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# On the representation of linear group equivariant operators

**Tesi di Laurea in Topologia Computazionale**

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*Ai miei amici*

*"Ὁ δὲ ἀνεξέταστος βίος
οὐ βιωτὸς ἀνθρώπῳ."*

*"Una vita senza ricerca
non è degna per l'uomo
di essere vissuta."*
Socrate

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# Introduction

In the last years, a strong interest in big data problems and machine learning has arisen. Machine learning is a field of research that aims to reproduce human learning by algorithms. An important mathematical branch, useful in machine learning, is topological data analysis (TDA). TDA studies big data through topology in order to recognize shapes within data, and, as a consequence, to convert data in more usable and explicit knowledge [3]. Its methods allow us to focus on the most significant properties of data by applying new topological and geometrical techniques. Important tools in TDA are persistent homology and group equivariant operators (GEOs).

**Persistent homology** [5]. Persistent homology is a variant of classical homology. It studies a topological space, usually compact, denoted by $X$, through a filtering function $\varphi$, i.e. a function $\varphi : X \to \mathbb{R}$. The filtering function gives us a filtration of topological subspaces of $X$, $\varphi^{-1}(]-\infty, c])$, where $c$ is a real number. For every subspace in the filtration it is possible to compute its homology. Persistent homology studies the evolution of the homology of $\varphi^{-1}(]-\infty, c])$ on varying of $c$. It allows us to understand how relevant a feature of $X$ is in the presence of noise.

Data are usually perturbed for several reasons. If we want to compare data, it is important to consider admissible transformations. Equivariance is the property that allows us to study data up to selected transformations. For this reason, equivariance is a key property in studying topological data.

**Equivariance**. An operator is called equivariant with respect to a group if the action of the group commutes with the operator. In machine learning, there is a growing interest in these operators, since they allow to insert pre-existing knowledge into the system. Furthermore, such operators allow us to reduce the complexity of problems in TDA and get more precise results [2]. Examples of equivariant operators are convolution operators used in building convolutional neural networks (CNNs). We recall that a CNN is a neural network where every "neuron" is built by using a suitable convolution [8].

Let us give an example in order to stress the importance of equivariance in our context. A grayscale image on $X = \mathbb{R}^2$, can be modelled by a function $\varphi : X \to [0, 1]$, where 0 corresponds to black and 1 to white. Let $\Phi$ be the set of all functions from $X$ to $[0, 1]$, i.e. the set of all grayscale images. We observe that an image and the same image translated are often considered equivalent to each other even if they are expressed by different functions. In order to face this problem, we can consider a group $G$ of homeomorphisms of $X$ preserving $\Phi$. We are interested in comparing functions by considering $\varphi$ and $\varphi \circ g$ equivalent to each other. If we wish to identify an image with its translated versions, we can set $G$ equal to the group of translations.

In this way, we stress that we are not interested in just one function, but in the whole space $\Phi$, representing our data. This approach is useful, since there are a lot of cases where data can be expressed by $\mathbb{R}$- or $\mathbb{R}^n$-valued functions. For example, a color image can be represented by an $\mathbb{R}^3$-valued function. On the other hand, the group $G$ describes how the space $\Phi$ can be modified without changing the meaning of data. For this reason, the choice of $G$ is strictly related to the choice of the observer.

Therefore, it comes naturally to compare functions up to the action of $G$. This rationale leads us to consider the natural pseudo-distance $d_G$ on the space $\Phi$, defined by setting $d_G(\varphi, \psi) = \inf_{g \in G} ||\varphi - \psi \circ g||$. This pseudo-distance is an important theoretical tool, but in general it is not simple to compute [6].

**Group Equivariant Operators.** Different methods have been introduced in order to approximate the natural pseudo-distance. One of these methods concerns the use of suitable operators defined on the space $\Phi$. These operators need to be equivariant under the action of the group $G$, and non-expansive. They are called *Group Equivariant Non-Expansive Operators* (GENEOs). By using them it is possible to approximate the natural pseudo-distance with arbitrary precision [6].

If the space $\Phi$ of admissible data is compact, then the space of GENEOs is a compact topological subspace of the space of GEOs (*Group Equivariant Operators*). This space is studied also for its relevance in the research concerning neural networks, since they can be indeed decomposed into elementary parts by using GEOs.

Recently, several methods to build GEOs have been studied, for their importance in TDA and machine learning. In this research, we have moved towards the study of techniques to build GEOs. An interesting method to build linear GEOs consists in computing a kind of average by making use of a particular measure called *permutant measure*, which is a signed measure on the space of homeomorphisms of $X$, invariant under the conjugation action of the elements of $G$.

In this thesis, we will first see how one can build GEOs by permutant measures. Later we will analyse the representability of linear GEOs as linear GEOs associated with permutant measures for $G$. Our main result is a representation theorem. In particular, we will prove that, if $G$ is a group acting transitively on a finite space $X$, and $\Phi = \mathbb{R}^X$, then every linear GEO $F : \Phi \to \Phi$ is representable by a permutant measure.

In more detail, in Chapter 1 we will introduce the concept of GEO and its mathematical setting. In Chapter 2 we will focus on the concept of permutant measure and its main properties. In Chapter 3 we will study the representability of GEOs as GEOs associated with permutant measures.

# Chapter 1

# Mathematical setting

In this thesis we will study particular operators named GEOs. This study can be carried out both in the infinite case and in the finite one. In this chapter we will study both cases. For more details about the definitions and the results cited in this section, we refer the interested reader to [2].

We will study data that are expressed by real functions on a set $X$. We consider a non-empty set $X$ and a set $\Phi \subseteq \mathbb{R}^X$, where $\mathbb{R}^X$ is the set of all real functions from $X$ to $\mathbb{R}$. In this section we will see how we can give topological structures to these sets. An example which explains why we use these sets, is given by considering $X = \mathbb{R}^2$ and $\Phi = [0,1]^X$, i.e. the set of all functions from $X$ to $[0,1]$. This set of functions can represent the set of grayscale images on $X$, where 0 corresponds to black and 1 to white.
In this section we will see that the sets $X$ and $\Phi$ are closely related to each other.
We can endow the set $\Phi$ with an extended metric by setting:

$$D_\Phi(\varphi, \psi) = \sup_{x \in X} |\varphi(x) - \psi(x)| = \|\varphi - \psi\|. \tag{1.1}$$

In this thesis, by using the symbol $\|\cdot\|$, we refer to the infinity-norm.
We can endow the set $X$ with an extended pseudo-metric too, by setting:

$$D_X(x, y) = \sup_{\varphi \in \Phi} |\varphi(x) - \varphi(y)|. \tag{1.2}$$

We can notice that we use the word *extended* since $X$ does not have to be compact, hence $D_X$ can take the infinite value. Moreover $D_X$ is an extended pseudo-metric but not an extended metric since we can have $x \neq y$ and $D_X(x, y) = 0$. For example, if $\Phi$ is the set of constant functions, $D_X(x, y) = 0$ for all $x, y$. On the other hand $D_\Phi$ is an extended metric.
By means of these definitions, we can endow $X$ and $\Phi$ with topologies. As a consequence, we can consider the group $Homeo(X)$ of all homeomorphisms of $X$.

In this work we are not interested in all homeomorphisms, but only in homeomorphisms that preserve $\Phi$, since $\Phi$ is the set of data and, as such, it is the most relevant information in our research. For this reason, we introduce the set $Homeo_\Phi(X)$, i.e. the set of homeomorphisms on $X$ that preserve the space $\Phi$. The formal definition of $Homeo_\Phi(X)$ is: $Homeo_\Phi(X) = \{g \in Homeo(X) \mid \forall \varphi \in \Phi,\ \varphi \circ g \in \Phi \text{ and } \varphi \circ g^{-1} \in \Phi\}$.

