \prod_{\mathbf{k}} Z_{\mathbf{k}}$$

In other words,

$$\log Z = V \int_{BZ} \frac{d^3\mathbf{k}}{(2\pi)^3} \log Z_{\mathbf{k}}$$

The single momentum partition function is composed of the four commuting number operators for the single fermion components of the Dirac spinor, and as such it reduces to the product of the single-component partition functions:

$$Z_k = \left(1 + e^{-\beta\lambda_+}\right)^2 \left(1 + e^{-\beta\lambda_-}\right)^2$$

After substituting and rearranging some terms we see that

$$Z_k = 2\beta\omega(\mathbf{k}) + 2\log\left(1 + e^{-\beta(\omega(\mathbf{k})-\mu)}\right) + 2\log\left(1 + e^{-\beta(\omega(\mathbf{k})+\mu)}\right)$$

So that we finally obtain

$$\log Z = 2V \int_{BZ} \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\beta\omega(\mathbf{k}) + \log\left(1 + e^{-\beta(\omega(\mathbf{k})-\mu)}\right) + \log\left(1 + e^{-\beta(\omega(\mathbf{k})+\mu)}\right)\right]$$

where we remind the reader that

$$\omega(\mathbf{k}) = \sqrt{m^2 + \sum_i \frac{\sin^2(k_i a)}{a^2}}$$

Naively, this expression reduces in the continuum limit to the finite-temperature partition function of the free relativistic fermion gas in the continuum [8]. However, it will in fact be contaminated by the fermion doublers and the naive continuum limit is invalid near the edges of the Brillouin zone, as we've seen in section 2.3. To compare with the result in [58], we take the same form of the energy density,

$$\epsilon = -\frac{1}{V}\frac{\partial}{\partial\beta} \log Z|_{\beta\mu=\text{fixed}}$$

which gives in our case

$$\epsilon = 2 \int_{BZ} \frac{d^3\mathbf{k}}{(2\pi)^3} \omega(\mathbf{k}) \left[-1 + \frac{1}{e^{\beta(\omega(\mathbf{k})-\mu)}+1} + \frac{1}{e^{\beta(\omega(\mathbf{k})+\mu)}+1}\right]$$

Now we remove the first term, which corresponds to the vacuum contribution, as they do in [58] and take the zero-temperature limit $\beta \to \infty$. The Fermi-Dirac distribution tends to a step function,

$$\epsilon = 2 \int_{BZ} \frac{d^3\mathbf{k}}{(2\pi)^3} \omega(\mathbf{k}) \left[\Theta(\mu - \omega(\mathbf{k})) + \Theta(-\mu - \omega(\mathbf{k}))\right]$$

where $\Theta(x)$ is the Heaviside step function, equal to 1 for positive x and zero otherwise. In order to take the continuum limit, we need to split the Brillouin zone into a central region, where ω is approximated by a linear function, and the edge regions. As described in section 2.3 for a similar case, appropriate transformations show that the two edge integrals give the same contribution of the central region. The end result is a factor of 2 per discretised dimension, giving

$$\epsilon = 16 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \omega(\mathbf{k}) \left[\Theta(\mu - \omega(\mathbf{k})) + \Theta(-\mu - \omega(\mathbf{k}))\right]$$

where now the integration is taken over only part of the Brillouin zone, $[-\pi/2a, \pi/2a]^3$. Now we can safely take the continuum limit $a \to 0$,

$$\epsilon = 16 \int_{\omega(\mathbf{k})^2 < \mu^2} \frac{d^3\mathbf{k}}{(2\pi)^3} \omega(\mathbf{k})$$

which is a finite contribution, and could be computed exactly. Therefore, the naive discretisation for the chemical potential term appears not to cause issues in the Hamiltonian formulation, at least for free fermions.

A possible explanation for this phenomenon can be found by looking at it as a regularisation issue. The two ways of putting the chemical potential in the action ("naive" and like the fourth component of a gauge field) can be seen as two possible time lattice regularisations of the corresponding term in the Hamiltonian, and only the gauge-field prescription is a proper regularisation. In fact, one can try to follow the procedure outlined by Kogut [49] to obtain the Hamiltonian from the two different actions. One first takes the continuum limit in the time direction, leaving the lattice spacing in the spatial directions fixed. Then the Hamiltonian is obtained via a "lattice" Legendre transformation. However, it is clear that the time continuum limit for the two different actions is identical, and as such they give the same Hamiltonian.

Chapter 3

Finite-Group Lattice Gauge Theory

In the previous chapter, we've seen how to formulate lattice gauge theories with a Lie gauge group in the Hamiltonian approach. As described in the present chapter, the formulation of the Hamiltonian in the modern language of representation theory allows the extension of Yang-Mills lattice gauge theories to cover the case of finite gauge groups. This was already noted in the literature. An outstanding unsolved problem with this approach is the determination of the eigenvalues of the electric energy density beyond the Abelian case. Here we propose a method to determine these eigenvalues based on the construction of a natural Laplacian operator for finite groups. The proposed method works for any finite gauge group and reproduces the available results in the literature in the Abelian case. In the final section we construct explicitly the gauge theories based on the cyclic groups $\mathbb{Z}_N$ and the dihedral groups D_N, the simplest non-Abelian groups. We also offer a discussion of other groups of interest, in particular various ways to approximate SU(2) lattice gauge theory.

3.1 Lattice Hamiltonian for a finite group

We've seen in the previous chapters that the action for the lattice gauge theory in the Euclidean path integral approach can be formulated entirely in terms of group-valued quantities. In particular, we've seen in section 2.1.1 that finite gauge groups have been previously studied in the path-integral formulation. However, the Hamiltonian formulation does not immediately admit a formulation in terms of discrete groups, as the lattice chromoelectric field operator $\hat{\ell}$ is a Lie algebra representation, but finite groups don't have associated Lie algebras. This is an inherent limitation, and not a feature of our specific approach. One way to see this is by considering the derivation of the Kogut-Susskind Hamiltonian from the Wilson action, for example with the transfer matrix formalism [2]. The lattice Hamiltonian is obtained from the lattice action via a Legendre transformation, in the limit where the lattice spacing goes to zero in the time

dimension, but is kept constant in the spatial dimensions. Since the chromoelectric field is associated with the time components of the curvature tensor, it becomes infinitesimal in the continuous time limit.

Despite this apparent limitation, there is a possible way out: while $\hat{\ell}$ is a Lie algebra representation, it still acts on the space $L^2(\mathrm{SU}(N))$ of square-integrable functions on $\mathrm{SU}(N)$, which can be defined for a general group and does not depend on the Lie algebra. Moreover, the Kogut-Susskind Hamiltonian eq. (2.7) does not contain $\hat{\ell}^a$ directly, but only through the combination $\hat{\ell}^a\hat{\ell}^a$, with the Lie algebra index a summed over. While the former contains a Lie algebra index, the latter does not, and it is then possible that its action on $L^2(\mathrm{SU}(N))$ could be described entirely in terms of group-related quantities, which would allow a generalisation to finite groups. This turns out to be the case.

As noted in section 2.2, the Hilbert space on each link can be identified with $L^2(G)$, i.e. the space of square-integrable functions on the gauge group G. The overall Hilbert space is then given by the tensor product

$$\mathcal{H} = \bigotimes_{\text{links}} L^2(G)$$

In what follows, we'll exploit a number of structural similarities between the representation theory of finite groups and of compact Lie groups. Appendix A gives an overview of the crucial mathematical results used in what follows.

The single-link Hilbert space admits a particularly useful decomposition. By the Peter-Weyl theorem (see appendix A.3), it can be written as a sum over representation subspaces,

$$L^2(G) = \bigoplus_j V_j^* \otimes V_j$$

where j indexes the (at most) countable set of inequivalent irreducible unitary representations of G. In this decomposition, the translation operator L_U takes a particularly simple form (see A.3):

$$L_U = \bigoplus_j \pi_j(U)^* \otimes \mathbb{1}$$

where the π_j are the finite dimensional inequivalent irreducible representations of the gauge group. Moreover, by the same theorem, we have another orthonormal basis, the "representation basis" $|jmn\rangle$, whose wavefunctions in the position basis are given by the matrix elements of representations of G. Here j indexes representations, while $m, n = 1, 2, \ldots, \dim(j)$ where $\dim(j)$ is the dimension of the j^{th} representation. More details and discussion can be found in appendix A.3. An operative definition is that it can be seen as the basis dual to the position basis, in the sense that

$$\langle U|jmn\rangle = \sqrt{\frac{\dim(j)}{\mathrm{Vol}(G)}}\,[\pi_j(U)]_{mn} \tag{3.1}$$

for a compact Lie group, while

$$\langle g|jmn\rangle = \sqrt{\frac{\dim(j)}{|G|}}\,[\pi_j(g)]_{mn} \tag{3.2}$$

for a finite group. This is a generalisation to cover the general case of compact Lie groups of the duality relation given in [23]. In the case where G is a finite group, we can reassure ourselves that the two bases have the same number of elements:

$$\#\,\text{elements of}\,\{|g\rangle\} = |G|$$

$$\#\,\text{elements of}\,\{|jmn\rangle\} = \sum_{j\in\text{irreps}} (\dim j)^2$$

as already noted in [23]. A standard result asserts that these two quantities are always equal (see appendix A.1). Now we have two bases:

1. the position basis $|U\rangle$, or $|g\rangle$, given by the elements of G. We will see that the magnetic part of the Hamiltonian is diagonal in this basis.

2. the representation basis $|jmn\rangle$ given by the matrix elements of representations of G. We will see that the electric part of the Hamiltonian is diagonal in this basis.

It is crucial that everything we've said so far is equally valid for both finite groups and for compact Lie groups, as emphasised in appendix A.

Now we can specialise to compact Lie groups, in order to write the action of the electric Hamiltonian in group-valued terms only. This is implicit in [23], but here we offer a full justification along the lines of the modern approach of [42]. While this is not too hard, to our knowledge it hasn't appeared previously in the literature.

First of all, note that on each representation subspace labelled by a specific value of j, the Laplacian Δ_e is the Casimir operator on the Lie algebra, as the Killing form of a compact group is always diagonal in a suitable basis. More precisely, we consider the action of $\hat{\ell}^a$ on $L^2(G)$, which can be obtained from the action of L_U by differentiation. Note that if π_j is a Lie group representation, then we can define the corresponding Lie algebra representation $\widetilde{\pi}_j$ via

$$\widetilde{\pi}_j(X) = -i\frac{d}{d\epsilon}\pi_j(e^{i\epsilon X})\bigg|_{\epsilon=0}$$

for any Lie algebra element X. Under some assumptions, the Lie group and Lie algebra representations are in one-to-one correspondence [52]. This is certainly the case for $\mathrm{SU}(N)$, which is simply-connected. In other cases, more care is needed, but we will only work with $\mathrm{SU}(N)$ here. We've seen that in the Peter-Weyl decomposition,

$$L_U = \bigoplus_j \pi_j(U)^* \otimes \mathbb{1}$$

The action of $\hat{\ell}^a$ is then obtained via differentiation. The dual Lie algebra representation (that is, the one corresponding to the dual of a Lie group representation) is then given by $\widetilde{\pi}^* = -\widetilde{\pi}^T$ [52], leading to

$$\hat{\ell}^a = \bigoplus_j -\widetilde{\pi}_j(T^a)^T \otimes \mathbb{1}_j$$

so that the Laplacian is then given by

$$\Delta \equiv \sum_a \left(\hat{\ell}^a\right)^2 = \bigoplus_j \sum_a \widetilde{\pi}_j(T^a)^T \widetilde{\pi}_j(T^a)^T \otimes \mathbb{1}_j = \bigoplus_j \sum_{a,b} \delta_{ab} \widetilde{\pi}_j(T^a) \widetilde{\pi}_j(T^b) \otimes \mathbb{1}_j$$

For a compact group with semi-simple Lie algebra, such as SU(N), the Killing form is proportional to δ_{ab} [52]. It follows that the sum on the right hand side is (proportional to) the Casimir operator on each representation subspace, which, by definition, takes the same value on each representation subspace [52], say $C(j)$. Therefore the Laplacian can also be written in the representation basis as

$$\Delta = \sum_j C(j) P_j$$

where P_j is the projector onto the jth representation subspace, or more explicitly,

$$P_j = \sum_{m,n} |jmn\rangle \langle jmn|$$

Crucially, the representation basis can be defined equally well for a finite group. Therefore, one can simply define the finite group Laplacian as

$$\Delta = \sum_j f(j) P_j$$

where now $f(j)$ is an arbitrary function of the representation j. We can interpret the representation basis as the electric flux basis, where $f(j)$ gives the energy density associated with the flux state j. We'll offer some discussion on the choice of f later. As long as f is a function of j only (and not of the specific matrix element m or n), then the above Laplacian is gauge-invariant. This can be shown as follows. Under a gauge

transformation, given by group elements g, h at the two ends of the link, we get

$$\begin{aligned}
|jmn\rangle \mapsto L_g R_h \, |jmn\rangle = \\
&= \sqrt{\frac{\dim(j)}{|G|}} \sum_{k\in G} [\pi^j(k)]_{mn} L_g R_h \, |k\rangle = \\
&= \sqrt{\frac{\dim(j)}{|G|}} \sum_{k\in G} [\pi^j(k)]_{mn} \, |gkh^{-1}\rangle = \\
&= \sqrt{\frac{\dim(j)}{|G|}} \sum_{k\in G} [\pi^j(g^{-1}kh)]_{mn} \, |k\rangle = \\
&= \sqrt{\frac{\dim(j)}{|G|}} \sum_{k\in G} \sum_{pq} [\pi^j(g^{-1})]_{mp} [\pi^j(k)]_{pq} [\pi^j(h)]_{qn} \, |k\rangle = \\
&= \sum_{pq} [\pi^j(g^{-1})]_{mp} [\pi^j(h)]_{qn} \, |jpq\rangle
\end{aligned}$$

where, in going from the first to the second line, and again in the final line, we have expanded the representation basis in terms of the group element basis using their duality relation. Therefore, under a gauge transformation,

$$\begin{aligned}
P_j &= \sum_{mn} |jmn\rangle \langle jmn| \\
&\mapsto \sum_{mn} \sum_{pq} \sum_{p'q'} [\pi^j(g^{-1})]_{mp} [\pi^j(h)]_{qn} [\pi^j(g^{-1})]^*_{mp'} [\pi^j(h)]^*_{q'n} \, |jpq\rangle \langle jp'q'| = \\
&= \sum_{mn} \sum_{pq} \sum_{p'q'} [\pi^j(g)]_{p'm} [\pi^j(g^{-1})]_{mp} [\pi^j(h)]_{qn} [\pi^j(h^{-1})]_{nq'} \, |jpq\rangle \langle jp'q'| = \\
&= \sum_{pq} \sum_{p'q'} \delta_{p'p} \delta_{q'q} \, |jpq\rangle \langle jp'q'| = \\
&= \sum_{pq} |jpq\rangle \langle jpq| = P_j
\end{aligned}$$

In going from the second to the third line we used the unitarity of the representations. This shows that the Laplacian is gauge-invariant. An alternative derivation, based on an interesting implementation of the operators which give gauge transformation, can be found in [23].

The Hamiltonian for lattice gauge theory with a finite gauge group is then simply given by

$$H = \lambda_E \sum_{\text{links}} \sum_j f(j) P_j - \lambda_B \sum_{\square} \left(\text{tr}(\hat{W}_{\square}) + \text{tr}(\hat{W}_{\square})^\dagger \right) \tag{3.3}$$

which is essentially the same as that given in [23]. The coefficients λ_E and λ_B can be taken to have the same form as in the continuum, eq. (2.7), or they can be treated as independent parameters. The function $f(j)$ is in principle undetermined, although we'll propose a way to determine it in the following section.

Note that with our choice of $\hat{u}$, the magnetic part of the Hamiltonian is expressed as a trace over the representation in which the fermions transform. Even if we don't include fermions, we choose one representation only and include only its character in the magnetic term. Some other popular models expand the magnetic term as a linear combination of all the irreducible group characters [23], but this is not of interest here.

If one wishes to include fermions to the theory, the fermionic Hamiltonian for staggered spinless fermions is given by [23]

$$H_F = m\sum_x (-1)^x \psi^\dagger(x)\psi(x) + \lambda \sum_{x,\hat{\mu}} \sum_{m,n} \left[\psi^\dagger_m(x)\hat{u}_{mn}(x,\hat{\mu})\psi_n(x+\hat{\mu}) + \text{h.c.}\right]$$

where the sums are taken over lattice sites, and then over lattice sites and directions. It is not hard to generalise this to the spinful case in a manner similar to that of section 2.3.

3.2 Finite group Laplacian

As we have seen above, the function $f(j)$, which gives the eigenvalues of the electric energy density, is in principle undetermined. In past literature, different options have appeared in the Abelian case. One may choose it in analogy with an infinitesimal transformation [19], or by seeing $\mathbb{Z}_N$ gauge theory as a truncation of U(1) gauge theory [22]. However, there is no obvious way to extend these results to the non-Abelian case. In [60], to our knowledge the only other work on the non-Abelian case, the coefficients for $f(j)$ are chosen for dihedral gauge theory in analogy with the Abelian case. However, this choice is arbitrary, and it is not generalisable to the general case of a finite non-Abelian group.

There are a number of requirements one may wish to impose on the function $f(j)$. For example, it is natural to require that it is a positive function, as it can be interpreted as an energy density. Moreover, one may impose the requirement that it is zero on the lowest-flux state, the singlet corresponding to the trivial representation, and that this zero-energy state is unique. Other requirements may be imposed in specific cases. Consider for example $\mathbb{Z}_N$ gauge theory as in section 3.3.1. Then as we only have a finite number of states, it is natural to require that f be periodic: $f(j+N) = f(j)$. Moreover, if one wishes to see the $\mathbb{Z}_N$ gauge theory as an approximation to the U(1) gauge theory, in which case the eigenvalues are given by j^2, then it is natural to require that $f(j) \to j^2$ at least in the $N \to \infty$ limit.

The freedom to choose the electric term should be seen as a feature, rather than a limit, of finite group gauge theories. In fact, it is conceivable that it could be chosen for ease of experimental implementation. However, in order to do so safely, a better understanding is required of the effect of different choices for the electric term. We attempt some steps in this direction in chapter 4.

