ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

SCUOLA DI SCIENZE

Corso di Laurea Magistrale in Matematica

ON THE CONCEPT OF PERMUTANT IN THE THEORY OF GROUP EQUIVARIANT NON-EXPANSIVE OPERATORS

Tesi di Laurea in Topologia Computazionale

Relatore:
Chiar.mo Prof.
Patrizio Frosini

Presentata da:
Martina Brasini

III Sessione
Anno Accademico 2018/2019

Contents

Introduction

Nowadays the challenging problem of handling huge amounts of digital data has become a major issue, therefore many new research fields are being explored in order to develop appropriate instruments, capable of processing the data and extrapolating meaningful information.
A significant contribution to this end is given by topological data analysis, whose approach is based on the fact that topology is concerned with studying qualitative geometric features of a space and provides significant dimensionality reduction. In particular, a fundamental tool when trying to detect structures in the datasets overlooking the presence of noise is represented by persistent homology. This theory studies the evolution of the k-dimentional holes when going through a filtration of sub-level sets of a space X, induced by a given continuous function $\varphi\colon X \to \mathbb{R}$. The distance that elapses between the time of birth and the time of death of a hole is called its persistence, and denotes its relevance in shape comparison. In fact, the topological properties that mainly characterize a space X should keep on being detected over a wide range of sub-level sets, while the ones that do not persist are supposed to be noise. Making the evidence of those main features available plays a key role in successful shape comparison. For more detailed information about the deployment of topology in data analysis we refer the reader to [4].
What enables us to apply topological data analysis' techniques to real-world situations, and thus makes them so appealing to us, is the idea that often collected data are the result of measurements, hence they can be represented

by continuous $\mathbb{R}^m$-valued functions defined on a topological space X.
For example any grayscale image can be thought of as a map from $\mathbb{R}^2$ to $[0,1]$, where the values taken in this interval indicate the grayscale levels. In this view, it is clear that we are interested in comparing those functions, providing a quantitative measure of their similarity. Nevertheless, we have to do so while taking into account that in some cases different functions should be considered equivalent, since according to the context they are essentially the same, that is they are linked to each other through a homeomorphism that we consider irrelevant. For instance, referring to the previous example, when comparing images by means of the corresponding functions, we clearly should be able to recognize a picture even if it is transformed through an isometry of the plane. On the other hand, when dealing with images depicting hand-written numbers, rotated pictures can not be considered the same since we want to distinguish 6 from 9, and hence the equivalences between the functions can be assimilated to homotheties.
There is a mathematical tool that perfectly complies with our requirements, indeed the *natural pseudo-distance* is a pseudo-metric whose definition is grounded on the purpose of finding the actual best correspondence between two functions, with respect to a group of homeomorphisms. Named Φ the set of functions to be compared and G the group of self-homeomorphisms of X that express the equivalences between data, assuming that G acts on Φ by composition on the right, we define the *natural pseudo-distance* d_G on Φ by setting $d_G(\varphi, \varphi') = \inf_{g\in G} \| \varphi - \varphi' \circ g \|_\infty$ for any $\varphi, \varphi' \in \Phi$.
Since in general the computation of d_G is quite difficult, it is necessary to approximate its value, and this is made possible, at least when working with real-valued functions, by an approach that exploits persistent homology.
In this context, in fact, the filtering functions can be compared through a particular metric d_{match}, called *bottleneck distance*, between the associated persistence diagrams, which has been proved to be a lower bound for the

natural pseudo-distance. Nevertheless, this instrument does not allow us to distinguish φ from $\varphi \circ f$, when f is a homeomorphism of X, meaning that the filtrations respectively defined turn out to have exactly the same topological properties. For this reason persistent homology has to be employed in a restricted setting, where we can accept the invariance expressed by G, but not a more extended invariance.
Since, as previously said, we want to consider $\varphi \circ g$ equivalent to φ only for g belonging to a specific subgroup G of $\mathrm{Homeo}(X)$, in our model we need to treat G as a variable to have control on it, and this leads us to refer to the theory of group equivariant non-expansive operators (GENEOs).
A GENEO for the pair (Φ, G) is basically a function on Φ that is equivariant with respect to the action of the homeomorphisms in G. We can think of any of these operators as an observer that manipulates the data in order to better analyze them, according to the equivalence she/he considers reasonable for comparisons, and in this perspective the non-expansivity reflects the attempt of simplifying the information we have to work with. The importance of introducing this kind of operators lies in the fact that they allow us to approximate the natural pseudo-distance d_G with arbitrarily high accuracy. Precisely, it has been proved that under suitable hypothesis $d_G(\varphi, \varphi')$ coincides with the supremum of the bottleneck distances between the persistence diagrams of $F(\varphi)$ and $F(\varphi')$, when F varies in the space of all G-equivariant non-expansive operators [1]. From this result we immediately understand how fundamental is the ability of finding methods to build these operators for the computation of the natural pseudo-distance. Some practical examples of the employment of this theory in shape comparison can be found in [3].

This thesis focuses on further exploring the concept of *permutant*, on which a constructional method for defining new GENEOs is based.
The work is organized as follows: the first chapter is devoted to describe the

mathematical setting where our study takes place, the second one concerns an extension of the notion of permutant and the construction of associated operators, while the third one examines their structure as a whole.
In particular, we will prove that the set of all permutants ordered by inclusion is a bounded lattice, whose maximum is a group.

