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Renormalization in non-relativistic quantum mechanics

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Abstract

A problem of non-relativistic quantum mechanics solved using regularization and renormalization techniques is presented in this thesis. After a general introduction of these techniques, they are applied to a problem in classical electromagnetism and to the bound state of a single quantum particle subjected to a two-dimensional δ-function potential, that is divergent if computed naively solving directly the Schrödinger equation or using the theory of propagators. The regularization techniques used are the cutoff regularization and the dimensional one and they both leads to the same outcome. An effective field theory approach, in which the potential is regularized through the real space scheme, is also presented. After regularization has been performed, the potential is renormalized re-defining the coupling constant. The running of the renormalized coupling constant is also found, i.e. the renormalization group equation.

Sommario

In questa tesi viene trattato un problema di meccanica quantistica non relativistica usando le tecniche di regolarizzazione e rinormalizzazione. Dopo una generale introduzione di queste tecniche, esse vengono applicate ad un problema di elettromagnetismo classico e al calcolo dell'energia di legame di una particella quantistica soggetta ad un potenziale contenente una delta di Dirac bidimensionale, la quale risulta divergente se calcolata risolvendo direttamente l'equazione di Schrödinger o usando la teoria dei propagatori. Le tecniche di regolarizzazione usate sono la regolarizzazione dimensionale e la regolarizzazione tramite cutoff, le quali conducono allo stesso risultato. Viene inoltre presentato un approccio di teoria di campo efficace, dove il potenziale è regolarizzato con lo schema a spazio reale. Dopo aver eseguito la regolarizzazione, il potenziale viene rinormalizzato ridefinendo la costante di accoppiamento e determinandone l'evoluzione, ossia trovando l'equazione del gruppo di rinormalizzazione.

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Introduction

The purpose of this thesis is to study a topic of theoretical physics that is not treated in a bachelor's degree, but that is fundamental for the quantum field theory and condensed matter physics: renormalization. Renormalization is a collection of techniques developed in order to treat infinities arising in calculated quantities. The first renormalized quantity was the energy gap between the $2s$ and $2p$ level of the hydrogen atom, i.e. the Lamb shift, that was found to be divergent even including relativistic correction [1]. Hans Bethe, in a 1947 paper [2], was the first to find a non divergent result: this was the born of renormalization. Since then, renormalization was largely used in particle physics to calculate physical quantities although its meaning was not deeply understood. Renormalization was developed independently by Kenneth G. Wilson studying continuous phase transition in condensed matter [3] (for this work Wilson was awarded the 1982 Nobel Prize in physics). Even thought historically renormalization appeared independently in particle physics and in phase transition theory, it was lately understood that both procedures have the same physical aspects. Then, a general description of renormalization techniques can be derived.

Although renormalization is an advanced theoretical physics topic, there is a classical physics example (the electric potential generated by a straight wire) and a non-relativistic quantum mechanics example that can be treated using renormalization techniques. The present work deals with those problems. The quantum mechanics problem is well known in literature and it is the bound state energy for a particle in a two-dimensional delta-function potential, which seems to diverge if calculated naively. The two-dimensional interaction is an important model in phenomenological physics since it can applied to approximate zero-range potential, that are important in nuclear physics, condensed matter physics and particle physics.

The thesis is divided in three chapters. In the first chapter the fundamental concepts of non-relativistic quantum mechanics are given. In the second one, the theory of propagators is presented. In this part, the Feynman's interpretation of quantum mechanics, where the propagators are represented as path integrals, is also discussed. The third chapter is the core of this work. It starts describing renormalization techniques applying them to a toy model. Then, those techniques are used to renormalize the electric potential generated by a straight wire and the two-dimensional δ-interaction of quantum mechanics.

Chapter1

Foundations of quantum mechanics

In this chapter, the fundamental concepts of quantum mechanics are introduced. This theory was born to explain an array of strange phenomena which demonstrated the inadequacy of classical physics. In particular, it explains why sometimes light, and electrons, behave as particles while in other as waves.

In the first part of this chapter, the formal structure of quantum mechanics is described. Then, the time evolution of a quantum system and the Schrödinger equation are derived. Finally, it is presented a short overview of wave-mechanics, that is the Schrödinger interpretation of quantum mechanics.

The Dirac notation is used.

1.1 The formal structure of quantum mechanics

Quantum mechanics is summarized in some postulates:

First postulate. *To every states of a quantum system is associated a normalized ket $|\psi\rangle$, defined up to a phase factor, in the space of states, that is a Hilbert's complex linear vector space $\mathcal{H}$.*

This postulate implies that the superposition of two states is again a state of the system. If $|\psi_1\rangle$ and $|\psi_2\rangle$ are two states of the system, so $|\psi\rangle = b\,|\psi_1\rangle + a\,|\psi_2\rangle$, where a and b are complex numbers, is a state of the system. In addition, the Hilbert's space comes with an inner product

$$\langle\psi_1|\psi_2\rangle = \int dx \psi_1^*(x)\psi_2(x), \tag{1.1.1}$$

which is hermitian $\langle\psi_1|\psi_2\rangle^* = \langle\psi_2|\psi_1\rangle$ and non-negative $\langle\psi|\psi\rangle \geq 0$. It is also imposed that the Hilbert's product is non-divergent, $\langle\psi|\psi\rangle < \infty$. $|\psi\rangle$ satisfies the normalization condition precisely when $\langle\psi|\psi\rangle = 1$.

Second postulate. *Every physical observable Q is represented by an associated selfadjoint operator $\hat{Q}$ which acts on the ket describing the system:*

$$\hat{Q} : |\psi\rangle \to \left|\psi'\right\rangle = \hat{Q}\,|\psi\rangle\,. \tag{1.1.2}$$

Usually, by acting with an operator on a state, the state changes. However, there are some states that do not change when an operator acts on them:

$$\hat{Q}\,|a\rangle = x\,|a\rangle\,. \tag{1.1.3}$$

These special states $|a\rangle$ are called eigenstates of the operator, while x are called eigenvalues of the operator. Equation (1.1.3) is an eigenvalue equation. A consequence of equation (1.1.3) is that a function of the operator $\hat{Q}$, that represents a functional relation between two observables, behaves as

$$f(\hat{Q})\,|a\rangle = f(a)\,|a\rangle \tag{1.1.4}$$

when acting on an eigenket of the operator.

The spectrum of an operator, that is the ensemble of all its eigenvalues, may be continuous, discrete or both. Eigenstates of a selfadjoint operator have the important property of forming a generalized orthogonal basis, so they fulfill the generalized orthogonal relation

$$\langle l,\lambda|m,\mu\rangle = K_{m,\mu}\delta(\lambda-\mu)\delta_{lm} \tag{1.1.5}$$

and the generalized completeness relation

$$\sum_l \int \frac{d\lambda}{K_{l,\lambda}}\,|l,\lambda\rangle\,\langle l,\lambda| = \hat{1}, \tag{1.1.6}$$

where $K_{m,\mu} > 0$ is a constant, $\delta(x)$ is the Dirac delta function and δ_{ij} is the Kronecker delta. The indexes l, m are discrete, while λ, μ are continuous. The indexes vary on a range where the subdivision of discrete and continuous indexes may changes. The notation includes all the possible distributions of those indexes. When the discrete and continuous indexes are arranged in only two independent subdivisions, the previous equation becomes:

$$\sum_l |l\rangle\,\langle l| + \int \frac{d\lambda}{K_\lambda}\,|\lambda\rangle\,\langle\lambda| = \hat{1}. \tag{1.1.7}$$

The eigenkets are indicated with the same letter of the indexes that label them. The generalized completeness relation allows to expand every state of the system $|\psi\rangle$ as a linear combination of generic basis vectors, in particular of eigenkets:

$$|\psi\rangle = \sum_l \int \frac{d\lambda}{K_{l,\lambda}}\,|l,\lambda\rangle\,\langle l,\lambda|\psi\rangle\,. \tag{1.1.8}$$

This last relation tells that the ket describing a state of the system is a superposition of different eigenstates $|l,\lambda\rangle$ of a generic observable associated to the operator. Rigorously, the eigenstates related to the continuous spectrum are not physical states because they

cannot be normalized. Even an operator $\hat{Q}$ can be expanded similarly:

$$\hat{Q} = \sum_l \sum_m \int \int \frac{d\lambda}{K_{l,\lambda}} \frac{d\mu}{K_{m,\mu}} |l,\lambda\rangle \langle l,\lambda| \hat{Q} |m,\mu\rangle \langle m,\mu| . \tag{1.1.9}$$

Since the base considered is made of eigenstates, which satisfy the equation (1.1.3), equation (1.1.9) comes in a simpler form:

$$\hat{Q} = \sum_l \int \frac{d\lambda}{K_{l,\lambda}} |l,\lambda\rangle x_{l,\lambda} \langle l,\lambda| . \tag{1.1.10}$$

This property is known as spectral theorem. If the spectrum is completely discrete/continuous, only the summation/integration on the discrete/continuous indexes of equations (1.1.6)-(1.1.10) survives.

If the observable is a vector $\mathbf{Q}$, like for example the position $\mathbf{q}$, the associated operator is a vector too. If the components of the observable $\mathbf{Q}$ are compatible, that means that they can be measured simultaneously, the components of the associated selfadjoint vector operator $\hat{\mathbf{Q}}$ commute, that is their commutator equals 0:

$$[\hat{Q}_i, \hat{Q}_j] = \hat{Q}_i \hat{Q}_j - \hat{Q}_j \hat{Q}_i = 0. \tag{1.1.11}$$

When the components of the vector operator commute, the second postulate can be generalized to a generic set of commuting selfadjoint operators. For example, to common physical states of the compatible observables Q_i correspond common eigenkets $|\psi\rangle$ of selfadjoint commuting operators $\hat{Q}_i$ and those eigenkets form a generalized orthonormal basis. This is particularly interesting when a set of selfadjoint commuting operators is complete, that is when it forms a basis of the space of the operator on the Hilbert space. In this situation, any other operator that commutes with that set can be written as a function of those operators and thus its eigenkets and its spectrum are completely determined by the equation (1.1.4).

Third postulate. *The only possible result of the measurement of an observable Q on a generic state $|\psi\rangle$ is one of the eigenvalues of the corresponding operator $\hat{Q}$.*

One may want to know how much is the probability of obtaining a measured value equals to the eigenvalue of the associated operator. If the eigenvalue $x_{l,\lambda}$ belongs to the discrete spectrum, the probability is $|\langle l,\lambda|\psi\rangle|^2$ and the quantity $\langle l,\lambda|\psi\rangle$ is called probability amplitude. For an eigenvalue belonging to the continuous spectrum, since the related eigenstates are not normalizable, the question is not well defined. However, it is possible to know the probability of measuring a value belonging to a numerical range $\mathcal{X}$ containing $x_{l,\lambda}$:

$$p(\mathcal{X}) = \sum_l \int_{x_{l,\lambda} \in \mathcal{X}} \frac{d\lambda}{K_{l,\lambda}} |\langle l,\lambda|\psi\rangle|^2. \tag{1.1.12}$$

Moreover, the expectation value of Q is given by

$$\langle Q \rangle = \sum_l \int \frac{d\lambda}{K_{l,\lambda}} x_{l,\lambda} |\langle l, \lambda | \psi \rangle|^2. \tag{1.1.13}$$

The sum of $|\langle l, \lambda | \psi \rangle|^2$ must be equal to 1.

If the operator is a vector $\hat{\boldsymbol{Q}}$, equations (1.1.12) and (1.1.13) turn into

$$p(\mathcal{X}) = \sum_l \int_{x_{i;l,\lambda} \in \mathcal{X}} \frac{d\lambda}{K_{l,\lambda}} |\langle l, \lambda | \psi \rangle|^2, \tag{1.1.14}$$

$$\langle Q \rangle = \sum_l \int \frac{d\lambda}{K_{l,\lambda}} x_{i;l,\lambda} |\langle l, \lambda | \psi \rangle|^2, \tag{1.1.15}$$

where $x_{i;l,\lambda}$ is the eigenvalue of the ith operator associated to the common eigenket $|l, \lambda\rangle$.

Fourth postulate. *Upon the measurement of an observable Q has yielded a value $x_{l,\lambda}$ of the discrete spectrum of the associated operator $\hat{Q}$, the state of the system collapses onto the normalized eigenstate $|l, \lambda\rangle$. If the measurement yields a continuous eigenvalue, the state of the system is a superposition of the eigenstates related to the eigenvalues belonging to a small range $\mathcal{X}$ that includes $x_{l,\lambda}$.*

This postulate explains a known principle of quantum mechanics, the Heisenberg's uncertainty principle. This principle stands that if the commutator of two operators is not zero, the correspondent observables cannot be measured simultaneously with any degrees of precision because they have no common eigenstates. Indeed, suppose to have two non-compatible observables A and B. After a measurement of A, the ket state $|\psi\rangle$ collapses onto an eigenstate s of A. Since A and B have no common eigenstates, s is not an eigenstate of B, but rather it can be represented as a sum of B's eigenstates. This means that B has not a defined value, but it changes every time the measurement is repeated. In other words, B must be less precise.

Fifth postulate. *The time evolution of a quantum system preserves the normalization of the associated ket. The time evolution of the state of a quantum system is described by*

$$|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle \tag{1.1.16}$$

for some unitary operator $\hat{U}$.

This last postulate will be investigated more carefully in the next section.

1.2 Time evolution and the Schrödinger equation

In quantum mechanics time is just a parameter and not an operator. In particular, time is not an observable in the language of section 1.1.

Suppose to have a physical system whose state is represented at t_0 by the ket $|\psi(t_0)\rangle$. The state of the system at time $t > t_0$ is denoted by $|\psi(t)\rangle$. The problem of this section is to find the time evolution from the state at t_0 to the one at t. Since time is assumed to be a continuous parameter, the relation

$$\lim_{t \to t_0} |\psi(t)\rangle = |\psi(t_0)\rangle \tag{1.2.1}$$

holds.

As a consequence of the fifth postulate enunciated in the previous section, the two states are related by an unitary operator, that is called time-evolution operator $\hat{U}(t_0, t)$, as in equation (1.1.16). The unitary property follows from probability conservation. Another feature required for the operator $\hat{U}$ is the composition property:

$$\hat{U}(t_2, t_0) = \hat{U}(t_2, t_1)\hat{U}(t_1, t_0), \tag{1.2.2}$$

where $t_1 < t_1 < t_2$.

Consider now the infinitesimal time evolution operator

$$|\psi(t_0 + dt)\rangle = \hat{U}(t_0 + dt, t_0)\,|\psi(t_0)\rangle\,, \tag{1.2.3}$$

because of the continuity property (1.2.1), the infinitesimal time-evolution operator might reduce to the identity operator $\hat{1}$ as dt goes to zero. The difference between $\hat{U}(t_0 + dt, t_0)$ and $\hat{1}$ is also expected to be of first order in dt.

It can be easily proved that all these asserts are satisfied by

$$\hat{U}(t_0 + dt, t_0) = \hat{1} - i\hat{\Omega}dt, \tag{1.2.4}$$

where $\hat{\Omega}$ is a hermitian operator ($\hat{\Omega} = \hat{\Omega}^\dagger$). The operator in the form of (1.2.4) satisfies the composition property

$$\hat{U}(t_0 + dt_1 + dt_2, t_0) = \hat{U}(t_o + dt_1 + dt_2, t_0 + dt_1)\hat{U}(t_0 + dt_1, t_0) \tag{1.2.5}$$

and, ignoring terms of order $(dt)^2$ or higher, the unitary property

$$\hat{U^\dagger}(t_0 + dt, t_0)\hat{U}(t_0 + dt, t_0) = (\hat{1} + i\hat{\Omega}^\dagger dt)(\hat{1} - i\hat{\Omega}dt) \simeq \hat{1}. \tag{1.2.6}$$

The operator $\hat{\Omega}$ has the dimension of frequency. By the Planck-Einstein relation[1]

$$E = \hbar\omega, \tag{1.2.7}$$

it is natural to relate $\hat{\Omega}$ to the Hamiltonian operator $\hat{H}$ as follow:

$$\hat{\Omega} = \hbar^{-1}\hat{H}. \tag{1.2.8}$$

A more detailed dissertation on the derivation of $\hat{\Omega}$ can be found in [4]. To sum up, the infinitesimal time-evolution operator is

$$\hat{U}(t_0 + dt, t_0) = \hat{1} - \frac{i}{\hbar}\hat{H}dt. \tag{1.2.9}$$

Now, one is able to derive the fundamental time-evolution operator from the infinitesimal one. Consider now the composition property of the operator from $t+dt$ to t_0 passing through a generic t. Using equation (1.2.2) and equation (1.2.9), it gets

$$\hat{U}(t + dt, t_0) = \hat{U}(t + dt, t)\hat{U}(t, t_0) = \left(\hat{1} - \frac{i}{\hbar}\hat{H}dt\right)\hat{U}(t, t_0). \tag{1.2.10}$$

The last relation can be written in a differential equation form:

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t, t_0) = \hat{H}\hat{U}(t, t_0). \tag{1.2.11}$$

By multiplying both sides of (1.2.11) by $|\psi(t_0)\rangle$,

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t, t_0)\,|\psi(t_0)\rangle = \hat{H}\hat{U}(t, t_0)\,|\psi(t_0)\rangle\,, \tag{1.2.12}$$

and using equation (1.1.16), the Schrödinger equation for a state ket comes out

$$i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle\,. \tag{1.2.13}$$

Equation (1.2.13) gives the time evolution of the state of a physical system. However, it is not necessary to bother with the Schrödinger equation (1.2.13) because, if the ket state at t_0 is known, it is also known the ket state at each time t. Then, it is only necessary to derive the solution for the Schrödinger equation for the time-evolution operator (1.2.11) and apply the relation (1.1.16).

As explained in [5], there are three cases to be treated separately. The first case deals with those Hamiltonian $\hat{H}$ that are time-independent. In this case the solution of (1.2.11)

[1] $\hbar$ is the reduced Planck constant: $\hbar = 1.055 \times 10^{-27}\,\mathrm{erg\,s} = 6.582\,119\,569 \times 10^{-16}\,\mathrm{eV\,s}$

is

$$\hat{U}(t,t_0) = \exp\left(-\frac{i}{\hbar}\hat{H}(t-t_0)\right). \tag{1.2.14}$$

The second case considers time-dependent Hamiltonian $\hat{H}$ and, in this case, the solution of (1.2.11) is a generalization of the previous case:

$$\hat{U}(t,t_0) = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} dt'\hat{H}(t')\right). \tag{1.2.15}$$

The last case studies Hamiltonian $\hat{H}$ of given system that at different times does not commute. The formal solution in such situation is given by the Dyson series:

$$\hat{H}(t,t_0) = \hat{1} + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_2} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n) \tag{1.2.16}$$

The first case is the most relevant because in non-relativistic quantum mechanics often the Hamiltonian of a system is time-independent. From equation (1.1.9), the operator (1.2.14) can be expanded as

$$\begin{aligned}\exp\left(-\frac{i}{\hbar}\hat{H}(t-t_0)\right) = \\ = \sum_l \sum_m \int\int \frac{d\lambda}{K_{l,\lambda}}\frac{d\mu}{K_{m,\mu}} |l,\lambda\rangle\langle l,\lambda| \exp\left(-\frac{i}{\hbar}\hat{H}(t-t_0)\right)|m,\mu\rangle\langle m,\mu|,\end{aligned} \tag{1.2.17}$$

where $|l,\lambda\rangle$ are the eigenkets of an operator $\hat{A}$ that commute with $\hat{H}$; so they are energy eigenkets, whose eigenvalues $E_{l,\lambda}$ satisfy the eigenvalue equation

$$\hat{H}|l,\lambda\rangle = E_{l,\lambda}|l,\lambda\rangle. \tag{1.2.18}$$

The energy eigenvalue equation allows to put (1.2.17) in a simpler form analogous to (1.1.10)

$$\exp\left(-\frac{i}{\hbar}\hat{H}(t-t_0)\right) = \sum_l \int \frac{d\lambda}{K_{l,\lambda}} \exp\left(-\frac{i}{\hbar}E_{l,\lambda}(t-t_0)\right)|l,\lambda\rangle\langle l,\lambda|. \tag{1.2.19}$$

The time-evolution operator written in this form allows to solve any initial-value problem. Indeed, if $\psi(t_0)$ is known, one has

$$|\psi(t)\rangle = \sum_l \int \frac{d\lambda}{K_{l,\lambda}} \exp\left(-\frac{i}{\hbar}E_{l,\lambda}(t-t_0)\right)|l,\lambda\rangle\langle l,\lambda|\psi(t_0)\rangle. \tag{1.2.20}$$

So, what change in time are the expansion coefficient. Let remind that $|l,\lambda\rangle$ are the energy eigenkets or the eigenkets of an operator that commute with $\hat{H}$.