The set $Homeo_\Phi(X)$ is of great interest in our research, since it allows us to compare different functions and to define an equivalence relation on $\Phi$. In general we are not directly interested in $Homeo_\Phi(X)$ itself, but in a subgroup $G$ of $Homeo_\Phi(X)$. For example, we can consider the set $\Phi = [0, 1]^{\mathbb{R}^2}$ of all functions from $\mathbb{R}^2$ to $[0, 1]$, where $\Phi$ represents the set of grayscale images that we presented before. It is natural to consider equivalent an image and the same image translated. In other terms we want to define an equivalence relation on $\Phi$. This is possible by taking a group $G \subseteq Homeo_\Phi(\mathbb{R}^2)$. We say that $\varphi$ is equivalent to $\psi$ if and only if $\psi = \varphi \circ g$ for some $g \in G$. The choice of the group $G$ depends on what we are interested to study. For example, if we consider images of digits, the group $G$ must not contain rotations, since we do not want to mistake a 6 for a 9. In other cases this problem cannot arise and $G$ can be the set of isometries.

In the following of this thesis we will see that $G$ acts on $\Phi$ by right composition and on $X$ by computing the image of $g(x)$.

**Definition 1.1.** A right action of a group $G$ on a set $A$, is a function $\varrho : A \times G \to A$ that, for all $a \in A$ and for all $g, h \in G$, satisfies the axioms:

- $\varrho(a, id) = a$,
- $\varrho(a, gh) = \varrho(\varrho(a, g), h)$,

where $id$ is the identity of $G$.

We can analogously define a left action $\bar{\varrho} : G \times A \to A$.

Finally, we can give a topological structure on each group $G \subseteq Homeo_{\Phi}(X)$. As we have done for $\Phi$ and $X$, we can define an extended pseudo-distance on $G$ by setting:

$$D_G(g_1, g_2) := \sup_{\varphi \in \Phi} D_{\Phi}(\varphi \circ g_1, \varphi \circ g_2) \tag{1.3}$$

for every $g_1, g_2 \in G$.

*Remark* 1. It is equivalent to define $D_G$ as above or as:

$$D_G(g_1, g_2) := \sup_{x \in X} D_X(g_1(x), g_2(x)). \tag{1.4}$$

Indeed these equalities hold:

$$\begin{aligned} \sup_{\varphi \in \Phi} D_{\Phi}(\varphi \circ g_1, \varphi \circ g_2) &= \sup_{\varphi \in \Phi} \sup_{x \in X} |(\varphi \circ g_1)(x) - (\varphi \circ g_2)(x)| \\ &= \sup_{x \in X} \sup_{\varphi \in \Phi} |\varphi(g_1(x)) - \varphi(g_2(x))| \\ &= \sup_{x \in X} D_X(g_1(x), g_2(x)) \end{aligned} \tag{1.5}$$

The pair $(\Phi, G)$ is called a *perception pair* (cf. [2]). In our framework, we want to endow $\Phi$ with another pseudo-metric that compares functions up to equivalences with respect to the elements of $G$. Therefore, we define a pseudo-distance, called *natural pseudo-distance* $d_G$, by setting:

$$d_G(\varphi, \psi) := \inf_{g \in G} D_{\Phi}(\varphi, \psi \circ g). \tag{1.6}$$

*Remark* 2. If $G_1, G_2$ are subgroups of $Homeo_\Phi(X)$ and $G_1 \subseteq G_2$, then the following inequalities hold for all $\varphi, \psi \in \Phi$:

$$d_{Homeo_\Phi(X)}(\varphi, \psi) \leq d_{G_2}(\varphi, \psi) \leq d_{G_1}(\varphi, \psi) \leq D_\Phi(\varphi, \psi) \tag{1.7}$$

The pseudo-distance $d_G$ is useful since it allows to compare functions with reference to a group $G$, but it is difficult to compute. The inequalities 1.7 provide an approximation of $d_G$, even if it is not a good approximation. Other methods to approximate the natural pseudo-distance consist in using particular operators. The theory of these operators has been widely developed in the last years and it is the fulcrum of this research.

## 1.1 Equivariant operators

Let us consider a set $X$ and a perception pair $(\Phi, G)$. We endow $\Phi$, $X$ and $G$ with the topological structures we described in the previous section. We can build operators from the space $\Phi$ to itself and we want these operators to respect the action of the group $G$, in a suitable sense. These operators represent how we can modify data, expressed by $\Phi$, without changing their "meaning". Now we are ready to define group equivariant operators:

**Definition 1.2.** Let us consider a set $X$ and a perception pair $(\Phi, G)$, where $\Phi \subseteq \mathbb{R}^X$ and $G \subseteq Homeo_\Phi(X)$. A *Group Equivariant Operator* (GEO) is a homeomorphism $F : \Phi \to \Phi$ such that, $\forall \varphi \in \Phi,\ g \in G$:

$$F(\varphi \circ g) = F(\varphi) \circ g. \tag{1.8}$$

The condition given above means that the operator $F$ commutes with the action of the group $G$.

We can define an extended pseudo-distance and, as a consequence, a topology, on the space of GEOs, by setting:

$$D_{GEO}(F_1, F_2) := \sup_{\varphi \in \Phi} D_\Phi(F_1(\varphi), F_2(\varphi)) \tag{1.9}$$

where $F_1$ and $F_2$ are GEOs.
With this pseudo-distance, the set of all GEOs is a topological space. Let us now introduce an interesting subspace of the space of GEOs.

**Definition 1.3.** Let $(\Phi, G)$ be a perception pair with $\Phi \subseteq \mathbb{R}^X$. An operator $F : \Phi \to \Phi$ is a *Group Equivariant Non-Expansive Operator* (GENEO), if $F$ is a GEO and the following condition holds:

$$D_\Phi(F(\varphi), F(\psi)) \leq D_\Phi(\varphi, \psi) \tag{1.10}$$

for all $\varphi, \psi \in \Phi$. In plain words, this condition means that the operator $F$ semplifies the data.

By using the theory of GEOs, it is possible to approximate $d_G$, as we can see in [2]. However, in this thesis we will not elaborate on such an approximation, but on methods to build and represent GEOs. In the next section we will see what the spaces and operators introduced before become when $X$ is a finite set. In the rest of the thesis we will study a method to build linear GEOs and the representation of linear GEOs in the finite case.

## 1.2 Finite case

Let $X$ be the finite set $\{x_1, \ldots, x_n\}$. Let us denote by $\chi_x$ the characteristic function of the singleton $\{x\}$ for $x \in X$. We may observe that the space $\mathbb{R}^X$ is a vector space with basis $\{\chi_{x_1}, \ldots, \chi_{x_n}\}$. Moreover we can see that $D_\Phi(\varphi, \psi)$ is a real number for all $\varphi, \psi \in \Phi$, since it is a sup on a finite set, whereas $D_X(x_i, x_j)$ can be infinite. For example, if $\Phi = \mathbb{R}^X$, the distance $D_X(x_i, x_j)$ is zero if $i = j$ and infinity otherwise. Hence, the topology induced by the extended pseudo-metric $D_X$ is the discrete topology. The value of $D_X(x_i, x_j)$ is closely related to the choice of $\Phi$. For example, if we consider $\Phi = [0, 1]^X \subseteq \mathbb{R}^X$, then $D_X(x_i, x_j)$ is zero if $i = j$ and 1 otherwise.

We can observe that, if for every $i, j$ with $i \neq j$, a function $\varphi \in \Phi$ exists such that $\varphi(x_i) \neq \varphi(x_j)$, then $X$ is endowed with the discrete topology.

In this case homeomorphisms are just permutations, thus we can replace the spaces $Homeo(X)$ and $Homeo_{\Phi}(X)$ with $Aut(X)$ and $Aut_{\Phi}(X)$, where $Aut(X)$ is the set of all permutations of $X$ and $Aut_{\Phi}(X)$ is the subset of $Aut(X)$ whose elements are the permutations that preserve $\Phi$.

In our work we will study the representation of linear GEOs when $X$ is finite. In the next chapter we will see how to build new GEOs by means of permutant measures. This construction is possible for both finite and infinite spaces. In the last chapter we will focus on a finite set $X$ and we will study when a linear GEO is representable by a permutant measure.

