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A higher derivative fermion model

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Sommario

Nel presente elaborato studiamo un modello fermionico libero ed invariante di scala con derivate di ordine elevato. In particolare, controlliamo che la simmetria di scala sia estendibile all'intero gruppo conforme. Essendoci derivate di ordine più alto il modello non è unitario, ma costituisce un nuovo esempio di teoria conforme libera. Nelle prime sezioni riguardiamo la teoria generale del bosone libero, partendo dapprima con modelli semplici con derivate di ordine basso, per poi estenderci a dimensioni arbitrarie e derivate più alte. In questo modo illustriamo la tecnica che ci permette di ottenere un modello conforme da un modello invariante di scala, attraverso l'accoppiamento con la gravità e richiedendo l'ulteriore invarianza di Weyl. Se questo è possibile, il modello originale ammette certamente l'intera simmetria conforme, che emerge come generata dai vettori di Killing conformi. Nel modello scalare l'accoppiamento con la gravità necessita di nuovi termini nell'azione, indispensabili affinchè la teoria sia appunto invariante di Weyl. La costruzione di questi nuovi termini viene ripetuta per un particolare modello fermionico, con azione contenente l'operatore di Dirac al cubo ($\nabla\!\!\!/^3$), per il quale dimostriamo l'invarianza conforme. Tale modello descrive equazioni del moto con derivate al terzo ordine. Dal momento che l'invarianza di Weyl garantisce anche l'invarianza conforme, ci si aspetta che il tensore energia-impulso corrispondente sia a traccia nulla. Per ogni modello introdotto controlliamo sistematicamente che tale condizione sia verificata, ed in particolar modo per il caso della teoria fermionica con $\nabla\!\!\!/^3$, che rappresenta il contributo originale di questa tesi.

Abstract

In this work we study a free scale invariant fermionic model with higher order derivatives. In particular, we verify that scale symmetry can be extended to the full conformal group. Due to the presence of higher order derivatives the model is not unitary, nevertheless it establishes a new example of free conformal theory. In the first sections we review the general free bosonic theory, starting at first with simple models with lower order derivatives and then extending them to arbitrary dimensions and higher order derivatives. In particular, we review the technique used to obtain a conformal model, starting from a scale invariant one, through the coupling with gravity and the Weyl invariance condition. The conformal symmetry arises as the one generated by the conformal Killing vectors. In the scalar case the gravity coupling needs new terms in the action, essential for the theory to be Weyl invariant. The construction of these new terms are repeated for a particular fermionic model, with an action containing a cubic Dirac operator ($\nabla\!\!\!\!/^3$). We demonstrate that this new fermionic model is conformal invariant. This model describes equations of motion with third order derivatives. Since the Weyl invariance implies conformal invariance, we expect that the corresponding stress tensor is traceless. For each model introduced we systematically make sure that this condition is verified, in particular for the $\nabla\!\!\!\!/^3$ fermionic theory, which represents the original contribution of this thesis.

Contents

1 Introduction

Conformal field theories (CFTs) constitute a central topic in modern theoretical physics and, over the last decades, our understanding of conformal field theories has advanced significantly. Consequently, conformal field theory is a very broad subject. By definition, a conformal field theory is a quantum field theory that is invariant under the conformal group. We are all familiar with the Poincarè group as the symmetry group of relativistic field theory in flat space. Poincarè transformations are isometries of flat spacetime and are a combination of Lorentz and translation transformations. In addition to these symmetries, CFTs have extra spacetime symmetries, that combined with those of the Poincarè group form the conformal group. The conformal group is defined by the set of transformations of spacetime that preserve angles. Thus, it obviously contains Poincarè transformations and scale transformations, the latter defined mathematically by $x \longrightarrow \lambda x$ and $t \longrightarrow \lambda t$. Scale transformations act by rescaling, or zooming in and out of some region of spacetime. In additions, the conformal group contains the so-called conformal boosts. Generally speaking, a conformal transformation is a coordinate transformation that produces a local rescaling of the metric.

Even if we expect that interacting quantum field theories can not be conformally invariant due to the presence of the running coupling constants, that become functions of some energy scale, CFTs theories describe critical points (as originally introduced in statistical physics) and give us a better understanding of the structure of general quantum field theory (QFT). QFT is the main theoretical framework describing most of nature, with applications including elementary particle physics, statistical physics, condensed matter physics and fluid dynamics. The description of a physical system very much depends on the energy scale one wishes to study and QFT comes equipped with some ultraviolet cutoff Λ, the energy scale beyond which new degrees of freedom are necessary. We do not know what is going on past this energy, but we can still calculate observable results which are applicable to low-energy physics. The program of the renormalization group in QFT is a way to parametrize this ignorance in terms of interactions or coupling constants that we measure between low-energy degrees of freedom. The renormalization group makes use of the renormalization group flow and the β-functions in order to study theories close to fixed points, where scale invariance is satisfied, and understand the behaviour of these theories close to these fixed points.

Examples of free CFTs are the scalar and the spin 1/2 fermion fields, which can be defined in arbitrary dimensions, and the theory of gauge p-forms, which are conformally invariant in $D = 2p + 2$ dimensions. CFTs exhibit scale invariance. In 2 dimensions scale invariance can be promoted to the full conformal symmetry under some very general assumptions, which includes unitarity [1, 2]. In 4 dimensions similar results have been derived in [3]. The case of the CFT of a free boson in 2 dimensions is of particular relevance: it is a central element of string theory, it is used to construct free field realizations of interacting conformal field theories (the so-called Coulomb gas approach [4]), and can be extended to the famous Liouville theory when an exponential interaction is added to the kinetic term [5, 6, 7]. The free two-dimensional boson has the peculiar characteristic of having vanishing canonical dimensions as well as a propagator that behaves logarithmically. An extension to higher even dimensions has been investigated recently in [8]. Requiring vanishing dimensions and a logarithmic propagator, the scalar field must have a higher derivative kinetic term in the action. It is a non unitary model, which appears in many contexts (see [9] [8]) and it is useful to study.

Inspired by these (non-unitary) free bosonic theories with scalar fields of vanishing canonical dimensions, that exist only in even dimensions, we wish to study similar theories that have fields with vanishing canonical dimensions in odd dimensions, too. As scalar fields in odd dimensions contain a non local kinetic term in the action, with the square root of a differential operator, an option is to study fermionic theories as natural candidates for free field with vanishing mass dimensions and local kinetic term. At first we analyze the simplest candidate, a fermion in three dimensions with a kinetic term given by the cube of the Dirac operator. We show that it is possible to make this model Weyl invariant in a curved space through the introduction of non-minimal terms. We also extend our analysis to arbitrary dimensions, where generically the fermion acquires some canonical mass dimension.

At first we review various relevant models to build up confidence with free CTFs and present some useful background material, and then we face the study of the cubic derivative fermion. In the latter case we show that three non minimal terms can be introduced, and it is possible to fix their dimensionless coupling constants to achieve the Weyl invariance of the action with the cubic $\nabla\!\!\!\!/^3$ operator. We then calculate the stress-tensor for such a theory, which is known to be a quite laborious task for higher derivative theories, and verify that it is symmetric and traceless, as expected for a CFT theory.

In the mathematical literature, there are many results on the Weyl properties of powers of the Dirac operators. Working on a semi Riemannian spin manifold (M^D, g) mathematicians have studied conformal powers of the Dirac and Laplacian operators. The cases of the Dirac operator and the Laplacian are known to be conformally covariant, with the Laplacian that has to be modified by a multiple of the scalar curvature in order to become conformally covariant, to obtain the so-called Yamabe operator (see [10]). In 1983, having these two examples of conformally covariant operators, Paneitz constructed a conformally second order power of the Laplacian with explicit curvature correction terms. This conformal second power of the Laplacian is called the Paneitz operator. During the first decade after the publication of the Paneitz operator, the interest in the community has increased and Graham, Jenne, Mason and Sparling [11] constructed a series of conformally covariant operatos $P_{2N}(g)$ acting on functions with leading part the N-th power of the Laplacian. N=1,2 are of course the Yamabe and the Paneitz operators. Beside this construction, there is another point of view describing the so-called GJMS operators, which uses the tractor machinery described by Gover and Peterson [12]. Although all these constructions are algorithimic, they have very rarely been used to produce explicit formulas due to their complexity. As for the Laplacian operator, powers of the Dirac operator must be supplied by lower order curvature correction terms in order to have Weyl covariance. In particular Fischmann [13] has derived an algorithmic construction in terms of associated tractor bundles to compute these correction terms. However this language is not of much practical use for physicists, and the invariance is not of immediate comprehension. Our explicit construction in terms of an action principle, allows a direct derivation of the conformal properties of the cubic Dirac operator, and allows to identify the stress tensor for the corresponding field theory.

2 Conformal transformation introduction

The conformal symmetry in a D-dimensional space-time is defined as the group of coordinate transformations $x \to x'$ which leave the metric $g_{\mu\nu}$ invariant up to a conformal factor:

$$g_{\mu\nu}(x) = \Omega(x')g'_{\lambda\sigma}(x')\frac{\partial x'^{\lambda}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} \tag{1}$$

If we wish to study the infinitesimal form of the transformations

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}$$

the relation (1) leads to the conformal Killing equations

$$\nabla_{\mu}\epsilon_{\nu} + \nabla_{\nu}\epsilon_{\mu} = \frac{2}{D}g_{\mu\nu}\nabla_{\sigma}\epsilon^{\sigma} \tag{2}$$

We denote with ∇ the metric compatible covariant derivative

$$\nabla_{\mu}\epsilon_{\nu} = \partial_{\mu}\epsilon_{\nu} - \Gamma^{\sigma}_{\mu\nu}\epsilon_{\sigma} \tag{3}$$

We will now focus on theories with scalars. The infinitesimal transformation of a scalar field ϕ with scaling dimension Δ under the full conformal group can be written as

$$\delta_c\phi = -\epsilon^{\mu}\nabla_{\mu}\phi - \frac{\Delta}{D}\nabla\epsilon\,\phi \tag{4}$$

We say that a system is conformally invariant if the variation of its action functional $S[g_{\mu\nu}, \phi]$ under the full group of conformal transformation (4) is zero

$$\delta_c S[g_{\mu\nu}, \phi] = \int d^D x \frac{\delta S}{\delta\phi}\delta_c\phi = 0 \tag{5}$$

Weyl transformations are different from conformal transformations, they constitute a pointwise rescaling transformations of the metric and fields. We will indicate this Weyl transformation as

$$\hat{g}_{\mu\nu}(x) = e^{2\sigma(x)}g_{\mu\nu}(x) \qquad \text{and} \qquad \hat{\phi}(x) = e^{-\Delta\sigma}\phi(x) \tag{6}$$

With their infinitesimal form

$$\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}\ , \quad \delta_\sigma\phi = -\Delta\sigma\phi \tag{7}$$

this leads to the following condition for a theory to be Weyl invariant

$$\delta_\sigma S[g_{\mu\nu}, \phi] = \int d^D x\sigma\Big(2\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\nu} - \Delta\frac{\delta S}{\delta\phi}\phi\Big) = 0 \tag{8}$$

Now we can observe that the two differential δ_c and δ_σ are related to each other. In fact, if we choose $\sigma = \frac{\nabla\epsilon}{D}$ we have

$$\delta_c\phi = \delta_d\phi + \delta_\sigma\phi \tag{9}$$

where $\delta_d\phi$ is the usual transformation of the scalar field under general coordinate transformations

$$\delta_d\phi = -\epsilon^\mu\partial_\mu\phi$$

In this way we can rewrite (5) as

$$0 = \int d^Dx\frac{\nabla\epsilon}{D}\Big(2\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\nu} - \Delta\frac{\delta S}{\delta\phi}\phi\Big) \tag{10}$$

At this point it is clear that Weyl invariance implies conformal invariance, but not the other way around, since $\nabla\epsilon$ is not an arbitrary function of coordinates. We will see in the next section that, if a theory is conformally invariant, it is possible to write all currents corresponding to the conformal group as j_c^μ (14) and that under several conditions (12)(15) we are able to construct the improved traceless energy momentum tensor $\Theta_{\mu\nu}$. However, it is not guaranteed that the theory can be made Weyl invariant.

2.1 Scale and conformal invariance in QFT: traceless of the stress-tensor

In this section we aim to investigate the relation between the invariance of a QFT under a local change of lenght scale and the invariance under a global change of scale. In particular we are interested in the conditions under which global invariance implies the local one. Taking a Poincarè invariant quantum field theory in D flat dimensions we are investigating two transformations:

1. scale transformations: $\delta x^\mu = \varepsilon x^\mu$;
2. special conformal transformation: $\delta x^\mu = \epsilon^\mu(x)$ with $\epsilon^\mu(x) = 2b\cdot x\, x^\mu - x^2\, b^\mu$ and b^μ a constant vector. As we know for $D \geq 3$ conformal transformation implies the assumed Poincarè invariance, the rigid scale transformation and D special conformal transformations. In $D = 2$ one sometimes is interested in the smaller algebra of the Möbius transformations.

From the literature [14] we know that a scale current must be of the form:

$$S^\mu(x) = x^\nu T^\mu_\nu(x) + K^\mu(x) \tag{11}$$

where $T_{\mu\nu}$ is the symmetric stress energy tensor and it follows from the Poincarè invariance, while K^μ is a local operator contributing to other scaling dimensions of the fields.

Imposing the conservation of S^μ we find the first necessary condition in order to have scale invariance :

$$T^\mu_\mu = -\partial_\mu K^\mu(x) \tag{12}$$

In general there are many symmetric stess tensors giving rise to the same hamiltonian. However we know a relation between the difference of two such tensors, namely:

$$T'_{\mu\nu}(x) - T_{\mu\nu}(x) = \partial^\sigma \partial^\rho Y_{\mu\sigma\nu\rho} \tag{13}$$

where $Y_{\mu\sigma\nu\rho}$ is antisymmetric on $\mu\sigma$ and on $\nu\rho$, and symmetric under exchange of $\mu\sigma$ with $\nu\rho$. The main point is that if $T_{\mu\nu}$ satisfies (12) then so does $T'_{\mu\nu}$ with K^μ replaced by $K^\mu - \partial^\rho Y^{\mu\nu}_{\rho\nu}$. In this way we have found out that the necessary and sufficient condition for existence of a conserved scale current is (12). After evaluating the condition on scale current we can write down the general form of a conserved conformal current, associated to the special conformal transformations, that we shall indicate with j^μ_c. This current will have 3 terms; the first one is determined by the space-time nature of the transformation, the second one will be related to the local nature of the conformal transformation view as a scale transformation with scale factor $\varepsilon\partial \cdot \epsilon$, while the third one is a correction due to the position dependence of the scale factor $\partial \cdot \epsilon$.

$$j^\mu_c = \epsilon^\nu T^\mu_\nu(x) + \partial \cdot \epsilon(x) K'^\mu(x) + \partial_\nu \partial \cdot \epsilon(x) + L^{\nu\mu} \tag{14}$$

where K'^μ is the same as K^μ up to the possible addition of conserved current, corresponding to an ambiguity in the original choice of scale current, and $L^{\nu\mu}$ is some local operator. Imposing the conservation of j^μ_c we obtain other two conditions whether the dimension of the system are D=2 or $D \geq 3$. We find that:

$$\begin{aligned} D \geq 3: &\qquad T^\mu_\mu = \partial_\nu \partial_\mu L^{\nu\mu}(x) \\ D = 2: &\qquad T^\mu_\mu = \partial^2 L(x) \\ D = 2: &\qquad T^\mu_\nu = \partial_{\nu\mu} L^{\nu\mu} \qquad \text{(Möbius transf.)} \end{aligned} \tag{15}$$

This is due to the fact that for $D \geq 3$ $\partial \cdot \epsilon$ is a general linear function of x^μ so conservation of j^μ_c implies $T^\mu_\mu = -\partial_\mu K'^\mu(x)$ plus $K'^\mu(x) = -\partial_\nu L^{\nu\mu}(x)$. Instead in 2 dimension $\partial \cdot \epsilon$ is a general harmonic function and conservation implies the additional relation $L^{\nu\mu} = g^{\nu\mu} L(x)$.

Also in this case the form of (15) is independent of the particular choice of $T_{\mu\nu}$ so the necessary and sufficient condition for the existence of conserved conformal currents is that the trace of the stress tensore have the form (15).

