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Loop Quantum Gravity: Quantum Space and New Coherent States from Twisted Geometries

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Abstract

La teoria della Gravità Quantistica a Loop (LQG) raffigura il concetto di spazio dinamico come un oggetto quantistico. Predice infatti una nozione discreta e quantizzata di geometria, la quale deve essere riconciliata in qualche modo con l'idea classica di spazio curvo. L'intento perseguito nel redigere questa tesi è quello di fornire uno strumento utile a studiare questa riconciliazione, il quale è dato dal risultato originale di una nuova classe di stati coerenti. In questo documento ci si pone dunque il problema di come una geometria che appare continua possa essere recuperata dagli stati fondamentali della LQG, e si presenta quindi una nuova famiglia di stati coerenti per la teoria a grafo fissato.

Il punto di partenza di questa tesi è un'introduzione generale alla teoria, con particolare attenzione agli aspetti legati ai quanti di spazio. La LQG segue un approccio non perturbativo, dando valore alle lezioni chiave della GR: indipendenza da un background e covarianza generale. La struttura cinematica della LQG non è altro che una quantizzazione canonica della formulazione di Ashtekar della GR, nello spirito di Wheeler - de Witt e seguendo le idee di Dirac sulla quantizzazione di teorie vincolate. Il risultato è uno spazio di Hilbert in cui vivono le cosiddette reti di spin, le quali hanno un'affascinante interpretazione in termini di una geometria spaziale quantizzata e dunque discreta. Questa descrizione viene recuperata anche a partire da un processo di quantizzazione formale di idee puramente di tipo matematico e geometrico.

Gli stati coerenti giocano un ruolo essenziale nell'analisi semiclassica della teoria. La famiglia di stati coerenti usata in letteratura è data dai cosiddetti stati coerenti del nucleo del calore (heat-kernel). La ricerca degli ultimi anni ha messo in evidenza una parametrizzazione alternativa dello spazio delle fasi della LQG, in termini di variabili che descrivono uno spazio metrico discreto, come generalizzazione delle geometrie di Regge. Questa alternativa è data dalle cosiddette Twisted Geometries, e suggerisce la definizione di nuovi stati coerenti con diverse proprietà di piccatezza. Il lavoro svolto tramite la tesi ha permesso di costruire questa nuova classe di stati coerenti, studiare le loro proprietà, e confrontarli con quelli del heat-kernel.

Il risultato è un set di stati coerenti che permettono di interpretare la natura discreta dello spazio in termini di poliedri semiclassici. Questi stati godono delle opportune proprietà di piccatezza e overcompletezza, e fungono dunque da ponte tra una teoria classica discretizzata, come approssimazione della continua, e la teoria quantistica. A livello applicativo, questi stati possono dimostrarsi molto utili per corroborare o eventualmente alterare i risultati ottenuti nelle letteratura adoperando gli stati coerenti del nucleo del calore, unici utilizzati fino ad ora.

Contents

Introduction

Loop Quantum Gravity depicts the very concept of physical space itself as a quantum object. It predicts in fact a discrete and quantized notion of geometry which must be reconciled in some manner to the classical idea of continuous space. The aim of this thesis is to provide a tool in order to bridge the quantum space picture emerging from LQG to the classical continuous geometry interpretation, as brought forth from General Relativity. This tool is provided by an appropriate use of coherent states, which in this scenario are quantum states of discrete space, that are peaked on a discrete geometry. Therefore the question we address here is understanding how a seemingly smooth classical geometry can be recovered from the fundamental quantum states of LQG. This is a very important theme for the semiclassical analysis of the theory, and as the path that leads there is rather long, it is now described step by step since the very beginning.

It is clear since a long time that a theory of quantum gravity is needed, and yet missing. Profound modifications of existing theoretical structures will be mandatory in order to find an answer to the problem. The success of the known frameworks such as Quantum Field Theory (QFT) or General Relativity (GR) has an astounding degree of accuracy, but each of them describes its respective intended domain of physical phenomena. Of course, a question arises quite naturally concerning whether an overlapping domain of quantum gravitational phenomena exists and if so, how we could describe and observe it. It is argued that an interface of both frameworks is needed to provide a satisfying description of the microstructure of spacetime together with matter at the Planck scale. This is the natural scale where effects of quantum gravity are expected to occur and it is relevant for instance to understand the fundamental singularities of GR such as the nature of black holes or the Big Bang. Furthermore, a careful analysis of the underlying elementary and universal assumptions of these frameworks leads to the observation that they are mutually incompatible from a conceptual point of view. The need to overcome this confusion in fundamental physics has spurred research on quantum gravity for more than 80 years and has led to the development of a plethora of approaches each with their individual strengths and weaknesses. Nevertheless a complete and consistent solution has remained yet elusive, as all candidates suffer from formal and conceptual problems. One of the main paths that is worth to mention here is the so called covariant, or functional integral, approach. This is a standard quantum field theory and as such, it involves the need of a fixed geometrical background where small fluctuations are treated in a perturbative way. While there is a number of very interesting results obtained by this approach, there also is a fundamental difficulty with it. As for any quantum field theory, divergences appear when considering the effects of arbitrarily small (ultraviolet) fields fluctuations. These infinities are usually dealt with the procedure of renormalization which, while succeeding for gauge theories, fails for gravity. This can be expected already from dimensional arguments, and the rigorous proof that perturbative quantization of GR fails because of non-renormalizable UV divergences, was obtained in the late eighties [19]. This means that this effective theory can be used for low-energy calculations but it would be inconsistent if taken seriously at all energy scales. A setting where the quantized gravitational field is represented as a graviton field propagating over a flat background spacetime seems rather questionable in light of the fact that spacetime is dynamical in general relativity, being the gravitational field nothing but the geometry of spacetime itself. This represents a valid reason to expect that a successful theory of quantum gravity should be formulated as a background independent quantum field theory. That is, the theory should be expressed in a way in which no reference is made to any fixed, non-dynamical background spacetime. Nevertheless there is a consistent representative insisting on the perturbative approach which is String Theory: this path attempts to provide a description which unifies all fundamental interactions through more fundamental objects living on a higher-dimensional target space. Essentially, the idea is to increase the amount of symmetries as compared to GR and QFT with the aim to regain perturbative renormalizability. This is strongly inspired by the replacement of the perturbatively non-renormalizable Fermi model for the weak interaction, by the renormalizable electroweak theory. We will not deal nor mention the jungle of technicalities that accompanies the partial successes or the difficulties of this path. This thesis focuses on non-perturbative approaches, that value the role of background independence and general covariance as unveiled by GR, and keep them as guiding principles for the construction of a quantum theory. In this perspective, the spacetime continuum is abandoned and is instead replaced by degrees of freedom of discrete and combinatorial nature. As a matter of fact the key lessons of general relativity are the following. The world is relational: only events independent from coordinates are meaningful and physics must be described by generally covariant theories. The geometry of spacetime, namely the gravitational field, is fully dynamical: gravity defines the geometry on top of which its own degrees of freedom and those of matter and other fields propagate. GR is not a theory of fields living on a (possibly curved) background geometry, it is a theory of fields moving on top of each other, namely it is background independent. This, together with the assumptions of quantum mechanics, is the starting point of the canonical approach to the problem, known as Loop Quantum Gravity (LQG).

As briefly reviewed in chapter 1, the kinematical framework of LQG is essentially the outcome of a canonical quantization of the Ashtekar formulation of GR, in the spirit of Wheeler - de Witt, following the ideas laid out by Dirac on the quantization of generally covariant theories. Thus, the only basic inputs which go into the construction of the theory are quantum mechanics and general relativity, including in particular the idea that background independence is of fundamental importance. Loop quantum gravity does not require the introduction of any radically new physical assumptions – such as additional spacetime dimensions which would have to be compactified in an ad hoc manner, or the existence of supersymmetric particles which continue to evade the best efforts of the experimentalists to detect them – for the internal consistency of the theory. At the kinematical level, the structure of loop quantum gravity is well understood. The kinematical Hilbert space of the theory is spanned by the so-called spin network states, which have a compelling physical interpretation as states describing discrete, quantized spatial geometries. Thanks to this, loop quantum gravity is able to provide a concrete realization of the idea of the quantized gravitational field as a dynamical object, whose excitations are the elementary quanta out of which spacetime itself is built.

Chapter 2 contains the precise meaning of the notion of quantum space, emerging both from a pure mathematical or geometrical point of view, and from the theory of LQG. Remarkably enough, the two paths lead to the same result as one might expect from the physical interpretation of GR. The quantum geometry appearing in LQG naturally cuts UV divergences due to discrete spectra of geometric operators and background independence. The Hamiltonian (constraint) of the theory, including standard model matter, is formally a finite and well defined operator without the need of renormalization, even though its spectrum and the dynamics it gives are not known. Loop quantum gravity is thus a candidate for a rigorous definition of quantum field theory, but much work remains to be done. The first thing is to improve the control over the dynamics. The task of deriving any non-trivial solutions of the Hamiltonian constraint in explicit form has turned out to be extremely challenging. The difficulties encountered in working with the Hamiltonian constraint have motivated many researchers of loop quantum gravity to look for alternative ways of formulating the dynamics of the theory. Perhaps the most popular among these is the spin foam formalism, also often referred to as covariant loop quantum gravity, which completely abandons the canonical formulation of the dynamics. It introduces instead a particular implementation of the path integral for general relativity, which enables one to define the dynamics by associating transition amplitudes to spin network states. This is a very interesting development of the theory, naively describing the quantum structure and dynamics of spacetime, although this thesis will not focus on these aspects. Another open question in this approach is the recovery of classical spacetime, diffeomorphism invariance and GR as an effective description for the dynamics of the geometry in an appropriate limit. This thesis tries to go a step further in order to address the first problem. In other words, the fundamental quantum geometry present in loop quantum gravity has to be coarse grained in order to yield a smooth classical spacetime. To be precise, the notion of quantum geometry has three main peculiar features: the quantization of spectra, the distributional nature, and the non-commutativity. The first one is historically the reason of why a concept of quantum geometry emerged in the first place. We shall see that geometric operators have a quantized spectrum as opposed to their classical counterparts. This is in fact a standard situation in quantum mechanics and to recover the classical results one usually needs to take some continuum limit which will involve some quantum number to go to infinity. With distributional nature we mean that the states of quantum geometry only capture a finite number of components of the original field, that is their values along paths and surfaces. As we will describe in detail, this happens because the states are defined on a graph and the full theory will be recovered only when a sum over all possible graph is considered. In other words, one would need a graph to be infinitely refined. This is reminiscent of what happens in lattice theories, where the continuous field gets discretized on a fixed lattice and only a finite number of degrees of freedom are captured. The difference here however lies in the fact that the lattice is in fact the quantum description of space itself, and the lattice spacing cannot really be sent to zero, due to the discreteness of geometry mentioned above. Finally, the non commutativity describes the fact that there are geometric operators which do not commute among themselves. This is also standard in quantum mechanics, and this is where coherent states come into play. In spite of these differences with the classical notion of geometry, a correct theory must admit a semiclassical regime where a smooth geometry emerges. Coherent states might help in the following sense: since they are a linear superposition of spin network states peaked on the classical phase space, they can be the bridge to a smooth geometry. The Ashtekar formulation describes GR in terms of variables called connection and triad, therefore a coherent state will be peaked on a point in the phase space spanned by them, which in turn define an intrinsic and extrinsic geometry. A family of coherent states with these properties has been introduced bt Thiemann and collaborators. They do minimize some uncertainty relations but in order to recover a smooth geometry everywhere, the coherent states will have to have support over an infinite number of graphs. This last point turned out to be a formidable task, and for practical purposes one works with a fixed graph, with the idea that it could be enough to address specific physical questions. The Hilbert space of a single graph represents a truncation of the full theory, which may still be sufficient to capture the physics of appropriate regimes. This is similar in spirit to introducing an upper bound ih the usual Fock space, and the associated truncation of higher energy degrees of freedom. However the analogy cannot be pushed too far since a concept of energy is not well defined in GR. Once the truncation has been made, the coherent states will live in the Hilbert space of the fixed graph. The question is then about how one can assign a classical geometrical interpretation to these states. They will be peaked on points on the phase space consisting of classical quantities that capture a finite number of degrees of freedom of a continuum geometry. In general, they cannot determine a smooth 3-geometry completely. The problem is analogue to interpolate a continuous function if one is given a finite number of its values. Of course there exist in general several interpolation procedures and each of these will have specific advantages. In the present case, the one which will give an approximation of a continuous geometry is based on the idea of a discrete metric space. A construction of discrete geometries determined by classical variables gave rise to the so called Twisted Geometries, which are a generalization of Regge geometries.

As described in chapter 3, the idea is to reparametrize the classical phase space of GR in terms of a suitable set of variables, and to define a class of discrete metric spaces defined over a cellular decomposition dual to the graph. The building blocks of space itself will be considered flat and equipped with an orthonormal reference frame. They will have a clear geometrical interpretation as in fact they will turn out to describe polyhedra attached to each other. There is therefore a map between the twisted geometries variables and the usual Ashtekar GR variables, and the space of twisted geometries will turn out to be symplectomorphic, other that isomorphic, to the classical phase space of GR. From these quantities one can construct now a metric, but this will be in general discontinuous, as it is interpreted as a bunch of polyhedra glued together. The name 'twisted' is in fact justified by two reasons: first they have a nice mathematical connection with Twistors, and moreover they define a metric which is locally flat but discontinuous, since the polyhedra will not have in general shape matching faces, even though they will have the same area. It is very interesting to notice also that when one imposes these shapes to match among themselves, one recovers a Regge model for discrete gravity in three dimensions, generalized to generic polyhedra defined by edge lengths. With all these information, one can now go back to the concept of coherent states (CS). As mentioned, Thiemann's states are properly labelled by a point in the discrete classical phase space of LQG on a single graph and they also fulfil a number of important properties. Nevertheless there are reasons to search for alternatives: the distributions to study are in fact horrible functions of the holonomy and the flux, and furthermore the expectation values are well peaked on the norm of the fluxes but they do not single out nicely their direction. Moreover Thiemann's states are heavily used in phenomenological applications and one might ask to what extent the results obtained in the literature depend on the choice of coherent states. On the other hand, the spin foam formalism suggests a different approach, where the phase space is described by quantities referring to discrete geometries. That is why the twisted geometries are the key to the construction of our new coherent states for LQG on a fixed graph. These will help to single out the directions more clearly and are indeed useful for a discrete geometrical interpretation.

To be able to construct and study the new family, some preliminary work is necessary. In chapter 4 some standard coherent states are revised and studied explicitly. These includes two well known examples motivated by the twisted geometries - the sphere and the particle on the circle - and also a new original class of coherent states for the harmonic oscillator, motivated by analogy with other structures of the LQG case. These states were a turning point for the final construction of the so called Twisted Geometries Coherent States. In fact this preliminary work has allowed us to understand the proper framework and thus, starting from there and combining the right ingredients, we have finally defined a new family of coherent states for LQG on a fixed graph, based on the twisted geometries parametrization.

In chapter 5, their definition and properties are presented. In particular we prove that they are an overcomplete basis on the correct space and we show their peakedness properties. We compare them to the known coherent states and highlight the origin of the improved peakedness in the direction. Then we discuss the gauge invariance. This is one of the main advantages of the twisted geometry parametrization, namely the locality at the level of the nodes. In particular, the new states naturally incorporate the so called coherent intertwiners, which have proven to be very useful in defining the dynamics of spin foam models and studying its semiclassical limit; this was in fact another motivation to introduce this new set of states. Coherent intertwiners are indeed nothing but the states of semiclassical polyhedra. Now due to the truncation mentioned above, coherent states on a fixed graph are not peaked on a smooth and continuous classical geometry. The twisted geometries offer a way to see them as peaked on a discrete geometry, to be viewed as an approximation of a smooth geometry on a cellular decomposition dual to the graph. The above results provide a compelling picture of these geometries in terms of polyhedra, and thus of coherent states intriguingly describing the structure of space itself as a collection of semiclassical fuzzy polyhedra.

We wish to point out that the new family of CS introduced here is in a sense weaker that the traditional ones. It is true that they resolve the identity and that they are peaked on the classical values, with relative uncertainties vanishing in the large spin limit. However, they are not eigenstates of a destruction operator, at least not one that we were able to identify. As a consequence, they do not define as they stand a holomorphic representation, nor it is know if and which Heisenberg relation they saturate. We believe they have enough interesting properties to make them useful in spite of these drawbacks, like their peakedness and vanishing of relative uncertainties in the large scale limit. Moreover, since the Hamiltonian constraint in the LQG dynamics is very hard to treat, recent works have been put forward concerning the use of coherent states in simplified models. In these mini-superspace models and effective dynamics theories, one usually needs to compute expectation values of the Hamiltonian. Up until now, all these computations have been made with Thiemann's CS, and since these result can in fact also give rise to interesting physical interpretations, we believe that a new set of coherent states is in a sense useful to check whether the same results would be confirmed or changed.

Chapter 1

The path to Loop Quantum Gravity

This chapter introduces the basic concepts needed to describe the theory of Loop Quantum Gravity [39]. Starting from General Relativity and its Hamiltonian formulation, a path is described following the major steps taken in the last decades, in order to build the theory of loops [15]. We will not review the covariant approach to quantum gravity which in short fails when it comes to renormalization procedures, due to the inadequacy of perturbative quantization for General Relativity. We will therefore follow the canonical approach and focus on the structuring of the theory from the beginning.

1.1 Canonical formulation of General Relativity

In this section the Hamiltonian formulation of General Relativity (GR) is introduced. This formalism is crucial to understand the symmetries of the classical theory and therefore, to build the quantum theory which respects those classical symmetries. We will use the standard ADM foliation of spacetime and we will see that the canonical analysis underlines the role played by the diffeomorphism symmetry of the theory, i.e. the background independence of GR.

1.1.1 Review of metric GR

Before proceeding to the canonical analysis, let us briefly recall the metric formulation. General Relativity asserts that spacetime is modelled as a four dimensional differentiable manifold $\mathcal{M}$, equipped with a metric tensor $g_{\mu\nu}$ with a Lorentzian signature (−,+,+,+) and a rule of parallel transport ∇ or equivalently, a spacetime connection. This rule of parallel transport is taken to be the unique torsionless and metric - compatible connection, which is the so called Levi - Civita connection. With these assumptions, the gravitational field is described by the Einstein-Hilbert action[1]

$$S_{EH}(g_{\mu\nu}) = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\,R \tag{1.1}$$

where G is Newton's constant and we have set $c = 1$ (see Appendix B for a comment on this). The dynamical field is the four dimensional metric tensor itself and the Ricci scalar R is built

[1]We will set the cosmological constant $\Lambda = 0$ in the whole analysis, without loss of generality.

contracting the Riemann tensor. It is in fact the only scalar leading to equation involving at most second order derivatives of the metric. Using the Levi Civita connection, the definitions are

$$R = g^{\mu\nu} R_{\mu\nu}(\Gamma(g)) \qquad\qquad R^{\sigma}{}_{\rho\mu\nu} v^{\rho} = [\nabla_\mu, \nabla_\nu]\, v^{\sigma} \tag{1.2}$$

for any vector field v, where the connection enters the covariant derivative as $\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^{\nu}_{\rho\mu}(g) v^\rho$. The Riemann tensor describes the curvature of the manifold. It satisfies geometrical identities which arise because of the invariance under coordinates transformations. Those are called Bianchi identities and using the Levi Civita connection they are simply

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \qquad\qquad \nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0 \tag{1.3}$$

As a matter of fact the Riemann tensor in terms of the Levi Civita connection is given by

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\rho\nu} - \partial_\nu \Gamma^{\sigma}_{\rho\mu} + \Gamma^{\sigma}_{\lambda\mu}\Gamma^{\lambda}_{\rho\nu} - \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\lambda}_{\rho\mu} \tag{1.4}$$

where the Levi Civita connection is completely determined by the first derivatives of the metric

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \tag{1.5}$$

and as a result of the torsionless requirement, Γ is symmetric in the last two indices. Putting some matter lagrangian $\mathcal{L}_M$ in the game, the Einstein Hilbert action becomes

$$S_{EH} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R + \mathcal{L}_M \right) \tag{1.6}$$

and from there one gets the Einstein's fields equation which describe the dynamic of the gravitational field coupled to matter

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1.7}$$

where $T_{\mu\nu}$ is the source of gravity, the energy momentum tensor which is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} \tag{1.8}$$

The contracted second Bianchi identity leads to the "conservation" law of the Einstein tensor $\nabla_\mu G^{\mu\nu} = 0$. However the Bianchi identities are purely geometrical constraints and do not refer to the dynamics. This means that one never uses the field equation to obtain them. The symmetry behind them is the invariance under coordinate transformation which is a non dynamical symmetry. That is why the covariant "conservation" of the Einstein tensor is not a true conservation law.

The field equation imply another conservation, namely the one of the energy momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ which on the contrary is obtained only when the dynamics is taken into account. So there is a dynamical symmetry behind it which is nothing else that the invariance of the theory under diffeomorphisms. This fact is usually called background independence.

The metric field which dictates the geometry and the causality of the spacetime has now the very same status as any field in physics. Within this new framework, the classical fields do not propagate in space through time, i.e. in a given spacetime, but they simply interact with another dynamical field, namely the metric tensor. Due to the gauge invariance of the theory under diffeomorphisms, one cannot speak about the value of a field at the point A in spacetime since this field can always be pushed forward to another point B and relabelled by a diffeomorphism which does not modify the physical content of the theory (see the hole argument originally proposed by Einstein). From this observation, one is led to a very unusual picture of reality, where spacetime as a fixed arena on top of which the other field live disappears, to let only the dynamical gravitational field interact with the other fundamental fields. There is no more a fixed background on which one can do physics. The physical reality is truly background independent and localization in presence of gravity is purely relational. For more on this "disappearance" of spacetime see [36]. This status of the diffeomorphisms in General Relativity becomes crystal clear in the Hamiltonian formulation of the theory, which was worked out in the sixties by Arnowitt, Deser and Misner (ADM) [4]. The canonical analysis of General Relativity unravels the true dynamical variables of the theory and the gauge symmetry under which the theory is invariant.

1.1.2 The ADM formalism

In order to put the action (1.1) into canonical form we need to identify the canonically conjugated variables, and then perform the Legendre transform. The idea is to make a $3+1$ decomposition, selecting a foliation of the four dimensional spacetime into a family of space-like Cauchy hypersurfaces Σ. Therefore we assume the topology $\mathcal{M} \simeq \mathbb{R} \times \Sigma$, where Σ is a three dimensional manifold with space-like signature. This assumption does not pose any restriction if $\mathcal{M}$ is globally hyperbolic. As a matter of fact, at each point of any space-like Σ there is a unit vector $n \in \mathcal{T}_p\mathcal{M}$ which is time-like and normal to Σ. So the vector n connects two events in $\mathcal{M}$ for which the time interval is greater than the space interval, i.e. $g_{\mu\nu}n^\mu n^\nu = -1$. It is known that if spacetime is globally hyperbolic than it must necessarily be diffeomorphic to the topology given by the decomposition above.

So $\mathcal{M}$ foliates into a one-parameter family of hypersurfaces $\Sigma_t = X_t(\Sigma)$. Thanks to this, one can identify the coordinate $t \in \mathbb{R}$ as a time parameter. However this time should not be thought as an absolute quantity, because of the diffeomorphism invariance of the theory. We can always work with a chosen foliation but the invariance will guarantee that physical quantities are independent of the choice. Given some local ADM coordinate adapted to the foliation, the time evolution vector field (or time flow) between two Cauchy hypersurfaces Σ_t and Σ_{t+dt} is defined by

$$\tau^\mu \equiv \frac{\partial x^\mu}{\partial t} \tag{1.9}$$

This should not be confused with the unit normal vector n^μ defined above, because they are both time-like, $g_{\mu\nu}\tau^\mu\tau^\nu = g_{00}$ and $g_{\mu\nu}n^\mu n^\nu = -1$, but not parallel in general. The time flow can be decomposed in its normal and tangential parts with respect to Σ

$$\tau^\mu(x) = N(x)n^\mu + N^\mu(x) \tag{1.10}$$

and if we parametrize the normal as $n^\mu = (1/N, -N^a/N)$, we have $N^\mu = (0, N^a)$ where latin indices are spatial, i.e. $a = 1, 2, 3$. N is called lapse function and N^a shift vector. Their geometrical meaning is the following: the lapse function gives the proper distance Ndt in the normal direction between the two hypersurfaces Σ_t and Σ_{t+dt}, whereas the shift vector gives the tangential deformation $N^a dt$ that is applied to the points of Σ_t if the embedding is changed from Σ_t to Σ_{t+dt}, with constant x.

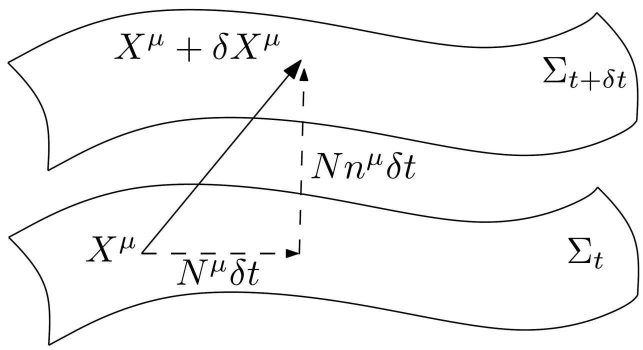


Figure 1.1: Spacetime foliation

It is easy to see that

$$g_{\mu\nu}\tau^\mu\tau^\nu = g_{00} = -N^2 + g_{ab}N^aN^b \qquad g_{0a} = g_{ab}N^b \equiv N_a \tag{1.11}$$

so that the ADM metric tensor reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (-N^2 + N_aN^a)dt^2 + 2N_adx^adt + g_{ab}dx^adx^b \tag{1.12}$$

where a are spatial indices and are contracted with the three dimesnional metric g_{ab}. Notice that this is *not* in general the intrinsic metric on Σ_t which is instead given by

$$q_{\mu\nu} = g_{\mu\nu} + \ n_\mu n_\nu \tag{1.13}$$

and is in fact the first fundamental form of Σ. It is the spatial metric in the sense that $q_{\mu\nu}n^\mu = 0$ thus tensors on Σ can equivalently be contracted with g or q. So now we can define tensorial calculus on the spatial slice starting from the one on $\mathcal{M}$ by projecting with q^μ_ν. One can in fact define the extrinsic curvature

$$K_{\mu\nu} \equiv q^\rho_\mu q^\sigma_\nu \nabla_\rho n_\sigma \tag{1.14}$$

which is nothing but the second fundamental form of Σ. This is also spatial since $K_{\mu\nu}n^\mu = 0$, it is symmetric and can be written as a Lie derivative of the intrinsic metric

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n q_{\mu\nu} \tag{1.15}$$

The extrinsic curvature of Σ is needed to express the usual four dimensional Ricci scalar R in terms of the three dimensional one $\mathcal{R}$ on the spatial submanifold plus additional terms. So in order to decompose the Riemann tensor it is convenient to construct a differential D_μ which is compatible with the metric $q_{\mu\nu}$[2]

$$D_\mu f \equiv q_\mu^\nu \nabla_\nu \tilde{f} \qquad\qquad D_\mu v_\nu \equiv q_\mu^\rho q_\nu^\sigma \nabla_\rho \tilde{v}_\sigma \tag{1.16}$$

so that it is the common covariant derivative which is spatially projected after its application. Here the function f and the vector v are defined on the spatial slice whereas $\tilde{f}$ and $\tilde{v}$ are arbitrarily smooth continuation of the formers in a neighbourhood of Σ in $\mathcal{M}$.

With the above definitions, one can express the Riemann curvature tensor of Σ_t associated with D_μ in terms of the usual four dimensional Riemann tensor of $\mathcal{M}$ associated with ∇_μ

$$\mathcal{R}^\mu{}_{\nu\rho\sigma} = q_\alpha^\mu q_\nu^\beta q_\rho^\gamma q_\sigma^\delta \, R^\alpha{}_{\beta\gamma\delta} + K_{\nu\sigma} K_\rho^\mu - K_{\nu\rho} K_\sigma^\mu \tag{1.17}$$

Due to the simetries of the Riemann tensor one can finally get the so called Gauss-Codazzi equation between the Ricci scalars

$$R = \mathcal{R} + K_{\mu\nu} K^{\mu\nu} - (K_\mu^\mu)^2 - 2\nabla_\mu (n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu) \tag{1.18}$$

where the last term is a total derivative and it will not affect the action. Plugging this into (1.1) one can finally write the so called ADM action

$$S_{ADM} = \frac{1}{16\pi G} \int_{\mathbb{R}} dt \int_\Sigma d^3x \sqrt{q} N \left[\mathcal{R} + K_{ab} K^{ab} - (K_a^a)^2 \right] \tag{1.19}$$

where $q = \det(q)$ and K_a^a is the trace of the extrinsic curvature. Notice that the time derivatives of the three dimensional metric only appears in terms involving the extrinsic curvature. The key point to notice is that the action (1.19) depends on N and N^a but not their time derivatives. This means that when performing the Legendre transform to go over a Hamiltonian formulation, the momentum maps will be non-invertible, being the Lagrange density singular[3].

The canonical conjugated momenta are

$$\pi \equiv \frac{\delta \mathcal{L}}{\delta \dot{N}} = 0 \qquad \pi_a \equiv \frac{\delta \mathcal{L}}{\delta \dot{N}^a} = 0 \qquad \pi^{ab} = \frac{\delta \mathcal{L}}{\delta \dot{q}_{ab}} = \frac{\sqrt{q}}{16\pi G} (K^{ab} - q^{ab} K_c^c) \tag{1.20}$$

so the true dynamical variables are the spatial components of the metric only, and the lapse and the shift must be considered as Lagrange multipliers. This is the case of a constrained Hamiltonian system. Following Dirac algorithm for such systems, we get the primary constraints

$$\mathcal{C} \equiv \pi = 0 \qquad\qquad \mathcal{C}^a \equiv \pi^a = 0 \tag{1.21}$$

called scalar and vector constrain respectively. So now one has to perform the Legendre transform for the remaining variables and neglecting boundary terms this yields

$$S = \frac{1}{16\pi G} \int \int d^3x \left[\pi^{ab} \dot{q}_{ab} - N^a H_a - NH \right] \tag{1.22}$$

[2]The differential compatible with $g_{\mu\nu}$ is denoted ∇_μ.

[3]This singularity is in fact generated by the diffeomorphism invariance of the theory.

where

$$H_a = -2q_{ac}\, D_b \pi^{bc} \tag{1.23}$$

is the (spatial) diffeomorphism constraint and

$$H = \frac{1}{\sqrt{q}} \left[q_{ac} q_{bd} - \frac{1}{2} q_{ab} q_{cd} \right] \pi^{ab} \pi^{cd} - \mathcal{R}\sqrt{q} \tag{1.24}$$

is the Hamiltonian constraint. Note that the above are nothing but the Hamilton-Jacobi equations. Requiring consistency of the primary constraints with the equation of motion, one has to vary the action with respect to the Lagrange multipliers to get

$$H_a(q, \pi) = 0 \qquad\qquad H(q, \pi) = 0 \tag{1.25}$$

and one can check that there are no further constraints. Physical configurations must satisfy (1.25) which encode the four Einstein equation $G_{0\mu} = 0$ whereas the others $G_{ab} = 0$ are encoded in the equations of motion of the spatial metric. From the action (1.22) one can easily see that the ADM Hamiltonian for General Relativity is

$$\mathbf{H}_{GR} = \frac{1}{16\pi G} \int d^3x \, [N^a H_a + NH] \tag{1.26}$$

and it clearly vanishes on shell! It is in fact proportional to the Lagrange multipliers and thus there is no true dynamics and no physical evolution in t. General Relativity is an example of a constrained Hamiltonian system, and this intriguing absence of a physical Hamiltonian is a consequence of the above mentioned diffeomorphism invariance. This means that t is a mere parameter devoid of absolute physical meaning. This is also known as *the problem of time* in GR [36].

The symplectic structure is given by definition by the canonical Poisson brackets among the coordinates of the Phase Space of General Relativity

$$\{\pi^{ab}(t, x), q_{cd}(t, x')\} = 16\pi G\, \delta^a_{(c}\, \delta^b_{d)} \delta^3(x - x') \tag{1.27}$$

the other brackets being null. From this relation one can evaluate the brackets among the constraints to find the algebra (we know choose units in which $16\pi G = 1$)

$$\begin{aligned} \{H_a(x), H_b(x')\} &= H_a(x')\partial_b \delta^3(x - x') - H_b(x)\partial'_a \delta^3(x - x') \\ \{H_a(x), H(x')\} &= H(x)\partial_a \delta^3(x - x') \\ \{H(x), H(x')\} &= H^a(x')\partial_a \delta^3(x - x') - H^a(x)\partial'_a \delta^3(x - x') \end{aligned} \tag{1.28}$$

The right hand sides of the above brackets vanish on the constraint surface (1.25) as expected from first class constraints. This means that the Poisson flow generated by the constraints preserve the constraint surface (1.25). As it is known, first class constraint generate gauge transformation on the surface. To see the explicit gauge transformation in the present case, one has to define the smearing of the constraints

$$H(\vec{N}) = \int_\Sigma d^3x H^a(x) N_a(x) \qquad\qquad H(N) = \int_\Sigma d^3x H(x) N(x) \tag{1.29}$$

and compute

$$\{H(\vec{N}), q_{ab}\} = \mathcal{L}_{\vec{N}} q_{ab} \qquad \{H(\vec{N}), \pi^{ab}\} = \mathcal{L}_{\vec{N}} \pi^{ab} \tag{1.30}$$

These show that the vector constraint is the generator of space-diffeomorphism on Σ. And then

$$\{H(N), q_{ab}\} = \mathcal{L}_{\vec{n}N} q_{ab} \qquad \{H(N), \pi^{ab}\} = \mathcal{L}_{\vec{n}N} \pi^{ab} + \frac{1}{2} q^{ab} N H - 2N\sqrt{q}\, q^{c[a}\, q^{b]d} R_{cd} \tag{1.31}$$

The first bracket is the action of the time or Hamiltonian diffeomorphism on q_{ab}. The second one is subtle: it gives the same on π_{ab} thanks to the first term but has two additional pieces. These extra terms vanish on the constraint surface ($H = 0$) and for physical solution ($R_{\mu\nu} = 0$). Therefore we can say that the constraints are the generators of spacetime diffeomorphism group *on physical configurations*. Therefore this invariance is a dynamical symmetry, i.e. it constraints the equations of motion.

In general, not referring only to physical trajectories of the phase space, equations (1.28) define the algebra of hypersurfaces deformation. This in particular is not a Lie algebra in fact the smeared Poisson brackets are

$$\{H(N_1), H(N_2)\} = H(q^{ab}(N_1 \partial_b N_2 - N_2 \partial_b N_1)) \tag{1.32}$$

The "structure constants" outside the constraint surface contain the field q_{ab} itself, so are not constant at all. What this all means is that, given a foliation, we have the symmetry of diffeomorphism as discussed above, which acts by changing the foliation. Moreover we have another symmetry, the last one pointed out, which act deforming the foliation. These two symmetries coincide on physical configurations. To summarize, when we consider on shell Poisson brackets, we discover that the constraints H_a and H are the generators of infinitesimal diffeomorphisms along spatial direction in Σ and along the normal vector n, respectively.

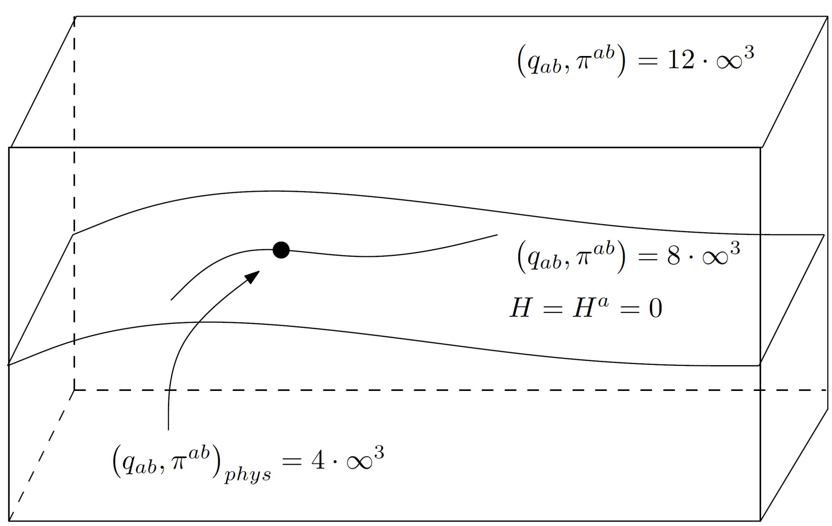


Figure 1.2: Counting dimensions and degrees of freedom

One last comment about degrees of freedom is now in order. In general, the number of degrees of freedom is defined to be half the dimension of the phase space. In constrained system, such as GR or gauge theories, one has to be careful. Let us distinguish between the physical phase space and the kinematical phase space. The latter one is defined by the symplectic structure of the theory so it has points (q_{ab}, π^{ab}) with Poisson brackets (1.27). Excluding the ∞^3 degeneracy given by space dependence, it has dimension $6 + 6 = 12$. On this space we have constraints that define a hypersurface, i.e. the space of (q_{ab}, π^{ab}) such that (1.25) holds. This is the constraint surface and it has dimension $12 - 4 = 8$. Being a first class constraint, we are guaranteed that gauge transformations generated by the constraints preserve the hypersurface. The trajectories obtained by gauge transformations are called orbits. Points along a orbit correspond to the same physical configuration therefore one must divide by the gauge orbits in order to obtain physical degrees of freedom. This is what happens in gauge theories. The orbits span a 4 dimensional manifold at each spatial point so we are left with $8 - 4 = 4$ dimensions. This is the physical phase space. It has dimension 4 per space point, so that the theory has 2 degrees of freedom which have physical meaning. This result is consistent with the linearised analysis done in the covariant perturbative approach, where the degrees of freedom correspond to the two helicities of a massless spin 2 particle. What are these degrees of freedom in the ADM formulation of GR?

To address that question one would have to control the general solution of the theory. This task has never been solved due to the highly non-linearity of the equations, in spite of many attempts.

Having a canonical Hamiltonian formulation of GR, one could build a quantum theory from the classical phase space. But following the Dirac quantization procedure, one encounters the Hamiltonian constraint (which is a gauge transformation and generates the whole GR dynamics) that is highly non-trivial. The fact that the constraints contain the whole dynamics is heavily exploited by the quantization program proposed by Dirac, which is based on the definition of dynamical physical states as the ones annihilated by the quantum constraints. One has first to build a representation of the quantum algebra generated by the canonical variable promoted to operators $\left(\hat{\pi}^{ab}, \hat{q}_{ab}\right)$ in an auxiliary "kinematical" Hilbert space $\mathcal{H}_{kin}$, where as usual $\{\cdot, \cdot\} \to (i\hbar)^{-1} [\cdot, \cdot]$. One has also to promote the constraints to the operators $\hat{H}^\mu \in \mathcal{H}_{kin}$. Now one imposes the first class constraints on the quantum states $\hat{H}^\mu \psi = 0 \quad \forall \psi \in \mathcal{H}_{phys}$ which extracts the physical quantum states from the whole set of states, that live now in $\mathcal{H}_{phys}$. These states respect the symmetries of the theory. This is all hypothetical: one would need to know explicitly the scalar product on $\mathcal{H}_{phys}$ to complete these steps and have a physical interpretation of the quantum observables. If we apply the above to the ADM formulation of GR, we look for a functional space carrying a representation of the quantum algebra

$$\left[\hat{q}_{ab}(x), \hat{\pi}^{cd}(x')\right] = i\hbar \delta^{cd}_{(ab)} \delta^3(x, x') \qquad \left[\hat{q}_{ab}(x), \hat{q}^{cd}(x')\right] = 0 \qquad \left[\hat{\pi}_{ab}(x), \hat{\pi}^{cd}(x')\right] = 0 \quad (1.33)$$

If we follow by analogies the known cases, for instance the scalar field theory, and consider a

Schrödinger representation

$$\hat{q}_{ab}(x) = q_{ab}(x) \qquad\qquad \hat{\pi}^{ab}(x) = -i\hbar\frac{\delta}{\delta q_{ab}(x)} \tag{1.34}$$

acting on wave functionals $\psi[q_{ab}(x)]$ of the three-metric, we encounter some difficulties. For instance one would need the scalar product for the Hilbert space which formally reads

$$\int dg\,\overline{\psi[g]}\psi'[g] \equiv \langle\psi|\psi'\rangle \tag{1.35}$$

but there is no Lebesgue measure on the space of metrics modulo diffeomorphisms that can be used to define dg. Without it one cannot even check whether $\hat{q}_{ab}$ and $\hat{\pi}^{ab}$ are hermitian, nor that $\hat{q}_{ab}$ has a positive definite spectrum as needed to be a space-like metric. But let's ignore these issues and try to proceed formally assuming that everything is well defined. One therefore comes to the constraints (1.25) transformed into operators. Their action characterizes the space of solutions

$$\mathcal{H}_{kin} \quad \xrightarrow{\hat{H}^a=0} \quad \mathcal{H}_{Diff} \quad \xrightarrow{\hat{H}=0} \quad \mathcal{H}_{phys} \tag{1.36}$$

The first constraint behaves very nicely. In the Schrödinger representation, the smeared version gives, after integration by parts

$$\hat{H}(N_a)\psi[q_{ab}] = 2i\hbar\int_\Sigma d^3x D_b N_a\frac{\delta\psi}{\delta q_{ab}} = 0 \tag{1.37}$$

and that implies

$$\psi[q_{ab} + 2D_{(a}N_{b)}] \equiv \psi[q_{ab}] \tag{1.38}$$

which means that the solution of the vector constraint are the functionals which are diffeomorphism invariant. This is exactly the correct action one expects at the quantum level. However, the space $\mathcal{H}_{Diff}$ is again ill defined since it inherits from $\mathcal{H}_{kin}$ the lack of a measure theory. Proceeding we have the Hamiltonian constraints which yields

$$\hat{H}\psi[q_{ab}] = \left[-\frac{\hbar^2}{2}G_{abcd} : \frac{1}{\sqrt{\det\hat{q}}}\frac{\delta^2}{\delta q_{ab}(x)\delta q_{cd}(x)} : -\sqrt{\det\hat{q}}R(\hat{q})\right]\psi[q_{ab}] \tag{1.39}$$

where $G_{abcd} = q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}$ is called supermetric, and the colon : means that the ordering of the operators needs a prescription. This is the so called Wheeler-DeWitt equation and it is more subtle than anything above. It requires the definition of products of operators at the same point, notoriously very ill defined objects. Even if one managed to give a suitable ordering prescription and regularize the differential operator, the problem with the equation is that we do not have any characterization of the solutions, not even formally as for the diffeomorphism constraint above. And of course, one again would have no clue on the knowledge of the physical Hilbert space and scalar product. In conclusion, given the highly non triviality of the scalar constraint, no one has never succeeded to build a quantum theory from the ADM phase space. It is therefore natural to look for another formulation of GR in order to have a proper classical theory from which we can launch the quantization program. One way to obtain such formulation is to provide the mathematical tools to reformulate General Relativity in terms of connection (and vielbein or tetrads) just as in Yang Mills theories. This is the key to Loop Quantum Gravity: instead of changing the gravitational theory or the quantization paradigm, we just use different variables to describe gravity.

1.2 Tetrads formulation

The standard GR formulation is described in the metric notation. This formalism, which is the one introduced by Einstein himself, is however incomplete. The whole theory can in fact be formulated in terms of the tetrad notation, which gives a clearer and more accurate understanding of the physical gravitational interaction. The latter is in fact a a 1-form $e^I(x) = e^I_\mu(x)dx^\mu$ with values in the Minkowski space. Geometrically speaking, this idea trades the manifold $\mathcal{M}$ with a new structure which is a G-principal bundle (just like for Yang-Mills theories) where G is the Lorentz group $SO(3,1)$.

A tetrad (or vielbein) is a quadruple of 1-forms $e^I_\mu(x)$ which is related to the metric as

$$g_{\mu\nu}(x) = e^I_\mu(x)e^J_\nu(x)\eta_{IJ} \tag{1.40}$$

where the indices $I, J = 0, 1, 2, 3$ are raised and lowered by the Minkowski metric η_{IJ}.

By definition, the tetrad provides a local isomorphism between a general reference frame and an inertial one, characterized by the flat metric η_{IJ}. A local inertial frame is defined up to Lorentz transformations, and in fact one can write

$$e^I_\mu(x) \to \tilde{e}^I_\mu(x) = \Lambda^I_J(x)e^J_\mu \tag{1.41}$$

Therefore the *internal* index I carries a representation of the Lorentz group. The isomorphism mentioned above is between the tangent bundle $T(\mathcal{M})$ and a Lorentz principal bundle. Geometrically speaking, on this bundle there is a connection ω^{IJ}_μ that is a 1-form with values in the Lorentz algebra. This is actually also a quite natural object from the gauge theories perspective: one asks there for a covariant derivative under the local Lorentz transformation seen above. As usual, the construction of this covariant derivative requires the introduction of a gauge field which is the connection ω so that the covariant differentiation of the fibre is

$$D_\mu v^I(x) = \partial_\mu v^I(x) + \omega^I_{\mu J}(x)v^J(x) \tag{1.42}$$

where $v^I = v^I_\mu dx^\mu \equiv e^I_\mu v^\mu$. The derivative (1.42) is the analogue of the covariant derivative $\nabla_\mu = \partial_\mu + \Gamma_\mu$ for vectors in $T(\mathcal{M})$. It is immediate to see that ω is antisymmetric, imposing the invariance of the Minkowski metric[4]. One can also define the derivative for objects with both indices, such as a tetrad

$$\mathcal{D}_\mu e^I_\nu = \partial_\mu e^I_\nu + \omega^I_{\mu J}e^J_\nu - \Gamma^\rho_{\nu\mu}e^I_\rho \tag{1.43}$$

[4]Namely $D_\mu\eta_{IJ} = 0$ which implies $\omega_{IJ} = -\omega_{JI}$ since $\nabla_\mu\eta_{IJ} = 0$.

The Levi-Civita connection (1.5) is metric-compatible, i.e. $\nabla_\mu g_{\nu\rho} = 0$, and similarly one requires the connection ω_μ to be tetrad-compatible, i.e. $\mathcal{D}_\mu e^I_\nu = 0$. We call such a connection spin connection. If we split indices into their symmetric and antisymmetric parts, the compatibility implies

$$\partial_{(\mu} e^I_{\nu)} + \omega^I_{(\mu J} e^J_{\nu)} = \Gamma^\rho_{(\mu\nu)} e^I_\rho \qquad \partial_{[\mu} e^I_{\nu]} + \omega^I_{[\mu J} e^J_{\nu]} = \Gamma^\rho_{[\mu\nu]} e^I_\rho \equiv 0 \tag{1.44}$$

Therefore we can obtain a relation between the spin and Levi-Civita connections

$$\omega^I_{\mu J} = e^I_\nu \nabla_\mu e^\nu_J \tag{1.45}$$

and

$$De^I = de^I + \omega^I{}_J \wedge e^J = (\partial_\mu e^I_\nu + \omega^I_{\mu J} e^J_\nu)\, dx^\mu \wedge dx^\nu = 0 \tag{1.46}$$

where d is the exterior derivative and D the covariant exterior derivative. As a matter of fact the above is the definition of the torsion 2-form

$$T^I = De^I \tag{1.47}$$

and a tetrad field determines uniquely a torsion-free spin connection imposing $T^I = 0$. The explicit solution of this equation is in fact

$$\omega[e]^I_{\mu J} = e^\nu_J (\partial_\mu e^I_\nu - \Gamma^\rho_{\mu\nu} e^I_\rho) \tag{1.48}$$

in accordance with (1.45).

Given a connection, one can define its curvature as

$$R^{IJ} = d\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ} \tag{1.49}$$

whose components are

$$R^{IJ}_{\mu\nu} = \partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + \omega^I_{\mu K} \omega^{KJ}_\nu - \omega^J_{\mu K} \omega^{KI}_\nu \tag{1.50}$$

Now, using (1.45) in the definition of the curvature, one can explicitly compute

$$R^{IJ}_{\mu\nu}(\omega(e)) = e^{I\rho} e^{J\sigma} R_{\mu\nu\rho\sigma}(e) \tag{1.51}$$

where $R_{\mu\nu\rho\sigma}(e)$ is the familiar Riemann tensor built out of the tetrad (from which one can define the metric tensor). This relations show that GR is a gauge theory whose local gauge group is the Lorentz group and the Riemann tensor is nothing but the field strength (which is just another name for the curvature form, just as in the electromagnetic case where it is called $F_{\mu\nu}$) of the spin connection. The definition of the Torsion and Curvature forms are in fact simply the first and second Cartan structure equations and in general the first and second Bianchi identities take the form

$$DT^I = R^{IJ} \wedge e_J \qquad\qquad DR^{IJ} = 0 \tag{1.52}$$

These definition are the general expression of the identities[5] (1.3) and the second one is valid more loosely for any connection in a principal bundle. It does not restrict the class of connections but implies that taking successive derivative of the curvature R^{IJ} does not generate new independent tensors.

Now the goal is to rewrite the Einstein Hilbert action (1.1) in terms of the tetrad, and setting $16\pi G = 1$ it is easy to see that

$$\begin{aligned} S_{EH}(g_{\mu\nu}(e)) &= \int d^4x\sqrt{-g}\, g^{\mu\nu}R_{\mu\nu} = \int d^4x\, e\, e^\mu_I e^{\nu I} R_{\mu\rho\nu\sigma} e^\rho_J e^{\sigma J} = \int d^4x\, e\, e^\mu_I e^\rho_J R^{IJ}_{\mu\rho}(\omega(e)) \\ &= \frac{1}{4}\int d^4x\, \epsilon_{IJKL}\epsilon^{\mu\rho\alpha\beta}\, e^K_\alpha e^L_\beta R^{IJ}_{\mu\rho}(\omega(e)) = \frac{1}{2}\int \epsilon_{IJKL}\, e^I \wedge e^J \wedge R^{KL}\omega(e) \end{aligned} \tag{1.53}$$

where we used the property $e\, e^\mu_I e^\nu_J = \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho e^L_\sigma$ and the fact that $g = \det g = -(\det e)^2 = -e^2$. This formulation not only has the invariance under diffeomorphisms, it also possesses an additional gauge symmetry under local Lorentz transformations. The key point now is to view the connection as an independent variable, and consider therefore the action

$$S(e^I_\mu, \omega^{IJ}_\mu) = \frac{1}{2}\epsilon_{IJKL}\int e^I \wedge e^J \wedge R^{KL}(\omega) \tag{1.54}$$

This is sometimes called Einstein-Palatini action. Remarkably enough, even if it depend on extra fields, this action gives the same equations of motion as the Einstein-Hilbert action. In fact, the extra field equation coming from varying the action with respect to ω simply imposes the form of the spin connection (i.e. tetrad compatible), and therefore GR is recovered (a completely analogous thing happens with g and Γ). Explicitly, since $\delta_\omega R^{KL}(\omega) = D\delta\omega^{KL}$ one has

$$\delta_\omega S = \frac{1}{2}\epsilon_{IJKL}\int e^I \wedge e^J \wedge D\delta\omega^{KL} = -\frac{1}{2}\int D\left(e^I \wedge e^J\right) \wedge \delta\omega^{KL} \tag{1.55}$$

and imposing the vanishing of the variation, one obtains the field equation

$$\epsilon_{IJKL} e^I \wedge De^J = 0 \tag{1.56}$$

If we define the inverse of the tetrad $e^\mu_I(x)$ and we assume it exists, then the above implies

$$De^J = 0 \tag{1.57}$$

which as a solution implies the form (1.45) of ω in terms of the Levi-Civita connection of the metric associated to $e^I_\mu(x)$. Therefore we are left with the other variation which gives the equation of motion

$$\epsilon_{IJKL} e^I \wedge R^{JK} = 0 \tag{1.58}$$

[5] To be precise, while the second one is straightforward, one has to impose $DT^I = dT^I + \omega^I{}_J \wedge T^J = R^I{}_J \wedge e^J \equiv 0$ to make it become $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\nu} = 0$, which means assume a Levi-Civita connection, as it is the case for GR.

This is nothing but Einstein equation in vacuum, i.e. $e^{I\nu}G_{\nu\alpha} = 0$ or equivalently $G_{\mu\nu} = 0$[6].

So, since it gives the same equations of motion, the Einstein-Palatini action (1.54) can be used as the action for GR. Notice that only first derivatives appear, this is in fact called first order formulation of GR. However, if one wants to build the most general action and insists on keeping the connection as a independent variable, there exist another term that is compatible with all the symmetries (in particular it's a Lorentz scalar) and has mass dimension equal to four, which is

$$\delta_{IJKL}e^I \wedge e^J \wedge R^{KL}(\omega) \tag{1.59}$$

where $\delta_{IJKL} = \delta_{I[K}\delta_{L]J}$. This term is invisible in the ordinary second order metric, since when (1.45) holds

$$\delta_{IJKL}e^I \wedge e^J \wedge R^{KL}(\omega(e)) = \epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}(e) = 0 \tag{1.60}$$

by the first Bianchi identity. Therefore it is not inconsistent and we are allowed to add it to the integrand in the action. Calling the coupling constant $1/\gamma$ one obtains the so called Holst action [21]

$$S_H(e,\omega) = \left(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{IJKL}\right)\int e^I \wedge e^J \wedge R^{KL} \tag{1.61}$$

Once again, it is remarkable that this action leads to the same field equations of general relativity. The first piece is in fact as above and gives

$$\omega_\mu^{IJ} = e_\nu^I \nabla_\mu e^{J\nu} \qquad\qquad G_{\mu\nu}(e) = 0 \tag{1.62}$$

The above result is independent of the value of γ which is irrelevant in classical vacuum GR (but it's important if one has torsion). It will play a crucial role in the quantum theory where it is known as Immirzi parameter.

Computing the variations explicitly and imposing them to vanish, one gets the field equation from the Holst action

$$\left(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{IJKL}\right)e^I \wedge De^J = 0 \qquad \left(\frac{1}{2}\epsilon_{IJKL} + \frac{1}{\gamma}\delta_{IJKL}\right)e^J \wedge R^{KL}(\omega) = 0 \tag{1.63}$$

Easily enough, from equation (1.60) and thanks again to the form of the connection, the Holst action gives the same identical equations of motion (1.62) of the Einstein-Palatini and therefore of the Einstein-Hilbert actions. The added term is not of topological nature (i.e. it cannot be written as a total derivative) but vanishes on histories on which ω is tetrad compatible.

To summarize, in the Palatini and Holst actions of GR, where the connection is treated as independent of the tetrad field, the extra equation of motion implies that the torsion form vanishes and selects the unique torsionless connection. Once this has been solved, the remaining equation reduces to the usual vacuum Einstein equation. This is true also with the extra

[6]Given the curvature form (1.49) $R^{IJ} = R^{IJ}_{\mu\nu}dx^\mu dx^\nu$ it is enough to define the Ricci tensor $R^I_\mu = R^{IJ}_{\mu\nu}e^\nu_J$ and scalar $R = R^I_\mu e^\mu_I$ in terms of the tetrad, so that equation (1.58) becomes $R^I_\mu - \frac{1}{2}Re^I_\mu = 0$.

Holst piece as shown above. However, contrary to what happens in the second order formalism, General Relativity is obtained only on shell, i.e. when the torsionless equation has been solved and assuming an invertible tetrad.

Now, performing the canonical analysis is quite difficult because it involves second class constraints. Moreover one obtains a new first class constraint which encodes the new Lorentz gauge invariance of the theory, the so called Gauss constraint. In order to simplify things and in view of the quantum theory it is convenient to introduce now the Ashtekar's formulation of GR which is the starting point for the quantization program of Loop Quantum Gravity. We will deal with that in the next section, but before doing so let us remark some physical aspects about the tetrad formulation and the interpretation of gravity.

1.2.1 The physical gravitational field

According to [36], there are several reasons to call ”gravitational field” the tetrad field rather that Einstein's metric field. First of all the standard model cannot be written in terms of g because fermions require the tetrad formalism; e represents the gravitational field in a more conceptually clean way than g thanks to the Local Lorentz invariance, and lastly, from a quantum gravity point of view, the tetrad field is simply more suitable than g. Les us talk about these aspects.

As Einstein understood, the gravitational field is the entity or the field that at each point of spacetime determines the preferred frames in which motion is inertial. If we pick arbitrary coordinates x^μ we can describe an event A in spacetime. In general, motion described by an arbitrary set of coordinates is not inertial. But we can always find a local inertial frame around A: let's denote it z^I and take the event at the origin. One can express the z coordinates as functions of the x ones

$$z^I = z^I(x) \tag{1.64}$$

In the arbitrary x coordinates, the non linearity of motion in A is gravity. Therefore gravity is the information of change of coordinates that brings to inertial ones. But only the value of the function (1.64) in a small neighbourhood of A are relevant, if one moves away the inertial frame will change. So we Taylor expand (1.64) and keep the first non-vanishing term

$$z^I(x) = e^I_\mu(x_A)x^\mu \tag{1.65}$$

where we have defined

$$e^I_\mu(x_A) \equiv \left.\frac{\partial z^I(x)}{\partial x^\mu}\right|_{x_A} \tag{1.66}$$

The field $e^I_\mu(x_A)$ contains all the information needed to know the local inertial frame. The construction can be done at each point x, where now $z^I(x)$ are local coordinates at x

$$e^I_\mu(x) \equiv \left.\frac{\partial z^I(x)}{\partial x^\mu}\right|_{x} \tag{1.67}$$

This is the gravitational field in x. It is the Jacobian matrix of the change of coordinates from a general frame x to an inertial local frame z^I. The field is called tetrad or vielbien or even 'soldering form' because solders a Minkowski vector bundle to the tangent bundle.

Of course, the index I transforms as a Lorentz index under a local Lorentz transformation, since any other local coordinate system $y^J = \Lambda^J{}_I z^I$ defines a local inertial frame too. So the filed $e^I_\mu(x)$ and

$$e'^J_\mu(x) = \Lambda^J{}_I e^I_\mu(x) \tag{1.68}$$

represent the same physical gravitational field. Moreover, if instead of x we had used coordinates $y = y(x)$, we would have obtained

$$e^I_\nu(y) = \frac{\partial x^\mu(y)}{\partial y^\nu} e^I_\mu(x(y)) \tag{1.69}$$

Now, the transformation properties of the tetrads (1.68) and (1.69) are exactly the ones under which the GR action is invariant: local Lorentz transformations and diffeomorphism gauge transformations. From a geometrical point of view, these are also precisely the transformation laws of a 1-form field valued in a vector bundle P over the spacetime manifold $\mathcal{M}$, whose fibre is Minkowski space, associated with a principal $SO(3,1)$ Lorentz bundle. The spin connection introduced above is the connection associated to this bundle. This setting realizes the picture of a patchwork of Minkowski spaces, carrying Lorentz frames. Better said, the gravitational field is a map $e : T(\mathcal{M}) \to P$ that sends tangent vector to Lorentz vectors.

At each point x of the spacetime manifold $\mathcal{M}$, the gravitational field e defines therefore a map from the tangent space $T_x(\mathcal{M})$ to the Minkowski space M

$$\begin{array}{ccc} T_x(\mathcal{M}) & \to & M \\ v^\mu & \mapsto & v^I = e^I_\mu v^\mu \end{array} \tag{1.70}$$

It is in this sense that the metric tensor is not a fundamental object but it is composite

$$ds^2 = \eta_{IJ} e^I e^J = \eta_{IJ} e^I_\mu e^J_\nu dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \tag{1.71}$$

The metric g is not affected by a local Lorentz transformation so the tetrad has more independent components. Indeed, while $g_{\mu\nu}$ has 10, the tetrad has 16 independent components. The additional 6 are simply given by the possible Lorentz transformations. They underline the infinite possible reference frames in a tangent space of a point that one can choose. Therefore, for one given metric tensor, there are infinitely many tetrads which reproduce it.

Now we can go back to the task of finding new suitable variables for the quantum theory of general relativity.

1.3 Ashtekar's variables

Proceeding as in section 1.1, one could now define the Hamiltonian formulation assuming again a 3+1 splitting of spacetime ($\mathcal{M} \simeq \mathbb{R}\times\Sigma$) and ADM coordinates (t, x). So one introduces again the lapse function and shift vector (N, N^a) and the ADM decomposition of the metric (1.12). It's necessary now to split both the spacetime and internal indices into their components. For the tetrad field, the decomposition is

$$e^0_\mu dx^\mu = e^0_0 dx^0 + e^0_a dx^a = N dx^0 + e^0_a dx^a \qquad e^i_\mu dx^\mu = e^i_0 dx^0 + e^i_a dx^a = N^i dx^0 + e^i_a dx^a \quad (1.72)$$

where we can write $N^i = N^a e^i_a$ and $N = e^0_0$ (the indices i and a run over $1, 2, 3$). Moreover

$$\delta_{ij} e^i_a e^j_b = q_{ab} \quad (1.73)$$

with the triad e^i_a being defined as the spacial part of the tetrad. The goal will be to use (a suitable modified version of) the triad as one half of the new canonical variables. The so called co-triad $e^i_a = q_{ab} e^{bi}$ is the inverse of e^a_i both with respect to the spatial and internal indices

$$e^i_a e^a_j = \delta^i_j \qquad e^i_a e^b_i = \delta^b_a \quad (1.74)$$

So the spatial metric (and its inverse) can be expressed in terms of the triad as $q_{ab} = e^i_a e_{bi}$. To simplify the discussion it is customary to work in the so called "time gauge" $e^I_\mu n^\mu = \delta^I_0$ in which

$$e^0_\mu = (N, 0), \qquad e^I_0 = (N, N^a e^i_a) = (N, N^i) \quad (1.75)$$

This gauge basically means that the pull-back of the tetrad components e^0_μ to the spacelike hypersurface Σ is zero, i.e. $e^0_a = 0$. So it reduces the non vanishing terms of the 4×4 matrix of the tetrad which transforms under $SO(3, 1)$, to the 3×3 matrix e^i_a which transforms under $SU(2)$, plus the lapse e^0_0 and the shift e^i_0. The time gauge selects therefore the compact subgroup $SU(2)$ from the initial non compact Lorentz group.

