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Dipartimento di Fisica e Astronomia
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QUANTUM ROTATING BLACK HOLES AND EXTRA DIMENSIONS

Relatore:
Prof. Roberto Casadio

Correlatore:
Dott. Andrea Giugno

Presentata da:
Sofia Di Gennaro

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Abstract

We employ the formalism of the Horizon Quantum Mechanics to describe the gravitational radius of compact sources by means of an operator and derive a Horizon Wave Function which will allow us to estimate the probability of formation of black holes in scattering processes. If the Planck scale is kept at its standard value, however, it will be impossible to test that regime with any foreseeable technology. We then review how the introduction of extra dimensions can potentially lower the Planck scale down to the TeV range in an attempt to solve the hierarchy problem. In this context, we proceed by studying black holes described by a generalisation of the Kerr metric for higher dimensional spacetime known as the Myers-Perry metric. Our computation of the probability that a rotating source in higher dimensions is a black hole suggests that, even if the fundamental Planck scale is as low as a few TeV's, we should not be able to detect any black hole in colliders as is indeed the case.

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Chapter 1

Introduction

The hierarchy problem is one of the major problems of the recent years and stimulates the research for physics beyond the Standard Model. It concerns the ratio of the weak scale mass and the Planck scale mass, questioning why it is so small. The weak scale mass is set by a cut-off $\Lambda \sim 1$ TeV necessary to make the quantum corrections to the Higgs' mass sufficiently small and maintain naturalness. The Planck scale, on the other hand, is the coupling of gravity and the theory becomes non-unitary when this value is crossed.

The answer to why gravity is weaker than gauge interactions by so many orders of magnitude might lie in extra dimensional models, which in recent years have become more and more popular, especially because string theory also requires additional dimensions. The Planck scale is then derived from the electroweak one, which is taken to be fundamental. Two main models concern extra dimensions: the ADD (Arkani-Hamed-Dimopoulos-Dvali) LXD (large extra dimensions) and the RS (Randall-Sundrum) models.

In the ADD model the Universe is described by a higher dimensional manifold, while the Standard Model is confined on a brane. Then, only gravity can propagate on the whole manifold and its strength is diluted over all d-dimensional spacetime. The experimental results for the Planck mass on the lower dimensional brane that contains the SM is given by

$$m_p^2 = M_d^{(n+2)} V_n, \tag{1.1}$$

with n indicating the number of extra dimensions, explaining why it is perceived as weaker if the volume V_n of compactification is big enough. Then, the fundamental d-dimensional Planck scale M_d is set to be ~ 1 TeV of the same order of magnitude as the electroweak scale and thus solve the hierarchy problem. This model requires more than one extra compactified dimension in order to avoid modifications at solar-system distances in Newton's potential. New dimensions could be potentially discovered already at the scale of a millimetre (if $n = 2$).

Meanwhile, the RS model is effectively 5-dimensional and thus only requires one warped extra dimension. The observable universe is confined on a 4-dimensional brane, embedded in a AdS_5 bulk. On another brane, gravity and all relevant scales are set to be ~ 1 TeV, reproducing the hierarchy on the observable brane thanks to the exponential warp factor that reduces only gravity to the measured value.

Successfully pushing m_p to values $M_d \sim 1$ TeV makes ultra-Planckian energies possible to investigate in the near future at the LHC, perhaps allowing us to probe quantum gravity. What we could investigate is the production of mini-Black Holes in scattering experiments. Indeed, when particles in the collider get closer to each other, the gravitational force is increased as distance is reduced and if the impact parameter is smaller than the horizon radius, a Black Hole is formed. The horizon radius is much smaller than the compactification scale, then these objects are effectively d-dimensional and it must also be greater than the Compton wavelength of the source (representing quantum uncertainty), making the mini-BH treatable in a semi-classical regime. In this case, we aim to study a rotating Black Hole, therefore a generalized version of the Kerr metric, the Myers-Perry metric, is needed.

Even though we would expect to be able to observe quantum gravity once we cross the M_d threshold, following [6], we argue that gravity is self-complete and quantum gravitational effects are strongly suppressed. Indeed, any attempt to probe it would end up forming a classical Black Hole instead, whose horizon would prevent examining the region inside. A mapping

$$L \leftrightarrow \frac{L_d^2}{L} \tag{1.2}$$

can be established between sub-Planckian distances and macroscopic distances. Then, any new degree of freedom in the Trans-Planckian regime does not modify physics as it can be written in terms of already existing classical degrees of freedom.

Following the idea hereby presented, this work is organized as follows:
the first chapter consists in this introduction; later, in Chapter 2, we study the hierarchy problem, self-UV completion and models with extra dimensions (Kaluza and Klein's original model, the ADD and RS models) more in detail. In Chapter 3 we study the Myers-Perry metric, focusing on the derivation of an expression for the horizon radius and area, which is then approximated in two main regimes (static and ultra-spinning). In Chapter 4, finally, employing a Gaußian source, we calculate the probability of formation of a Black Hole and Generalized Uncertainty Principle (GUP) in the two regimes.

Throughout the paper we will use the mostly-plus convention for the Minkowski metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ and we will denote vectors $\vec{a}$ as made up of purely spatial components, regardless of how many these may be. Moreover, we indicate the whole spacetime dimensionality with d, covered by indices like $\mu, \nu, \ldots$. We choose to keep $c = 1$ always and $\hbar = 1$ only when it is not necessary (it will be specified in the text).

Chapter 2

Extra dimensions

The vast gap between electroweak and Planck scale is also known as hierarchy problem and it is the one that most directly motivates the study of theories beyond the Standard Model, as it might be the hint that there is new physics beyond the probed energies. Indeed, it is solved by introducing extra dimensions that would possibly lower the Planck scale to the electroweak scale, effectively nullifying the problem.

2.1 Hierarchy Problem

The ratio of the couplings of the two above-mentioned theories gives a very small value, indicating gravity is much weaker than electroweak gauge interactions:

$$\frac{G_N}{G_F} \sim 10^{-34}. \tag{2.1}$$

The same problem can be formulated in terms of characteristic masses: the Planck mass $m_p \sim 10^{19}$ GeV and the electroweak characteristic mass $M_{ew} \sim \frac{1}{\sqrt{G_F}} \sim 250$ GeV. Of the two, M_{ew} is taken to be the fundamental one, since, contrary to the Planck mass, it has been probed and the Standard Model is confirmed with outstanding precision. On the other hand, the Planck mass is still far from being reached by our experimental capacity.

The mass scale M_{ew} comes from the mechanism of EWSB (Electro-Weak Symmetry Breaking), that occurs when the gauge group of the SM spontaneously breaks to electromagnetism: $SU(3)_c \times SU(2)_L \times U(1)_y \to U(1)_{em}$. The value of the Higgs mass is not protected by symmetries and, as such, quantum corrections are very large and diverge quadratically. Requiring the smallness of these corrections with respect to the Higgs mass, implies introducing a cut-off $\Lambda \sim 1$ TeV, which is the M_{ew}. After this value is crossed, the theory loses naturalness.

On the other hand, the Planck scale m_p is the limit after which the theory of gravity loses unitarity and a new theory, like quantum gravity, is necessary. Indeed, the fact that m_p is much bigger than the electroweak cut-off means that the SM would lose naturalness much before this value is reached and the only two conclusions we can make out of this are: either there is a new theory after M_{ew} that keeps the SM natural or there is no other fundamental scale except for 1 TeV. This second solution is explored in the extra dimensional models we are going to present.

In this section we are going to formulate the hierarchy problem more in detail, following mainly [8].

2.1.1 Planck scale

The first thing we define is the Planck scale, which represents the natural cut-off for any quantum theory.
It is defined as a combination of the fundamental constants $G_N, c, \hbar$ so that the final quantity has the units of a mass:

$$m_p = G_N^\alpha \hbar^\beta c^\gamma. \tag{2.2}$$

The powers α, β, γ are easily determined by dimensional analysis.
In generic d dimensions this gives:

$$[M] = \left(\frac{[L]^{d-1}}{[M][T]^2}\right)^\alpha \left(\frac{[M][L]^2}{[T]}\right)^\beta \left(\frac{[L]}{[T]}\right)^\gamma, \tag{2.3}$$

where for the constant G_N we used the intuitive relation $F_N = ma = G_N \frac{m^2}{r^{d-2}}$ that comes from the requirement that the gravitational field should have zero divergence.
The following system is obtained:

$$\begin{cases} \beta - \alpha = 1; \\ \gamma + \beta + 2\alpha = 0; \\ \gamma + 2\beta + \alpha(d-1) = 0. \end{cases} \tag{2.4}$$

The solution of the system is:

$$m_p = \left(\frac{\hbar^{d-3}}{G_N c^{d-5}}\right)^{\frac{1}{d-2}}. \tag{2.5}$$

In four dimensions it gives the usual Planck mass:

$$m_p = \sqrt{\frac{c\hbar}{G_N}}. \tag{2.6}$$

If we set $c = \hbar = 1$, we obtain the proportionality:

$$G_N = \frac{1}{m_p^2}. \tag{2.7}$$

In a totally analogous way, we can obtain the Planck length, by requiring that a combination of the fundamental constants gives a length:

$$l_p = G_N^\alpha \hbar^\beta c^\gamma, \tag{2.8}$$

which in dimensional analysis becomes:

$$[L] = \left(\frac{[L]^{d-1}}{[M][T]^2}\right)^\alpha \left(\frac{[M][L]^2}{[T]}\right)^\beta \left(\frac{[L]}{[T]}\right)^\gamma. \tag{2.9}$$

This equation generates the following system:

$$\begin{cases} \beta - \alpha = 0; \\ \gamma + \beta + 2\alpha = 0; \\ \gamma + 2\beta + \alpha(d-1) = 1. \end{cases} \tag{2.10}$$

The solution is:

$$l_p = \left(\frac{G_N \hbar}{c^3} \right)^{\frac{1}{d-2}}, \tag{2.11}$$

which in 4 dimensions recovers the usual value for the Planck length:

$$l_p = \sqrt{\frac{G_N \hbar}{c^3}}. \tag{2.12}$$

By setting again $c = \hbar = 1$, we check its proportionality with the constant G_N:

$$G_N = l_p^2. \tag{2.13}$$

Moreover, we notice that the relation $\hbar = l_p m_p$ is valid for any dimensionality. Another remark we can make is that the Planck scale is fundamentally quantum, as it goes to 0 in the classical limit $\hbar \to 0$.

2.1.2 Naturalness and the fundamental scale

We employ the principle of naturalness as guiding line in our description. It originated from the reasoning that if a parameter is small, it must not be due to fine tuning but it should rather come as a consequence of an underlying symmetry.
The principle, as formulated by ’t Hooft ([9]), states:
At any energy scale μ, a set of parameters, $\alpha_i(\mu)$ describing a system can be small, if and only if, in the limit $\alpha_i(\mu) \to 0$ for each of these parameters, the system exhibits an enhanced symmetry.
For instance, we apply it to Quantum Electrodynamics:
We take the theory of electromagnetic interaction of a fermion:

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{f=e,\mu,\ldots} \bar{\psi}_f \left[i(\partial_\mu - ieq_f A_\mu)\gamma^\mu - m_f \right] \psi_f. \tag{2.14}$$

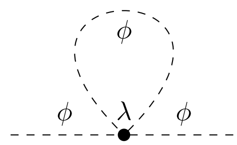
In this case, a null mass (like $m_e \to 0$ for the electron) makes the theory invariant under chiral symmetry (that transforms left- and right-handed electron-like leptons separately). This symmetry protects the mass from getting diverging quantum corrections and allows it to keep its small value. Indeed, from the self-energy diagram we conclude the divergence is only logarithmic instead of power-law and the correction to the (electron) mass is proportional to the mass itself and thus can not be much larger. Moreover, we see that each mass can be taken to be 0 separately.
If we take the same limit for the coupling $e \to 0$, we get a theory of free particles, whose number is thus conserved. Also, the coupling e can be small independently to the masses. The divergence of one-loop diagrams is logarithmically divergent and proportional to e^2,

thus protecting its smallness.

Now, if we take the theory of a scalar field:

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{2.15}$$

we notice that the limit $\lambda \to 0$, as before, makes the theory non-interacting and consequently the total number of particles is conserved. Taking $m \to 0$, though, does not enhance any symmetry, as it only leads to scale invariance which is broken at the quantum level. Indeed, it can be seen that the corrections to the mass diverge quadratically.
For instance, at one-loop level the correction term is given by a scalar loop with incoming and outgoing scalars:



The correction to the mass is calculated from this diagram and corresponds to:

$$\delta m^2 = \frac{i}{2}\int \frac{dk^4}{(2\pi)^4}(-i\lambda)\frac{i}{k^2 - m^2} = \frac{\lambda}{2}\int \frac{dk_E^4}{(2\pi)^4}\frac{1}{k_E^2 + m^2}, \tag{2.16}$$

where we used a Wick rotation $k^0 = ik_E^0$ that implies $k_E^2 = -k^2$.
Now we use spherical coordinates and denote with k the modulus of k_E, then we change variables to $r = \frac{k}{m}$:

$$\delta m^2 = \frac{2\pi^2\lambda}{2}\int_0^\Lambda \frac{dk}{(2\pi)^4}\frac{k^3}{k^2 + m^2} = \frac{\lambda}{16\pi^2}\int_0^{\frac{\Lambda}{m}} drm\frac{(mr)^3}{m^2(1 + r^2)}, \tag{2.17}$$

where $\Lambda \gg m$ is the cut-off of the theory.
Now we perform another change of variable: $t = r^2$:

$$\delta m^2 = \frac{\lambda m^2}{16\pi^2}\int_0^{\frac{\Lambda^2}{m^2}} \frac{dt}{2\sqrt{t}}\frac{t^{\frac{3}{2}}}{1 + t} = \frac{\lambda m^2}{32\pi^2}\int_0^{\frac{\Lambda^2}{m^2}} dt\frac{t}{1 + t}. \tag{2.18}$$

Next, we change variables to $u = \frac{1}{1+t}$, assuming $= \frac{m^2}{m^2+\Lambda^2} \simeq \frac{m^2}{\Lambda^2}$:

$$\delta m^2 = -\frac{\lambda m^2}{32\pi^2}\int_1^{\frac{m^2}{\Lambda^2}} \frac{du}{u}\left(\frac{1}{u} - 1\right) = -\frac{\lambda m^2}{32\pi^2}\int_1^{\frac{m^2}{\Lambda^2}} \frac{du}{u^2}(1 - u). \tag{2.19}$$

The result is straightforward:

$$\delta m^2 = \frac{\lambda m^2}{32\pi^2}\left[-1 + \frac{\Lambda^2}{m^2} - \ln\left(\frac{\Lambda^2}{m^2}\right)\right] \sim \lambda\Lambda^2. \tag{2.20}$$

We confirm that the quantum corrections are quadratically divergent and thus the the smallness of masses is not protected against quantum effects.

We now move on to consider a theory of a complex scalar field, a left-handed fermion and a right-handed one. The first two interact with an electromagnetic field, while the other does not. We also add Yukawa couplings of the fermions with the scalar:

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\partial_\mu\gamma^\mu\psi_R \\ &+ |D_\mu\phi|^2 + \mu^2|\phi|^2 - \lambda|\phi|^4 - Y\phi\bar{\psi}_L\psi_R + \text{ h.c. } ,\end{aligned} \tag{2.21}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative and $\not{D}_\mu = D_\mu\gamma^\mu$ is the slash notation. The Lagrangian (2.21) has gauge invariance under:

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta, \qquad \phi' = e^{i\theta}\phi, \qquad \psi'_L = e^{i\theta}\psi_L, \qquad \psi'_R = \psi_R. \tag{2.22}$$

The limits for the couplings $e \to 0$ and $Y \to 0$ respectively give no gauge interaction and no Yukawa coupling, enhancing the symmetry. These two parameters are natural and their corrections are proportional to themselves. Conversely, the parameter λ behaves differently. Even by taking the limit $\lambda \to 0$, Yukawa and gauge interactions give fermionic and photonic one-loop corrections to the $|\phi|^4$ vertex in the form e^4M+Y^4N, where M and N are constants. This means that the coupling λ can not be much smaller than e and Y.
Now we look at the mass term: its sign indicates that symmetry can be spontaneously broken.
The minima are:

$$\phi^\dagger\phi = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}. \tag{2.23}$$

We choose one as:

$$\phi = \frac{1}{\sqrt{2}}e^{i\sigma}(v+h), \tag{2.24}$$

where the phase can be rotated away by gauge transformations.
Expanding about this vacuum gives:

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R + \frac{1}{2}\partial_\mu h\partial^\mu h \\ &+ \frac{e^2v^2}{2}A_\mu A^\mu - \frac{Yv}{\sqrt{2}}(\bar{\psi}_L\psi_R\bar{\psi}_R\psi_L) + \frac{\mu^2}{2}h^2 + \text{ interaction terms,}\end{aligned} \tag{2.25}$$

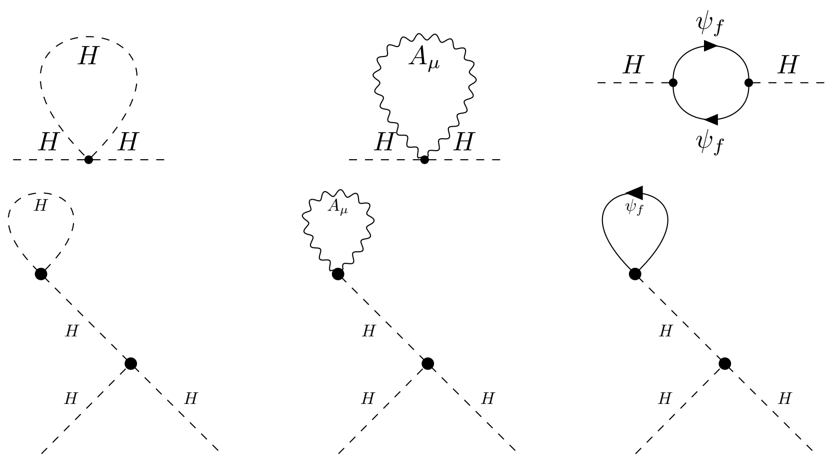
that is a theory of a massive vector field A_μ, a massive scalar h and a massive fermion ψ. Their masses are given in the second line of (2.25), by:

$$m_A = ev, \qquad m_h = \sqrt{2\lambda}v, \qquad m_\phi = \frac{1}{\sqrt{2}}Yv. \tag{2.26}$$

The symmetries of (2.25) are not exact symmetries, as they are broken at the quantum level. Therefore, the masses are not natural parameters and their corrections should diverge. In particular, we are interested in the Higgs mass, so, first, we look at the possible vertices in the interaction part of (2.25) that include the Higgs scalar:

$$\mathscr{L}_{\text{int}} = e^2vA^2h + \frac{e^2}{2}A^2h^2 + \left(\frac{\mu^2}{2} - \frac{3\lambda}{2}v^2\right)h^2 - \frac{\lambda}{4}h^4 - \frac{3\lambda}{4}vh^3 - \frac{Y}{\sqrt{2}}h\bar{\psi}_L\psi_R + \text{ h.c. }. \tag{2.27}$$

Now we can draw all the possible one-loop diagrams that give quantum corrections to the Higgs mass:



The first diagram was already calculated in (2.20) and, from it, we can also derive the expression for the forth diagram, the one that contains a scalar tadpole.
Denoting it with t_1, we can write:

$$t_1 = \left[\left(-i\frac{3\lambda}{4}v \right)^2 \left(-\frac{i}{m_h^2} \right) \right] \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_h^2}, \tag{2.28}$$

in which the propagator that connects the loop to the incoming and outgoing scalars is $-\frac{i}{m^2}$ as the momentum is 0 to satisfy the conservation of momenta at the vertex.
Since we already calculated the integral in (2.20), we can straightforwardly write:

$$t_1 \propto \frac{\lambda^2 v^2 \Lambda^2}{m_h^2} = \frac{\lambda \Lambda^2}{2}, \tag{2.29}$$

using the expression (2.26) for m_h.
Then, the third diagram, the one with the fermionic loop, is processed analogously to QED's vacuum polarization.
It can be seen that all the diagrams above diverge quadratically in the cut-off and we can thus parametrize the whole correction as:

$$\delta m^2 \propto \frac{1}{16\pi^2}(A\lambda + Be^2 - CY^2)\Lambda^2 \equiv \alpha\Lambda^2, \tag{2.30}$$

where A, B and C are constants that respectively correspond to Higgs, gauge and fermion loops.
The same discussion can be applied to the Standard Model, where $SU(2)_L \times U(1)_Y$ is spontaneously broken to $U(1)_{em}$. We end up obtaining the same result of quadratic divergence for the corrections to the Higgs mass.
Since $m_h \sim 100$ GeV and the coupling $\alpha \sim \frac{1}{100}$ GeV, in order to require that the corrections to the mass are small $\delta m_h^2 \sim m_h^2$, we get:

$$\Lambda^2 = \frac{\delta m_h^2}{\alpha} \sim \frac{m_h^2}{\alpha} \sim \frac{(100 \text{ GeV})^2}{(100 \text{ GeV})^{-1}} \sim 10^6 \text{ GeV}^2. \tag{2.31}$$

Therefore the cut-off of the theory must be $\Lambda \sim 1$ TeV, after which naturalness is lost. In fact, the observed small value would result in a fine tuning of the quantum corrections. The

Planck mass is 10^{19} GeV, which is much higher than Λ and this poses a problem in the connection of gravity with the Standard Model.
The idea to solve this problem is to hypothesize the Planck mass is not a fundamental scale, but rather a derived quantity, that can be obtained from the only characteristic scale $\Lambda \sim 1$ TeV.
In the following, we shall present two possible solutions to this problem: the ADD model of large extra dimensions and the Randall-Sundrum model of warped metric.

2.2 UV self-completeness

In this section we present a theory, formulated by Dvali and Gomez in [6], which proposes that a UV completion of gravity is unnecessary. In the trans-Planckian regime, where a quantum theory would be needed, we would expect new degrees of freedom need to be integrated in, but actually we can show that these are classical already existing degrees of freedom. In fact, in the process of probing energies beyond the Planck scale, a classical larger Black Hole is forming, making it impossible to investigate inside its horizon.
We are now going to see this, starting from the Einstein-Hilbert action associated to the four-dimensional theory of gravity:

$$S_{EH} = m_p^2 \int d^4x \sqrt{-g} \mathcal{R}. \tag{2.32}$$

At linearised level the metric is written in the canonically normalised form as:

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, \qquad \delta g_{\mu\nu} = \frac{h_{\mu\nu}}{m_p}. \tag{2.33}$$

Therefore the graviton couples to generic matter sources as:

$$\frac{1}{m_p} h_{\mu\nu} T^{\mu\nu}. \tag{2.34}$$

It appears clear that the coupling constant is then given by the Planck mass. This property remains valid at all orders in the non-linear interactions of the graviton. It results that the theory violates unitarity above m_p, so we could argue that it needs UV-completion. However, we claim that gravity could be self-complete, i.e. its properties at high energy are equivalent to the low energy ones, while energies $M \gg m_p$ cannot be probed.

We consider a scattering process in which we localize an energy E in a limited space L.
The Schwarzschild radius of this portion of space is:

$$R_s = 2GM = \frac{2l_p^2}{L}. \tag{2.35}$$

Then, if $L \ll l_p$, the Schwarzschild radius becomes much bigger than L and l_p. Therefore, any attempt to probe a distance smaller than the Planck length will result in the formation of a black hole with horizon $R_s \gg L$. This corresponds to a macroscopic black hole that is completely classical and its horizon forbids to access small distances.

2.2.1 Trans-Planckian degrees of freedom cannot be probed

We argue that any new degree of freedom localized at Sub-Planckian length is not accessible. We start by assuming new degrees of freedom appear at distance $L \ll l_p$ and with masses $m \sim L^{-1} \gg m_p$. We consider the interaction term between two sources $T^{\mu\nu}$ and $t^{\mu\nu}$:

$$\frac{1}{m_p^2} T^{\mu\nu} \langle h_{\mu\nu} h_{\rho\sigma} \rangle t^{\rho\sigma}. \tag{2.36}$$

To compute the propagator, we proceed in analogy to electromagnetism and write it in the form:

$$\frac{f_{\mu\nu\alpha\beta}}{k^2 + m^2 - i\epsilon}, \tag{2.37}$$

where we considered a massive gravitational field.
Gravity, like electromagnetism, is characterized by polarization vectors $\epsilon^\lambda_{\mu\nu}$, where $\lambda = 1, \ldots, 5$. In fact, as gravity is coupled to the energy-momentum tensor, which, due to diffeomorphism and scale invariance, must be symmetric and traceless. Then, the tensor coupled to it must carry the same properties. Moreover, from the Ward identity, we know that $k^\mu \epsilon^\lambda_{\mu\nu}$, where k^μ is the momentum transfer. Indeed, we can always choose $k^\mu = (m, 0, 0, 0)$ in the rest frame and since the polarizations $\epsilon^\lambda_{\mu\nu}$ are all spatial, the result of $k^\mu \epsilon^\lambda_{\mu\nu}$ is 0 in this frame and in all frames, for Lorentz invariance.
In total we have:

$$\begin{cases} \epsilon^\lambda_{\mu\nu} = \epsilon^\lambda_{\nu\mu} \\ g^{\mu\nu} \epsilon^\lambda_{\mu\nu} = 0 \\ k^\mu \epsilon^\lambda_{\mu\nu} = 0. \end{cases} \tag{2.38}$$

From this we conclude that it contains: $\frac{d(d+1)}{2} - 1 - 4$ degrees of freedom. In $d = 4$ a rank-2 tensor has 5 possible values of λ, as we anticipated.
When computing the scattering amplitude, we realise that $f_{\mu\nu\alpha\beta} = \sum_\lambda \epsilon^\lambda_{\mu\nu} \epsilon^\lambda_{\alpha\beta}$. Then, we wish to express this quantity as Lorentz invariant, and when doing so we realise we can only use combinations of $g_{\mu\nu}$ and k_μ. Furthermore, the last condition in (2.38) restricts these possible combinations to those of a quantity that we choose to express as:

$$\tilde{G}_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \tag{2.39}$$

Writing the possible symmetric combinations gives:

$$\sum_\lambda \epsilon^\lambda_{\mu\nu} \epsilon^\lambda_{\alpha\beta} = A \left[G_{\mu\nu} G_{\alpha\beta} \right] + B \left[G_{\mu\alpha} G_{\nu\beta} + G_{\nu\alpha} G_{\mu\beta} \right]. \tag{2.40}$$

When applying the tracelessness condition in (2.38), we get:

$$g^{\mu\nu} \sum_\lambda \epsilon^\lambda_{\mu\nu} \epsilon^\lambda_{\alpha\beta} = 0 = G_{\alpha\beta} \left[3A + 2B \right], \tag{2.41}$$

where, as can be easily verified:

$$G^\mu_n u G_{\mu\beta} = G_{\nu\beta} \qquad \text{and} \qquad G^\alpha_\alpha = 3. \tag{2.42}$$

Then we are left with only one constant:

$$A = -\frac{2B}{3}. \tag{2.43}$$

Now, we impose the orthonormality condition:

$$\epsilon^{\lambda}_{\mu\nu}\epsilon^{\mu\nu}_{\rho} = \delta^{\lambda}_{\rho} \tag{2.44}$$

which gives:

$$\sum_{\lambda} \epsilon^{\lambda}_{\mu\nu}\epsilon^{\lambda}_{\mu\nu} \stackrel{!}{=} 5. \tag{2.45}$$

This condition furthermore allows us to fix the constant:

$$\sum_{\lambda} \epsilon^{\lambda}_{\mu\nu}\epsilon^{\lambda}_{\mu\nu} = \sum_{\lambda} g^{\mu\alpha}g^{\nu\beta}\epsilon^{\lambda}_{\mu\nu}\epsilon^{\lambda}_{\alpha\beta} = 10B \stackrel{!}{=} 5. \tag{2.46}$$

Then:

$$\sum_{\lambda} \epsilon^{\lambda}_{\mu\nu}\epsilon^{\lambda}_{\alpha\beta} = -\frac{1}{3}\left[G_{\mu\nu}G_{\alpha\beta}\right] + \frac{1}{2}\left[G_{\mu\alpha}G_{\nu\beta} + G_{\nu\alpha}G_{\mu\beta}\right]. \tag{2.47}$$

We notice that it is valid only in 4 dimensions and in asymptotically flat spacetime. The massless propagator is obtained from the massive one. By inverting the propagator we construct the Lagrangian and then add a gauge breaking term.
Therefore, the propagator for a massless graviton field is:

$$< h_{\alpha\beta}h_{\mu\nu} >= \frac{1}{2}\frac{g_{\alpha\mu}g_{\beta\nu} + g_{\beta\mu}g_{\alpha\nu} - g_{\mu\nu}g_{\alpha\beta}}{k^2} \tag{2.48}$$

and it gives in (2.36):

$$\frac{1}{m_p^2}T^{\mu\nu} < h_{\mu\nu}h_{\rho\sigma} > t^{\rho\sigma} = \frac{1}{m_p^2}\frac{T^{\mu\nu}t_{\mu\nu} - \frac{1}{2}T^{\mu}_{\mu}t^{\nu}_{\nu}}{k^2}. \tag{2.49}$$

Now we try to modify it by adding a new particle with mass $\sim L^{-1}$. Then:

$$\frac{1}{m_p^2}T^{\mu\nu} < h_{\mu\nu}h_{\rho\sigma} > t^{\rho\sigma} = \frac{1}{m_p^2}\left[\frac{T^{\mu\nu}t_{\mu\nu} - \frac{1}{2}T^{\mu}_{\mu}t^{\nu}_{\nu}}{k^2} + \frac{aT^{\mu\nu}t_{\mu\nu} - \frac{b}{3}T^{\mu}_{\mu}t^{\nu}_{\nu}}{k^2 - (L)^{-2}}\right], \tag{2.50}$$

where $a = 0, b < 0$ if the new particle has spin 0 and $a = b$ if it has spin 2.
The second term shows a pole at $k^2 = L^{-2}$, but we remember that, according to our hypothesis $L \ll l_p$, therefore $k \gg m_p$. Indeed, if we scatter gravitons with centre of mass energy $\sim L^{-1}$, we must localize them within a distance $\sim L \ll l_p$, according to Heisenberg's indetermination principle, where L is the impact parameter of the process.
On the other hand, the Schwarzschild radius is much bigger than the impact parameter:

$$R_s = 2GM = 2\frac{l_p^2}{L} \gg L. \tag{2.51}$$

This indicates that a classical black hole would form before being able to access this distance, as the region inside the event horizon is impossible to probe.
We can thus conclude that an additional pole in gravity would not add any new information, as it is completely inaccessible or at least as much as a classical black hole.

2.2.2 Trans-Planckian degrees of freedom do not hold new information

We argue that these Trans-Planckian degrees of freedom are equivalent to classical ones located further away than the Planck length.
The contribution from the newly introduced Trans-Planckian poles in (2.50) is inconsistent. In fact, we tried to add new propagating degrees of freedom but we obtained a classical black hole. Then, this means the new particles' propagation must hold an exponential suppression. We express it as $e^{-(l_p/L)^2}$. Indeed, being a thermodynamic object it carries an entropy suppression factor, which can be seen as either Bekenstein-Hawking entropy or as the Boltzmann suppression of a classical BH.
The Bekenstein-Hawking entropy is, in fact:

$$S = \frac{k_B A}{4 l_p^2} \sim \frac{l_p^4}{l_p^2 L^2} = \frac{l_p^2}{L^2}, \tag{2.52}$$

where $A \propto R^2$ is the area of the horizon and the radius is given by (2.51).
The other interpretation is explained as follows.
We call the newly introduced degrees of freedom $\phi_{\mu\nu}$, without specifying its spin (0 or 2) and we consider a particle q whose energy momentum tensor is $T_{\mu\nu}$. The interaction between the two describes the formation of a pair of particle-antiparticle:

$$\phi \to q + \bar{q}.$$

If $L^{-1} \ll m_p$ it is a normal decay of a heavy particle into lighter ones. If, however, $L^{-1} \gg m_p$, what is described is the evaporation of a classical BH, which is suppressed as $e^{\frac{1}{LT}}$, where $T = \frac{1}{8\pi GM} = \frac{L}{l_p^2}$ is the Hawking temperature.

2.2.3 Trans-Planckian degrees of freedom do not contribute

Now, we want to see that any hypothetical trans-Planckian degrees of freedom does not give any contribution to external physics.
We consider an usual massless graviton to which we add a scalar massive degree of freedom ϕ. That means the metric is:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_p} + \eta_{\mu\nu}\frac{\phi}{m_p} \tag{2.53}$$

and at the start we consider the mass of ϕ as $m \leq m_p$.
At large distances the equation of motion is given by the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.54}$$

To linear order and in the harmonic gauge $2\partial^\mu h_{\mu\nu} = \partial_\nu h$.
We obtain, as calculated in the appendix (A.24):

$$\Box h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right). \tag{2.55}$$

Then we consider a static point-like source $T_{\mu\nu} = \delta_\mu^0\delta_\nu^0 M\delta(r)$ and substitute it:

$$\Box\frac{h_{\mu\nu}}{m_p} = -8\pi GM\delta(r)\delta_{\mu\nu}. \tag{2.56}$$

The solution is easily found to be:

$$\frac{h_{\mu\nu}}{m_p} = \frac{2GM}{r}\delta_{\mu\nu} = \frac{R_S}{r}\delta_{\mu\nu}, \tag{2.57}$$

where $R_S = 2GM$ is the Schwarzschild radius. So, when we get close to the horizon, $h_{\mu\nu}$ becomes of order 1. At the same time, higher order corrections in G become equally important as the linear one. The series then has to be resummed, signalling the proximity of the horizon.
These non-linear corrections are found by considering a second order expansion in $h_{\mu\nu}$ of (2.54), taking into account also the interaction of gravity with its own energy-momentum tensor. Moreover, the source energy-momentum tensor must be further corrected, as rather than a point static mass, it is better modelled as a fluid ball with radius equal to the Compton length.
Doing all this, at second order the result that can be found is:

$$\frac{h_{00}^{(1)}}{m_p} = \frac{R_S}{r}\left(1 + a\frac{M^2}{m_p^2}\right) \qquad \text{and} \qquad \frac{h_{00}^{(2)}}{m_p} = -\frac{1}{2}\frac{R_S^2}{r^2}, \tag{2.58}$$

where a is a factor of order one and $M \ll m_p$ is the source mass. We see that taking into account these effects gives a different effective gravitational mass, but as it is much smaller than m_p, this additional contribution can be neglected. From (2.58) we see what we had anticipated, that at $r = R_S$ the series expansion is made up of terms of order one and must be resummed.
Now, when introducing a massive scalar degree of freedom, this gives origin to a energy-momentum tensor:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\lambda\phi\partial^\lambda\phi - m^2\phi^2\right). \tag{2.59}$$

To first order, taking a point-like static mass as source, the equations of motions of a scalar are given by the Klein-Gordon equation with delta-like source. The first order solution of which is $\phi^{(1)} = \frac{R_S}{r}e^{-mr}$. Using this into (2.59) and substituting in (2.54) one can see that the corrections to $h_{\mu\nu}^{(2)}$ that are obtained are proportional to e^{-mr} and thus exponentially suppressed.
The correction that gives the highest contribution is the one from the non-minimal coupling:

$$\frac{\phi\partial^n h^k}{m_p^{n+k-3}}. \tag{2.60}$$

From the equations of motion, an effective source appears in the form:

$$\left(\Box + m^2\right)\phi = \frac{\partial^n h^k}{m_p^{n+k-3}} + \dots \tag{2.61}$$

If we evaluate this expression with (2.57) and considering $m \gg r^{-1}$ so that the left-hand side $\left(\Box + m^2\right)\phi \simeq m^2\phi$, we obtain:

$$m^2\phi^{(k)} \propto \frac{R_S^k}{r^{k+n}m_p^{n-3}} \qquad \rightarrow \qquad \frac{\phi^{(k)}}{m_p} \propto \frac{R_S^k}{r^k}\frac{1}{(rm_p)^{n-2}(rm)^2}. \tag{2.62}$$

In this correction we do not observe an exponential suppression, as it is rather of power-law type. These are due to the fact that we did not require the degree of freedom ϕ to propagate further than its Compton length, therefore the exponential suppression does not appear. Indeed, the massive mode can be integrated out and its effects are understood as modifications of the massless metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_p} + \eta_{\mu\nu}\frac{\partial^n h^k}{m_p^{n+k-3} m^2} + \dots \tag{2.63}$$

In short, we have seen that the contribution from massive modes is suppressed either exponentially or by inverse power of m and therefore it cannot influence Einstein gravitation at distances $r \gg m^{-1}$. Nevertheless, these modes give small corrections to long-distance physics that can, in principle, be measured.
These considerations were made at perturbative level and are therefore not applicable to Trans-Planckian scales, where $m \gg m_p$. Indeed, the Compton wavelength m^{-1} is smaller than the radius of the event horizon of ϕ: $R_\phi = 2Gm$.
As we have also seen before in (2.51), the black hole that is formed when trying to probe sub-Planckian distances is a fully classical one. Therefore, we do not need the non-perturbative approach to treat it. Indeed, ϕ is non-propagating and the description of the trans-Planckian region can be made completely in terms of light already-existing degrees of freedom.
ϕ only covers the role, through the vertex (2.60), of controlling the decay of a classical Black Hole by Hawking evaporation into k-massless gravitons.
Being a decay of a thermal object, it is suppressed by a Boltzmann factor $e^{-\frac{m}{T}}$, where $T = \frac{m_p^2}{m}$ is the Hawking temperature.
Then, whenever ϕ is a virtual state, its contribution is exponentially suppressed.