We can say even more, we can write an equivalent stress tensor $\Theta_{\mu\nu}$ and assert that conformal invariance is equivalent to the existence of a traceless stress tensor. The equivalent stress tensor $\Theta_{\mu\nu}$ will have the form:

$$\begin{aligned} \Theta_{\mu\nu} =& T_{\mu\nu}(x) + \frac{1}{D-2}\big(\partial_\mu \partial_\alpha L^\alpha_\nu(x) + \partial_\nu \partial_\alpha L^\alpha_\mu(x) - \partial^2 L_{\mu\nu}(x) - g_{\mu\nu}\partial_\alpha \partial_\beta L^{\alpha\beta}(x)\big) \\ &+ \frac{1}{(D-2)(D-1)}\big(g_{\mu\mu}\partial^2 L^\alpha_\alpha(x) - \partial_\mu \partial_\nu L^\alpha_\alpha(x)\big) \qquad \text{in} \quad D \geq 3 \\ \Theta_{\mu\nu}(x) =& T_{\mu\nu}(x) + \frac{1}{D-1}\big(\partial_{\mu\nu} L(x) - g_{\mu\nu}\partial^2 L(x)\big) \qquad \text{in} \quad D = 2 \end{aligned}$$

and is indeed traceless:

$$\Theta^\mu_\mu = 0 \tag{16}$$

We see that a system will be scale invariant without being conformally invariant if the trace of the stress tensor is the divergence of a local operator K^μ which is not itself a conserved current plus a divergence in $D \geq 3$ or a gradient in $D = 2$. The work of Zamolodchikov and [2] have shown that for $d = 2$ under very broad conditions, namely unitarity plus a discrete spectrum of operator dimensions, scale invariance implies conformal invariance. However, this is not always the case. So, to recap, we know that if a theory is Weyl invariant it is also conformal invariant, and if a theory is conformal invariant we expect a traceless stress-tensor.

This is how we proceed in our thesis, at first we present a theory in flat space-time, we couple it to gravity and see if it is Weyl invariant through the eventual addition of non minimal terms in the action. If Weyl invariance can be achieved, we know that the theory in flat space is conformal invariant. Then we proceed to calculate the stress-tensor which is expected, and indeed verified, to be traceless.

3 The Klein-Gordon field

The action of a real, massless scalar field ϕ (the Klein-Gordon field) in D dimensions is

$$S[\phi] = \int d^D x \, \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \, . \tag{17}$$

For definiteness we use an euclidean signature. Dimensional analysis fixes the mass dimension of ϕ to be $\Delta = \frac{D-2}{2}$. There are no dimensionful parameters in the action, and the model enjoys the scale invariance induced by a constant rescaling of the coordinates:

$$\begin{aligned} x'^\mu &= \lambda x^\mu \\ \phi'(x') &= \lambda^{-\Delta} \phi(x) = \lambda^{\frac{2-D}{2}} \phi(x) \, . \end{aligned} \tag{18}$$

The propagator is fixed by Poincarè and scale invariance for $D \geq 3$, up to a normalization constant α

$$\langle \phi(x) \phi(y) \rangle = \frac{\alpha}{|x-y|^{D-2}} \, , \tag{19}$$

while in $D = 2$ infrared divergences require an infrared cutoff determined by a mass scale μ, and the propagator takes the form

$$\langle \phi(x) \phi(y) \rangle = -\frac{1}{2\pi} \ln(\mu |x-y|) \, . \tag{20}$$

The constant α may be found by deducing the two-point function from the path integral

$$\langle \phi(x) \phi(y) \rangle = \frac{1}{Z} \int D\phi \, \phi(x) \phi(y) \, e^{-S[\phi]} \tag{21}$$

where $Z = \int D\phi \, e^{-S[\phi]}$. This two-point function is contained in the generating functional of correlation functions

$$Z[J] = \int D\phi \, e^{-S[\phi] + J\phi} \tag{22}$$

where we used the shorthand notation $J\phi = \int d^Dx\, J(x)\phi(x)$, with $J(x)$ an arbitrary function known as the source. Completing squares one finds (with a similar shorthand notation)

$$Z[J] = Ne^{-\frac{1}{2}J\Box^{-1}J} \tag{23}$$

where $N = Det^{-\frac{1}{2}}(-\Box)$ is a normalization constant, and $\Box^{-1}$ is the Green function of $\Box = \partial^\mu\partial_\mu$. Then, the second functional derivative produces the two-point function

$$\langle\phi(x)\phi(y)\rangle = -\Box^{-1}_{(x,y)} = \int \frac{d^Dp}{(2\pi)^D}\frac{e^{ip\cdot(x-y)}}{p^2} = \frac{\alpha}{|x-y|^{D-2}} \tag{24}$$

with the last result valid for $D \geq 3$. The normalization may be fixed by using the Gauss law in D dimensions, which gives

$$\alpha = \frac{1}{(D-2)\Sigma(S^{D-1})} = \frac{\Gamma(\frac{D}{2})}{2(D-2)\pi^{\frac{D}{2}}} \tag{25}$$

where $\Sigma(S^{D-1}) = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$ is the area of a sphere of unit radius in D dimensions.

To verify that the model is conformally invariant, we couple it to background gravity and study if the coupling can be extended to achieve Weyl invariance. In order to do so we have to follow the minimal coupling prescription. This consists of the following three rules:

1. Replace the Minkowski metric $\eta_{\mu\nu}$ by the spacetime metric tensor $g_{\mu\nu}$
2. Replace each derivative ∂_μ by the appropriate covariant derivative ∇_μ with connection $\Gamma^\rho_{\mu\nu}$
3. Use the canonical volume form with the square root determinant of the metric $dV = d^Dx\sqrt{g}$

These rules incorporate the equivalence principle of general relativity and the principle of general covariance. In this way the Lagrangian transforms as a scalar under coordinate transformations, so the matter action is invariant. The second rule is not really needed for the scalar case since $\partial_\mu\phi = \nabla_\mu\phi$. The minimal coupling is standard and given by the lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\ . \tag{26}$$

Possible non minimal terms with dimensionless coupling constants are fixed by general covariance and dimensional analysis. These terms can only depend linearly on the curvature (two derivatives can be substituted by a curvature), and one finds only a possible non minimal term of the form

$$\mathcal{L}_{nm} = \frac{1}{2}\sqrt{g}\xi R\phi^2 \tag{27}$$

where R is the scalar curvature and ξ a dimensionless coupling constant. The value of ξ is fixed by demanding the Weyl invariance of $\mathcal{L}_0 + \mathcal{L}_{nm}$, i.e. of the action

$$S[\phi;g] = S_0[\phi;g] + S_{nm}[\phi;g] = \int d^Dx\sqrt{g}\ \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi R\phi^2)\ . \tag{28}$$

Let us derive its value: under an infinitesimal Weyl transformation

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu} \ , \qquad \delta\phi = \frac{(2-D)}{2}\sigma\phi \tag{29}$$

with the scaling of ϕ obtained from the rigid scale invariance in (18), one finds

$$\delta S_0 = \int d^D x \sqrt{g}\ \frac{(D-2)}{4}\phi^2 \Box\sigma \tag{30}$$

where $\Box = \nabla^\mu \nabla_\mu$, while using the formulae in appendix for the Weyl variation of R, see eq. (158), one calculates

$$\delta S_{nm} = \int d^D x \sqrt{g}\, \xi(1-D)\phi^2 \Box\sigma \ . \tag{31}$$

The sum is invariant for

$$\xi = \frac{(D-2)}{4(D-1)} \tag{32}$$

known as the conformal value. The corresponding Weyl covariant field equation

$$(-\Box + \xi R)\phi = 0 \tag{33}$$

contains the Weyl covariant scalar operator $-\Box_0 + \xi R$, where we inserted the subfix to remind that the laplacian acts on scalar fields, and is known as the Yamabe operator in the mathematical literature. Under Weyl transformations it scales as

$$(-\Box'_0 + \xi R') = e^{-\frac{D+2}{2}\sigma} \, (-\Box_0 + \xi R) \, e^{\frac{D-2}{2}\sigma} \ . \tag{34}$$

An important operator of the theory is the stress tensor (with properties discussed earlier). It is defined by

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \ .$$

One way to compute it in flat space is to vary the action under $\delta g_{\mu\nu} = h_{\mu\nu}$, while restricting soon after to flat space, so that it is read off from

$$\delta S = -\frac{1}{2} \int d^D x \, h^{\mu\nu} T_{\mu\nu} \ . \tag{35}$$

The emerging expression may be simplified by using the equations of motion (*eom*) in flat space. Varying S_0 we find

$$\begin{aligned} \delta S_0 &= \int d^D x \Big(-\frac{1}{2} h^{\mu\nu} \partial_\mu\phi \partial_\nu\phi + \frac{1}{4} h (\partial\phi)^2 \Big) \\ &= -\frac{1}{2} \int d^D x \, h^{\mu\nu} \Big(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 \Big) \end{aligned} \tag{36}$$

while varying R in S_{nm} with the formula in (159), and integrating by parts, gives

$$\begin{aligned} \delta S_{nm} &= \int d^D x \, \frac{\xi}{2} (\partial^\mu \partial^\nu h_{\mu\nu} - \Box h)\phi^2 \\ &= -\frac{1}{2} \int d^D x \, h^{\mu\nu} \xi (\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\phi^2 \\ &= -\frac{1}{2} \int d^D x \, h^{\mu\nu} 2\xi \Big(\eta_{\mu\nu}(\partial\phi)^2 - \partial_\mu\phi\partial_\nu\phi - \phi\partial_\mu\partial_\nu\phi + eom \Big) \end{aligned} \tag{37}$$

Collecting all terms one finds

$$T_{\mu\nu} = (1-2\xi)\partial_\mu\phi\partial_\nu\phi - 2\xi\phi\partial_\mu\partial_\nu\phi + \frac{1}{2}(4\xi-1)\eta_{\mu\nu}(\partial\phi)^2 \ . \tag{38}$$

It is conserved, $\partial^\mu T_{\mu\nu} = 0$, as verified using the *eom*, and with an on-shell trace

$$T^\mu{}_\mu = \frac{1}{2}\Big(2-D+4\xi(D-1)\Big)(\partial\phi)^2 \tag{39}$$

that vanishes precisely for the conformal coupling $\xi = \frac{D-2}{4(D-1)}$, as guaranteed by Weyl invariance. All general properties of the stress tensor are thus verified by the explicit expression of $T_{\mu\nu}$.

4 The higher derivative scalar field

A free scale invariant bosonic theory in even $D = 2n$ dimensions with the boson having vanishing mass dimension is given by

$$S[\varphi] = \int d^Dx \ \frac{1}{2}\varphi(-\Box)^n\varphi \tag{40}$$

Because of infrared divergences the propagator is again expected to be logarithmic, as in $D = 2$. Scale invariance of the action is manifest.

To prove conformal invariance it is again useful to minimally couple to gravity, and then study possible nonminimal terms that could allow for Weyl invariance. One first considers $D = 4$. From the covariant

$$S_0[\varphi] = \int d^4x\sqrt{g} \ \frac{1}{2}\Box\varphi\Box\varphi \tag{41}$$

one computes the Weyl variation

$$\delta S_0[\varphi] = \int d^4x\sqrt{g} \ \Box\varphi \, 2\, \nabla^\mu\varphi\nabla_\mu\sigma \tag{42}$$

which can be rewritten placing two derivatives on σ

$$\delta S_0[\varphi] = \int d^4x\sqrt{g} \ \left(\Box\sigma\nabla_\mu\varphi\nabla^\mu\varphi - 2\nabla^\mu\nabla^\nu\sigma\nabla_\mu\varphi\nabla_\nu\varphi\right) \ . \tag{43}$$

We want to compensate this variation through the incorporation of non minimal terms with dimensionless coupling constants, exactly as we have done for the free boson action given in (17). How do we find this new counter-terms? Looking at our action we have a total of four derivatives and the first geometrical object formed by two derivatives is the Riemann tensor $R_{\mu\nu\lambda\rho}$ within his contractions: the Ricci tensor $R_{\mu\nu}$ and the scalar Ricci tensor R. Remember that every index has to be contracted in the action in order to have Lorentz covariance. So two derivatives can be substited by a curvature, leading us to the following new possible terms

$$R_{\mu\nu\lambda\rho}\phi\phi \qquad \partial^\mu\partial^\nu R_{\mu\nu}\phi\phi \qquad \Box R\phi\phi \tag{44}$$

The first improvement terms has too many indices and cannot be contracted properly with only two derivatives so, he is not a good candidate. If we look at the Ricci tensor $R_{\mu\nu}$ term we can construct a term like $R_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$,which is a good candidate. Since we want only independent terms the construction we disregard $R_{\mu\nu}\phi\partial^\mu\partial^\nu\phi$, being equivalent through integration by parts to $R_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$. The third option for this case would be $\nabla^\mu\nabla^\nu R_{\mu\nu}\phi\phi$, but this case falls within the Ricci tensor R proportional terms case, since the Bianchi identity provides the relation $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = 0$. Analogously for the scalar tensor R dependent term we don't have many choices, basically only the term $g^{\mu\nu}R\partial_\mu\phi\partial_\nu\phi$ is truly independent since all other combinations can be, by integration by parts, be brought back to this case.

We deduce that the variation of S_0 is compensated by the Weyl variation of

$$S_1[\varphi] = \int d^4x\sqrt{g}\ (\alpha R^{\mu\nu} + \beta g^{\mu\nu}R)\partial_\mu\varphi\partial_\nu\varphi \tag{45}$$

for $\alpha = -1$ and $\beta = -\frac{1}{3}$. Thus the complete Weyl invariant action is given by

$$S[\varphi; g] = \int d^4x\sqrt{g}\left(\frac{1}{2}\Box\varphi\Box\varphi + (-R^{\mu\nu} - \frac{1}{3}g^{\mu\nu}R)\partial_\mu\varphi\partial_\nu\varphi\right) \tag{46}$$

also known as the local Riegert action. The resulting equations of motion contains the so-called Paneitz operator

$$\left(\Box^2 + 2\nabla_\mu(R^{\mu\nu} + \frac{1}{3}g^{\mu\nu}R)\partial_\nu\right)\varphi = 0 \tag{47}$$

a Weyl invariant, fourth order, scalar operator (where all derivatives act through till reaching φ).

In order to get familiar with the calculations we can check that the variation of (45) is indeed related to $\delta S_0[\phi]$. We use the formulas (158) for the variations $R_{\mu\nu}$ and R and we find that

$$\delta S_1[\phi] = \int d^4x\sqrt{g}\Big[(-2\alpha\nabla^\mu\partial^\nu\sigma - g^{\mu\nu}\Box\sigma)\partial_\mu\phi\partial_\nu\phi - 6\beta g^{\mu\nu}\Box\sigma\partial_\mu\phi\partial_\nu\phi\Big] \tag{48}$$

which lead us to the system of equation

$$\begin{aligned} -2\alpha\nabla_\mu\nabla_\nu\sigma\nabla_\mu\phi\nabla_\nu\phi - 2\nabla^\mu\nabla^\nu\sigma\nabla_\mu\phi\nabla_\nu\phi = 0 \\ (-\alpha g^{\mu\nu}\Box\sigma - 6\beta\Box\sigma + g^{\mu\nu}\Box\sigma)\nabla_\mu\phi\nabla_\nu\phi = 0 \end{aligned}$$

In this way we have 2 equations for two unknown variables which fix $\alpha = -1$ and $\beta = -\frac{1}{3}$.

It might be useful in the next sections to explore more examples of higher derivative scalar field theories, which are well known in the literature and have been largely studied.

4.1 Scalar field theory in D dimension with $\Box^2$ operator

As an example consider the theory in flat space time given by the following action:

$$S[\phi] = \int d^Dx\frac{1}{2}\Box\phi\Box\phi \tag{49}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the D'Alambertian operator. Imposing the scaling dimension $\Delta = \frac{D}{2} - 1$ we find out that the variation of this action under the conformal transformation (1) is

$$\delta S[\phi] = -\int d^D x\, \partial^\mu \Big[\epsilon_\mu \frac{1}{2}\Box\phi\Box\phi - \frac{2}{d}\partial^\nu\partial\cdot\epsilon\Big(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2\Big)\Big] \tag{50}$$

The system is indeed conformally invariant for $D \neq 2$ since the relations (15) are satisfied if we use the following definitions

$$\begin{aligned} T_{\mu\nu} &= \eta_{\mu\nu}\Big(\partial_\lambda\partial^\lambda\Box\phi + \frac{1}{2}(\Box\phi)^2\Big); \\ K_\mu &= \frac{1}{2}\Box\phi\partial_\mu\phi + \frac{\Delta}{D}\phi\partial_\mu\Box\phi; \\ L_{\mu\nu} &= \frac{1}{D}\Big(2\partial_\nu\phi\partial_\mu\phi - \eta_{\mu\nu}(\partial\phi)^2 + \Delta\eta_{\mu\nu}\phi\Box\phi\Big) \end{aligned}$$

We are now able to couple the (49) theory with gravity in a Weyl invariant way. In order to do so we use the conformally invariant operator with four derivatives, largely studied by [15]. As a result we get

$$S[\phi; g^{\mu\nu}] = \int d^D \sqrt{g}\frac{1}{2}\phi P_4(g)\phi \tag{51}$$

with P_4 being the Paneitz operator.

$$P_4(g) = \nabla^4 + \nabla^\mu\Big[\Big(\frac{4}{D-2}S_{\mu\nu} - S g_{\mu\nu}\Big)\nabla^\nu\Big] - \frac{D-2}{2(D-2)}\nabla^2 S - \frac{D-4}{(D-2)^2}S_{\mu\nu}S^{\mu\nu} + \frac{D(D-4)}{4(D-2)^2}S^2 \tag{52}$$

Remember that $S^{\mu\nu}$ is the Schouten tensor (defined in A) that vanishes for $D = 2$, this mean that the case $D = 2$ has to be treated differently. We remark that this $D = 2$ case is the first example of a theory conformally invariant that cannot be made Weyl invariant. Indeed for the scalar theory, as demonstrated in [18] and [19] the Weyl covariant analogs of $\Box^n$ exist unless the number of space-time dimensions D is even and less than $\frac{n}{2}$. Truly, this is due to the presence of divergent terms in the conformally invariant operators for $D = 2, 4, 6, ...$ We are going to show some examples of this cases.