So as before, one has to identify the canonically conjugated variables and perform the Legendre transform. But as mentioned, the first difference with section 1.1 is that the tetrad formulation has introduced a new symmetry in the action: the invariance under local Lorentz transformations. As a consequence a richer class of constraints it's expected, containing also the generators of the new local symmetry[7]. Another important new feature which complicates the analysis is the use of the tetrad and connection as independent fields. Therefore the conjugate variables are now function of both e^I_a and ω^{IJ}_a (and their derivatives) instead of being functions of the metric q_{ab} only. One gets a much more complicated structure that in particular has a second class constraint algebra. Luckily enough, there is a particular choice of variables which

[7] In particular note that equation (1.73) is invariant under rotations (or $SU(2)$ transformations). This means that there are three rotational degrees of freedom contained in e^i_a and not q_{ab}: we are using the nine e^i_a to describe the six independent components of q_{ab}. If one acts on the internal index of the triads in (1.73) with rotations it will not change the metric, hence to formulate GR in terms of these redundant variables one has to impose an additional constraint.

simplifies the analysis and brings back the constraint algebra to being first class. These are the famous Ashtekar's variables which we now introduce.

The first definition of variable that we make is of the densitized triad

$$E^a_i \equiv \sqrt{q} e^a_i = \mathfrak{e}\, e^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}e^j_b e^k_c \tag{1.76}$$

where $\mathfrak{e}$ is the determinat of the triad, and from here it's easy to see that the inverse spatial metric is related to the densitized triad by

$$q\, q_{ab} = E^i_a E^j_b \delta_{ij} \tag{1.77}$$

By taking the time derivative of this equation one finds

$$\dot{q}^{ab} = \frac{1}{q}\left(\dot{E}^a_i E^{bi} + E^a_i \dot{E}^{bi} - q^{ab}\dot{E}^c_i E^{ci}\right) \tag{1.78}$$

which can be used to show that the canonical term in the ADM action becomes

$$\int d^4x\, \pi^{ab}\dot{q}_{ab} = \int d^4x\, E^a_i \dot{K}^i_a \tag{1.79}$$

where K^i_a is related to the extrinsic curvature by

$$K^i_a = K_{ab}e^{bi} = \frac{1}{\sqrt{\det E}}K_{ab}E^b_j\delta^{ij} \tag{1.80}$$

This suggests that the denzitized triad can indeed be considered a canonical momentum, with the corresponding configuration variable being K^i_a. As a matter of fact one should show that the Poisson brackets

$$\{E^a_j(x), K^i_b(x')\} = 8\pi G\, \delta^a_b\delta^i_j\delta(x,x') \qquad \{E^a_j(x), E^b_i(x')\} = 0 = \{K^j_a(x), K^i_b(x')\} \tag{1.81}$$

which define a symplectic structure for the new canonical pair of coordinates, are equivalent to the ones of the ADM variables, namely equation (1.27). It turns out that they are satisfied if the following holds

$$E^i_a K^{bi} = E^b_i K^{ai} \tag{1.82}$$

which is equivalent to say that K_{ab} is symmetric. The above constraint can also be written in the following form

$$G_i \equiv \epsilon_{ijk}E^{aj}K^k_a = 0 \tag{1.83}$$

and it suggests that it is the generator of rotations associated with the invariance of the spatial metric under internal $SU(2)$ transformation of the triad, as mentioned above.

One could now express the Hamiltonian and Spatial diffeomorphism constraints in terms of the new variable and a bit of algebra shows that

$$H_a = \frac{1}{8\pi G}\left(D_b(K^i_a E^b_i) - D_a(K^i_b E^b_i)\right) \tag{1.84}$$

and

$$H = -\frac{1}{16\pi G}\left(\frac{E^a_i E^b_j}{\sqrt{\det E}}(K^i_a K^j_b - K^j_a K^i_b) + \sqrt{\det E}\,\mathcal{R}\right) \tag{1.85}$$

We can therefore write the action as

$$S = \frac{1}{16\pi G}\int_{\mathbb{R}} dt \int_{\Sigma} d^3x \left(2\dot{K}^j_a E^a_j - \left[\Lambda^j G_j + N^a H_a + NH\right]\right) \tag{1.86}$$

This will generate the same dynamics as the ADM action up to $SU(2)$ gauge transformations. Thus, as long as rotationally invariant observables are concerned (i.e. on the constraint hypersurface $G_i = 0$) this extended formulation is identical to the physical ADM system.

Although we now have an internale $SU(2)$ gauge freedom, none of our variables transforms as a connection. There is however a natural choice for this. We recall here the covariant derivative (1.43) projected in the three dimensional hypersurfaces (i.e. with spatial indices)

$$\mathcal{D}_a e^i_b = \partial_a e^i_b - \Gamma^c_{ab} e^i_b + \omega^i_{ak} e^k_b \tag{1.87}$$

Now since the connection ω is antisymmetric (even before being called spin connection, i.e. before requiring the compatibility $\mathcal{D}_a e^i_b = 0$), we see that we can rewrite it as

$$\omega^{ij}_\mu = \epsilon^{ij}{}_k \omega^k_\mu \qquad \Leftrightarrow \qquad \omega^k_\mu = \frac{1}{2}\epsilon^k{}_{ij}\omega^{ij}_\mu \equiv -\Gamma^k_\mu \tag{1.88}$$

The last definition is due to some notations in which the derivative (1.87) is written

$$\mathcal{D}_a e^i_b = \partial_a e^i_b - \Gamma^c_{ab} e^i_b + \epsilon^i{}_{jk}\Gamma^j_a e^k_b \tag{1.89}$$

where, if $\Gamma^j_a = -\omega^j_a$, the last term is in fact equivalent to $\omega^i_{ak} e^k_b$ thanks to equations (1.88) and the antisymmetry of ϵ. So now we require triad-compatibility, i.e. that (1.87) must be zero, then we call Γ^i_a (or sometimes ω^i_a which differs only by a sign) spin connection. Its explicit expression in terms of the densitized triad reads

$$\begin{aligned}\Gamma^i_a &= \frac{1}{2}\epsilon^{ijk} e^b_k \left(\partial_b e_{aj} - \partial_a e_{bj} + e^c_j e^l_a \partial_b e_{cl}\right) \\ &= \frac{1}{2}\epsilon^{ijk} E^b_k \left(\partial_b E_{aj} - \partial_a E_{bj} + E^c_j E^l_a \partial_b E_{cl} + E_{aj}\frac{\partial_b \det E}{\det E}\right)\end{aligned} \tag{1.90}$$

We can finally define the celebrated Ashtekar connection

$$A^i_a = \Gamma^i_a + \gamma K^i_a \tag{1.91}$$

where γ is the same encountered in the above: it is a free parameter constituting an ambiguity in the construction of the connection variable. In the Ashtekar's original formulation it could only take the values $\gamma = \pm i$ because this choice turns out to significantly simplify the form of the Hamiltonian constraint. However this was coming at a price of having to impose reality conditions later, which are in fact very difficult to deal with. The connection (1.91) with arbitrary values of γ was consider later by Immirzi and Barbero. For this reason we will call it Barbero-Immirzi parameter and we will only consider the case in which it is real-valued. Of course, thanks to what we have seen above, one could also define the connection with a global minus sign, expliciting the dependence on ω

$$\tilde{A}^i_a = \gamma\omega^{0i}_a + \frac{1}{2}\epsilon^i_{jk}\omega^{jk}_a \tag{1.92}$$

where $\omega^{0i}_a = -K^i_a$ and the only irrelevant difference would be in a sign of the Poisson brackets. Notice also that the Ashtekar variables could be derived performing a $3+1$ decomposition of the Holst action (1.61). In fact, the introduction of the second or 'Holst' term is required for the canonical formalism in order to have a dynamical theory of connections. Without it, as shown by Ashtekar [5], the connection variable would not survive the Legendre transform.

As a matter of fact the Ashtekar connection turns out to be canonically conjugate to the densitized triad, although this is not obvious at first sight. A quick way to see this is to take the antisymmetric part of equation (1.87) or equivalently (1.89), exploiting the fact that the Levi-Civita term is not going to count. Then one takes the time derivative of that equation and contracts suitably with a triad, to get

$$E^a_i\dot{\Gamma}^i_a = -\frac{1}{2}\epsilon_i^{\ jk}\, e\, e^a_j e^b_k \mathcal{D}_a \dot{e}^i_a \tag{1.93}$$

This shows that $E^a_i\dot{\Gamma}^i_a$ is a total derivative so that up to boundary term the canonical piece in the action is

$$\frac{1}{8\pi G}\int d^3x\, E^a_i \dot{K}^i_a = \frac{1}{8\pi\gamma G}\int d^3x\, E^a_i \dot{A}^i_a \tag{1.94}$$

suggesting that A and E are canonically conjugate. One can in fact check that the symplectic structure

$$\{A^i_a(x), E^b_j(x')\} = 8\pi\gamma G\, \delta^b_a\delta^i_j\delta^3(x,x') \qquad \{A^i_a(x), A^j_b(x')\} = 0 = \{E^a_i, E^b_j(x')\} \tag{1.95}$$

follows from (1.81), which in turn followed from the ADM structure (1.27). The canonical variables A^i_a and E^a_i are are called Ashtekar variables or sometimes Barbero-Ashtekar variables.

We now translate the constraint in the new variables. The clever definition of the Ashtekar connection allows to to put the constraint (1.83) in the exact form of a Gauss law used in a $SU(2)$ gauge theory. For this reason G_j is called Gauss constraint. Since E^i_a is constant with respect to the covariant derivative of the spin connection (because it is both triad- and spatial metric- compatible), one can write the Gauss constraint as

$$G_i = \frac{1}{8\pi\gamma G} D_a E^a_i \tag{1.96}$$

where $D_a v^i = \partial_a v^i + \epsilon^i_{\ jk} A^j_a v^k$ is now the covariant derivative of the Ashtekar connection.

In order to re-express the diffeomorphism and Hamiltonian constraint, the key identity is

$$F^i_{ab} = R^i_{ab} + \gamma(D_a K^i_b - D_b K^i_a) + \gamma^2 \epsilon^i{}_{jk} K^j_a K^k_b E^b_i \tag{1.97}$$

which is the relation between the curvature tensors of the spin connection and the Ashtekar connection

$$\begin{aligned} R^i_{ab} &= \partial_a \Gamma^i_b - \partial_b \Gamma^i_a + \epsilon^i{}_{jk} \Gamma^j_a \Gamma^k_b \\ F^i_{ab} &= \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b \end{aligned} \tag{1.98}$$

Notice that the first of (1.98) is simply (1.50) rearranged and projected on the spatial hypersurface. If we contract the relation (1.97) with E^b_i, we get an equation that allows to rewrite the diffeomorphism constraint (1.84) as

$$H_a = \frac{1}{8\pi\gamma G}\left(F^i_{ab}E^b_i - (1+\gamma^2)\epsilon^i{}_{jk}K^j_a K^k_b E^b_i\right) = \frac{1}{8\pi\gamma G}\left(F^i_{ab}E^b_i - (1+\gamma^2)K^i_a G_i\right) \tag{1.99}$$

where we used the Gauss constraint expressed as (1.83). One actually does not have to consider it because $G_i = 0$, and it is there only for completeness. From now on we will ignore any term proportional to the Gauss constraint. Contracting similarly the relation (1.97) with $\epsilon_i^{\ jk} E^a_j E^b_k$ one gets an equation which allows to rewrite the Hamiltonian (1.85) constraint as follows

$$H = \frac{1}{16\pi G}\frac{\epsilon_j^{kl} E^a_k E^b_l}{\sqrt{\det E}}\left(F^j_{ab} - (1+\gamma^2)\epsilon^j_{mn} K^m_a K^n_b\right) \tag{1.100}$$

again up to terms proportional to the Gauss constraint. Let us recall that a constraint is called first class if the Poisson brackets of it with other constraints can be written as a linear combination of the constraints again. This means that the parts of the scalar and vector constraints which do not depend on G_i will describe the same system on the hypersurface $G_i = 0$, that is why one can work with them disregarding the Gauss constraint. Summarising the Holst action (1.61) can be rewritten in the new variables

$$S[A, E, N, N^a] = \frac{1}{8\pi\gamma G}\int_{\mathbb{R}} dt \int_{\Sigma} d^3x \left[\dot{A}^i_a E^a_i - A^i_0 D_a E^a_i - N^a H_a - NH\right] \tag{1.101}$$

This action is similar to (1.22) with (A, E) as canonically conjugated variables instead of (q, π). Lapse and shift are still Lagrange multipliers and consistently we still refer to $H(A, E)$ and $H_a(A, E)$ as the Hamiltonian and space-diffeomorphism constraints. The new formulation has introduced the extra constraint (1.96) $G_j = D_a E^a_j = \partial_a E^a_j + \epsilon_{jkl} A^j_a E^{al} = 0$ which is exactly a Gauss constrain as in the gauge theories and it generates gauge transformations. It is in fact easy to check that E^b_j and A^i_a transform respectively as an $SU(2)$ vector and an $SU(2)$ connection under such transformations.
First we define the smeared Gauss constraint

$$G(\lambda) = \int d^3x \, G_i(x)\lambda^i(x) \tag{1.102}$$

and then evaluate its Poisson brackets with the canonical variables (putting $8\pi G = 1$)

$$\{E_i^a, G(\lambda)\} = \gamma \epsilon_{ij}{}^k \lambda^j E_k^a \qquad \{A_a^i, G(\lambda)\} = -D_a \lambda^i \tag{1.103}$$

where the right hand sides are the variations of $A_a = A_a^i \tau_i$ and $E^a = E_i^a \tau^i$ under un infinitesimal version of the gauge transformations $A_a \to g A_a g^{-1} + g \partial_a g^{-1}$ and $E^a \to g E^a g^{-1}$. These are the standard transformations of the connection and electric field in Yang-Mills theories.
The algebra of the constraints is still first class and the algebra of the new Gauss constraint is

$$\{G(\lambda_1), G(\lambda_2)\} = \frac{\gamma}{2} G([\lambda_1, \lambda_2]) \tag{1.104}$$

where $[\lambda_1, \lambda_2]^i = \epsilon^i{}_{jk} \lambda_1^j \lambda_2^k$ is the $SU(2)$ commutator.

Even though we should not be surprised about the appearance of the new Gauss constraint due to the tetrad formalism, one might be perplexed by the fact that the local gauge invariance of the action was the full Lorentz group, but the action (1.101) is only invariant under $SU(2)$. The origin of this lies exactly in the change of variables: the Ashtekar connection is an $SU(2)$ connection, not a Lorentz one. It is therefore an auxiliary variable useful to cast the algebra in a first class form. It is only on the case $\gamma = i$ that the link with the original Lorentz group is manifest: in this case $SU(2)$ corresponds to the selfdual subgroup of the the Lorentz group. Even though that choice simplifies the constraints, it yields complex variable which are difficult at the quantum level. That is why LQG mostly focuses on real γ, as we do here.

Summarizing the theory of General Relativity is described by an extended phase space of dimension 18 per space point, with fundamental Poisson brackets (1.95). The old 12-dimensional phase space can be recovered on the constraint surface $G_i = 0$ dividing by gauge orbits generated by G_i. We are almost ready to quantize the theory.

1.4 Holonomy-flux algebra

Due to the usual complications, one cannot quantize all functions on phase space, but only a subset. The choice of such a preferred subset in loop quantum gravity is, as we shall see, closely related to the choice of variables in lattice gauge theory. In the case of general relativity, these variables should be defined in a background-independent manner, without reference to a metric or any other fixed background structures on the spatial manifold Σ. The Ashtekar variables (A_a^i, E_i^a) do not provide a suitable starting point for quantization. However, natural geometric objects related to those variables are holonomies of the connection along curves, and fluxes of the (densitized) triad through surfaces in the spatial manifold. Similar variables are well-known in the context of lattice gauge theories but the main difference is that we do not consider only a given set of holonomies and fluxes specified by a choice of lattice, but all possible holonomies and fluxes obtained by choosing arbitrary curves and surfaces. The key point is that the Ashtekar variables have a different tensorial nature. Being the non-trivial geometry of spacetime the main goal, we have to smear these variables with extra care.

Looking at the definition of the densitized triad (1.76), we see that it is a 2-form and therefore it is natural to smear it on a surface. Introducing coordinates (σ_1, σ_2) on the surface S, the flux is defined as

$$E_i(S) \equiv \int_S d^2\sigma \, n_a(\sigma) \, E_i^a(\sigma) \tag{1.105}$$

where $n_a = \epsilon_{abc} \frac{\partial x^b}{\partial \sigma_1} \frac{\partial x^c}{\partial \sigma_2}$ is the normal 1-form to the surface.

The connection on the other hand is a 1-form, so it is natural to smear it along a path. Recall that a connection defines the notion of parallel transport of the fibre over the base manifold. Then the holonomy is the $SU(2)$-valued parallel propagator of the Ashtekar connecton along a curve e in the spatial manifold. In other words, if e is parametrized by $s \in [0,1]$ and if u is any constant vector then the covariant derivative of $u(s)$ along e must vanish. From this it follows that the holonomy satisfies the differential equation

$$\frac{d}{ds} h_e[A] - h_e[A] A(e) = 0 \tag{1.106}$$

with $h_{e(0)} = 0$ and $A_a = A_a^i \tau_i$. Integrating one gets, calling $\int A = \int ds \, A_a^i \, \frac{dx^a}{ds} \tau_i$

$$h_e[A] = 1 + \int_0^s ds' \, A(e(s)) h_e[A] \tag{1.107}$$

and by iterating repeatedly

$$h_e[A] = \sum_n \int_0^1 ds_1 \int_{s_1}^1 ds_2 \cdots \int_{s_{n-1}}^1 ds_n \, A(e(S_1)) A(e(s_2)) \cdots A(e(s_n)) \tag{1.108}$$

or in a compact form

$$h_e[A] = \mathcal{P} \exp\left(\int_e A \right) \tag{1.109}$$

where $\mathcal{P}$ stands for the path-ordered product. The holonomy gives the parallel transport for points at finite distance.

The important thing to notice here is that since the algebra of A and E is singular (a delta function appears) one needs to smear the basic variables with certain test functions, as usual in field theories. In standard field theories on a background, the fields are usually smeared over 3-dimensional spatial regions but here the functional of A and E are obtained such that the total smearing is just enough to absorb the delta function in the Poisson brackets (1.95), making the one between the holonomy and the flux non-singular.

To obtain the Poisson brackets between the holonomy and the flux, one needs to find the derivative of the holonomy with respect to the connection. After a bit of algebra and calling

$e(s, s_0)$ the segment of e extending from the value s_0 to s of the parameter, one obtains

$$\frac{\delta h_e[A]}{\delta A_a^i(x)} = \int_0^1 ds\, \dot{x}^a \delta^{(3)}(x, e(s)) h_{e(1,s)} \tau_i h_{e(s,0)} \tag{1.110}$$

if x is inside e. Hence, one may now compute the Poisson bracket between the holonomy and the flux

$$\{h_e[A], E_i(S)\} = 8\pi\gamma G \int_0^1 ds \int_S d^2\sigma\, \dot{x}^a(s) n_a(\sigma) \delta^{(3)}(x(\sigma), e(s)) h_{e(1,s)} \tau_i h_{e(s,0)} \tag{1.111}$$

where σ are the coordinates of the surface. Now notice that the delta function has support only in the intersection point. The integral clearly vanishes if the curve e does not intersect the surface S, and also if e intersects S tangentially, in which case $\dot{x}^a$ is orthogonal to n_a at the intersection point. In the case of a single transversal intersection the factor

$$\dot{x}^a n_a = \epsilon_{abc} \frac{\partial x^a(s)}{\partial s} \frac{\partial x^b(\sigma)}{\partial \sigma^1} \frac{\partial x^c(\sigma)}{\partial \sigma^2} \tag{1.112}$$

is the Jacobian of the coordinate transformtaion $(\sigma, s) \to x^a(\sigma) + x^a(s)$ around the intersection point. After the change of variable we see that we can get rid of the delta function and the integral is equal to ± 1 depending on the relative orientation of S and e

$$\int_S \int_0^1 d\sigma_1 d\sigma_2 ds\, \epsilon_{abc} \frac{\partial x^a(s)}{\partial s} \frac{\partial x^b(\sigma)}{\partial \sigma^1} \frac{\partial x^c(\sigma)}{\partial \sigma^2} \delta^{(3)}(x(s), x(\sigma)) = \pm \int d^3x \delta^{(3)}(x) = \pm 1 \tag{1.113}$$

Thus[8]

$$\{h_e[A], E_i(S)\} = 8\pi\gamma G \kappa(S, e) h_{e(1,s_0)} \tau_i h_{e(s_0,0)} \tag{1.114}$$

where s_0 is the value of the curve parameter at the intersection point, and the factor $\kappa(S, e)$ is $+1$ if the orientation of S agrees with the one of e, -1 if the orientations are opposite and 0 if e intersects S tangentially or not at all. If S and e intersect at multiple points, then each intersection contributes with a term of the form (1.114). For the purpose of choosing appropriate classical variables for the construction of the quantum theory, the most important feature of the result (1.114) is that the algebra of the holonomy and the flux closes, in the sense that their Poisson bracket depends on a finite number of the basic variables. This would not be the case if one had chosen a three-dimensional smearing of the densitized triad.

Other important properties of the holonomy are:

- The holonomy of the path e taken with opposite orientation, namely e^{-1} is the inverse of the holonomy of e

$$h_{e^{-1}}[A] = h_e^{-1}[A] \tag{1.115}$$

[8] Here the point of intersection is not the beginning nor the ending point of e. In those cases, the result (1.114) gets multiplied by $\frac{1}{2}$ because the delta function in (1.110) gets integrated over half the domain.

- The holonomy of the composition of two paths is the product of the holonomies of each path
$$h_\alpha[A]h_\beta[A] = h_{\alpha\circ\beta}[A] \tag{1.116}$$

- Under a local gauge transformation $g(x) \in SU(2)$, the holonomy transforms as
$$h_e^{(g)} \equiv h_e[A^{(g)}] = g_{s(e)}h_e[A]g^{-1}_{t(e)} \tag{1.117}$$
where $s(e)$ and $t(e)$ are the source and target points of the path e.

- Under the action of a diffeomorphism ϕ on the spatial manifold Σ, the holonomy transforms as
$$h_e[A^{(\phi)}] \equiv h_e[\phi^* A] = h_{\phi\circ e}[A] \tag{1.118}$$

The proofs of these properties follow almost straightforwardly from the definitions above.

Summarizing: the most natural regular (i.e. with no delta functions) version of the Poisson algebra (1.95) is the smeared algebra of $h_e[A]$ and $E_i(S)$ which is called *holonomy-flux algebra*.

1.5 Kinematical Hilbert space of LQG

Following the Dirac steps for quantizing a gauge theory, one has first to define an Hilbert space on which the Poisson brackets among the elementary classical variables are represented by commutation relations between the corresponding quantum operators. However the difference between a gauge theory and GR is that the former is defined on a background and one can use a metric to define a integration measure. In GR the metric is a fully dynamical quantity and we do not have a background metric at disposal, hence we need to define a measure on the space of connection without relying on any fixed background.

In order to do so one has to introduce the notion of **cylindrical functions** which are functionals of the connection that depends only on some subset of the field itself. Here the field is the connection and the cylindrical function will depend on it only through the holonomies along some finite paths.

Consider a graph Γ defined as a collection of oriented paths $e \subset \Sigma$ meeting at most at their endpoints. Given $\Gamma \subset \Sigma$ we call L the total number of paths (also called edges or links) that it contains. A Cylindrical function is a couple (Γ, f) of a graph and a smooth function $f : SU(2)^L \to \mathbb{C}$ such that

$$\langle A|\Gamma, f\rangle \equiv \psi_{(\Gamma,f)}[A] = f(h_{e_1}[A], ..., h_{e_L}[A]) \in Cyl_\Gamma \tag{1.119}$$

The space of functionals Cyl_Γ can be turned into a Hilbert space if we equip it with a scalar product, and the fact that we passed from the connection to the holonomy is crucial to this point. The holonomy is an element of $SU(2)$ and the integration over this group is well defined. In particular there is a unique gauge-invariant and normalized measure $d\mu_H$ called Haar measure. Using L copies of the Haar measure, we define on Cyl_Γ the scalar product

$$\langle\psi_{(\Gamma,f)}|\psi_{(\Gamma,f')}\rangle \equiv \int\prod_e d\mu_H \overline{f(h_{e_1}[A],...,h_{e_L}[A])}f'(h_{e_1}[A],...,h_{e_L}[A]) \tag{1.120}$$

This define an Hilbert $\mathcal{H}_\Gamma$ space associated to the graph Γ. The Hilbert space of all cylindrical functions for all graphs will be defined as the direct sum of Hilbert spaces on a given graph

$$\mathcal{H}_{kin} = \bigoplus_{\Gamma\in\Sigma}\mathcal{H}_\Gamma \tag{1.121}$$

The scalar product on $\mathcal{H}_{kin}$ is easily induced from the one on $\mathcal{H}_\Gamma$. If ψ and ψ' have different graphs, say Γ_1 and Γ_2 we use the freedom to take any other graph Γ_{12} which contains both the others as subgraphs and view the functions as cylindrical on Γ_{12}. Then we extend trivially the functions f_1 and f_2 to Γ_{12} and define

$$\langle\psi_{(\Gamma_1,f_1)}|\psi'_{(\Gamma_2,f_2)}\rangle = \langle\psi_{(\Gamma_{12},f_1)}|\psi'_{(\Gamma_{12},f_2)}\rangle \tag{1.122}$$

Now, the key result and next step is to notice that (1.121) defines an Hilbert space over gauge connections A on Σ, namely

$$\mathcal{H}_{kin} = L_2[A, d\mu_{AL}] \tag{1.123}$$

This result is due to Ashtekar and Lewandowski, and the integration measure over the space of connections takes its name from them. This means that the scalar product (1.122) above can be seen as a scalar product between cylindrical functions of the connection with respect to the Ashtekar-Lewandowski measure

$$\langle\psi_{(\Gamma_1,f_1)}|\psi_{(\Gamma_2,f_2)}\rangle \equiv \int d\mu_{AL}\overline{\psi_{(\Gamma_1,f_1)}(A)}\,\psi'_{(\Gamma_2,f_2)}(A) \tag{1.124}$$

In order to find a representation of the holonomy-flux algebra, it is convenient to introduce now an orthogonal basis in the Hilbert space (which does not require a background metric). We can use the Peter-Weyl theorem, which states that an orthonormal basis on $L_2[G, d\mu_H]$ with G any compact Lie group, is given by the matrix elements of the irreducible representation of the group. In the $SU(2)$ case the functions

$$\sqrt{d_j}D^{(j)\,m}{}_{n}(h_e) \tag{1.125}$$

provide an orthonormal basis, where the $D^{(j)}(h_e)$ are the Wigner matrices. If we forget about the normalization, this means that a function on the group can be expanded as

$$f(g) = \sum_j \tilde{f}^j_{mn}D^{(j)}_{mn}(g) \tag{1.126}$$

where

$$\tilde{f}^j_{mn} = \int_{SU(2)} dg D^{(j)}_{mn}(g)f(g) \qquad j = 0, \frac{1}{2}, 1, ... \qquad m = -j, ..., j \tag{1.127}$$

and the Wigner matrices $D^j_{mn}(g)$ give the spin j irreducible matrix representation of the group element g. This applies also to $\mathcal{H}_{kin}$ which is essentially a tensor product of $L_2[SU(2), d\mu_H]$, therefore the (non-normalized) basis elements are

$$\langle A \,|\, \Gamma; j_e, m_e, n_e\rangle \equiv D^{(j_1)}_{m_1 n_1}(h_{e_1}) \cdots D^{(j_n)}_{m_n n_n}(h_{e_n}) \tag{1.128}$$

and a general function $\psi_{(\Gamma,f)}[A] \in \mathcal{H}_{kin}$ can be written as

$$\psi_{(\Gamma,f)}[A] = \sum_{j_e,m_e,n_e} \tilde{f}^{j_1,\dots j_n}_{m_1,\dots m_n,n_1,\dots n_n} D^{(j_1)}_{m_1 n_1}(h_{e_1}[A]) \cdots D^{(j_n)}_{m_n n_n}(h_{e_n}[A]) \tag{1.129}$$

On this basis, one can give a Schrödinger representation. In general, one can in fact introduce several basic operators on the kinematical Hilbert space. Let's start with the quantum counterparts of the classical elementary variables. For simplicity we consider the fundamental representation $j = 1/2$ in which $h_e \equiv D^{(\frac{1}{2})}(h_e)$ The holonomy operator acts by multiplication

$$\hat{h}_\gamma[A]\, h_e[A] = h_\gamma[A]\, h_e[A] \tag{1.130}$$

and the flux through the derivative (1.110)

$$\hat{E}_i(S) h_e[A] = -i\hbar\gamma \int_S d^2\sigma\, n_a \frac{\delta h_e[A]}{\delta A^i_a(x(\sigma))} = \pm i\hbar\gamma h_{e_1}[A]\tau_i h_{e_2}[A] \tag{1.131}$$

where e_1 and e_2 are the two new edges defined by the point at which the triad acts and the sign depends on the relative orientation of e and S, as described in the previous section. Moreover, as mentioned above, the action of the flux vanishes when e is tangential to S or $S \cap e = 0$.

Now, for instance, let's consider the action of the scalar product between two fluxes

$$\hat{E}_i(S)\hat{E}^i(S) h_e[A] = -\hbar^2\gamma^2 h_{e_1}[A]\tau^i\tau_i h_{e_2}[A] \tag{1.132}$$

where the scalar contraction of the algebra generators in the fundamental representation $\tau^i\tau_i \equiv C^2$ defines the Casimir operator of the algebra, which in particular commutes with all group elements. Hence

$$\hat{E}_i(S)\hat{E}^i(S) h_e[A] = -\hbar^2 C^2\gamma^2 h_{e_1}[A] h_{e_2}[A] \tag{1.133}$$

This will be useful to study the area operator.

On the other hand, if two fluxes act on the same endpoint we get

$$\hat{E}_i(S)\hat{E}_j(S) h_e[A] = -\hbar^2\gamma^2 h_e[A]\tau_i\tau_j \tag{1.134}$$

from which one immediately sees that two flux operators do not commute

$$\left[\hat{E}_i(S), \hat{E}_j(S)\right] h_e[A] = -\hbar^2\gamma^2 h_e[A]\, [\tau_i, \tau_j] = -\hbar^2\gamma^2 \epsilon_{ij}^{\ \ k} h_e[A]\tau_k \tag{1.135}$$

The action of the holonomy-flux algebra (1.130),(1.131) trivially extends to a generic base element $D^{(j)}(h_e)$ in a generic representation j. The action of the holonomy is the same and in the right hand side of (1.131) one has to replace the τ_i with the generators J_i in the arbitrary irreducible j representation. Therefore the Casimir will be $C^2 = -j(j+1)\mathbb{1}_{2j+1}$ so that for instance equation (1.133) becomes

$$\hat{E}_i(S)\hat{E}^i(S) = \hbar^2\gamma^2\, j(j+1)\, D^{(j)}(h_e) \tag{1.136}$$

Thanks to the linearity, the action is also extended to the whole $\mathcal{H}_{kin}$.

The construction of this representation may seem somewhat arbitrary, so it is natural to ask how much freedom there would be in building different, inequivalent kinematical representations for loop quantum gravity. The answer to that is remarkable and is provided by a uniqueness theorem called LOST theorem (Lewandowski, Okolow, Sahlmann, Thiemann). The representation described above of the holonomy-flux algebra is in fact unique on $\mathcal{H}_{kin}$ and this result can be compared to the Von Neumann theorem in quantum mechanics about the uniqueness of the Schrödinger representation. As it is known, this result does not extend to interacting field theories on flat spacetime, but remarkably, insisting on background-independence reintroduces such uniqueness also for a field theory.

We have defined a proper Kinematical Hilbert space for General Relativity and the elementary operators have been set on it. It carries a representation of the canonical algebra and this is unique. The next stage of quantization consist of imposing the constraints

$$\mathcal{H}_{kin} \quad \xrightarrow{\hat{G}_i=0} \quad \mathcal{H}_{kin}^{G} \quad \xrightarrow{\hat{H}^a=0} \quad \mathcal{H}_{Diff} \quad \xrightarrow{\hat{H}=0} \quad \mathcal{H}_{phys} \tag{1.137}$$

as we shall do.

1.6 Gauge invariance and Spin networks

The first constraint is the Gauss law. Solution to the Gauss constraints are those states in $\mathcal{H}_{kin}$ which are $SU(2)$ gauge invariant. These states will define a new Hilbert space, called $\mathcal{H}_{kin}^{G}$ because we have to remember that there are still constraints to solve before arriving to the physical space.

The approach most in line with Dirac's ideas on quantizing a constrained theory is to construct a quantum operator corresponding to the Gauss constraint and then find all the states annihilated by that operator. This turns out to give the same result as the one we outline here below, simply imposing gauge invariance. The only thing to do then is to show that imposing the gauge invariance amounts to solve the Gauss constraint $\hat{G}_i\psi = 0$. To this end consider a gauge invariant node n and a surface S centred in n of radius ϵ. The action of the total flux operator through S on n vanishes identically

$$\lim_{\epsilon\to 0} \hat{E}(S)|n\rangle = 0 \tag{1.138}$$

In fact, using (1.131) at each link one notices that the above limit produces the infinitesimal gauge transformation $g_\alpha = 1 - \alpha_i \tau^i \in SU(2)$ at the node and since the latter is gauge invariant, such action vanishes.

Now, recall that under a gauge transformation the holonomy transforms as

$$h_e \to h_e^g = g_{s(e)} h_e g_{t(e)}^{-1} \tag{1.139}$$

where $g \in SU(2)$. Similarly in a generic irrep j one has

$$D^{(j)}(h_e) \to D^{(j)}(h_e^g) = D^{(j)}(g_{s(e)} h_e g_{t(e)}^{-1}) = D^{(j)}(g_{s(e)}) D^{(j)}(h_e) D^{(j)}(g_{t(e)}^{-1}) \tag{1.140}$$

and this means that gauge transformations act on the sources and targets of the links, namely only on the *nodes* of a graph. As a consequence, imposing gauge invariance implies requiring the cylindrical functions to be invariant under the group at the nodes

$$f(h_1, \ldots, h_L) = f(g_{s_1} h_e g_{t_1}^{-1}, \ldots, g_{s_L} h_L g_{t_L}^{-1}) \tag{1.141}$$

and we will see this can be implemented via group averaging. Before doing that, one could also notice that the solution of the Gauss constraint can be done by inspection.

What we see from the above, is that in order to construct a gauge invariant state, we have to choose the cylindrical functions such that the transformations at the endpoints of holonomies cancel each other. This means that we need to look for tensors which are invariant w.r.t. the action of $SU(2)$ and contract all holonomies ending or starting at a given vertex with such a tensor, in a way that no free indices remain.

The simplest example is known as a Wilson loop: one there takes a single curve with coinciding endpoints (a loop) and simply trace over the holonomy. This corresponds to contracting its group indices with the metric tensor of the group, which is of course an invariant tensor of $SU(2)$. The next simplest example is a three-valent vertex: here the invariant tensors turn out to be the familiar Clebsch-Gordan coefficients or the related Wigner 3j-symbols (see appendix A) which enjoy a higher symmetry. From there on, all the invariant tensors can be built contracting the 3-j symbols in a suitable way.

To be more precise, let's consider a single node of a graph Γ with N incoming edges and N' outgoing edges. The ingoing links $(e_1, ..., e_N)$ carry the spins $(j_1, ..., j_N)$ and similarly for the outgoing ones. To this node we associate a tensor of the form $t^{n_1 \cdots n_N}_{m_1 \cdots m_{N'}}$ which has a lower index for each edge coming into the node and un upper index for each edge going out of the node. These tensors are required to form an orthonormal basis of the space

$$\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N} \otimes \overline{\mathcal{H}}_{j'_1} \otimes \cdots \otimes \overline{\mathcal{H}}_{j'_{N'}} \tag{1.142}$$

but are otherwise arbitrary. Now one can define the basis functions

$$f^{\Gamma}_{(j_e, m_e, n_e)}(h_{e_1}, \cdots, h_{e_N}) = \left(\prod_{v \in \Gamma} (t_v)^{n_1 \cdots n_{N'}}_{m_1 \cdots m_N} \right) \left(\prod_{e \in \Gamma} D^{(j_e) m_e}{}_{n_e}(h_e) \right) \tag{1.143}$$

where the contraction of $SU(2)$ indices is carried out according to the structure of the graph Γ. This is nothing but a generalization of the basis (1.128) and in fact it is a basis of $\mathcal{H}_{kin}$ itself. Of course if one wants an orthonormal basis, it's enough to put in the second brackets the factor $\sqrt{d_{j_e}}$, as seen in (1.125).

From the equations above it follows that each edge ending in the node v contributes with a $D^{(j_e)}(g_v)$ while each edge starting from v with a $D^{(j_e)}(g_v^{-1})$ where g_v is the value of the group element at the node. Hence the gauge transformation effectively replaces the tensors t_v with the transformed one

$$(t_v^{(g)})^{n_1\cdots n_{N'}}_{m_1\cdots m_N} = D^{(j_1)\mu_1}{}_{m_1}(g_v)\dots D^{(j_N)\mu_N}{}_{m_N}(g_v)\, D^{(j'_1)n_1}{}_{\nu_1}(g_v^{-1})\dots D^{(j'_{N'})n_{N'}}{}_{\nu_{N'}}(g_v^{-1})(t_v)^{\nu_1\cdots\nu_{N'}}_{\mu_1\cdots\mu_N} \tag{1.144}$$

at each node of the graph, but a part from this the form of the states $\langle A\,|\,\Gamma; j_e, t_v\rangle$ is preserved. From these considerations it is clear that gauge invariant states can be obtained by restricting the tensors to satisfy $t_v^{(g)} = t_v$ for every $g \in SU(2)$. In particular, if we call

$$\mathrm{Inv}\left(\mathcal{H}_{j_1}\otimes\cdots\otimes\mathcal{H}_{j_N}\otimes\overline{\mathcal{H}}_{j'_1}\otimes\cdots\otimes\overline{\mathcal{H}}_{j'_{N'}}\right) \tag{1.145}$$

the subspace of (1.142) whose elements are invariant under $SU(2)$ in the sense of equation (1.144), then we can take the tensors $\imath_v$ from any orthonormal basis of (1.145) and build an orthonormal basis on the gauge invariant Hilbert subspace for a fixed graph as

$$\psi_{(\Gamma,j_e,\imath_v)}[h_e] \equiv \langle A\,|\,\Gamma; j_e, \imath_v\rangle = \left(\prod_{v\in\Gamma}(\imath_v)^{m_1\cdots m_N}_{n_1\cdots n_N}\right)\left(\prod_{e\in\Gamma}\sqrt{d_{j_e}}\,D^{(j_e)m_e}{}_{n_e}(h_e)\right) \tag{1.146}$$

The entire gauge invariant Hilbert space $\mathcal{H}^G_{kin}$ is then spanned by the states (1.146) on all possible graphs.

If one is not convinced about the method outlined above, let us return to the group averaging approach and show that it will give the same result. Let us consider the (formal) projection operator

$$\mathcal{P}_G = \int dg\, U(g) \tag{1.147}$$

where $U(g)$ is the unitary operator which implements the gauge transformation on cylindricl functions and dg is some appropriate measure constructed using the $SU(2)$ Haar measure. Now given any cylindrical function ψ, the function $\mathcal{P}_G\psi$ has to be gauge invariant. Since the gauge transformations are completely determined by the value of g at the nodes of Γ, the proper form of the projector on a graph will be

$$\mathcal{P}^\Gamma_G = \int dg_{v_1}\dots dg_{v_M}\, U(g) \tag{1.148}$$

and recalling (1.139) one sees that the function

$$\mathcal{P}_G\psi(h_{e_1},\dots,h_{e_L}) = \int\prod_v dg_v\,\psi(g_{s_1}h_e g_{t_1}^{-1},\dots,g_{s_L}h_L g_{t_L}^{-1}) \tag{1.149}$$

is indeed gauge invariant and satisfies (1.141). In order to see the explicit expression one now has to expand the function ψ on the basis (1.143) as

$$\psi(h_{e_1},\dots,h_{e_L}) = \sum_{j_e,m_e,n_e} c_{(j_e,m_e,n_e)} f^{\Gamma}_{(j_e,m_e,n_e)}(h_{e_1},\dots,h_{e_L}) \tag{1.150}$$

Inserting above and using the identity

$$\int dg D^{(j_1)m_1}{}_{n_1}(g)\cdots D^{(j_N)m_N}{}_{n_N}(g)\, D^{(j_1')m_1'}{}_{n_1'}(g^{-1})\cdots D^{(j_N')m_N'}{}_{n_N'}(g^{-1}) = \sum_\alpha (\imath_\alpha)^{m_1\cdots m_N}_{m_1'\cdots m_N'} \overline{(\imath_\alpha)^{n_1'\cdots n_N'}_{n_1\cdots n_N'}} \tag{1.151}$$

where the sum runs over any orthonormal basis of the so called intertwiner space $\mathcal{H}_v$ which is defined as $\mathrm{Inv}\Big(\mathcal{H}_{j_1}\otimes\cdots\otimes\mathcal{H}_{j_N}\otimes\overline{\mathcal{H}}_{j_1'}\otimes\cdots\otimes\overline{\mathcal{H}}_{j_{N'}'}\Big) \equiv \mathrm{Inv}\big[\otimes_e\mathcal{H}_{(j_e)}\big]$, we find that group averaging indeed produces

$$\mathcal{P}_G\psi(h_{e_1},\dots,h_{e_L}) = \sum_{j_e,\imath_v} c_{(j_e,\imath_v)} \left(\prod_v \imath_v\right)\left(\prod_{e\in\Gamma}\sqrt{d_{j_e}}D^{(j_e)}_{m_e n_e}(h_e\right) \tag{1.152}$$

which consistently is a linear combinations of the states (1.146).

From a more geometrical point of view, what we have done is projecting down on the gauge invariant Hilbert space, using at each node the operator

$$P = \int dg \prod_e D^{(j_e)}(g) \tag{1.153}$$

where the integrand is an element in the tensor product of the $SU(2)$ irreducible representations

$$\prod_e D^{(j_e)}_{m_e n_e}(h_e) \in \bigotimes_e \mathcal{H}_{j_e} \tag{1.154}$$

and therefore transforms non-trivially under gauge transformations and it is in general reducible

$$\bigotimes_e \mathcal{H}_{j_e} = \bigoplus_i \mathcal{H}_{j_i} \tag{1.155}$$

This means that the integrand in (1.153) selects the gauge invariant part of $\bigotimes_e \mathcal{H}_{(j_e)}$ which is the singlet $\mathcal{H}_0$, if the latter exists. One can decompose (1.153) in terms of a basis of this $\mathcal{H}_0$

$$P = \sum_{\alpha=1}^{\dim\mathcal{H}_{(0)}} \imath_\alpha\imath_\alpha^* = \sum_\alpha |\imath_\alpha\rangle\langle\imath_\alpha| \tag{1.156}$$

and the invariants appearing here are the same intertwiners as above: invariant tensors residing at the nodes of a graph. Equation (1.156) is in fact nothing but (1.151). Therefore the facts that the action is only on the nodes of the graph that labels the basis of $\mathcal{H}_{kin}$, and the decomposition (1.156), imply that the action of the projector on elements of $\mathcal{H}_{kin}$ can be written as a linear combination of products of matrices $D^{(j)}(h_e)$ contracted with intertwiners.

So as mentioned above (1.146), the states labeled with a graph, an irreducible representation $D^{(j)}$ of spin j of the holonomy along each link, and an element $\imath$ of the intertwiner space $\mathcal{H}_v$ in each node, are called spin network states and can be written compactly (forgetting about the normalization) as

$$\psi_{(\Gamma,j_e,\imath_v)}[h_e] = \bigotimes_v \imath_v \bigotimes_e D^{(j_e)}(h_e) \tag{1.157}$$

where v are the vertices (or nodes) and e are the edges (or links) of Γ. The indices of the matrices and itertwiners are hidden for simplicity of notation, their contraction pattern can be easily recognized from the connectivity of Γ.

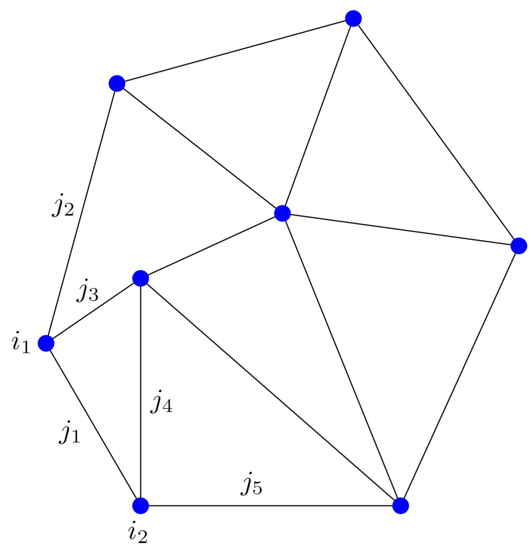


Figure 1.3: Example of Spin Network

As previously, different graphs select different orthogonal spaces thus the full gauge invariant Hilbert space of Loop Quantum Gravity decomposes as a direct sum over spaces on a fixed graph

$$\mathcal{H}^G_{kin} = \bigoplus_{\Gamma\in\Sigma} \mathcal{H}^G_\Gamma \tag{1.158}$$

The spin network represents the basis of the cylindrical functions living in this Hilbert space. They are built over graphs which are embedded in the three dimensional manifold Σ therefore a vertex remains located at a fixed point of the manifold so there is no diffeomorphism invariance yet.

The gauge invariant Hilbert space associated to a graph $\mathcal{H}^G_\Gamma$ is denoted

$$\mathcal{H}^G_\Gamma = L_2\left[SU(2)^L/SU(2)^N, d\mu_H\right] \tag{1.159}$$

and it decomposes as a sum over intertwiner spaces

$$\mathcal{H}_\Gamma^G = \bigoplus_{j_e} \left(\bigotimes_v \mathcal{H}_v \right) \tag{1.160}$$

where $\mathcal{H}_v = \text{Inv}[\otimes_e \mathcal{H}_{j_e}]$ is the gauge invariant Hilbert space associated to each vertex. To understand this, one has to remember that associating a Wigner matrix $D^{(j_e)}_{m_e n_e}$ to an edge means that the Wigner matrix is a linear map from $\mathcal{H}_{j_e}$ to $\overline{\mathcal{H}_{j_e}}$. Graphically, the source $s(e)$ carries the Hilbert space $\mathcal{H}_{j_e}$ and the target $t(e)$ carries $\overline{\mathcal{H}_{j_e}}$. The edge is therefore the linear map between them. For a general n valent vertex, the space $\mathcal{H}_v$ will be the tensorial product of all the Hilbert spaces associated to the sources and dual Hilbert spaces associated to the targets.

Concluding, spin network states form a complete basis of the Hilbert space of solution to the quantum Gauss law, $\mathcal{H}^G_{kin}$. The structure of this space is in fact nicely organized by the spin network basis. The physical interpretation of these as states of *quantum space* will be given in the next chapter. For details about the intertwiners space see Appendix A or the following example.

1.6.1 Theta Graph

Let's consider the so called theta graph Γ_θ. It contains three edges (e_1, e_2, e_3) which recouple at two vertices (v_1, v_2)

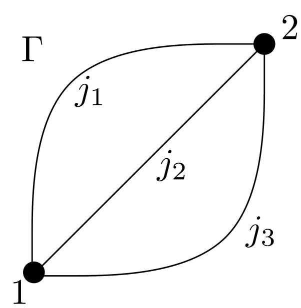


Figure 1.4: The simple Theta graph

Thanks to the Peter-Weyl theorem, a generic cylindrical function can be written as

$$\psi_{\Gamma_\theta}(h_1, h_2, h_3) = \sum_{j_i, m_i, n_i} \tilde{f}^{j_1, j_2, j_3}_{m_1, m_2, m_3, n_1, n_2, n_3} D^{(j_1)}_{m_1 n_1}(h_1) D^{(j_2)}_{m_2 n_2}(h_2) D^{(j_3)}_{m_3 n_3}(h_3) \tag{1.161}$$

If one applies a gauge transformation, one would see that it is obviously not $SU(2)$ invariant. Since the gauge transformations act only on the group elements, the gauge invariant part of (1.161) will be obtained by looking at the gauge invariant part of the product of the Wigner matrices. Using group averaging one sees that the invariant part of the basis is

$$\left[D^{(j_1)}_{m_1 n_1}(h_1) D^{(j_2)}_{m_2 n_2}(h_2) D^{(j_3)}_{m_3 n_3}(h_3) \right]_{inv} = \int dg_1 dg_2 \, D^{(j_1)}_{m_1 n_1}(g_1 h_1 g_2^{-1}) D^{(j_2)}_{m_2 n_2}(g_1 h_2 g_2^{-1}) D^{(j_3)}_{m_3 n_3}(g_1 h_3 g_2^{-1}) \tag{1.162}$$

where we integrate with the Haar measure over $g_{v_1} \equiv g_1$ and $g_{v_2} \equiv g_2$ to get rid of the dependence on gauge transformations at the vertices. Now we introduce the tensorial projector on the gauge invariant space (more precisely on the tensor product of the $SU(2)$ irreducible representations)

$$\begin{aligned} P_{m_1m_2m_3\,\alpha_1\alpha_2\alpha_3} &= \int dg_1, D^{(j_1)}_{m_1\alpha_1}(g_1) D^{(j_2)}_{m_2\alpha_2}(g_1) D^{(j_3)}_{m_3\alpha_3}(g_1) \\ &= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \overline{\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}} \\ &= \imath_{m_1m_2m_3\alpha_1\alpha_2\alpha_3} \end{aligned} \tag{1.163}$$

where we used the property of the Wigner matrices to be written in terms of the Clebsch-Gordan coefficient or Wigner 3-j symbols, which give the components of the $SU(2)$ invariant tensor $\imath$. Therefore, the projectors to be inserted in (1.162) act as the successive action of two intertwiners. Considering two representations and their associated spaces $\mathcal{H}_{j_1}$ and $\mathcal{H}_{j_2}$, the intertwiners map the tensorial product of those spaces into the $\mathcal{H}_j$, i.e. they provide usual recouping of two representations into another one. The interwiners live in

$$[\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}]_{\text{inv}} \tag{1.164}$$

and this space is non-empty only when $|j_2 - j_3| \leq j_1 \leq j_2 + j_3$ as the theory of angular momentum teaches us (see appendix A for more details on this). Equation (1.162) becomes

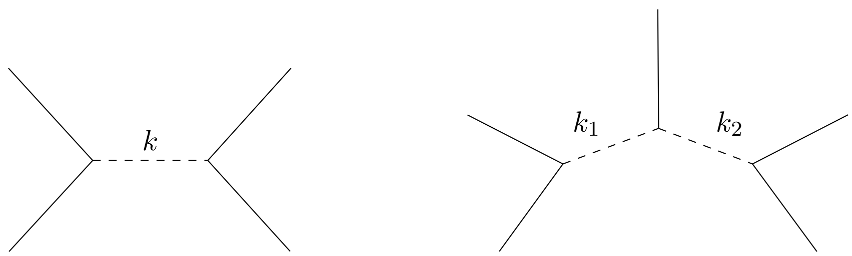
$$\begin{aligned} &\left[D^{(j_1)}_{m_1n_1}(h_1) D^{(j_2)}_{m_2n_2}(h_2) D^{(j_3)}_{m_3n_3}(h_3)\right]_{\text{inv}} = P_{m_1m_2m_3\alpha_1\alpha_2\alpha_3} P_{\beta_1\beta_2\beta_3n_1n_2n_3}\, D^{(j_1)}_{\alpha_1\beta_1}(h_1) D^{(j_2)}_{\alpha_2\beta_2}(h_2) D^{(j_3)}_{\alpha_3\beta_3}(h_3) \\ &= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \overline{\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}} \begin{pmatrix} j_1 & j_2 & j_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \overline{\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_2 \end{pmatrix}} D^{(j_1)}_{\alpha_1\beta_1}(h_1) D^{(j_2)}_{\alpha_2\beta_2}(h_2) D^{(j_3)}_{\alpha_3\beta_3}(h_3) \\ &= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \overline{\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_2 \end{pmatrix}} \prod_e D^{(j_e)}(h_e) \prod_v \imath_v \end{aligned} \tag{1.165}$$

where $\imath_v$ is not casually a shorthand notation for the 3-j symbols. They are in fact the invariant tensors in the space of all the spins that enter the node v. From here it follows that the invariant part of (1.161) is

$$\begin{aligned} \psi^{\text{inv}}_{\Gamma_\theta}(h_1,h_2,h_3) &= \sum_{j_e} \prod_e D^{(j_e)}(h_e) \prod_v \imath_v \sum_{m_e n_e} \tilde{f}^{j_1,j_2,j_3}_{m_1,m_2,m_3,n_1,n_2,n_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \overline{\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_2 \end{pmatrix}} \\ &= \sum_{j_e} \tilde{f}^{j_1,j_2,j_3} \prod_e D^{(j_e)}(h_e) \prod_v \imath_v \end{aligned} \tag{1.166}$$

where the new coefficient $\tilde{f}^{j_1,j_2,j_3}$ include the sums over the magnetic indices m_e and n_e. This equation generalises to (1.157). To each edge (here 3) one can associate a given representation of the holonomy, and in order to obtain a gauge invariant quantity, one associates to each vertex an intertwiner which couples the ingoing representations with the outgoing ones. In the three-valent case, the dimension of the singlet space $\mathcal{H}_0$ is 1 and the unique intertwiner is given by the 3-j Wigner symbol.

For a n-valent vertex the space can have larger dimension. To build higher valence intertwiner one can use the basic one, and the epsilon tensor. It is in fact easy to visualize it adding firstly two irreps only, and then a third one and so on. This gives rise to a decomposition over virtual links (see appendix A)



To conclude the section let us summarize the main results. We have built the kinematical Hilbert space of LQG and imposed the first constraint, namely the Gauss law, to reduce the space $\mathcal{H}_{kin}$ to the gauge invariant $\mathcal{H}^G_{kin}$. This space decomposes as a direct sum over spaces on fixed graphs and we have found that the spin network represent the basis for this Hilbert space. The quantum numbers of these states are the graph Γ, a spin j_e on each link of Γ and an intertwiner i_v at each node. Furthermore, the space (1.159) decomposes as (1.160) and this is nothing but the analogue in LQG of the Fock decomposition of the Hilbert space of a free field in Minkowski spacetime into a direct sum of n-particle states, and play an equally important fundamental role. Before turning to the physical interpretation in terms of quantized spatial geometries, we briefly outline how to implement the two missing constraints which will give rise to the dynamics of the theory.

1.7 Dynamics of LQG: outline

We want to give here a very concise overview regarding the LQG dynamics. Since the present work will focus on kinematical aspects only, there will be no details at all about the solution of diffeomorphisms and Hamiltonian constraints, nor about other approaches. We will only sketch, for completeness reasons, the various paths and developments achieved in these essential aspects of the theory.

1.7.1 Spatial diffeomorphisms constraint

To be precise, the solution of the spatial diffeomorphisms constraint, does not concern the dynamics yet. It is nevertheless often separated from the construction of the kinematical Hilbert space, playing an intermediate (and fundamental) role.

We have learned that spin network states $\psi_{(\Gamma,j_e,\imath_n)}[A]$ live in $\mathcal{H}^G_{kin}$, namely where $\hat{G}^i\psi_{(\Gamma,j_e,\imath_n)}[A] = 0$ holds. The next step in the Dirac program is to implement the spatial diffeomorphisms constraint, namely to find invariant states such that

$$\hat{H}^a\,\psi[A] = 0 \tag{1.167}$$

Let us consider a finite diffeomorphism ϕ, its action on the holonomy being (1.118). This induces an operator $\hat{\phi}$ on the space of cylindrical functions such that

$$\hat{\phi} \quad : \quad \mathrm{Cyl}_\Gamma \mapsto \mathrm{Cyl}_{\phi\circ\Gamma} \tag{1.168}$$

This is an action onto the graph structure of the spin network states such that

$$\hat{\phi}\,\psi_\Gamma = \psi_{\phi\circ\Gamma} \tag{1.169}$$

Since the Ashtekar-Lewandowski measure $d\mu_{AL}$ is also diffeomorphism invariant, this action is well-defined and unitary. However, Cyl_Γ and $\mathrm{Cyl}_{\phi\circ\Gamma}$ are orthogonal Hilbert spaces, regardless of what the diffeomorphism is. This means that we can not define the action of an infinitesimal diffeomorphism but that they are all finite from the perspective of cylindrical functions. We can nevertheless proceed with the construction of $\mathcal{H}_{Diff}$ by group averaging as we did for the Gauss constraint, and we will build states invariant under finite diffeomorphisms. There are two subtleties to take into account now. The first one has to do with the existence of symmetries of the graphs. Namely, for each graph there are always some diffeomorphisms that act trivially on it, leaving it unchanged. Let us distinguish two cases : the diffeomorphisms that exchange the links among themselves without changing Γ, called GS_Γ, and those that also preserve each link, and merely shuffle the points inside the link, called TDiff_Γ. The latter have to be taken out, because their infinite-dimensional trivial action would spoil the group averaging procedure. Thus the group of graph symmetries is

$$GS_\Gamma = \mathrm{Diff}_\Gamma/\mathrm{TDiff}_\Gamma \tag{1.170}$$

where Diff_Γ is the group of diffeomorphisms preserving the labelled graph. The group (1.170) is finite and acts non-trivially on $\mathcal{H}^G_{kin}$.

The next thing to notice is that unlike imposing the Gauss law, imposing the invariance under diffeomorphisms will not result in a subspace of $\mathcal{H}^G_{kin}$ since the group (1.170) is non-compact. So we first define a projection map which averages states in $\mathcal{H}^G_{kin}$ with respect to GS_Γ

$$\hat{P}^\Gamma_{Diff}\,\psi_\Gamma = \frac{1}{n_\Gamma}\sum_{\phi\in GS_\Gamma}\hat{\phi}\,\psi_\Gamma \tag{1.171}$$

where n_Γ denotes the number of elements of GS_Γ. The second averaging is now done with respect the diffeomorphism which move the graph Γ. To achieve this, we can consider the

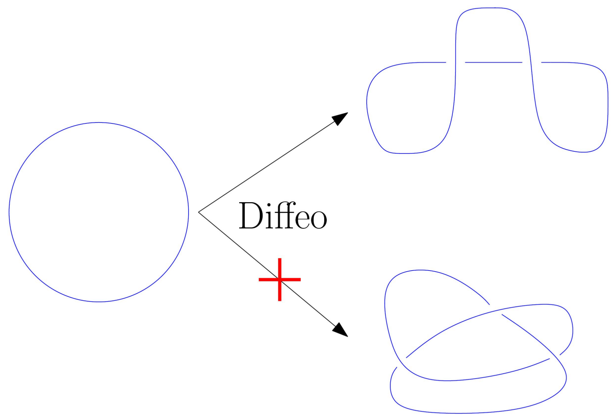


Figure 1.5: A diffeomorphism can change the way a graph is embedded in Σ, but not the presence of *knots* within the graph.

algebraic dual space $\mathcal{H}^{G*}_{kin}$ consisting of elements $\eta(\psi_\Gamma)$. The latter are diffeomorphic invariant functionals if

$$\eta\,[\hat{\phi}\,\psi] = \eta[\psi] \qquad \forall\psi \in \mathcal{H}^{H}_{kin} \tag{1.172}$$

The subspace of such functionals is denoted $\mathcal{H}^*_{Diff}$ and it is the space of diffeomorphism invariant functionals due to the diffeomorphism invariance of the scalar product on the kinematical Hilbert space. Hence η defines a map

$$\eta \quad : \quad \mathcal{H}^{G}_{kin} \mapsto \mathcal{H}^*_{Diff} \tag{1.173}$$

and on $\mathcal{H}^*_{Diff}$ the Hermitian inner product reads

$$\langle \eta(\psi_\Gamma)|\eta(\psi'_{\Gamma'})\rangle = \eta(\psi_\Gamma)[\psi'_{\Gamma'}] \tag{1.174}$$

So now we can implement the group averaging as

$$\eta(\psi_\Gamma)\,[\psi'_{\Gamma'}] = \sum_{\phi\in\mathrm{Diff}(\Sigma)/\mathrm{Diff}_\Gamma} \langle \hat{\phi}\,\hat{P}^{\Gamma}_{Diff}\,\psi_\Gamma|\psi'_{\Gamma'}\rangle \tag{1.175}$$

where the angular brackets denote the inner product on the kinematical Hilbert space. Thanks to the map (1.173), this dual space is the space of diffeomorphism invariant functionals.