Chapter 1

GENEOs

In our mathematical model the dataset X is thought as the space where measurements are made through a set Φ of admissible functions, that are the real-valued functions that represent the output of measuring instruments. X is equipped with an extended pseudo-metric D_X that reflects our inability to know data if not by looking at them using the functions we have in place. And besides, the topology induced by D_X allows us to assess data stability. We assume that data can be transformed through a certain group G of homeomorphisms of X, each of which clearly needs to preserve Φ, since, as said before, we are working with a functional viewpoint based on that specific set of admissible functions. These transformations are the ones we consider allowed for agents, in other words we require agents to be equivariant with respect to the homeomorphisms of G while acting on the data. In addition to this, we want our agents to be able to compress the data, in order to simplify the information we are interested in. Our aim is to define operators that simulate the transformation operated by agents, namely group equivariant non-expansive operators (GENEOs).

What makes these operators so worthy of interest to us is the fact that the space of GENEOs formalizes the space of the agents acting on a dataset in a topological framework, where a rigorous study of its properties is possible.

It has already been proved that the topological space of G-equivariant non-expansive operators benefits from characteristics that make its employment particularly convenient in deep learning context. A more exhaustive discussion about these topics is provided in [1].

1.1 Mathematical setting

Let X be a non-empty set and let Φ be a topological subspace of $\mathbb{R}_b^X$, which denotes the space of all bounded functions $\varphi : X \to \mathbb{R}$ endowed with the topology induced by the sup-norm. Thus on Φ we have the metric:

$$D_\Phi\,(\varphi, \varphi') := \|\,\varphi - \varphi'\,\|_\infty = \sup_{x\in X} |\,\varphi(x) - \varphi'(x)\,| \qquad \forall\,\varphi, \varphi' \in \Phi.$$

In this work we assume that Φ is convex as a subspace of $\mathbb{R}_b^X$.
We define the following extended pseudo-metric D_X on X:

$$D_X\,(x, x') := \sup_{\varphi\in\Phi} |\,\varphi(x) - \varphi(x')\,| \qquad \forall\,x, x' \in X.$$

This definition basically remarks the fact that we compare the values taken by the admissible functions $\varphi \in \Phi$ to quantify the distance between points in X.
If Φ is bounded, then D_X is a pseudo-metric on X. We consider the pseudo-metric space (X, D_X) as a topological space by fixing as a base for the topology τ_{D_X} the collection of all the sets $B_X(x, \varepsilon) = \{x' \in X \mid D_X(x, x') < \varepsilon\}$ where $x \in X$ and $\varepsilon > 0$.

Theorem 1.1.1. [1]. *The topology τ_{D_X} on X induced by the pseudo-metric D_X is finer than the initial topology with respect to Φ τ_{in} on X.*

The importance of the previous result lies in its usefulness for applications, in fact, since τ_{in} is the coarsest topology on X that makes each function of Φ continuous, it guarantees that each $\varphi \in \Phi$ is automatically continuous with respect to the topology τ_{D_X}.

Remark 1. All functions in Φ are 1-Lipschitz functions with respect to the topology τ_{D_X}, since trivially $\forall\, \varphi \in \Phi$ the following inequality holds:

$$|\,\varphi(x) - \varphi(x')\,| \;\leq\; \sup_{\varphi \in \Phi} |\,\varphi(x) - \varphi(x')\,| = D_X(x, x') \qquad \forall\, x, x' \in X.$$

Theorem 1.1.2. [1]. *If Φ is compact and X is complete, then X is also compact.*

In this work we assume that Φ is compact with respect to the topology induced by D_Φ, and that X is complete and hence compact with respect to the topology induced by D_X.

Let Homeo(X) be the group of the homeomorphisms of X with respect to D_X, and let Homeo$_\Phi(X)$ denote the subgroup of the Φ-preserving homeomorphisms of X with respect to D_X, that is the elements $g \in$ Homeo(X) such that $\varphi \circ g \in \Phi$ and $\varphi \circ g^{-1} \in \Phi \quad \forall \varphi \in \Phi$.
The next proposition shows that Homeo$_\Phi(X)$ actually coincides with the set of all the bijections from X to X that preserve Φ.

Proposition 1.1.3. [1]. *If g is a bijection from X to X such that $\varphi \circ g \in \Phi$ and $\varphi \circ g^{-1} \in \Phi \quad \forall \varphi \in \Phi$, then g is an isometry with respect to D_X, and hence g is a homeomorphism with respect to D_X.*

Remark 2. The previous proposition implies that Homeo$_\Phi(X)$ is a subgroup of the isometry group of X with respect to D_X.

Let G be a subgroup of Homeo$_\Phi(X)$, where it can be $G =$ Homeo$_\Phi(X)$. We define the following pseudo-distance D_G on G:

$$D_G\,(g, g') := \sup_{\varphi \in \Phi} D_\Phi\,(\varphi \circ g, \varphi \circ g') = \sup_{\varphi \in \Phi} \|\, \varphi \circ g - \varphi \circ g'\,\|_\infty \quad \forall\, g, g' \in G.$$

This definition is motivated by our will to lead back the comparisons between the elements of G to the comparisons between the corresponding measurements carried out by the admissible functions of Φ.

