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Class LM-29

STUDY OF THE TORQUE VECTORING CONTROL PROBLEM

Master's Thesis in
Applied Automatic Control

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Contents

Abstract

The aim of this thesis is to address the design of a torque vectoring system for an electric All-Wheel-Drive car. The torque vectoring objective is to substitute the need of a mechanical differential in case of an electric car with four individual hub motors. It can be exploited also to perform various forms of electronic stability, such as traction and launch control, improving safety, sports performance and off-road capabilities of vehicles. These technologies involve individual control of each wheel's drive torque or braking force in response to the dynamics of the driving conditions and the driver's intentions. This work present a non-linear model derived from a Lagrangian approach to the solution of the control problem, that is then studied at the equilibrium point along a curvilinear trajectory.

List of Symbols

$4WD$	Four-Wheel drive
$6DOF$	Six degree of freedom
UV	Under-vehicle
$\dot{p}$ $[\frac{m}{s}]$	velocity
$\ddot{p}$ $[\frac{m}{s^2}]$	acceleration
t $[s]$	time
F $[N]$	force

Chapter 1

Introduction

With the increasing problem of air pollution and global warming, most of the car manufacturers are starting to develop and sell cars with different topologies of electric powertrains. In this context of changes also in the racing world it is possible to see an increasing effort to make likable also the electric races, as it is possible to see with FIA Formula E. In this category, the chassy is the same for all cars while the team has to push above all on the powertrain configuration and the energy management.

The energy management in the case of electric vehicles is really important since now the benchmark for this new technology is to have the same performances and ranges with the internal combustion engine cars, considering a trade off with the costs. What is important to highlight is that considering the introduction of a new technology in the market, there will always be some advantages and disadvantages, but it is really important to let people be aware of the capabilities of the electric

Figure 1.1: Formula E MercedesEQ car for the season 2019/20

vehicles.

1.1 Traction architecture

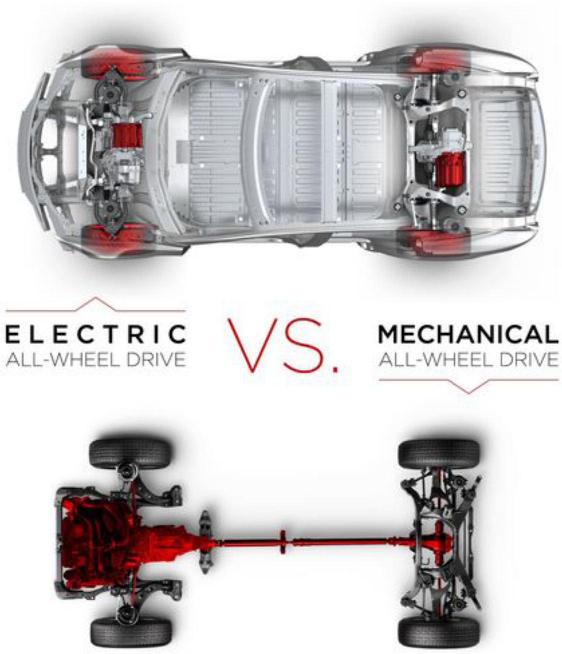


Figure 1.2: Electric All-Wheel drive architecture versus an mechanical all wheel drive

It is possible to make a comparison between the traction architecture of a four wheel drive internal combustion engine vehicle and a four wheel drive full electric vehicle. In case of a full electric vehicle it is possible to exploit the fact that the power-train can be designed with four independent electric motors, each one with its own inverter.

At this point the vehicle control unit can distribute four different torque demands to the wheels independently, allowing an improvement in performance and stability of the vehicle. This chance of having four independently controlled wheels leads to avoid the need of a mechanical differential, that is normally used with internal combustion engines vehicle.

A huge advantage of this architecture is the simplification in the transmission chain, thus leading to the reduction of the total weight of the vehicle and to less moving mechanical parts.

Reducing the weight of a mechanical part, it is really important for the vehicle performance, because instead of having the addition of weight in a fixed point, it can be distribute, through a proper design, to have a better vehicle dynamics.

1.2 Torque vectoring

The torque vectoring, in case of a four wheel drive full electric vehicle, is the torque distribution on each wheel that can be exploited to have optimal vehicle dynamics. The way of how the control is designed depends on the type of vehicle needs. For instance, for a road car it can be exploited to have a better stability, while in case of a race car, the performance can be stressed as the main goal.

The torque vectoring control can be also designed to allow traction and launch control of the vehicle, acting again on the wheel torque. It depends on how it is decided to design the control law.

When the vehicle goes around the corner, the weight distribution is not equal on both sides. During the corner the weight get transferred from the inside to the outside, thus leading to more load on the outside tyres. This is way it is important to have a different torque on each wheel.

Furthermore, while the vehicle is performing a straight, then during the acceleration, there is a longitudinal weight transfer. The result is that the rear tyres experience more load than the rear one, while they are all running at the same speed. The same happens during the deceleration of breaking, but in the opposite way.

According to the previous considerations, it is necessary to have on the vehicle a central control unit that calculate, in each instant of time, which is the better torque distribution on each wheel according to different measurable parameters. It is possible to exploit both the power of a feed-forward and feed-back control law. The first one is used to calculate the differentiation of wheel torque during a normal trajectory, while the second one can be exploited to keep the vehicle on the trajectory in case of disturbances.

Chapter 2

Mathematical Introspection

In this chapter, it will be described the mathematical tools used in the following model solution.

2.1 Euler-Lagrange Equations

Definition: The Euler-Lagrange equations

They are second-order partial differential equation that gives functional stationary solutions. Where a functional, in mathematical analysis, refers to a mapping from a space $\mathcal{X}$ into real or complex numbers, with a purpose of having more computation like structures. While, a stationary point of a differentiable function on a variable, is a point in which the derivative of the function is zero.

Given the functional:

$$S(q) = \int_a^b L(t.q(t), \dot{q}(t)\, dt \tag{2.1}$$

where q is a function of real argument t, then the Euler-Lagrange equation is the one that has a stationary point given by it.

The Euler-Lagrange equation, then, is given by:

$$L_x(t.q(t), \dot{q}(t) - \frac{d}{dt} L_v(t.q(t), \dot{q}(t) = 0 \tag{2.2}$$

where L_x and L_v denote the partial derivative of L with respect to the second and third arguments, respectively.

If the dimension of the space $\mathcal{X}$ is greater than one, it becomes a system of differential equations, one for each components:

$$\frac{\partial L}{\partial q_i}(t.q(t),\dot{q}(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}(t.q(t),\dot{q}(t) = 0 \qquad for \qquad i = 1,...,n \tag{2.3}$$

In this specific case, it has been exploited the formulation for **several functions of single variable with single derivative**.

In general, if the problem involves finding several functions of a single independent variable. Which in the specific case, the function are the motion equation of the vehicle and the single variable is the time t. Then, the formulation of the Euler-Lagrange equations is:

$$\frac{\partial L}{\partial f_i} - \frac{d}{dx}(\frac{\partial L}{\partial f_i'} = 0_i \tag{2.4}$$

2.2 Lagrangian Mechanics

The Lagrangian approach, in classical mechanic, can be used to describe the evolution in time of a physical system, since it is equivalent to the solution of Newton law of motion. A huge advantage of this approach is that these equations take the same form in any system of generalized coordinate systems, thus leading for it to be better suited for generalization.