2.3 Kaluza-Klein Theory

The extra dimensions were first introduced in 1919 by Kaluza in an attempt to unify the gravitational and electromagnetic theories. He introduced a fifth dimension, amounting to a total of 15 degrees of freedom: 10 for the space-time metric, 4 for the electromagnetic vector potential and one more for a new scalar field called *radion*. However, he imposed *a priori* that the fifth coordinate should not appear in the laws of physics, using the *cylinder condition* of vanishing derivatives in the fifth component. Only later, in 1926, Klein quantized this theory and added the compactification condition, arguing that the extra dimensions should be curled into a circle and microscopic enough to justify why it has never been observed. This relieved the cylinder condition, making it more naturally justified and, furthermore, it allowed a Fourier expansion, explaining the quantization of charges. The usual fields are recovered identifying them with the ground state Fourier modes. Then, from the equations of motion of the full d-dimensional theory, they successfully recovered the four-dimensional Einstein's equations, electromagnetism and a scalar field. The latter is taken to be too large to be observed and it is thus explained.

Following the realization that string theory requires extra dimensions, Kaluza-Klein's extra dimensions were revived in the 80s and brought forward new interest in extra dimensional models, which originated the ADD and RS models, both based on compactification. This section mostly follows [10].

2.3.1 The model

The initial setup of the Kaluza-Klein theory involves the construction of a 5-dimensional metric.
A 5d interval can be written in the general form:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{44}(dy + A_\mu dx^\mu)^2, \tag{2.64}$$

where $M, N = 0\ldots4$ and $x^4 \equiv y$. The parameter A_μ indicates: $A_\mu = \frac{g_{\mu4}}{2g_{44}}$. Moreover, we denote by $\phi^2 \equiv g_{44}$.
The metric is thus:

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix}.$$

The inverse is found thanks to the property $g_{AB}g^{BC} = \delta^C_A$:

$$g^{MN} = \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & \frac{1}{\phi} + A^2 \end{pmatrix}.$$

By making the assumption that the metric does not depend on the extra coordinate y (so-called cylinder condition, assumed by Kaluza) $g_{MN,4} = 0$, we can calculate the corresponding Einstein equations.
The Christoffel symbols are thus given by:

$$\Gamma^4_{44} = 0; \tag{2.65}$$

$$\Gamma^4_{\mu4} = \phi_{,\mu} + A^\rho A_\mu \phi\phi_{,\rho} - \frac{1}{2}\phi^2 A^\rho F_{\mu\rho}; \tag{2.66}$$

$$\Gamma^\nu_{\mu4} = \frac{1}{2}\phi^2 F^\nu_\mu - \phi A_\mu \phi^{,\nu}; \tag{2.67}$$

$$\Gamma^\nu_{\mu\nu} = \frac{1}{2}g^\nu_{\mu\nu,} - A^\nu\phi^2 F_{\nu\mu} + 2\phi\phi_{,\mu}A^2 - \phi^{,\nu}A_\mu A_\nu\phi + \frac{1}{2}\phi^2 A^\nu\left(A_{\mu,\nu} + A_{\nu,\mu}\right), \tag{2.68}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the field strength tensor.
We apply the empty space condition $R_{AB} = 0$ for the Riemann tensor:

$$R_{AB} = \Gamma^C_{AB,C} - \Gamma^C_{AC,B} + \Gamma^C_{AB}\Gamma^D_{CD} - \Gamma^C_{AD}\Gamma^D_{BC} \tag{2.69}$$

and we get:

$$R_{\mu\nu} = 0 \quad \rightarrow \quad \tilde{G}_{\mu\nu} = \frac{\phi^2}{2}\tilde{T}_{\mu\nu} - \frac{1}{\phi}\left[\nabla_\mu\left(\phi_\nu\right) - g_{\mu\nu}\Box\phi\right]; \tag{2.70}$$

$$R_{\mu4} = 0 \quad \rightarrow \quad \nabla^\mu F_{\mu\nu} = -3\frac{\phi^{,\mu}}{\phi}F_{\mu\nu}; \tag{2.71}$$

$$R_{44} = 0 \quad \rightarrow \quad \Box\phi = \frac{\phi^3}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.72}$$

where

$$\tilde{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{2.73}$$

is the usual Einstein tensor, and

$$\tilde{T}_{\mu\nu} = \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} - F^\rho_\mu F_{\nu\rho} \tag{2.74}$$

is the electromagnetic stress-energy tensor.
If the scalar field ϕ is constant, we recover Einstein and Maxwell equations from (2.70) and (2.71):

$$\tilde{G}_{\mu\nu} = \frac{\phi^2}{2}\tilde{T}_{\mu\nu}, \qquad \nabla^\mu F_{\mu\nu} = 0. \tag{2.75}$$

We can rescale $A_\mu \to kA_\mu$ so as to get the correct coefficient in the Einstein equations: $k = 4\sqrt{\pi G}$ and $\phi = 1$.
This is exactly what Kaluza did, but, as was later pointed out by Thiry and Jordan, this is inconsistent with the third field equation (2.72) unless $F^{\mu\nu}F_{\mu\nu} = 0$. In that case, we would simply obtain th Klein-Gordon equation for a massless scalar.
We saw how the source-less 5-dimensional equations $R_{AB} = 0$ give rise to the field equations (2.70),(2.71),(2.72) which are instead sourced. Therefore, we can conclude that it was proven how 4-dimensional radiation can be originated simply by empty 5d space geometrical properties.
If we do not set ϕ =constant and do not consider the cylinder condition, we can instead choose the coordinates so that $A_\mu = 0$. This is only valid in homogeneous and isotropic situations, as the off-diagonal components of the metric vanish:

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \phi^2 \end{pmatrix}.$$

In these cases only scalar fields dominate and we are in the so-called "graviton-scalar sector" of the theory.
This model corresponds to a Brans-Dicke theory with a parameter $\omega \gtrsim 500$ which is ruled out by experiments, at least in the present era.

2.3.2 Compactification

The assumption that physics is not influenced by the fifth coordinate which is encoded in Kaluza's cylinder condition was justified by Klein in 1926. He explained the fact that we do not see the effects of the extra dimension by making it very small, so that its contribution would be negligible.
The fifth dimension is taken to be of circular topology (S^1) and small scale. Therefore, if y is the extra coordinate and R is the radius of the circle:

$$y \widehat{=} y + 2\pi R. \tag{2.76}$$

From this property of periodicity, we can Fourier expand in the y coordinate all the fields that are involved:

$$\phi(x,y) = \sum_{n=-\infty}^{n=+\infty} \phi^{(n)}(x)e^{\frac{iny}{R}}; \tag{2.77}$$

$$g_{\mu\nu}(x,y) = \sum_{n=-\infty}^{n=+\infty} g_{\mu\nu}^{(n)}(x)e^{\frac{iny}{R}}; \tag{2.78}$$

$$A_\mu(x,y) = \sum_{n=-\infty}^{n=+\infty} A_\mu^{(n)}(x)e^{\frac{iny}{R}}, \tag{2.79}$$

with the superscript (n) to indicate the nth Fourier mode.
Therefore, these modes carry a momentum n/R and if R is small, as assumed by Klein, then even for $n = 1$ they would be undetectable and only the 0th mode (independent of y) would remain, as it was in Kaluza's theory. Then, if the scale R is small enough, there is an invariance for translations along the extra dimension, which corresponds to moving around the circle:

$$y \to y' = y + f(x). \tag{2.80}$$

So, by applying the tensor transformation law:

$$g_{AB} \to g'_{AB} = \frac{\partial x^C}{\partial x'^A}\frac{\partial x^D}{\partial x'^B} g_{CD} \tag{2.81}$$

and using the metric (2.64), we see that the only component that is transformed under the translation (2.80) is:

$$g'_{\mu 4} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial y}{\partial y'} g_{\nu 4} + \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\rho}{\partial y'} g_{44} = \delta^\nu_\mu g_{\nu 4} + \delta^\nu_\mu \partial^\rho f(x) g_{\rho\nu} = g_{\mu 4} + \partial_\mu f(x). \tag{2.82}$$

This implies then:

$$A_\mu \to A_\mu + \partial_\mu f(x), \tag{2.83}$$

which is a U(1) gauge transformation. So, as expected, A_μ really corresponds to the electromagnetic field, inserted in a 5-dimensional theory.

2.3.3 Masses and charge

We can see an application of the above described mechanism on the free real scalar field:

$$S_5 = M_5 \int d^5x \sqrt{-g}\, \partial_M \Phi \partial^M \Phi^*, \tag{2.84}$$

where M_5 is a parameter with dimensions of a mass, as we conclude from the requirement that the action must be dimensionless. In fact, from the definition $S = \int d^d x \mathscr{L}$, we know that every term in the Lagrangian must have mass-dimensions d.
The kinetic term for a scalar field:

$$d = [\partial_\mu \Phi \partial^\mu \Phi^*] = 2 + 2\,[\Phi] \quad \to \quad [\Phi] = \frac{d}{2} - 1 = 1 \quad \text{if d=4} \,. \tag{2.85}$$

In order to keep the same mass dimensions for Φ even in 5 dimensions, we introduce the scale M_5 so that:

$$d = [M_5 \partial_\mu \phi \partial^\mu \phi^*] = 4 + [M_5] \quad \to \quad [M_5] = d - 4 = 1 \quad \text{if d=5} \,. \tag{2.86}$$

By plugging in the Fourier expanded field, normalized by the volume of the extra dimension, and using the inverse metric of (2.64), we get:

$$\begin{aligned} S_5 = \frac{M_5}{2\pi R} \sum_n \int d^4x dy \sqrt{-g} \Big[& \partial_\mu \Phi^{(n)} \partial^\mu \Phi^{(n)} - \frac{in}{R} A^\mu \Phi^{(n)} \partial_\mu \Phi^{(n)} \\ & + \frac{in}{R} A^\mu \partial_\mu \Phi^{(n)} \Phi^{(n)} - \left(\frac{1}{\phi} + A^2\right) \frac{n^2}{R^2} \Phi^{(n)2} \Big] \\ = \frac{M_5}{\sum_n} \int d^4x & \left[\left[\left(\partial_\mu + \frac{in}{R} A_\mu \right) \Phi^{(n)} \right]^2 - \frac{n^2}{\phi R^2} \Phi^{(n)2} \right]. \end{aligned} \tag{2.87}$$

We notice that we can define a covariant derivative as $\left(\partial_\mu + \frac{in}{R} A_\mu\right)$, in an analogous way to QED, where $\partial_\mu \to \partial_\mu + iqA_\mu$. The role of the charge q is taken up by $\frac{n}{R}$ and thus, by a convenient normalization, we can identify the electronic charge with the first Kaluza-Klein mode q_1.
We can use it to predict the coupling constant for QED:

$$\alpha \equiv \frac{q_1^2}{4\pi} \simeq 4, \tag{2.88}$$

if we consider $R\sqrt{\phi} \sim l_p = \sqrt{G}$ and $q_n = \frac{n}{R\sqrt{\phi}} 4\sqrt{\pi G}$.
Taking a better approximation for $R\sqrt{\phi}$ would probably get closer to the actual value, but the masses pose the real problem. Indeed, as the action is now made up of a kinetic and a mass term, we can read off the mass from the quadratic term:

$$m_n = \frac{|n|}{R\sqrt{\phi}}. \tag{2.89}$$

Then the mass of the electron m_1 would be, assuming $R\sqrt{\phi} \sim l_p$ as before, $m_1 \sim l_p^{-1} = 10^{19}$ GeV $\neq 0.5$ MeV by several orders of magnitude.
As a consequence to this result the theory was first abandoned and later fixed by assuming the observed parameters are instead the $n = 0$ ones. Even though it would imply the mass of the particles is null, the experimental value for the masses can be recovered by considering a SSB mechanism. The problem of giving a non-zero charge to SM particles can be shown to be instead solved by considering higher-dimensional theories.
Now we can consider another example with a complex scalar field, described by the action:

$$S_5 = M_5 \int d^4x dy \sqrt{-g} \left[|\partial_\mu \Phi|^2 + |\partial_y \Phi|^2 + \lambda_5 |\Phi|^4 \right]. \tag{2.90}$$

Next, we require the compactification of the extra dimension y and Fourier expand Φ:

$$\Phi(x, y) = \frac{1}{\sqrt{2\pi R M_5}} \sum_{n=-\infty}^{n=+\infty} \Phi^{(n)}(x) e^{\frac{iny}{R}} \qquad \Phi^{(n)\dagger}(x) = \Phi^{(-n)}(x). \tag{2.91}$$

By inserting it in (2.90) with the simple metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu + dy^2$ we get:

$$S_5 = \int d^4x dy \sqrt{-g} \frac{1}{2\pi R} \sum_n \left[\left|\partial_\mu \Phi^{(n)}\right|^2 - \frac{n^2}{R^2} \left|\Phi^{(n)}\right|^2 + \frac{\lambda_5}{2\pi R} \left|\Phi^{(n)}\right|^4 \right]. \tag{2.92}$$

We redefine $\lambda_4 \equiv \frac{\lambda_5}{2\pi R}$ and after integrating over y (there is no y dependence in the Lagrangian now so it only gives a factor $2\pi R$) we see that it is possible to decompose (2.90) in 2 parts:

$$S_5 = S_4^{(0)} + S_4^{(n)} \tag{2.93}$$

defined as:

$$S_4^{(0)} = \int d^4x \sqrt{-g} \left[\left|\partial_\mu \Phi^{(0)}\right|^2 + \lambda_4 \left|\Phi^{(0)}\right|^4 \right]; \tag{2.94}$$

$$S_4^{(n)} = \int d^4x \sqrt{-g} \sum_{n \neq 0} \left[\left|\partial_\mu \Phi^{(n)}\right|^2 - \frac{n^2}{R^2} \left|\Phi^{(n)}\right|^2 + \lambda_4 \left|\Phi^{(n)}\right|^4 \right]. \tag{2.95}$$

It is evident that (2.94) corresponds to the 4d scalar theory for a massless field, while the term (2.95) refers to an infinite *Kaluza-Klein tower* of massive modes $\phi^{(n)}(x)$.
The equation of motion for (2.84) is:

$$\partial_M \partial^M \Phi = 0, \tag{2.96}$$

which corresponds to a 5-dimensional Klein-Gordon equation.
By substituting the Kaluza-Klein expansion (2.91), this becomes:

$$\sum_n \left(\partial^\mu \partial_\mu - \frac{n^2}{R^2} \right) \Phi^{(n)}(x) e^{\frac{iny}{R}} = 0 \tag{2.97}$$

and it corresponds to:

$$\Box \Phi^{(n)}(x) = \frac{n^2}{R^2} \Phi^{(n)}(x), \tag{2.98}$$

where $m_n^2 \equiv \frac{n^2}{R^2}$ is the mass of the Kaluza-Klein modes. Therefore, it is possible to treat a theory with $d > 4$ as if it were 4 dimensional but with an infinite amount of fields. Even though the theory is non-renormalizable due to the presence of infinite modes, we can achieve renormalizability by truncating the series and using only the lowest Kaluza-Klein excitations. In fact, the heavy modes would only be significant at high energies $E \gg R^{-1}$. In particular, if R is very small, the Kaluza-Klein modes are heavy (since $m_n^2 = \frac{n^2}{R^2}$) and basically only the 0 mode contributes to the action, i.e. the model is effectively 4 dimensional. Conversely, if R is big enough, the extra dimensions can be regarded as flat (the radius $R \to \infty$) and the spacetime is seen as d-dimensional. Also, in this limit $\lambda_4 \to 0$ and thus the scalar field is weakly coupled.
Another thing we notice is that if we had taken the extra dimension y to be temporal instead of spatial, in addition to having an unclear interpretation, it would be problematic in the fact that it would generate tachyonic modes in the Klein-Gordon equation. Indeed, the sign of $g_4 4$ would be opposite, giving a negative mass. It would, in fact, appear in the metric with an opposite sign and the KG equation derived from (2.96) would give negative mass modes.

2.4 ADD Model

The idea to solve the hierarchy problem came in 1998 by Nima Arkani-Hamed, Savas Dimopoulos and Georgi Dvali and formulated in [12]-[13]. To introduce extra dimensions while at the same time explaining why they remain undetected, they are taken to be finite and compactified, analogously to Kaluza-Klein's. To keep these dimensions hidden, the simplest idea is to take them small so that in order to probe them one would need a very large energy. Nevertheless, this model concerns large extra dimensions in order to recover the experimental value of the Planck mass, as the observed physics should be a four-dimensional effective theory of a more fundamental general model.

We start by introducing n extra dimensions, so that spacetime is effectively d-dimensional with $d = 4 + n$. Then, the reduced Planck mass $M_{pl} = 2 \cdot 10^{18}$ GeV is found to be given in terms of the d-dimensional Planck mass M_d by the relation $M_{pl}^2 = M_d^{n+2} V_n$, where V_n is the volume of the compact extra dimensions. Therefore, the large value of M_{pl} can be explained without affecting M_d, which can be taken to be ~ 1 TeV in a natural way, by assuming the extra dimensions are very large, increasing the volume V_n.

Since the Standard Model works well in the usual four dimensions, only gravity is taken to be able to propagate on the whole d-dimensional spacetime, called *bulk*, while the SM is confined on a *3-brane*, that is a four-dimensional subspace. Indeed the mass scale of compactification $\mu_c \sim R_n^{-1} \propto V_n^{-\frac{1}{n}}$ is much smaller than M_{ew} and should be able to change the electroweak theory we measure. Since we do not detect any changes, it is reasonable to assume the SM is not affected by a d-dimensional space, but rather lives only on a four-dimensional subspace. Then, only gravity, as it is defined as a distortion of spacetime itself, is able to enter the extra dimensions and new physics that is given by these additional dimensions would only affect the gravitational sector. This is why we were not able to detect anything despite having large extra dimensions. The true bond on the size of these is given by experimental tests on deviations from Newton's law that have been measured so far ([17]-[18]).

To avoid confusion, in this section we use a different notation for the Planck mass. To clarify it once more: M_{pl} indicates the four-dimensional reduced Planck mass; M_p is the four-dimensional Planck mass: $M_p = \frac{M_{pl}}{\sqrt{8\pi}}$ and finally M_d indicates the d-dimensional Planck scale.

2.4.1 The model

We start by constructing the higher dimensional action by generalizing the Einstein-Hilbert one. The general metric is given by $ds^2 = g_{MN}dx^M dx^N$ where the indices are $M, N = 0 \dots 4$. Moreover, if we take g_{MN} to be dimensionless, then the Christoffel symbols' mass units are

$$\left[\Gamma^P_{MN}\right] = \left[g^{PR} g_{MN,R}\right] = [\partial_R] = 1. \tag{2.99}$$

The Riemann tensor, therefore, has units:

$$[R_{MN}] = \left[\Gamma^P_{MN,P}\right] = 2. \tag{2.100}$$

To make the action dimensionless and in the same form as the four-dimensional Einstein-Hilbert action, we need to add a parameter with dimensions $d-2$ (since $[\mathcal{R}] = [R_{MN}] = 2$ and $\left[d^d x\right] = -d$). This parameter represents a new mass scale and we denote it as $\frac{\bar{M}_d^{2+n}}{2}$ for later convenience.
Furthermore, as the Standard Model has been accurately tested at distances $\sim M_{ew}$, we can assume the extra dimensions have a negligible effect on it, as we said in the introduction. Therefore, the SM particles are confined to a 3-brane, with induced metric g_4.
Taking all this into consideration, we obtain:

$$S_{EH} = \frac{\bar{M}_d^{2+n}}{2} \int d^4x d^n y \sqrt{-g}\mathcal{R} + \int d^4 x \sqrt{-g_4}\mathscr{L}_{SM}, \tag{2.101}$$

where x has been used for the four-dimensional spacetime coordinates and y for the n extra ones.
We consider the n extra dimensions to be compactified on a torus of radius R, so that the volume takes the simple form $V_n = (2\pi R)^n$.

Next, we proceed to Kaluza-Klein-expanding the graviton field:

$$g_{\mu\nu}(x,y) = g^{(0)}_{\mu\nu}(x) + \frac{1}{\sqrt{V_n}} \sum_{\vec{n}\neq 0} g^{(n)}_{\mu\nu}(x) e^{\frac{i\vec{n}\vec{y}}{R}}. \tag{2.102}$$

By performing a Kaluza-Klein reduction as was done in the dedicated section, at low energies we have that only the 0-th mode is relevant:

$$S^{(0)}_{EH4} = \frac{\bar{M}_d^{2+n}}{2} V_n \int d^4x \sqrt{-g^{(0)}} \mathcal{R}^{(0)} + \int d^4x \sqrt{-g^{(0)}_4} \mathscr{L}_{SM} \tag{2.103}$$

and it must match the four-dimensional theory:

$$S_{EH4} = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g_4} \mathcal{R}_4 + \int d^4x \sqrt{-g_4} \mathscr{L}_{SM}, \tag{2.104}$$

where the subscript 4 indicates the four-dimensional spacetime.
The low energy correspondence with the four-dimensional theory implies a relation between the usual Planck mass M_{pl} and the mass scale $\bar{M}_d$. Indeed, remembering the relation found in the specific section: $M_p^{-2} = G_4$, we can write a correspondence in terms of the Planck mass

$$\frac{1}{16\pi G_4} = \frac{M_{pl}^2}{2} \overset{!}{=} \frac{\bar{M}_d^{2+n}}{2} V_n. \tag{2.105}$$

Therefore

$$M_{pl}^2 = \bar{M}_d^{2+n} V_n \equiv M_d^{2+n} R^n, \tag{2.106}$$

where we redefined $M_d \equiv \bar{M}_d (2\pi)^n$.
To solve the hierarchy problem, we assumed there is only one relevant mass scale, that is $M_{ew} \sim 1$ TeV, therefore we impose $M_d \sim 1$ TeV. To recover the experimental value for the four-dimensional reduced Planck mass M_{pl}, which is:

$$\tilde{M}_{pl} = 2.435 \cdot 10^{18} \text{GeV}/c^2, \tag{2.107}$$

we impose a constraint on R. Expressing it from the above condition (2.106), we can predict possible values for the compactification radius:

$$\frac{1}{R} = M_d \left(\frac{M_d}{M_{pl}}\right)^{\frac{2}{n}} = (1\text{TeV}) 10^{-\frac{30}{n}}. \tag{2.108}$$

By using the conversion factor 1 $\text{GeV}^{-1} = 2 \cdot 10^{-14}$ cm, we get the following results:

For $n = 1$, $R \sim 10^{13}$ cm, which would imply deviations from Newtonian gravity on the scales of the Solar systems (one astronomic unit is $1.5 \cdot 10^{13}$ cm), but since no such thing has been observed, it is ruled out.

For $n = 2$, $R \sim 0.1$ mm and this case is particularly interesting since it would be possible to probe these distances in the near future. So far, gravity has only been tested at scales of 1 mm.

For $n \geq 2$ the scale R of the extra dimensions gets smaller and smaller $R \leq 10^{-6}$ cm in the range not yet investigated by experiments and very unlikely to be probed soon.

The effective value of the coupling constant, which contains the Planck mass, gets lowered from proportional to M_{pl} to a value proportional to M_d, as a result of the dependence on the radius of compactification. This solves the hierarchy problem, as M_d is set to be equal to the Electroweak scale and it also contributes to build expectations for the future of colliders, as they would be able to probe energies close to the new threshold for quantum gravity.

Next, we can apply the same reasoning to the electromagnetic field, while removing the assumption that the Standard Model is localized on the 3-brane:

$$S = \int d^4x d^n y \sqrt{-g^{(4+n)}} \frac{1}{4g_d^{*2}} F_{MN} F^{MN}, \tag{2.109}$$

where g_d^{*2} is the d-dimensional coupling constant and the theory is take to be non-canonically normalized.
Now we perform a Kaluza-Klein reduction, keep the 0-th mode and integrate over the extra dimensions, like we did before:

$$S = \frac{V_n}{4g_d^{*2}} \int d^4x \sqrt{-g^{(4)}} F_{\mu\nu} F^{\mu\nu}, \tag{2.110}$$

where $F_{\mu\nu}$ is the usual four-dimensional electromagnetic field strength.
The matching with the corresponding four-dimensional action is thus:

$$\frac{V_n}{g_d^{*2}} = \frac{1}{g^{*2}}. \tag{2.111}$$

It is now evident that the d-dimensional coupling constant is not dimensionless any more: $[g_d^*] = [V_n]^{1/2} = -\frac{n}{2}$. Consequently, the theory is non-renormalizable and can thus be thought of as an effective theory for a higher dimensional more fundamental one.
To solve this, we apply the same reasoning we used for the gravitational scale to the gauge field and we assume:

$$g_d^* \sim \frac{1}{\bar{M}_d^{n/2}}, \tag{2.112}$$

using $M_d \sim 1$ TeV as the only mass scale.
Then the above relation (2.111) becomes:

$$\frac{1}{g^{*2}} \sim V_n \bar{M}_d^n = R^n M_d^n. \tag{2.113}$$

From this, we can obtain M_d and substitute it in (2.106), getting:

$$R \sim \frac{g_4^{\frac{n-2}{n}}}{M_{pl}}. \tag{2.114}$$

This would imply $R \sim \frac{1}{M_{pl}}$ but, as such, it would rule out the possibility of finding extra dimensions in the future. This was the main obstacle to the development of theories with extra dimensions until the '90s.

2.4.2 Observational evidence

Experiments on gravity are difficult to perform, because at small scales intermolecular and electromagnetic interactions are prevalent and gravity is negligible. This is the reason why Newton's law has only been tested up to the scale ~ 0.1 mm by essentially Cavendish-type experiments. The method to investigate the presence of extra dimensions is by studying the deviation from the Newtonian potential $\Phi(r) \sim \frac{1}{r}$.

To see this, we first derive the expression for the gravitational potential in d dimensions. To start with, we define the gravitational force per unit test mass:

$$\vec{g} = -\nabla\Phi, \tag{2.115}$$

where Φ is the potential.
We can infer $g(r) \propto \frac{1}{r^{d-2}}$ from the request that the gravitational field is a coulombian field, i.e. its divergence is null if $r \neq 0$. We use the fact that we can write $\vec{g}(r) = g(r)\hat{r}$ as it is a radial field and write $\hat{r} = \frac{\vec{r}}{r}$. Then, in the second line, we make use of $\boldsymbol{\nabla} \cdot r = d-1$ and $\vec{\nabla} r = \frac{\vec{r}}{r}$

$$\begin{aligned} 0 = \vec{\nabla}\cdot\vec{g} = \vec{\nabla}\cdot[g(r)\hat{r}] = \vec{\nabla}\cdot\left[\frac{g(r)}{r}\vec{r}\right] = \frac{g(r)}{r}\vec{\nabla}\cdot\vec{r} + \vec{\nabla}\left[\frac{g(r)}{r}\right]\cdot\vec{r} \\ = \frac{g(r)}{r}(d-1) + \left(\frac{dg}{dr}\frac{1}{r} - \frac{g(r)}{r^2}\right)\frac{\vec{r}}{r}\vec{r} = \frac{g(r)}{r}(d-2) + \frac{dg}{dr} \\ = \frac{1}{r^{d-2}}\frac{d}{dr}(r^{d-2}g(r)). \end{aligned} \tag{2.116}$$

Therefore the potential is:

$$\Phi(r) = -\frac{GM}{r^{d-3}}. \tag{2.117}$$

So, roughly, the potential should behave this way, with G representing a d-dimensional gravitational constant. We can see it more precisely in another way.

In the usual four-dimensional spacetime, the potential Φ is determined by the Poisson equation:

$$\nabla^2_{(3)}\Phi^{(4)} = 4\pi G^{(4)}\rho^{(4)}, \tag{2.118}$$

where the number in brackets indicates the dimensionality of the space considered. By inverting the Laplacian we find the solution:

$$\Phi = \frac{M}{r}G^{(4)} = \frac{M}{M_p^2}\frac{1}{r} \tag{2.119}$$

for a mass M and $G^{(4)} = M_p^{-2}$.
The equation (2.118) is derived from the weak field limit of the Einstein equation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \text{ everywhere} \tag{2.120}$$

with conditions $|T_{00}| \gg |T_{0i}| \gg |T_{ij}|$. We can generalize the construction to d arbitrary dimensions, as follows.
By using the harmonic gauge:

$$h^{\mu\nu}_{,\nu} = \frac{1}{2}\eta^{\mu\nu}h^{\lambda}_{\lambda,\nu}, \tag{2.121}$$

where the index $\mu = 0, \dots, d$ covers all dimensions, the Ricci tensor and scalar take the form:

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}, \qquad R = -\frac{1}{2}\Box h. \tag{2.122}$$

The Einstein's equation is thus:

$$-\frac{1}{2}\left[\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h\right] \equiv -\frac{1}{2}\Box \bar{h}_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.123}$$

Taking only the dominant component h_{00} and neglecting the temporal derivatives $\Box \simeq \nabla^2$:

$$\nabla^2 \bar{h}_{00} = -16\pi G\rho. \tag{2.124}$$

We argue that this must be the d-dimensional Poisson equation:

$$\nabla^2_{(d-1)}\Phi^{(d)} = 4\pi G^{(d)}\rho^{(d)}. \tag{2.125}$$

Therefore we establish the correspondence $\bar{h}_{00} = -4\Phi^{(d)}$.
Since

$$h = h^\lambda_\lambda \simeq h^0_0 \simeq \eta^{00}h_{00} = -h_{00}, \tag{2.126}$$

we get

$$\bar{h}_{00} = h_{00}\left(1 - \frac{1}{2}\right) \quad \rightarrow \quad h_{00} = -2\Phi^{(d)}. \tag{2.127}$$

By inverting the Laplacian we find:

$$\Phi^{(d)} = -\frac{8\pi G}{(d-2)\Omega_{d-2}}\frac{M}{r^{d-3}}, \tag{2.128}$$

which is exactly the behaviour we already mentioned and which we aimed to find. The constant differs from the one in (2.119) due to a redefinition of G.

Now, we inspect the behaviour of the potential under toroidal compactification of the extra dimensions, also re-deriving equation (2.106) through the use of Gauß law. We introduce $d-4$ compact spacial extra dimensions with period $2\pi R \equiv L$:

$$y_i \widehat{=} y_i + L, \qquad i = 1 \dots n. \tag{2.129}$$

We place a mass m in the origin and unfold the compactified extra dimensions, so that it corresponds to repeating the mass every distance L from the previous one. A test mass placed at $r \ll L$ will feel the d dimensional force law, since the mirror masses give a negligible contribution. On the other hand, if $r \gg L$, the mirror masses will be too close to each other to be distinguished and together they will form a $d-4$ dimensional line of homogeneous distribution.

The gravitational field, in the first case, is:

$$g(r) = -\frac{Gm}{r^{d-2}}, \tag{2.130}$$

where G is understood as d-dimensional.