We have seen that in the case of the Paneitz operator the coefficients in front of the Schouten tensor are divergent for $D = 2$. In this case the general ansatz for the conformally invariant operator in 2 dimension P(g) has the following form:

$$P(g) = \nabla^4 + \alpha_1\nabla^\mu(R\nabla_\mu) + \alpha_2\nabla^2 R + \alpha_3 R^2 \tag{53}$$

where all the αs are constants. It turns out that the variation of ∇^4 cannot be cancelled by the variations of the R-dependent terms, due to the presence of $(\nabla^\mu\nabla_\nu\sigma)\nabla_\mu\nabla_\nu$, this means that in 2 dimension there is no Weyl covariant generalization of the fourth-order differential operator. The system in two dimension is invariant only under global conformal transformations.

4.2 Scalar field theory in D dimension with $\Box^3$ operator

Here we give another example of a theory that is conformally invariant but not Weyl invariant. Remember that, according to [19] and [18], for an even number of dimensions D there are Weyl invariant generalizations of $\Box^n$ theories only for $n \leq \frac{D}{2}$. Therefore considering a theory with six derivatives with an action given by:

$$S[\phi] = \int d^D x \frac{1}{2} (\partial_\mu \Box \phi)^2 \tag{54}$$

cannot be made Weyl invariant for $D = 2$ and $D = 4$. This impossibility is reflected in the Weyl covariant analog of the $(\partial_\mu \Box \phi)^2$ that we shall call P_6(g). According to [16] it contains terms proportional the Schouten tensor and the Bach tensor $B_{\mu\nu}$ (this tensors are defined in A), namely:

$$P_6(g) \propto \frac{1}{(D-2)(D-4)} B_{\mu\nu} S^{\mu\nu} + \frac{1}{D-4} \nabla^\mu (B_{\mu\nu} \nabla^\nu) \tag{55}$$

As we can see this terms are divergent for $D = 2$ and $D = 4$.

However this theory is conformal in flat space time, in this case the scaling dimension of the field is $\Delta = \frac{D}{2} - 3$ and the conformal variation of the action is also a total derivative:

$$\delta S[\phi] = -\int d^D x \partial^\mu \Big[\epsilon_\mu \frac{1}{2} (\partial_\nu \Box \phi)^2 - \frac{1}{D} \partial^\nu \partial \epsilon \Big(4 \partial_\mu \partial_\nu \phi \Box \phi - \frac{1}{2} (\Box \phi)^2 (\frac{D}{2} + 3) \eta_{\mu\nu} \Big) \Big] \tag{56}$$

Moreover one can build the energy-momentum tensor for this theory $T_{\mu\nu}$ along with the tensors K_μ and $L_{\mu\nu}$. They appear in a quite long form but the important remark is that we are able to construct the improved (traceless) energy-momentum tensor $\Theta_{\mu\nu}$. We write down the form of this terms

$$\begin{aligned} T_{\mu\nu} &= \Box^2 \phi \partial_\mu \partial_\nu \phi - \left(\partial_\mu \phi \partial_\nu \Box^2 \phi + \partial_\nu \phi \partial_\mu \Box^2 \phi \right) \\ &+ \partial^\lambda \phi \partial_\mu \partial_\nu \partial_\lambda \Box \phi + \Box \phi \partial_\mu \partial_\nu \Box \phi + \partial^\lambda \Box \phi \partial_\mu \partial_\nu \partial_\lambda \phi \\ &- \partial_\mu \Box \phi \partial_\nu \Box \phi - \eta_{\mu\nu} \left[\frac{1}{2} (\partial_\lambda \Box \phi)^2 + \partial^\lambda \partial^\sigma \phi \partial_\lambda \partial_\sigma \Box \phi \right] \end{aligned}$$

as well as the operators

$$\begin{aligned} K_\mu &= \alpha \partial_\mu \partial_\nu \phi \partial_\nu \Box \phi - (D + \alpha) \partial^\nu \phi \partial_\mu \partial_\nu \Box \phi - \left(\frac{D}{2} + \alpha \right) \partial_\mu \Box \phi \Box \phi \\ &+ \left(\alpha + \frac{D}{2} + 2 \right) \partial_\mu \phi \Box^2 \phi + \left(\frac{D}{2} - 3 \right) \phi \partial_\mu \Box^2 \phi \end{aligned}$$

and

$$\begin{aligned} L_{\mu\nu} &= \left(\alpha - \frac{D-10}{4} \right) \partial_\mu \phi \partial_\nu \Box \phi - \left(\alpha + \frac{3D-10}{4} \right) \partial_\nu \phi \partial_\mu \Box \phi \\ &+ \frac{D-10}{4} \partial_\mu \partial_\nu \phi \Box \phi - \frac{D+10}{4} \phi \partial_\mu \partial_\nu \Box \phi + \frac{3D-2}{4} \eta_{\mu\nu} \phi \Box^2 \phi \end{aligned}$$

α can take arbitrary values but if we want to have $L_{\mu\nu}$ symmetric we have to set $\alpha = -\frac{D}{4}$.

4.3 Generalization for $\Box^n$ scalar field theories

The examples we considered clearly show that not any conformally invariant theory, both in flat and curved space-time, can be made Weyl invariant. As we have already mentioned the Weyl covariant generalizations of $\Box^n$ exist until for an even number of dimension the relation $n \leq \frac{D}{2}$ is provided. We said that the impossibility to construct the corresponding operators in an even number of dimensions manifest itself through the presence of terms singular at $D = 2, 4, 6, ...$ However, it seems plausible that those singular terms vanish, or at least become regular, once the geometry is restricted to that of Einstein spaces, namely:

$$R_{\mu\nu} = \frac{R}{D} g_{\mu\nu} \tag{57}$$

As a result, the corresponding limit $D \to 4, 6, ...$ exists and is invariant under conformal transformations (or only global conformal transformations for $D \to 2$).
Since flat spaces are a particular case of Einstein ones, according to the above argument, the theories whose dynamics is described by action in flat space-time:

$$S[\phi] = \int d^D x \frac{1}{2} \phi \Box^n \phi \tag{58}$$

are conformal for $D \neq 2$. This has been proven considering the variation of the action respect to conformal transformations. We can distinguish two cases, one with $n = 2m$ and the other one with $n = 2m + 1$ as:

$$S_{2m}[\phi] = \int d^D x \frac{1}{2} (\Box^m \phi)^2 \tag{59}$$

and

$$S_{2m+1}[\phi] = \int d^D x \frac{1}{2} (\partial_\mu \Box^m \phi)^2 \tag{60}$$

with their variations are respectively given by:

$$\delta_c S_{2m}[\phi] = -\int d^D x \partial_\mu \left\{ \varepsilon^\mu \frac{1}{2} (\Box^m \phi)^2 - \frac{2m^2}{D} \partial_\nu \partial \varepsilon \left[\partial^\mu \Box^{m-1} \phi \partial^\nu \Box^{m-1} \phi - \frac{1}{2} \eta^{\mu\nu} \left(\partial \Box^{m-1} \phi \right)^2 \right] \right\}$$

and

$$\begin{aligned} \delta_c S_{2m+1}[\phi] = -\int d^D x \partial_\mu \{ \varepsilon^\mu \frac{1}{2} (\partial_\mu \Box^m \phi)^2 - \frac{1}{D} \partial_\nu \partial \varepsilon [2m(m+1) \partial^\mu \partial^\nu \Box^{m-1} \phi \Box^m \phi \\ - \frac{1}{2} \eta^{\mu\nu} (\frac{D}{2} - 1 + 2m(m+1)) (\Box^m \phi)^2] \} \end{aligned}$$

being, as we can see, a total derivative in the action respect to the conformal variation.

4.4 Stress-energy tensor and the coefficient C_T for scalar theories

In any CFT, the coefficient C_T is given by the two-point function of the stress tensor, and plays a crucial role as the measure of the number of degrees of freedom, in the sense that it determines the contribution of the energy-momentum tensor in the conformal partial-wave expansion, and so is readily determined in bootstrap calculations. With our conventions the coefficient C_T appears in the formula

$$S_D^2 \langle T^{\mu\nu}(x)T^{\sigma\rho}(0)\rangle = C_T \frac{1}{(x^2)^D}\mathcal{I}^{\mu\nu,\sigma\rho}(x) \tag{61}$$

where $S_D = 2\pi^{\frac{1}{2}D}/\Gamma\left(\frac{1}{2}D\right)$ and I is the inversion tensor for symmetric traceless tensors, constructed as

$$\mathcal{I}^{\mu\nu,\sigma\rho} = \frac{1}{2}\left(I^{\mu\sigma}I^{\nu\rho} + I^{\mu\rho}I^{\nu\sigma}\right) - \frac{1}{d}\eta^{\mu\nu}\eta^{\sigma\rho}, \quad I^{\mu\nu}(x) = \eta^{\mu\nu} - \frac{2}{x^2}x^\mu x^\nu \tag{62}$$

Following the work of [16], we calculate the contributions to C_T to higher-derivative scalar theories. The energy-momentum tensor is determined from the corresponding Weyl-invariant action on curved space. As already seen, the construction of such actions is equivalent to obtaining conformal differential operators starting from powers of the Laplacian. The actions for higher-derivative free scalars considered here have the form

$$S_4[\varphi] = -\int d^Dx \frac{1}{2}\partial^2\varphi\partial^2\varphi, \quad S_6[\varphi] = -\int d^Dx \frac{1}{2}\partial^\mu\partial^2\varphi\partial_\mu\partial^2\varphi \tag{63}$$

We can calculate a symmetric traceless energy-momentum tensor by the usual Noether procedure or by exteding these actions to a general curved space background, so as to be invariant under Weyl rescalings of the metric and then reducing to flat space assuming diffeomorphism invariance. The extension to a Weyl invariant form in a general curved space is equivalent, as seen in the previous sections, to construct the Paneitz operator for S_4 and the Branson operator for S_6. Varying the metric around flat space we obtain these tensors

$$\begin{aligned} T^{\mu\nu}_{\varphi,4} =& 2\partial^\mu\partial^\nu\varphi\partial^2\varphi - \frac{1}{2}\eta^{\mu\nu}\partial^2\varphi\partial^2\varphi - \partial^\mu\left(\partial^\nu\varphi\partial^2\varphi\right) - \partial^\nu\left(\partial^\mu\varphi\partial^2\varphi\right) + \eta^{\mu\nu}\partial_\rho\left(\partial^\rho\varphi\partial^2\varphi\right) \\ &+ 2\mathcal{D}^{\mu\nu\sigma\rho}\left(\partial_\sigma\varphi\partial_\rho\varphi\right) - \frac{1}{D-1}\left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2\right)\left(\partial^\rho\varphi\partial_\rho\varphi - \frac{1}{2}(D-4)\partial^2\varphi\varphi\right) \end{aligned} \tag{64}$$

for

$$\begin{aligned} \mathcal{D}^{\mu\nu\sigma\rho} =& \frac{1}{D-2}\left(\eta^{\mu(\sigma}\partial^{\rho)}\partial^\nu + \eta^{\nu(\sigma}\partial^{\rho)}\partial^\mu - \eta^{\mu(\sigma}\eta^{\rho)\nu}\partial^2 - \eta^{\mu\nu}\partial^\sigma\partial^\rho\right) \\ &- \frac{1}{(D-2)(D-1)}\left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2\right)\eta^{\sigma\rho} \end{aligned} \tag{65}$$

where $\partial_\mu\mathcal{D}^{\mu\nu\sigma\rho} = 0, \eta_{\mu\nu}\mathcal{D}^{\mu\nu\sigma\rho} = -\partial^\sigma\partial^\rho$ and

$$\begin{aligned} T^{\mu\nu}_{\varphi,6} =& \partial^\mu\partial^2\varphi\partial^\nu\partial^2\varphi - 2\partial^\mu\partial^\nu\varphi\partial^2\partial^2\varphi - \frac{1}{2}\eta^{\mu\nu}\partial^\sigma\partial^2\varphi\partial_\sigma\partial^2\varphi \\ &+ \partial^\mu\left(\partial^\nu\varphi\partial^2\partial^2\varphi\right) + \partial^\nu\left(\partial^\mu\varphi\partial^2\partial^2\varphi\right) - \eta^{\mu\nu}\partial_\rho\left(\partial^\rho\varphi\partial^2\partial^2\varphi\right) \\ &+ 8\mathcal{D}^{\mu\nu\sigma\rho}\left(\partial_\sigma\partial_\rho\varphi\partial^2\varphi\right) - \frac{1}{D-1}\left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2\right)O \\ &+ \lambda\mathcal{D}_B^{\mu\nu\sigma\rho}\left(\partial_\sigma\varphi\partial_\rho\varphi\right) \end{aligned} \tag{66}$$

where

$$\mathcal{D}_B^{\mu\nu\sigma\rho} = \mathcal{D}^{\mu\nu\sigma\rho}\partial^2 - \frac{1}{D-1}\left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2\right)\partial^\sigma\partial^\rho \tag{67}$$

The terms in the expressions for $T^{\mu\nu}_{\varphi,4}, T^{\mu\nu}_{\varphi,6}$ involving the second and higher-order derivative operators $\mathcal{D}^{\mu\nu\sigma\rho}$, $\sigma^\mu\partial^\nu - \eta^{\mu\nu}\partial^2$ and $\mathcal{D}_B^{\mu\nu\sigma\rho}$ arise from explicit curvature dependent terms in the curved-space action and represent improvement terms to be added to the canonical energy-momentum tensor. When we generalise in a general curved space-time we obtain various terms. One of them is proportional to $\partial_\mu\varphi\partial_\nu\varphi B^{\mu\nu}$, where $B^{\mu\nu}$ is the Bach tensor. The contribution of $\mathcal{D}_B^{\mu\nu\sigma\rho}$ comes from the reduction of $B^{\mu\nu}$ and this gives

$$\lambda = -\frac{8}{D-4} \tag{68}$$

The results (65) and (66) obey the more general conservation and trace conditions,

$$\partial_\mu T^{\mu\nu}_{\varphi,2p} = (-1)^{p-1}\left(\partial^2\right)^p\varphi\partial^\nu\varphi, \quad \eta_{\mu\nu}T^{\mu\nu}_{\varphi,2p} = (-1)^{p-1}\Delta_{2p}\left(\partial^2\right)^p\varphi\varphi, \quad \Delta_{2p} = \frac{1}{2}(D-2p) \tag{69}$$

which vanish on the relevant equations of motion $(\partial^2)^p\phi = 0$. It is worthwhile writing down the correlators and operator products in these free field theories

$$\begin{aligned}\langle\varphi(x)\varphi(0)\rangle_4 &= \frac{1}{2(D-4)(D-2)S_D}\frac{1}{(x^2)^{\frac{1}{2}(D-4)}}\\ \langle\varphi(x)\varphi(0)\rangle_6 &= \frac{1}{8(D-6)(D-4)(D-2)S_D}\frac{1}{(x^2)^{\frac{1}{2}(D-6)}}\end{aligned} \tag{70}$$