# Chapter 2

# Permutant measures

In this chapter we will introduce the concept of permutant measure, which is a signed measure on the set $Homeo_{\Phi}(X)$, invariant under the conjugation action of a group $G$. These measures represent an important tool to build GEOs.

**Definition 2.1.** Let $X$ be a non-empty set and let $\Phi \subseteq \mathbb{R}^X$, where $\mathbb{R}^X$ is the set of all functions from $X$ to $\mathbb{R}$. We can endow $X$ with the topology induced by the pseudo-distance $D_X$ introduced by Equality (1.2). Let $G \subseteq Homeo_{\Phi}(X)$ be a group of homeomorphisms that preserves $\Phi$. For any $g \in G$ we define the conjugation action $\alpha_g$ as:

$$\alpha_g : Homeo_{\Phi}(X) \to Homeo_{\Phi}(X)$$

$$\alpha_g(h) := g \circ h \circ g^{-1}.$$

*Remark* 3. We can observe that $\alpha_g$ is an homeomorphism, and its inverse is $\alpha_g^{-1} = \alpha_{g^{-1}}$.

We remind that $Homeo_{\Phi}(X)$ is a topological group, where the topology is induced by the extended pseudo-distance $D_{Homeo_{\Phi}(X)}$ introduced by Equality (1.3). Therefore, we can endow $Homeo_{\Phi}(X)$ with the Borel $\sigma$-algebra. Moreover, we have to bear in mind that the Borel $\sigma$-algebra is the smallest $\sigma$-algebra that contains the open sets. Now we are going to introduce the concept of signed measure. In plain words, a signed measure is a measure that can assume also negative values.

**Definition 2.2.** Let $(\Omega, \Sigma)$ be a measurable space. A (finite) signed measure on $(\Omega, \Sigma)$ is a function

$$m : \Sigma \to \mathbb{R}$$

such that $m(\emptyset) = 0$; and

$$m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$$

for all sequences $(A_i)$ of disjoint sets in $\Sigma$, where the sum converges absolutely if the value of the left-hand side is finite.

**Definition 2.3.** A Borel signed measure on $(\Omega, \Sigma)$ is a signed measure on $(\Omega, \Sigma)$, where $\Omega$ is a topological space and $\Sigma$ is its Borel $\sigma$-algebra.

**Definition 2.4.** Let $m$ be a Borel signed measure on $Homeo_{\Phi}(X)$ and let $G \subseteq Homeo_{\Phi}(X)$. The signed measure $m$ is a *permutant measure with respect to $G$* if and only if for all $g \in G$ and for all $A \subseteq Homeo_{\Phi}(X)$ measurable with respect to $m$, $m(\alpha_g(A)) = m(A)$.

*Remark* 4. In the following, if the singleton $\{h\}$ is measurable with respect to a measure $m$, we will write $m(h)$ instead of $m(\{h\})$.

**Theorem 2.0.1.** *Assume that $X$ is a non-empty set, $\Phi \subseteq \mathbb{R}^X$ is a vector space, and $G \subseteq Homeo_{\Phi}(X)$. Let $m$ be a permutant measure on $Homeo_{\Phi}(X)$ with respect to $G$. Assume that the support $H$ of $m$ is finite, and that $\{h\}$ is measurable with respect to $m$ for every $h \in H$.*
*Then the operator $F_m : \Phi \to \Phi$, defined by:*

$$F_m(\varphi) = \sum_{h \in H} \varphi \circ h \; m(h) \tag{2.1}$$

*if $H \neq \emptyset$, and the zero function if $H = \emptyset$, is a linear GEO.*

*Proof.* Since $h$ preserves $\Phi$ for all $h \in H$, and $\Phi$ is a vector space, then we can assert that $F_m(\Phi) \subseteq \Phi$. The linearity of $F_m$ is trivial.
Now we prove that $F_m$ is equivariant, i.e. for all $g \in G$ and $\varphi \in \Phi$, $F_m(\varphi \circ g) = F_m(\varphi) \circ g$. If $H = \emptyset$ the statement is trivial. Let us assume $H = \{h_1, \dots, h_n\} \neq \emptyset$. We can denote by $\tilde{\alpha}_g$ the permutation of $\{1, \dots, n\}$ such that $\alpha_g(h_i) = h_{\tilde{\alpha}_g(i)}$, hence we have $g \circ h_i \circ g^{-1} = h_{\tilde{\alpha}_g(i)}$ and $g \circ h_i = h_{\tilde{\alpha}_g(i)} \circ g$. Moreover we have $m(h_i) = m(h_{\tilde{\alpha}_g(i)})$ for all $g \in G$, since $m$ is a permutant measure. Hence,

$$\begin{aligned} F_m(\varphi \circ g) &= \sum_{i=1}^{n} \varphi \circ g \circ h_i \ m(h_i) \\ &= \sum_{i=1}^{n} \varphi \circ h_{\tilde{\alpha}_g(i)} \circ g \ m(h_{\tilde{\alpha}_g(i)}) \\ &= \sum_{j=1}^{n} \varphi \circ h_j \circ g \ m(h_j) \\ &= \left( \sum_{j=1}^{n} \varphi \circ h_j \ m(h_j) \right) \circ g \\ &= F_m(\varphi) \circ g, \end{aligned} \tag{2.2}$$

where $j = \tilde{\alpha}_g(i)$.
Therefore, $F_m$ is a GEO. □

**Definition 2.5.** If for a GEO $F$ a permutant measure $m$ exists, such that $F = F_m$, then we say that $F$ is *representable by the permutant measure $m$*.

Now we will extend Theorem 2.0.1 to GENEOs.

**Theorem 2.0.2.** *Assume that $X$ is a non-empty set, $\Phi \subseteq \mathbb{R}^X$ is a convex space that contains the zero function, and $G \subseteq Homeo_\Phi(X)$. Let $m$ be a permutant measure on $Homeo_\Phi(X)$ with respect to $G$. Assume that the support $H$ of $m$ is finite, and that $\{h\}$ is measurable with respect to $m$ for every $h \in H$. If the measure $m$ is non-negative and $m(H) \leq 1$, then $F_m : \Phi \to \Phi$,* *defined by:*

$$F_m(\varphi) = \sum_{h \in H} \varphi \circ h \; m(h) \tag{2.3}$$

*if $H \neq \emptyset$, and the zero function if $H = \emptyset$, is a linear GENEO.*

*Proof.* Our hypotheses on $\Phi$ guarantee that $F_m(\Phi) \subseteq \Phi$. The linearity is trivial and the proof of equivariance is the same as in Theorem 2.0.1. Let us now prove the non-expansivity of $F_m$, i.e. $D_\Phi(F_m(\varphi), F_m(\psi)) \leq D_\Phi(\varphi, \psi)$, where $D_\Phi(\varphi, \psi) = \|\varphi - \psi\|$. If $H = \emptyset$ the statement is trivial. Let us assume $H = \{h_1, \ldots, h_n\} \neq \emptyset$.

Let $\varphi, \psi \in \Phi$:

$$\begin{aligned}
\|F_m(\varphi) - F_m(\psi)\| &= \left\| \sum_{i=1}^{n} \varphi \circ h_i \; m(h_i) - \sum_{i=1}^{n} \psi \circ h_i \; m(h_i) \right\| \\
&= \left\| \sum_{i=1}^{n} (\varphi \circ h_i - \psi \circ h_i) \; m(h_i) \right\| \\
&\leq \sum_{i=1}^{n} \|\varphi \circ h_i - \psi \circ h_i\| \; |m(h_i)| \qquad (2.4) \\
&\leq \sum_{i=1}^{n} \|\varphi - \psi\| \; |m(h_i)| \\
&\leq m(H) \; \|\varphi - \psi\| \\
&\leq \|\varphi - \psi\|
\end{aligned}$$

Therefore, we have proved that the operator $F_m$ is non-expansive and hence a GENEO. □

# Chapter 3

# Representation of linear GEOs by permutant measures

In this chapter we will show a correspondence between linear GEOs and GEOs representable by permutant measures. In particular we will study this problem when $X$ is a finite set.