A special case of interest is when the initial ket state $|\psi(t_0)\rangle$ is one of the energy eigenkets $|l,\lambda\rangle$. By considering (1.2.20), it is clear that at a later time the state is again an eigenket of $\hat{H}$. Indeed, it differs from the initial one only by a phase factor. It is in this sense that an observable compatible with $\hat{H}$, which means that it commutes with the Hamiltonian, is a constant of the motion. Such a state is called stationary state. Finally, a ket state at a generic time t is the superposition of stationary states defined by equation (1.2.18), called the Schrödinger time independent equation. The superposition equation is the (1.2.20).

1.3 Wave mechanics

Originally, Schrödinger formulated quantum mechanics in an equivalent way called wave-mechanics, where a states of a physical system is encoded in a wave-function $\psi(\boldsymbol{x},t)$ and observables are represented by differential operators that act on the wave-function. Wave-mechanics is equivalent to the abstract Dirac's formulation if one defines, for each state $|\psi(t)\rangle$ of the Hilbert's space, the position basis wave-function $\psi(\boldsymbol{x})$ corresponding to that state as $\psi(\boldsymbol{x},t)=\langle\boldsymbol{x}|\psi(t)\rangle$, where $\boldsymbol{x}$ belongs to the configuration space. In the momentum space representation, the wave-function is defined by the expression $\psi(\boldsymbol{y},t)=\langle\boldsymbol{y}|\psi(t)\rangle$, where obviously $\boldsymbol{y}$ varies in the momentum space.

In this formulation of quantum mechanics, the Schrödinger equation (1.2.13) is written as:

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x},t)=\hat{H}\psi(\boldsymbol{x},t). \tag{1.3.1}$$

$\hat{H}$ is the Hamiltonian, that is a differential operator. In non-relativistic quantum mechanics, the Hamiltonian is always in the form:

$$\hat{H}=-\frac{\hbar^2}{2m}\nabla^2+V(\boldsymbol{x}). \tag{1.3.2}$$

The linearity of the problem suggests that the solution is a linear combination of wave-functions ψ_n that solve the Schrödinger equation. n is a multiplet of indexes made of both a discrete and a continuous part. When $\hat{H}$ is time independent, it is natural to search solution in the form $\psi_n(t,\boldsymbol{x})=\alpha_n(t)\phi_n(\boldsymbol{x})$. By inserting $\psi_n(t,\boldsymbol{x})$ in equation (1.3.1), it turns that the solution for the time-dependent part is

$$\alpha_n(t)=e^{-\frac{i}{\hbar}E_n\cdot t}, \tag{1.3.3}$$

while the ϕ_n are given by the time independent Schrödinger equation

$$\hat{H}\phi_n(\boldsymbol{x})=E_n\phi_n(\boldsymbol{x}). \tag{1.3.4}$$

To sum up, accordingly to equation (1.2.20), the solution is:

$$\psi(t, \boldsymbol{x}) = \sum_{n_d} \int dn_c \phi_n(\boldsymbol{x}) e^{-\frac{i}{\hbar} E_n \cdot t}, \tag{1.3.5}$$

where the summation is extended on the discrete part of the multiplet n, while the integration is over the continuous part. The ϕ_n are called energy eigenfunctions and the E_n are the energy eigenvalues. (ϕ_n, E_n) is the solution of the eigenvalue problem (1.3.4).

Since in equation (1.3.2) the differential operator ∇^2 appears, the wave function should satisfy the following boundary conditions:

- $\psi_n(t, \boldsymbol{x})$ must vanish on the frontier of the domain $\mathcal{D}$ of the problem
- $\psi_n(t, \boldsymbol{x})$ must be bounded for $|\boldsymbol{x}| \to \infty$ in $\mathcal{D}$
- $\psi_n(t, \boldsymbol{x})$ must be continuous for all $\boldsymbol{x} \in \mathcal{D}$
- $\frac{d}{dx}\psi_n(t, \boldsymbol{x})$ must be continuous for all $\boldsymbol{x} \in \mathcal{D}$
- $\psi_n(t, \boldsymbol{x})$ must be non identically vanishing

The energy eigenfunctions should fulfill the same boundary conditions.

Chapter2

Propagators and path integrals

In the first part of this chapter, the theory of Green's function and the use of propagators in quantum mechanics are introduced. Then, it is described a conceptual experiment in order to present a new way to compute probability amplitudes developed by Richard Feynman in 1948. This method, that generalizes the least action principle of classical mechanics to quantum one, allows to represent propagators as "path integrals".

2.1 Green's functions

Green's functions play a fundamental role in physics, especially in electrodynamics [6] and in quantum physics [7]. They are a powerful tool for solving inhomogeneous differential equations, such as the Poisson equation of electrostatic or the Schrödinger equation, and also homogeneous one.

2.1.1 General theory

Time-independent Green's functions. Green's functions can be defined as solutions of inhomogeneous differential equations of the type:

$$[z - \hat{L}(\boldsymbol{x})]G(\boldsymbol{x}, \boldsymbol{x'}; z) = \delta^{(D)}(\boldsymbol{x} - \boldsymbol{x'}),$$

that in the Dirac notation is written as [1]

$$[z - \hat{L}]G(z) = 1. \tag{2.1.1}$$

$\delta^{(D)}(\boldsymbol{x} - \boldsymbol{x'})$ is the Dirac delta function (see appendix B.1). Here it is assumed that z is a complex variable with $\mathrm{Re}\{z\} \equiv a$ and $\mathrm{Im}\{z\} \equiv b$ and that $\hat{L}(\boldsymbol{x})$ is an hermitian selfadjoint and time-independent operator. Then, $\hat{L}(x)$ has a generalized orthonormal basis of eigenfunction and its spectrum consists in the ensemble of all its eigenvalues. Considering only the case where the discrete and the continuous part of the spectrum of

[1] Because in Dirac notation one has $\phi(\boldsymbol{x}) \equiv \langle \boldsymbol{x}|\phi\rangle$, $\delta(\boldsymbol{x} - \boldsymbol{x'})L(\boldsymbol{x}) \equiv \langle \boldsymbol{x}| L \left|\boldsymbol{x'}\right\rangle$, $G(\boldsymbol{x}, \boldsymbol{x'}; z) \equiv \left\langle \boldsymbol{x'}\right| G(z) \left|\boldsymbol{x}\right\rangle$, $\left\langle \boldsymbol{x}\middle|\boldsymbol{x'}\right\rangle = \delta(\boldsymbol{x} - \boldsymbol{x'})$, $\int d^D x \left|\boldsymbol{x}\right\rangle \langle \boldsymbol{x}| = \hat{1}$.

$\hat{L}$ are separable, it is possible to summarize these properties in the following equations:

$$\begin{gathered}\hat{L}\left|\phi_{l,\lambda}\right\rangle = \zeta_{l,\lambda}\left|\phi_{l,\lambda}\right\rangle; \\ \langle\phi_l|\phi_m\rangle = \delta_{l,m}, \qquad \left\langle\phi_\lambda\middle|\phi_\mu\right\rangle = K_\lambda\delta(\lambda-\mu), \qquad \left\langle\phi_l\middle|\phi_\mu\right\rangle = \langle\phi_\lambda|\phi_m\rangle = 0; \\ \sum_l |\phi_l\rangle\langle\phi_l| + \int \frac{d\lambda}{K_\lambda}|\phi_\lambda\rangle\langle\phi_\lambda| = \hat{1}.\end{gathered} \tag{2.1.2}$$

If all eigenvalues $z-\hat{L}$ are not zero, i.e. $z \neq \{\zeta_{l,\lambda}\} \quad \forall l,\lambda$, equation (2.1.1) can be formally solved:

$$G(z) = \frac{1}{z-L}. \tag{2.1.3}$$

By multiplying (2.1.3) by the last equation of the set (2.1.2), it gets:

$$G(z) = \sum_l \frac{|\phi_l\rangle\langle\phi_l|}{z-\zeta_l} + \int \frac{d\lambda}{K_\lambda}\frac{|\phi_\lambda\rangle\langle\phi_\lambda|}{z-\zeta_\lambda}. \tag{2.1.4}$$

This last equation is called spectral representation of G and in the configuration space representation it is written as:

$$G(\boldsymbol{x},\boldsymbol{x'};z) = \sum_l \frac{\phi_l(\boldsymbol{x})\phi_l^*(\boldsymbol{x'})}{z-\zeta_l} + \int \frac{d\lambda}{K_\lambda}\frac{\phi_\lambda(\boldsymbol{x})\phi_\lambda^*(\boldsymbol{x'})}{z-\zeta_\lambda}. \tag{2.1.5}$$

Since $\hat{L}$ is hermitian, all its eigenvalues $\{\zeta_{l,\lambda}\}$ are real. Hence, if $\mathrm{Im}\{z\} \neq 0$, then $z \neq \zeta_l$. This means that G is an analytic function in the complex plane, except for those points or part of the real axis that correspond to the eigenvalues of $\hat{L}$. From (2.1.4) it is clear that the discrete eigenvalues of $\hat{L}$ are the simple poles of $G(z)$. The residue at each pole ζ_l gives the quantity $\sum_i \phi_i(\boldsymbol{x})\phi_i^*(\boldsymbol{x'})$, where i runs over all the (possibly) degenerate eigenfunctions $\phi_l(\boldsymbol{x})$ corresponding to ζ_l. If z is equal to a continuous eigenvalues ζ of $\hat{L}$, $G(\zeta)$ is not well defined since the integrand of equation (2.1.4) has a pole. For ζ belonging to such a spectrum, one can define $G(\zeta)$ using two Green's function defined through a limiting procedure:

$$G^+(z,\zeta) = \lim_{b\to 0^+} G(z,\zeta+ib), \tag{2.1.6}$$

$$G^-(z,\zeta) = \lim_{b\to 0^+} G(z,\zeta-ib). \tag{2.1.7}$$

Usually the limits of $G(\zeta \pm ib)$ as $b \to 0$ exist, but they are different from each other. Thus, this type of continuous spectrum produces a branch cut in $G(z)$ along parts of the real axis.

When $G(z)$ is known, the solution of the differential inhomogeneous problem

$$[z-\hat{L}(\boldsymbol{x})]u(\boldsymbol{x}) = f(\boldsymbol{x})$$

is simply:

$$u(\boldsymbol{x}) = \begin{cases} \int d^D x' G(\boldsymbol{x}, \boldsymbol{x'}; z) f(\boldsymbol{x'}), & \text{if } z \neq \{\zeta_n\}; \\ \phi(\boldsymbol{x}) + \int d^D x' G^{\pm}(\boldsymbol{x}, \boldsymbol{x'}; z) f(\boldsymbol{x'}), & \text{if } z = \zeta, \end{cases} \tag{2.1.8}$$

where ζ is a continuous eigenvalue and $\phi(\boldsymbol{x})$ is the solution of the corresponding homogeneous equation for the given value of ζ.

Time-dependent Green's functions. A time-dependent Green's function associated to an homogeneous and inhomogeneous first order in time differential equations, that are

$$\left[\frac{i}{c}\frac{\partial}{\partial t} - \hat{L}(\boldsymbol{x})\right] \phi(\boldsymbol{x}, t) = 0 \tag{2.1.9}$$

and

$$\left[\frac{i}{c}\frac{\partial}{\partial t} - \hat{L}(\boldsymbol{x})\right] \psi(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \tag{2.1.10}$$

is a function $g(\boldsymbol{x'}, \boldsymbol{x}, t' - t)$ that satisfies

$$\left[\frac{i}{c}\frac{\partial}{\partial t} - \hat{L}(\boldsymbol{x})\right] g(\boldsymbol{x'}, \boldsymbol{x}, t' - t) = \delta^{(D)}(\boldsymbol{x} - \boldsymbol{x'})\delta(t' - t), \tag{2.1.11}$$

where c is a constant and $\hat{L}(\boldsymbol{x})$ is an hermitian selfadjoint operator similar to the one studied in the time-dependent case.

Expressing $g(\boldsymbol{x'}, \boldsymbol{x}; \tau)$, with $\tau = t' - t$, through a Fourier transform

$$g(\boldsymbol{x'}, \boldsymbol{x}; \tau) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} g(\boldsymbol{x'}, \boldsymbol{x}; \omega) \tag{2.1.12}$$

and substituting it in (2.1.11), it gets:

$$\left(\frac{\omega}{c} - L(\boldsymbol{x})\right) = \delta^{(D)}(\boldsymbol{x} - \boldsymbol{x'}). \tag{2.1.13}$$

Comparing this expression with (2.1.1) it is evident that

$$g(\omega) = G\left(\frac{\omega}{c}\right), \tag{2.1.14}$$

where $G(z)$ is the time independent Green's function. Then g is an analytic function with singularities (poles and cut brunches) on the real axis. So, the Fourier transform of g is not well defined as it stands in (2.1.12). However, similarly at the time-independent case, it can be obtained with a limiting procedure

$$g^{\pm}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G^{\pm}\left(\frac{\omega}{c}\right) e^{-i\omega\tau}. \tag{2.1.15}$$

What is important for the application in quantum mechanics is the difference between $g^{\pm}$ (indicated as $\tilde{g}$) since it can be expanded as [7]

$$\tilde{g}(\tau) = -ic\sum_l e^{-ic\tau\zeta_l}\phi_l(\boldsymbol{x})\phi_l^*(\boldsymbol{x'}) \equiv -ice^{-icL}, \tag{2.1.16}$$

where ϕ_l and ζ_l are respectively the eigenvalues and the eigenfunctions of L. $\tilde{g}$ is analogous to the time-evolution operator (1.2.14), but with $\hat{L}$ instead of $\hat{H}$. From relation (2.1.15), it can be concluded that $\tilde{g}$ is nothing but the Fourier transform of the time-independent Green's function. $\tilde{g}$ allows to solve the first equation of (2.1.9), once the initial condition $\phi(\boldsymbol{x'}, t')$ is known:

$$\phi(\boldsymbol{x}, t) = \frac{i}{c}\int d^D x' \tilde{g}(\boldsymbol{x'}, \boldsymbol{x}; t - t')\phi(\boldsymbol{x'}, t'). \tag{2.1.17}$$

It can be proved that $\tilde{g}$ allows also to know the solution of the inhomogeneous equation:

$$\psi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, t) + \int d^D x' \int_{-\infty}^{t} dt' \tilde{g}(\boldsymbol{x'}, \boldsymbol{x}; t - t')f(\boldsymbol{x'}, t). \tag{2.1.18}$$

2.1.2 Green's functions in quantum mechanics

The time-independent and time-dependent Schrödinger equations are differential and inhomogeneous. For the search of their solutions, the general theory of Green's function can be applied. The complex variable z is substituted by the energy E and the operator $\hat{L}$ is replaced the Hamiltonian $\hat{H}$ (1.3.2); therefore the eigenfunctions and the eigenvalues of $\hat{L}$ are the energy eigenfunctions and the energy eigenvalues. It is posited as follow:

$$G(\boldsymbol{x}, \boldsymbol{x'}; E) = \sum_l \frac{\psi_l(\boldsymbol{x})\psi_l^*(\boldsymbol{x'})}{E - E_l} + \int \frac{d\lambda}{K_\lambda}\frac{\psi_\lambda(\boldsymbol{x})\psi_\lambda^*(\boldsymbol{x'})}{E - E_\lambda}. \tag{2.1.19}$$

From the general theory the following points may be deduced:

1. The position of the poles of G coincides with the discrete energy eigenvalues of $\hat{H}$
2. The residue at each pole E_l equals $\sum_i \psi_i(\boldsymbol{x})\psi_i^*(\boldsymbol{x'})$, where the summation is over all the possibly degenerate energy eigenfunctions corresponding to E_l.
3. The branch cuts of G along the real axis coincides with the continuous spectrum of $\hat{H}$.
4. Since the time-dependent Schrödinger equation is an homogeneous differential equation, its solution is given by

$$\psi(\boldsymbol{x}, t) = i\hbar\int d^D x' \tilde{g}(\boldsymbol{x}, \boldsymbol{x'}; t - t_0)\psi(\boldsymbol{x'}, t_0) \tag{2.1.20}$$

in analogy with (2.1.17). In quantum mechanics the constant c used in the general dissertation on time-dependent Green's function is simply $\hbar^{-1}$. $\tilde{g}$ is sometimes

called the retarded Green's function and it is related to the time-dependent Green's function, often called the energy Green's function, by a Fourier transform (2.1.12):

$$G(\boldsymbol{x}, \boldsymbol{x'}; E) = \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{\frac{i}{\hbar}E\tau} \tilde{g}(\boldsymbol{x}, \boldsymbol{x'}; \tau), \tag{2.1.21}$$

where $\tau = t'' - t'$.

2.1.3 Green's fuctions and perturbation theory

When the Hamiltonian of a system is separable in two parts

$$\hat{H} = \hat{H}_0 + \hat{H}' \tag{2.1.22}$$

and the eigenvalues and eigenfunctions of $\hat{H}_0$ (and thus the associated Green's function $G_0(z)$) are known, the perturbation theory can be used. The Hamiltonian $\hat{H}_0$ is called unperturbed Hamiltonian, while $\hat{H}'$ is a perturbation term. The Hamiltonian $\hat{H}$ is said perturbed. It is possible to write the Green's function $G(z)$ associated to the perturbed Hamiltonian in function of the unperturbed one $G_0(z)$ and the perturbation term $\hat{H}'$. Indeed,

$$G(z) = \frac{1}{z - \hat{H}_0 - \hat{H}'} = \frac{1}{(z - \hat{H}_0)[1 - \hat{H}'(z - \hat{H}_0)^{-1}]} = \frac{G_0(z)}{1 - \hat{H}' G_0(z)} \tag{2.1.23}$$

Expanding the denominator in power series, one obtains an equation for $G(z)$

$$\begin{aligned} G(z) = G_0(z) + G_0(z)\hat{H}'G_0(z) + G_0(z)\hat{H}'G_0(z)\hat{H}'G_0(z) + \cdots = \\ = G_0(z) + G_0(z)\hat{H}'G(z), \end{aligned} \tag{2.1.24}$$

that in the configuration space it is written as:

$$G(\boldsymbol{x}, \boldsymbol{x'}; z) = G_0(\boldsymbol{x}, \boldsymbol{x'}; z) + \int d^D k G_0(\boldsymbol{x}, \boldsymbol{k}; z) \hat{H}'(\boldsymbol{k}) G(\boldsymbol{k}, \boldsymbol{x'}; z) \tag{2.1.25}$$

$G(z)$ can be expressed in another way:

$$G(z) = G_0(z)\hat{T}(z)G_0(z) \tag{2.1.26}$$

where $\hat{T}$ is the t-matrix and it is defined as

$$\hat{H}'G(z)(z - \hat{H}_0) \tag{2.1.27}$$

This definition is valid for all z except for $z = \{E_n\}$, where $\{E_n\}$ are the eigenvalues of $\hat{H}$. When z is one of the continuous eigenvalues of $\hat{H}$, one defines

$$\hat{T}^{\pm}(z) = \hat{H}' G^{\pm}(z)(z - \hat{H}_0). \tag{2.1.28}$$

$\hat{T}(z)$ is particularly useful since it allows to write a solution for the time-independent Schrödinger equation $(E - \hat{H})\,|\psi\rangle = 0$, that is equivalent to $(E - \hat{H}_0)\,|\psi\rangle = \hat{H}'\,|\psi\rangle$, as an integral equation:

$$\psi^{\pm}(\boldsymbol{x}) = \phi(\boldsymbol{x}) + \int d^D k G_0^{\pm}(\boldsymbol{x}, \boldsymbol{k}; E)\hat{H}'(\boldsymbol{k})\psi^{\pm}(\boldsymbol{k}), \tag{2.1.29}$$

where $\phi(\boldsymbol{x})$ is the unperturbed solution of the Schrödinger equation. The last equation is known as the Lippman-Schwinger equation.