Next, we study the second case as an application of Gauß' law to a line homogeneous distribution.
We choose a hyper-cylinder C of length l as a Gaußian surface placed around the multi-line.
Gauß' law in d dimensions is:

$$\int_{\partial C} d^{d-2}x \quad \vec{g}\cdot\hat{n} = \int_C d^{d-1}x \quad \vec{\nabla}\cdot\vec{g}. \tag{2.131}$$

We start by calculating the flux of $\vec{g}$ in $d-1$ dimensions:
We consider a mass M placed in the origin and a (d-1)-sphere of radius R as Gaußian surface:

$$\begin{aligned}\int_{S^{d-1}} d^{d-2}x \quad \vec{g}\cdot\hat{n} &= \int d\Omega_{d-2}R^{d-2}g(R) = -\Omega_{d-2}R^{d-2}\frac{GM}{R^{d-2}} \\ &= -\Omega_{d-2}GM.\end{aligned} \tag{2.132}$$

Now we surround the mass by an arbitrary surface Σ and a sphere S. Then, we link them together by two paths ($P1$ and $P2$) distant ϵ from each other.
Calling the total volume V:

$$\begin{aligned}\int_V d^{d-1}x \quad \vec{\nabla}\cdot\vec{g} = 0 = &\int_S d^{d-2}x \quad \vec{g}\cdot\hat{n} + \int_\Sigma d^{d-2}x \quad \vec{g}\cdot\hat{n} \\ &+ \left(\int_{P1} - \int_{P2}\right) d^{d-2}x \quad \vec{g}\cdot\hat{n},\end{aligned} \tag{2.133}$$

where the contribution of the two paths cancels because $\vec{g}$ is the same, since their distance is infinitesimal, but the normal unit vector $\hat{n}$ is opposite.
We conclude:

$$-\int_S d^{d-2}x \quad \vec{g}\cdot\hat{n} = \int_\Sigma d^{d-2}x \quad \vec{g}\cdot\hat{n}, \tag{2.134}$$

where the minus sign is due to the orientation of the surfaces.
Then we just proved that the result (2.132) is independent of the shape of the Gaußian surface. Then we consider that since the divergence of $\vec{g}$ is null except on $r=0$, if the mass is placed outside of the Gaußian surface, the flux is 0.
Next, we can express the total mass M as:

$$M = m\left(\frac{l}{L}\right)^{d-4}, \tag{2.135}$$

which means the total mass is given by the number of mirror masses m contained in the cylinder in every extra dimension.
Then the r.h.s of (2.131) is:

$$-\Omega_{d-2}Gm\left(\frac{l}{L}\right)^{d-4}. \tag{2.136}$$

To calculate the l.h.s of (2.131):

$$\int_{\partial C} d^{d-2}x \quad \vec{g}\cdot\hat{n} = g(r)\cdot\text{Area of C}, \tag{2.137}$$

we need the area of the cylinder.
The hyper-cylinder can be written as the Cartesian product $B^3 \times L^{d-4}$. L^{d-4} is the multiline, while B^3 is the 3-ball of volume:

$$V = \frac{\pi^{\frac{3}{2}} r^3}{\Gamma\left(\frac{5}{2}\right)} \tag{2.138}$$

and area

$$A = \frac{\partial}{\partial r} V. \tag{2.139}$$

The area of the hyper-cylinder is thus given by:

$$4\pi r^2 l^{d-4} \tag{2.140}$$

and therefore equation (2.131) is:

$$g(r) 4\pi r^2 l^{d-4} = -\Omega_{d-2} G m \left(\frac{l}{L}\right)^{d-4}. \tag{2.141}$$

This allows us to conclude:

$$g(r) = -\frac{\Omega_{d-2} G}{4\pi L^{d-4}} \frac{m}{r^2}, \tag{2.142}$$

which is the typical behaviour of a four-dimensional gravitational field $g(r) \sim \frac{1}{r^2}$.
We can identify:

$$G_4 = \frac{\Omega_{d-1} G}{4\pi V_{(d-4)}}, \tag{2.143}$$

where $V_{(d-4)}$ is the volume of the compactified extra dimensions.
We can conclude the following:

$$\begin{cases} g(r) = -\dfrac{Gm}{r^{d-2}} & \text{if } r \ll L; \\ g(r) = -\dfrac{G_4 m}{r^2} & \text{if } r \gg L. \end{cases} \tag{2.144}$$

Now, in order to write the same relation in terms of the Planck masses, we need to use the d-dimensional Planck mass.
In the dedicated section we already found:

$$G = \frac{1}{M_d^{d-2}}, \tag{2.145}$$

so that:

$$\frac{1}{M_p^2} = G_4 = \frac{\Omega_{d-2} G}{4\pi V_{(d-4)}} = \frac{\Omega_{d-2}}{4\pi} \frac{1}{M_d^{d-2} V_{(d-4)}}. \tag{2.146}$$

From this we get again

$$M_p^2 \propto M_d^{d-2} V_{(d-4)} \tag{2.147}$$

and:

$$\begin{cases} g(r) = -\dfrac{m}{M_d^{d-2}} \dfrac{1}{r^{d-2}} & \text{if } r \ll L; \\ g(r) = -\dfrac{\Omega_{d-2}}{4\pi} \dfrac{m}{M_d^{d-2} V_{(d-4)}} \dfrac{1}{r^2} & \text{if } r \gg L. \end{cases} \tag{2.148}$$

We can now study the deviation from Newton's law on toroidal compactification and analyse the corrections to the four-dimensional potential.
They can be parametrized as:

$$\Phi(r) = -\frac{G^{(4)}M}{r}\left(1 + \alpha e^{-\frac{r}{\lambda}}\right). \tag{2.149}$$

We infer that the potential grows stronger as the distance r between the masses decreases. The parameter α determines the strength of the deviation, while λ signals the scale at which this becomes relevant. The value of λ is found to be at most 1 mm, leaving open the possibility of discovering submillimeter new forces.
The potential for the gravitational force calculated before is:

$$\Phi(r) = -\frac{GM}{r^{n+1}}, \tag{2.150}$$

where n represents the number of extra dimensions.
On toroidal compactification, we perform the same trick of unfolding the extra dimensions and replicating the masses every L. The square distance between the test mass and the mass M becomes $\sum_{b\in\mathbb{Z}}\left(r^2 + \sum_{i=1}^{n}(Lb)^2\right)$ where b is an integer and r is the distance of the test mass from the centre.
In general, we can consider a different radius of compactification for every extra dimensions and thus:

$$\Phi(r) = -\sum_{\vec{b},b^i\in\mathbb{Z}}\frac{GM}{\left[r^2 + \sum_{i=1}^{n}(L_i b^i)^2\right]^{\frac{n+1}{2}}}. \tag{2.151}$$

In the limit $r \gg L$ we can approximate the sum over b with an integral:

$$\Phi(r) = -\int_{-\infty}^{\infty} d^n b \frac{GM}{\left[r^2 + \sum_{i=1}^{n}(L_i b^i)^2\right]^{\frac{n+1}{2}}}. \tag{2.152}$$

Now we make the substitution $x_i = \frac{L_i b^i}{r}$ and obtain:

$$\Phi(r) = -\int_{-\infty}^{\infty} d^n x \frac{r^n}{V_n}\frac{GM}{\left[1 + \sum_{i=1} x_i^2\right]^{\frac{n+1}{2}}}\frac{1}{r^{n+1}}, \tag{2.153}$$

where $V_n = (2\pi)^n \prod_i R_i$.
Now we change to polar coordinates:

$$\Phi(r) = -\frac{GM}{rV_n}\int_0^{\infty} d\rho\,\Omega_{n-1}\frac{\rho^{n-1}}{(1+\rho^2)^{\frac{n+1}{2}}}. \tag{2.154}$$

Subsequently, we perform a new change of variables $u = \rho^2$:

$$\Phi(r) = -\frac{GM}{2rV_n}\Omega_{n-1}\int_0^{\infty} du \frac{u^{\frac{n-1}{2}}}{(1+u)^{\frac{n+1}{2}}}. \tag{2.155}$$

In this expression, we recognize the Beta function, defined as:

$$B(p-1,q+1)=\int_0^\infty du\frac{u^p}{(1+u)^{p+q+2}}=\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}, \tag{2.156}$$

so that we get:

$$\Phi(r)=-\frac{GM}{2rV_n}\Omega_{n-1}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}=-\frac{GM}{2rV_n}\Omega_n. \tag{2.157}$$

This is the Newtonian behaviour we would expect, with the identification:

$$G^{(4)}=\frac{G}{2V_n}\Omega_n. \tag{2.158}$$

Now, to see the corrections to the potential, we must also take the limit $r \ll L$ and investigate the short distance regime of (2.151). To do so, we employ the Poisson summation formula: for a periodic function $f(nR)$ with $n \in \mathbb{Z}$ it allows us to write it in another way by using its Fourier transform $\tilde{f}$:

$$\sum_{n=-\infty}^{\infty} f(nR)=\sum_{n=-\infty}^{\infty}\frac{\tilde{f}\left(\frac{n}{R}\right)}{2\pi R}. \tag{2.159}$$

We can generalize it to the case of periodicity in n directions.
First, we define the quantities: $\vec{m}=\left(\frac{b^1}{R_1},\dots,\frac{b^n}{R_n}\right)$ and $\vec{y}=\left(L_1b^1,\dots,L_nb^n\right)$ and then obtain:

$$\Phi(r)=-\frac{GM}{V_n}\int d^ny\sum_{\vec{b},b^i\in\mathbb{Z}}\frac{e^{-i\vec{m}\cdot\vec{y}}}{\left[r^2+\sum_{i=1}^n\left(x_i^2-2\pi R_ib^i\right)^2\right]^{\frac{n+1}{2}}}. \tag{2.160}$$

Now we perform a change of variables by shifting the y coordinate $y \to y+x$, and we notice the measure is invariant:

$$\Phi(r)=-\frac{GM}{V_n}\int d^ny\sum_{\vec{b},b^i\in\mathbb{Z}}e^{-i\vec{m}\cdot\vec{x}}\frac{e^{-i\vec{m}\cdot\vec{y}}}{\left[r^2+\sum_{i=1}^n(y_i)^2\right]^{\frac{n+1}{2}}}. \tag{2.161}$$

Then we change to spherical coordinates, but making sure to keep out the θ dependence, as in:

$$\int d^ny=\int d\rho d\theta\Omega_{n-2}\rho^{n-1}\sin^{n-2}\theta. \tag{2.162}$$

The result is, denoting $m=|\vec{m}|$:

$$\Phi(r)=-\frac{GM}{V_n}\Omega_{n-2}\sum_{\vec{b},b^i\in\mathbb{Z}}e^{-i\vec{m}\cdot\vec{x}}\int_0^\infty d\rho\rho^{n-1}\frac{1}{\left[r^2+\rho^2\right]^{\frac{n+1}{2}}}\int_0^\pi d\theta e^{-im\rho\cos\theta}\sin^{n-2}\theta. \tag{2.163}$$

We change the integration variable in the second integral to $u\equiv\cos\theta$:

$$\Phi(r)=-\frac{GM}{V_n}\Omega_{n-2}\sum_{\vec{b},b^i\in\mathbb{Z}}e^{-i\vec{m}\cdot\vec{x}}\int_0^\infty d\rho\rho^{n-1}\frac{1}{\left[r^2+\rho^2\right]^{\frac{n+1}{2}}}\int_{-1}^1 due^{im\rho u}(1-u^2)^{\frac{n-3}{2}}. \tag{2.164}$$

Then we notice that $e^{-im\rho u} = \cos(m\rho u) + i\sin(m\rho u)$, of which, on the interval $[-1,1]$, the sine part does not contribute.
We thus write the angular integral as:

$$I \equiv \int_{-1}^{1} du \cos(m\rho u)(1-u^2)^{\frac{n-3}{2}}. \tag{2.165}$$

By renaming the quantities $z \equiv m\rho$ and $\nu \equiv \frac{n}{2} - 1$, we recognize the Bessel function:

$$I = J_{\frac{n}{2}-1}(m\rho)\frac{\Gamma(\frac{1}{2})\Gamma\left(\frac{n-1}{2}\right)}{\left(\frac{m\rho}{2}\right)^{\frac{n}{2}-1}}. \tag{2.166}$$

Then, renaming again $b \equiv m$ and $r^2 \equiv a^2$, we take the first integral and obtain:

$$\begin{aligned}
&\int_0^\infty d\rho\rho^{n-1}\frac{1}{\left[r^2+\rho^2\right]^{\frac{n+1}{2}}}J_{\frac{n}{2}-1}(m\rho)\frac{\Gamma(\frac{1}{2})\Gamma\left(\frac{n-1}{2}\right)}{\left(\frac{m\rho}{2}\right)^{\frac{n}{2}-1}}\\
=&\frac{\Gamma(\frac{1}{2})\Gamma\left(\frac{n-1}{2}\right)}{\left(\frac{m}{2}\right)^{\frac{n}{2}-1}}\int_0^\infty d\rho\frac{\rho^{\frac{n}{2}}}{\left[r^2+\rho^2\right]^{\frac{n+1}{2}}}J_{\frac{n}{2}-1}(m\rho)\\
=&\frac{\Gamma(\frac{1}{2})\Gamma\left(\frac{n-1}{2}\right)}{\left(\frac{m}{2}\right)^{\frac{n}{2}-1}}\frac{m^{\frac{n}{2}-1}\sqrt{\pi}}{2^{\frac{n}{2}}re^{rm}\Gamma\left(\frac{n+1}{2}\right)}.
\end{aligned} \tag{2.167}$$

Putting it all together:

$$\begin{aligned}
\Phi(r) &= -\frac{GM}{V_n}\Omega_{n-2}\sum_{\vec{b}\in\mathbb{Z}} e^{-i\vec{m}\cdot\vec{x}}\frac{\Gamma(\frac{1}{2})\Gamma\left(\frac{n-1}{2}\right)}{\left(\frac{m}{2}\right)^{\frac{n}{2}-1}}\frac{m^{\frac{n}{2}-1}\sqrt{\pi}}{2^{\frac{n}{2}}re^{rm}\Gamma\left(\frac{n+1}{2}\right)}\\
&= -\frac{GM}{2V_n}\Omega_n\sum_{\vec{b}\in\mathbb{Z}} e^{-i\vec{m}\cdot\vec{x}}\frac{1}{re^{rm}}\\
&= -\frac{G^{(4)}M}{r}\sum_{\vec{b}\in\mathbb{Z}} e^{-i\vec{m}\cdot\vec{x}}e^{-rm}.
\end{aligned} \tag{2.168}$$

We can take the points in the distribution to be in the origin $\vec{x} = 0$ and get:

$$\Phi(r) = -\frac{G^{(4)}M}{r}\sum_{\vec{b},b^i\in\mathbb{Z}} e^{-rm}. \tag{2.169}$$

The term with $\vec{b} = 0$ corresponds to the Newtonian potential and the others are the Kaluza-Klein modes, so we find the first correction by analysing the lightest mode $\left|\vec{b}\right| = 1$:

$$\Phi(r) = -\frac{G^{(4)}M}{r}\left(1 + 2n_0 e^{-\frac{r}{R_0}}\right), \tag{2.170}$$

where n_0 is the number of dimensions with the same radius R_0. So that $\frac{r}{R_0}$ is the mass of the $2n_0$ states (2 for every extra dimension, since two states $b_i = \pm 1$ have the same modulus).

2.5 Randall-Sundrum Model

This model was developed in 1999 as an alternative to the ADD scenario, criticized because instead of solving the hierarchy, it rather pushes the problem to a new hierarchy between the compactification scale and the Planck mass. Indeed, the size of extra dimensions would be around $\mu_c \sim 8 \cdot 10^3$ eV, without a justification to explain why it is so different from the value of the d-dimensional Planck scale M_d. Moreover, in that model, it was required the four-dimensional metric was independent of the extra dimensions, which is not a general assumption. Indeed, we are going to take a non-factorizable metric, with the four-dimensional components multiplied by an exponential factor that carries the extra dimensional dependence, called *warp factor*. Another difference with the ADD model is that this time we are going to use two 3-branes instead of one, which are located at the boundaries of an orbifold $S^1/\mathbb{Z}$ compactification on the extra dimension.

The Standard Model is, again, taken to be confined only to a 3-brane, while gravitons are able to propagate on the bulk. While the four-dimensional Planck mass takes the same value on both branes, to reach one of the two branes mass scales undergo a rescaling by the warp factor, thus generating the hierarchy on the brane we live on. Therefore, it was not necessary to assume extra dimensions are very large. The only unnatural parameter is the warp factor's exponential, which is set to ~ 30 while its natural value would be of the order of unity. This problem is fixed by setting this parameter dynamically through radius stabilization. In this discussion we follow [19]-[23].

2.5.1 The model

We consider a 5-dimensional model, i.e. with only one extra dimension that we call y. It is compactified on an orbifold, that is a circle in which we identify the upper and lower halves: $\frac{S^1}{\mathbb{Z}_2}$, where S^1 is the circle and $\mathbb{Z}_2 = \{1, -1\}$. There are two fixed points: 0 and $\pi R \equiv L$. This means $f(y) \in [0, \pi R]$, with periodicity $f(y) = f(y + 2\pi R)$ and even condition $f(y) = f(-y)$. Alternatively, we can consider even functions on the interval $[-L, L]$ on the circle.
The generalized Einstein-Hilbert action is therefore:

$$S = \int d^4x \int_{-L}^{L} dy \sqrt{-g} \left(M^3 \mathcal{R} - \Lambda \right), \tag{2.171}$$

where $M^3 = \frac{1}{16\pi G}$ is proportional to the reduced d-dimensional Planck mass, $\mathcal{R}$ is the d-dimensional Ricci scalar and Λ is a cosmological constant.
For the metric, we use the Ansatz:

$$g_{\mu\nu} = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{2.172}$$

where we take the 4-dimensional space to be Minkowski for simplicity.
Now, we use the Einstein equations:

$$G_{MN} = 8\pi G T_{MN}, \tag{2.173}$$

where the capital indices run from 0 to 4 and the index 4 corresponds to y.
The Christoffel symbols are given by:

$$\Gamma^P_{MN} = \frac{1}{2} g^{PQ} \left(g_{MQ,N} + g_{QN,M} - g_{MN,Q} \right). \tag{2.174}$$

The only non-vanishing ones are:

$$\begin{aligned}
\Gamma^4_{\mu\nu} &= \frac{1}{2} g^{44}(-g_{\mu\nu,4}) \\
&= A' e^{-2A} \eta_{\mu\nu}; \\
\Gamma^\nu_{\mu 4} &= \frac{1}{2} g^{\rho\nu}(g_{\mu\rho,4}) \\
&= \frac{1}{2} \eta^{\rho\nu} e^{2A}(\eta_{\mu\rho}(-2A' e^{-2A})) \\
&= -A' \delta^\nu_\mu,
\end{aligned} \tag{2.175}$$

where the prime indicates the derivative with respect to y.
The Ricci tensor is:

$$R_{MN} = \Gamma^P_{MN,P} - \Gamma^P_{MP,N} + \Gamma^P_{PQ}\Gamma^Q_{MN} - \Gamma^P_{NQ}\Gamma^Q_{MP}. \tag{2.176}$$

Therefore its components are:

$$\begin{aligned}
R_{\mu\nu} &= \Gamma^4_{\mu\nu,4} + \Gamma^\rho_{\rho 4}\Gamma^4_{\mu\nu} - \Gamma^\rho_{\nu 4}\Gamma^4_{\mu\rho} - \Gamma^4_{\nu\rho}\Gamma^\rho_{\mu 4} \\
&= (-2A'^2 + A'')e^{-2A}\eta_{\mu\nu} - A'^2 \delta^\rho_\rho e^{-2A}\eta_{\mu\nu} + A'^2 \delta^\rho_\nu e^{-2A}\eta_{\mu\rho} + A'^2 \delta^\rho_\mu e^{-2A}\eta_{\nu\rho} \\
&= (-4A'^2 + A'')e^{-2A}\eta_{\mu\nu} \\
&= (-4A'^2 + A'')g_{\mu\nu}; \\
R_{44} &= -\Gamma^\rho_{\rho 4,4} - \Gamma^\nu_{\mu 4}\Gamma^\mu_{\nu 4} = \\
&= A''\delta^\rho_\rho - A'^2 \delta^\nu_\mu \delta^\mu_\nu = 4A'' - 4A'^2.
\end{aligned} \tag{2.177}$$

From these we also construct the Ricci scalar as:

$$\begin{aligned}
\mathcal{R} &= g^{MN} R_{MN} \\
&= R_{44} + g^{\mu\nu} R_{\mu\nu} \\
&= 4A'' - 4A'^2 + 4(-4A'^2 + A'') \\
&= 8A'' - 20A'^2.
\end{aligned} \tag{2.178}$$

The Einstein tensor is:

$$G_{MN} = R_{MN} - \frac{1}{2} R g_{MN} \tag{2.179}$$

and now we can calculate it:

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \\
&= \left[-4A'^2 + A'' - (4A'' - 10A'^2)\right] g_{\mu\nu} \\
&= 3\left[2A'^2 - A''\right] g_{\mu\nu} \\
G_{44} &= R_{44} - \frac{1}{2} R \\
&= 4A'' - 4A'^2 - (4A'' - 10A'^2) \\
&= 6A'^2.
\end{aligned} \tag{2.180}$$

By a variation of the action (2.171) with respect to the metric, we obtain the Einstein equation as equation of motion:

$$G_{MN} = \frac{1}{2M^3} T_{MN}, \tag{2.181}$$

where T_{MN} is the energy-momentum tensor given by:

$$T_{MN} = -2 \frac{\delta(\sqrt{-g}\mathscr{L})}{\sqrt{-g}\delta g^{MN}}. \tag{2.182}$$

In this action (2.171), the only contribution to T_{MN} comes from the cosmological constant Λ.
Therefore, the Einstein equation can be written as:

$$G_{MN} = -\frac{1}{2M^3} \Lambda g_{MN} \tag{2.183}$$

and this gives the equations:

$$\begin{aligned} G_{\mu\nu} &= 3\left[2A'^2 - A''\right] g_{\mu\nu} = -\frac{1}{2M^3}\Lambda g_{\mu\nu}; \\ G_{44} &= 6A'^2 = -\frac{1}{2M^3}\Lambda. \end{aligned} \tag{2.184}$$

From the 44 component we get:

$$A'^2 = -\frac{1}{12M^3}\Lambda \equiv k^2. \tag{2.185}$$

In this equation we defined k and we notice that if it is real then Λ must be negative ($\implies$ the spacetime is AdS). In this model we are precisely interested in k real so that the exponential is decaying and not oscillating.
Integrating the equation (2.185) over y gives:

$$A(y) = \pm ky. \tag{2.186}$$

Since we need to maintain the orbifold $\mathbb{Z}_2$ symmetry $y \to -y$, we express it like this:

$$A(y) = k|y|. \tag{2.187}$$

The metric can then be written as:

$$g_{MN} = e^{-2k|y|}\eta_{\mu\nu} + \delta_M^4 \delta_n^4 \tag{2.188}$$

and its inverse is taken to be:

$$g_{MN} = e^{2k|y|}\eta^{\mu\nu} + \delta_4^M \delta_4^n. \tag{2.189}$$

From the expression for A we can calculate its derivatives as:

$$\begin{aligned} A' &= sgn(y)k = \left[\theta(y) - \theta(-y)\right]k; \\ A'' &= 2k\delta(y). \end{aligned} \tag{2.190}$$

This second derivative comes only from the delta in $y = 0$ but there is actually another one in $y = L$, so by putting them together:

$$A'' = 2k(\delta(y) - \delta(y - L)). \tag{2.191}$$

Next, we plug this into the four dimensional component of the Einstein equations and get:

$$G_{\mu\nu} = 6k(k - \delta(y) + \delta(y - L))g_{\mu\nu}. \tag{2.192}$$

The term $6k^2$ cancels with what we had previously calculated in the l.h.s., but the deltas do not seem to be matched to anything. The reason for this is that our setup was not completely correct: we had to include the contribution from the branes, which are in fact *dynamical* objects and must contribute to the action. There are two branes: one in $y = 0$ and one in $y = L$, i.e. at the boundaries of our theory. We can consider other two cosmological constant terms that live on these branes and insert them in the action (2.171):

$$S_j = -\int d^4x\sqrt{-g_{(j)}}\lambda_j = -\int d^4x \int_{-L}^{L} dy\sqrt{-g}\lambda_j\delta(y - j), \tag{2.193}$$

where $j = 0, L$, because $\sqrt{-g_{(j)}} = \sqrt{-g}$ since $g_44 = 1$.
So the action becomes:

$$S = \int d^4x \int_{-L}^{L} dy\sqrt{-g}\left[M^3\mathcal{R} - \Lambda - \lambda_0\delta(y) - \lambda_L\delta(y - L)\right]. \tag{2.194}$$

These two new terms modify the Einstein equations:

$$G_{MN} = -\frac{1}{2M^3}\left[\Lambda + \lambda_0\delta(y) + \lambda_L\delta(y - L)\right]g_{MN}. \tag{2.195}$$

By taking (2.188) as the metric, the same definition (2.185) for k and the expressions we have calculated for A' and A'' we can write:

$$\begin{aligned}
G_{\mu\nu} &= \frac{1}{2M^3}\left[\Lambda + \lambda_0\delta(y) + \lambda_L\delta(y - L)\right]g_{\mu\nu} \\
3(2A'^2 - A'')g_{\mu\nu} &= \frac{1}{2M^3}\left[\Lambda + \lambda_0\delta(y) + \lambda_L\delta(y - L)\right]g_{\mu\nu} \\
6k^2g_{\mu\nu} - 6\left[\delta(y) - \delta(y - L)\right]g_{\mu\nu} &= 6k^2g_{\mu\nu} - \frac{1}{2M^3}\left[\lambda_0\delta(y) + \lambda_Le^{-2kL}\delta(y - L)\right]\eta_{\mu\nu} \\
12kM^3\left[\delta(y) - \delta(y - L)e^{-2kL}\right]\eta_{\mu\nu} &= \left[\lambda_0\delta(y) + \lambda_Le^{-2kL}\delta(y - L)\right] \\
12kM^3 &= \lambda_0 = -\lambda_L.
\end{aligned} \tag{2.196}$$

This satisfies the condition of flat and static four-dimensional spacetime, as the two cosmological constant "cancel" each other resulting in a null effective four-dimensional cosmological constant. This requires a fine-tuning of the bulk cosmological constant, which, from the aforementioned definition of k (2.185), is given by:

$$\Lambda = -\frac{\lambda_0^2}{12M^3}. \tag{2.197}$$

We are going to see later how this fine-tuning can be solved dynamically.

2.5.2 Mass scale

Now that we have successively fixed the metric, we investigate what mass scales are generated by this model. To do so, we assume the SM is confined in the $y = L$ brane, the one with negative tension, and take the Higgs field on this subspace:

$$S_H = \int d^4x\sqrt{-g_L}\left[g_L^{\mu\nu}(D_\mu H)^\dagger D_\nu H - \lambda(H^\dagger H - v^2)^2\right], \tag{2.198}$$

where the induced metric on the brane is $g_L = \eta_{\mu\nu}e^{-2kL} + 1$.
We plug it into the action to get:

$$S_H = \int d^4x e^{-4kL}\left[e^{2kL}\eta^{\mu\nu}(D_\mu H)^\dagger D_\nu H - \lambda(H^\dagger H - v^2)^2\right]. \tag{2.199}$$

Now we rescale the Higgs field as: $\tilde{H} = e^{-kL}H$.

$$\begin{aligned} S_H &= \int d^4x\left[\eta^{\mu\nu}(D_\mu\tilde{H})^\dagger D_\nu\tilde{H} - \lambda e^{-4kL}(e^{2kL}\tilde{H}^\dagger\tilde{H} - v^2)^2\right] \\ &= \int d^4x\left[\eta^{\mu\nu}(D_\mu\tilde{H})^\dagger D_\nu\tilde{H} - \lambda(\tilde{H}^\dagger\tilde{H} - e^{-2kL}v^2)^2\right]. \end{aligned} \tag{2.200}$$

This gives exactly the Higgs theory in flat space, but with a different VEV:

$$v_{eff} = e^{-kL}v. \tag{2.201}$$

This indicates that the $y = L$ brane will observe a lower value v_{eff} for the VEV, while on the $y = 0$ brane it will remain identified as v. Since the Higgs VEV is what sets the mass of the particles in the Standard Model, the exponential suppression on v is applied to all the masses on the $y = L$ brane. Then, the $y = 0$ brane is called the *Planck brane* as the fundamental mass scale is taken to be of the order of the Planck scale, while the negative tension brane is called the *TeV brane*, since the relevant mass scale is on the TeV scale. Indeed, we set the bare Higgs mass to be $\sim m_p$ on the $y = 0$ brane, while the physical mass to be generated by the exponential suppression, reaching the expected value of $\sim$ TeV. Then, we must fix k in order to reproduce this situation where there are 16 orders of magnitude of discrepancy between the two masses on the branes. This gives:

$$kL \sim ln(10^{16}) \simeq 37. \tag{2.202}$$

Even though this looks like a fine-tuning, we shall see later how this value is recovered.

Next, we should make sure this exponential suppression really solves the hierarchy. To do so, we also investigate the dependence of the effective scale of gravity on the parameter L. We perturb the metric (2.188) so that the 4-dimensional graviton is given by $h_{\mu\nu}$. The interval is then:

$$ds^2 = e^{-2k|y|}\left(\eta_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu + dy^2. \tag{2.203}$$

To calculate the four-dimensional action we only consider the parts coming from $h_{\mu\nu}$, so that:

$$\sqrt{-g_{(5)}} \quad\rightarrow\quad \sqrt{-g_{(4)}}e^{-4k|y|} \tag{2.204}$$

and since we know that the Riemann tensor is invariant under rescaling of the metric, the Ricci scalar is:

$$\mathcal{R}^{(5)} \quad \rightarrow \quad h^{\mu\nu} e^{2k|y|} R_{\mu\nu} = e^{2k|y|} \mathcal{R}^{(4)}. \tag{2.205}$$

So, the four-dimensional action is only a part of the 5-dimensional one, such as:

$$S_{(5)} = \int d^4x dy M^3 \sqrt{-g} \mathcal{R}^{(5)} \supset S_{(4)} = \int d^4x dy M^3 e^{-4k|y|} \sqrt{-g^{(4)}} e^{2k|y|} \mathcal{R}^{(4)}. \tag{2.206}$$

Now we integrate out the extra coordinate:

$$\begin{aligned} S_{(4)} &= M^3 \int d^4x \int_{-L}^{L} dy e^{-2k|y|} \sqrt{-g^{(4)}} \mathcal{R}^{(4)} \\ &= 2M^3 \int d^4x \int_0^L dy e^{-2k|y|} \sqrt{-g^{(4)}} \mathcal{R}^{(4)} \\ &= M^3 \frac{1 - e^{-2kL}}{k} \int d^4x \sqrt{-g^{(4)}} \mathcal{R}^{(4)}. \end{aligned} \tag{2.207}$$

From this we can find an explicit expression for the Planck mass by matching with the Einstein-Hilbert 4-action:

$$S_{(4)} = M^3 \frac{1 - e^{-2kL}}{k} \int d^4x \sqrt{-g^{(4)}} \mathcal{R}^{(4)} \stackrel{!}{=} S^{(4)}_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g^{(4)}} \mathcal{R}^{(4)}. \tag{2.208}$$

Therefore, since $(8\pi G)^{-2} = M_{pl}$ where M_{pl} is the reduced Planck mass:

$$M_{pl} = 2M^3 \frac{1 - e^{-2kL}}{k}. \tag{2.209}$$

From this expression, it is apparent that the parameter L is not so relevant here, as it is only present in the exponential, contrary to what happened in the ADD scenario (2.106). It is apparent here how a moderately large value of L can introduce the hierarchy between masses. Indeed, assuming M^3 of the order of the Planck mass, we see it remains essentially unchanged, while the VEV in (2.201) is exponentially rescaled, thus developing a hierarchy on the Planck brane we live on.

2.5.3 Radion stabilization

We have seen that L must be fixed to a value of $L \sim 37/k$, but that is not a condition required by any dynamical process, it is just a quantity we need to match the experimental result. Indeed, as a natural choice we would expect the typical size of the extra dimension to be $L \sim 1/k$. So it appears we are introducing a new hierarchy when trying to solve the Standard Model one.

The presence of a new degree of freedom corresponds to the introduction of a new scalar field called *radion*, which represents the fluctuations of the radius L on the extra dimension. The fact that it is an arbitrary parameter, it means it is a flat direction and thus has no potential, therefore we assume it is massless. This is not acceptable, as it would generate a fifth force that is not observed, then the radius must be *stabilized* to an unnatural value that would reproduce $L \sim 37/k$ as we argued before. The problem is solved by the Goldberger-Wise mechanism, which provides radius stabilization. Its idea is as follows:

The radius is stabilized if there is a balance between the kinetic and potential terms: the former tends to make the radius bigger (to have smaller values for the derivatives), while the latter prefer smaller ones in order to minimize the action. In order to achieve a non-trivial minimum, we introduce a massive radion (so that it has a potential) and admit the existence of brane potentials at the fixed points, with different minima each to guarantee a non-trivial minimum.
Now we implement what we have discussed so far:

$$\begin{aligned} S_\phi = & \int d^4x \int_{-L}^{L} dy \sqrt{-g} \left(M^3 \mathcal{R} + \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{m_\phi^2}{2} \phi^2 \right) \\ & - \int d^4x dy \sqrt{-g} \lambda_0 \left(\phi^2 - v_0^2 \right)^2 \delta(y) - \int d^4x dy \sqrt{-g} \lambda_L \left(\phi^2 - v_L^2 \right)^2 \delta(y-L). \end{aligned} \tag{2.210}$$

For simplicity we denote:

$$\begin{aligned} V_\phi(\phi) &\equiv \frac{m_\phi^2}{2} \phi^2 \\ V_0(\phi) &\equiv \lambda_0 \left(\phi^2 - v_0^2 \right)^2 \\ V_L(\phi) &\equiv \lambda_L \left(\phi^2 - v_L^2 \right)^2 \\ V(\phi) &\equiv -V_\phi - V_0(\phi)\delta(y) - V_L(\phi)\delta(y-L) \end{aligned} \tag{2.211}$$

so that the action takes the simpler form:

$$S_\phi = \int d^4x \int_{-L}^{L} dy \sqrt{-g} \left(M^3 \mathcal{R} + \frac{1}{2} \partial_M \phi \partial^M \phi + V(\phi) \right). \tag{2.212}$$

The fact that the radion field is not visible in the four-dimensional theory implies $\phi(x,y) = \phi(y)$.
By taking the Ansatz that the metric is the same as before (2.188), we can write the Einstein's equations. They are very similar to those we have already calculated, except that this time the energy-momentum tensor of the matter Lagrangian is not null. T_{MN} can be calculated from its definition (2.182):

$$\begin{aligned} \delta S &= \int d^4x dy \sqrt{-g} \left[\frac{1}{2} g^{AB} \delta g_{AB} \left(\frac{1}{2} g^{CD} \partial_C \phi \partial_D \phi + V(\phi) \right) + \delta g^{AB} \frac{1}{2} \partial_A \phi \partial_B \phi \right] \\ &= - \int d^4x dy \frac{1}{2} \sqrt{-g} \left[g_{AB} \left(\frac{1}{2} g^{CD} \partial_C \phi \partial_D \phi + V(\phi) \right) - \partial_A \phi \partial_B \phi \right] \delta g^{AB} \\ &\stackrel{!}{=} \int d^4x dy \frac{1}{2} \sqrt{-g} T_{AB} \delta g^{AB}, \end{aligned} \tag{2.213}$$

where we used the relations:

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{AB} \delta g_{AB}, \qquad g^{AB} \delta g_{AB} = -g_{AB} \delta g^{AB}. \tag{2.214}$$

Therefore, keeping in mind that ϕ only depends on y:

$$\begin{aligned} T_{\mu\nu} &= g_{\mu\nu} \left(\frac{\phi^{2\prime}}{2} + V(\phi) \right); \\ T_{44} &= \phi'^2 - \left(\frac{1}{2} \phi'^2 - V_\phi(\phi) \right), \end{aligned} \tag{2.215}$$

where we obviously did not variate the extra dimension on the brane part of the action. Finally, the Einstein's equations are:

$$\begin{aligned} G_{MN} &= \frac{1}{2M^3} T_{MN}; \\ G_{\mu\nu} = 3(2A'^2 - A'')g_{\mu\nu} &= \frac{1}{2M^3} \left[\frac{1}{2}\phi'^2 + V(\phi)\right] g_{\mu\nu}; \\ G_{44} = 6A'^2 &= \frac{1}{2M^3} \left[\frac{\phi'^2}{2} - V_\phi(\phi)\right]. \end{aligned} \tag{2.216}$$

To obtain A'' we use the expression for A' from the second equation, and get:

$$A'' = \frac{1}{6M^3} \left[V_0 \delta(y) + V_L \delta(y - L)\right]. \tag{2.217}$$

We also need to consider the equations of motion for the scalar field:

$$\nabla_M \frac{\delta \mathscr{L}_\phi}{\delta \nabla_M \phi} = \frac{\delta \mathscr{L}_\phi}{\delta \phi}. \tag{2.218}$$

Then, by using the fact that the covariant derivative is the partial derivative for a scalar field $\nabla_M \phi = \partial_M \phi$, $\phi = \phi(y)$ and the expression for the covariant divergence, the l.h.s is:

$$\begin{aligned} \nabla_M \frac{\delta \mathscr{L}_\phi}{\delta \partial_M \phi} = \partial_M \left(\sqrt{-g} g^{MN} \partial_N \phi\right) &= \sqrt{-g} \left(\frac{g^{AB}}{2} g_{AB,M} \partial^M \phi + \partial_M \partial^M \phi\right) \\ &= \sqrt{-g} \left(\frac{g^{\mu\nu}}{2} g'_{\mu\nu} \phi' + \phi''\right) = \sqrt{-g} \left(\phi'' - 4A'\phi'\right), \end{aligned}$$

since $\phi = \phi(y)$.
Putting it together with the r.h.s. and eliminating the common $\sqrt{-g}$ factor, gives the EOMs:

$$-4\phi' A' + \phi'' = -\frac{\partial V(\phi)}{\partial \phi}. \tag{2.219}$$

Now we integrate (2.217) and (2.219) over a small interval $[j - \epsilon, j - \epsilon]$ where $j = 0, L$ and $\epsilon \to 0$, to get the boundary conditions.
The equation for ϕ' is obtained from (2.219) by integrating by parts the following term:

$$-4A'\phi' = -4 \left[A\phi'\right]_{j \pm \epsilon} + 4 \int dy A \phi''. \tag{2.220}$$

Since A'' is proportional to a delta function, A is proportional to the ramp function, which is continuous and therefore $A(j + \epsilon) = A(j - \epsilon)$.

$$-4A(j + \epsilon) \left[\phi'(j + \epsilon) - \phi'(j - \epsilon)\right] + 4A(j + \epsilon) \int dy \phi'' = 0. \tag{2.221}$$

Then we can straightforwardly obtain the boundary conditions:

$$\begin{aligned} A'|_{j \pm \epsilon} &= \frac{1}{6M^3} V_j; \\ \phi'|_{j \pm \epsilon} &= V'_j. \end{aligned} \tag{2.222}$$

In general, these equations are hard to solve so we only consider one particular case:

$$V_\phi(\phi) = \frac{1}{8}\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{1}{12M^3}W^2(\phi), \tag{2.223}$$

where the function W is called *super potential*.
Thus, we can rewrite the 44 component of the Einstein equations as:

$$V_\phi(\phi) = -12M^3 A'^2 + \frac{1}{2}\phi'^2 \tag{2.224}$$

and match it with the potential (2.223), to get:

$$A' = \frac{1}{12M^3}W, \qquad\qquad \phi' = \frac{1}{2}\frac{\partial W}{\partial \phi}. \tag{2.225}$$

Now, we remember that $V_\phi(\phi)$ is supposed to be a mass term for the scalar ϕ and that, in addition, we need to recover a bulk cosmological constant term Λ to give null contribution to the effective four-dimensional cosmological constant.
To obtain this, we choose the super potential in the form:

$$W = 12M^3 k - m_\phi \phi^2, \tag{2.226}$$

so that:

$$\phi' = \frac{1}{2}\frac{\partial W}{\partial \phi} = -m_\phi \phi. \tag{2.227}$$

The solution is:

$$\phi = e^{-m_\phi y}\phi_0, \tag{2.228}$$

which, on the Planck brane, becomes:

$$\phi_L = e^{-m_\phi L}\phi_0. \tag{2.229}$$

So, we can write the following relation:

$$L = \frac{1}{m_\phi}\ln\left(\frac{\phi_0}{\phi_L}\right), \tag{2.230}$$

where the value of ϕ_0 is given by the boundary conditions (2.222).
Then it is only the parameter m_ϕ that has to be tuned to give the experimental result of $kL \simeq 37$. This tuning, though, is very modest compared to the original one from the hierarchy problem and can be thus considered as a solution, successfully setting the radius dynamically.