As we've said, the issue of fixing the eigenvalues of the electric energy density, that is the function $f(j)$ in eq. (3.3), is as of yet unresolved. Here we propose a new method to determine $f(j)$ based on the construction of a natural Laplacian operator on a finite group. The key observation is that in the continuum theory, the (chromo)electric energy density is given, at each link, by the Laplacian on the Hilbert space, as explained in section 3.1.

It turns out to be possible to define a meaningful "Laplacian" even for a finite group, which satisfies many of the properties that we require of $f(j)$. In particular, it is a positive semi-definite operator on the group algebra $\mathbb{C}[G]$ with the correct decomposition into representation subspaces. Moreover, it always has a non-degenerate zero eigenvalue corresponding to the subspace of the trivial representation. The Laplacian is not unique, but the choices are strongly limited by the conjugacy structure of the group. We'll now review its construction.

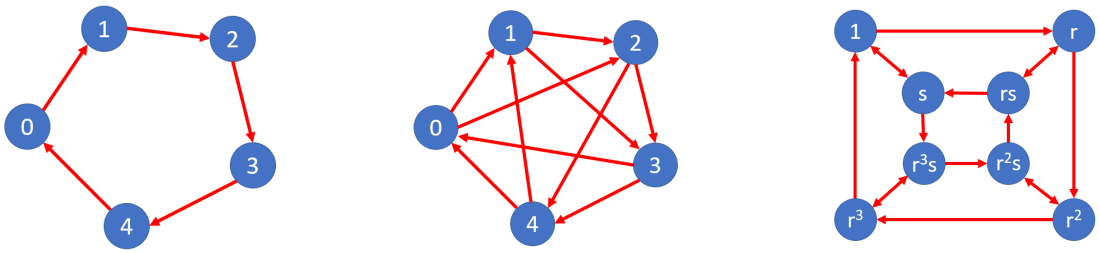
The basic idea is to construct the Cayley graph corresponding to the finite group. Any graph is associated to a graph Laplacian, a well-known operator which mimics many of the properties of the continuum Laplacian [61]. For example, the heat equation on a graph is naturally associated with the graph Laplacian. We then define the Laplacian of the group as the graph Laplacian of its Cayley graph.

The Cayley graph is constructed as follows [61]. Starting from a finite group G, we choose a generating subset $\Gamma \subset G$. This is of course non-unique, and we will obtain a different Cayley graph $\mathrm{Cay}(G, \Gamma)$ for different choices of generating subset. We will require that Γ is symmetric, that is $\Gamma^{-1} = \Gamma$. The vertices of the Cayley graph $\mathrm{Cay}(G, \Gamma)$ are the elements of G. We add edges between two group elements if one can be reached from the other by right multiplication by an element of the generating set Γ. Whether one chooses right or left multiplication is a convention. In other words, we place an edge between $g \in G$ and $h \in G$ if $hg^{-1} \in \Gamma$. The result is an undirected graph, and we do not allow multiple edges. Typically, we choose $1 \notin \Gamma$, so that self-loops are not allowed and the graph is therefore simple. Examples of *directed* Cayley graphs for D_4 and for $\mathbb{Z}_5$ with two different choices of non-symmetric generating sets are shown in figure 3.1.

Given a Cayley graph, its adjacency matrix can be seen as an operator on the group algebra as follows. Given a function $f : G \to \mathbb{C}$, that is given an assignment of a complex number to each group element, one can define

$$Af(g) = \sum_{h \in G} A_{gh} f(h)$$

for group elements g, h. In other words, one can imagine f as a vector whose components



(a) Cayley graph for $\mathbb{Z}_5$ with generating set $\{1\}$

(b) Cayley graph for $\mathbb{Z}_5$ with generating set $\{1, 2\}$

(c) Cayley graph for D_4 with generating set $\{r, s\}$

Figure 3.1: Some examples of Cayley graphs

are its values on group elements, and each group element corresponds to a basis vector. Then the action of A is simply given by matrix multiplication. The definition of the action of the adjacency matrix as written above is actually clearly valid in the case of a general graph. For a Cayley graph $\mathrm{Cay}(G, \Gamma)$ the action simplifies. In fact A_{gh} is non-zero only for those pairs g, h such that $gh^{-1} \in \Gamma$. In other words, given g the only non-zero elements of the adjacency matrix are $A_{gh} = 1$ for $h = gk$ where $k \in \Gamma$. It follows that

$$Af(g) = \sum_{k \in \Gamma} f(gk)$$

However, this can be immediately written in terms of the right regular representation of G,

$$Af(g) = \sum_{k \in \Gamma} R_k f(g)$$

See appendix A.3 for the action of the regular representations on functions. If we had chosen the convention that edges are placed between group elements which can be reached by *left* (rather than right) multiplication by an element of Γ, we would have had here the left-regular representation instead. Therefore, as an operator,

$$A = \sum_{k \in \Gamma} R_k$$

From the Peter-Weyl theorem (see section A.3) the right regular representation decomposes as a sum over representation subspaces, where each irreducible representation appears with multiplicity equal to its dimension:

$$R = \bigoplus_j \pi_j^{\oplus \mathrm{dimj}}$$

where the π_j are inequivalent irreducible representations of G. It follows that, up to equivalence,

$$A = \sum_{k\in\Gamma} R_k = \sum_{k\in\Gamma} \bigoplus_j \dim(j)\pi_j(k) = \bigoplus_j \dim(j) \sum_{k\in\Gamma} \pi_j(k)$$

Now suppose that Γ is a union of conjugacy classes, so that it is closed under conjugation. Then we can find a simple formula for the eigenvalues of A in each representation subspace [61, 62]. In fact, for any $g \in G$ we find

$$AR_g = \sum_{k\in\Gamma} R_k R_g = \sum_{k\in\Gamma} R_{kg} = \sum_{k\in\Gamma} R_{(gkg^{-1})g} = \sum_{k\in\Gamma} R_{gk} = R_g A$$

where we crucially used the closure of Γ under conjugation. This means that A commutes with R_g on any representation subspace. Since R_g restricted to each subspace is irreducible, it follows by Schur's lemma that on each representation subspace A is proportional to the identity matrix:

$$A = \bigoplus_j \lambda_j P_j$$

where j labels irreps and P_j is the projector onto the j^{th} representation subspace. The eigenvalue can be computed by noting that R_g restricted to each representation subspace is given by the irreducible representation. Therefore,

$$A|_j = \sum_{k\in\Gamma} R_k|_j = \sum_{k\in\Gamma} \pi_j(k)$$

Taking traces, it follows that

$$\lambda_j = \frac{1}{\dim(j)} \sum_{k\in\Gamma} \chi_j(k)$$

where χ_j is the character of the j^{th} representation.

Now, the Laplacian of a graph is defined as [61]

$$L = D - A$$

where A is the adjacency matrix and D is the degree matrix. The degree matrix is diagonal, and each diagonal element is the degree of the corresponding vertex; the degree of a vertex is the number of edges that it is connected to. In the case of an undirected Cayley graph, the degree of each vertex is given by the size of the generating set, $|\Gamma|$. Therefore D is proportional to the identity matrix and the Laplacian admits a decomposition

$$L = \bigoplus_j f(j) P_j \tag{3.4}$$

where its eigenvalues are given by

$$f(j) = |\Gamma| - \frac{1}{\dim(j)} \sum_{k \in \Gamma} \chi_j(k) \tag{3.5}$$

This is precisely the decomposition that we obtained previously for the electric energy density, only that now we have a specific formula for the eigenvalues. By the general properties of the Laplacian of a graph [61], there is always a zero eigenvalue with an eigenvector whose components are all equal to 1. This is precisely the expansion of the basis element of the trivial representation in the group element basis.

It is clear that different choices of Γ result in different eigenvalues. However, given that we must choose Γ to be both a generating set and the union of conjugacy classes, there aren't many choices.

3.2.1 $\mathbb{Z}_N$ Laplacian

As an example, we can compute the Laplacian in the $\mathbb{Z}_N$ case. If g is a generator of the group, then the smallest symmetric generating set is $\{g, g^{-1}\}$. This is also a union of conjugacy classes. For a summary of the relevant representation theory see section 3.3.1. The eigenvalues of the Laplacian for the j^{th} representation are then given by

$$f(j) = 2\left[1 - \cos\left(\frac{2\pi j}{N}\right)\right] = 4\sin^2\left(\frac{\pi j}{N}\right)$$

This is is the same function obtained by other authors with different methods [19, 20]. Moreover, note that the eigenvalues of the U(1) theory (see section 4.2) are precisely given by j^2 on the j^{th} representation subspace. We expect the $\mathbb{Z}_N$ theory to approach the U(1) theory in the large N limit. With our choice of Laplacian, we see that the $\mathbb{Z}_N$ eigenvalues approach the U(1) eigenvalues in the appropriate limit:

$$f(j) \to \frac{4\pi^2}{N^2} j^2 \qquad N \text{ large}$$

The proportionality constant in front may be absorbed into the coupling constant, which may show a different behaviour in the different theories. This reinforces our belief that the above construction of the Laplacian of a finite group may give the correct eigenvalues for the electric energy density. In other words, the choice of $f(j) = j^2$ even in the finite group theory (see for example [22]) corresponds to a truncation of U(1) gauge theory, while the $\sin^2$ choice corresponds to proper $\mathbb{Z}_N$ gauge theory.

One can make make a different choice of generating set Γ, which due to its required properties must be a union of sets of the form $\{g^k, g^{-k}\}$. Each of these sets contributes a $\cos\left(\frac{2\pi kj}{N}\right)$ term in $f(j)$.

Apart from the smallest generating subsets, the other interesting example is the case where Γ is as large as possible. This occurs when $\Gamma = G - \{1\}$, and independently of the underlying group the resulting graph is the complete graph on $|G|$ elements (that is, the graph with all possible edges). The Laplacian corresponding to the complete graph has eigenvalues given by $f(j) = 0$ for the trivial representation and $f(j) =$ constant otherwise [61].

3.2.2 Laplacian in the group element basis

As a final calculation, which will turn out to be useful later, we'll compute the matrix elements of the Laplacian in the group element basis. By making use of the duality relation between the two bases, eq. (3.2). For generic eigenvalues $f(j)$ we find

$$\sum_{jmn} f(j) \, |jmn\rangle \, \langle jmn| = \sum_{g,h \in G} \widetilde{f}(gh^{-1}) \, |g\rangle \, \langle h|$$

where the function $\widetilde{f}$ is given by:

$$\widetilde{f}(gh^{-1}) = \frac{1}{|G|} \sum_{j} f(j) \dim(j) \, \chi_j(gh^{-1})$$

where χ_j is the character of the j^{th} irreducible representation of G. Thus in general $\widetilde{f}$ is similar to the (inverse) discrete Fourier transform of f, but not quite the same. If we choose $f(j)$ as the eigenvalues of the group Laplacian for a certain finite generating subset Γ, then as we just saw,

$$f(j) = |\Gamma| - \frac{1}{\dim(j)} \sum_{g \in \Gamma} \chi_j(g)$$

Substituting into the formula for $\widetilde{f}$ leads to

$$\widetilde{f}(g) = \frac{|\Gamma|}{|G|} \sum_{j} \dim(j) \, \chi_j(g) - \frac{1}{|G|} \sum_{h \in \Gamma} \sum_{j} \chi_j(h) \chi_j(g)$$

The sum in the first term is equal to the character of the left regular representation [51], which is equal to $|G|$ on the identity and zero otherwise. Moreover, since we required $\Gamma^{-1} = \Gamma$, we can replace h with h^{-1} in the character in the second term. Therefore

$$\widetilde{f}(g) = |\Gamma| \, \delta(g, 1) - \frac{1}{|G|} \sum_{h \in \Gamma} \sum_{j} \chi_j(h)^* \chi_j(g)$$

Now the sum over representations in the second term can be computed using the orthogonality relations for the characters (see appendix A.2). It is non-zero only if g and h are

conjugate, in which case it equals $|G| / |C(g)|$ where $|C(g)|$ is the size of the conjugacy class of g. Since Γ was defined to be a union of conjugacy classes, the sum will therefore be non-zero only if $g \in \Gamma$. Therefore

$$\widetilde{f}(g) = |\Gamma|\, \delta(g,1) - \frac{1}{|G|} \sum_{h \in C(g) \subset \Gamma} \frac{|G|}{|C(g)|}$$

So that, in the end,

$$\widetilde{f}(g) = \begin{cases} |\Gamma| & g = 1 \\ -1 & g \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

which is nothing other than the Laplacian in the group element basis. Therefore, once Γ has been determined, $\widetilde{f}$ can be computed easily and directly.

3.3 Some examples of finite-group gauge theory

We'll now explicitly construct the Hamiltonian for some finite gauge groups. In general, one needs to know the full representation theory of the finite group. This can be looked up in a reference, or it can be computed using software such as *GAP* [63]. In the special case of pure gauge theory (i.e. without fermions), there is actually no need to know the specific form of the representation matrices, but it is sufficient to know the group's character table. Appendix A.1 contains a short review of the relevant representation theory.

3.3.1 $\mathbb{Z}_N$ gauge theory

In this section, we'll study $\mathbb{Z}_N$ gauge theory and show that it is described by our theory in a manner equivalent to that studied by other authors [19, 20].

First of all, $\mathbb{Z}_N$ is Abelian, and as such, its conjugacy classes are singlets. By a fundamental result in representation theory, this implies that $\mathbb{Z}_N$ has N inequivalent irreducible representations, and they are all one dimensional. If g is a generator of $\mathbb{Z}_N$, then $\mathbb{Z}_N = \{1, g, g^2, \ldots, g^{N-1}\}$. It's not hard to see that all of the irreps are given by

$$\pi_j(g^k) = e^{i\frac{2\pi j}{N}k} \quad j = 0, 1, \ldots, N-1$$

As all the representations are one dimensional, the π_j all have only one element, the one given above.

In order to give an explicit realisation for the coupling matrices $\hat{u}$, which in this case also only have one element, we must choose a representation under which fermions transform. We'll pick the $j = 1$ representation. Then the operator $\hat{u}$ acts as

$$\hat{u}\, |g^k\rangle = e^{i\frac{2\pi}{N}k}\, |g^k\rangle$$

In other words, the position basis is an eigenbasis for $\hat{u}$ whose eigenvalues are given by the 1×1 matrices of the corresponding element of the representation in which the fermions transform. Using the duality relation between the two bases, it is not hard to check that

$$\hat{u}\,|j\rangle = |j+1\rangle \qquad \text{(cyclically)}$$

that is, on each link, $\hat{u}$ acts as a flux raising operator for the "flux" indexed by the representation basis vector $|j\rangle$ [19]. The magnetic Hamiltonian is given by

$$H_B = \lambda_B \sum_{\text{plaquettes}\,\square} \hat{u}_{\square,1}\hat{u}_{\square,2}\hat{u}^\dagger_{\square,3}\hat{u}^\dagger_{\square,4} + \text{h.c.}$$

where the indices on the $\hat{u}$ operators denote the operators assigned to each of the four links in the plaquette. The fermionic Hamiltonian for spinless fermions is given by

$$H_F = m\sum_x (-1)^x \psi^\dagger(x)\psi(x) + \sum_{x,\hat{\mu}} \left[\psi^\dagger(x)\hat{u}(x,\hat{\mu})\psi(x+\hat{\mu}) + \text{h.c.}\right]$$

as in [19], where $\hat{u}(x,\hat{\mu})$ is the $\hat{u}$ operator on the link between x and $x+a\hat{\mu}$. The fermions only have one colour component, which in this case is a finite analog of the $U(1)$ electric charge. The interpretation of the coupling term is in this case straightforward: a fermion is destroyed at site $x+\hat{\mu}$, a fermion is created at site x, and the electric flux on the link connecting the two sites is raised by 1. The electric Hamiltonian is

$$H_E = \lambda_E \sum_{\text{links}} \sum_j f(j)\,|j\rangle\langle j|$$

If one wishes to do so, it is possible to introduce a unitary operator $\hat{v}$ which tells us on which irrep we're on,

$$\hat{v}\,|j\rangle = e^{i\frac{2\pi}{N}j}\,|j\rangle$$

this is the same operator introduced in [19]. Then the electric Hamiltonian can be expressed more compactly as

$$H_E = \lambda_E \sum_{x,\hat{\mu}} f(\hat{v})$$

which is the same form as the one given in [19]. It should be noted that $\hat{v}\hat{u}\hat{v}^\dagger = e^{i\frac{2\pi}{N}}\hat{v}$, so that $\hat{u}$ and $\hat{v}$ satisfy the Schwinger-Weyl algebra [19, 25].

Making use of the duality relation, we can compute the matrix elements of the Hamiltonian in the group element basis. With this choice, only the electric Hamiltonian needs to be converted as the magnetic Hamiltonian is already diagonal. Inserting the completeness relation in the group element basis, a short calculation leads to

$$\sum_j f(j)\,|j\rangle\langle j| = \sum_{k,l} \widetilde{f}(k-l)\,|g^k\rangle\langle g^l|$$

where $\widetilde{f}$ is the discrete Fourier transform of f:

$$\widetilde{f}(k) = \frac{1}{N}\sum_j f(j)e^{\frac{2\pi i}{N}jk}$$

With any choice, the electric Hamiltonian on each site contains hopping between any combination of first group element "neighbours", second neighbours, etc. For example, choosing $f(j) = 4\sin^2\left(\frac{2\pi j}{N}\right)$ as outlined in section 3.2 leads to (cyclic) first neighbours coupling:

$$\sum_{k,l}\widetilde{f}(k-l)\left|g^k\right\rangle\left\langle g^l\right| = \sum_{g^k}\left(2\left|g^k\right\rangle\left\langle g^k\right| - \left|g^k\right\rangle\left\langle g^{k+1}\right| - \left|g^{k+1}\right\rangle\left\langle g^k\right|\right)$$

in the same way as in [19].

3.3.2 D_N gauge theory

The dihedral groups are some of the simplest examples of finite non-Abelian groups, and have attracted attention in the past [60]. They are of particular interest as the dihedral group with six elements is the smallest non-Abelian group. As such, one expects that it should be easier to simulate than other non-Abelian groups. In the large N limit, one naively expects D_N gauge theory to approach $O(2)$ gauge theory [60].