Therefore one can define a projector $\mathcal{P}_{Diff}$ on $\mathcal{H}_{Diff}$ such that

$$\langle\psi|\psi'\rangle_{Diff} \equiv \langle\psi|\mathcal{P}_{Diff}|\psi'\rangle = \sum_{\phi\in\mathrm{Diff}/\mathrm{TDiff}_\Gamma} \langle\hat{\phi}\psi|\psi'\rangle \tag{1.176}$$

where the sum is over all the diffeomorphism mapping Γ to Γ' except those corresponding to the trivial ones TDiff_Γ, and now we have a new scalar product between diffeomorphisms invariant states.

The result of this procedure are spin network states defined on equivalence classes of graphs under diffeomorphisms. These equivalence classes are called *knots*, see figure 1.6. The study of knots forms an elegant branch of mathematics and one can find interesting applications in physics [6]. The diff-invariant Hilbert space of loop quantum gravity is spanned by knotted spin networks.

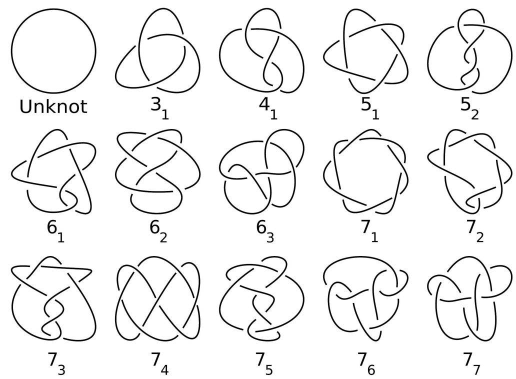


Figure 1.6: The first few knots, without nodes.

The physical interpretation coming out of this is very precise: passing from a spin network state to a knot state we preserve all the information, expect for its localisation in the 3-dimensional manifold. This is precisely the implementation of the diffeomorphism invariance also in the classical theory, where the physical geometry is an equivalence class of metrics under diffeomorphisms. In other words, two spin network states are equal if their graphs lie in the same equivalence class, and the latter differ from one another if their underlying graphs are differently knotted. This scheme represents a discrete quantized geometry, which is formed by abstract quanta of space not living on a 3-dimensional manifold. They are only localized one respect to another.

The fact that all information about the embedding in Σ has been washed out in this construction, suggests that the smooth manifold structure, on which LQG is originally built, can be replaced by that of a piecewise linear manifold. Spin network graphs are then defined using abstract graphs which are combinatorial objects dual to cellular decompositions. This perspective is central to the spin foam approach for the covariant quantisation of LQG.

1.7.2 Hamiltonian constraint

Finally one approaches the last step of Dirac's program. So we want to define the Hamiltonian constraint on the space of knotted spin networks $\mathcal{H}_{Diff}$, and study its solution. The classical scalar constraint is given by equation (1.100), and can be written

$$H(N) = H^E(N) - (1+\gamma^2)T(N) \tag{1.177}$$

introducing the shorthand notation $H^E(N)$ (E stands for Euclidean part) and $T(N)$. As with the ADM Hamiltonian constraint, this expression is non-linear. This anticipates difficulties to turn it into an operator. However, a trick due to Thiemann allows us to rewrite it in a way amenable to quantization. Denoting $V = \int \sqrt{\det(E)}$ the volume of Σ and using the classical brackets (1.95), the Thiemann's trick [45] lets us write

$$H^E(N) = \int d^3x N \epsilon^{abc} \delta_{ij} F^i_{ab} \{A^j_c, V\} \tag{1.178}$$

$$T(N) = \int d^3x \frac{N}{\gamma^3} \epsilon^{abc} \epsilon_{ijk} \{A^i_a, \{H^E(1), V\}\} \{A^j_b, \{H^E(1), V\}\} \{A^k_c, V\} \tag{1.179}$$

(1.180)

The advantage of this reformulation is that the non-linearity is mapped into Poisson brackets. The next step is to rewrite these expressions in terms of holonomies and fluxes, so that we can turn them into operators. Notice that we will see the volume operator in the next chapter, and its spectrum can be explicitly computed. This is very promising towards the prospect of knowing the action of the Hamiltonian constraint. Next, the connection and curvature have to be written in terms of holonomies. This requires a regularization procedure. We describe it here only for the Euclidean part, the regularisation of the remaining terms in the Hamiltonian constraint works similarly, although the resulting expression is more cumbersome (see [48]).

The connection can be easily expressed in terms of the holonomies. From (1.108), we have that for a path e_a of length ϵ along the x^a coordinate

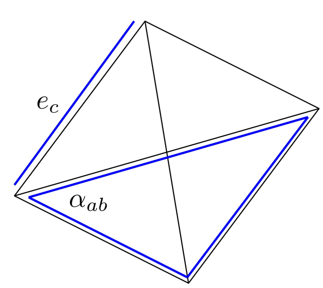
$$h_{e_a}[A] \simeq 1 + \epsilon A^i_a \tau_i + O(\epsilon^2) \quad \Rightarrow \quad h^{-1}_{e_a} \{h_{e_a}, V\} = \epsilon \{A^i_a, V\} + O(\epsilon^2) \tag{1.181}$$

For the curvature, consider an infinitesimal triangulation loop α_{ab} lying on the ab - plane and with coordinate area ϵ^2. At the lowest order we have

$$h_{\alpha_{ab}} = 1 + \frac{1}{2} \epsilon^2 F^i_{ab} \tau_i + O(\epsilon^4) \quad \Rightarrow \quad h_{\alpha_{ab}} - h^{-1}_{\alpha_{ab}} = \epsilon^2 F^i_{ab} \tau_i + O(\epsilon^4) \tag{1.182}$$

At this point we introduce a cellular decomposition of Σ and regularize the integral as a Riemann sum over the cells C_I

$$\begin{aligned} H^E &= \lim_{\epsilon \to 0} \sum_I \epsilon^N_I 3 \epsilon^{abc} \, \mathrm{Tr}(F_{ab} \{A_c, V\}) \\ &= \lim_{\epsilon \to 0} \sum_I N_I \epsilon^{abc} \, \mathrm{Tr}\big((h_{\alpha_{ab}} - h^{-1}_{\alpha_{ab}}) h^{-1}_{e_c} \{h_{h_c}, V\}\big) \end{aligned} \tag{1.183}$$

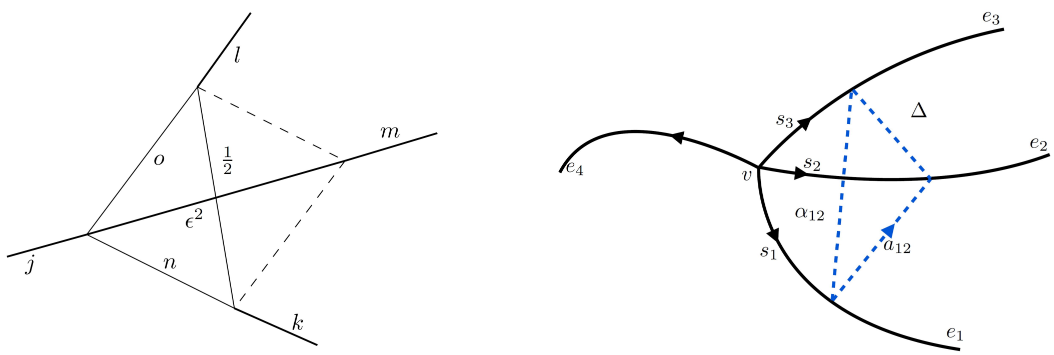


It is more convenient to specify the cellular decomposition in terms of a triangulation, namely a collection of tetrahedral cells. The loop α_{ab} can then be adapted to the triangular faces of this decomposition, as in the above figure.

Expression (1.183) can now be promoted to an operator in the quantum theory,

$$\hat{H}^E = \lim_{\epsilon \to 0} \sum_I N_I \epsilon^{abc} \, \mathrm{Tr} \left((\hat{h}_{\alpha_{ab}} - \hat{h}^{-1}_{\alpha_{ab}}) \hat{h}^{-1}_{e_c} \left[\hat{h}_{e_c}, \hat{V} \right] \right) \tag{1.184}$$

This is a well-defined operator, whose action is explicitly known. It inherits the property of the volume operator of acting only on the nodes of the spin network, which we will see explicitly in chapter 2. From the holonomies, it modifies the spin network by creating new links carrying spin 1/2 around the node, see figure 1.7a.



(a) Creation of links

(b) Construction of a triangulation

Figure 1.7: Action of the Hamiltonian on a node

Finally, its amplitude depends on the details of the action of the volume operator. This is the so called graph changing Hamiltonian originally constructed by Thiemann.

To better understand this let us look at the example above with more details. In order to prescribe the edges on which the holonomies in the operator are defined, one first constructs a triangulation of Σ adapted to the graph Γ underlying the spin network acted upon. Then, around a vertex, the triangulation is used to prescribe a segment s of an edge as well as a loop α, as shown in figure 1.7b. One averages over all possible such prescriptions. The size (fineness) of the triangulation works so far as a finite regulator in this definition. However, when one evaluates the result on a diffeomorphism-invariant state, this regulator can be removed, i.e. the triangulation infinitely refined, since two arcs of different size are related by a diffeomorphism. These precise choices lead to a certain notion of on-shell anomaly-freedom.

An alternative approach to defining a Hamiltonian constraint is to prescribe a graph-preserving regularisation. In doing this, one fixes once and for all an underlying graph on which quantum states can have support. The regularisation then strongly resembles that of lattice gauge theory, with the difference that the underlying metric is an operator. From a fundamental point of view however, one would prefer a graph-changing operator that creates new generic vertices, in particular to be able to describe an expanding universe only in terms of low spin.

To summarize, we have a perfectly well-defined Hamiltonian constraint, whose action is explicitly known and finite. An infinite number of states solutions of $\hat{H}$ are known, and the Dirac algebra is anomaly-free on physical states. Comparing this with the old-fashioned Wheeler-De Witt equation, which was badly ill-defined, one sees the full force of the use of the Ashtekar variables to quantize general relativity. We do not mention here other past an current attempts to improve the understanding of the dynamics, such as the Master constraint or the Group field theory approach [29, 30, 31], and several others. Since the present work focuses on the kinematics, we invite the reader interested in the dynamics to use the literature [48, 47, 37].

To conclude, it is necessary to say that although the kinematics of loop quantum gravity is beautifully under control, the dynamics is still work in progress.

1.7.3 Spinfoams

This section is literally just a glimpse of the concept of spinfoam and it will not contain any detail since they will not be needed in this work.

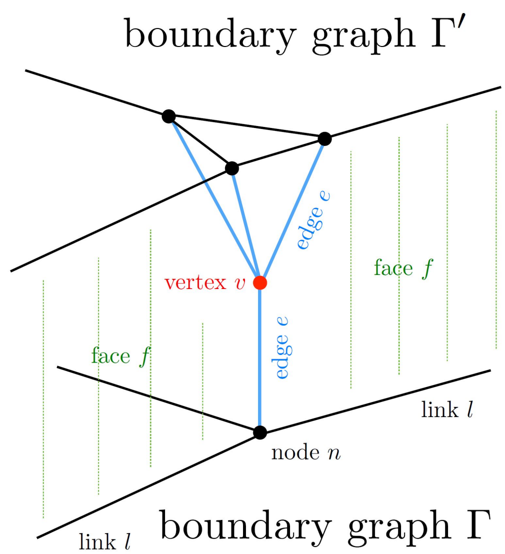


Figure 1.8: A spin network containing a three valent vertex evolves under the path integral into a new spin network, containing an additional arc reminiscent of the action of the Hamiltonian constraint seen above.

Spinfoam models are a covariant path integral approach to defining the dynamics of loop quantum gravity. They grew out of state-sum models and their development was influenced by the dynamics defined by the Hamiltonian constraint as well as the quantum kinematics. The first important model was the Barrett-Crane model, followed by the improved EPRL/FK model, which cured problems with the graviton propagator.

There are two basic strategies to arrive at the currently known spinfoam models. In the first, one formally tries to define a projector on the physical Hilbert space by giving sense to the expression

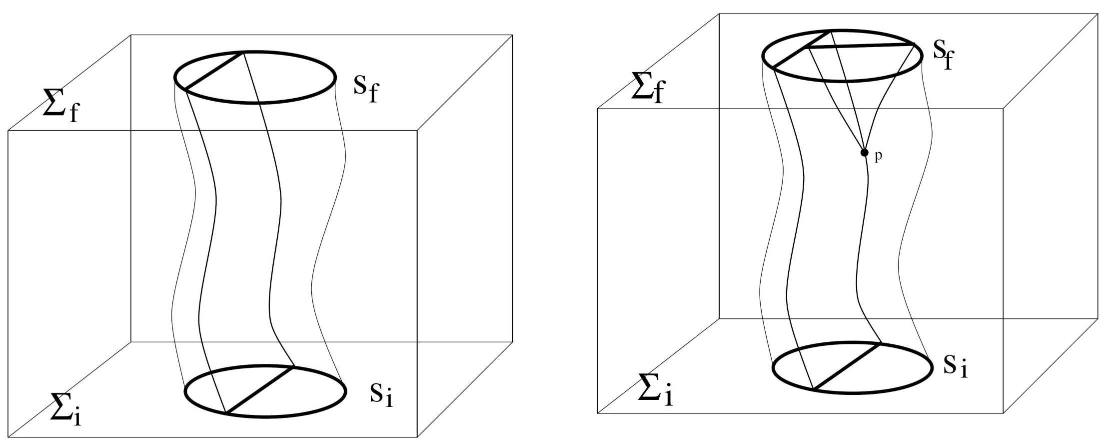
$$|\psi_{\text{phys}}\rangle \equiv \delta(\hat{H})|\psi\rangle = \int [DN] \exp\left(i \int d^3x \, N(x)\hat{H}(x)\right) |\psi\rangle \tag{1.185}$$

In practise, one then computes a path integral between two kinematical (or diff-invariant) boundary states whose value defines the physical scalar product as

$$\langle\psi_{\text{phys}}|\psi'_{\text{phys}}\rangle \equiv \langle\psi_{\text{diff}}|\delta(\hat{H})|\psi'_{\text{diff}}\rangle_{Diff} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int [DN] \, \langle\psi_{\text{diff}}|\hat{H}[N]^n|\psi'_{\text{diff}}\rangle_{Diff} \tag{1.186}$$

Different terms in this sum than can be interpreted as Feynman graphs, with the simplest example shown in figure 1.8.

The aim of the spinfoam formalism is to provide an explicit tool to compute transition amplitudes in quantum gravity. These are expressed as a sum over paths and here the 'paths' summed over are spinfoams. A spinfoam can be thought as the world-surface swept by a spin network. Spinfoams are background independent combinatorial objects: they do not need a spacetime to live in, they represent a quantum spacetime themselves, in the same sense in which a spin network describes space.



(a) The worldsheet of a spin network

(b) An edge cuts off in three edges, like in figure 1.8 giving a spinfoam with one vertex

The second way of seeing this is to consider the formulation of general relativity as a constrained BF-theory. A BF-theory is a topological theory with action

$$S_{BF} = \int_{\mathcal{M}} d^4x \ \text{Tr}\,[B \wedge F(A)] \tag{1.187}$$

where F is the curvature of a connection A and B is a Lie algebra valued two-form.

The equations of motion of this action tells us that $F(A) = 0$ so that the theory is topological. Moreover, calling $D(A)$ the covariant derivative, we see that

$$D(A)B = 0 \tag{1.188}$$

which corresponds to a Gauss law. Using the gauge group $SO(1,3)$, we recover General Relativity provided that

$$B_{IJ} = \epsilon_{IJKL}\, e^K \wedge e^L \tag{1.189}$$

where $e^I = e^I_\mu dx^\mu$ is the co-tetrad. The reason is that equation (1.188) turns into the torsion-free condition with respect to the tetrad, and the action reduces to the Einstein-Hilbert action. The condition (1.189) can conveniently be expressed in terms of a quadratic expression

$$\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \propto \epsilon_{\mu\nu\rho\sigma} \tag{1.190}$$

up to a topological sector, known as simplicity constraints. The main idea is then to quantize (1.187) as a topological quantum field theory and to impose the simplicity constraints at the quantum level. Much care has to be taken here since the simplicity constraints turn out to be non-commuting and imposing all of them strongly seems to restrict the physical degrees of freedom too much.

The spinfoam formalism provides an independent approach to the dynamics. It is clearly motivated by the canonical framework and with a (roughly) identical kinematics, but it is not known whether the dynamics is the same. In the context of large spins, a promising relation to discretized GR has been highlighted.

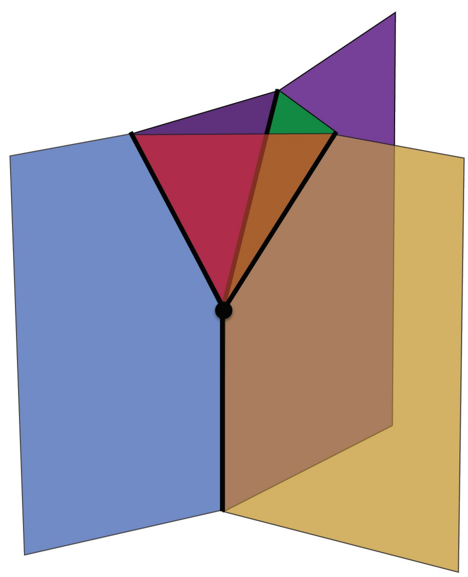


Figure 1.10: A spinfoam is a two-complex with coloured faces and edges, namely a spin representation associated to each face, which is the evolution of a link, and an intertwiner to each edge, which is ethe evolution of a node.

Chapter 2

Quantum Geometry

In chapter 1 we introduced the concept of spin network states which are in fact the building blocks of LQG. In this chapter we will give a very fascinating geometrical interpretation as we will find that the space itself is quantized. Each node of a spin network represents in fact a quantum of volume and these ”chunks” of space that define the fabric of the manifold, are separated from each other by elementary surfaces. The latter are governed by the link that crosses them and are in fact quantized as well. Therefore there will be quantum numbers associated with the nodes (the intertwiners), and with the links (the spins). In short we will see that a spin network states determines a discrete quantized spatial geometry. Inspired by these results of LQG, and trusting the physical interpretation of GR, we will also compare them with a formal quantization approach applied to the pure mathematical concept of geometry. Fascinatingly enough, it will turn out that the results of a quantum notion of space itself will be the same.

2.1 Geometric operators in LQG

The basic strategy for constructing operators in loop quantum gravity is to re-express the classical functions in terms of holonomies and fluxes, which are the elementary variables that can be promoted into well-defined operators in LQG.

2.1.1 Area operator

The area of a two-dimensional surface is probably the simplest geometric operator of LQG. The standard definition of the classical area in in terms of the metric is

$$A(S) = \int_S d\sigma_1 \sigma_2 \sqrt{\det\left(g_{ab}\frac{\partial x^a}{\partial \sigma^\alpha}\frac{\partial x^b}{\partial \sigma^\beta}\right)} \tag{2.1}$$

where $\sigma^\alpha = (\sigma^1, \sigma^2)$ are the coordinates on the surface. In order to write it in terms of densitized triads (or better, fluxes) we have to expand the determinant as

$$\det\left(g_{ab}\frac{\partial x^a}{\partial \sigma^\alpha}\frac{\partial x^b}{\partial \sigma^\beta}\right) = g_{ab}g_{cd}\left[\partial_1 x^a \partial_1 x^b \partial_2 x^c \partial_2 x^d - \partial_1 x^a \partial_2 x^b \partial_1^c \partial_2 x^d\right] \tag{2.2}$$

where ∂_1 stands for $\frac{\partial}{\partial\sigma^1}$. Rearranging and after a bit of algebra one gets

$$\det\left(g_{ab}\frac{\partial x^a}{\partial\sigma^\alpha}\frac{\partial x^b}{\partial\sigma^\beta}\right) = 2g_{a[b}g_{c]d}\partial_1 x^a\partial_1 x^b\partial_2 x^c\partial_2 x^d = g\, g^{ef} n_e n_f \tag{2.3}$$

where we used the identities

$$g_{a[b}g_{c]d} = \frac{1}{2}\epsilon_{ace}\epsilon_{bdf}\, g\, g^{ef} \qquad\qquad n_a = \epsilon_{ab}\frac{\partial x^a}{\partial\sigma^1}\frac{\partial x^b}{\partial\sigma^2} \tag{2.4}$$

Using the definitions of the densitized triads (1.76) one finds

$$A(S) = \int_S d^2\sigma\sqrt{e^2\, e^a_i e^{bi} n_a n_b} = \int_S d\sigma^1 d\sigma^2\sqrt{E^a_i E^{bi} n_a n_b} \tag{2.5}$$

At this stage, the area is not expressed in terms of fluxes yet. At the quantum level, we know that the flux acts as a functional derivative (1.131). Moreover, we have also seen the action of the scalar product of two fluxes, equation (1.133), for the case in which the surfaces is intersected only once by the holonomy path. The general case can be dealt with if one regularizes the area expression in the following way. We introduce a decomposition of S in N cells, a set of infinitesimal surfaces. Then the integral can be written as a limit

$$A(S) = \lim_{N\to\infty} A_N(S) \tag{2.6}$$

where the Riemann sum is

$$A_N(S) = \sum_{I=1}^{N}\sqrt{E_i(S_I)E^i(S_I)} \tag{2.7}$$

and N is the number of cells, and $E_i(S_I)$ is the flux of E_i through the I-th cell. This concludes the classical preparation for constructing the area operator since now the area is expressed as a function of fluxes, and therefore can be easily promoted into an operator in the quantum theory. Accordingly the area operator is defined to be

$$\hat{A}(S) = \lim_{N\to\infty}\hat{A}_N(S) \tag{2.8}$$

where the $E_i(S_I)$ is simply replaced by $\hat{E}_i(S_I)$. This operator now acts on a generic spin network state ψ_Γ where the graph Γ is generic and can intersect S multiple times. We already know from the previous chapter that $\hat{E}_i(S_I)\hat{E}^i(S_I)$ gives zero if S_I is not intersected by any link of Γ. Therefore once the decomposition is sufficiently fine so that each surface S_I is punctured once and only once, taking further refinement has no consequences. This means that the above limit amounts to simply sum the contributions of the finite number of punctures p of S caused by the edges of the graph.

Ergo, using equation (1.136)

$$\hat{A}(S)\psi_\Gamma = \lim_{N\to\infty}\sum_{I=0}^{N}\sqrt{\hat{E}_i(S_I)\hat{E}^i(S_I)}\,\psi_\Gamma = \sum_{p\in S\cup\Gamma}\hbar\gamma\sqrt{j_p(j_p+1)}\,\psi_\Gamma \tag{2.9}$$

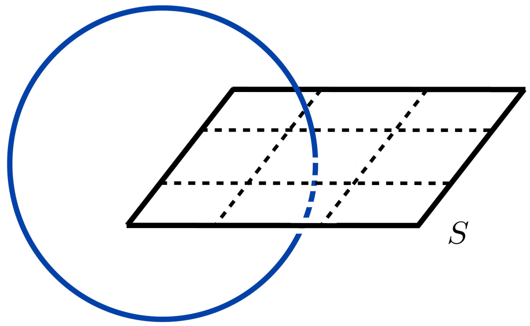


Figure 2.1: Closed loop intersecting a quantum of area

where we assumed that each puncture is caused by a link crossing the surface. The key remarks to make about this formula are two: first of all, the spectrum of the area operator is completely known and quantized, i.e. the area can only take discrete values with minimal excitations proportional to the squared Planck length (easy to see also restoring $8\pi G \neq 1$). Secondly, the operator has a diagonal action on spin network, therefore the latter are eigenstates of this operator with eigenvalue $8\pi\hbar\gamma G\sqrt{j(j+1)}$ (see Appendix B for a comment on the Planck length).

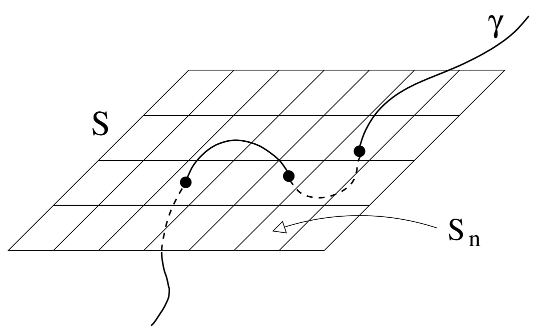


Figure 2.2: A partition of S and multiple intersections

Another thing to notice is that the regulator introduced in the definition of the area is such that the limit $N \to \infty$ is reached already at some finite value of N. The fact that a regulator involved in constructing a quantum operator out of a classical function can be removed trivially at the end of the construction is a recurring theme in loop quantum gravity; it is one of the distinctive, powerful features of the background-independent framework of the theory. Concluding we have a well defined area operator which is evidently gauge invariant but of course not diffeomorphism invariant. Its spectrum is quantized even though discreteness of geometry is never imposed by hand at any point in the development of the kinematical structure, and it comes out naturally.

2.1.2 Volume operator

The classical definition of the volume of a give region $R \subset \Sigma$ of space is

$$V(R)) \int_R d^3x \sqrt{g} = \int_R d^3g \sqrt{\left| \frac{1}{3!} \epsilon_{abc} \epsilon^{ijk} E_i^a E_j^b E_k^c \right|} \tag{2.10}$$

where the quantity in absolute value can be identified with the determinant of the densitised triad. The steps leading from this expression to the quantum volume operator[1] are similar to those required for the area operator. First we replace the integral over R by the limit of a Riemann sum. We consider a partition in cubic cells C_I so that $R \subset \cup_I C_I$ and the integral can be approximated from above by the sum of the volume of the cells. This partition allows us to rewrite the operator in terms of fluxes. Let's in fact consider the integral

$$W_I = \frac{1}{48} \int_{\partial C_I} d^2\sigma_1 \int_{\partial C_I} d^2\sigma_2 \int_{\partial C_I} d^2\sigma_3 \left| \epsilon_{ijk} E_i^a(\sigma_1) n_a(\sigma_1) E_j^b(\sigma_2) n_b(\sigma_2) E_k^c(\sigma_3) n_c(\sigma_3) \right| \tag{2.11}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ is a suitably chosen set of surfaces associated with the cell. In the continuum limit we send the size ϵ of the cell to zero, shrinking it to a point x, so that

$$W_I = \frac{1}{48} \epsilon^{abc} n_a n_b n_c \det E_i^a(x) \epsilon^6 \simeq \det E_i^a(x) \epsilon^6 \tag{2.12}$$

which roughly speaking is the square of the volume of the cell C_I. Therefore

$$V(R) = \lim_{\epsilon \to 0} \sum_I \sqrt{W_I} \tag{2.13}$$

We can never stress enough the fact that the regularized expression (2.13) does not depend on the parameter ϵ. This is a reflection of the fact that the integrand in (2.10) is a density of weight 1, and guarantees that the regulator can be removed so that (2.13) can be promoted into a well defined operator in the quantum theory.

For the sake of notation let us subdivide each ∂C_I into surfaces S^α so that $\partial C_I = \cup_\alpha S_I^\alpha$. Then one can write W_I as a sum of fluxes over three surfaces and thus

$$V(R) = \lim_{\epsilon \to 0} \sum_I \sqrt{\frac{1}{48} \sum_{\alpha,\beta,\gamma} \left| \epsilon^{ijk} E_i(S_I^\alpha) E_j(S_I^\beta) E_k(S_I^\gamma) \right|} \tag{2.14}$$

Finally one can simply turn the classical fluxes to operators and obtain the explicit expression for the quantum volume operator

$$\hat{V}(R) = \lim_{\epsilon \to 0} \sum_I \sqrt{\frac{1}{48} \sum_{\alpha,\beta,\gamma} \left| \epsilon^{ijk} \hat{E}_i(S_I^\alpha) \hat{E}_j(S_I^\beta) \hat{E}_k(S_I^\gamma) \right|} \tag{2.15}$$

As for the area, one finds that there is an optimal subdivision after which the result remains unchanged with any further refinement, so that the limit can be safely taken. In the area case,

[1] This is the Rovelli-Smolin volume operator. There exist another well defined volume operator due to Ashtekar and Lewandowski, but it is not discussed here.

this consisted in the surfaces being punctured only once at most. Something similar happens here for the volume. The nodes of the graph Γ can fall only in the interior of the cells, and a cell C_I contains at most one node. If a cell contains no nodes, then we assume it contains at most one link. Also, the partition of the surfaces ∂C_I in S_I^α is refined so that edges of Γ can intersect a cell S_I^α only in its interior and each S_I^α is punctured at most by one link.

Let us turn to study the action of this operator. The first thing to notice is that the presence of the epsilon tensor requires all three fluxes to be different. This means that the volume operator does not act on links since if no nodes is present then two of the S_I^α have to be the same. The important result here is therefore that *the volume operator acts only on nodes of the graphs.*

Let's start considering a single node, i.e. the I-th contribution to (2.15). The cubic operator for each cell is

$$\hat{U} = \frac{1}{48} \sum_{\alpha,\beta,\gamma} \left| \epsilon^{ijk} \hat{E}_i(S_\alpha) \hat{E}_j(S^\beta) \hat{E}_k(S^\gamma) \right| \tag{2.16}$$

We now look at the gauge invariant spin networks, so each node is labeled by an intertwiner $|\imath\rangle$. First of all, the action of (2.16) on a three-valent node is zero. In fact the Gauss law tells us that the sum of the fluxes thorough a surface around a gauge invariant vertex is zero. In the three-valent case, only three S^α give non vanishing fluxes, thus

$$\left[\hat{E}_i(S^\alpha) + \hat{E}_i(S^\beta) + \hat{E}_i(S^\gamma)\right] |\imath\rangle = 0 \quad \Rightarrow \quad \hat{E}_i(S^\alpha)|\imath\rangle = -\left[\hat{E}_i(S^\beta) + \hat{E}_i(S^\gamma)\right] |\imath\rangle \tag{2.17}$$

and using this in (2.16) we get zero because one always has two identical fluxes

$$\epsilon^{ijk} \hat{E}_i(S\alpha) \hat{E}_j(S^\beta) \hat{E}_k(S^\gamma)|\imath\rangle = -\epsilon^{ijk} \left[\hat{E}_i(S^\beta) + \hat{E}_i(S^\gamma)\right] \hat{E}_j(S^\beta) \hat{E}_k(S^\gamma)|\imath\rangle = 0 \tag{2.18}$$

Therefore non-zero contribution to the volume only come from nodes of valence four or higher. In fact those are the case in which the intertwiner is not unique but a genuine independent quantum number. This means that (2.16) probes exactly the degrees of freedom hidden in the intertwiners. Let's consider the four-valent case. Of all the surface cells S^α, only four are punctured by the links. So the Gauss law has four contributions and one can eliminate one flux in favour of the remaining three

$$\hat{E}_i(S^4) = -\hat{E}_i(S^1) - \hat{E}_i(S^2) - \hat{E}_i(S^3) \tag{2.19}$$

The sum in (2.16) reduces to the contributions from the four punctured surfaces and thanks to the last equation, these contribution are all equal. A simple combinatorial exercise shows that there are 48 identical terms, therefore

$$\hat{U} = \left| \epsilon^{ijk} \hat{E}_i(S^1) \hat{E}_j(S^2) \hat{E}_k(S^3) \right| = \left| \hbar^3 \gamma^3 \epsilon^{ijk} J_i^1 J_j^2 J_k^3 \right| \tag{2.20}$$

where in the last step we used the action of the fluxes in terms of the $SU(2)$ generators $\vec{J}_a$ in the spin j_a representation, as given by the generalization of equation (1.131)[2]. Notice that the

[2] Namely, $\hat{E}_i(S)\, D^{(j)}(h_e) = \pm i\hbar\gamma D^{(j)}(h_{e_1})\, J_i\, D^{(j)}(h_{e_2})$ where e is separated into two parts $e = e_1 \cup e_2$ by the intersection point.

orientation sign $\pm$ (between the edge and surface) is irrelevant due to the modulus.

Thus (2.15) is a well defined operator, whose action spectrum is again discrete, with minimal excitation proportional to $(8\pi\gamma G\hbar)^{3/2}$ namely the Planck length cube (see Appendix B). The general case will look more complicated and can be cast as

$$\hat{V}(R)|\Gamma; j_e, \imath_v\rangle = (8\pi\gamma G\hbar)^{3/2} \sum_{v\in R\subset\Gamma} \sqrt{|q_v|}\, |\Gamma; j_e, \imath_v\rangle \tag{2.21}$$

where

$$q_v = \frac{1}{48} \sum_{e_I, e_J, e_K} \kappa(e_I, e_J, e_K)\epsilon^{ijk} J_i^{(e_I)} J_j^{(e_J)} J_k^{(e_K)} \tag{2.22}$$

Here each sum runs over all edges at the node and the orientation factor κ equals $+1$ is the triple of tangent vectors $(\dot{e}_I, \dot{e}_J, \dot{e}_K)$ at the node v is positively oriented, -1 if it is negatively oriented and 0 if the tangent vectors are not linearly independent.

Summarizing, the volume operator acts only on nodes of the graph. Its matrix elements vanish between different intertwiner spaces and since every intertwiner space is finite-dimensional, its spectrum is discrete with minimal excitation proportional to the Plank length cube.

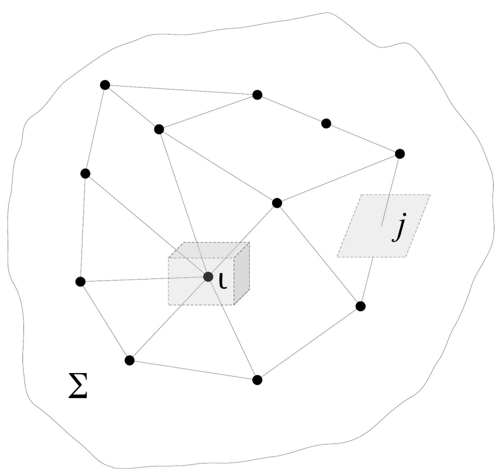


Figure 2.3: At the vertices of the graph there can be non vanishing action of the volume operator, determined by the invariant map $\imath$. The area operator is non trivial when the intersection is by edges labeled with $j \neq 0$. The emerging picture is that of a discrete geometry of the spatial slide Σ, where quanta of volume are connected by the graph edges, which, at the same time, determine the area quanta separating two volume quanta.

Together with the discreteness of the area operator, this result shows that in Loop Quantum Gravity *the space geometry is discrete at the Plank scale.* The point is that the quantum area of a surface receives a contribution from each edge of a spin network intersecting the surface, while the quantum volume of a region receives a contribution from each node of a spin network contained inside the region. This naturally suggests a physical interpretation where a spin network is a state of discrete, quantized spatial geometry, consisting of quantized excitations of volume (at the nodes) separated from each other by quantized excitations of area (at the edges). Within this interpretation, the graph of a spin network state is seen as dual to the quantized geometry defined by the state, each node of the graph being dual to an elementary quantum of volume, while each edge being dual to an elementary quantum of area.

It is important to stress that this concept of quantum geometry described by spin networks is not a built-in discretization, as in lattice approaches to quantum gravity. Nowhere it was imposed as a postulate or assumed. It is a pure result of the quantum theory of General Relativity, similar to the quantization of the energy levels of an harmonic oscillator or the radii of the atomic orbitals. Thanks to this fundamental discreteness, the theory is expected to have no ultraviolet divergences and to resolve the problem of the classical singularities of GR.

One last remark is that after the diffeomorphism constraint is imposed, the excitations of quantum geometry are not localized in any background manifold. Hence the picture is truly background independent, the only physically meaningful information being the relative localization of the quanta of geometry with respect to each other.

In conclusion, a spin network corresponds to a quantum state where the geometry is excited in such a way that there are quanta of volume at the vertices of the graph, as well as quanta of area on surfaces intersected by it. The edges of the graph thus define a certain notion of connectedness for two neighbouring quanta of volume, associated with the magnitude of a surface separating them.

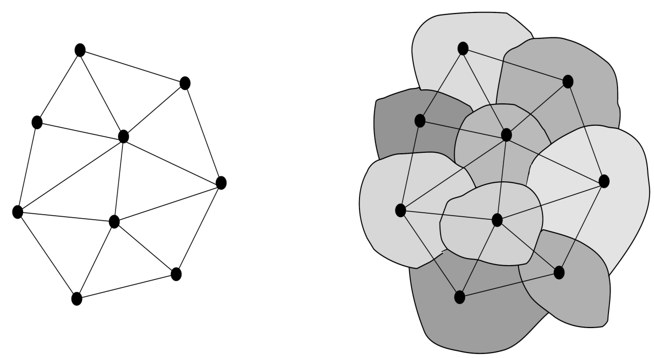


Figure 2.4: Spin network and chunks of space

2.2 Quantum Polyhedra

Studying the spectral problems associated to some geometrical operators, we found two families of quantum numbers which have a direct geometrical interpretation: $SU(2)$ spins, labeling the links of the spin network, and $SU(2)$ intertwiners, labeling its nodes. A four-valent node, for instance, can be interpreted as a "quantum tetrahedron" as we will see below: an elementary "atom of space" whose face areas, volume and dihedral angles are determined by the spin and intertwiner quantum numbers.

The connections between geometric objects and angular momenta in Quantum Mechanics have been observed in many ways since long ago, and the properties of some invariants which can be obtained from $SU(2)$ representations have been used by Ponzano and Regge [35] to build a quantum gravity model in three dimensions.

The remarkable fact here is that the same geometrical interpretation outlined in the previous sections resulting from the LQG theory, can be obtained from a formal quantization of the degrees of freedom of the geometry of polyhedra (for instance the tetrahedron) without any reference to the complete quantization of General Relativity. We will see as an example that one can obtain the Hilbert space describing a single quantum tetrahedron [9] which will contain states describing its quantum geometry, the latter being coincident with the one defined by LQG. Then we move to general polyhedra.

2.2.1 Classical Tetrahedron

As a first example let us look at the simplest case which was first introduced in [9]. A tetrahedron can be seen as the complex envelope of four points in the three-dimensional Euclidean space.

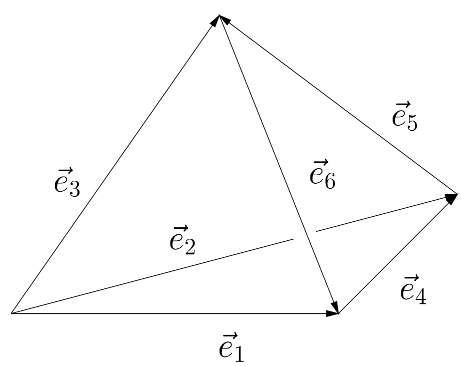


Figure 2.5: Classical tetrahedron

We can see from the figure 2.5 that a triad $\vec{e}_1, \vec{e}_2, \vec{e}_3$ of independent vectors (nine numbers) defines completely the tetrahedron. If we are interested in its geometry, meaning only in properties independent from space orientation, then the relevant independent parameters become six due to the factorization of the rotation group.

Let us now consider the vectorial areas of the tetrahedron given by

$$\vec{n}_1 = -\vec{e}_1 \times \vec{e}_2 \qquad \vec{n}_2 = -\vec{e}_2 \times \vec{e}_3 \qquad \vec{n}_3 = -\vec{e}_3 \times \vec{e}_1 \qquad \vec{n}_4 = \vec{e}_4 \times \vec{e}_5 \equiv -\vec{n}_1 - \vec{n}_2 - \vec{n}_3 \quad (2.23)$$

where the last equation is simply the closure condition[3] and it shows that only three of these vectors called normals are independent. Now we want to see whether the geometry of the tetrahedron can be reconstructed from the normals rather than the edges.

[3]In the form $\int_S \vec{n} da = 0$ it holds for any closed surface S

The independent parameters must belong to the set of invariants which can be built from $\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4$, namely: their squares and their mutual scalar products. We also notice that the triple product $\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3$ is invariant and we shall add it to our set.

We have therefore ten numbers which are not independent due to the last equation in (2.23). Taking the scalar products of the triple product with the normals we obtain four independent equations

$$\begin{aligned} (n_1)^2 + n_{12} + n_{13} + n_{14} &= 0 \\ (n_2)^2 + n_{12} + n_{23} + n_{24} &= 0 \\ (n_3)^2 + n_{13} + n_{23} + n_{34} &= 0 \\ (n_4)^2 + n_{14} + n_{24} + n_{34} &= 0 \end{aligned} \tag{2.24}$$

where $(n_i)^2 = \vec{n}_i \cdot \vec{n}_i$ and $n_{ij} = \vec{n}_i \cdot \vec{n}_j$. It is now easy to verify that the independent parameters are the four squared areas $(n_i)^2$ and two dihedral angles associated to edges sharing a vertex, e.g. n_{12} and $n_2 3$. With this choice, and using the relation between couples of angles associated to opposite edges, the second equation in (2.24) gives the relation between n_{13} and the chosen variables

$$n_{13} = \frac{1}{2}\left[(n_4)^2 - (n_1)^2 - (n_2)^2 - (n_3)^2\right] - n_{12} - n_{23} \tag{2.25}$$

As it is known from the literature, taking various scalar products one can obtain an algebraic system with just quadratic and constants terms. Provided some geometrical non-holonomic restrictions such as $(n_i)^2 > 0$ and $|n_{ij}| < [(n_i)^2 (n_j)^2]^{1/2}$ are satisfied, the system has two sets of opposite roots. The correct solution is clearly the one with $\vec{e}_i \cdot \vec{e}_i > 0$, the other one corresponding to pure imaginary edges.

One last thing to point out regarding the classical geometry, is that the volume of the classical tetrahedron in terms of the normals is given by

$$V^2 = -\frac{1}{36}\epsilon_{abc} n_1^a n_2^b n_3^c = -\frac{1}{36}\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 \tag{2.26}$$

To summarize, the values of the four areas and of two ”non opposite” dihedral angles actually describe and define completely the six-dimensional tetrahedron geometry.

2.2.2 Quantum Tetrahedron

Now we want to quantize only the degrees of freedom of a tetrahedron, instead of referring to the full LQG theory. The first step is to associate to the four faces of the tetrahedron four unitary irreducible representations of $SU(2)$ acting on the spaces $\mathcal{H}_{j_i}$. Here $i = 1, 2, 3, 4$ labels the face and j is the spin of the representation. The tensor products of those spaces carries a reducible representation of $SU(2)$ that can be decomposed in its irreducible components (recall (1.155)). Let's denote the ensemble of the spin-zero components, i.e. the $SU(2)$ invariant components of the tensor product, as

$$\mathcal{I}_{j_1 \ldots j_4} = \mathrm{Inv}\left[\bigotimes_{i=1}^{4} \mathcal{H}_{j_i}\right] \tag{2.27}$$

This space can be interpreted as the space of quantum states of a quantum tetrahedron whose i-th triangle has area given by the (square root of the) $SU(2)$ Casimir operator, $A_i = l_P^2\, C(j_i)$. The areas A_i are nothing but the normals n_i described in the previous section, up to a factor. The Hilbert space $\mathcal{H} = \bigotimes_j \mathcal{I}_{j_1 \dots j_4}$ describes the degrees of freedom associated to the volume and the dihedral angles of this atom of quantum geometry.

The geometric quantization of the classical degrees of freedom is based on the identification of the $SU(2)$ generators $\vec{J}_i$ as the quantum operators corresponding to the $\vec{n}_i$. The squared normals will therefore be the $SU(2)$ Casimirs $C^2(j)$, as in LQG. As mentioned, this construction gives directly the same quantum geometry that one finds via a much longer path by quantising the phase space of general relativity. A quantum state of a tetrahedron with fixed areas must live in the tensor product $\bigotimes_{i=1}^4 \mathcal{H}_{j_i}$ of the spin j_i representations spaces. The closure constraint now reads

$$\sum_{i=1}^{4} \vec{J}_i = 0 \tag{2.28}$$

and imposes that the state of the quantum tetrahedron is invariant under global rotations. This means that it will be a singlet state, namely an intertwiner map

$$\bigotimes_{i=1}^{4} \mathcal{H}_{j_i} \to \mathcal{H}_{j=0} \equiv \mathbb{C} \tag{2.29}$$

which is nothing but the Hilbert space of intertwiners $\mathcal{I}_{j_1 \dots j_4}$ given in (2.27).

The operators J_i^2,$\vec{J}_i \cdot \vec{J}_j$ are well defined on this space and so is the operator

$$\hat{U} = -\epsilon_{abc} J_1^a J_2^b J_3^c \tag{2.30}$$

which is the quantum counterpart of (2.26). To be precise, the absolute value of U can be identified with the classical squared volume $36V^2$, which is again in agreement with the standard LQG results.

To find the angle operators, we have to introduce the quantities $\vec{J}_{ij} = \vec{J}_i + \vec{J}_j$. Their geometrical interpretation can be found applying the same arguments with the classical counterparts $\vec{n}_i + \vec{n}_j$ (not to be confused with the notation of the previous paragraph n_{ij}). It turns out that $\sqrt{J_{ij}^2}$ is proportional to the area A_{ij} of the internal parallelogram, whose vertices are given by the midpoints of the segments belonging to either the face i or the face j but not both. Given these quantities, the angle operators $\hat{\theta}_{ij}$ can be obtained from

$$J_i J_j \cos \hat{\theta}_{ij} = \vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2} \left(J_{ij}^2 - J_i^2 - J_j^2 \right) \tag{2.31}$$

Thus we can conclude that the quantum geometry of a tetrahedron is encoded in the operators J_i^2, J_{ij}^2, and U, acting on $\mathcal{H}$.

An important point to make here is that out of the six independent classical variables, only five commute in the quantum theory. In fact it's easy to see that we have of course

$$\left[J_k^2 \,,\, \vec{J}_i \cdot \vec{J}_j \right] = 0 \tag{2.32}$$

but we also have

$$\left[\vec{J}_1 \cdot \vec{J}_2\,,\,\vec{J}_1 \cdot \vec{J}_3\right] = \frac{1}{4}\left[J_{12}^2\,,\,J_{13}^2\right] = i\epsilon_{abc}J_1^a J_2^b J_3^c \equiv -i\hat{U} \neq 0 \tag{2.33}$$

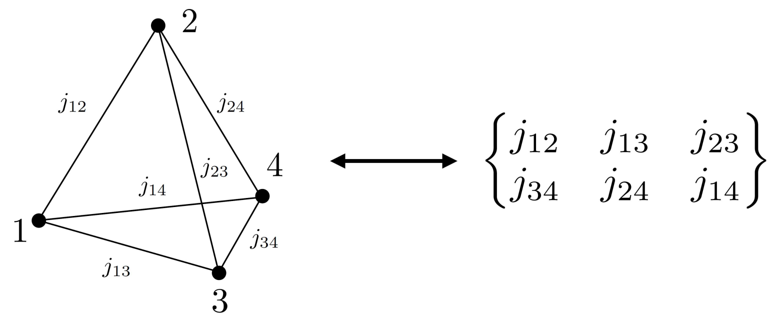
A complete set of commuting operators is given by $\{J_i^2, J_{12}^2\}$. Said differently, a basis for $\mathcal{I}_{j_1\ldots j_4}$ is provided by the eigenvectors of any one of the operators J_{ij}^2. The corresponding eigenbasis is denoted $|j\rangle_{ij}$ so that, for instance, the basis $|j\rangle_{12}$ diagonalizes the four triangle areas and the dihedral angle θ_{12} (or equivalently, the area A_{12} of the internal parallelogram).

As it is known [49], the relation between different basis is easily obtained from $SU(2)$ recoupling theory: the matrix describing the change of basis in the space of intertwiners is the usual Wigner $\{6j\}$ symbol (see Appendix A)

$$W_{jk} = {}_{12}\langle j|k\rangle_{13} = (-1)^{\sum_i j_i}\sqrt{d_j d_k}\begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & k \end{Bmatrix} \tag{2.34}$$

so that

$$|k\rangle_{13} = \sum_j W_{jk}|j\rangle_{12} \tag{2.35}$$



Notice that the 6j symbol can be associated to a labelling of the six edges of a tetrahedron by irreducible representations of SU(2). This abstract association is traditionally used simply to express the symmetry of the 6j –symbol as an algebraic object but it has a deeper geometrical meaning [34].

The physical interpretation in terms of a quantum language of these geometrical result is the following: the states $|j\rangle_{12}$ are eigenvectors of the five commuting geometrical operators $\{J_i^2, J_{12}^2\}$, thus the average value of the operator corresponding to the sixth observable, say J_{13}^2, is maximally spread on these states. This means that a basis state has an undetermined classical geometry or, in other words, is not an eigenstate of the geometry. In order to study the semiclassical limit of the geometry, one is led to consider superpositions of states. Suitable ones could be constructed for instance requiring that they minimise the uncertainty relations between non–commuting observables, such as

$$(\Delta J_{12}^2)^2\,(\Delta J_{13}^2)^2 \geq \frac{1}{4}\left|\langle\left[J_{12}^2\,,\,J_{13}^2\right]\rangle\right|^2 \equiv 4|\langle\hat{U}\rangle|^2 \tag{2.36}$$

and they would be in fact coherent states. In general, equation (2.36) implies that the geometry of the quantum tetrahedron cannot be defined exactly, because of quantum fluctuations, so when we speak about geometry we mean it in a semiclassical sense, or ”on average”. Semiclassical states for the tetrahedron have been studied in [38] where they are built in $\mathcal{I}_{j_1 \ldots j_4}$ such that all the relative uncertainties vanish in the large scale limit[4]. This is something that we will also use later in the work for new semiclassical states, built out of the so called Twisted Geometries which will be discussed in the next chapter.

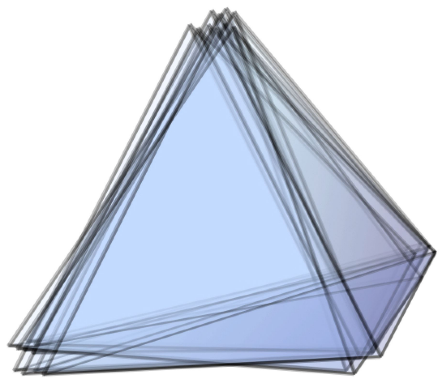


Figure 2.6: The geometry of a tetrahedron cannot be sharp in the quantum theory, in the same sense in which the three components of angular momentum can never be all sharp. Due to the non-commutativity, the geometry is fuzzy at the Planck scale.

To conclude, in the quantum geometry seen above not all the variables describing the geometry of the tetrahedron turn out to commute. Consequently, in general there is no state in $\mathcal{H}$ that corresponds to a given classical geometry of the tetrahedron. This fact raises immediately the problem of finding semiclassical quantum states in $\mathcal{H}$ that approximate a given classical geometry, in the sense in which wave packets or coherent states approximate classical configurations in ordinary quantum theory.

2.2.3 Semiclassical geometry

We have seen that spin network states form a basis in the kinematical Hilbert space and diagonalize geometric operators. In particular, the quantum numbers carried by a spin network define a notion of quantum geometry[5], as outlined above. One would like now to compare these quantum labels with the kinematical[6] metric g_{ab} defining the classical geometry of space.

[4]This does not necessarily mean that they minimize equation (2.36), so those states would be coherent in a different sense.

[5]the graph Γ determines the contiguity relations between chunks of space and it is the dual graph of a decomposition of the physical space.

[6]We use the word kinematical to mean that it is an arbitrary metric, not necessarily a solution of Einstein's equations, just like an arbitrary spin network spans the kinematical Hilbert space, not necessarily solving the diffeomorphisms and Hamiltonian constraints.

However the quantum geometry is very different from the classical one and it seems largely insufficient to reconstruct a metric.

The three main features of quantum geometry are:

(i) Quantized spectra: the spectra of geometric operators are discrete, as opposed to the continuum values of their classical counterparts. This is a standard situation in quantum mechanics, not different from the discretization of energy levels of the harmonic oscillator, for instance.

(ii) Non commutativity: not all geometric operators commute among themselves. This is a consequence of the non-commutativity of the fluxes. This is also standard, like the incompatibility of position and momentum observables.

(iii) Distributional nature : the states capture only a finite number of components of the original fields, that is their values along paths (for the connection) and surfaces (for the triad). This is reminiscent (but not physically identical) of what happens in lattice theories, where the continuum field theory is discretized on a fixed lattice and only a finite number of degrees of freedom are captured.

Despite these differences, the theory must admit a semiclassical regime in which a smooth geometry emerges. The first point (i) is easier to deal with: also the orbitals of the hydrogen atoms are quantized, placed at distances labelled by an integer n. The classical Keplerian behaviour is recovered if we look at the large n limit. Similarly, continuum spectra are recovered in the large spin limit $j_l \to \infty$. Points (ii) and (iii) are more subtle and the key to deal with them is the use of coherent states, namely linear superpositions of spin network states peaked on a smooth geometry. We will focus on these later on the work but roughly speaking and by analogy with simple systems, coherent states for loop quantum gravity are peaked on a point $(A_a^i(x), E_i^a(x))$ in the classical phase space, which defines an intrinsic (through the triad) and extrinsic (through the connection) 3-geometry. State that minimize some uncertainties between operators where introduced by Thiemann and collaborators and will be discussed in the next section. They address the second point (ii). In order to address (iii) and recover a smooth geometry everywhere on the spatial submanifold Σ, the coherent states must have support over an infinite number of graphs. This is a formidable task and for practical purposes one needs to approximate the theory.

The convenient move is to allow only states living on a fixed graph Γ. Then the Hilber space $\mathcal{H}_\Gamma$ provides a *truncation* of the theory which may still be sufficient to capture the physics of appropriate regimes. However the coherent states in $\mathcal{H}_\Gamma$ will capture only a *finite* number of components of a continuum geometry therefore, to be able to interpret these truncated coherent states in a physical sense, one needs to use these data to approximate a continuum geometry. Fortunately, a notion of interpolating geometry emerges naturally. The insight comes from the structure of the space $\mathcal{H}_\Gamma$. As we have seen (1.160), it decomposes in terms of $SU(2)$ invariant spaces which we now call $\mathcal{H}_F$. As shown above, for a 4-valent node, the intertwiner represents the state of a quantum tetrahedron. For a generic F-valent node one would expect a relation to polyhedra with F faces, as it is the case.

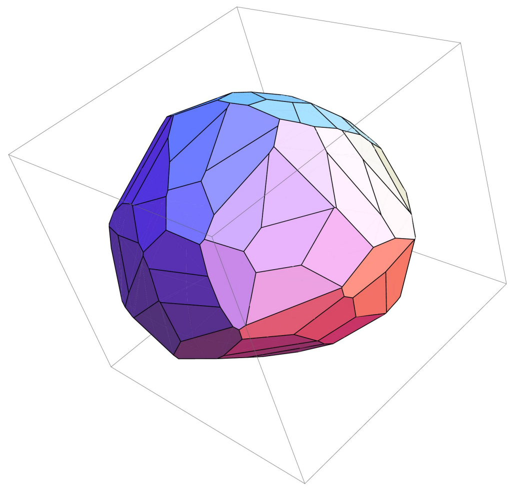


Figure 2.7: Random polyhedron with many faces

The phase space of polyhedra

The first important fact is that $\mathcal{H}_F$ is the quantization of some classical phase space, $\mathcal{S}_F$, introduced by Kapovich and Millson in [23]. As it turns out, to each point in $\mathcal{S}_F$ there is associated a unique convex polyhedron with F faces of given areas. This is guaranteed by an old theorem by Minkowski. Therefore

$$\text{polyhedra with } F \text{ faces} \quad \longleftrightarrow \quad \text{classical phase space } \mathcal{S}_F \quad \longleftrightarrow \quad \text{intertwiner space } \mathcal{H}_F \tag{2.37}$$

An immediate consequence of this is a complete characterization of coherent states at a fixed graph: they uniquely define a collection of polyhedra associated to each node of the graph. This provides a simple and compelling picture of the degrees of freedom of $\mathcal{H}_\Gamma$ in terms of discrete geometries. As we will see, these are associated with a parametrization of the classical holonomy-flux variables in terms of twisted geometries, described in the next chapters.

Classically speaking, a convex polyhedron is the convex hull of a finite set of points in 3d Euclidean space. It can be represented as the intersection of finitely many half-spaces as

$$\mathcal{P} = \{x \in \mathbb{R}^3 | n_i \cdot x \leq a_i, \;\; i = 1, \ldots, m\} \tag{2.38}$$

where n_i are arbitrary vectors and a_i are real numbers. That definition is not unique and it is in fact redundant: the minimal set of half-spaces needed to describe a polyhedron corresponds to taking their number m equal to the number of faces F of the polyhedron. We want to express the polyhedron in terms of the areas of its faces and the unit normals to the planes that support such faces. So let's consider a set of unit vectors $n_i \in \mathbb{R}^3$ and a set of positive real numbers A_i such that the following *closure* condition is satisfied

$$C = \sum_{i=1}^{F} A_i n_i = 0 \tag{2.39}$$

Now it's easy to obtain a convex polyhedron with F faces having areas A_i and normals n_i. Consider the plane orthogonal to each vector n_i, then translate this plane at a distance a_i from the origin. The intersection of the half-spaces bounded by the planes defines the polyhedron, $n_i \cdot x \leq a_i$. One can then adjust the heights $a_i = a_i(A)$ so that the faces have area A_i. Remarkably, a convex polyhedron with such areas and normals always exists. Moreover, it is unique, up to rotations and translations. These results are established by the following theorem due to H. Minkowski [27]

If $n_1, \ldots, n_F$ are non-coplanar unit vectors and $A_1, \ldots, A_F$ are positive numbers such that the closure condition (2.39) *holds, then there exists a convex polyhedron whose faces have outwards normals n_i and areas A_i. If each face of a convex polyhedron is equal in area to the corresponding face with parallel external normal of a second convex polyhedron and conversely, then the two polyhedra are congruent by translation.*

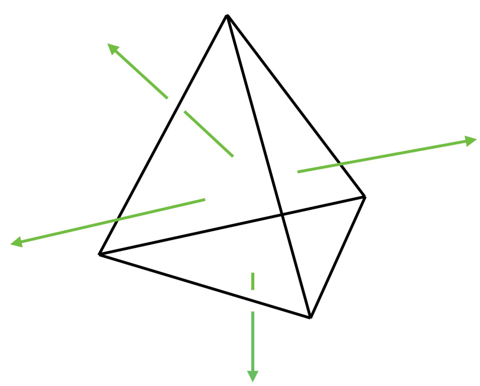


Figure 2.8: In the case $F = 4$ we recover the case seen above

Let us now consider F vectors that have given norms $A_1, \ldots, A_F$ and such that they sum up to zero. The space of such vectors modulo rotations has the structure of a symplectyc manifold and it is the phase space introduced by Kapovich and Millson

$$\mathcal{S}_F = \{n_i \in (S^2)^F \mid \sum_{i=1}^{F} A_i n_i = 0\}/SO(3) \tag{2.40}$$

The Poisson structure on this $2(F-3)$ -dimensional space is the one that descends via symplectic reduction from the natural $SO(3)$-invariant Poisson structure on each of the F spheres S^2. To see where the dimension comes from, it is enough to notice that from F we go to $2F - 3$ due to the closure condition. Then, we have to further consider the rotation symmetry which decreases the degrees of freedom of 3 more, so that we only need $2(F-3)$ numbers to specify a polyhedron.

It's easy to see why this is interesting for us: polyhedra can be glued together to approximate a smooth manifold and their geometry will induce a discrete metric of some sort. So the intertwiners seen in the LQG theory will be quantized polyhedra.

We will call (2.40) the space of shapes of polyhedra with fixed areas and thanks to Minkowski's theorem, a point in $\mathcal{S}_F$ with non coplanar normals identifies a unique such polyhedron. Leaving aside the details about degeneracy (configurations corresponding to coplanar normals) and the reconstruction procedure of the polyhedra from holonomies and fluxes[7] we are now interested in the connection with the quantum theory.

Relation to LQG

In the first chapter we have seen that intertwiners are the building blocks of spin network states, an orthonormal basis of the Hilbert space of loop quantum gravity. Moreover, it is a fact that intertwiners are also the quantization of the phase space $\mathcal{S}_F$. It is therefore understood that an intertwiner is the state of a *quantum polyhedron*, and spin network states are a collection of adjacent quantum polyhedra associated with each vertex.