Chapter 3

The Myers-Perry metric

We propose to study rotating Black Holes in arbitrary dimensions and the metric that describes them is the Myers-perry metric. Then, we are going to derive it as a generalization of the four-dimensional Kerr metric and study some of the parameters. The most important of these are the horizon radius, the mass and the angular momentum. First, we derive an equation to define the horizon through its radius, which is a very important quantity as it will give origin to the Horizon Wave Function through its quantization. Since the equation for the horizon can not be solved analytically, we will need to make some approximations and distinguish two main regimes, which will be treated separately. Next, we shall address the problem of instabilities and examine more closely which values of the angular momentum are allowed before the Black Hole has a high chance of splitting itself. These are important data we shall use in our chapter on the probabilities.

In this discussion, we shall follow mainly [27]-[28],[31]-[32].
We consider asymptotically flat metrics and a non-relativistic ($v \ll c$), weakly gravitating field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The energy-momentum tensor is taken to be such as $|T_{00}| \gg |T_{0i}| \gg |T_{ij}|$. Although the condition $|T_{0i}| \gg |T_{ij}|$ might not be true in general, it is usually verified in rotating systems, so we take it to be true.

3.1 The metric at linear order

Preliminarily, we derive an expression for the first correction to the Minkowski metric as it will be used later in expressing the mass and angular momentum. We refer to the appendices on Einstein equation and Green's function for more details on how to get the following result for $h_{\mu\nu}$:

$$h_{\mu\nu}(x^i) = \frac{16\pi G}{(d-3)\Omega_{d-2}} \int d^{d-1}y \frac{\tilde{T}_{\mu\nu}(y^i)}{|\vec{x}-\vec{y}|^{d-3}}, \tag{3.1}$$

where G indicates the d-dimensional gravitational constant.
Now we take the asymptotic limit $r \equiv |\vec{x}| \gg |\vec{y}|$ and perform a Taylor expansion on the denominator:

$$\frac{1}{|\vec{x}-\vec{y}|^{d-3}} = \frac{1}{|\vec{x}|^{d-3}} \frac{1}{\left|1-\frac{y^k}{x^k}\right|^{d-3}} \simeq \frac{1}{r^{d-3}} \left(1 + (d-3)\frac{y^k}{x^k}\right). \tag{3.2}$$

Then we obtain:

$$h_{\mu\nu}(x^i) = \frac{16\pi G}{(d-3)\Omega_{d-2}} \left[\int d^{d-1}y \frac{\tilde{T}_{\mu\nu}(y^i)}{r^{d-3}} + (d-3)\frac{x_k y^k}{r^{d-1}} \tilde{T}_{\mu\nu}(y^i) \right]. \tag{3.3}$$

Then we proceed to calculate each component separately:

$$\begin{aligned} h_{00} = &\frac{16\pi G}{(d-3)\Omega_{d-2}} \frac{1}{r^{d-3}} \int d^{d-1}y \left(T_{00} + \frac{g_{00}}{2-d} T \right) \\ &+ \frac{16\pi G}{\Omega_{d-2}} \frac{x_k}{r^{d-1}} \int d^{d-1}y \left(T_{00} + \frac{g_{00}}{2-d} T \right) y^k. \end{aligned} \tag{3.4}$$

Next we consider the definitions:

$$\begin{cases} \int d^{d-1}y T^{00} \equiv M; \\ \int d^{d-1}y^k T^{00} = 0 \quad \text{by choosing the center of mass to be in the origin.} \end{cases} \tag{3.5}$$

By neglecting the T_{ij} components of the energy-momentum tensor when calculating its trace T, we obtain the first result:

$$h_{00} = \frac{16\pi G}{(d-2)\Omega_{d-2}} \frac{M}{r^{d-3}}. \tag{3.6}$$

Now we move on to the h_{oi} component:

$$h_{0i} = \frac{16\pi G}{(d-3)\Omega_{d-2}} \frac{1}{r^{d-3}} \int d^{d-1}y T_{0i} + \frac{16\pi G}{\Omega_{d-2}} \frac{x_k}{r^{d-1}} \int d^{d-1}y T_{0i} y^k. \tag{3.7}$$

To express it in a more convenient form, we use these other definitions:

$$\begin{cases} \int d^{d-1}y T^{0i} \equiv P^i = 0 \quad \text{working in the rest frame;} \\ \int d^{d-1} \left(y^\mu T^{\nu 0} - y^\nu T^{\mu 0} \right) \equiv J^{\mu\nu} \quad \text{that in this frame satisfies } J^{00} = J^{0i} = 0. \end{cases} \tag{3.8}$$

At the end we obtain:

$$h_{0i} = -\frac{8\pi G}{\Omega_{d-2}} \frac{x_k}{r^{d-1}} J^{ki}, \tag{3.9}$$

where the minus sign is obtained by raising the temporal index in the stress energy tensor. Finally, the spatial-only component:

$$\begin{aligned} h_{ij} = &\frac{16\pi G}{(d-3)\Omega_{d-2}} \frac{1}{r^{d-3}} \int d^{d-1}y \left(T_{ij} + \frac{\delta_{ij}}{2-d} T \right) \\ &+ \frac{16\pi G}{\Omega_{d-2}} \frac{x_k}{r^{d-1}} \int d^{d-1}y y^k \left(T_{ij} + \frac{\delta_{ij}}{2-d} T \right). \end{aligned} \tag{3.10}$$

This time, it is sufficient to neglect the purely spatial components of the energy-momentum tensor:

$$h_{ij} = \frac{16\pi G}{(d-2)(d-3)\Omega_{d-2}} \frac{M\delta_{ij}}{r^{d-3}}. \tag{3.11}$$

So, to summarize, the components of $h_{\mu\nu}$ in these approximations are:

$$\begin{cases} h_{00} = \dfrac{16\pi G}{(d-2)\Omega_{d-2}} \dfrac{M}{r^{d-3}}; \\ h_{0i} = -\dfrac{8\pi G}{\Omega_{d-2}} \dfrac{x_k}{r^{d-1}} J^{ki}; \\ h_{ij} = \dfrac{16\pi G}{(d-2)(d-3)\Omega_{d-2}} \dfrac{M\delta_{ij}}{r^{d-3}}. \end{cases} \tag{3.12}$$

3.2 The metric

The Myers-Perry metric is the solution of the Einstein vacuum equations for a rotating, d dimensional black hole.
Since it is too complicated to solve the Einstein equations directly, the metric is obtained by generalization of the $d = 4$ Kerr solution.
Therefore we first write the Kerr metric in Boyer-Lindquist coordinates is:

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 \tag{3.13}$$

$$+ \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\sin^2\theta\Delta\right], \tag{3.14}$$

where we defined:

$$\rho^2 \equiv r^2 + a\cos^2\theta, \qquad \Delta \equiv r^2 + a^2 - \frac{\mu}{r^{d-5}}. \tag{3.15}$$

Its generalization for arbitrary dimension, the Myers-Perry metric, is as follows:

$$ds^2 = -dt^2 + \frac{\mu}{r^{d-5}\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$
$$+ (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\Omega^2_{d-4}. \tag{3.16}$$

We immediately notice that in the 4-dimensional case the last term in (3.16) vanishes and $\mu = 2GM$ as we will see.
Next, we want to compare the temporal component of the Myers-Perry metric (3.16) with g_{00} calculated in the weak field approximation using (3.6):

$$-\left(1 - \frac{\mu}{r^{d-5}\rho^2}\right) = g_{00} \simeq \eta_{00} + h_{00} = -1 + \frac{16\pi GM}{(d-2)\Omega_{d-2}r^{d-3}}. \tag{3.17}$$

In the asymptotic limit $r \to \infty$ we have that $\rho \simeq r$ and we obtain the value of the Black Hole physical mass:

$$M = \frac{(d-2)\Omega_{d-2}}{16\pi G\mu}. \tag{3.18}$$

From the inverse of this expression we can verify the correspondence to the Kerr metric we anticipated:

$$\mu = \frac{16\pi GM}{(d-2)\Omega_{d-2}} = 2GM \quad \text{ if } d = 4. \tag{3.19}$$

Now, in order to derive an expression for the angular momentum, we start by changing to Cartesian coordinates, as they provide a more convenient parametrization:

$$\begin{cases} x = \sqrt{r^2+a^2}\sin\theta\cos\left[\phi - \tan^{-1}\left(\frac{a}{r}\right)\right]; \\ y = \sqrt{r^2+a^2}\sin\theta\sin\left[\phi - \tan^{-1}\left(\frac{a}{r}\right)\right]. \end{cases} \tag{3.20}$$

We are interested in computing the particular combination $xdy - ydx$, as it will be useful:

$$xdy - ydx = (r^2 + a^2) \sin^2 \theta \left(d\phi + \frac{dr \frac{a}{r^2}}{\sin^2 \left(\frac{a}{r}\right)} \right) \simeq r^2 \sin^2 \theta d\phi, \tag{3.21}$$

where in the last passage we used the asymptotic limit $r \to \infty$.
Now we examine more closely the off-diagonal part of the metric and we substitute $d\phi$ from (3.21):

$$-\frac{2\mu a \sin^2(\theta)}{r^{d-5} \rho^2} dt d\phi \simeq -\frac{2\mu a}{r^{d-1}} dt(xdy - ydx). \tag{3.22}$$

We are now ready to compare it to the weak field metric, with h_{0i} given by (3.9):

$$\frac{\mu a}{r^{d-1}} y = g_{0x} \simeq \eta_{0x} + h_{0x} = \frac{8\pi G}{\Omega_{d-2} r^{d-1}} y J^{yx}. \tag{3.23}$$

Then from this result we read off the angular momentum:

$$J^{yx} = \frac{\Omega_{d-2} \mu a}{8\pi G}. \tag{3.24}$$

An alternative expression can be obtained by using (3.19) to express it as a function of the mass M:

$$J^{yx} = \frac{2Ma}{d-2}. \tag{3.25}$$

Then, the parameter a can be thought of as an angular momentum per unit mass, similarly to the $d = 4$ case where it is exactly $a = \frac{J}{M}$.

3.3 The horizon

We start this section by defining what exactly an horizon is:

An *event horizon* is the boundary of a region from which null rays cannot escape to future null infinity (therefore, it needs the concept of asymptotic flatness to define ”future null infinity”).

This corresponds to the boundary of a BH in the future, while an *apparent horizon* describes it in a certain instant of time (they coincide if the horizon is not perturbed).

The apparent horizon was defined by Penrose through the concept of *closed trapped surfaces*: closed space-like surfaces such that their area decreases along any possible future direction, i.e. the null future-pointing geodesics orthogonal to the surfaces are everywhere converging. This means that every time we try to send a light signal either towards the outside or the inside of the surface, it eventually ends up going to the inside.

Therefore, consider Σ a closed (compact with no boundary) space-like submanifold of co-dimension 2, i.e. $d - 2$ dimensional, with coordinates λ^A, where $A = 2, \ldots, d - 1$. The embedding in the spacetime is achieved by the parametric equations $x^\mu = \Phi^\mu(\lambda^A)$, where

as usual $\mu = 0, \dots, d-1$.
The tangent unit vectors to Σ are:

$$\vec{e}_A = e^\mu_A \frac{\partial}{\partial x^\mu} = \frac{\partial \Phi^\mu}{\partial \lambda^A} \frac{\partial}{\partial x^\mu}. \tag{3.26}$$

The induced metric on the surface Σ is:

$$\gamma_{AB} = g_{\mu\nu} \frac{\partial \Phi^\mu}{\partial \lambda^A} \frac{\partial \Phi^\nu}{\partial \lambda^B}, \tag{3.27}$$

which has to be positive definite for Σ to be spacelike.
The normal space $(T_p\Sigma)^\perp$ is timelike ($g_{ab} < 0$) and 2-dimensional ($a, b = 0, 1$), hence it admits two future-directed null directions orthogonal to Σ: k^+, considered "outward pointing" and k^-, "inward pointing".
We can choose them to satisfy the normalization condition:

$$k^\pm_\mu k^{\mu\pm} = 0, \qquad k^+_\mu k^{\mu-} = -1 \tag{3.28}$$

and the condition of orthogonality: $k^\pm_\mu e^\mu_A = 0$.
There still remains the freedom to rescale these vectors by a function f so that:

$$k^\pm \to f^{\pm 2} k^\pm. \tag{3.29}$$

Next, we define the null second fundamental forms as:

$$K^\pm_{AB} = -k^\pm_\mu e^\nu_A \nabla_\nu e^\mu_B, \tag{3.30}$$

where the symbol ∇_ν indicates a covariant derivative along the direction x^ν. This tensor measures the change of the normal vector along the surface and is thus related to the Riemann tensor.
The trace of the second fundamental form is denoted as:

$$K^\pm = \gamma^{AB} K^\pm_{AB}. \tag{3.31}$$

Then, we are able to define the mean curvature of the surface:

$$H_\mu = -K^- k^+_\mu - K^+ k^-_\mu, \tag{3.32}$$

that is, as an expansion on the space $(T_p\Sigma)^\perp$ spanned by the two vectors $k^\pm_\mu$.
Its norm is the curvature scalar:

$$\kappa = g^{\mu\nu} H_\mu H_\nu = 2K^+ K^-. \tag{3.33}$$

Σ, by definition, is a trapped surface if $\kappa > 0$, while the condition $\kappa = 0$ determines the existence of an apparent horizon, as the boundary of a trapped surface. It comes from the fact that when $\kappa > 0$ the null orthogonal vectors have the same direction. Clearly, both κ and H^μ are invariant under (3.29). Indeed, as long as we rescale by a positive quantity, the sign of $K^\pm$ does not change, thus leaving κ invariant.

3.3.1 A useful formula

We now want to apply the concepts of the last section to derive a useful practical formula. First, we choose the embedding space to be characterized by $x^a = X^a$ =constant, $a = 0, 1$ and the surface by $x^A = \lambda^A$.
The metric can then be represented in this form:

$$ds^2 = g_{ab}dx^a dx^b + g_{aA}dx^a dx^A + g_{AB}dx^A dx^B. \tag{3.34}$$

The null vectors $k^{\pm}_{\mu}$ only have the components $k^{\pm}_{a}$ because they are orthogonal to Σ.
By using the definition of the Christoffel symbols:

$$\Gamma^{\rho}_{\mu\nu} e_{\rho} = \nabla_{\mu} e_{\nu}, \tag{3.35}$$

the second fundamental form is straightforwardly:

$$K^{\pm}_{AB} = -k^{\pm}_{c} \Gamma^{c}_{AB}. \tag{3.36}$$

Then, we write another definition for the second fundamental form that is equivalent to (3.30). Indeed, the change in curvature can also be expressed as the variation of the surface metric when the area is allowed to evolved for a short time.
Defining:

$$\sqrt{\det g_{AB}} \equiv G \equiv e^{U}, \tag{3.37}$$

we can thus state:

$$K^{\pm} = k^{\pm a} \left(\frac{G_{,a}}{G} - \frac{1}{G} (G\gamma^{AB} g_{aA})_{,B} \right). \tag{3.38}$$

Hence:

$$H_{\mu} = \delta^{a}_{\mu} (U_{,a} - \boldsymbol{\nabla} \cdot \vec{g}_{a}), \tag{3.39}$$

where $\vec{g}_a \equiv g_{aA}$.
To prove this last formula, we make first a general discussion. Let g_{ij} be the metric with determinant g and V^i an arbitrary vector.
Then, we start the proof from the end result:

$$\frac{(\sqrt{g} V^i)_{,i}}{\sqrt{g}} = \boldsymbol{\nabla} \cdot V^i. \tag{3.40}$$

The divergence of the vector is defined as:

$$\boldsymbol{\nabla} \cdot V = \nabla_i V^i = V^i_{,i} + \Gamma^i_{ik} V^k, \tag{3.41}$$

where the Γ^i_{ik} are the Christoffel symbols:

$$\Gamma^i_{ik} = \frac{1}{2} g^{ij} (g_{ij,k} + g_{jk,i} - g_{ik,j}). \tag{3.42}$$

Using the Leibniz rule on (3.40) we get:

$$\frac{(\sqrt{g} V^i)_{,i}}{\sqrt{g}} = \frac{1}{\sqrt{g}} \left[(\sqrt{g})_{,i} V^i + \left(V^i\right)_{,i} \sqrt{g} \right] = V^i_{,i} + \frac{\left(\sqrt{g}\right)_{,i}}{\sqrt{g}} V^i. \tag{3.43}$$

Next, we make use of the relation (A.10) for the determinant, to calculate the second term:

$$\frac{\left(\sqrt{g}\right)_{,i}}{\sqrt{g}}V^i = \frac{1}{2}\frac{g_{,i}}{g}V^i = \frac{1}{2}\,\mathrm{tr}\left(g^{-1}g_{,i}\right)V^i = \frac{1}{2}\,\mathrm{tr}\left(g^{jk}g_{kl,i}\right)V^i = \frac{1}{2}(g^{jk}g_{kj,i})V^i. \tag{3.44}$$

Now we use the Christoffel symbols to express $g_{kj,i}$:

$$g_{kj,i} = \Gamma^l_{ik}g_{lj} + \Gamma^l_{ij}g_{lk}. \tag{3.45}$$

We then get the final result:

$$V^i_{,i} + \frac{1}{2}\left[\Gamma^k_{ik} + \Gamma^j_{ij}\right]V^i = V^i_{,i} + \Gamma^i_{ik}V^k = \boldsymbol{\nabla}\cdot V, \tag{3.46}$$

that is what we wished to show. Indeed, we only need to substitute $V^i = \gamma^{AB}g_{aA}$ to get the result (3.39).

3.3.2 Application to the Myers-Perry metric

Now we want to apply (3.39) to calculate the curvature of the surface Σ with constant t and r in order to find an expression that characterizes the horizon.
Since Σ has t and r constant, it follows that:

$$x^a = \{t, r\}, \qquad x^A = \{\phi, \theta, \theta_1, \ldots, \theta_{d-4}\}; \tag{3.47}$$

$$\vec{g}_r = 0, \qquad \vec{g}_t = -\frac{a\mu\sin^2\theta}{r^{d-5}\rho^2}d\phi. \tag{3.48}$$

So that $\boldsymbol{\nabla}\cdot\vec{g}_a = \nabla_\phi g_{t\phi} = 0$ and (3.39) reduces to a simple form:

$$H_a = U_{,a}. \tag{3.49}$$

Next, we calculate the square of the determinant of the metric on the surface, according to the definition (3.37):

$$e^{2U} = r^2\rho^2\left[(r^2+a^2) + \frac{\mu a^2\sin^2\theta}{r^{d-5}\rho^2}\right]\sin^2\theta\cos^2\theta\left(\sin^{d-3}\theta_1\ldots\sin\theta_{d-3}\right)^2, \tag{3.50}$$

where we used the fact that the round metric for a $d-4$ dimensional surface is:

$$d\Omega_{d-4} = \sin^{d-3}(\theta_1)\ldots\sin(\theta_{d-3})d\theta_{d-4}d\theta_{d-3}\ldots d\theta_1. \tag{3.51}$$

Then, the mean curvature is:

$$H_a = H_r = U_{,r} = \frac{(e^{2U})_{,r}}{2e^{2U}}. \tag{3.52}$$

Since the last part of equation (3.50) is constant in r, we can rename it for convenience:

$$\sin^2\theta\cos^2\theta\left(\sin^{d-3}\theta_1\ldots\sin\theta_{d-3}\right)^2 \equiv C. \tag{3.53}$$

We perform the derivation on r of equation (3.50):

$$(e^{2U})_{,r} = 2rC\left[(r^2+a^2)(\rho^2+r^2)+\rho^2r^2+\frac{\mu a^2\sin^2\theta}{r^{d-5}}\left(1-\frac{d-5}{2}r^2\right)\right] \tag{3.54}$$

and the quantity that we actually needed can be easily recovered:

$$(e^{2U})_{,r} = 2U_{,r}e^{2U} \quad \rightarrow \quad U_{,r} = \frac{(e^{2U})_{,r}}{2e^{2U}}. \tag{3.55}$$

Then what we obtain is:

$$\begin{aligned} H_r = &\frac{1}{r}\left[(r^2+a^2)(\rho^2+r^2)+\rho^2r^2+\frac{\mu a^2\sin^2\theta}{r^{d-5}}\left(1-\frac{d-5}{2}r^2\right)\right] \\ &\times\frac{1}{\left[(r^2+a^2)\rho^2+\dfrac{\mu a^2\sin^2\theta}{r^{d-5}}\right]}. \end{aligned} \tag{3.56}$$

Now we are ready to calculate the curvature:

$$\begin{aligned} \kappa = g^{rr}H_rH_r = &g^{rr}\frac{1}{r^2}\left[(r^2+a^2)(\rho^2+r^2)+\rho^2r^2+\frac{\mu a^2\sin^2\theta}{r^{d-5}}\left(1-\frac{d-5}{2}r^2\right)\right]^2 \\ &\times\frac{1}{\left[(r^2+a^2)\rho^2+\dfrac{\mu a^2\sin^2\theta}{r^{d-5}}\right]^2}. \end{aligned} \tag{3.57}$$

The condition $\kappa = 0$ to have a horizon can only be fulfilled if $g^{rr} = 0$ since H_r cannot be null.
Therefore, we conclude that the horizon is set at:

$$g^{rr} = \frac{\Delta}{\rho^2} = 0 \qquad \rightarrow \qquad \Delta = r^2+a^2-\frac{\mu}{r^{d-5}} = 0. \tag{3.58}$$

Finally, we have an equation for the horizon, which is:

$$r_0^2+a^2-\frac{\mu}{r_0^{d-5}} = 0. \tag{3.59}$$

Moreover, we can calculate its area:

$$\begin{aligned} A_H &= \int d\theta d\phi A_{d-4}\sqrt{detg_{AB}} \\ &= \Omega_{d-4}r_0^{d-4}\int d\theta d\phi\sqrt{(r_0^2+a^2)\rho^2+\frac{\mu a^2\sin^2\theta}{r_0^{d-5}}}\sin\theta. \end{aligned} \tag{3.60}$$

By making use of equation (3.59) and the definition $\rho^2 = a^2\cos^2\theta + r^2$, we obtain:

$$\begin{aligned} A_H &= \Omega_{d-4}r_0^{d-4}\int d\theta d\phi\sqrt{(r_0^2+a^2)\rho^2+(r^2+a^2)a^2\sin^2\theta}\sin\theta \\ &= \Omega_{d-4}r_0^{d-4}\int d\theta d\phi(r_0^2+a^2)\sin\theta = \Omega_{d-4}r_0^{d-4}(r_0^2+a^2)\Omega_2. \end{aligned} \tag{3.61}$$

The final result that is obtained is:

$$A_H = \Omega_{d-2} r_0^{d-4}(r_0^2 + a^2). \tag{3.62}$$

3.4 Spinning regimes

The equation (3.59) cannot be solved explicitly for r_0 as a function of M and a, unless we make some approximations.
Therefore, we separate the problem in different cases that may occur:

- **static:** characterized by $a = 0$, it effectively corresponds to the Schwarzschild solution;
- **weakly spinning:** characterized by $a \ll r_0$;
- **ultra-spinning:** characterized by $a \gg r_0$.

As the weakly-spinning solution of the equation (3.59) is the one that requires more work, we explicit it. We make use of the parameter $\frac{a}{r} \equiv \epsilon \ll 1$ and write $r_0 \equiv r$ to avoid confusion:

$$r^{d-3}(1 + \epsilon^2) = \mu. \tag{3.63}$$

Next, we expand r in powers of ϵ, up to second order:

$$r = r_0 + \epsilon r_1 + \epsilon^2 r_2. \tag{3.64}$$

Then we use a Taylor expansion to express the term r^{d-3}:

$$r^{d-3} = r_0^{d-3}\left(1 + \epsilon\frac{r_1}{r_0} + \epsilon^2\frac{r_2}{r_0}\right)^{d-3} \simeq r_0^{d-3}\left[1 + (d-3)\left(\epsilon\frac{r_1}{r_0} + \epsilon^2\frac{r_2}{r_0}\right)\right]. \tag{3.65}$$

Now we substitute this result into the expression for the horizon (3.59), obtaining:

$$r_0^{d-3}\left[1 + (d-3)\left(\epsilon\frac{r_1}{r_0} + \epsilon^2\frac{r_2}{r_0}\right)\right] + \epsilon^2 r_s^{d-3} \simeq \mu, \tag{3.66}$$

where third order terms have been neglected.
The next step is to equate the terms which are the same order in ϵ:

$$\begin{cases} r_0^{d-3} = \mu; \\ (d-3)r_0^{d-4} r_1 = 0; \\ (d-3)r_0^{d-4} r_2 + r_0^{d-3} = 0. \end{cases} \tag{3.67}$$

We immediately notice r_0 is the Schwarzschild radius, as we expected, as it is the solution for $a = 0$.
By solving for r_1 and r_2 we get:

$$r = \mu^{\frac{1}{d-3}}\left(1 - \frac{\epsilon^2}{d-3}\right). \tag{3.68}$$

This weakly spinning limit reduces to the static case quite easily, since the correction to the Schwarzschild radius is of order 2 in ϵ and it goes to 0 even faster as the dimension increases.

The next observation we want to make is that when taking a high-dimensional limit $d \gg 5$ we get:

$$d-3 \simeq d-5 \quad \rightarrow \quad r_0 = \left(\frac{\mu}{1+a^2}\right)^{\frac{1}{d-5}}, \tag{3.69}$$

which for $a \gg r_0$ or $a \simeq 0$ gives the same result as the ultra-spinning or static limits. This allows us to conclude there are fundamentally two main regimes, which are the static and the ultra-spinning one.

To summarize, then, when solving equation (3.59) in the two approximations we have:

- **static:** $r_0 = \mu^{\frac{1}{d-3}}$;
- **ultra-spinning:** $r_0 = \left(\frac{\mu}{a^2}\right)^{\frac{1}{d-5}}$,

making the expression for the horizon radius invertible in terms of the mass.
Indeed, using (3.19) we can express the radius as a function of the mass M. For simplicity of notation, we denote the constant multiplying M as C, so that: $\mu = CM$.
Then we can write:

$$r_0 = (CM)^{\frac{1}{d-3}} \text{ for the static case;} \tag{3.70}$$

$$r_0 = \left(\frac{CM}{a^2}\right)^{\frac{1}{d-5}} \text{ for the ultra-spinning case,} \tag{3.71}$$

where

$$C = \frac{16\pi G}{(d-2)\Omega_{d-2}}. \tag{3.72}$$

3.4.1 Ultra-spinning regime

We want to see which constraints the angular momentum is subject to and to do so, we first analyse what shape the horizon takes.

While taking the static limit in the Myers-Perry metric gives exactly the Schwarzschild one in general d dimensions. We can see now what happens when substituting the ultra-spinning limit in (3.16).
As we want to keep the area and horizon finite, together with $a \to \infty$, we have to take the limit $\mu \to \infty$, with $r_0^{d-5} \simeq \frac{\mu}{a^2}$ fixed. Also, we consider the metric near $\theta = 0$, otherwise the rotation would grow without restraints. We introduce a new coordinate $\sigma \equiv a \sin\theta$, that is kept finite in the limit.
First, we have the quantities:

$$\rho^2 \simeq a^2(1+\frac{r^2}{a^2}) \simeq a^2, \qquad \Delta \simeq a^2\left(1-\mu\frac{1}{a^2 r^{d-5}}\right) \tag{3.73}$$

and the mixed terms $g_{t\phi} = g_{\phi t}$:

$$\frac{\mu}{r^{d-5}\rho^2} a \sin^2\theta \simeq \frac{\mu}{r^{d-5}a^2}\frac{\sigma^2}{a} \to 0. \tag{3.74}$$

Moreover:

$$\rho^2 d\theta^2 = \left(\frac{r^2}{a^2\cos^2\theta} + 1\right) a^2\cos^2\theta d\theta^2 \simeq a^2\cos^2\theta d\theta^2 = d\sigma^2 \tag{3.75}$$

and

$$\begin{aligned}\left(\frac{\mu a^2 \sin^4\theta}{r^{d-5}\rho^2} + (r^2+a^2)\sin^2\theta\right) d\phi^2 &\simeq \left(\frac{\mu}{a^2}\frac{\sin^2\theta}{r^{d-5}} + \frac{r^2}{a^2} + 1\right) a^2 \sin^2\theta d\phi^2 \\ &\simeq a^2\sin^2\theta d\phi^2 = \sigma^2 d\phi^2. \end{aligned} \tag{3.76}$$

So, at the end the metric (3.16) in the ultra-spinning limit is:

$$ds^2 = -\left(1 - \frac{\mu}{r^{d-5}a^2}\right) dt^2 + \left(1 - \frac{\mu}{r^{d-5}a^2}\right)^{-1} dr^2 + r^2 d\Omega^2_{d-4} + d\sigma^2 + \sigma^2 d\phi^2. \tag{3.77}$$

We notice that this version of the metric is not dependent of a but rather of the quantity μ/a^2. It describes a black membrane on the plane (σ, ϕ) with topology $\mathbb{R}^2 \times S^{d-4}$.
Indeed, we can confirm the occurrence of this type of symmetry by analysing the area of the horizon.
We consider the "parallel" component of the area, at a fixed position on the $d-4$ dimensional sphere:

$$A_\parallel = 4\pi(r^2 + a^2) \simeq 4\pi a^2, \tag{3.78}$$

obtained from (3.16) in an analogous way to what we previously did when calculating the horizon area in (3.60).
Now, examining the "transverse" component at fixed θ and ϕ, we get:

$$A_\perp = \Omega_{d-4}(r\cos\theta)^{d-4}. \tag{3.79}$$

The components have a characteristic size:

$$\begin{aligned} A_\parallel \sim l_\parallel^2 \quad &\to \quad l_\parallel = a; \\ A_\perp \sim l_\perp^{d-4} \quad &\to \quad l_\perp = r. \end{aligned} \tag{3.80}$$

In the ultra-spinning limit, since $a \gg r$, the component of the area in the (θ, ϕ) plane is much larger than along the directions transverse to the plane of rotation and grows as a increases, making the Black Hole progressively flatter and in the shape of a black membrane. Conversely, as $a \to 0$ the geometry resembles more that of a sphere, recovering the Schwarzschild geometry.