## 3.1 Representation of linear GEOs

The aim of this section is to prove the following theorem:

**Theorem 3.1.1.** *Let $X$ be a finite set and let $\Phi = \mathbb{R}^X$ be the set of all real-valued functions on $X$. Let $G$ be a permutation group acting transitively on the set $X$. Then, $F : \Phi \to \Phi$ is a linear GEO with respect to $G$ if and only if $F$ is a GEO associated with a permutant measure for $G$.*

In Chapter 2 we have already seen that every operator $F$ associated with a permutant measure for $G$ is a linear GEO (Theorem 2.0.1).
Now we have to show that, under the hypotheses of the theorem, every linear GEO is associated with a permutant measure for $G$.

*Remark* 5. Let us introduce some notations, that will come in handy later. If we set $X = \{x_1, \ldots, x_n\}$, we may observe that every function $\varphi : X \to \mathbb{R}$

can be written as $\varphi = \sum_{j=1}^{n} a_j \chi_{x_j}$, where $\chi_x$ is the characteristic function of the singleton $\{x\}$. $\Phi$ is the vector space with basis $\{\chi_{x_1}, \ldots, \chi_{x_n}\}$. We can easily verify that every permutation group acting on the set $X$ preserves $\Phi = \mathbb{R}^X$ under right composition.
If $g$ is a permutation of $X$, we can consider the permutation $\sigma_g$ of the set of indices $\{1, ..., n\}$, such that $g(x_i) = x_{\sigma_g(i)}$.
Finally, if $F$ is a linear GEO, $F(\chi_{x_j}) = \sum_{i=1}^{n} b_{ij} \chi_{x_i}$, for suitable coefficients $b_{ij} \in \mathbb{R}$.

First of all, we remind the definition of transitive action and we explain why the hypothesis of transitivity is fundamental.

**Definition 3.1.** A right action $\varrho$ of the group $G$ on the set $A$ is *transitive* if for every $a, b \in A$ there is an element $g \in G$ such that $\varrho(a, g) = b$. Analogously we can define transitivity for a left action.

In the following example, we will consider a set $X$ and a GEO $F$ for the perception pair $(\Phi, G)$, where the action of $G$ is not transitive on $X$. We will see that $F$ is not representable by a permutant measure.

**Example 3.1.** Let $X = \{1, 2\}$, $\Phi \cong \mathbb{R}^2$ the set of functions from $X$ to $\mathbb{R}$, $G = \{id\}$. Since $G$ contains only the identity, every linear operator $F : \Phi \to \Phi$ is a GEO. $Aut_\Phi(X) = \{id, h\}$, where $h$ is the permutation that switches 1 and 2.
Let us define $F : \Phi \to \Phi$ by setting $F(\chi_i) = \chi_1 \ \forall i \in X$. $F$ is a linear GEO with respect to $G$, but it is not representable by a permutant measure. Indeed, if we assume that such a measure $m$ exists,

$$F(\chi_1) = \chi_1 \circ id \ m(id) + \chi_1 \circ h \ m(h) = \chi_1 \ m(id) + \chi_2 \ m(h),$$

but $F(\chi_1) = \chi_1 = 1\chi_1 + 0\chi_2$. Hence $m(id) = 1$ and $m(h) = 0$.
On the other hand,

$$F(\chi_2) = \chi_2 \circ id \ m(id) + \chi_2 \circ h \ m(h) = \chi_2 \ m(id) + \chi_1 \ m(h),$$

but $F(\chi_2) = \chi_1 = 1\chi_1 + 0\chi_2$. Hence $m(id) = 0$ and $m(h) = 1$. This is a contradiction. Therefore, such a measure $m$ does not exist and $F$ is not representable by a permutant measure.

*Remark* 6. If $\chi_x$ is the characteristic function of the singleton $\{x\}$, and $g$ is a permutation of $X$, then $\chi_x \circ g = \chi_{g^{-1}(x)}$.

**Proposition 3.1.2.** *Let $F : \Phi \to \Phi$ be a linear GEO with respect to the group $G$. Then, for every $i$ and $j$:*

$$b_{ij} = b_{\sigma_g(i)\sigma_g(j)} \ \forall g \in G. \tag{3.1}$$

*Proof.* The equivariance of $F$ with respect to $G$ implies that, by setting $\varphi = \sum_{j=1}^{n} a_j \chi_{x_j} \in \Phi$ and $r = \sigma_{g^{-1}}(j)$,

$$\begin{aligned} F(\varphi \circ g) &= F\left(\sum_{j=1}^{n} a_j \chi_{x_j} \circ g\right) \\ &= F\left(\sum_{j=1}^{n} a_j \chi_{g^{-1}(x_j)}\right) \\ &= \sum_{j=1}^{n} a_j F(\chi_{g^{-1}(x_j)}) \\ &= \sum_{r=1}^{n} a_{\sigma_g(r)} F(\chi_{x_r}) \\ &= \sum_{r=1}^{n} a_{\sigma_g(r)} \sum_{i=1}^{n} b_{ir} \chi_{x_i} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} a_j b_{i\sigma_g^{-1}(j)} \chi_{x_i}. \end{aligned} \tag{3.2}$$

On the other hand, by setting $s = \sigma_{g^{-1}}(i)$,

$$\begin{aligned} F(\varphi) \circ g &= F\left(\sum_{j=1}^{n} a_j \chi_{x_j}\right) \circ g \\ &= \sum_{j=1}^{n} a_j F(\chi_{x_j}) \circ g \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} a_j b_{ij} \chi_{x_i} \circ g \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} a_j b_{ij} \chi_{g^{-1}(x_i)} \\ &= \sum_{j=1}^{n} \sum_{s=1}^{n} a_j b_{\sigma_g(s)j} \chi_{x_s}. \end{aligned} \tag{3.3}$$

Therefore, we can say that the equivariance of $F$ with respect to $G$ is equivalent to the following relation:

$$b_{ij} = b_{\sigma_g(i)\sigma_g(j)} \quad \forall g \in G. \tag{3.4}$$

□

**Lemma 3.1.3.** *Assume that the action of $G$ on $X$ is transitive and that $B = (b_{ij})$ is the $n \times n$ matrix with $b_{ij}$ the entry in the $i$-th row and $j$-th column, representing the linear GEO $F$ with respect to the basis $\{\chi_{x_1}, \dots, \chi_{x_n}\}$. Then, all the lines of $B$ are permutations of the same $n$-tuple and hence have the same sum $c$.*

*Proof.* First of all we will prove that all the rows are permutations of the first row, and all the colums are permutations of the first column.
Since $G$ is transitive, for every $h, k \in \{1, \dots, n\}$ a $g_{hk} \in G$ exists, such that $g_{hk}(x_h) = x_k$, and hence $\sigma_{g_{hk}}(h) = k$. Now we consider the $\bar{\imath}$-th row. From Proposition 3.1.2 we know that $b_{\bar{\imath}j} = b_{\sigma_{g_{\bar{\imath}1}}(\bar{\imath})\sigma_{g_{\bar{\imath}1}}(j)} = b_{1\sigma_{g_{\bar{\imath}1}}(j)}$. Since $\sigma_{g_{\bar{\imath}1}}$ is bijective, this proves that the $\bar{\imath}$-th row is a permutation of the first row. Since $\bar{\imath}$ is arbitrary, every row is a permutation of the first row. By the same arguments we can assert that every column is a permutation of the

first column.
Now we want to prove that rows and columns are permutations of each other. For every real number $y$, we denote the number of times $y$ occurs in each row by $r(y)$, and the number of times $y$ occurs in each column by $s(y)$. Since $nr(y) = ns(y)$, then $r(y) = s(y)$, and hence every line of $B$ has the same elements counted with multiplicity. Therefore, the lines of $B$ are permutations of each other and the lemma is proved. $\square$

We decompose $F$ in two linear GEOs, $F^+ : \Phi \to \Phi$ and $F^- : \Phi \to \Phi$, such that $F = F^+ - F^-$, by setting:

- $F^+(\chi_{x_j})(x_i) := \max\{F(\chi_{x_j})(x_i), 0\}$,
- $F^-(\chi_{x_j})(x_i) := \max\{-F(\chi_{x_j})(x_i), 0\}$,

for all $i$ and $j$, and extending by linearity.