Remark 3. The following equality holds:

$$D_G\,(g,g') = \sup_{\varphi\in\Phi} \|\,\varphi\circ g - \varphi\circ g'\,\|_\infty = \sup_{\varphi\in\Phi}\,\sup_{x\in X}\,|\,\varphi(g(x)) - \varphi(g'(x))\,| =$$
$$= \sup_{x\in X}\,\sup_{\varphi\in\Phi}\,|\,\varphi(g(x)) - \varphi(g'(x))\,| = \sup_{x\in X} D_X(g(x),g'(x)) \qquad \forall\, g,g'\in G.$$

Theorem 1.1.4. [1]. *G is a topological group with respect to the topology induced by D_G, and besides, the action of G on Φ through right composition is continuous.*

The homeomorphisms in G are the transformations on data for which we require invariance of the natural pseudo-distance to be respected.
The pair (Φ, G) is called a *perception pair*.

Definition 1.1. A *Group Equivariant Non-Expansive Operator* (GENEO) for the pair (Φ, G) is a function $F\colon \Phi \to \Phi$ such that:

- $F(\varphi\circ g) = F(\varphi)\circ g \qquad \forall\,\varphi\in\Phi, \forall\, g\in G$
- $D_\Phi(F(\varphi_1),F(\varphi_2)) = \|F(\varphi_1) - F(\varphi_2)\|_\infty \le \|\varphi_1-\varphi_2\|_\infty = D_\Phi(\varphi_1,\varphi_2)$
  $\forall\,\varphi_1,\varphi_2\in\Phi$.

The first requirement expresses our wish to work with operators that commute with the action of a selected group G of homeomorphisms, while the condition on the norm describes the claim for non-expansivity on Φ.

Note that the definition of GENEO can be broaden to the case of two distinct perception pairs, nevertheless in this work we will just consider the case of them coinciding.

We must point out that, as proven in [1], having available a sufficiently large set of GENEOs, which describes with reasonable accuracy the topological space of all G-equivariant non-expansive operators, allows us to approximate

the natural pseudo-distance d_G with arbitrary precision. Since this is exactly our final goal we aim to find procedures to build this kind of operators.
The next chapter is concerned with the description of one effective method to build GENEOs.

Chapter 2

Permutants

In this chapter we will focus on the description of a method that allows us to built new GENEOs. Specifically this construction exploits the existence of structures in $\mathrm{Homeo}_{\Phi}(X)$, namely *permutants*, that naturally commute with the elements of G, in view of ensuring the equivariance of the operator with respect to the group. Although the method presents no difficulty when working with finite permutants, its generalization to non-finite permutants is not trivial, indeed it requires several assumptions in order for the operator to be possibly defined.

2.1 A method to build GENEOs

Let G be a subgroup of $\mathrm{Homeo}_{\Phi}(X)$.

For any $g \in G$, let α_g be the conjugacy action of g on $\mathrm{Homeo}_{\Phi}(X)$, that is $\alpha_g \colon \mathrm{Homeo}_{\Phi}(X) \to \mathrm{Homeo}_{\Phi}(X)\,, \quad \alpha_g(h) := g \circ h \circ g^{-1} \quad \forall h \in \mathrm{Homeo}_{\Phi}(X).$

For now, let us just stick to see how it is possible to build a GENEO when given a finite subset of $\mathrm{Homeo}_{\Phi}(X)$ that is stable under the conjugacy action of all $g \in G$.

Proposition 2.1.1. [2] *Let $H \subseteq \mathrm{Homeo}_{\Phi}(X)$, $H = \{h_1, \ldots, h_n\}$, such that $\alpha_g(H) = H \quad \forall g \in G$, that is $\alpha_g(h_i) = g \circ h_i \circ g^{-1} \in H \quad \forall h_i \in H, \forall g \in G$.*

Then $\quad F_H \colon \Phi \longrightarrow \Phi \,, \quad F_H(\varphi) := \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} (\varphi \circ h_i) \quad$ *is a GENEO for (Φ, G).*

Proof. First of all, let us notice that the convexity of the function space Φ together with the fact that the elements of H are Φ-preserving homeomorphisms of X ensures that $F_H(\Phi) \subseteq \Phi$.
Let us verify that the operator F_H defined above is G-equivariant. In fact, $\forall g \in G$ named σ_g the permutation on the set of index $I = \{1, \ldots, n\}$ such that $g \circ h_i \circ g^{-1} = h_{\sigma_g(i)}$ hence $g \circ h_i = h_{\sigma_g(i)} \circ g \quad \forall i \in I$, we have that:

$$
\begin{aligned}
F_H(\varphi \circ g) &= \frac{1}{n} \sum_{i=1}^{n} (\varphi \circ g \circ h_i) = \frac{1}{n} \sum_{i=1}^{n} (\varphi \circ h_{\sigma_g(i)} \circ g) = \frac{1}{n} \sum_{i=1}^{n} (\varphi \circ h_i \circ g) \\
&= \frac{1}{n} \sum_{i=1}^{n} (\varphi \circ h_i) \circ g = F_H(\varphi) \circ g \qquad \forall\, \varphi \in \Phi,\ \forall\, g \in G.
\end{aligned}
$$