2.2 Propagators in wave mechanics

In section 1.2, it is shown that the most general time-evolution problem with a time-independent Hamiltonian can be solved expanding the initial ket in terms of the eigenkets $|l, \lambda\rangle$ of an observable that commutes with $\hat{H}$

$$|\psi(t_0)\rangle = \sum_l \int \frac{d\lambda}{K_{l,\lambda}} |l, \lambda\rangle \langle l, \lambda|\psi(t_0)\rangle \tag{2.2.1}$$

and then applying the time evolution operator $\hat{U}(t, t_0) = \exp\Big(-(i/\hbar)\hat{H}(t - t_0)\Big)$ to the initial state ket:

$$|\psi(t)\rangle = \hat{U}(t, t_0)\,|\psi(t_0)\rangle = \sum_l \int \frac{d\lambda}{K_{l,\lambda}} |l, \lambda\rangle \langle l, \lambda|\psi(t_0)\rangle\, e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}. \tag{2.2.2}$$

To translate the last equation in the wave mechanics language, it is needed to multiply both sides by $\langle \boldsymbol{x}|$

$$\psi(\boldsymbol{x}, t) = \sum_l \int \frac{d\lambda}{K_{l,\lambda}} c_{l,\lambda} u_{l,\lambda}(\boldsymbol{x}) e^{-\frac{i}{\hbar}E_{l,\lambda}(t-t_0)}, \tag{2.2.3}$$

where

$$u_{l,\lambda}(\boldsymbol{x}) = \langle \boldsymbol{x}|l, \lambda\rangle \tag{2.2.4}$$

and the expansion coefficients $c_{l,\lambda}(t_0)$ are given by

$$c_{l,\lambda}(t_0) = \langle l, \lambda|\psi(t_0)\rangle = \int d^D x \langle l, \lambda|\boldsymbol{x}\rangle \langle \boldsymbol{x}|\psi(t_0)\rangle \equiv \int d^D x u^*_{l,\lambda}(\boldsymbol{x})\psi(\boldsymbol{x}, t_0). \tag{2.2.5}$$

Equation (2.2.3) can be written also as

$$\psi(\boldsymbol{x''}, t'') = \int d^D x' K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t')\psi(\boldsymbol{x'}, t'), \tag{2.2.6}$$

where $K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t')$ is the propagator in wave mechanics. The propagator allows to know completely the wave function at an arbitrary time t'' and position $\boldsymbol{x''}$, once the initial value problem $\psi(\boldsymbol{x'}, t')$ is known. In this sense, the Schrödinger wave mechanics is a causal theory and the time development of a wave function is perfectly deterministic as anything else in classical mechanics. What changes radically is that when a measurement intervenes, the wave function changes abruptly as it collapses in one of the eigenstates of the observable being measured.

Comparing equation (2.2.6) with equation (2.2.3), it is possible to find an expression for the propagator

$$K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \sum_l \int \frac{d\lambda}{K_{l,\lambda}} \left\langle \boldsymbol{x''} \middle| l, \lambda \right\rangle \left\langle l, \lambda \middle| \boldsymbol{x'} \right\rangle e^{-\frac{i}{\hbar} E_{l,\lambda}(t''-t')}. \tag{2.2.7}$$

The propagator must solve the Schrödinger time-dependent wave equation for $t'' > t'$ in the variables $\boldsymbol{x''}$ and $t'i$

$$\left[\frac{\hbar^2}{2m}\nabla^2_{\boldsymbol{x''}} - V(\boldsymbol{x''}) + i\hbar\frac{\partial}{\partial t''}\right] K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = i\hbar\delta^{(D)}(\boldsymbol{x''} - \boldsymbol{x'})\delta(t'' - t'), \tag{2.2.8}$$

with the boundary condition

$$K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = 0 \quad \text{for} \quad t'' < t', \tag{2.2.9}$$

and for $t'' \to t'$ it has to tend to a D-dimensional Dirac delta:

$$\lim_{t''\to t'} K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \delta^{(D)}(\boldsymbol{x''} - \boldsymbol{x'}). \tag{2.2.10}$$

Because of these two properties the propagator, regarded as a function of $\boldsymbol{x''}$, is the wave function at t'' of a particle detected in an earlier instant t' in $\boldsymbol{x'}$. Indeed, this interpretation follows even noticing that equation (2.2.7) can also be written as

$$K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \left\langle \boldsymbol{x''} \middle| e^{-\frac{i}{\hbar}\hat{H}(t''-t')} \middle| \boldsymbol{x'} \right\rangle, \tag{2.2.11}$$

where the time-evolution operator acting on $|\boldsymbol{x'}\rangle$ is just the state ket $|\boldsymbol{x''}\rangle$ at t'' of a system that was localized at $\boldsymbol{x'}$ at time $t' < t''$.

There is a relation between the propagator and the Green's function $\tilde{g}(\boldsymbol{x''}, t''; \boldsymbol{x'}, t')$

(2.1.11) that solves the Schrödinger time-dependent equation. The relation is:

$$\tilde{g}(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \frac{1}{i\hbar}\theta(t'' - t')K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t'), \tag{2.2.12}$$

where $\theta(x)$ is the Heaviside step function:

$$\theta(x) = \begin{cases} 0, & \text{if} \quad x < 0; \\ 1, & \text{if} \quad x > 0. \end{cases} \tag{2.2.13}$$

In some references the propagator and the Green's function for the Schrödinger time-dependent equation are the same thing. However, the propagator and the Green's function, even if they are considered as different entities, show the same analytic behaviour. So, the poles on the real axis of the propagator are the discrete energy eigenvalues, while the branch cuts along the real axis are the continuous parts of the energy spectrum of the system to which the propagator is associated.

The propagator can be identified with the quantity $\langle \boldsymbol{x''}, t'' | \boldsymbol{x'}, t' \rangle$, which is the probability amplitude for the particle prepared at t' with position eigenvalue $\boldsymbol{x'}$ to be detected at t'' with position eigenvalue $\boldsymbol{x''}$. Then, the probability to find the particle at $(\boldsymbol{x''}, t'')$ is $P(\boldsymbol{x''}, t'') = |K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t')|^2$. Indeed, in the Schrödinger picture the wave-function $\psi(\boldsymbol{x''}, t'')$ is known if it is known the initial wave-function $\psi(\boldsymbol{x'}, t')$:

$$\psi(\boldsymbol{x''}, t'') = \left\langle \boldsymbol{x''} \middle| \psi(t'') \right\rangle = \left\langle \boldsymbol{x''} \middle| e^{-(i/\hbar)\hat{H}(t''-t')} \middle| \psi(t') \right\rangle. \tag{2.2.14}$$

Defining the time-dependent bra vector[2]

$$\left\langle \boldsymbol{x''}, t'' \right| = \left\langle \boldsymbol{x''} \right| e^{-(i/\hbar)\hat{H}(t''-t')} \tag{2.2.15}$$

and using the completeness of that bra, equation (2.2.14) turns into:

$$\psi(\boldsymbol{x''}, t'') = \int d^3x' \left\langle \boldsymbol{x''}, t'' \middle| \boldsymbol{x'}, t' \right\rangle \left\langle \boldsymbol{x'}, t' \middle| \psi(t') \right\rangle. \tag{2.2.16}$$

Comparing this equation for $\psi(\boldsymbol{x''}, t'')$ with (2.2.6), it results that

$$K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \left\langle \boldsymbol{x''}, t'' \middle| \boldsymbol{x'}, t' \right\rangle. \tag{2.2.17}$$

Roughly speaking, the propagator is the probability amplitude for the particle to go from a space-time point $(\boldsymbol{x'}, t')$ to another space-time point $(\boldsymbol{x''}, t'')$. For this reason, sometimes K is called transition amplitude. The transition amplitude, i.e. the propagator, fulfills

[2]This is the Heisenberg picture of quantum mechanics, where the eigenkets of the operator vary in time while the physical state $|\psi\rangle$ remains constant. In the Heisenberg picture the quantity $\langle a, t'' | b, t' \rangle$ is the probability amplitude to find a particle prepared at t' in the state b at t'' in the state a

the composition property

$$K(\boldsymbol{x}'''', t''''; \boldsymbol{x}', t') = \int d^D x''' \int d^D x'' K(\boldsymbol{x}'''', t''''; \boldsymbol{x}''', t''') K(\boldsymbol{x}''', t'''; \boldsymbol{x}'', t'') K(\boldsymbol{x}'', t''; \boldsymbol{x}', t') \tag{2.2.18}$$

and so on. The propagator summarises the quantum mechanics of the system. In the usual formulation of quantum mechanics, given an initial wave-function, one can find the final wave -function by solving the time-dependent Schrödinger equation. In this formulation, the propagator gives the solution immediately.

If it is known the propagator for an infinitesimal time interval $K(\boldsymbol{x}'', t' + dt; \boldsymbol{x}', t')$, then it is possible to obtain the probability amplitude for a finite time interval by compounding the appropriate transition amplitudes for infinitesimal time intervals similarly to (2.2.18). This kind of reasoning leads to a different formulation of quantum mechanics due to Richard Feynman [8], which will be investigated in detail in the next section. The Feynman formulation is based on expressing propagators as "path integrals".

2.3 Feynman's path integral

2.3.1 Double slit experiment

In figure 2.1 it is shown the apparatus set of a conceptual experiment where a source S emits electrons with the same energy. The electrons come out in all directions to impinge on a screen C with two holes, through which electrons can pass. Behind the screen, at plane P, there is a detector of electrons which can be placed at different distances x from the center of the plane.

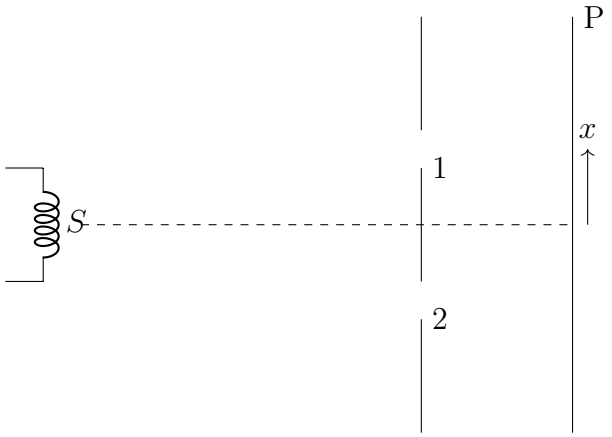

Fig. 2.1: Apparatus set of the double slit experiment.

If the detector is sufficiently sensible, it will measure a rain of particles and not a continuous current. If there is a large number of detectors all over the plane, only one detector at each time will detect an electron. This is the reason why electrons are con-

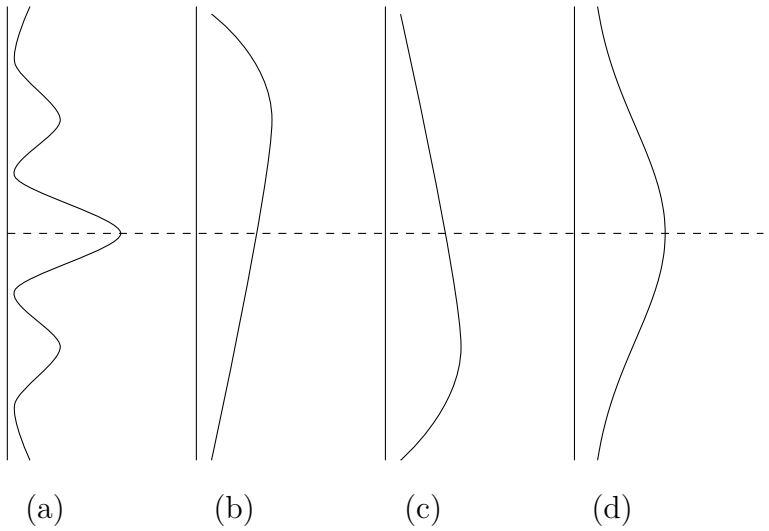

Fig. 2.2: Probability P(x) measured in different case: (a) either hole 1 and 2 open, (b) only hole 1 open, (c) only hole 2 open, (d) sum of curve (b) and (c).

sidered particles. What it can be measured for different positions x of the detector is the probability that an electron reaches that point. The probability as a function of x is shown in curve (a) of figure 2.2. Since the electrons behave as particles, the following points might be supposed :

1. Each electrons that reach the plane P in x must go through hole 1 or hole 2.

2. The chance of arrival at x is the sum of two terms: P_1 (the chance of arrival at x coming through hole 1) and P_2 (the chance of arrival at x coming through hole 2).

These points can be experimentally proved by closing hole 1 and measuring the chance of arriving at x through hole 2 and vice versa. The outcomes of this experiment is shown in curve (d) of figure 2.2, which clearly is not the same as curve (a). It is then possible to say that $P \neq P_1 + P_2$. Actually, the curve P(x) shown in 2.2 is exactly the intensity distribution of waves coming from S and passing through the two holes.

This fact suggests that P(x) is the absolute square of a certain complex quantity $\phi(x)$ which it is called the probability amplitude. Furthermore, $\phi(x)$ is the contribution of two terms: the amplitude of arrival through hole 1 ($\phi_1(x)$) and the amplitude of arrival through hole 2 ($\phi_2(x)$). As a consequence, a third point might be supposed:

3. There are complex numbers $\phi_1(x)$, $\phi_2(x)$ such that:

$$P(x) = |\phi(x)|^2 = |\phi_1(x) + \phi_2(x)|^2. \tag{2.3.1}$$

Then, it is possible to identify $P_1(x) = |\phi_1(x)|^2$ and $P_2(x) = |\phi_2(x)|^2$. So, clearly $P \neq P_1 + P_2$.

Since $P \neq P_1 + P_2$, it is not true that electrons pass through either hole 1 or 2. Given that electrons scatter light, if a light source is placed behind the holes, it can be determined if an electron passes through hole 1 or 2 just watching where the light is scattered. The result seems a paradox: it shows unequivocally that electrons really pass through either hole 1 or hole 2. So physical systems exhibit either a corpuscular or a undulatory properties, depending on conditions (wave-particle duality). However, it is not possible to make an experience where there are both corpuscular and undulatory aspects (complementarity principle).

If, among all electrons arriving at x, are selected only those that are scattered through hole 1, a distribution P_1 nearly to curve (b) of figure 2.2 is found. If only electrons passed through hole 2 are selected, there is distribution P_2 similar to curve (c) of figure 2.2 . In this case, the probability is $P = P_1 + P_2$ and so the process of measurement has changed the arrival at x of the electrons. The reason is that the collision of light with the electron may change their motion and so their chance of arrival at x.

Since the momentum carried by the light is h/λ, weaker effects could be produced using light of longer wavelength λ. But there is a limit to this: if it is used a too long wavelength light, one cannot say if electron was scattered from behind hole 1 or hole 2 (in general, a light of wavelength λ cannot be located in space with precision greater than λ). So, there is a limitation to the subtlety according to which experiments could be performed. This is strictly connected with the Heisenberg's uncertainty principle, which stands that the product of the uncertainties of momentum and position involved in any experiment can not be smaller than a number with the order of magnitude of the reduced Planck constant:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}. \tag{2.3.2}$$

In the case presented, the equation (2.3.2) tells us that any apparatus with the purpose of determining trough which hole electrons may pass, without destroying the interference pattern, must fail.

To sum up, when both holes are open and no attempt is made to determine through which hole the electron passes, it is not possible to know if the electron has passed through hole 1 or 2 and the two alternatives are said interfering. In this case the total probability is the sum of the probability amplitudes associated to the interfering alternatives in a "wave manner" ($P = |\phi_1 + \phi_2|^2 = |\phi_1|^2 + |\phi_2|^2 + 2\,\mathrm{Re}(2\phi_1\phi_2)$). Only when the apparatus is operating to determine through which hole the electron passes, it is possible to infer if it passes through one or the other. In this case the probability is given by $P = |\phi_1 + \phi_2|^2 = |\phi_1|^2 + |\phi_2|^2$. There are cases where the alternatives are both interfering and exclusive, for example when one tries to know the probability for the electron to reach a point in the plane within 1 cm from the center. In this case the probability of reach a point x is obtained summing the alternatives as they are interfering, then summing the probability

for each point x within 1 cm as they are exclusive alternatives.

This experiment can be interpreted in the Schrödinger formulation of quantum mechanics. When one has no intention to determine through which hole the electron passes, the wave-function of the system is the superposition of two states. When one tries to locate the electron, the wave-function collapses onto a precise state.

2.3.2 The quantum-mechanical propagator

Now, to proceed in the Feynman's interpretation, it is necessary to find a way to compute the amplitudes. The total amplitude for an event is the sum of the amplitudes for the various alternative ways that the event can occur. A particle going from a to b can do this going through different paths in space-time. The basic idea of Feynman is to associate to each path a different amplitude. The total amplitude is the sum of all the possible amplitudes associated to all the possible paths. This idea arises considering a modified version of the double slit experiment where there is a couple of additional screen between the source and the holes. These two additional screens have different holes (figure 2.3).

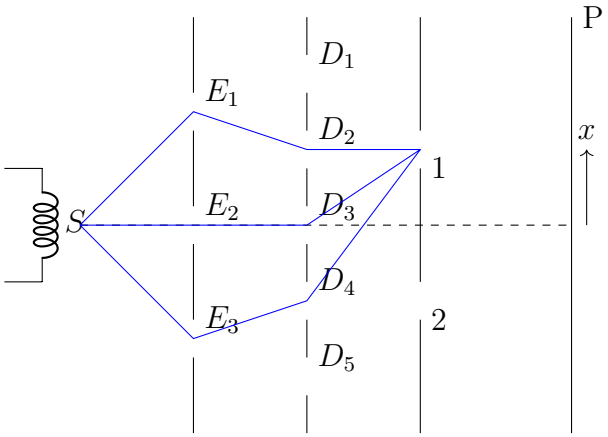

Fig. 2.3: Double slit experiment modified. Some of the different path through which the electron can reach the hole 1 are presented.

There are several alternatives for the electron to reach the hole 1. For example, it can pass through hole E_1 then D_2 and finally through the hole 1; or it may pass through holes E_3, D_4 and then hole 1. In this way, an electron that arrives at hole 1 might have travelled through any path whose extremities are the source and hole. Any path contributes with a difference amplitude, while the complete amplitude is the sum or the integral of these amplitudes over all the possible paths. This is a difference from classical physics, where only a precise path contributes, i.e. the one who results from the least action principle [9]; in quantum mechanics all the paths contribute.