We choose the convention where the dihedral group with $2N$ elements is called D_N. Its representation theory depends on whether N is even or odd. We will focus on the case N odd, but the other case is similar. Recall that D_N contains a $\mathbb{Z}_N$ subgroup of rotations $\{1, r, r^2, \ldots, r^{N-1}\}$ and another subset of reflections $\{s, rs, r^2s, \ldots, r^{N-1}s\}$, which together exhaust the group. We have the following representations [51]:

- The trivial representation, which we index by $j = 0$, given by $\pi_0(g) = 1$ for all group elements. This is one-dimensional.
- The sign representation, which we index by $j = -1$, given by $\pi_{-1}(r^l) = 1$ for rotations and $\pi_{-1}(r^ls) = -1$ for reflections. This is also one-dimensional.
- The non-Abelian representations, which we index by $j = 1, \ldots, \frac{N-1}{2}$, given by

$$\pi_j(r^l) = \begin{pmatrix} e^{\frac{2\pi ij}{N}l} & 0 \\ 0 & e^{-\frac{2\pi ij}{N}l} \end{pmatrix}$$

and

$$\pi_j(r^ls) = \begin{pmatrix} 0 & e^{\frac{2\pi ij}{N}l} \\ e^{-\frac{2\pi ij}{N}l} & 0 \end{pmatrix}$$

All these are two-dimensional.

We can check that the sum of the dimensions squared of these representations is $2N$ in total.

One can choose the fermions to live in the $j = 1$ representation. There is essentially no simplification that one can make on the general formula, except expressing the electric Hamiltonian in the group element basis. A straightforward computation shows that the single-link electric Hamiltonian is equal to

$$\sum_{k,l} \left[\Delta f + \widetilde{f}(k-l)\right] \left(|r^k\rangle \langle r^l| + |r^k s\rangle \langle r^l s| \right) - \Delta f \sum_{k,l} \left(|r^k\rangle \langle r^l s| + |r^k s\rangle \langle r^l| \right)$$

where

$$\Delta f = \frac{f(-1) - f(0)}{2N}$$

and

$$\widetilde{f}(k) = \frac{1}{N} \left[f(0) + 2 \sum_{j=1}^{(N-1)/2} f(j) \cos\left(\frac{2\pi kj}{N}\right) \right]$$

the same as in [60], although in different language. The first term corresponds to hopping within the subspace of rotations and within the subspace of reflections, while the second term corresponds to hopping between the two different subspaces. We see that the first term takes the same form as two copies of $\mathbb{Z}_N$ gauge theory. One is not free to set the various couplings: for example the hopping terms within rotations and within reflections must have the same coefficients.

The fermions live in the $j = 1$ two-dimensional representation, which means that they have two colour components. The fermionic action now has a more complicated interpretation than in the Abelian case. First of all, now $\hat{u}$ cannot simply be interpreted as a flux raising operator, as each flux state is degenerate. We can still provide an alternative interpretation, by looking at the form of the $j = 1$ representation matrices. If, in the group element basis, the link between x and $x + a\hat{\mu}$ is a rotation, then a fermion of one type is created at x and a fermion of the same type is destroyed at $x + a\hat{\mu}$ or viceversa. On the other hand, if the group element on the link is a reflection, fermions of different types are created and destroyed on neighbouring lattice sites.

As for the function $f(j)$, there is no obvious choice. Some prefer to let Δf be an independent coupling, and then choose the other couplings to mimic $\mathbb{Z}_N$ gauge theory [60]. The finite-group Laplacian method that we've exposed in section 3.2 offers a different perspective: the trivial representation should be the unique zero energy state, $f(0) = 0$, while the sign representation -1 should correspond to the highest energy state. With most choices of generating sets, the Laplacian gives a large degeneracy in the eigenvalues, even among different representations.

3.3.3 Other interesting examples

As outlined in the Introduction, one might one day want to use a finite-group gauge theory to obtain predictions for a non-Abelian gauge theory in the continuum. The simplest concrete example of such a gauge theory is based on the Lie group SU(2), and as such it is interesting to consider its finite non-Abelian subgroups and their gauge theories. In this specific case, all of its finite subgroups have been long classified, and they belong to one of five families [64]:

- The Abelian cyclic groups $\mathbb{Z}_n$ for any n.
- The non-Abelian binary dihedral groups $2D_n$ for any n.
- The binary tetrahedral group $2T$.
- The binary octahedral group $2O$.
- The binary icosahedral group $2I$.

The Abelian subgroups are also subgroups of U(1) and as such they are unlikely to capture the properties of a non-Abelian theory. A binary group is the preimage of a subgroup of SO(3) with respect to the universal covering map SU(2) $\to$ SO(3). For example, SO(3) has a D_n subgroup for any n, which is pulled back to SU(2) to give the binary dihedral groups with twice the elements. There's more hope that $2D_n$ be a useful approximation, but this still looks unlikely. It is known that as a manifold SU(2) $\cong S^3$, the 3−sphere. However, the D_n are symmetry groups of flat polygons, and as such they are "flat" and unlikely to capture the "three-dimensional" structure of SU(2).

The other three groups are more interesting. They are the preimage of the tetrahedral, octahedral and icosahedral groups in SO(3), and have twice their elements. These are the symmetry groups of polyhedra, and as SO(3) subgroups they are isomorphic to A_4, S_4 and A_5 respectively. Their binary versions inside SU(2) are truly "three-dimensional" and as such they can be expected to give a good approximation of SU(2). Both the binary octahedral group [31] and binary icosahedral group [32] has been successfully used in numerical simulations in the path-integral approach. We will consider their gauge theories in chapter 4. Appendix B contains a summary of their representation theory.

The general SU(N) case with $N > 2$ is far more complicated. Even for SU(3) the classification of its finite subgroups is much harder [64, 65]. As such, we'll mostly be concerned with the SU(2) case.

Chapter 4

Some Calculations in Finite-Group Gauge Theory

After formulating lattice gauge theories with a general finite gauge group in the previous chapter, we now turn our attention to extracting predictions from such theories. We limit ourselves to the case of pure gauge theory, without fermions. After some remarks on the structure of the Hamiltonian, the Hilbert space and the general features of these theories and their phase diagram, we turn to the analytical calculation of the string tension. This is nothing but the coefficient of the linear potential of a confined quark-antiquark pair, and it can be efficiently computed in the Hamiltonian approach. In the next section, we study the theory on a single plaquette, obtaining a full description of the physical, gauge-invariant Hilbert space and the Hamiltonian. After solving the theory exactly, albeit numerically, we comment on the results. In the final section, we investigate a variational ansatz for the ground state of the theory. We are able to carry out the calculations analytically in $2+1$ dimensions, and only resort to numerical calculations to compute the resulting finite sums and we comment on our results. As a check on our analytical results, we set up and perform Monte Carlo simulations, in a way that can be extended to any number of dimensions.

4.1 General outline

The Hamiltonians formulated in the previous chapter are in general very hard to solve exactly. In the pure gauge case the Hamiltonian is given by eq. (3.3):

$$H = \lambda_E \sum_{\text{links}} \sum_j f(j) P_j - \lambda_B \sum_{\square} \left(\text{tr}(\hat{W}_{\square}) + \text{tr}(\hat{W}_{\square})^\dagger \right)$$

In the continuum (eq. (2.7)) the coefficients are given in terms of the coupling as

$$\lambda_E \propto g^2 \qquad \lambda_B \propto 1/g^2$$

Therefore in the *strong-coupling limit* of large g, the magnetic term is highly suppressed and the Hamiltonian is diagonal in the representation basis [42]. We then assume that, like in the theory with a Lie gauge group, the function $f(j)$ has a non-degenerate minimum at 0, which it achieves at the trivial representation, whose only matrix element we represent as $|0\rangle$. This is a gauge invariant state, as

$$|0\rangle = \sum_g \langle g|0\rangle \, |g\rangle = \frac{1}{\sqrt{|G|}} \sum_g |g\rangle$$

owing to the fact that the matrix elements of the trivial representation are all one-dimensional and equal to 1. It follows that

$$L_h R_k \, |0\rangle = \frac{1}{\sqrt{|G|}} \sum_g |hgk^{-1}\rangle = \frac{1}{\sqrt{|G|}} \sum_g |g\rangle = |0\rangle$$

for any $h, k \in G$. It therefore follows that the strong coupling ground state is given by

$$|S\rangle = \bigotimes_{e \in \text{links}} |0\rangle_e$$

As we saw, this indeed belongs to the physical Hilbert space, and is in fact the only gauge-invariant product state.

On the other hand, in the *weak-coupling limit* of small g, the magnetic term dominates and the ground state is given by a superposition of those states such that all plaquette Wilson loops are equal to the identity [42]. One might think that this is given by those configurations in the group element basis for which every link is in the identity element; while this would indeed make all Wilson loops equal to the identity, it is not a gauge-invariant state.

As the lattice theory is trivial in both the lattice strong and weak coupling limits, these do not reproduce the correct behaviour for continuum Yang-Mills. In general, the coupling constant g will be renormalised, and will run as a function of a, in order to match the value of, say, the fundamental excitation of the theory [42]. In order to obtain the correct continuum limit, one would then look for a point in the g phase diagram with a divergent correlation length (i.e. a second order phase transition [2, 42]), which would force $a \to 0$, giving a continuum limit for the theory [2]. If one believes that lattice non-Abelian Yang-Mills in four dimensions is asymptotically-free like its continuum counterpart, then this requirement means that the phase transition can only occur at lattice coupling $g = 0$ [42].

It is interesting to note that while classically one cannot have a gauge theory with a finite gauge group in the continuum, as one necessarily needs a Lie algebra valued gauge field, some quantum gauge theories with finite gauge group show second-order phase transitions, and as such, have a well-defined continuum limit [66].

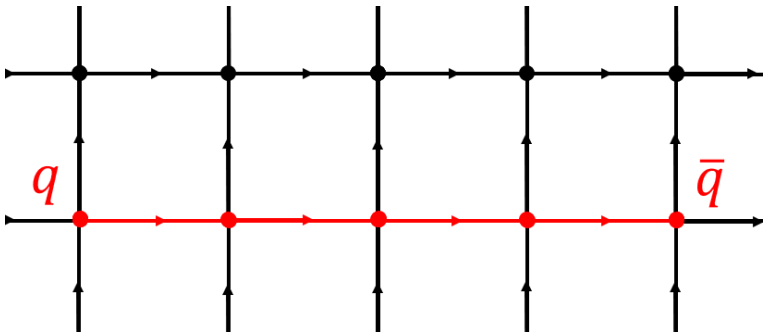


Figure 4.1: A static quark antiquark pair connected by a line of excited states

4.2 String tension

As detailed in section 2.1.1, in the strong coupling regime of both Abelian and non-Abelian gauge theories one expects to find a confined phase, where the potential between quarks increases linearly with distance. The signature of confinement in the path-integral approach is the area-law behaviour of Wilson loops [3]. In the Hamiltonian formalism, on the other hand, the energy of a quark-antiquark pair in the strong-coupling limit can be computed directly, treating the magnetic term as a small perturbation [15, 18]. This can be used to compute the string tension for the theory, i.e. the coefficient of the linear potential between the quarks. As the string tension is an observable, the typical renormalisation group prescription would require that it is kept constant as the coupling varies with the cutoff a [2, 18].

4.2.1 String tension in Lie-group gauge theories

We will now review the calculation of the string tension in Lie group gauge theories. In the strong-coupling limit, the Hamiltonian reduces to its electric part:

$$H = \frac{g^2}{2a^{d-2}} \sum_{\text{links}} \sum_{a} \left(\hat{\ell}^a\right)^2$$

One can imagine having two infinitely heavy quarks connected by a flux line [15], in other words a Wilson line C, made of consecutive links $\langle e_1, e_2, \dots e_k \rangle$ in a single direction, applied to the strong-coupling vacuum:

$$\hat{W}_{mn}(C)\,|S\rangle \equiv \sum_{j_1, j_2, \dots j_{k-1}} \hat{u}_{mj_1}(e_1)\hat{u}_{j_1 j_2}(e_2)\cdots\hat{u}_{j_{k-1}n}(e_k)\,|S\rangle$$

The lattice chromoelectric field $\hat{\ell}^a$ satisfies $\hat{\ell}^a\,|0\rangle_e = 0$ on each link e. The above Wilson line is the lowest excited state between the two static quarks. Note that $\hat{W}_{mn}$

is not gauge invariant by itself, but one should imagine it coupled to fermions so that $\psi_m^\dagger \hat{W}_{mn} \psi_n$, which does give a gauge-invariant state. The action of the strong-coupling Hamiltonian splits as the action of the electric field operator on each link. In fact, since $[\hat{\ell}^a, \hat{u}_{ij}] = -T^a \hat{u}_{ij}$, on each link we have

$$\hat{\ell}^a \hat{u}_{ij} |0\rangle = -[T^a \hat{u}]_{ij} |0\rangle$$

which then implies

$$\sum_a \left(\hat{\ell}^a\right)^2 \hat{u}_{ij} |0\rangle = \sum_a (T^a)_{ik} (T^a)_{kl} \hat{u}_{lj} |0\rangle$$

However, $(T^a)_{ik}(T^a)_{kl} = (T^a T^a)_{il}$ (with the sum over a now understood) and the square of the generators of representation matrices is the Casimir operator of the chosen representation for the fermions [1]. We therefore find,

$$\sum_a \left(\hat{\ell}^a\right)^2 \hat{u}_{ij} |0\rangle = \frac{N^2-1}{2N} \hat{u}_{ij} |0\rangle$$

where the numerical coefficient is nothing but the eigenvalue of the Casimir operator on the defining representation of SU(N). In the SU(2) case, it reduces to the familiar $\frac{3}{4} = \frac{1}{2}(\frac{1}{2}+1)$. Then the above Wilson line is an eigenstate of the Hamiltonian with energy

$$E = \frac{g^2}{2a^{d-2}} \frac{N^2-1}{2N} \frac{r}{a} \qquad G = \mathrm{SU}(N)$$

where r is the distance between the quarks. T In the U(1) case the calculation is even easier, as $T^a = 1$, giving

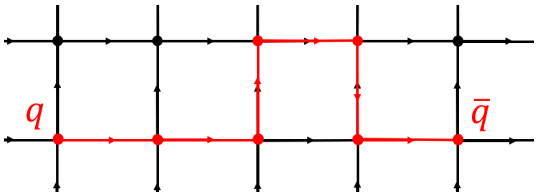
$$E = \frac{g^2}{2a^{d-2}} \frac{r}{a} \qquad G = \mathrm{U}(1)$$

This is a linear potential, which gives rise to a constant force between the quarks.

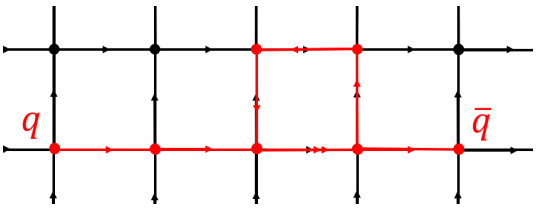
It is important to note that in the above calculations a crucial role is played by the generators of $\mathfrak{su}(N)$. In our case, since we chose the fermions to transform in the defining representation of SU(N) (the fundamental representation), the $\hat{u}$ also output the matrices in the fundamental representation. However, one can choose the fermions to live in *any* representation of the gauge group, and the above formula would have the generators in the chosen representations. The product relation that we used above is *not* in general satisfied by the generators of any representation, and as such the calculation of the potential between the two quarks crucially depends on the representation in which they live.

We can then compute the correction due to the magnetic Hamiltonian,

$$H_B = -\frac{1}{g^2 a^{4-d}} \sum_\square \left(\mathrm{tr}(\hat{W}_\square) + \mathrm{tr}(\hat{W}_\square)^\dagger\right)$$



(a) One possible orientation of a plaquette intersecting the Wilson line. In this case the line is deformed.



(b) The other possible orientation of a plaquette intersecting the Wilson line. Now one link is doubly excited.

Figure 4.2: Lattice orientation and gauge fields

The calculation for the non-Abelian case is harder than in the Abelian case and it depends on specific details of the representation theory of SU(N) [54]. As such, it is not very informative for the finite-group case and we will restrict ourselves to $G = \mathrm{U}(1)$.

In this case, the single-link electric Hilbert space is very easy to describe, as there is only one type of charge. It is given by the eigenstates $|n\rangle = \hat{u}^n |0\rangle$ of the electric Hamiltonian, which satisfy

$$\hat{\ell}^2 |n\rangle = n^2 |n\rangle$$

Note that each plaquette intersects the Wilson line either not at all, or precisely once, and in one link only. A short computation shows that

$$\langle 0| \hat{u} |0\rangle = \frac{1}{|G|} \sum_g \pi_F(g) = 0$$

This is zero because of the orthogonality theorem (see A.1). Therefore the first-order contribution in perturbation theory vanishes: we need at least second-order. Physically, this is because we need to excite at least a whole plaquette in order to get a non-zero magnetic flux. In order to compute the second-order contribution, we need to compute all the non-zero matrix elements of the form

$$\sum_{\square} \langle \psi | \operatorname{Re} \operatorname{tr} \hat{W}_{\square} \hat{W}_{mn}(C) |S\rangle$$

for any gauge invariant eigenstate $|\psi\rangle$ of the electric Hamiltonian. Most of these contributions will be constants independent of r, and can be ignored; in other words, we're only interested in excitations above the perturbed vacuum, which will be given by a sea of fluctuating plaquettes [15]. Therefore we are only interested in those single-plaquette states which share exactly one link with the Wilson line. This can happen in two ways: the shared link is in the positive direction for both, or it is in the positive direction for the Wilson line and in the negative direction for the plaquette loop.