Indeed, in Newton's laws when it is necessary to include the non-conservative forces, it is better to express it in Cartesian coordinates. Lagrangian mechanics at this poi becomes very useful because it allows to bypass any specific coordinates system. At this point it is possible to express dissipative and driven forces with the sum of potential and non-potential forces, leading to a set of modified equations. The main convenience of this approach cam from the fact that the generalized coordiantes can be chosen to suite any requirement, like simplification due to symmetry which may help in solve a motion problem.

In general, in order to use this approach, it is necessary to define the *Lagrangian* as the combination of kinetic and potential energy of the considered system. This can be done in the following way:

$$L = K - T \tag{2.5}$$

where K is the potential energy of the system and

Now exploiting the definition given in the equation 2.4, it can be found the *equation of motion*, which is actually a second-order differential equation, not the actual variable as function of t that can be derived by integrating twice the equation.

This instrument is really powerful since, if the problem involves more than one coordinate, than it would just be necessary to apply the equation 2.4 to each of it.

This solution of the problem comes from the **principle of stationary action** described in the previous paragraph.

2.3 Moore-Penrose pseudo-inverse

In linear algebra, the pseudo-inverse is a generalization of the inverse of a matrix.

It is widely used to find the solution of a system of linear equations which do not have a unique solution, indeed it peaks the one with the smallest norm. is is unique and defined if the entries of the matrix are real or complex and it can be computed using the *singular value decomposition*.

2.3.1 Notation

Conventions adopted in the following discussion:

- K denotes one of the fiels of real or complex numbers, R o C. The vector space of $m \times n$ matrices over K is denoted by $K^m \times n$.
- For $A \in K^m \times n$, $A^\top$ and $A*$ denote the transpose and the conjugate transpose.
- For $A \in K^m \times n$, $\text{im}(A)$ denotes the image of A, the space spanned by the column vectors of A.
- For $A \in K^m \times n$, $\ker(A)$ denotes the kernel of A.
- For any positive integer n, $I_n \in K^n \times n$ denotes the $K^n \times n$ identity matrix.

Definition: Moore-Penrose pseudo-inverse

For $A \in K^m \times n$, a pseudo-inverse of A is defined as a matrix $A^+ \in K^m \times n$ that satisfies all the following criteria:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)* = AA^+$
4. $(A^+A)* = A^+A$

A^+ always exist for any matrix A, but when it has full rank (that is defined as the $min\{m, n\}$), it can be expressed with a simple algebraic formula.

At this point it is necessary to distinguish two cases:

- n<m The A matrix in this case has linearly independent column, thus leading to an injective function. This means, it is dealing with an under-sized system:

$$A^{+} = (A * A)^{-1} A* \tag{2.6}$$

This is called *left pseud-inverse.*

- n>m The A matrix in this case has linearly independent row, thus leading to a surjective function. This means, it is dealing with an over-sized system:

$$A^{+} = A * (A * A)^{-1} \tag{2.7}$$

This is called *right pseud-inverse.*

For this specific case, it will be used the right pseudo-inverse.

2.4 Equilibrium condition study

Given a dynamic system, with a finite dimension, continuous in time, non-linear and stationary, that can be described by the state differential equation:

$$\dot{x}(t) = f(x(t), u(t) \tag{2.8}$$

It is possible to consider two different temporal evolution:

1. nominal movement of equilibrium $\widetilde{x}(t) = \bar{x}$. This can be obtained applying the nominal entry of equilibrium $\widetilde{u}(t) = \bar{u}$ to the system, that is in the nominal inital state $\widetilde{x}(t = 0) = \bar{x} \Rightarrow \widetilde{x}(t)$ has to satisfy the following system of equations:

$$\dot{\widetilde{x}}(t) = \dot{\bar{x}} = 0 = f(\bar{x}, \bar{u}) \tag{2.9}$$

2. disrupted movement $x(t)$ obtained applying the nominal entry $\widetilde{u}(t) = \bar{u}$ to the system in a initial state that is different form the nominal one $x_0 \not\equiv \bar{x} \Rightarrow x(t)$ has to satisfy the following system of equations:

$$\begin{cases} \dot{x}(t) = f(x(t), \bar{u}) \\ x(t_0 = 0) = x_0 \end{cases} \quad (2.10)$$

The difference between the two movements is the perturbation on the system state:

$$\delta x(t) = x(t) - \bar{x} \in \mathbb{R}^n \Rightarrow x(t) = \bar{x} + \delta x(t) \tag{2.11}$$

The time evolution of the perturbation is on the state $\delta x(t)$ is the solution of the following differential equation:

$$\begin{aligned} \delta \dot{x}(t) = \frac{d(\delta x(t))}{dt} = \\ = \frac{d(x(t) - \bar{x}}{dt} = \\ = \dot{x}(t) - \dot{\bar{x}} = \\ = f(x(t), \bar{u}) = \\ = f(\bar{x} + \delta x(t), \bar{u}) \end{aligned} \tag{2.12}$$

that is a non-linear equation in the variable $\delta x(t)$ and it has the following initial condition:

$$\delta x(t_0 = 0) = x(t_0 = 0) - \bar{x} = x_0 - \bar{x} = \delta x_0 \not\equiv 0 \tag{2.13}$$

In general the solution of a non linear differential equation $\delta \dot{x}(t) = f(\bar{x} + \delta x(t), \bar{u}), \delta x(t_0 = 0) = x_0 - \bar{x} = \delta x_0$ is really difficult to find. Furthermore it depends from both the initial nominal equilibrium state $\bar{x}$ and from the nominal equilibrium entry $\bar{u}$. This means, it depends on the considered equilibrium point.

In case of non-linear and stationary dynamic system, the property of stability can be studied only on a small neighbourhood of a chosen equilibrium state (local stability).

In many cases, with the **indirect method of Lyapunov**, known also as linearization method, it is possible to study the local stability at the equilibrium point without having to solve the non-linear differential equation.

$$\delta \dot{x}(t) = f(\bar{x} + \delta x(t), \bar{u}), \delta x(t_0 = 0) = x_0 - \bar{x} = \delta x_0 \tag{2.14}$$

The function $f(\bar{x} + \delta x(t), \bar{u})$ can be developed in Taylor series around the equilibrium point.

According to the linearization method, if it is possible to discard all the terms that contain power of grade greater than one, than the analysis of the stability of the equilibrium can be done through the study of the internal stability of an LTI dynamic system.

Chapter 3

Model

3.0 Reference Frames

Definition: Rigid body

It is a solid body with no deformation and if it as a deformation, it is so small that it can be neglected. In a rigid body the distance among each point remains constant regardless any external force.

Definition: Reference frame

In physics, it consist of an abstract coordinate frame and reference points that define uniquely, in term of position and orientation, the behaviour of an object in the space. Sometimes the reference frame is attached to the modifier.