What remains to determine is how much each trajectory contributes to the total amplitude. Feynman supposed that each path contributes with the same magnitude, but at different phases. The phase of the contribution for a given path is the classical

action S associated to that path in unit of the reduced Planck constant $\hbar$. Then, the total transition amplitude, i.e the propagator, from a point of the space-time $(\boldsymbol{x''}, t'')$ to another one $(\boldsymbol{x'}, t)$, is the sum over all the possible paths of those phases:

$$K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \sum \phi[\boldsymbol{x}(t)] = \sum const \cdot e^{(i/\hbar)S[\boldsymbol{x}(t)](\boldsymbol{x''}, t''; \boldsymbol{x'}, t')}, \tag{2.3.3}$$

where the allowed paths $\boldsymbol{x}(t)$ satisfy the boundary conditions $\boldsymbol{x}(t'') = \boldsymbol{x''}$ and $\boldsymbol{x}(t') = \boldsymbol{x'}$. The constant must be chosen properly in order to normalize K. The probability associated to the process is the absolute square of (2.3.3). The classical action is given by

$$S[\boldsymbol{x}(t)](\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \int_{t'}^{t''} dt L(\dot{\boldsymbol{x}}, \boldsymbol{x}, t), \tag{2.3.4}$$

where L is the Lagrangian function. The classical Lagrangian is defined as

$$L(\dot{\boldsymbol{x}}, \boldsymbol{x}, t) = \frac{1}{2}\dot{\boldsymbol{x}}^2 - V(\boldsymbol{x}). \tag{2.3.5}$$

Now, it has to be defined the "sum over paths". In order to do that, consider a subset of paths obtained dividing the time into N steps of width ϵ:

$$\{t_i | t_{i+1} - t_i = \epsilon, N\epsilon = t'' - t'; t_0 = t', t_N = t''\}. \tag{2.3.6}$$

At each time t_i, consider a special point $\boldsymbol{x_i}$, with boundary conditions $\boldsymbol{x_0} = \boldsymbol{x'}$ and $\boldsymbol{x_N} = \boldsymbol{x''}$. Using the analogy with the Riemann integral, it is natural to hypothesize:

$$K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = const \cdot \int d^D x_1 \int d^D x_2 \cdots \int \ldots d^D x_{N-1} \phi[\boldsymbol{x}(t)]. \tag{2.3.7}$$

The integration over $\boldsymbol{x_0}$ and $\boldsymbol{x_N}$ is excluded since they are fixed points. Making ϵ smaller, a more representative complete set of all paths can be found. However, the limit for $\epsilon \to 0$ of (2.3.7) does not exist until a normalization factor depending on ϵ^{-N} is provided. If the Lagrangian is the classical one (2.3.5), the normalization constant is [8]:

$$const = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{ND}{2}}, \tag{2.3.8}$$

where D is the dimension of the space considered.

$$K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \lim_{\epsilon \to 0} \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{N\frac{D}{2}} \int d^D x_1 \int d^D x_2 \cdots \int d^D x_{N-1} e^{\frac{i}{\hbar}S[\boldsymbol{x}(t)](\boldsymbol{x''}, t''; \boldsymbol{x'}, t')}. \tag{2.3.9}$$

The limit for $\epsilon \to 0$ is equivalent to the one for $N \to \infty$. If it is defined the following

multi-dimensional integral operator

$$\int_{\boldsymbol{x}'=\boldsymbol{x}(t')}^{\boldsymbol{x}''=\boldsymbol{x}(t'')} \mathcal{D}[\boldsymbol{x}(t)] = \lim_{\epsilon\to 0}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{ND}{2}} \cdot \int d^D x_1 \int d^D x_2 \cdots \int d^D x_{N-1}, \tag{2.3.10}$$

the propagator, i.e. the transition amplitude, is written in the form of a path integral

$$K(\boldsymbol{x}'', t''; \boldsymbol{x}', t') = \int_{\boldsymbol{x}'=\boldsymbol{x}(t')}^{\boldsymbol{x}''=\boldsymbol{x}(t'')} \mathcal{D}[\boldsymbol{x}(t)] e^{(i/\hbar)S[\boldsymbol{x}(t)](\boldsymbol{x}'', t''; \boldsymbol{x}', t')}. \tag{2.3.11}$$

As for as the transition amplitude, the path integral fulfills the composition property:

$$\begin{aligned} K(\boldsymbol{x}''', t'''; \boldsymbol{x}', t') &= \int_{\boldsymbol{x}'=\boldsymbol{x}(t')}^{\boldsymbol{x}'''=\boldsymbol{x}(t''')} \mathcal{D}[\boldsymbol{x}(t)] e^{(i/\hbar)(S[\boldsymbol{x}(t)](\boldsymbol{x}''', t'''; \boldsymbol{x}'', t'') + S[\boldsymbol{x}(t)](\boldsymbol{x}'', t''; \boldsymbol{x}', t'))} = \\ &= \int d^D x'' \int_{\boldsymbol{x}''=\boldsymbol{x}(t'')}^{\boldsymbol{x}'''=\boldsymbol{x}(t''')} \mathcal{D}[\boldsymbol{x}(t)] e^{(i/\hbar)(S[\boldsymbol{x}(t)](\boldsymbol{x}''', t'''; \boldsymbol{x}'', t'')} K(\boldsymbol{x}'', t''; \boldsymbol{x}', t') = \\ &= \int d^D x'' K(\boldsymbol{x}''', t'''; \boldsymbol{x}'', t'') K(\boldsymbol{x}'', t''; \boldsymbol{x}', t'). \end{aligned} \tag{2.3.12}$$

This property can be interpreted as following: the probability amplitude for the particle to go from $(\boldsymbol{x}''', t''')$ to $(\boldsymbol{x}', t')$ is the sum over all the possible points $(\boldsymbol{x}'', t'')$ of the probability amplitude for the particle to go from $(\boldsymbol{x}', t')$ to $(\boldsymbol{x}'', t'')$ and then to $(\boldsymbol{x}''', t''')$. This property can be generalized dividing the time scale into N intervals. The time slicing produces a division of all the possible paths from $\boldsymbol{x}''$ to $\boldsymbol{x}'$ into N segments $\boldsymbol{x}_i$, $i = 0, \ldots, N$ with the constraints $\boldsymbol{x}_0 = \boldsymbol{x}'$, $\boldsymbol{x}_N = \boldsymbol{x}''$. Therefore

$$K(\boldsymbol{x}'', t''; \boldsymbol{x}', t') = \lim_{\epsilon\to 0}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{N\frac{D}{2}} \left(\prod_{j=1}^{N-1} \int d^D x_j\right) \prod_{i=0}^{N-1} K(\boldsymbol{x}_{i+1}, t_{i+1}; \boldsymbol{x}_i, t_i), \tag{2.3.13}$$

where the propagator for the particle to go between two points separated by an infinitesimal time interval ϵ is

$$K(\boldsymbol{x}_{i+1}, t_{i+1}; \boldsymbol{x}_i, t_i) = \exp\left(\frac{i}{\hbar}\epsilon L\left(\frac{\boldsymbol{x}_{i+1} - \boldsymbol{x}_i}{\epsilon}, \frac{\boldsymbol{x}_{i+1} - \boldsymbol{x}_i}{2}, \frac{t_{i+1} - t_i}{2}\right)\right). \tag{2.3.14}$$

By combining the two previous equations, it is found a representation for the path integral completely equivalent to the (2.3.11):

$$\begin{aligned} K(\boldsymbol{x}'', t''; \boldsymbol{x}', t') = \lim_{\epsilon\to 0}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{N\frac{D}{2}} &\left(\prod_{j=1}^{N-1} \int d^D x_j\right) \times \\ &\times \sum_{i=0}^{N-1} \exp\left(\frac{i}{\hbar}\epsilon L\left(\frac{\boldsymbol{x}_{i+1} - \boldsymbol{x}_i}{\epsilon}, \frac{\boldsymbol{x}_{i+1} - \boldsymbol{x}_i}{2}, \frac{t_{i+1} - t_i}{2}\right)\right). \end{aligned} \tag{2.3.15}$$

Chapter3

Renormalization

In this chapter regularization and renormalization are introduced. In the first part, some techniques of regularization and some uses of renormalization in Quantum Electrodynamics are described. Then, a more mathematical introduction to renormalization is given.

In section 3.2 a classical physics problem is treated using renormalization. It consists in the electric potential generated by an uniform charged straight wire, that has to be renormalized since it is divergent if computed directly. This problem is treated using the cutoff regularization and the dimensional one.

The section 3.3 deals with quantum mechanics and the solution of the Schrödinger equation for a single particle subjected to an attractive δ-function potential. In particular, the aim is to find the bound state energy. First, it is studied the one-dimensional case, then it is generalized to D dimensions. In doing this, a problem arises when $D = 2$. Indeed, the eigenvalue equation for the two-dimensional case is not well defined as the bound state energy diverges. This problem is solved through renormalization techniques, both with the moment-cutoff scheme and the dimensional one. The problem is treated also with an effective field theory approach, where the renormalization scheme consists in replacing the $\delta^{(2)}$-potential with one known at short distance r_0, then solving directly the Schrödinger equation and finally taking r_0 to zero (real space regularization).

3.1 Introduction to the subject

Nature provides a huge amount of phenomena, which can be described by different theories depending on the scale of the physical process. For example, for the "every day" physics the Newtonian physics is sufficient, while at atomic level quantum mechanics is necessary. At the cosmological scale, the general relativity is used. To each scale corresponds a set of parameters which are useful for the description of the process. For example, in fluid dynamics the Reynolds's number $Re = UL/\nu$ (where U and L are the characteristic velocity and length of the fluid and ν is its viscosity) plays an important role as it determines which parameters are helpful in describing the system at different scales. Suppose U and ν are fixed; for a short distance fluid, i.e. $Re << 1$, the viscosity effects are dominant and inertia forces are negligible, while for $Re >> 1$ the dominant effects are the inertial ones. Therefore, it is clear that, depending on the scale in which the fluid is

observed, different parameters are used for the physical description. Renormalization, in a general definition, deals with the evolution of the physical description with the scale of observation.

Renormalization was born as a general framework for phase transition studies and particle physics calculation. A good dissertation on the scale dependence for a ferromagnetic phase transition is discussed in reference [10]. In this section, the focus is on the particle physics side. Quantum Electrodynamics (QED) explains why a propagating photon (emitted by an electron) may create a pair electron-positron e^-e^+ which later annihilates to create again a photon. This process is made possible by the relativistic equivalence mass-energy, that allows to create massive particles from a photon (that at least has energy equals two times the mass of the electron/positron, $10^6\,\mathrm{eV}$), and the quantum time-energy uncertainty principle, that allows the pair to exist from a time proportional to the inverse of their mass. This time is of order:

$$\Delta E \Delta t \sim \hbar \rightarrow \Delta t \sim \frac{\hbar}{m_e c^2} \sim \frac{6.582\,119\,569 \times 10^{-16}\,\mathrm{eV\,s}}{0.5\,\mathrm{MeV}} \sim 10^{-21}\,\mathrm{s}. \tag{3.1.1}$$

Since the pair e^-e^+ exists only for a short time and its existence is probabilistic, it is said virtual. Although the pair is virtual, it influences the strength of the interaction between the pair and the original electron. Therefore, it contributes to the initial electron's total charge which is measured. Indeed, the charge is a parameter that describes the strength of interaction between two electrons. For a general case, the world "charge" must be replaced with "coupling constant" and the reasoning is analogous. The computation of the corrections to the initial charge involves an integration over the Fourier modes of the electron. The straightforward calculation results in divergent corrections and the origin of the divergence lies in the point-like structure of the electron/positron. To avoid this problem, one has to regularize the integration using different techniques; some of these are:

1. Moment-cutoff regularization. This technique comes natural since it simply puts a cutoff Λ in the Fourier space of the electron. However, this procedure does not conserve the translational invariance of the system, which is an important gauge invariance that ensures the charge conservation (for this reason it is not used in QED).

2. Dimensional regularization. It consists in solving the problem not in the right number of dimensions D, but rather in $D \pm \epsilon$ dimensions, with $\epsilon \ll 1$. From the computational point of view this technique is quite simple. It was developed by Giambiagi and Bollini [11] and independently by Veltman and T'Hooft [12]. The latter were awarded by the 1999 Nobel for Physics.

3. Real-space regularization. To regularize the problem it is given to the electron a

finite radius r_0, which eliminates its point-like structure and thus the origin of the divergences. This technique is equivalent to the moment-cutoff regularization.

4. Pauli-Villars method. It consists in adding massive particles to the system with coupling constants that are constrained by physical requirement. The coupling constants must be chosen properly to eliminate the divergence and decouple the fictitious particles in the limit where their masses tend to infinity.

Only the first three techniques will be investigated. Regularization introduces a parameter, called regulator, that makes finite the correction computed: Λ for the cutoff scheme, ϵ for the dimensional one and r_0 for the real space technique. However, the correction is again infinite in the limit where the system comes back to the original situation, i.e. in the limit $\Lambda \to \infty$, $\epsilon \to 0$ and $r_0 \to 0$ respectively.

Once the regularization has been made, one has to introduce an arbitrary scale μ (in the QED example it is an energy or a mass scale[1]) that makes the correction finite as the regulator tends to what it has to. In general, one has to define a "renormalized" coupling constant, which is function of the regulator, the scale and the initial coupling constant, which is finite in the proper limit of the regulator. After the limit is performed, the renormalized coupling constant will be depending only on μ. Then the corrections are finite and known, although they depend on the arbitrary scale introduced. Going on the renormalization, one has to find the flow of the charge (or in general of the coupling constant) with the scale μ introduced before. Doing that, it is found a differential equation, called renormalization group equation (RGE from now on), that determines the evolution of the coupling constant/charge with μ. The final step is to fix experimentally the value of the coupling constant/charge for a precise scale. Then, the theory is complete and predictive. For the QED example, the electron charge is measured experimentally in the deep IR region where its energy is minimum. The energy scale dependence of the electron charge is not a formal results, but it has a confirmation in particle accelerators where the strength of the interaction between two electrically charged particles increases as the collision energy increases. The physical interpretation of this fact is the following. Since the vacuum is actually full of virtual pairs e^-e^+, the original electron can polarize them. As a consequence, the electron is surrounded by a cloud of virtual dipoles that screen the electron charge. Thus, one observes a decreasing charge. The “bare” charge is the one that would be measured at a distance of the order of the classical electron radius ($\sim 10^{-15}\,\mathrm{m}$). The physical charge, measured in the laboratory, is “dressed” by quantum corrections.

To sum up, the original parameters that define the bare theory become scale dependent after renormalization and have different values in the IR and in the UV region. Their

[1]Note that length and energy scale are related by the Heisenberg’s uncertainty principle, which implies that to small distances corresponds high energies (ultraviolet - UV - case) and vice versa, to large distances corresponds low energy (infrared - IR - case).

running is predicted by the RGE. Finally, a theory is said renormalizable if the number of parameters that have to be re-defined in this way is finite. QED is an example of renormalizable theory.

3.1.1 A toy model for renormalization

In this section, renormalization is studied more mathematically through a "toy model", a theory that has only one free parameter g_0 in terms of the which a physical quantity, represented by a function $F(x)$, is expanded perturbatively. In QED, g_0 may be the electron charge and F can represent, for example, the cross section of a scattering process. The coupling constant g_0 is defined by the Hamiltonian of the system. It is supposed that F can expanded in powers of g_0:

$$F(x) = g_0 F_0 + g_0^2 F_1(x) + g_0^3 F_2(x) + \dots . \tag{3.1.2}$$

Up to a suitable redefinition of F, the previous form is general. For studying renormalization, the perturbative expansion of F may be not well-defined and the elements of the series F_i $(i \neq 0)$ are divergent. An example is

$$F_1(x) = \alpha \int_0^\infty \frac{dt}{t+x} = \ln(t+x)|_0^\infty . \tag{3.1.3}$$

Since the theory has only one free parameter, it is sufficient a measurement at a point $x = \mu$ to fix the value of g_0 in order to reproduce the value of $F(\mu)$. Here, μ is the arbitrary scale cited previously. In general, if the theory has N parameters, N measurements are required to fix the coupling constants. Usually, one can parametrize the theory with the free parameter g_0 or directly with the measured physical quantity $F(\mu)$. In the case presented, since the expansion of F is ill-defined, the relationship between g_0 and $F(\mu)$ is also singular. Then, it seems necessary to re-parametrize F in terms of $F(\mu)$ and not in terms of g_0. The renormalizability hypothesis is that the re-parametrization of a theory in terms of a physical quantity $F(\mu)$ instead of g_0 is sufficient to make the expansions of F well-defined [1]. This hypothesis is equivalent to say that the divergence does not come from the expansion, i.e. from the $F_i(x)$, but from the choice of the parametrization. Then, from now on, it is assumed that $F(x)$ is known at a certain point μ and that

$$F(\mu) = g_R. \tag{3.1.4}$$

g_R is called renormalized coupling constant and the previous equation is known as a "renormalization prescription". The renormalization program consists in re-parametrizing the theory in function of g_R in such way that the expansion of F is well-defined and that it reproduces the renormalization prescription (3.1.4). Before proceeding through

renormalization, one has to make the F's expansion well-defined by introducing some parameters that makes the $F_i(x)$ finite. This process is known as "regularization" and the parameters are said "regulators". Then, in the moment-cutoff scheme $F_1(x)$ is regularized in the following way:

$$F_{1,\Lambda}(x) = \alpha \int_0^\Lambda \frac{dt}{t+x} = \alpha \ln\left(\frac{\Lambda + x}{x}\right), \tag{3.1.5}$$

while in the dimensional regularization it has the form

$$F_{1,\epsilon,\mu}(x) = \alpha\mu^\epsilon \int \frac{d^{1-\epsilon}t}{t+x}. \tag{3.1.6}$$

α is necessary in order to keep the right dimensions of the problem. It is evident that the cutoff scheme introduces a regulator Λ, which later will tend to ∞, while the dimensional regularization introduces two regulators: ϵ, which is dimensionless and it will tend to 0, and the dimensional parameter μ, introduced in order to keep the right dimensions of the problem. Usually, at the end of the dimensional scheme the parameter μ plays the same role of the renormalization point $x = \mu$ and this is why they are represented by the same letter. From now on the regulator will be indicated generally as Λ and it will tend to ∞ independently from the regularization method used, even the real space regularization which will be examined in section 3.3.3. This assumption will not lose out on generalization.