*Remark* 7. $F^+$ and $F^-$ are GEOs, since the maximum of GEOs is still a GEO, [7].

At this point it comes naturally to use the Birkhoff-von Neumann decomposition [4].

**Lemma 3.1.4** (Birkhoff-von Neumann decomposition)**.** *Let $A = (a_{ij})$ be a real square matrix with non-negative entries, such that the sum of entries in every line is equal to a positive number c. Then, there exist positive coefficients $\alpha_1, \ldots, \alpha_k$ with sum equal to c and permutation matrices $P_1, \ldots, P_k$ such that*

$$A = \sum_{k=1}^{t} \alpha_k P_k. \tag{3.5}$$

Let $B^+ = (b^+_{ij})$ and $B^- = (b^-_{ij})$ the matrices associated with $F^+$ and $F^-$ with respect to the basis $\{\chi_{x_1}, \ldots, \chi_{x_n}\}$. We observe that $b^+_{ij} = \max\{b_{ij}, 0\} \geq 0$ and $b^-_{ij} = \max\{-b_{ij}, 0\} \geq 0$.

If we apply Lemma 3.1.4 to $B^+$ and $B^-$, then $B^+ = \sum_{k=1}^{t_1} c^+_k P^+_k$ and $B^- = \sum_{k=1}^{t_2} c^-_k P^-_k$ for suitable positive coefficients and permutation matrices.

We recall that $B = B^{+} - B^{-}$, and hence we can decompose the matrix $B$ as:

$$B = \sum_{k=1}^{t} c_k P_k \tag{3.6}$$

where $P_k$ are permutation matrices and $c_k$ are real coefficients, not necessarily positive.

**Example 3.2.** The decomposition $B = \sum_{k=1}^{t} c_k P_k$ is not unique. For example, we can consider the following matrix:

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We observe that:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

but also

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Any $n \times n$ permutation matrix $P_k$ is associated with a permutation $\pi_k : X \to X$. In particular, if $p_k^{ij}$ is the element of $P_k$ corresponding to the $i$-th row and the $j$-th column, then $\pi_k(x_j) = x_i \Leftrightarrow p_k^{ij} = 1$.

With respect to these choices, the following statement holds.

**Lemma 3.1.5.** *If the action of $G$ on $X$ is transitive, then*

$$F(\varphi) = \sum_{k=1}^{t} c_k \varphi \circ \pi_k^{-1}. \tag{3.7}$$

*Proof.* Let $\varphi = \sum_{j=1}^{n} a_j \chi_{x_j} \in \Phi$. By reminding that $b_{ij} = \sum_{k=1}^{t} c_k p_k^{ij}$, we get:

$$\begin{aligned} F(\varphi) &= F\left(\sum_{j=1}^{n} a_j \chi_{x_j}\right) \\ &= \sum_{j=1}^{n} a_j F(\chi_{x_j}) \\ &= \sum_{j=1}^{n} a_j \sum_{i=1}^{n} b_{ij} \chi_{x_i} \\ &= \sum_{j=1}^{n} a_j \sum_{i=1}^{n} \sum_{k=1}^{t} c_k p_k^{ij} \chi_{x_i} \\ &= \sum_{j=1}^{n} a_j \sum_{k=1}^{t} c_k \sum_{i=1}^{n} p_k^{ij} \chi_{x_i}. \end{aligned} \tag{3.8}$$

We observe that $\sum_{i=1}^{n} p_k^{ij} \chi_{x_i}(x_s) = p_k^{sj}$, and $p_k^{sj} = 1$ if $\pi_k(x_j) = x_s$ and 0 otherwise. But also $\chi_{\pi_k(x_j)}(x_s) = 1$ if $\pi_k(x_j) = x_s$ and 0 otherwise. Hence $\sum_{k=1}^{t} p_k^{ij} \chi_{x_i} = \chi_{\pi_k(x_j)}$. Therefore, Equality (3.8) implies:

$$\begin{aligned} F(\varphi) &= \sum_{j=1}^{n} a_j \sum_{k=1}^{t} c_k \chi_{\pi_k(x_j)} \\ &= \sum_{k=1}^{t} \sum_{j=1}^{n} a_j \chi_{x_j} \circ \pi_k^{-1} c_k \\ &= \sum_{k=1}^{t} \varphi \circ \pi_k^{-1} c_k. \end{aligned} \tag{3.9}$$

□

Unfortunately neither $H = \{\pi_k | k = 1, \ldots, t\}$ nor $\{\pi_k^{-1} | k = 1, \ldots, t\}$ are closed under conjugation action, as we can see in the following example:

**Example 3.3.** Let $X$ be $\{1, 2, 3, 4\}$ and $G$ be the set of all the permutations of $X$. Let us consider the matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We can write $B$ as a sum of permutation matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We can observe that the first permutation matrix is associated with the identity, the second with a permutation that we can call $\pi$, the third with $\pi^2$ and the fourth with $\pi^3$. $H = \{id, \pi, \pi^2, \pi^3\}$ is a cyclic group, but is not closed under the conjugation action of $G$. In order to prove this statement we can consider $\pi^2$, which is the permutation that switches 1 and 3, and 2 and 4. Let $\tau$ be the transposition that switches 1 and 2, then $\tau \circ \pi^2 \circ \tau^{-1}$ is the permutation that switches 1 and 4, and 2 and 3, but this permutation does not belong to $H$.

With reference to the statement of Theorem 3.1.1, this example shows that we cannot consider $c_k$ as the measure of $\pi_k^{-1}$, since the function taking $\pi$ to $c_k$ if $\pi = \pi_k^{-1}$ and to 0 if $\pi^{-1} \notin H$ might not be a permutant measure for the conjugation action of $G$.

**Lemma 3.1.6.** *For all $g \in G$, if $Q_g$ is the permutation matrix associated with $g$, the equality $BQ_g = Q_gB$ holds.*

*Proof.* Since $F$ is a GEO, $F(\varphi \circ g) = F(\varphi) \circ g$, and hence $F(\chi_{x_j} \circ g) = F(\chi_{x_j}) \circ g$. Lemma 3.1.5 proves that $F(\varphi) = \sum_{k=1}^{t} c_k \varphi \circ \pi_k^{-1}$, and therefore

$$\sum_{k=1}^{t} c_k \varphi \circ g \circ \pi_k^{-1} = F(\varphi \circ g) = F(\varphi) \circ g = \sum_{k=1}^{t} c_k \varphi \circ \pi_k^{-1} \circ g.$$

In particular, for all $j$,

$$\sum_{k=1}^{t} c_k \chi_{x_j} \circ g \circ \pi_k^{-1} = \sum_{k=1}^{t} c_k \chi_{x_j} \circ \pi_k^{-1} \circ g,$$

i.e.

$$\sum_{k=1}^{t} c_k \chi_{\pi_k \circ g^{-1}(x_j)} = \sum_{k=1}^{t} c_k \chi_{g^{-1} \circ \pi_k(x_j)},$$

hence

$$\sum_{k=1}^{t} c_k \chi_{\pi_k \circ g^{-1}(x_j)}(x_i) = \sum_{k=1}^{t} c_k \chi_{g^{-1} \circ \pi_k(x_j)}(x_i)$$

for all $i$ and $j$.