First of all, it is necessary to define two parts of the car:

1. *Body*: representation of a **rigid body** with 6 degree of freedoms (three traslations and three rotatios), which is attached to the road by means of an

Figure 3.1: Definition of the *Body* and *Under-Vehicle* systems

equivalent system of suspension and tyres. On this part act the aerodynamic forces ${}^{i}F_a$ and the gravity g.

2. *Under-Vehicle*: ideal part of the vehicle modeled as a **rigid body** with 6 degree of freedoms (three traslations and three rotatios) which is attached to the road, which is plane if the suspention are in rest position and it coincides with the *body*. On this part act the wheel forces $\sum {}^{i}F_w$ and momenta $\sum {}^{u}\tau_w$).

These two parts are connected each other by means of both kinematic and dynamic constraints, which will be described in the section 3.2.3. It it possible to see a graphical representation of the definition of these system in figure 3.1.

This is the list of reference frame that has been exploited to describe the behaviour of the car in this case:

- Inertial: $\mathcal{F}_I(O_I, x_I, y_I, z_I)$ where the origin O_I is centered in a plane tangential to the Earth surface, while the axis x_I and y_I lay on it with the following directions:

  - x_I: oriented toward North
  - y_I: oriented toward West

  While the z_I axis is perpendicular to this plane and points upward.

  It is possible to see a graphical representation of this reference frame in fig. 3.2.

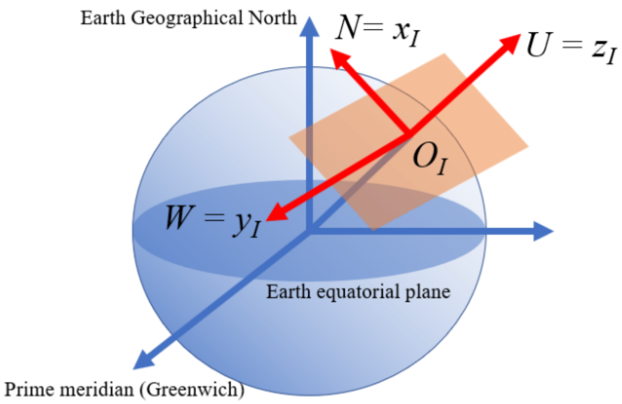


Figure 3.2: Definition of the *Inertial reference frame*

- Body: $\mathcal{F}_B(O_B, x_B, y_B, z_B)$ where the origin O_B is centered in centre of gravity of the car and the axis are defined in the following way:

- x_B: oriented along the longitudinal symmetry axis
- z_B: points upward and lays on the longitudinal symmetry plane
- y_B: complete the reference frame according to the right-hand rule

It is possible to see a graphical representation of this reference frame in fig. 3.3.

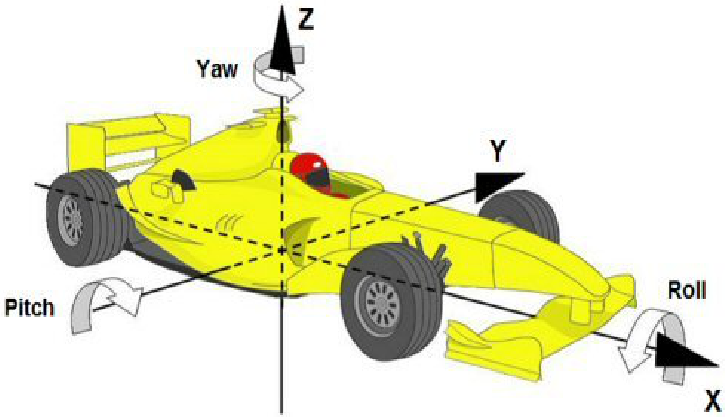


Figure 3.3: Definition of the *Body reference frame*

- Under-vehicle: $\mathcal{F}_U(O_U, x_U, y_U, z_U)$ where the centre coincides with the one of the body reference frame, $O_U \equiv O_B$. The axis are defined by means of a rotation of the body reference frame considering the definition of road-relative angles that will be given at page 24. If the suspensions are in rest position the two reference frames are totally coincident.

  It is possible to see a graphical representation of this reference frame in fig. 3.4.

- *Navigation*: $\mathcal{F}_N(O_N, x_N, y_N, z_N)$ where the centre coincides with the one of the body reference frame, $O_N \equiv O_B$. The important characteristic of this reference frame is that the axes x_N is aligned with the inertial speed of the vehicle. The y_N lies on the under-vehicle and it is perpendicular to x_N and the axis z_N completes the reference frame according to the right-hand rule.

  It is possible to see a graphical representation of this reference frame in fig. 3.5.

- *i*-th Wheel: $\mathcal{F}_{W_i}(O_{W_i}, x_{W_i}, y_{W_i}, z_{W_i})$ is a fixed reference frame attached to the wheel. Where the origin O_{W_i} is centered in the i-th wheel centre of gravity and the axis are defined in the following way:

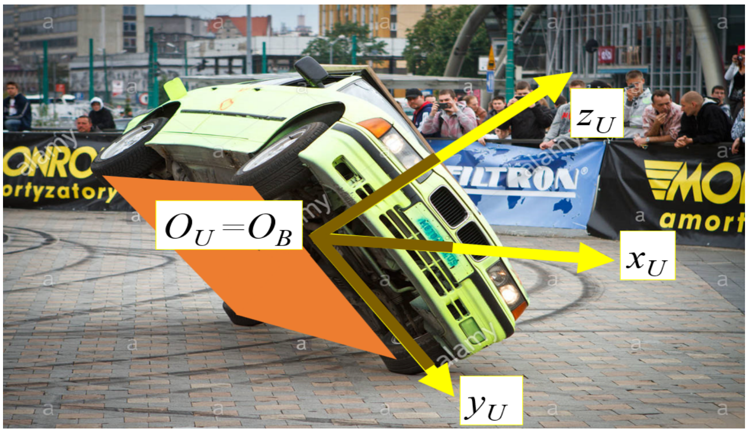


Figure 3.4: Definition of the *Under-vehicle reference frame*

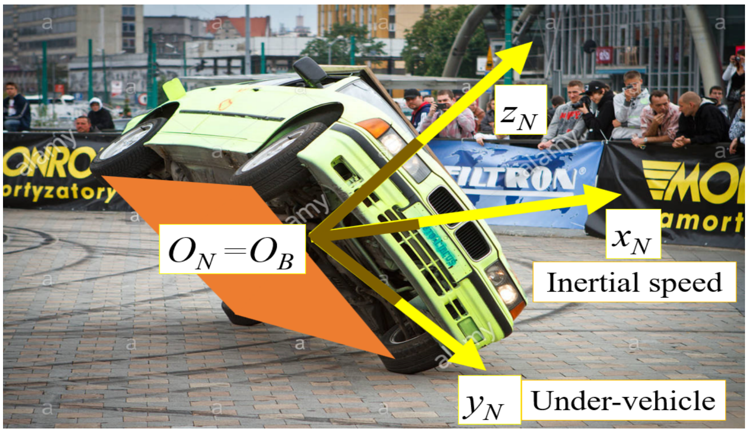


Figure 3.5: Definition of the *Navigation reference frame*

– y_{W_i}: aligned with the revolution axis of the wheel towards the left side of the vehicle

– z_{W_i}: points upward and it is aligned with the axis z_U

– x_{W_i}: complete the reference frame according to the right-hand rule

It is possible to see a graphical representation of this reference frame in fig. 3.14.