In the case where the plaquette intersects the Wilson line in opposite directions, on the shared link the Wilson line acts by $\hat{u}$, while the plaquette by $\hat{u}^\dagger$. Since $\hat{u}^\dagger \hat{u} = 1$ we see

that these cancel each other out, and overall effect is that the Wilson line has fluctuated from its original position [15], as shown in figure 4.2a. There is a single state which has a non-zero overlap with the above state, which is the perturbed Wilson line itself. In this case the electric field energy has increased by 2 units in total. Moreover, we can also compute the number of plaquettes which intersect the Wilson line, which is given $2(d-1)$ in d spatial dimensions. This is because the Wilson line is contained in $d-1$ independent planes, each of which contains two plaquettes which contain the given link. In each dimension, one plaquette will traverse the link in the positive direction, while the other one will traverse it in the negative direction. However, since both $\hat{W}_\square$ and $\hat{W}_\square^\dagger$ are included in the Hamiltonian, the two give the same contribution. Also note that all Wilson lines are automatically normalised in the U(1) theory. Remembering the extra factor of 1/2 in the magnetic term for U(1), the contribution of this case in perturbation theory is then given by

$$2(d-1)\frac{r}{a}\frac{\left(\frac{1}{2g^2a^{4-d}}\right)^2}{2\frac{g^2}{2a^{d-2}}} = (d-1)\frac{1}{2g^6a^{11-3d}}r$$

In the case where the plaquette intersects the Wilson line in the same direction, the shared link now gets excited to the second-highest level [15], as shown in figure 4.2b. Thus the electric energy increases by 6 units in total. All the states are still normalised, which leads to a contribution which is a third of the above. In total, we find the corrected energy to be

$$E \approx \frac{g^2}{2a^{d-1}}\left[1 - \frac{4}{3}(d-1)\frac{1}{g^8a^{12-4d}}\right]r \qquad G = \mathrm{U}(1)$$

The coefficient of the linear potential is the *string tension*

$$\sigma \approx \frac{g^2}{2a^{d-1}}\left[1 - \frac{4}{3}(d-1)\frac{1}{g^8a^{12-4d}}\right] \qquad G = \mathrm{U}(1)$$

As a consequence, the strong-coupling renormalisation group equation for the dimensionless coupling $\widetilde{g} = ga^{(3-d)/2}$ is

$$\frac{a}{\widetilde{g}}\frac{d\widetilde{g}}{da} = 1 - \frac{16\,(d-1)}{3}\frac{1}{\widetilde{g}^8}$$

which is obtained by imposing that the string tension is unchaged as g varies with a. We'll now consider the finite group case.

4.2.2 String tension in finite-group gauge theories

The setting is the same as in the Lie-group case, and the calculations can be performed in a similar manner. We will not repeat here all the details, except where they differ

from the previous case. Consider first $\mathbb{Z}_N$ gauge theory. Then the $\hat{u}$ operator generates the eigenstates of the electric Hamiltonian, as in section 3.3.1:

$$\hat{u}\,|j\rangle = |j+1\rangle$$

where $\{|j\rangle\}$ is the representation basis. Therefore, if $f(j)$ is the function describing the electric energy density, then the leading order potential between two quarks is given by

$$E = \lambda_E f(1)\frac{r}{a} \qquad G = \mathbb{Z}_N$$

this essentially boils down to our choice of $j = 1$ as the fermion representation in section 3.3.1. We can then compute the first order correction via perturbation theory. The main difference with the Lie-group case is that in the first case (where the Wilson line and the plaquette intersect in opposite directions) the increase in electric energy is now given by $2f(1)$, while in the second case (where they intersect in the same direction) it is given by $f(2) + 2f(1)$. This leads to the corrected energy

$$E = \lambda_E \frac{r}{a}\left[f(1) - 2(d-1)\left(\frac{\lambda_B}{\lambda_E}\right)^2\left(\frac{1}{2f(1)} + \frac{1}{2f(1)+f(2)}\right)\right]$$

The string tension follows immediately and one could compute the strong-coupling beta function, although it appears to be of limited use.

In the general non-Abelian case the calculation is harder. In order to describe the Hilbert space we need to generalise the $\hat{u}_{mn}$ operators to a general representation. This is done immediately, by defining $\hat{u}^j_{mn}$ for the j^{th} representation by the requirement that

$$\hat{u}^j_{mn}\,|g\rangle = [\pi^j(g)]_{mn}\,|g\rangle$$

In this language the previous $\hat{u}_{mn}$ operators coincide with $\hat{u}^F_{mn}$ where F is the chosen representation for the fermions. We see that we can represent these new operators as

$$\hat{u}^j_{mn} = \sum_g [\pi^j(g)]_{mn}\,|g\rangle\,\langle g|$$

It then follows that

$$\hat{u}^j_{mn}\,|0\rangle = \frac{1}{\sqrt{|G|}}\sum_g [\pi^j(g)]_{mn}\,|g\rangle = \frac{1}{\sqrt{\dim j}}\,|jmn\rangle$$

where $|0\rangle$ is the trivial representation singlet. Therefore the $\hat{u}^j_{mn}\,|0\rangle$ states are all the eigenstates of the single-link Hamiltonian, with eigenvalue $f(j)$. Since the Wilson line is formed using the $\hat{u}_{mn}$ operator, i.e. those in the chosen representation for the fermions, it follows that it is an eigenstate of the electric Hamiltonian with eigenvalue

$$E = \lambda_E f(F)\frac{r}{a}$$

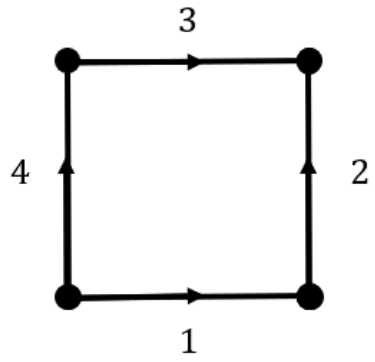


Figure 4.3: The four links on a single plaquette

so we see that here, as in the Lie group case, the coefficient of the potential depends on the fermionic representation. One can then compute corrections to the above energy, and these will also start at $\mathcal{O}(1/g^6)$ like in the Lie-group case. However, there is a complication: if the F representation is non-Abelian as one would expect, at each link the eigenstate with energy $\lambda_E f(F)$ has degeneracy equal to the square of the dimension of the F representation. The matrix elements would still vanish, and this means that the correction would be second order in degenerate perturbation theory. While the calculation can in principle be done, it is complicated and the end result is of limited use.

4.3 Pure gauge theory on a single plaquette

We will now study pure gauge theory on a single plaquette. Even in the single-plaquette case, the solution in the SU(N) case is non-trivial [67]. Here we will consider the case of a generic finite group. Label the four links $1, 2, 3, 4$ as in figure 4.3.

Then a generic state is given by

$$|\psi\rangle = \sum_{\{g\}} \psi(g_1, g_2, g_3, g_4) \, |g_1\rangle \, |g_2\rangle \, |g_3\rangle \, |g_4\rangle$$

where $\{g\} = g_1, g_2, g_3, g_4$. Under a general gauge transformation given by group elements

$a, b, c, d \in G$ in the four sites, we get

$$|\psi\rangle \mapsto \sum_{\{g\}} \psi(g_1, g_2, g_3, g_4) \, |ag_1b^{-1}\rangle \, |bg_2c^{-1}\rangle \, |dg_3c^{-1}\rangle \, |ag_4d^{-1}\rangle =$$
$$= \sum_{\{g\}} \psi(a^{-1}g_1b, b^{-1}g_2c, d^{-1}g_3c, a^{-1}g_4d) \, |g_1\rangle \, |g_2\rangle \, |g_3\rangle \, |g_4\rangle$$

The transformation on the last two links is opposite what one might naively expect, as they are in the negative orientation. Therefore, if $|\psi\rangle$ is a gauge-invariant state, its wavefunction must satisfy

$$\psi(a^{-1}g_1b,\, b^{-1}g_2c,\, d^{-1}g_3c,\, a^{-1}g_4d) = \psi(g_1, g_2, g_3, g_4)$$

for all $a, b, c, d \in G$. In order to form a valid function, we must therefore take products of the arguments so that the various group elements cancel. For a single plaquette, there is only one way to do that. We get that

$$\psi(g_1, g_2, g_3, g_4) = \psi(g_1g_2g_3^{-1}g_4^{-1})$$

Now, under a gauge transformation,

$$\psi(g_1g_2g_3^{-1}g_4^{-1}) \mapsto \psi(a^{-1}g_1g_2g_3^{-1}g_4^{-1}a)$$

which means that gauge invariance requires ψ to be a class function. In other words, it must satisfy $\psi(a^{-1}xa) = \psi(x)$ for any $a, x \in G$. It turns out that such functions can be expanded in a basis of irreducible characters (see section A.2),

$$\psi(g) = \sum_i \psi_i \chi_i(g)$$

where the $\{\chi_i\}$ are the irreducible characters of G and ψ_i are constants. It follows that

$$|\psi\rangle = \sum_{\{g\}} \sum_i \psi_i \chi_i(g_1g_2g_3^{-1}g_4^{-1}) \, |g_1\rangle \, |g_2\rangle \, |g_3\rangle \, |g_4\rangle$$

We can now break up the $\{g\}$ sum over the group elements as a sum over plaquette states,

$$|\psi\rangle = \sum_g \sum_i \phi_i \chi_i(g) \, |g\rangle_\square$$

where we have defined

$$|g\rangle_\square = \sum_{\{g\}'} |g_1\rangle \, |g_2\rangle \, |g_3\rangle \, |g_4\rangle$$

where with $\{g\}'$ we mean the sum over all the g_1, g_2, g_3, g_4 such that $g_1 g_2 g_3^{-1} g_4^{-1} = g$. In other words, $|g\rangle_\square$ is the superposition of all those group element states whose plaquette product is g. A short calculation shows that the $\{|g\rangle_\square\}$ states are orthonormal, and as such they form a basis for the overall, non-physical Hilbert space.

Since the wavefunction can be expanded in a basis of characters, we now see that the single-plaquette Hilbert space has the same dimension as the the number of irreducible representations of G, or equivalently, as the number of conjugacy classes of G. This corresponds to two possible bases for the "physical" Hilbert space, dual to each other. The first one is the *character basis* given by the states

$$|i\rangle = \frac{1}{\sqrt{|G|}} \sum_g \chi_i(g) \, |g\rangle_\square$$

A short calculation using the orthogonality theorem for characters shows that this basis is orthonormal. A state can then be expanded as

$$|\psi\rangle = \sum_i a_i \, |i\rangle$$

with $\sum_i |a_i|^2 = 1$. The character basis is analogous to the representation basis, and it turns out that it diagonalises the electric Hamiltonian.

On the other hand, as the characters are class functions they are constant on conjugacy classes, which means that one can also write

$$|\psi\rangle = \sum_C b_C \, |C\rangle$$

where $\{C\}$ are the conjugacy classes of G, with the orthonormal basis elements defined via

$$|C\rangle = \frac{1}{\sqrt{|C|}} \sum_{g \in C} |g\rangle_\square$$

where $|C|$ is the size of the conjugacy class and the constants are related to the previous ones via

$$b_C = \sqrt{|C|} \sum_i \psi_i \chi_i(C)$$

The duality relation between the two bases is given by

$$\langle C|i\rangle = \sqrt{\frac{|C|}{|G|}} \chi_i(C)$$

where $|C|$ is the size of the conjugacy class C.

The Kogut-Susskind Hamiltonian is given by

$$H = \lambda_E \sum_{\text{links}} \sum_{l} f(l) P_l - 2\lambda_B \sum_{\square} \text{Re}\,\text{tr}\,\hat{W}_{\square} = H_E + H_B$$

where the constants λ_E and λ_B can be chosen to be the same as the continuum constants, or they can be chosen to be independent parameters. The function $f(l)$ is for now arbitrary.

We can now compute the matrix elements of the Hamiltonian in the character basis. We first compute the matrix elements of the electric Hamiltonian. We see that

$$\langle i| H_E |j\rangle = \lambda_E \sum_{l} f(l) \langle i| \sum_{e\in\text{links}} P_l(e) |j\rangle$$

where as usual $P_l(e)$ is the projector onto the l^{th} representation subspace at link e. So we need to compute the matrix elements of the projector operator. These are given by

$$\langle i| \sum_{e\in\text{links}} P_l(e) |j\rangle = \frac{1}{|G|} \sum_{g,h} \chi_i(h)^* \chi_j(g)\, {}_{\square}\langle h| \sum_{e\in\text{links}} P_l(e) |g\rangle_{\square}$$

A short calculation shows that on each link,

$$P_l |g\rangle = \sum_{k\in G} \chi_l(g^{-1}k) |k\rangle$$

so that, on each plaquette group element state, the projector acts as

$$\begin{aligned}\sum_{e\in\text{links}} P_l(e) |g\rangle_{\square} = \sum_{\{g\}'} \sum_{k} \big[&\chi_l(g_1^{-1}k) |k\rangle |g_2\rangle |g_3\rangle |g_4\rangle + \\ &+ \chi_l(g_2^{-1}k) |g_1\rangle |k\rangle |g_3\rangle |g_4\rangle + \\ &+ \chi_l(g_3^{-1}k) |g_1\rangle |g_2\rangle |k\rangle |g_4\rangle + \\ &+ \chi_l(g_4^{-1}k) |g_1\rangle |g_2\rangle |g_3\rangle |k\rangle\big]\end{aligned}$$

Then, since

$${}_{\square}\langle h| |g_1\rangle |g_2\rangle |g_3\rangle |g_4\rangle = \delta(h, g_1 g_2 g_3^{-1} g_4^{-1})$$

we finally obtain

$${}_{\square}\langle h| \sum_{e\in\text{links}} P_l(e) |g\rangle_{\square} = 2|G|^3 \left[\chi_l(hg^{-1}) + \chi_l(h^{-1}g)\right]$$

This leads to

$$\langle i| \sum_{e\in\text{links}} P_l(e) |j\rangle = 2|G|^2 \sum_{g,h} \chi_i(h)^* \chi_j(g) \left[\chi_l(hg^{-1}) + \chi_l(h^{-1}g)\right]$$

which can be rewritten as a convolution of characters (see section A.2):

$$\langle i| \sum_{e\in\text{links}} P_l(e) |j\rangle = 2|G|^2 \sum_g \chi_j(g) \left[\chi_l \star \chi_i(g^{-1}) + \chi_l \star \chi_i^*(g)\right]$$

which then leads to (see section A.2)

$$\langle i| \sum_{e\in\text{links}} P_l(e) |j\rangle = \frac{2|G|^3}{\dim(l)} \left[\delta_{il}\delta_{lj} + \delta_{i^*l}\delta_{j^*l}\right]$$

where i^* is the index associated with the representation π_i^* dual to the representation π_i indexed by i. Then, finally we obtain

$$\langle i| H_E |j\rangle = 2\lambda_E |G|^3 \delta_{ij} \frac{f(i) + f(i^*)}{\dim(i)}$$

where we used the fact that $\delta_{i^*j^*} = \delta_{ij}$ and that $\dim(i^*) = \dim(i)$.

The matrix elements of the magnetic Hamiltonian can then be easily computed, as

$$\langle i| H_B |j\rangle = -2\lambda_B \frac{1}{|G|} \sum_{g,h} \chi_i(h)^* \chi_j(g) \, {}_\square\langle h| \operatorname{Re}\operatorname{tr}\left(\hat{u}_1\hat{u}_2\hat{u}_3^\dagger\hat{u}_4^\dagger\right) |g\rangle_\square$$

and

$$\operatorname{Re}\operatorname{tr}\left(\hat{u}_1\hat{u}_2\hat{u}_3^\dagger\hat{u}_4^\dagger\right) |g\rangle_\square = \operatorname{Re}\chi_F(g) \, |g\rangle_\square$$

where F is the chosen representation for the fermions. Then

$$\langle i| H_B |j\rangle = -2\lambda_B \frac{1}{|G|} \sum_g \chi_i(g)^* \chi_j(g) \operatorname{Re}\chi_F(g)$$

so that overall

$$\langle i| H |j\rangle = 2\lambda_E |G|^3 \delta_{ij} \frac{f(i) + f(i^*)}{\dim(i)} - 2\lambda_B \frac{1}{|G|} \sum_g \chi_i(g)^* \chi_j(g) \operatorname{Re}\chi_F(g)$$

Using the duality transformation, we can also express the above Hamiltonian in terms of the conjugacy class basis. The matrix element between two conjugacy classes A, B is given by

$$\langle A| H |B\rangle = 2|G|^2 \lambda_E \sqrt{|A|\,|B|} \sum_i \chi_i(A)\chi_i(B)^* \left[f(i) + f(i)^*\right] - 2\lambda_B \frac{|A|^4}{|G|^2} \operatorname{Re}\chi_F(A)\delta_{AB}$$

As we can see, now the magnetic term is diagonal, while the electric term is not, the opposite than in the previous case.

4.3.1 $\mathbb{Z}_N$ gauge theory

We can now construct the Hamiltonian explicitly for some simple gauge groups. In the $\mathbb{Z}_N$ case (compare with section 3.3.1) the representations are all one-dimensional, so the gauge-invariant subspace has the same size as the overall Hilbert space. The characters are simply given by the representations themselves,

$$\chi_j(g^k) = \exp\left(\frac{2\pi i}{N} jk\right)$$

the dual representation to π_j is given by $\pi_j^* = \pi_{N-j}$ (where this is interpreted mod N, so that $0^* = 0$).

Taking $f(j)$ with the group Laplacian like in section 3.2 we get

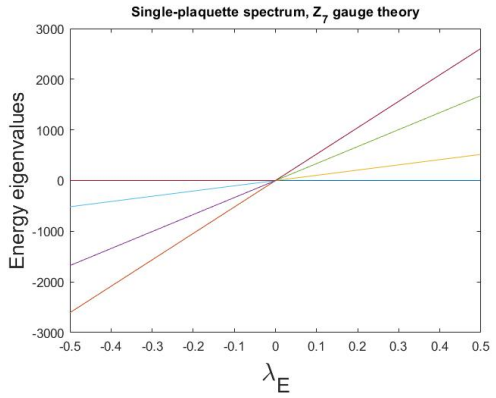
$$f(j) = f(N-j) = 4\sin^2\left(\frac{\pi j}{N}\right)$$

so that overall

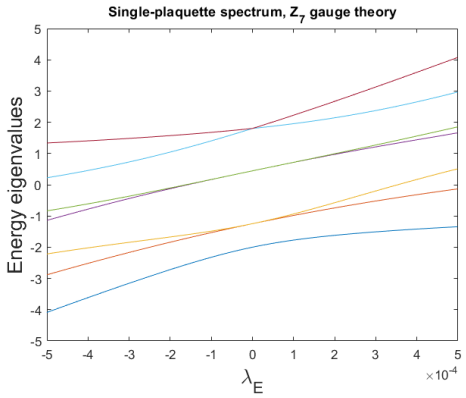
$$\langle i|\, H\, |j\rangle = 16\lambda_E N^3 \delta_{ij} \sin^2\left(\frac{\pi j}{N}\right) - \lambda_B \left[\delta_{i,j+1} + \delta_{i,j-1} + \delta_{i,0}\delta_{j,N-1} + \delta_{i,N-1}\delta_{j,0}\right]$$

This Hamiltonian can be solved exactly in both the electric and magnetic limit. Keeping only the electric term, the Hamiltonian is diagonal with energies $16\lambda_E N^3 \sin^2\left(\frac{\pi j}{N}\right)$ for $j = 0, 1, \ldots, N-1$. These are not all different, as they are symmetric under $j \to N - j$. In the case N odd, therefore, the eigenvalues are degenerate in pairs except for the $j = 0$ eigenvalue. For N even, they are again degenerate in pairs, except for the $j = 0$ and $j = N/2$ eigenvalues. In the magnetic limit, we set $\lambda_E = 0$. The resulting matrix is circulant, which allows the calculation of the eigenvalues immediately, $-2\lambda_B \cos\left(\frac{2\pi j}{N}\right)$. This is also symmetric under $j \to N - j$ and therefore in the magnetic limit we find a degeneracy similar to the one in the electric limit.