We can observe that if $\pi$ is a permutation of $X$, then $\pi$ is represented by $(p_{ij}) := (\chi_{\pi(x_j)}(x_i))$, since both $p_{ij}$ and $\chi_{\pi(x_j)}(x_i)$ are equal to 1 if $\pi(x_j) = x_i$ and 0 otherwise. For this reason, the matrices that represent $\pi_k \circ g^{-1}$ and $g^{-1} \circ \pi_k$ are $P_k Q_g^{-1}$ and $Q_g^{-1} P_k$, and hence:

$$\sum_{k=1}^{t} c_k P_k Q_g^{-1} = \sum_{k=1}^{t} c_k Q_g^{-1} P_k,$$

i.e.

$$\left( \sum_{k=1}^{t} c_k P_k \right) Q_g^{-1} = Q_g^{-1} \left( \sum_{k=1}^{t} c_k P_k \right),$$

therefore, $BQ_g^{-1} = Q_g^{-1} B$ and $Q_g B = BQ_g$. □

**Lemma 3.1.7.** *The following equalities hold:*

- $B = \sum_{k=1}^{t} \sum_{g \in G} \frac{c_k}{|G|} Q_g P_k Q_g^{-1}$
- $F(\varphi) = \sum_{k=1}^{t} \sum_{g \in G} \frac{c_k}{|G|} \varphi \circ g \circ \pi_k^{-1} \circ g^{-1}$.

*Proof.* For the previous lemma $Q_g B Q_g^{-1} = B$, for all $g \in G$, and hence:

$$\begin{aligned} B &= \overbrace{\frac{1}{|G|}B + \cdots + \frac{1}{|G|}B}^{|G| \text{ summands}} \\ &= \frac{1}{|G|}\sum_{g\in G} Q_g B Q_g^{-1} \\ &= \frac{1}{|G|}\sum_{g\in G} Q_g \left(\sum_{k=1}^{t} c_k P_k\right) Q_g^{-1} \\ &= \sum_{k=1}^{t}\sum_{g\in G}\frac{c_k}{|G|} Q_g P_k Q_g^{-1}. \end{aligned} \tag{3.10}$$

Since the entry of position $(i, j)$ in $Q_g P_k Q_g^{-1}$ is $\chi_{g\circ\pi_k\circ g^{-1}(x_j)}(x_i)$, we have that:

$$b_{ij} = \sum_{k=1}^{t}\sum_{g\in G}\frac{c_k}{|G|}\chi_{g\circ\pi_k\circ g^{-1}(x_j)}(x_i) = \sum_{k=1}^{t}\sum_{g\in G}\frac{c_k}{|G|}\chi_{x_j}\circ g\circ\pi_k^{-1}\circ g^{-1}(x_i). \tag{3.11}$$

Hence,

$$\begin{aligned} F(\chi_{x_j})(x_i) &= \sum_{s=1}^{n} b_{sj}\chi_{x_s}(x_i) \\ &= \sum_{s=1}^{n}\sum_{k=1}^{t}\sum_{g\in G}\frac{c_k}{|G|}\chi_{x_j}\circ g\circ\pi_k^{-1}\circ g^{-1}(x_s)\chi_{x_s}(x_i) \\ &= \sum_{k=1}^{t}\sum_{g\in G}\frac{c_k}{|G|}\chi_{x_j}\circ g\circ\pi_k^{-1}\circ g^{-1}(x_i). \end{aligned} \tag{3.12}$$

For the linearity of $F$ we can assert:

$$F(\varphi) = \sum_{k=1}^{t}\sum_{g\in G}\frac{c_k}{|G|}\varphi\circ g\circ\pi_k^{-1}\circ g^{-1}. \tag{3.13}$$

□

We consider a permutation $\pi$ and its orbit $O(\pi) = \{g\circ\pi\circ g^{-1} | g \in G\}$. We endow $Aut(X)$ with a measure that is constant on the orbit.

*Remark* 8. For all $h, \pi \in Aut(X)$, $h \in O(\pi^{-1}) \Leftrightarrow h^{-1} \in O(\pi) \Leftrightarrow \pi \in O(h^{-1})$. Hence $|O(\pi)| = |O(\pi^{-1})|$.

Now we are ready to define a permutant measure. For all $\pi \in Aut(X)$ we define $c(\pi) = c_k$ if $\pi = \pi_k$ and 0 otherwise. We can define:

$$m(\pi) := \sum_{h \in O(\pi^{-1})} \frac{c(h)}{|O(h)|} = \sum_{h \in O(\pi^{-1})} \frac{c(h)}{|O(\pi)|}. \tag{3.14}$$

We can observe that $m$ is a permutant measure by definition. Let us introduce the stabilizer $G_\pi = \{g \in G \mid g \circ \pi \circ g^{-1} = \pi\}$. We know that $|G| = |G_\pi||O(\pi)|$ (cf. [1]). We remind that we obtain each element of the orbit of $\pi$ exactly $G_\pi$ times, by conjugating $\pi$ with respect to every element of $G$.

Now, let $\delta(h, \pi^{-1}) = 1$ if $h$ and $\pi^{-1}$ belong to the same orbit, and 0 otherwise, then:

$$\begin{aligned}
F(\varphi) &= \sum_{k=1}^{t} \sum_{g \in G} \frac{c_k}{|G|} \varphi \circ g \circ \pi_k^{-1} \circ g^{-1} \\
&= \sum_{\pi \in Aut(X)} \sum_{g \in G} \frac{c(\pi)}{|G|} \varphi \circ g \circ \pi^{-1} \circ g^{-1} \\
&= \sum_{\pi \in Aut(X)} \frac{c(\pi)}{|G|} |G_{\pi^{-1}}| \sum_{h \in O(\pi^{-1})} \varphi \circ h \\
&= \sum_{\pi \in Aut(X)} \frac{c(\pi)}{|O(\pi^{-1})|} \sum_{h \in O(\pi^{-1})} \varphi \circ h \\
&= \sum_{\pi \in Aut(X)} \sum_{h \in Aut(X)} \frac{c(\pi)}{|O(\pi)|} \varphi \circ h \ \delta(h, \pi^{-1}) \\
&= \sum_{h \in Aut(X)} \left( \sum_{\pi \in O(h^{-1})} \frac{c(\pi)}{|O(\pi)|} \right) \varphi \circ h \\
&= \sum_{h \in Aut(X)} m(h) \ \varphi \circ h.
\end{aligned} \tag{3.15}$$

Therefore, $F$ is the linear GEO associated with the permutant measure $m$.

□

## 3.2 Representation of linear GENEOs

In this section, we will see what happens if we consider the space of GENEOs. We recall that a GENEO is a non-expansive GEO. Let $X$ be a non-empty set and let $\Phi \subseteq \mathbb{R}^X$, endowed with the topologies induced by $D_X$ and $D_\Phi$, respectively. If $G \subseteq Homeo_\Phi(X)$, a *Group Equivariant Non-Expansive Operator* (GENEO) is a map $F : \Phi \to \Phi$ such that, $\forall \varphi, \psi \in \Phi, g \in G$:

- $F(\varphi \circ g) = F(\varphi) \circ g$,
- $\|F(\varphi) - F(\psi)\| \leq \|\varphi - \psi\|$.

If we consider GENEOs instead of GEOs, we can rephrase Theorem 3.1.1 as it follows.

**Theorem 3.2.1.** *Let $X$ be a finite set, let $\Phi$ be the set of all functions from $X$ to the closed interval $[0, 1]$. Let $G$ be a permutation group acting transitively on the set $X$. Then, $F : \Phi \to \Phi$ is a linear GENEO with respect to $G$ if and only if $F$ is a GENEO associated with a permutant measure for $G$.*

*Proof.* It immediately follows from Theorem 3.1.1 by observing that every GENEO $F : \Phi \to \Phi$ is the restriction of exactly one GEO $\bar{F} : \mathbb{R}^X \to \mathbb{R}^X$. We notice that $F$ is non-expansive if and only if $\bar{F}$ is non-expansive. □

In the following $\bar{F} : \mathbb{R}^X \to \mathbb{R}^X$ will denote the GEO that extends the GENEO $F : \Phi \to \Phi$. Proposition 3.2.2 expresses the link between the non-expansivity of $F$ and the coefficients of the matrix $B = (b_{ij})$, associated with both $F$ and $\bar{F}$ with respect to the basis $\{\chi_{x_1}, \ldots, \chi_{x_n}\}$.

*Remark* 9. If $\varphi \in \Phi$, we can write $\varphi = \sum_{j=1}^{n} a_j \chi_{x_j}$, where $a_j \in [0, 1]$.