3.1 Rotation matrices

When it is necessary to consider two different reference frame, it is also necessary to define the relationship between them. Given two reference frames $\mathcal{F}_1$ and $\mathcal{F}_2$ and a vector v_1 with the coordinate defined in $\mathcal{F}_1$, if it is necessary to represent it in

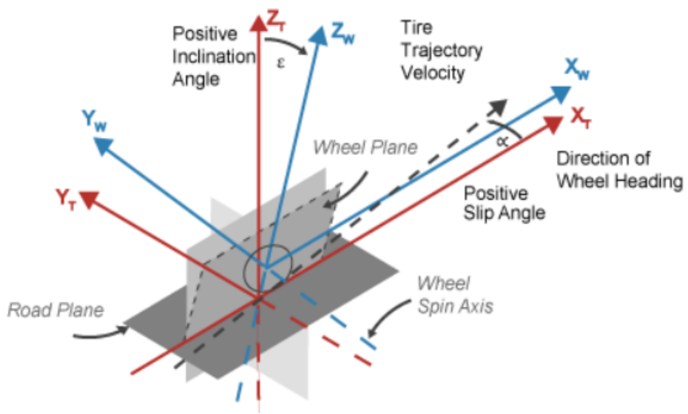


Figure 3.6: Definition of the *i-th wheel reference frame*

the second reference frame, this can be done by means of a linear transformation obtained through the rotational matrix used to define the projection of v_1 on the axis of $\mathcal{F}_2$.

One of the most common way of performing rotation in different reference frame is done using the **Euler angles** that are defined to describe the orientation of a rigid body in a fixed or moving coordinate reference frame.

The rotation is the combination of three different rotation performed along the axis, so it necessary also to define three angles of rotation with the corresponding rotational matrices:

- *yaw angle* (ψ): represents a rotation around the axes z

$$v' = R_3(\psi)v_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} v_1 \tag{3.1}$$

- *pitch angle* (ψ): represents a rotation around the axes y

$$v'' = R_2(\theta)v' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} v' \tag{3.2}$$

- *roll angle* (ψ): represents a rotation around the axes x

$$v_2 = R_1(\phi_b)v'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} v'' \tag{3.3}$$

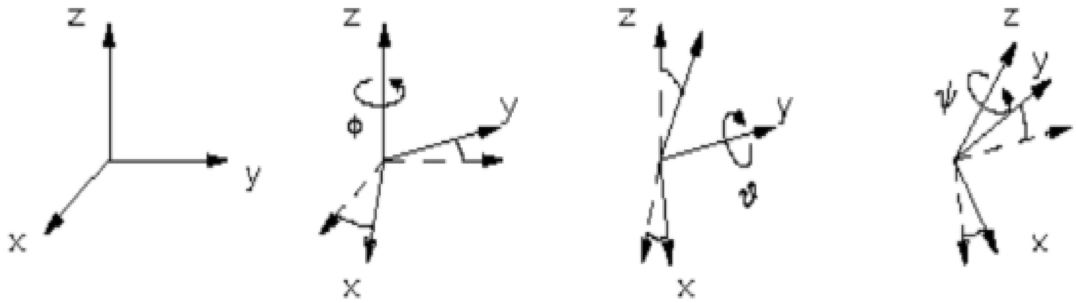


Figure 3.7: Sequence of rotations used to pass from $\mathcal{F}_1$ to $\mathcal{F}_2$

The composition of these three rotations results in:

$$v_2 = R_1(\phi)R_2(\theta)R_3(\psi)v_1 = {}^2R_1(\phi, \theta, \psi)v_2 \tag{3.4}$$

Where ${}^2R_1(\phi, \theta, \psi)v_2$ represent the total rotation matrix between, as it is possible to see in fig. 3.11. After these consideration it is possible to define all the rotational angles and matrices among the different reference frame defined in the paragraph 3.0:

- From *Inertial* to *Body*:

$$^BR_I(\phi, \theta, \psi) = R_1(\phi)R_2(\theta)R_3(\psi) \tag{3.5}$$

where ϕ, θ and ψ correspond to the Euler angles of pitch, yaw and roll, respectively;

- From *Body* to *Under-vehicle*:

$$^UR_B(\phi_r, \theta_r, \psi_r) = R_1(\phi_r)R_2(\theta_r)R_3(\psi_r) \tag{3.6}$$

where ϕ_r, θ_r and ψ_r correspond to the relative Euler angles of pitch, yaw and roll, respectively. Indeed they describe the relative position of the under-vehicle system with respect to the body part, as it is defined in the paragraph 3.0. The complete expression of this matrix can be found in the appendix A.8.

- From *Inertial* to *Under-vehicle*:

$$^UR_I(\phi_u, \theta_u, \psi_u) = R_1(\phi_u)R_2(\theta_u)R_3(\psi_u) \tag{3.7}$$

where ϕ_u, θ_u and ψ_u correspond to the Euler angles of pitch, yaw and roll, respectively. For this description it will be necessary actually to use the opposite transformation, so from the *Under-vehicle* to *Inertial* reference frame. In order to obtain the proper rotational matrix, it is necessary to perform the transpose of the matrix previously defined:

$$^{I}R_U(\phi_u,\theta_u,\psi_u) = {}^{U}R_I(\phi_u,\theta_u,\psi_u)^\top = R_3^\top(\psi_u)R_2^\top(\theta_u)R_1^\top(\phi_u) \tag{3.8}$$

This come from the fact that it can be demonstrated that the rotational matrix are orthonormal, which means $R^\top = R^{-1}$. The complete expression of this matrix can be found in the appendix A.2.

- From the *Navigation* to the *Under-vehicle*:

$$^{U}R_N(0,\gamma_U,\beta_U) = R_2(\gamma_U)R_3(\beta_U) \tag{3.9}$$

where β_U indicates the **side-slip angle**. The parameter is really important for this discussion and the definition of it will be explained in the chapter 4.1. While γ_U is the under-vehicle climb angle. The complete expression of this matrix can be found in the appendix A.3.

- From *i-th wheel* to *Under-vehicle*:

$$^{U}R_{W_i}(\phi_{W_i},0,\delta_{W_i}) = R_1(\phi_{W_i})R_3(\delta_{W_i}) \tag{3.10}$$

where δ_{W_i} indicates the angle of the wheel with respect to the rest position, during a straight trajectory, needed to perform a curvilinear trajectory. While ϕ_{W_i} indicates the i-th wheel roll angle, that is its camber angle. The complete expression of this matrix can be found in the appendix A.4.