These are respectively singular when $D = 4, 6$. This singularity is not a problem when we want to verify the leading term in the operator product

$$S_D T^{\mu\nu}_{\varphi,2p}(x)\varphi(0) \sim -\frac{D\Delta_{2p}}{D-1}\frac{1}{(x^2)^{\frac{1}{2}D}}\left(\frac{x^\mu x^\nu}{x^2} - \frac{1}{D}\eta^{\mu\nu}\right)\varphi(0) \tag{71}$$

because the only terms in (65) and (66) not involving $\partial\varphi$ have overall factors $D-4, D-6$. Now, to calculate C_T for both theories we need the OPE of

$$\left\langle T^{\mu\nu}_{\varphi,4}(x)T^{\sigma\rho}_{\varphi,4}(0)\right\rangle \tag{72}$$

and

$$\left\langle T^{\mu\nu}_{\varphi,6}(x)T^{\sigma\rho}_{\varphi,6}(0)\right\rangle \tag{73}$$

but in free field theories any local operator form from φ with derivatives at the same point can be decomposed in terms of conformal primaries and descendants, or derivatives, of conformal primaries of lower dimension. This ensure that, since $T^{\mu\nu}$ is a conformal primary the result is unchanged for $T^{\sigma\rho} \to T^{\sigma\rho} + \partial_\tau X^{\sigma\rho\tau}$ and dropping terms which vanish on the equations of motion we obtain

$$\begin{aligned}\left\langle T^{\mu\nu}_{\varphi,4}(x)T^{\sigma\rho}_{\varphi,4}(0)\right\rangle &= 2\left\langle T^{\mu\nu}_{\varphi,4}(x)\partial^\sigma\partial^\rho\varphi\partial^2\varphi(0)\right\rangle\\ \left\langle T^{\mu\nu}_{\varphi,6}(x)T^{\sigma\rho}_{\varphi,6}(0)\right\rangle &= -3\left\langle T^{\mu\nu}_{\varphi,6}(x)\partial^\sigma\partial^\rho\varphi\partial^2\partial^2\varphi(0)\right\rangle = 3\left\langle T^{\mu\nu}_{\varphi,6}(x)\partial^\sigma\partial^2\varphi\partial^\rho\partial^2\varphi(0)\right\rangle\end{aligned} \tag{74}$$

and remembering the relationships (61) and (68) we obtain

$$C_{T,\varphi,4} = -\frac{2D(D+4)}{(D-2)(D-1)}, \quad C_{T,\varphi,6} = \frac{3D(D+4)(D+6)}{(D-4)(D-2)(D-1)} \tag{75}$$

The generalisation of this formula for actions with more derivatives, S_{2p} is

$$C_{T,\varphi,2p} = C_{T,S}\frac{p\left(\frac{1}{2}D+2\right)_{p-1}}{\left(-\frac{1}{2}D+1\right)_{p-1}} \tag{76}$$

5 The Dirac field

In the scalar field case ϕ we have seen how to construct Weyl invariant actions through the minimal coupling prescription. We have discussed the presence of new independent counter-terms in the curved space action needed to provide Weyl invariance and presented some examples of scalar field theories in arbitrary dimension with their stress-tensors. Now we investigate if an analogous model can be introduced with the spin $\frac{1}{2}$ fermion field. At first, we present the easier case of the Dirac field with the $\nabla\!\!\!/$ operator, then we study the higher derivative cubic case $\nabla\!\!\!/^3$. We prove that the $\nabla\!\!\!/^3$ case in 3 dimension is Weyl invariant for a fermion that does not scale and with the right choices of improvement terms in the curved space action. In fact, we directly promote the $\nabla\!\!\!/^3$ theory to arbitrary dimension and determine the correct values of the coupling constants in the non minimal terms which make it Weyl invariant.

The action of a massless Dirac fermion is

$$S[\psi,\overline{\psi}] = \int d^Dx\, \overline{\psi}\partial\!\!\!/\psi \tag{77}$$

where $\partial\!\!\!/ = \gamma^a\partial_a$ (we use again a euclidean signature for definiteness). The fermion has mass dimension $\Delta = \frac{D-1}{2}$, and the action is scale invariant under

$$\begin{aligned} x'^{\mu} &= \lambda x^{\mu} \\ \psi'(x') &= \lambda^{-\Delta}\psi(x) = \lambda^{\frac{1-D}{2}}\psi(x)\ . \end{aligned} \tag{78}$$

Scale invariance extends to an invariance under the full conformal group. To see that, one may couple the model to background gravity and verifies its Weyl invariance. The coupling to gravity is obtained by using the vielbein $e_\mu{}^a$, and reads

$$S[\psi,\overline{\psi};e] = \int d^Dx e\,\overline{\psi}\nabla\!\!\!/\psi \tag{79}$$

where $\nabla\!\!\!/ = \gamma^\mu\nabla_\mu$, $\gamma^\mu = e^\mu{}_a\gamma^a$ are the gamma matrices with curved indices, $e^\mu{}_a$ is the inverse of the vielbein and e the determinant of the vielbein. The covariant derivative ∇_μ when acting on spinors contains the spin connection $\omega_{\mu ab}$ only, and reads

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\ . \tag{80}$$

The action is invariant under a Weyl rescaling

$$e'_\mu{}^a = e^\sigma e_\mu{}^a\ , \qquad \psi' = e^{\frac{(1-D)}{2}\sigma}\psi \tag{81}$$

as the Dirac operator itself scales as

$$\nabla\!\!\!/' = e^{-\frac{(D+1)}{2}\sigma}\, \nabla\!\!\!/\, e^{\frac{(D-1)}{2}\sigma} \ . \tag{82}$$

Some useful formulae for the Dirac operator are

$$\begin{aligned} \nabla\!\!\!/^2 &= \Box - \frac{1}{4}R \\ \nabla\!\!\!/^3 &= \nabla\!\!\!/(\Box - \frac{1}{4}R) = \nabla\!\!\!/\Box - \frac{1}{4}R\nabla\!\!\!/ - \frac{1}{4}(\partial_\mu R)\gamma^\mu \\ &= (\Box - \frac{1}{4}R)\nabla\!\!\!/ = \Box\nabla\!\!\!/ - \frac{1}{4}R\nabla\!\!\!/ \\ \nabla\!\!\!/\Box - \Box\nabla\!\!\!/ &= \frac{1}{4}(\partial_\mu R)\gamma^\mu \end{aligned} \tag{83}$$

where here $\Box = \Box_{\frac{1}{2}}$ is the laplacian acting on spinors.

Let us now deduce the stress tensor, now defined from varying the inverse vielbein $e^{a\mu}$ (or, equivalently, the vielbein $e_{\mu a}$ with a change of sign for consistency)

$$\delta S = \int d^Dx e\, \delta e^{a\mu}\, T_{a\mu} = -\int d^Dx e\, \delta e_{\mu a}\, T^{\mu a} \tag{84}$$

i.e.

$$T_{a\mu} = \frac{1}{e}\frac{\delta S}{\delta e^{a\mu}} \ , \qquad T^{\mu a} = -\frac{1}{e}\frac{\delta S}{\delta e_{\mu a}} \ , \qquad T^{\mu a} = e^{b\mu} T_{b\nu} e^{a\nu} \ . \tag{85}$$

For a review of the Clifford algebra and Gamma matrix manipulation see B. The energy momentum tensor, or stress tensor, is covariantly conserved, symmetric, and traceless on-shell, as consequence of diffeomorphisms, local Lorentz invariance, and Weyl symmetry, respectively

$$\nabla_\mu T^{\mu a} = 0 \ , \qquad T^{ab} = T^{ba} \ , \qquad T^a{}_a = 0 \tag{86}$$

(indices are made "curved" or "flat" by using the vierbein and its inverse). The derivation is somewhat tricky, so that it is useful to report the various steps leading to the explicit expression of the stress tensor. One has to vary the vielbein in the action

$$S = \int d^Dx e\, \overline{\psi} \nabla\!\!\!/ \psi = \int d^Dx e\, \overline{\psi} \gamma^a e_a{}^\mu \Big(\partial_\mu + \frac{1}{4}\omega_{\mu bc}\gamma^{bc}\Big)\psi \tag{87}$$

but since the fermion equations of motion can be used to simply the final answer, the variation of the vielbein determinant e does not contribute, and one is left with

$$\delta S = \int d^Dx e \Big\{ \underbrace{\delta e_a{}^\mu\, \overline{\psi}\gamma^a \nabla_\mu \psi}_{\delta S_1} + \underbrace{\frac{1}{4}\delta\omega_{\mu ab}\, \overline{\psi}\gamma^\mu\gamma^{ab}\psi}_{\delta S_2} \Big\} \tag{88}$$

that can be expressed in terms of the vielbein variation $\delta e_{\mu a}$ as

$$\begin{aligned} \delta S_1 &= \int d^Dx e\, (-\delta e_{\mu a}) \overline{\psi}\gamma^\mu \nabla^a \psi \\ \delta S_2 &= \int d^Dx e\, \frac{1}{4}\Big(e^{a\nu}(\nabla_\mu \delta e_\nu{}^b - \nabla_\nu \delta e_\mu{}^b) - e^{a\nu} e^{b\rho}(\nabla_\nu e_\rho{}^c) e_{\mu c}\Big) \overline{\psi}\gamma^\mu \gamma_{ab}\psi \\ &= \int d^Dx e\, \frac{1}{4}(\nabla_\mu \delta e_{\nu a}) \overline{\psi}\, \underbrace{(\gamma^\mu\gamma^{\nu a} - \gamma^\nu \gamma^{\mu a} - \gamma^a \gamma^{\mu\nu})}_{\tilde{\gamma}^{\mu\nu a}}\, \psi \Big\} \\ &= \int d^Dx e\, \frac{1}{4}(-\delta e_{\nu a}) \Big(\nabla_\mu \overline{\psi}\, \tilde{\gamma}^{\mu\nu a}\, \psi + \overline{\psi}\, \tilde{\gamma}^{\mu\nu a}\, \nabla_\mu \psi \Big) \end{aligned} \tag{89}$$

where in the last line we have integrated by parts. Now, one can use the identities

$$\begin{aligned}\tilde{\gamma}^{\mu\nu a} &= \gamma^\mu\gamma^\nu\gamma^a - \eta^{\mu\nu}\gamma^a - \eta^{\mu a}\gamma^\nu + \eta^{\nu a}\gamma^\mu \\ &= \gamma^\nu\gamma^a\gamma^\mu + \eta^{\nu a}\gamma^\mu + \eta^{\nu\mu}\gamma^a - 3\eta^{\mu a}\gamma^\nu\end{aligned} \tag{90}$$

where η indicates either the metric or the vielbein depending on the indices it contains. Then one can use the equations of motion and get

$$\delta S_2 = \int d^Dxe\,\frac{1}{4}\delta e_{\nu a}\Big(\nabla^\nu\overline{\psi}\,\gamma^a\psi + \nabla^a\overline{\psi}\,\gamma^\nu\psi - \overline{\psi}\,\gamma^a\nabla^\nu\psi + 3\overline{\psi}\,\gamma^\nu\nabla^a\psi\Big) \tag{91}$$

so that adding all pieces

$$\begin{aligned}\delta S = \delta S_1 + \delta S_2 &= \int d^Dxe\,\delta e_{\mu a}\frac{1}{4}\Big(\nabla^\mu\overline{\psi}\,\gamma^a\psi + \nabla^a\overline{\psi}\,\gamma^\mu\psi - \overline{\psi}\,\gamma^\mu\nabla^a\psi - \overline{\psi}\,\gamma^a\nabla^\mu\psi\Big)\\ &= \int d^Dxe\,\delta e_{\mu a}\Big[-\frac{1}{4}\overline{\psi}(\gamma^\mu\overset{\leftrightarrow}{\nabla}{}^a + \gamma^a\overset{\leftrightarrow}{\nabla}{}^\mu)\psi\Big] = -\int d^Dxe\,\delta e_{\mu a}\,T^{\mu a}\end{aligned} \tag{92}$$

and the stress tensor is

$$T^{\mu a} = \frac{1}{4}\overline{\psi}(\gamma^\mu\overset{\leftrightarrow}{\nabla}{}^a + \gamma^a\overset{\leftrightarrow}{\nabla}{}^\mu)\psi\ . \tag{93}$$

One may check that it is symmetric, conserved and traceless on-shell.

The flat space limit is obvious, but it may be useful to rederive it directly in flat space, as this procedure could be used in more demanding models with higher derivatives. One varies the vielbein $\delta e_\mu{}^a \equiv c_\mu{}^a$, as in the A, and restricts soon after to flat space. As before we use the equations of motion, so that it is not necessary to vary the vielbein determinant e. We find (all indices are now equivalent, corresponding to the cartesian coordinates of flat space)

$$\delta S = \int d^Dx\Big\{(-c^{a\mu})\overline{\psi}\gamma_a\partial_\mu\psi + \underbrace{\frac{1}{4}\delta\omega_{\mu ab}\,\overline{\psi}\gamma^\mu\gamma^{ab}\psi}_{\delta S_2}\Big\} \tag{94}$$

with δS_2 calculated using (173) as

$$\begin{aligned}\delta S_2 &= \int d^Dx\,\frac{1}{4}(\partial_\mu c_{ab} - \partial_a c_{\mu b} - \partial_a c_{b\mu})\overline{\psi}\gamma^\mu\gamma^{ab}\psi\\ &= \int d^Dx\,\frac{1}{4}(\partial_\mu c_{ab})\overline{\psi}\,\underbrace{(\gamma^\mu\gamma^{ab} - \gamma^a\gamma^{\mu b} - \gamma^b\gamma^{\mu a})}_{\tilde{\gamma}^{\mu ab}}\,\psi\Big\}\\ &= \int d^Dx\,\frac{1}{4}(-c_{ab})\Big(\partial_\mu\overline{\psi}\,\tilde{\gamma}^{\mu ab}\,\psi + \overline{\psi}\,\tilde{\gamma}^{\mu ab}\,\partial_\mu\psi\Big)\ .\end{aligned} \tag{95}$$

Now in the first term we use the identity

$$\tilde{\gamma}^{\mu ab} = \gamma^\mu\gamma^a\gamma^b - \eta^{\mu a}\gamma^b - \eta^{\mu b}\gamma^a + \eta^{ab}\gamma^\mu\ , \tag{96}$$

and in the second one the equivalent identity

$$\tilde{\gamma}^{\mu ab} = \gamma^a\gamma^b\gamma^\mu + \eta^{ab}\gamma^\mu + \eta^{a\mu}\gamma^b - 3\eta^{\mu b}\gamma^a\ , \tag{97}$$

and imposing the equations of motion one is left with

$$\delta S_2 = \int d^D x \, \frac{1}{4} c_{ab} \Big(\partial^a \overline{\psi} \gamma^b \psi + \partial^b \overline{\psi} \gamma^a \psi + \overline{\psi} (3\gamma^a \partial^b - \gamma^b \partial^a) \psi \Big) \, . \tag{98}$$

Thus, the full action varies as

$$\begin{aligned} \delta S &= \int d^D x \, \frac{1}{4} c_{ab} \Big(\partial^a \overline{\psi} \gamma^b \psi + \partial^b \overline{\psi} \gamma^a \psi - \overline{\psi} (\gamma^a \partial^b + \gamma^b \partial^a) \psi \Big) \\ &= -\frac{1}{4} \int d^D x \, c_{ab} \, \overline{\psi} (\gamma^a \overset{\leftrightarrow}{\partial^b} + \gamma^b \overset{\leftrightarrow}{\partial^a}) \psi = - \int d^D x \, c_{ab} \, T^{ab} \end{aligned} \tag{99}$$

so that

$$T^{ab} = \frac{1}{4} \overline{\psi} (\gamma^a \overset{\leftrightarrow}{\partial^b} + \gamma^b \overset{\leftrightarrow}{\partial^a}) \psi \tag{100}$$

consistently with (93). Evidently, it is conserved, symmetric, and traceless on-shell.