Once the regularization scheme has been done, the perturbative expansion is written as:

$$F_\Lambda(x) \equiv F_\Lambda(x, g_0, \Lambda) = g_0 + g_0^2 F_{1,\Lambda}(x) + g_0^3 F_{2,\Lambda}(x) + \dots. \tag{3.1.7}$$

Now, it is possible to use the renormalization prescription in order to obtain a well-defined expansion of F_Λ in terms of the physical quantity g_R. Thus, renormalization consists in rewriting the regularized expansion in terms of g_R and μ,

$$F_\Lambda(x, g_0, \Lambda) \to F_\Lambda(x, g_R, \mu), \tag{3.1.8}$$

and then taking the limit $\Lambda \to \infty$ at fixed g_R and μ. By hypothesis, $F(x)$ is the result of that limit:

$$F(x) \equiv F(x, g_R, \mu) = \lim_{\Lambda\to\infty} F_\Lambda(x, g_R, \mu). \tag{3.1.9}$$

The divergence is still somewhere, usually in the relationship between g_0 and g_R (for fixed g_R, g_0 diverges as $\Lambda \to \infty$). This divergence is not significant since g_0 is not supposed to be a physical quantity. Expanding g_0 in power of g_R

$$g_0 = g_R + \delta_2 g + \delta_3 g + \dots, \tag{3.1.10}$$

where $\delta_n g \sim O(g_R^n)$, the expansion at order 2 of $F_\Lambda(x)$ (equation (3.1.7)) becomes

$$F_\Lambda(x) = g_R + \delta_2 g + g_R^2 F_{1,\Lambda}(x) + O(g_R^3). \tag{3.1.11}$$

The last equation is based on the fact that $g_0^2 = g_R^2 + O(g_R^3)$. Imposing the renormalization prescription (3.1.4) at this order, it is found:

$$\delta_2 g = -g_R^2 F_{1,\Lambda}(\mu). \tag{3.1.12}$$

By substituting this value in (3.1.11), one has:

$$F_\Lambda(x) = g_R + g_R^2 (F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)) + O(g_R^3). \tag{3.1.13}$$

The divergence is absorbed only if the divergent part of $F_{1,\Lambda}(x)$ is exactly cancelled by $F_{1,\Lambda}(\mu)$ in the limit $\Lambda \to \infty$, that is only if $F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)$ is regular in x and μ for $\Lambda \to \infty$. This means that the divergent part of $F_{1,\Lambda}(\mu)$, at this order, must be x-independent, i.e. constant. It is convenient to split $F_{1,\Lambda}(x)$ in a singular and in a regular part

$$F_{1,\Lambda}(x) = F^s_{1,\Lambda}(x) + F^r_{1,\Lambda}(x). \tag{3.1.14}$$

The singular part is defined up to a regular function, $r(x)$, that can be chosen in order to have

$$\lim_{\Lambda\to\infty} F^s_{1,\Lambda}(x) - F^s_{1,\Lambda}(\mu) = 0 \tag{3.1.15}$$

as a regularity condition. Re-defining again the singular part with an appropriate regular function, one can have $F^s_{1,\Lambda}$ Λ-dependent only:

$$F^s_{1,\Lambda} = f(\Lambda). \tag{3.1.16}$$

Dimensionless coupling constant. Usually, the coupling constant is dimensionless (QED is an instance of this case). From the dimensional analysis it is clear that $F(x)$ must depend only by the ratio x/Λ. Since the singular part of $F_{1,\Lambda}$ can be chosen as Λ-dependent only and it is defined up to a regular function $r(x)$, it has a logarithmic form

$$F^s_{1,\Lambda} = f(\Lambda) + r(x) = \alpha \ln\left(\frac{\Lambda}{x}\right), \tag{3.1.17}$$

since the logarithm is the only function to obey such a property. Thus, at the first perturbation order the divergence is logarithmic. By definition, $F^r_{1,\Lambda}$ tends to a number that depends only on g_0 for $\Lambda \to \infty$. With a suitable choice of $r(x)$, one can assume

$F^r_{1,\Lambda} \overset{\Lambda\to\infty}{\to} 0$. By doing this, expansion (3.1.7) becomes

$$F_\Lambda(x) = g_0 + g_0^2 F_{1,\Lambda}(x) + O(g_0^2) = g_0 + g_0^2(F^s_{1,\Lambda}(x) + F^r_{1,\Lambda}(x)) + O(g_0^2) = \\ = g_0 + g_0^2 F^s_{1,\Lambda}(x) + O(g_0^2) = g_0 + \alpha g_0^2 \ln\left(\frac{\Lambda}{x}\right) + O(g_0^2). \quad (3.1.18)$$

By applying the renormalization prescription (3.1.4)

$$F_\Lambda(\mu) = g_R = g_0 + \alpha g_0^2 \ln\left(\frac{\Lambda}{\mu}\right) + O(g_0^2) \quad (3.1.19)$$

and remembering that $g_0^2 = g_R^2 + O(g_R^2)$, one is able to formally invert the previous equation and find an expansion for g_0

$$g_0 = g_R - \alpha g_R^2 \ln\left(\frac{\Lambda}{\mu}\right) + O(g_R^2), \quad (3.1.20)$$

Replacing this expansion in (3.1.18), a final expansion for $F_\Lambda(x)$ is obtained

$$F_\Lambda(x) = g_R + \alpha g_R^2 \ln\left(\frac{\Lambda}{x}\right) - \alpha g_R^2 \ln\left(\frac{\Lambda}{\mu}\right) + O(g_R^2) = g_R + \alpha g_R^2 \ln\left(\frac{\mu}{x}\right) + O(g_R^2). \quad (3.1.21)$$

The expansion for $F_\Lambda(x)$ shows unequivocally that the renormalization process does not change the functional form of the expansion (see equation (3.1.19)), but rather it consists in replacing (g_0, Λ) with (g_R, μ).

Running coupling. Now, in order to have a predictive theory, one has to find the "running" of the renormalized coupling constant g_R with the arbitrary scale μ. In the renormalization prescription (3.1.4), a fixed point μ was chosen, but another fixed point μ' can be chosen, so that another renormalized coupling constant g'_R is defined. Because there is only one free parameter, the various renormalized coupling constant $g_R = g_R(\mu)$ and $g'_R = g_R(\mu')$ should be related in a such way to satisfy $F(x, \mu, g_R) = F(x, \mu', g'_R)$. The invariance of the physical theory due to the chose of renormalization prescription means that there is an equivalence class of parametrization of the same theory and that changes of the parametrization should follow a group law. This law comes naturally at the first order for a dimensionless coupling constant. Indeed, from (3.1.19) it follows

$$g_R = g_0 + \alpha g_0^2 \ln\left(\frac{\Lambda}{\mu}\right) + O(g_0^2), \quad (3.1.22)$$

$$g'_R = g_0 + \alpha g_0^2 \ln\left(\frac{\Lambda}{\mu'}\right) + O(g_0^2). \quad (3.1.23)$$

Subtracting side by side the two previous equations and remembering that $g_0^2 = g_R^2 + O(g_R^2)$, the equation that determines the evolution of g_R emerges:

$$g'_R = g_R + \alpha g_R^2 \ln\left(\frac{\mu}{\mu'}\right). \tag{3.1.24}$$

In general, the running of the renormalized coupling constant is determined by the β-function [2]:

$$\beta(g_R) = \frac{\partial g_R}{\partial \ln(\mu)} = \mu \frac{\partial g_R}{\partial \mu}. \tag{3.1.25}$$

The computation of $\beta(g_R)$ gives the differential variation of the renormalized coupling constant. For the dimensionless coupling constant the β function is

$$\beta(g_R) = -\alpha g_R^2, \tag{3.1.26}$$

which integrated gives

$$g'_R = \frac{g_R}{1 - \alpha g_R \ln(\mu/\mu')}. \tag{3.1.27}$$

If the solution is expanded at order g_R^2 [3], equation (3.1.24) is recovered

$$g'_R = g_R(1 + \alpha g_R^2 \ln\big(\mu/\mu'\big) + \dots) = g_R + \alpha g_R^2 \ln\big(\mu/\mu'\big). \tag{3.1.28}$$

However the β-function is more general and it is exact at any order expansion [1]. Moreover, the β-function is independent from the regulator Λ as it depends only on g_R, μ' and μ.

Another important thing to underline is that the physical observable $F(x)$ must be independent from the the arbitrary scale μ, that is

$$\mu \frac{\partial F}{\partial \mu} = 0. \tag{3.1.29}$$

This equation means that the observable measurement that produce the renormalization prescription can be made at any scale.

3.2 Classical physics example

In this section it is presented a well known problem ([13], [14]) of classical physics that can be treated using regularization and renormalization.

It is known from classical electrodynamics that the electric field generated by an

[2] It must not be confused with the β-function of mathematicians (see appendix B.3)

[3] the denominator is seen as a geometric power series $(1-x)^{-1} = 1 + x + x^2 + \dots$ given that $|x| = |\alpha g_R \ln(\mu/\mu')| < 1$

uniform charged straight line along its perpendicular direction is:

$$\boldsymbol{E}(x) = \frac{\lambda}{2\pi\epsilon_0 x}\hat{\boldsymbol{x}}_{\mathbf{1}}, \tag{3.2.1}$$

where $\hat{\boldsymbol{x}}_{\mathbf{1}}$ is the unit vector along the x direction of a Cartesian coordinate system with the z-axis parallel to the wire (see figure 3.1). λ is the charge density.

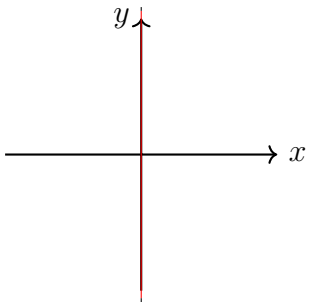

Fig. 3.1: Uniform charged straight wire

It is quite natural to associate the following differential potential:

$$dV = \frac{\lambda dy}{4\pi\epsilon_0\sqrt{x^2+y^2}}. \tag{3.2.2}$$

$V(x)$ is obtained by integrating dV from $-\infty$ to ∞

$$V(x) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dy}{\sqrt{x^2+y^2}} = \\ = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dt}{\sqrt{1+t^2}} = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{\sqrt{1+t^2}+t}{\sqrt{1+t^2}-t}\Bigg|_{-\infty}^{\infty}. \tag{3.2.3}$$

$V(x)$ is logarithmically divergent as shown in appendix A.1. The origin of the divergence is the point-like structure of the wire, as well as the electron's charge divergence lies in the point-like structure of the particle.

Cutoff scheme. The cutoff scheme proceeds as follow. First of all, a cutoff parameter Λ is introduced in order to redefine the potential as

$$V_\Lambda(x) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\Lambda}^{\Lambda}\frac{dy}{\sqrt{x^2+y^2}} = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{\sqrt{\Lambda^2+x^2}+\Lambda}{\sqrt{\Lambda^2+x^2}-\Lambda}. \tag{3.2.4}$$

Obviously $V_\Lambda(x)$ is divergent in the limit of $\Lambda \to \infty$. However, the electric field is non-divergent in that limit:

$$\boldsymbol{E}(x) = -\lim_{\Lambda\to\infty}\nabla V_\Lambda(x) = -\lim_{\Lambda\to\infty}\frac{dV_\Lambda(x)}{dx}\hat{\boldsymbol{x}}_{\mathbf{1}} = \frac{\lambda}{2\pi\epsilon_0 x}\hat{\boldsymbol{x}}_{\mathbf{1}}. \tag{3.2.5}$$

The derivative is performed in appendix A.2. The expression founded is the same of (3.2.1). In order to define a non-divergent potential, it has been necessary to introduce a parameter Λ that disappears in the physical relevant quantity (the electric field).

Now, it is time to renormalize the potential in order to make it Λ-independent and finite in the limit $\Lambda \to \infty$. In classical physics the electrostatic potential is defined up to a constant. So, to renormalize the problem, it is sufficient to subtract $V_\Lambda(x)$ by its value

at some reference point μ and take the limit $\Lambda \to \infty$:

$$V(x) = \lim_{\Lambda\to\infty} V_\Lambda(x) - V_\Lambda(\mu) = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{\mu}{x}. \tag{3.2.6}$$

The last limit is performed in appendix A.3

Dimensional scheme. There is another way to regularize and renormalize the potential, the dimensional scheme. In this scheme, the integral is computed in a D-dimension space, where D is the regularizing parameter. In order to keep the dimensions of the potential fixed, the integral has to be divided by an $(D-1)$-dependent length scale (μ).

$$V_D(x) = \frac{\lambda}{4\pi\epsilon_0} \mu^{1-D} \int d^D y \frac{1}{\sqrt{x^2+y^2}}, \tag{3.2.7}$$

where $y^2 = y_1^2 + y_2^2 + \ldots + y_D^2$. Using spherical coordinates

$$d^D y = d\Omega_D \rho^{D-1} d\rho, \tag{3.2.8}$$

where ρ is the radial coordinate running from 0 to ∞ and $d\Omega_D$ is the solid angle element, equation (3.2.7) turns into:

$$V_D(x) = \frac{\lambda}{4\pi\epsilon_0} \mu^{1-D} \int d\Omega_D \int_0^\infty d\rho \frac{\rho^{D-1}}{\sqrt{\rho^2+x^2}} = \frac{\lambda}{4\pi\epsilon_0} \mu^{1-D} \Omega_l x^{D-1} \int_0^\infty d\chi \frac{\chi^{D-1}}{\sqrt{1+\chi^2}}. \tag{3.2.9}$$

The variable $\chi = \rho/x$ was introduced in the last equality.

Ω_D is calculated in appendix A.4 and its value (A.4.4) is here reported

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{3.2.10}$$

while the integral is evaluated in [15] and it is

$$\int_0^\infty d\chi \frac{\chi^{D-1}}{\sqrt{1+\chi^2}} = \frac{\Gamma(D/2)\Gamma((1-D)/2)}{2\sqrt{\pi}}, \tag{3.2.11}$$

where $\Gamma(z)$ is the Γ-function (see appendix B.2).

By inserting (3.2.10) and (3.2.11) into (3.2.9), $V_D(x)$ becomes

$$V_D(x) = \frac{\lambda}{4\pi\epsilon_0} \pi^{(D-1)/2} \left(\frac{\mu}{x}\right)^{1-D} \Gamma\left(\frac{1-D}{2}\right). \tag{3.2.12}$$

The electric potential is now finite, except when D is an odd integer. So, expression (3.2.12) is the regularized form of $V(y)$ obtained with the dimensional regularization, in analogy with equation (3.2.4) that was the cutoff regularized potential. In the dimensional

regularization scheme there is a dimensionless regulator D and a dimensional regulator μ, while in the cutoff regularization Λ is the only regulator.

Now, it is time to renormalize the potential. To do that, $V_D(x)$ has to be expanded around the correct dimension of the problem $D = 1 + \epsilon$. Then, for $\epsilon \to 0$ one has:

$$\begin{cases} \left(\frac{\mu}{x}\right)^{1-D} = \exp\left(-\epsilon \ln \frac{\mu}{x}\right) = 1 - \epsilon \ln \frac{\mu}{x} + O(\epsilon^2), \\ \pi^{(D-1)/2} = \exp\left(\frac{\epsilon}{2} \ln \pi\right) = 1 + \frac{\epsilon}{2} \ln \pi + O(\epsilon^2), \\ \Gamma\left(\frac{1-D}{2}\right) = -\frac{2}{\epsilon}\Gamma\left(1 - \frac{\epsilon}{2}\right) = -\frac{2}{\epsilon}\left(1 + \frac{\gamma}{2}\epsilon + O(\epsilon^2)\right), \end{cases} \tag{3.2.13}$$

where γ is the Euler-Mascheroni constant and the last expansion is the same of (B.2.15).

By substituting (3.2.13) in (3.2.12) one has:

$$V_\epsilon(x) = \frac{\lambda}{2\pi\epsilon_0}\left(\ln \frac{\mu}{x} - \frac{1}{\epsilon} - \frac{\gamma}{2} - \frac{\ln \pi}{2} + O(\epsilon)\right), \tag{3.2.14}$$

where terms of order ϵ or higher were ignored because they vanish as ϵ vanishes.

To renormalize it is necessary to add a constant to the potential (3.2.14), similarly to the renormalization of the cutoff regularized potential. It is important to note that the divergent part of equation (3.2.14) is a constant, that is x-independent as expected by the general theory discussed in section 3.1.1. Since electrostatic potential are defined up to a constant, that constant is not physically relevant and it can be removed. The constant that has to be add might assume different forms. It could just be equal to $(\lambda/(2\pi\epsilon_0))\epsilon^{-1}$ and in that case it is a Minimal Subtraction scheme (MS). If the terms $\gamma/2$ and $\ln(\pi)/2$ are also included in the constant, it is the case of a Modified Minimal Subtraction Scheme ($\overline{\text{MS}}$). So, in the $\overline{\text{MS}}$-case the constant that has to be added is:

$$C_\epsilon = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{\epsilon} + \frac{\gamma}{2} + \frac{\ln \pi}{2}\right), \tag{3.2.15}$$

In this way, the potential is renormalized exactly as in equation (3.2.6) and it has the form:

$$V^{(\overline{\text{MS}})}(x) = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{\mu}{x}. \tag{3.2.16}$$

After the renormalization, the regulator ϵ is removed, but not the regulator μ, that plays the same role of the renormalization point μ of equation (3.2.6). μ is the point where $V^{(\overline{\text{MS}})}(\mu) = 0$. Whenever a constant is added to a potential the boundary conditions change. For example, in equation (3.2.12) V_D is equal to zero when $x \to \infty$ (if $D < 1$) or when $x \to 0$ (if $D > 1$), while in the renormalized expression the potential vanishes when $x = \mu$.

RGE. The physical observables, e.g. the electric field $\boldsymbol{E}$, are independent from the arbitrary length scale μ. This is a consequence of the renormarlization group equation (RGE):

$$\mu \frac{dF}{d\mu} = 0, \tag{3.2.17}$$

where F is any physical observable. Thus, in the electrostatic example, the RGE is

$$\mu \frac{d\boldsymbol{E}}{d\mu} = 0. \tag{3.2.18}$$

Indeed

$$\boldsymbol{E} = -\frac{dV^{(\overline{\text{MS}})}(x)}{dx}\boldsymbol{x_1} = -\frac{d}{dx}\frac{\lambda}{2\pi\epsilon_0}\ln\frac{\mu}{x}\boldsymbol{x_1} = \frac{\lambda}{2\pi\epsilon_0}\frac{x}{\mu}\frac{\mu}{x^2}\boldsymbol{x_1} = \frac{\lambda}{2\pi\epsilon_0 x}\boldsymbol{x_1} \tag{3.2.19}$$

or

$$\mu \frac{d\boldsymbol{E}}{d\mu} = 0. \tag{3.2.20}$$

This equation means that physical observables are not dependent on the arbitrary scale length introduced. In the case presented there are no dimensionless coupling constants to renormalize, so there is not any renormalization group equation that determines the differential change of the coupling constant, i.e. the running of the coupling constant.

Translation invariance It is interesting to notice that the cutoff regularization brakes the translational symmetry, while the dimensional regularization does not.

The original potential (3.2.3) is invariant for a translation transformation $y \to k + y$:

$$\tilde{V}(x) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dy}{\sqrt{x^2+(k+y)^2}} = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dy'}{\sqrt{x^2+y'^2}} = V(x),$$

where it was used the variable $y' = k + y$. So, the original potential is translational invariant because it is extended from $-\infty$ to ∞. This is not true for the cutoff regularized potential:

$$\tilde{V}_\Lambda(x) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\Lambda}^{\Lambda}\frac{dy}{\sqrt{(y+k)^2+x^2}} = \frac{\lambda}{4\pi\epsilon_0}\int_{-\Lambda+k}^{\Lambda+k}\frac{dy'}{\sqrt{y'^2+x^2}} =$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\frac{\sqrt{(\Lambda+k)^2+x^2}+(\Lambda+k)}{\sqrt{(\Lambda+k)^2+x^2}-(\Lambda+k)} \neq V_\Lambda(x).$$

From equation (3.2.7) it is clear that a transformation $y \to y + k$ does not change the

dimensional regularized potential.

$$\tilde{V_D}(x) = \frac{\lambda}{4\pi\epsilon_0}\Lambda^{1-D}\int d^D y \frac{1}{\sqrt{x^2+(y+k)^2}} = \frac{\lambda}{4\pi\epsilon_0}\Lambda^{1-D}\int d^D y' \frac{1}{\sqrt{x^2+y'^2}} = V_D(x).$$

Then, dimensional regularization does not brake the translation invariance of the system.

3.3 δ-interactions

In this section, the δ-function potentials in one and two dimensions are studied. In particular, the aim is to calculate the bound state energy of a single particle subjected to that potential. The problem in one dimension is treated with a simply solution of the Schrödinger equation and it is not problematic, while the two-dimensional case is treated with two different procedures. The first method is based on finding the solution of the Schrödinger equation for the problem in D-dimensions and then extract the results for $D = 1$ (in order to compare the result already obtained) and $D = 2$. The second method uses a two-dimensional Fourier transform of the Schrödinger equation. This latter case is more problematic since it has a divergent bound state energy.