Next, we look at equation (3.59) and see which values for the angular momentum are allowed. Starting with 4 dimensions, we have:

$$r_0^2 - r_0\mu + a^2 = 0. \tag{3.81}$$

The horizon is identified with the largest real solution of the following resulting equation:

$$r_0 = \frac{\mu \pm \sqrt{\mu^2 - 4a^2}}{2}, \tag{3.82}$$

which is real for $\mu \geq 2a$.
The extremal value $\mu = 2a$ gives one degenerate horizon such that:

$$r_0 = a = \frac{\mu}{2} \tag{3.83}$$

and its area is:

$$A_H = \Omega_2(r_0^2 + a^2) = 4\pi(2a^2), \tag{3.84}$$

which corresponds to the area of a sphere of radius $\sqrt{2}a$.
The values for which $\mu > 2a$ give $\Delta > 0$ so that the metric is regular except in $\rho = 0$, which represents a singularity. Since there is no horizon in this case, the singularity is called naked and the solution is discarded by Penrose's cosmic censorship hypothesis.
The area for $\mu > 2a$ in four dimensions is given as:

$$A_H = 2\pi(\mu^2 + \mu\sqrt{\mu^2 - 4a^2}). \tag{3.85}$$

Now, we can try to apply this reasoning and calculate the conditions for the existence of the horizon in the $d = 5$ case.
The equation (3.59) becomes:

$$r_0^2 + a^2 - \mu = 0, \tag{3.86}$$

so that it gives

$$r_0 = \sqrt{\mu - a^2}, \tag{3.87}$$

which is real iff $\mu \geq a^2$.
The extremal solution, $\mu = a^2$, in this case has zero area:

$$r_0 = 0 \quad \rightarrow \quad A_H = \Omega_3 r_0(r_0^2 + a^2) = 0 \tag{3.88}$$

and it consists in a naked ring singularity. Since it has zero area but non zero angular momentum, it cannot be a point singularity.
For $\mu > a^2$ then the area is:

$$A_H = 2\pi^2\mu\sqrt{\mu - a^2}. \tag{3.89}$$

Now, we we examine what happens if $d \geq 6$, Δ. From Descartes' rule of signs:

The number of positive real zeroes in a polynomial function f(x) is the same as the number of changes in the sign of the coefficients or less than it by an even number.

Then, we can conclude that (3.59) has exactly one positive real root, so there is always a horizon, independently of the value of a.
Thus, the conclusion would be that a can be made arbitrarily large if $d \geq 6$.

3.4.2 Instabilities

Even though the value of a should be unbounded, after the Black Hole starts to resemble a black membrane, gravitational perturbations might kick in, developing instabilities and fragmenting the membrane into smaller Black Holes with a bigger area.
Indeed, as we have seen in (3.80), increasing a would increase only the "parallel" size of the horizon, while the other would be comparatively small. Having one direction much bigger than the others gives rise to gravitational perturbations, as better explained in [33]. Thermodynamical considerations lead us to think that these perturbations would act by breaking the Black Hole into two or more non-spinning fragments.
Indeed, we can show that breaking the initial Black Hole into non-rotating fragments will increase the area, thus maximizing the entropy according to the second law of thermodynamics.

First, we see that when $a \gg r_0$ the horizon (3.59) is approximated as:

$$\frac{\mu}{a^2 r_0^{d-5}} = \frac{r_0^2}{a^2} + 1 \simeq 1 \quad \to \quad r_0^{d-5} \simeq \frac{\mu}{a^2}. \tag{3.90}$$

Therefore the area (3.62) becomes:

$$A_H = \Omega_{d-2} a^2 r_0^{d-4} (1 + \frac{r_0^2}{a^2}) \simeq \Omega_{d-2} a^2 r_0^{d-4} \simeq \Omega_{d-2} \left(\frac{\mu^{d-4}}{a^2} \right)^{\frac{1}{d-5}}. \tag{3.91}$$

Provided we keep the mass fixed at a constant value, the area decreases as the angular momentum parameter a increases, until it goes to 0 as $a \to \infty$.
The area shows there is a change in behaviour from static and compact to ultra-spinning and flat. Since the area is a thermodynamic quantity, we can use the entropy to see this transition.
First, then, we look at the entropy of the system, which is calculated using (3.62):

$$S = \frac{k_B A_H}{4G} = \frac{k_B}{4G} \Omega_{d-2} r_0 \mu, \tag{3.92}$$

where k_B is the Boltzmann constant.
Then, we use temperature to analyse the transition from Kerr-like to membrane regime. It is defined in thermodynamics as:

$$T = \frac{\partial M}{\partial S} = \frac{\partial M}{\partial r_0} \frac{\partial r_0}{\partial S}. \tag{3.93}$$

Then, first we express the mass using (3.19) and (3.59):

$$\mu = \frac{16\pi G M}{(d-2)\Omega_{d-2}} = r_0^{d-5} \left(r_0^2 + a^2 \right);$$
$$\implies M = \frac{(d-2)\Omega_{d-2}}{16\pi G} r_0^{d-5} \left(r_0^2 + a^2 \right). \tag{3.94}$$

The result we get is:

$$T = \frac{d-2}{4\pi k_b} \left[\frac{2 r_0^{d-4}}{\mu} + \frac{d-5}{r_0} \right]. \tag{3.95}$$

Keeping the mass and consequently μ fixed and increasing the angular momentum parameter a progressively decreases r. Then, starting from $a = 0$ and moving to $a \to \infty$, the first term goes from dominant to negligible and the behaviour of the Black Hole goes from Kerr-like to that typical of a membrane ($T \sim \frac{1}{r}$).
The change, for $d \geq 6$, is given evidently at the point in which the temperature has a minimum given by the equation:

$$\frac{\partial T}{\partial\left(a/r_0\right)} = \frac{d-5+\frac{2}{1+\left(\frac{a}{r_0}\right)^2}}{r_0^2}\frac{r_0^2}{a} \overset{!}{=} 0 \tag{3.96}$$

and it corresponds to:

$$\frac{a}{r_0} = \sqrt{\frac{d-3}{d-5}}. \tag{3.97}$$

The membrane regime can thus be expressed by the following dimensionless quantity, calculated at the minimum for the temperature:

$$\frac{a^{d-3}}{\mu} = \frac{1}{2(d-4)}\sqrt{\frac{(d-3)^{d-3}}{(d-5)^{d-5}}}. \tag{3.98}$$

For $d = 6$ it takes the not so large value $\frac{a^{d-3}}{\mu} = 1.30$ and then it slightly increases with the dimensions.
Then, the instabilities occur at quite low values of the angular momentum and not far into the ultra-spinning regime.

To correct this value, we can actually analyse the fragmentation process, by taking an initial Myers-Perry Black Hole which divides into two fragments of equal mass for simplicity. The system is taken to be in the centre of momentum frame and the final products are taken to be non-rotating (as it would increase their area).
The total mass M is obtained from the mass-shell relation:

$$M = 2\sqrt{p^2+m^2} = 2\sqrt{m^2+\left(\frac{J}{2R}\right)^2}, \tag{3.99}$$

where m is the mass of the fragments and R is the impact parameter.
Then, we define the parameter μ_1 for one of the fragments:

$$\mu_1 = \frac{16\pi G}{(d-2)\Omega_{d-2}}m \tag{3.100}$$

and substitute m found in (3.99), the definition of the angular momentum (3.25) and of the mass M in (3.94):

$$\mu_1 = \frac{r_0^{d-5}\left(r_0^2+a^2\right)}{2}\sqrt{1-\frac{4a^2}{(d-2)^2R^2}}. \tag{3.101}$$

Then, the area of a static Black Hole is:

$$A_1 = \Omega_{d-2}r_0^{d-2} = \Omega_{d-2}\mu_1^{\frac{d-2}{d-3}} = \Omega_{d-2}\left[\frac{r_0^{d-5}\left(r_0^2+a^2\right)}{2}\sqrt{1-\frac{4a^2}{(d-2)^2R^2}}\right]^{\frac{d-2}{d-3}}. \tag{3.102}$$

Then, the condition to be satisfied for the second law of thermodynamics is that the area must increase, then $2A_1/A_H > 1$:

$$\left[\frac{1+\frac{a^2}{r_0^2}}{2}\left[1-\frac{4a^2}{(d-2)^2R^2}\right]^{\frac{d-2}{2}}\right]^{\frac{1}{d-3}} > 1. \tag{3.103}$$

The maximum value of the l.h.s. is obtained for large R, that is when the fragments carry the least kinetic energy and therefore the maximum rest mass, also contributing to get a larger area. However, a priori we do not know the process under which the Black Hole divides itself, so it is not safe to assume a large value for R. However, we can still assume R is inside the horizon of the initial Black Hole, for it to be a fragmentation, and, furthermore, since the dominant direction in the ultra-spinning regime is $l_\parallel \sim a$, R would scale approximately like a. In this case, the inequality is satisfied, provided a is large enough. Actually, we could just assume R grows faster than $\frac{2a}{d-2}$ to satisfy the inequality.
Imposing the condition that the impact parameter should not be bigger than the horizon radius of the initial Black Hole allows us to impose:

$$R \leq \sqrt{r_0^2 + a^2}. \tag{3.104}$$

Then, substituting this value into the inequality (3.103) gets:

$$\frac{1+\frac{a^2}{r_0^2}}{2}\left[1-\frac{4}{(d-2)^2}\frac{a^2}{r_0^2}\frac{1}{1+\frac{a^2}{r_0^2}}\right]^{\frac{d-2}{2}} > 1, \tag{3.105}$$

which can be solved for various dimensions, getting an estimate of the parameter $\frac{a}{r_0}$. This results to be not much larger than unity and in particular for $d = 6$ $\frac{a}{r_0} \gtrsim 1.36$. The correspondent $\frac{a^{d-3}}{\mu}$ is then set to be:

$$\frac{a^{d-3}}{\mu} = \frac{a^{d-3}}{r_0^{d-5}(r_0^2+a^2)} = \frac{a^{d-3}}{r_0^{d-3}(1+\frac{a^2}{r_0^2})}. \tag{3.106}$$

These values should not be taken as more than rough estimates as they are obtained from inaccurate models with approximations.
However, from this discussion we can conclude that in the ultra-spinning limit the Black Hole becomes similar to a membrane and already when the angular momentum gets slightly larger than the horizon radius, the systems starts developing instabilities that further lead it to fragmentation. Then, even though in principle the angular momentum could grow unbounded, in practice it must be kept to be not so large when compared to the horizon radius in order to avoid unstable configurations. Indeed, the instabilities seem to appear at a relatively early stage in the ultra-spinning regime, signalled by a change in the behaviour of the temperature.

Chapter 4

Horizon Wave Function

Higher dimensions lower the Planck scale from $m_p \simeq 10^{16}$ TeV to values that could, in principle, be tested in accelerators and might be around the TeV scale. Indeed, a lower Planck scale signifies we might be able to produce mini-Black Holes, which could be useful in the study of quantum gravity, as well as in confirming the existence of extra dimensions. In this chapter we are going to analyse the production of these Black Holes through their probability of formation.

To see better how these are formed, we take a scattering experiment and consider the "hoop conjecture" [37] (numerically verified for $d > 4$ in [35]), according to which a Black Hole will form if the colliding particles have impact parameter smaller than the horizon radius. The horizon radius is considered to be much less than the radius of the compact extra dimensions, which is generally true. We notice that in case this condition was not complied, we would simply end up with an effectively 4-dimensional Black Hole.

Another inequality we must take into account is related to the quantum nature of the formed particle. We take the uncertainty in the position of the source to be approximately in the form of the Compton wavelength λ:

$$\lambda = \frac{h}{mc}. \tag{4.1}$$

This quantity is, indeed, intrinsically quantum given the dependence on $\hbar$, which would make it vanish in the classical formal limit $\hbar \to 0$. It indicates the scale at which quantum fluctuations become of the order of the mass of the source. Therefore, the horizon radius is defined only if it is larger than this uncertainty:

$$R_0 \geq \lambda. \tag{4.2}$$

It implies that, classically, Black Holes arbitrarily small are allowed and this limit is achieved for $R_0 \gg \lambda$. From the condition (4.2), using (4.1) and the expression for the static horizon $R_0 = \mu^{\frac{1}{d-3}}$, with $\mu = Cm$ given by (4.65), it implies

$$\left[\frac{2}{d-3}\left(\frac{m}{m_p}\right)^{d-2}\right]^{\frac{1}{d-3}} \geq 1, \tag{4.3}$$

so that we expect approximately $m \gtrsim m_p$. For instance, in $d = 6$ it gives $m \gtrsim 1.10 m_p$. Moreover, since in this case $\lambda < l_p$, we have $R_0 \gtrsim \lambda > l_p$, strong gravity effects appear before the Planck length is probed. On the other hand, if $m < m_p$, $\lambda > l_p$ and $R_0 \lesssim< l_p < \lambda$ and quantum effects affect the measurement, making it impossible to determine whether it is a Black Hole.
Then, we can conclude that a static Black Hole is formed if its mass is above the Planck scale, but we can only take it as an estimate value, as we are not sure about the modifications to gravity that might occur at this scale.

However, if we take into account the self-UV completeness of gravity, when considering masses above the Planck scale we might not need a theory for quantum gravity since trans-Planckian physics corresponds to classical theories. Then, a semi-classical description is still possible, as after the Planck threshold is crossed, a large, classical Black Hole is formed.

The scattering cross section of two particles with centre of mass energy $\sqrt{s} = m$, then, is given semi-classically by:

$$\sigma \sim P_{BH} 4\pi m^2 \tag{4.4}$$

and it depends strongly on the probability we are going to calculate.

To treat classical horizons and Quantum Mechanical objects such as the colliding particles, we need a new formalism, which has been implemented in [38]-[41] and it is called Horizon Quantum Mechanics. It allows us to introduce operators whose expectation value give the classical quantity and, by defining a Horizon Wave Function corresponding to the horizon, we can calculate the probability of obtaining a Black Hole instead of a regular particle from the above-mentioned scattering experiment. We are also going to see how this formalism naturally leads to a Generalized Uncertainty Principle (GUP) for the particle position.

In this chapter, we shall employ the notation m_p to indicate the generally d-dimensional Planck mass. As the RS model is five-dimensional and the ADD model accounts for more extra dimensions, we implicitly assume m_p refers to one of these two, according to the dimensionality considered.

4.1 Horizon Quantum Mechanics

In this section we develop the formalism of Horizon Quantum Mechanics, which will allow us to study the horizon of a Black Hole as a quantum mechanical object, near the Planck scale, where gravity starts being affected by quantum effects. We are going to construct an operator for the horizon radius and an horizon wave function to describe it.

4.1.1 The Formalism of Horizon Quantum Mechanics

Our goal is to construct a Horizon Wave Function that would represent the Myers-Perry Black Hole, in order to calculate the probability that it is formed in a scattering experiment. As the type of Black Hole we are considering is axisymmetric and stationary, we consider sources localised in space and subject to pure rotation in a specific frame.
The Hilbert space of the source is described by a set of commuting operators $\{\hat{H}, \hat{J}^2, \hat{J}_z\}$.

We consider only the discrete energy spectrum with quantum numbers $\{a, j, m\}$ and we write the spectral decomposition of the source function as:

$$\psi_S \in L_2(\mathbb{R}^D), \qquad |\psi_S\rangle = \sum_{a,j,m} C(E_{aj}, \lambda_j, \xi_m)\,|a\ j\ m\rangle\,. \tag{4.5}$$

The coefficient in the expansion above depends on the eigenvalues of the following operators:

$$\hat{H} = \sum_{a,j,m} E_{aj}\,|a\ j\ m\rangle\,\langle a\ j\ m|\,; \tag{4.6}$$

$$\hat{J}^2 = \sum_{a,j,m} \lambda_j\,|a\ j\ m\rangle\,\langle a\ j\ m| \equiv \sum_{a,j,m} j(j+1)\,|a\ j\ m\rangle\,\langle a\ j\ m|\,; \tag{4.7}$$

$$\hat{J}_z = \sum_{a,j,m} \xi_m\,|a\ j\ m\rangle\,\langle a\ j\ m| \equiv \sum_{a,j,m} m\,|a\ j\ m\rangle\,\langle a\ j\ m|\,. \tag{4.8}$$

For an asymptotically flat metric, we can define the mass m of the source. Then, its quantum mechanical counterpart is given by the expectation value of the Hamiltonian:

$$\begin{aligned}\langle\psi_S|\,\hat{H}\,|\psi_S\rangle &= \sum_{a,j,m}\sum_{b,k,n} C^*(E_{aj}, \lambda_j, \xi_m)C(E_{bj}, \lambda_k, \xi_n)\,\langle a\ j\ m|\,\hat{H}\,|b\ k\ n\rangle \\ &= \sum_{a,j,m} |C(E_{aj}, \lambda_j, \xi_m)|^2 E_{aj}.\end{aligned} \tag{4.9}$$

Similarly, we calculate the expectation values for the angular momentum operators:

$$\langle\psi_S|\,\hat{J}^2\,|\psi_S\rangle = \sum_{a,j,m} |C(E_{aj}, \lambda_j, \xi_m)|^2 \lambda_j; \tag{4.10}$$

$$\langle\psi_S|\,\hat{J}_z\,|\psi_S\rangle = \sum_{a,j,m} |C(E_{aj}, \lambda_j, \xi_m)|^2 \xi_m. \tag{4.11}$$

The other important classical quantity is the horizon, to which, following an analogous reasoning, we can associate an operator:

$$\hat{R}_0\,|\alpha\rangle = R_{0\alpha}\,|\alpha\rangle\,. \tag{4.12}$$

Then, the system in a generic state can be described as an entangled state, depending on the energy, angular momentum and horizon radius:

$$|\psi\rangle = \sum_{a,j,m}\sum_{\alpha} C(E_{aj}, \lambda_j, \xi_m, R_{0\alpha})\,|a\ j\ m\rangle\,|\alpha\rangle\,. \tag{4.13}$$

4.1.2 Application to the Myers-Perry Black Hole

Now we want to apply the operatorial formalism above to the Myers-Perry radius.
According to the expression (3.59) we found in the first chapter, it is given by the following equation:

$$R_0^2 + a^2 - \frac{\mu}{R_0^{d-5}} = 0. \tag{4.14}$$

Since we need to define it in quantum mechanical terms, we must substitute the mass with the Hamiltonian operator.
As the mass appears raised to a fractional power, we have to make sure that it is allowed to do so with the corresponding operator.
First, we assume $\hat{H}$ is invertible, so that the inverse of its counterpart m is well-defined. Then a theorem states that as $\hat{H}$ is self-adjoint and positive semi-definite, its root is well-defined as well.
We can, therefore, conclude that any fractional power of $\hat{H}$ is allowed.
Next, in order to find an explicit expression for R_0, we need to proceed separately in the two approximations: static and ultra-spinning.

Static case

Using (3.70), we can define a corresponding operator $\hat{O}$ by substituting $\hat{H}$ to the classical mass:

$$R_0 = (Cm)^{\frac{1}{d-3}} \quad \rightarrow \quad \hat{O} = \left(k\hat{H}\right)^{\frac{1}{d-3}} . \tag{4.15}$$

The physical states, then, must satisfy the following Gupta-Bleuler condition:

$$\left(\hat{R}_0 - \hat{O}\right) |\psi_{\text{phys}}\rangle = 0. \tag{4.16}$$

Expressing the state as in (4.13), we get:

$$\sum_{a,j,m} \sum_{\alpha} \left[R_{0\alpha} - (CE_{aj})^{\frac{1}{d-3}}\right] C(E_{aj}, \lambda_j, \xi_m, R_{0\alpha}) |a\ j\ m\rangle |\alpha\rangle = 0, \tag{4.17}$$

which implies

$$R_{0\alpha} = (CE_{aj})^{\frac{1}{d-3}} . \tag{4.18}$$

Ultra-spinning case

We proceed analogously to the static case, but this time, in addition to (3.71), we also have to consider the angular momentum parameter:

$$a = \frac{d-2}{2}\frac{J}{m}. \tag{4.19}$$

We assume that also $\hat{J}^2$ is invertible, self-adjoint and positive semi-definite, so that we can raise it to any power.
We are ready to define the corresponding operator:

$$R_0 = \left(\frac{4C}{(d-2)^2}\frac{m^2}{J^2}\right)^{\frac{1}{d-5}} \quad \rightarrow \quad \hat{O}' = \left[\frac{4k}{(d-2)^2}\hat{H}^2\left(\hat{J}^2\right)^{-1}\right]^{\frac{1}{d-5}} . \tag{4.20}$$

Again we impose the Gupta-Bleuler condition on the physical states:

$$\left(\hat{R}_0 - \hat{O}'\right) |\psi_{\text{phys}}\rangle = 0. \tag{4.21}$$

When substituting (4.13), we get:

$$\sum_{a,j,m}\sum_{\alpha}\left(R_{0\alpha}-\frac{4C}{(d-2)^2}\frac{E_{aj}^2}{\lambda_j^2}\right)^{\frac{1}{d-5}}C(E_{aj},\lambda_j,\xi_m,R_{0\alpha})\left|a\ j\ m\right\rangle\left|\alpha\right\rangle=0, \tag{4.22}$$

which implies

$$R_{0\alpha}=\left[\frac{4C}{(d-2)^2}\frac{E_{aj}^2}{\lambda_j^2}\right]^{\frac{1}{d-5}}. \tag{4.23}$$

For both cases we can conclude a generic state, in the form (4.13), is:

$$\left|\psi\right\rangle=\sum_{a,j,m}\sum_{\alpha}C(E_{aj},\lambda_j,\xi_m,R_{0\alpha}(E_{aj}))\left|a\ j\ m\right\rangle\left|\alpha\right\rangle, \tag{4.24}$$

where $R_{0\alpha}(E_{aj})$ is given by the appropriate relation, depending on the regime we are considering.
Then, by tracing out the gravitational radius part, we recover ψ_S:

$$\begin{aligned}\left|\psi_S\right\rangle&=\sum_{\beta}\left\langle\beta|\psi\right\rangle=\sum_{a,j,m}\sum_{\beta\alpha}C(E_{aj},\lambda_j,\xi_m,R_{0\alpha}(E_{aj}))\left|a\ j\ m\right\rangle\delta_{\alpha\beta}\\&=\sum_{a,j,m,\alpha}C(E_{aj},\lambda_j,\xi_m,R_{0\alpha}(E_{aj}))\left|a\ j\ m\right\rangle\overset{!}{=}C_S(E_{aj},\lambda_j,\xi_m)\left|a\ j\ m\right\rangle.\end{aligned} \tag{4.25}$$

This allows us to conclude that:

$$C_S(E_{aj},\lambda_j,\xi_m)=C(E_{aj},\lambda_j,\xi_m,R_{0\alpha}(E_{aj})). \tag{4.26}$$

Similarly, ψ_H, the horizon state, is obtained by integrating out the energy eigenstates, leaving only the expansion on the radial quantum states:

$$\begin{aligned}\left|\psi_H\right\rangle&=\sum_{bkn}\left\langle b\ k\ n|\psi\right\rangle=\sum_{a,j,m}\sum_{bkn}\sum_{\alpha}C(E_{aj},\lambda_j,\xi_m,R_{0\alpha}(E_{aj}))\delta_{ajm,bkn}\left|\alpha\right\rangle\\&=\sum_{a,j,m,\alpha}C(E_{aj},\lambda_j,\xi_m,R_{0\alpha}(E_{aj}))\left|\alpha\right\rangle.\end{aligned} \tag{4.27}$$

The normalization is fixed by the internal product:

$$\left\langle\psi_H|\phi_H\right\rangle=\int_0^\infty\psi_H^*(r_0)\phi_H(r_0)A_H dr_0, \tag{4.28}$$

where A_H is the area of the event horizon.
Finally, the horizon wave function is defined as:

$$\begin{aligned}\psi_H(R_0)=\left\langle\beta|\psi_H\right\rangle&=\sum_{a,j,m,\alpha}C(E_{aj},\lambda_j,\xi_m,R_{0\alpha}(E_{aj}))\\&=C_S(E_{aj}(R_{0\alpha}),\lambda_j,\xi_m).\end{aligned} \tag{4.29}$$

4.2 Static case

We make a brief recap of the results we found for the static case, characterized by $a \simeq 0$:

$$\mu \equiv Cm = r_0^{d-3}, \qquad A_H = \Omega_{d-2} r_0^{d-2}. \tag{4.30}$$

We shall use these later to actually calculate the probability of obtaining a Black Hole in a scattering experiment.

4.2.1 Wave functions

We decide to model to source using a Gaußian distribution, with extra dimensions localized on the origin:

$$\psi_S(r,y) = \mathcal{N} e^{-\frac{r^2}{2l^2}} \delta(y_1)\ldots\delta(y_{d-2}). \tag{4.31}$$

The width l in the particle position must be greater than the Compton length of the particle for it to make sense:

$$l \geq \lambda = \frac{\hbar}{m}, \tag{4.32}$$

where $l = \Lambda$ signals the maximum localization of the particle.
The probability of finding the particle source inside the horizon is given by:

$$P_S = \int d^d x |\psi_S(r,y)|^2 = \int dA_H \int_0^{r_0} dr |\psi_S(r)|^2. \tag{4.33}$$

The normalization constant $\mathcal{N}$ in (4.31) is fixed by requiring that $P_S = 1$ if $r_0 = \infty$:

$$1 = \int d\Omega_{d-2} \int_0^\infty dr r^{d-2} |\psi_S(r,y)|^2. \tag{4.34}$$

To get the square of (4.31) we need to give meaning to the Dirac delta squared:

$$[\delta(y_1)\ldots\delta(y_{d-2})]^2 = \delta(y_1)\ldots\delta(y_{d-2})\delta(0)^{d-2} = \delta(y_1)\ldots\delta(y_{d-2})\frac{V_{(d-2)}}{(2\pi)^{d-2}}. \tag{4.35}$$

So that

$$\begin{aligned} 1 &= \frac{V_{(d-2)}}{(2\pi)^{d-2}} \int d\Omega_{d-2} \int_0^\infty dr \mathcal{N}^2 e^{-\frac{r^2}{l^2}} r^{d-2} \delta(y_1)\ldots\delta(y_{d-2}) \\ &= \frac{V_{(d-2)}}{(2\pi)^{d-2}} \int d\theta_1 \ldots d\theta_{d-2} |J| \frac{\delta(\theta_1)\ldots\delta(\theta_{d-2})}{|J|} \int_0^\infty dr \mathcal{N}^2 e^{-\frac{r^2}{l^2}} r^{d-2} \\ &= \frac{V_{(d-2)}}{(2\pi)^{d-2}} \int_0^\infty dr \mathcal{N}^2 e^{-\frac{r^2}{l^2}} r^{d-2}, \end{aligned} \tag{4.36}$$

where in the second line we used the change of variable for the Dirac delta function:

$$\delta(y_1)\ldots\delta(y_{d-2}) = \frac{\delta(\theta_1)\ldots\delta(\theta_{d-2})}{|J|} \tag{4.37}$$

and the definition:

$$d\Omega_{d-2} = d\theta_1 \ldots d\theta_{d-2} |J|. \tag{4.38}$$

Now, we change the integration variable to s as:

$$\begin{cases} s = \dfrac{r^2}{l^2} \quad \rightarrow \quad r = \sqrt{s}l; \\ ds = \dfrac{2r}{l^2}dr \quad \rightarrow \quad dr = \dfrac{l^2}{2r}ds \end{cases} \tag{4.39}$$

and we obtain:

$$\begin{aligned} 1 &= \frac{V_{(d-2)}}{(2\pi)^{d-2}}\mathcal{N}^2 \int_0^\infty ds \frac{l^2}{2r} r^{d-2} e^{-s} \\ &= \frac{V_{(d-2)}}{(2\pi)^{d-2}}\mathcal{N}^2 \frac{l^2}{2} \int_0^\infty ds e^{-s} (\sqrt{s}l)^{d-3} \\ &= \frac{V_{(d-2)}}{(2\pi)^{d-2}}\mathcal{N}^2 \frac{l^{d-1}}{2} \Gamma\left(\frac{d-1}{2}\right), \end{aligned} \tag{4.40}$$

using the definition of the Gamma functions:

$$\Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x}. \tag{4.41}$$

In conclusion, the normalization function is:

$$\mathcal{N}^2 = \frac{(2\pi)^{d-2}}{V_{(d-2)}} \frac{2}{\Gamma\left(\frac{d-1}{2}\right) l^{d-1}} \tag{4.42}$$

and the wave function of the source is:

$$\psi_S = \left[\frac{(2\pi)^{d-2}}{V_{(d-2)}} \frac{2}{\Gamma\left(\frac{d-1}{2}\right) l^{d-1}}\right]^{\frac{1}{2}} e^{-\frac{r^2}{2l^2}} \delta(y_1) \ldots \delta(y_{d-2}). \tag{4.43}$$

Now, we want to express (4.31) in momentum space, so we Fourier transform it:

$$\begin{aligned} \psi_S(p, p_j) &= \mathcal{N} \int dr dy_1 \ldots dy_{d-2} \delta(y_1) \ldots \delta(y_{d-2}) e^{-\frac{r^2}{2l^2}} e^{-\frac{i}{\hbar}pr} e^{-\frac{i}{\hbar}p^j y_j} \\ &= \mathcal{N} \int_0^\infty dr e^{-\left(\frac{r^2}{2l^2} + \frac{i}{\hbar}pr\right)}. \end{aligned} \tag{4.44}$$

By completing the square:

$$-\left(\frac{r^2}{2l^2} + \frac{i}{\hbar}pr\right) = -\frac{1}{2}\left[\left(\frac{r}{l} + \frac{i}{\hbar}pl\right)^2 + \frac{p^2 l^2}{\hbar^2}\right] \tag{4.45}$$

and changing the variable to k:

$$\begin{cases} k = \frac{r}{l} + \frac{i}{\hbar}pl \\ dr = l dk \end{cases} \tag{4.46}$$

we get a Gaußian integral:

$$\psi_S(p) = \mathcal{N} e^{-\frac{p^2 l^2}{2\hbar^2}} l \int dk e^{-\frac{k^2}{2}} = \mathcal{N} l \sqrt{\frac{\pi}{2}} e^{-\frac{p^2 l^2}{2\hbar^2}} \equiv \mathcal{N}' e^{-\frac{p^2}{2\Delta^2}}. \tag{4.47}$$

Here, we introduced the width $\Delta = \frac{\hbar}{l}$.
Now we can assume the mass shell relation for flat space $E^2 = p^2 + m^2$, since corrections to it would be negligible, and construct the Horizon Wave Function.
In (4.47) we substitute $p^2 = E^2 - m^2$ where $m = \frac{\mu}{C} = \frac{R_0^{d-3}}{C}$ as stated above. Moreover, we insert a Heaviside step function $\theta(r_0 - R_0)$ that is necessary to implement the condition $E \geq m$ and we use another generic normalization constant $\mathcal{N}_0$.

$$\psi_H(r_0) = \mathcal{N}_0 \theta(r_0 - R_0) e^{-\frac{1}{2(C\Delta)^2}\left(r^{2(d-3)} - R^{2(d-3)}\right)}. \tag{4.48}$$

The constant $\mathcal{N}_0$ is fixed similarly to what was done for the source function (4.31).
We take the probability of finding the horizon at r_0:

$$P_H = \int d\Omega_{d-2} \int_0^{r_0} dr |\psi_H(r)|^2 r^{d-2} \tag{4.49}$$

and we require that at $r_0 = \infty$ it is $P_H = 1$. Therefore:

$$\begin{aligned} 1 &= \Omega_{d-2} \int_{R_0}^{\infty} dr_0 \mathcal{N}_0^2 e^{-\frac{r_0^{2(d-3)}}{(C\Delta)^2}} e^{\frac{R_0^{2(d-3)}}{(C\Delta)^2}} r_0^{d-2} \\ &= \mathcal{N}_0^2 \Omega_{d-2} e^{\frac{m^2}{\Delta^2}} \int_{R_0}^{\infty} dr_0 e^{-\frac{r_0^{2(d-3)}}{(C\Delta)^2}} r_0^{d-2}. \end{aligned} \tag{4.50}$$

Where, again, we used $m = \frac{\mu}{C} = \frac{R_0^{d-3}}{C}$, and also we made use of the fact that $r_0 > R_0$ given by the Heaviside function to change the lower extreme of integration.
Now, we perform a convenient change of variable:

$$\begin{cases} s = \dfrac{r_0^{2(d-3)}}{(C\Delta)^2} \qquad \rightarrow \qquad r_0 = \left(C^2\Delta^2 s\right)^{\frac{1}{2(d-3)}}; \\ ds = 2(d-3)\dfrac{r_0^{2(d-3)-1}}{(C\Delta)^2} dr_0. \end{cases} \tag{4.51}$$

So that by substituting we obtain:

$$\begin{aligned} \frac{\mathcal{N}_0^{-2} e^{-\frac{m^2}{\Delta^2}}}{\Omega_{d-2}} &= \int_{\frac{m^2}{\Delta^2}}^{\infty} ds \frac{(C\Delta)^2}{2(d-3) r_0^{2(d-3)-1}} r_0^{d-2} e^{-s} \\ &= \frac{(C\Delta)^2}{2(d-3)} \int_{\frac{m^2}{\Delta^2}}^{\infty} ds \left(C^2\Delta^2 s\right)^{\frac{-d+5}{2(d-3)}} e^{-s} \\ &= (C\Delta)^{\frac{d-1}{d-3}} \frac{1}{2(d-3)} \Gamma\left[\frac{d-1}{2(d-3)}, \frac{m^2}{\Delta^2}\right]. \end{aligned} \tag{4.52}$$

where we used the function called upper incomplete gamma function:

$$\Gamma(s, x) = \int_x^{\infty} dx x^{s-1} e^{-x}. \tag{4.53}$$

So then

$$\mathcal{N}_0^2 e^{\frac{m^2}{\Delta^2}} = \frac{2(d-3)}{\Omega_{d-2}(C\Delta)^{\frac{d-1}{d-3}}} \frac{1}{\Gamma\left[\frac{d-1}{2(d-3)}, \frac{m^2}{\Delta^2}\right]} \tag{4.54}$$

and we can insert it in (4.48) to finally obtain:

$$\psi_H(r_0) = \left\{ \frac{2(d-3)}{\Omega_{d-2}} \frac{(C\Delta)^{-\frac{d-1}{d-3}}}{\Gamma\left[\frac{d-1}{2(d-3)}, \frac{m^2}{\Delta^2}\right]} \right\}^{\frac{1}{2}} \theta(r_0 - R_0) e^{-\frac{r^{2(d-3)}}{2(C\Delta)^2}}. \tag{4.55}$$

4

4.2.2 Probabilities

The probabilities involved are the same that were mentioned in the previous section.
First, we calculate P_S, the probability that the source is inside r_0, that was defined in (4.33). We employ the function for the source (4.43) and omit some of the passages we used when calculating the normalization constant, as they are exactly the same.

$$\begin{aligned} P_S &= \int d\Omega_{d-2} \int_0^\infty dr r^{d-2} |\psi_S(r,y)|^2 \\ &= \frac{(2\pi)^{d-2}}{V_{(d-2)}} \frac{2}{\Gamma\left(\frac{d-1}{2}\right) l^{d-1}} \int d\Omega_{d-2} \int_0^{r_0} dr e^{-\frac{r^2}{l^2}} \frac{V_{(d-2)}}{(2\pi)^{d-2}} \delta(y_1) \dots \delta(y_{d-2}) r^{d-2} \\ &= \frac{2}{\Gamma\left(\frac{d-1}{2}\right) l^{d-1}} \int_0^{r_0} dr e^{-\frac{r^2}{l^2}} r^{d-2} \\ &= \frac{\gamma\left(\frac{d-1}{2}, \frac{r_0^2}{l^2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}. \end{aligned} \tag{4.56}$$