**Proposition 3.2.2.** *Let $F : \Phi \to \Phi$ be a linear GENEO with respect to the group $G$. Let $B = (b_{ij})$ the matrix associated with $F$ with respect to the basis $\{\chi_{x_1}, \ldots, \chi_{x_n}\}$. Then, for every $i$ and $j$:*

1. $0 \leq b_{ij} \leq 1$;

*2.* $\sum_{j=1}^{n} b_{ij} \leq 1$.

*Proof.* 1. Since $F(\chi_{x_j}) \in \Phi$ and $F(\chi_{x_j})(x_i) = b_{ij}$, it follows that $b_{ij} \in [0, 1]$.
2. We observe that, since $F$ is linear, the non-expansivity of $F$ is equivalent to the inequality

$$\|F(\varphi)\| \leq \|\varphi\|.$$

If $F \equiv 0$ our statement is trivial. Let us assume that $F \not\equiv 0$.
If $\varphi = \sum_{j=1}^{n} a_j \chi_{x_j} \in \Phi$ with $a_j \in [0, 1]$, we have:

$$F(\varphi) = F\left(\sum_{j=1}^{n} a_j \chi_{x_j}\right) = \sum_{j=1}^{n} a_j F(\chi_{x_j}) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_j b_{ij} \chi_{x_i}, \tag{3.16}$$

and hence:

$$\|F(\varphi)\| = \left|\max_{s=1,\ldots,n} F(\varphi)(x_s)\right| = \max_{s=1,\ldots,n}\left|\sum_{j=1}^{n}\sum_{i=1}^{n} a_j b_{ij} \chi_{x_i}(x_s)\right| = \max_{s=1,\ldots,n} \sum_{j=1}^{n} a_j b_{sj}. \tag{3.17}$$

We notice that we can remove the absolute value since $a_j, b_{ij} \geq 0$. In the same way, we can write:

$$\|\varphi\| = \max_{t=1,\ldots,n}\left|\sum_{j=1}^{n} a_j \chi_{x_j}(x_t)\right| = \max_{t=1,\ldots,n} a_t. \tag{3.18}$$

Therefore, the non-expansivity of $F$ is equivalent to the inequality:

$$\max_{s=1,\ldots,n} \sum_{j=1}^{n} a_j b_{sj} \leq \max_{t=1,\ldots,n} a_t. \tag{3.19}$$

By setting $\bar{a} = \max_{t=1,\ldots,n} a_t$, we can rewrite the inequality above as:

$$\forall s \in \{1, \ldots, n\}, \quad \sum_{j=1}^{n} \frac{a_j}{\bar{a}} b_{sj} \leq 1. \tag{3.20}$$

This inequality must hold for all functions $\varphi \in \Phi$, i.e. for all $n$-tuples $a = (a_1, \ldots, a_n)$. Such an inequality is equivalent to $\sum_{j=1}^{n} b_{sj} \leq 1$. For one implication we can choose $a = (\bar{a}, \ldots, \bar{a})$ in (3.20), whereas the second follows

since $\frac{a_i}{\bar{a}} \leq 1$.
As a consequence, the non-expansivity is equivalent to:

$$\sum_{j=1}^{n} b_{sj} \leq 1 \quad \forall s \in \{1, \dots, n\}. \tag{3.21}$$

□

*Remark* 10. We can decompose a GENEO $F : \Phi \to \Phi$ as $F = F^+ - F^-$, where $F^+$ and $F^-$ are the restrictions of $\bar{F}^+$ and $\bar{F}^-$, respectively. Since $\Phi = [0,1]^X$, we get $F^- \equiv 0$ and $F = F^+$.

## 3.3 Examples

This section is devoted to show some applications of Theorem 3.1.1 in different cases. In the following we will always consider the set $X = \{1, \dots, n\}$ and $\Phi = \mathbb{R}^X$. Every summation will be considered modulo $n$, and we will replace $k+n$ with $k$. We can observe that $Aut(X)$ is the symmetric group $S_n$. For each $k \in X$, we define the permutation $g_k \in S_n$ by setting $g_k(i) = i + k$ for every $i \in X$. We observe that $g_n = id$. The aim of the following examples is to see how the representation changes when the group $G$ changes. In our examples we will write $b_{i,j}$ instead of $b_{ij}$, in order to avoid misunderstandings.

**Example 3.4.** Let $G = \{g_1, \dots, g_n\} \subseteq S_n$. $G$ is a cyclic group that acts transitively on $X$.
Let $F$ be a linear GEO with respect to $G$. Let $B = (b_{i,j})$ be the matrix associated with $F$ with respect to the basis $\{\chi_1, \dots, \chi_n\}$. Equality (3.1) in Proposition 3.1.2, implies that $b_{i,j} = b_{g_k(i),g_k(j)} = b_{i+k,j+k}$ holds for all $i, j, k$.
Let us now build a signed measure $m$ on the power set $\mathcal{P}(S_n)$ of $S_n$.

Take $\varphi = \sum_{j=1}^{n} a_j \chi_j$. Then:

$$
\begin{aligned}
F(\varphi) &= F\left(\sum_{j=1}^{n} a_j \chi_j\right) \\
&= \sum_{j=1}^{n} a_j F(\chi_j) \\
&= \sum_{j=1}^{n} a_j \sum_{i=1}^{n} b_{i,j} \chi_i \\
&= \sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_{j-k,j} \chi_{j-k} \\
&= \sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_{n,k} \chi_j \circ g_k \\
&= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_j \chi_j \circ g_k\right) b_{n,k} \\
&= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_j \chi_j\right) \circ g_k \; b_{n,k} \\
&= \sum_{k=1}^{n} \varphi \circ g_k \; b_{n,k}
\end{aligned}
\tag{3.22}
$$

where we have set $i = j - k$.

For all $h \in Aut_{\Phi}(X)$, we can set $m(h) = b_{n,k}$ if $h = g_k \in G$, and $m(h) = 0$ if $h \notin G$. In this example, the support of the signed measure $m$ is $G$ itself and $m$ is a permutant measure, since $G$ is an Abelian group and the orbit of each element $g_k$, under the conjugation action of the elements of $G$, is the singleton $\{g_k\}$.

$F$ results to be the linear GEO associated with the permutant measure $m$.

**Example 3.5.** Let $s \in S_n$ be such that $s(i) = n - i$ for every $i \in X$. Let us consider the dihedral group $G = \{id, g_1, \ldots, g_{n-1}, s, g_1 \circ s, \ldots, g_{n-1} \circ s\}$, where $g_k$ are rotations and $g_k \circ s$ reflections. The group $G$ transitively acts on $X$.

We recall that the dihedral group $D_n$ is the group of the symmetries of a regular polygon with $n$ vertices. For example, Figure 3.1 shows a hexagon: the group of its symmetries is the dihedral group $D_6$.

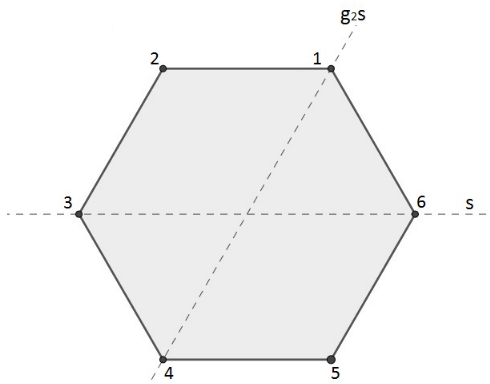


Figure 3.1: The dihedral group $D_6$ is given by the symmetries of a regular hexagon. Two of the six axes of reflection are displayed.

Let $F$ be a linear GEO with respect to $G$. Let $B = (b_{i,j})$ be the matrix associated with $F$ with respect to the basis $\{\chi_1, \ldots, \chi_n\}$.