One-dimensional case The physical system that is studied is a single particle subjected to the following one dimensional potential

$$V(x) = \frac{\hbar^2}{m}\lambda\delta(x), \tag{3.3.1}$$

where λ is a "coupling constant" with the dimensions of $[L]^{-1}$ and $\delta(x)$ the Dirac delta (see appendix B.1), that has the same dimensions of λ. The time-independent Schrödinger equation in one dimension is:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{\hbar^2}{m}\lambda\delta(x)\psi(x) = E\psi(x). \tag{3.3.2}$$

Multiplying both sides by $2m/\hbar^2$, it turns into a rescaled form:

$$-\frac{d^2}{dx^2}\psi(x) + 2\lambda\delta(x)\psi(x) = B^2\psi(x), \tag{3.3.3}$$

where

$$B^2 = \frac{2mE}{\hbar^2}. \tag{3.3.4}$$

B has the same dimension of λ. The solution of (3.3.3) is not defined in $x = 0$, otherwise it is

$$\psi(x) = \begin{cases} A_1 e^{iBx} + A_2 e^{-iBx}, & \text{if } x < 0; \\ C_1 e^{iBx} + C_2 e^{-iBx}, & \text{if } x > 0. \end{cases} \tag{3.3.5}$$

In order to find the value of the constants A_1,A_2,C_1,C_2 and the equation for the wave number (and therefore for the energy eigenvalues) one has to impose some boundary conditions. In particular, $\psi(x)$ must be continuous in $x = 0$. Then

$$\psi(0^+) = \psi(0^-) \quad \Rightarrow \quad A_1 + A_2 = C_1 + C_2. \tag{3.3.6}$$

The first derivative of the energy eigenfunction has also to be continuous in $x = 0$, but, when the potential is a one-dimensional Dirac delta, this condition has the form [16]:

$$\frac{d}{dx}\psi(0^+) - \frac{d}{dx}\psi(0^-) = 2\lambda\psi(0^+) \quad \Rightarrow \quad iB(A_1 - A_2 - C_1 + C_2) = 2\lambda(A_1 + A_2). \tag{3.3.7}$$

Finally, $\psi(x)$ must be bounded when $|x| \to \infty$. If the initial potential is attractive, there is a bound state, which is characterized by a negative energy E. Then, from equation (3.3.4), one notes that $B^2 < 0$ or rather that B is imaginary. So, the eigenfunctions (3.3.5) are bounded for $|x| \to \infty$ precisely when $A_2 = C_1 = 0$. From the first boundary condition (3.3.6) follows that $A_1 = C_2$. The value of A_1 is determined imposing the normalization of the eigenfunction. Due to the second condition (3.3.7), one obtains the relation for the wave number

$$B = -i\lambda. \tag{3.3.8}$$

In order to make B positive, λ must be negative. This means that to create a bound state the potential must be attractive as expected. Finally, from the relation (3.3.4), one finds the energy of the only bound state:

$$E = -\frac{\hbar^2\lambda^2}{2m}. \tag{3.3.9}$$

Two-dimensional case In this case, the bound state energy can be find out by solving directly the Schrödinger equation for the δ-function potential in D-dimensions:

$$\left[-\frac{\hbar}{2m}\nabla_D^2 + \lambda\frac{\hbar^2}{m}\delta^{(D)}(\boldsymbol{x})\right]\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x}). \tag{3.3.10}$$

Here, λ has the dimensions of $[L]^{D-2}$ since $[\delta(\boldsymbol{x})^{(D)}] = [L]^{-D}$. The only case where λ is dimensionless is when $D = 2$. The potential $(\hbar^2/m)\lambda\delta^{(D)}(\boldsymbol{x})$ is central, then it is convenient to study it in hyper-spherical coordinates (3.2.8). Furthermore, it does not depend on the angular part of the coordinates. As derived in appendix C.1 and in the

reference [17], the radial part is not defined in $r = 0$, while for $r \neq 0$ it reduces to

$$\left[\frac{d^2}{dr^2} - B^2 - \frac{(l+\nu)^2 + 1/4}{r^2}\right] R_{E,l}(r) r^{\nu+1/2} = 0, \tag{3.3.11}$$

where

$$\begin{cases} B^2 = -\dfrac{2mE}{\hbar^2}, \\ \nu = D/2 - 1. \end{cases} \tag{3.3.12}$$

The term $((l+\nu)^2+1/4)/r^2$ represents a centrifugal potential and its magnitude is defined by the quantum number l. Upon doing the substitution

$$\begin{cases} z = Br, \\ R_{E,l}(z/B)(z/B)^{\nu+1/2} = z \cdot w(z), \end{cases} \tag{3.3.13}$$

equation (3.3.11) turns into

$$\frac{d^2w}{dz^2} + \frac{2}{z}\frac{dw}{dz} - \left(1 + \frac{\alpha(\alpha+1)}{z^2}\right) w = 0, \tag{3.3.14}$$

where $\alpha = l + \nu - 1/2$. The previous equation is an instance of the modified spherical Bessel's equation. Its solution is a linear combination of the modified spherical Bessel's function B.4:

$$w(z) = A_l'' i_\alpha(z) + C_l'' k_\alpha(z). \tag{3.3.15}$$

Using the relation between the modified spherical Bessel's functions and the modified Bessel's function (B.4.18) and replacing the value of w,z and α it is found that the solution of the radial part is a linear combination of the modified Bessel's function of the first and the second kind as follow:

$$\frac{R_{l,E}(r) r^\nu \sqrt{r}}{Br} = \frac{1}{\sqrt{r}}(A_l' I_{l+\nu}(Br) + C_l' K_{l+\nu}(Br)). \tag{3.3.16}$$

Since $R_{l,E}(r)$ must be bounded at the infinity, one has to reject the Bessel's function of the first kind, which is divergent for $r \to \infty$.

$$R_{l,E}(r) = C_l r^{-\nu} K_{l+\nu}(Br). \tag{3.3.17}$$

The behaviour of the Bessel's function for a small argument is [15]:

$$K_{l+\nu}(z) \overset{(r\to 0)}{\sim} \frac{1}{2}\left[\Gamma(l+\nu)\left(\frac{z}{2}\right)^{-(l+\nu)} + \Gamma(-(l+\nu))\left(\frac{z}{2}\right)^{l+\nu}\right]. \tag{3.3.18}$$

It is also needed to impose the bound of the eigenfunction for $r \to 0$. For $l \neq 0$, equation (3.3.17) shows a singular term r^{-l}, according to equation (3.3.18). Therefore, this boundary can be satisfied only for $l = 0$, which physically means that the $\delta^{(D)}$-potential admits a bound state only if there are not centrifugal terms. Comparing the boundary condition for the origin obtained in appendix C.2

$$R_{0,E}(r) \overset{(r\to 0)}{\sim} R_{0,E}(0)\left[1 + \frac{2\lambda\Gamma(\nu)}{4\pi^{\nu+1}} r^{-2\nu}\right]. \tag{3.3.19}$$

with relation (3.3.18) and (3.3.17) in the case $l = 0$, two equations emerge:

$$C_0 = 2\frac{R_0(0)}{\Gamma(-\nu)}\left(\frac{B}{2}\right)^{-\nu}, \tag{3.3.20}$$

$$B^{-2\nu} = \frac{\lambda\Gamma(-\nu)}{2^{2\nu+1}\pi^{\nu+1}}. \tag{3.3.21}$$

The second equation is the eigenvalue equation. It has the right dimensions since both sides have the dimensions of $[L]^{D-2}$. Setting $D = 1$, that is equivalent to setting $\nu = -1/2$, one has again the bound energy (3.3.9). For $D = 2$, or $\nu = 0$, the eigenvalue equation is not well defined.

There is a second method that points out the problem of the two-dimensional δ-interaction. It is the method used in [18], [19],[20] and it consists in a Fourier transform of the rescaled Schrödinger equation:

$$(-\nabla^2 + 2\lambda\delta^{(2)}(\boldsymbol{x}))\psi(\boldsymbol{x}) = -B^2\psi(\boldsymbol{x}), \tag{3.3.22}$$

where $B^2 = -2mE/\hbar^2$. The Fourier transform, performed in detail in appendix A.5, leads to

$$((y/\hbar)^2 + B^2)\varphi(\boldsymbol{y}) = \frac{2\lambda}{(2\pi\hbar)^2}\psi(\boldsymbol{0}), \tag{3.3.23}$$

where $\varphi(\boldsymbol{y})$ is the Fourier transform in two dimensions of $\psi(\boldsymbol{x})$. Then

$$\psi(\boldsymbol{x}) = \int \frac{d^2y}{(2\pi\hbar)^2} e^{(i/\hbar)\boldsymbol{y}\cdot\boldsymbol{x}}\varphi(\boldsymbol{y}). \tag{3.3.24}$$

Integrating both sides of (3.3.23) over $\boldsymbol{y}$, remembering that [18]

$$\int d^2y\varphi(\boldsymbol{y}) = \frac{1}{(2\pi\hbar)^2}, \tag{3.3.25}$$

substituting the value of $\psi(\boldsymbol{0}) = 1/(2\pi\hbar)^2$ and using the variable $t = y/\hbar$, that has the same dimensions of B, one obtains the eigenvalue equation

$$\frac{2\pi^2}{\lambda} = \int d^2t \frac{1}{B^2 + t^2}, \tag{3.3.26}$$

that is divergent as shown below:

$$\frac{2\pi^2}{\lambda} = \int d^2t \frac{1}{t^2+B^2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr \frac{r}{r^2+B^2} = 2\pi \left.\frac{\ln\left(B^2+r^2\right)}{2}\right|_0^{\infty} = \infty. \tag{3.3.27}$$

3.3.1 Path integral approach

The problem of $\delta^{(2)}$-interaction also arises using the theory of the propagators [21], [22], [23]. As for the Schrödinger equation, the problem is studied in D dimensions case. The propagator of the system admits the usually representation as a path integral (2.3.11)

$$K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \int_{\boldsymbol{x'}=\boldsymbol{x}(t')}^{\boldsymbol{x''}=\boldsymbol{x}(t'')} \mathcal{D}[\boldsymbol{x}(t)] e^{(i/\hbar)S[\boldsymbol{x}(t)](\boldsymbol{x''},t'';\boldsymbol{x'},t')} \tag{3.3.28}$$

and it is related to the Green's function by the relation (2.2.12)

$$\tilde{g}(\boldsymbol{x''}, t''; \boldsymbol{x'}, t') = \frac{1}{i\hbar}\theta(t''-t')K(\boldsymbol{x''}, t''; \boldsymbol{x'}, t'). \tag{3.3.29}$$

The Fourier transform of the Green's function, i.e. of the propagator, is the energy Green's function (2.1.21)

$$G(\boldsymbol{x}, \boldsymbol{x'}; E) = \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{\frac{i}{\hbar}E\tau} \tilde{g}(\boldsymbol{x}, \boldsymbol{x'}; \tau). \tag{3.3.30}$$

Thus,

$$\begin{aligned} G(\boldsymbol{x}, \boldsymbol{x'}; E) &= \frac{1}{i\hbar}\int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{\frac{i}{\hbar}E\tau}\theta(\tau)K(\boldsymbol{x''}, \boldsymbol{x'}, \tau) = \\ &= \frac{1}{i\hbar}\int_0^{\infty} \frac{d\tau}{2\pi} e^{\frac{i}{\hbar}E\tau} \int_{\boldsymbol{x'}=\boldsymbol{x}(t')}^{\boldsymbol{x''}=\boldsymbol{x}(t'')} \mathcal{D}[\boldsymbol{x}(t)] e^{(i/\hbar)S[\boldsymbol{x}(t)](\boldsymbol{x''},\boldsymbol{x'};\tau)}. \end{aligned} \tag{3.3.31}$$

The energy Green's function fulfills the spectral representation

$$G(\boldsymbol{x}, \boldsymbol{x'}; E) = \sum_l \frac{\psi_l(\boldsymbol{x})\psi_l^*(\boldsymbol{x'})}{E-E_l} + \int \frac{d\lambda}{K_\lambda} \frac{\psi_\lambda(\boldsymbol{x})\psi_\lambda^*(\boldsymbol{x'})}{E-E_\lambda}. \tag{3.3.32}$$

The $\delta^{(D)}$-potential is considered as a perturbation of the free particle's Hamiltonian $\hat{H}_0 = -\hbar^2/(2m)\nabla^2$. Then, from equation (2.1.25), where $\hat{H}'$ has to be replaced with $(\hbar^2/m)\lambda\delta^{(D)}(\boldsymbol{x})$, it follows that

$$\begin{aligned} G(\boldsymbol{x}, \boldsymbol{x'}; E) &= G_0(\boldsymbol{x}, \boldsymbol{x'}; E) + \int d^D k G_0(\boldsymbol{x}, \boldsymbol{k}; E) \frac{\hbar^2}{m} \lambda\delta(\boldsymbol{k}) G(\boldsymbol{k}, \boldsymbol{x'}; E) = \\ &= G_0(\boldsymbol{x}, \boldsymbol{x'}; E) + \frac{\hbar^2}{m}\lambda G_0(\boldsymbol{x}, 0; E) G(0, \boldsymbol{x'}; E). \end{aligned} \tag{3.3.33}$$

$G(0, \boldsymbol{x}'; E)$ can be obtained putting $\boldsymbol{x} = 0$ in the previous equation. It is then found

$$G(0, \boldsymbol{x}'; E) = \frac{G_0(0, \boldsymbol{x}'; E)}{1 - \frac{\hbar^2}{m}\lambda G_0(0, 0; E)}. \tag{3.3.34}$$

Replacing this value in (3.3.33), it gets:

$$G(\boldsymbol{x}, \boldsymbol{x}'; E) = G_0(\boldsymbol{x}, \boldsymbol{x}'; E)\frac{G_0(\boldsymbol{x}, 0; E)G_0(0, \boldsymbol{x}'; E)}{1 - \frac{\hbar^2}{m}\lambda G_0(0, 0; E)}. \tag{3.3.35}$$

From the general theory of the Green's function (see section 2.1.2), it is possible to derive that the energy eigenvalues of a quantum system coincide with the poles of the associated Green's function. Then, the last equation tells that the energy bound state of the system is

$$\frac{m}{\hbar^2}\frac{1}{\lambda} = G_0(0, 0; E). \tag{3.3.36}$$

Finally, $G_0(0, 0; E)$ is obtained by a Fourier transform of the equation defining $G(\boldsymbol{x}, \boldsymbol{x}'; E)$, that is $(E - \hat{H}_0)G_0(\boldsymbol{x}, \boldsymbol{x}'; E) = \delta(\boldsymbol{x} - \boldsymbol{x}')$, or equivalently $(B^2 + \nabla^2)G_0(\boldsymbol{x}, \boldsymbol{x}'; E) = (2m/\hbar^2)\delta(\boldsymbol{x} - \boldsymbol{x}')$ (see app A.5):

$$G_0(\boldsymbol{x}, \boldsymbol{x}'; E) = \frac{2m}{\hbar^2}\int \frac{d^D y}{(2\pi\hbar)^D}\frac{e^{(i/\hbar)\boldsymbol{y}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{B^2 + (y/\hbar)^2}, \tag{3.3.37}$$

where B^2, given by (3.3.12), is positive since E is the bound state energy. Equation (3.3.36) has the general form

$$\frac{m}{\hbar^2}\frac{1}{\lambda} = \frac{2m}{\hbar^2}\int \frac{d^D y}{(2\pi\hbar)^D}\frac{1}{B^2 + (y/\hbar)^2}. \tag{3.3.38}$$

The last integral can be performed analytically using the hyper-spherical coordinates (3.2.8) and the variable $t = y/\hbar$.

$$\int d^D t \frac{1}{B^2 + t^2} = \hbar^D \Omega_D B^{D-2}\int_0^\infty d\rho \frac{\rho^{D-1}}{1 + \rho^2} = \hbar^D \frac{\Omega_D}{2} B^{D-2}\int_0^\infty dt \frac{t^{D/2-1}}{1 + t} =$$
$$= \hbar^D \frac{\pi^{D/2}}{\Gamma(D/2)} B^{(D-2)}\beta(D/2, 1 - D/2), \tag{3.3.39}$$

where Ω_D is calculated in A.4, $\Gamma(z)$ is the Γ-function (see appendix B.2) and $\beta(x, y))$ is the β-function (see appendix B.3). Using the basic property of the β-function (B.3.2) and inserting the value of Ω_D (A.4.4), equation (3.3.39) turns into

$$\int d^D t \frac{1}{B^2 + t^2} = \hbar^D \pi^{D/2} B^{D-2}\Gamma(1 - D/2). \tag{3.3.40}$$

Whence the eigenvalue equation

$$\frac{\pi^{D/2}}{(2\pi)^D} 2\lambda\Gamma(1-D/2)B^{D-2} = 1, \tag{3.3.41}$$

that can be written in function of the parameter ν (3.3.12):

$$\frac{\Gamma(-\nu)\lambda}{2^{2\nu+1}\pi^{\nu+1}} B^{-2\nu} = 1. \tag{3.3.42}$$

Fortunately, the eigenvalue equation is the same of (3.3.21). Setting $D = 1$, one finds the same result obtained previously (3.3.9). In two dimensions, a problem that is absent in the one-dimensional case arises. Indeed, if $D = 2$ the eigenvalue equation (3.3.41) is not well defined because $\Gamma(z)$ diverges for $z \to 0$. The explicit computation for $D = 2$ shows a logarithmic divergence (see equation (3.3.27)).

3.3.2 Renormalization of the bound state energy

The origin of the bound state divergence for the two-dimensional δ-function lies in the zero-range nature of the potential, as well as the divergence of the electron's charge is caused by the point-like structure of the electron. In this section, the bound state energy of the δ-potential is renormalized using first the cutoff scheme and then the dimensional regularization.

Cutoff scheme. A regularized expression of equation (3.3.26) (or equivalently of equation (3.3.38) in the case $D = 2$) is found by introducing a cutoff Λ with the dimension of $[L]^{-1}$

$$\frac{2\pi^2}{\lambda} = \int d^2t \frac{1}{t^2 + B^2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr \frac{r}{r^2 + B^2} = \\ = 2\pi \left.\frac{\ln\left(B^2 + r^2\right)}{2}\right|_0^{\Lambda} = \pi \ln\left(1 + \frac{\Lambda^2}{B^2}\right). \tag{3.3.43}$$

This equation has a simple solution

$$B^2 = \frac{\Lambda^2}{e^{2\pi/\lambda} - 1}. \tag{3.3.44}$$

The result is finite, but in the limit $\Lambda \to \infty$ diverges as expected from the general theory.

In order to renormalize the moment-cutoff regularized expression (3.3.44), it is necessary to redefine the dimensionless coupling constant in function of the regulator Λ and an

arbitrary scale μ (with the dimensions of Λ, or equivalently B):

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} - \frac{1}{2\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right). \tag{3.3.45}$$

μ is introduced in order to have a dimensionless quantity inside the logarithm. In this way, the bound state energy turns into

$$B^2 = \lim_{\Lambda\to\infty}\left(1+\frac{B^2}{\Lambda^2}\right)\mu^2 e^{-2\pi/\lambda_R} = \mu^2 e^{-2\pi/\lambda_R} \tag{3.3.46}$$

and finally from equation (3.3.12)

$$E = -\frac{\hbar^2}{2m}\mu^2 e^{-2\pi/\lambda_R}. \tag{3.3.47}$$

Dimensional scheme. Another method to regularize the bound state energy of the two-dimensional δ-interaction consists in replacing the original D-dimensional space with a $(D-\epsilon)$-dimensional space, where ϵ acts as the regulator, and then taking the limit for $\epsilon \to 0$. In the case presented, where the space is two-dimensional, the regularizing substitution is

$$D = 2-\epsilon \to \nu = -\frac{\epsilon}{2}. \tag{3.3.48}$$

In doing this, the energy eigenvalue equation (3.3.41) turns into

$$\frac{\lambda\mu^\epsilon\Gamma(\epsilon/2)}{2^{1-\epsilon}\pi^{1-\epsilon/2}}B^{-\epsilon} = 1, \tag{3.3.49}$$

where μ^ϵ is a constant introduced to keep the right dimension of the problem. This expression is regularized for all $\epsilon > 0$, but it diverges for $\epsilon = 0$ (that corresponds to $D = 2$) as expected by the general theory.