Consider again the space of vectors in 3d-Euclidean space with norm j. This is a phase space, the Poisson structure being the rotationally invariant one, typical of the 2-sphere S^2_j of radius j. As it is known, its quantization is the representation space $\mathcal{H}^j$ of $SU(2)$, with the half-integer spin j quantizing the norm of the vectors. We are interested in the phase space $\mathcal{S}_F$, that is the space of F vectors that sum to zero, up to rotations. The Poisson structure on $\mathcal{S}_F$ is obtained via the symplectic reduction of the Poisson structure on the product of F spheres of given radius. Thanks to Guillemin-Sternberg's theorem that quantization commutes with reduction, we can quantize first the unconstrained phase space $\otimes_i S^2_{j_i}$, and then reduce it at the quantum level exctracting the subspace of $\otimes_i \mathcal{H}^{j_i}$ that is invariant under rotations. This gives precisely the intertwiner space $\mathcal{H}_F = \mathrm{Inv}\big[\otimes_{i=1}^F \mathcal{H}^{j_i}\big]$. The situation is summarized by the commutativity of the following diagram,

$$
\begin{array}{ccccc}
 & \bigtimes_i S^2_{j_i} & \longrightarrow & \otimes_i \mathcal{H}^{j_i} & \\
\text{Symplectic reduction} & \downarrow & & \downarrow & \text{Quantum reduction} \\
 & \mathcal{S}_F & \longrightarrow & \mathcal{H}_F &
\end{array}
$$

As for the tetrahedron, the correspondence between classical quantities and their quantization is the following: up to a constant, the generators $\vec{J}_i$ of $SU(2)$ acting on each representation

[7] The details can be found in [10]

space $\mathcal{H}^{j_i}$ are understood as the quantization of the vectors $A_i n_i$. The mentioned constant in LQG is the Immirzi parameter γ times Plank's area $8\pi l_p^2$

$$A_i n_i \qquad \rightarrow \qquad \hat{E}_i = 8\pi\gamma l_p^2 \vec{J}_i \tag{2.41}$$

The closure condition (2.39) on the normals is promoted to an operator equation (as seen for the 4-valent case)

$$\sum_{i=1}^{F} \vec{J}_i = 0 \tag{2.42}$$

This equation defines the space of intertwiners and corresponds to the Gauss constraint of classical General Relativity in Ashtekar variables. Then one can proceed to associate operators to the observable we are interested in. In agreement with what we studied in the first chapter, we find again that the area of a face of the quantum polyhedron is

$$\hat{A}_i = \sqrt{\hat{E}_i \cdot \hat{E}^i} = 8\pi\gamma l_p^2 \sqrt{j_i(j_i+1)} \tag{2.43}$$

The scalar product between two generators of $SU(2)$ associated to two faces of the polyhedron measures the angle θ_{ij} between them

$$\hat{\theta}_{ij} = \arccos \frac{\vec{J}_i \cdot \vec{J}_j}{\sqrt{j_i(j_i+1) j_j(j_j+1)}} \tag{2.44}$$

in agreement with (2.31). Notice that as mentioned above, the angle operators do not commute among themselves, therefore it is not possible to find a state for a quantum polyhedron that has a definite value of all the angles between its faces. In fact an eigenstate of a maximal commuting set of angles is far from the state of a classical polyhedron: it is an infinite superposition of polyhedra of different shapes (including different combinatorial classes).

2.2.4 Coherent intertwiners

Semiclassical states for a quantum polyhedron were introduced in [25] and are called Livine-Speziale (LS) coherent intertwiners. They naturally describe the concept of a semiclassical polyhedron: the areas are in fact sharp and the expectation value of the non-commuting angle operators θ_{ij} reproduces the classical angles between faces in the large spin limit.

Their definition is quite simple: in order to desribe a semiclassical polyhedron with F faces, one first considers the tensor product of F $SU(2)$ coherent states[8]

$$|j_1, \vec{n}_1\rangle \cdots |j_F, \vec{n}_F\rangle \equiv |j_1, \vec{n}_1 \cdots j_F, \vec{n}_F\rangle \qquad \in \qquad \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_F} \tag{2.45}$$

and then one projects down onto the invariant part $\mathcal{H}_F = \mathrm{Inv}\big[\otimes_{i=1}^F \mathcal{H}^{j_i}\big]$. The resulting state will describe a semiclassical polyhedron. To implement the projection explicitly, one has to

[8]these are described with a little more detail in chapter 4.

integrate over the $SU(2)$ actions on the state, proceeding by group averaging as explained in chapter 1 (see also appendix A). Therefore the LS coherent intertwiner are defined as

$$||\underline{j},\underline{n}\rangle_{LS} = \int dh \bigotimes_{e=1}^{F} h|j_e,\vec{n}_e\rangle \tag{2.46}$$

and it can easily be seen that expanding in the intertwiner basis one gets

$$||\underline{j},\underline{n}\rangle_{LS} = \sum_k c_k(j,n)|k\rangle \tag{2.47}$$

with coefficient $c_k(j,\vec{n}) = \langle j_1,\cdots,j_F\,;\,k|j_1,\vec{n}_1\cdots j_F,\vec{n}_F\rangle \equiv \langle k|j_1,\vec{n}_1\cdots j_F,\vec{n}_F\rangle$, where the intertwiner basis is

$$\imath^{(k)}_{m_1,\ldots,m_F}|j_1,m_1\rangle\cdots|j_F,m_F\rangle = |j_1,\cdots,j_F\,;\,k\rangle \equiv |k\rangle \tag{2.48}$$

As a matter of fact, the projection implemented via group averaging in equation (2.46) is nothing but the action of the operator

$$P = \sum_k |k\rangle\langle k| \tag{2.49}$$

which, if applied to the tensor product state made of coherent states of $SU(2)$, gives

$$P|j_1,\vec{n}_1\rangle\cdots|j_N,\vec{n}_N\rangle = \sum_k c_k(j,n)|k\rangle \tag{2.50}$$

The coherent intertwiners have a number of properties, and have been very important for the development of the theory. For a more complete discussion see [25]. Notice however that they are coherent states for the space of intertwiners only and should not be confused with coherent spin network states for loop quantum gravity. The latter will be presented in the rest of this work, first following the literature [48, 46, 40] and then proposing a new set of states based on twisted geometries [12].

2.3 Thiemann's Coherent states

The first coherent states that were proposed for Loop Quantum Gravity are called Heat Kernel Coherent States and were introduced and studied by Thiemann and collaborators [46, 42, 43, 44]. Coherent states describe the nearly classical configurations of a quantum system. The spin network states are not good candidate since, being eigenstates of geometric operators they will be maximally spread with respect to the corresponding conjugated variables.

2.3.1 The complexifier method

A systematic algorithm for constructing coherent states is provided by the so-called complexifier method. The complexifier $\mathcal{C}(q,p)$ is a function on the classical phase space, which is assumed to satisfy certain requirements. In particular, the complexifier must be a positive function, and must have a stronger than linear dependence on the momentum variable. By canonically quantizing the classical function $\mathcal{C}(q,p)$ the corresponding quantum operator C, which we also refer to as the complexifier. The complexifier is used to construct coherent states in the following way. We start by applying the operator e^{-C} to the delta function of the configuration variable, obtaining the function

$$\psi_{q_0}(q) = e^{-C}\delta(q,q_0) \tag{2.51}$$

The effect of the operator e^{-C} is to smooth out the delta function, producing a function which has a peak of finite width concentrated around the point $q = q_0$. The next step is to make coherent the functions (2.51) by complexifying the label q_0, i.e. analytically extending them to complex values of q_0. Symbolically

$$\psi_{z_0}(q) = \left[e^{-C}\delta(q,q_0)\right]_{q_0\to z_0} \tag{2.52}$$

where the complexification rule is specified by

$$z(q,p) = \sum_n \frac{i^n}{n!}\{q,\mathcal{C}(q,p)\}^{(n)} \tag{2.53}$$

where $\{q,\mathcal{C}\}^{(n)}$ is the Poisson brackets iterated n times, i.e. $\{q,\mathcal{C}\}^{(n+1)} = \{\{q,\mathcal{C}\}^{(n)},\mathcal{C}\}$ with $\{q,\mathcal{C}\} = q$. Equation (2.53) also provides the relation between the label z and the variables of the classical phase space, thereby defining the point in the classical phase space on which the coherent state (2.52) is supposedly peaked.

The construction of the states (2.52) guarantees that they are eigenstates of an annihilation operator a, which is defined through a quantum analog of the classical relation as

$$a = e^{-C}qe^{C} = \sum_n \frac{1}{n!}\,[q,C]^{(n)} \tag{2.54}$$

with $[q,C]^{(n)}$ the n times iterated commutator. A simple calculation shows that the state ψ_{z_0} is an eigenstate of the operator a with complex eigenvalue z_0

$$a\,\psi_{z_0} = z_0\,\psi_{z_0} \tag{2.55}$$

This result provides the main motivation for why the states (2.52) can be expected to be legitimate coherent states. Indeed the eigenvalue equation above implies that they are optimally peaked with respect to the Hermitian operators

$$Q = \frac{1}{2}(a + a^\dagger) \qquad\qquad P = \frac{1}{2i}(a - a^\dagger) \tag{2.56}$$

in the sense that the uncertainty relation

$$\Delta Q \Delta P \geq \frac{1}{2}|\langle[Q,P]\rangle| \tag{2.57}$$

is minimized when both sides are evaluated in the state (2.52). However, it is not guaranteed that the states also posses other semiclassical properties such as the vanishing of the relative uncertainty, and they should be checked separately in each case.

2.3.2 Heat Kernel Coherent States

In order to derive coherent states from the complexifier construction, one must start by considering the form of the delta function on the appropriate Hilbert space. For instance, on the gauge invariant Hilbert space $\mathcal{H}^G_\Gamma$ associated to a fixed graph, the delta function δ_Γ should satisfy

$$\int d\mu_\Gamma(g_e)\delta_\Gamma(g_e, q_e)f(g_e) = f(q_e) \tag{2.58}$$

where $d\mu_\Gamma$ denotes the Haar measure on $\mathcal{H}^G_\Gamma$ defined for functions of the same graph. It's easy to verify that the expansion of the delta function in the spin network basis is given by

$$\delta_\Gamma(g_e, q_e) = \sum_{j_e, \imath_v} \psi_{(\Gamma, j_e, \imath_v)}[q_e]\overline{\psi_{(\Gamma, j_e, \imath_v)}[g_e]} \tag{2.59}$$

Thus, an explicit expression for coherent states on $\mathcal{H}^G_\Gamma$ can now be written down, assuming that the spin network states are eigenstates of the complexifier. Denoting the eigenvalues by $\lambda(j_e, \imath_v)$, we have

$$\psi^\Gamma_{\{z_e\}}(g_e) = e^{-\lambda(j_e,\imath_v)} \sum_{j_e, \imath_v} \psi_{(\Gamma, j_e, \imath_v)}\, z_e(A, E)\, \overline{\psi_{(\Gamma, j_e, \imath_v)}} \tag{2.60}$$

This expression is of course not of much practical use, unless an explicit expression is available for the eigenvalues of the complexifier. Amongst the operators whose spectra are known in closed form, the area operator seems like a natural candidate for a complexifier, due to the simplicity of its action on spin networks. Coherent states based on using (a suitable version of) the area operator as the complexifier are commonly known as heat kernel coherent states and where introduced by Thiemann [7, 8] who was inspired by a different context studied by Hall [20].

The specific complexifier chosen by Thiemann is a variant of the squared area operator, defined in terms of the parallel transported flux operator. To each edge e of a spin network there is associated a corresponding surface S_e, which intersects e but does not intersect any other edges of the spin network. Using the surface S_e, one then defines a parallel transported flux $E_i^{(x_0)}(S_e)$, in which points on the surface are transported to a fixed point x_0 on the edge through a path inside the surface that first goes to the edge and then along it until it reaches x_0. The complexifier associated to the edge is then defined as

$$\mathcal{C}_e = \frac{t}{2}(p_{x_0})^2 \tag{2.61}$$

where $(p_{x_0})_i = (8\pi G\gamma)^{-1}E_i^{(x_0)}(S_e)$ and t is a parameter which will control the peakedness properties of the resulting coherent states.

Before continuing, and also in order to convince ourself about the method that is being used here, let us briefly look at the simple case of the Harmonic Oscillator. Applying this methods in standard quantum mechanics with a complexifier $\mathcal{C} = tp^2/2$, the action of the operator

$e^{-tp^2/2}$ on the delta function $\delta(x-x_0) = (2\pi)^{-1}\int dk e^{ik(x-x_0)}$ gives the well known Gaussian $\psi_{x_0}(x) \sim e^{-(x-x_0)^2/2t}$. Then, complexifying $x_0 \to z_0 = x_0 + itp_0$ one gets the familiar Gaussian coherent state $\psi_{(x_0,p_0)}(x) \sim e^{ip_0 x} e^{-(x-x_0)^2/2t}$. Of course in this simple example equations (2.55), (2.56) and (2.57) are easy and very well known.

Going back to Thiemann's states, now we see that the operator corresponding to $\mathcal{C}_e$ acts only on the edge e, so it is enough to carry out the construction of coherent states within the Hilbert space $\mathcal{H}_e = L_2[SU(2)]$ of a single edge. After the construction is completed, coherent states on a fixed graph can be obtained at the non-gauge invariant level as tensor products of the single-edge coherent states, and gauge invariant coherent states will be given by projections of such tensor products onto the gauge invariant Hilbert space.

The expansion of the delta function on $\mathcal{H}_e$ is given by

$$\delta(g,h) = \sum_j d_j \chi^{(j)}(h\, g^{-1}) \tag{2.62}$$

where $\chi^{(j)} = \mathrm{Tr}\, D^{(j)}(g)$ is the character of the spin j representation of $SU(2)$, and $d_j = 2j+1$. Now since the complexifier operator acts diagonally on the spin j subspace, with eigenvalue $(t/2)j(j+1) \equiv (t/2)\lambda_j$, one immediately finds

$$\psi_H(g) = \sum_j d_j\, e^{-\frac{t}{2}\lambda_j}\, \chi^{(j)}(H\, g^{-1}) \tag{2.63}$$

as the expression for coherent states on $\mathcal{H}_e$. Here H is the complexification of the $SU(2)$ element h and is therefore a element of $SL(2,\mathbb{C})$, as can be verified computing H from equation (2.53). In fact a bit of algebra shows that computing

$$H = \sum_n \frac{i^n}{n!}\{h_e\,,\, \frac{t}{2}(p_{x_0})^2\}^{(n)} \tag{2.64}$$

gives the two equivalent following expressions

$$H = h\, e^{\frac{t}{2}\vec{p}_0\cdot\vec{\sigma}} \qquad\qquad H = e^{\frac{t}{2}\vec{p}_0'\cdot\vec{\sigma}}\, h \tag{2.65}$$

where we have relabeled $h_e \to h$, $(p_{s(e)})_i \to (p_0)_i$ and $(p_{t(e)})_i \to (p_0')_i$, $s(e)$ and $t(e)$ being the source and the target of the edge. This confirms that H is indeed an element of $SL(2,\mathbb{C})$, since the expressions above are the standard decomposition of an $SL(2,\mathbb{C})$ element into a $SU(2)$ rotation and a boost. The variables of the two decompositions are related to each other by

$$\vec{p}_0' \cdot \vec{\sigma} = h\,(\vec{p}_o\cdot\vec{\sigma})\, h^{-1} \tag{2.66}$$

reflecting the relation $E^{t(e)}(S) = h_e\, E^{s(e)}(S)\, h_e^{-1}$ between the flux variables at the endpoints of the edge e.

2.3.3 Properties

From the Complexifier method one gets the Heat Kernel CS and one finds the expression (2.63), for the coherent states on $\mathcal{H}_e$, where $H \in SL(2,\mathbb{C})$ is the complexification of $h \in SU(2)$. As it is known, an orthonormal basis in $\mathcal{H}_e$ is given by the states $|j,a,b\rangle$, where now we consider them as already normalized such that $\langle g|j,a,b\rangle = \sqrt{d_j} D^{(j)a}{}_b(g)$. So one sees that from (2.63) the CS $\psi_H(g) = \langle g|h,\vec{p}_0\rangle$ can be expanded as

$$|h,\vec{p}_0\rangle = \sum_{j,a,b} \sqrt{d_j} e^{-\frac{t}{2}j(j+1)} D^{(j)a}{}_b(h\, e^{i\frac{t}{2}\vec{p}_0\cdot\sigma})\,|j,a,b\rangle \tag{2.67}$$

Resolution of the Identity

An important property of the HKCS $|h,\vec{p}_0\rangle$ is that they resolve the identity n the space $\mathcal{H}_e = L_2[SU(2),dg]$

$$\mathbb{1} = \int d\mu(g,p)|h,\vec{p}_0\rangle\langle h,\vec{p}_0| \tag{2.68}$$

and therefore provide an overcomplete basis in $\mathcal{H}_e$. In equation (2.68), the integral is taken over the classical phase space and theintegration measure has the factorized form [42]

$$d\mu(g,p) = dg\, d\nu(p) \tag{2.69}$$

where dg is the Haar measure and the factor involving p is

$$d\nu(p) = d^3\, p e^{-\frac{t}{4}} \left(\frac{t}{\pi}\right)^{3/2} \frac{\sinh(tp)}{tp} e^{-tp^2} \tag{2.70}$$

where we denote $|\vec{p}| = p$. In order to check that (2.68) is the unit operator on $\mathcal{H}_e$ it is enough to compute its matrix elements $\langle j,a,b|\mathbb{1}|j',a',b'\rangle$. After inserting the expression (2.67), the integral over g can be calculated immediately using the orthogonality of the Wigner matrices, and one is left with

$$\langle j,a,b|\mathbb{1}|j',a',b'\rangle = \delta_{jj'}\delta^a_{a'} e^{-tj(j+1)} \int d\nu(p) D^{(j)b'}{}_b(e^{t\vec{p}\cdot\vec{\sigma}}) \tag{2.71}$$

Let us call the remaining integral on the RHS $I^{b'}{}_b$. The key to evaluate it is to view it as a tensor in the space $\mathcal{H}_e \otimes \overline{\mathcal{H}_e}$ and observe that the rotation invariance of the measure $d\nu(p)$ implies that $I^{b'}{}_b$ is invariant under the action of $SU(2)$

$$D^{(j)a'}{}_{b'}(g^{-1}) I^{b'}{}_b D^{(j)b}{}_a(g) = I^{a'}{}_a \tag{2.72}$$

So $I^{b'}{}_b$ is an element of the space $\mathrm{Inv}\left[\mathcal{H}_e \otimes \overline{\mathcal{H}_e}\right]$ and must be proportional to $\delta^{b'}_b$ which is the only invariant tensor carrying one upper and one lower index in that space. The coefficient of proportionality $I^{b'}{}_b = c(j)\,\delta^{b'}_b$ can be determined by contracting both sides with $\delta^b_{b'}$

$$c(j) = \frac{1}{d_j}\int d\nu(p)\chi_j(e^{t\vec{p}\cdot\vec{\sigma}}) = \frac{1}{d_j}\int d\nu(p)\frac{\sinh(d_j tp)}{\sinh(tp)} = e^{tj(j+1)} \tag{2.73}$$

where the trace $\chi_j(e^{t\vec{p}\cdot\vec{\sigma}})$ was evaluated in the basis where $D^{(j)}(e^{t\vec{p}\cdot\vec{\sigma}})$ is diagonal with eigenvalues
$e^{tjp}, e^{t(j-1)p}, \ldots, e^{e^{-tjp}}$. Putting everything together we have that

$$\langle j, a, b|\mathbb{1}|j', a', b'\rangle = \delta_{jj'}\delta^b_{b'}\delta^{a'}_a \tag{2.74}$$

from which one can conclude that (2.68) is an expression for the identity operator on $\mathcal{H}_e$.

Peakedness properties

The construction of the heat kernel coherent states (2.63) guarantees that they are sharply peaked with respect to the operators (2.56), where a is given here by a quantization of the classical variable $g\,e^{\frac{t}{2}\vec{p}\cdot\vec{\sigma}}$. It is therefore not immediately obvious that the states are also properly peaked on the holonomy and the flux. These properties were established by direct calculations [42] and we here summarize their results.

The possible peakedness of the state (2.63) with respect to the holonomy is described by the probability distribution in 'holonomy space'

$$\rho_H(g) = \frac{|\psi_H(g)|^2}{\langle\psi_H|\psi_H\rangle} \tag{2.75}$$

where the denominator is necessary because the state $|\psi_H\rangle$ is in general not normalized. If the state $|\psi_H\rangle$ is properly peaked on the holonomy, then the function (2.75) should have a sharp peak concentrated around the point $g = h$ (recall that $H = h\,e^{\frac{t}{2}\vec{p}\cdot\vec{\sigma}}$). Now we notice that the following relation

$$\psi_H(g) = \psi_{e^{\vec{p}_0\cdot\sigma/2}}(h^{-1}g) \tag{2.76}$$

allows us to replace the problem of showing that (2.75) is peaked on $g = h$ with the equivalent of showing that the distribution

$$\rho_{e^{\vec{p}_0\cdot\sigma/2}}(g) \tag{2.77}$$

is peaked on the identity $g = \mathbb{1}$ independently of the value of $\vec{p}_0$. A long and tedious calculation, Thiemann and collaborators managed to show that this is the case. The width of the peak is characterized by the parameter t with the peak becoming sharp as $t \to 0$.

The only thing that we recall here, is the key role played by the Poisson summation formula

$$\sum_{-\infty}^{\infty} f(tn) = \frac{2\pi}{t}\sum_{-\infty}^{\infty}\tilde{f}\left(\frac{2\pi n}{t}\right) \tag{2.78}$$

where $\tilde{f}$ denotes the Fourier transform of f. The importance of the Poisson summation formula to the present problem is that it allows one to convert a sum like (2.63), which converges extremely slowly in the limit $t \to 0$, into a sum which typically is converging very rapidly when $t \to 0$, and which can be approximated very well by keeping only the leading term.

In order to study the peakedness of the state $|\psi_H\rangle$ with respect to the flux operator, one need to expand the state in the momentum basis

$$|\psi_H\rangle = \sum_{j,a,b} \psi_H(j,a,b)\,|j,a,b\rangle \tag{2.79}$$

where the coefficient of the expansion are of course

$$\psi_H(j,a,b) = \sqrt{d_j} e^{-\frac{t}{2}j(j+1)} D^{(j)a}{}_b(H) \tag{2.80}$$

The probability of the state $|\psi_H\rangle$ to have a specific momentum configuration is then described by the (discrete) probability distribution

$$\rho_H(j,a,b) = \frac{|\psi_H(j,a,b)|^2}{\langle\psi_H|\psi_H\rangle} \tag{2.81}$$

By making use of estimates based on the explicit expression for the Wigner matrix elements $D^{(j)a}{}_b(H)$, Thiemann and collaborators showed that

$$\rho_H(j,a,b) \lesssim \frac{\sqrt{t}}{4\sqrt{\pi}p_0} \exp\left[-t\left(j+\frac{1}{2}-p_0\right)^2 - \frac{j}{2}\frac{(a/j-p'_{0z}/p_0)^2}{1-(p'_{0z}/p_0)^2} - \frac{j}{2}\frac{(b/j-p_{0z}/p_0)^2}{1-(p_{0z}/p_0)^2}\right] \tag{2.82}$$

where p_{0z} and p'_{0z} refer to the two decompositions of the $SL(2,\mathbb{C})$ element

$$H = h\, e^{\frac{t}{2}\vec{p}_0\cdot\vec{\sigma}} \qquad H = e^{\frac{t}{2}\vec{p}'_0\cdot\vec{\sigma}}\, h \tag{2.83}$$

So the peakedness properties of the state $|\psi_H\rangle$ with respect to the flux operator can be read off from equation (2.82) and one can see that the probability distribution (2.81) is peaked on the values

$$j = p_0 - \frac{1}{2} \qquad a \simeq p'_{0_z} \qquad b \simeq p_{0_z} \tag{2.84}$$

and the peak becomes sharp in the limit of large p_0, specifically when $p_0 \gg 1/\sqrt{t}$ (where the value of t is fixed for example requiring that the state is sufficiently well peaked on the holonomy).

They also showed that the overlap function for these coherent states

$$i(H_1,H_2) = \frac{|\langle\psi_{H_1}|\psi_{H_2}\rangle|^2}{\langle\psi_{H_1}|\psi_{H_1}\rangle\langle\psi_{H_2}|\psi_{H_2}\rangle} \tag{2.85}$$

is peaked on $H_1 = H_2$, with the peak falling off exponentially fast around the maximum (at least in the limit if small t). Intuitively, if one sees the CS $|\psi_H\rangle = |h,\vec{p}_0\rangle$ as a state-vector valued function on the classical phase space, then the peakedness of the state function means that $|h,\vec{p}_0\rangle$ differs significantly from zero only within a small neighbourhood (controlled by t) around the point $(h,\vec{p}_0)$ in $T^*SU(2)$.

2.3.4 Gauge invariance

So far the discussion was restricted to a single spin network edge. The generalization of the coherent states on a fixed graph is rather trivial, at least at the non gauge-invariant level. Coherent states for a fixed graph Γ are simply given by tensor products of the single edge coherent states (2.63) over the edges of the graph (where now we call $H_e = h_e$ to make the notation lighter)

$$\phi^{\Gamma}_{h_e}(g_{e_1},\dots,g_{e_N}) = \prod_{e\in\Gamma}\sum_{j_e} d_{j_e}\, e^{-\frac{t_e}{2}\lambda_{j_e}}\chi^{(j_e)}(h_e g_e^{-1}) \tag{2.86}$$

On the other hand, gauge invariant states are not so easy to construct, at least not explicitly. The operation to do is group averaging the tensor product states (2.86) with respect to the gauge transformations at each node of the graph[9]

$$\Phi^{\Gamma}_{h_e}(g_{e_1},\dots,g_{e_N}) = \int da_1\dots da_M\, \phi^{\Gamma}_{h_e}(a_{t(e_1)}g_{e_1}a^{-1}_{s(e_1)},\dots,a_{t(e_M)}g_{e_M}a^{-1}_{s(e_M)}) \tag{2.87}$$

Inserting in (2.86) we obtain

$$\begin{aligned}\Phi^{\Gamma}_{h_e}(g_{e_1},\dots,g_{e_N}) = &\prod_{e\in\Gamma}\left(\sum_{j_e} d_{j_e}\, e^{-\frac{t_e}{2}\lambda_{j_e}}\, D^{(j_e)m_e}{}_{n_e}(h_e) D^{(j_e)m'_e}{}_{n'_e}(g_e^{-1})\right)\\ &\times\int da_1\dots da_M \prod_{e\in\Gamma}\left(D^{(j_e)n_e}{}_{m'_e}(a_{s(e)})D^{(j_e)n'_e}{}_{m_e}(a^{-1}_{t(e)})\right)\end{aligned} \tag{2.88}$$

So at each node of the graph we have an integral of the form

$$\int da_v \left(\prod_{e^{out}} D^{(j_e)n_e}{}_{m'_e}(a_v)\right)\left(\prod_{e^{in}} D^{(j_e)n'_e}{}_{m_e}(a_v^{-1})\right) \tag{2.89}$$

which can be viewed as an $SU(2)$ tensor and it is essentially a normalized projection operator onto the intertwiner space of the node. Therefore it can be written as

$$\sum_{\imath} \overline{\imath^{n_1\cdots n_o}_{m_1\cdots m_i}}\; \imath^{n'_1\cdots n'_i}_{m'_1\cdots m'_o} \tag{2.90}$$

where the sum runs over any orthonormal basis of the intertwiner space, and the indices refer to the edges coming into the node (i) and going out of the node (o). So now from equation (2.88) we see that the gauge invariant coherent states can be written in the form

$$\Phi^{\Gamma}_{h_e}(g_e) = \sum_{j_e,\imath_v}\left(\prod_e e^{-\frac{t_e}{2}\lambda_{j_e}}\right)\psi_{(\Gamma,j_e,\imath_v)}(h_e)\overline{\psi_{(\Gamma,j_e,\imath_v)}(g_e)} \tag{2.91}$$

where $\psi_{(\Gamma,j_e,\imath_v)}$ are the standard spin network states.

[9] Recall that the holonomy transforms as $h_e \to a(t(e))\, h_e\, a^{-1}(s(e))$ where $a(x)\in SU(2)$ is a gauge function.

It is not immediately obvious how the group averaging operation affects the peakedness properties of the coherent states. In their paper Thiemann and Winkler argue that peakedness with respect of holonomies at the gauge invariant level follows from the corresponding properties of the non gauge invariant coherent states. On the other hand, the situation is much more transparent with respect to the flux operators, although to see it clearly, one needs another parametrization that links the Heat Kernel coherent states with the Livine-Speziale coherent intertwiners. The alternative decomposition of the $SL(2,\mathbb{C})$ element was given in [11] and was inspired by the twisted geometries to which now we turn.

Chapter 3

Twisted Geometries

The main task is now to assign a geometry to the states on a graph Γ. Since this is a truncation of the theory, it captures only a finite number of degrees of freedom and thus the semiclassical states in the Hilbert space $\mathcal{H}_\Gamma$ do not represent smooth geometries. But they can represent a *discrete* geometry that is an approximation of a smooth one, on the given graph. The problem is basically a choice of interpolation. So the idea is to capture this finite amount of information with a discrete metric space. In this chapter we show how to parameterize the phase space of LQG in terms of quantities describing the intrinsic and extrinsic geometry of the triangulation dual to the graph. These are defined by the assignment to each face of its area, the two unit normals as seen from the two polyhedra sharing it, and an additional angle related to the extrinsic curvature. These quantities do not define a Regge geometry, since they include extrinsic data, but a looser notion of discrete geometry which is *twisted* [16] in the sense that it is locally well-defined, but the local patches lack a consistent gluing among each other. It is only thanks to these twisted geometries that we can see each classical holonomy-flux configuration on a fixed graph as a collection of adjacent polyhedra with extrinsic curvature between them. The name of these geometries refers to both their link with the concept of twistors and to some properties that justify the term twisted: they define a metric which is locally flat but *discontinuous* since two adjacent polyhedra are attached by faces with same area (since they share a link) but different shape.

3.1 Motivation and definition

As we have seen the geometrical operators have discrete spectra, with minimal excitation proportional to the Planck length. In spite of the key role played in the theory, spin network states lack a low-energy physical interpretation. How can we bridge from the Planck scale quantum geometry they describe, to a smooth and classical three dimensional geometry? To answer this question, one is interested in the construction of coherent states, namely superpositions of spin networks peaked on classical geometries labeling the phase space of the theory. We will construct them in the following chapters, using the parametrization introduced in this one.

3.1.1 The phase space of LQG

As seen in the first chapter, the continuum phase space of loop quantum gravity is defined by the Ashtekar-Barbero connection A_a^i and the densitized triad field E_i^a, satisfying the Poisson algebra (1.95)

$$\{A_a^i(x), E_j^b(x')\} = 8\pi\gamma G\, \delta_a^b \delta_j^i \delta^3(x, x') \tag{3.1}$$

The link with the ADM phase space of General Relativity is establish thanks to the split of the connection (1.91)

$$A_a = \Gamma_a + \gamma\, K_a \quad \in \quad \mathfrak{su}(2) \tag{3.2}$$

As known, an important step towards quantization is the smearing of the algebra (3.1). This is achieved introducing a graph Γ embedded in the spatial manifold Σ and replacing (A, E) with the pair $(g_e, X_e) \in SU(2) \times \mathfrak{su}(2)$ on each edge. These variables are holonomies $g_e = \mathcal{P}\exp\big(\int_e A\big)$ and fluxes $X_e = \int_{e^*}(gE)^a N_a d^S$ where $\mathcal{P}$ denotes the path-ordered product, e^* is the face dual to the edge e with normal N^a and g is the parallel transport from a vertex to the point of integration on a path adapted to Γ.

Since $SU(2) \times \mathfrak{su}(2) \cong T^*SU(2)$ we see that the phase space of LQG on a fixed graph is the direct product of $SU(2)$ cotangent bundles. The complete phase space of the theory will be recovered taking the union over all possible graphs.

Usually, the new variables (g_e, X_e) are seen as a distributional version of the continuum geometric interpretation. However, one might wonder whether there exists also an interpretation of these variables in terms of discrete geometries. In particular, an interpretation which would include the equivalent of the splitting (3.2) with a clear separation between intrinsic and extrinsic geometry, and possibly, a nice description of the gauge invariant reduction of the phase space.

Immirzi was the first one [22] to think about labeling the states of loop quantum gravity in terms of some notion of discrete geometries: he suggested a connection with Regge calculus as the analogue of the lattice description of QCD. However we don't show here a connection with Regge geometries, but with a looser notion of them. Twisted geometry are in fact locally well defined but with a notion of extrinsic curvature too, that makes them more general. The difference between the two is captured by a natural *closure condition* which can be satisfied or not by the labels.

The motivation for these labels come from the coherent intertwiners which have proved useful to construct the new EPR-FK-LS spin foam models, but also spin foam graviton calculations and area-angle Regge calculus. Twisted geometries will also be crucial to study the semiclassical limit of the theory as it turns out that they allow a interpretation of the states in terms of convex polyhedra, as seen in the last chapter. The new parametrization provides a a direct and simple route towards quantisation of the LQG phase space in terms of coherent states labelled by twisted geometries. This will be the subject of chapter 5.

3.1.2 Towards twisted geometries

Consider an oriented graph and assume for simplicity that it is four-valent so that it is dual to a triangulation. Assign a real number j_e to each edge which represents the oriented area of the dual triangle, and four unit vectors $N_e(v)$ to each vertex representing the normals to the four triangles in the tetrahedron dual to the node. The graph Γ carries the space

$$P'_\Gamma \equiv \bigtimes_e \mathbb{R}_e \bigtimes_v P_v \qquad \text{where} \qquad P_v \equiv \bigtimes_{e|v\subset e} S^2_e \tag{3.3}$$

where we used the fact that unit vectors define elements on the two-sphere. So by assumption, each edge is labeled by j_e and two unit vectors, $N_e = N_e(s)$ and $\tilde{N}_e = N_e(t)$ where s and t are the source and target of the edge e. Thus one can factorize the edges

$$P'_\Gamma = \bigtimes_e P'_e \qquad \text{where} \qquad P'_e = S^2_e \times S^2_e \times \mathbb{R}_e \tag{3.4}$$

So the variables associated to each edge are the triple $(N_e, \tilde{N}_e, j_e)$. The goal is to try and use these to define a notion of (discrete) metric. Regge showed long ago that in order to do so one has to assign the edges lengths. So we need suitable conditions to reconstruct them starting from the variables above. As it turns out, there are two such conditions called closure and gluing constraints. The closure constraint is defined on each vertex by

$$C_v = \sum_{e\supset v} j_e N_e(v) = 0 \tag{3.5}$$

which of course is not surprising compared to what we have seen in the last chapter. When (3.5) is satisfied, the variables $(N_e(v, j_e))$ in the constrained space

$$T_v = \{N_e(v \in P_v | C_v = 0)\} \tag{3.6}$$

define the geometry of a *flat* tetrahedron (we are dealing with a four valent node) embedded in $\mathbb{R}^3$. This geometry is unique up to rotations and in fact it is useful to recall the space of shapes of the tetrahedron (2.40) already encountered

$$S_v = T_v / SU(2) \tag{3.7}$$

which is the space of closed normals modulo rotations. At fixed areas the space is two dimensional and it can be parametrized by two (non-opposite) dihedral angles. Moreover it is known [23, 13] that this space is a symplectic manifold isomorphic to S^2, and that the $SU(2)$ orbits in T_v are generated precisely by the closure condition (3.5); therefore S_v can be obtained imposing it and dividing out the action of the gauge transformations that it generates. We will denote this symplectic reduction by a double quotient $S_v = P_v // C_v$. If one now considers the space on the whole graph (3.3), one can apply the reduction and impose the closure at each node. The result is

$$\mathcal{K}_\Gamma \equiv P'_\Gamma // \mathcal{C} = \bigtimes_e \mathbb{R}_e \bigtimes_v S_v \tag{3.8}$$

where $\mathcal{C} = \prod_v C_v$. The constrained space $\mathcal{K}_\Gamma$ of oriented areas and angles defines a precise classical 3d geometry on each tetrahedron. The next step would be to ensure that the individual tetrahedra glue together to form a consistent geometry on the whole triangulation. By construction, two neighbouring tetrahedra induce different geometries on the shared triangle, with same area but, in general, different shape.

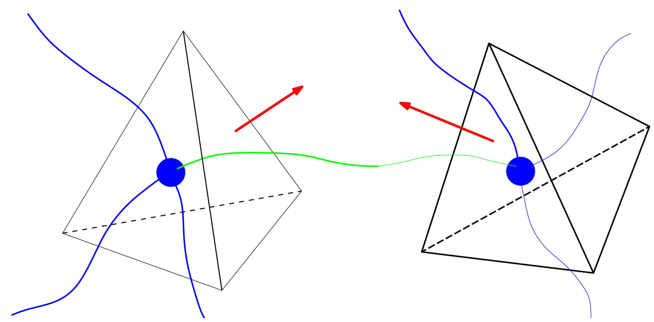


Figure 3.1: Two tetrahedra sharing a link

To match the shapes one needs additional gluing constraints, which involve only the dihedral angles, and are local on each pair of tetrahedra. They will not be discussed here as they don't play any role for our purposes. It is enough to say that together, the closure and gluing constraints guarantee that a Regge geometry (given by a unique set of edge lengths) can be reconstructed from areas and angles.

A similar construction can be generalized to vertices of any valency. As seen above, each n-valent vertex can be thought of as a flat convex polyhedron with n faces. As described in the last chapter, there is a phase space of shapes like (3.7) associated to the n-vertex, this time $2(n-3)$-dimensional, which is still a symplectic quotient generated by the closure condition, and whose geometrical data can be used to label the quantum states.

But now the question is more ambitious: can we relate the new variables not only to Regge geometries, but to the full LQG phase space variables? As we recalled above, also the loop phase space takes a form factorized on edges, thus one can directly look at the edge contributions, respectively P'_e and $T^*SU(2)$. The problem is however not easy: the dimension of those spaces are different, as well as the topology and the former does not seem to carry any information about the connection. If areas and angles are complete geometrical data, we should be able to reconstruct a notion of local frame, and the corresponding rotations mapping one local frame onto the next. *Where is the information on such rotation?*

The answer to that question lies at the heart of the definition of the **Twisted Geometries.**

Consider two adjacent vertices and the edge connecting them. The five variables $(N_e, \tilde{N}_e, j_e)$ represent the area of the triangle and its normals in the two frames sharing it (see figure 3.1). The crucial presence of two normals allows us to write down a natural compatibility condition for a finite connection g_e as the group element rotating one normal into the other, i.e.

$$N_e = R(g_e)\tilde{N}_e \tag{3.9}$$

This equation can be solved for $g_e \in SU(2)$, with R the rotation matrix in the adjoint representation. The connection g_e so introduced defines a notion of parallel transport by which the normal n_e of the triangle in the frame of its source tetrahedron is mapped into the frame of its target tetrahedron. This is exactly the rotation that answer our question.

Notice however that (3.9) does not fully determine g_e because it gives only two independent equations, therefore $R(g_e)$ will be determined up to rotations along the $\tilde{N}$ axis. This means that if $\bar{g}_e$ is a solution, then $g_e(N_e, \tilde{N}_e, \xi_e) = \bar{g}_e e^{\xi_e \tilde{N}_e^i \tau_i}$ is also a solution[1], for an arbitrary angle $\xi \in [-\pi, \pi]$. It is easy to find a solution of the compatibility condition (3.9) by constructing a group element n_e rotating τ_3 into $N_e^i \tau_i$, that is

$$R(n_e)\tau_3 = N_e^i \tau_i \tag{3.10}$$

and similarly $\tilde{n}_e$ for $\tilde{N}_e$. Then, once n_e and $\tilde{n}_e$ are found, the most general solution will be

$$g_e(N_e, \tilde{N}_e, \xi_e) = \tilde{n}_e e^{\xi_e \tau_3} n_e^{-1} \tag{3.11}$$

Therefore, in order to uniquely define a connection associated to $(N_e, \tilde{N}_e)$ one needs an extra angle per edge ξ_e. Including it, the space of variables associated with an edge of the graph is 6-dimensional, $(N_e, \tilde{N}_e, j_e, \xi_e)$. Accordingly, one can define the extended space (which justifies the previous symbols P' for (3.3))

$$P_\Gamma = \bigtimes_e P_e \qquad \text{where} \qquad P_e = S_e^2 \times S_e^2 \times T^* S_e^1 \tag{3.12}$$

where we used the obvious isomorphism of $\mathbb{R}_e \times S_e^1$ with the cotangent bundle to the circle $T^* S_e^1$. This is the space of twisted geometries, which is in fact lacking the required constraints to read a Regge geometry off these variables. The first thing to notice is that the edge component P_e of the space of twisted geometries has the same dimension of $T^*SU(2)$. More precisely, the following claims were shown by Freidel and Speziale in [16]:

(1) P_Γ is a *presymplectic*[2] manifold and its reduction $\bar{P}_\Gamma$ exists and is a phase space vith conjugate variables (j_e, ξ_e)

(2) as a phase space, it is g*lobally symplectomorphic* to the non-gauge-invariant phase space of loop quantum gravity on a fixed graph,

$$\bar{P}_\Gamma \cong \bigtimes_e T^*SU(2)_e \tag{3.13}$$

[1]We use the $SU(2)$ fundamental representation with generators $\tau_i = -i\sigma_i/2$.

[2]A manifold is presymplectic if it is equipped with a closed but possibily degenerate 2-form Ω. The reduction is then the quotient by the kernel of Ω.

If one considers the closure constraint on the extended space $\bar{P}_\Gamma$, one gets the notion of *closed* twisted geometries associated to

$$S_\Gamma = \bigtimes_e T^* S^1_e \bigtimes_v S_v \tag{3.14}$$

where S_v is the space of the shapes of the polyhedron corresponding to the valency of the vertex. The last result then is that

(3) S_Γ is presymplectic and its reduction $\bar{S}_\Gamma$ is symplectomoprhic to the gauge-invariant space of loop quantum gravity

$$\bar{S}_\Gamma = \bigtimes_e T^* SU(2)_e // SU(2)^V \tag{3.15}$$

where V is the number of vertices in the graph and the double quotient means again imposing Gauss law at each vertex and dividing out the action of the $SU(2)$ gauge transformation it generates.

We will not give the complete proofs here (see [16] for details) but we will quote the key points which are useful to understand and to carry on with the work. Before doing that let us remark a few things.

Twisted geometries are particularly useful because they naturally parametrize also the gauge-invariant phase space: the only thing to take into account is the closure condition on the labels, and this can be implemented going from the normals to suitable cross sections. The result (3.14) implies that the parametrization in terms of twisted geometries factorises the gauge-invariant phase space of gravity as a product of phase spaces associated with edges and vertices. This factorization provides a classical analogue to the well known factorization of the $SU(2)$ spin network Hilbert space associated with a graph Γ as a sum over intertwiner spaces, $\mathcal{H}_\Gamma = \bigoplus_{j_e} \left(\bigotimes_v \mathcal{H}_v\right)$ where $\mathcal{H}_v = \mathrm{Inv}\left[\otimes_e \mathcal{H}_{j_e}\right]$.

Those results mean that both the non-gauge-invariant and the gauge-invariant phase spaces of loop gravity can be parametrized in terms of a notion of discrete geometry. These twisted geometries are then candidate labels for full coherent states of the theory. These results are in fact the starting point for the construction of new coherent states, described in details later.

3.2 Phase space of Twisted Geometries

Consider the edge space (dropping the label e)

$$P \equiv S^2 \times S^2 \times T^* S^1 \tag{3.16}$$

with variables $(N, \tilde{N}, j, \xi)$. Given the cartesian factorization (3.12), it is enough to show a symplectomorphism to $T^* SU(2)$. The first step is noticing that each factor of the Cartesian product (3.16) is a symplectic manifold on its own:

- T^*S^1 has a symplectic 2-form $\Omega_{T^*S^1} = d\xi \wedge dj$ and Poisson brackets $\{\xi, j\} = 1$.
- S^2_R, where R is the radius, has a symplectic 2-form given by the area form $\Omega_{S^2} = \pm R \sin\theta d\theta \wedge d\phi$ where θ and ϕ are olar and azimuthal angles and the sign depends on the orientation of the sphere. The Poisson bracket is conveniently given in terms of the components of the unit vector $N = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$ as $\{RN^i, RN^j\} = \pm\epsilon^{ij}{}_k RN^k$.

Now we extend the above brackets to the space P taking the to sphere in (3.16) to have radius j and opposite orientation[3], and such that the brackets between N and $\tilde{N}$ vanish. At this point one has still to chose the brackets between ξ and N, $\tilde{N}$. Let's give them in terms of a certain function $L : S^2 \to \mathbb{R}^3$, so that the whole Poisson structure is

$$\begin{gathered}\{jN^i, jN^j\} = \epsilon^{ij}{}_k jN^k \qquad \{j\tilde{N}^i, j\tilde{N}^j\} = -\epsilon^{ij}{}_k j\tilde{N}^k \qquad \{N^i, \tilde{N}^j\} = 0 \qquad \{\xi, j\} = 1 \\ \{N^i, j\} = 0 \qquad \{\tilde{N}^i, j\} = 0 \qquad \{\xi, jN^i\} \equiv L^i(N) \qquad \{\xi, j\tilde{N}^i\} \equiv L^i(\tilde{N})\end{gathered} \tag{3.17}$$

In [16] it is shown that L^i is in fact unique up to canonical transformations, such that the Poisson algebra closes an P is locally symplectomorphic to $T^*SU(2)$. As it turns out, L^i is in fact unique up to a change of sections and for the Hopf section described below, it will be $L^i(-\bar{z}, z, 1)$.

3.2.1 Hopf map and section

The Hopf map is a projection $\pi : S^3 \to S^2$, such that, roughly speaking, every point on S^2 comes from a circle on S^3. Since $SU(2) \cong S^3$ and $S^2 \cong SU(2)/U(1)$, where $U(1)$ is for instance generated by τ_3, the map can be defined in terms of group elements. In the fundamental representation of $SU(2)$, the Hopf map is

$$\begin{aligned}\pi : \quad & SU(2) \to S^2 \\ & g \longmapsto N(g) = g\tau_3 g^{-1}\end{aligned}$$

The vector $N(g) \in S^2$ is a function of two variables and we can parametrize g with two complex numbers z_0 and z_1 as

$$g = \frac{1}{\sqrt{|z_0|^2 + |z_1|^2}} \begin{pmatrix} \bar{z}_1 & z_0 \\ -\bar{z}_0 & z_1 \end{pmatrix} \tag{3.18}$$

Then, calling $\zeta \equiv z_0/z_1$, one can calculate directly

$$N(g) = g\tau_3 g^{-1} = \frac{1}{1+|\zeta|^2}\left[\left(1-|\zeta|^2\right)\tau_3 - 2\tau_+ - \bar{\zeta}\tau_-\right] = \frac{i}{2(1+|\zeta|^2)} \begin{pmatrix} |\zeta|^2 - 1 & 2\zeta \\ 2\bar{\zeta} & 1-|\zeta|^2 \end{pmatrix} \tag{3.19}$$

where $\tau_\pm = \tau_1 \pm i\tau_2$. Notice that z is the Hopf map for the stereographic projection of S^2 from the hemisphere with $z_1 = 0$.

[3] This is just a matter of convention, one could also take the same orientation.

Equation (3.19) shows that $SU(2)$ can be seen as a bundle (called Hopf bundle) over S^2 with a $U(1)$ fibre. Now, on this bundle, one can define an inverse map, that is a section

$$\begin{aligned} n: \quad & S^2 \to SU(2) \\ & N \longmapsto n(N) \end{aligned} \tag{3.20}$$

such that $\pi(n(N)) \equiv N$, and it is given by

$$n(N(z)) = \frac{1}{\sqrt{1+|\zeta|^2}}\left[\mathbb{1} + i\zeta\tau_+ - i\bar{\zeta}\tau_-\right] = \frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix} 1 & \zeta \\ -\bar{\zeta} & 1 \end{pmatrix} \tag{3.21}$$

This section associates an $SU(2)$ element n to each point of the S^2 sphere, the latter being parametrized by the stereographical projection[4] ζ. The choice (3.21) is not unique but that is the one we will use in the rest of the work, calling it simply n with abuse of notation. Moreover we will write the Hopf map simply as $N = n\tau_3 n^{-1}$.

As it turns out, the function L^i mentioned above that closes the algebra (3.17) acts in fact as a connection in preserving the Hopf section. But this is not enough to say that the algebra closes and in order to do so, one hs to discuss the properties of the symplectic 2-form.

3.2.2 Symplectic potential

It is conveninet to introduce the symplectic potential Θ, or canonical 1-form, such that $\Omega = -d\Theta$. For the cotangent bundle to the circle, we have seen that the canonical 2-form is

$$\Omega_{T^*S^1}(\xi, j) = d\xi \wedge dj \tag{3.22}$$

therefore it is straightforward to see that the symplectic potential will be

$$\Theta_{T^*S^1} = j d\xi \tag{3.23}$$

In the case of a two-sphere of radius j, say with right-handed orientation, we can use the Hopf section to write the potential as

$$\Theta_{S^2}(N) = j\ \mathrm{Tr}\big(N dn n^{-1}\big) = j(\cos\theta - 1)d\phi \tag{3.24}$$

and in fact one can easily verify that the associated 2-form reads

$$\Omega_{S^2} = -d\Theta_{S^2} = j\,\mathrm{Tr}\big(N dn n^{-1} \wedge dn n^{-1}\big) = j \sin\theta d\theta \wedge d\phi \tag{3.25}$$

Let us come to the Poisson algebra (3.17) of the twisted geometries and to its symplectic 2-form $\Omega_P = -d\Theta_P$. By inspection we see that Ω_P should contain $\Omega_{T^*S^1}$ and two Ω_{S^2} with radius j and opposite orientation. Remarkably, it can be shown [16] that Θ_P turns out to be

[4] For instance, taking the projection from the south pole, one has $\zeta = -\tan\frac{\theta}{2}e^{-i\phi}$ in terms of the familiar (θ, ϕ).

a sum of the elementary symplectic potentials of each factor in the cartesian decomposition $P = S^2 \times S^2 \times T^*S^1$

$$\begin{aligned}\Theta_P \equiv \quad & \Theta_{S^2}(N) + \Theta_{S^2}(\tilde{N}) + \Theta_{T^*S^1} \\ = \quad & j \operatorname{Tr}\big(N dn n^{-1}\big) - j \operatorname{Tr}\Big(\tilde{N} d\tilde{n}\tilde{n}^{-1}\Big) + j d\xi\end{aligned} \tag{3.26}$$

This is equivalent to say that the brackets (3.17) are given by

$$\begin{aligned}\Omega_P = -d\Theta_P =& j \operatorname{Tr}\big(N dn n^{-1} \wedge dn n^{-1}\big) - j \operatorname{Tr}\Big(\tilde{N} d\tilde{n}\tilde{n}^{-1} \wedge d\tilde{n}\tilde{n}^{-1}\Big) \\ & - dj \wedge \Big[d\xi + \operatorname{Tr}\big(N dn n^{-1}\big) - \operatorname{Tr}\Big(\tilde{N} d\tilde{n}\tilde{n}^{-1}\Big)\Big]\end{aligned} \tag{3.27}$$

The detailed calculations can be found in [16]. This was the construction of the phase space of twisted geometries, and now we have to link it to the standard phase space of loop quantum gravity.

3.3 Symplectomorphism with SU(2) phase space

It is very well known that a Lie group G is a manifold and as such its cotangent bundle T^*G has a symplectic structure associated with it [3, 28]. For instance, the harmonic oscillator's configuration space is the real line $\mathbb{R}$, and in fact the phase space is simply the plane $T^*\mathbb{R}$. Another example encountered in the above is a particle in the circle S^1 which has the phase space given by the $T^*S^1 \cong \mathbb{R} \times S^1$. Now in LQG one is interested in the edge phase space which is simply $T^*SU(2)$ so let's see how to deal with this cotangent bundle.

The Lie algebra $\mathfrak{su}(2)$ is isomorphic to the set of right-invariant vector fields on $SU(2)$. As a matter of fact the group acts on itself by either left or right multiplication. Both actions can be used to get an isomorphism[5] of vector fields with the Lie algebra, and to trivialize the cotangent bundle. In other words, the tangent space of a Lie group, seen as a differential manifold, has a natural Lie algebra structure at the identity. Here we choose the right multiplication, but a similar construction can be carried over choosing the left multiplication. Then a right-invariant vector field in the direction of $X \in \mathfrak{su}(2)$, which is denoted ∇^L_X, acts on functions of the group as the *left* derivative

$$\nabla^L_X f(g) \equiv \frac{d}{dt} f(e^{-tX}g)|_{t=0} \tag{3.28}$$

The right derivative is obtained under the adjoint transformation $g \longmapsto gXg^{-1}$

$$\nabla^R_X f(g) \equiv \frac{d}{dt} f(ge^{tX})|_{t=0} = -\nabla^L_{(gXg^{-1})} f(g) \tag{3.29}$$

Similarly, the map from the vector fields to element X of the algebra is provided by algebra-valued right-invariant 1-forms dgg^{-1}, such that

$$i_{\hat{X}}(dgg^{-1}) = (\mathcal{L}_{\hat{X}} g)g^{-1} = -X \tag{3.30}$$

[5] To be a little more formal, one can identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via $X^i = \operatorname{Tr}(X\tau^i)$ where the cyclic trace $\operatorname{Tr}(XY) \equiv -2\operatorname{tr}_{1/2}(XY)$ for any $(X, Y) \in \mathfrak{su}(2)$. Of course $\operatorname{tr}_{1/2}$ is the trace in the fundamental representation and $\tau_i = -i\sigma_i/2$ satisfy $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$.

The set of right-invariant 1-forms is isomorphic to the *dual* algebra $\mathfrak{su}(2)^*$, thus the cotangent bundle trivialises as $T^*SU(2) = SU(2) \times \mathfrak{su}(2)^*$. To study the functions on $T^*SU(2)$, recall that each element $X \in \mathfrak{su}(2)$ of the algebra determines a linear function h_X on the dual algebra $\mathfrak{su}(2)^*$. With the positive pairing given by the trace one can define the linear action as $h_X(Y^*) = \mathrm{Tr}(XY) = X_i Y^i$ and identify $\mathfrak{su}(2)$ with its dual $\mathfrak{su}(2)^*$. Thanks to this identification one can parametrize $\mathfrak{su}(2)^*$ with elements X of the algebra. So the six-dimensional cotangent bundle $T^*SU(2)$ is then trivialized by the following symplectic potential

$$\begin{array}{ccc} SU(2) \times \mathfrak{su}(2)^* & \rightarrow & T^*SU(2) \\ (g, X) & \longmapsto & \Theta = \mathrm{Tr}\big(X dg g^{-1}\big) \end{array} \tag{3.31}$$

The symplectic 2-form is then computed to be

$$\Omega = -d\,\mathrm{Tr}\big(X dg g^{-1}\big) = \frac{1}{2}\,\mathrm{Tr}\Big(d\tilde{X} \wedge g^{-1}dg - dX \wedge dg g^{-1}\Big) \tag{3.32}$$

where $\tilde{X} \equiv -g^{-1}Xg$. From the symplectic 2-form one gets the following Poisson brackets

$$\{h_Y, h_Z\} = h_{[Y,Z]} \qquad \{h_Y, f(g)\} = \nabla^L_Y f(g) \qquad \{f(g), h(g)\} = 0 \tag{3.33}$$

Now we have to show explixicty the symplectomorphism. The key to construct the right isomorphism is the Hopf map (3.19). We consider the sections $n(N)$ and $\tilde{n}(\tilde{N})$ such that

$$n(N) = n\tau_3 n^{-1} \qquad \tilde{n}(\tilde{N}) = \tilde{n}\tau_3\tilde{n}^{-1} \tag{3.34}$$

Then we define the map

$$(N, \tilde{N}, j, \xi) \rightarrow (X, g): \quad \begin{array}{ccc} X & = & j\, n\tau_3 n^{-1} = jN \\ g & = & \tilde{n}e^{\xi\tau_3}n^{-1} \end{array} \tag{3.35}$$

which also implies that $\tilde{X} = -gXg^{-1} = -j\,\tilde{n}\tau_3\tilde{n}^{-1} = -j\tilde{N}$. The map is two-to-one[6], as the configurations $(N, \tilde{N}, j, \xi)$ and $(-N, -\tilde{N}, -j, -\xi)$ give the same pair (X, g), and it can be inverted at each branch provided that $|X| \neq 0$

$$j = \pm|X| \qquad N = \pm\frac{X}{|X|} \qquad \tilde{N} = \pm\frac{gXg^{-1}}{|X|} \qquad \xi = \pm\,\mathrm{Tr}\big(\tau_3 \ln\big(\tilde{n}^{-1}gn\big)\big) \tag{3.36}$$

Then the map gives the isomorphism

$$\bar{P}/\mathbb{Z}_2 \cong T^*SU(2) \tag{3.37}$$

where we recall $\bar{P} = P/\mathrm{Ker}\Omega_P$ is the true phase space, being P presymplectic, and the $\mathbb{Z}_2$ symmetry is the identification of the two configurations with opposite signs.

Now the main result here is that it can be shown that the isomorphism (3.37) is also a symplectomorphism, namely it preserves the Poisson structure of the symplectic spaces. As a

[6]This point will be important for considerations about coherent states in the following

matter of fact this is a direct consequence of the identification of the two symplectic potentials (3.26) and (3.31). So the map (3.35) provides an invertible symplectomorphism between the space $\bar{P}_e$ of twisted geometries with Poisson brackets (3.17) and $T^*SU(2)$ with Poisson brackets (3.33).

It is very straightforward to prove this: a direct computations gives

$$\begin{aligned}\Theta_{T^*SU(2)} &= j\ \mathrm{Tr}\left[n\tau_3 n^{-1}\left(dnn^{-1} + nd\xi\tau_3 n^{-1} - ne^{\xi\tau_3}\tilde{n}^{-1}d\tilde{n}\tilde{n}^{-1}e^{-\xi\tau_3}n^{-1}\right)\right] \\ &= j\ \mathrm{Tr}\left(Ndnn^{-1}\right) + jd\xi - j\ \mathrm{Tr}\left(\tilde{N}d\tilde{n}\tilde{n}^{-1}\right) = \Theta_P\end{aligned} \tag{3.38}$$

It is nevertheless instructive to check the symplectomorphism at the level of Poisson brackets. To that end, it is convenient to use the identification seen above of $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via $X^i = \mathrm{Tr}(\tau^i X)$ and write the Poisson brackets (3.33) of linear function on $T^*SU(2)$ in the equivalent and simple form

$$\{X^i, X^j\} = \epsilon^{ij}{}_k X^k \qquad \{X^i, g\} = -\tau^i g \qquad \{\tilde{X}^i, g\} = g\tau^i \tag{3.39}$$

Now, the first Poisson bracket is immediately verified using (3.17) with the definition (3.35). The others can be checked without too much effort, using the defining map (3.35),the algebra (3.17) and the usual properties outlined above.

Let's conclude this section by recapitulating where we stand. We have introduced the phase space $\bar{P}$ of twisted geometries with Poisson brackets (3.17) and we have seen that it is symplectomorphic to $T^*SU(2)$. This extends straightforwardly to the whole graph, so one can conclude that

$$\bar{P}_\Gamma \equiv \bigtimes_e \bar{P}_e/\mathbb{Z}_2 \cong \bigtimes_e T^*SU(2) \tag{3.40}$$

This symplectomorphism allows us to give a completely new parametrization of the kinematical phase space of loop quantum gravity on a fixed graph as the space of twisted geometries. This result shows that there is a natural discrete geometry associated with the space of holonomy-flux variables: the latter space can be written in terms of areas, normals and an abelian "connection" $\xi \in S^1$. Notice that this discrete geometry is not a Regge geometry, in particular, it is discontinuous because the shapes of the triangles do not match in general.

3.3.1 Gauge invariance

The description of the kinematical phase space of loop gravity in terms of the discrete, discontinuous twisted geometries is particularly useful when one works at the gauge-invariant level. As anticipated in the previous chapter, one expects 2 variables for each edge and $2(n-3)$ variables for each n-valent vertex. The question is then how to conveniently extract this set of variables from the initial (g_e, X_e), and the answer lies on twisted geometries.

Recall that the gauge-invariant phase space of loop gravity is obtained imposing the Gauss law constraint at each vertex, and then dividing out the action of the $SU(2)$ gauge transformation it generates,

$$\bar{S}_\Gamma \equiv \bigtimes_e T^*SU(2)_e//SU(2)^{V_\Gamma} \tag{3.41}$$

where V_Γ is the total number of vertices in the graph. In order to impose the Gauss law one has to recall that the kinematical space is given by the assignment of the holonomy-flux variables (g_e, X_e) on a fixed, oriented graph. Under reversal of the orientation of an edge, $g_{-e} = g_e^{-1}$ and $X_{-e} = -g_e^{-1}X_e g_e \equiv \tilde{X}_e$. This means that under reversal of the orientation we get the left-invariant vector fields, because we are trivializing $T^*SU(2)$ with the right-invariant ones. Thanks to this fact, the Gauss law can be defined as

$$C_v = \sum_{e|s(e)=v} X_e + \sum_{e|t(e)=v} \tilde{X}_e = 0 \tag{3.42}$$

at each vertex. Notice the "non-local" nature of the quotient in (3.41): each edge subspace is affected by the two vertices it connects. This is a complicate feature that our new parametrization (3.35) simplifies, since we assign to every edge two unit vectors, N_e to the source vertex and $\tilde{N}_e$ to the target one, moreover the relation $X_{-e} = -g_e^{-1}X_e g_e$ is automatically taken into account. Then the Gauss law can be imposed and the quotient by $SU(2)^{V_\Gamma}$ can be taken separately at each vertex.