Equality (3.1) in Proposition 3.1.2, implies that $b_{i,j} = b_{g_k(i),g_k(j)} = b_{i+k,j+k}$ and $b_{i,j} = b_{s(i),s(j)} = b_{n-i,n-j}$ for all $k$. By applying first $s$ and then $g_{i+j}$ we obtain:

$$b_{i,j} = b_{(g_{i+j}\circ s)(i),(g_{i+j}\circ s)(j)} = b_{g_{i+j}(n-i),g_{i+j}(n-j)} = b_{n+j,n+i} = b_{j,i},$$

hence $B$ is symmetric.
As seen in Example 3.4, we have that

$$F(\varphi) = \sum_{k=1}^{n} \varphi \circ g_k \; b_{n,k}. \tag{3.23}$$

We may define a signed measure $m$ on the power set $\mathcal{P}(S_n)$ of $S_n$, by setting

$$m(h) = \begin{cases} b_{n,k}, & \text{if } h = g_k; \\ 0, & \text{if } h \notin \{g_1, \ldots, g_n\}. \end{cases} \tag{3.24}$$

Now, we will see that $m$ is constant on each orbit of the conjugation action of the elements of $G$, and hence it is a permutant measure. Consider the orbit $O(g_k) = \{g \circ g_k \circ g^{-1} | g \in G\}$, of $g_k$. We can observe that $g_r^{-1} = g_{-r}$ and $(g_r \circ s)^{-1} = g_r \circ s$. It is easy to check that $g_r \circ g_k \circ g_{-r} = g_k$ and $g_r \circ s \circ g_k \circ g_r \circ s = g_{-k}$ for all $r$. Therefore, $O(g_k) = \{g_k, g_{-k}\}$. Since $b_{i,j} = b_{n-i,n-j}$, we have that $b_{n,k} = b_{n,n-k}$. This argument proves that the signed measure $m$ is a permutant measure.

$F$ results to be the linear GEO associated with the permutant measure $m$.

**Example 3.6.** Let $G$ be the symmetric group $S_4$. Let $F$ be a linear GEO with respect to $G$. Let $B = (b_{i,j})$ be the matrix associated with $F$ with respect to the basis $\{\chi_1, \chi_2, \chi_3, \chi_4\}$.
Since $G$ contains every permutation of $X$, Equality (3.1) in Proposition 3.1.2 implies that, for all $i, j, h, k \in X$, with $i \neq j$, $h \neq k$, $b_{ij} = b_{hk}$. Moreover, $b_{ii} = b_{jj}$ for all $i, j$, since the action of $G$ on $X$ is transitive. Therefore, $b_{ii} = \alpha$ and $b_{ij} = \beta$ for $i \neq j$, for some $\alpha, \beta \in \mathbb{R}$. As seen in Example 3.4, we have that

$$F(\varphi) = \sum_{k=1}^{4} \varphi \circ g_k \ b_{4,k} = \sum_{k=1}^{4} \varphi \circ g_k \ \mu(g_k) \tag{3.25}$$

where $\mu(g_4) := b_{4,4} = \alpha$ and $\mu(g_k) := b_{4,k} = \beta$ for $k = 1, 2, 3$. We set $\mu(g) := 0$ for every $g \notin \{g_1, g_2, g_3, g_4\}$.

We will see that the signed measure $\mu$ is not invariant under the conjugation action of $G$. Therefore, according to the proof of Theorem 3.1.1, we will build a permutant measure $m$ by computing an average on the orbits. For a classical result, we know that the conjugacy class of each element $g \in S_n$ is given by all permutations with the same cycle type of $g$ (cf. [9]).

First of all, we can notice that the orbit of the identity $g_4$ contains only the identity, hence we set $m(id) = \sum_{h \in O(id)} \frac{\mu(h)}{|O(id)|} = \alpha$. By using the cyclic notation, $g_1 = (1\ 2\ 3\ 4)$, $g_2 = (1\ 3)(2\ 4)$ and $g_3 = (1\ 4\ 3\ 2)$. Hence

$$O(g_1) = O(g_3) = \{g_1, g_3, (1\ 2\ 4\ 3), (1\ 3\ 2\ 4), (1\ 3\ 4\ 2), (1\ 4\ 2\ 3)\}.$$

Moreover, $\mu(g_1) = \mu(g_3) = \beta$ and $\mu(g) = 0$ for the other elements of $O(g_1)$ and therefore $\mu$ is not a permutant measure for $G$.

We set

$$m(g) := \sum_{h \in O(g_1)} \frac{\mu(h)}{|O(g)|} = \frac{\beta}{3}, \tag{3.26}$$

for every $g \in O(g_1)$. In the same way, we observe that $|O(g_2)| = 3$, and hence we set $m(g) := \frac{\beta}{3}$ for all $g \in O(g_2) = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Finally, we set $m(g) := 0$ for every $g \in S_4$ with $g \in S_4 \backslash (O(g_1) \cup O(g_2) \cup O(g_4))$. From Equality (3.25), since

$$\begin{aligned}
\varphi \circ g_1\ \mu(g_1) + \varphi \circ g_3\ \mu(g_3) &= \sum_{g \in O(g_1)} \varphi \circ g\ m(g) \\
\varphi \circ g_2\ \mu(g_2) &= \sum_{g \in O(g_2)} \varphi \circ g\ m(g) \\
\varphi \circ g_4\ \mu(g_4) &= \sum_{g \in O(g_4)} \varphi \circ g\ m(g) \\
0 &= \sum_{g \in S_4 \backslash (O(g_1) \cup O(g_2) \cup O(g_4))} \varphi \circ g\ m(g),
\end{aligned}$$

it follows that $F$ is the linear GEO associated with the permutant measure $m$.

# Conclusion

In this thesis, we have focused on the study of linear group equivariant operators (linear GEOs). Their use in topological data analysis (TDA), is motivated by the fact that they allow us to approximate the natural pseudo-distance and can be considered as building blocks in the construction of equivariant neural networks. We recall that TDA is a branch of mathematics that studies datasets by means of topological and geometrical tools, in order to recognize shape within data and to reduce the computational complexity of many applied problems.

In equivariant TDA, a dataset can be expressed by a collection $\Phi$ of admissible signals defined on a set $X$, and a group $G$ of transformations acting on $X$ (and hence also on $\Phi$). In order to study the space of GEOs for the perception pair $(\Phi, G)$, we have introduced the concept of permutant measure, i.e. a measure constant on the orbit in $Homeo_{\Phi}(X)$ with respect to the conjugation action of the group $G$. In this context, we have presented two main results. In Chapter 2, we have shown how one can build a GEO by a permutant measure $m$. The invariance of $m$ with respect to the conjugation action of $G$ guarantees the equivariance of the operator. We have also seen that if the value of $m$ is non-negative and at most 1, then the operators we have built are group equivariant non-expansive operators (GENEOs). In Chapter 3, we have studied the inverse problem, giving a representation theorem for linear GEOs. We have seen that, if $X$ is finite, and the action of $G$ on $X$ is transitive, then every GEO $F$ can be represented by a permutant measure. Along the proof of this result of representation (Theorem 3.1.1), we have shown how to build a suitable permutant measure. By using the Birkhoff-von Neumann decomposition, we have proved that we can associate each GEO $F$ with a signed measure. Then, by computing an average on the conjugacy classes, we have built the permutant measure that allows us to represent $F$. Finally, we extended these results to linear GENEOs.

As one can see in [8], if the group $G$ is compact and the action of $G$ on $X$ is transitive, every linear GEO can be written as a convolution by integrating on the whole group $G$. In general $G$ can be large, and hence the computational cost can be high. One of the main aims of this thesis is to obtain a method to build equivariant operators with a lower computational cost, since the computation of GEOs associated with permutant measures requires to sum over sets that are often much smaller than the group $G$.

It would be interesting to study the possible extension of the results described in this thesis, in several directions.

First of all, we could examine the possibility of reformulating Theorems 3.1.1 and 3.2.1 under the assumption that $G$ is a compact group that transitively acts on a topological space $X$. In this case we should consider the operator

$$F(\varphi) = \int_{Homeo_{\Phi}(X)} \varphi \circ h \; m(h), \tag{3.27}$$

where $m$ is a permutant measure on $Homeo_{\Phi}(X)$.

Secondly, we could try to extend our results to the case of operators taking a perception pair $(\Phi, G)$ to a different perception pair $(\Psi, G')$. However, this extension could require the development of new approaches and techniques.

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