FROM	TO	SYMBOLS	ROTATIONS
Inertial	Body	$^{B}R_I$	ϕ,θ,ψ
Body	Under-vehicle	$^{U}R_B$	ϕ_r,θ_r,ψ_r
Inertial	Under-vehicle	$^{U}R_I$	ϕ_u,θ_u,ψ_u
Navigation	Under-vehicle	$^{U}R_N$	γ_U,β_U
i-th wheel	Under-vehicle	$^{U}R_{W_i}$	ϕ_{W_i},δ_{W_i}

Table 3.1: Summary of all the required coordinate transformation, rotation matrices and angles

3.2 Kinematics

3.2.1 Justification to the model

Considering the definition of the vehicle as the assembly of two parts, given at page 3.0, it is necessary at this point to highlight the main idea that is behind the description of this model. Starting from a general idea, it will be given the description of all the assumption made for this specific case.

Definition: Inverted pendulum

An inverted pendulum is a particular type of pendulum which has its centre of mass above its pivot point, that is the point that should allow the body to keep a null displacement when a rotation is applied on it.

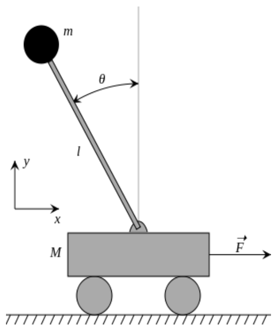


Figure 3.8: Schematic of a cart inverted pendulum

In this case the system considers more degree of freedom with respect to the normal inverted pendulum that usually as only one. Indeed there are two masses, one of the body and the other one of the under-vehicle system, that are connected by the means of a torsional spring. This way of outline the vehicle allows the body to have both relative roll and pitch with respect to the under-vehicle.

Together with the rotational degrees of freedom, it is necessary also to describe the relative translations that can occur between the body and the under-vehicle idealization. This is done assuming that the only possible translation can be performed by means of the suspensions is along the z axis.

The following list is the description of all the considered distances within the model:

- $L_x > 0$ is the arm linking the centre of mass of the body to the rotation joint in the lateral direction;
- $L_z > 0$ is the arm linking the centre of mass of the body to the rotation joint in the vertical direction;

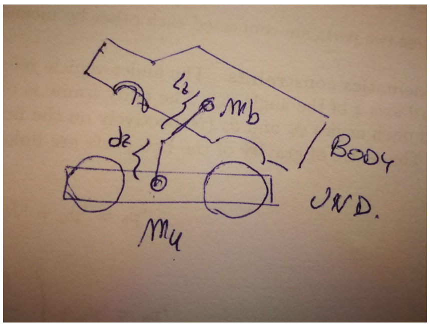


Figure 3.9: Schematic of the proposed system

- $d_z(t)$ represents the relative vertical movement between the two parts (in the under-vehicle coordinates).

3.2.2 Position Description

The following equation describes the position of the vehicle in the inertial reference frame. From this equation it is possible to understand also how it is defined the geometry that centre of gravity of the two parts.

$$ {}^{i}p_b = {}^{i}p_u + {}^{i}R_u(\Theta_u) \left({}^{u}\begin{bmatrix} 0 \\ 0 \\ d_z \end{bmatrix} + {}^{u}R_b(\phi_r, \theta_r)^{b} \begin{bmatrix} L_x \\ 0 \\ L_z \end{bmatrix} \right) \tag{3.11} $$

Where:

- ${}^{i}p_u$ represents the centre of gravity of the under-vehicle (in the inertial coordinates)
- ${}^{i}p_b$ represents the centre of gravity of the body (in the inertial coordinates)
- $\Theta_u = [\phi_u, \theta_u, \psi_u]$ are the roll, pitch and yaw angles of the under-vehicle
- ϕ_r, θ_r are the relative roll and pitch angles

The distances, along the different axis, between the centre of gravity are given according to the definition highlighted in the previous paragraph. At this point it is possible to see that the relative yaw angle between the body and the under-vehicle systems has been considered always zero. This means that the two parts can't rotate one on the other along the z axis.

Considering this equation, it is necessary to substitute the rotational matrices according to the definition given at paragraph 3.1. The complete expression of this system is to complicated to be handle manually, so it can be found, just for the reader to know, in appendix A.5.

Small angle approximation

Let us now consider the angles ϕ_r and θ_r sufficiently small:

- $\phi_r \approx 0$

  such that

  - $\sin\phi_r \approx \phi_r$
  - $\cos\phi_r \approx 1$

- $\theta_r \approx 0$

  such that

  - $\sin\theta_r \approx \theta_r$
  - $\cos\theta_r \approx 1$

This approximation comes from the fact that it is possible to assume that all the relative rotations generated by the suspension systems can be neglected.

It is possible to find the complete result of this approximation in the appendix A.6. Since also this expression is not as easy to be handle, it has been tried to shrink the system using the following definitions:

$$A = \begin{bmatrix} (\sin\phi_u \sin\psi_u + \cos\phi_u \cos\psi_u \sin\theta_u) \\ -(\cos\psi_u \sin\phi_u - \cos\phi_u \sin\psi_u \sin\theta_u) \\ \cos\phi_u \cos\theta_u \end{bmatrix} \tag{3.12}$$

$$B = \begin{bmatrix} \cos\psi_u \cos\theta_u \\ \cos\theta_u \sin\psi_u \\ -\sin\theta_u \end{bmatrix} \tag{3.13}$$

$$C = \begin{bmatrix} (\cos\phi_u \sin\psi_u - \cos\psi_u \sin\phi_u \sin\theta_u) \\ -(\cos\phi_u \cos\psi_u + \sin\phi_u \sin\psi_u \sin\theta_u) \\ -\cos\theta_u \sin\phi_u \end{bmatrix} \tag{3.14}$$

This come from the equation that can be found in the appendix A.7.

Thus leading to the far more compact equation:

$$ {}^{i}p_b \approx {}^{i}p_u + A\left(d_z - L_x\,\theta_r + L_z\right) + B\left(L_x + L_z\,\theta_r\right) + C\left(L_z\,\phi_r\right) \tag{3.15} $$

At this point it is necessary to compute the first derivative of ${}^{i}p_b$ to express the vehicle speed in the inertial reference frame, that is given by:

$$ \begin{aligned} {}^{i}\dot{p}_b \;=\; & {}^{i}\dot{p}_u + \dot{A}\left(d_z - L_x\,\theta_r + L_z\right) + A\left(\dot{d}_z - L_x\,\dot{\theta}_r\right) + \\ & + \dot{B}\left(L_x + L_z\,\theta_r\right) + B\dot{\theta}_r + \dot{C}L\left(L_z\,\phi_r\right) + C\dot{\phi}_r \end{aligned} \tag{3.16} $$

where the following substitutions holds:

- $$ \dot{A} = \frac{\partial A}{\partial \phi_u}\dot{\phi}_u + \frac{\partial A}{\partial \theta_u}\dot{\theta}_u + \frac{\partial A}{\partial \psi_u}\dot{\psi}_u \tag{3.17} $$
- $$ \dot{B} = \frac{\partial B}{\partial \phi_u}\dot{\phi}_u + \frac{\partial B}{\partial \theta_u}\dot{\theta}_u + \frac{\partial B}{\partial \psi_u}\dot{\psi}_u \tag{3.18} $$
- $$ \dot{C} = \frac{\partial C}{\partial \phi_u}\dot{\phi}_u + \frac{\partial C}{\partial \theta_u}\dot{\theta}_u + \frac{\partial C}{\partial \psi_u}\dot{\psi}_u \tag{3.19} $$

The complete expression of these derivatives can be found in the appendix A.8.