Finally, it is not difficult to check the non-expansivity of F_H:

$$
\begin{aligned}
\| F_H(\varphi) - F_H(\varphi') \|_\infty &= \left\| \frac{1}{n} \sum_{i=1}^{n} (\varphi \circ h_i) - \frac{1}{n} \sum_{i=1}^{n} (\varphi' \circ h_i) \right\|_\infty = \\
= \left\| \frac{1}{n} \sum_{i=1}^{n} (\varphi \circ h_i - \varphi' \circ h_i) \right\|_\infty &\leq \frac{1}{n} \sum_{i=1}^{n} \| \varphi \circ h_i - \varphi' \circ h_i \|_\infty = \\
= \frac{1}{n} \sum_{i=1}^{n} \| \varphi - \varphi' \|_\infty &= \| \varphi - \varphi' \|_\infty \qquad \forall\, \varphi, \varphi' \in \Phi.
\end{aligned}
$$

And hence we can conclude that F_H is a GENEO for the pair (Φ, G). $\square$

What we are actually interested in at this point is to extend the construction described above to the case of non-finite subsets of $\mathrm{Homeo}_{\Phi}(X)$ that are stable under the conjugacy action of G. In practice, our long-term aim is to be able to produce operators, which under proper conditions are still GENEOs for (Φ, G), in the following way:

$F_H \colon \Phi \longrightarrow \Phi \,, \quad F_H(\varphi) := \dfrac{1}{\mu(H)} \displaystyle\int_H (\varphi \circ h)\, d\mu \,, \quad$ for a suitable measure μ.

Furthermore, we want to give a structure to the space of these operators, in order to be able to relate them to each other and possibly generate more of them. In this perspective we need to provide a general theoretical framework that allows us to deal with operators similar to F_H, where though the sum is replaced by an integral, and this means firstly that we need to come up with a suitable definition of *permutants*, that on one hand takes into account our need to equip them with a measure, and on the other hand is global enough to allow us to consider them as a whole.

Remark 4. Let K be a subgroup of $\mathrm{Homeo}_{\Phi}(X)$. Since $\mathrm{Homeo}_{\Phi}(X)$ is a topological group, K is a topological group with respect to the subspace topology.

Let $K \neq \emptyset$ be a compact Hausdorff topological subgroup of $\mathrm{Homeo}_{\Phi}(X)$, equipped with a finite Borel measure μ which is invariant under the conjugacy action of all $g \in G$, that is for any $A \subseteq K$ measurable with respect to μ, we require $\mu(\alpha_g(A)) = \mu(A) \quad \forall g \in G$.

Remark 5. For any $g \in G$ the conjugacy action α_g is a self-homeomorphism of $\mathrm{Homeo}_{\Phi}(X)$, and therefore it maps Borel sets to Borel sets.
Let us prove this statement.
First of all we can observe that $\forall g \in G \quad \alpha_g \colon \mathrm{Homeo}_{\Phi}(X) \to \mathrm{Homeo}_{\Phi}(X)$ is a bijection, in fact it is sufficient to notice that α_g is clearly invertible, since $\forall g \in G \quad \alpha_g \circ \alpha_{g^{-1}} = \alpha_{g^{-1}} \circ \alpha_g = id_{\mathrm{Homeo}_{\Phi}(X)}$.
Furthermore we have to prove that $\forall g \in G \ \alpha_g$ and its inverse are continuous.
In order to do that we can point out that $\forall g \in G$ and $\forall h, h' \in \mathrm{Homeo}_{\Phi}(X)$:

$$D_{\mathrm{Homeo}_{\Phi}(X)}(\alpha_g(h), \alpha_g(h')) = \sup_{\varphi \in \Phi} \| \varphi \circ g \circ h \circ g^{-1} - \varphi \circ g \circ h' \circ g^{-1} \|_\infty =$$

$$\sup_{\varphi \in \Phi} \| \varphi \circ g \circ h - \varphi \circ g \circ h' \|_\infty = \sup_{\varphi' \in \Phi} \| \varphi' \circ h - \varphi' \circ h' \|_\infty = D_{\mathrm{Homeo}_{\Phi}(X)}(h, h').$$

Therefore we can conclude that $\forall g \in G \ \alpha_g \colon \mathrm{Homeo}_{\Phi}(X) \to \mathrm{Homeo}_{\Phi}(X)$ is an isometry, and hence a homeomorphism, with respect to $D_{\mathrm{Homeo}_{\Phi}(X)}$.

We can now introduce the main new concept in this thesis.

Definition 2.1. We say that a closed subset $H \subseteq K$ is a *permutant for G in K* if $\alpha_g(H) \subseteq H \quad \forall g \in G$, that is $g \circ h \circ g^{-1} \in H \quad \forall h \in H, \forall g \in G$.

Remark 6. If H is a permutant for G, then it turns out $\alpha_g(H) = H \quad \forall g \in G$, since α_g is a bijection.

Remark 7. $H = \emptyset$ is a permutant for G in K.

Remark 8. $H = \{id_X\}$ is always a permutant for G in K.