It is important to point out here also that, through the map (3.35), *the Gauss law coincides with the closure constraint* discussed above [13].

$$C_v = \sum_{e|s(e)=v} j_e N_e - \sum_{e|t(e)=v} j_e \tilde{N}_e = 0 \tag{3.43}$$

In details, consider the presymplectic kinematical space (3.12), parametrized by the twisted geometries. We can factorize it as a product over edges *and* vertices, analogously to what we did with (3.3) at the beginning

$$P_\Gamma = \bigtimes_e T^*S^1 \bigtimes_v \left(\bigtimes_{e\supset v} S^2_{j_e} \right) \tag{3.44}$$

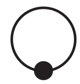
We now identify the Gauss law with the closure condition, and take the symplectic quotient locally at each vertex. As discussed at the beginning of this chapter, this amounts to impose the classical closure condition (3.5) and to divide by the $SU(2)$ rotations it generates. The result on each vertex is the space of shapes of the polyhedron, $S_v \equiv \bigtimes_e S^2_e//SU(2)$, which is a $2(n-3)$-dimensional phase space.

On each edge, although the closure does not affect the ξ_e directly as a constraint, it does as the generator of $SU(2)$ transformations, since

$$\{\xi_e, C^i_{s(e)}\} = L^i(z_e) \qquad\qquad \{\xi_e, C^i_{t(e)}\} = -L^i(\tilde{z}_e) \tag{3.45}$$

This double action on each ξ_e has the role of shifting the Hopf sections. Hence, the reduction requires a gauge-fixing of the choice of Hopf sections in the ξ_e variable.

Now a tricky question could be how to find a gauge-invariant reduction of the ξ_e variables. To understand this problem, consider first the simple example of a single edge closed on itself.



In this degenerate case the unique closure $C = jN - j\tilde{N} = 0$ implies only $N = \tilde{N}$ i.e. $\tilde{z} = z$. Then it's easy to see from (3.45) that $\{\xi, C\}$ vanishes on shell, hence ξ is a gauge invariant variable. However, this vanishing would be lost had we chosen a non-matching section between n and $\tilde{n}$. In other words, although we are free to choose the sections of the Hopf bundle at the kinematical level, imposing the gauge invariance removes this freedom. Hence, finding a gauge-invariant angle ξ_e requires also finding a consistent choice of sections throughout the graph. This is something yet to be studied. For now, let us assume that the graph is such that this fixing can be done globally without ambiguities. Then, denoting ξ_e^0 the gauge-fixed variables, one has

$$S_\Gamma = \bigtimes_e T^* S^1 \bigtimes_v S_v \tag{3.46}$$

This is a factorization of the presymplectic gauge-invariant phase space in terms of a 2-dimensional phase space assigned to each edge, and a $2(n-3)$-dimensional phase spaces assigned to each n-valent vertex. The procedure is then completed as before, dividing by the kernel of the gauge-reduced symplectic 2-form. This results in a symplectic space $\overline{S}_\Gamma$ isomorphic by construction to the gauge-invariant phase space (3.41), namely we obtain (3.14).

This decomposition shows that the gauge-invariant space is described by closed twisted geometries, and factorises as a product of phase spaces associated with edges and vertices. This factorization offers a classical analogue to the decomposition $\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N] = \oplus_{j_e} (\otimes_v \mathcal{H}_v)$ of the gauge-invariant Hilbert space on a fixed graph of the quantum theory. In particular, closed twisted geometries realize explicitly the counting described earlier: 2 variables per edge and $2(n-3)$ per vertex. The edge variables are still the abelian pairs (j_e, ξ_e^0), whereas the vertex variables are suitable cross ratios parametrized by $n-3$ complex variables. This complete factorisation and the related abelianization of the loop quantum gravity gauge-invariant phase space is the most remarkable property of the new parametrisation introduced here. It will play a key role in the construction of coherent states.

3.4 Spinors and Twistors

In this chapter we have introduced a parametrization of the phase space of loop quantum gravity. The latter was given by holonomies of the gravitational connection and fluxes of the triad field, and the new parametrization is in terms of quantities descriving the intrinsic and extrinsic discrete geometry of a cellular decomposition dual to the graph. The name *twisted* was meant to stress the discontinuous nature, but also to imply the existence of a relation to twistors as we now shall see [17].

3.4.1 Symplectic reduction

For the purpose of finding the connection with the edge phase space of LQG, the twistor space is simply

$$\mathbb{T} = \mathbb{C}^2 \times \mathbb{C}^2 \tag{3.47}$$

with coordinates $(z_A, \tilde{z}_A)$, where the components z_A and $\tilde{z}_A$ $(A = 0, 1)$ are 2-dimensional spinors. In these $\mathbb{C}^2$ spaces we introduce the two standard Poisson algebras

$$\{z_A, \overline{z}_B\} = i\delta^{AB} \qquad \text{and} \qquad \{\tilde{z}_A, \overline{\tilde{z}}_B\} = -i\delta^{AB} \tag{3.48}$$

in terms of spinors and of course we endow $\mathbb{C}^2$ with the standard positive inner product

$$\langle w|z\rangle = \overline{w}_A z^A = \overline{w}_0 z_0 + \overline{w}_1 z_1 \tag{3.49}$$

and norm

$$||z||^2 = \langle z|z\rangle = \overline{z}_A z^A = \overline{z}_0 z_0 + \overline{z}_1 z_1 = |z_0|^2 + |z_1|^2 \tag{3.50}$$

In each of the two $\mathbb{C}^2$ we can also introduce the two (not independent) spinors

$$|z\rangle = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \qquad |z] = \begin{pmatrix} -\overline{z}_1 \\ \overline{z}_0 \end{pmatrix} \tag{3.51}$$

and we will also have the $\tilde{z}$ counterpart. In general a spinor can be used to define a Minkowski 4-vector which is future-pointing and null. Choosing[7] for example $|\tilde{z}\rangle$ we can construct $\tilde{X}^\mu = (\tilde{X}^0, \tilde{X}^i)$ such that

$$|\tilde{z}\rangle\langle\tilde{z}| = \tilde{X}^0 \mathbb{1}_2 + \tilde{X}^i \sigma_i \tag{3.52}$$

where $\mathbb{1}_2$ is the 2×2 Identity matrix and σ_i are the Pauli matrices. We also have

$$\tilde{X}^i = \langle\tilde{z}|\frac{\sigma^i}{2}|\tilde{z}\rangle \qquad \tilde{X}^0 = \frac{1}{2}\langle\tilde{z}|\tilde{z}\rangle = |\vec{\tilde{X}}| \tag{3.53}$$

whose explicit components are easy to compute

$$\vec{\tilde{X}} = \frac{1}{2}\begin{pmatrix} \overline{\tilde{z}}_1\tilde{z}_0 + \overline{\tilde{z}}_0\tilde{z}_1 \\ i(\overline{\tilde{z}}_1\tilde{z}_0 - \overline{\tilde{z}}_0\tilde{z}_1) \\ |\tilde{z}_0|^2 - |\tilde{z}_1|^2 \end{pmatrix} = \begin{pmatrix} \mathrm{Re}[\tilde{z}_0\overline{\tilde{z}}_1] \\ \mathrm{Im}[\tilde{z}_0\overline{\tilde{z}}_1] \\ \frac{1}{2}(|\tilde{z}_0|^2 - |\tilde{z}_1|^2) \end{pmatrix} \qquad \tilde{X}^0 = \frac{1}{2}(|\tilde{z}_0|^2 + |\tilde{z}_1|^2) \tag{3.54}$$

It is also often convenient to use $\tilde{X}^3$ and $\tilde{X}^\pm \equiv \tilde{X}^1 \pm i\tilde{X}^2$ where $\tilde{X}^+ = \overline{\tilde{z}}_0\tilde{z}_1$ and $\tilde{X}^- = \tilde{z}_0\overline{\tilde{z}}_1$. Notice that these are nothing but the classical version of Schwinger representation of angular

[7] See appendix B for conventions.

momentum in terms of two harmonic oscillators. An analogous definition will hold for X^μ, with a global minus sign according to the algebras (3.48).

It's easy to see that both X^i and $\tilde{X}^i$ satisfy the algebra

$$\{\tilde{X}^i, \tilde{X}^j\} = \epsilon^{ij}{}_k \tilde{X}^k \tag{3.55}$$

in fact using $\tilde{X}^i = \frac{1}{2}\sigma^i_{AB}\bar{\tilde{z}}_A\tilde{z}_B$

$$\begin{aligned}
\{\tilde{X}^i, \tilde{X}^j\} &= \frac{1}{4}\sigma^i_{AB}\sigma^j_{CD}\{\bar{\tilde{z}}_A\tilde{z}_B, \bar{\tilde{z}}_C\tilde{z}_D\} \\
&= \frac{1}{4}\sigma^i_{AB}\sigma^j_{CD}(\bar{\tilde{z}}_A\tilde{z}_D\{\tilde{z}_B, \bar{\tilde{z}}_C\} + \tilde{z}_B\bar{\tilde{z}}_C\{\bar{\tilde{z}}_A, \tilde{z}_D\}) \\
&= \frac{1}{4}\sigma^i_{AB}\sigma^j_{CD}\left(-i\delta^{BC}\bar{\tilde{z}}_A\tilde{z}_D + i\delta^{AD}\tilde{z}_B\bar{\tilde{z}}_C\right) \\
&= \frac{-i}{4}\left(\sigma^i_{AC}\sigma^j_{CD}\bar{\tilde{z}}_A\tilde{z}_D - \sigma^j_{CA}\sigma^i_{AC}\tilde{z}_B\bar{\tilde{z}}_C\right) \\
&= -\frac{i}{4}\langle\tilde{z}|\left[\sigma^i, \sigma^j\right]|\tilde{z}\rangle = \frac{1}{2}\epsilon^{ij}{}_k\langle\tilde{z}|\sigma^k|\tilde{z}\rangle = \epsilon^{ij}{}_k\tilde{X}^k
\end{aligned} \tag{3.56}$$

being $\left[\sigma^i, \sigma^j\right] = 2i\epsilon^{ij}{}_k\sigma^k$. This will hold also for the right vector field since the minus sign in its definition is compensated by the spinor algebra. This is nothing but a first piece of the twisted geometries algebra.

So going back to the original task, we wish to show the connection between these spinor variables and the edge phase space of LQG. We have now two spinors which encode 8 degrees of freedom. The space we are interested in is $T^*SU(2)$ which is only 6 dimensional. Thus we wish to make a reduction like

$$\mathbb{C}^4 : 8d \longrightarrow 6d : T^*SU(2) \tag{3.57}$$

To do so we need a constraint, namely the norm matching constraint

$$H \equiv X^0 - \tilde{X}^0 = 0 \tag{3.58}$$

which will be interpreted as an area matching constraint in the twisted geometries language. Naively speaking, the requirement that the spacial vectors $\vec{X}$ and $\vec{\tilde{X}}$ have the same norm, generates the action of $U(1)$ transformations. So it first removes a degree of freedom being a 1-dimensional constraint but then we also have to "divide" by the $U(1)$ orbits to get in fact $(8-1)/U(1) = 6$ degrees of freedom left.

$$\mathbb{T} :\ 8d \xrightarrow{H=0} 7d \xrightarrow{/U(1)} 6d :\ T^*SU(2) \tag{3.59}$$

Without going into much details, it is enough to define the reduced variables

$$j \equiv \frac{1}{2}\left(X^0 + \tilde{X}^0\right) \qquad\qquad \xi_A \equiv -i\left(\ln\frac{\bar{z}_A}{z_A} + \ln\frac{\tilde{z}_A}{\bar{\tilde{z}}_A}\right) \tag{3.60}$$

The first one is the common norm of the spacial vectors and for which it is useful to introduce the unit vectors

$$N^i = \frac{X^i}{j} \qquad \tilde{N}^i = -\frac{\tilde{X}^i}{j} \tag{3.61}$$

which, once parametrized with stereographic complex coordinates (3.19), are the ingredient of the twisted geometries. The second variable ξ in (3.60) is needed because $(j, N^i, \tilde{N}^i)$ span a 5-dimensional subspace commuting with the constraint that is not enough. The reduced phase space needs also a sixth angular variable such that altogether they satisfy

$$\{j, H\} = 0 \qquad \{\xi_A, H\} = 0 \qquad \{\xi_A, j\} = 1 \tag{3.62}$$

which means that the new variables commute with the constraint and both ξ_0 and ξ_1 are conjugated to j. They are thus equally valid choices for the reduced space, related by a the canonical transformation. One could now extract the same algebra of the twisted geometries with variables $(N^i, \tilde{N}^i, j, \xi_A)$ so that the space reduction is complete. The key point is that this algebra comes naturally from a reduction of the canonical Poisson structure of two spinor spaces or equivalently a twistor space. So far we have the brackets (3.55) written in terms of (3.61), and the last bracket in (3.62). It is also immediate to see that j commutes with both N and $\tilde{N}$. The only remaining brackets to evaluate are

$$\{\xi_A \,, jN^i\} = L^i_A(N) \tag{3.63}$$

which give, in cylindrical components $i = 3, -, +$

$$L^i_0(N(z)) = (1, -z, -\bar{z}) \qquad L^i_1(N(z)) = (1, 1/\bar{z}, 1/z) \tag{3.64}$$

Here $L(N) \equiv L_0(N(z))$ is exactly the function introduced in section 3.2 and $L_1(N) = L(N(-1/\bar{z})) = L(-N(z))$. From now on, we take $\xi = \xi_1$ as the reduced variable. As explained in [16], the existence of canonical transformations which shift the ξ variable and L are related to changes of section in the Hopf map. Collecting the brackets we find

$$\{jN^i, jN^j\} = \epsilon^{ij}{}_k\, jN^k \qquad \{j\tilde{N}^i, j\tilde{N}^j\} = -\epsilon^{ij}{}_k\, j\tilde{N}^k \qquad \{N^i, \tilde{N}^j\} = 0 \qquad \{\xi, j\} = 1$$
$$\{N^i, j\} = 0 \qquad \{\tilde{N}^i, j\} = 0 \qquad \{\xi, jN^i\} = L^i(N) \qquad \{\xi, j\tilde{N}^i\} = L^i(\tilde{N}) \tag{3.65}$$

exactly as (3.17). Notice that in the original paper [17] a slightly different choice was used: the orientation of the spheres were in fact chose to be the same, so that the second brackets would have a plus sign exactly as the first ones. As mentioned in the above, this is nothing but convention. In the present case it would change the definition $\tilde{N}^i$ in (3.61) simply removing the minus sign, and it would affect the isomorphism (3.35) with $T^*SU(2)$ in a minor way

$$(N, \tilde{N}, j, \xi) \to (X, g): \quad \begin{aligned} X &= jN \\ g &= \tilde{n} e^{\xi \tau_3}\, \epsilon\, n^{-1} \end{aligned} \tag{3.66}$$

namely, by the introduction of $\epsilon = i\sigma_2$ which is nothing but the metric tensor in the spinor space. Here one would therefore have $\tilde{X} = j\tilde{N} = -gXg^{-1}$, but in fact one could make even different choices[8].

In any case, we have shown that the twisted geometries algebra descends in a simple way from the canonical Poisson brackets on twistor space. Another thing that could have been done is to show another symplectic reduction, namely from twistor space to the cotangent bundle of $SU(2)$. Of course we already know from the previous sections that this is isomorphic to the twisted geometries space but for completeness and later convenience, we also explore this very simple alternative reduction.

If we forget about the twisted geometries parametrization, we recall that we can trivialize $T^*SU(2) \cong \mathfrak{su}(2) \times SU(2)$ as the pair (X, g). If X is a right-invariant vector field then the adjoint

$$\tilde{X} = -gXg^{-1} \tag{3.67}$$

is left-invariant. So the Poisson brackets read

$$\{X^i, X^j\} = \epsilon^{ij}{}_k X^k \qquad \{\tilde{X}^i, \tilde{X}^j\} = \epsilon^{ij}{}_k \tilde{X}^k \qquad \{X^i, g\} = -\tau^i g \qquad \{g, \tilde{X}^i\} = g\tau^i \tag{3.68}$$

The first two brackets hold automatically in the reduction of the twistor space, since X^i and $\tilde{X}^i$ commute with the constraint (3.58) and satisfy (3.55). To close the algebra we now need to find a $g(z_A, \tilde{z}_A) \in SU(2)$ such that commutes with the constraint H and satisfy the above algebra. It is easy to see that

$$g(z_A, \tilde{z}_A) = \frac{|\tilde{z}\rangle\langle z| + |z]\,[\tilde{z}|}{||z||\,||\tilde{z}||} = \frac{|\tilde{z}\rangle\langle z| + |z]\,[\tilde{z}|}{\sqrt{\langle z|z\rangle\langle\tilde{z}|\tilde{z}\rangle}} \tag{3.69}$$

fulfils those requirements. It is a proper element of $SU(2)$ and the following properties hold

$$g|z\rangle = |\tilde{z}\rangle \qquad\qquad g\,|z] = |\tilde{z}] \tag{3.70}$$

thanks to $gg^\dagger = g^\dagger g = \mathbb{1}$. It can also be written in terms of components

$$g^A{}_B = \frac{\tilde{z}^A \delta_{B\dot{B}} \bar{z}^{\dot{B}} + \delta^{A\dot{A}} \bar{\tilde{z}}_{\dot{A}} z_B}{||z||\,||\tilde{z}||} = \frac{1}{||z||\,||\tilde{z}||} \begin{pmatrix} \tilde{z}^0 \bar{z}^0 + \bar{\tilde{z}}^1 z^1 & \tilde{z}^0 \bar{z}^1 - \bar{\tilde{z}}^1 z^0 \\ \tilde{z}^1 \bar{z}^0 - \bar{\tilde{z}}^0 z^1 & \tilde{z}^1 \bar{z}^1 + \bar{\tilde{z}}^0 z^0 \end{pmatrix} \tag{3.71}$$

The commutation with the constraint H is straightforward and less trivially one could also show that the matrix elements commute among themselves in the surfaces where $H = 0$ is satisfied. Furthermore, the property (3.67) follows from (3.70) and the brackets (3.68) from the

[8] It is worth to stress that all such choices are equivalent and just a matter of conventions: one could also define the map $g = \tilde{n}\epsilon e^{\xi\tau_3} n^{-1}$ that together with $X = jN$ would also imply $\tilde{X} = -gXg^{-1} = j\tilde{N}$. Moreover one could use two epsilons in $g = \tilde{n}\epsilon e^{\xi\tau_3}\epsilon n^{-1}$ and get the same convention as (3.35) with no epsilons at all, namely $\tilde{X} = -gXg^{-1} = -j\tilde{N}$. Of course, the important point is to be consistent with the choice made. This liberty will be reflected in the choice of the sphere coherent states that compose the twisted geometries coherent states.

definitions (3.54) and the algebra (3.48). Finally, under local $SU(2)$ transformations $g(z,\tilde{z})$ transforms exactly as the holonomy of an $SU(2)$ connection

$$(|z\rangle, |\tilde{z}\rangle) \xrightarrow{SU(2)} (h_1|z\rangle, h_2|\tilde{z}\rangle) \qquad \Rightarrow \qquad g(z,\tilde{z}) \xrightarrow{SU(2)} h_1 g(z,\tilde{z}) h_2^{-1} \tag{3.72}$$

Summarizing, in terms of the standard holonomy-flux parametrization which is given by (3.69) and (3.53), we obtain a spinorial representation such that the space reduction considered reads

$$\mathbb{T} \ni (|z\rangle, |\tilde{z}\rangle) \longrightarrow (X(\mathbf{z}), g(\mathbf{z})) \in T^*SU(2) \tag{3.73}$$

To be precise, starting from the natural symplectic structure on $\mathbb{T}$ one can derive the reduced symplectic structure for g and X which (on the constraint hypersurface, i.e. when $H = 0$ is satisfied) turns out to be

$$\{X^i(z), X^j(z)\} = \epsilon^{ij}{}_k X^k(z) \qquad \{g_{IJ}(z,\tilde{z}), g_{KL}(z,\tilde{z})\} = 0 \qquad \{X^i(z), g_{IJ}(z,\tilde{z})\} = -\tau^i g_{IJ}(z)$$

$$\{\tilde{X}^i(z), \tilde{X}^j(z)\} = \epsilon^{ij}{}_k X^k(z) \qquad \{\tilde{X}^i(z), g_{IJ}(z,\tilde{z})\} = g_{IJ}(z)\tau^i \tag{3.74}$$

which is identical to the standard symplectic structure on $T^*SU(2)$ (3.39). This symplectic reductions gives a spinorial representation of holonomies and fluxes.

Applied to loop gravity this gives an interesting picture: consider a graph consisting simply of one edge e between vertex v_1 and vertex v_2. Then, instead of assigning a group element g and a Lie algebra element X to the edge, one can equally well assign doublet of spinors $|z\rangle$ and $|\tilde{z}\rangle$ to the vertices. Thus, the dynamical degrees of freedom are shifted to the nodes of the graph where these new spinor variables "are sitting", in this interpretation.

3.4.2 Null twistors

Up to now, we have connected twisted geometries to a pair of spinors in $\mathbb{C}^4$. The true relation is in fact with twistors and in particular null ones. Let us very briefly review some basic notions about twistor, following the literature [32].

A twistor $Z^\alpha \in \mathbb{C}^4$ can be viewed as a pair of spinors $Z^\alpha = (|\omega\rangle, |\pi\rangle)$, where $|\pi\rangle$ defines a null direction $p_\pi = |\pi]\,[\pi|$ in Minkowski space, while $|\omega\rangle$ defines a point x in complexified Minkowski space via $|\omega\rangle = ix|\pi\rangle$.

On twistor space there is a natural hermitian pairing given by

$$\overline{Z}_\alpha Z^\alpha = \langle\omega|\pi\rangle + \langle\pi|\omega\rangle \tag{3.75}$$

and the quantity $s = \overline{Z}_\alpha Z^\alpha/2$ is called helicity of the twistor. When a twistor is null, namely $s = 0$, the matrix x is hermitian and thus identifies a point in real Minkowski space. However, x is defined only up to the addition of a null momentum p_π, since $p_\pi|\pi\rangle = 0$. The resulting null ray $x + \lambda\, p_\pi$ can be explicitly obtained as

$$x(\lambda) = \frac{|\omega\rangle\langle\omega|}{i\langle\omega|\pi\rangle} + \lambda\,|\pi]\,[\pi| \qquad\qquad \lambda \in \mathbb{R} \tag{3.76}$$

Hence, a null twistor defines a null generator p_π and a null ray in Minkowski space. These together are called "ruled" null ray in the following, since the ray has a specific generator.

Now, the relation with the twisted geometries is established by the map

$$|\omega\rangle \equiv |z\rangle + |\tilde{z}] \qquad\qquad |\pi\rangle \equiv |z\rangle - |\tilde{z}] \tag{3.77}$$

Under this map, the twistor hermitian pairing (3.75) becomes

$$s = \frac{1}{2}\left(\langle\omega|\pi\rangle + \langle\pi|\omega\rangle\right) = \langle z|z\rangle - [\tilde{z}|\tilde{z}] \tag{3.78}$$

Then, the constraint $H = 0$ in (3.58) is equivalent to the statement that $Z^\alpha(z, \tilde{z})$ is a null twistor, and its $U(1)$ action translates into a global rescaling of Z^α. Therefore, the space $\{(|z\rangle, |\tilde{z}\rangle)\}$ reduced by $H = 0$ can be interpreted as a phase space of null twistors $\mathbb{TN}$ up to a global phase. This is the connection between (null) twistors and twisted geometries. This mathematical correspondence shows that we can think of an element of the edge phase space of loop quantum gravity, as a ruled null ray in Minkowski space. Whether this is just a mathematical correspondence, or it has a deeper geometrical origin, is still to be understood.

One remark is now in order, regarding the geometrical interpretation of the constraint H_e associated to the edge. Consider a cellular decomposition dual to the graph. A twisted geometry assigns to each face (dual to the edge e) its oriented area j_e, the two unit normals N_e and $\tilde{N}_e$ as seen from the two vertex frames sharing it, and an additional angle ξ_e related to the extrinsic curvature between the frames. Working with $\mathbb{C}^4_e = \{(z_A, \tilde{z}_A)\}$ corresponds to relaxing the uniqueness of the area and assigning to each face two areas X^0_e and $\tilde{X}^0_e$, one for each polyhedra sharing the link. The constraint H_e then imposes the matching of these areas. What we have shown is that the phase space of loop quantum gravity on a fixed graph can be obtained starting from a geometric intepretation of twistors and imposing an area matching condition, which is equivalent to say that the twistors are null. For a detailed analysis on the relation with twistors see [41].

3.5 From twistors to Regge geometries

We have seen that twisted geometries describe a notion of discrete geometry associated to the kinematical phase space of loop quantum gravity. Then we have noticed that after the closure condition is imposed over the whole graph, one has the space of closed twisted geometries corresponding to gauge-invariant phase space of LQG. This reduced space still describes discontinuous metrics, because of the shape-matching problem discussed above.

At this point, one might also wonder what happens if the shapes are made to match. To make the shapes match and the geometries continuous, one needs to add gluing constraints. Since the reduction by the gluing constraints of areas and normals alone corresponds to Regge calculus, the reduction of the closed twisted geometries gives a notion of phase space for Regge calculus, described by the (now continuous) piecewise flat Regge metrics, and their extrinsic

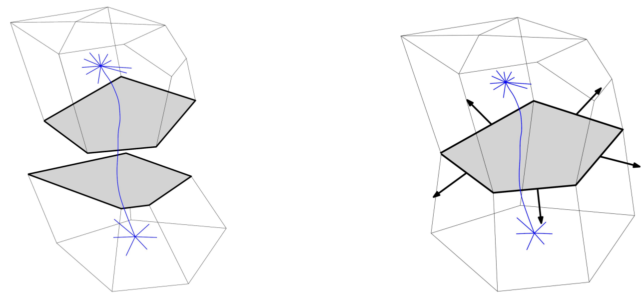


Figure 3.2: Two high valency nodes and the link connecting them. Left panel: in general two adjacent polyhedra do not glue well: even if the area of the grey face is the same because there is a unique spin associated with the link, the shapes will be different in general. Right panel: if one imposes the appropriate conditions on the polyhedra the shapes can be made to match.

curvature. As these constraints provide a further restriction than the Gauss law, such a Regge phase space is smaller than the gauge-invariant loop gravity phase space on a fixed graph. This should not come as a surprise: each configuration of holonomies and fluxes in corresponds in fact to infinite possible continuous metrics, but in general none of these will be piecewise flat. Such characterization requires additional conditions, which thanks to the twisted geometry parametrization of the gauge-invariant phase space, are manifestly identified precisely by the gluing constraints.
Moreover we have also pointed out a connection with a ”bigger” space, namely the twistor space. We know that the space of twisted geometries, isomorphic to $T^*SU(2)$, is related to null twistors in $\mathbb{C}^4$. Since the phase space of LQG on a fixed graph is just the cartesian product $\bigtimes_e T^*SU(2)$, it then can be derived from the larger space $\bigtimes_e \mathbb{C}^4$, imposing the area (or norm) matching constraint (3.58) at each edge. The derivation can be done in both the usual holonomy-fux parametrization (g_e, X_e) or in the twisted geometries parametrization $(N_e, \tilde{N}_e, j_e, \xi_e)$, as seen above. An interesting aspect of the twistor description is that it admits a complete factorization over the *vertices*, as opposed to the edges

$$\bigtimes_e \mathbb{C}^4 = \bigtimes_v \mathbb{C}^{2F(v)} \tag{3.79}$$

where $F(v)$ is the valency of the vertex v. This result follows straightforwardly once we use the orientation of the edges to assign $|z\rangle$ to say the source vertex and $|\tilde{z}\rangle$ to the target one.

Twistors and twisted geometries form natural spaces that can be associated to a graph. They admit simple geometric interpretations, and are related to the kinematical (i.e. prior to imposing the Gauss law implementing gauge-invariance) phase space of loop gravity on a fixed graph. We can see a picture forming here, going down from the twistor space to the Regge geometries. Let us recall that gauge-invariance is achieved reducing the space of twisted geometries by the closure condition (3.5)

$$C_v = \sum_{e \supset v} j_e N_e(v) = 0 \tag{3.80}$$

at each vertex. The resulting space of *closed* twisted geometries is isomorphic to the gauge-invariant space of LQG, $\bigtimes_e T^*SU(2)//SU(2)^V$. The variables parametrize it as $\bigtimes_e T^*S^1_e \bigtimes_v S_v$, where T^*S^1 is the cotangent bundle to a circle and S_v is the space of shapes of polyhedra. Closed twisted geometries define a local flat metric on each polyhedron. However, this metric is discontinuous: although each face has a unique area, it acquires a different shape when determined from the variables associated to the two polyhedra sharing it, since there is nothing enforcing a consistent matching of the faces. This discontinuity can be traced back to the fact that the normals carry both intrinsic and extrinsic geometry. Finally, as anticipated here above, for graphs dual to triangulations, the space of closed twisted geometries can be related to the phase space of Regge calculus[9] when one further imposes the gluing or shape matching conditions.

The following scheme summarizes the hierarchy in which twisted geometries fit in.

$$
\begin{array}{ccc}
\text{Twistor space} & & \\
\big\downarrow \;\textit{area matching} & & \\
\text{Twisted geometries} & \Longleftrightarrow & \text{Phase space of LQG} \\
\big\downarrow \;\textit{closure} & & \\
\text{Closed Twisted Geometries} & \Longleftrightarrow & \text{Gauge invariant LQG} \\
\big\downarrow \;\textit{shape matching} & & \\
\text{Regge phase space} & &
\end{array}
$$

From top to bottom, we move from larger and simpler spaces, with less intuitive geometrical meaning, to smaller and more constrained spaces, with clearer geometrical meaning. The results establish a path between twistors and Regge geometries, via loop gravity.

3.6 Geometrical picture

The unique correspondence between closed normals and polyhedra studied in chapter 3, together with the decomposition of the space of twisted geometries (3.14), mean that *each classical holonomy flux configuration on a fixed graph can be visualized as a collection of polyhedra, with a notion of parallel transport between them.* Just as the intertwiners are the building blocks of the quantum geometry of spin networks, polyhedra are the building blocks of the classical phase space in the twisted geometries parametrization.

[9]Regge was dealing with tetrahedra only, so for an arbitrary graph one would have a generalization of a 3d Regge geometry to arbitrary cellular decomposition. Notice however that the variables are not equivalent any more to edge lengths, since these do not specify uniquely the geometry of a generic polyhedron. Rather, these general Regge geometries must use areas and normals as fundamental variables.

These results will be fundamental for the constructions of coherent states. As we know, coherent states for loop quantum gravity have been introduced and extensively studied by Thiemann. It turns out that in practical terms, it is often convenient to truncate the theory to a single graph. This truncation provides a useful computational tool, to be compared to a perturbative expansion, and has found many applications, from the study of propagators to cosmology. In many of these applications, control of the semiclassical limit requires a notion of semiclassical states in the truncated space. The truncation can only capture a finite number of degrees of freedom, thus coherent states are not peaked on a smooth classical geometry. Twisted geometries offer a way to see them as peaked on a discrete geometry, to be viewed as an approximation of a smooth geometry on a cellular decomposition dual to the graph. In fact the reparametrization in terms of them achieves precisely a prescription for an interpolation procedure: starting from holonomies and fluxes on a graph, we can assign to them a specific twisted geometry which is nothing but a bunch of fuzzy polyhedra. This chapter allows one to see the geometrical picture of twisted geometries in terms of quantum polyhedra, and thus of coherent states as a collection of semiclassical polyhedra.

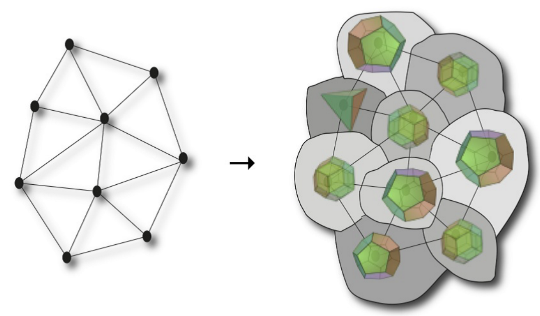


Figure 3.3: Geometrical interpretation of a spin network in terms of adjacent quantum polyhedra, to be compared to the generic chunks of figure 2.4

Chapter 4

Coherent states

This chapter contains bibliographical content but also some original work. The concept of coherent states (CS) goes back to the year 1926 when Schrödinger himself introduced them. Since then, several mathematical generalizations have been proposed and the many approaches to coherent states that we have nowadays almost transmit the impression of purely arbitrary definitions. Sometimes CS are introduced as the states such that the expectation values of some suitable operators, taken with respect to them, satisfy the classical equations of motion. Furthermore they are often introduced as eigenstates of some annihilation operator and this is related to the minimization of some Heisenberg uncertainties relations. In the simplest case of the Harmonic Oscillator, more formally, CS can be seen as generated by the action of the Heisenberg group over the vacuum state. Only for this simple system however, all these aspects and definitions coincide; in general this will not happen. It is clear now that for general systems, one always need to be careful when talking about coherent states. Whether they are defined through a modern geometrical or algebraic formulation, one has to clarify in which sense among the above, some states are considered coherent. In general CS are wanted to be "quasi-classical", namely the quantum states which best approximate the classical behaviour. The weakest requirement for CS is probably the fact that they form an overcomplete basis and they resolve the identity operator, a feature that will be used in what follows. This chapter is not a comprehensive review of coherent states, it is merely the description of the ones needed to build the Twisted Geometries CS. In general, there is no better definition of coherent states and this depends on the context and on what they will be used for.

In the case of LQG, as described in chapter 2, the existing heat kernel coherent states are in fact eigenvectors of an certain destruction operator and there are some peakedness proofs. However, the destruction operator for instance is a horrible function of holonomies and fluxes and at the same time the various properties are very hard to prove and one needs very cumbersome techniques to argue the peakedness. For example the direction of the fluxes is not so nicely singled out. Here lie some of the motivations for a new set of coherent states which have similar properties but have a nicer and clear geometrical interpretation, since for example they single out clearly the relevant things useful to have a discrete geometrical description.

In this chapter we recall the existing coherent states for quantum systems useful for our purpose, namely the $SU(2)$ and T^*S^1 CS motivated by the twisted geometries parametrization, and we also introduce a new set of CS for the harmonic oscillators. These new states are needed due to the $\mathbb{Z}_2$ symmetry technical problem featuring the twisted geometries parametrization, which exclude the possibility to use the standard T^*S^1 CS. The known states will be only reviewed whereas the new harmonic oscillator states, which will play a key role for the construction of the final Twisted Geometries LQG coherent states, will be discussed in more detail.

4.1 Sphere

The sphere coherent states are very well known objects, therefore this section will be just a short review on the topic, where we recall the useful properties which will be needed in the following.

Following [33], the so called $SU(2)$ coherent states (also spin or Bloch CS) are those that minimize the uncertainty $\Delta = |\langle J_i^2\rangle - \langle J\rangle^2|$ in the direction of the angular momentum. To be precise, $SU(2)$ is the exponentiation of the real Lie algebra $\mathfrak{su}(2)$ (which has generators J_i such that $[J_i, J_j] = i\epsilon_{ijk}J_k$), and its unitary irreducible representations of are the Hilbert spaces $\mathcal{H}_j$. These are labelled by a half integer spin j and spanned by the magnetic basis $|j,m\rangle$ with $m \in \{-j, \cdots, j\}$. This basis diagonalises simmultaneously J_z and the Casimir $\vec{J}^2$

$$J_z\,|j,m\rangle = m\,|j,m\rangle \qquad\qquad \vec{J}^2\,|j,m\rangle = j(j+1)\,|j,m\rangle \tag{4.1}$$

A simple calculation shows that on basis state, the uncertainty $\Delta = j(j+1) - m^2$ is minimized when $j = \pm m$. The maximal and minimal weight vectors, $|j,j\rangle$ and $|j,-j\rangle$, are thus coherent states for arbitrary choice of spin j and angular momentum axis J_z.

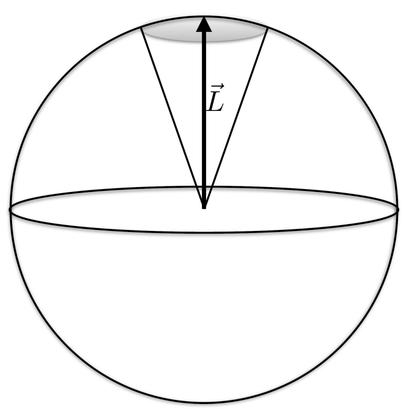


Figure 4.1: The cone represents the spread which scales as $1/\sqrt{j}$

It is in fact elementary to check that for example the state $|j,j\rangle$ for which

$$J_z|j,j\rangle = j|j,j\rangle \qquad \Longrightarrow \qquad \langle J_z\rangle = j \tag{4.2}$$

holds, saturates the Heisenberg relation

$$\Delta J_x \Delta J_y = \frac{1}{2}|\langle J_z \rangle| \tag{4.3}$$

since $\Delta J_x = \frac{j}{2} = \Delta J_y$. Another immediate property is that in the large j limit the relative uncertainty scales like $1/\sqrt{j}$ and there fore the state becomes sharp for $j \to \infty$. The same arguments hold for the state $|j, -j\rangle$.

Starting from the two families above, an infinite set of coherent states on the sphere are constructed through the group action, and to keep track of the difference we use the notations

$$|j, n\rangle = n(\zeta)\,|j, -j\rangle \qquad\qquad |j, n] = n(\zeta)\,|j, j\rangle \tag{4.4}$$

Note that they are sometimes denoted $|j, \vec{n}\rangle$ but we omit the arrows[1] upon the vectors inside the bra and ket. We will use the common notation $|j, \vec{n}\rangle = n\,|j, \pm j\rangle$ to represent both the highest and lowest weight together.
The group element n can be decomposed as $n(\zeta) : S^2 \rightarrowtail SU(2)$ times a $U(1)$ phase. The unit sphere S^2 can be parametrized by a complex number $\zeta \in \mathbb{C}$ (expect from one point) by the stereograhical projection

$$\vec{n}(\zeta) = \frac{1}{1 + |\zeta|^2} \begin{pmatrix} -\zeta - \bar{\zeta} \\ i(\bar{\zeta} - \zeta) \\ 1 - |\zeta|^2 \end{pmatrix} \tag{4.5}$$

which defines a point or direction on the sphere. Formally, in [33] they show that the group $U(1)$ is the so called isotropy subgroup of $SU(2)$ which stabilises the states up to a phase, so we have the isomorphism $SU(2)/U(1) \cong S^2$. We denote the representative for each class $\vec{n}(\zeta) \in S^2$ with the same symbol $n \in SU(2)$

$$n(\zeta) = \frac{1}{\sqrt{1 + |\zeta|^2}} \begin{pmatrix} 1 & \zeta \\ -\bar{\zeta} & 1 \end{pmatrix} \tag{4.6}$$

which are the group elements that enter the definitions (4.4). Equation (4.6) is nothing but the Hopf section defined in chapter three, see equation (3.21).

Simply put, the group action rotates the z direction in the $\vec{n}$ direction. Explicitly, taking $\vec{n} =$
$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we choose $n = \exp\{i\theta m \cdot \vec{J}\}$ where $m = (\sin\phi, -\cos\phi, 0)$ is a unit vector orthogonal to both the directions z and $\vec{n}$. So just as $|j, j\rangle$ has direction z with minimal uncertainty, $|j, n]$ has direction $\vec{n}$ with minimal uncertainty as can be easily verified. The same holds for the lowest weight case for which, for example,

$$\langle j, n|J_i|j, n\rangle = \langle j, -j|J'_i|j, -j\rangle \tag{4.7}$$

where $J'_i = nJ_in^{-1}$ is the rotated generator.

[1] This implies that the same notation is used for the group and the sphere element.

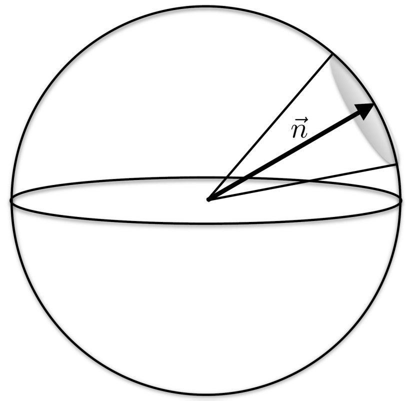


Figure 4.2: Generic direction on the sphere

A coherent state can be decomposed in terms of the usual magnetic basis as

$$
\begin{aligned}
|j, n(\zeta)\rangle &= \frac{1}{(1+|\zeta|^2)^j} \sum_{m=-j}^{j} \begin{pmatrix} 2j \\ j+m \end{pmatrix}^{\frac{1}{2}} \zeta^{j+m} |j, m\rangle \\
|j, n(\zeta)] &= \frac{1}{(1+|\zeta|^2)^j} \sum_{m=-j}^{j} \begin{pmatrix} 2j \\ j+m \end{pmatrix}^{\frac{1}{2}} \zeta^{j-m} |j, m\rangle
\end{aligned}
\tag{4.8}
$$

These states are normalized but not orthogonal. Consequently they provide an overcomplete basis for the irreducible representations j, and the resolution of the identity can be written as

$$
\mathbb{1}_j = d_j \int_{S^2} d^2n \, |j, n\rangle\langle j, n| = d_j \int_{S^2} d^2n \, |j, n] \, [j \ , n| \tag{4.9}
$$

where d^2n denotes the normalized measure on the sphere (namely it includes the factor $(4\pi)^{-1}$).

Among the important properties that these states satisfy, we point out that the $SU(2)$ CS are eigenstates of $\vec{n} \cdot \vec{J}$ (here $\pm$ stands for highest and lowest respectively)

$$
\vec{n} \cdot \vec{J} |j, \vec{n}\rangle = \pm j |j, n\rangle \qquad \Longrightarrow \qquad \langle j, \vec{n}| J_i |j, \vec{n}\rangle = \pm j n_i \tag{4.10}
$$

This is the analogous relation of (4.2) in an arbitrary direction. Another important property concerns the square of a component of a $SU(2)$ generator

$$
\langle j, \vec{n}| J_i^2 |j, \vec{n}\rangle = \frac{j}{2} + j\left(j - \frac{1}{2}\right) n_i^2 \tag{4.11}
$$

which is identical for both families of CS and follow from the more general formula

$$
\langle j, \vec{n}| J_i J_j |j, \vec{n}\rangle = \frac{j}{2}(\delta_{ij} \pm i\epsilon_{ijk} n_k) + j\left(j - \frac{1}{2}\right) n_i^2 \tag{4.12}
$$

depending whether $|j, \vec{n}\rangle$ highest (+) or lowest (-) weight. Equation (4.11) must not be confused with

$$
\langle j, \vec{n}| \vec{J}^2 |j, \vec{n}\rangle = j(j+1) \tag{4.13}
$$

which involves the Casimir operator.

From all these properties one immediately sees that the uncertainty

$$\Delta^2 \equiv \langle j, \vec{n}|\vec{J}^2|j, \vec{n}\rangle - \langle j, \vec{n}|\vec{J}|j, \vec{n}\rangle\langle j, \vec{n}|\vec{J}|j, \vec{n}\rangle = j \tag{4.14}$$

as anticipated and that for instance the Heisenberg relation

$$\Delta J_1 \Delta J_2 \geq \frac{1}{2}|\langle J_3\rangle| \tag{4.15}$$

is minimized, similarly to what we had in equation (4.3).

To make a contact with twisted geometries we also notice that thanks to the properties listed above, these coherent states satisfy

$$\langle j, \pm j|n^{-1}(\zeta)J_i n(\zeta)|j, \pm j\rangle = \pm j\, N_i \tag{4.16}$$

where $N \equiv n\tau_3 n^{-1}$ is one of the Twisted Geometries variables introduced in chapter 3. It is a unit vector identifying a point on the sphere with spherical components

$$N_+(\zeta) = -\frac{2\bar{\zeta}}{1+|\zeta|^2}, \quad N_-(\zeta) = -\frac{2\zeta}{1+|\zeta|^2}, \qquad N_3(\zeta) = \frac{1-|\zeta|^2}{1+|\zeta|^2} \tag{4.17}$$

Finally from equations (4.10) and (4.11), they minimize the uncertainty in the angular momentum direction, in the sense that the relative uncertainty vanishes in the large spin limit

$$\frac{(\Delta J_i)^2}{\langle J_i\rangle^2} = \frac{(1-n_i^2)}{2jn_i^2} \xrightarrow[j\to\infty]{} 0 \tag{4.18}$$

where they become sharp. This is exactly the property we will look for in the new LQG coherent states, to look at their peakedness. The SU(2) coherent states were used in the quantum gravity community since the introduction of the coherent intertwiner discussed in chapter 2, and they turn out to be a very natural object for this subject.

Let us also recall that the two families of CS are related by the $SU(2)$ metric tensor $\epsilon_{ab} = (-1)^{j-a}\delta_{a,-b}$ which satisfy $\epsilon|j,-j\rangle = |j,j\rangle$, where in the fundamental representation $\epsilon = i\sigma_2$.

4.2 Particle on the circle

We describe here the coherent state for the space T^*S^1 which is the classical phase space of a particle on a circle. We do not derive all the results since they are known and can be found in the literature [24], nevertheless we show in a little more details the calculations, compared to what we did with the $SU(2)$ CS, in order to clarify some interesting topics not clearly discussed elsewhere.

We call the conjugated variables $\xi \in [-2\pi, 2\pi]$ which is the angle and $j \in \mathbb{R}$ which is the momentum. As it is well known they satisfy the following Poisson bracket at the classical level which gets quantized by the operator algebra given by the commutator

$$\{\xi, j\} = 1 \qquad \xrightarrow{\hbar} \qquad \left[\hat{\xi}, \hat{j}\right] = i \tag{4.19}$$

where of course $\hbar$ is set to 1. Note that we have chosen for future convenience the periodicity in ξ to be 4π, so that the spectrum of $\hat{j}$ is quantized in half-integer steps.

The Hilbert space is spanned by states $|\psi\rangle$ which are 4π-periodic functions of the angle $\psi(\xi + 4\pi n) = \psi(\xi)$, with scalar product

$$\langle\psi|\psi\rangle = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} d\xi |\psi(\xi)|^2 \tag{4.20}$$

Let us denote $|\xi\rangle$ the eigenstate of $e^{\frac{i}{2}\hat{\xi}}$ and $|j\rangle$ those of $\hat{j}$ such that

$$e^{\frac{i}{2}\hat{\xi}}\,|\xi\rangle = e^{\frac{i}{2}\xi}\,|\xi\rangle \qquad \hat{j}\,|j\rangle = j\,|j\rangle \tag{4.21}$$

From the canonical commutator one has

$$\left[e^{\frac{i}{2}\hat{\xi}}\,,\,\hat{j}\right] = -\frac{1}{2}e^{\frac{i}{2}\hat{\xi}} \tag{4.22}$$

And from there one can derive

$$e^{\frac{i}{2}\hat{\xi}}\,|j\rangle = |j + 1/2\rangle \tag{4.23}$$

since

$$e^{\frac{i}{2}\hat{\xi}}|j\rangle = -2\left(e^{\frac{i}{2}\hat{\xi}}\hat{j} - \hat{j}e^{\frac{i}{2}\hat{\xi}}\right)|j\rangle = -2j\,e^{\frac{i}{2}\hat{\xi}}|j\rangle + 2\hat{j}e^{\frac{i}{2}\hat{\xi}}|j\rangle \tag{4.24}$$

Therefore

$$\hat{j}e^{\frac{i}{2}\hat{\xi}}|j\rangle = \left(j + \frac{1}{2}\right)e^{\frac{i}{2}\hat{\xi}}|j\rangle \tag{4.25}$$

which implies (4.23). So from now on we take $j \in \mathbb{Z}/2$. We are using the same notation for the classical variable and the eigenvalue of its operator, as usually done. We will use extra care when needed.

Coherent states for this space have been studied e.g. in [7, 8, 24] and more recently in [18]. They can be defined as eigenstates $|\omega\rangle$ of the (non-unitary) operator $e^{i\hat{\omega}}$, where $\hat{\omega} \equiv \hat{\xi} + it\hat{j}$ and t is just a real parameter for later convenience

$$e^{i\hat{\omega}}\,|\omega\rangle = e^{i\omega}\,|\omega\rangle \tag{4.26}$$

and they are defined to be

$$|\omega\rangle \equiv \sum_{j\in\mathbb{Z}/2} e^{-\frac{t}{2}j^2}e^{-i\omega j}\,|j\rangle \tag{4.27}$$

To check this, it is enough to compute

$$e^{i\hat{\omega}}\,|j\rangle = e^{i\hat{\xi}-t\hat{j}}|j\rangle = e^{-\frac{t}{2}}e^{i\hat{\xi}}e^{-t\hat{j}}|j\rangle \tag{4.28}$$

where we used the BCH formula. Now, using the known action of the operators we get

$$e^{i\hat{\omega}}\,|j\rangle = e^{-\frac{t}{2}}e^{-tj}|j+1\rangle = e^{\frac{t}{2}j^2}e^{-\frac{t}{2}(j+1)^2}|j+1\rangle \tag{4.29}$$

where we rearranged the terms in the last equality. Now, plugging it in (4.26) it is straightforward to see that

$$e^{i\hat{\omega}}|\omega\rangle = \sum_j e^{-\frac{t}{2}j^2}e^{-i\omega j}\,e^{i\hat{\omega}}|j\rangle = e^{i\omega}|\omega\rangle \tag{4.30}$$

4.2.1 Properties

The scalar product of these states computes to

$$\langle\omega'|\omega\rangle = \sum_{j\in\mathbb{Z}/2} e^{-tj^2}e^{-i(\omega-\bar{\omega}')j} \tag{4.31}$$

Notice that they are not normalized. They provide a resolution of the identity

$$\mathbb{1} = \int d\mu(\omega)e^{-\frac{\mathrm{Im}(\omega)^2}{t}}|\omega\rangle\langle\omega| \qquad \text{where} \qquad \int d\mu(\omega) \equiv \int_{-2\pi}^{2\pi}\frac{d\,\mathrm{Re}(\omega)}{4\pi}\int_{-\infty}^{\infty}\frac{d\,\mathrm{Im}(\omega)}{\sqrt{\pi t}} \tag{4.32}$$

as it is easy to verify

$$\begin{aligned}
\mathbb{1} &= \int d\mu(\omega)\,e^{-\frac{\mathrm{Im}(\omega)^2}{t}}\sum_{j,k}e^{-\frac{t}{2}(j^2+k^2)}e^{-i\,\mathrm{Re}(\omega)(j-k)}e^{\mathrm{Im}(\omega)(j+k)}|j\rangle\langle k| \\
&= \sum_{j,k}e^{-\frac{t}{2}(j^2+k^2)}\int_{-2\pi}^{2\pi}\frac{d\,\mathrm{Re}(\omega)}{4\pi}e^{-i\,\mathrm{Re}(\omega)(j-k)}\int_{-\infty}^{\infty}\frac{d\,\mathrm{Im}(\omega)}{\sqrt{\pi t}}e^{\mathrm{Im}(\omega)(j+k)}e^{-\frac{\mathrm{Im}(\omega)^2}{t}}|j\rangle\langle k| \\
&= \sum_j e^{-tj^2}\int_{-\infty}^{\infty}\frac{d\,\mathrm{Im}(\omega)}{\sqrt{\pi t}}e^{2\,\mathrm{Im}(\omega)j}e^{-\frac{\mathrm{Im}(\omega)^2}{t}}|j\rangle\langle j| \\
&= \sum_j |j\rangle\langle j|
\end{aligned} \tag{4.33}$$

where we used $\int_{-2\pi}^{2\pi}dx\,e^{-ix(j-k)} = 4\pi\delta_{jk}$ and then $\int_{-\infty}^{\infty}dx\,e^{2xj}e^{-\frac{x^2}{t}} = \sqrt{\pi t}\,e^{tj^2}$.

Now these states are semiclassical in the sense that they reproduce the classical variables in phase space. First of all we have the to compute the normalized expectation values

$$\frac{\langle\omega|\hat{j}|\omega\rangle}{\langle\omega|\omega\rangle} \qquad\qquad \frac{\langle\omega|e^{i\hat{\xi}}|\omega\rangle}{\langle\omega|\omega\rangle} \tag{4.34}$$

where, from (4.31) we have

$$\langle\omega|\omega\rangle = \sum_j e^{-tj^2}e^{2\,\mathrm{Im}(\omega)j} \;\equiv\; D \tag{4.35}$$

So we obtain

$$\begin{aligned}\frac{\langle\omega|\hat{j}|\omega\rangle}{\langle\omega|\omega\rangle} &= \frac{\sum_j j\, e^{-tj^2+2\,\mathrm{Im}(\omega)j}}{\sum_j e^{-tj^2+2\,\mathrm{Im}(\omega)j}} = \frac{\sum_j j\, e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2+\frac{\mathrm{Im}(\omega)}{t}}}{\sum_j e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2+\frac{\mathrm{Im}(\omega)}{t}}} = \frac{\sum_j j\, e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2}}{\sum_j e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2}} \\ &\simeq \frac{\int dj\, j\, e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2}}{\int dj e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2}} = \frac{\sqrt{\frac{\pi}{t}}}{\sqrt{\frac{\pi}{t}}}\frac{\mathrm{Im}(\omega)}{t} = \frac{\mathrm{Im}(\omega)}{t}\end{aligned} \tag{4.36}$$

where we approximated the sum with continuous integrals and used the Gauss integral. Moreover

$$\begin{aligned}\frac{\langle\omega|e^{i\hat{\xi}}|\omega\rangle}{\langle\omega|\omega\rangle} &= \frac{1}{D}\sum_{j,k} e^{-\frac{t}{2}j^2}e^{i\bar{\omega}j}e^{-\frac{t}{2}k^2}e^{-i\omega k}\langle j|k+1\rangle = \frac{1}{D}\sum_k e^{-\frac{t}{2}(k+1)^2}e^{i\bar{\omega}k}e^{i\bar{\omega}}e^{-\frac{t}{2}k^2-i\omega k} \\ &= \frac{1}{D}\sum_k e^{-tk^2}e^{-tk}e^{-\frac{t}{2}}e^{2\,\mathrm{Im}(\omega)k}e^{i\bar{\omega}} = \frac{1}{D}e^{-\frac{t}{4}+i\bar{\omega}}\sum_k e^{-t\left(k+\frac{1}{2}\right)^2}e^{2\,\mathrm{Im}(\omega)k} \\ &= \frac{1}{D}e^{-\frac{t}{4}+i\bar{\omega}}\sum_{k'} e^{-tk'^2}e^{2\,\mathrm{Im}(\omega)k'-\mathrm{Im}(\omega)} = e^{-\frac{t}{4}}e^{i\bar{\omega}-\mathrm{Im}(\omega)}\frac{\sum_{k'} e^{-tk'^2}e^{2\,\mathrm{Im}(\omega)k'}}{\sum_j e^{-tj^2}e^{2\,\mathrm{Im}(\omega)j}} \\ &= e^{i\,\mathrm{Re}(\omega)-\frac{t}{4}}\end{aligned} \tag{4.37}$$

where we completed the square at the exponent and we renamed $k+1/2 \to k'$. This last result does not belong to the circle but one should not be alarmed by this fact: an expectation value can normally be ill-defined even for Hermitian operators. A well-defined expectation value can be defined via the reference ket $|\omega_0\rangle$

$$\frac{\langle\omega|e^{i\hat{\xi}}|\omega\rangle/\langle\omega|\omega\rangle}{\langle\omega_0|e^{i\hat{\xi}}|\omega_0\rangle/\langle\omega_0|\omega_0\rangle} = \frac{e^{i\,\mathrm{Re}(\omega)-\frac{t}{4}}}{e^{i\,\mathrm{Re}(\omega_0)-\frac{t}{4}}} = e^{i[\mathrm{Re}(\omega)-\mathrm{Re}(\omega_0)]} = e^{i\,\mathrm{Re}(\omega)} \qquad \text{where} \quad \omega_0 = i \tag{4.38}$$

With these definitions of expectation values, the states are peaked on the classical pair $(\xi, j) = (\mathrm{Re}(\omega), \mathrm{Im}(\omega)/t)$. In order to look at the uncertainties (without using the reference ket for the expectation value) we need

$$\left(\Delta\widehat{e^{i\xi}}\right)^2 = \langle(\widehat{e^{i\xi}})^2\rangle - \langle\widehat{e^{i\xi}}\rangle^2 \tag{4.39}$$

where of course from (4.37)

$$\langle\widehat{e^{i\xi}}\rangle^2 = e^{2i\,\mathrm{Re}(\omega)-\frac{t}{2}} \tag{4.40}$$

and recalling (4.35)

$$\begin{aligned}\langle(\widehat{e^{i\xi}})^2\rangle &= \frac{\langle\omega|(e^{i\hat{\xi}})^2|\omega\rangle}{\langle\omega|\omega\rangle} = \frac{1}{D}\sum_k e^{-\frac{t}{2}(k+2)^2}e^{i\bar{\omega}(k+2)}e^{-\frac{t}{2}k^2}e^{-i\omega k} = \frac{1}{D}e^{2i\bar{\omega}-2t}\sum_k e^{-tk^2}e^{2\,\mathrm{Im}(\omega)k}e^{-2kt} \\ &= \frac{1}{D}e^{2i\bar{\omega}-t}\sum_k e^{-t(k+1)^2}e^{2\,\mathrm{Im}(\omega)k} = \frac{1}{D}e^{2i\bar{\omega}-t-2\,\mathrm{Im}(\omega)}\sum_j e^{-tj^2}e^{2\,\mathrm{Im}(\omega)j} = e^{2i\,\mathrm{Re}(\omega)-t}\end{aligned} \tag{4.41}$$

where we completed the squared and renamed $j = k + 1$. So

$$\left(\Delta\widehat{e^{i\xi}}\right)^2 = e^{2i\,\mathrm{Re}(\omega)-t} - e^{2i\,\mathrm{Re}(\omega)-\frac{t}{2}} = e^{2i\,\mathrm{Re}(\omega)-t}\left(1 - e^{\frac{t}{2}}\right) \tag{4.42}$$

Now coming to $\hat{j}$ we have

$$(\Delta\hat{j})^2 = \langle\hat{j}^2\rangle - \langle\hat{j}\rangle^2 \tag{4.43}$$

where from (4.36)

$$\langle\hat{j}\rangle^2 = \frac{\mathrm{Im}(\omega)^2}{t^2} \tag{4.44}$$

and

$$\begin{aligned}\langle\hat{j}^2\rangle &= \frac{\langle\omega|\hat{j}^2|\omega\rangle}{\langle\omega|\omega\rangle} = \frac{\sum_j j^2\, e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2}}{\sum_j e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2}} \\ &\simeq \frac{\int dj\, j^2\, e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2}}{\int dj e^{-t\left(j-\frac{\mathrm{Im}(\omega)}{t}\right)^2}} = \frac{2\,\mathrm{Im}(\omega)^2 + t}{2t^2}\end{aligned} \tag{4.45}$$

where again we used Gaussian integral in the limit. Therefore

$$(\Delta\hat{j})^2 = \frac{2\,\mathrm{Im}(\omega)^2 + t}{2t^2} - \frac{\mathrm{Im}(\omega)^2}{t^2} = \frac{1}{2t} \tag{4.46}$$

4.2.2 Minimization of uncertainties relations

So one now expects that the Heisenberg relation

$$(\Delta\hat{j})^2(\Delta\widehat{e^{i\xi}})^2 \geq \frac{1}{4}\langle\widehat{e^{i\xi}}\rangle^2 \tag{4.47}$$

is minimized. However this is not the case, in general. If one insists and consider the small t limit in $(\Delta e^{\hat{i\xi}})^2$

$$\left|(\Delta\widehat{e^{i\xi}})^2\right| \sim \frac{t}{2}e^{2i\,\mathrm{Re}(\omega)} \tag{4.48}$$

it would be closer but still not satisfying

$$(\Delta\hat{j})^2(\Delta\widehat{e^{i\xi}})^2 = \frac{1}{4}e^{2i\,\mathrm{Re}(\omega)} \neq \frac{1}{4}e^{2i\,\mathrm{Re}(\omega)-\frac{t}{2}} \tag{4.49}$$

As a matter of fact there is a case in which this holds. As suggested by Kowalski and collaborators in [24], one could pick the reference ket $|\omega_0\rangle$ with $\omega_0 = i$ such that the expectation value $\langle e^{i\xi}\rangle = e^{i\,\mathrm{Re}(\omega)}$, as seen above. In that case, the minimization would hold although one should justify the specific choice for the normalization of the expectation value.