This equation for the small angles approximation previously described, i.e. $\phi_r,\, \theta_r \approx 0$, is simplified as follows:

$$ \begin{aligned} {}^{i}\dot{p}_b \;=\; & {}^{i}\dot{p}_u + \dot{A}\left(d_z - L_x\,\theta_r + L_z\right) + A\left(\dot{d}_z - L_x\,\dot{\theta}_r\right) + \\ & + \dot{B}\left(L_x + L_z\,\theta_r\right) + B\dot{\theta}_r + \dot{C}L\left(L_z\,\phi_r\right) + C\dot{\phi}_r \end{aligned} \tag{3.20} $$

3.2.3 Kinematic Constrains

In this paragraph it will be described which are the kinematic constrains that link all the systems used to describe the vehicle behaviour. The links are above all between the body and the under-vehicle systems. It is important to highlight that it has been assumed that a movement of the body, as a consequence in the motion of the under-vehicle and vice-versa. As a consequence, this leads to necessity of describing how the single kinematic of the two system influence actually the overall behaviour of the vehicle.

This comes from the fact that many reference system are exploited for the description of the vehicle behaviour and the complete model need to be coherent with all the assumption that links all the definitions.

1. *Under-vehicle* reference frame with respect to *Inertial* reference frame
   The under-vehicle plane, (O_U, x_U, y_U), is coincident with the corresponding inertial, $O_I, x_I, y_I)$. It means that the vehicle is leveled with the North-West plane and the under-vehicle reference frame is aligned with the inertial one, with except for a rotation around the third axis z of the angle ψ_u. This assumption will be exploited after the computation of the Eulero-Lagrange equations, since to calculate it it is necessary to consider all the possible degree of freedom.

   - $\phi_U = 0$
   - $\theta_U = 0$

   From the physical point of view, it means that the wheels are always attached to the ground and the vehicle is not jumping or flying.

2. *Body* reference frame with respect to *Under-vehicle* reference frame:
   The body frame is obtained by rotating the under-vehicle frame by means of the relative roll and pitch angles ϕ_r and θ_r (respectively on the first and second axis), that has been defined at page 24. While the relative yaw angle is null and the z axis are coincident.

   - $\psi_r = 0$

   From the physical point of view, it means that according suspensions design, they don't allow the relative rotations of the two system along the z axis.

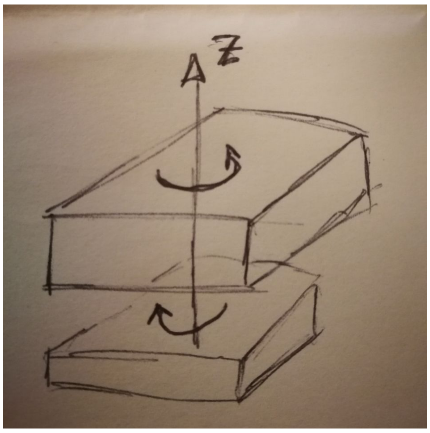


Figure 3.10: Not allowed relative rotation

3.2.4 Car Trim

In this paragraph, it will be described which are the further assumption on the vehicle attitude.

- Null *camber angle* $\phi_{W_i} = 0$:
  The camber angle of a wheel is the inclination that is can have along x, so with respect to the frontal section. In this case, it is considered that the suspension kinematic chain doesn't allow the following behaviour, but onth the first case.

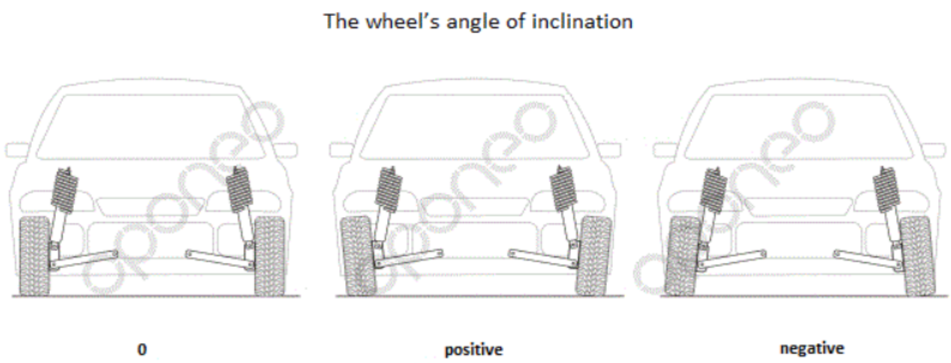


Figure 3.11: Camber angle examples

- Null *climb angle* γ_U:
  The climb angle describe the road inclination according the speed of the vehicle. It come as a consequence of the considered type of road, as for example the vehicle climbing a mountain or coming down from a descent. Assuming a null climb angle means that in this case the vehicle is moving on a flat road.

3.3 Dynamics

3.3.1 Masses Definition

As the vehicle as been arranged as the sum of two subsystem, the body and the under-vehicle, it is also important to define which are the masses that belongs two the two sub-system and where the centre of gravity are located.

The masses of the two sub-systems includes the following vehicle parts:

- Sprung mass (m_b): body and chassy

- Unsprung mass (m_u): wheels, brakes, hubs, axles, A-frames, springs and shock absorbers

The position of the centre of gravity of the two subsystem considering the position of the total centre of gravity is important also for the definition used at page 27.

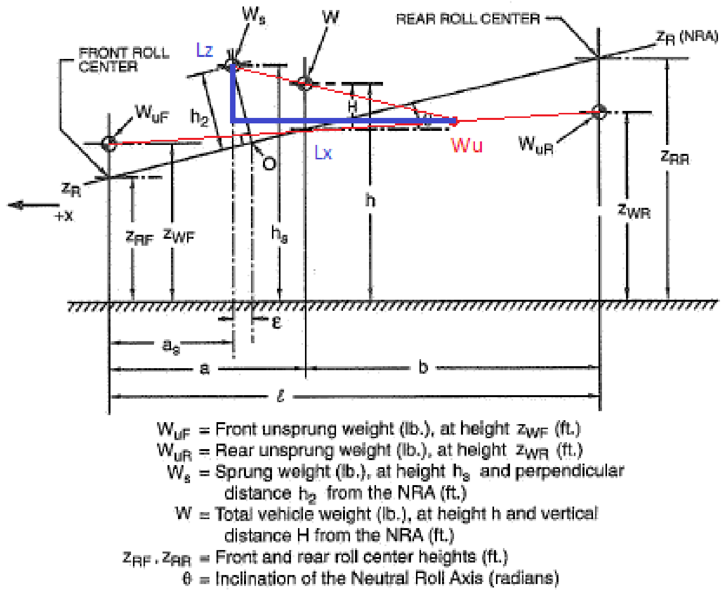


Figure 3.12: Weight distribution of the vehicle

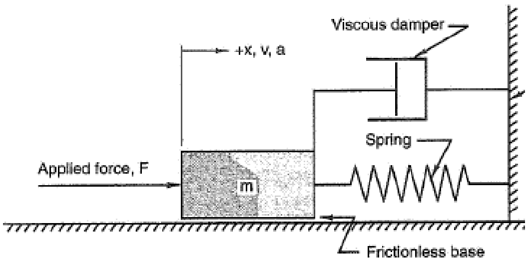
Through this image, it is possible to understand also where the dynamic constrains have been applied, from the schematic point of view.