It is useful to report the infinitesimal form of the background local symmetries that guarantees the above properties, working directly in curved space. They take the form

$$\begin{aligned} \delta e_\mu{}^a &= \epsilon^\nu \partial_\nu e_\mu{}^a + (\partial_\mu \epsilon^\nu) e_\nu{}^a + \omega^a{}_b e_\mu{}^b + \sigma e_\mu{}^a \\ \delta \psi &= \epsilon^\mu \partial_\mu \psi + \frac{1}{4} \omega_{ab} \gamma^{ab} \psi + \frac{1-D}{2} \sigma \psi \end{aligned} \tag{101}$$

where $\epsilon^\mu, \omega_{ab}$, and σ are the infinitesimal local parameters of the Einstein, local Lorentz, and Weyl symmetries, respectively. Under the Weyl symmetry with local parameter $\sigma(x)$, the invariance of the action gives

$$\begin{aligned} \delta_\sigma S &= \int d^D x \left(\frac{\delta S}{\delta e_\mu{}^a(x)} \delta_\sigma e_\mu{}^a(x) + \frac{\delta_R S}{\delta \psi(x)} \delta_\sigma \psi(x) + \delta_\sigma \overline{\psi}(x) \frac{\delta_L S}{\delta \overline{\psi}(x)} \right) \\ &= \int d^D x e \, T^\mu{}_a(x) \delta_\sigma e_\mu{}^a(x) = \int d^D x e \, T^\mu{}_a(x) \sigma(x) e_\mu{}^a(x) \\ &= \int d^D x e \, T^a{}_a(x) \sigma(x) = 0 \end{aligned} \tag{102}$$

where the equations of motion of the spinor field have been used (employing left and right derivatives for the Grassmann valued fields). Thus local Weyl invariance implies tracelessness of the stress tensor (recall that the infinitesimal function $\sigma(x)$ is arbitrary). Similarly, the Lorentz symmetry with local parameters $\omega_{ab}(x)$ implies

$$\begin{aligned} \delta_\omega S &= \int d^D x \left(\frac{\delta S}{\delta e_\mu{}^a(x)} \delta_\omega e_\mu{}^a(x) + \frac{\delta_R S}{\delta \psi(x)} \delta_\omega \psi(x) + \delta_\omega \overline{\psi}(x) \frac{\delta_L S}{\delta \overline{\psi}(x)} \right) \\ &= \int d^D x e \, T^\mu{}_a(x) \delta_\omega e_\mu{}^a(x) = \int d^D x e \, T^\mu{}_a(x) \omega^a{}_b(x) e_\mu{}^b(x) \\ &= \int d^D x e \, T^{ba}(x) \omega_{ab}(x) = 0 \end{aligned} \tag{103}$$

which constrains the antisymmetric part of the stress tensor to vanish on-shell. Finally, conservation of the stress tensor arises as consequence of the infinitesimal diffeomorphism invariance

$$
\begin{aligned}
\delta_\epsilon S &= \int d^D x \left(\frac{\delta S}{\delta e_\mu{}^a(x)} \delta_\epsilon e_\mu{}^a(x) + \frac{\delta_R S}{\delta \psi(x)} \delta_\epsilon \psi(x) + \delta_\epsilon \overline{\psi}(x) \frac{\delta_L S}{\delta \overline{\psi}(x)} \right) \\
&= \int d^D x e\, T^\mu{}_a(x) \mathcal{L}_\epsilon e_\mu{}^a(x) = \int d^D x e\, T^\mu{}_a(x) \nabla_\mu \epsilon^a(x) \\
&= -\int d^D x e\, \epsilon_a(x) \nabla_\mu T^{\mu a}(x) = 0
\end{aligned}
\tag{104}
$$

where in the second line we have added to the Lie derivative (the transformation rule of the vierbein) a spin connection term (as it drops out once the stress tensor is symmetric), and then integrated by parts.

Thus, the stress tensor in flat space in is conserved, symmetric, and traceless

$$
\partial_a T^{ab} = 0 \ , \qquad T^{ab} = T^{ba} \ , \qquad T^a{}_a = 0 \tag{105}
$$

and can be used to construct the charges of the full conformal group.

The Einstein and local Lorentz invariant of the action are obvious, because of the tensor formalism used. Invariance under the Weyl symmetry must instead be verified by direct computation. The Weyl transformation rules are contained in (101) (the σ dependent terms). On the spin connection they induce the transformation

$$
\delta \omega_\mu{}^{ab} = (e_\mu{}^a e^{\nu b} - e_\mu{}^b e^{\nu a}) \partial_\nu \sigma \tag{106}
$$

that is used to verify the Weyl invariance of the action (note that $\gamma_\mu \gamma^{\mu\nu} = (D-1)\gamma^\nu$).

6 The higher derivative Dirac field: the cubic case

The lagrangian of a free massless higher derivative Dirac fermion is given by

$$
\mathcal{L} = \overline{\psi} \partial\!\!\!/^n \psi \ . \tag{107}
$$

where $n > 1$. The choice of $n = D$ is particularly interesting, as then ψ has vanishing mass dimension, which reminds of the property of the Liouville field and its higher dimensional extension.

Could these models be conformal? To start with, we may consider the special case $n = D$. For $n = 1$ the model is certainly conformal, while for $n = 2$ it is not conformal: eliminating the gamma matrices and using well-known 2D Weyl formulas one cannot construct a Weyl invariant action, even using non-minimal terms.

The next interesting case concerns $n = 3$. Let us consider directly the model for arbitrary D

$$
S[\psi, \overline{\psi}] = \int d^D x \ \overline{\psi} \partial\!\!\!/^3 \psi \tag{108}
$$

which is covariantized to curved space by

$$
S_0[\psi, \overline{\psi}] = \int d^D x e \ \overline{\psi} \nabla\!\!\!\!/^3 \psi \ . \tag{109}
$$

We wish to study how it varies under the Weyl symmetry, and test if there exist improvement terms that make the model Weyl invariant. Iterating (82) one finds the following Weyl transformation for the n-th power of the Dirac operator

$$\nabla\!\!\!\!/'^{n} = e^{-\frac{D+1}{2}\sigma}\nabla\!\!\!\!/ e^{-\sigma}\nabla\!\!\!\!/ e^{-\sigma}\cdots\nabla\!\!\!\!/ e^{-\sigma}\nabla\!\!\!\!/ e^{\frac{D-1}{2}\sigma} . \tag{110}$$

The infinitesimal Weyl symmetry can be written as

$$\delta e_{\mu a} = \sigma e_{\mu a} \ , \quad \delta\omega_{\mu ab} \ = e_{\mu a}\partial_b\sigma - e_{\mu b}\partial_a\sigma \ , \quad \delta\psi = -\Delta\sigma\psi = \frac{(3-D)}{2}\sigma\psi \tag{111}$$

Note also that the derivative term on σ from the variation of $\nabla\!\!\!\!/$ arises from

$$\gamma^\mu\frac{1}{4}\delta\omega_{\mu ab}\gamma^{ab} = \frac{1}{2}\gamma^\mu e_{\mu a}(\partial_b\sigma)\gamma^{ab} = \frac{(D-1)}{2}(\partial\!\!\!/\sigma) . \tag{112}$$

Then, we compute the following Weyl variations

$$\begin{aligned}
\delta\psi &= \frac{(3-D)}{2}\sigma\psi \\
\delta\nabla\!\!\!\!/\psi &= \frac{(1-D)}{2}\sigma\nabla\!\!\!\!/\psi + (\partial\!\!\!/\sigma)\psi \\
\delta\nabla\!\!\!\!/^2\psi &= -\frac{(1+D)}{2}\sigma\nabla\!\!\!\!/^2\psi + \nabla\!\!\!\!/[(\partial\!\!\!/\sigma)\psi] \\
\delta\nabla\!\!\!\!/^3\psi &= -\frac{(3+D)}{2}\sigma\nabla\!\!\!\!/^3\psi - (\partial\!\!\!/\sigma)\nabla\!\!\!\!/^2\psi + \nabla\!\!\!\!/^2[(\partial\!\!\!/\sigma)\psi] \\
&= -\frac{(3+D)}{2}\sigma\nabla\!\!\!\!/^3\psi - (\partial\!\!\!/\sigma)\Box\psi + \Box[(\partial\!\!\!/\sigma)\psi] \\
&= -\frac{(3+D)}{2}\sigma\nabla\!\!\!\!/^3\psi + (\Box\partial\!\!\!/\sigma)\psi + 2(\nabla^\mu\partial\!\!\!/\sigma)\nabla^\mu\psi
\end{aligned} \tag{113}$$

from which we calculate the variation of S_0. Despite the dependence on the dimension D the variation result independent of D.

$$\delta S_0[\psi,\overline{\psi}] = \int d^Dxe[(\Box\partial_\mu\sigma)\overline{\psi}\gamma^\mu\psi + 2(\nabla_\mu\partial_\nu\sigma)\overline{\psi}\gamma^\mu\nabla^\nu\psi] \tag{114}$$

In order to have Weyl invariance we add new terms to the action and calculate their variations, these new terms have to be a combination of curvature tensor which count as two derivatives and covariant derivatives with each index contracted, in order to keep the Lorentz invariance. One finds that the only non minimal terms are given by the action

$$S_{nm} = \int d^Dxe[\alpha_1 R\overline{\psi}\nabla\!\!\!\!/\psi + \alpha_2(\nabla_\mu R)\overline{\psi}\gamma^\mu\psi + \alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi] \tag{115}$$

The variation of this non minimal terms action is quite complicated and laborious but we divide it step by step. At first we have the variation of the determinant of the vielbein term which gives us

$$\int d^DxD\sigma e\Big[\alpha_1 R\overline{\psi}\nabla\!\!\!\!/\psi + \alpha_2(\nabla_\mu R)\overline{\psi}\gamma^\mu\psi + \alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi\Big] \tag{116}$$

while each variation of the alphas term can be computed according to the (158) and the variation for the ψ and $\overline{\psi}$ given above. For the α_1 term we find

$$\begin{aligned}\delta S_{nm(1)} = \int d^D x e\alpha_1 \Big[(-2\sigma R + 2(1-D)\Box\sigma)\overline{\psi}\nabla\!\!\!\!/\,\psi + \frac{(3-D)}{2}\sigma\alpha_1 R\overline{\psi}\nabla\!\!\!\!/\,\psi \\ + \alpha_1 R\overline{\psi}(\frac{1-D}{2}\sigma\nabla\!\!\!\!/\,\psi + (\partial\!\!\!/\sigma)\psi\Big]\end{aligned} \tag{117}$$

For the α_2 term we have

$$\delta S_{nm(2)} = \int d^D x e\alpha_2 \Big[\nabla_\mu(-2\sigma R + 2(1-D)\Box\sigma)\overline{\psi}\gamma^\mu\psi - \sigma\nabla_\mu R\overline{\psi}\gamma^\mu\psi + (3-D)\sigma\nabla_\mu R\overline{\psi}\gamma^\mu\psi\Big] \tag{118}$$

this can be rewritten remembering that $\partial_\mu\Box\sigma = \Box\partial_\mu\sigma - R_{\mu\nu}\partial^\nu\sigma$ and the last and more tricky α_3 term

$$\begin{aligned}\delta S_{nm(3)} = \int d^D x e\alpha_3 \Big[(2-D)\nabla_\mu\partial_\nu\sigma\overline{\psi}\gamma^\mu\nabla^\nu\psi - g_{\mu\nu}\Box\sigma\overline{\psi}\gamma^\mu\nabla^\nu\psi \\ + \frac{3-D}{2}\sigma R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi - \sigma R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi + R_{\mu\nu}\overline{\psi}\gamma^\mu\delta(\nabla^\nu\psi)\Big]\end{aligned} \tag{119}$$

It is the case to investigate the variation of the last term in the $\delta S_{nm(3)}$ action

$$e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\delta(\nabla^\nu\psi) \tag{120}$$

this can be written using (173) as

$$\begin{aligned}e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\delta(\nabla^\nu\psi) &= e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\delta(g^{\nu\rho}\nabla_\rho\psi) \\ &= e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\delta(g^{\nu\rho})\nabla_\rho\psi + e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu g^{\nu\rho}\delta(\partial_\rho\psi + \frac{1}{4}\omega_{\rho ab}\gamma^{ab}\psi) \\ &= -2\sigma e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi + \frac{1}{4}e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu g^{\nu\rho}\Big(e_{\rho a}\nabla_b\sigma\gamma^{ab}\psi - e_{\rho b}\nabla_a\sigma\gamma^{ab}\psi\Big) \\ &+ \frac{3-D}{2}e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu(\partial^\nu\sigma)\psi + \frac{3-D}{8}\sigma e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\omega_{\rho ab}\gamma^{ab}\psi \\ &+ \frac{3-D}{2}\sigma e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\partial^\nu\psi\end{aligned} \tag{121}$$

Now, using the relationship for the product of two gamma matrices

$$\gamma^\mu\gamma^{ab} = \gamma^b\eta^{\mu a} - \gamma^a\eta^{\mu b} \tag{122}$$

we finally obtain

$$\begin{aligned}e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\delta(\nabla^\nu\psi) = -2\sigma e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi + \frac{1}{2}e\alpha_3 R\overline{\psi}(\nabla\!\!\!\!/\,\sigma)\psi \\ - \frac{1}{2}e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\nu(\nabla^\mu\sigma)\psi + \frac{3-D}{2}e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu(\partial^\nu\sigma)\psi \\ + \frac{3-D}{2}\sigma e\alpha_3 R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi\end{aligned} \tag{123}$$

Now we add all this variations for δS_{nm} to (114) and see if the total variation can be set to vanish. We get a system of five equations for three unknown coefficients

$\alpha_1, \alpha_2, \alpha_3$

$$\Big[\alpha_1 2(1-D)\sigma - \alpha_3\sigma\Big]\Box\sigma\overline{\psi}\gamma^\mu\psi = 0$$
$$\Big[\alpha_1 + \frac{1}{2}\alpha_3 - 2\alpha_2\Big]R\overline{\psi}\partial\!\!\!/\psi = 0$$
$$\Big[1 + \alpha_2 2(1-D)\Big](\Box\partial_\mu\sigma)\overline{\psi}\gamma^\mu\psi = 0$$
$$\Big[2 + \alpha_3(2-D)\Big]\nabla_\mu\partial_\nu\sigma\overline{\psi}\gamma^\mu\nabla^\nu\psi = 0$$
$$\Big[-\frac{1}{2}\alpha_3 + \frac{3-D}{2}\alpha_3 - 2(1-D)e\alpha_2\Big]R_{\mu\nu}\overline{\psi}\partial^\nu\sigma\gamma^\mu\psi = 0$$

We get complete cancellation for

$$\alpha_1 = -\frac{1}{(D-2)(D-1)} \ , \quad \alpha_2 = \frac{1}{2(D-1)} \ , \quad \alpha_3 = \frac{2}{(D-2)} \ , \quad D = 3 \tag{124}$$

The last of the five equations gives us the ratio between α_2 and α_3, which is

$$\frac{\alpha_2}{\alpha_3} = \frac{(D-2)}{4(D-1)} \tag{125}$$

For $D = 3$ these values reduce to

$$\alpha_1 = -\frac{1}{2} \ , \quad \alpha_2 = \frac{1}{4} \ , \quad \alpha_3 = 2 \tag{126}$$

so that the action

$$S = \int d^3xe \left[\overline{\psi}\nabla\!\!\!\!/^3\psi - \frac{1}{2}R\overline{\psi}\nabla\!\!\!\!/\psi + \frac{1}{4}(\partial_\mu R)\overline{\psi}\gamma^\mu\psi + 2R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi\right] \tag{127}$$

is Weyl invariant (with ψ that does not scale). The equations of motion are

$$\left(\nabla\!\!\!\!/^3 - \frac{1}{2}R\nabla\!\!\!\!/ + \frac{1}{4}(\partial\!\!\!/ R) + 2R_{\mu\nu}\gamma^\mu\nabla^\nu\right)\psi = 0 \tag{128}$$

In general, for $D \geq 3$, one gets the Weyl invariant field equations

$$\left(\nabla\!\!\!\!/^3 - \frac{1}{(D-2)(D-1)}R\nabla\!\!\!\!/ + \frac{1}{2(D-1)}(\partial\!\!\!/ R) + \frac{2}{(D-2)}R_{\mu\nu}\gamma^\mu\nabla^\nu\right)\psi = 0 \tag{129}$$

that follow from the Weyl invariant action

$$\boxed{S = \int d^Dxe \left[\overline{\psi}\nabla\!\!\!\!/^3\psi - \frac{1}{(D-2)(D-1)}R\overline{\psi}\nabla\!\!\!\!/\psi + \frac{1}{2(D-1)}(\partial_\mu R)\overline{\psi}\gamma^\mu\psi + \frac{2}{(D-2)}R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi\right]} \ . \tag{130}$$

Alternatively, using the identities in (83) one can write the equations (or the action) as

$$\left(\Box_{\frac{1}{2}}\nabla\!\!\!\!/ - \Big(\frac{1}{4} + \frac{1}{(D-2)(D-1)}\Big)R\nabla\!\!\!\!/ + \frac{1}{2(D-1)}(\partial\!\!\!/ R) + \frac{2}{(D-2)}R_{\mu\nu}\gamma^\mu\nabla^\nu\right)\psi = 0 \tag{131}$$

or

$$\left(\nabla\!\!\!\!/\Box_{\frac{1}{2}} - \Big(\frac{1}{4} + \frac{1}{(D-2)(D-1)}\Big)R\nabla\!\!\!\!/ + \Big(\frac{1}{2(D-1)} - \frac{1}{4}\Big)(\partial\!\!\!/ R) + \frac{2}{(D-2)}R_{\mu\nu}\gamma^\mu\nabla^\nu\right)\psi = 0 \ . \tag{132}$$

This last form reproduces the $D = 4$ result of [20] and [21], which appears also in the supersymmetric version of the $D = 4$ Liouville theory of [22].