So strictly speaking, the states do not saturate the Heisenberg inequality between those operators. Nevertheless, as shown in [24], one can actually find and build new operators for which these coherent states do minimize some uncertainties relations. They are

$$Q := \frac{X + X^\dagger}{2} \qquad P := \frac{X - X^\dagger}{2i} \tag{4.50}$$

where $X = e^{i\hat{\omega}}$ and $\hat{\omega} = \hat{\xi} + it\hat{j}$. One can in fact show that

$$\Delta Q \Delta P = \frac{1}{2}\langle [Q, P] \rangle \tag{4.51}$$

To show this we have to compute

$$\begin{aligned}
[Q, P] &= \frac{1}{4i}\left[\left(X + X^\dagger\right), \left(X - X^\dagger\right)\right] \\
&= \frac{1}{4i}\left(-\left[X, X^\dagger\right] + \left[X^\dagger, X\right]\right) \\
&= \frac{1}{2i}\left(-XX^\dagger + X^\dagger X\right) \\
&= \frac{1}{2i}\left(-e^{2t}X^\dagger X + X^\dagger X\right) \\
&= \frac{1}{2i}\left(1 - e^{2t}\right)X^\dagger X = -\frac{1}{2i}\left(e^{2t} - 1\right)X^\dagger X
\end{aligned} \tag{4.52}$$

where we used $X^\dagger X = e^{-2tj-t}$ and $XX^\dagger = e^{-2tj+t} = e^{2t}X^\dagger X$. So now

$$\frac{\langle \omega | \, [Q, P] \, | \omega \rangle}{\langle \omega | \omega \rangle} = -\frac{1}{2i}\left(e^{2t} - 1\right) e^{-2t\,\mathrm{Im}(\omega)} \tag{4.53}$$

It is in fact easier t check the square of the uncertainties so we will need

$$\frac{1}{4}\langle [Q, P] \rangle^2 = -\frac{1}{16} e^{-4\,\mathrm{Im}(\omega) t}\left(1 + e^{4t} - 2e^{2t}\right) \tag{4.54}$$

to confront with

$$(\Delta Q)^2 (\Delta P)^2 = \left(\langle Q^2 \rangle - \langle Q \rangle^2\right)\left(\langle P^2 \rangle - \langle P \rangle^2\right) \tag{4.55}$$

where

$$\begin{aligned}
\langle Q \rangle \equiv \frac{\langle \omega | Q | \omega \rangle}{\langle \omega | \omega \rangle} &= \frac{1}{2}\langle e^{i\xi} \rangle_\omega + \frac{1}{2}\langle (e^{i\omega})^\dagger \rangle_\omega \\
&= \frac{1}{2}\left(e^{i\omega} + e^{-i\bar{\omega}}\right) = \frac{1}{2}\left(e^{i\,\mathrm{Re} - t\,\mathrm{Im}(\omega)} + e^{-i\,\mathrm{Re} - t\,\mathrm{Im}(\omega)}\right) = e^{-t\,\mathrm{Im}(\omega)} \cos(\mathrm{Re}(\omega))
\end{aligned} \tag{4.56}$$

Similarly

$$\langle P \rangle \equiv \frac{\langle \omega | P | \omega \rangle}{\langle \omega | \omega \rangle} = e^{-t\,\mathrm{Im}(\omega)} \sin(\mathrm{Re}(\omega)) \tag{4.57}$$

Thus

$$\langle Q \rangle^2 = e^{-2\,\mathrm{Im}(\omega) t} \cos^2(\mathrm{Re}(\omega)) \qquad \langle P \rangle^2 = e^{-2\,\mathrm{Im}(\omega) t} \sin^2(\mathrm{Re}(\omega)) \tag{4.58}$$

Moreover

$$\langle Q^2 \rangle \equiv \frac{\langle \omega | Q^2 | \omega \rangle}{\langle \omega | \omega \rangle} = \frac{e^{-2\,\mathrm{Im}(\omega) t}}{4}\left[2\cos(2\,\mathrm{Re}(\omega)) + e^{2t} + 1\right] \tag{4.59}$$

$$\langle P^2 \rangle \equiv \frac{\langle \omega | P^2 | \omega \rangle}{\langle \omega | \omega \rangle} = -\frac{e^{-2\,\mathrm{Im}(\omega) t}}{4}\left[2\cos(2\,\mathrm{Re}(\omega)) - (e^{2t} + 1)\right] \tag{4.60}$$

Using the above ingredient one can explicitly compute (4.55), in fact thanks to various trigonometric identities

$$(\Delta Q)^2 (\Delta P)^2 = -\frac{e^{-4It}}{16}\left[1 - 2e^{2t} + e^{4t}\right] \tag{4.61}$$

in agreement with (4.54).

4.3 New harmonic oscillator CS

The twisted geometries parametrization describes holonomies and fluxes in terms of variables $(j, \xi, N, \tilde{N}) \in T^*S^1 \times S^2 \times S^2$. This is analogous to the $SL(2,\mathbb{C})$ polar decomposition $(X, g) \rightarrowtail H = e^{iX}g \in SL(2,\mathbb{C})$ which is used for the HK CS. Similarly, we can take $(j, \xi, N, \tilde{N})$ as a starting point to construct new coherent states based on the tensor product of the CS for T^*S^1 and S^2. The $SU(2)$ CS are nicely peaked on the direction and related to the coherent intertwiners, and as a bonus, these useful properties will be inherited by the new family. However there is a subtlety. As briefly mentioned in chapter 3, the twisted geometry map is 2 to 1, namely the pair (g, X) is invariant under the $\mathbb{Z}_2$ transformation

$$(j, \xi, N, \tilde{N}) \quad \mapsto \quad (-j, -\xi, -N, -\tilde{N}) \tag{4.62}$$

The isomorphism (3.37) means in fact the space of twisted geometries is in some sense the double cover of $T^*SU(2)$. Therefore we cannot use directly the CS for the particle on the circle, but we have to restrict the momenta to have positive values.

Below we show how this can be done . Namely, one can start from the T^*S^1 CS, reduce by parity to only even functions, and this will provide a basis for the harmonic oscillator. The basis is coherent in the sense that it is overcomplete and provides a resolution of the identity as an integral over the phase space. It is not however in the sense of being eigenstates of some destruction operator. We thus not know if they saturate inequalities, but we prove peakedness and vanishing of the relative uncertainties.

4.3.1 Proposal

Inspired by the coherent states for the particle on the circle, we propose new coherent states for the harmonic oscillator

$$|\omega\rangle = \sum_{n \in \mathbb{N}/2} e^{-\frac{t}{2}n^2} \cos(n\omega)|n\rangle, \qquad \omega \in \mathbb{C} \tag{4.63}$$

Let's also recall the basic property:

$$\begin{aligned} \cos(n\omega) &= \cos\left(n\,\mathrm{Re}(\omega) + in\,\mathrm{Im}(\omega)\right) \\ &= \cosh(nI)\cos(nR) - i\sinh(nI)\sin(nR) \end{aligned} \tag{4.64}$$

which will be used below. Here and in the following $I = \mathrm{Im}(\omega)$ and $R = \mathrm{Re}(\omega)$.

Let's compute the normalization

$$\begin{aligned} \langle\omega|\omega\rangle &= \sum_{m,n} e^{-\frac{t}{2}(n^2+m^2)} \cos(m\bar{\omega})\cos(n\omega)\langle m|n\rangle \\ &= \sum_{n} e^{-tn^2} \cos(n\bar{\omega})\cos(n\omega) \end{aligned} \tag{4.65}$$

Now it is easy to manipulate the product $\cos(n\bar{\omega})\cos(n\omega)$ as follows

$$\cos(n\bar{\omega})\cos(n\omega) = \frac{1}{2}\left[\cos(2n\,\mathrm{Re}(\omega)) + \cos(2in\,\mathrm{Im}(\omega))\right] = \frac{1}{2}\left[\cos(2nR) + \cosh(2nI)\right] \quad (4.66)$$

and then, treating the sum as an integral $\sum_n \to \int_0^\infty dx$, we can get the result

$$\langle\omega|\omega\rangle \simeq \frac{1}{2}\int_0^\infty dx e^{-tx^2}\left[\cos(2Rx) + \cosh(2Ix)\right] = \frac{1}{4}\sqrt{\frac{\pi}{t}}\left(e^{\frac{I^2}{t}} + e^{\frac{-R^2}{t}}\right) \quad (4.67)$$

To solve these integrals it's in fact enough to complete the squares at the exponents, in the definition of the cosine and hyperbolic cosine functions. Then the direct integration gives error functions which vanish in the domain. So the two contributions of (4.67) are

$$\int_0^\infty dx e^{-tx^2}\cos(2Rx) = \frac{1}{4}\sqrt{\frac{\pi}{t}}e^{-\frac{R^2}{t}}\left[\mathrm{erf}\left(\frac{tx+iR}{\sqrt{t}}\right) + \mathrm{erf}\left(\frac{tx-iR}{\sqrt{t}}\right)\right]_0^\infty = \frac{1}{2}\sqrt{\frac{\pi}{t}}e^{-\frac{R^2}{t}} \quad (4.68)$$

and similarly

$$\int_0^\infty dx e^{-tx^2}\cosh(2Ix) = \frac{1}{2}\sqrt{\frac{\pi}{t}}e^{\frac{I^2}{t}} \quad (4.69)$$

Note also that we have another way to express the product (4.66) which relies in the property (4.64), that is $\cos(n\bar{\omega})\cos(n\omega) = (\cosh^2(nI)\cos^2(nR) + \sinh^2(nI)\sin^2(nR))$. Of course integrating this gives the same result (4.67) but we write it here because we might need an analogy later.

So the normalized coherent state would be

$$|\omega\rangle_N = \mathcal{N}|\omega\rangle \qquad \mathcal{N} = 2\left(\frac{\pi}{t}\right)^{-\frac{1}{4}}\left(e^{\frac{I^2}{t}} + e^{-\frac{R^2}{t}}\right)^{-\frac{1}{2}} \quad (4.70)$$

in such a way that

$${}_N\langle\omega|\omega\rangle_N = \mathcal{N}^2\langle\omega|\omega\rangle = 1 \quad (4.71)$$

4.3.2 Resolution of Identity

Denoting $d^2\omega = dRdI$ where $R \in [-2\pi, 2\pi]$ and $I \in [0, \infty[$, we have

$$\begin{aligned}
\mathbb{1} &= \int d^2\omega\mu(\omega)|\omega\rangle\langle\omega| \\
&= \sum_{n,m}\int d^2\omega\mu(\omega)e^{-\frac{t}{2}(n^2+m^2)}|n\rangle\langle m|\times \\
&\times\left[\cosh(nI)\cos(nR)\cosh(mI)\cos(mR) + i\cosh(nI)\cos(nR)\sinh(mI)\sin(mR)\right. \\
&\quad\left.-i\sinh(nI)\sin(nR)\cosh(mI)\cos(mR) + \sinh(nI)\sin(nR)\sinh(mI)\sin(mR)\right]
\end{aligned} \quad (4.72)$$

If we call $L^t_{m,n}(\omega) = \mu(\omega)e^{-\frac{t}{2}(m^2+n^2)}|n\rangle\langle m| \equiv L(\omega)$, we get

$$\begin{aligned}
\mathbb{1} = &\sum_{m,n}\int_{-2\pi}^{2\pi} dR\cos(nR)\cos(mR)\int_0^\infty dI\cosh(nI)\cosh(mI)L(\omega)\\
&+ i\sum_{m,n}\int_{-2\pi}^{2\pi} dR\cos(nR)\sin(mR)\int_0^\infty dI\cosh(nI)\sinh(mI)L(\omega)\\
&- i\sum_{m,n}\int_{-2\pi}^{2\pi} dR\sin(nR)\cos(mR)\int_0^\infty dI\sinh(nI)\cosh(mI)L(\omega)\\
&+ \sum_{m,n}\int_{-2\pi}^{2\pi} dR\sin(nR)\sin(mR)\int_0^\infty dI\sinh(nI)\sinh(mI)L(\omega)
\end{aligned} \tag{4.73}$$

Now we will focus on the R integrals and we assume that the 'measure function' $\mu(\omega)$ can only depend on $\mathrm{Im}(\omega)$ as in fact it will be the case. Now using the trigonometric addition formulae it's easy to get

$$\int_{-2\pi}^{2\pi} dR\cos(nR)\cos(mR) = \frac{\sin(2\pi(n-m))}{(n-m)} = 2\pi\delta_{2m,2n} \tag{4.74}$$

where we used properly the limit $\sin(2\pi x)/x \to 2\pi$ for $x \to 0$. Notice that although the notation $\delta_{2n,2m}$ might seem redundant or useless, it is necessary to remind us that m, n are semi-integers and the Kronecker delta is only defined for integers. Now the R integrals in the second and third lines of (4.73) are identical and immediately equal to zero, since the integrand is odd and the domain of integration even. The last integral is trivially equal to the first one (4.74) and the result is the same $2\pi\delta_{m,n}$. Therefore we are left with

$$\begin{aligned}
\mathbb{1} &= 2\pi\sum_n\int_0^\infty dI\cosh^2(nI)e^{-tn^2}\mu(\omega)|n\rangle\langle n|\\
&+ 2\pi\sum_n\int_0^\infty dI\sinh^2(nI)e^{-tn^2}\mu(\omega)|n\rangle\langle n|\\
&= 2\pi\sum_n e^{-tn^2}|n\rangle\langle n|\int_0^\infty dI\left[\cosh^2(nI)+\sinh^2(nI)\right]\mu(\omega)
\end{aligned} \tag{4.75}$$

Finally, it is easy to check that if $\mu(\omega) = e^{-\frac{I^2}{t}}$ the integral above is

$$\int_0^\infty dI\left[\cosh^2(nI)+\sinh^2(nI)\right]e^{-\frac{I^2}{t}} = \frac{1}{2}\sqrt{\pi t}e^{tn^2} \tag{4.76}$$

So that the resolution of the Identity reads

$$\mathbb{1} = \int_{-2\pi}^{2\pi}\frac{dR}{\pi}\int_0^\infty\frac{dI}{\sqrt{\pi t}}e^{-\frac{I^2}{t}}|\omega\rangle\langle\omega| = \sum_n|n\rangle\langle n| \tag{4.77}$$

Let us show explicitly the integral (4.76). We first rewrite

$$\cosh^2(nI)+\sinh^2(nI) = \cosh(2nI) = \frac{1}{2}\left(e^{2nI}+e^{-2nI}\right) \tag{4.78}$$

Then we will just have to complete the squares in the exponents, as we get

$$\begin{aligned}
&\int_0^\infty dI \left[\cosh^2(nI) + \sinh^2(nI)\right] e^{-\frac{I^2}{t}} \\
&= \frac{1}{2}\int_0^\infty dI \left(e^{2nI-\frac{I^2}{t}} + e^{-2nI-\frac{I^2}{t}}\right) \\
&= \frac{1}{2}\int_0^\infty dI e^{tn^2 - t\left(\frac{I}{t}-n\right)^2} + \frac{1}{2}\int_0^\infty dI e^{tn^2 - t\left(\frac{I}{t}+n\right)^2} \\
&= \frac{1}{2}\left(\frac{\sqrt{\pi t}}{2}e^{tn^2}\,\mathrm{erf}(n\sqrt{t}) + \frac{\sqrt{\pi t}}{2}e^{tn^2} - \frac{\sqrt{\pi t}}{2}e^{tn^2}\,\mathrm{erf}(n\sqrt{t}) + \frac{\sqrt{\pi t}}{2}e^{tn^2}\right) = \frac{1}{2}\sqrt{\pi t}e^{tn^2}
\end{aligned} \tag{4.79}$$

Normalized version

We can do the same computation using normalized coherent states, and this will only change the measure of the integrals

$$\mathbb{1} = \int d^2\omega \mu(\omega) |\omega\rangle_{N\,N}\langle\omega| = \int d^2\omega\mu(\omega)\mathcal{N}^2|\omega\rangle\langle\omega| \tag{4.80}$$

where as above

$$\mathcal{N}^2 = 4\left(\frac{\pi}{t}\right)^{-\frac{1}{2}}\left(e^{\frac{I^2}{t}} + e^{-\frac{R^2}{t}}\right)^{-1} \tag{4.81}$$

And now it will be enough to choose $\mu(\omega)$ such that $\mu(\omega)\mathcal{N}^2 = e^{-\frac{I^2}{t}}$ to get something similar to the above resolution, thus

$$\mathbb{1} = \int d^2\omega\mu(\omega)|\omega\rangle_{N\,N}\langle\omega| \qquad\qquad \mu(\omega) = \frac{1}{4}\sqrt{\frac{\pi}{t}}e^{-\frac{I^2}{t}}\left(e^{\frac{I^2}{t}} + e^{-\frac{R^2}{t}}\right) \tag{4.82}$$

4.3.3 Expectation values

So now that we have our coherent states (4.63) that resolve the Identity, we are interested in another property which is the minimization of some uncertainty relations. Let's start computing the expectation values of the number operator $\hat{N}$.

We first need to know the action of $a, a^\dagger$ but since they are acting directly on $|n\rangle$

$$a^\dagger|\omega\rangle = \sum_n e^{-\frac{t}{2}n^2}\cos(n\omega)\sqrt{n+1}|n+1\rangle \tag{4.83}$$

$$a|\omega\rangle = \sum_n e^{-\frac{t}{2}n^2}\cos(n\omega)\sqrt{n}|n-1\rangle \tag{4.84}$$

it is pretty straightforward to get

$$\hat{N}|\omega\rangle = \sum_n e^{-\frac{t}{2}n^2}\cos(n\omega)n|n\rangle \tag{4.85}$$

We will be interested in normalized expectation values, so we have to compute

$$\frac{\langle\omega|\hat{N}|\omega\rangle}{\langle\omega|\omega\rangle} = \frac{\sum_n n e^{-\frac{t}{2}n^2}\cos(n\bar{\omega})\cos(n\omega)}{\sum_n e^{-\frac{t}{2}n^2}\cos(n\bar{\omega})\cos(n\omega)} \tag{4.86}$$

In the same 'continuum' approximation used in the above, this becomes

$$\frac{\langle\omega|\hat{N}|\omega\rangle}{\langle\omega|\omega\rangle} \simeq \frac{\int_0^\infty dx x e^{-tx^2}\left[\cos(2Rx)+\cosh(2Ix)\right]}{\int_0^\infty dx e^{-tx^2}\left[\cos(2Rx)+\cosh(2Ix)\right]} \tag{4.87}$$

and, computing the integral at the numerator, using (4.67) for the denominator, we get

$$\frac{\langle\omega|\hat{N}|\omega\rangle}{\langle\omega|\omega\rangle} \simeq \frac{2\sqrt{t}+iR\sqrt{\pi}e^{-\frac{R^2}{t}}\,\mathrm{erf}(\frac{iR}{\sqrt{t}})+I\sqrt{\pi}e^{\frac{I^2}{t}}\,\mathrm{erf}(\frac{I}{\sqrt{t}})}{t\sqrt{\pi}\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)} \equiv \langle\hat{N}\rangle \tag{4.88}$$

Now we want to consider the limit $I \to \infty$, so we first rewrite

$$\langle\hat{N}\rangle = \frac{2}{\sqrt{\pi t}\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)} + \frac{iRe^{-\frac{R^2}{t}}\,\mathrm{erf}\left(\frac{iR}{\sqrt{t}}\right)}{t\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)} + \frac{I}{t}\frac{e^{\frac{I^2}{t}}\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)}{\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)} \tag{4.89}$$

and notice that the last one is the only non vanishing term. Since the error functions expands like

$$\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right) = \frac{1}{\sqrt{t}}\sqrt{t} + e^{-\frac{I^2}{t}}\left(-\frac{\sqrt{t}}{\sqrt{\pi}I}\right) + O(2) \tag{4.90}$$

we get

$$\langle\hat{N}\rangle \xrightarrow[I\to\infty]{} \frac{I}{t} \tag{4.91}$$

like in the T^*S^1 case (see (4.36)).

In order to continue the analogy with the particle on the circle, we need an operator that plays the role of $e^{i\xi}$. For the harmonic oscillator this can be achieved by ladder operators defined[2] as

$$\hat{E}_+ = \hat{a}^\dagger(\hat{N}+1)^{-1/2} \qquad \Rightarrow \qquad \hat{E}_+|n\rangle = |n+1\rangle \tag{4.92}$$

$$\hat{E}_- = (\hat{N}+1)^{-1/2}\hat{a} \qquad \Rightarrow \qquad \hat{E}_-|n\rangle = |n-1\rangle \tag{4.93}$$

so that

$$\hat{E}_\pm|\omega\rangle = \sum_n e^{-\frac{t}{2}n^2}\cos(n\omega)|n\pm1\rangle \tag{4.94}$$

[2]Formally, one has to quantize directly the cosine and sine functions appearing in the description of the classical harmonic oscillator, described in terms of action and angle variables. Then the exponential operators $E_\pm$ will be proper composition of the cosine and sine operators. Given their definition in terms of the number operator N as well as creation and annihilation operators a and $a^\dagger$, their action on the states $|n\rangle$ is straightforward.

Given the similarity with the action of $e^{i\xi}$ on $|j\rangle$ we focus on E_+, and we compute

$$\begin{aligned}\langle\omega|\hat{E}_+|\omega\rangle &= \sum_{m,n} e^{-\frac{t}{2}(n^2+m^2)}\cos(m\bar{\omega})\cos(n\omega)\langle m|n+1\rangle \\ &= e^{-\frac{t}{2}}\sum_n e^{-tn^2-tn}\cos[(n+1)\bar{\omega}]\cos(n\omega)\end{aligned} \tag{4.95}$$

As we did before, we now have to rewrite the product $\cos[(n+1)\bar{\omega}]\cos(n\omega)$ in a convenient way

$$\begin{aligned}\cos[(n+1)\bar{\omega}]\cos(n\omega) &= \frac{1}{2}\left[\cos(2nR+\bar{\omega})+\cos(2niI-\bar{\omega})\right] \\ &= \frac{1}{2}\left[\cos[(2n+1)R-iI]+\cos[-R+iI(2n+1)]\right] \\ &= \frac{1}{2}[\cosh(I)\cos(R(2n+1))+\cosh(I(2n+1))\cos(R) \\ &\quad + i\sinh(I)\sin(R(2n+1))+i\sinh(I(2n+1))\sin(R)]\end{aligned} \tag{4.96}$$

So we get, again taking the 'continuum limit',

$$\begin{aligned}\langle\omega|\hat{E}_+|\omega\rangle \simeq{}& \frac{1}{2}e^{-\frac{t}{2}}\cosh(I)\int_0^\infty dx e^{-tx^2-tx}\cos((2x+1)R) \\ &+ \frac{1}{2}e^{-\frac{t}{2}}\cos(R)\int_0^\infty dx e^{-tx^2-tx}\cosh((2x+1)I) \\ &+ \frac{i}{2}e^{-\frac{t}{2}}\sinh(I)\int_0^\infty dx e^{-tx^2-tx}\sin((2x+1)R) \\ &+ \frac{i}{2}e^{-\frac{t}{2}}\sin(R)\int_0^\infty dx e^{-tx^2-tx}\sinh((2x+1)I)\end{aligned} \tag{4.97}$$

which can also be written as

$$\begin{aligned}\langle\omega|\hat{E}_+|\omega\rangle \simeq{}& \frac{1}{4}e^{-\frac{t}{2}}\cosh(I)\int_0^\infty dx e^{-tx^2-tx}\left(e^{i(2x+1)R}+e^{-i(2x+1)R}\right) \\ &+ \frac{1}{4}e^{-\frac{t}{2}}\cos(R)\int_0^\infty dx e^{-tx^2-tx}\left(e^{(2x+1)I}+e^{-(2x+1)I}\right) \\ &+ \frac{1}{4}e^{-\frac{t}{2}}\sinh(I)\int_0^\infty dx e^{-tx^2-tx}\left(e^{i(2x+1)R}-e^{-i(2x+1)R}\right) \\ &+ \frac{i}{4}e^{-\frac{t}{2}}\sin(R)\int_0^\infty dx e^{-tx^2-tx}\left(e^{(2x+1)I}-e^{-(2x+1)I}\right)\end{aligned} \tag{4.98}$$

And the result is

$$\begin{aligned}\langle\omega|\hat{E}_+|\omega\rangle \simeq{}& \frac{1}{8}e^{-\frac{t}{4}}\cosh(I)e^{-\frac{R^2}{t}}\sqrt{\frac{\pi}{t}}\left(2+\operatorname{erf}\left(\frac{2iR-t}{2\sqrt{t}}\right)-\operatorname{erf}\left(\frac{2iR+t}{2\sqrt{t}}\right)\right) \\ &+ \frac{1}{8}e^{-\frac{t}{4}}\cos(R)e^{\frac{I^2}{t}}\sqrt{\frac{\pi}{t}}\left(2+\operatorname{erf}\left(\frac{2I-t}{2\sqrt{t}}\right)-\operatorname{erf}\left(\frac{2I+t}{2\sqrt{t}}\right)\right) \\ &+ \frac{1}{8}e^{-\frac{t}{4}}\sinh(I)e^{-\frac{R^2}{t}}\sqrt{\frac{\pi}{t}}\left(\operatorname{erf}\left(\frac{2iR-t}{2\sqrt{t}}\right)+\operatorname{erf}\left(\frac{2iR+t}{2\sqrt{t}}\right)\right) \\ &+ \frac{i}{8}e^{-\frac{t}{4}}\sin(R)e^{\frac{I^2}{t}}\sqrt{\frac{\pi}{t}}\left(\operatorname{erf}\left(\frac{2I-t}{2\sqrt{t}}\right)+\operatorname{erf}\left(\frac{2I+t}{2\sqrt{t}}\right)\right)\end{aligned} \tag{4.99}$$

Now using these relations concerning Real and Imaginary parts of the erf function

$$\mathrm{Re}[\mathrm{erf}(x+iy)] = \frac{\mathrm{erf}(x+iy)+\mathrm{erf}(x-iy)}{2} \qquad \mathrm{Im}[\mathrm{erf}(x+iy)] = \frac{\mathrm{erf}(x+iy)-\mathrm{erf}(x-iy)}{2i} \tag{4.100}$$

and organizing the terms, (4.99) can be rewritten as follows

$$\begin{aligned}\langle\omega|\hat{E}_+|\omega\rangle = &\frac{1}{4}e^{-\frac{t}{4}}\sqrt{\frac{\pi}{t}}\left(e^{-\frac{R^2}{t}}\cosh(I)+e^{\frac{I^2}{t}}\cos(R)\right)\\ &-\frac{1}{4}e^{-\frac{t}{4}-\frac{R^2}{t}}\sqrt{\frac{\pi}{t}}\left(\cosh(I)\,\mathrm{Re}\left[\mathrm{erf}\left(\frac{2iR+t}{2\sqrt{t}}\right)\right]-i\sinh(I)\,\mathrm{Im}\left[\mathrm{erf}\left(\frac{2iR+t}{2\sqrt{t}}\right)\right]\right)\\ &+\frac{1}{8}e^{-\frac{t}{4}+\frac{I^2}{t}}\sqrt{\frac{\pi}{t}}\left(e^{iR}\,\mathrm{erf}\left(\frac{2I-t}{2\sqrt{t}}\right)-e^{-iR}\,\mathrm{erf}\left(\frac{2I+t}{2\sqrt{t}}\right)\right)\end{aligned} \tag{4.101}$$

In this way it is simpler to look at the limit $I\to\infty$ which will be taken in a moment. As a matter of fact we want to consider normalized expectation values, exactly as above with N, so we have to divide by (4.67) to get

$$\begin{aligned}\frac{\langle\omega|\hat{E}_+|\omega\rangle}{\langle\omega|\omega\rangle} = &e^{-\frac{t}{4}}\frac{e^{-\frac{R^2}{t}}\cosh(I)+e^{\frac{I^2}{t}}\cos(R)}{e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}}\\ &-e^{-\frac{t}{4}}e^{-\frac{R^2}{t}}\frac{\cosh(I)\,\mathrm{Re}\left[\mathrm{erf}\left(\frac{2iR+t}{2\sqrt{t}}\right)\right]-i\sinh(I)\,\mathrm{Im}\left[\mathrm{erf}\left(\frac{2iR+t}{2\sqrt{t}}\right)\right]}{e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}}\\ &+\frac{1}{2}e^{-\frac{t}{4}}e^{\frac{I^2}{t}}\frac{e^{iR}\,\mathrm{erf}\left(\frac{2I-t}{2\sqrt{t}}\right)-e^{-iR}\,\mathrm{erf}\left(\frac{2I+-t}{2\sqrt{t}}\right)}{e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}}\end{aligned} \tag{4.102}$$

This might look ugly at first sight, but it is in fact not difficult to notice that the limit $I\to\infty$ of the second line vanishes, as well as thee first bit in the first line, and using the limits of the erf we are left with

$$\frac{\langle\omega|\hat{E}_+|\omega\rangle}{\langle\omega|\omega\rangle}\xrightarrow[I\to\infty]{} e^{-\frac{t}{4}}\cos(R)+e^{-\frac{t}{4}}i\sin(R)=e^{iR}e^{-\frac{t}{4}} \tag{4.103}$$

just like (4.37) for the particle on the circle!

4.3.4 Uncertainties

The next step would be, aiming to compute the uncertainties $\Delta\hat{E}_+$ and $\Delta\hat{N}$, to calculate the squares of the above results and also the (normalized) expectation values of the squares of those operators so that

$$(\Delta\hat{N})^2 = \langle\hat{N}^2\rangle-\langle\hat{N}\rangle^2 \tag{4.104}$$

$$(\Delta\hat{E}_+)^2 = \langle\hat{E}_+^2\rangle-\langle\hat{E}_+\rangle^2 \tag{4.105}$$

Let's start with computing $\langle \hat{N}^2 \rangle$

$$\frac{\langle \omega | \hat{N}^2 | \omega \rangle}{\langle \omega | \omega \rangle} = \frac{\sum_n n^2 e^{-\frac{t}{2}n^2} \cos(n\bar{\omega}) \cos(n\omega)}{\sum_n e^{-\frac{t}{2}n^2} \cos(n\bar{\omega}) \cos(n\omega)} \tag{4.106}$$

These are not known sums so we pass to the continuum approximation

$$\frac{\langle \omega | \hat{N}^2 | \omega \rangle}{\langle \omega | \omega \rangle} \simeq \frac{\int_0^\infty dx\, x^2 e^{-tx^2} \left[\cos(2Rx) + \cosh(2Ix)\right]}{\int_0^\infty dx e^{-tx^2} \left[\cos(2Rx) + \cosh(2Ix)\right]} \tag{4.107}$$

and we get

$$\langle \hat{N}^2 \rangle \simeq \frac{e^{-\frac{R^2}{t}} \left(-2R^2 + t\right) + e^{\frac{I^2}{t}} \left(2I^2 + t\right)}{2t^2 \left(e^{-\frac{R^2}{t}} + e^{\frac{I^2}{t}}\right)} \tag{4.108}$$

Now, we need also $\langle \hat{N} \rangle^2$, and starting from (4.88), we will just write (for the moment)

$$\langle \hat{N} \rangle^2 = \left[\frac{2\sqrt{t} + iR\sqrt{\pi} e^{-\frac{R^2}{t}} \operatorname{erf}(\frac{iR}{\sqrt{t}}) + I\sqrt{\pi} e^{\frac{I^2}{t}} \operatorname{erf}(\frac{I}{\sqrt{t}})}{t\sqrt{\pi} \left(e^{-\frac{R^2}{t}} + e^{\frac{I^2}{t}}\right)}\right]^2 \tag{4.109}$$

so that the uncertainty can be written in this cumbersome way

$$(\Delta \hat{N})^2 = \frac{e^{-\frac{R^2}{t}} \left(-2R^2 + t\right) + e^{\frac{I^2}{t}} \left(2I^2 + t\right)}{2t^2 \left(e^{-\frac{R^2}{t}} + e^{\frac{I^2}{t}}\right)} - \left[\frac{2\sqrt{t} + iR\sqrt{\pi} e^{-\frac{R^2}{t}} \operatorname{erf}(\frac{iR}{\sqrt{t}}) + I\sqrt{\pi} e^{\frac{I^2}{t}} \operatorname{erf}(\frac{I}{\sqrt{t}})}{t\sqrt{\pi} \left(e^{-\frac{R^2}{t}} + e^{\frac{I^2}{t}}\right)}\right]^2 \tag{4.110}$$

which eventually has to be simplified or rewritten in a clever way if possible.

We start by rewriting (4.110)

$$\begin{aligned}(\Delta \hat{N})^2 &= \frac{e^{-\frac{R^2}{t}} \left(-2R^2 + t\right) + e^{\frac{I^2}{t}} \left(2I^2 + t\right)}{2t^2 \left(e^{-\frac{R^2}{t}} + e^{\frac{I^2}{t}}\right)} - \left[\frac{2\sqrt{t} + iR\sqrt{\pi} e^{-\frac{R^2}{t}} \operatorname{erf}(\frac{iR}{\sqrt{t}}) + I\sqrt{\pi} e^{\frac{I^2}{t}} \operatorname{erf}(\frac{I}{\sqrt{t}})}{t\sqrt{\pi} \left(e^{-\frac{R^2}{t}} + e^{\frac{I^2}{t}}\right)}\right]^2 \\ &= \frac{e^{-\frac{R^2}{t}} \left(-2R^2 + t\right) + e^{\frac{I^2}{t}} \left(2I^2 + t\right)}{2t^2 \left(e^{-\frac{R^2}{t}} + e^{\frac{I^2}{t}}\right)} - \frac{1}{d}\left[4t - R^2 \pi e^{-\frac{2R^2}{t}} \operatorname{erf}^2\left(\frac{iR}{\sqrt{t}}\right) + I^2 \pi e^{\frac{2I^2}{t}} \operatorname{erf}^2\left(\frac{I}{\sqrt{t}}\right)\right. \\ &\left. + 4iR\sqrt{\pi t} e^{-\frac{R^{2}{''}}{t}} \operatorname{erf}\left(\frac{iR}{\sqrt{t}}\right) + 4I\sqrt{\pi t} e^{\frac{I^2}{t}} \operatorname{erf}\left(\frac{I}{\sqrt{t}}\right) + iRI\pi e^{-\frac{R^2}{t} + \frac{I^2}{t}} \operatorname{erf}\left(\frac{iR}{\sqrt{t}}\right) \operatorname{erf}\left(\frac{I}{\sqrt{t}}\right)\right]\end{aligned} \tag{4.111}$$

where

$$d = t^2 \pi \left(e^{-\frac{R^2}{t}} + e^{\frac{I^2}{t}}\right)^2 \tag{4.112}$$

Now we take the common denominator and we call it D

$$D = 2t^2 \pi \left(e^{-\frac{R^2}{t}} + e^{\frac{I^2}{t}}\right)^2 \tag{4.113}$$

and the above (4.111) becomes

$$
\begin{aligned}
(\Delta\hat{N})^2 = \frac{1}{D}\Big[&\pi\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)e^{-\frac{R^2}{t}}\left(-2R^2+t\right)+\pi\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)e^{\frac{I^2}{t}}\left(I^2+t\right)-8t\\
&+2R^2\pi e^{-\frac{2R^2}{t}}\operatorname{erf}^2\left(\frac{iR}{\sqrt{t}}\right)-2I^2\pi e^{\frac{2I^2}{t}}\operatorname{erf}^2\left(\frac{I}{\sqrt{t}}\right)-8iR\sqrt{\pi t}e^{-\frac{R^2}{t}}\operatorname{erf}\left(\frac{iR}{\sqrt{t}}\right)\\
&-8I\sqrt{\pi t}e^{\frac{I^2}{t}}\operatorname{erf}\left(\frac{I}{\sqrt{t}}\right)-2iRI\pi e^{-\frac{R^2}{t}+\frac{I^2}{t}}\operatorname{erf}\left(\frac{iR}{\sqrt{t}}\right)\operatorname{erf}\left(\frac{I}{\sqrt{t}}\right)\Big]
\end{aligned}
\tag{4.114}
$$

Now we separate conveniently the terms in order to get an expression ready for the limit

$$
\begin{aligned}
(\Delta\hat{N})^2 = &\frac{e^{-\frac{R^2}{t}}\left(-2R^2+t\right)}{2t^2\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)}+\frac{e^{\frac{I^2}{t}}\left(2I^2+t\right)}{2t^2\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)}-\frac{4}{\pi t\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)^2}\\
&+\frac{R^2e^{-\frac{2R^2}{t}}\operatorname{erf}^2\left(\frac{iR}{\sqrt{t}}\right)}{t^2\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)^2}-\frac{I^2e^{\frac{2I^2}{t}}\operatorname{erf}^2\left(\frac{I}{\sqrt{t}}\right)}{t^2\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)^2}-\frac{4iRe^{-\frac{R^2}{t}}\operatorname{erf}\left(\frac{iR}{\sqrt{t}}\right)}{\sqrt{\pi}\,t^{\frac{3}{2}}\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)}\\
&-\frac{4Ie^{\frac{I^2}{t}}\operatorname{erf}\left(\frac{I}{\sqrt{t}}\right)}{\sqrt{\pi}\,t^{\frac{3}{2}}\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)^2}-\frac{iRIe^{-\frac{R^2}{t}+\frac{I^2}{t}}\operatorname{erf}\left(\frac{iR}{\sqrt{t}}\right)\operatorname{erf}\left(\frac{I}{\sqrt{t}}\right)}{t^2\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)^2}
\end{aligned}
\tag{4.115}
$$

Finally, we note that every term except for the second and fifth vanishes in the limit $I\to\infty$, and the two that remain seem to diverge, individually. Nevertheless if we put them together we see that

$$
\frac{e^{\frac{I^2}{t}}\left(2I^2+t\right)}{2t^2\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)}-\frac{I^2e^{\frac{2I^2}{t}}\operatorname{erf}^2\left(\frac{I}{\sqrt{t}}\right)}{t^2\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right)^2}\xrightarrow[I\to\infty]{}\frac{2I^2+t}{2t^2}-\frac{I^2}{t^2}=\frac{1}{2t}
\tag{4.116}
$$

which promisingly enough, is again what we had in the T^*S^1 case (4.46). As a matter of fact we could have decomposed the problem since even the single contributions coming from the limit of $\langle\hat{N}^2\rangle$ and $\langle\hat{N}\rangle^2$ individually, coincide already with the results of the particle on the circle, (4.44),(4.45). Therefore when put together we expect nothing new but the known result of (4.116) for T^*S^1, which was $(\Delta\hat{j})^2 = 1/2t$.

But now let's turn to the other operator. We need $\langle\hat{E}_+^2\rangle$ and we start from

$$
\hat{E}_+^2|\omega\rangle = \sum_n e^{-\frac{t}{2}n^2}\cos(n\omega)|n+2\rangle
\tag{4.117}
$$

So

$$
\langle\omega|\hat{E}_+^2|\omega\rangle = \sum_n e^{-\frac{t}{2}[(n+2)^2+n^2]}\cos\left[(n+2)\bar{\omega}\right]\cos(n\omega) = e^{-2t}\sum_n e^{-tn^2-2tn}\cos\left[(n+2)\bar{\omega}\right]\cos(n\omega)
\tag{4.118}
$$

Now we rewrite the product of those cosines in a similar way as in (4.96)

$$\begin{aligned}\cos[(n+2)\bar{\omega}]\cos(n\omega) &= \frac{1}{2}\left[\cos(2nR+2\bar{\omega})+\cos(2niI-2\bar{\omega})\right]\\ &= \frac{1}{2}\left[\cos[2R(n+1)-2iI]+\cos[-2R+2iI(n+1)]\right]\\ &= \frac{1}{2}[\cosh(2I)\cos(2R(n+1))+\cosh(2I(n+1))\cos(2R)\\ &\quad + i\sinh(2I)\sin(2R(n+1))+i\sinh(2I(n+1))\sin(2R)]\end{aligned} \tag{4.119}$$

And we take again the continuum limit to compute $\langle\omega|\hat{E}_+^2|\omega\rangle$ as

$$\begin{aligned}\langle\omega|\hat{E}_+^2|\omega\rangle \simeq& \frac{1}{2}e^{-2t}\cosh(2I)\int_0^\infty dx e^{-tx^2-2tx}\cos(2R(x+1))\\ &+\frac{1}{2}e^{-2t}\cos(2R)\int_0^\infty dx e^{-tx^2-2tx}\cosh(2I(x+1))\\ &+\frac{i}{2}e^{-2t}\sinh(2I)\int_0^\infty dx e^{-tx^2-2tx}\sin(2R(x+1))\\ &+\frac{i}{2}e^{-2t}\sin(2R)\int_0^\infty dx e^{-tx^2-2tx}\sinh(2I(x+1))\end{aligned} \tag{4.120}$$

the calculation is similar to the one we did above and the result is

$$\begin{aligned}\langle\omega|\hat{E}_+^2|\omega\rangle \simeq& \frac{1}{8}e^{-t}\cosh(2I)e^{-\frac{R^2}{t}}\sqrt{\frac{\pi}{t}}\left(2+\operatorname{erf}\left(\frac{iR-t}{\sqrt{t}}\right)-\operatorname{erf}\left(\frac{iR+t}{\sqrt{t}}\right)\right)\\ &+\frac{1}{8}e^{-t}\cos(2R)e^{\frac{I^2}{t}}\sqrt{\frac{\pi}{t}}\left(2+\operatorname{erf}\left(\frac{I-t}{\sqrt{t}}\right)-\operatorname{erf}\left(\frac{I+t}{\sqrt{t}}\right)\right)\\ &+\frac{1}{8}e^{-t}\sinh(2I)e^{-\frac{R^2}{t}}\sqrt{\frac{\pi}{t}}\left(\operatorname{erf}\left(\frac{iR-t}{\sqrt{t}}\right)+\operatorname{erf}\left(\frac{iR+t}{\sqrt{t}}\right)\right)\\ &+\frac{i}{8}e^{-t}\sin(2R)e^{\frac{I^2}{t}}\sqrt{\frac{\pi}{t}}\left(\operatorname{erf}\left(\frac{I-t}{\sqrt{t}}\right)+\operatorname{erf}\left(\frac{I+t}{\sqrt{t}}\right)\right)\end{aligned} \tag{4.121}$$

It is then convenient to write it in this way as well

$$\begin{aligned}\langle\omega|\hat{E}_+^2|\omega\rangle =& \frac{1}{4}e^{-t}\sqrt{\frac{\pi}{t}}\left(e^{-\frac{R^2}{t}}\cosh(2I)+e^{\frac{I^2}{t}}\cos(2R)\right)\\ &-\frac{1}{4}e^{-t-\frac{R^2}{t}}\sqrt{\frac{\pi}{t}}\left(\cosh(2I)\operatorname{Re}\left[\operatorname{erf}\left(\frac{iR+t}{\sqrt{t}}\right)\right]-i\sinh(2I)\operatorname{Im}\left[\operatorname{erf}\left(\frac{iR+t}{\sqrt{t}}\right)\right]\right)\\ &+\frac{1}{8}e^{-t+\frac{I^2}{t}}\sqrt{\frac{\pi}{t}}\left(e^{2iR}\operatorname{erf}\left(\frac{I-t}{\sqrt{t}}\right)-e^{-2iR}\operatorname{erf}\left(\frac{I+t}{\sqrt{t}}\right)\right)\end{aligned} \tag{4.122}$$

and dividing by (4.67)

$$\begin{aligned}\frac{\langle\omega|\hat{E}_+^2|\omega\rangle}{\langle\omega|\omega\rangle} &= e^{-t}\frac{e^{-\frac{R^2}{t}}\cosh(2I)+e^{\frac{I^2}{t}}\cos(2R)}{e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}} \\ &- e^{-t}e^{-\frac{R^2}{t}}\frac{\cosh(2I)\,\mathrm{Re}\left[\mathrm{erf}\left(\frac{iR+t}{\sqrt{t}}\right)\right]-i\sinh(2I)\,\mathrm{Im}\left[\mathrm{erf}\left(\frac{iR+t}{\sqrt{t}}\right)\right]}{e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}} \\ &+ \frac{1}{2}e^{-t}e^{\frac{I^2}{t}}\frac{e^{2iR}\,\mathrm{erf}\left(\frac{I-t}{\sqrt{t}}\right)-e^{-2iR}\,\mathrm{erf}\left(\frac{I+t}{\sqrt{t}}\right)}{e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}}\end{aligned} \tag{4.123}$$

We'll come to this later, taking the familiar limit. That was only the expectation value of the squared operator $\langle\hat{E}_+^2\rangle$. We also need the square of the expectation value (4.102)

$$\begin{aligned}\langle\hat{E}_+\rangle^2 = \Bigg[& e^{-\frac{t}{4}}\frac{e^{-\frac{R^2}{t}}\cosh(I)+e^{\frac{I^2}{t}}\cos(R)}{e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}} \\ &- e^{-\frac{t}{4}}e^{-\frac{R^2}{t}}\frac{\cosh(I)\,\mathrm{Re}\left[\mathrm{erf}\left(\frac{2iR+t}{2\sqrt{t}}\right)\right]-i\sinh(I)\,\mathrm{Im}\left[\mathrm{erf}\left(\frac{2iR+t}{2\sqrt{t}}\right)\right]}{e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}} \\ &+ \frac{1}{2}e^{-\frac{t}{4}}e^{\frac{I^2}{t}}\frac{e^{iR}\,\mathrm{erf}\left(\frac{2I-t}{2\sqrt{t}}\right)-e^{-iR}\,\mathrm{erf}\left(\frac{2I+t}{2\sqrt{t}}\right)}{e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}}\Bigg]^2\end{aligned} \tag{4.124}$$

And then compute $(\Delta\hat{E}_+)^2$. One might expect

$$\Delta\hat{N}\Delta\hat{E} \geq \frac{1}{2}\langle\hat{E}\rangle \tag{4.125}$$

is minimized by the new states. But exactly as in the T^*S^1 case, this will not be true.

In order to compute the above expressions, we first keep the two contribution coming from $\langle\hat{E}^2\rangle$ and $\langle\hat{E}\rangle^2$ separate, to simplify the calculation. We begin by considering the limit $I\to\infty$ of $\langle\hat{E}^2\rangle$ which is similar to what we've seen for $\langle\hat{E}\rangle$. As a matter of fact, looking at (4.102), (4.103) and comparing with (4.123) it's easy to see that

$$\frac{\langle\omega|\hat{E}_+^2|\omega\rangle}{\langle\omega|\omega\rangle} \xrightarrow[I\to\infty]{} e^{2iR}e^{-t} \tag{4.126}$$

which is again the same as for the particle on the circle $\langle\widehat{(e^{i\xi})^2}\rangle$, (4.41). Then, in order to calculate (4.124) let us call

$$d=\left(e^{-\frac{R^2}{t}}+e^{\frac{I^2}{t}}\right) \qquad A=\mathrm{erf}\left(\frac{2iR+t}{2\sqrt{t}}\right) \qquad B_\pm=\mathrm{erf}\left(\frac{2I\pm t}{2\sqrt{t}}\right) \tag{4.127}$$

so we get

$$\begin{aligned}\langle\hat{E}_+\rangle^2 = \frac{1}{d^2}\Big[&e^{-\frac{t}{2}}\Big(e^{-\frac{2R^2}{t}}\cosh^2(I)+e^{\frac{2I^2}{t}}\cos^2(R)+2e^{-\frac{R^2}{t}+\frac{I^2}{t}}\cosh(I)\cos(R)\Big)\\ &+e^{-\frac{t}{2}}e^{-\frac{2R^2}{t}}\left(\cosh^2(I)\operatorname{Re}^2[A]-\sinh^2(I)\operatorname{Im}^2[A]-2i\cosh(I)\sinh(I)\operatorname{Re}[A]\operatorname{Im}[A]\right)\\ &+\frac{1}{4}e^{-\frac{t}{2}}e^{\frac{2I^2}{t}}\left(e^{2iR}B_-^2+e^{-2iR}B_+^2-2B_-B_+\right)\\ &-2e^{-\frac{t}{2}}e^{-\frac{2R^2}{t}}\Big(e^{-\frac{R^2}{t}}\cosh(I)+e^{\frac{I^2}{t}}\cos(R)\Big)(\cosh(I)\operatorname{Re}[A]-i\sinh(I)\operatorname{Im}[A])\\ &-e^{\frac{t}{2}}e^{-\frac{R^2}{t}}e^{\frac{I^2}{t}}(\cosh(I)\operatorname{Re}[A]-i\sinh(I)\operatorname{Im}[A])\left(e^{iR}B_--e^{-2iR}B+\right)\\ &+e^{-\frac{t}{2}}e^{\frac{I^2}{t}}\Big(e^{-\frac{R^2}{t}}\cosh(I)+e^{\frac{I^2}{t}}\cos(R)\Big)\left(e^{iR}B_--e^{-iR}B_+\right)\Big]\end{aligned} \tag{4.128}$$

And now, noticing that in the limit $I\to\infty$ very few terms survive (the ones in the first and third line with $\exp\left(2I^2/t\right)$, which also appears in the last line implicitly), we are left with

$$\begin{aligned}\langle\hat{E}_+\rangle^2 \xrightarrow[I\to\infty]{} & e^{-\frac{t}{2}}\cos^2(R)+\frac{1}{4}e^{-\frac{t}{2}}\left(e^{2iR}+e^{-2iR}-2\right)+e^{-\frac{t}{2}}\cos(R)\left(e^{iR}-e^{-iR}\right)\\ &=\frac{1}{2}e^{-\frac{t}{2}}(1+\cos(2R))+\frac{1}{4}e^{-\frac{t}{2}}(2\cos(2R)-2)+2ie^{-\frac{t}{2}}\cos(R)\sin(R)\\ &=e^{-\frac{t}{2}}\cos(2R)+ie^{-\frac{t}{2}}\sin(2R)=e^{-\frac{t}{2}}e^{2iR}\end{aligned} \tag{4.129}$$

that is identical to the T^*S^1 result $\langle\widehat{e^{i\xi}}\rangle^2$, (4.40). Finally we put the pieces together to get $(\Delta\hat{E}_+)^2=\langle\hat{E}_+^2\rangle-\langle\hat{E}_+\rangle^2$

$$(\Delta\hat{E}_+)^2 \xrightarrow[I\to\infty]{} e^{-t}e^{2iR}-e^{-\frac{t}{2}}e^{2iR}=e^{2iR}e^{-t}\left(1-e^{\frac{t}{2}}\right) \tag{4.130}$$

that with no surprise is again what happens with the particle on the circle, see equation (4.42).

Thanks to the close analogy, it is not even worth spending time trying to check whether (4.125) is minimized and in fact just looking at (4.130) and (4.116) we see that it does not happen.

This is nothing new. It is just like what happens with the particle on the circle with $(\Delta\hat{j})^2$ and $(\Delta\widehat{e^{i\xi}})^2$. So every calculation done above behaves, in the limit considered ($I\to\infty$), exactly as in the T^*S^1 case. For this reason, it's not surprising at all that the uncertainties relations between $\hat{N}$ and $\hat{E}$ are not properly minimized, it is actually a well known result.

Now an interesting point would be to extend the analogy also to the new operators that were found [24] for the particle on the circle. Namely one might wonder if operators like (4.50) exist in our case here. We have different states (4.63) so we expect different operators. In fact our X in the definitions above (4.50) must be the operator for which (4.63) is the eigenstate. However, it turns out the this is yet to be found, if it exists.

So even though we don't know what exactly is minimized by the new harmonic oscillator coherent states discussed above (4.63), we still trust the fact that they behave just like the T^*S^1 case and so they will probably work just fine as coherent states. Plus, we have a resolution of the Identity which is enough to carry on and use them for the Twisted Geometries coherent states, outlined in the next chapter.

4.3.5 Inexistence of Holomorphic representation

Concerning these new HO coherent states, it is easy to show that the holomrphic representation of the algebra does not exist, and since this is one of the building blocks of the twisted geometries CS, the same argument will hold for the latter ones. This fact is also connected to the inexistence of the operators for which some uncertainties relations are minimized $\Delta Q \Delta P = \frac{1}{2}\langle [Q, P]\rangle$, namely

$$Q = \frac{X + X^\dagger}{2} \qquad\qquad P = \frac{X - X^\dagger}{2i} \tag{4.131}$$

where X is the destruction operator for which the coherent state is eigenstate. This operator does not exist for the HO CS $|\omega\rangle = \sum_n \exp\big(-tn^2/2\big)\cos(n\omega)|n\rangle$, and this is the key difference between this scenario and the particle on the circle case, where all the above operators exist and a Heisenberg relation is minimized.

In order to prove that there are no such operators, it is enough to show that the basis operators which span the whole algebra do not exist in the so called Fock-Bargmann space. So for example let us show that the annihilation operator a such that $a|n\rangle = \sqrt{n}|n-1\rangle$ doesn't exist on the space of functions

$$\psi^t_n(\omega) = \langle\bar\omega|n\rangle = e^{-\frac{t}{2}n^2}\cos(n\omega) \tag{4.132}$$

on which its action must be

$$A(\omega)\psi^t_n(\omega) = \sqrt{n}\psi^t_{n-1}(\omega) \tag{4.133}$$

The most general form of $A(\omega)$ is a linear combination of functions of ω and derivates with respect to it

$$A(\omega) = \sum_i f_i(\omega)\partial^{(i)}_\omega \tag{4.134}$$

so that its action on (4.132)

$$A\,\psi^t_n(\omega) = e^{-\frac{t}{2}n^2}\sum_{i\,\text{even}} f_i(\omega)(-n^2)^{\frac{i}{2}}\cos(n\omega) + e^{-\frac{t}{2}n^2}\sum_{i\,\text{odd}} f_i(\omega)(-n^2)^{\frac{i-1}{2}}(-n)\sin(n\omega) \tag{4.135}$$

must coincide with

$$\sqrt{n}\,\psi^t_{n-1}(\omega) = \sqrt{n}\,e^{-\frac{t}{2}n^2}e^{-\frac{t}{2}+tn}\cos((n-1)\omega) \tag{4.136}$$

We already see that for this to be true, the functions $f_i(\omega)$ must depend on t. Now it is easy to see that the equality reduces to

$$\sum_{i\,\mathrm{even}} f_i^t(\omega)(-n^2)^{\frac{i}{2}}\mathrm{c}(n\omega) + \sum_{i\,\mathrm{odd}} f_i^t(\omega)(-n^2)^{\frac{i-1}{2}}(-n)\mathrm{s}(n\omega) = \sqrt{n}\, e^{-\frac{t}{2}+tn}\left[\mathrm{c}(n\omega)\mathrm{c}(\omega) + \mathrm{s}(n\omega)\mathrm{s}(\omega)\right] \tag{4.137}$$

which has to be true for all n and ω, and we called $c(x) = \cos(x)$ and $s(x) = \sin(x)$. But this is not the case for instance if we pick a value, say $\omega = 0$ because the equation

$$\sum_{i\in 2\mathbb{N}} f_i^t(0)(-n^2)^{\frac{i}{2}} = \sqrt{n}e^{-\frac{t}{2}}e^{tn} \tag{4.138}$$

is not solvable for any functions f. One can convince himself by expanding the exponential as a sum and then square both sides to see explicitly that there can never be a matching of even and odd powers of those polynomials. The same arguments hold for the creation operator and therefore for every other operators, a and $a^\dagger$ being a basis. Since we wanted to impose a formula valid for all n and ω, it was enough to find a value for which that was not true. So the claim is that it is impossible to find the homomorphic representation of the annihilation and creations operators, i.e. there are no A and $A^\dagger$ such that (4.133) and its counterpart hold in general. Therefore we conclude that the homomorphic representation does not exist at all.

Chapter 5

Twisted Geometries CS for LQG

In chapters 1 and 2 we have seen that spin network states are the building blocks of LQG: they are a basis of the kinematikal Hilbert space and diagonalize some geometrical operators. They also carry a notion of quantum geometry. In order to recover a smooth (intrinsic and exstrinsic) classical geometry, one needs a superposition of spin network states which are suitbaly peaked. Examples of such coherent states for LQG were introduced by Thiemann and discussed in chapter 2. They satisfy a number of important properties, in particular they are peaked on a point in phase space, that is a configuration of holonomies of the gravitational connection and fluxes of the triad fields. However, when performing explicit calculations, one typically considers one single graph at a time, and the associated Hilbert space $\mathcal{H}_\Gamma$. As discussed above this truncation captures only a finite number of degrees of freedom. One then can associate a classical phase space to $\mathcal{H}_\Gamma$ and view its points as distributional configurations of classical holonomies and fluxes. It would be useful to have a picture of the classical degrees of freedom captured by $\mathcal{H}_\Gamma$ in terms of discrete geometries, to provide some approximate description of smooth 3d geometries. This is what we studied in chapter 3: the parametrization of this same phase space can be done in terms of twisted geometries, which are discrete and possibly discontinuous 3-geometries assigned to a cellular decomposition dual to the graph. This provides a simple and compelling picture of the degrees of freedom of $\mathcal{H}_\Gamma$ in terms of discrete geometries which can be seen as a collection of quantum polyhedra associated to the nodes with non-trivial extrinsic curvature among them. Therefore coherent spin network states in $\mathcal{H}_\Gamma$ can be interpreted as a collection of semiclassical polyhedra, instead of distributional holonomies and fluxes.

Coherent states on a fixed graph have played an important role in the spin foam formalism for the dynamics of the theory. More recently, they have found many applications in minisuperspace models and effective dynamics, both in cosmological and black hole contexts. All applications are so far based on the heat kernel coherent states. It would be nice to include the structure of the coherent intertwiners in the concept of coherent states, as they proved to be very useful when analysing the theory. This was mentioned and partially done in [11] but the question we would like to ask is whether one can introduce an altogether new family of coherent states, such that they always include the coherent intertwiners, and not just in some limiting case. The answer is suggested by the twisted geometry parametrization, and was indeed one of the original motivations to study that. One advantage of this parametrization with respect to the holonomy flux is the locality at the level of the nodes. This advantage will be inherited by the new class of coherent states. Their reduction to the gauge invariant class is simpler than for the HK states, and we can show that the LS coherent intertwiners always appear. The family of CS that we will introduce is however weaker than the traditional HK one: the new ones are coherent in the sense that they provide a resolution of the identity. They furthermore are peaked on the classical values, with relative uncertainty vanishing in the large spin limit. On the other hand, they are not eigenstates of a destruction operator, at least not one that we were able to identify. As a consequence, they do not define a holomorphic representation, and we don't know if and which Heisenberg relation they saturate.