The results of the simulation are shown in figure 4.4 for $\mathbb{Z}_7$ gauge theory for different ranges of λ_E, while keeping $\lambda_B = 1$. If we choose λ_E like in the continuum, it must be positive; however, here we chose to vary it from negative to positive to highlight the behaviour at $\lambda_E = 0$. The qualitative behaviour of the theory is the same for any N. Due to the N^3 term in the Hamiltonian, the electric part dominates already for relatively small values of λ_E, and the magnetic part can be treated as a perturbation. This is shown in figure 4.4a, where we would expect seven states and we find only four. This is because the eigenvalues of the electric Hamiltonian are degenerate in pairs except for the one closest to zero; the degeneracy is broken by the magnetic term, but the scale of the picture is too large to appreciate this. In order to see the effect of the magnetic term, one has to go to very small values of λ_E, as shown in figure 4.4b. Here we find that all the excited states undergo level crossings at $\lambda_E = 0$, where their energies become



(a) Eigenvalues for $\mathbb{Z}_7$ gauge theory (large scale)



(b) Eigenvalues for $\mathbb{Z}_7$ gauge theory (near zero)

Figure 4.4: Exact solution on a single plaquette $\mathbb{Z}_7$ gauge theory for different ranges of λ_E. We set $\lambda_B = 1$.

exactly degenerate in pairs, as seen in the previous section. However, there is no level crossing for the ground state, and as such, no phase transition. The case N even is similar, except now there are two singlet states. There is again no phase transition.

We also tried keeping $\lambda_E = 1$ and varying λ_B, as shown in figure 4.5a for $\mathbb{Z}_7$ gauge theory. Due to the N^3 factor in the electric Hamiltonian, here we need to go to large values of λ_B to observe interesting structure. The situation is similar to the previous case, with level crossings in the excited states, but no phase transition.

4.3.2 D_N gauge theory

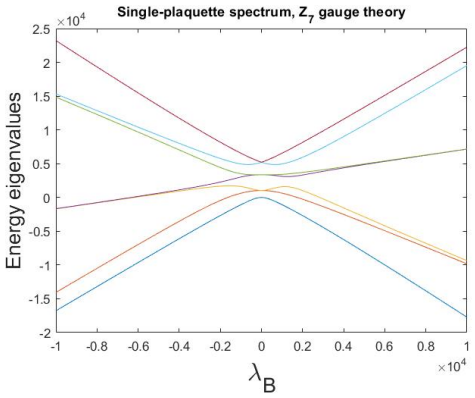
We'll construct explicitly another case, that of D_N gauge theory with odd N (again, compare with section 3.3.2). In this case, there are $2N$ elements, but only $(N+1)/2$ conjugacy classes, which is also the dimension of the Hilbert space. We order the matrix elements starting from -1. In this case, all the characters are real, and given by

$$\chi_0(g) = 1 \qquad \chi_{-1}(r^k) = 1, \;\; \chi_{-1}(r^k s) = 1$$

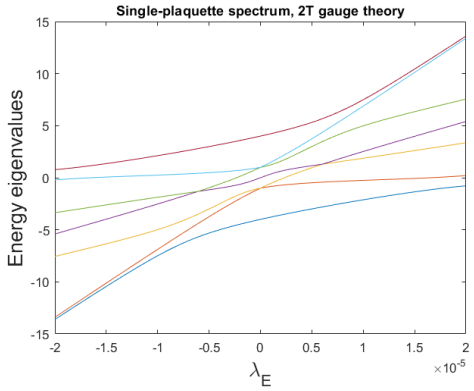
$$\chi_j(r^k) = w \cos\left(\frac{2\pi jk}{N}\right) \qquad \chi_j(r^k s) = 0$$

We will leave f undetermined here. However, all D_N representations are self-conjugate, so that $i^* = i$. In the character basis the Hamiltonian becomes a block matrix

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$



(a) Varying λ_B instead of λ_E for $\mathbb{Z}_7$



(b) Eigenvalues for 2T gauge theory

Figure 4.5: More exact solutions on a single plaquette.

with $H_{21} = H_{12}^\dagger$, where H_{11} is 2×2 (corresponding to the two one-dimensional irreps), H_{12} is $2 \times (N-1)/2$ and H_{22} is $(N-1)/2 \times (N-1)/2$ (corresponding to the non-Abelian irreps). We find

$$(H_{11}) = 32N^3\lambda_E \begin{pmatrix} f(-1) & 0 \\ 0 & f(1) \end{pmatrix}$$

Choosing the fermionic representation to be the non-Abelian $j = 1$ representation, we see that

$$(H_{12})_{-1j} = (H_{12})_{0j} = -2\lambda_B \delta_{j1}$$

for $j = 1, \ldots (N-1)/2$. Moreover,

$$(H_{22})_{ij} = 16N^3\lambda_E \delta_{ij} f(i) - 2\lambda_B \left(\delta_{i,j+1} + \delta_{i,j-1} + \delta_{i,(N-1)/2}\delta_{j,(N-1)/2}\right)$$

for $i, j = 1, \ldots (N-1)/2$.

4.3.3 Binary tetrahedral (2T) gauge theory

Finally, we consider the gauge theory of the binary tetrahedral group, one of the finite subgroups of SU(2). A summary of the relevant group properties can be found in appendix B. This group has seven complex inequivalent irreps, and as such, the physical Hilbert space on a single plaquette will have dimension 7. We set $\lambda_B = 1$ and vary λ_E as in the previous cases. The results are shown in figure 4.5b. There is no significant difference with the previous cases. There are a number of level crossings among excited states at $\lambda_E = 0$ and for very small non-zero values of λ_E, but again no phase transition.

4.4 Variational ground state

Even in the case of pure gauge theory, the complex interplay between the electric term and the plaquette magnetic term make the Hamiltonians that we've considered very hard to solve exactly. In the Lie group case, some progress has been made in the understanding of the eigenstates of the Hamiltonian via the variational method [17, 18, 68]. It can be used to investigate both the ground state and the excited states of the pure gauge theory [17]. While this technique works in principle for any number of dimensions and any gauge group, the calculations are in general still hard, and the minimisation of the expectation value must be performed numerically in the general case [17]. In the next sections we consider a specific ansatz in $2+1$ dimensions, for which we are able to compute the expectation values analytically. It should be emphasised that due to the different structure of the electric term, calculations of expectation values are harder in finite group gauge theories than in their continuum counterpart. We analyse the results for various finite gauge groups and compare to the continuum case. Finally, we discuss the implications for the ground state structure of our theories.

4.4.1 The one-plaquette ansatz

It has been argued [68] that the vacuum wave functional for continuum Yang-Mills theory should take the approximate form

$$\psi = \exp\left[-\text{const} \times \int d^d x\, \mathbf{B}^2\right]$$

where $\mathbf{B}$ is the chromomagnetic field. The corresponding form on the lattice is given by the exponential of the plaquette action acting on the strong-coupling vacuum [18]:

$$|\psi_\alpha\rangle = \exp\left(\alpha \sum_{\square} \mathrm{Re}(\mathrm{tr}\, \hat{W}_\square)\right) |S\rangle \tag{4.1}$$

where α is a variational parameter, fixed by minimising the expectation value of the Hamiltonian. This is known as the "one-plaquette ansatz" [17]. We now turn to the computation of the expectation values.

The expectation value of the magnetic Hamiltonian boils down to that of a single plaquette. Since the above ansatz is translationally invariant, the expectation value of any plaquette will be the same. Let

$$P = \sum_{\square} \mathrm{Re}(\mathrm{tr}\, \hat{W}_\square)$$

Then if we call $P_\star = \mathrm{Re}\,\mathrm{tr}\, \hat{W}_\star$ for a specific plaquette $\star$, since $H_B = -2\lambda_B P$, we find that

$$\langle\psi_\alpha|\, H_B\, |\psi_\alpha\rangle = -2\lambda_B \langle S|\, e^{\alpha P} P e^{\alpha P}\, |S\rangle = -2\lambda_B \sum_{\square} \langle S|\, e^{2\alpha P} P_\star\, |S\rangle$$

The single-plaquette expectation value can then be computed by inserting twice a complete basis of states $|\{g\}\rangle$ in the group element basis for each lattice link, where $\{g\}$ denotes the collection of group elements on the links. We have that

$$\langle S|\{g\}\rangle = \frac{1}{|G|^{N_L/2}}$$

where N_L is the number of links. Since the plaquette operators are diagonal in the group element basis, we obtain

$$\langle S|\, e^{2\alpha P} P_\star \,|S\rangle = \frac{1}{|G|^{N_L}} \sum_{\{g\}} e^{2\alpha P(\{g\})} P_\star(\{g\})$$

Now each summation may be interpreted as the Haar measure on a finite group:

$$\frac{1}{|G|}\sum_g = \int dg$$

It has been shown [35, 69] that the Jacobian of the change of variables from links to plaquettes is given by a lattice Bianchi identity, similarly to what was seen in section 1.1. In particular, this means that in two discretised dimensions, the change of variables has unit Jacobian [18, 36]. Therefore in the above we can sum over plaquette variables rather than over link variables, leading to

$$\langle S|\, e^{2\alpha P} P_\star \,|S\rangle = \frac{1}{|G|^{N_P}} \sum_{\{g_\square\}} e^{2\alpha P(\{g_\square\})} \operatorname{Re}\chi_F(g_\star)$$

where N_P is the number of plaquettes, χ_F is the character of the chosen representation of the fermions and $\{g_\square\}$ is a complete set of plaquette variables. We can then factorise the exponential into the product of single-plaquette exponentials, each of which will be integrated individually. Since $g_\star$ is a specific plaquette, it is only included in one integral, leading to

$$\langle S|\, e^{2\alpha P} P_\star \,|S\rangle = \left[\frac{1}{|G|}\sum_{g_\square} e^{2\alpha \operatorname{Re}\chi_F(g_\square)}\right]^{N_P-1} \frac{1}{|G|}\sum_{g_\star} e^{2\alpha \operatorname{Re}\chi_F(g_\star)} \operatorname{Re}\chi_F(g_\star)$$

A similar calculation shows that

$$\langle \psi_\alpha|\psi_\alpha\rangle = \langle S|\, e^{2\alpha P} \,|S\rangle = \left[\frac{1}{|G|}\sum_{g_\square} e^{2\alpha \operatorname{Re}\chi_F(g_\square)}\right]^{N_P}$$

Overall, the Rayleigh-Ritz ratio for the magnetic Hamiltonian will then be given by

$$\frac{\langle \psi_\alpha|\, H_B \,|\psi_\alpha\rangle}{\langle \psi_\alpha|\psi_\alpha\rangle} = -2\lambda_B N_P \frac{\sum_g e^{2\alpha \operatorname{Re}\chi_F(g)} \operatorname{Re}\chi_F(g)}{\sum_g e^{2\alpha \operatorname{Re}\chi_F(g)}}$$

It appears that in general the above sum has no simple analytic expression, but it is nonetheless simple to compute numerically.

The electric field term is more complicated. However, its expectation value can be computed similarly. By again twice inserting a complete basis of lattice group element states,

$$\langle\psi_\alpha|\, H_E\,|\psi_\alpha\rangle = \frac{1}{|G|^{N_L}} \sum_{\{g\},\{h\}} e^{\alpha P(\{g\})+\alpha P(\{h\})}\, \langle\{g\}|\, H_E\,|\{h\}\rangle$$

On a single link we use the result from the end of section 3.2, valid for general f:

$$\langle g|\, H_E\,|h\rangle = \lambda_E \frac{1}{|G|} \sum_j f(j)\dim(j)\chi_j(gh^{-1}) \equiv \lambda_E \widetilde{f}(gh^{-1})$$

Then on the whole lattice

$$\langle\psi_\alpha|\, H_E\,|\psi_\alpha\rangle = \frac{1}{|G|^{N_L}}\lambda_E \sum_{e\in\text{links}} \sum_{\{g\},\{h\}} e^{\alpha P(\{g\})+\alpha P(\{h\})}\, \widetilde{f}(g_e h_e^{-1}) \prod_{e'\neq e} \delta(g_{e'}, h_{e'})$$

where g_e, h_e are the group elements at link e and the δ are delta functions. Evaluating the delta functions and setting $k = g_e h_e^{-1}$ we get

$$\langle\psi_\alpha|\, H_E\,|\psi_\alpha\rangle = \frac{1}{|G|^{N_L}}\lambda_E \sum_{e\in\text{links}} \sum_{\{g\},k} e^{\alpha P(\{g\})+\alpha P(\{g\}')}\, \widetilde{f}(k)$$

where now $\{g\}'$ is the same configuration as $\{g\}$ except at link e, where the element g_e is replaced by $k^{-1}g_e$. We will now go again from link variables to plaquette variables for the $\{g\}$. In order to do so, we recognise that there are two plaquettes which intersect link e, one in the same direction as the link, and the other one in the opposite direction. Upon a proper reordering of the link variables of the two plaquettes, the end result is that one will be multiplied by k, and the other one by k^{-1}. The sum factorises again:

$$\langle\psi_\alpha|\, H_E\,|\psi_\alpha\rangle = \left[\frac{1}{|G|}\sum_{g_\square} e^{2\alpha\,\mathrm{Re}\,\chi_F(g_\square)}\right]^{N_P-2} \lambda_E N_L \frac{1}{|G|^2}\sum_{g,h,k} e^{\alpha A(g,h,k)}\widetilde{f}(k)$$

with

$$A(g,h,k) = \mathrm{Re}\,\chi_F(g) + \mathrm{Re}\,\chi_F(h) + \mathrm{Re}\,\chi_F(kg) + \mathrm{Re}\,\chi_F(k^{-1}h) \tag{4.2}$$

where g and h are the two plaquettes variables intersecting e and as the result is independent of the link chosen, we have replaced the sum over links with the number of links. Therefore the Rayleigh-Ritz ratio is now

$$\frac{\langle\psi_\alpha|\, H_E\,|\psi_\alpha\rangle}{\langle\psi_\alpha|\psi_\alpha\rangle} = \lambda_E N_L \frac{\sum_{g,h,k} e^{\alpha A(g,h,k)}\widetilde{f}(k)}{\left[\sum_g e^{2\alpha\,\mathrm{Re}\,\chi_F(g)}\right]^2}$$

The variational energy density in units of the inverse lattice spacing is

$$\epsilon_0(\alpha) = \frac{a}{N_P} \frac{\langle\psi_\alpha| H |\psi_\alpha\rangle}{\langle\psi_\alpha|\psi_\alpha\rangle}$$

In other words,

$$\epsilon_0(\alpha) = \frac{1}{\beta} \frac{2\sum_{g,h,k} e^{\alpha A(g,h,k)} \widetilde{f}(k)}{\left[\sum_g e^{2\alpha \operatorname{Re}\chi_F(g)}\right]^2} - \beta \frac{\sum_g e^{2\alpha \operatorname{Re}\chi_F(g)} \operatorname{Re}\chi_F(g)}{\sum_g e^{2\alpha \operatorname{Re}\chi_F(g)}} \tag{4.3}$$

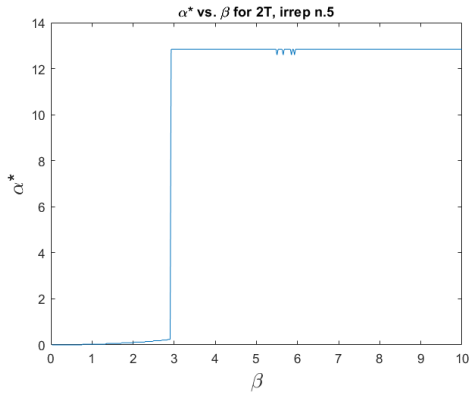
where A is given in eq. (4.2) and $\beta = 2a\lambda_B = 1/(a\lambda_E) = 2/(g^2 a)$ is a dimensionless parameter which follows from taking λ_E, λ_B as in eq. (2.7). The same parameter was adopted in previous works [17]. To obtain this formula we used $N_L = 2N_P$, which is true for an infinite lattice in two spatial dimensions. Now $\epsilon_0(\alpha)$ is minimised numerically for each β, so that α is determined as a function of β. The result of the minimisation procedure is a function $\alpha^\star(\beta)$, which gives the α at which the minimum is achieved as a function of β.

4.4.2 Finite subgroups of SU(2)

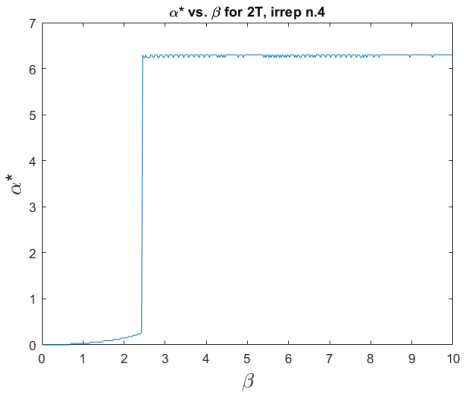
We now perform the minimisation procedure in some cases of interest. Here we discuss the case of the three "exceptional" finite subgroups of SU(2), the binary tetrahedral, octahedral and icosahedral groups. These are denoted by $2T$, $2O$ and $2I$ respectively. We choose the group Laplacian for the electric term, as described in section 3.2. Appendix B contains a summary of the main properties of these finite groups, together with their conjugacy classes and character tables. The choice of generating set Γ for the group Laplacian, together with the notation, the description of the group elements and the labels of irreps are also outlined in appendix B.