6.1 Construction method for improvement terms in the action

Here we aim to explain the construction of non-minimal terms for the action

$$S_0[\psi,\overline{\psi}] = \int d^D x e \overline{\psi} \nabla\!\!\!\!/^3 \psi \tag{133}$$

These counter-terms provide Weyl invariance through their variations. Not only these new terms have to contain, of course, the Dirac matter field and the equivalent of three derivatives (for scale invariance), but also they have form a scalar under general coordinate and local Lorentz transformations: this means that every index has to be properly contracted. Indeed we know that the current-term $\overline{\psi}\gamma^\mu\psi$ for the Dirac field scales as $-\Delta = \frac{3-D}{2}$, while the vielbein scales as D. For this reason we need three derivatives for scale invariance. On a Riemannian manifold, the only useful geometrical objects containing a pair of derivatives are the Riemann tensor $R^{\mu\alpha\beta\gamma}$, the Ricci tensor $R^{\mu\nu}$ and the scalar tensor R. Starting from the scalar tensor we can write down a term like

$$\nabla_\mu R \overline{\psi}\gamma^\mu\psi \tag{134}$$

where the derivative ∇_μ can act on R, ψ and $\overline{\psi}$ (while covariant derivatives of γ^μ vanish), so that we obtain three different terms

$$R(\nabla_\mu\overline{\psi})\gamma^\mu\psi \ , \quad R\overline{\psi}\gamma^\mu(\nabla_\mu\psi) \ , \quad (\partial_\mu R)\overline{\psi}\gamma^\mu\psi \tag{135}$$

However, one can always perform partial integration in the action and remove the derivative acting on $\overline{\psi}$, leaving us with two possible terms

$$T_1 = R\overline{\psi}\gamma^\mu\nabla_\mu\psi \qquad T_2 = (\partial_\mu R)\overline{\psi}\gamma^\mu R \ . \tag{136}$$

Now we consider the Ricci tensor $R^{\mu\nu}$, we need two more gamma-matrices to provide Lorentz invariance. This means that in this case we can write down a term like

$$\nabla_\lambda R_{\mu\nu} \ \overline{\psi}\gamma^\mu\gamma^\nu\gamma^\lambda\psi \tag{137}$$

where lower indices can be contracted with upper indices also in different ways, and the derivative can act again on all fields including $R_{\mu\nu}$. We can again free $\overline{\psi}$ from derivatives by partial integration, finding structures of the form

$$R_{\mu\nu}\overline{\psi}\gamma^\mu\gamma^\nu\gamma^\lambda\nabla_\lambda\psi \ , \qquad (\nabla_\lambda R_{\mu\nu})\overline{\psi}\gamma^\mu\gamma^\nu\gamma^\lambda\psi \tag{138}$$

Now, by gamma matrices properties we know that $\gamma^\mu\gamma^\nu\gamma^\lambda = \gamma^{\mu\nu\lambda} + g^{\mu\nu}\gamma^\lambda - g^{\mu\lambda}\gamma^\nu + g^{\nu\lambda}\gamma^\mu$. The Ricci tensor is symmetric, which means that in the first term, when contracted with the completely antisymmetric matrix $\gamma^{\mu\nu\lambda}$ is zero, while the contractions with the metric $g_{\mu\nu}$ give one term proportional to T_1 and a new term

$$T_3 = R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi \tag{139}$$

Analogously for second structre, we have the contraction of an antisymmetric object with the Ricci tensor that gives zero, while the contractions with the metric $g_{\mu\nu}$ are proportional to T_2 through the Bianchi identity $\nabla^\mu\big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\big) = 0$. The last possible geometrical object is given by the Riemann tensor $R_{\mu\nu\lambda\rho}$ but this case gives no more independent terms. We have terms from

$$\nabla_\alpha R_{\mu\nu\lambda\rho}\ \overline{\psi}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma^\alpha\psi \tag{140}$$

with the derivative that can only on ψ and the Riemann tensor. Again, indices can be contracted otherwise. The product of five gamma matrices

$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma^\alpha \tag{141}$$

can be reduced to linear combinations of

$$\gamma^{\mu\nu\lambda\rho\alpha}\ ,\gamma^{\mu\nu\lambda}g^{\rho\alpha}\ ,\gamma^\mu g^{\nu\lambda}g^{\rho\alpha} \tag{142}$$

The first term with the totally antisymmetric $\gamma^{\mu\nu\lambda\rho\alpha}$ with five indices vanishes because of the symmetries of the Riemann tensor, while the terms with a γ^μ reduces to the previous structures. Similarly the terms with three indices $\gamma^{\mu\nu\lambda}$ vanish since it is always reduced to a contraction with the last three indices of the Riemann tensor, namely $R_{\mu[\nu\alpha\rho]} = 0$. Analogously, if the derivative had acted on $R_{\mu\nu\lambda\rho}$ we would have not found more independent terms.

6.2 Stress tensor for the higher derivative Dirac field: cubic case

The action (130) defines in flat space a CFT. Being a higher derivative theory, it is expected to be non-unitary. Here we wish to identify its stress tensor. It is a very demanding and laborious calculation, which we simplify a bit by restricting to flat space. We start considering (109), we vary it and restrict soon after to flat space. Since the fermion equations of motion can be used to simply the final answer, the variation of the vielbein determinant e does not contribute, and one is left with

$$\begin{aligned}\delta S = \int x^D e\Big\{ & c^\mu_a\overline{\psi}\gamma^a\nabla_a\Box\psi + \overline{\psi}\gamma^\mu\partial_\mu c^\nu_b\gamma^b\nabla_\nu\nabla\!\!\!\!/\,\psi + \overline{\psi}\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu c^\rho_c\gamma^c\nabla_c\psi \\ & + \underbrace{\frac{1}{4}\overline{\psi}\Box\delta\omega_{\rho ab}\gamma^{ab}\psi}_{\delta S_{\omega_1}} + \underbrace{\frac{1}{4}\overline{\psi}\gamma^\mu\delta\omega_{\mu ab}\gamma^{ab}\Box\psi}_{\delta S_{\omega_2}} + \underbrace{\frac{1}{4}\overline{\psi}\gamma^\mu\partial_\mu(\gamma^\nu\delta\omega_{\nu ab}\gamma^{ab}\nabla\!\!\!\!/\,\psi)}_{\delta S_{\omega_3}}\Big\}\end{aligned} \tag{143}$$

At first we consider the δS_ω terms. Using the relation (172) for $\delta\omega$, several usages of integrations by parts and the techique used in the Dirac field for the gamma-matrices manipulation, we reach the formulas

$$\delta S_{\omega_1} = \int d^Dx\frac{1}{4}c_{ab}\Big(\partial^a\overline{\psi}\gamma^b\Box\psi + \partial^b\overline{\psi}\gamma^a\Box\psi + 3\overline{\psi}\gamma^a\partial^b\Box\psi - \overline{\psi}\gamma^b\partial^a\Box\psi\Big)$$

$$\begin{aligned}\delta S_{\omega_2} = \int d^Dx - \frac{1}{4}c_{ab}\Big(&\partial^a\partial_\mu\overline{\psi}\gamma^\mu\gamma^b\gamma^\rho\partial_\rho\psi + \partial^b\partial_\mu\overline{\psi}\gamma^\mu\gamma^a\gamma^\rho\partial_\rho\psi + 3\partial_\mu\overline{\psi}\gamma^\mu\gamma^a\partial^b\gamma^\rho\partial_\rho\psi \\ & - \partial_\mu\overline{\psi}\gamma^\mu\gamma^b\partial^a\gamma^\rho\partial_\rho\psi\Big)\end{aligned}$$

$$\delta S_{\omega_3} = \int d^D x \frac{1}{4} c_{ab} \Big(\partial^a \Box \overline{\psi} \gamma^b \psi + \gamma^b \Box \overline{\psi} \gamma^a \psi + 3 \Box \psi \gamma^a \partial^b \psi - \Box \overline{\psi} \gamma^b \partial^a \psi \Big)$$

Now we need to evaluate the variation of the improvement terms using (157). This procedure leads us to the formula

$$\delta S_{nm} = \int d^D x \alpha_1 \delta R \overline{\psi} \gamma^\mu \nabla_\nu \psi + \alpha_2 (\partial_\mu \delta R) \overline{\psi} \gamma^\mu \psi + \alpha_3 \delta R_{\mu\nu} \overline{\psi} \gamma^\mu \nabla^\nu \psi \tag{144}$$

Eventually, after several integrations by parts, used in order to leave underivated c_{ab}, and omitting terms proportional to the equations of motion, we obtain

$$\begin{aligned}
\delta S_{nm} = - \int d^D x \ c^{\rho\sigma} \Big\{ & \alpha_1 (\partial_\rho \partial_\sigma \overline{\psi}) \gamma^\mu \partial_\mu \psi + \overline{\psi} \partial_\rho \partial_\sigma \gamma^\mu \partial_\mu \psi - g_{\rho\sigma} \Box \overline{\psi} \gamma^\mu \partial_\mu \psi \\
& - \alpha_2 (\partial_\rho \partial_\sigma \overline{\psi} \gamma^\mu \partial_\mu \psi + \partial_\mu \partial_\sigma \overline{\psi} \gamma^\mu \partial_\rho \psi + \partial_\sigma \overline{\psi} \gamma^\mu \partial_\mu \partial_\rho \psi \\
& + \partial_\mu \partial_\rho \overline{\psi} \gamma^\mu \partial_\sigma \psi + \partial_\rho \overline{\psi} \gamma^\mu \partial_\mu \partial_\sigma \psi + \partial_\mu \overline{\psi} \gamma^\mu \partial_\rho \partial_\sigma \psi) \\
& + \alpha_2 \eta_{\rho\sigma} \Big[\partial_\mu (\Box \overline{\psi} \gamma^\mu \psi + \overline{\psi} \gamma^\mu \Box \psi + 2 \partial_\mu \overline{\psi} \gamma^\mu \partial_\nu \psi) \Big] \\
& + \frac{\alpha_3}{2} (\partial_\mu \partial_\rho \overline{\psi} \gamma^\mu \nabla_\sigma \psi + \partial_\rho \overline{\psi} \gamma^\mu \partial_\mu \partial_\sigma \psi + \partial_\mu \overline{\psi} \gamma^\mu \partial_\rho \partial_\sigma \psi) \\
& + \frac{\alpha_3}{2} (\partial_\mu \partial_\rho \overline{\psi} \gamma_\sigma \partial^\nu \psi + \partial_\nu \overline{\psi} \gamma^\sigma \partial_\rho \partial_\nu \psi + \partial_\rho \overline{\psi} \gamma^\sigma \partial_\mu \partial^\mu \psi) \\
& - \frac{\alpha_3}{2} (\Box \overline{\psi} \gamma_\rho \partial_\sigma \psi + 2 \partial_\mu \overline{\psi} \gamma_\rho \partial^\mu \partial_\sigma \psi) \\
& - \frac{\alpha_3}{2} \eta_{\rho\sigma} (\partial_\mu \partial_\nu \overline{\psi} \gamma^\mu \partial^\nu \psi + \partial_\nu \overline{\psi} \gamma^\mu \partial_\mu \partial^\nu \psi + \partial_\mu \overline{\psi} \gamma^\mu \partial_\nu \partial^\nu \psi) \Big\} \\
+ (\rho \leftrightarrow \sigma))
\end{aligned} \tag{145}$$

From the definition of stress tensor

$$\delta S = \int d^D x e \, \delta e^{a\mu} \, T_{a\mu} = - \int d^D x e \, \delta e_{\mu a} \, T^{\mu a} \tag{146}$$

we find

$$
\begin{aligned}
T_{ab} =& -\overline{\psi}\gamma_a\partial_b\Box\psi + \partial_c\overline{\psi}\gamma_c\gamma_a\partial_b\gamma^d\partial_d\psi - \Box\overline{\psi}\gamma_a\partial_b\psi \\
&+ \frac{1}{4}\Big(\partial_a\overline{\psi}\gamma_b\Box\psi + \partial_b\overline{\psi}\gamma_a\Box\psi + 3\overline{\psi}\gamma_a\partial_b\Box\psi - \overline{\psi}\gamma_b\partial_a\Box\psi\Big) \\
&- \frac{1}{4}\Big(\partial_a\partial_c\overline{\psi}\gamma^c\gamma_b\gamma^d\partial_d\psi + \partial^b\partial_c\overline{\psi}\gamma^c\gamma^a\gamma^d\partial_d\psi + 3\partial_c\overline{\psi}\gamma^c\gamma^a\partial^b\gamma^d\partial_d\psi \\
&- \partial_c\overline{\psi}\gamma^c\gamma^b\partial^a\gamma^d\partial_d\psi\Big) \\
&+ \frac{1}{4}\Big(\partial_a\Box\overline{\psi}\gamma_b\psi + \partial_b\Box\overline{\psi}\gamma_a\psi + 3\Box\psi\gamma_b\partial_a\psi - \Box\overline{\psi}\gamma_b\partial_a\psi\Big) \\
&- \Big\{\alpha_1(\partial_a\partial_b\overline{\psi})\gamma^c\partial_c\psi + \overline{\psi}\partial_a\partial_b\gamma^c\partial_c\psi - g_{ab}\Box\overline{\psi}\gamma^c\partial_c\psi \\
&- \alpha_2(\partial_a\partial_b\overline{\psi}\gamma^c\partial_c\psi + \partial_c\partial_b\overline{\psi}\gamma^c\partial_a\psi + \partial_b\overline{\psi}\gamma^c\partial_c\partial_a\psi \\
&+ \partial_c\partial_a\overline{\psi}\gamma^c\partial_b\psi + \partial_a\overline{\psi}\gamma^c\partial_c\partial_b\psi + \partial_c\overline{\psi}\gamma^c\partial_a\partial_b\psi) \\
&+ \alpha_2\eta_{ab}\Big[\partial_c(\Box\overline{\psi}\gamma^c\psi + \overline{\psi}\gamma^c\Box\psi + 2\partial_d\overline{\psi}\gamma^c\partial^d\psi)\Big] \\
&+ \frac{\alpha_3}{2}(\partial_c\partial_a\overline{\psi}\gamma^c\nabla_b\psi + \partial_a\overline{\psi}\gamma^c\partial_c\partial_b\psi + \partial_c\overline{\psi}\gamma^c\partial_a\partial_b\psi) \\
&+ \frac{\alpha_3}{2}(\partial_c\partial_a\overline{\psi}\gamma_b\partial^c\psi + \partial_c\overline{\psi}\gamma^b\partial_a\partial_c\psi + \partial_a\overline{\psi}\gamma^b\partial_c\partial^c\psi) \\
&- \frac{\alpha_3}{2}(\Box\overline{\psi}\gamma_a\partial_b\psi + 2\partial_c\overline{\psi}\gamma_a\partial^c\partial_b\psi) \\
&- \frac{\alpha_3}{2}\eta_{ab}(\partial_c\partial_d\overline{\psi}\gamma^c\partial^d\psi + \partial_d\overline{\psi}\gamma^c\partial_c\partial^d\psi + \partial_c\overline{\psi}\gamma^c\partial_d\partial^d\psi)\Big\} + (a \leftrightarrow b)
\end{aligned}
\tag{147}
$$

Reporting the stress-tensor T_{ab} we have decided not to group similar terms; in this way it is possible to see the form of each single term. This could help the reader, since the calculation is really long. We note that (147) is symmetric and we wish to verify that the trace of the stress-tensor vanishes, as we expect from the theory studied in section 2.1. In order to do so we calculate T^a_a and compare similar terms. A lot of them are proportional to the equations of motion when contracted to find T^a_a and they will be set to zero. We obtain 4 relations involving the coefficients α_1, α_2 and α_3, which we expect vanish in order to provide $T^a_a = 0$.

$$
\begin{aligned}
&(2\alpha_1 - 2\alpha_1 D - 2\alpha_2 + 2\alpha_2 D - \alpha_3 - 1)(\Box\overline{\psi}\gamma^a\partial_a\psi) \\
&(4\alpha_1 - 4D\alpha_1 - 4\alpha_2 + 4\alpha_2 D - D\alpha_3)(\partial_a\overline{\psi}\gamma^b\partial_b\partial^b\psi) \\
&(-4\alpha_2 + 4\alpha_2 D + 2\alpha_3 - D\alpha_3)(\partial_a\partial_b\overline{\psi}\gamma^a\partial^b\psi) \\
&(-2\alpha_2 + 2\alpha_2 D + 2\alpha_3 - D\alpha_3 + 1)(\partial_c\overline{\psi}\gamma^c\Box\psi)
\end{aligned}
\tag{148}
$$

Doing the calculation every single relation vanishes, and we conclude that the stress-tensor is traceless. This result was expected from general arguments, of course, but its verification constitutes a good check on our identification of the stress tensor. The stress tensor is an important operator in CFT, and one could start studying its correlation functions.