5.1 Preliminaries

The Hilbert space on a fixed graph decomposes as a tensor product over edge contributions, $\mathcal{H}_\Gamma = \otimes_e \mathcal{H}_e$ which are the building blocks of loop quantum gravity. The edge contribution $\mathcal{H}_e = L_2[SU(2), d\mu_H] = \oplus_j \left[\mathcal{H}_j \otimes \overline{\mathcal{H}_j}\right]$ is the space of functions $\psi(u) : SU(2) \to \mathbb{C}$ which are square integrable with respect to the Haar measure $d\mu_H$. The edge space is associated to each oriented link of a graph and carries a representation of the holonomy-flux algebra. We define the fluxes as right-invariant vector fields $\hat{R}$ associated with the source node of the link and the adjoint representation $\hat{L} = -\hat{g}\hat{R}\hat{g}^{-1}$, associated with the target node of the link[1]. Two orthonormal basis are the holonomy basis $|g\rangle$ of eigenvectors of the holonomy operator, and the momentum basis $|j, m, n\rangle$ which diagonalizes $\hat{L}^2$, $\hat{L}_z$ and $\hat{R}_z$. The transformation between the two is given by the Wigner matrices

$$\langle g|j, m, n\rangle = \sqrt{d_j} D^{(j)}_{mn}(g) = \sqrt{d_j}\langle j, m|g|j, n\rangle \tag{5.1}$$

where $d_j = 2j + 1$. With that normalization convention, the property

$$\int dg\, D^{(j)}_{ab} \overline{D^{(k)}_{cd}(g)} = \frac{1}{d_j} \delta^{jk} \delta_{ac} \delta_{bd} \tag{5.2}$$

becomes simply

$$\langle j, a, b\,|\,k, c, d\rangle = \delta^{jk}\delta_{ac}\delta_{bd} \qquad \Rightarrow \qquad \mathbb{1} = \int dg\, |g\rangle\langle g| = \sum_{j,a,b} |j, a, b\rangle\langle j, a, b| \tag{5.3}$$

Expression (5.1) invites the convenient interpretation $|j, m, n\rangle = \sqrt{d_j}|j, n\rangle \otimes \langle j, m|$, in line with the Peter-Weyl decomposition $L_2[SU(2), d\mu_H] = \oplus_j \left[\mathcal{H}_j \otimes \overline{\mathcal{H}_j}\right]$. This Hilbert space with the holonomy-flux algebra provides a quantization of the classical phase space $T^*SU(2)$, with its canonical $SU(2)$ invariant symplectic structure. Since $T^*SU(2) \simeq SU(2) \times \mathfrak{su}(2) \simeq SL(2, \mathbb{C})$, a point on this space phase can be identified by an $SL(2, \mathbb{C})$ group element in the

[1] These fields were called X and $\tilde{X}$ in chapter 3 and in the original papers [16],[17]. See appendix B for conventions and notations adopted.

polar decomposition $H = e^{iL}g$. Thus one can label a point with the left invariant vector field and the holonomy $(g, L) \in T^*SU(2)$. In chapter 2 we summarized the work of Thiemann on coherent states and we recall here only the expression

$$\Psi^t_H(g) = \langle g|H,t\rangle = \sum_j d_j e^{-\frac{t}{2}j(j+1)}\chi^{(j)}(Hg^{-1}) \tag{5.4}$$

In chapter 3 we also showed that the same space can be parametrized by a complex number $\omega \in \mathbb{C}$ and two unit vectors N and $\tilde{N}$ in $\mathbb{R}^3$. This map allowed us to interpret the holonomy-flux variables in terms of discrete and possibly discontinuous 3-geometries. The parametrization also suggests a new complex structure for the space space, the one induced by the natural complex structures of the building blocks of the twisted geometries phase space. Let us briefly recall the twisted geometries parametrization (3.35)

$$R = An\tau_3 n^{-1} \qquad\qquad g = \tilde{n}e^{\xi\tau_3}n^{-1} \qquad\qquad L = -A\tilde{n}\tau_3\tilde{n}^{-1} \tag{5.5}$$

where $\tau_i = -(i/2)\sigma_i$, $n = n(\zeta)$ is the Hopf section defined in chapter 3, and A is what was called j in the previous chapters: the new name is used to avoid confusion and to remember its area interpretation. As known, this parametrization replaces the pair (g, L) on each link with the three pairs

$$(A, \xi) \in \mathbb{R}^+ \times [-2\pi, 2\pi), \qquad \zeta \in \mathbb{C}P^1, \qquad \tilde{\zeta} \in \mathbb{C}P^1, \tag{5.6}$$

which we collectively denote Ω (and sometimes we write $n(\zeta) = n$ instead of ζ, labelling the same thing). The use of this parametrization is convenient with a certain geometric interpretation. First, fix once and for all the set of surfaces to which the fluxes are associated, by taking a cellular decomposition dual to the graph. Each face of the cell is associated to the link. We can then construct a piecewise flat geometry using these data. The geometric interpretation that turns out is that A is the area of the link, ζ the direction of the flux in the source frame, and $\tilde{\zeta}$ the direction of the flux in the target frame. Upon imposing the closure constraint, the normals are outgoing and we have the nice polyhedral picture described in the above with the twist angle ξ which is pure gauge. Now we can write

$$H = ge^{iR} = \tilde{n}e^{\omega\tau_3}n^{-1} \qquad \text{where} \qquad \omega = \xi + iA \tag{5.7}$$

It is worth to mention that one can straightforwardly see from (5.4) that

$$\Psi^t_H(g) \overset{A\mapsto\infty}{\longrightarrow} e^{\frac{1}{2t}(A-\frac{t}{2})^2} \sum_j d_j^{\frac{1}{2}} e^{-\frac{t}{2}(\frac{d_j}{2}-\frac{A}{t})^2 - i\xi j} D^{(j)}_{jj}(n^{-1}g^{-1}\tilde{n}) \tag{5.8}$$

This shows that in the limit of large area, the magnetic part of the heat kernel coherent states behaves like an $SU(2)$ coherent states. This fact was used in [11] to make a first link between these coherent states and the coherent intertwiners. In fact, it follows that at the gauge invariant level, $\mathcal{H}_0 = L_2[SU(2)^L/SU(2)^N]$

$$\begin{aligned}\mathcal{H}_0 \ni |H,t\rangle_0 &= \sum_{j_l,i_n}\prod_l d_{j_l}^{\frac{1}{2}} e^{-\frac{t}{2}j_l(j_l+1)}\,\mathrm{tr}_\Gamma\left[\otimes_l D^{(j_l)}(H_l)\otimes_n i_n\right]|\Gamma, j_l, i_n\rangle \\ &\overset{A_l\mapsto\infty}{\longrightarrow} \sum_{j_l,i_n}\prod_l d_{j_l}^{\frac{1}{2}} e^{-\frac{t}{2}j_l(j_l+1)+(A_l-i\xi_l)j_l}\prod_n c_{i_n}(n_l)\,|\Gamma, j_l, i_n\rangle\end{aligned} \tag{5.9}$$

where $c_i(n_l)$ are the coefficients of the coherent intertwiners and $|\Gamma, j_l, i_n\rangle$ are spin network states. Careful as we use n for node, for the magnetic index, and for the $SU(2)$ matrix. This limit shows that the spread is not minimal for the states with finite A. The properties (5.8) and (5.9) only show that the limiting set $A \mapsto \infty$ of heat-kernel coherent states has minimal spread in those directions. This is not enough to claim that coherent intertwiners can be used to construct coherent states in $\mathcal{H}^0_\Gamma$. Notice however that one can take the right-hand side of (5.8) as the definition of a new family of coherent states, for any $A \in \mathbb{R}^+$

$$\Psi^t_G(g) := \sum_j d_j e^{-\frac{t}{2}(j-A)^2 - i\xi j} D_{jj}(n^{-1} g^{-1} \tilde{n}) \tag{5.10}$$

where G stands for Gaussian. This family is not anymore coherent in the stronger sense of being eigenvectors of a destruction operator, but they still have satisfactory peakedness properties, and provide a resolution of the identity, with measure this time given by

$$\int_{T^*SU(2)} d\mu(G) := \frac{e^{-\frac{t}{4}}}{(\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}^+} \frac{dA}{1 + \mathrm{erf}(A\sqrt{t})} \int_{S^2} d^2\Omega \int_{SU(2)} dg \tag{5.11}$$

What we propose is in a similar spirit, but with a broader mathematical structure, a more elegant integration measure, and nicer peakedness properties.

5.2 Definition and first properties

As discussed in the previous chapters, the $\mathbb{Z}_2$ symmetry problem has led us to consider the new set of coherent states of the harmonic oscillator introduced in chapter 4, instead of the T^*S^1 ones. Using these and the sphere coherent states we are ready to build a new family of coherent states for LQG. Consider an oriented edge of the graph, and associate n with the source vertex and $\tilde{n}$ with the target. We first 'make coherent' the magnetic indices (a, b) by taking the sphere coherent states $|j, \tilde{n}]$ and $|j, n\rangle$

$$|j, a, b\rangle \equiv \sqrt{d_j} |j, b\rangle \otimes \langle j, a| \rightarrowtail \sum_{a,b} \langle j, b | j, \tilde{n}] \langle j, n | j, a\rangle \, |j, a, b\rangle \equiv \sqrt{d_j} |j, \tilde{n}] \otimes \langle j, n| \equiv |j, n, \tilde{n}\rangle \tag{5.12}$$

Note that we have made a specific choice in doing so regarding the highest or lowest weight $SU(2)$ coherent state. This is one of the conventions but all the possible combination[2] will work just the same and we take the liberty to consider the most convenient one when needed. This liberty is what we anticipated in chapter 3 and it is related to the simple property $\epsilon|j, -j\rangle = |j, j\rangle$. Then we also make coherent the remaining part, summing over $j \in \mathbb{N}/2$ with the weights

[2] On the course of this work we realised that a more generic construction can be made. It is nevertheless instructive to show the results with this simpler form of coherent states, and we will also provide the most general expression in the last section. The latter will include a combination of more than one of such choices.

of the new harmonic oscillator coherent states (4.63), and we define

$$
\begin{aligned}
|\Omega\rangle \equiv |\omega, n, \tilde{n}\rangle &= \sum_{j\in\mathbb{N}/2} d_j^{\frac{3}{2}}\, e^{-\frac{t}{2}j^2} \cos(j\omega)\, |j,\tilde{n}] \otimes \langle j, n| \\
&\equiv \sum_{j\in\mathbb{N}/2} d_j\, e^{-\frac{t}{2}j^2} \cos(j\omega)|j,n,\tilde{n}\rangle
\end{aligned}
\tag{5.13}
$$

where we chose the factor d_j for later convenience.

We recall that $|j,n\rangle = n|j,-j\rangle$ and $|j,n] = n|j,j\rangle$ and by definition one has

$$
\langle j,n|j,a\rangle = \langle j,-j|n^{-1}|j,a\rangle = D^{(j)}_{-ja}(n^{-1}) \qquad\qquad \langle j,b\,|j,\tilde{n}] = \langle j,b|\tilde{n}|j,j\rangle = D^{(j)}_{bj}(\tilde{n}) \tag{5.14}
$$

Therefore one can write

$$
\begin{aligned}
|\Omega\rangle &= \sum_{j,a,b} d_j e^{-\frac{t}{2}j^2} \cos(j\omega)\langle j,b\,|j,\tilde{n}]\,\langle j,n|j,a\rangle|j,a,b\rangle \\
&= \sum_{j,a,b} d_j e^{-\frac{t}{2}j^2} \cos(j\omega) D^{(j)}_{bj}(\tilde{n}) D^{(j)}_{-ja}(n^{-1})|j,a,b\rangle
\end{aligned}
\tag{5.15}
$$

From here it follows that

$$
\begin{aligned}
\Psi^t_\Omega(g) = \langle g|\Omega\rangle &= \sum_j d_j^{\frac{3}{2}} e^{-\frac{t}{2}j^2} \cos(j\omega)\langle j,n|g\,|j,\tilde{n}] \\
&= \sum_{j,a,b} d_j^{\frac{3}{2}} e^{-\frac{t}{2}j^2} \cos(j\omega) D^{(j)}_{bj}(\tilde{n}) D^{(j)}_{-ja}(n^{-1}) D^{(j)}_{ab}(g) \\
&= \sum_j d_j^{\frac{3}{2}} e^{-\frac{t}{2}j^2} \cos(j\omega) D^{(j)}_{-jj}(n^{-1}g\tilde{n})
\end{aligned}
\tag{5.16}
$$

Where we used the basic properties $\sum_a D^{(j)}_{-ja}(n^{-1}) D^{(j)}_{ab}(g) = D^{(j)}_{-jb}(n^{-1}g)$ and

$$
\sum_b D^{(j)}_{-jb}(n^{-1}g) D^{(j)}_{bj}(\tilde{n}) = D^{(j)}_{-jj}(n^{-1}g\tilde{n}) = \langle j,-j|n^{-1}g\tilde{n}|j,j\rangle = \langle j,n|g\,|j,\tilde{n}] \tag{5.17}
$$

In a very similar way one can see that

$$
\Psi^t_{\overline{\Omega}}(g) = \langle\overline{\Omega}|g\rangle = \sum_j d_j^{\frac{3}{2}} e^{-\frac{t}{2}j^2} \cos(j\omega)\,[j,\tilde{n}|\,g^{-1}|j,n\rangle = \sum_j d_j^{\frac{3}{2}} e^{-\frac{t}{2}j^2} \cos(j\omega) D^{(j)}_{j-j}(\tilde{n}^{-1}g^{-1}n) \tag{5.18}
$$

where $[j,\tilde{n}|\,g^{-1}|j,n\rangle = \langle j,j|\tilde{n}^{-1}g^{-1}n|j,-j\rangle = D^j_{j,-j}(\tilde{n}^{-1}g^{-1}n)$.

The squared norm of the coherent state (5.16) is

$$
||\Psi^t_\Omega(g)||^2 = \langle\Omega|\Omega\rangle = \sum_j d_j^2 e^{-tj^2} \cos(j\bar{\omega}) \cos(j\omega) \tag{5.19}
$$

This expression has a (subleading, but not vanishing) dependence on the real part of ω and it cannot be simplified any further. This will be instead done with the generalized family of coherent states of section 5.6, where the Jacobi's theta functions appear. Nevertheless, thanks to the Poisson resummation formula, one can approximate for small t the sum in (5.19) with an integral, to get[3]

$$\begin{aligned}\langle\Omega|\Omega\rangle &\simeq \frac{1}{2}\int_0^\infty dx(2x+1)^2 e^{-tx^2}\left[\cos(2\,\mathrm{Re}\,[\omega]\,x)+\cosh(2\,\mathrm{Im}\,[\omega]\,x)\right]\\ &= \frac{e^{-\frac{R^2}{t}}}{4t^{\frac{5}{2}}}\Bigg[-4R^2\sqrt{\pi}+2\sqrt{\pi}t+8e^{\frac{R^2}{t}}t^{\frac{3}{2}}+\sqrt{\pi}t^2+4iR\sqrt{\pi}t\,\mathrm{erf}\left(\frac{iR}{\sqrt{t}}\right)\\ &\qquad +\sqrt{\pi}e^{\frac{R^2}{t}+\frac{I^2}{t}}\left(4I^2+t(t+2)+4It\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)\right)\Bigg]\end{aligned} \tag{5.20}$$

This formula is what will be used for practical calculations.

5.2.1 Resolution of the Identity

The states provide the following resolution of the identity in $\mathcal{H}_e$, which is a weak but necessary condition to require for coherent states

$$\mathbb{1}=\int_{S^2}d^2n\int_{S^2}d^2\tilde{n}\int_D d\mu(\omega)e^{-\frac{I^2}{t}}|\Omega\rangle\langle\Omega|,\qquad\qquad \int_D d\mu(\omega)=\int_{-2\pi}^{2\pi}\frac{dR}{\pi}\int_0^\infty\frac{dI}{\sqrt{\pi t}} \tag{5.21}$$

where R and I denote the real and imaginary part of ω, thus they in fact are ξ and A respectively. Let's check it by explicitly computing the integral

$$\begin{aligned}&\int_{S^2}d^2n\int_{S^2}d^2\tilde{n}\int_D d\mu(\omega)e^{-\frac{I^2}{t}}|\Omega\rangle\langle\Omega|\\ &=\sum_{j,k}d_jd_ke^{-\frac{t}{2}(j^2+k^2)}\int_{S^2}d^2n\int_{S^2}d^2\tilde{n}\int_D d\mu(\omega)e^{-\frac{I^2}{t}}\cos(j\bar{\omega})\cos(k\omega)|j,n,\tilde{n}\rangle\langle j,n,\tilde{n}|\\ &=\sum_{j,k}d_jd_ke^{-\frac{t}{2}(j^2+k^2)}\sum_{a,b,c,d}\int_{S^2}d^2n\langle k,c|k,n\rangle\langle j,n|j,a\rangle\int_{S^2}d^2\tilde{n}\;\langle j,b|j,\tilde{n}]\,[k,\tilde{n}|k,d\rangle\\ &\quad\times\int_{-2\pi}^{2\pi}\frac{dR}{\pi}\int_0^\infty\frac{dI}{\sqrt{\pi t}}e^{-\frac{I^2}{t}}\cos(j\bar{\omega})\cos(k\omega)|j,a,b\rangle\langle k,c,d|\end{aligned} \tag{5.22}$$

Now, as we did in the harmonic oscillator case, we write the product of cosines as follows

$$\begin{aligned}\cos(j\bar{\omega})\cos(k\omega)=&[\cosh(kI)\cos(kR)\cosh(jI)\cos(jR)+i\cosh(kI)\cos(kR)\sinh(jI)\sin(jR)\\ &-i\sinh(kI)\sin(kR)\cosh(jI)\cos(jR)+\sinh(kI)\sin(kR)\sinh(jI)\sin(jR)]\end{aligned} \tag{5.23}$$

Notice that the R dependance is only here, and upon integration over $\int_{-2\pi}^{2\pi}dR$, the only non vanishing terms of (5.23) are the first and the fourth ones which are both equal to $2\pi\delta_{2n,2m}$

[3] Using the property (4.66).

(see section 4.3). Thus (5.22) is

$$\begin{aligned}
&\sum_j d_j^2 e^{-tj^2} \sum_{a,b,c,d} \int_{S^2} d^2n \langle j,c|j,n\rangle\langle j,n|j,a\rangle \int_{S^2} d^2\tilde{n}\; \langle j,b|j,\tilde{n}]\,[j,\tilde{n}|j,d\rangle \\
&\times \int_0^\infty \frac{2dI}{\sqrt{\pi t}} e^{-\frac{I^2}{t}} \left[\cosh^2(jI)+\sinh^2(jI)\right] |j,a,b\rangle\langle j,c,d| \\
&= \sum_j e^{-tj^2} \sum_{a,b,c,d} \delta_{a,c}\delta_{b,d} \int_0^\infty \frac{2dI}{\sqrt{\pi t}} e^{-\frac{I^2}{t}} \left[\cosh^2(jI)+\sinh^2(jI)\right] |j,a,b)\,(j,c,d| \\
&= \sum_{j,a,b} |j,a,b\rangle\langle j,a,b| = \mathbb{1}
\end{aligned} \tag{5.24}$$

where we used the integral computed in section 4.3, equation (4.79). This proves that (5.16) is an overcomplete basis in $L_2[SU(2), d\mu_H]$.

5.3 Expectation values and peakedness properties

In order to study the peakedness properties of the coherent states, we have to study the operator algebra. We expect the states to be peaked on the phase space variables (g, L).

5.3.1 Action of the algebra

The operator algebra is spanned by $\hat{g}$, $\hat{L}^i$ and $\hat{R}^i$, where the holonomy acts by multiplication and the vector fields by left and right derivatives which, given an element of the algebra $X = X_i\tau^i$, are defined as

$$\nabla^R_X f(g) = \frac{d}{dt} f\left(g e^{tX}\right)\Big|_{t=0} \qquad \nabla^L_X f(g) = \frac{d}{dt} f\left(e^{-tX} g\right)\Big|_{t=0} \tag{5.25}$$

So we introduce the operators

$$\begin{aligned}
\hat{R}_i\, f(g) &= i\nabla^L_i = -\sum_{j,a,b} \tilde{f}_{jab} D^{(j)}_{ac}(J_i) D^{(j)}_{cb}(g) \\
\hat{L}_i\, f(g) &= i\nabla^R_i = \sum_{j,a,b} \tilde{f}_{jab} D^{(j)}_{ac}(g) D^{(j)}_{cb}(J_i)
\end{aligned} \tag{5.26}$$

in terms of the hermitian generators $J_i = i\tau_i$. From the above expression and the structure of a generic function

$$\psi(g) = \sum_{j,a,b} \hat{f}_{jab} \langle g|j,a,b\rangle \quad \in \quad \mathcal{H}_e = \oplus_j \left[\mathcal{H}^j \otimes \overline{\mathcal{H}^j}\right] \tag{5.27}$$

we can see how these operators act on the momentum basis of states $|j,a,b\rangle$. Denoting also the Casimir operator as $\hat{C} = \hat{L}_i\hat{L}^i = \hat{R}_i\hat{R}^i$, we read

$$\hat{C}|j,m,n\rangle = j(j+1)|j,m,n\rangle \tag{5.28}$$

as well as

$$\hat{R}_3|j,m,n\rangle = -m|j,m,n\rangle \qquad \hat{L}_3|j,m,n\rangle = n|j,m,n\rangle \tag{5.29}$$

and

$$\hat{R}_\pm|j,m,n\rangle = -c_\mp(m)|j,m\mp 1,n\rangle \qquad \hat{L}_\pm|j,m,n\rangle = c_\pm(n)|j,m,n\pm 1\rangle \tag{5.30}$$

where $c_\pm(a) = \sqrt{(j\mp a)(j\pm a+1)} = \sqrt{j(j+1)-a(a\pm 1)}$.

Concerning the holonomy operator, things will be a little bit more complicated. However as mentioned, the twisted geometries parametrization will help to single out the only relevant operators. We can immediately write

$$\hat{g}^A_{\ B}\, D^{(j)}_{ab}(g) = g^A_{\ B} D^{(j)}_{ab}(g) = D^{(\frac{1}{2})}_{AB}(g) D^{(j)}_{ab}(g) = \sum_{J,M,N} C^{JM}_{\frac{1}{2}Ajm} C^{JN}_{\frac{1}{2}Bjn}\, D^{(J)}_{MN}(g) \tag{5.31}$$

where the C are Clebsh Gordan coefficient, $J = j-\frac{1}{2}, j+\frac{1}{2}$ and $M,N = -J,...,J$. We can rename labels and indices and consider the product

$$\begin{aligned} D^{(\frac{1}{2})}_{m_1n_1}(g) D^{(j)}_{m_2n_2}(g) &= \sum_{k=j-\frac{1}{2}}^{k=j+\frac{1}{2}} \sum_{m,n=-k}^{k} C^{km}_{\frac{1}{2}m_1jm_2}\, C^{kn}_{\frac{1}{2}n_1jn_2}\, D^{(k)}_{mn}(g) \\ &= \sum_{m,n} C^{j-\frac{1}{2},m}_{\frac{1}{2}m_1jm_2} C^{j-\frac{1}{2},n}_{\frac{1}{2}n_1jn_2}\, D^{(j-\frac{1}{2})}_{mn}(g) + \sum_{m,n} C^{j+\frac{1}{2},m}_{\frac{1}{2}m_1jm_2} C^{j+\frac{1}{2},n}_{\frac{1}{2}n_1jn_2}\, D^{(j+\frac{1}{2})}_{mn}(g) \end{aligned} \tag{5.32}$$

but we already see the complication here. If one wants to write down the explicit action on the momentum basis, looking at the equations (5.31) and (5.32) we get

$$\begin{aligned} \hat{g}^{\,a_1}{}_{b_1}|j,a_2,b_2\rangle &= g^{\,a_1}{}_{b_1}|j,a_2,b_2\rangle \\ &= \sum_{m,n} C^{j+\frac{1}{2},m}_{\frac{1}{2}a_1jam_2} C^{j+\frac{1}{2},n}_{\frac{1}{2}b_1jb_2}\, |j+1/2,m,n\rangle + \sum_{m,n} C^{j-\frac{1}{2},m}_{\frac{1}{2}a_1jam_2} C^{j-\frac{1}{2},n}_{\frac{1}{2}b_1jb_2}\, |j-1/2,m,n\rangle \end{aligned} \tag{5.33}$$

It's already pretty clear that this is not easy to treat and in general will not give nice and clean results for the expectation value. That is why for the holonomy we will consider the spinorial approach to single out the only useful operators that appear in the parametrization of the twisted geometries.

5.3.2 Flux peakedness

Using the action of the operator and taking advantage of the $SU(2)$ coherent states and their properties (4.10) and (4.16), one can easily compute the expectation value

$$\langle\Omega|\hat{L}_i|\Omega\rangle = \sum_j j\, d_j^2 e^{-tj^2} \cos(j\bar{\omega}) \cos(j\omega) \tilde{N}_i \tag{5.34}$$

and in the same integral approximation used for the norm one can show

$$
\begin{aligned}
\langle\Omega|\hat{L}_i|\Omega\rangle \simeq& \frac{1}{2}\int_0^\infty dx\, x(2x+1)^2 e^{-tx^2}\left[\cos(2xR)+\cosh(2xI)\right]\tilde{N}_i \\
=& \frac{e^{-\frac{R^2}{t}}}{4t^{\frac{7}{2}}}\Big[2\sqrt{\pi}t(-2R^2+t)+2e^{\frac{R^2}{t}+\frac{I^2}{t}}\sqrt{\pi}t(2I^2+t)+2e^{\frac{R^2}{t}}\sqrt{t}(-2R^2+2I^2+t(4+t)) \\
&+\sqrt{\pi}\left(Ie^{\frac{R^2}{t}+\frac{I^2}{t}}(4I^2+t(6+t))\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)-iR(4R^2-t(6+t))\,\mathrm{erf}\left(\frac{iR}{\sqrt{t}}\right)\right)\Big]\tilde{N}_i
\end{aligned}
\tag{5.35}
$$

where as always we call $\omega=\xi+iA=R+iI$. Now we normalize it, dividing by (5.20)

$$
\begin{aligned}
\langle\hat{L}_i\rangle_\Omega :=& \frac{\langle\Omega|\hat{L}^i|\Omega\rangle}{\langle\Omega|\Omega\rangle} \\
=& \frac{1}{t}\frac{1}{\mathcal{D}}\Big[2\sqrt{\pi}t(-2R^2+t)+2e^{\frac{R^2}{t}+\frac{I^2}{t}}\sqrt{\pi}t(2I^2+t)+2e^{\frac{R^2}{t}}\sqrt{t}(-2R^2+2I^2+t(4+t)) \\
&+\sqrt{\pi}\left(Ie^{\frac{R^2}{t}+\frac{I^2}{t}}(4I^2+t(6+t))\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)-iR(4R^2-t(6+t))\,\mathrm{erf}\left(\frac{iR}{\sqrt{t}}\right)\right)\Big]\tilde{N}_i
\end{aligned}
\tag{5.36}
$$

where $\mathcal{D}$ is

$$
\begin{aligned}
\mathcal{D}=&-4R^2\sqrt{\pi}+2\sqrt{\pi}t+8e^{\frac{R^2}{t}}t^{\frac{3}{2}}+\sqrt{\pi}t^2+4iR\sqrt{\pi}t\,\mathrm{erf}\left(\frac{iR}{\sqrt{t}}\right) \\
&+\sqrt{\pi}e^{\frac{R^2}{t}+\frac{I^2}{t}}\left(4I^2+t(t+2)+4It\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)\right)
\end{aligned}
\tag{5.37}
$$

These expressions look a bit chaotic, but now we are about to take the large spin (or area) limit $I\to\infty$, and so it is easy to notice that given the last term in $\mathcal{D}$, only few term are going to survive in the numerator (5.36), namely

$$
\begin{aligned}
\langle\hat{L}_i\rangle_\Omega \sim& \frac{1}{t}\frac{2e^{\frac{R^2}{t}+\frac{I^2}{t}}\sqrt{\pi}t(2I^2+t)+\sqrt{\pi}Ie^{\frac{R^2}{t}+\frac{I^2}{t}}(4I^2+t(6+t))\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)\tilde{N}_i}{\sqrt{\pi}e^{\frac{R^2}{t}+\frac{I^2}{t}}\left[4I^2+t(t+2)+4It\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)\right]} \\
=& \frac{4I^2t+2t^2+4I^3+It(6+t)\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)}{4tI^2+2t^2+t^3+4It^2\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)}\tilde{N}_i \xrightarrow[I\to\infty]{} \frac{I}{t}\tilde{N}_i
\end{aligned}
\tag{5.38}
$$

So to summarize, in the large area (or spin) limit

$$
\langle\hat{L}_i\rangle_\Omega = f(\omega)\tilde{N}_i \qquad f(\omega)\equiv\frac{\sum_j j\, d_j^2 e^{-tj^2}\cos(j\bar{\omega})\cos(j\omega)}{\sum_j d_j^2 e^{-tj^2}\cos(j\bar{\omega})\cos(j\omega)}\simeq\frac{\mathrm{Im}(\omega)}{t}=\frac{A}{t}
\tag{5.39}
$$

From the basic properties of the $SU(2)$ CS depicted in chapter 4, it also follows that

$$
\langle\Omega|\hat{R}_i|\Omega\rangle=-\sum_j j\, d_j^2 e^{-tj^2}\cos(j\bar{\omega})\cos(j\omega)N_i \qquad \langle\hat{R}_i\rangle_\Omega=-f(\omega)N_i\simeq-\frac{\mathrm{Im}(\omega)}{t}N_i=-\frac{A}{t}N_i
\tag{5.40}
$$

These result are exactly the same as the ones encountered in the previous chapter. To see the peakedness one has now to compute the whole uncertainty $(\Delta_\Omega \hat{L}_i)^2 = \langle \hat{L}_i^2 \rangle_\Omega - \langle \hat{L}_i \rangle_\Omega^2$. Let's start with the square of (5.36) and denote it

$$\langle \hat{L}_i \rangle_\Omega^2 := \left[\frac{\langle \Omega | \hat{L}_i | \Omega \rangle}{\langle \Omega | \Omega \rangle} \right]^2 \equiv \frac{\mathcal{N}^2}{t^2 \mathcal{D}^2} \tilde{N}_i^2 \tag{5.41}$$

The first thing to notice is that when we square the denominator $\mathcal{D}$, we get many terms that will not be relevant in the limit $I \to \infty$, since in that regime it will be controlled by the squared of the very last term in (5.37). So, to the leading order we have

$$\begin{aligned} \mathcal{D}^2 = & \left[-4R^2\sqrt{\pi} + 2\sqrt{\pi}t + 8e^{\frac{R^2}{t}} t^{\frac{3}{2}} + \sqrt{\pi}t^2 + 4iR\sqrt{\pi}t \operatorname{erf}\left(\frac{iR}{\sqrt{t}}\right) \right. \\ & \left. + \sqrt{\pi} e^{\frac{R^2}{t} + \frac{I^2}{t}} \left(4I^2 + t(t+2) + 4It \operatorname{erf}\left(\frac{I}{\sqrt{t}}\right) \right) \right]^2 \\ \sim & \, \pi e^{2\frac{R^2}{t} + 2\frac{I^2}{t}} \left[16I^4 + t^2(t+2)^2 + 16I^2t^2 \operatorname{erf}^2\left(\frac{I}{\sqrt{t}}\right) + 32I^3 t \operatorname{erf}\left(\frac{I}{\sqrt{t}}\right) \right. \\ & \left. + 8I^2 t(t+2) + 8It^2(t+2) \operatorname{erf}\left(\frac{I}{\sqrt{t}}\right) \right] \end{aligned} \tag{5.42}$$

At the same time, while squaring the numerator in (5.36) one has to notice that everything not proportional at least to $e^{2\frac{I^2}{t}}$ will vanish in the limit, due to (5.42). So again to the leading order we have

$$\mathcal{N}^2 \sim 4e^{2\frac{R^2}{t} + 2\frac{I^2}{t}} \pi t^2 (4I^4 + t^2 + 4I^2 t) + \pi I^2 e^{2\frac{R^2}{t} + 2\frac{I^2}{t}} (16I^4 + t^2(6+t)^2 + 8I^2 t(t+6)) \operatorname{erf}^2(I/\sqrt{t}) \tag{5.43}$$

If we now put everything together according to (5.41), and we consider the correct limit of the error functions, we get

$$\langle \hat{L}_i \rangle_\Omega^2 \sim \frac{1}{t^2} \frac{16I^4t^2 + 4t^4 + 16I^2t^3 + 16I^6 + I^2t^2(6+t)^2 + 8I^4t(6+t)}{16I^4 + t^2(t+2)^2 + 16I^2t^2 + 32I^3t + 8I^2t(t+2) + 8It(t+2)} \tilde{N}_i^2 \xrightarrow[I \to \infty]{} \frac{I^2}{t^2} \tilde{N}_i^2 \tag{5.44}$$

which is in fact what was expected.

In order to compute $\langle \Omega | \hat{L}_i^2 | \Omega \rangle$, we need the slightly more complicated property (4.11) of the $SU(2)$ CS. In the same spirit as above, one finds the much more cumbersome expression

$$
\begin{aligned}
\langle\Omega|\hat{L}_i^2|\Omega\rangle &= \sum_j j\left(j\tilde{N}_i^2+\frac{1}{2}(1-\tilde{N}_i^2)\right)d_j^2 e^{-tj^2}\cos(j\bar{\omega})\cos(j\omega)\\
&\simeq \frac{1}{2}\int_0^\infty dx\, x\left(x\tilde{N}_i^2+\frac{1}{2}(1-N_i^2)\right)(2x+1)^2 e^{-tx^2}\left[\cos(2Rx)+\cosh(2Ix)\right]\\
&= \frac{e^{-\frac{R^2}{t}}}{8t^{9/2}}\Big[8R^4\tilde{N}_i^2\sqrt{\pi}+2R^2t\left(\sqrt{\pi}\left(\tilde{N}_i^2(-12+t)-2t\right)-2(1+\tilde{N}_i^2)e^{\frac{R^2}{t}}\sqrt{t}\right)\\
&\quad+\sqrt{\pi}t^2\left(-\tilde{N}_i^2(t-6)+2t\right)+2e^{\frac{R^2}{t}}t^{3/2}\left(2(1+\tilde{N}_i^2)I^2+t(4-\tilde{N}_i^2(-4+t)+t)\right)\\
&\quad+e^{\frac{R^2}{t}+\frac{I^2}{t}}\sqrt{\pi}\left(2t^2(2I^2+t)+\tilde{N}_i^2(8I^4-2I^2(-12+t)t-(t-6)t^2)\right)\\
&\quad+\sqrt{\pi}t\left(Ie^{\frac{R^2}{t}+\frac{I^2}{t}}(4(1+\tilde{N}_i^2)I^2+t(6+6\tilde{N}_i^2+t-\tilde{N}_i^2t))\,\mathrm{erf}\left(\frac{I}{\sqrt{t}}\right)\right.\\
&\quad\left.-iR(4R^2(1+\tilde{N}_i^2)+t(-6(1+\tilde{N}_i^2)+(\tilde{N}_i^2-1)t))\,\mathrm{erf}\left(\frac{iR}{\sqrt{t}}\right)\right)\Big]
\end{aligned}
\tag{5.45}
$$

Once again one has to normalize it using (5.20), and in the usual limit we get to the leading order

$$
\frac{\langle\Omega|\hat{L}_i^2|\Omega\rangle}{\langle\Omega|\Omega\rangle}\sim\frac{1}{2t^2}\frac{8\tilde{N}_i^2I^2+4tI^3(1+\tilde{N}_i^2)}{4I^2}=\frac{1}{2t^2}\left(2\tilde{N}_i^2I^2+tI(1+\tilde{N}_i^2)\right)\equiv\langle\hat{L}_i^2\rangle \tag{5.46}
$$

because as above, only the pieces proportional to $e^{\frac{I^2}{t}}$ matter, and when they simplify out with the denominator, that is what is left at the leading order.

Finally, the uncertainty can be read from the above expressions

$$
(\Delta_\Omega\hat{L}_i)^2=\frac{1}{2t^2}\left(2\tilde{N}_i^2I^2+tI(1+\tilde{N}_i^2)\right)-\frac{I^2}{t^2}\tilde{N}_i^2=\frac{I(1+\tilde{N}_i^2)}{2t}\propto\mathrm{Im}(\omega) \tag{5.47}
$$

and this is enough to prove that the relative uncertainty vanishes, namely

$$
\frac{(\Delta_\Omega\hat{L}_i)^2}{\langle\hat{L}_i\rangle_\Omega^2}\simeq\frac{t(1+\tilde{N}_i^2)}{2I\tilde{N}_i^2}\propto\frac{1}{\mathrm{Im}(\omega)}\xrightarrow[I\to\infty]{}0 \tag{5.48}
$$

which is very very similar to the simple angular momentum case for $SU(2)$, (4.18) and it behaves exactly like it. Similarly we'll have something identical for $\hat{R}_i$, namely

$$
(\Delta_\Omega\hat{R}_i)^2\propto A\qquad\Rightarrow\qquad\frac{(\Delta_\Omega\hat{R}_i)^2}{\langle\hat{R}_i\rangle_\Omega^2}\propto\frac{1}{A}\xrightarrow[A\to\infty]{}0 \tag{5.49}
$$

where we recall that the imaginary part $\mathrm{Im}(\omega)=A$ is the area.

We are still investigating whether the same results hold in different limits involving t other than $\mathrm{Im}(\omega)=A$, concerning the validity of our integral approximation and possibly a truncation of the above sums. The final words about this will appear in [12], where we explore the new and more general family of TGCS, briefly previewed in section 5.6.

5.3.3 Holonomy peakedness

In order to consider the holonomy operator, it is necessary to recall some aspects of its spinorial representation introduced in chapter 3. As mentioned there, the spinor variables realize the classical counterpart of the Schwinger representation (see Appendix B for details on this). Here we are only interested in the part concerning the holonomy, since we have already dealt with the fluxes even without the spinorial representation. Starting from (3.71) , we exploit the twisted geometry parametrization

$$g = \tilde{n} e^{\xi \tau_3} n^{-1} \tag{5.50}$$

and focus only on the twist angle part. Recall in fact that the twisted geometry picture suggests to shift attention from the holonomy-flux operators to the right and left-invariant fluxes plus the twist angle operator. We therefore wish to quantize $e^{i\xi}$ in terms of harmonic oscillators. Let us first find its classical explicit expression. Starting from (5.50) and using the conventions defined in chapter 3, we can write

$$\begin{aligned} g &= \tilde{n} e^{\xi \tau_3} n^{-1} \\ &= \frac{1}{\sqrt{1+|\zeta|^2}} \frac{1}{\sqrt{1+|\tilde{\zeta}|^2}} \begin{pmatrix} 1 & \bar{\tilde{\zeta}} \\ -\bar{\tilde{\zeta}} & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{i}{2}\xi} & 0 \\ 0 & e^{\frac{i}{2}\xi} \end{pmatrix} \begin{pmatrix} 1 & -\zeta \\ \bar{\zeta} & 1 \end{pmatrix} \\ &= \frac{|z^1|}{||z||} \frac{|\tilde{z}^1|}{||\tilde{z}||} \begin{pmatrix} e^{-\frac{i}{2}\xi} & e^{\frac{i}{2}\xi} \bar{\tilde{\zeta}} \\ -e^{-\frac{i}{2}\xi} \bar{\tilde{\zeta}} & e^{\frac{i}{2}\xi} \end{pmatrix} \begin{pmatrix} 1 & -\zeta \\ \bar{\zeta} & 1 \end{pmatrix} \\ &= \frac{1}{||z||} \frac{1}{||\tilde{z}||} \begin{pmatrix} |\tilde{z}^1| e^{-\frac{i}{2}\xi} & \tilde{z}^0 e^{-i \arg \tilde{z}^1} e^{\frac{i}{2}\xi} \\ -\bar{\tilde{z}}^0 e^{i \arg(\tilde{z}^1)} e^{-\frac{i}{2}\xi} & |\tilde{z}^1| e^{\frac{i}{2}\xi} \end{pmatrix} \begin{pmatrix} |z^1| & -z_0 e^{-\arg z^1} \\ \bar{z}^0 e^{i \arg z^1} & |z^1| \end{pmatrix} \end{aligned} \tag{5.51}$$

so that with the choice

$$\xi = 2 \arg \tilde{z}^1 - 2 \arg(z^1) = -i \ln \left(\frac{\tilde{z}^1 \, \bar{z}^1}{\bar{\tilde{z}}^1 \, z^1} \right) \tag{5.52}$$

(which justifies (3.60) with $A = 1$) one recovers

$$g = \begin{pmatrix} \bar{\tilde{z}}^1 z^1 + \tilde{z}^0 \bar{z}^0 & -z^0 \bar{\tilde{z}}^1 + \tilde{z}^0 \bar{z}^1 \\ -z^1 \bar{\tilde{z}}^0 + \tilde{z}^0 \tilde{z}^1 & \bar{\tilde{z}}^0 z^0 + \tilde{z}^1 \bar{z}^1 \end{pmatrix} \tag{5.53}$$

which is exactly what we had (3.71). It follows that

$$e^{i\xi} = e^{2i(\arg \tilde{z}^1 - \arg z^1)} = \frac{\tilde{z}^1 \, \bar{z}^1}{\bar{\tilde{z}}^1 \, z^1} \tag{5.54}$$

Now, the quantization of (5.54) has to have the correct action on $|j, m, n\rangle$ (respecting the range of $m, n = -j \cdots, j$), which in particular has to vanish on $|0, 0, 0\rangle$. Moreover, since we classically have $\{A, \xi\} = 1 \Rightarrow \{A, e^{i\xi}\} = i e^{i\xi}$, at the quantum level we should recover $\left[\hat{A}, \widehat{e^{i\xi}}\right] = -\widehat{e^{i\xi}}$. It turns out that, calling $z \to a$, $\bar{z} \to a^\dagger$, a definition consistent with all the requirements is

$$\widehat{e^{i\xi}} = (a^{1\dagger})^2 (n^1)^{-1} (\tilde{n}^1)^{-1} (\tilde{a}^1)^2 \tag{5.55}$$

The first thing to notice is that classically it reproduces exactly (5.54). Its complete action on the basis states is

$$\widehat{e^{i\xi}}|j,m,n\rangle = \frac{\sqrt{(j-n)(j-n-1)(j-m)(j-m-1)}}{(j-m)(j-n)}|j-1,m+1,n+1\rangle \tag{5.56}$$

which is written in this way because one has to remember that the operator ordering in (5.55) is important. In fact if m or n are equal to j, $j-1$, the action vanishes thanks to the square roots at the numerator given by the action of the operators $a^{1\dagger}$, $\tilde{a}^1$. The latter act before the operators $\tilde{n}^1$ and n^1 which will end up acting on zero before the denominator can give troubles. Moreover

$$\widehat{e^{i\xi}}|0,0,0\rangle = 0 \tag{5.57}$$

and one can also easily check that

$$\left[\hat{A},\widehat{e^{i\xi}}\right] = \left[\frac{\tilde{n}^0+\tilde{n}^1}{2},\widehat{e^{i\xi}}\right] = -(a^{1\dagger})^2(n^1)^{-1}(\tilde{n}^1)^{-1}(\tilde{a}^1)^2 = -\widehat{e^{i\xi}} \tag{5.58}$$

Therefore, given the action on the basis (5.56) one can write the complete expression of the expectation value

$$\begin{aligned}\langle\Omega|\widehat{e^{i\xi}}|\Omega\rangle = \sum_j \sqrt{\mathcal{C}(j)}\, d_{j-1}^{\frac{3}{2}} d_j^{\frac{3}{2}} e^{-\frac{t}{2}-tj^2+tj}\cos\left[(j-1)\omega\right]\cos(j\omega)\\ \times D^{(j-1)}_{j-1,m+1}(\tilde{n}^{-1})D^{(j-1)}_{m+1,-(j-1)}(n)D^{(j)}_{jn}(\tilde{n})D^{(j)}_{m,-j}(n^{-1})\end{aligned} \tag{5.59}$$

where we denoted $\sqrt{\mathcal{C}(j)} = \sqrt{\frac{(j-n-1)(j-m-1)}{(j-m)(j-n)}}$. Now we call

$$W(j) = D^{(j-1)}_{j-1,m+1}(\tilde{n}^{-1})D^{(j-1)}_{m+1,-(j-1)}(n)D^{(j)}_{jn}(\tilde{n})D^{(j)}_{m,-j}(n^{-1}) \tag{5.60}$$

for the sake of notation, and we notice that the following consideration will not depend on these numeric coefficients. Now we complete the square at the exponent, we write the cosines in terms of exponentials and we rearrange everything that does not depend on j in front of the sum, to get

$$\begin{aligned}\langle\Omega|\widehat{e^{i\xi}}|\Omega\rangle = & e^{-\frac{t}{4}}e^{-i\bar{\omega}}\sum_j \sqrt{\mathcal{C}(j)}\,(2j-1)^{\frac{3}{2}}(2j+1)^{\frac{3}{2}}e^{-t\left(j-\frac{1}{2}\right)^2}\left(e^{2ij\xi}+e^{2jA}\right)W(j)\\ &+ e^{-\frac{t}{4}}e^{i\bar{\omega}}\sum_j \sqrt{\mathcal{C}(j)}\,(2j-1)^{\frac{3}{2}}(2j+1)^{\frac{3}{2}}e^{-t\left(j-\frac{1}{2}\right)^2}\left(e^{-2ij\xi}+e^{-2jA}\right)W(j)\end{aligned} \tag{5.61}$$

Even if it is not clear at first sight, this result does look promising. First of all we have to consider the normalized expectation value

$$\langle\widehat{e^{i\xi}}\rangle_\Omega := \frac{\langle\Omega|\widehat{e^{i\xi}}|\Omega\rangle}{\langle\Omega|\Omega\rangle} \tag{5.62}$$

Now one might recognize that a very similar problem was analysed in chapter 4, while dealing with the particle on the circle or even better with the new harmonic oscillator coherent states. In the latter, we recovered the result of the particle on the circle in the large $\mathrm{Im}(\omega)$ limit which is here the large area or spin limit $A \to \infty$. Instead of treating the sums as integrals and then go to the large A regime, for now we limit ourselves to the following considerations. Since $i\bar{\omega} = A + i\xi$, in the large area limit we immediately see that the first piece of (5.61) will not contribute. Moreover, when considering the normalized expectation value, we will have something very similar to what happened in the above examples, and in the same limit the only contribution that will survive will be the one with $e^{i\xi - t/4}$, namely one might expect

$$\langle \widehat{e^{i\xi}} \rangle_\Omega \sim e^{i\xi - \frac{t}{4}} \qquad \text{for} \qquad A \to \infty \tag{5.63}$$

This would be nothing but the same result obtained in (4.103) and (4.37).

This is still work in progress, the difficulty arising from both the analytical and numerical analysis needed for the above non trivial coefficients. Nevertheless, given the similarities and the analogies, if that will be the case it would mean that our coherent states can both be peaked also in the holonomy (or its twist angle component) as well as on the fluxes, with clear directions. These complete results, especially concerning the new and more general family of CS introduced below in section 5.6, will appear in [12] and we will discuss all the methods and investigations used.

5.4 Relation with Heat Kernel CS

Thiemann's coherent states can be associated with the complex structure induced by the parametrization

$$\begin{aligned} (g, X) \quad \to \quad & H \in SL(2,\mathbb{C}) \\ & H \equiv e^{iL} g = g e^{iR} \end{aligned} \tag{5.64}$$

The corresponding coherent states are given by (the analytic continuation of) the heat kernel over SU(2)

$$\psi_H^t(u) = \sum_{j \in \mathbb{N}/2} d_j e^{-\frac{t}{2} j(j+1)} \chi^{(j)}(H u^{-1}) \tag{5.65}$$

where $\chi^{(j)}$ is the character in the irreducible representation j. Notice that if the Twisted Geometries parametrization is chosen to be[4]

$$g = \tilde{n}\, \epsilon\, e^{\xi\tau_3} n^{-1} \qquad R = A n \tau_3 n^{-1} \qquad L = -g R g^{-1} = A \tilde{n} \tau_3 \tilde{n}^{-1} \tag{5.66}$$

where $\epsilon = i\sigma_2$, then the element H can be written as[5]

$$H = \tilde{n}\, \epsilon\, e^{\omega\tau_3} n^{-1} \tag{5.67}$$

[4]It's easy to see that since $N = n\tau_3 n^{-1}$ and $\tilde{N} = \tilde{n}\tau_3\tilde{n}^{-1}$, then $L = -A\tilde{n}e^{-\xi\tau_3}\epsilon\tau_3\,\epsilon^{-1}\, e^{\xi\tau_3}\tilde{n}^{-1} = A\tilde{N}$ because $\epsilon\tau_3\epsilon^{-1} = -\tau_3$.

[5]$H = e^{iL}g = e^{iA\tilde{n}\tau_3\tilde{n}^{-1}}\,\tilde{n}\,\epsilon\, e^{\xi\tau_3} n^{-1} = \tilde{n}\,\epsilon\, e^{ij\tau_3} e^{\xi\tau_3} n^{-1} = \tilde{n}\,\epsilon\, e^{\omega\tau_3} n^{-1} = g e^{iR}$.

where $\tau_3 = -i\sigma_3/2$ and $\omega = \xi + iA$. This choice is made to make the comparison more transparent and as it is explained in the above, it only represent a different choice of $SU(2)$ coherent states composing (5.13), or a different orientations of the two spheres. For instance, we choose the combination in (5.68) to match Thiemann's state. For more comment in this regard see the end of this section. Thanks to (5.67) we see that the two parametrizations are related. The first difference with the new coherent states

$$\psi^t_\Omega(u) = \sum_{j\in\mathbb{N}/2} d_j^{3/2}\, e^{-\frac{t}{2}j^2} \cos(j\omega)\, [j,n|\, u^{-1}|j,\tilde{n}\rangle \tag{5.68}$$

is the prefactor, namely the spin weights. But remembering that

$$[j,n|\, u^{-1}|j,\tilde{n}\rangle = \langle j,j|n^{-1}u^{-1}\tilde{n}|j,-j\rangle = D^{(j)}_{j,-j}(n^{-1}u^{-1}\tilde{n}) = D^{(j)}_{j,j}(n^{-1}u^{-1}\tilde{n}\,\epsilon) \tag{5.69}$$

we see that the true key difference lies in the dependence on the labels. In the HK coherent states we have

$$\chi^{(j)}(Hu^{-1}) = D^{(j)}_{ab}(H)D^{(j)}_{ba}(u^{-1}) = D^{(j)}_{ab}(\tilde{n}\,\epsilon\, e^{\omega\tau_3}n^{-1})D^{(j)}_{ba}(u^{-1}) \tag{5.70}$$

In the new states we have instead

$$\begin{aligned}
\cos(j\omega)\, [j,n|\, u^{-1}|j,\tilde{n}\rangle &= \cos(j\omega) \sum_{a,b} \langle j,a|j,\tilde{n}\rangle\, [j,n|\, j,b\rangle D^{(j)}_{ba}(u^{-1}) \\
&= \frac{1}{2}\sum_{a,b} \langle j,a|\tilde{n}|j,-j\rangle\langle j,-j|e^{\omega\tau_3}|j,-j\rangle\langle j,-j|\epsilon^{-1}n^{-1}|j,b\rangle D^{(j)}_{ba}(u^{-1}) \\
&\quad + \frac{1}{2}\sum_{a,b} \langle j,a|\tilde{n}\,\epsilon\,|j,j\rangle\langle j,j|e^{\omega\tau_3}|j,j\rangle\langle j,j|n^{-1}|j,b\rangle D^{(j)}_{ba}(u^{-1}) \\
&= \frac{1}{2}\sum_{a,b} D^{(j)}_{a,-j}(\tilde{n})D^{(j)}_{-j,-j}(e^{\omega\tau_3})D^{(j)}_{-j,b}(\epsilon^{-1}n^{-1})D^{(j)}_{ba}(u^{-1}) \\
&\quad + \frac{1}{2}\sum_{a,b} D^{(j)}_{aj}(\tilde{n}\,\epsilon)D^{(j)}_{jj}(e^{\omega\tau_3})D^{(j)}_{jb}(n^{-1})D^{(j)}_{ba}(u^{-1})
\end{aligned} \tag{5.71}$$

where we used

$$2\cos(j\omega) = e^{i\omega j} + e^{-i\omega j} = \langle j,-j|e^{\omega\tau_3}|j,-j\rangle + \langle j,j|e^{\omega\tau_3}|j,j\rangle = D^{(j)}_{-j,-j}(e^{\omega\tau_3}) + D^{(j)}_{jj}(e^{\omega\tau_3}) \tag{5.72}$$

since $\epsilon|j,-j\rangle = |j,j\rangle$. A comparison of (5.70) and (5.71) shows that the difference boils down to a different matrix structure

$$D^{(j)}_{ab}(\tilde{n}\,\epsilon\, e^{\omega\tau_3}n^{-1}) \neq \frac{1}{2}\sum_{a,b} D^{(j)}_{a,-j}(\tilde{n})D^{(j)}_{-j,-j}(e^{\omega\tau_3})D^{(j)}_{-j,b}(\epsilon^{-1}n^{-1}) + \frac{1}{2}\sum_{a,b} D^{(j)}_{aj}(\tilde{n}\,\epsilon)D^{(j)}_{jj}(e^{\omega\tau_3})D^{(j)}_{jb}(n^{-1}) \tag{5.73}$$

This comparison highlights the role of the different complex structures used: in the new states, the complex structures of S^2 and T^*S^1 stay separated, while they are mixed up in the $SL(2,\mathbb{C})$

complex structure, namely in Thiemann states.

We will show however that the two sets of states *coincide* in the limit of large area $A = \mathrm{Im}(\omega)$. The first thing to notice is that in the new states, namely the RHS of (5.73), it will survive only the second part since $e^{i\omega j} = D^{(j)}_{-j,-j}(e^{\omega\tau_3}) \xrightarrow[I\to\infty]{} 0$. Next, concerning the Thiemann states, we first rewrite

$$D^{(j)}_{ab}(\tilde{n}\,\epsilon\, e^{\omega\tau_3}n^{-1}) = \sum_{c,d} D^{(j)}_{ac}(\tilde{n}\epsilon)D^{(j)}_{cd}(e^{\omega\tau_3})D^{(j)}_{db}(n^{-1}) \tag{5.74}$$

Then notice that

$$D^{(j)}_{cd}(e^{\omega\tau_3}) = \langle j,c|e^{\omega\tau_3}|j,d\rangle = \langle j,d|j,c\rangle e^{-i\omega d} = \delta_{cd}e^{-i\omega d} \tag{5.75}$$

will effectively project on the maximal spin j since in the limit $\mathrm{Im}(\omega)\to\infty$ I have

$$\lim_{I\to\infty} D^{(j)}_{ab}(\tilde{n}\,\epsilon\, e^{\omega\tau_3}n^{-1}) = \sum_{c} D^{(j)}_{ac}(\tilde{n}\epsilon)\left(\lim_{I\to\infty} e^{-i\omega c}\right)D^{(j)}_{cb}(n^{-1}) \tag{5.76}$$

where

$$e^{-i\omega c} = e^{-i\omega j}e^{-i\omega(c-j)} \sim e^{-i\omega j}e^{(c-j)\,\mathrm{Im}(\omega)} \tag{5.77}$$

now since c runs over $\{-j,j\}$ then $(c-j)\leq 0$ and therefore in the sum over c, when $\mathrm{Im}(\omega)\to\infty$, it will only survive the first contribution. Thus, being $e^{-i\omega j} = D^{(j)}_{jj}(e^{\omega\tau_3})$ we have

$$\lim_{I\to\infty} D^{(j)}_{ab}(\tilde{n}\,\epsilon\, e^{\omega\tau_3}n^{-1}) = D^{(j)}_{aj}(\tilde{n}\epsilon)D^{(j)}_{jj}(e^{\omega\tau_3})D^{(j)}_{jb}(n^{-1}) \tag{5.78}$$

which is exactly what remains in the same limit on the RHS of equation (5.73), again up to factors.

A remark concerning the conventions is now in order. First of all, the choice of the parametrization (5.66) and (5.67) is simply made in order to see clearly the comparison between the two sets of CS. Any other choice would have been possible, providing that the suitable adjustment had been taken in (5.68). For example, the element H could be parametrized without the use of the epsilon tensor or with two of them, but in that case one would need two highest or two lowest weight $SU(2)$ coherent states, rather than a mixed combination, in (5.68). As a matter of fact all these choices define coherent states and as we will see, there is indeed a more general definition that will include more than one of them, defining a generalized family of CS. These states will be easily compared to Thiemann's states and the comparison will not have any sort of dependence upon the above choices.

Concluding, as we anticipated, the HK coherent states show a nice and separated structure only in the large area limit, whereas our states are already factorized in vertices and edges contributions.

5.5 Gauge invariance

An advantage of the new coherent states is that the geometrical interpretation associated to the labels is carried through quite naturally at the gauge invariant level. In fact as discussed in chapter 2, an application of the Guillermin-Sternberg theorem guarantees us that imposing the Gauss constraint amounts to impose the closure conditions on the labels and then divide out by the rotation group. This symplectic reduction provides exactly $2(n-3)$ classical parameters in place of the $n-3$ intertwiners. As we know, this leaves us in the space of shapes of polyhedra, hence the interpretation of the coherent states is naturally in terms of flat polyhedra associated to each vertex.

Implementing the Gauss constraint will result in a well factorized state, in terms of the coherent intertwiners discussed in chapter 2. The final result is nothing but a superposition of spin network states with suitable coefficients

$$|\Omega\rangle_0 = \sum_{j_e} \mathbb{A}_{j_e} \prod_v c_v(n)\, |\Gamma, j_e, \imath_v\rangle \tag{5.79}$$

where we have denoted

$$\mathbb{A}_{j_e} = d_{j_e}^{3/2} e^{-\frac{t}{2} j_e^2} \cos(j_e \omega_e) \tag{5.80}$$

and the coefficient $c_v(n)$ are the ones appearing in the coherent intertwiners.
Let us see how to obtain that result, and prove it in the holonomy representation. Starting from equation (5.16), we can express the edge coherent state as

$$\Psi_\Omega(g) = \sum_j d_j^{\frac{3}{2}} e^{-\frac{t}{2}j^2} \cos(j\omega) \langle j, n | g\, | j, \tilde{n}] = \sum_{j_e} \mathbb{A}_{j_e} D^{(j)}_{-j,j}(n^{-1} g \tilde{n}) \tag{5.81}$$

Then the coherent states on a fixed graph Γ are simply given by tensor products of the single-edge CS over the links of the graph

$$\Psi_{\Gamma,\Omega}(g_1, \ldots, g_L) = \bigotimes_{e=1}^{L} \sum_{j_e} \mathbb{A}_{j_e} D^{(j_e)}_{-j_e, j_e}(n_e^{-1} g_e \tilde{n}_e) \tag{5.82}$$

where $g_1, \ldots, g_L$ stands for $g_{e_1}, \ldots, g_{e_L}$, so the index e runs over the links from 1 to L. Now by definition, the gauge invariant states will be given by group averaging

$$\Psi^0_{\Gamma,\Omega}(g_{e_1}, \ldots, g_{e_L}) = \int \prod_v dh_v \bigotimes_e \sum_{j_e} \mathbb{A}_{j_e} D^{(j_e)}_{-j_e, j_e}(n_e^{-1} h_{s(e)} g_e h^{-1}_{t(e)} \tilde{n}_e) \tag{5.83}$$

The edge e is an index in the equation, and therefore also each subscript of the Wigner matrix, has implicitly another index, which will be taken into account in the tensor product over all the edges.

Let us focus on the the last matrix, and decompose it as

$$D^{(j_e)}_{-j_e, j_e}(n_e^{-1} h_{s(e)} g_e h^{-1}_{t(e)} \tilde{n}_e) = D_{-j_e a_e}(n_e^{-1}) D_{a_e b_e}(h_{s(e)}) D_{b_e c_e}(g_e) D_{c_e d_e}(h^{-1}_{t(e)}) D_{d_e j_e}(\tilde{n}_e) \tag{5.84}$$

Now recall that the last and first pieces of that multiplication of matrices are nothing but

$$D_{-j_e a_e}(n_e^{-1}) = \langle j_e, n_e | j_e, a_e \rangle \qquad D_{d_e j_e}(\tilde{n}_e) = \langle j_e, d_e | j_e, \tilde{n}_e] \tag{5.85}$$

and of course there will be many of such pieces "tensor producted" together, due to the product over e. We also need to recall the property

$$\int_{SU(2)} dh D^{(j_1)m_1}{}_{n_1}(h) \cdots D^{(j_N)m_N}{}_{n_N}(h) = \sum_{\imath} \imath^{m_1 \cdots m_N} \overline{\imath_{n_1 \cdots n_n}} \tag{5.86}$$

which of course holds also if the arguments are h^{-1} thanks to the invariance of the Haar measure. In view of the decomposition (5.84), this means that in each v-th integral of equation (5.83), we can recognize the integrand as a product of Wigner matrices evaluated $h_{s(e)}$ or $h_{t(e)}^{-1}$. So thanks to (5.86) we can introduce the invariant tensors i on their behalves, the other matrices being untouched by the integration over h_v. Remember that these tensors are exactly the ones that define the intertwiner basis

$$\imath^{(k)}_{a_1,\ldots,a_N} |j_1, a_1\rangle \cdots |j_N, a_N\rangle = |j_1, \cdots, j_N\,;\, k\rangle \equiv |k\rangle \tag{5.87}$$

and in fact the projector onto the gauge invariant subspace is nothing but

$$P = \sum_k |k\rangle\langle k| \tag{5.88}$$

which, if applied to the tensor product state made of coherent states[6] of $SU(2)$, gives

$$P|j_1, \vec{n}_1\rangle \cdots |j_N, \vec{n}_N\rangle = \sum_k c_k(j,n)|k\rangle \tag{5.89}$$

where

$$c_k(j,n) = \langle j_1, \ldots, j_N\,;\, k | j_1 \vec{n}_1, \cdots, j_N \vec{n}_N \rangle \tag{5.90}$$

We will recognize the definition of the coherent intertwiners discussed in chaper 2

$$||\underline{j}, \underline{n}\rangle_{LS} = \int dh \bigotimes_e h|j_e, \vec{n}_e\rangle = \sum_k c_k(j,n)|k\rangle \tag{5.91}$$

which naturally emerge when group averaging the new states. Let us show it explicitly, going back to the original task. We see that when we consider the tensor product over the edges of the Wigner matrix with argument $h_{s(e)}$ and $h_{t(e)}^{-1}$, we obtain a sum over the intertwiners. So leaving the explicit tensor product for the non integrated part, we have

$$\left(\bigotimes_e \langle j_e, n_e | j_e, a_e \rangle D_{b_e, c_e}(g_e) \langle j_e, d_e | j_e\ \tilde{n}_e] \right) \sum i^{(v)}_{a_1,\ldots,a_N} \overline{i^{b_1,\ldots,b_N}_{(v)}} \sum i^{(v)}_{c_1,\ldots,c_N} \overline{i^{d_1,\ldots,d_N}_{(v)}} \tag{5.92}$$

[6] $|j, \vec{n}\rangle$ can be highest weight $|j, n]$ or lowest weight $|j, n\rangle$

and using what said above we recognize the coefficients (5.90) appearing in the last expression, namely

$$\overline{i_{(v)}^{d_1,\ldots,d_N}}(\otimes_e\langle j_e, d_e|j_e\;\tilde{n}_e]) = \langle j_1,\ldots,j_N\,;\,v|j_1\tilde{n}_1,\cdots,j_N\;\tilde{n}_N] = c_v \tag{5.93}$$

and

$$(\otimes_e\langle j_e, n_e|j_e, a_e\rangle)i^{(v)}_{a_1,\ldots,a_N} = \langle j_1 n_1,\ldots,j_N n_N|j_1,\ldots,j_N\,;\,v\rangle = c_v \tag{5.94}$$

Something is left untouched in this operation. But this is exactly the contraction dictated by the connectivity of the graph (here between indices b_e and c_e) of the intertwiners tensors with a Wigner matrix of the original variables $D_{b_e,c_e}(g_e)$. When the product over the edges is taken into account, this is nothing but a spin network state as it was expected, being the most natural gauge invariant basis.