For each of the three groups, $\epsilon_0(\alpha)$ was minimised numerically, with different choices of non-Abelian representations. Some sample results of the minimisation procedure are shown in figure 4.6. As explained in the previous section, $\alpha^\star(\beta)$ is the α at which a minimum of $\epsilon_0(\alpha)$ is achieved for a given value of β. The results show great consistency for all the three groups and the choice of representation. For small β the critical α is found near 0, while a sharp transition occurs at a finite β_c, where α reaches a large finite value. The qualitative features of the behaviour of $\alpha^\star(\beta)$ are found to be the same across the whole range of the three exceptional finite subgroups of SU(2) and the choice of their non-Abelian representations, although the value of β_c and that of α at the plateau differ. We find that, using representations of the same size, β_c increases with the order of the group. A similar result was found in [31]. This is consistent with the fact that SU(2) shows a single, confined phase in $2+1$ dimensions [17, 46, 31].

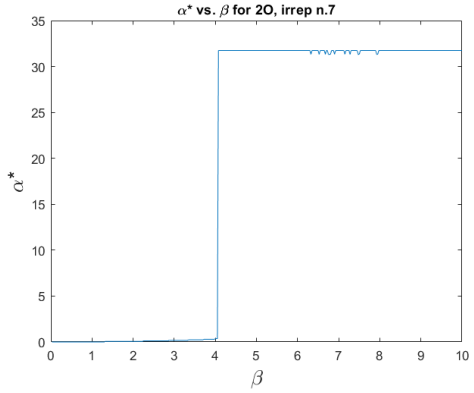
The situation is further elucidated by plotting $\epsilon_0(\alpha)$ against α for values of β before and after the transition, in figure 4.7. We see that while for small β the expectation



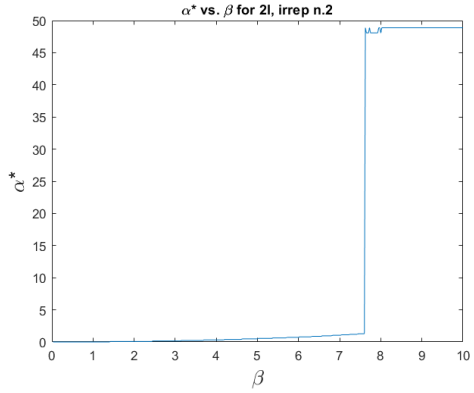
(a) Binary tetrahedral group, irrep 5



(b) Binary tetrahedral group, irrep 4

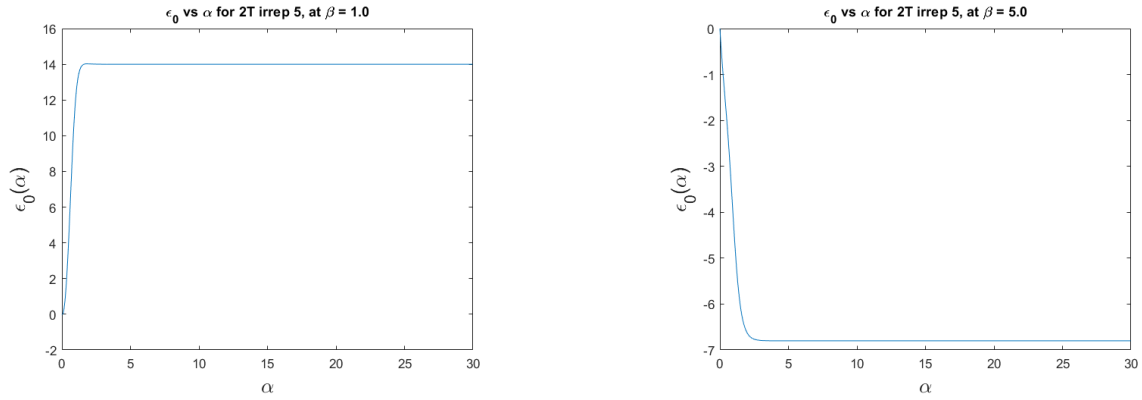


(c) Binary octahedral group, irrep 7



(d) Binary icosahedral group, irrep 2

Figure 4.6: $\alpha^\star(\beta)$ vs. β for various combinations of finite subgroups of SU(2) and their irreps. For the notation, see appendix B.



(a) Before the transition, at $\beta = 1.0$ (b) After the transition, at $\beta = 5.0$

Figure 4.7: $\epsilon_0(\alpha)$ vs. α for the Binary Tetrahedral group 2T with irrep 5, before and after the transition at $\beta_c \approx 3$.

value $\epsilon_0(\alpha)$ does achieve a minimum (and in fact at $\alpha = 0$ it is the exact ground state for $\beta \to 0$), the same is not true for β beyond the critical value; in this case the energy asymptotes on its minimum value, rendering the exact value of α in this region essentially meaningless. It thus appears that the one-plaquette variational wavefunction is unable to capture the structure of the ground-state beyond a possible transition. Similar evidence of a phase transition based on a trial wavefunction is found in the Abelian case in [70]. We will discuss possible improvements on the one-plaquette ansatz in section 4.4.5

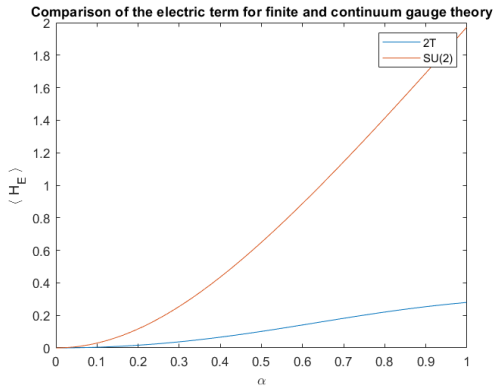
To compare, let us summarise the results for SU(2) in its defining two-dimensional representation. In that case [17] one finds

$$-\frac{1}{2N_P\lambda_B}\langle H_B\rangle = 2\frac{I_2(4\alpha)}{I_1(4\alpha)}$$

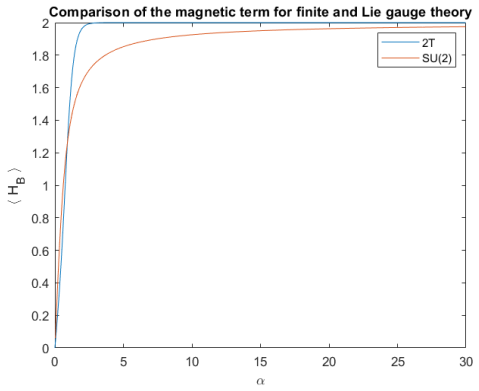
where I_ν are modified Bessel functions, and

$$\frac{1}{N_P\lambda_E}\langle H_E\rangle = 3\alpha\frac{I_2(4\alpha)}{I_1(4\alpha)}$$

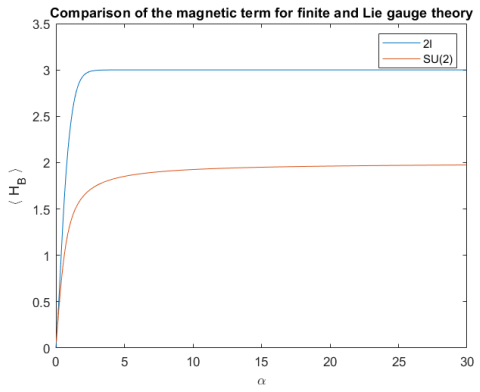
for an infinite lattice. Comparing with the finite group theory, one finds that the SU(2) magnetic term is well-approximated by all three finite gauge groups as long as one chooses their two-dimensional representation. Figure 4.8b shows that the even the smallest of the three groups, the Binary Tetrahedral group, approximates the SU(2) magnetic term well. For the higher order groups the approximation is better, but only as long as their two-dimensional representation is chosen. For example, figure 4.8c shows that even the Binary Icosahedral group fails in approximating the SU(2) results if its irrep n.5, which is three-dimensional, is chosen. On the other hand, as shown in figure 4.8a for 2T, the



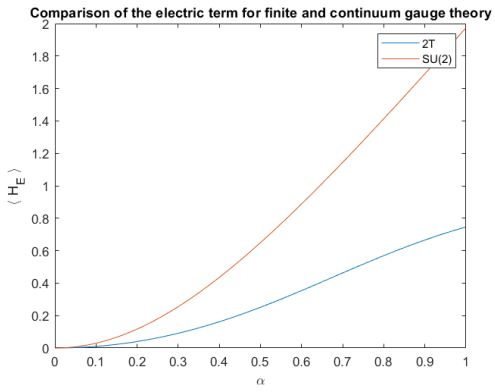
(a) 2T irrep 5 electric term



(b) 2T irrep 5 magnetic term



(c) 2I irrep 5 magnetic term



(d) 2T irrep 5 electric term, modified Γ

Figure 4.8: Comparison of the electric and magnetic one-plaquette expectation values for SU(2) and its finite subgroups in various cases.

electric term is not well-approximated by any finite gauge group. In fact, the finite groups asymptote to a finite value for large α, while for SU(2) the electric term grows linearly. This is because in the Lie group case the electric term is a differential operator capable of "bringing down" a factor of α from the exponential, unlike in the finite group case. It therefore does not appear possible to reproduce this behaviour with any choice of finite-group electric term. We investigated whether changing the generating set Γ for the finite group Laplacian (see section 3.2) might improve the electric term. Figure 4.8d shows the electric term for 2T, with a modified Γ made of all group elements except the identity. This should be compared with 4.8a. A slight improvement in the approximation is observed, but the general qualitative behaviour is unchanged.

A variational calculation for SU(2) shows that the one-plaquette ansatz is good for all β, in the sense that it achieves a minimum at a finite value of α. One finds that [17]

$$\alpha^\star(\beta) \approx \begin{cases} \frac{2}{3}\beta^2 & \beta \,\text{small} \\ \beta & \beta \,\text{large} \end{cases}$$

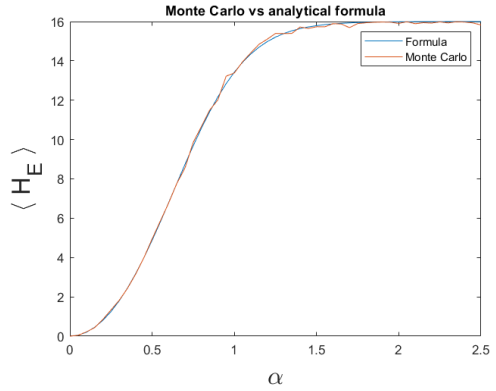
and the function interpolates smoothly between the two and there is no phase transition. In fact, SU(2) lattice gauge theory is known to have a single confining phase in 3 or less spatial dimensions [43, 46, 47]. However, it has been shown in the path-integral approach in both $3+1$ [31] and $2+1$ dimensions [43] that finite-group gauge theories, including those based on subgroups of SU(2), may have more phases than their parent Lie group. The situation was also discussed in section 2.1.1. It is therefore not surprising to find that the phase structure of a finite-group gauge theory may be more complicated than that of its parent Lie group, and that therefore a ground state ansatz that is valid for the latter may not be valid everywhere for the former.

4.4.3 Monte Carlo simulation

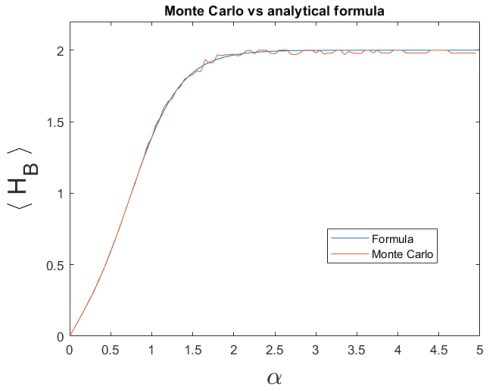
In order to obtain the formula for the variational energy we made use of the transformation from link variables to plaquette variables. To make sure that this is appropriate, we also computed the expectation value directly from the formula in terms of a sum over the links, by means of a Monte Carlo simulation. In general, given a basis x of the Hilbert space, one can write the expectation value of the Hamiltonian as

$$\langle H \rangle = \frac{\int dx \; |\psi_\alpha(x)|^2 \, (H\psi_\alpha(x)/\psi_\alpha(x))}{\int dx \; |\psi_\alpha(x)|^2}$$

Thus one can evaluate the expectation value $\langle H \rangle$ by sampling the distribution $|\psi_\alpha(x)|^2$ and computing the average of $(H\psi_\alpha(x)/\psi_\alpha(x))$. The comparison between the analytical formula with the link to plaquette transformations and the Monte Carlo simulation is shown for irrep 5 of the Binary Tetrahedral group in figure 4.9. As one can see, the two



(a) Expectation value of electric Hamiltonian



(b) Expectation value of magnetic Hamiltonian

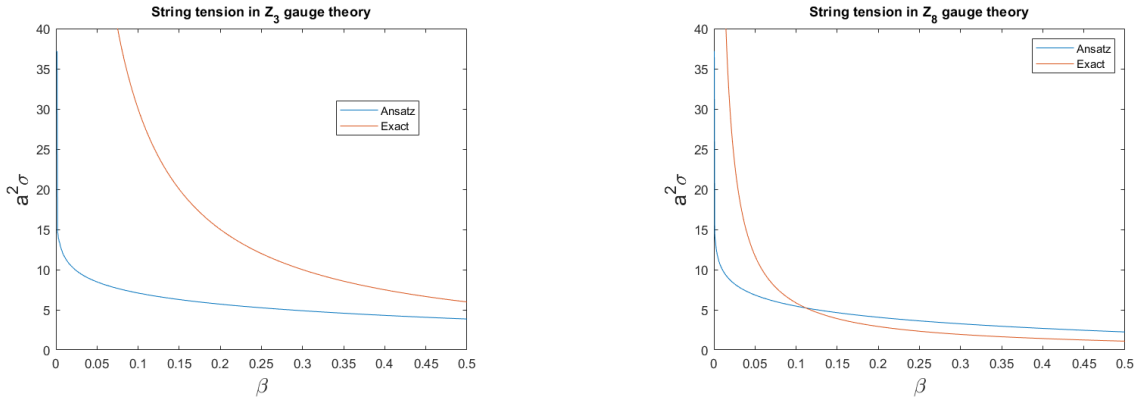
Figure 4.9: Comparison of the analytical formula for $\epsilon_0(\alpha)$ with a Monte Carlo simulation, for the electric and magnetic part separately, for the Binary Tetrahedral group irrep 5.

are in excellent agreement, although the analytical formula is computationally much less expensive and doesn't suffer from statistical fluctuations. These simulations were run on a 10×10 lattice using the Metropolis algorithm. Simulations on larger lattices and for other combinations of Lie groups and representations confirm the above picture.

4.4.4 Abelian case and string tension

The one-plaquette ansatz was also studied in the $\mathbb{Z}_N$ case in $2+1$ dimensions. The qualitative features of the results are the same as in the non-Abelian cases we've already seen. The ansatz leads to a minimum of $\epsilon_0(\alpha)$ as long as $\beta < \beta_c$ for some critical value β_c, while for $\beta > \beta_c$ we found that $\epsilon_0(\alpha)$ does not achieve a true minimum at any finite value of α. In this case, however, the increase of $\alpha^\star$ versus β in the strong-coupling phase is more marked. For $\mathbb{Z}_N$ gauge theory it was found that β_c increases roughly linearly with N. This confirms our expectations that for large N the $\mathbb{Z}_N$ theory should approach U(1) gauge theory, which in $2+1$ dimensions has a single, confined phase [43, 71]. In the $2+1$ path-integral calculations for $\mathbb{Z}_N$ gauge theory, a two phase structure is found, with the two phases separated by a second order transition [43]. In our case, with the one-plaquette ansatz, we find the transition to be of first order, but it is unclear whether the two calculations should agree. This is because the phase structure is not simply a property of the gauge group, but it strongly depends on the precise form of the action or Hamiltonian [43].

In the $\mathbb{Z}_N$ case, we can compare the string tension obtained from spatial Wilson loops with the one-plaquette ansatz with that obtained via the strong-coupling expansion in section 4.2. Actually, it is not entirely established that these should give the same



(a) $\mathbb{Z}_3$ gauge theory

(b) $\mathbb{Z}_8$ gauge theory

Figure 4.10: Comparison of analytical string tension vs numerical calculation with one-plaquette ansatz for $\mathbb{Z}_N$ gauge theories

result. The coefficient of the area law should give the string tension for timelike Wilson loops only, but there is no difference between timelike and spacelike Wilson loops in the Euclidean formulation. As briefly reviewed in section 2.2.1 results obtained in the Euclidean and Hamiltonian formulations may not be directly compared. However, in Lie group cases, one typically finds that the conversion factor is of order unity [2, 17, 56]. Renormalisation group studies are not available in the finite group case, so we will simply assume that the two give the same result. In the $\mathbb{Z}_N$ case, we took $\Gamma = \{g, g^{-1}\}$ where g is a generator of $\mathbb{Z}_N$. Up to second order in perturbation theory, the string tension obtained via the strong-coupling expansion is therefore given by:

$$a^2\sigma \approx \frac{4}{\beta}\sin^2\left(\frac{\pi}{N}\right) - \frac{\beta^3}{16\sin^2\left(\frac{\pi}{N}\right)}\left(1 + \frac{1}{1+2\cos^2\left(\frac{\pi}{N}\right)}\right)$$

We will now compute the coefficient of the area-law for spatial Wilson loops with the one-plaquette ansatz. The following method works for any closed loop without self-intersections, however consider for simplicity a $I \times J$ rectangle in the lattice. We denote the $I \times J$ Wilson loop operator with $\hat{W}(I, J)$. Since in the Abelian case $uu^\dagger = 1$, the rectangle can be tiled with plaquettes of definite orientation in a way that they cancel each other, leaving only the rectangular Wilson loop. With the one-plaquette ansatz, all plaquettes oscillate independently. Therefore we can compute the expectation value in a manner similar to the calculation of the magnetic Hamiltonian in the previous sections. There are two differences: in this case we have a product of exactly IJ plaquette Wilson loops, instead of only one. Moreover, since the plaquettes are of definite orientation, we have $\chi_F(g)$ rather than its real part. However, summing over g^{-1} instead of g shows that

only the real part gives a non-zero contribution. We find:

$$\langle \hat{W}(I,J) \rangle = \left[\frac{\sum_g e^{2\alpha \operatorname{Re} \chi_F(g)} \operatorname{Re} \chi_F(g)}{\sum_g e^{2\alpha \operatorname{Re} \chi_F(g)}} \right]^{IJ}$$

Which then implies

$$a^2 \sigma = -\log \left[\frac{\sum_g e^{2\alpha \operatorname{Re} \chi_F(g)} \operatorname{Re} \chi_F(g)}{\sum_g e^{2\alpha \operatorname{Re} \chi_F(g)}} \right]$$

Here $\alpha = \alpha^\star(\beta)$ is determined by minimising the ground state energy $\epsilon_0(\alpha)$ at each β. The result of the comparison is shown for small values of β in figure 4.10 for two Abelian gauge theories. We see that for larger N the one-plaquette ansatz gives a better approximation for the string tension.