2.2 Examples of permutants

In this section we present some examples of permutants in various settings, in order to describe a range of different cases for what concerns the measure μ defined on K and the action of the homeomorphisms α_g on the elements of a permutant.

Example 2.1. Let $X = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, let Φ be the set of the functions $\varphi_v \colon S^2 \to [0, 1]$ with $v \in S^2$, so defined: $\varphi_v(p) := \langle p, v\rangle$ for every $p \in S^2$, where $\langle \cdot, \cdot\rangle$ denotes the standard inner product in $\mathbb{R}^3$. Let K be the cyclic group generated by the rotation ρ_θ of angle $\theta = \frac{\pi}{n}$, where $n \in \mathbb{N}^+$, with respect to the z axis. The pseudo-metric D_K induces the discrete topology on K, and we consider the Borel measure μ on K such that $\mu(\{\rho_{k\theta}\}) = \frac{1}{n} \quad \forall k = 1, \dots, 2n$. Let us consider the group $G = \{r, id_{S^2}\}$, where r is the reflection with respect to the xy plane, that is $r \colon S^2 \to S^2$, $r(x, y, z) = (x, y, -z)$. Since $\varphi_v \circ r = \varphi_{r(v)}$ the action of G takes Φ to Φ. Moreover any subset H of K is a permutant for G. In fact, it is sufficient to notice that the conjugacy action of all $g \in G$ on the elements of K is trivial.

Example 2.2. Let $X = \{1, 2, 3, 4\}$ and let Φ be the set of all functions $\varphi\colon X \to [0, 1]$. Let $K = S_4$ be the symmetric group over X, which is equipped with the discrete topology. We can consider the Borel measure μ on S_4 such that $\mu(\{\sigma\}) = \frac{1}{4!} \quad \forall \sigma \in S_4$.
Let $G = \{(1, 2, 3, 4), (1, 3, 2, 4), (2, 1, 3, 4), (2, 3, 1, 4), (3, 1, 2, 4), (3, 2, 1, 4)\}$ be the subgroup of S_4 made of the permutations of X that fix the last element. Then the set $H = \{(4, 2, 3, 1), (1, 4, 3, 2), (1, 2, 4, 3)\}$, which consists of the permutations that only switch the fourth element with one of the others, is a permutant for G in K.
In particular we can notice that even in this case clearly $\alpha_g(H) = H \; \forall \, g \in G$, but here the action of α_g is not trivial, unless $g = id_{S_4}$.

Example 2.3. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and let Φ be the set of all functions $\varphi\colon X \to [0, 1]$. Let $K = S_7$ be the symmetric group over X, with the discrete topology. Now let $K_{1,2,3}$ be the subset of S_7 made of the non trivial permutations that fix the last four elements, namely $K_{1,2,3} = \{(2, 1, 3, 4, 5, 6, 7), (3, 2, 1, 4, 5, 6, 7), (1, 3, 2, 4, 5, 6, 7), (2, 3, 1, 4, 5, 6, 7), (3, 1, 2, 4, 5, 6, 7)\}$ and let $K_{4,5,6}$ be the subset of S_7 made of the non trivial permutations that fix the first three elements and the last one, that is $K_{4,5,6} = \{(1, 2, 3, 4, 6, 5, 7), (1, 2, 3, 6, 5, 4, 7), (1, 2, 3, 5, 4, 6, 7), (1, 2, 3, 5, 6, 4, 7), (1, 2, 3, 6, 4, 5, 7)\}$.
Finally let $G = K_{1,2,3} \cup K_{4,5,6} \cup \{id_{S_7}\}$. Since $K_{1,2,3}$ and $K_{4,5,6}$ are stable under the conjugacy action of all $\sigma \in G$, the measure μ on S_7 defined by setting $\mu(\{\sigma\}) = 1$ for $\sigma \in K_{1,2,3}$, $\mu(\{\sigma\}) = \frac{1}{2}$ for $\sigma \in K_{4,5,6}$ and $\mu(\{\sigma\}) = 0$ otherwise, is invariant under the conjugacy action of G.
Then the set $H = \{ (7, 2, 3, 4, 5, 6, 1), (1, 7, 3, 4, 5, 6, 2), (1, 2, 7, 4, 5, 6, 3), (1, 2, 3, 7, 5, 6, 4), (1, 2, 3, 4, 7, 6, 5), (1, 2, 3, 4, 5, 7, 6) \}$, which consists of the permutations that only switch the last element with one of the others, is a permutant for G in K.

2.3 Permutants and versatile groups

In this section we raise the point that for groups G that fulfil a particular property, namely *weakly versatility*, we are able to state a priori that no interesting operator can be produced by means of the procedure described in Proposition 2.1.1, since the only non-empty finite permutant available for G is the trivial one, that only consists of the identity function.

Definition 2.2. [2]. We say that G is *versatile* if for any triple $(x, y, z) \in X^3$, where $x \neq z$, and for any finite $S \subseteq X$ at least one homeomorphism $g \in G$ exists such that $g(x) = y,\ g(z) \notin S$.