In conclusion, since we have v such integrals, the gauge invariant coherent states are

$$\Psi^0_{\Gamma,\Omega}(g_{e_1},\ldots,g_{e_L}) = \sum_{j_e}\mathbb{A}_{j_e}\prod_v c_v(n)\,\psi^{SN}_\Gamma(g_{e_1},\ldots,g_{e_N}) \tag{5.95}$$

a coherent superposition of gauge invariant spin network states.

It is worth to recall that this is similar to the so called "coherent spin network" [11], the difference being in the coefficient characterizing our new coherent state, $\mathbb{A}$. However notice the the new set of states introduced here is truly coherent thanks to the properties discussed above. The coherent spin networks (CSN) just mentioned were simply defined as the gauge invariant projection of a product over links of the heat-kernel for the cotangent bundle of SU(2). The labels of these states are written in terms of the twisted geometries variables which, as we know, can be easily mapped to an element of $SL(2, C)$. As a matter of fact the CSN simply use an alternative parametrization but coincide with the heat-kernel coherent states. It is true that they can be interpreted as a cluster of semiclassical polyhedra, instead of distributional holonomies and fluxes associated to the graph. However, they reproduce a superposition of spin networks with nodes labelled by Livine-Speziale coherent intertwiners, only in the large spin limit. On the contrary, the TGCS are truly different and coincide with the others only in the large spin limit.

Thus we see from (5.95) that after imposing the Gauss law, the result is a factorization on vertex and edge contributions. This factorization is exact, for any spin, i.e. it does not require a large spin limit, contrary to what happens with the states known so far.

5.6 Generalization and new TGCS

As anticipated, on the course of this work it was realized that a more general definition was possible. This resolves some of the arbitrariness of the so far discussed coherent states. Moreover it will have more palatable features, dealing with nicer mathematical structures and it will have an elegant relation to known special functions such as the Jacobi's theta functions. In general, they will have similar (if not identical) properties to the one discussed above, therefore we will not detail the calculations here, some of which are yet to be completed. This general version of twisted geometries coherent states will appear in [12] where all the details can be found. Here we give a brief review and picture of the work in progress.

5.6.1 A new family

The first step in this realization was made by including a simple shift in the ω parameter of the original coherent states, in order to look slightly more similar to Thiemann CS. Then we noticed that since the original idea was to deal with the $\mathbb{Z}_2$ symmetry of the twisted geometry parametrization, this could be done both using the coherent states introduced in section 5.1, or more generally considering the following family of CS

$$\Psi^t_\Omega(g) := e^{-\frac{t}{8}} \sum_j d_j^{\frac{3}{2}} e^{-\frac{t}{2}j(j+1)} \Big(e^{\frac{i}{2}d_j\omega} D^{(j)}_{-j,-j}(n^{-1}g\tilde{n}) + e^{-\frac{i}{2}d_j\omega} D^{(j)}_{jj}(n^{-1}g\tilde{n}) \Big). \tag{5.96}$$

which can be written in the momentum basis as

$$|\Omega, t\rangle = e^{-\frac{t}{8}} \sum_{jmn} d_j e^{-\frac{t}{2}j(j+1)} \Big(e^{\frac{t}{2}d_j\omega} D^{(j)}_{-jm}(n^{-1}) D^{(j)}_{n,-j}(\tilde{n}) + e^{-\frac{i}{2}d_j\omega} D^{(j)}_{jm}(n^{-1}) D^{(j)}_{nj}(\tilde{n}) \Big). \tag{5.97}$$

This more generic definitions will be called from now on twisted geometries coherent states, whereas we will refer to the old ones as the simplified version. The main difference between the heat-kernel and the twisted geometries coherent states (5.96) is that the trace in (5.4) has been replaced by a projection on highest and lowest weights. The fact that this is possible while still providing an over-complete basis in $L_2[SU(2)]$ is one of the main point of this family of CS, and will be proved below.

The states coincide with the heat-kernel ones for large A, as can be seen comparing (5.8) and the same limit on (5.96). The only difference is in the prefactor, but this is only because we kept it explicitly to have the norms given precisely by the theta functions. The states as defined are in fact not normalized; their squared norm can be computed to be

$$\begin{aligned} \left\|\Psi^t_\Omega\right\|^2 &= 2e^{-\frac{t}{4}} \sum_j d_j^2 e^{-tj(j+1)} \cosh(d_j A) \\ &= (1-4\partial_t)\Big(\vartheta_2(iA, e^{-t}) + \vartheta_3(iA, e^{-t}) - 1\Big) \end{aligned} \tag{5.98}$$

where $\vartheta_i(u,q)$ are Jacobi's theta functions in the conventions of Wolfram Mathematica. The differential operator acting on the theta functions is a consequence of the d_j^2 factor. Without this factor, the norm of the states would be directly given by the theta functions. The factor is however important and cannot be removed from the definition, because it encodes the radius of the spheres. We remark that the second theta is there merely to take into account the half-integer representations. If one were interested in $L_2[SO(3)]$ instead of $L_2[SU(2)]$, the sum in (5.96) would be over the natural numbers only. Finally, let us comment on the pre-factor $\exp(-t/8)$ in (5.96). This would naturally be present also in the heat-kernel coherent states (5.4), but it can be always removed modifying the normalization of the measure in the resolution of the identity. Keeping it permits the match with the theta functions.

The reason for the two terms is due to the $\mathbb{Z}_2$ symmetry of the twisted geometry parametrization of $T^*SU(2)$, and the existence of two families of $SU(2)$ coherent states, those associated with the lowest or highest weight.

For most practical purposes, it is possible to consider the simplified family given by

$$\Psi^t_{\Omega_s}(g) := 2\sum_j d_j^{\frac{3}{2}} e^{-\frac{t}{2}j(j+1)} \cos\left(\tfrac{1}{2}d_j\omega\right) D^{(j)}_{-j-j}(n^{-1}g\tilde{n}), \tag{5.99}$$

or in the momentum basis

$$|\Omega_s, t\rangle = \sum_{jmn} d_j e^{-\frac{t}{2}j(j+1)} \cos\left(\frac{d_j}{2}\omega\right) D^{(j)}_{-jm}(n^{-1}) D^{(j)}_{n,-j}(\tilde{n}). \tag{5.100}$$

This family is in fact a simple modification to the coherent states discussed before this section (5.16). It also has the property of being a holomorphic function in $(\omega, \bar{\zeta}, \tilde{\zeta})$[7], even though unfortunately this does not lead to the possibility of constructing a new holomorphic represetation for the holonomy-flux algebra. It also has slightly less desirable features, for instance its norm is

$$\left\|\mathring{\Psi}^t_\Omega\right\|^2 = 4\sum_j d_j^2 e^{-tj(j+1)}(\cosh(d_jA) + \cos(d_j\xi)) \tag{5.101}$$

This expression has a subleading dependence on the real part of ω, and lacks the elegant connection to the theta functions. The latter is the reason why we removed the $e^{-t/8}$ prefactor in their definition to put it in the measure.[8] All the properties of these simplified family can be deducted by the analysis carried in the first sections of this chapter.

The convergence of the sum in (5.98) depends mainly on t, improving for larger t, and secondarily on A, improving for smaller A. For $t \leq 1$, numerical investigations show that the sum converges within an error smaller than 10^5 for a cut off that scales roughly with $t^{-1/2}$. This is an upper bound that permits to include large ranges of A. For small t, it is also possible to approximate the sum with an integral. We are still investigating what are the best regimes for which the approximations hold, and we will give a complete report in [12]. The same arguments will hold for the various expectation values we illustrate below.

[7] A version holomorphic in $(\omega, \zeta, \tilde{\zeta})$ is obtained taking mixed highest/lowest weights D_{j-j}.

[8] As it is done with the heat-kernel coherent states.

5.6.2 Resolution of identity

The necessary, and weakest condition that one can require of coherent states, is that they provide a resolution of the identity as an integral over the classical phase space [2]. With the new family of coherent states, the resolution of the identity is given by the following measure

$$\mathbb{1} = \int_{S^2} d^2n \int_{S^2} d^2\tilde{n} \int_{-2\pi}^{2\pi} \frac{d\xi}{4\pi} \int_0^\infty \frac{dA}{\sqrt{\pi t}} e^{-\frac{I^2}{t}} |\omega, \zeta, \tilde{\zeta}\rangle\langle\omega, \zeta, \tilde{\zeta}| \tag{5.102}$$

where the notation $|\Omega, t\rangle = |\omega, \zeta, \tilde{\zeta}\rangle$ is simply a way to make it different from $|\Omega\rangle = |\omega, n, \tilde{n}\rangle$ which was used with the previous states, but it labels the same things. To prove the resolution of the unity, it is sufficient to evaluate

$$\begin{aligned}\langle j, m, n|\, k, p, q\rangle = e^{-\frac{t}{4}} d_j d_k e^{-\frac{t}{2}C_j - \frac{t}{2}C_k} \int_{S^2} d^2n \int_{S^2} d^2\tilde{n} \int_{-2\pi}^{2\pi} \frac{d\xi}{4\pi} \int_0^\infty \frac{dA}{\sqrt{\pi t}} e^{-\frac{A^2}{t}} \\ \left(e^{-\frac{1}{2}(A+i\xi)d_j} \overline{D^{(j)}_{m,-j}(\tilde{n}) D^{(j)}_{-j,n}(n^{-1})} + e^{\frac{1}{2}(A+i\xi)d_j} \overline{D^{(j)}_{m,-j}(\tilde{n}) D^{(j)}_{-j,n}(n^{-1})} \right) \\ \left(e^{-\frac{1}{2}(A-i\xi)d_k} D^{(k)}_{p,-k}(\tilde{n}) D^{(j)}_{-k,q}(n^{-1}) + e^{\frac{1}{2}(A-i\xi)d_k} D^{(k)}_{p,-k}(\tilde{n}) D^{(k)}_{-k,q}(n^{-1}) \right)\end{aligned} \tag{5.103}$$

and verify that it results in a product of Kronecker deltas. To proceed, it is simpler to first integrate over ξ. There are four possible terms, and the relevant integrals give

$$\int_{-2\pi}^{2\pi} \frac{d\xi}{4\pi} e^{-i\xi(j+\frac{1}{2})} e^{\pm i\xi(k+\frac{1}{2})} = \begin{cases} \delta_{2j,2k} \\ \delta_{2j+2,-2k} \end{cases} \tag{5.104}$$

With the chosen parametrization of $SU(2)$, the period of ξ is 4π and this includes the half-integer representations. We will from now on use a shorter notation d_{jk} including the half-integers, as customary in $SU(2)$ theory. Since the spins are only positive, this integral kills the mixed terms between the second and third line of (5.103). This leaves sphere integrals, that give deltas on the magnetic indices thanks to the resolution of the identity satisfied by the $SU(2)$ coherent states,

$$\int \overline{D^{(j)}_{m,-j}(n)} D^{(k)}_{p,-k}(n) = \frac{1}{d_j} \delta_{jk} \delta_{mp} \tag{5.105}$$

and similarly δ_{nq} comes from the second sphere integral. The final integration gives

$$\langle j, m, n|\, k, p, q\rangle = \delta_{jk}\delta_{mp}\delta_{nq} e^{-\frac{t}{4}} e^{-tC_j} \int_0^\infty \frac{dA}{\sqrt{\pi t}} e^{-\frac{A^2}{t}} \left(e^{\frac{1}{2}Ad_j} + e^{-\frac{1}{2}Ad_j} \right) = \delta_{jk}\delta_{mp}\delta_{nq}. \tag{5.106}$$

This proves that (5.96) is an overcomplete basis in $L_2[\mathrm{SU}(2), d\mu_{\mathrm{H}}]$. The resolution of the identity for the alternative, simpler definition (5.99) can be proved along similar lines (see section 5.2), and with the same measure. Finally, a similar procedure shows that the Gaussian ansatz works with the less elegant measure (5.11).

5.6.3 Expectation values and peakedness

The peakedness properties of these states will follow almost directly from the ones obtained with the simplified version. However this piece of work is yet to be completed, especially regarding the limit in which some approximation are used.

The holonomy peakedness, and therefore the one of the operator associated to the exponential of the twist angle ξ, will be done in close analogy with the simplified version sketched in section 5.3.

Regarding the fluxes, of course for example a simple calculation shows immediately that

$$\langle \Omega, t | \hat{\vec{L}} | \Omega, t \rangle = f(A,t) \vec{n}_{\mathrm{L}}, \qquad f(A,t) := 2e^{-\frac{t}{4}} \sum_j d_j^2 e^{-tj(j+1)} \sinh(d_j A) j. \tag{5.107}$$

and that dividing by the norm of the coherent states, we arrive at

$$\langle \hat{\vec{L}} \rangle := \frac{\langle \Omega, t | \hat{\vec{L}} | \Omega, t \rangle}{\left\| \Psi_\Omega^t \right\|^2} = \frac{f(A,t)}{\left\| \Psi_\Omega^t \right\|^2} \vec{n}_{\mathrm{L}} \sim \frac{A}{t} \vec{n}_{\mathrm{L}} \qquad \text{for} \qquad t \ll 1. \tag{5.108}$$

This simple expression shows the advantage of using the SU(2) coherent states to peak on the direction. Similarly for the right-invariant fluxes,

$$\langle \Omega, t | \hat{\vec{R}} | \Omega, t \rangle = -f(A,t) \vec{n}_{\mathrm{R}}, \qquad \langle \hat{\vec{R}} \rangle \sim -\frac{A}{t} \vec{n}_{\mathrm{R}} \qquad \text{for} \qquad t \ll 1. \tag{5.109}$$

These expression look pretty much the same as we obtained before, but we are still probing via numerical investigation what are the right expressions to use while doing the calculations. Moreover we will need

$$\langle \Omega, t | \hat{L}_i^2 | \Omega, t \rangle = 2e^{-\frac{t}{4}} \sum_j d_j^2 e^{-tj(j+1)} \sinh(d_j A) \left(\frac{j}{2} + j \left(j - \frac{1}{2} \right) n_i^2 \right) \tag{5.110}$$

and we will exhibit all the details to arrive at the vanishing of the relative uncertainty.

In the forthcoming article [12], we will show all the details missing here, and also investigate other properties of these new states concerning the gauge invariance, the relation with the other existing coherent states and other interesting features as well as future applications.

Conclusion and outlook

To conclude, we briefly comment on the results obtained in this work, elucidating the motivations and explaining what is left to do together with possible new interesting applications that can follow the present thesis.

The task of building coherent states for Loop Quantum Gravity dates back to quite a long time ago. Even though the spin network states play a key role in the theory, they lack a low energy physical interpretation and therefore are not suitable for a semiclassical analysis. If one wishes to bridge the Planck scale quantum geometry they describe to a smooth and classical geometry, one needs coherent states which are superposition of spin networks peaked on classical geometries labelling the phase space of general relativity. After an early attempt with the weave states, a proper answer to the problem was extensively developed by Thiemann and collaborators. Their heat-kernel coherent states (HKCS) are superpositions of spin networks with the same graph, and are properly labelled by a point in the discrete phase space of loop gravity associated to the graph. These states fulfil a number of important properties, however they have some limitations which provide a first motivation for our new family of coherent states. While the HKCS are well peaked on the norm of the fluxes, they do not single out so nicely their directions. This instead is very clearly obtained with the twisted geometries coherent states (TGCS). Other motivations date back to the spin foam graviton calculations as well as other progresses in the spin foam formalism, since they were a first hint suggesting a classical phase space described by quantities referring to discrete geometries. Another incentive to search for alternatives was the work of [11], where it was shown that Thiemann's CS reproduce a superpositions of spin networks with nodes labelled by Livine-Speziale coherent intertwiners, but only some of them. The new family has this property for all of them.

A useful idea for the new family of coherent states came from the structure underlying the twisted geometries parametrization. To clarify this point, recall that in the spirit of geometric quantization, coherent states are associated to complex polarizations of the phase space. Different polarizations lead to different sets of coherent states with their specific properties. The parametrization in terms of twisted geometries comes with a natural complex structure which differs from the one behind Thiemann's states. It is using this new complex structure that we were in fact able to introduce the new set.

The main achievements of this thesis are the following. The new family of states is coherent in the sense that they provide a resolution of the identity in $T^*SU(2)$, are peaked on classical values and the relative uncertainties of the concerned operators can be shown to vanish in the large area or spin limit, as it happens with the simple $SU(2)$ case. Even though they do not sharply minimize some uncertainties relation nor are eigenvector for some destruction operator, an important feature is that they automatically and naturally incorporate the coherent intertwiners at the gauge invariant level. This does not happen in a certain limit as for the coherent spin network, but it is always true thanks to the locality at the nodes of the twisted geometries. In this sense the reduction to the gauge invariant level is much simpler, even compared to the heat kernel coherent states. In fact the TGCS are truly different from the previous ones and they reconcile only in the large spin limit. This is why, after imposing the Gauss law, the result will be a factorization on vertex and edge contributions. This factorization is exact, for any spin, i.e. it does not require a large spin limit, contrary to what happens with the states known so far.

We conclude recalling that a generalization of the present thesis is still work in progress and will appear in [12]. This will contain all the details missing in the last section of this thesis where we only present the new family, and it will generalize the results obtained here. Finally, several future applications are possible regarding the twisted geometries coherent states. Interest was in fact recently put into new models and phenomenological applications as an alternative to the complicated dynamics proposed by LQG. Since the Hamiltonian constraint is very difficult to treat, mini-superspace models regarding cosmology and black holes are have recently been considered (for example [14, 26, 1], but also others). The basic logic behind their idea is that one starts with the Hamiltonian operator and defines a mini superspace depending on the context. Then one uses coherent states to compute expectation vale of the constraint in order to obtain an effective Hamiltonian, which will be considered generator of an effective dynamics. So far, everything is done using the HKCS, and we believe that a new family of coherent states is a way to test the extent to which we can trust the predictions we have until now.

Appendix A

SU(2): recoupling and intertwiners

The quantum degrees of freedom of the discrete, quantized spatial geometries of loop quantum gravity are encoded in intertwiners, or invariant tensors of $SU(2)$. An indispensable technical tool for carrying out such calculations quickly and efficiently is provided by the graphical formalism of $SU(2)$ recoupling theory. However since this is not needed for the present work, we will not give here a description of it. This formalism consists of a diagrammatic notation for the basic elements of $SU(2)$ recoupling theory, together with a set of simple rules according to which diagrams appearing in a graphical calculation can be manipulated. We refer the reader to the main references for the graphical techniques, as we here only introduce the basic concept on basic representation and recoupling theory of $SU(2)$, emphasizing the crucial role of the intertwiners.

A.1 Clebsch-Gordan coefficients

Consider the tensor product space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$. The familiar basis of this space is provided by the tensor product states $|j_1, m_1\rangle|j_2, m_2\rangle$, which are eigenstates of the commuting operators

$$(J^{(1)})^2 \qquad (J^{(2)})^2 \qquad J_z^{(1)} \qquad J_z^{(2)} \tag{A.1}$$

As it is known from the theory of angular momentum, another set of commuting operators can be chosen to be

$$(J^{(1)})^2 \qquad (J^{(2)})^2 \qquad (J^{(1)} + J^{(2)})^2 \qquad J_z^{(1)} + J_z^{(2)} \tag{A.2}$$

Denoting their eigenstates $|j_1 j_2; jm\rangle$, they must be related to the first set since they both span the space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$, by a unitary transformation

$$|j_1, m_1\rangle|j_2, m_2\rangle = \sum_{j,m} C^{(j_1 j_2\, j)}{}_{m_1 m_2}{}^{m} |j_1 j_2; jm\rangle \tag{A.3}$$

and vice versa

$$|j_1 j_2; jm\rangle = \sum_{m_1, m_2} C^{(j_1 j_2\, j) m_1 m_2}{}_{m} |j_1, m_1\rangle|j_2, m_2\rangle \tag{A.4}$$

The coefficient are known as Clebsch-Gordan coefficients which are denoted in the literature as $C^{jm}_{j_1m_1,j_2m_2} = \langle j_1m_1, j_2m_2|jm\rangle$. The particular index notation used above is not necessary but it is designed to display the tensorial structure of the coeffieicents when seen as $SU(2)$ tensors. From the theory of addition of angular momenta, it is known that the following properties hold

- They are trivially zero unless the condition for the spins j_1, j_2 and j
$$|j_1 - j_2| \leq j \leq j_1 + j_2 \tag{A.5}$$
holds. This is called Clebsch-Gordan or triangular condition.
- If $m \neq m_1 + m_2$ then $C^{jm}_{j_1m_1,j_2m_2} = 0$
- Since $|jm\rangle$ is an orthonormal basis, one also has the orthogonality relations
$$\sum_{j,m} C^{(j_1j_2\,j)}{}_{m_1m_2}{}^{m}\, C^{(j_1j_2\,j)m_1'm_2'}{}_{m} = \delta^{m_1'}_{m_1}\delta^{m_2'}_{m_2} \tag{A.6}$$
and
$$\sum_{m_1,m_2} C^{(j_1j_2\,j)}{}_{m_1m_2}{}^{m}\, C^{(j_1j_2\,j')m_1m_2}{}_{m'} = \delta_{jj'}\delta^{m}_{m'} \tag{A.7}$$
- The Condon-Shortley phase convention also fixes them such that $C^{jm}_{j_1m_1,j_2m_2} \in \mathbb{R}$.

Under this convention the numerical value between $C^{(j_1j_2\,j)}{}_{m_1m_2}{}^{m}$ and the inverse $C^{(j_1j_2\,j)m_1m_2}{}_{m}$ is the same, and that is why they are usually not distinguished in the physics literature.

It is also important to show another property that justify the index notation chosen above. Let us consider the effect of a $SU(2)$ action on equation (A.3). On the LHS, the rotation acts as $D^{(j_1)}(g) \otimes D^{(j_2)}(g)$ where $D^{(j)}(g)$ are the Wigner matrices[1] whereas on the RHS the terms with a given j tranform according to $D^{(j)}(g)$
$$D^{(j_1)}(g)|j_1, m_1\rangle D^{(j_2)}(g)|j_2, m_2\rangle = \sum_{j,m} C^{(j_1j_2\,j)}{}_{m_1m_2}{}^{m} D^{(j)}(g)|j_1j_2; jm\rangle \tag{A.8}$$
Now taking the product from the left of this equation with $\langle j_1, n_1|\langle j_2, n_2|$ and using (A.3) one gets the Clebsch-Gordan series
$$D^{(j_1)m_1}{}_{n_1}(g) D^{(j_2)m_2}{}_{n_2}(g) = \sum_{j,m,n} C^{(j_1j_2\,j)m_1m_2}{}_{m}\, C^{(j_1j_2\,j)}{}_{n_1n_2}{}^{n}\, D^{(j)m}{}_{n}(g) \tag{A.9}$$
Finally contracting the last equation with a Clebsch-Gordan coefficient and using the orthonormality relation (A.7), one has
$$D^{(j_1)m_1}{}_{n_1}(g) D^{(j_2)m_2}{}_{n_2}(g) C^{(j_1j_2\,j)}{}_{n_1n_2}{}^{m} = C^{(j_1j_2\,j)}{}_{m_1m_2}{}^{n} D^{(j)m}{}_{n}(g) \tag{A.10}$$
which shows that the coefficient itself behaves under $SU(2)$ transformations and justifies the index structure used in the notation above.

[1] The action of $SU(2)$ over $\mathcal{H}_j$ defines a linear action of the group by exponentiation. The Wigner matrix $D^{(j_2)}(g)$ represents the action of $g \in SU(2)$ in the $|j, m\rangle$ basis. It is thus a square matrix of size $2j + 1$ with elements $D^{(j)}_{mn}(g) = \langle j, m|g|j, n\rangle$.

A.2 The 3-j symbols

The Wigner $3-j$ symbols are defined by lowering the upper index of the Clebsch-Gordan coefficients using the antisymmetric epsilon tensor., and multiplying with a suitable factor (in order to optimize the symmetries properties)

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \frac{1}{\sqrt{d_{j_3}}}(-1)^{j_1-j_2+j_3} C^{(j_1 j_2 j_3)}{}_{m_1 m_2}{}^{n}\, \epsilon^{(j_3)}_{n m_3} \\ &= \frac{1}{\sqrt{d_{j_3}}}(-1)^{j_1-j_2-m_3} C^{(j_1 j_2 j_3)}{}_{m_1 m_2}{}^{-m_3}\end{aligned} \tag{A.11}$$

These objects are usually introduced as a more symmetric version of the Clebsch-Gordan coefficient, but the relevance to LQG follows from their behaviour under $SU(2)$ transformation. As a matter of fact, starting from (A.10) one can show that the $3-j$ symbol is invariant under the action of $SU(2)$

$$D^{(j_1)m_1}{}_{n_1}(g) D^{(j_2)m_2}{}_{n_2}(g) D^{(j_3)m_3}{}_{n_3}(g) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \tag{A.12}$$

In the language of the angular momentum, the Clebsch-Gordan coefficient couples two angular momenta j_1 and j_2 to a total angular momentum j, whereas the $3-j$ symbol couples three angular momenta j_1, j_2 and j_3 to zero. The $SU(2)$ invariance of the $3j$ symbols implies that

$$|\Psi_0\rangle = \sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1, m_1\rangle |j_2, m_2\rangle |j_3, m_3\rangle \tag{A.13}$$

is rotationally invariant and hence is eigenstate of the total angular momentum with eigenvalue zero.

The properties of the Wigner 3jm symbols follow from those of the Clebsch-Gordan coefficient:

- The value of the 3j symbol can be non zero only if

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2 \qquad \text{and} \qquad m_1 + m_2 + m_3 = 0 \tag{A.14}$$

- The orthogonality relations (A.6) and (A.7) read

$$\sum_{j,m} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ m'_1 & m'_2 & m \end{pmatrix} = \frac{1}{d_j} \delta^{m'_1}_{m_1} \delta^{m'_2}_{m_2} \tag{A.15}$$

and

$$\sum_{m_1,m_2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix} = \frac{1}{d_j} \delta_{jj'} \delta^{m'}_{m} \tag{A.16}$$

- When the Condon-Shortley phase convention is chosen the 3j-symbol is real valued and thanks to the definition (A.11) it also satisfy a number of symmetry properties.

For example, interchanging any two columns in the symbol produces the factor $(-1)^{j_1+j_2+j_3}$

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{A.17}$$

This in particular implies that the symbol is invariant under cyclic permutations of ot columns

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} \tag{A.18}$$

The same phase factor arises from reversing the sign of all the magnetic numbers

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{A.19}$$

Notice also that the coefficient $C^{(j0j)}{}_{m0}^{\;\;\;n}$ is simply equal to δ^n_m. Using this result in (A.11), one also obtains that when one of the angular momenta is zero , the 3j symbol reduces to the epsilon tensor

$$\begin{pmatrix} j & j' & 0 \\ m & n & 0 \end{pmatrix} = \delta_{jj'} \frac{1}{\sqrt{d_j}} \epsilon_{mn} \tag{A.20}$$

One last remark which will be useful is that it can be shown that

$$\int_{SU(2)} dg D^{(j_1)}_{m_1 n_1}(g) D^{(j_2)}_{m_2 n_2}(g) D^{(j_3)}_{m_3 n_3}(g) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \tag{A.21}$$

for which a proof will be give shortly. Further properties of the 3j symbol, relations satisfied by it, and explicit expression for particular values can be found in any of the standard references on angular momentum theory. The most comprehensive source is the book by Varshalovich, Moskalev and Khersonskii [49].

A.3 Three-valent intertwiners

The invariant tensors of $SU(2)$, or intertwiners, play a crucial role in LQG as they represent the building blocks of the spin network state (see main text). Equation (A.12) shows that the 3j-symbol is nothing more than such an invariant tensor. To emphasise the tensorial nature one can introduce the notation

$$\imath_{m_1 m_2 m_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{A.22}$$

As a matter of fact, the 3j symbol is the only (up to normalization) three-valent invariant tensor with indices in three given representations j_1, j_2 and j_3. So the symbol alone spans the

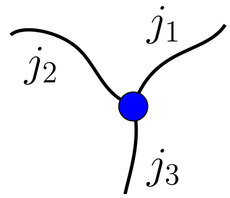


Figure A.1: A three valent intertwiner in its graphical representation

one-dimensional space of three-valent intertwiners denoted $\mathrm{Inv}\,[\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}]$. This is in fact the invariant subspace of the tensor product of three $SU(2)$ irreducible representations. It is easy to see that the orthogonal projector over this space is nothing but

$$P = \int_{SU(2)} dg\, D^{(j_1)}(g) D^{(j_2)}(g) D^{(j_3)}(g) \tag{A.23}$$

in fact thanks to the invariance and normalization of the Haar measure this integral is invariant under the action of $SU(2)$ and satisfies $P^2 = P$ and $P^\dagger = P$. Therefore, seen as an operator on $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$, P is the projection operator onto the invariant subspace $\mathrm{Inv}\,[\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}]$, namely the space of intertwiners. Recall that if $|\imath\rangle$ is an orthonormal basis of $\mathrm{Inv}\,[\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}]$, then P can also be written as

$$P = \sum_{\imath} |\imath\rangle\langle\imath| \tag{A.24}$$

Putting together the (A.23) and (A.24) nd expressing them in the magnetic basis, one sees that

$$\int_{SU(2)} dg D^{(j_1)m_1}{}_{n_1}(g) D^{(j_2)m_2}{}_{n_2}(g) D^{(j_3)m_3}{}_{n_3}(g) = \sum_{\imath} \imath^{m_1m_2m_3} \overline{\imath_{n_1n_2n_3}} \tag{A.25}$$

where the sum runs over any orthonormal basis of the intertwiner space. Notice that thanks to the notation (A.22), the last equation (A.25) is exactly (A.21).

Now, for instance, by using the epsilon tensor to raise an index of (A.22), one obtains the tensor

$$\imath^{m_1}{}_{m_2m_3} = \epsilon^{m_1m}\,\imath_{m\,m_2m_3} \tag{A.26}$$

which spans the intertwiner space $\mathrm{Inv}\left[\overline{\mathcal{H}}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}\right]$. The epsilon tensor therefore is simply a map between the spaces $\mathcal{H}_j$ and its dual $\overline{\mathcal{H}_j}$. Up to a prefactor, the tensor $\imath^{m_1}{}_{m_2m_3}$ is equal to the Clebsch-Gordan coefficient $C^{(j_2j_3j_1)}{}_{m_2m_3}{}^{m_1}$ seen above. Elements of a space such as this one are invariant under the action of $SU(2)$ in the sense of equation (A.10), namely when a matrix $D^{(j)}(g)$ acts on each lower index while an inverse $D^{(j)}(g^{-1})$ acts on each upper index.

The symmetry relation (A.19) and the condition $m_1 + m_2 + m_3 = 0$ imply that the tensor obtained raising all the indices of the intertwiner (A.22) is numerically equivalent to $\imath_{m_1m_2m_3}$

$$\imath^{m_1m_2m_3} = \epsilon^{m_1n_2}\epsilon^{m_2m_3}\epsilon^{m_3n_3}\imath_{n_1n_2n_3} \tag{A.27}$$

The orthogonality relation (A.16) then implies that the three-valent (A.22) is normalized

$$\imath^{m_1m_2m_3}\imath_{m_1m_2m_3} = 1 \tag{A.28}$$

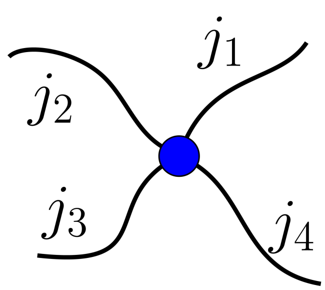
A.4 Higher valence intertwiners

The invariant tensors $\imath_{m_1m_2m_3}$ and ϵ_{mn} are the basic objects from which intertwiners of higher valence can be built. For instance one can consider the tensor product space of the addition of four angular momenta $\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}\otimes\mathcal{H}_{j_4}$ which can be decomposed as a direct sum of irreducible representations. Then one can consider the invariant subspace Inv $\left[\bigotimes_{i=1}^4 \mathcal{H}_{j_i}\right]$ and it turns out that an orthonormal basis of this space is given by

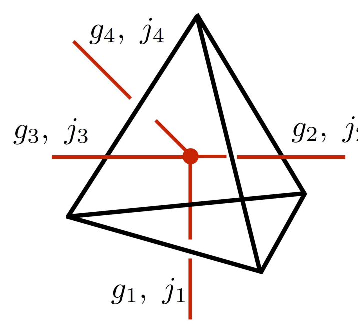
$$|\imath_{12}^{(j)}\rangle = \sum_{m_1,m_2,m_3,m_4} \sqrt{d_j} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j)} \bigotimes_{i=1}^{4} |j_i, m_i\rangle \tag{A.29}$$

with $j \in \{\max(|j_1-j_2|\,,\,|j_3-j_4|),\ldots,\min(j_1+j_2\,,\,j_3+j_4)\}$ and

$$\begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j)} \equiv \left(\imath_{12}^{(j)}\right)_{m_1m_2m_3m_4} = \sum_m (-1)^{j-m} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j & j_3 & j_4 \\ -m & m_3 & m_4 \end{pmatrix} \tag{A.30}$$



(a) The graphical 4-valent intertwiner



(b) In the case a coherent intertwiner, we also have a geometrical interpretation

It is easy to see that the four valent intertwiner was obtained contracting the three-valent one with an epsilon, coupling the spins j_1 and j_2

$$\left(\imath_{12}^{(j)}\right)_{m_1m_2m_3m_4} = \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \epsilon^{mn} \begin{pmatrix} j & j_3 & j_4 \\ n & m_3 & m_4 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j)} \tag{A.31}$$

where the last notations denotes the so called Wigner 4jm symbol and can be used as well as the tensor notation, similarly to (A.22).

The invariance of the three valent intertwiner and the epsilon tensor, implies that the four-valent intertwiner is indeed invariant under the action of $SU(2)$ on its indices

$$D^{(j_1)m_1}{}_{n_1}(g)D^{(j_2)m_2}{}_{n_2}(g)D^{(j_3)m_3}{}_{n_3}(g)D^{(j_4)m_4}{}_{n_4}(g) \left(\imath_{12}^{(j)}\right)_{m_1m_2m_3m_4} = \left(\imath_{12}^{(j)}\right)_{n_1n_2n_3n_4} \tag{A.32}$$

As the ”internal” spin j runs over all the values allowed by the Clebsch-Gordan conditions, the tensors (A.31) span the intertwiner space Inv $\left[\bigotimes_{i=1}^4 \mathcal{H}_{j_i}\right]$. Using the orthogonality relations of the 3j symbols one finds

$$\langle \imath_{12}^{(j)} | \imath_{12}^{(j')} \rangle = \sqrt{d_j}\sqrt{d_{j'}} \left(\imath_{12}^{(j)}\right)^{m_1 m_2 m_3 m_4} \left(\imath_{12}^{(j')}\right)_{m_1 m_2 m_3 m_4} = \delta_{jj'} \tag{A.33}$$

showing that the basis (A.29) is orthonormal (whereas the basis expressed in terms of the tensor defined in (A.31) needs to be multiplied by $\sqrt{d_j}$ to be normalized).

Among the other things, it is interesting to note that this basis diagonalises the operator $(\vec{J}_1 + \vec{J}_2)^2$

$$(\vec{J}_1 + \vec{J}_2)^2 \, |\imath_{12}\rangle = j(j+1) \, |\imath_{12}\rangle \tag{A.34}$$

Finally, similarly to (A.21) or (A.25) one also has that

$$\int_{SU(2)} dg D^{(j_1)}_{m_1 n_1}(g) D^{(j_2)}_{m_2 n_2}(g) D^{(j_3)}_{m_3 n_3}(g) D^{(j_4)}_{m_4 n_4}(g) = \sum_j d_j \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(j)} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ n_1 & n_2 & n_3 & n_4 \end{pmatrix}^{(j)} \tag{A.35}$$

Now the basis (A.29) or its tensorial counterpart (A.31) was a choice. It was built from one possible decomposition of the tensor product space into irreps. Another possible decomposition leads to another basis on the four valent intertwiner space in which the spins j_1 and j_3 are coupled to the internal spin

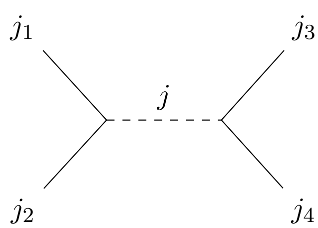
$$|\imath_{13}^{(k)}\rangle = \sum_{m_1, m_2, m_3, m_4} \sqrt{d_j} \begin{pmatrix} j_1 & j_3 & j_2 & j_4 \\ m_1 & m_3 & m_2 & m_4 \end{pmatrix}^{(k)} \bigotimes_{i=1}^{4} |j_i, m_i\rangle \tag{A.36}$$

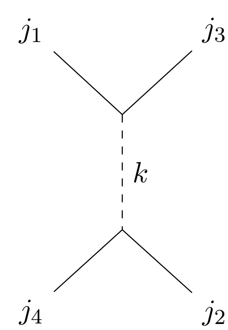
where this time

$$\begin{pmatrix} j_1 & j_3 & j_2 & j_4 \\ m_1 & m_3 & m_2 & m_4 \end{pmatrix}^{(k)} \equiv \left(\imath_{13}^{(k)}\right)_{m_1 m_2 m_3 m_4} = \begin{pmatrix} j_1 & j_3 & k \\ m_1 & m_3 & m \end{pmatrix} \epsilon^{mn} \begin{pmatrix} k & j_2 & j_4 \\ n & m_2 & m_4 \end{pmatrix} \tag{A.37}$$

Again notice that this basis diagonalizes the operator $(\vec{J}_1 + \vec{J}_3)^2$.

So one has two inequivalent bases for the four-valent intertwiner space





and they simply represent the different spins one chooses to recouple. This freedom was of course not present in the three-valent case. Notice that according to the equations above, the label of the intertwiner is given by the virtual spin (or spins for higher valence). The change of basis is given by

$$W_{jk} \equiv \langle {}^{(j)}_{12}\imath | \imath^{(k)}_{13} \rangle = (-1)^{j_2+j_3+j+k} \sqrt{d_j d_k} \begin{Bmatrix} j_1 & j_2 & j \\ j_4 & j_3 & k \end{Bmatrix} \tag{A.38}$$

so that

$$|\imath^{(k)}_{13}\rangle = \sum_j W_{jk} |\imath^{(j)}_{12}\rangle \tag{A.39}$$

where the 6j symbol was introduced

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \sum_{m_1,\ldots,m_6} (-1)^{\sum_{i=1}^{6}(j_i - m_i)} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ m_1 & -m_5 & m_6 \end{pmatrix} \times \\ \times \begin{pmatrix} j_4 & j_2 & j_6 \\ m_4 & m_2 & -m_6 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ m_3 & -m_4 & m_5 \end{pmatrix} \tag{A.40}$$

or equivalently as a contraction of four three-valent intertwiners

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \imath^{m_1 m_2 m_3}\, \imath_{m_1}{}^{m_5}{}_{m_6}\, \imath_{m_4 m_2}{}^{m_6}\, \imath_{m_3}{}^{m_4}{}_{m_5} \tag{A.41}$$

This actually shows that the 6j vanishes unless the triples of spins indicted by the little circles

$$\begin{Bmatrix} \circ & \circ & \circ \\ & & \end{Bmatrix} \qquad \begin{Bmatrix} \circ & & \\ & \circ & \circ \end{Bmatrix} \qquad \begin{Bmatrix} & \circ & \\ \circ & & \circ \end{Bmatrix} \qquad \begin{Bmatrix} & & \circ \\ \circ & \circ & \end{Bmatrix} \tag{A.42}$$

satisfy the Clebsch-Gordan conditions. Moreover the 6j symbol satisfy the orthogonality relation

$$\sum_l d_l \begin{Bmatrix} j_1 & j_2 & l \\ j_3 & j_4 & j \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & l \\ j_3 & j_4 & k \end{Bmatrix} = \frac{1}{d_j} \delta_{jk} \tag{A.43}$$

which in turn implies

$$\sum_i W_{ij} W_{jk} = \delta_{jk} \tag{A.44}$$

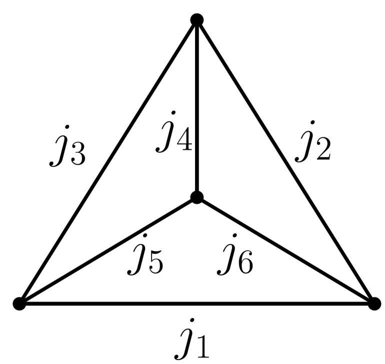
The Wigner 6j symbols satisfy several symmetry properties, for instance a symbol is unchanged by any permutation of its column

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_3 & j_2 \\ j_4 & j_6 & j_5 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_3 & j_1 \\ j_5 & j_6 & j_0 \end{Bmatrix} = \ldots \tag{A.45}$$

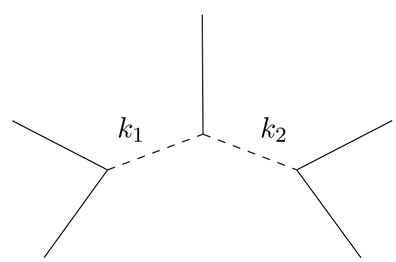
and by interchanging the upper and lower spins simultaneously in any two columns

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_5 & j_6 \\ j_4 & j_2 & j_3 \end{Bmatrix} = \ldots \tag{A.46}$$

Let us also mention the relation between a 6j symbol and a tetrahedron. It is true that they share the same algebraic symmetries (in a way that can be easily understood), but there is also a purely geometric significance behind it. Ponzano and Regge, expanding on work of Wigner, gave a striking asymptotic formula relating the value of the 6j – symbol, when the dimensions of the representations are large, to the volume of an genuine Euclidean tetrahedron whose edge lengths are these dimensions. This played a very important role in the three-dimensional theory [34].



Now that the game is understood, it is straightforward to continue. The Wigner 9j symbols will arise when changes of bases between five-valent intertwiners spaces are performed. From here it will follow that a 9j symbol is in fact defined as a contraction of six three-valent intertwiners, or 3j symbols. And one can also see that the 9j symbol can have a non zero value only if the Clebsch-Gordan conditions are satisfied by the spins in each row and each column. Of course this symbol will have a high degree of symmetry. As an example, notice that a five-valent intertwiner will be labelled by two virtual spin



(A.47)

Everything concerning these and also 12j or 15j symbol can be found in the literature [49], or easily deducted from there.

The point here is that intertwiner of arbitrarily high valency can be derived by continuing to attach three-valent intertwiners by contraction with epsilon. An N-valent intertwiner will be labelled by $N-2$ internal spins, which determine the eigenvalues of the operators $(J_1+J_2)^2, (J_1+J_2+J_3)^2, \ldots, (J_1+\cdots+J_{N-2})^2$.

A useful fact of the N-valent intertwiner space concerns the integral

$$P^{(j_1,\dots,j_N)} = \int_{SU(2)} dg \bigotimes_{i}^{N} D^{(j_i)} \tag{A.48}$$

which is easily recognized as the projection operator acting on $\bigotimes_{i=1}^{N} \mathcal{H}_{j_i}$ which projects over the invariant subspace Inv $\left[\bigotimes_{i=1}^{N} \mathcal{H}_{j_i}\right]$, i.e. the intertwiner space. In fact the integral (A.48) is invariant under the action of $SU(2)$ thanks to the Haar measure and satisfies $P^2 = P = P^\dagger$. Generalizing the above results for three-valent intertwiners (A.25) one sees that the integral can be expressed as $P^{(j_1,\dots,j_N)} = \sum_\imath |\imath\rangle\langle\imath|$, or

$$\int_{SU(2)} dg D^{(j_1)m_1}{}_{n_1}(g) \cdots D^{(j_N)m_N}{}_{n_N}(g) = \sum_\imath \imath^{m_1\cdots m_N} \overline{\imath_{n_1\cdots n_N}}$$

where the sum runs over an orthonormal basis of Inv $[\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}]$.

Appendix B

Conventions and Notations

B.1 Physical constants

In the main text we usually consider the most convenient units depending on the context. However, it is always important to remember the physical dimensions of the quantities, and this is especially true in a theory of quantum gravity where all the fundamental constants come into play.

The first thing to remark is that in the whole thesis the speed of light is set to be $c = 1$, as customary. The only exception is this appendix. Of course if this were not the case, we would have differences starting from the very first equation of this document (1.1), which would propagate from there into all the calculations. Without the speed of light one is left with the factor $8\pi G$ that again sometimes is killed in the main text, choosing units where it amounts to unity.

The second comment concerns the Planck length

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}\, m \tag{B.1}$$

which is fundamental for a theory of quantum gravity, since this tries to understand what happens at that extreme short-distance scale. This is in fact *the* fundamental length scale. It not only sets the scale of quantum gravitational phenomena, it also determines a physical *limit*. It literally sets a lower limit to the divisibility of physical space. Quantum gravity in this sense is a realisation that space (-time) itself is not continuum. There is a finite discrete granular structure, in the same way as for instance there is a limit to the divisibility of matter.

Regarding the equation in the main text, expression (B.1) dictates for example that considering the correct physical dimension, the area and volume operators will have spectra with eigenvalues

$$\frac{8\pi G\hbar\gamma}{c^3} = 8\pi\gamma l_P^2 \qquad \text{and} \qquad \left(\frac{8\pi G\hbar\gamma}{c^3}\right)^{\frac{3}{2}} = (8\pi\gamma)^{\frac{3}{2}}\, l_P^3 \tag{B.2}$$

respectively, the first one being in accordance with (2.43). Notice that in units where $8\pi G = 1 = c$ they simply become $\hbar\gamma$ and $\sqrt{\hbar^3\gamma^3}$ as in equations (2.9), (2.21).

B.2 Schwinger representation and Twisted Geometries

We briefly report here the conventions adopted in the main text, particular choices or changes in notations, regarding the link between twisted geometries, spinors and the Schwinger representation.

Fluxes

We start with the right- and left- invariant vector fields. They were called X and $\tilde{X}$ respectively in the original papers [16, 17]. However, in order to avoid a tilde symbol flying around, in chapter 5 we decided to intuitively call them R and L at the classical level, so that the operators will simply be $\hat{R}$ and $\hat{L}$. In accordance with chapter 3, their classical algebra is

$$\{R_i, R_j\} = \epsilon_{ijk} R_k \qquad \{L_i, L_j\} = \epsilon_{ijk} L_k \qquad (\text{recall } X = R\,, \tilde{X} = L) \tag{B.3}$$

and they are in fact defined in terms of the derivatives

$$\nabla_i^R f(g) = \frac{d}{dt} f\left(g e^{tX}\right)\Big|_{t=0} \qquad \nabla_i^L f(g) = \frac{d}{dt} f\left(e^{-tXg}\right)\Big|_{t=0} \tag{B.4}$$

simply as

$$R_i = i\nabla_i^L \qquad L_i = i\nabla_i^R \tag{B.5}$$

Consistently, they are related by

$$L = -gRg^{-1} \qquad (\text{or for the last time } \tilde{X} = -gXg^{-1}) \tag{B.6}$$

as one can easily check

$$g^{-1}\nabla^R g = g^{-1} g X = X = -(\nabla^L g) g^{-1} \quad \Rightarrow \quad g^{-1} L = -R g^{-1} \tag{B.7}$$

The quantization rule chosen $[\cdot,\cdot] = i\{\cdot,\cdot\}$ is therefore such that

$$\left[\hat{R}_i, \hat{R}_j\right] = i\epsilon_{ijk}\hat{R}_k \qquad \left[\hat{L}_i, \hat{L}_j\right] = i\epsilon_{ijk}\hat{L}_k \tag{B.8}$$

The explicit construction of these quantum operators is given below in terms of the Schwinger representation of the angular momentum, for which one has to introduce spinorial variables.

Spinors

In agreement with the main text, the spinor variables are chosen to satisfy the classical algebra

$$\{z^A, \bar{z}^B\} = i\delta^{AB} \qquad \{\tilde{z}^A, \bar{\tilde{z}}^B\} = -i\delta^{AB} \tag{B.9}$$

and in view of the twisted geometries parametrization we choose to call the ratios

$$\tilde{\zeta} = \frac{\tilde{z}^0}{\tilde{z}^1} \qquad \zeta = \frac{z^0}{z^1} \tag{B.10}$$

The relation between these variables and the fluxes is the straightforward link between spinors and vectors. For the left - invariant vector field the one has

$$|\tilde{z}\rangle\langle\tilde{z}| = A\mathbb{1} + L^i\sigma_i \tag{B.11}$$

where A is, in accordance to chapter 5, the name of the twisted geometries variable which was originally called j in [16]. Thus

$$L^i = \langle\tilde{z}|\frac{\sigma^i}{2}|\tilde{z}\rangle \quad \Rightarrow \quad L^i = \frac{1}{2}\begin{pmatrix} \overline{\tilde{z}}_1\tilde{z}_0 + \overline{\tilde{a}z}_0\tilde{z}_1 \\ i(\overline{\tilde{z}}_1\tilde{z}_0 - \overline{\tilde{z}}_0\tilde{z}_1) \\ |\tilde{z}_0|^2 - |\tilde{z}_1|^2 \end{pmatrix} = \begin{pmatrix} \mathrm{Re}[\tilde{z}_0\overline{\tilde{z}}_1] \\ \mathrm{Im}[\tilde{z}_0\overline{\tilde{z}}_1] \\ \frac{1}{2}(|\tilde{z}_0|^2 - |\tilde{z}_1|^2) \end{pmatrix} \tag{B.12}$$

or equivalently, using $L^{\pm} = L^1 \pm iL^2$

$$L^+ = \overline{\tilde{z}}^0\tilde{z}^1 \qquad\qquad L^- = \tilde{z}^0\overline{\tilde{z}}^1 \tag{B.13}$$

This is of course the classical version of the Schwinger representation. Moreover the following identification holds in agreement with chapter 3

$$A = |\vec{L}| = \frac{1}{2}\langle\tilde{z}|\tilde{z}\rangle = \frac{1}{2}(|\tilde{z}_0|^2 + |\tilde{z}_1|^2) \tag{B.14}$$

Similarly we will have for the right - invariant vector field

$$R^i = -\langle z|\frac{\sigma^i}{2}|z\rangle \quad \Rightarrow \quad R^i = -\frac{1}{2}\begin{pmatrix} \overline{z}_1 z_0 + \overline{z}_0 z_1 \\ i(\overline{z}_1 z_0 - \overline{z}_0 z_1) \\ |z_0|^2 - |z_1|^2 \end{pmatrix} = -\begin{pmatrix} \mathrm{Re}[z_0\overline{z}_1] \\ \mathrm{Im}[z_0\overline{z}_1] \\ \frac{1}{2}(|z_0|^2 - |z_1|^2) \end{pmatrix} \tag{B.15}$$

or equivalently, again using $R^{\pm} = R^1 \pm iR^2$ (á la Schwinger)

$$R^+ = -\overline{z}^0 z^1 \qquad\qquad R^- = -z^0\overline{z}^1 \tag{B.16}$$

Remember that when using two spinors, one is describing something bigger than the edge space of LQG (see chapter 3). In fact from 8 degrees of freedom one goes to only 6 associated to the $T^*SU(2)$ imposing the area matching constraint, which is nothing but the matching between the norms of the spinors. Therefore we will also have

$$A = |\vec{R}| = \frac{1}{2}\langle z|z\rangle = \frac{1}{2}(|z_0|^2 + |z_1|^2) \tag{B.17}$$

Notice also that due to the above definitions (B.9) one can explicitly check the algebras (B.3) since, being for example $L^i = \frac{1}{2}\sigma^i_{AB}\overline{\tilde{z}}_A\tilde{z}_B$

$$\begin{aligned} \{L^i, L^j\} &= \frac{1}{4}\sigma^i_{AB}\sigma^j_{CD}\{\overline{\tilde{z}}_A\tilde{z}_B, \overline{\tilde{z}}_C\tilde{z}_D\} \\ &= \frac{1}{4}\sigma^i_{AB}\sigma^j_{CD}(\overline{\tilde{z}}_A\tilde{z}_D\{\tilde{z}_B, \overline{\tilde{z}}_C\} + \tilde{z}_B\overline{\tilde{z}}_C\{\overline{\tilde{z}}_A, \tilde{z}_D\}) \\ &= \frac{1}{4}\sigma^i_{AB}\sigma^j_{CD}\left(-i\delta^{BC}\overline{\tilde{z}}_A\tilde{z}_D + i\delta^{AD}\tilde{z}_B\overline{\tilde{z}}_C\right) \\ &= \frac{-i}{4}\left(\sigma^i_{AC}\sigma^j_{CD}\overline{\tilde{z}}_A\tilde{z}_D - \sigma^j_{CA}\sigma^i_{AC}\tilde{z}_B\overline{\tilde{z}}_C\right) \\ &= -\frac{i}{4}\langle\tilde{z}|\left[\sigma^i, \sigma^j\right]|\tilde{z}\rangle = \frac{1}{2}\epsilon^{ij}{}_k\langle\tilde{z}|\sigma^k|\tilde{z}\rangle = \epsilon^{ij}{}_k L^k \end{aligned} \tag{B.18}$$

because $\left[\sigma^i, \sigma^j\right] = 2i\epsilon^{ij}{}_k\sigma^k$. This will hold also for the vector R since the minus sign in (B.15) is compensated by the algebra (B.9).

The holonomy in terms of the spinors is chosen to be

$$g^A{}_B = \frac{\tilde{z}^A \delta_{B\dot{B}} \bar{z}^{\dot{B}} + \delta^{A\dot{A}} \bar{\tilde{z}}_{\dot{A}} z_B}{||z||\,||\tilde{z}||} = \frac{1}{||z||\,||\tilde{z}||} \begin{pmatrix} \tilde{z}^0 \bar{z}^0 + \bar{\tilde{z}}^1 z^1 & \tilde{z}^0 \bar{z}^1 - \bar{\tilde{z}}^1 z^0 \\ \tilde{z}^1 \bar{z}^0 - \bar{\tilde{z}}^0 z^1 & \tilde{z}^1 \bar{z}^1 + \bar{\tilde{z}}^0 z^0 \end{pmatrix} \tag{B.19}$$

such that

$$g|z\rangle \propto |\tilde{z}\rangle \qquad\qquad g\,|z] \propto |\tilde{z}] \tag{B.20}$$

Thanks to the twisted geometries, we will not need to quantize the expression (B.19) but a convenient constituent of it. This is done in the main text and reported below within the Schwinger representation quantizing the spinorial variables.

Twisted Geometries

Regarding the fluxes, for the Twisted Geometries parametrization we make the choice

$$L = -A\tilde{n}\tau_3\tilde{n}^{-1} \equiv -A\tilde{N} \qquad\qquad R = An\tau_3 n^{-1} \equiv AN \tag{B.21}$$

which is consistent with the above ones. In fact, to check whether the signs in (B.21) are correct, one can for example see explicitly that

$$L^3 = \frac{|\tilde{z}^0|^2 - |\tilde{z}^1|^2}{2} = \frac{\tilde{n}^0 - \tilde{n}^1}{2} \tag{B.22}$$

where in the last equality we called $\tilde{n}^0 = \bar{\tilde{z}}^1 \tilde{z}^1 = |\tilde{z}^0|^2$ and $\tilde{n}^1 = \bar{\tilde{z}}^1 \tilde{z}^1 = |\tilde{z}^1|^2$ in view of the Schwinger representation for the angular momentum in terms of harmonic oscillators. Starting from $L = L^i\tau_i = -\frac{i}{2}L^i\tau_i$ which implies $L^i = i\,\mathrm{tr}(\sigma_i L)$, we have

$$L^3 = i\,\mathrm{tr}(\sigma_3 L) = -\frac{1}{2}A\,\mathrm{tr}\big(\sigma_3\tilde{n}\sigma_3\tilde{n}^{-1}\big) = -A\frac{1-|\tilde{\zeta}|^2}{1+|\tilde{\zeta}|^2} = -A\frac{|\tilde{z}^1|^2 - |\tilde{z}^0|^2}{||\tilde{z}||^2} = \frac{|\tilde{z}^0|^2 - |\tilde{z}^1|^2}{2} = \frac{\tilde{n}^0 - \tilde{n}^1}{2} \tag{B.23}$$

where we used $||\tilde{z}||^2 = 2A$ and $\mathrm{tr}\big(\sigma_3\tilde{n}\sigma_3\tilde{n}^{-1}\big) = 2\frac{1-|\tilde{\zeta}|^2}{1+|\tilde{\zeta}|^2}$. Given the area matching constraint discussed above which implies also $||z||^2 = 2A$, one now clearly sees that R^3 will have a global minus sign in accordance with (B.21) and (B.15). The same will hold of course for the other components of the fluxes.

Regarding the holonomy, in the twisted geometries parametrization the choice is

$$g = \tilde{n}e^{\xi\tau_3}n^{-1} \tag{B.24}$$

where $n = n(\zeta)$ and $\tilde{n} = \tilde{n}(\tilde{\zeta})$ are the Hopf sections of chapter 3. It is shown in the main text that in order to recover (B.19), one has to define

$$\xi = 2\arg\tilde{z}^1 - 2\arg(z^1) = -i\ln\left(\frac{\tilde{z}^1\,\bar{z}^1}{\bar{\tilde{z}}^1\,z^1}\right) \tag{B.25}$$

which implies

$$e^{i\xi} = e^{2i(\arg\tilde{z}^1 - \arg z^1)} = \frac{\tilde{z}^1\,\bar{z}^1}{\bar{\tilde{z}}^1\,z^1} \tag{B.26}$$

To quantize this we need the following representation.

Schwinger representation

Calling $z \to a$ and $\bar{z} \to a^\dagger$, one obtains the commutators of harmonic oscillators starting from the classical algebras (B.9) and the usual quantization rule

$$\left[\tilde{a}^A\,,\,\tilde{a}^{B\dagger}\right] = \delta^{AB} \qquad\qquad \left[a^A\,,\,a^{B\dagger}\right] = -\delta^{AB} \tag{B.27}$$

Therefore, calling as customary $n^A = a^{A\dagger}a^A$, the Schwinger representation is realized as follows

$$|n^0, n^1, \tilde{n}^0, \tilde{n}^1\rangle = |j, m, n\rangle \qquad \text{where} \quad |j,m,n\rangle = \sqrt{d_j}|j,n\rangle \otimes \langle j,m| \tag{B.28}$$

with

$$j = \frac{n^0 + n^1}{2} = \frac{\tilde{n}^0 + \tilde{n}^1}{2} = \tilde{j} \qquad\qquad m = \frac{n^0 - n^1}{2} \qquad\qquad n = \frac{\tilde{n}^0 - \tilde{n}^1}{2} \tag{B.29}$$

or equivalently

$$\tilde{n}^1 = j - n, \qquad \tilde{n}^0 = j + n, \qquad n^1 = j - m, \qquad n^0 = j + m \tag{B.30}$$

We are not putting a hat above the "a" operators since they already differ from their classical "z" spinor counterparts. In this representation, according to the above results, the fluxes are

$$\hat{L}^{\pm,3} = \left(\tilde{a}^{0\dagger}\tilde{a}^1, \quad \tilde{a}^0\tilde{a}^{1\dagger}, \quad \frac{\tilde{n}^0 - \tilde{n}^1}{2}\right) \tag{B.31}$$

and

$$\hat{R}^{\pm,3} = -\left(a^{0\dagger}a^1, \quad a^0 a^{1\dagger}, \quad \frac{n^0 - n^1}{2}\right) \tag{B.32}$$

and their actions can indeed be checked to be

$$\hat{R}_3|j,m,n\rangle = -m|j,m,n\rangle \qquad\qquad \hat{L}_3|j,m,n\rangle = n|j,m,n\rangle \tag{B.33}$$

and

$$\hat{R}_\pm|j,m,n\rangle = -c_\mp(m)|j,m\mp 1,n\rangle \qquad\qquad \hat{L}_\pm|j,m,n\rangle = c_\pm(n)|j,m,n\pm 1\rangle \tag{B.34}$$

where $c_\pm(a) = \sqrt{(j \mp a)(j \pm a + 1)} = \sqrt{j(j+1) - a(a\pm 1)}$.

Those action are rather quick to check. It is very easy to see that $\hat{L}_3$ acts on the ket $|j,n\rangle$ of (B.28) from the left, and given the identifications (B.29) and (B.30), it clearly is an eigenstate with eigenvalue n. Similarly, since $\hat{R}_3$ has a sign of difference and it acts from the right, it will act on the bra $\langle j, m|$ with eigenvalue $-m$. This is true because the number operator acts identically from the left and from the right, since the creation and annihilation operators switch role (other than position) passing from the 'ket' to the 'bra' action. Concerning $\hat{L}_{\pm}$, from (B.31) and the usual identifications (B.29), (B.30), its action immediately follows from the standard action of the harmonic oscillators on the ket $|j, n\rangle$. On the contrary, when dealing with $\hat{R}_{\pm}$, we have to remember that we are acting from the right on the bra $\langle j, m|$ with (B.32), so that again the oscillator operators exchange roles. That is why one gets the coefficient $-c_{\mp}$, with the minus sign coming from the definitions.

Concerning the operator (B.26), it is already shown in the main text that a suitable quantization in accordance with all the requirements is given by

$$\widehat{e^{i\xi}} = (a^{1\dagger})^2 (n^1)^{-1} (\tilde{n}^1)^{-1} (\tilde{a}^1)^2 \tag{B.35}$$

with action and features described in chapter 5.

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