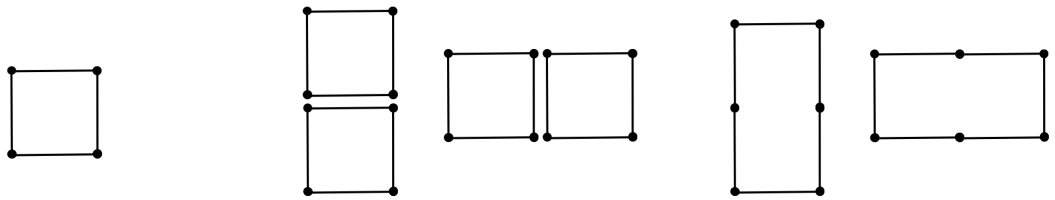
4.4.5 Beyond the one-plaquette ansatz

From the above discussion, we see that the one-plaquette ansatz is unable to capture the structure of the ground state of our finite-group gauge theories beyond a certain critical value of β. In $2+1$ dimensions, the absence of lattice Bianchi identities implies that the one-plaquette ansatz eq. (4.1) describes completely independent magnetic fluctuations of each plaquette. This is a sort of "Hartree" approximation, in that the ground state wavefunction is taken to be a product of the same wavefunction for each plaquette. In order to improve on the one-plaquette ansatz, we could include some correlations. We then consider a "two-plaquette ansatz", of the form

$$|\psi_\alpha\rangle = \exp\left[\alpha_1 \hat{\phi}_1 + \alpha_2 \hat{\phi}_2 + \alpha_3 \hat{\phi}_3 + \alpha_4 \hat{\phi}_4\right] |0\rangle$$

where $\hat{\phi}_1$ is the one-plaquette term that we already saw, $\hat{\phi}_2$ contains the square of each one-plaquette term, $\hat{\phi}_3$ includes the sum over all 1×2 and 2×1 Wilson loops, and $\hat{\phi}_4$ includes the sum over all products of two neighbouring plaquettes, either in the horizontal or vertical directions. The situation is illustrated in figure 4.11. The α_i are variational parameters; these are taken to be the same for 1×2 and 2×1 Wilson loops (and similarly for ϕ_4) by assuming invariance under rotations of the ground state.

The two-plaquette wavefunction has been studied in the $2+1$ dimensional SU(2) case in [17] and in the $2+1$ dimensional U(1) case in [71]. In the SU(2) case it was found that the two-plaquette wavefunction strongly improves the scaling behaviour of the string tension and of the glueball masses; moreover, the values of α_3 and α_4 were always found to be small compared to α_1 and α_2, consistent with the fact that the one-plaquette ansatz is already a good approximation. In the $U(1)$ case it was found that the two-plaquette ground state correctly describes the physics of the ground state for all β. However, both of these theories show a single, confined phase [43]. Moreover, the



(a) A ϕ_1 term (b) Two ϕ_3 terms (c) Two ϕ_4 terms

Figure 4.11: Graphical representation of three of the four terms of the two-plaquette ansatz. The missing term, ϕ_2, is simply the square of ϕ_1 terms.

two-plaquette vacuum gives a finite string tension for spatial Wilson loops, implying a confined phase [17]. As such, if there is indeed a phase transition in the finite-group gauge theories under consideration, it is likely to be a transition to some kind of deconfined phase. This was observed to be the case in $2+1$ dimensional $\mathbb{Z}_N$ gauge theory in the path-integral approach [43]. In fact, we are not aware of a theory showing two different confined phases. If there is indeed a phase transition to a deconfined phase, then we expect that not even the two-plaquette vacuum should be able to correctly describe the physics of the ground state beyond the critical β. Moreover, due to the presence of the ϕ_3 terms, the transformation from links to plaquettes is not applicable anymore, so one must investigate the variational energy with a Monte Carlo simulation.

Another promising alternative could be to adapt to the finite-group case the trial wavefunction proposed in [70] for the $U(1)$ theory in $3+1$ dimensions. This is particularly interesting because $U(1)$ theory in $3+1$ dimensions shows a phase transition between a strong-coupling confined phase and a weak-coupling deconfined phase (see section 2.1.1) and this is similar to what we find in our finite-group cases with the one-plaquette ansatz. Since the wavefunction proposed in [70] is able to capture the physics of both the strong and weak-coupling phases, it looks like a promising alternative. The basic idea of its construction is to approach the ground state from the weak-coupling phase, rather than from the strong-coupling phase as in the one-plaquette ansatz. In U(1) gauge theory, one assigns a periodic angle θ_e to each link, and a corresponding plaquette angle variable $\theta_\square$. The corresponding group element is then $e^{i\theta}$. In the group element basis, the wavefunction on a link configuration $\{\theta\}$ is taken to be

$$\psi(\{\theta\}) = \sum_{\{n_\square\}} F(\{n_\square\}) \prod_\square e^{i\theta_\square n_\square}$$

where the product is taken over all plaquettes $\square$ and $n_\square$ is an assignment of a positive or negative integer to each plaquette. One then sums over all configurations $\{n_\square\}$, which

correspond to all possible plaquette excitations, and these are weighted by the function F, which is Gaussian and includes two-plaquette correlations:

$$F(\{n_\Box\}) = \exp\left(-\alpha \sum_{\Box,\Box'} n_\Box R(\Box,\Box') n_{\Box'}\right)$$

where the kernel R is chosen based on the weak-coupling limit of the theory and α is a variational parameter. This ansatz approaches the ground state from weak-coupling, since the weak-coupling ground state is given by the superposition of all the configurations with the plaquette variables equal to the identity. In the case of finite (non-Abelian) groups, the obvious modification would be to replace $e^{i\theta_\Box n_\Box}$ with the character $\chi_F(g_\Box^{n_\Box})$ where $g_\Box$ is the plaquette group element. Moreover, the range of $n_\Box$ needs to be reduced as each plaquette can now only support a finite number of excitations. However, it is not clear how the kernel R should be chosen in the general non-Abelian case, and whether the calculations can be performed as efficiently as in the U(1) case. It is likely however that only minimal modification should be required to make the above ansatz useful in the $\mathbb{Z}_N$ case.

Conclusions

We started this thesis with a brief review of the basic features of Yang-Mills theory in the continuum, highlighting the Hamiltonian formulation and the Euclidean path-integral approach. Moreover, we fully constructed SU(N) Yang-Mills theory with fermions in 0+1 dimensions as an example, which turns out to be equivalent to N fermionic harmonic oscillators together with Gauss' law.

The basic ideas were carried over to the second chapter, were lattice gauge theory was developed first in the path-integral approach, and then in the Hamiltonian formalism. Starting from the basic setup of [42], we offered our own full derivation of the Hamiltonian for lattice gauge theory. This formulation, based on representation theory, leads directly to the Hamiltonian for the case of a finite gauge group, equivalently to what was done in [23]. In appendix A.3 we gave a heuristic justification of the main result underpinning this approach, the Peter-Weyl theorem, based on the rigorous results proved in [50, 52]. We also discussed the well-known issue of "fermion doubling" on the lattice in both the path-integral and Hamiltonian formalisms, and showed how to mitigate it using so-called *staggered fermions*, as proposed in [15]. Moreover, we illustrated some known issues with the chemical potential on the lattice in the path-integral approach. Our own calculations in the Hamiltonian formalism show that no such issue occurs in this case, and the "naive" chemical potential term is perfectly valid. This is particularly important as one would like to use quantum simulators described by Hamiltonians precisely in the region with a non-zero chemical potential.

After formulating the Hamiltonians for finite group gauge theory, we considered the open problem of fixing the eigenvalues of the electric energy density. We proposed an approach to determine these via the construction of a natural Laplacian operator for any finite group. We found that it has the correct decomposition in terms of representation subspaces, and that it shares many of the desired properties of the electric energy density. After formulating their Hamiltonian and constructing some interesting examples, we investigated a number of properties of pure gauge theories with finite gauge group. We first computed the string tension in the strong-coupling limit for any finite gauge group, and also the first non-trivial correction in perturbation theory for the $\mathbb{Z}_N$ case. We then investigated the general pure gauge finite-group theory on a single plaquette, a simplified case which can be solved exactly. After obtaining two dual bases for the

physical, gauge-invariant Hilbert space, we generically found quantum phase transitions at coupling $g = 0, \infty$, but no transition for finite values of the coupling, independently of the chosen gauge group. We then investigated the ground state of our theories via a variational ansatz in $2 + 1$ dimensions. We carry out the calculations analytically, and only computed the resulting finite sums numerically. A Monte Carlo simulation confirms the validity of our analytical results. We found that the ansatz gives a phase transition between a strong-coupling phase and a weak-coupling phase; the ansatz works well in the former, but not in the latter. We explained why this is to be expected if our theories have a deconfinement transition. In the $\mathbb{Z}_N$ case, a comparison of the string tension is made with that computed via the ansatz, and good agreement is found in the strong-coupling phase. The ansatz gives a better approximation for larger N.

The present work may be extended in many directions. A direct continuation of the investigation of the variational ground state may lead to the calculation of glueball masses in the confining phase of finite group lattice gauge theory, adapting the techniques of [17, 55]. In order to do so, however, a better understanding is required of the ground state in both the strong-coupling and weak-coupling regions of finite-group lattice gauge theories. It appears hard that this may be achieved analytically, but it should be amenable to numerical analysis. Different kinds of trial wavefunctions, which could be Gaussian [72, 70] or inspired by recent theoretical developments [37] may possibly provide a better approximation for the ground state. Further techniques which have proved successful in the Lie group case, such as the $t-$expansion [73, 74], the plaquette expansion [75, 76] and the coupled cluster method [77, 78, 79, 80] could possibly be adapted to the finite group case, leading to a better understanding of the ground state and the excited states of finite group lattice gauge theories. One would hope to find that for the calculation of physical quantities one may replace the Lie group with one of its finite subgroups. It would also be interesting to extend results about the spectrum of SU(2) via its finite subgroups to the finite subgroups of SU(3), and in dimensions higher than $2 + 1$. Moreover, a long-term goal should be the introduction of fermions, together with a chemical potential term. This could be done along the lines of [81].

Another question which may have an interesting answer involves the role of centre symmetry in finite-group gauge theories, considering its importance in the Lie group case [45]. One may also want to systematically investigate the effect of the choice of the eigenvalues of the electric energy density. Although from our limited investigation it would appear that different choices would not affect the qualitative behaviour of the theory, further investigation is needed. This topic is intimately connected with that of the renormalisation group behaviour of finite-group lattice gauge theories, which is however extremely difficult to study [66]. An interesting related possibility is the comparison of finite-group results obtained in the Hamiltonian formalism with those obtained by means of the path-integral approach. If a precise correspondance is found, like in the Lie group case [56], some calculations which are hard to perform in the Hamiltonian approach may be done in the path-integral approach and viceversa.

A final suggestion involves the physical, gauge-invariant Hilbert space; it is conceivable, for example by looking at the exact solution on a single plaquette of section 4.3, that the description of the physical Hilbert space could be much simpler in the finite group case than in the Lie group case; it is also unclear if there is an equivalent of the so-called "Mandelstam constraints" [53] in the finite group case. The finite dimensionality of the finite group Hilbert space could considerably simplify the description of the gauge-invariant Hilbert space, for example allowing the development of numerical algorithms which work entirely inside the physical Hilbert space.

Appendix A

Some Results in Representation Theory

In this appendix we sum up some of the basic results of the representation theory of finite groups and compact Lie groups. All the representation theory here presented is over $\mathbb{C}$.

A.1 Some basic results

We first state a few important results of the representation theory of both finite groups (based on [51]) and compact Lie groups (based on [52]). The exact formulation of the theorems is our own.

First, a finite group only has finitely many representations up to equivalence, and they are all unitary:

Theorem 1 *Let G be a finite group and Σ the set of equivalence classes of irreducible representations of G. Then Σ is finite, and the representative of each class can be chosen to be unitary.*

We can then state the following:

Theorem 2 *(Burnside) Let G be a finite group. Then:*

(i) If n_m is the dimension of the m^{th} inequivalent irreducible representation of G, and there are M such, then

$$\sum_{m=1}^{M} n_m^2 = |G|$$

(ii) The number of inequivalent irreducible representations of G is equal to the number of conjugacy classes of G.

An immediate consequence follows:

Corollary 2.1 *If G is a finite Abelian group, then it has precisely $|G|$ inequivalent irreducible representations.*

Similar results apply to compact groups. First of all,

Theorem 3 *The irreducible representations of a compact Lie group are finite-dimensional.*

Moreover,

Theorem 4 *Let G be a compact Lie group and Σ the set of equivalence classes of irreducible representations of G. Then Σ is countable, and the representative of each class can be chosen to be unitary.*

Given the irreducible representations $\{\pi_j\}$ of a group (compact Lie or finite), these satisfy the so-called *orthogonality theorem*:

Theorem 5 *(Orthogonality theorem) Let $\{\pi_j\}$ be irreducible, unitary representations of G. Then*

$$\int dg\, [\pi_j(g)]^*_{nm} [\pi_{j'}(g)]_{n'm'} = \frac{\mathrm{Vol}(G)}{\dim(j)} \delta_{jj'} \delta_{nn'} \delta_{mm'}$$

The above is stated for Lie groups; the corresponding statement for finite groups is obtained by replacing $\int dg \to \sum_{g \in G}$ and $\mathrm{Vol}(G) \to |G|$. The volume of the group in the above is the volume corresponding to the chosen Haar measure. A useful corollary is that the sum of all matrices of a non-trivial irreducible representation j is zero:

$$\sum_g \pi^j(g) = 0$$

This follows by taking j' equal to the trivial representation, whose matrix elements are all equal to the identity. Then if j is non-trivial, the right-hand side of the orthogonality theorem is always zero. Taking $m' = n'$ on the left-hand side gives the claim.

These results form the basis of our understanding of representation theory for finite groups and for compact Lie groups.

A.2 Character theory

In this section, we'll only be concerned with finite groups. The irreducible characters of a finite group G are the functions $\chi : G \to \mathbb{C}$ defined as the traces of irreducible representations of G:

$$\chi_j(g) = \operatorname{tr} \pi_j(g)$$

There are as many irreducible characters as there are irreducible representations. We'll use the following result:

Theorem 6 *The characters $\{\chi_j\}$ of a group G form a basis for the space of class functions on G.*

A class function f satisfies $f(axa^{-1}) = f(x)$ for all $a, x \in G$. In other words, it is constant on conjugacy classes. We'll also need the following:

Theorem 7 *(Orthogonality theorem for characters) The irreducible characters of a finite group are orthonormal, in the sense that*

$$\frac{1}{|G|} \sum_{g \in G} \chi_i^*(g) \chi_j(g) = \delta_{ij}$$

The characters also satisfy a different kind of orthogonality relation, where one sums over characters rather than over group elements:

Theorem 8 *The irreducible characters $\{\chi_i\}$ of a finite group satisfy*

$$\sum_i \chi_i^*(g) \chi_i(h) = \begin{cases} \frac{|G|}{|C(g)|} & g \text{ and } h \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases}$$

where i indexes the irreducible characters and $|C(g)|$ is the size of the conjugacy class of g.

Finally, we can define the *convolution* of two class functions ϕ and ψ:

$$\phi \star \psi(g) = \sum_{h \in G} \phi(gh^{-1}) \psi(h)$$

The convolution is symmetric, $\phi \star \psi = \psi \star \phi$. We'll use the fact that the convolution of two characters is again a character,

$$\chi_i \star \chi_j = \frac{|G|}{\dim(j)} \delta_{ij} \chi_j$$

This can be shown by direct computation,

$$\begin{aligned}\chi_i \star \chi_j(g) &= \sum_{h\in G} \chi_i(gh^{-1})\chi_j(h) = \\ &= \sum_{h\in G}\sum_{m,n} [\pi_i(gh^{-1})]_{mm}[\pi_j(h)]_{nn} = \\ &= \sum_{m,n,l} [\pi_i(g)]_{ml} \sum_{h\in G} [\pi_i(h^{-1})]_{lm}[\pi_j(h)]_{nn} = \\ &= \sum_{m,n,l} [\pi_i(g)]_{ml} \frac{|G|}{\dim(j)} \delta_{ij}\delta_{ln}\delta_{mn} = \frac{|G|}{\dim(j)}\delta_{ij}\chi_j\end{aligned}$$

where we used the orthogonality theorem for representations. This concludes our discussion of character theory.

A.3 The Peter-Weyl theorem

The Peter-Weyl theorem is instrumental in the formulation of the Hamiltonian in section 3.1. See [50, 52] for the Lie group case and [51] for the finite group case. The statement is:

Theorem 9 *(Peter-Weyl) Let G be a compact Lie group. Then*

(i) The space of square-integrable functions on G can be decomposed as a sum of representation spaces. More precisely, if V_π is the vector space for the irreducible representation π, then

$$L^2(G) = \bigoplus_{\pi\in\Sigma} V_\pi^* \otimes V_\pi$$

(ii) The matrix elements of all the inequivalent irreducible representations of G form an orthogonal basis for $L^2(G)$.

(iii) If $\{|g\rangle\}$ is the orthonormal group element basis for $L^2(G)$, then the orthonormal matrix element basis $\{|jmn\rangle\}$ satisfies

$$\langle g|jmn\rangle = \sqrt{\frac{\dim(j)}{\mathrm{Vol}(G)}}\,[\pi_j(g)]_{mn}$$

Note that there are multiple ways of writing the Peter-Weyl decomposition, as

$$V^* \otimes V \cong \mathrm{End}V \cong V^{\oplus \dim V}$$

As we'll see later, these correspond to different ways of seeing $L^2(G)$ as a representation space.