In the last passage we used the lower incomplete gamma function, defined as:

$$\gamma(s,x) = \int_0^x dx x^{s-1} e^{-x}. \tag{4.57}$$

This can be expressed as $\gamma(s,x) = \Gamma(s) - \Gamma(s,x)$. By using this, we write the probability in a more convenient form:

$$P_S = 1 - \frac{\Gamma\left(\frac{d-1}{2}, \frac{r_0^2}{l^2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}. \tag{4.58}$$

With this, we can calculate the probability that a black hole is formed, P_{BH}, by combining the probability that the horizon is located at r_0 (P_H, defined in (4.49)) and the probability that the source particle is inside the horizon (P_S, defined in (4.33)).
Therefore, we define it:

$$P_{BH} = \int_0^\infty dr_0 P_S \mathcal{P}_H, \tag{4.59}$$

where $\mathcal{P}_H$ is the probability density corresponding to P_H.
Then, we can write:

$$\begin{aligned} P_{BH} &= \int_0^\infty dr_0 \left[1 - \frac{\Gamma\left(\frac{d-1}{2}, \frac{r_0^2}{l^2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}\right] \int d\Omega_{d-2} r_0^{d-2} |\psi_H|^2 \\ &= 1 - \frac{\Omega_{d-2}}{\Gamma\left(\frac{d-1}{2}\right)} \left\{ \frac{2(d-3)}{\Omega_{d-2}} \frac{(C\Delta)^{-\frac{d-1}{d-3}}}{\Gamma\left[\frac{d-1}{2(d-3)}, \frac{m^2}{\Delta^2}\right]} \right\} \int_{R_0}^\infty dr_0 \Gamma\left[\frac{d-1}{2}, \frac{r_0^2}{l^2}\right] r_0^{d-2} e^{-\frac{r_0^{2(d-3)}}{(C\Delta)^2}} \\ &= 1 - \frac{1}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left[\frac{d-1}{2(d-3)}, \frac{m^2}{\Delta^2}\right]} \frac{2(d-3)}{(C\Delta)^{\frac{d-1}{d-3}}} \int_{R_0}^\infty dr_0 \Gamma\left(\frac{d-1}{2}, \frac{r_0^2}{l^2}\right) r_0^{d-2} e^{-\frac{r_0^{2(d-3)}}{(C\Delta)^2}}. \end{aligned} \tag{4.60}$$

Then we change variable exactly as was done in (4.51)

$$\begin{cases} s = \dfrac{r_0^{2(d-3)}}{(C\Delta)^2} \qquad \rightarrow \qquad r_0 = \left(C^2\Delta^2 s\right)^{\frac{1}{2(d-3)}}; \\ ds = 2(d-3)\dfrac{r_0^{2(d-3)-1}}{(C\Delta)^2} dr_0 \end{cases} \tag{4.61}$$

and by substituting it we obtain:

$$P_{BH} = 1 - \frac{1}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left[\frac{d-1}{2(d-3)}, \frac{m^2}{\Delta^2}\right]} \int_{\frac{m^2}{\Delta^2}}^\infty ds \Gamma\left(\frac{d-1}{2}, \frac{\left(C^2\Delta^2 s\right)^{\frac{1}{d-3}}}{l^2}\right) e^{-s} s^{\frac{d-1}{2(d-3)}-1}. \tag{4.62}$$

Now, we are interested in expressing this probability as a function of $\frac{l}{l_p}$. To this end, we employ a more convenient expression for the constant $C = \frac{16\pi G}{(d-2)\Omega_{d-2}}$.
Indeed, as we have seen in the previous chapters, the gravitational force per unit mass is obtained by $\vec{F} = -\nabla\phi$ and $\phi = -\frac{1}{2}h_{00}$.
Therefore,

$$\vec{F} = \frac{(d-3)8\pi G}{(d-2)\Omega_{d-2}} \frac{m}{r^{d-2}} \hat{r}. \tag{4.63}$$

We can, alternatively, redefine the gravitational constant G so that the force law is similar to what we normally use in $d = 5$:

$$\vec{F} = G' \frac{m}{r^{d-2}} \hat{r}. \tag{4.64}$$

With this new definition we get:

$$C = \frac{16\pi}{(d-2)\Omega_{d-2}} \frac{(d-2)\Omega_{d-2}}{(d-3)8\pi} G' = \frac{2G'}{(d-3)} = \frac{2 l_p^{d-3}}{(d-3)m_p}. \tag{4.65}$$

By substituting it in (4.62), together with the definition of $\Delta = \frac{m_p l_p}{l}$, we have:

$$\begin{aligned} P_{BH} = 1 &- \frac{1}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left[\frac{d-1}{2(d-3)}, \left(\frac{m}{m_p}\frac{l}{l_p}\right)^2\right]} \\ &\times \int_{\left(\frac{m}{m_p}\frac{l}{l_p}\right)^2}^\infty ds e^{-s} s^{\frac{d-1}{2(d-3)}-1} \Gamma\left\{\frac{d-1}{2}, \left[\frac{2}{d-3}\left(\frac{l_p}{l}\right)^{d-2}\right]^{\frac{2}{d-3}} s^{\frac{1}{d-3}}\right\}. \end{aligned} \tag{4.66}$$

Now, we wish to plot this quantity and we show how it changes when fixing one parameter and varying another. Below, the probability of Black Hole formation from equation (4.66) is displayed as a function $\frac{l}{l_p}$. The plots are realised using a numerical computation of the function, as an analytic approach is not viable.
The first plot shows the probability for fixed $m = m_p$ and varying dimensionality:

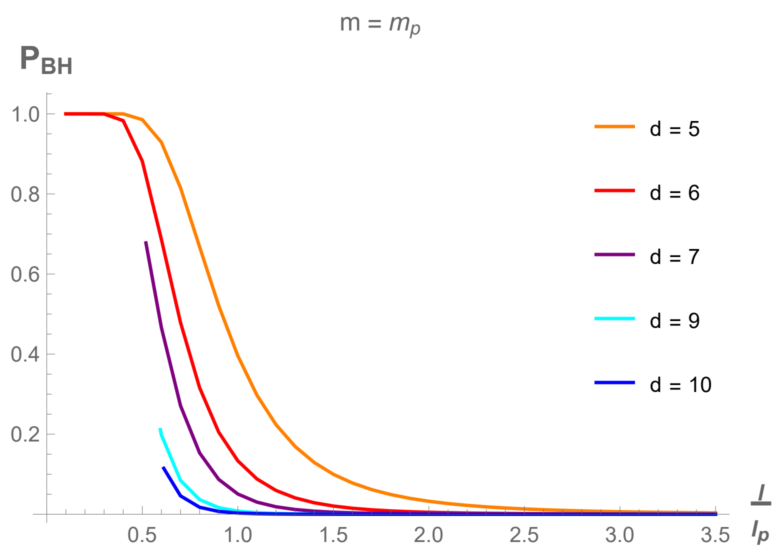


Figure 4.1: P_{BH} for $m = m_p$

It shows the probability decreases for increasing dimensionality. Indeed, in smaller dimensions it has a maximum at around $l = \frac{1}{2}l_p$, and it is pretty small at the Planck length already at $d = 6$. For higher dimensions it is very suppressed even at the Planck length and it requires to reach much smaller scales to experience a relevant probability of measuring a Black Hole. It could be thought that the diminution in the probability is linked to the decrease in the horizon radius for high dimensionality. We argue this is not the case, as taking a very small gravitational radius $R_0 \to 0$ does not affect much the probability when $d = 6$, as it shown in the following numerical plot:

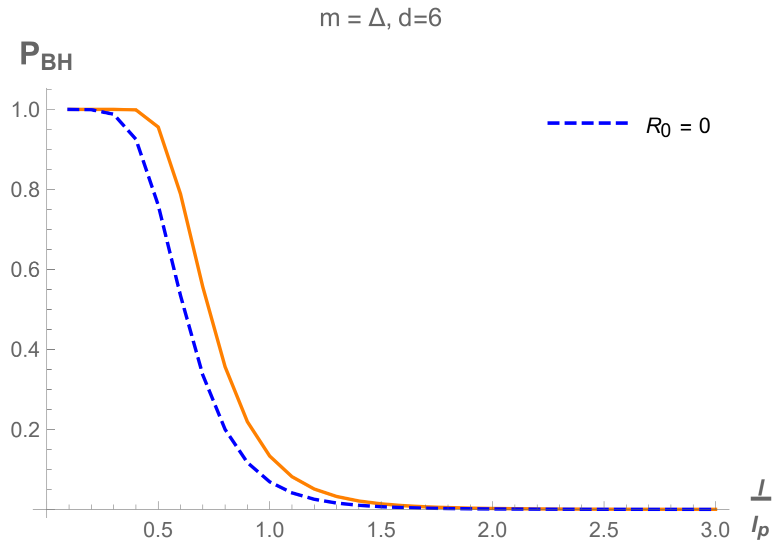


Figure 4.2: P_{BH} for $d = 6$ and $m = \Delta$ in the case when $R_0 \to 0$ compared to the probability with no such approximation

Next, we show the dependence of the Probability on the mass, by plotting for fixed dimensionality and varying $\frac{m}{m_p}$:

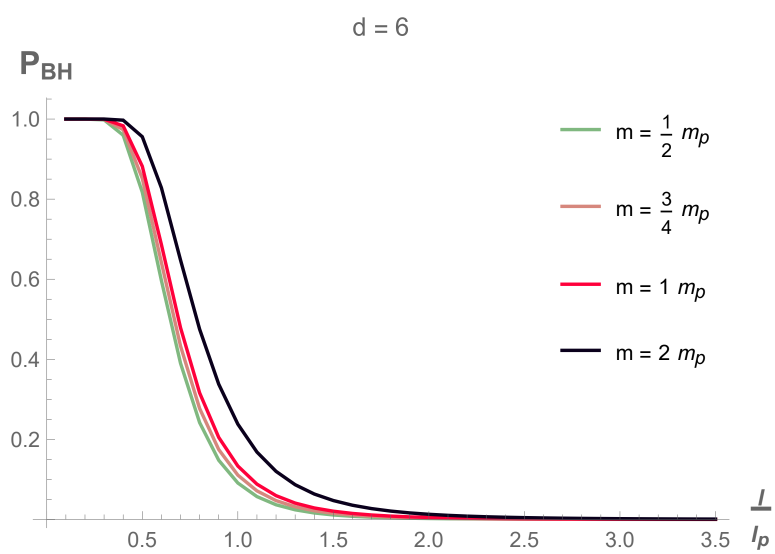


Figure 4.3: P_{BH} for $d = 6$

The behaviour portrayed in this figure shows that the smaller the ratio $\frac{m}{m_p}$ the lower the probability is. The maximum probability is obtained at around $l = \frac{1}{2}l_p$ for all the values considered here. Among the different curves, the one for $m = 2m_p$ gives the highest prob-

ability and corresponds to trans-Planckian masses.
The next plot we shall present refers to the case when $m = \Delta$, meaning $l = \lambda$, which is the extremal condition when the Gaußian width is exactly the Compton length. In this case, $\frac{m}{m_p} = \frac{l_p}{l}$ and then the probability is only a function of $\frac{m}{m_p}$ and dimensionality d.

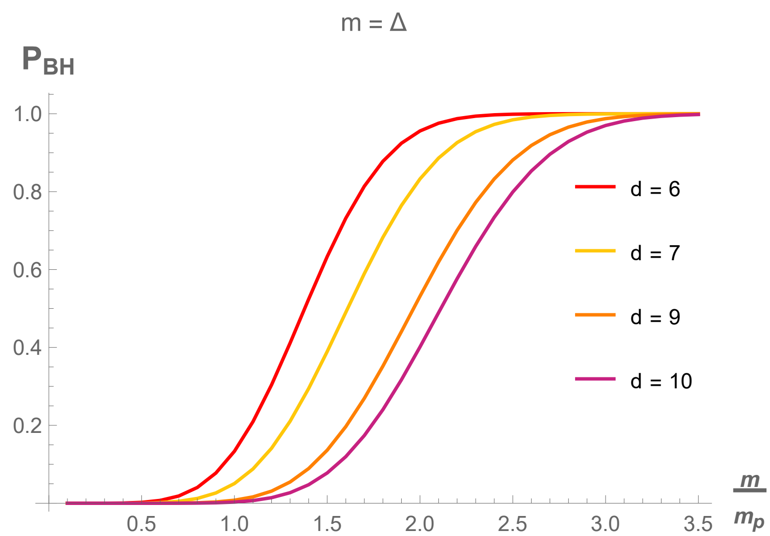


Figure 4.4: P_{BH} with $m = \Delta$

The result is in accordance to what we expected at the beginning of this section, the maximum probability is reached only after the Planck mass threshold is crossed and for $d = 6$ it indeed starts growing after $m \sim 1.10 m_p$. With increasing dimensionality of spacetime the probability is lowered. At $d = 10$ the maximum probability is reached only at $m \sim 3 m_p$.

4.2.3 Generalized Uncertainty Principle

We can find the uncertainty in the size of the horizon:

$$\Delta r_0 = \sqrt{\langle r_0^2 \rangle - \langle r_0 \rangle^2}. \tag{4.67}$$

The average is calculated as:

$$\langle \hat{O} \rangle = \Omega_{d-2} \int_0^\infty dr r^{d-2} \psi_H^* \hat{O} \psi_H. \tag{4.68}$$

By using the horizon wave function (4.48), with normalization constant (4.54) and proceeding with the change of variable (4.51) we obtain:

$$
\begin{aligned}
\langle r_0\rangle &= \Omega_{d-2}\mathcal{N}^2\frac{(C\Delta)^{\frac{d}{d-3}}}{2(d-3)}\int_{\frac{m^2}{\Delta^2}}^{\infty} dse^{-s}s^{\frac{d}{2(d-3)}} \\
&= \frac{\Omega_{d-2}}{2(d-3)}(C\Delta)^{\frac{d}{d-3}}\left\{\frac{2(d-3)}{\Omega_{d-2}}\frac{(C\Delta)^{-\frac{d-1}{d-3}}}{\Gamma\left[\frac{d-1}{2(d-3)},\frac{m^2}{\Delta^2}\right]}\right\}\Gamma\left[\frac{d}{2(d-3)},\frac{m^2}{\Delta^2}\right] \\
&= \frac{\Gamma\left[\frac{d}{2(d-3)},\frac{m^2}{\Delta^2}\right]}{\Gamma\left[\frac{d-1}{2(d-3)},\frac{m^2}{\Delta^2}\right]}(C\Delta)^{\frac{1}{d-3}} = \frac{\Gamma\left(\frac{d}{2(d-3)},\frac{m^2}{\Delta^2}\right)}{\Gamma\left(\frac{d-1}{2(d-3)},\frac{m^2}{\Delta^2}\right)}\left[\frac{\Delta}{m}\right]^{\frac{1}{d-3}}R_0, \qquad (4.69)
\end{aligned}
$$

where in the last passage we used the fact that $Cm = R_0^{d-3}$.
Similarly,

$$
\langle r_0^2\rangle = \frac{\Gamma\left[\frac{d+1}{2(d-3)},\frac{m^2}{\Delta^2}\right]}{\Gamma\left[\frac{d-1}{2(d-3)},\frac{m^2}{\Delta^2}\right]}\left(\frac{\Delta}{m}\right)^{\frac{2}{d-3}}R_0^2. \qquad (4.70)
$$

Now we use the expression:

$$
\Gamma\left(s,x\right) = x^s E_{1-s}(x), \qquad E_n(x) = \int_1^{\infty} dt\frac{e^{-xt}}{t^n}, \qquad (4.71)
$$

where $E_n(x)$ is the generalised exponential integral.
The result (4.67) is thus:

$$
\begin{aligned}
\Delta r_0 &= \sqrt{\frac{\left(\frac{m^2}{\Delta^2}\right)^{\frac{d+1}{2(d-3)}}E_{\frac{d-7}{2(d-3)}}\left(\frac{m^2}{\Delta^2}\right)}{\left(\frac{m^2}{\Delta^2}\right)^{\frac{d-1}{2(d-3)}}E_{\frac{d-5}{2(d-3)}}\left(\frac{m^2}{\Delta^2}\right)} - \frac{\left(\frac{m^2}{\Delta^2}\right)^{\frac{d}{d-3}}E^2_{\frac{d-6}{2(d-3)}}\left(\frac{m^2}{\Delta^2}\right)}{\left(\frac{m^2}{\Delta^2}\right)^{\frac{d-1}{d-3}}E^2_{\frac{d-5}{2(d-3)}}\left(\frac{m^2}{\Delta^2}\right)}\left(\frac{\Delta}{m}\right)^{\frac{1}{d-3}}}R_0 \\
&= R_0\sqrt{\frac{E_{\frac{d-7}{2(d-3)}}\left(\frac{m^2}{\Delta^2}\right)}{E_{\frac{d-5}{2(d-3)}}\left(\frac{m^2}{\Delta^2}\right)} - \frac{E^2_{\frac{d-6}{2(d-3)}}\left(\frac{m^2}{\Delta^2}\right)}{E^2_{\frac{d-5}{2(d-3)}}\left(\frac{m^2}{\Delta^2}\right)}}. \qquad (4.72)
\end{aligned}
$$

Expressing the radius as $R_0 = (Cm)^{\frac{1}{d-3}}$ and using (4.65) it can be rewritten as:

$$
\frac{\Delta r_0}{l_p} = \left(\frac{2}{d-3}\frac{m}{m_p}\right)^{\frac{1}{d-3}}K(d), \qquad (4.73)
$$

where the square root was indicated with $K(d)$ for simplicity of notation.
Now we can plot this quantity as a function of $\frac{l}{l_p}$ (contained in $K(d)$, in the dependence of the exponential integrals):

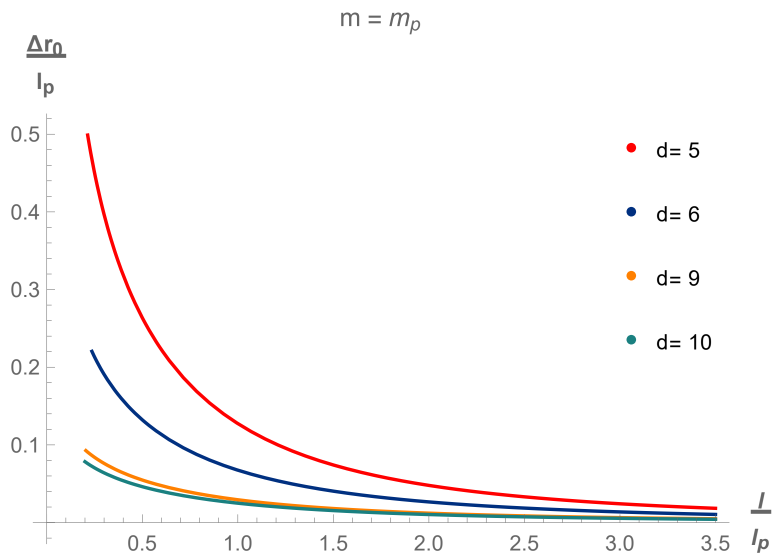


Figure 4.5: $\frac{\Delta r_0}{l_p}$ with $m = m_p$

The plot shows the uncertainty on the horizon radius as a function of the width of the source when m is kept fixed at m_p and d takes different values. This figure makes it clear that if $l \gg l_p$ the classical limit is reproduced:

$$\langle r_0 \rangle \simeq R_0 \qquad \text{and} \qquad \Delta r_0 \simeq 0. \tag{4.74}$$

Also, for higher dimensions the uncertainty is reduced, in line with the low probability of forming a Black Hole. This could hint to the formation of classical particles before the Planck scale is reached.
The next plot shows the dependence on the mass when d is kept fixed:

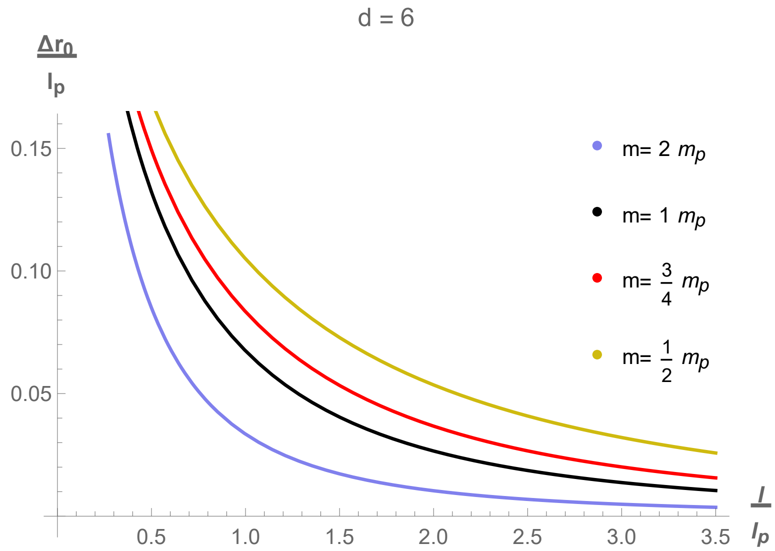


Figure 4.6: $\frac{\Delta r_0}{l_p}$ with $d = 6$

In this figure we notice that increasing the mass the uncertainty is reduced, making the horizon more sharply defined. This could hint to the fact that trans-Planckian do indeed generate classical Black Holes, as masses greater than the Planck scale give more classical results.
Next, we plot (4.73) when $l = \lambda$ and we end up with an equation that only depends on $\frac{m}{m_p}$ and on the dimensions of spacetime.

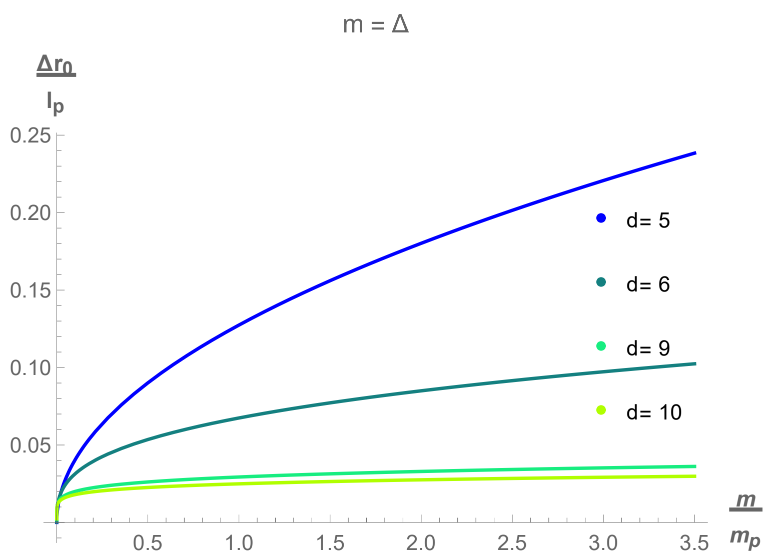


Figure 4.7: $\frac{\Delta r_0}{l_p}$ with $m = \Delta$

Here, we notice the behaviour of the uncertainty gets more and more constant as the dimensionality increases and as m is increased. Classicality is no longer retrieved for $l \gg l_p$ which corresponds to $m < m_p$. Moreover, if $m > m_p$ the curves are either constant or diverge, not resulting in the semi-classical behaviour we would have expected. This could be due to the fact that this time $K(d)$ is a constant and therefore $\Delta r_0 \propto R_0$. Having an uncertainty on the horizon proportional to it is not what we want for a classical Black Hole. This can be solved changing model and considering the Black Hole as a condensate of gravitons, each of which is described by a Gaußian source. Then it can be shown (refer to [36]) that the uncertainty becomes $\Delta r_0 \propto \frac{R_0}{N}$, where N is the number of gravitons and since it is very large, the uncertainty goes to 0, giving the expected classical result.

Subsequently, we obtain the total GUP by adding to the uncertainty Δr_0 the Heisenberg one on the source Δr_S:

$$\Delta r = \Delta r_s + \alpha \Delta r_0. \tag{4.75}$$

The coefficient α should be measured experimentally, but for simplicity we can set it to 1. The term Δr_s can be found analogously to (4.67) but using ψ_S from (4.43) instead of ψ_H:

$$\langle r \rangle = \Omega_{d-2} \int_0^\infty dr r^{d-2} \psi_S^* r \psi_S. \tag{4.76}$$

By using the same substitution as in (4.39):

$$\begin{aligned} \langle r \rangle &= \frac{V_{(d-2)}}{(2\pi)^{d-2}} \frac{l^2}{2} \int_0^\infty ds \mathcal{N}^2 e^{-s} (l\sqrt{s})^{d-2} \\ &= \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} l \end{aligned} \tag{4.77}$$

and

$$\langle r^2 \rangle = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} l^2. \tag{4.78}$$

The property of gamma function $\Gamma(z+1) = z\Gamma(z)$ can be applied to get:

$$\langle r^2 \rangle = \frac{d-1}{2} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} l^2 = \frac{d-1}{2} l^2. \tag{4.79}$$

For the other average we use the properties:

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), \qquad \Gamma(n) = (n-1)! \tag{4.80}$$

and obtain:

$$\langle r \rangle = \frac{\Gamma\left(\frac{d-1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} l = \frac{2^{2-d}\sqrt{\pi}\Gamma\left(d-1\right)}{\Gamma^2\left(\frac{d-1}{2}\right)} l = \frac{2^{2-d}\sqrt{\pi}\left(d-2\right)!}{\Gamma^2\left(\frac{d-1}{2}\right)} l. \tag{4.81}$$

So that:

$$\Delta r_S = \sqrt{\langle r^2 \rangle - \langle r \rangle^2} = \sqrt{\frac{d-1}{2} - \frac{2^{2(2-d)}\pi\left[(d-2)!\right]^2}{\Gamma^4\left(\frac{d-1}{2}\right)}} l \equiv A(d) l. \tag{4.82}$$

In total, (4.75) gives:

$$\Delta r = \sqrt{\frac{d-1}{2} - \frac{2^{2(2-d)}\pi\left[(d-2)!\right]^2}{\Gamma^4\left(\frac{d-1}{2}\right)}}\, l + \alpha\left[R_0 K(d)\right]. \tag{4.83}$$

By working in momentum space with the Fourier transform (4.47) and proceeding analogously to what we did in (4.82), we obtain:

$$\Delta p = A(d)\Delta = A(d)\frac{l_p m_p}{l}. \tag{4.84}$$

So we can write equation (4.83) in a more homogeneous form as a sole function of Δp, by making use of (4.73):

$$\Delta r = A(d)^2\frac{l_p m_p}{\Delta p} + \alpha K(d)\left(\frac{2}{d-3}\frac{m}{m_p}\right)^{\frac{1}{d-3}}. \tag{4.85}$$

We can further simplify this by taking $m = \Delta = \frac{\Delta p}{A(d)}$, so that we obtain:

$$\Delta r = A(d)^2\frac{l_p m_p}{\Delta p} + \alpha K(d)\left(\frac{2}{d-3}\frac{1}{A(d)}\frac{\Delta p}{m_p}\right)^{\frac{1}{d-3}}, \tag{4.86}$$

carrying the dependence on Δp as:

$$\frac{\Delta r}{l_p} = C_s\frac{1}{\Delta p} + C_H(\Delta p)^{\frac{1}{d-3}}, \tag{4.87}$$

where C_s, C_H are constants independent of Δp.
We can plot (4.86) as a function of $\frac{\Delta p}{m_p}$ and get:

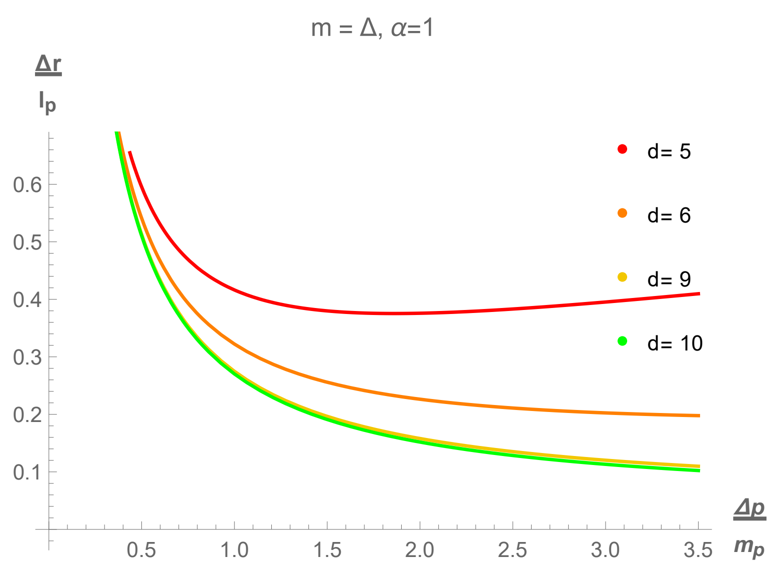


Figure 4.8: $\frac{\Delta r}{l_p}$ with $\alpha = 1$ and $m = \Delta$

This plot shows the total uncertainty in length as a function of the uncertainty in momentum for different dimensions. When the dimension is increased the uncertainty is reduced, analogously to what we already saw in figure 4.5 for the uncertainty on the horizon.

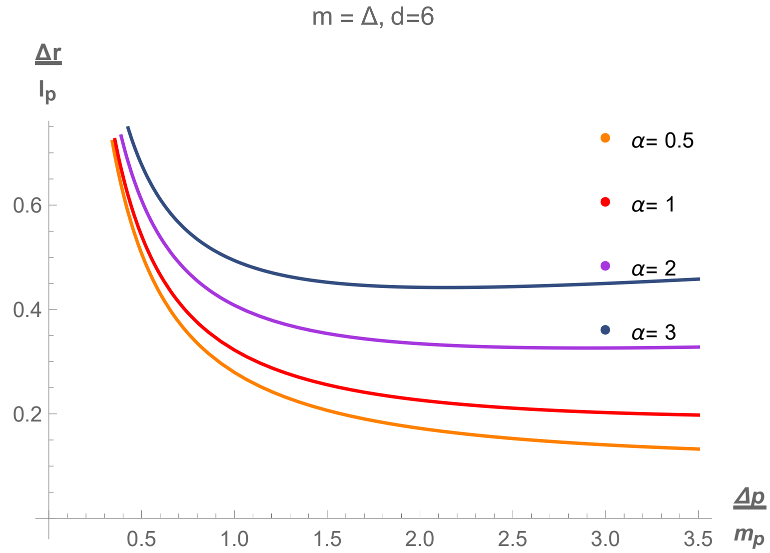


Figure 4.9: $\frac{\Delta r}{l_p}$ with $d = 6$

This time the parameter α is varied, keeping the dimensions fixed at six. We notice that lower values of the parameter give lower uncertainties. This implies that when the quantum fluctuations of the source are more important with respect to the fluctuations of the horizon, the total uncertainty is lower. Moreover, the curves tend to be almost constant if Δp is large enough.

It can be noticed in the last two figures that there is a minimum in the resolution length, which we shall proceed to calculate. We denote:

$y \equiv \frac{\Delta r}{l_p}, x \equiv \frac{\Delta p}{m_p}, C = \frac{2}{d-3}\frac{l_p^{d-3}}{m_p}$ and the square root in (4.83) as $K(d)$.

$$\frac{dy}{dx} = 0 \quad \rightarrow \quad \frac{A(d)^2}{x^2} = \frac{\alpha K(d)}{d-3}\left(\frac{2}{d-3}\frac{x}{A(d)}\right)^{\frac{4-d}{d-3}}\frac{2}{(d-3)A(d)}. \tag{4.88}$$

The solution is:

$$x = \frac{(d-3)A(d)^{\frac{2d-5}{d-2}}}{2^{\frac{1}{d-2}}(\alpha K(d))^{\frac{d-3}{d-2}}}. \tag{4.89}$$

Then we substitute this value to find the minimum of y:

$$\begin{aligned} y &= \frac{\left\{2A(d)[\alpha K(d)]^{d-3}\right\}^{\frac{1}{d-2}}}{d-3} + \left\{2A(d)\left[\alpha K(d)\right]^{d-3}\right\}^{\frac{1}{d-2}} \\ &= \left\{2A(d)[\alpha K(d)]^{d-3}\right\}^{\frac{1}{d-2}}\frac{d-2}{d-3}. \end{aligned} \tag{4.90}$$

Therefore, there is a minimum uncertainty for the length, given by:

$$L = l_p \frac{d-2}{d-3} \left\{ [2\sqrt{\frac{d-1}{2} - \frac{2^{2(2-d)}\pi\left[(d-2)!\right]^2}{\Gamma^4\left(\frac{d-1}{2}\right)}} \right\}^{\frac{1}{d-2}} \times \left[\left(\alpha \sqrt{\frac{E_{\frac{d-7}{2(d-3)}}(1)}{E_{\frac{d-5}{2(d-3)}}(1)} - \frac{E^2_{\frac{d-6}{2(d-3)}}(1)}{E^2_{\frac{d-5}{2(d-3)}}(1)}} \right)^{d-3} \right]^{\frac{1}{d-2}} . \tag{4.91}$$

We can thus plot the minimum resolvable length in units of Planck length $\frac{L}{l_p}$ as a function of the parameter α:

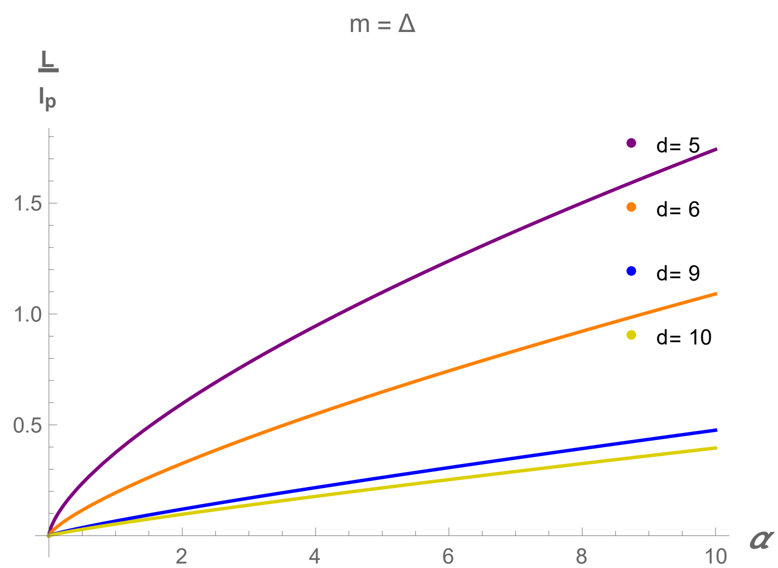


Figure 4.10: $\frac{L}{l_p}$ as a function of α

From this figure we deduce the minimum length resolution increases for increasing α, in an almost linear behaviour. Taking L at the Planck length clearly favours small values of the parameter α, while the opposite would have happened had we taken $m \simeq m_p$. Moreover, higher dimensions favour larger values for the parameter, for fixed L. This means that the larger the number of dimensions the more the horizon influences the uncertainty, rather than the source. Furthermore, the more the dimensionality increases, for lower values of α, the more classical the Black Hole becomes, especially at sub-Planckian lengths.