Example 2.4. Let $X = \mathbb{P}^2 = S^2/\sim$, where $\sim$ is the equivalence relation that identifies the antipodal points on the sphere, and let Φ be the set of the functions $\varphi_{[v]} \colon \mathbb{P}^2 \to [0, 1]$ with $[v] = [-v] = \{v, -v\} \in \mathbb{P}^2$, so defined: $\varphi_{[v]}([p]) := |\langle p, v\rangle|$ for every $[p] \in \mathbb{P}^2$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^3$. We observe that the definition of $\varphi_{[v]}([p])$ does not depend on the representatives of the equivalence classes $[v], [p]$. Now, let us consider the group G of the maps that can be obtained by taking an isometry f of S^2 with respect to the Euclidean distance and considering the induced map $[f] : \mathbb{P}^2 \to \mathbb{P}^2$ defined by setting $[f]([p]) := [f(p)]$ for every $[p] \in \mathbb{P}^2$ (this definition does not depend on the representative of the equivalence class $[p]$, since f takes antipodal points to antipodal points). It is easy to check that if $[f_1], [f_2] \in G$, then $[f_1] \circ [f_2] = [f_1 \circ f_2]$. Since $\varphi_{[v]}([f(p)]) = \varphi_{[f^{-1}(v)]}([p])$ for every $v, p \in S^2$ and every $[f] \in G$, the maps in G are Φ-preserving bijections. The definition of G easily implies that G is versatile.

Definition 2.3. We say that G is *weakly versatile* if for any pair $(x, z) \in X^2$, where $x \neq z$, and for any finite $S \subseteq X$ at least one homeomorphism $g \in G$ exists such that $g(x) = x,\ g(z) \notin S$.

Remark 9. If G is versatile then G is weakly versatile.
In fact it is sufficient to choose $y = x$ to obtain the weakly versatility condition from the versatility one.

Remark 10. If G is weakly versatile then G is not necessarily versatile.

Example 2.5. Let us define $\mathbb{P}^2$, [v], $\langle \cdot, \cdot \rangle$ and G as in Example 2.4.
Let $\hat{\Phi}$ be the set of the functions $\hat{\varphi}_{[v]} \colon \mathbb{P}^2 \times [0,1] \to [0,2]$ so defined: $\hat{\varphi}_{[v]}(([p],t)) := |\langle p, v \rangle| + t$ for every $[p] \in \mathbb{P}^2$ and every $t \in [0,1]$. We observe that the definition of $\hat{\varphi}_{[v]}(([p],t))$ does not depend on the representatives of the equivalence classes $[v], [p]$. Let $\hat{G}$ be the group of the maps $\hat{g} : \mathbb{P}^2 \times [0,1] \to \mathbb{P}^2 \times [0,1]$ such that $\hat{g}([p],t) = (g([p]),t)$ for a suitable $g \in G$ independent of t. The action of the group $\hat{G}$ takes $\hat{\Phi}$ to $\hat{\Phi}$. Moreover, $\hat{G}$ is weakly versatile, but not versatile, since no point $([p],t) \in \mathbb{P}^2 \times [0,1]$ can be taken to a point $([q],t') \in \mathbb{P}^2 \times [0,1]$ by a map $\hat{g} \in \hat{G}$, for $t' \neq t$.

Proposition 2.3.1. *Let $H \neq \emptyset$ be a finite permutant for G in K. If G is weakly versatile, then $H = \{id_X\}$.*

Proof. Let us suppose that $H \neq \{id_X\}$, this means that an element $h \in H$ exists such that $h \neq id_X$. Let $H = \{h_1, \ldots, h_n\}$, we can suppose without loss of generality that $h_1 \neq id_X$, hence there exists $\bar{x} \in X$ such that $h_1(\bar{x}) \neq \bar{x}$. Since H is a finite permutant for G we have that $\forall g \in G \quad g \circ h_1 \circ g^{-1} = h_i$, hence $g \circ h_1 = h_i \circ g$ for some $i \in \{1, \ldots, n\}$. If we consider the pair of distinct points $(\bar{x}, h_1(\bar{x}))$ and the finite set $S = \{h_1(\bar{x}), \ldots, h_n(\bar{x})\}$, we obtain that $\forall g \in G$ such that $g(\bar{x}) = \bar{x} \quad g(h_1(\bar{x})) = h_i(g(\bar{x})) = h_i(\bar{x}) \in S$, but this is not possible since we supposed the group G was weakly versatile. $\square$

Chapter 3

The lattice of permutants

In this last chapter we investigate the relationships that exist among the permutants for a given group G in K. In particular we show that the set of all permutants for G in K is a bounded lattice, and then we prove that the top element of this lattice, that is called the *maximal permutant for G in K*, turns out to be a group.

Proposition 3.0.2. *The set of all permutants for G in K is a lattice.*

Proof. It is sufficient to prove that if H, H' are permutants for G in K then $H \cap H'$ and $H \cup H'$ are too. Let H, H' be permutants for G in K, then $H \cap H'$ and $H \cup H'$ clearly are closed subsets of K, thus we only have to verify that they are invariant under the conjugacy action of all $g \in G$.
If $H \cap H' = \emptyset$ the result is trivial, hence suppose $H \cap H' \neq \emptyset$. Let us consider $h \in H \cap H'$, in particular $h \in H$ and $h \in H'$, but now, since H and H' are permutants for G we have that $\forall g \in G \;\; g \circ h \circ g^{-1} \in H$ and $g \circ h \circ g^{-1} \in H'$, and thus $g \circ h \circ g^{-1} \in H \cap H'$. Hence $H \cap H'$ is a permutant for G in K.
In the same way, let $h \in H \cup H'$, if $h \in H$ then $\forall g \in G$ we have $g \circ h \circ g^{-1} \in H$, and hence $g \circ h \circ g^{-1} \in H \cup H'$; on the other hand if $h \notin H$ necessarily $h \in H'$, and so we have $g \circ h \circ g^{-1} \in H'$, from which we get $g \circ h \circ g^{-1} \in H \cup H'$. This proves that $H \cup H'$ is a permutant for G in K. □