The renormalization of the dimensional regularized energy eigenvalue expression (3.3.49) follows some steps. First, it is required to expand for $\epsilon \to 0$ the following "problematic" terms (they are said "problematic" because they disappears for $\epsilon \to 0$):

$$\begin{cases} \mu^\epsilon = 1 + \epsilon\ln(\mu) + O(\epsilon^2), \\ B^{-\epsilon} = 1 - \epsilon\ln(B) + O(\epsilon^2), \\ \Gamma(\epsilon/2) = \dfrac{2}{\epsilon} - \gamma + O(\epsilon^2). \end{cases} \tag{3.3.50}$$

Second, the eigenvalue equation (3.3.49) is re-written ignoring terms of order ϵ or higher

$$\frac{\lambda\left(\frac{2}{\epsilon} + 2\ln\left(\frac{\mu}{B}\right) - \gamma\right)}{2^{1-\epsilon}\pi^{1-\epsilon/2}} = 1. \tag{3.3.51}$$

Finally, to renormalize the problem it is sufficient to define a new "renromalized" coupling constant that absorbs the divergent part, i.e $1/\lambda_R = \alpha/\lambda - \alpha/\epsilon$, where $\alpha = 2^{1-\epsilon}\pi^{1-\epsilon/2}$. This is the case of the Minimal Subtraction scheme (MS). If also the constant γ, that appears with the dimensional regularization procedure, is absorbed, it is the case of the Modified Minimal Subtraction Scheme ($\overline{\text{MS}}$). In the $\overline{\text{MS}}$-case the renormalized coupling constant is

$$\frac{2^{1-\epsilon}\pi^{1-\epsilon/2}}{\lambda_R} = \frac{2^{1-\epsilon}\pi^{1-\epsilon/2}}{\lambda} + \gamma - \frac{2}{\epsilon} \tag{3.3.52}$$

and the renormalized bound energy is obtained taking the limit for $\epsilon \to 0$:

$$\lim_{\epsilon\to 0} \frac{\lambda_R}{2^{1-\epsilon}\pi^{1-\epsilon/2}} \ln\left(\frac{\mu^2}{B^{2(\overline{\text{MS}})}}\right) = \frac{\lambda_R}{2\pi} \ln\left(\frac{\mu^2}{B^{2(\overline{\text{MS}})}}\right) = 1, \tag{3.3.53}$$

from which

$$B^{2(\overline{\text{MS}})} = \mu^2 e^{-2\pi/\lambda_R} \tag{3.3.54}$$

This last expression, that is equivalent to equation (3.3.47) provided that $B^2 = -(2m/\hbar^2)E$, is the renormalized bound state energy of the two-dimensional δ-function interaction. Note that this expression, as well as equation (3.3.47), must be independent from μ since E is a physical observable.

RGE. Now, it is time to study in details the dependence on μ of E. Since E is a physical observable, it must not depend on the arbitrary scale length μ. Indeed, from equation (3.3.45) and (3.3.46), it follows that

$$\mu\frac{dE}{d\mu} = \frac{\hbar^2}{2m}\mu\left(2\mu e^{-2\pi/\lambda_R} - 2\pi\mu^2\frac{d(1/\lambda_R)}{d\mu}e^{-2\pi/\lambda_R}\right) = \frac{\hbar^2}{2m}\mu\left(2\mu - 2\pi\mu^2\frac{1}{\pi\mu}\right)e^{-2\pi/\lambda_R} \tag{3.3.55}$$

or

$$\mu\frac{dE}{d\mu} = 0. \tag{3.3.56}$$

Then μ may be fixed experimentally through the measured value of E.

From equation (3.3.53), one has in the dimensional regularization scheme

$$\frac{1}{\lambda_R} = \frac{1}{2\pi}\ln\frac{\mu^2}{B^2}. \tag{3.3.57}$$

It follows that the running of the renormalized coupling constant is given by

$$\mu\frac{d(1/\lambda_R)}{d\mu} = \mu\frac{d}{d\mu}\left(\frac{1}{2\pi}\ln\frac{\mu^2}{B^2}\right) = \mu\left(\frac{2\mu}{2\pi}\frac{1}{\mu^2}\right) = \frac{1}{\pi}, \tag{3.3.58}$$

but, since

$$\mu\frac{d(1/\lambda_R)}{d\mu} = \mu\frac{d(1/\lambda_R)}{d\lambda_R}\frac{d\lambda_R}{d\mu} = -\frac{1}{\lambda_R^2}\frac{d\lambda_R}{d\mu}, \tag{3.3.59}$$

one concludes that the renormalization group equation is

$$\mu\frac{d\lambda_R}{d\mu} = -\frac{\lambda_R^2}{\pi}. \tag{3.3.60}$$

The same result comes out starting from the equation for the renormalized coupling constant obtained with the cutoff scheme (3.3.45)

$$\mu\frac{d(1/\lambda_R)}{d\mu} = \mu\frac{d}{d\mu}\left(\frac{1}{\lambda} - \frac{1}{2\pi}\ln\frac{\Lambda^2}{\mu^2}\right) = \mu\left(\frac{2}{2\pi}\frac{1}{\mu}\right) = \frac{1}{\pi} \to \mu\frac{d\lambda_R}{d\mu} = -\frac{\lambda_R^2}{\pi}. \tag{3.3.61}$$

3.3.3 Effective field theory program

This section deals with another technique of regularization and renormalization, that is the real space regularization [24]. Probably this is the most physical way of regularize. It consists in modifying the theory by changing the short distance behaviour of the potentials involved. Indeed, the theory is assumed to be true at a short distance r_0, which is a parameter artificially introduced as a regulator, and then taking the limit for $r_0 \to 0$. In the case of the two-dimensional δ-function potential, the regularization consists in replacing that potential with a circular wall of radius r_0 and then take r_0 to 0. From this point of view, it can be deduced directly why the two-dimensional δ-interaction is problematic [25]. Indeed, from the Schrödinger equation in D dimensions for a particle in a $\delta^{(D)}$-function potential

$$\nabla^2\psi(\boldsymbol{x}) - 2\lambda\delta^{(D)}(\boldsymbol{x})\psi(\boldsymbol{x}) = B^2\psi(\boldsymbol{x}), \tag{3.3.62}$$

where $B^2 = -(2m/\hbar^2)E$, it is clear that the kinetic term scales as $\nabla^2 \sim |\boldsymbol{x}|^{-2}$, while the potential, since $\int d^Dx\delta^{(D)}(\boldsymbol{x}) = 1$, scales as $\delta^{(D)}(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-D}$. If $D = 1$, the kinetic term dominates at short distances and then the particle behaves as a free one. On the contrary, for $D \geq 3$, where the interaction must be attractive (if it is not, the particle would not reach short distances), the potential energy prevails over the kinetic energy and there is an infinite number of bound states. Obviously, the case $D = 2$ is special since it is the only one where the kinetic and the potential term scale the same way. Surprisingly, this is also the only case when the coupling constant λ is dimensionless.

So, in the real space regularization scheme, the $\delta^{(2)}$-potential is replaced by the potential

$$V_0 = \frac{\hbar^2}{m}\frac{\lambda}{\pi r_0^2}, \quad \text{for} \quad r < r_0; \tag{3.3.63}$$

where r_0 represents a radius and it has the dimensions of $[L]^{-1}$. As known, λ is dimensionless in order to keep the right dimensions of the potential. The rescaled Schrödinger

equation, written in polar coordinates, splits in two parts:

$$\begin{cases} \nabla^2\psi(r,\theta) - \dfrac{2\lambda}{\pi r_0^2}\psi(r,\theta) = B^2\psi(r,\theta), & \text{for} \quad r < r_0; \\ \nabla^2\psi(r,\theta) = B^2\psi(r,\theta), & \text{for} \quad r > r_0. \end{cases} \tag{3.3.64}$$

Using the Laplacian operator in polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{3.3.65}$$

and supposing the ansatz

$$\psi(r,\theta) = a(r)b(\theta), \tag{3.3.66}$$

the Schrödinger equation becomes

$$\begin{cases} \dfrac{1}{a}\dfrac{\partial^2 a}{\partial r^2} + \dfrac{1}{r}\dfrac{1}{a}\dfrac{\partial a}{\partial r} + \dfrac{1}{r^2}\dfrac{1}{b}\dfrac{\partial^2 b}{\partial\theta^2} - B'^2 = 0, & \text{for} \quad r < r_0; \\ \dfrac{1}{a}\dfrac{\partial^2 a}{\partial r^2} + \dfrac{1}{r}\dfrac{1}{a}\dfrac{\partial a}{\partial r} + \dfrac{1}{r^2}\dfrac{1}{b}\dfrac{\partial^2 b}{\partial\theta^2} - B^2 = 0, & \text{for} \quad r > r_0; \end{cases} \tag{3.3.67}$$

where

$$B'^2 = B^2 + \frac{2\lambda}{\pi r_0^2}. \tag{3.3.68}$$

Since the previous set of equation holds for r and θ varying independently, one can keep r fixed and vary only θ. By doing that, the angular part of the Schrödinger equation is equals to a constant $-m^2$

$$\frac{1}{b}\frac{\partial^2 b}{\partial\theta^2} = -m^2. \tag{3.3.69}$$

The solution of this last equation is simply a linear combination of complex exponential:

$$b(\theta) = e^{\pm im\theta}. \tag{3.3.70}$$

Inserting the expression (3.3.69) in (3.3.67), two radial Schrödinger equations are obtained:

$$\begin{cases} \dfrac{1}{a}\dfrac{\partial^2 a}{\partial r^2} + \dfrac{1}{r}\dfrac{1}{a}\dfrac{\partial a}{\partial r} - (B'^2 + \dfrac{m^2}{r^2}) = 0, & \text{for} \quad r < r_0; \\ \dfrac{1}{a}\dfrac{\partial^2 a}{\partial r^2} + \dfrac{1}{r}\dfrac{1}{a}\dfrac{\partial a}{\partial r} - (B^2 + \dfrac{m^2}{r^2}) = 0, & \text{for} \quad r > r_0. \end{cases} \tag{3.3.71}$$

Using the variable $z = Br$ for the first equation and $z' = B'r$ for the second one, one immediately recognizes two instances of the modified Bessel's equation (see appendix

B.4.1)

$$\begin{cases} \frac{1}{a}\frac{\partial^2 a}{\partial r^2} + \frac{1}{r}\frac{1}{a}\frac{\partial a}{\partial r} - (B'^2 + \frac{m^2}{r^2}) = 0, & \text{for} \quad r < r_0; \\ \frac{1}{a}\frac{\partial^2 a}{\partial r^2} + \frac{1}{r}\frac{1}{a}\frac{\partial a}{\partial r} - (B^2 + \frac{m^2}{r^2}) = 0, & \text{for} \quad r > r_0. \end{cases} \tag{3.3.72}$$

Then, the solution is a linear combination of the modified Bessel's functions of first and second kind:

$$a(r) = \begin{cases} a_-(B'r) = A_1 I_m(B'r) + A_2 K_m(B'r), & \text{for} \quad r < r_0; \\ a_+(Br) = A_3 I_m(Br) + A_4 K_m(Br), & \text{for} \quad r > r_0. \end{cases} \tag{3.3.73}$$

Here, m has the same interpretation of the quantum number l used in section 3.3. Then, it represents the magnitude of a centrifugal barrier. Since the $\delta^{(2)}$-function potential, being a zero-range potential, can sustain a bound state only in the absence of a centrifugal barrier, m must be equal to zero. Thus, the modified Bessel's functions are of order zero. Moreover, since the solution must be bounded for $r \to \infty$, the modified Bessel's function of the first kind must be avoided in the solution for $r > r_0$. Analogously, the modified Bessel's function of the second kind must vanish when $r < r_0$. These conditions on the boundedness of $a(r)$ implicate $A_2 = A_3 = 0$. Finally, $a(r)$ must be continuous at $r = r_0$:

$$a_-(B'r_0) = a_+(Br_0) \to \frac{A_4}{A_1} = \frac{I_0(B')}{K_0(B}. \tag{3.3.74}$$

This relation allows to express A_1 in function of A_4 or vice versa. A_4 (or A_1) is determined through the normalization condition. It is assumed for simplicity that $A_1 = 1$. Then, the solution is

$$a(r) = \begin{cases} a_-(B'r) = I_0(B'r), & \text{for} \quad r < r_0; \\ a_+(Br) = \dfrac{I_0(B'r_0)}{K_0(Br_0)} K_0(Br), & \text{for} \quad r > r_0. \end{cases} \tag{3.3.75}$$

The regularized eigenvalue equation is obtained by imposing the continuity of the first order derivative of $a(r)$ in $r = r_0$:

$$\frac{B'}{B} = \frac{K_0'(Br_0)}{I_0'(B'r_0)} \frac{I_0(B'r_0)}{K_0'(Br_0}. \tag{3.3.76}$$

Using the recurrence relations for $K_0'(z)$ and $I_0'(z)$ (B.4.15) and the limiting form for $z \to 0$ (B.4.18), the eigenvalue equation turns into:

$$\frac{B'}{B} = \frac{2}{BB'r_0^2} \frac{1}{\ln(Br_0)}. \tag{3.3.77}$$

By inserting the value of B'^2 (3.3.68), the eigenvalue equation results to be

$$1 + \frac{2\lambda}{\pi B^2 r_0^2} = \frac{2}{B^2 r_0^2} \frac{1}{\ln(Br_0)}, \tag{3.3.78}$$

from which it follows immediately that

$$\frac{1}{\lambda} = \frac{2}{\pi} \frac{\ln(Br_0)}{2 - \ln(Br_0) B^2 r_0^2}. \tag{3.3.79}$$

The previous equation is the regularized equation for the bound state energy and it diverges for $r_0 \to 0$ as expected. Thus, the problem can be renormalized by introducing a renormalized dimensionless coupling constant depending on an arbitrary scale μ with the dimensions of $[L]^{-1}$

$$\frac{1}{\lambda_R} = \frac{2}{\pi} \frac{\ln(\mu r_0)}{2 - \ln(Br_0) B^2 r_0^2} - \frac{1}{\lambda}, \tag{3.3.80}$$

that in the limit for $r_0 \to 0$ tends to

$$\frac{1}{\lambda_R} = \frac{\ln(\mu/B)}{\pi}, \tag{3.3.81}$$

from which it follows the renormalized bound state energy

$$B = \mu e^{-\pi/\lambda_R} \tag{3.3.82}$$

or

$$E = -\frac{\hbar^2}{2m} \mu^2 e^{-2\pi/\lambda_R}. \tag{3.3.83}$$

Fortunately, the bound state energy is exactly the same obtained through the dimensional regularization and the cutoff regularization (3.3.47). E, as a physical observable, is independent from the arbitrary scale μ:

$$\mu \frac{dE}{d\mu} = 0. \tag{3.3.84}$$

The proof is the same of equation (3.3.55). Finally, from equation (3.3.81), it is possible to get again the renormalization group equation, that determines the running of the renormalized coupling constant:

$$\mu \frac{d\lambda_R}{d\mu} = -\frac{\lambda_R^2}{\pi}. \tag{3.3.85}$$

Conclusion

In this thesis, some advanced topics in theoretical physics connected to the idea and method of renormalization are discussed. To this aim, a fictitious model is introduced that allows to present the general techniques used to renormalize. These methods are then applied to a problem in classical physics and to a non-relativistic quantum mechanics one. The first example concerns the determination of the electric potential generated by a uniform charged straight wire, while the quantum one the bound state energy of a single particle in presence of a two-dimensional δ-function potential.

The essence of the problem hides in the point-structure of the potentials involved. Another case where there is a problem due to a point-structure is when one tries to calculate the electron mass in the Quantum Electrodynamics. Even in this case, to find a finite quantity renormalization is necessary. For this reason, the examples presented are helpful to understand the renormalization techniques used in quantum field theory since they are more accessible for a first approach to the subject. The non-relativistic quantum mechanics example is analyzed both solving directly the Schrödinger equation and using propagators. So, the thesis deals also with the theory of propagators, specially Feynman's path integral, which are largely used in quantum field theory.

The problem of the straight wire electric potential arises when one integrates over all the position-space of the wire, as the $\delta^{(2)}$-interaction bound state energy diverges when one tries to integrate over all the Fourier-space of the particle. Then, the more intuitive idea is to set a cutoff in the position/Fourier-space, then add some constants to the potential (or re-defining the coupling constant), that is allowed since physical potential are defined up to a constant, and finally take the cutoff to infinity. This is the cutoff renormalization scheme, that was applied to both the problems. Even if this scheme is really simple, it breaks the translational symmetry of the problem and for this reason it is not the preferred choice in quantum field theory calculation. A regularization scheme that does not break translational invariance is dimensional regularization, where the number of dimensions of the problem is a parameter used to regularized the potential. The results found with this technique are the same of the previous one.

For the $\delta^{(2)}$-interaction bound state energy it was also used an effective field theory approach, which consists in replacing the singular $\delta^{(2)}$-potential with a circular wall of finite radius, computing the bound-state energy and then re-defining the coupling constants in order to make the result finite in the limit for the radius vanishing. Again, the results found are the same. Finally, the running of the renormalized coupling constants was determined, i.e. the renormalization group equation.

AppendixA

Some computation

A.1 Limit of equation 3.2.3

$$\lim_{t\to\infty} \ln \frac{\sqrt{1+t^2}+t}{\sqrt{1+t^2}-t} = \lim_{t\to\infty} \ln \frac{t(\sqrt{1+t^{-2}}+1)}{t(\sqrt{1+t^{-2}}-1)} =$$
$$= \lim_{t\to\infty} \ln \frac{(\sqrt{1+t^{-2}}+1)}{(\sqrt{1+t^{-2}}-1)} = \left[\ln \frac{2}{0}\right] = \infty. \quad \text{(A.1.1)}$$

A.2 Derivative of equation 3.2.5

$$\frac{d}{dx} \ln \frac{\sqrt{\Lambda^2+x^2}+\Lambda}{\sqrt{\Lambda^2+x^2}-\Lambda} =$$
$$= \frac{\sqrt{\Lambda^2+x^2}-\Lambda}{\sqrt{\Lambda^2+x^2}+\Lambda} \cdot \frac{1}{(\sqrt{\Lambda^2+x^2}-\Lambda)^2} \cdot \frac{x}{\sqrt{\Lambda^2+x^2}} \left(\sqrt{\Lambda^2+x^2}-\Lambda+\sqrt{\Lambda^2+x^2}+\Lambda\right) =$$
$$= \frac{2x}{(\sqrt{\Lambda^2+x^2}+\Lambda)(\sqrt{\Lambda^2+x^2}-\Lambda)} =$$
$$= \frac{2x}{(\sqrt{\Lambda^2+x^2})^2-\Lambda^2} = \frac{2}{x}. \quad \text{(A.2.1)}$$

A.3 Limit of equation 3.2.6

$$\lim_{\Lambda\to\infty} V_\Lambda(x) - V_\Lambda(\mu) = -\frac{\lambda}{4\pi\epsilon_0} \ln \frac{\sqrt{\Lambda^2+x^2}+\Lambda}{\sqrt{\Lambda^2+x^2}-\Lambda} \frac{\sqrt{\Lambda^2+\mu^2}-\Lambda}{\sqrt{\Lambda^2+\mu^2}+\Lambda} =$$
$$= -\frac{\lambda}{4\pi\epsilon_0} \ln \frac{\sqrt{1+(x/\Lambda)^2}+1}{\sqrt{1+(x/\Lambda)^2}-1} \frac{\sqrt{1+(\mu/\Lambda)^2}-1}{\sqrt{1+(\mu/\Lambda)^2}+1}. \quad \text{(A.3.1)}$$

Using Taylor's series $(1+x)^\alpha = 1+\alpha x+O(x^2)$ for the squared root around $x/\Lambda = 0$ and

$\mu/\Lambda = 0$, the limit turns into

$$\lim_{\Lambda\to\infty} V_\Lambda(x) - V_\Lambda(\mu) = \lim_{\Lambda\to\infty} -\frac{\lambda}{4\pi\epsilon_0} \ln \frac{2+\frac{x^2}{2\Lambda^2}}{2+\frac{\mu^2}{2\Lambda^2}} \frac{\mu^2}{2\Lambda^2} \frac{2\Lambda^2}{x^2} = \\ = -\frac{\lambda}{4\pi\epsilon_0} \ln \frac{\mu^2}{x^2} = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{\mu}{x}. \quad \text{(A.3.2)}$$

A.4 Computation of Ω_D

To compute Ω_D, let me consider the integral of $\exp\left(-x^2\right)$ in D-dimensions both in rectangular and polar coordinates:

$$\int d^D x e^{-x^2} = \left[\int_0^\infty e^{-x^2}\right]^D = \pi^{D/2}, \quad \text{(A.4.1)}$$

$$\int d^D x e^{-x^2} = \int d\Omega_D \int_0^\infty d\rho \rho^{D-1} e^{-\rho^2}. \quad \text{(A.4.2)}$$