7 Conclusions and speculations

We have studied higher derivative fermionic theories, focusing on the case of a model with cubic derivatives. Our original motivation was inspired by the work of Tom Levi and Yaron Oz [8], that analyzed a scalar field of vanishing mass dimensions in higher dimensions, having a kinetic term with higher derivatives in the action, which defines a conformally invariant field theory. After having revisited Weyl invariant scalar theories, and described how a scale invariant theory in flat space can be proven to be conformally invariant, we have focused on fermionic theories with higher derivatives. In particular, we have noticed how a fermion with vanishing scaling dimension is scale invariant in three dimensions if it contains a kinetic term cubic in derivatives. This model can be extended to higher dimensions, maintaining scale invariance by assigning a scaling transformation also to the fermion fields. Thus, considering the action

$$S[\psi,\overline{\psi}] = \int d^D x \ \overline{\psi}\partial\!\!\!/^3\psi \tag{149}$$

we have shown how to couple it to curved space and make it Weyl invariant by finding appropriate non minimal terms. The resulting action is given by

$$\begin{aligned} S = \int d^D x e \bigg[& \overline{\psi}\nabla\!\!\!\!/^3\psi - \frac{1}{(D-2)(D-1)} R\overline{\psi}\nabla\!\!\!\!/\psi + \frac{1}{2(D-1)}(\partial_\mu R)\overline{\psi}\gamma^\mu\psi \\ & + \frac{2}{(D-2)} R_{\mu\nu}\overline{\psi}\gamma^\mu\nabla^\nu\psi \bigg] \end{aligned} \tag{150}$$

The corresponding Weyl invariant field equations are

$$\left(\nabla\!\!\!\!/^3 - \frac{1}{(D-2)(D-1)} R\nabla\!\!\!\!/ + \frac{1}{2(D-1)}(\partial\!\!\!/ R) + \frac{2}{(D-2)} R_{\mu\nu}\gamma^\mu\nabla^\nu\right)\psi = 0 \tag{151}$$

The singularity at $D = 2$ indicates that such a model is not Weyl invariant in two dimensions. Restricting this model to flat space, we find a new example of free CFT that, though non-unitary, might find interesting applications in physics.

In CFT a crucial operator is given by the stress tensor, so that we have calculated it from the above action obtaining the expression in (147). It contains many terms, but we have verified its consistency by checking that it is indeed traceless, as required in CFTs.

It could be useful to extend our work, by computing the two-point functions of the stress tensor in the quantum theory to identify the central charge coefficient C_T of our theory, as has been done in [16] for analogous scalar field models.

Greetings

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A Curvatures and variational formulae

A.1 Metric

Given a metric $g_{\mu\nu}$, we define the curvature tensors by

$$[\nabla_\mu, \nabla_\nu]V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma \ , \quad R_{\mu\nu} = R_{\rho\mu}{}^\rho{}_\nu \ , \quad R = R_\mu{}^\mu \tag{152}$$

where the covariant derivative is defined on vector fields by

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu \ , \qquad \nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho \tag{153}$$

with the connection fixed by requiring the metric to be covariantly constant $\nabla_\rho g_{\mu\nu} = 0$. The latter gives

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{154}$$

and a useful formula for the Riemann curvature is

$$R_{\mu\nu}{}^\rho{}_\sigma = \overline{\nabla}_\mu \Gamma^\rho_{\nu\sigma} - \overline{\nabla}_\nu \Gamma^\rho_{\mu\sigma} \tag{155}$$

where $\overline{\nabla}_\mu$ contains a connection only for the upper index (the connection is not a tensor, and thus its covariant derivative is not defined).

These explicit expressions allow to derive variational formulae induced by a metric deformation $\delta g_{\mu\nu}$. We consider $\delta g_{\mu\nu} = g'_{\mu\nu} - g_{\mu\nu}$ as the fundamental variation, and define the tensor $h_{\mu\nu} = \delta g_{\mu\nu}$ on which indices can be raised (and lowered) by the metric, e.g.

$$h^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} h_{\rho\sigma} = g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} \tag{156}$$

so that, for example, the variation of the inverse metric is $\delta g^{\mu\nu} = -h^{\mu\nu}$.

With this notation, we have the following variations (with $\Box = \nabla^\mu \nabla_\mu$ and $h = h^\mu{}_\mu$)

$$\boxed{\begin{aligned}
\delta\Gamma^\rho_{\mu\nu} &= \frac{1}{2}(\nabla_\mu h_\nu{}^\rho + \nabla_\nu h_\mu{}^\rho - \nabla^\rho h_{\mu\nu}) \\
\delta R_{\mu\nu}{}^\rho{}_\sigma &= \nabla_\mu \delta\Gamma^\rho_{\nu\sigma} - \nabla_\nu \delta\Gamma^\rho_{\mu\sigma} \\
&= \frac{1}{2}(\nabla_\mu \nabla_\nu h_\sigma{}^\rho + \nabla_\mu \nabla_\sigma h_\nu{}^\rho - \nabla_\mu \nabla^\rho h_{\nu\sigma} - (\mu \leftrightarrow \nu)) \\
\delta R_{\mu\nu} &= \frac{1}{2}(\nabla^\rho \nabla_\mu h_{\nu\rho} + \nabla^\rho \nabla_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \nabla_\mu \nabla_\nu h) \\
\delta R &= -h^{\mu\nu} R_{\mu\nu} + \nabla^\mu \nabla^\nu h_{\mu\nu} - \Box h \ .
\end{aligned}} \tag{157}$$

They may be specialized to a Weyl variation

$$\boxed{\begin{aligned}
\delta g_{\mu\nu} &\equiv h_{\mu\nu} = 2\sigma g_{\mu\nu} \\
\delta\Gamma^\rho_{\mu\nu} &= \delta^\rho_\mu \partial_\nu \sigma + \delta^\rho_\nu \partial_\mu \sigma - g_{\mu\nu} \nabla^\rho \sigma \\
\delta R_{\mu\nu}{}^\rho{}_\sigma &= \delta^\rho_\nu \nabla_\mu \partial_\sigma \sigma + g_{\mu\sigma} \nabla_\nu \nabla^\rho \sigma - (\mu \leftrightarrow \nu) \\
\delta R_{\mu\nu} &= (2 - D) \nabla_\mu \partial_\nu \sigma - g_{\mu\nu} \Box \sigma \\
\delta R &= 2(1 - D) \Box \sigma - 2\sigma R \ .
\end{aligned}} \tag{158}$$

Finally, one may restrict the formula in (157) to flat space after variation, which is useful for example to get the stress tensor directly in flat space (with metric in cartesian coordinates denoted by $\eta_{\mu\nu}$). We find

$$\boxed{\begin{aligned}\delta\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}(\partial_\mu h_\nu{}^\rho + \partial_\nu h_\mu{}^\rho - \partial^\rho h_{\mu\nu}) \\ \delta R_{\mu\nu}{}^\rho{}_\sigma &= \partial_\mu \delta\Gamma^\rho_{\nu\sigma} - \partial_\nu \delta\Gamma^\rho_{\mu\sigma} \\ &= \frac{1}{2}(\partial_\mu\partial_\nu h_\sigma{}^\rho + \partial_\mu\partial_\sigma h_\nu{}^\rho - \partial_\mu\partial^\rho h_{\nu\sigma} - (\mu \leftrightarrow \nu)) \\ \delta R_{\mu\nu} &= \frac{1}{2}(\partial_\mu\partial^\rho h_{\rho\nu} + \partial_\nu\partial^\rho h_{\rho\mu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h) \\ \delta R &= \partial^\mu\partial^\nu h_{\mu\nu} - \Box h\ .\end{aligned}} \tag{159}$$

Let me now define some standard tensor fields useful for the creation of conformally invariant operators:

1. the Schouten tensor $S_{\mu\nu} = \frac{1}{D-2}(R_{\mu\nu} - \frac{R}{2(D-1)} g_{\mu\nu})$,
2. the Weyl tensor

$$\begin{aligned}W_{iklm} = R_{iklm} &+ \frac{1}{D-2}\left(R_{im}g_{kl} - R_{il}g_{km} + R_{kl}g_{im} - R_{km}g_{il}\right) \\ &+ \frac{1}{(D-1)(D-2)} R\left(g_{il}g_{km} - g_{im}g_{kl}\right)\end{aligned} \tag{160}$$

3. the Bach tensor

$$B_{ab} = S_{cd}W^{cd}_{ab} + \nabla^c\nabla_c S_{ab} - \nabla^c\nabla_a S_{bc} \tag{161}$$

A.2 Vielbein

Similar formulae can be written down for the vielbein $e_\mu{}^a$, which fixes the metric by

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}\ . \tag{162}$$

One gains a local Lorentz symmetry in tangent space, acting on frame indices (or flat indices) $a, b...$, etc. The covariant derivative acting on a generic Lorentz tensor V requires a connection $\omega_{\mu ab}$, called the spin connection

$$\nabla_\mu V = \partial_\mu V + \frac{1}{2}\omega_{\mu ab}M^{ab}V \tag{163}$$

where M^{ab} are the generators of the Lorentz group, normalized as

$$[M^{ab}, M^{cd}] = \eta^{bc}M^{ad} + \dots \tag{164}$$

For vectors $(M^{ab})^c{}_d = \eta^{ac}\delta^b_d - \eta^{bc}\delta^a_d$, while for Dirac spinors $M^{ab} = \frac{1}{2}\gamma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$. We have assumed the tensor V to be a scalar under diffeomorphisms, otherwise additional connections should be added. The spin connection is fixed by requiring the vielbein to be covariantly constant (the "vielbein postulate")

$$\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma^\rho_{\mu\nu} e_\rho{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \tag{165}$$

form which one derives the explicit formula

$$\omega_\mu{}^{ab} = \frac{1}{2}e^{a\nu}(\partial_\mu e_\nu{}^b - \partial_\nu e_\mu{}^b) - (a \leftrightarrow b) - \frac{1}{2}e^{a\nu}e^{b\rho}(\partial_\nu e_\rho{}^c - \partial_\rho e_\nu{}^c)e_{\mu c}\,. \tag{166}$$

The corresponding curvature is defined by

$$[\nabla_\mu, \nabla_\nu] = \frac{1}{2}R_{\mu\nu ab}(\omega)M^{ab} \tag{167}$$

and is equivalent to the Riemann tensor since by the vielbein postulate it follows that

$$[\nabla_\mu, \nabla_\nu]e_\rho{}^a = 0 \quad \rightarrow \quad R_{\mu\nu\rho a}(\Gamma) = R_{\mu\nu\rho a}(\omega) \tag{168}$$

with the nature of indices (flat or curved) transformed as usual by the vielbien and its inverse.

Let us now derive variational formulae induced by a deformation of the vielbein $\delta e_\mu{}^a$. In the same spirit as before, we consider

$$\delta e_\mu{}^a = e'_\mu{}^a - e_\mu{}^a \equiv c_\mu{}^a \tag{169}$$

as the fundamental variation, whose indices may be raised, lowered and transformed by the metric and vielbein. In particular, the variation of the inverse vielbein $e_a{}^\mu$ is given by

$$\delta e_a{}^\mu = -c_a{}^\mu\,. \tag{170}$$

The metric variation $h_{\mu\nu}$ is related to the vielbein variation $c_\mu{}^a$ by

$$h_{\mu\nu} = c_{\mu\nu} + c_{\nu\mu} \tag{171}$$

and in addition we find

$$\boxed{\delta\omega_{\mu ab} = \frac{1}{2}(\nabla_\mu c_{ab} - \nabla_a c_{\mu b} - \nabla_a c_{b\mu}) - (a \leftrightarrow b)} \tag{172}$$

with an obvious flat space limit. Finally, this formula may be specialized to Weyl variations as

$$\boxed{\begin{aligned} \delta e_\mu{}^a &\equiv c_\mu{}^a = \sigma e_\mu{}^a \\ \delta\omega_{\mu ab} &= e_{\mu a}\partial_b\sigma - e_{\mu b}\partial_a\sigma \end{aligned}} \tag{173}$$

where $\partial_a = e_a{}^\mu\partial_\mu$, of course. For convention we will refer to the determinant of the metric $g_{\mu\nu}$ as g and to the determinant of the vielbein as e, the following relationship is valid

$$e := \det\left(e^a_\mu\right) = \sqrt{|g|} \tag{174}$$

with their variations given by

$$\boxed{\begin{aligned} \delta\sqrt{g} &= \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu} \\ \delta e &= e\delta e^a_\mu e^\mu_a \end{aligned}} \tag{175}$$

We also define the stress-tensor as

$$\boxed{T^{\mu\nu} = -\frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g_{\mu\nu}}, \quad T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}}} \tag{176}$$

this form fermions is given by the formula

$$\boxed{T^{\mu v} \equiv \frac{\delta S}{\delta e_{\alpha\mu}} e^v_\alpha} \tag{177}$$

B Clifford algebras and spinors

In this section we describe the main features of the Clifford algebra and the most useful relationships used in this work. The Clifford algebra is generated by a set of γ-matrices, which satisfies the anti-commutaion relations

$$\gamma^\mu\gamma^\nu + \gamma^v\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1} \tag{178}$$

and which plays an important role in supersymmetric and supergravity theories. We discuss the Clifford algebra associated with the Lorentz group in D dimensions and we start with a general and explicit construction of the generating γ-matrices. It is easier to construct Euclidean γ matrices, which satisfy (178) using the tensor product $\otimes$

$$\begin{aligned}
\gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes 1 \otimes \ldots \\
\gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \ldots \\
\gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \ldots
\end{aligned} \tag{179}$$

These matrices are all hermitian with squares equal to the identity matrix and they mutually anti-commute. If $D = 2m$ is even, then we need m factors in this construction to obtain γ^μ. This implies that we obtain a representation of dimension $2^{\frac{D}{2}}$. If $D = 2m+1$ we need one additional matrix and we take γ^{2m+1} from the list above, but we keep only the first m factors deleting a σ_1. The construction (178) gives us Euclidean γ-matrices and in order to obtain Lorentzian γ matrices, we just multiply γ^0 it by i. The hermiticity properties of the Lorentzian γ matrices are summarized by

$$\gamma^{\mu^\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{180}$$

The full Clifford algebra consists of the identity $\mathbb{1}$, the D generating elements γ^μ, plus all independent matrices formed from products of the generators. By (178) symmetric products of γ-matrices reduce to fewer γ-matrices, so the new elements must be antysimmetric products. For this reason we define

$$\gamma^{\mu_1 \ldots \mu_r} = \gamma^{[\nu_1} \ldots \gamma^{\mu_r]} \tag{181}$$

For example

$$\gamma^{\mu v} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \tag{182}$$

Each new product contains an overall factor of $\frac{1}{r!}$ which is usually factoralized. There are C^D_r(binomial coefficients) independent index choices at rank r so that, for even dimension, the Clifford algebra is of dimension 2^D.