4.3 Ultra-spinning case

The ultra-spinning regime is characterized by $a \gg r$, which implies, as we saw in the other chapter:

$$\mu \equiv Cm = a^2 r_0^{d-5}, \qquad A_H = \Omega_{d-2} a^2 r_0^{d-4}. \tag{4.92}$$

4.3.1 Wave functions

Again, we consider a Gaußian source, as in (4.31), but this time the normalization is fixed differently, because in this approximation the area is different:

$$
\begin{aligned}
1 &= \frac{V_{(d-2)}}{(2\pi)^{d-2}} a^2 \int d\Omega_{d-2} \int_0^\infty dr \mathcal{N}^2 e^{-\frac{r^2}{l^2}} r^{d-4} \delta(y_1) \dots \delta(y_{d-2}) \\
&= \frac{V_{(d-2)}}{(2\pi)^{d-2}} a^2 \int_0^\infty dr \mathcal{N}^2 e^{-\frac{r^2}{l^2}} r^{d-4}.
\end{aligned}
\tag{4.93}
$$

We change the integration variable:

$$
\begin{cases}
s = \dfrac{r^2}{l^2} \quad \rightarrow \quad r = \sqrt{s} l; \\
ds = \dfrac{2r}{l^2} dr \quad \rightarrow \quad dr = \dfrac{l^2}{2r} ds
\end{cases}
\tag{4.94}
$$

and obtain:

$$
\begin{aligned}
1 &= \frac{V_{(d-2)}}{(2\pi)^{d-2}} a^2 \mathcal{N}^2 \int_0^\infty ds \frac{l^2}{2r} r^{d-4} e^{-s} \\
&= \frac{V_{(d-2)}}{(2\pi)^{d-2}} a^2 \mathcal{N}^2 \frac{l^2}{2} \int_0^\infty ds e^{-s} (\sqrt{s} l)^{d-5} \\
&= \frac{V_{(d-2)}}{(2\pi)^{d-2}} a^2 \mathcal{N}^2 \frac{l^{d-3}}{2} \Gamma\left(\frac{d-3}{2}\right).
\end{aligned}
\tag{4.95}
$$

Then the normalisation constant is fixed to:

$$
\mathcal{N}^2 = \frac{(2\pi)^{d-2}}{V_{(d-2)}} \frac{2}{\Gamma\left(\frac{d-3}{2}\right) a^2 l^{d-3}}
\tag{4.96}
$$

and the function for the source is:

$$
\psi_S = \left[\frac{(2\pi)^{d-2}}{V_{(d-2)}} \frac{2}{\Gamma\left(\frac{d-3}{2}\right) a^2 l^{d-3}} \right]^{\frac{1}{2}} e^{-\frac{r^2}{2l^2}} \delta(y_1) \dots \delta(y_{d-2}).
\tag{4.97}
$$

Proceeding analogously to the static case, we move on to momentum space by Fourier transforming:

$$
\begin{aligned}
\psi_S(p, p_j) &= \mathcal{N} \int dr dy_1 \dots dy_{d-2} \delta(y_1) \dots \delta(y_{d-2}) e^{-\frac{r^2}{2l^2}} e^{-\frac{i}{\hbar} pr} e^{-\frac{i}{\hbar} p^j y_j} \\
&= \mathcal{N} \int_0^\infty dr e^{-\left(\frac{r^2}{2l^2} + \frac{i}{\hbar} pr\right)}.
\end{aligned}
\tag{4.98}
$$

We perform exactly the same passages as the previous section and obtain:

$$
\psi_S(p) = \mathcal{N} e^{-\frac{p^2 l^2}{2\hbar^2}} l \int dk e^{-\frac{k^2}{2}} = \mathcal{N} l \sqrt{\frac{\pi}{2}} e^{-\frac{p^2}{2\Delta^2}}.
\tag{4.99}
$$

So that now we are able to construct the horizon wave function, assuming again the flat-space mass-shell relation $E^2 = p^2 + m^2$.

$$
\psi_H(r_0) = \mathcal{N}_0 \theta(r_0 - R_0) e^{-\frac{1}{2(C\Delta)^2} a^4 \left(r^{2(d-5)} - R^{2(d-5)}\right)}.
\tag{4.100}
$$

But in which this time we used $Cm = \mu = a^2 R_0^{d-5}$ that was mentioned in the introduction. The next step involves fixing the normalisation constant $\mathcal{N}_0$ and we do it by, again, imposing the condition $P_H = 1$ for $r_0 = \infty$.

$$\begin{aligned} 1 &= \Omega_{d-2} a^2 \int_{R_0}^{\infty} dr_0 \mathcal{N}_0^2 e^{-a^4 \frac{r_0^{2(d-5)}}{(C\Delta)^2}} e^{a^4 \frac{R_0^{2(d-5)}}{(C\Delta)^2}} r_0^{d-4} \\ &= \mathcal{N}_0^2 \Omega_{d-2} a^2 e^{\frac{m^2}{\Delta^2}} \int_{R_0}^{\infty} dr_0 e^{-a^4 \frac{r_0^{2(d-5)}}{(C\Delta)^2}} r_0^{d-4} \end{aligned} \tag{4.101}$$

and we perform a change of variable:

$$\begin{cases} y = \dfrac{a^4 r_0^{2(d-5)}}{(C\Delta)^2} \qquad \rightarrow \qquad r_0 = \left(\dfrac{C\Delta}{a^2} y^{\frac{1}{2}} \right)^{\frac{1}{d-5}} ; \\ dy = 2(d-5) \dfrac{a^4 r_0^{2(d-5)-1}}{(C\Delta)^2} dr_0. \end{cases} \tag{4.102}$$

This gets:

$$\begin{aligned} \frac{\mathcal{N}_0^{-2} e^{-\frac{m^2}{\Delta^2}}}{\Omega_{d-2} a^2} &= \int_{\frac{m^2}{\Delta^2}}^{\infty} dy \frac{(C\Delta)^2}{2(d-5) a^4 r_0^{2(d-5)-1}} r_0^{d-4} e^{-y} \\ &= \left(\frac{C\Delta}{a^2} \right)^2 \frac{1}{2(d-5)} \int_{\frac{m^2}{\Delta^2}}^{\infty} dy \left(\frac{C\Delta}{a^2} \right)^{\frac{-d+7}{d-5}} y^{\frac{-d+7}{2(d-5)}} e^{-y} \\ &= \left(\frac{C\Delta}{a^2} \right)^{\frac{d-3}{d-5}} \frac{1}{2(d-5)} \Gamma \left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2} \right]. \end{aligned} \tag{4.103}$$

Then, finally, the constant is fixed to:

$$\mathcal{N}_0^2 e^{\frac{m^2}{\Delta^2}} = \left(\frac{C\Delta}{a^2} \right)^{-\frac{d-3}{d-5}} \frac{2(d-5)}{a^2 \Omega_{d-2}} \frac{1}{\Gamma \left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2} \right]} \tag{4.104}$$

and the horizon wave function is:

$$\psi_H(r_0) = \left\{ \left(\frac{C\Delta}{a^2} \right)^{-\frac{d-3}{d-5}} \frac{2(d-5)}{a^2 \Omega_{d-2}} \frac{1}{\Gamma \left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2} \right]} \right\}^{\frac{1}{2}} \theta(r_0 - R_0) e^{-\frac{a^4 r^{2(d-5)}}{2(C\Delta)^2}}. \tag{4.105}$$

4.3.2 Probabilities

First we calculate P_S, defined as in (4.33):

$$\begin{aligned} P_S &= \left[\frac{(2\pi)^{d-2}}{V_{(d-2)}} \frac{2}{\Gamma\left(\frac{d-3}{2}\right) a^2 l^{d-3}} \right] a^2 \int d\Omega_{d-2} \int_0^{r_0} dr \delta(y_1) \ldots \delta(y_{d-2}) e^{-\frac{r^2}{l^2}} r^{d-4} \\ &= \left[\frac{2}{\Gamma\left(\frac{d-3}{2}\right) l^{d-3}} \right] \int d\Omega_{d-2} \int_0^{r_0} dr e^{-\frac{r^2}{l^2}} r^{d-4} \\ &= \frac{\gamma\left(\frac{d-3}{2}, \frac{r_0^2}{l^2}\right)}{\Gamma\left(\frac{d-3}{2}\right)} = 1 - \frac{\Gamma\left(\frac{d-3}{2}, \frac{r_0^2}{l^2}\right)}{\Gamma\left(\frac{d-3}{2}\right)}, \end{aligned} \tag{4.106}$$

where we used again the representation of the lower incomplete gamma function: $\gamma(s,x) = \Gamma(s) - \Gamma(s,x)$.
Then we calculate P_{BH}, that was defined in (4.59):

$$\begin{aligned}
P_{BH} = \int_0^\infty dr_0 \left[1 - \frac{\Gamma\left(\frac{d-3}{2}, \frac{r_0^2}{l^2}\right)}{\Gamma\left(\frac{d-3}{2}\right)}\right] a^2 \int d\Omega_{d-2} r_0^{d-4} |\psi_H|^2 \\
= 1 - \frac{a^2\Omega_{d-2}}{\Gamma\left(\frac{d-3}{2}\right)} \left\{ \left(\frac{C\Delta}{a^2}\right)^{-\frac{d-3}{d-5}} \frac{2(d-5)}{a^2\Omega_{d-2}} \frac{1}{\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \right\} \\
\times \int_{R_0}^\infty dr_0 \Gamma\left(\frac{d-3}{2}, \frac{r_0^2}{l^2}\right) r_0^{d-4} e^{-a^4 \frac{r_0^{2(d-5)}}{(C\Delta)^2}} \\
= 1 - \frac{2(d-5)}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \left(\frac{C\Delta}{a^2}\right)^{-\frac{d-3}{d-5}} \int_{R_0}^\infty dr_0 \Gamma\left(\frac{d-3}{2}, \frac{r_0^2}{l^2}\right) r_0^{d-4} e^{-a^4 \frac{r_0^{2(d-5)}}{(C\Delta)^2}}.
\end{aligned} \tag{4.107}$$

Now we change the variable to y, exactly as in (4.102):

$$\begin{cases} y = \dfrac{a^4 r_0^{2(d-5)}}{(C\Delta)^2} \qquad \rightarrow \qquad r_0 = \left(\dfrac{C\Delta}{a^2} y^{\frac{1}{2}}\right)^{\frac{1}{d-5}}; \\ dy = 2(d-5)\dfrac{a^4 r_0^{2(d-5)-1}}{(C\Delta)^2} dr_0 \end{cases} \tag{4.108}$$

and perform the substitution:

$$\begin{aligned}
P_{BH} = 1 - \frac{2(d-5)}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \left(\frac{C\Delta}{a^2}\right)^{-\frac{d-3}{d-5}} \\
\times \int_{\frac{m^2}{\Delta^2}}^\infty dy \frac{1}{2(d-5)} \left(\frac{C\Delta}{a^2}\right)^2 e^{-y} \Gamma\left(\frac{d-3}{2}, \frac{r_0^2}{l^2}\right) r_0^{d-4+11-2d} \\
= 1 - \frac{1}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \left(\frac{C\Delta}{a^2}\right)^{2-\frac{d-3}{d-5}} \\
\times \int_{\frac{m^2}{\Delta^2}}^\infty dy e^{-y} \Gamma\left[\frac{d-3}{2}, \frac{1}{l^2}\left(y^{\frac{1}{2}}\frac{C\Delta}{a^2}\right)^{\frac{2}{d-5}}\right] \left(y^{\frac{1}{2}}\frac{C\Delta}{a^2}\right)^{\frac{7-d}{d-5}} \\
= 1 - \frac{1}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \int_{\frac{m^2}{\Delta^2}}^\infty dy e^{-y} \Gamma\left[\frac{d-3}{2}, \frac{1}{l^2}\left(y^{\frac{1}{2}}\frac{C\Delta}{a^2}\right)^{\frac{2}{d-5}}\right] y^{\frac{d-3}{2(d-5)}-1}.
\end{aligned} \tag{4.109}$$

At this point, we explicit the quantity $\frac{1}{l^2}\left(\frac{C\Delta}{a^2}\right)^{\frac{2}{d-5}}$ using (4.65):

$$\frac{1}{l^2}\left(\frac{C\Delta}{a^2}\right)^{\frac{2}{d-5}} = \left(\frac{2}{d-3}\frac{l_p^{d-3}}{m_p}\frac{m_p l_p}{l}\frac{1}{l^{d-5}a^2}\right)^{\frac{2}{d-5}} = \left(\frac{2}{d-3}\frac{l_p^{d-4}}{l^{d-4}}\frac{l_p^2}{a^2}\right)^{\frac{2}{d-5}}. \tag{4.110}$$

Now, it is necessary to know the limit for which the ratio $\frac{a}{l_p}$ generates instabilities. We make use of (3.106), from which we calculate, for example, that $\frac{a^{d-3}}{\mu}$ should be smaller than 0.88 for $d = 6$, 0.97 for $d = 7$ and 1.02 for $d = 8$ and so on. We want to use this quantity to express the ratio $\frac{a}{l_p}$ and we use (4.65) to do so:

$$\frac{a^{d-3}}{\mu} = \frac{a^{d-3}}{Cm} = \left(\frac{a}{l_p}\right)^{d-3}\left(\frac{d-3}{2}\frac{m_p}{m}\right). \tag{4.111}$$

From this:

$$\frac{a}{l_p} = \left(\frac{a^{d-3}}{\mu}\frac{2}{d-3}\frac{m}{m_p}\right)^{\frac{1}{d-3}}. \tag{4.112}$$

Thanks to this expression, we can rewrite the probability:

$$P_{BH} = 1 - \frac{1}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left[\frac{d-3}{2(d-5)}, \left(\frac{m}{m_p}\frac{l}{l_p}\right)^2\right]} \times \int_{\left(\frac{m}{m_p}\frac{l}{l_p}\right)^2}^{\infty} dy e^{-y} y^{\frac{d-3}{2(d-5)}-1} \Gamma\left\{\frac{d-3}{2}, \left(\frac{2}{d-3}\right)^{\frac{2}{d-3}}\left[y\left(\frac{l_p}{l}\right)^{2(d-4)}\left(\frac{\mu}{a^{d-3}}\frac{m_p}{m}\right)^{\frac{4}{d-3}}\right]^{\frac{1}{d-5}}\right\} \tag{4.113}$$

This expression can be simplified taking $m = \Delta$, which is implied if the width of the source is exactly equals to the Compton length $l = \lambda$. In that case we have:

$$P_{BH} = 1 - \frac{1}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left[\frac{d-3}{2(d-5)}, \left(\frac{m}{m_p}\frac{l}{l_p}\right)^2\right]} \times \int_{\left(\frac{m}{m_p}\frac{l}{l_p}\right)^2}^{\infty} dy e^{-y} y^{\frac{d-3}{2(d-5)}-1} \Gamma\left\{\frac{d-3}{2}, \left[\left(\frac{2}{d-3}\right)^2\left(\frac{m}{m_p}\right)^{d-2}\left(\frac{\mu}{a^{d-3}}\right)^{\frac{4}{d-5}}\right]^{\frac{1}{d-3}} y^{\frac{1}{d-5}}\right\} \tag{4.114}$$

The probability (4.114), taken at the Planck scale, is shown in numerical plots below, as a function of the ratio $\frac{l}{l_p}$. It depends on three parameters: mass ratio, angular momentum and dimensionality. These are varied separately.

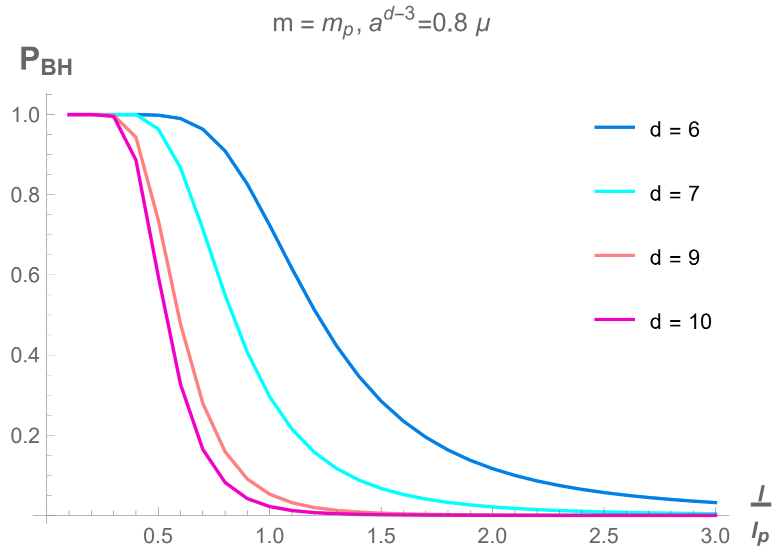


Figure 4.11: P_{BH} with $m = m_p$ and $\frac{a^{d-3}}{\mu} = 0.8$

Here we vary the dimensionality and we notice that increasing the number of dimensions drastically decreases the probability of forming Black Holes. While the maximum probability for $d = 6$ is found at the Planck length, for $d = 7, 9, 10$ it is around $l = \frac{1}{2}l_p$, while at l_p the probability is very low, especially in $d = 9, 10$. Similarly to the static case, we could infer that the diminution of the horizon radius afflicts the probability, so we plot numerically the case when $R_0 \to 0$ in $d = 6$ and $m = m_p$:

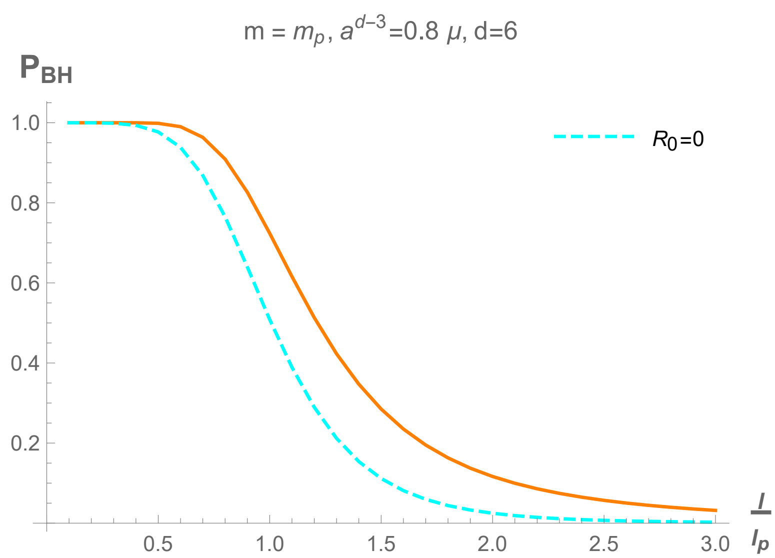


Figure 4.12: P_{BH} with $m = m_p$ and $d = 6$, compared to the case when $R_0 \to 0$

We notice the probability is not lowered so drastically and we conclude the effect of

dimensionality is the principal factor of the decrease in the probability.
Next, we vary the angular momentum:

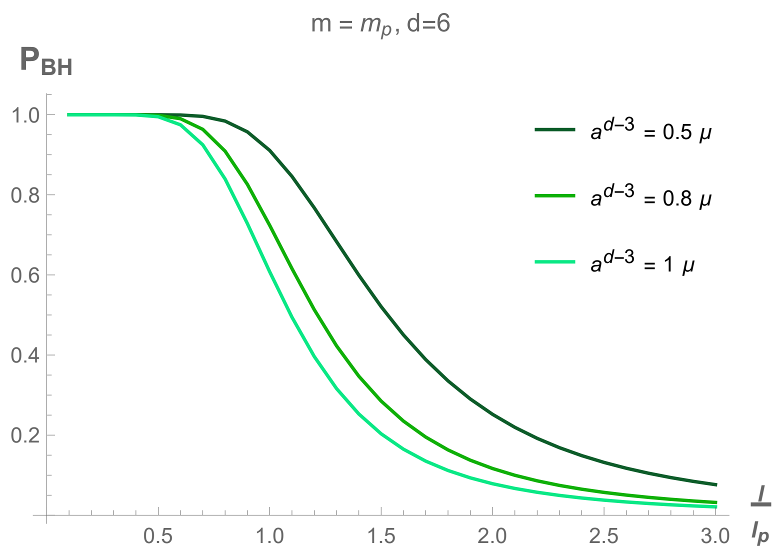


Figure 4.13: P_{BH} with $m = m_p$ and $d = 6$

From this plot, it is apparent that higher momenta give smaller probabilities, with the possibly unstable value $a = l_p$ giving the smallest values, among those considered here. While $\frac{a^{d-3}}{\mu}$ results in the probability reaching its maximum when the width l is around the Planck scale, higher momenta reduce it to around $l \simeq \frac{1}{2} l_p$.
Next, when varying the mass we obtain:

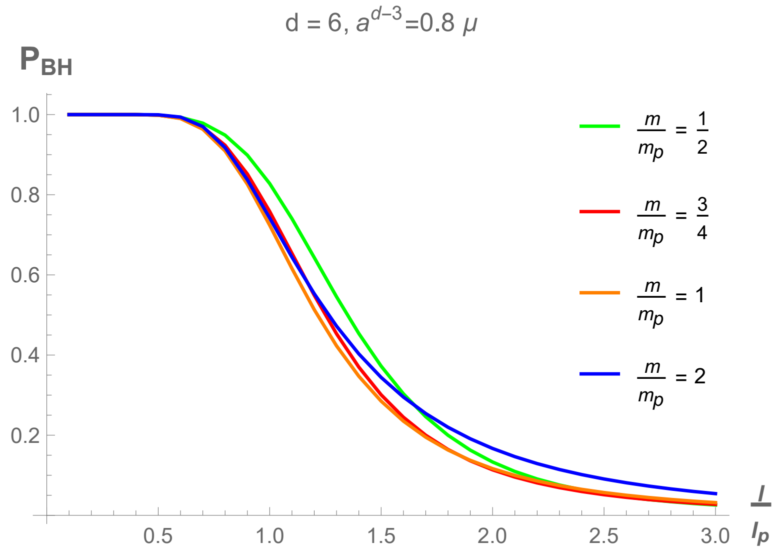


Figure 4.14: P_{BH} with $m = \Delta$ and $\frac{a^{d-3}}{\mu} = 0.4$

We can see that, contrary to the previous cases analysed, when it is the mass ratio that is varied, the change of the probability in $d = 6$ is negligible, giving an overall higher probability for $m = 2m_p$, except in the range around the Planck scale where $m = \frac{1}{2}m_p$ gives slightly higher values. The maximum probability is reached when $l \simeq 0.7l_p$.

Next, we can simplify (4.114) by considering the extremal case $m = \Delta$, obtaining numerically the following plots. The parameters on which we can act now are only the dimensionality and angular momentum, while plotting P_{BH} as a function of $\frac{m}{m_p}$.

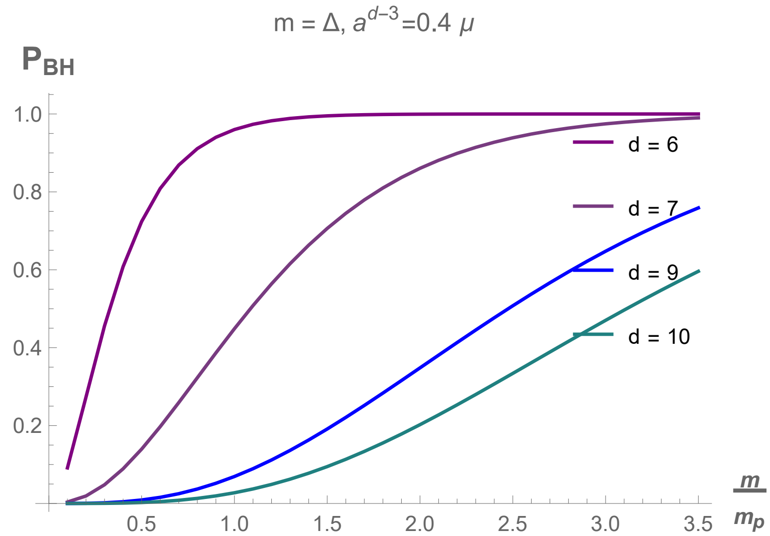


Figure 4.15: P_{BH} with $m = \Delta$ and $\frac{a^{d-3}}{\mu} = 0.4$

This first plot represents the probability P_{BH} from equation (4.114), obtained considering $m = \Delta$ and for fixed momentum over mass $\frac{a^{d-3}}{\mu} = 0.4$, well below the instability threshold (the lowest one is for $d = 6$ and it corresponds to $\frac{a^{d-3}}{\mu} = 0.88$). Using (4.114) instead of (4.113) makes it so that the probability only depends on dimensionality and the mass ratio $\frac{m}{m_p}$. Then, in this plot we expressed P_{BH} as a function of $\frac{m}{m_p}$ at different values of d. We see the maximum probability is obtained for smaller masses and with less dimensions, reaching $P_{BH} = 1$ slightly before the Planck scale in the $d = 6$ case and at around three times the Planck scale for $d = 7$. The higher the dimensionality, the further the highest probability lies, in terms of energy. If $d = 7$ at the Planck scale the probability of having a Black Hole is around 50% and it gets significantly lower for higher dimensions.
Now, we plot the probability as a function of the ratio $\frac{a}{l_p}$, with fixed dimensionality $d = 6$ and for different fixed values of $\frac{m}{m_p}$.

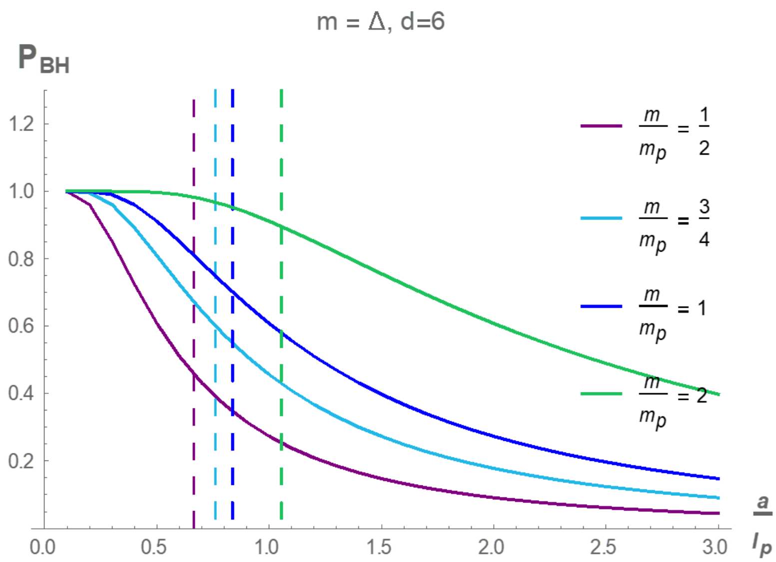


Figure 4.16: P_{BH} with $m = \Delta$ and $d = 6$

The vertical dashed line roughly represents the maximum value for the angular momentum allowed by instabilities, calculated for each curve. We notice that for smaller masses the instability limit is reached when the probability is already small, while for masses twice the Planck mass the system becomes unstable not much further than the maximum. The general behaviour of the curves indicates that higher angular momenta give smaller values for the probability, signalling the ultra-spinning Black Hole is an unstable configuration and will not be a favoured state for the system. Moreover, higher spins mean smaller radius (for fixed mass), lowering the cross section and probability.
We also drew a plot of the case when the dimensionality is varied and the angular momentum is set at the maximum value corresponding to each curve.

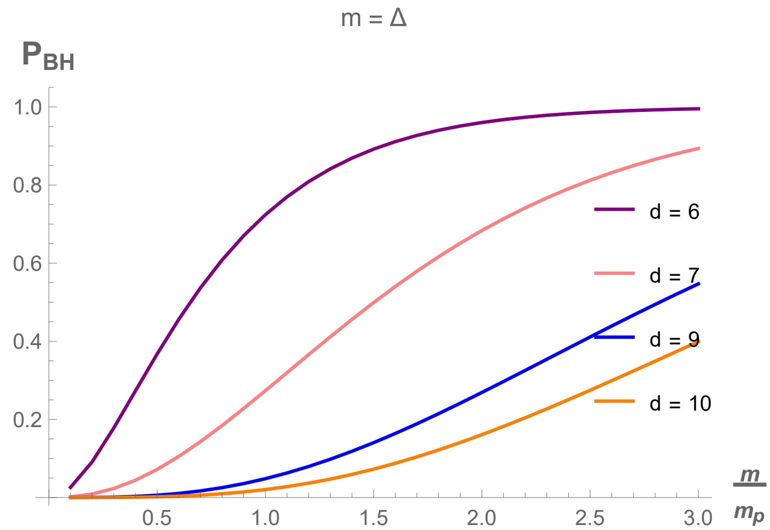


Figure 4.17: P_{BH} with $m = \Delta$ and $\frac{a^{d-3}}{\mu}$ max for every value of d

Here we see the case when the angular momentum is maximum (before reaching possible instabilities) and thus the probability is the lowest in each dimensionality. The behaviour is similar to the case analysed in figure 4.15, with the curves reaching the maximum probability a little later in energy.

4.3.3 Generalized Uncertainty Principle

We calculate the uncertainty in the size of the horizon:

$$\Delta r_0 = \sqrt{\langle r_0^2 \rangle - \langle r_0 \rangle^2} \tag{4.115}$$

(as stated in equation (4.67)). We use the horizon wave function (4.105), with the same change of variables as in equation (4.102).

$$\begin{aligned}
\langle r_0^b \rangle &= \frac{\Omega_{d-2} a^2 \mathcal{N}^2}{2(d-5)} \left(\frac{C\Delta}{a^2}\right)^2 \int_{\frac{m^2}{\Delta^2}}^{\infty} ds e^{-s} r^{d-4+b-2(d-5)+1} \\
&= \frac{\Omega_{d-2} a^2}{2(d-5)} \left(\frac{C\Delta}{a^2}\right)^2 \left\{ \left(\frac{C\Delta}{a^2}\right)^{-\frac{d-3}{d-5}} \frac{2(d-5)}{a^2 \Omega_{d-2}} \frac{1}{\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \right\} \\
&\quad \times \int_{\frac{m^2}{\Delta^2}}^{\infty} ds e^{-s} \left(\sqrt{s}\frac{C\Delta}{a^2}\right)^{\frac{-d+7+b}{d-5}} \\
&= \frac{\Gamma\left[\frac{d-3+b}{2(d-5)}, \frac{m^2}{\Delta^2}\right]}{\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \left(\frac{C\Delta}{a^2}\right)^{\frac{b}{d-5}} = \frac{\Gamma\left[\frac{d-3+b}{2(d-5)}, \frac{m^2}{\Delta^2}\right]}{\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \left(\frac{\Delta}{m}\right)^{\frac{b}{d-5}} R_0^b,
\end{aligned} \tag{4.116}$$

where in the last passage we used the fact that $Cm = a^2 R_0^{d-5}$. Therefore,

$$\langle r_0^2 \rangle = \frac{\Gamma\left[\frac{d-1}{2(d-5)}, \frac{m^2}{\Delta^2}\right]}{\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \left(\frac{\Delta}{m}\right)^{\frac{2}{d-5}} R_0^2 \tag{4.117}$$

and

$$\langle r_0 \rangle = \frac{\Gamma\left[\frac{d-2}{2(d-5)}, \frac{m^2}{\Delta^2}\right]}{\Gamma\left[\frac{d-3}{2(d-5)}, \frac{m^2}{\Delta^2}\right]} \left(\frac{\Delta}{m}\right)^{\frac{1}{d-5}} R_0. \tag{4.118}$$

We can express these quantities using the exponential integral:

$$\Gamma(s,x) = x^s E_{1-s}(x), \qquad E_n(x) = \int_1^\infty dt \frac{e^{-xt}}{t^n}. \tag{4.119}$$

Then, (4.67) gives:

$$\begin{aligned} \Delta r_0 &= R_0 \sqrt{\frac{\left(\frac{m^2}{\Delta^2}\right)^{\frac{d-1}{2(d-5)}} E_{\frac{d-9}{2(d-5)}}\left(\frac{m^2}{\Delta^2}\right)}{\left(\frac{m^2}{\Delta^2}\right)^{\frac{d-3}{2(d-5)}} E_{\frac{d-7}{2(d-5)}}\left(\frac{m^2}{\Delta^2}\right)} - \left[\frac{\left(\frac{m^2}{\Delta^2}\right)^{\frac{d-2}{2(d-5)}} E_{\frac{d-8}{2(d-5)}}\left(\frac{m^2}{\Delta^2}\right)}{\left(\frac{m^2}{\Delta^2}\right)^{\frac{d-3}{2(d-5)}} E_{\frac{d-7}{2(d-5)}}\left(\frac{m^2}{\Delta^2}\right)}\right]^2} \left(\frac{\Delta}{m}\right)^{\frac{1}{d-5}} \\ &= R_0 \sqrt{\frac{E_{\frac{d-9}{2(d-5)}}}{E_{\frac{d-7}{2(d-5)}}} - \frac{E^2_{\frac{d-8}{2(d-5)}}}{E^2_{\frac{d-7}{2(d-5)}}}}. \end{aligned} \tag{4.120}$$

The horizon radius R_0 can be expressed as a function of the angular momentum parameter $\frac{a^{d-3}}{\mu}$ in the following way. First, we use (3.106) and obtain μ:

$$\mu = a^{d-3} \frac{2}{d-3} \frac{m}{m_p} \left(\frac{l_p}{a}\right)^{d-3} \tag{4.121}$$

and we substitute it into the expression for the horizon radius:

$$R_0 = \left(\frac{\mu}{a^2}\right)^{\frac{1}{d-5}} = a\left[\frac{2}{d-3}\frac{m}{m_p}\left(\frac{l_p}{a}\right)^{d-3}\right]^{\frac{1}{d-5}}. \tag{4.122}$$

Next, we use (4.112) to express (4.120) as a function of $\frac{a^{d-3}}{\mu}$ and indicate the square root as $K(d)$:

$$\frac{\Delta r_0}{l_p} = K(d)\left[\frac{2}{d-3}\frac{m}{m_p}\frac{a^{d-3}}{\mu}\right]^{\frac{1}{d-3}}\left(\frac{a^{d-3}}{\mu}\right)^{-\frac{1}{d-5}} = K(d)\left[\frac{2}{d-3}\frac{m}{m_p}\left(\frac{a^{d-3}}{\mu}\right)^{-\frac{2}{d-5}}\right]^{\frac{1}{d-3}}. \tag{4.123}$$

We can plot this quantity as a function of $\frac{l}{l_p}$, where the dependence on this ratio is contained in the exponential integral (that depends on $\frac{m}{\Delta}$). Now we plot the same equation (4.120) as a function of $\frac{l}{l_p}$, for a general $l > \lambda$:

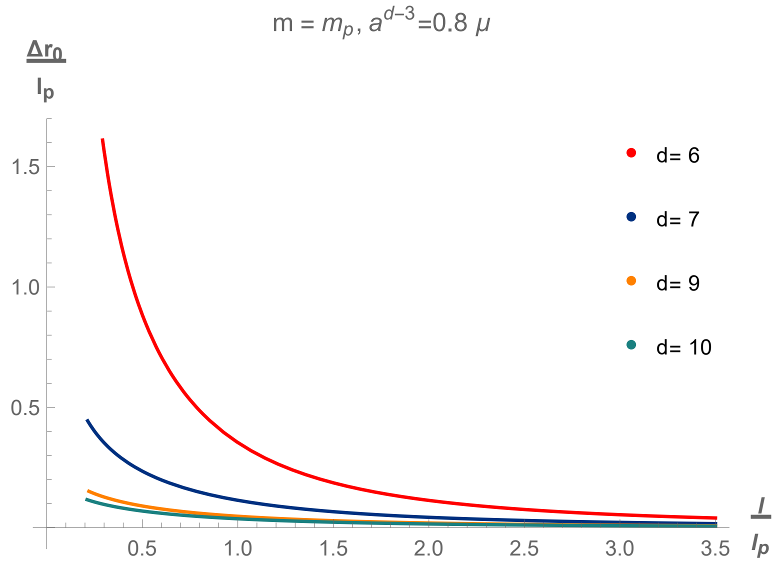


Figure 4.18: $\frac{\Delta r_0}{l_p}$ with $m = m_p$ and $\frac{a^{d-3}}{\mu} = 0.8$

The plot shows the uncertainty on the horizon radius as a function of the source's width, both in units of the Planck length. As the width l is increased, the uncertainty approaches 0, thus reproducing a classical behaviour:

$$\langle r_0 \rangle \simeq R_0 \qquad \text{and} \qquad \Delta r_0 \simeq 0. \tag{4.124}$$

Moreover, higher dimensionality seems to hint to negligible uncertainties and thus to a classical behaviour. This could be due to the fact that also the probability of forming a Black Hole is very small, making it more likely to be a classical particle.