Introducing the variable $r = \rho^2$ ($d\rho = (r^{(-1/2)}/2dr)$), the previous integral becomes:

$$\int d^D x e^{-x^2} = \frac{\Omega_D}{2} \int_0^\infty dr r^{\frac{D}{2}-1} e^{-r} = \frac{\Omega_D}{2}\Gamma(D/2), \quad \text{(A.4.3)}$$

where $\Gamma(x)$ is the Γ-function. Ω_l is given by equating the two forms of the integral

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \quad \text{(A.4.4)}$$

A.5 Fourier transform of Schrödinger equation

The Fourier transform of the Schrödinger time-independent equation

$$(B^2 - \nabla^2)\psi(\boldsymbol{x}) = 2\lambda\delta^{(2)}(\boldsymbol{x})\psi(\boldsymbol{x}) \quad \text{(A.5.1)}$$

is

$$\int \frac{d^D x}{(2\pi\hbar)^D}(B^2 - \nabla^2)e^{(-i/\hbar)\boldsymbol{x}\cdot\boldsymbol{y}}\psi(\boldsymbol{x}) = 2\lambda \int \frac{d^D x}{(2\pi\hbar)^D}\delta^{(2)}(\boldsymbol{x})e^{(-i/\hbar)\boldsymbol{x}\cdot\boldsymbol{y}}\psi(\boldsymbol{x}), \quad \text{(A.5.2)}$$

$$B^2 \int \frac{d^D x}{(2\pi\hbar)^D} e^{(-i/\hbar)\boldsymbol{x}\cdot\boldsymbol{y}}\psi(\boldsymbol{x}) - \int \frac{d^D x}{(2\pi\hbar)^D}\nabla^2 e^{(-i/\hbar)\boldsymbol{x}\cdot\boldsymbol{y}}\psi(\boldsymbol{x}) = \frac{2\lambda}{(2\pi\hbar)^D}\psi(\boldsymbol{0}). \quad \text{(A.5.3)}$$

Using the definition of the momentum-space wave-function

$$\int \frac{d^D x}{(2\pi\hbar)^D} e^{(-i/\hbar)\boldsymbol{x}\cdot\boldsymbol{y}}\psi(\boldsymbol{x}) = \varphi(\boldsymbol{y}), \quad \text{(A.5.4)}$$

which satisfies [18]

$$\int d^D y \varphi(\boldsymbol{y}) = \frac{1}{(2\pi\hbar)^D} \tag{A.5.5}$$

and calculating explicitly

$$\nabla^2 e^{(-i/\hbar)\boldsymbol{x}\cdot\boldsymbol{y}} = -\frac{y^2}{\hbar^2} e^{(-i/\hbar)\boldsymbol{x}\cdot\boldsymbol{y}}, \tag{A.5.6}$$

the Fourier transform results to be

$$\left(B^2 + \frac{y^2}{\hbar^2}\right)\varphi(\boldsymbol{y}) = \frac{2\lambda}{(2\pi\hbar)^D}\psi(\mathbf{0}). \tag{A.5.7}$$

Dividing both sides by $B^2 + y^2/\hbar^2$ and integrating over $\boldsymbol{y}$, one gets

$$1 = 2\lambda \int d^D y \frac{\psi(\mathbf{0})}{B^2 + \frac{y^2}{\hbar^2}}, \tag{A.5.8}$$

Finally, by the Fourier transform of the configuration space wave-function, it is clear that $\psi(\mathbf{0}) = 1/(2\pi\hbar)^D$, so

$$1 = \frac{2\lambda}{(2\pi\hbar)^D} \int d^D y \frac{1}{B^2 + \frac{y^2}{\hbar^2}} = \frac{2\lambda}{(2\pi)^D} \int d^D t \frac{1}{B^2 + t^2}, \tag{A.5.9}$$

AppendixB

Special functions[1]

B.1 Dirac δ-distribution

The Dirac δ-function is a generalized function or distribution which is defined as

$$\delta(x) = \begin{cases} \infty, & x = 0; \\ 0 \quad , & x \neq 0. \end{cases} \tag{B.1.1}$$

Moreover, it satisfies the relation

$$\delta(x) = \int_{-\infty}^{\infty} dx \delta(x) f(x) = f(0). \tag{B.1.2}$$

The Fourier representation (in D dimensions) of the Dirac δ-distribution is

$$\delta(\boldsymbol{x}) = \int \frac{d^D y}{(2\pi)^D} e^{i\boldsymbol{y}\cdot\boldsymbol{x}}. \tag{B.1.3}$$

B.2 Γ-function

Γ-function is defined as

$$\Gamma(z) = \int_0^{\infty} dr r^{z-1} e^{-r}, \tag{B.2.1}$$

where z is a complex number. Γ-function is an analytic function except for $z = 0, -1, -2, \ldots$. An important property of the Γ-function is

$$\Gamma(z+1) = z\Gamma(z). \tag{B.2.2}$$

It follows that

$$\Gamma(n+1) = n!. \tag{B.2.3}$$

There are special cases where the value of the Γ-function is analytically known:

$$\Gamma(1) = 1, \tag{B.2.4}$$

$$\Gamma(1/2) = \sqrt{\pi}. \tag{B.2.5}$$

[1] Chapter written using [15] and https://dlmf.nist.gov

From these values and relation (B.2.2), it follows

$$\Gamma(2) = 1, \tag{B.2.6}$$
$$\Gamma(3) = (3-1)! = 2, \tag{B.2.7}$$
$$\Gamma(4) = 6, \tag{B.2.8}$$
$$\Gamma(3/2) = \sqrt{\pi}/2, \tag{B.2.9}$$
$$\Gamma(5/2) = 3\sqrt{\pi}/4 \tag{B.2.10}$$

ans so on. Other special values are[2]

$$\Gamma(1/3) = 2.67893853470774763365\ldots, \tag{B.2.11}$$
$$\Gamma(2/3) = 1.35411793942640041694\ldots, \tag{B.2.12}$$
$$\Gamma(1/4) = 3.62560990822190831193\ldots, \tag{B.2.13}$$
$$\Gamma(3/4) = 1.22541670246517764512\ldots. \tag{B.2.14}$$

Moreover, if $\epsilon \ll 1$

$$\Gamma(1+\epsilon) = 1 - \gamma\epsilon + O(\epsilon^2), \tag{B.2.15}$$

where γ is the Euler-Mascheroni constant $\gamma = 0.5772$. By the relation $\Gamma(1+z) = z\Gamma(z)$ and the previous approximation, one deduces the expansion

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon^2). \tag{B.2.16}$$

B.3 β-function

The β-function can be defined as [15]:

$$\beta(x,y) = \int_0^\infty dt \frac{t^{x-1}}{(1+t)^{x+y}}. \tag{B.3.1}$$

A fundamental relation involving the β-function and the Γ-function is:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{B.3.2}$$

[2] see https://dlmf.nist.gov/5.4

B.4 Bessel's function

Given the Bessel's equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \alpha^2)w = 0, \tag{B.4.1}$$

its solutions are the Bessel's function of the first kind $J_{\pm\alpha}(z)$, of the second kind $Y_\alpha(z)$ (also called Weber's function) and of the third kind $H_\alpha^{(1),(2)}(z)$ (also named Hankel's function). Important features of the solution are as following: $J_{\pm\alpha}(z)$ is bounded as $|z| \to 0$, $J_\alpha(z)$ and $J_{-\alpha}(z)$ are linearly independent except when α is an integer, $J_\alpha(z)$ and $Y_\alpha(z)$ are always linearly independent. However, $Y_\alpha(z)$ and $J_{\pm\alpha}(z)$ are not all linearly independent since they are related:

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos(\pi\alpha) - J_{-\alpha}(z)}{\sin(\pi\alpha)}. \tag{B.4.2}$$

$H_\alpha^{(1)}(z)$ tend to zero for $|z| \to \infty$ if $0 < \arg z < \pi$ and $H_\alpha^{(2)}(z)$ tends to zero for $|z| \to \infty$ if $-\pi < \arg z < 0$. They are linearly independent for all values of α. There is a relation between the Hankel's functions and the Bessel's function of the first and second kind, that is:

$$H_\alpha^{(1),(2)}(z) = J_\alpha(z) \pm iY_\alpha(z). \tag{B.4.3}$$

When α is fixed and $|z| \to 0$ the Bessel's function show the following behaviour [3]:

$$J_0(z) \to 1; \tag{B.4.4}$$

$$Y_0(z) \sim (2/\pi)\ln(z); \tag{B.4.5}$$

$$H_0^{(1)} \sim -H_0^{(2)} \sim (2i/\pi)\ln(z); \tag{B.4.6}$$

$$J_\alpha(z) \sim \frac{\left(\frac{1}{2}z\right)^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \neq -1, -2, -3, \ldots; \tag{B.4.7}$$

$$Y_\alpha(z) \sim -(1/\pi)\Gamma(\alpha)\left(\frac{1}{2}z\right)^{-\alpha}, \quad \alpha = -1/2, -3/2, -5/2, \ldots; \tag{B.4.8}$$

$$Y_{-\alpha(z)} \sim -(1/\pi)\cos(\pi\alpha)\Gamma(\alpha)\left(\frac{1}{2}z\right)^{-\alpha}, \quad \alpha \neq 1/2, 3/2, 5/2, \ldots. \tag{B.4.9}$$

B.4.1 Modified Bessel's function

The modified Bessel's equation is

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2 + \alpha^2)w = 0. \tag{B.4.10}$$

[3] see https://dlmf.nist.gov/10.7

Solutions of the modified Bessel's equation are the modified Bessel's function of the first, second and third kind: $I_{\pm\alpha}(z)$ and $K_\alpha(z)$. Each is a regular function of z. $I_\alpha(z)$ is bounded as $z \to 0$, but diverges as $z \to \infty$, while $K_\alpha(z)$ tends to zero as $z \to \infty$. The behaviour of $I_{\pm\alpha}(z)$ and $K_\alpha(z)$ for small argument is[4]

$$I_\alpha(z) = \frac{\left(\frac{1}{2}z\right)^\alpha}{\Gamma(\alpha+1)}, \tag{B.4.11}$$

$$K_\alpha(z) \sim \frac{1}{2}\left(\Gamma(\alpha)\left(\frac{1}{2}z\right)^{-\alpha} + \Gamma(-\alpha)\left(\frac{1}{2}z\right)^{\alpha}\right), \tag{B.4.12}$$

$$K_0(z) = -\ln(z). \tag{B.4.13}$$

$I_\alpha(z)$ and $I_{-\alpha}(z)$ are linearly independent just as $I_\alpha(z)$ and $K_\alpha(z)$. There is a relation between the various solutions:

$$K_\alpha(z) = \frac{\pi}{2}\frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin(\alpha\pi)} \tag{B.4.14}$$

Moreover a fundamental property of the modified Bessel's function of the second kind is $K_\alpha(z) = K_{-\alpha}(z)$.

Finally, there are two important recurrence relations that involve the derivatives of the order 0 modified Bessel's functions of first and second kind [5]:

$$I_0'(z) = I_1(z), \tag{B.4.15}$$

$$K_0'(z) = -K_1(z). \tag{B.4.16}$$

B.4.2 Modified spherical Bessel's function

The modified spherical Bessel's equation is

$$z^2\frac{d^2w}{dz^2} + 2z\frac{dw}{dz} - (z^2 + \alpha(\alpha+1))w = 0. \tag{B.4.17}$$

Note that equation (3.3.11) can be put in this form by replacing $z = Kr\rho$ and setting $\alpha(\alpha+1) = (l+\nu)^2 + 1/4$ or equivalently $\alpha = l + \nu - 1/2$. The solution of the modified spherical Bessel's equation can be of the the first and second kind $i_\alpha^{(1),(2)}(z)$ or of the third kind $k_\alpha(z)$. These solution are related to the modified Bessel's functions by the following

[4] see https://dlmf.nist.gov/10.30

[5] see https://dlmf.nist.gov/10.29

relations:

$$i_\alpha^{(1),(2)}(z) = \sqrt{\frac{\pi}{2z}} I_{(\pm(\alpha+1/2))}(z), \tag{B.4.18}$$

$$k_\alpha(z) = \sqrt{\frac{\pi}{2z}} K_{\alpha+1/2}(z). \tag{B.4.19}$$

AppendixC

Schrödinger equation for a central potential in D-dimensions

C.1 Radial and angular part

For the solution of the Schrödinger equation for a particle in a D-dimensional central field, it is common to introduce the orbital angular momentum operator $\hat{L}^2$, that written in D-dimensional hyper-spherical coordinates is

$$\hat{L_j}^2 = -\hbar^2 \sum_{k=j}^{D-1} \left(\prod_{i=1}^{D-1} \sin^2(\theta_j) \right)^{-1} \left[\frac{\partial^2}{\partial \theta_k^2} + (D-k-1)\cot(\theta_k) \frac{\partial}{\partial \theta_k} \right] \tag{C.1.1}$$

where j runs from 1 to $D-1$. This operator satisfies the eigenvalue equation:

$$\hat{L_j}^2 |E, L\rangle = |E, L\rangle \, l_j(l_j + D - j - 1)\hbar^2 \tag{C.1.2}$$

The eigenkets are labelled by the energy E and the set of all the eigenvalues of $\hat{L}^2$, $L = \{l \equiv l_1, l_2, l_3, \ldots, l_{D-2}, m \equiv l_{D-1}\}$. To solve the Schrödinger problem one usually supposes the same factorization of the 3-dimensional case

$$\psi_{E,L}(r, \Omega_D) = \langle r, \Omega_D | E, L \rangle = R_{E,l}(r) Y_L \Omega_D \tag{C.1.3}$$

in terms of the hyper-spherical harmonics $Y_L(\Omega_D) = \langle \Omega_D | L \rangle$. The peculiar number m is associated with the rotation on the (x_{D-1}, x_D) plane, which is characterized by the azimuthal angle $\phi \equiv \theta_{D-1}$; the corresponding operator is usually chosen to be

$$\hat{L}_{D-1} = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \tag{C.1.4}$$

instead of $\hat{L}^2_{D-1}$, with eigenvalue $m\hbar$. In addition the generalized orbital angular momentum quantum numbers satisfy the constraints $0 \leq |m| \leq l_{D-2} \leq l_{D-3} \leq \cdots \leq l_2 \leq l$.

It is known [26] that the Laplacian in D-dimensions is

$$\nabla^2_D = \Delta^{(D)}_r + \frac{\Delta^{(D)}_\Omega}{r^2} \tag{C.1.5}$$

where the radial part is:

$$\Delta_r^{(D)} = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left(r^{D-1} \frac{\partial}{\partial r} \right) \tag{C.1.6}$$

while the angular one is

$$\Delta_\Omega^{(D)} = \sum_{j=1}^{D-1} \left[\left(\prod_{k=1}^{j-1} \sin^2(\theta_k) \right) \sin^{D-j-1}(\theta_j) \right]^{-1} \frac{\partial}{\partial \theta_j} \left(\sin^{D-j-1}(\theta_j) \frac{\partial}{\partial \theta_j} \right). \tag{C.1.7}$$

The angular part can be expressed in terms of the angular momentum $\hat{L}^2 \equiv \hat{L_1}^2$:

$$\Delta_\Omega^{(D)} = -\frac{\hat{L}^2}{\hbar^2}. \tag{C.1.8}$$

The eigenvalue equation of $\hat{L}^2$ leads to

$$\Delta_\Omega^{(D)} Y_L \Omega_D = -l(l+D-2) Y_L \Omega_D. \tag{C.1.9}$$

Then, the Schrödinger equation for a central potential in D dimensions turns into:

$$\left[-\Delta_r^{(D)} + \frac{l(l+D-2)}{r^2} + \frac{2m}{\hbar^2} V(r) \right] R_{El}(r) = -K^2 R_{El}(r), \tag{C.1.10}$$

where $K^2 = -2mE/\hbar^2$ as usually. The term $l(l+D-2)/r^2$ represents a centrifugal potential and its magnitude is defined by the quantum number l. The radial Laplacian can be expanded as

$$\Delta_r^{(D)} = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left(r^{D-1} \frac{\partial}{\partial r} \right) = \frac{1}{r^{(D-1)/2}} \frac{d^2}{dr^2} \left[r^{(D-1)/2} \right] + \frac{(D-1)(D-3}{4r^2}. \tag{C.1.11}$$

Finally, the function $R_{E,l}(r) r^{(D-1)/2}$ satisfies the one-dimensional Schrödinger equation:

$$\left[-\frac{d^2}{dr^2} + V(r) + \frac{l(l+D-2) + (D-1)(D-3)/4}{r^2} \right] R_{E,l}(r) r^{(D-1)/2} = \\ = -K^2 R_{E,l}(r) r^{(D-1)/2}, \tag{C.1.12}$$

which is equivalent to equation (3.3.11) once one has replaced $\nu = D/2 - 1$ and $V(r) = \frac{\hbar^2}{m} \delta(r)$

C.2 Boundary condition at the origin

The Schrödinger equation for a δ-potential in D-dimensions is

$$\left[-\nabla_D^2 + 2\lambda\delta^{(D)}(\boldsymbol{x})\right]\psi(\boldsymbol{x}) = -B^2\psi(\boldsymbol{x}). \tag{C.2.1}$$

The Schrödinger equation behaviour for small $\boldsymbol{x}$ is

$$-\nabla_D^2\psi(\boldsymbol{x}) \overset{(\boldsymbol{x}\to 0)}{\sim} 2\lambda\delta^{(D)}(\boldsymbol{x})\psi(\boldsymbol{0}). \tag{C.2.2}$$

Expressing $\boldsymbol{x}$ in hyper-spherical coordinates (r, Ω_D) and using the representation of the δ-function

$$\delta^{(D)}(\boldsymbol{x}) = -\nabla_D^2\left[\frac{r^{-(D-2)}}{(D-2)\Omega_D}\right], \tag{C.2.3}$$

one finds the behaviour of the wave-function around the origin

$$\psi(\boldsymbol{x}) = \psi_o(\boldsymbol{x}) + \psi(\boldsymbol{0})\frac{2\lambda r^{-(D-2)}}{(D-2)\Omega_D}, \tag{C.2.4}$$

where $\psi_o(\boldsymbol{x})$ is the solution of the correspondent homogeneous equation. From (3.2.10), it follows:

$$(D-2)\Omega_D = 2(D/2-1)\frac{2\pi^{D/2}}{\Gamma(D/2)} = 4\nu\frac{\pi^{\nu+1}}{\nu\Gamma(\nu)} = \frac{4\pi^{\nu+1}}{\Gamma(\nu)}, \tag{C.2.5}$$

where $\nu = D/2 - 1$. Then,

$$\psi(\boldsymbol{x}) = \psi_o(\boldsymbol{x}) + \psi(\boldsymbol{0})\frac{2\lambda r^{-2\nu}\Gamma(\nu)}{4\pi^{\nu+1}}. \tag{C.2.6}$$

$\psi_o(\boldsymbol{x})$ must be equal to $\psi(\boldsymbol{0})$ as required by self-consistency of the value of the wave function at the origin. Then

$$\psi(\boldsymbol{x}) \overset{(\boldsymbol{x}\to 0)}{\sim} \psi(\boldsymbol{0})\left[\frac{\lambda r^{-2\nu}\Gamma(\nu)}{2\pi^{\nu+1}} + 1\right]. \tag{C.2.7}$$

Finally, since $\boldsymbol{x}$ was expressed in hyper-spherical coordinates, the general solution is

$$\psi(\boldsymbol{x}) = \sum_{l=0}^{\infty} R_l(r)Y_L(\Omega_D). \tag{C.2.8}$$

Therefore, for $l = 0$, that is the interesting case treated in 3.3, one can deduce

$$R_0(r) \overset{(r\to 0)}{\sim} R_0(0)\left[1 + \frac{\lambda\Gamma(\nu)}{2\pi^{\nu+1}}r^{-2\nu}\right]. \tag{C.2.9}$$

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