B.1 Practical γ-matrix manipulation

We list some useful tricks to multiply γ-matrices, D will denote the dimension of the Clifford Algebra. Consider the first products with index contractions

$$\gamma^{\mu\nu}\gamma_\nu = (D-1)\gamma^\mu \tag{183}$$

This is valid because ν runs over all values except μ, so there are $D-1$ terms in the sum. In the same way we have

$$\gamma^{\mu\nu\rho}\gamma_\rho = (D-2)\gamma^{\mu\nu} \tag{184}$$

and so on until we recognize the general formula

$$\gamma^{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}\gamma_{\nu_s\ldots\nu_1} = \frac{(D-r)!|}{(D-r-s)!}\gamma^{\mu_1\ldots\mu_r} \tag{185}$$

Note that the second γ on the left side has its indices in opposite order,so that no signs appear when contracting the indices. It is useful to remember the general order reversal symmetry, which is

$$\gamma^{\nu_1\ldots\nu_r} = (-)^{r(r-1)/2}\gamma^{\nu_r\ldots\nu_1} \tag{186}$$

In the case of products with indices not contracted we have similar combinatorial tricks such as

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots,\nu_D} = D(D-1)\delta^{\mu_2\mu_1}_{[\nu_1\nu_2}\gamma_{\nu_3\ldots\nu_d]} \tag{187}$$

Indeed, the index μ_1 and μ_2 appear in the sef of $\{\nu_i\}$. There are D possibilities for μ_2 and D-1 possibilities for μ_1, since μ_1 should be different from μ_2. δ functions are always normalized with weight 1, i.e.

$$\delta^{\mu_2\mu_1}_{\nu_1\nu_2} = \frac{1}{2}\left(\delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\right) \tag{188}$$

C Conformal change of the Riemann curvature

In this appendix we derive the basic identities which describe the behaviour of the Levi-Civita connection ∇ , the Riemann curvature tensor R, the Ricci curvature tensor Ric and the scalar curvature τ with respect the conformal changes of the metric. Here we give up on the notation used until now to introduce a new mathematical one, since it is more pratical. We first recall how the Riemann curvature tensor of a manifold (M^D, g) transforms under $g \longrightarrow \hat{g} = e^{2\phi}g$. A torsion-free metric connection on a Riemannian manifold is necessarily given by the Koszul formula:

$$\begin{aligned}2g(\nabla_X Y, Z) =& X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) \\ &+ g([Z,Y],X)\end{aligned} \tag{189}$$

For all vector fields X,Y $\in \chi(M)$. At this point it is straightforward to demonstrate the transformation of the curvature:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad R(X,Y,Z,W) = g(R(X,Y)Z,W) \tag{190}$$

It follows that :

$$\begin{aligned}\hat{R}(X,Y)Z = R(X,Y)Z + g(\nabla_X(grad\,\phi), Z)Y - g(\nabla_Y(grad\,\phi), Z)X \\ + \nabla_Y(grad\,\phi)g(X,Z) - \nabla_X(grad\,\phi)g(Y,Z) \\ + (Y(\phi)Z(\phi) - g(Y,Z)|grad\,\phi|^2)X \\ - (X(\phi)Z(\phi) - g(X,Z)|grad\,\phi|^2)Y \\ + (X(\phi)g(Y,Z) - Y(\phi)g(X,Z))grad\,\phi\end{aligned} \tag{191}$$

Using (191) it is possible to deduce the transformation of objects like Ric or the scalar curvature τ. In this way we see that :

$$\begin{aligned}e^{-2\phi}\hat{R}(X,Y,Z,W) &= e^{-2\phi}\hat{g}(\hat{R}(X,Y)Z,W) \\ &= R(X,Y,Z,W) + \xi(X,Z)g(Y,W) - \xi(Y,Z)g(X,W) \\ &+ \xi(Y,W)g(X,Z) - \xi(X,W)g(Y,Z)\end{aligned} \tag{192}$$

here ξ is:

$$\xi(X,Y) = \xi_{g,\phi}(X,Y) \equiv g(\nabla_X(grad\phi), Y) - X(\phi)Y(\phi) + \frac{1}{2}|grad\phi|^2 g(X,Y) \tag{193}$$

In this way we obtain for the Ricci Tensor:

$$Ric(X,Y) = \sum_i R(X,e_i,e_i,Y) = \sum_i g(R(X,e_i)e_i,Y)$$

$$\begin{aligned}\hat{Ric}(X,Y) &= Ric(X,Y) - (D-2)\xi(X,Y) - g(X,Y)\sum_i \chi(e_i,e_i) \\ &= Ric(X,Y) - (D-2)g(\nabla_X grad\,\phi, Y) + g(X,Y)\Delta\phi - (D-2)|grad\,\phi|^2 g(X,Y) \\ &+ (D-2)X(\phi)Y(\phi)\end{aligned} \tag{194}$$

where:

$$-\Delta\phi = \sum_i g(\nabla_{e_i}(grad\,\phi), e_i)$$

(194) yields to the transformation rules of the scalar $\tau = \sum_i Ric(e_i, e_i)$:

$$\hat{\tau} = e^{-2\phi}(\tau + (2D-2)\Delta\phi - (D-2)(D-1)|grad\,\phi|^2) \tag{195}$$

At this point i can finally introduce the Schouten tensor S

$$S = \frac{1}{D-2}(Ric - \frac{\tau}{2(D-1)}g) \tag{196}$$

and his transformation rule:

$$\begin{aligned}\hat{S}(X,Y) &= S(X,Y) - \frac{1}{2}|grad\,\phi|^2 g(X,Y) - g(\nabla_X(grad\,\phi), Y) + X(\phi)Y(\phi) \\ &= S(X,Y) - \xi(X,Y)\end{aligned} \tag{197}$$

Let me now define some standard tensor fields, starting from the curvature, useful for the creation of conformally invariant operators:

1. Riemann curvature tensor $R(X, Y, Z, W) := g(R(X, Y)Z, W)$,
2. the Ricci tensor $Ric(X, Y) := Tr_g(g(R(X, -)-, Y)$,
3. the scalar curvature $\tau := Tr_g(Ric(-.-))$.
4. the normalized scalar curvature $J := \frac{\tau}{2(D-1)}$
5. the Schouten tensor $S(X, Y)$,
6. the Weyl tensor W(X,Y,Z,W):=$R(X, Y, Z, W) + (S \otimes g)(X, Y, Z, W)$,
7. the Cotton tensor $C(X, Y, Z) := (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$,
8. the Bach tensor $B(X, Y) = Tr_g((\nabla C)(-, X, Y) + g(S, W(-, X, Y, -))$

where in the definition of the Weyl tensor we have introduced the Kulkarni-Nomizu product $\otimes$ of two symmetric (0,2) tensors K_1, K_2 , defined as:

$$
\begin{aligned}
(K_1 \otimes K_2) :=& K_1(X, Z)K_2(Y, W) + K_1(Y, W)K_2(X, Z) \\
&- K_1(X, W)K_2(Y, Z) - K_1(Y, Z)K_2(X, W)
\end{aligned}
$$

C.1 Conformally covariant operators

Conformal transformations in Riemann geometry preserve angles between tangent vectors at every point x on a Riemann manifold M. A Riemann metric g_1 is conformally equivalent to a metric g_0 if :

$$g_{1ij}(x) = e^{w(x)} g_{0ij}(x) \tag{198}$$

$e^{w(x)}$ is a positive function on the manifold M and is called a conformal factor. We define a conformal class $[g_0]$ of a metric g_0 as the set of all the metrics of the form $\{(e^{w(x)} g_0 : w(x) \in C^\infty(M))\}$. As explained in [15] the Uniformization theorem for compact Riemann surfaces says that on such a surface, in every conformal class there exists a metric of constant Gauss curvature: the corresponding statement in dimension $D \geq 3$, known as the Yamabe Problem, stipulates that in every conformal class there exists a metric of costant scalar curvature.

Conformally covariant differential operators include the Laplacian in dimension two, as well as the conformal Laplacian, Paneitz operator and other higher order operators in dimension $D \geq 3$.

Their defining property is the transformation law under a conformal change of the metric: there exist a,b $\in \mathbb{R}$ such that if g_1 and g_0 are related as in (C.1),then :

$$P_{g_1} = e^{aw} P_{g_0} e^{bw} \tag{199}$$

These operators have been widely studied and today we know that the Laplace-Beltrami operator Δ_g of a Riemann manifold (M, g) is invariant respect to isometries and in two dimensions is also invariant respect to conformal changes $g \to e^{\phi} g$ of the metric.

In dimension $D \geq 3$ this is not true, so we define the Yamabe operator as :

$$P_2(g) = \Delta_g - (\frac{D}{2} - 1) J_g \tag{200}$$

The Yamabe operator is conformally invariant , its transformation rule is :

$$e^{(\frac{D}{2}+1)\phi} \circ P_2(e^{2\phi}g) = P_2(g) \circ e^{(\frac{D}{2}-1)\phi} \tag{201}$$

In the 1980's a conformally covariant operator of the form $\Delta^2 + ...$ terms with fewer than four derivatives was discovered by Paneitz,Eastwood-Singer and Riegert.

D Correlation functions for general dimensions free field theories

Following the work of [23] we aim to review the requirements of conformal invariance for two point functions of conserved currents for general dimension D. We will review the main relations between the two point propagator and the coefficients C_V and C_T for free field theories in both the scalar and fermionic case. We are looking for general requirements that a two-point function has to satisfy under conformal transformations. In order to do so, we remember that conformal transformations may be defined as coordinate transformations preserving the infinitesimal euclidean lenght element up to a local scale factor Ω^g,

$$x_\mu \to x'_\mu(x) = (gx)_\mu, \quad dx'_\mu dx'_\mu = \Omega^g(x)^{-2} dx_\mu dx_\mu \tag{202}$$

where g is a conformal transformation. Using the language of group theory we note that for any conformal transformation g we may define a local orthogonal transformation by

$$\mathcal{R}^g_{\mu\alpha}(x) = \Omega^g(x)\frac{\partial x'_\mu}{\partial x_\alpha}, \quad \mathcal{R}^g_{\mu\alpha}(x)\mathcal{R}^g_{\nu\alpha}(x) = \delta_{\mu\nu} \tag{203}$$

which in D dimensions is an element of $O(D)$,$\mathcal{R}^{g'}(gx)\mathcal{R}^g(x) = \mathcal{R}^{g'g}(x), \mathcal{R}^g(x)^{-1} = \mathcal{R}^{g^{-1}}(gx)$. The set of conformal transformations $\{g\}$ forms the conformal group which is isomorphic to $O(D+1,1)$ and is composed by convential constant rotations and traslations plus scale transformations and special conformal transformations. Constant rotations and translations form the group $O(D) \propto T_D$

$$x'_\mu = R_{\mu\nu}x_\nu + a_\mu, \quad R_{\mu\alpha}R_{\nu\alpha} = \delta_{\mu\nu} \tag{204}$$

Constant scale transformations form the Dilatation group K

$$x'_\mu = \lambda x_\mu, \quad \Omega^g(x) = \lambda^{-1} \tag{205}$$

and special conformal transformations

$$x'_\mu = \frac{x_\mu + b_\mu x^2}{\Omega^g(x)}, \quad \Omega^g(x) = 1 + 2b\cdot x + b^2x^2 \tag{206}$$

The full conformal group may be generated by combining rotations and translations with an inversion through the origin represented by the discrete element $i, i^2 = 1$. We may write

$$x'_\mu = (ix)_\mu = \frac{x_\mu}{x^2}, \quad \mathcal{R}^i_{\mu\nu}(x) = I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}, \quad \Omega^i(x) = x^2 \tag{207}$$

Inversions are not elements of the component of the conformal group connected to the identity since $\mathcal{D}\rceil\sqcup I=-1$, but special conformal transformations are formed by an inversion, a translation and another inversion. By definition, for a quasi-primary quantum field $\mathcal{O}(x)$ of scale dimension Δ, a finite dimensional representation under conformal transformations is induced by a representation of group $(O(D)\otimes K)\bowtie T_D$, if $\mathcal{O}\to T(g)\mathcal{O}$ where

$$(T(g)\mathcal{O})^i\left(x'\right)=\Omega^g(x)^\eta K^i_j\left(\mathcal{R}^g(x)\right)\mathcal{O}^j(x) \tag{208}$$

Here $\mathcal{R}^g(x)$ denotes transformation as (203), while the index i denotes the components in somme representation of the rotation group $O(D)$ so that, for $R_{\mu\nu}$ any orthogonal rotation matrix, $K^i_j(R)$ is the corresponding element in this representation acting on the fields $\mathcal{O}^i$. In this way we are able to define a conformally invariant two point function by

$$\left\langle\mathcal{O}_1^{i_1}\left(x_1\right)\mathcal{O}_2^{i_2}\left(x_2\right)\right\rangle=\frac{1}{\left(x_{12}^2\right)^\Delta}P^{i_1i_2}\left(x_{12}\right),\quad x_{12}=x_1-x_2 \tag{209}$$

if $P^{i_1i_2}(x)$ is required to satisfy

$$K_1^{i_1}\left(\mathcal{R}\left(x_1\right)\right)K_2^{i_2}\left(\mathcal{R}\left(x_2\right)\right)P^{j_1j_2}\left(x_{12}\right)=P^{i_1i_2}\left(x'_{12}\right),\quad P^{i_1i_2}(\lambda x)=P^{i_1i_2}(x) \tag{210}$$

using $x'^2_{12}=x^2_{12}/\left(\Omega^g\left(x_1\right)\Omega^g\left(x_2\right)\right)$. A solution of this condition is provided by

$$P^{i_1i_2}\left(x_{12}\right)=K_1^{i_1}\left(I\left(x_{12}\right)\right)g^{j_1i_2} \tag{211}$$

where $g^{i_1i_2}$ is an invariant tensor for the representations K_1 and K_2. We can apply this formalism to cases involving vector fields $V_\mu(x)$, of dimension $D-1$, and the energy momentum tensor $T_{\mu\nu}(x)$,which is symmetric and traceless and of dimension D. Accordingly with the above result the two point functions are.

$$\begin{aligned}
\langle V_\mu(x)V_\nu(0)\rangle&=\frac{C_V}{x^{2(D-1)}}I_{\mu\nu}(x)\\
\langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\rangle&=\frac{C_T}{x^{2D}}\mathcal{I}_{\mu\nu,\sigma\rho}(x)\\
\mathcal{I}_{\mu\nu,\sigma\rho}(x)&=\frac{1}{2}\left(I_{\mu\sigma}(x)I_{\nu\rho}(x)+I_{\mu\rho}(x)I_{\nu\sigma}(x)\right)-\frac{1}{d}\delta_{\mu\nu}\delta_{\sigma\rho}
\end{aligned} \tag{212}$$

with C_V and C_T constants determing the overall scale of these two point functions. $I_{\mu\nu,\sigma\rho}$ represents the inversion operator on symmetric traceless tensors. It is possible to derive an analogous result for the three point functions. For general D dimensions the only completely explicit conformal field theories are those provided by free scalr and free fermion fields (38) (100). In the scalar case we may write

$$\begin{aligned}
T_{\mu\nu}&=\partial_\mu\phi\partial_\nu\phi-\frac{1}{4}\frac{1}{D-1}\left((D-2)\partial_\mu\partial_\nu+\delta_{\mu\nu}\partial^2\right)\phi^2\\
V_\mu^a&=\phi t_\phi^a\partial_\mu\phi,\quad\left(t_\phi^a\right)^T=-t_\phi^a,\quad\left[t_\phi^a,t_\phi^b\right]=f^{abc}t_\phi^c
\end{aligned} \tag{213}$$

while in the fermion case

$$\begin{aligned}
T_{\mu\nu}&=\frac{1}{2}\bar{\psi}\left(\gamma_\mu\overleftrightarrow{\partial_\nu}+\gamma_\nu\overleftrightarrow{\partial_\mu}\right)\psi,\quad\overleftrightarrow{\partial_\mu}=\frac{1}{2}\left(\partial_\mu-\overleftarrow{\partial}_\mu\right)\\
V_\mu^a&=\bar{\psi}t_\psi^a\gamma_\mu\psi,\quad\left(t_\psi^a\right)^\dagger=-t_\psi^a,\quad\left[t_\psi^a,t_\psi^b\right]=f^{abc}t_\psi^c
\end{aligned} \tag{214}$$

The basic two point functions for the massless scalar,fermion fields are

$$\langle \phi(x)\phi(0)\rangle = \frac{1}{(D-2)S_D}\frac{1}{x^{D-2}}, \quad \langle \psi(x)\bar{\psi}(0)\rangle = \frac{1}{S_D}\frac{\gamma\cdot x}{x^D} \tag{215}$$

with $S_D = 2\pi^{\frac{1}{2}D}/\Gamma\left(\frac{1}{2}D\right)$. These results allows us to determine the form of the two point functions of V_μ^a and $T_{\mu\nu}$ which are

$$\left\langle V_\mu^a(x)V_\nu^b(0)\right\rangle = \delta^{ab}\frac{C_V}{x^{2(D-1)}}I_{\mu\nu}(x) \tag{216}$$

In the scalar case, if ϕ has n_ϕ components and $\operatorname{tr}\left(t_\phi^a t_\phi^b\right) = -N_\phi\delta^{ab}$,then

$$C_V = \frac{N_\phi}{D-2}\frac{1}{S_D^2}, \quad C_T = n_\phi\frac{D}{D-1}\frac{1}{S_D^2} \tag{217}$$

while in the fermion case, if there are n_ψ Dirac fields and $\operatorname{tr}\left(t_\psi^a, t_\psi^b\right) = -N_\psi\delta^{ab}$, then

$$C_V = N_\psi 2^{\frac{1}{2}D}\frac{1}{S_D^2}, \quad C_T = n_\psi\frac{1}{2}D2^{\frac{1}{2}D}\frac{1}{S_D^2}. \tag{218}$$

In our fermion theory given by the action, in the flat space

$$S[\psi,\overline{\psi}] = \int d^Dx\overline{\psi}\partial\!\!\!/^3\psi \tag{219}$$

the propagator in the Fourier space is

$$\langle\psi(x)\overline{\psi}(0)\rangle = -i\int\frac{d^Dp}{(2\pi)^D}e^{ipx}\frac{ip^2 p\!\!\!/}{p^6} \tag{220}$$

while in the configuration space we expect according to the conformal dimension analysis made above

$$\langle\psi(x)\overline{\psi}(0)\rangle = \frac{1}{S_D}\frac{x\!\!\!/^3}{x^D} = \frac{1}{S_D}\frac{x\!\!\!/}{x^{D-2}} \tag{221}$$

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