Note that part (i) can be understood as a generalisation of the Fourier decomposition. In fact, since $U(1)$ is Abelian, all of its irreps are one-dimensional and are given by matrix elements of the form $\{e^{inx}\}$ for $x \in S^1 = [0, 2\pi)$. Then the Peter-Weyl theorem states that any square-integrable function on $U(1) \cong S^1$ can be written as a Fourier series.

Recall that matrix elements are defined as follows. Consider the example of SU(2) [42], but the generalisation is easy. As we know, the irreps of SU(2) are labeled by a half integer $j \in \frac{1}{2}\mathbb{Z}^+$. Then in this case

$$L^2(G) = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}^+} V_j^* \otimes V_j$$

where $V_j = \mathbb{C}^{2j+1}$. We have irreps π_j for each j and the matrix elements are literally the elements of the matrices representing a certain $U \in \mathrm{SU}(2)$ as a function of U. More precisely, they are the functions

$$[\pi_j(\,\cdot\,)]_{mn} : \mathrm{SU}(2) \to \mathbb{C}$$

$$g \mapsto [\pi_j(g)]_{mn}$$

where $-j \leq m, n \leq j$ in integer steps. In the general case, it is more natural to take $1 \leq m, n \leq \dim(j)$.

In part (iii) $\mathrm{Vol}(G)$ is the volume of the group given by the chosen Haar measure. The result of part (iii) can be readily derived as a consequence of part (ii) and the orthogonality theorems for representations. The non-trivial statement is that the matrix elements of representations span $L^2(G)$, while the orthogonality is an algebraic statement. In fact, by (ii) the matrix elements π^j_{mn} form a basis for the space of wavefunctions $L^2(G)$. The corresponding states are then given by

$$|jmn\rangle = C_{jmn} \int dU\, [\pi_j(U)]_{mn}\, |U\rangle$$

where the constant C_{jmn} can be chosen to ensure that the $|jmn\rangle$ are normalised. Then we can compute their inner product,

$$\langle j'm'n'|jmn\rangle = C^*_{j'm'n'} C_{jmn} \int dU\ [\pi_{j'}(U)]^*_{m'n'}\, [\pi_j(U)]_{mn}$$

where we used the orthonormality of the $|U\rangle$. The orthogonality theorem from the previous section gives us precisely what we need:

$$\langle j'm'n'|jmn\rangle = C^*_{j'm'n'} C_{jmn} \frac{\mathrm{Vol}(G)}{\dim(j)} \delta_{jj'} \delta_{mm'} \delta_{nn'}$$

It follows that the representation basis $\{|jmn\rangle\}$ is orthonormal with an appropriate choice of constants,

$$C_{jnm} = \sqrt{\frac{\dim(j)}{\mathrm{Vol}(G)}}$$

for compact Lie groups. Everything we've said here also holds for finite groups, with the replacement $\mathrm{Vol}(G) \to |G|$.

Crucially, the Peter-Weyl theorem also holds for finite groups ([51] Proposition 10):

Theorem 10 *(Peter-Weyl for finite groups) Let G be a finite group. Then*

(i) The group algebra on G can be decomposed as a sum of representation spaces. More precisely, if V_π is the vector space for the irreducible representation π, then

$$\mathbb{C}[G] = \bigoplus_{\pi \in \Sigma} V_\pi^* \otimes V_\pi$$

(ii) The matrix elements of all the inequivalent irreducible representations of G form an orthogonal basis for $\mathbb{C}[G]$.

(iii) If $\{|g\rangle\}$ is the orthonormal group element basis for $\mathbb{C}[G]$, then the orthonormal matrix element basis $\{|jmn\rangle\}$ can be chosen to satisfy

$$\langle g|jmn\rangle = \sqrt{\frac{\dim(j)}{|G|}}\,[\pi_j(g)]_{mn}$$

The result is essentially the same as in the compact case. Note that in the finite group case there is no issue of convergence, and as such we don't need to specify further information (i.e. L^2) on the group algebra. The duality relation can be shown to hold in the same manner as for compact Lie groups.

In order to show some of the results that we need in section 3.1, we'll study in more detail the various forms of the Peter-Weyl theorem for compact Lie groups. Useful references are [50, 52].

We've mentioned that $L^2(G)$ is the space of wavefunctions. The Peter-Weyl theorem is intimately connected with the representation theory of the left regular representation on G, which we defined in section 2.2 as the map $U \to L_U$ where L_U is defined by $L_U |V\rangle = |UV\rangle$. It is useful to study the left-regular representation on wavefunctions. We see that

$$L_V |\psi\rangle = \int dU\, \psi(U) L_V |U\rangle = \int dU\, \psi(V^{-1}U) |U\rangle$$

where in the last line we used the definition of L and the left invariance of the Haar measure. It follows that the left regular representation on wavefunctions is given by

$$L_V \psi(U) = \psi(V^{-1}U)$$

Since the left-regular representation is infinite dimensional, it must be irreducible. It turns out that its decomposition into irreducible representations contains all of the irreducible representations of G with multiplicity equal to their dimension. This gives the decomposition of $L^2(G)$ into the subspaces of the form $V_j^{\oplus \dim V_j}$.

We'll see how this works in a heuristic manner. Given G and its representations $\{\pi_j\}$, we can define the subset $L^2(G)_j$ of $L^2(G)$ as the subset spanned by functions of the form

$$f(g)_{j,u,v} \equiv u \cdot \pi_j(g)v$$

for $g \in G$ and $u, v \in V_j$, where $\cdot$ is the inner product on the vector space V_j corresponding to the representation π_j. The $L^2(G)_j$ are left-invariant subspaces of the left-regular representation, as

$$\begin{aligned} L_h f(g)_{j,u,v} &= f(h^{-1}g)_{j,u,v} = \\ &= u \cdot \pi_j(h^{-1}g)v = \\ &= u \cdot \pi_j(h^{-1})\pi_j(g)v = \\ &= \pi_j(h)u \cdot \pi_j(g)v = f(g)_{j,\pi_j(h)u,v} \end{aligned}$$

which is again an element of $L^2(G)_j$. In going to the last line, we used the unitarity of the π_j. Since they are left-invariant, the subspaces $L^2(G)_j$ induce subrepresentations of the left-regular representation. We can then define further subspaces $L^2(G)_{j,m}$ spanned by the functions of the form $f(g)_{j,u,e_m}$ where $\{e_m\}$ is a basis of V_j. Then it is clear that

$$L^2(G)_j = \bigoplus_m L^2(G)_{j,m}$$

Each of the $L^2(G)_{j,m}$ is also an invariant subspace of the left-regular representation, by the same argument above. Moreover, the restriction $L|_{j,m}$ of the left-regular representation to $L^2(G)_{j,m}$ is isomorphic to π_j, via the map

$$\begin{aligned} u &\leftrightarrow f_{j,u,e_m} \\ \pi^j(g) &\leftrightarrow L_g|_{j,m} \end{aligned}$$

It follows that as a vector space, $L^2(G)_{j,m} \cong V_j$ and that therefore

$$L^2(G)_j = \bigoplus_m V_j \equiv V_j^{\oplus \dim V_j}$$

The non-trivial mathematical statement is then that

$$L^2(G) = \bigoplus_j L^2(G)_j$$

but this requires more analysis to establish [50, 52]. What is the action of L_g in the decomposition above? We can write any function $\psi \in L^2(G)$ in the above decomposition,

$$\psi(g) = \sum_{jnm} c_{jnm} f_{j,e_n,e_m}(g)$$

so that then, as we have seen above,

$$L_h\psi(g) = \sum_{jnm} c_{jnm} f_{j,\pi_j(h)e_n,e_m}$$

In other word, the left-regular representation acts on each component $L^2(G)_{j,m}$ via left-multiplication by $\pi_j(h)$. In terms of the decomposition

$$L^2(G) = \bigoplus_j V_j^{\oplus \dim V_j}$$

we have

$$L_g = \bigoplus_j \pi_j(g)^{\oplus \dim V_j}$$

There is also another way of seeing the Peter-Weyl decomposition. One can view $L^2(G)$ as a representation of $G \times G$, where the group acts by the left and right regular representations simultaneously,

$$(g, h) \mapsto L_g R_h$$

so that on wavefunctions,

$$L_g R_h \psi(x) = \psi(g^{-1}xh)$$

Similarly to what we've just seen, we can then define the space $L^2(G)|_j$ as the subspace of $L^2(G)$ spanned by functions of the form

$$f(g)_{j,\phi,v} = \phi(\pi_j(g)v)$$

with $g \in G$ and $v \in V_j$ and $\phi \in V_j^*$. This the same decomposition as the previous one, but written in a different language; so again the $L^2(G)|_j$ together exhaust $L^2(G)$. In fact, as vector spaces with this specific form, we can identify $L^2(G)|_j = V_j^* \otimes V_j$ so that we get a different form of the Peter-Weyl decomposition, $L^2(G) = \bigoplus_j V_j^* \otimes V_j$ In components, we can write $\phi = \phi_i \widetilde{e}_i$ and $v = v_i e_i$, where $\{\widetilde{e}_i\}$ is the basis dual to $\{e_i\}$, and this leads to

$$f(g)_{j,\phi,v} = \phi(\pi_j(g)v) = \phi_i \pi_j(g)_{ik} v_k$$

where we used the fact that for a linear map A, its action on a basis vector is given by $Ae_i = A_{ji}e_j$ in order to get the correct action on the components of a generic vector.

Therefore the action of $G \times G$ is given by

$$\begin{aligned} L_h R_k f(g)_{j,\phi,v} &= f(h^{-1}gk)_{j,\phi,v} = \\ &= \phi_i \pi_j(h^{-1}gk)_{ik} v_k = \\ &= \phi_i \pi_j(h^{-1})_{im} \pi_j(g)_{mn} \pi_j(k)_{nk} v_k = \\ &= (\pi_j(h^{-1})^T \phi)_m \pi_j(g)_{mn} (\pi_j(k)v)_n = \\ &= f(g)_{j,\pi^j(h)^*\phi,\pi^j(k)v} \end{aligned}$$

where $\pi^j(g)^*$ is the dual representation to $\pi^j(g)$, and we exploited the unitarity of the irreps. This shows that $L^2(G)|_j$ is an invariant subspace, and gives the combined action of the left and right regular representations on each representation subspace as

$$(L_g R_h)|_j \leftrightarrow \pi^j(g)^* \otimes \pi^j(h)$$

In other words,

$$L_g R_h = \bigoplus_j \pi^j(g)^* \otimes \pi^j(h)$$

so that in particular,

$$L_g = \bigoplus_j \pi^j(g)^* \otimes \mathbb{1}_j$$

where $\mathbb{1}_j$ is the $\dim(j) \times \dim(j)$ identity matrix. This is the decomposition that we use in section 3.1.

Appendix B

Finite subgroups of SU(2)

In this appendix we sum up the relevant properties of the finite subgroups of SU(2) that we employ throughout this thesis. As already outlined in section 3.3.3 the finite subgroups of SU(2) have been completely classified in five families [64]:

- The Abelian cyclic groups $\mathbb{Z}_n$ for any n.
- The non-Abelian binary dihedral groups $2D_n$ for any n.
- The binary tetrahedral group $2T$.
- The binary octahedral group $2O$.
- The binary icosahedral group $2I$.

All of these subgroups arise from subgroups of SO(3), being pulled back to SU(2) by the universal covering map SU(2) $\to$ SO(3). The three binary groups that don't fit in a family are for this reason sometimes called "exceptional subgroups" [64]. These are the preimage of the groups of symmetries of polyhedra, which are subgroups of SO(3) isomorphic to A_4, S_4 and A_5 for the tetrahedron, octahedron and icosahedron respectively. Since the map SU(2) $\to$ SO(3) is two-to-one, the binary groups have twice the elements of the corresponding subgroups of SO(3).

In the present work, and in particular in section 4.4, we have made use of a number of properties of $2T$, $2O$ and $2I$. In the following, we'll describe their group elements, their conjugacy classes and their character tables. We'll follow [64, 82, 83] but the labels of the representations is our own and not standard.

In order to describe the group elements, we note that SU(2) can be seen as a subset of the real quaternion algebra. In these terms, any $X \in \mathrm{SU}(2)$ can be written as

$$X = a + bi + cj + dk \qquad\qquad a^2 + b^2 + c^2 + d^2 = 1$$

where $a, b, c, d \in \mathbb{R}$ and $1, i, j, k$ are quaternions, which satisfy

$$i^2 = j^2 = k^2 = ijk = -1$$

as usual. This defines multiplication and inversion inside the group. The finite subgroups are then defined by specific choices of the coefficients a, b, c, d.

B.1 The binary tetrahedral group

The 24 elements of the binary tetrahedral group are given as quaternions by

$$\pm 1,\ \pm i,\ \pm j,\ \pm k,\ \frac{\pm 1 \pm i \pm j \pm k}{2}$$

with the group structure given by quaternion multiplication. Now define

$$a = \frac{1+i+j+k}{2} \qquad b = \frac{1+i+j-k}{2}$$
$$c = \frac{-1+i+j+k}{2} \qquad d = \frac{-1+i+j-k}{2}$$

Under conjugation, the real part of a quaternion is unchanged and exactly two signs of the three imaginary parts are flipped. The conjugacy classes are then as follows:

rep	1	-1	i	a	b	c	d
size	1	1	6	4	4	4	4

The first line contains the representative of the class, and the second line the size of the class. The character table is as follows. Let $\zeta = (-1+\sqrt{3}i)/2$. Then

	1	-1	i	a	b	c	d
1	1	1	1	1	1	1	1
2	1	1	1	ζ	ζ^2	ζ^2	ζ
3	1	1	1	ζ^2	ζ	ζ	ζ^2
4	3	3	-1	0	0	0	0
5	2	-2	0	ζ	ζ^2	$-\zeta^2$	$-\zeta$
6	2	-2	0	ζ^2	ζ	$-\zeta$	$-\zeta^2$
7	2	-2	0	1	1	-1	-1

Each row represents an irreducible character, while each column represents a conjugacy classes. Note that, as usual, the character computed on 1 gives the dimension of the representation. The representation labels are not standard.

The last thing we need to find is the subset Γ, which must be a symmetric generating set which is a union of conjugacy classes (as explained in section 3.2). It turns out that group elements a and b generate the group, and that a^{-1} is in the conjugacy class of b, while b^{-1} is in the conjugacy class of a. As such, Γ is taken to be the union of the conjugacy classes of a and b unless otherwise specified.

B.2 The binary octahedral group

The 48 elements of the binary octahedral group are all the 24 elements of the binary tetrahedral group together with the 24 elements of the form

$$\frac{1}{\sqrt{2}}(\pm 1 \pm i), \quad \frac{1}{\sqrt{2}}(\pm 1 \pm j), \quad \frac{1}{\sqrt{2}}(\pm 1 \pm k)$$

$$\frac{1}{\sqrt{2}}(\pm i \pm j), \quad \frac{1}{\sqrt{2}}(\pm i \pm k), \quad \frac{1}{\sqrt{2}}(\pm j \pm k)$$

Define a, b, c, d as in the previous section and moreover

$$e = \frac{1}{\sqrt{2}}(1+i), \qquad f = \frac{1}{\sqrt{2}}(-1+i), \qquad g = \frac{1}{\sqrt{2}}(i+j)$$

The conjugacy classes and their sizes are

rep	1	-1	i	a	c	e	f	g
size	1	1	6	8	8	6	6	12

The character table is given by

	1	-1	i	a	c	e	f	g
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	-1	-1	-1
3	2	2	2	-1	-1	0	0	0
4	3	3	-1	0	0	1	1	-1
5	3	3	-1	0	0	-1	-1	1
6	2	-2	0	1	-1	$\sqrt{2}$	$-\sqrt{2}$	0
7	2	-2	0	1	-1	$-\sqrt{2}$	$\sqrt{2}$	0
8	4	-4	0	-1	1	0	0	0

Unless otherwise specified, Γ is taken to be the union of conjugacy classes a and e. In fact the elements a and e generate the whole group.

B.3 The binary icosahedral group

The 120 elements of the binary icosahedral group are all the 24 elements of the binary tetrahedral group together with the 96 quaternions obtained from $\frac{1}{2}(\pm 0 \pm 1i \pm \varphi^{-1} j \pm \phi k)$ by an even permutation of $(0, 1, \varphi^{-1}, \varphi)$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. For completeness, the even permutations on four elements $(1, 2, 3, 4)$ are given by

$$(1,2,3,4),\ (1,3,4,2),\ (1,4,2,3),\ (2,1,4,3),\ (2,3,1,4),\ (2,4,3,1),$$

$$(3,1,2,4),\ (3,2,4,1),\ (3,4,1,2),\ (4,1,3,2),\ (4,2,1,3),\ (4,3,2,1)$$

The conjugacy classes are particularly simple in this case, as they are determined by the real part of the group element:

Re	1	−1	0	−1/2	1/2	$-\varphi/2$	$\varphi^{-1}/2$	$\varphi/2$	$-\varphi^{-1}/2$
size	1	1	30	20	20	12	12	12	12

The character table is given by

	1	−1	0	−1/2	1/2	$-\varphi/2$	$\varphi^{-1}/2$	$\varphi/2$	$-\varphi^{-1}/2$
1	1	1	1	1	1	1	1	1	1
2	2	−2	0	−1	1	$-\varphi$	φ^{-1}	φ	$-\varphi^{-1}$
3	2	−2	0	−1	1	φ^{-1}	$-\varphi$	$-\varphi^{-1}$	φ
4	3	3	−1	0	0	φ	$-\varphi^{-1}$	φ	$-\varphi^{-1}$
5	3	3	−1	0	0	$-\varphi^{-1}$	φ	$-\varphi^{-1}$	φ
6	4	4	0	1	1	−1	−1	−1	−1
7	4	−4	0	1	−1	−1	−1	1	1
8	5	5	1	−1	−1	0	0	0	0
9	6	−6	0	0	0	1	1	−1	−1

The group is generated by elements a and $\frac{1}{2}(\varphi + \varphi^{-1}i + j)$. For unit quaternions, inversion simply flips the sign of the three imaginary parts. Since the conjugacy classes are determined by the real part of the group element, they are all symmetric. We can take Γ to be the union of the conjugacy classes of real part $1/2$ and of real part $\varphi/2$.

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