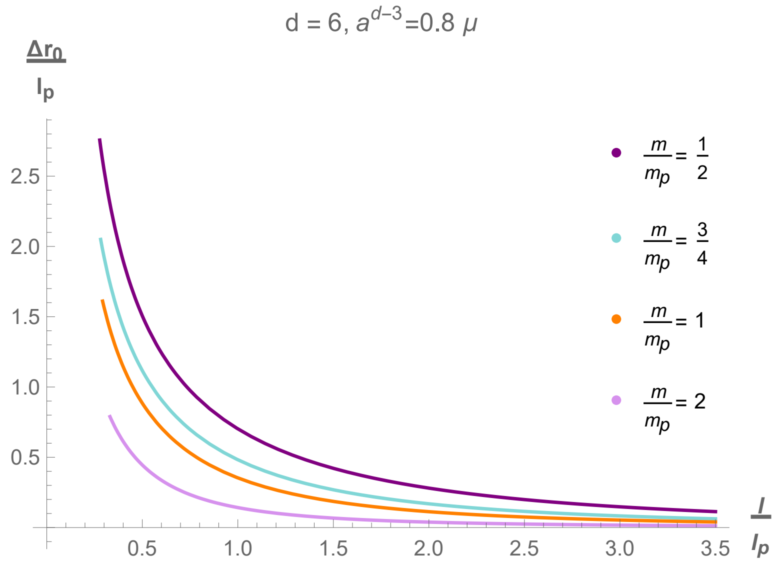


Figure 4.19: $\frac{\Delta r_0}{l_p}$ with $d = 6$ and $\frac{a^{d-3}}{\mu} = 0.8$

In this plot, $\frac{m}{m_p}$ takes different values while d is kept fixed. We can see that the higher the mass is, the more classical the behaviour of the Black Hole. This could be related to self-completeness, as, supposedly, at trans-Planckian scales a classical Black Hole is produced. We can now plot the uncertainty as a function of $\frac{m}{m_p}$ in the extremal case $m = \Delta$:

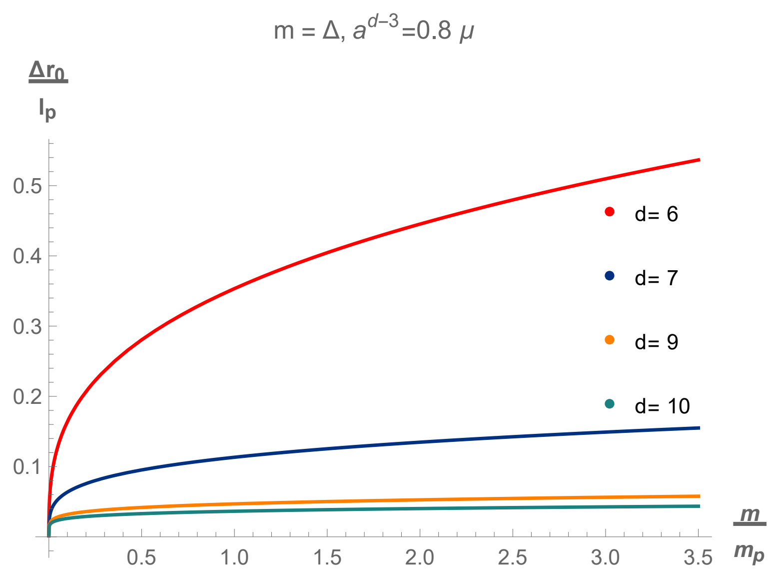


Figure 4.20: $\frac{\Delta r_0}{l_p}$ with $m = \Delta$ and $\frac{a^{d-3}}{\mu} = 0.8$

The behaviour is very similar to the static case, indeed, we notice the uncertainty gets more

and more constant as the dimensionality increases and as m is increased, not reaching 0 unless $m = 0$. Then, classicality is not retrieved for $m < m_p$ as we would expect. Moreover, if $m > m_p$ the curves are either constant or diverge, not resulting in the semi-classical behaviour we would have expected. We can take the same conclusion as the static case, that is, if $m = \Delta$, then $K(d)$ is a constant and therefore $\Delta r_0 \propto R_0$. Having an uncertainty on the horizon proportional to it is not what we want for a classical Black Hole and we should be studying another model to describe it more effectively. The corpuscular model could help in this direction, as it would also allow to have an axisymmetric function for the source instead of a symmetric Gaußian like the one we used.
In the following plot we express the uncertainty on the horizon as a function of $\frac{a}{l_p}$, using a dashed line to indicate roughly the maximum value of this ratio before the configuration becomes unstable. The number of dimensions is kept fixed at $d = 6$ and $\frac{m}{m_p}$ takes different fixed values.

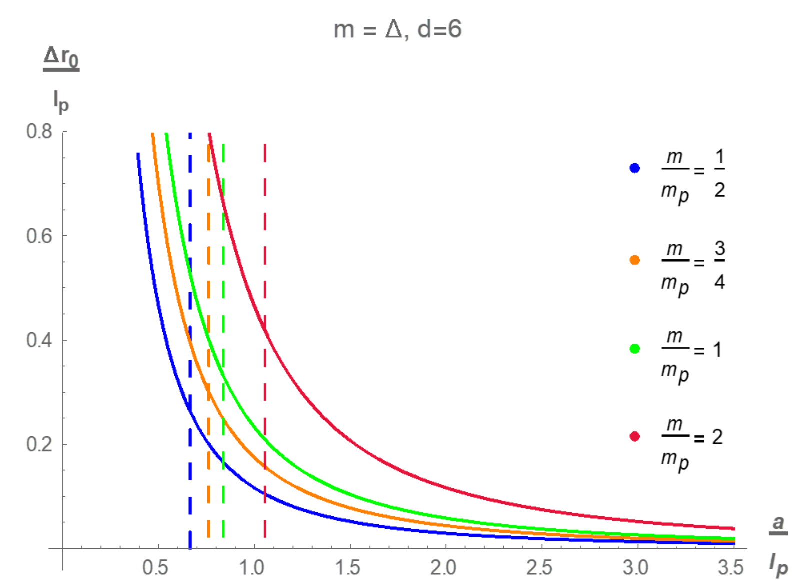


Figure 4.21: $\frac{\Delta r_0}{l_p}$ with $m = \Delta$ and $d = 6$

First, we can see that higher momenta give lower uncertainties and instabilities are reached at higher values of the uncertainty for higher masses. The smallness of the uncertainty for high momenta possibly refers to the lack of Black Holes in favour of normal particles, as its behavior corresponds to the probability of forming a Black Hole in figure 4.13.

The total GUP is obtained by:

$$\Delta r = \Delta r_s + \alpha \Delta r_0 \tag{4.125}$$

as in (4.75). The term Δr_s can be found analogously to (4.67), using ψ_S from (4.97) instead of ψ_H:

$$\langle r^a \rangle = \Omega_{d-2} a^2 \int_0^\infty dr r^{d-4} \psi_S^* r^a \psi_S. \tag{4.126}$$

By using the same substitution as in (4.39):

$$\begin{aligned}\langle r^a \rangle &= \frac{V_{(d-2)}}{(2\pi)^{d-2}} a^2 \frac{l^2}{2} \int_0^\infty ds \mathcal{N}^2 e^{-s} (l\sqrt{s})^{d-5+a} \\ &= \frac{V_{(d-2)}}{(2\pi)^{d-2}} a^2 \left[\frac{2}{a^2 l^{d-3}} \frac{(2\pi)^{d-2}}{V_{(d-2)}} \frac{1}{\Gamma\left(\frac{d-3}{2}\right)} \right] \frac{l^{d-3+a}}{2} \Gamma\left(\frac{d-3+a}{2}\right) \\ &= \frac{\Gamma\left(\frac{d-3+a}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right)} l^a. \end{aligned} \tag{4.127}$$

So that

$$\langle r \rangle = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right)} l, \tag{4.128}$$

by using the properties of gamma function:

$$\begin{aligned}\Gamma\left(\frac{d-2}{2}\right) = \Gamma\left(\frac{d-3}{2} + \frac{1}{2}\right) &= \frac{1}{\Gamma\left(\frac{d-3}{2}\right)} 2^{4-d} \sqrt{\pi} \Gamma(d-3) \\ &= \frac{1}{\Gamma\left(\frac{d-3}{2}\right)} 2^{4-d} \sqrt{\pi} (d-4)!, \end{aligned} \tag{4.129}$$

becomes:

$$\langle r \rangle = \frac{2^{4-d} \sqrt{\pi} (d-4)!}{\Gamma^2\left(\frac{d-3}{2}\right)} l. \tag{4.130}$$

On the other hand we have:

$$\langle r^2 \rangle = \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right)} l^2. \tag{4.131}$$

We now use the property $\Gamma(z+1) = z\Gamma(z)$:

$$\langle r^2 \rangle = \frac{\frac{d-3}{2} \Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right)} l^2 = \frac{d-3}{2} l^2. \tag{4.132}$$

Therefore, the end result is:

$$\Delta r_S = \sqrt{\langle r^2 \rangle - \langle r \rangle^2} = \sqrt{\frac{d-3}{2} - \frac{2^{2(4-d)} \pi \left[(d-4)!\right]^2}{\Gamma^4\left(\frac{d-3}{2}\right)}} l \equiv B(d) l. \tag{4.133}$$

In total, (4.75) gives:

$$\Delta r = \sqrt{\frac{d-3}{2} - \frac{2^{2(4-d)} \pi \left[(d-4)!\right]^2}{\Gamma^4\left(\frac{d-3}{2}\right)}} l + \alpha \left[R_0 \sqrt{\frac{E_{\frac{d-9}{2(d-5)}}}{E_{\frac{d-7}{2(d-5)}}} - \frac{E^2_{\frac{d-8}{2(d-5)}}}{E^2_{\frac{d-7}{2(d-5)}}}} \right]. \tag{4.134}$$

Repeating the steps we used in calculating the uncertainty on the horizon gives:

$$\frac{\Delta r}{l_p} = B(d)\frac{l}{l_p} + \alpha K(d)\left[\frac{2}{d-3}\frac{m}{m_p}\left(\frac{a^{d-3}}{\mu}\right)^{-\frac{2}{d-5}}\right]^{\frac{1}{d-3}}. \tag{4.135}$$

Moreover, in momentum space we obtain, analogously to (4.133):

$$\Delta p = B(d)\Delta = B(d)\frac{l_p m_p}{l}. \tag{4.136}$$

Then, (4.135) can be written in a more homogeneous form as a sole function of Δp, by substituting l from (4.136) and taking $m = \Delta = \frac{\Delta p}{B(d)}$:

$$\frac{\Delta r}{l_p} = B(d)^2\frac{m_p}{\Delta p} + \alpha K(d)\left[\frac{2}{d-3}\frac{\Delta p}{m_p}\frac{1}{B(d)}\left(\frac{a^{d-3}}{\mu}\right)^{-\frac{2}{d-5}}\right]^{\frac{1}{d-3}}, \tag{4.137}$$

which carries the dependence on Δp in the following way:

$$\frac{\Delta r}{l_p} = C_s\frac{1}{\Delta p} + C_H(\Delta p)^{\frac{1}{d-5}}, \tag{4.138}$$

where C_s, C_H are constants independent of Δp.
We can plot this uncertainty as a function of $\frac{\Delta p}{m_p}$:

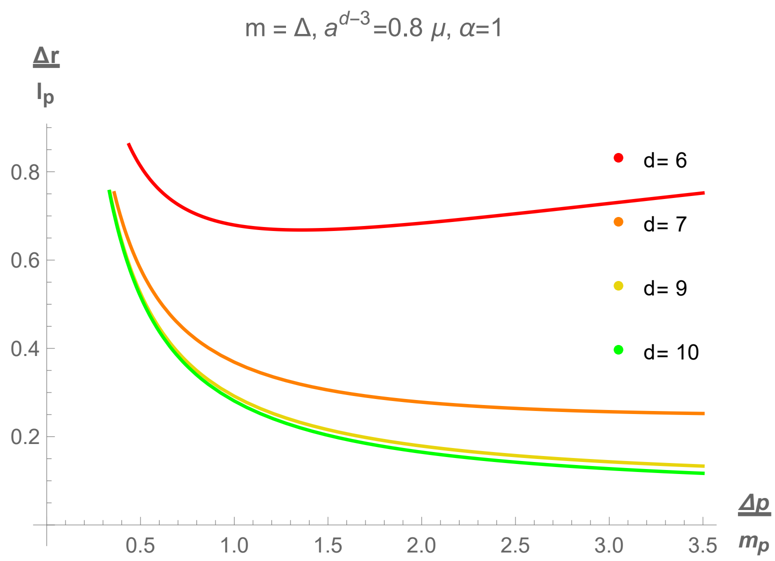


Figure 4.22: $\frac{\Delta r_0}{l_p}$ with $m = \Delta$ and $d = 6$

This plot shows the total uncertainty in length as a function of the uncertainty in momentum for different dimensions. When the dimension is increased the uncertainty is reduced, analogously to the static case portrayed in figure 4.8. Then, higher uncertainties on the momentum and higher dimensions give smaller values for the uncertainty on lengths. While the former

is due to Heisenberg's uncertainty principle, the latter may be due to the fact that higher dimensions are more likely to produce regular particles instead of Black Holes.
Next, what we vary is the angular momentum:

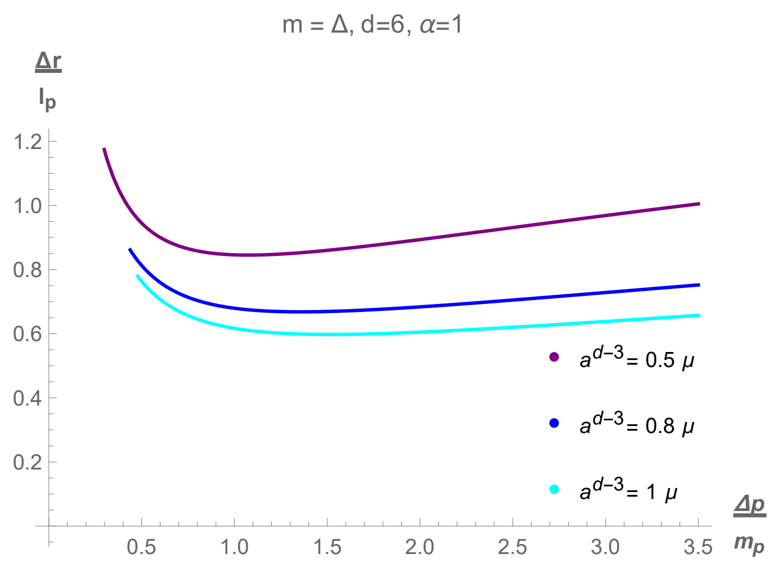


Figure 4.23: $\frac{\Delta r}{l_p}$ with $m = \Delta$, $\alpha = 1$ and $d = 6$

It is apparent that the possibly unstable configuration $a^{d-3} = \mu$ gives the least uncertainty, while $a^{d-3} = \frac{1}{2}\mu$ gives the highest value. As Black Holes are more favoured in stable configurations, in the other cases more regular particles are produced, thus lowering the total uncertainty.
Lastly, we examine the uncertainty for different values of the parameter α:

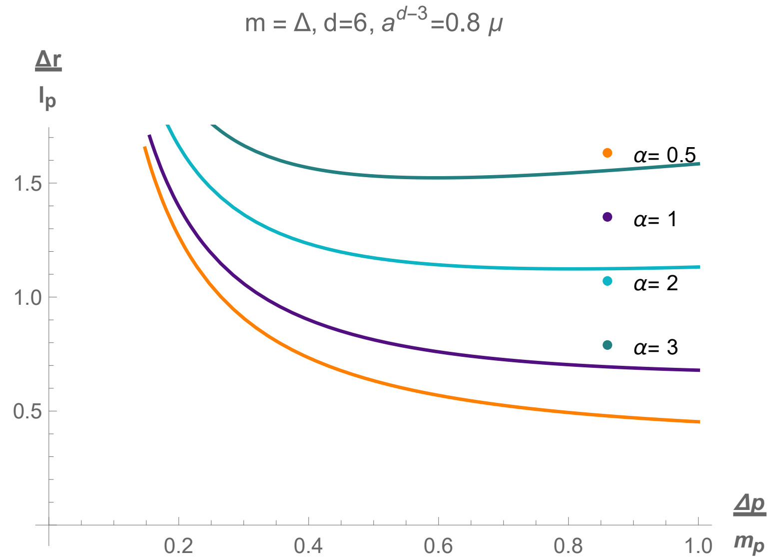


Figure 4.24: $\frac{\Delta r}{l_p}$ with $m = \Delta$,$d = 6$ and $\frac{a^{d-3}}{\mu} = 0.8$

We notice that lower values of the parameter give lower uncertainties. This implies that when the quantum fluctuations of the source are more important than the fluctuations of the horizon, the total uncertainty on lengths is lower. Moreover, the curves tend to be almost constant if Δp is large enough.

Similarly to the static case, we proceed to calculate the minimum of equation (4.137). We denote: $y \equiv \frac{\Delta r}{l_p}$ and $x \equiv \frac{\Delta pr}{m_p}$. Then:

$$\frac{dy}{dx} = 0 \quad \rightarrow \quad \frac{B(d)^2}{x^2} = \frac{\alpha K(d)}{d-3} \left[\frac{2}{d-3} \frac{1}{B(d)} \left(\frac{a^{d-3}}{\mu} \right)^{-\frac{2}{d-5}} \right]^{\frac{1}{d-3}} x^{\frac{1}{d-3}-1}. \tag{4.139}$$

The solution is:

$$x = (d-3) \left\{ \frac{B(d)^{2d-5}}{2} [\alpha K(d)]^{-(d-3)} \left(\frac{a^{d-3}}{\mu} \right)^{\frac{2}{d-5}} \right\}^{\frac{1}{d-2}}. \tag{4.140}$$

Then we substitute this value to find the minimum of y:

$$
\begin{aligned}
y &= \frac{B(d)^2}{x} + \alpha K(d) \left(\frac{2}{d-3} \frac{x}{B(d)} \right)^{\frac{1}{d-3}} \left(\frac{a^{d-3}}{\mu} \right)^{-\frac{2}{(d-3)(d-5)}} \\
&= B(d)^{2-\frac{2d-5}{d-2}} \left[\alpha K(d)\right]^{\frac{d-3}{d-2}} 2^{\frac{1}{d-2}} (d-3) \left(\frac{a^{d-3}}{\mu} \right)^{-\frac{2}{(d-2)(d-5)}} \\
&+ \left[\alpha K(d)\right]^{1-\frac{1}{d-2}} 2^{\frac{1}{d-3}-\frac{1}{(d-2)(d-3)}} B(d)^{\frac{2d-5}{(d-2)(d-3)}-\frac{1}{d-3}} \left(\frac{a^{d-3}}{\mu} \right)^{\frac{2}{(d-3)(d-5)(d-2)}-\frac{2}{(d-3)(d-5)}} \\
&= \left(\frac{d-2}{d-3} \right) \left\{ 2B(d) \left[\alpha K(d)\right]^{d-3} \left(\frac{a^{d-3}}{\mu} \right)^{-\frac{2}{d-5}} \right\}^{\frac{1}{d-2}} .
\end{aligned} \tag{4.141}
$$

This represents the minimum uncertainty in length, given by:

$$
\begin{aligned}
L = l_p &\left[\frac{2 l_p^2}{(d-3) a^2 (d-5)^{d-6}} \right]^{\frac{1}{d-4}} \\
&\times \left\{ \left(\alpha \sqrt{ \frac{E_{\frac{d-9}{2(d-5)}}}{E_{\frac{d-7}{2(d-5)}}} - \frac{E^2_{\frac{d-8}{2(d-5)}}}{E^2_{\frac{d-7}{2(d-5)}}} } \right)^{d-5} \sqrt{ \frac{d-3}{2} - \frac{2^{2(4-d)} \pi \left[(d-4)!\right]^2}{\Gamma^4\left(\frac{d-3}{2}\right)} } \right\}^{\frac{1}{d-4}} .
\end{aligned} \tag{4.142}
$$

We can thus plot this minimum length as a function of α:

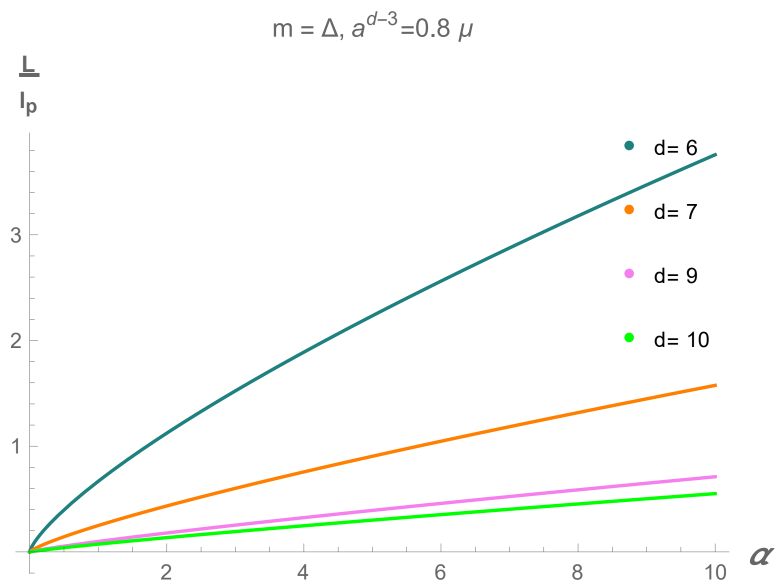


Figure 4.25: $\frac{L}{l_p}$ with $m = \Delta$ and $\frac{a^{d-3}}{\mu} = 0.8$

From this figure we see the minimum length has an almost linear behaviour, which becomes constant and smaller as the dimension increases. This, again, reflects the fact that higher dimensions give more classical results. Similarly to the static analogous plot, taking L at the Planck length clearly favours small values of the parameter α, while the opposite would have happened had we taken $m \simeq m_p$.

Chapter 5

Conclusions

In this work we extended the results of Ref. [41] by generalizing the discussion to rotating d-dimensional Myers-Perry Black Holes. We explored two different regimes: the static and ultra-spinning one. Since the static case is basically a d-dimensional Schwarzschild, the real extension to [41] comes from the ultra-spinning regime.
The main result of this discussion is that despite the Planck scale is lowered when considering extra dimensions, if we want a significant probability of forming a Black Hole it could not be sufficient to just reach the Planck scale but we might need to exceed it, especially for higher dimensional models. This could explain why we still have not detected Black Holes at LHC, despite reaching energies of order ~ 10 TeV. Therefore, increasing the energy at which the investigation takes place might prove the existence of higher dimensions, determining more precisely their number by fitting the data of Black Hole production with the probabilities here calculated. Through the investigation of trans-Planckian energies we might also be able to ascertain whether there is a theory of quantum gravity and research it or, alternatively, it may be true that gravity is self-complete and in that case we would observe only classical large Black Holes.
The results found in this work show the probability of getting a Black Hole resulting from a scattering experiment. It confirms that relevant probabilities are reached for trans-Planckian masses and lower angular momenta, indeed making the static case more favourable for the system. From the analysis of the GUP it emerged that the use we made of a Gauian distribution might be inaccurate, while a statistical treatment of a Black Hole as an ensemble of gravitons would probably give more truthful results. Moreover, it emerges that trans-Planckian masses give lower uncertainties on the horizon, possibly hinting to the formation of a classical Black Hole beyond the Planck scale and thus reinforcing the self-completeness theory.

Appendix A

The Einstein equation

In this section we are going to present how to obtain the Einstein equation. We are going to employ asymptotically flat metrics and a non-relativistic, weakly gravitating field:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \text{ everywhere.} \tag{A.1}$$

Also, the condition of non-relativistic system implies $v \ll c$ and that temporal derivatives can be considered much smaller than the spatial ones. Moreover, in this approximation, the components of the energy-momentum tensor are typically ranked as $|T_{00}| \gg |T_{0i}| \gg |T_{ij}|$. Although the condition $|T_{0i}| \gg |T_{ij}|$ might not be true in general, it is usually verified in rotating systems, so we take it to be true.
Gravity is contained in the Einstein-Hilbert action, which is:

$$S_{EH} = \frac{1}{16\pi G}\int d^d x\sqrt{-g}\mathcal{R} + \int d^d x\sqrt{-g}\mathscr{L}_{\text{matter}}. \tag{A.2}$$

The symbols used in (A.2) correspond to: G is the d-dimensional gravitational constant, g is the determinant of the metric, $\mathscr{L}_{\text{matter}}$ is the Lagrangian of other fields of the theory and $\mathcal{R}$ is the Ricci scalar.
The Ricci scalar is defined as:

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu}, \tag{A.3}$$

where $R_{\mu\nu}$ is called Ricci tensor and is a contraction of the Riemann tensor: $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$. The expression of the latter in terms of Christoffel symbols is as follows:

$$R^{\rho}_{\mu\lambda\nu} = \Gamma^{\rho}_{\nu\mu,\lambda} - \Gamma^{\rho}_{\lambda\mu,\nu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}. \tag{A.4}$$

We want to minimize the action (A.2) and obtain the equations of motion. We shall see that we end up getting the Einstein equation. We start by calculating the variation of each term that appears in (A.2). First, we variate $\mathcal{R}$:

$$\delta\mathcal{R} = \delta R_{\mu\nu}g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}. \tag{A.5}$$

To calculate $\delta R_{\mu\nu}$ we first notice that $\delta\Gamma^{\rho}_{\mu\nu}$ is a tensor, because it is obtained as the difference of the varied Γ' and the original Γ. As such, the covariant derivative acts as expected for a tensor:

$$\left(\delta\Gamma^{\rho}_{\mu\nu}\right)_{;\lambda} = \left(\delta\Gamma^{\rho}_{\mu\nu}\right)_{,\lambda} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\mu\sigma}. \tag{A.6}$$

From this expression, we obtain $\left(\delta\Gamma^{\rho}_{\mu\nu}\right)_{,\lambda}$ to substitute in the variation of (A.4):

$$\begin{aligned}\delta R^{\rho}_{\mu\lambda\nu} &= \left[\left(\delta\Gamma^{\rho}_{\nu\mu}\right)_{;\lambda} - \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}\right] \\ &\quad - \left[\left(\delta\Gamma^{\rho}_{\lambda\mu}\right)_{;\nu} - \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} + \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma}\right] \\ &\quad + \left[\delta\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu}\right] \\ &\quad - \left[\delta\Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu}\right] \\ &= \left(\delta\Gamma^{\rho}_{\nu\mu}\right)_{;\lambda} - \left(\delta\Gamma^{\rho}_{\lambda\mu}\right)_{;\nu}. \end{aligned} \tag{A.7}$$

Then, we notice that it is a total derivative and therefore it does not contribute to the action:

$$\begin{aligned}\delta S &= \int d^dx\sqrt{-g}g^{\mu\nu}\left[\left(\delta\Gamma^{\rho}_{\nu\mu}\right)_{;\rho} - \left(\delta\Gamma^{\rho}_{\rho\mu}\right)_{;\nu}\right] \\ &= \int d^dx\sqrt{-g}\left[g^{\mu\nu}\delta\Gamma^{\rho}_{\nu\mu} - g^{\mu\rho}\delta\Gamma^{\rho}_{\rho\mu}\right]_{;\rho} = 0. \end{aligned} \tag{A.8}$$

So, next, we calculate the variation of the measure $\sqrt{-g}$. First, we consider a generic matrix M and notice that:

$$\ln(\det M) = \mathrm{Tr}(\ln M) \tag{A.9}$$

and we differentiate it

$$\frac{\delta(\det M)}{\det M} = \mathrm{Tr}\left(M^{-1}\delta M\right). \tag{A.10}$$

Now we apply it to the metric, by taking $M = g_{\mu\nu}$:

$$\frac{\delta g}{g} = (g^{\mu\nu}\delta g_{\mu\nu}) \tag{A.11}$$

and since these are also valid:

$$\delta(g^{\mu\nu}g_{\mu\nu}) = 0 \quad \rightarrow \quad \delta g_{\mu\nu} = -g_{\mu\rho}\delta g^{\rho\sigma}g_{\sigma\nu}, \tag{A.12}$$

all together we obtain:

$$\begin{aligned}\delta g &= g(g^{\mu\nu}\delta g_{\mu\nu}) = -g(g^{\mu\nu}g_{\mu\rho}\delta g^{\rho\sigma}g_{\sigma\nu}) \\ &= -g(g_{\mu\nu}\delta g^{\mu\nu}). \end{aligned} \tag{A.13}$$

Then, finally, the variation of the measure is given by:

$$\delta\sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g}(g_{\mu\nu}\delta g^{\mu\nu}), \tag{A.14}$$

which we have all the elements to calculate.
Now, we use (A.14), (A.8) and (A.5) to obtain the variation of the action:

$$\delta S_{EH} = \frac{1}{16\pi G}\int d^dx\left[\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} - 8\pi G T_{\mu\nu}\right)\delta g^{\mu\nu}\right] \overset{!}{=} 0. \tag{A.15}$$

In this result, we used the definition of the energy-momentum tensor as obtained from the Euler-Lagrange equation applied only to the non-gravitational part of the action:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathscr{L}_{\mathrm{matter}}\right)}{\delta g^{\mu\nu}}. \tag{A.16}$$

Therefore, at the end we see that by extremizing the action (A.2) we obtain the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu}. \tag{A.17}$$

A.1 The first order correction

To first order in the metric perturbation, we can explicitly write an expression for $h_{\mu\nu}$. Indeed, in this approximation, (A.4) can be written more simply as:

$$R^{\rho}_{\mu\lambda\nu} = \Gamma^{\rho}_{\nu\mu,\lambda} - \Gamma^{\rho}_{\lambda\mu,\nu}. \tag{A.18}$$

Next, we use the definition of the Christoffel symbol and the weak field approximation (A.1), we can write:

$$\Gamma^{\rho}_{\nu\mu} = \frac{1}{2} g^{\rho\alpha} \left(g_{\nu\alpha,\mu} + g_{\alpha\mu,\nu} - g_{\nu\mu,\alpha} \right) \simeq \frac{1}{2} \eta^{\rho\alpha} \left(h_{\nu\alpha,\mu} + h_{\alpha\mu,\nu} - h_{\nu\mu,\alpha} \right). \tag{A.19}$$

Now we employ the so-called harmonic gauge, which consists in a particular gauge choice:

$$h^{\mu\nu}_{,\nu} = \frac{1}{2} \eta^{\mu\nu} h^{\lambda}_{\lambda,\nu}. \tag{A.20}$$

Then, to first order, the Riemann tensor is calculated to be:

$$R_{\mu\nu} = \Gamma^{\rho}_{\nu\mu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} = -\frac{1}{2} \Box h_{\mu\nu} \tag{A.21}$$

and the corresponding Ricci scalar is:

$$\mathcal{R} = -\frac{1}{2} \Box h \qquad \text{where } h \text{ is } h^{\mu}_{\mu}. \tag{A.22}$$

Moreover, by contracting (A.17) with $g^{\mu\nu}$ we obtain an expression for the Ricci scalar:

$$\mathcal{R} - \frac{d}{2} \mathcal{R} = 8\pi G T \quad \rightarrow \quad \mathcal{R} = \frac{8\pi G T}{1 - \frac{d}{2}}. \tag{A.23}$$

This allows us to rewrite (A.17) as:

$$\Box h_{\mu\nu} = -16\pi G \left(T_{\mu\nu} + \frac{g_{\mu\nu}}{2 - d} T \right) \equiv -16\pi G \tilde{T}_{\mu\nu}. \tag{A.24}$$

According to the approximations we decided to make, the temporal derivatives are negligible with respect to the spatial ones, therefore, the D'Alembert operator can be approximated to a Laplacian.

Appendix B

Green's function in arbitrary dimensions

The Green's function for the Laplacian is defined as follows:

$$\nabla^2 G(\vec{x} - \vec{x}') = \delta(\vec{x} - \vec{x}'). \tag{B.1}$$

To simply it, we take $\vec{x}' = 0$ and the dependence of the Green's function only on $|\vec{x}| = r$:

$$\nabla^2 G(r) = \delta(r). \tag{B.2}$$

Next we integrate on a sphere in $d-1$ dimensions of radius R, to get:

$$\int_{r<R} d^{d-1}x \nabla^2 G(r) = \int_{r<R} d^{d-1}x \delta(r) = 1. \tag{B.3}$$

Then we can apply the divergence theorem to the left-hand side:

$$\int_{r<R} d^{d-1}x \nabla^2 G(r) = \int_{r=R} d^{d-2}x \vec{\nabla} G \cdot \hat{r}. \tag{B.4}$$

Now we use the definition of ∇ and write:

$$\frac{\partial G}{\partial \vec{x}} = \frac{\partial G}{\partial r}\frac{\partial r}{\partial \vec{x}} = \frac{\partial G}{\partial r}\frac{r}{\vec{x}}, \tag{B.5}$$

where $\hat{r} = \frac{\vec{x}}{r}$. So then:

$$\int_{r=R} d^{d-2}x \frac{\partial G}{\partial r}\frac{\hat{r}}{\hat{r}} = \Omega_{d-2} R^{d-2} \frac{\partial G}{\partial r}(R), \tag{B.6}$$

where $\Omega_{d-2}R^{d-2}$ is the area of the $d-2$ dimensional sphere:

$$\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{B.7}$$

Now we can integrate on r the end result of equation (B.6) and obtain:

$$G(R) = \int_0^R \frac{dr}{\Omega_{d-2} r^{d-2}} = -\frac{1}{R^{d-3}\Omega_{d-2}(d-3)}. \tag{B.8}$$

So that in general it gives:

$$G(\vec{x}-\vec{y}) = \frac{1}{(d-3)\Omega_{d-2}|\vec{x}-\vec{y}|^{d-3}}. \tag{B.9}$$

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