Proposition 3.0.3. *Let $H \subseteq K$, and let $\overline{H}$ denote the topological closure of H in K. If $\alpha_g(H) \subseteq H \quad \forall g \in G$, then $\alpha_g(\overline{H}) \subseteq \overline{H} \quad \forall g \in G$.*

Proof. We need to verify that $\forall \bar{h} \in \overline{H} \quad \alpha_g(\bar{h}) \in \overline{H}$, $\forall g \in G$. Since $\overline{H}$ is the topological closure of H, $\forall \bar{h} \in \overline{H}$ we have that $\forall \varepsilon > 0$ there exists $h_\varepsilon \in H$ such that $\sup_{\varphi \in \Phi} \|\varphi \circ \bar{h} - \varphi \circ h_\varepsilon\|_\infty < \varepsilon$. This implies that $\forall g \in G$, $\forall \varepsilon > 0$

$$\sup_{\varphi \in \Phi} \|\varphi \circ (\alpha_g(\bar{h})) - \varphi \circ (\alpha_g(h_\varepsilon))\|_\infty = \sup_{\varphi \in \Phi} \|\varphi \circ g \circ \bar{h} \circ g^{-1} - \varphi \circ g \circ h_\varepsilon \circ g^{-1}\|_\infty$$
$$= \sup_{\varphi \in \Phi} \|\varphi \circ g \circ \bar{h} - \varphi \circ g \circ h_\varepsilon\|_\infty = \sup_{\varphi' \in \Phi} \|\varphi' \circ \bar{h} - \varphi' \circ h_\varepsilon\|_\infty < \varepsilon.$$

But now, since we were assuming that for $h_\varepsilon \in H \quad \alpha_g(h_\varepsilon) \in H$, $\forall g \in G$, the previous inequality implies that $\alpha_g(\bar{h}) \in \overline{H}$, for any $\bar{h} \in \overline{H}$, $\forall g \in G$. $\square$

Corollary 3.0.4. *Let $H \subseteq K$ such that $\alpha_g(H) \subseteq H \quad \forall g \in G$, then $\overline{H}$ is a permutant for G in K.*

Proof. It is sufficient to notice that $\overline{H} \subseteq K$, since K is closed, then the statement immediately follows from Proposition 3.0.3. $\square$

Proposition 3.0.5. *Let $\{H_i\}_{i \in \mathcal{I}}$ be the lattice of all permutants for G in K with respect to the inclusion, then $\mathcal{H} = \bigcup_{i \in \mathcal{I}} H_i$ is a permutant for G in K.*

Proof. It is sufficient to observe that $\mathcal{H} \subseteq K$ is invariant under the conjugacy action of all $g \in G$, hence Corollary 3.0.4 claims that its topological closure $\overline{\mathcal{H}}$ is a permutant for G in K, but then from the maximality of $\mathcal{H}$ follows that $\mathcal{H} \equiv \overline{\mathcal{H}}$. $\square$

Definition 3.1. We say that $\mathcal{H}$ is the maximal permutant for G in K.

Remark 11. We can notice that the lattice of all permutants for G in K is a bounded lattice, since it has a top element, represented by $\mathcal{H}$, and a bottom element which is the empty-set.

Proposition 3.0.6. *Let* $H = \{h_i\}_{i\in I}$ *and denote* $H^{-1} := \{h_i^{-1}\}_{i\in I}$. *If* H *is a permutant for* G *in* K *then* H^{-1} *is too.*

Proof. First of all let us notice that $H \subseteq K$ implies $H^{-1} \subseteq K$, since K is a group. Moreover, thanks to the fact that K is a topological group we have that the inversion operation is continuous and hence is a homeomorphism on K, thus it maps closed sets to closed sets, and from this we get that H^{-1} is a closed set. Hence we only have to verify that H^{-1} is invariant under the conjugacy action of all $g \in G$. Now, assuming that H is a permutant for G in K means that $\forall g \in G$ a permutation σ of the index set I exists such that $g \circ h_i \circ g^{-1} = h_{\sigma(i)} \quad \forall i \in I$. It follows that $\forall g \in G$ we have $g \circ h_i^{-1} \circ g^{-1} = (g \circ h_i \circ g^{-1})^{-1} = h_{\sigma(i)}^{-1} \quad \forall i \in I$, that is overall $\alpha_g(H^{-1}) \subseteq H^{-1}$. Therefore we can conclude that H^{-1} is a permutant for G in K. $\square$

Proposition 3.0.7. *Let* $H = \{h_i\}_{i\in I}$ *and denote* $H^2 := \{(h_i \circ h_j)\}_{i,j\in I}$. *If* H *is a permutant for* G *in* K *then* H^2 *is too.*

Proof. First of all let us notice as before that $H \subseteq K$ implies $H^2 \subseteq K$, since K is a group. In addition, H is closed and hence compact in K, thus $H \times H$ is compact with respect to the product topology, and since K is a topological group we have that the composition map $f : H \times H \to H^2$, $f(h_i, h_j) = h_i \circ h_j$ is continuous, thus it maps compact sets to compact sets, and from this we get that H^2 is a compact set. Now the only thing left to prove is that H^2 is invariant under the conjugacy action of all $g \in G$. Since we assume that H is a permutant for G in K, we have that $\forall g \in G$ a permutation σ of the index set I exists such that $g \circ h_i \circ g^{-1} = h_{\sigma(i)}$ $\forall i \in I$. It follows that $\forall g \in G$ $g \circ (h_i \circ h_j) \circ g^{-1} = (g \circ h_i \circ g^{-1}) \circ (g \circ h_j \circ g^{-1}) = h_{\sigma(i)} \circ h_{\sigma(j)}$ $\forall i, j \in I$ that is overall $\alpha_g(H^2) \subseteq H^2$. Therefore we can conclude that H^2 is a permutant for G in K. $\square$

Proposition 3.0.8. *Let $\mathcal{H}$ be the maximal permutant for G in K, then $\mathcal{H}$ is a group.*

Proof. Trivially $\{id_X\}$ is a permutant for G in K, hence $id_X \in \mathcal{H}$.
Let us prove that $\forall h \in \mathcal{H}$ also $h^{-1} \in \mathcal{H}$. Since $h \in \mathcal{H}$ there will be a permutant H such that $h \in H$, but then thanks to Proposition 3.0.6 we can assure that there exists a permutant for G in K that contains h^{-1}, and so thanks to the maximality of $\mathcal{H}$ it must be $h^{-1} \in \mathcal{H}$.
Finally let us show that $\forall h, h' \in \mathcal{H}$ also $h \circ h' \in \mathcal{H}$. Since the set of all permutants for G in K is closed under union, there will be a permutant H such that $h, h' \in H$, but now Proposition 3.0.7 guarantees the existence of a permutant for G in K that contains $h \circ h'$, and hence for the maximality of $\mathcal{H}$ we get $h \circ h' \in \mathcal{H}$.
Therefore we can conclude that $\mathcal{H}$ is a group. $\square$

Conclusions

In order to motivate our work it is important to recall that the real goal of this study is to extend our knowledge about the space of all the G-equivariant non-expansive operators defined on a given set of functions Φ, and hence reach an approximation of the natural pseudo-distance d_G as accurate as possible, since this would give us a powerful and effective instrument for shape comparison. We can say that this corresponds to the ability of simulating the performance of an observer who is trying to analyze some data, acting on them according to reasonable criteria, with the purpose of bringing useful information out of them.

In particular, this thesis revolves around a deeper examination of the concept of permutant, since it appears to be a key element in the construction of new GENEOs, especially because it seems to be suitable for generalization in several directions and this is precisely what we are interested in.

For instance, we have started exploring the way of non-finite permutants, since we perceive this extension as necessary in order to make things work well other than a natural evolution, and we have provided a general definition that includes the non-finite case and that allows us to structure the collection of permutants, but still a lot of work has to be done in order to develop a general method to define new operators starting from non-finite permutants.

Another natural extension goes in the direction of applying the permutant-procedure in a slightly different context, that is to construct GENEOs between two different perception pairs.

Another attempt that perhaps can lead to some interesting results is changing our approach towards the conjugacy action, since it is possible that looking at it globally as a topological group action can help us identify useful algebraic features regarding the group G or permutants.
Finally there are some metric aspects we did not go into, which are worthy of investigation and concern the metric properties of the space of homeomorphisms $\mathrm{Homeo}_{\Phi}(X)$ where we work.

Bibliography

[1] Mattia G. Bergomi, Patrizio Frosini, Daniela Giorgi, Nicola Quercioli, *Towards a topological-geometrical theory of group equivariant non-expansive operators for data analysis and machine learning*, Nature Machine Intelligence, vol. 1, n. 9, pages 423-433 (2 September 2019).

[2] Francesco Camporesi, Patrizio Frosini, Nicola Quercioli, *On a new method to build group equivariant operators by means of permutants*, Lecture Notes in Computer Science, Proceedings of the International Cross-Domain Conference, CD-MAKE 2018, Hamburg, Germany, August 27-30, 2018, MAKE Topology, Springer, Cham, A. Holzinger et al. (Eds.), LNCS 11015, 265-272, 2018.

[3] Patrizio Frosini, Nicola Quercioli, *Some remarks on the algebraic properties of group invariant operators in persistent homology*, Lecture Notes in Computer Science, Proceedings of the International Cross-Domain Conference, CD-MAKE 2017, Reggio, Italy, August 29-September 1, 2017, MAKE Topology, Springer, Cham, Holzinger A., Kieseberg P., Tjoa A M., Weippl E. (Eds.), LNCS 10410, 14-24, 2017.

[4] Gunnar Carlsson, *Topology and data*, American Mathematical Society, Bulletin, New Series, Volume 46, Number 2, Pages 255-308, 2009.