3.3.2 Dynamic Constrains

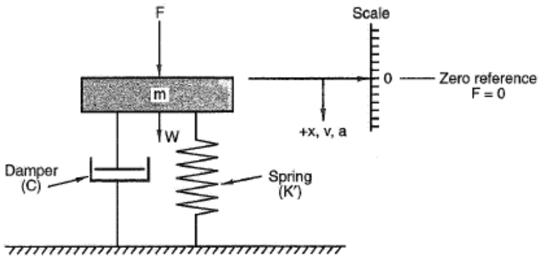
As it has been highlighted at page 20, some of the external forces are ideally applied on the body and others are applied to the under-vehicle. As a consequence, it is really important to describe how the forces applied to one system influence the other. For instance, if the vehicle is accelerating the under-vehicle is going to move faster, while the body is initially dragged by it.

The dynamical link between the two subsystem is defined by a system of both translational and rotational assembly of springs and dumpers that follows the same relative degree of freedom left from the kinematic constrains.

The forces and momenta provided by the dumper-spring system are the following:



(a) Horizontal system of spring and dumper



(b) Vertical system of spring and dumper

- Force along the z_U axis:

$$F_z = -k_z d_z - \beta_z \dot{d}_z \tag{3.21}$$

- Momentum along the x_U axis:

$$M_x = -k_{\phi_r}\phi_r - \beta_{\phi_r}\dot{\phi}_r \tag{3.22}$$

- Momentum along the y_U axis:

$$M_y = -k_{\theta_r}\theta_r - \beta_{\theta_r}\dot{\theta}_r \tag{3.23}$$

3.3.3 Kinetic Energy

The kinetic energy associated to the system is:

$$T = \frac{1}{2}m_u\, {}^i\dot{p}_u^\top\, {}^i\dot{p}_u + \frac{1}{2}m_b\, {}^i\dot{p}_b^\top\, {}^i\dot{p}_b + \frac{1}{2}\begin{bmatrix} \dot{\phi}_u & \dot{\theta}_u & \dot{\psi}_u \end{bmatrix}\begin{bmatrix} J_{x_u} & J_{xy_u} & J_{xz_u} \\ J_{xy_u} & J_{y_u} & J_{yz_u} \\ J_{xz_u} & J_{yz_u} & J_{z_u} \end{bmatrix}\begin{bmatrix} \dot{\phi}_u \\ \dot{\theta}_u \\ \dot{\psi}_u \end{bmatrix} +$$

$$+\frac{1}{2}\begin{bmatrix} \dot{\phi}_r & \dot{\theta}_r & \dot{\psi}_u \end{bmatrix}\begin{bmatrix} J_{x_b} & J_{xy_b} & J_{xz_b} \\ J_{xy_b} & J_{y_b} & J_{yz_b} \\ J_{xz_b} & J_{yz_b} & J_{z_b} \end{bmatrix}\begin{bmatrix} \dot{\phi}_r \\ \dot{\theta}_r \\ \dot{\psi}_u \end{bmatrix} \tag{3.24}$$

In this formula, it is necessary to substitute the expression of the body speed ${}^i\dot{p}_u$, with the one found at paragraph 3.2.2 as function of the under-vehicle, in order to consider the geometry of the system.

The contribution to the kinetic energies come from all the masses and inertias of the system according to the previous definition of the two sub-system.

3.3.4 Potential Energy

From the assumption given at paragraph 3.2.3, it is possible to deduce that the under-vehicle is always at ground level (zero level). Thus leading to the fact that there is not potential energy associated with the mass m_u, *i.e.* ${}^{i}p_u(3) \equiv 0$). As a consequence the potential energy depends only on m_b and the energy associated to the spring-dumper systems, according to the following formula:

$$K = \frac{1}{2}k_z d_z^2 + \frac{1}{2}k_{\phi_r}\phi_r^2 + \frac{1}{2}k_{\theta_r}\theta_r^2 + m_b g(L_z + d_z) \tag{3.25}$$

3.3.5 Work of Non-Conservative Forces

In this paragraph, it is going to be described the non-conservative forces that acts on the the vehicle. This is necessary in order to find the work of these forces, that will be used for the equalities in the Eulero-Lagrange equations, at paragraph 3.3.6.

Wheel Forces

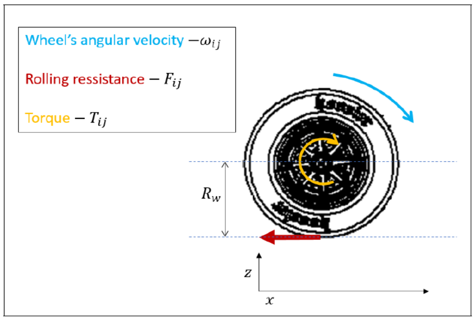


Figure 3.14: Free body diagram of a driven wheel represented in two dimension

In order to have the generation of forces between the pneumatic and the road, it is necessary to have a relevant slip in the zone where the contact happens. In this case it is necessary to have a null relative speed between the two parts. The slip takes into account the scenario where the deformation of the pneumatic compensate the difference of speed between the wheel and the road.

For this model the forces between the tyres and the ground on the three axis can be expressed all as function of $F^W_{W_i z}$. This is possible because the vertical force is the one that determines the load on the road depending on different coefficients.

The force produced by tires in contact with the ground can be expressed as follows:

$$F_{W_i}^{W} = F_{W_i z}^{W} \begin{bmatrix} \frac{\lambda_{L_i}}{\lambda_{T_i}} \mu\left(\lambda_{T_i}\right) + C_R\left(V_{ix}^{W_i}\right) \\ \frac{\lambda_{S_i}}{\lambda_{T_i}} \mu\left(\lambda_{T_i}\right) \\ 1 \end{bmatrix} \tag{3.26}$$

where the *total*, *longitudinal* and *side slip ratio* are defined by means of the following equations:

- λ_{L_i} *longitudinal slip ratio*

$$\lambda_{L_i} = \frac{\omega_i r - V_{ix}^{W}}{v_{max}} \tag{3.27}$$

- λ_{S_i} *side slip ratio*

$$\lambda_{S_i} = \frac{-V_{iy}^{W}}{v_{max}} \tag{3.28}$$

- λ_{T_i} *total slip ratio*

$$\lambda_{T_i} = \lambda_{L_i}^2 + \lambda_{S_i}^2 \tag{3.29}$$

Where the definition of v_{max} is given by:

$$v_{max} = \sqrt{\left(V_{iy}^{W}\right)^2 + \left(max\left\{|V_{ix}^{W} - \omega_i r|, |V_{ix}^{W}|, |\omega_i r|\right\}\right)} \tag{3.30}$$

while the wheel speed expressed in the wheel reference frame, V_i^W, depends on the vehicle speed according to the following rotation:

$$V_i^{W_i} =^{U} R_{W_i}(0, 0, \delta_{W_i}) V_i^U \tag{3.31}$$

while the definition of V_i^U is:

$$V_i^U = \omega_U^U \times r_i^U + {}^{U}R_N(0, 0, \beta_U) V_{O_B}^N \tag{3.32}$$

It is also necessary to show the behavior of the following coefficients through some graphs that are obtained with successive experiments, this is due to the fact that is not possible to describe it through a model because they depends on too many variable:

- $\mu\left(\lambda_{T_i}\right)$ *friction coefficient*

  This graph is actually strictly dependent on the road conditions, as it is possible to see in figure 3.16. From this graph it is possible to see how for certain condition the friction coefficient decreases a lot, hence leading to high difficulty in controlling the vehicle.

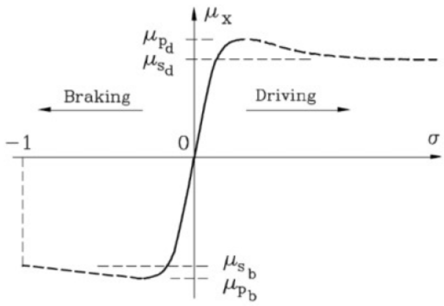


Figure 3.15: Longitudinal friction coefficient at various longitudinal slip

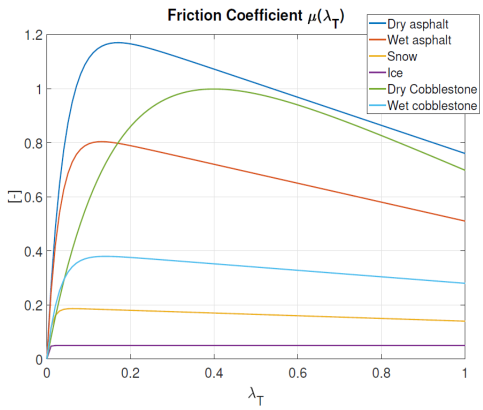


Figure 3.16: Longitudinal friction coefficient for various road conditions

- $C_R\left(V_{ix}^{W_i}\right)$ *rolling coefficient*

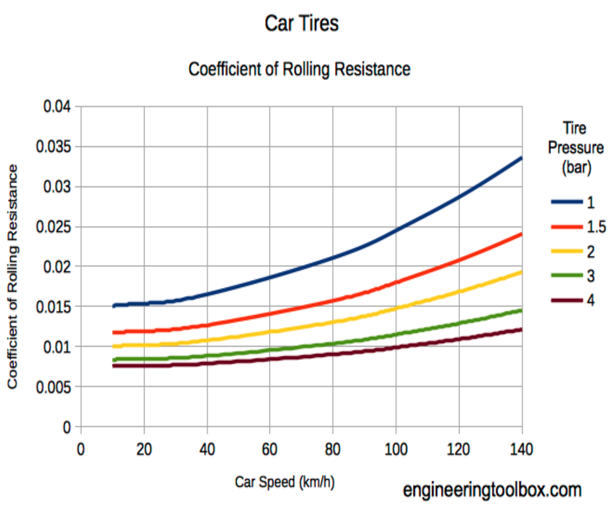


Figure 3.17: Rolling resistance as function of the vehicle speed

This parameter takes into account the penetration of the tyre and the road.

In figure 3.18, it is possible to see a schematic of the forces acting on the wheels.

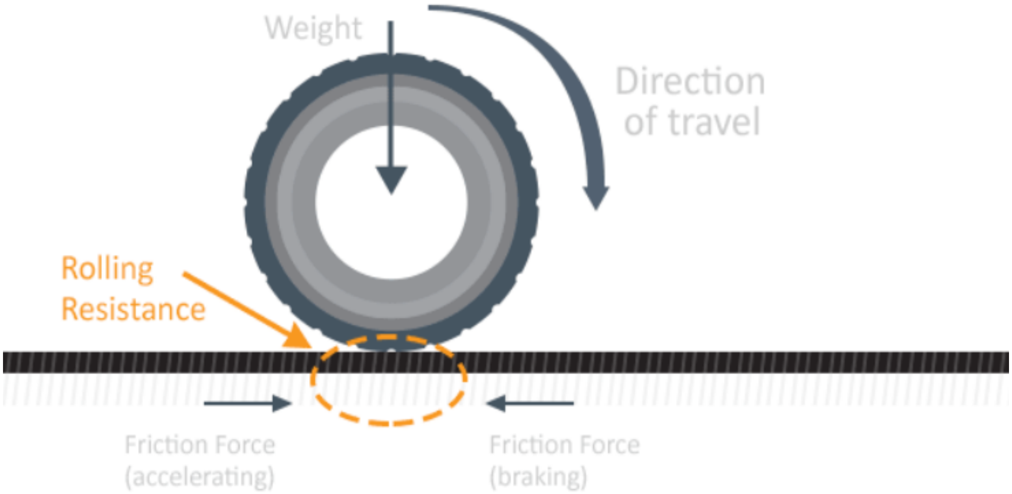


Figure 3.18: Wheel forces

Aerodynamic Forces

The aerodynamic forces in this cases is described in the body reference frame $\mathcal{F}_{\mathcal{B}}$, while the speed from which it depends at the beginning is defined in the inertial reference frame $\mathcal{F}_{\mathcal{B}}$.

Now the most important parameter to be defined is the *air relative speed*, that describes the speed of the vehicle taking into account the wind speed, as it show in the following formula:

$$V_a^I = V_{O_B}^I - W^I \tag{3.33}$$

where:

- $V_{O_B}^I$ is the *inertial vehicle speed*, considered as the speed of the centre of gravity;
- W^I is the *wind speed* measured in the inertial reference frame

It is necessary, at this point, to apply a reference transformation since the force needs to be described in the body reference frame. This is done in the following way:

$$V_a^B = {}^B R_I(\phi, \theta, \psi) V_a^I = {}^B R_I(\phi, \theta, \psi) \left(V_{O_B}^I - W^I \right) \tag{3.34}$$

It is necessary now to define two angles, starting from this velocity, in order to explain the dependence of the force on it.

The two angles are defined according to the following formulas:

- α is the *aerodynamic angle of attack*:

$$\alpha = \tan^{-1}\left(\frac{V_{a_x}^B}{\sqrt{\left(V_{a_x}^B\right)^2 + \left(V_{a_y}^B\right)^2}}\right) \tag{3.35}$$

  It describes the angle that the air relative speed has with the $x_b - y_b$ plane of the body reference frame

- β is the *aerodynamic side slip angle*:

$$\beta = \sin^{-1}\left(\frac{V_{a_y}^B}{\sqrt{\left(V_{a_x}^B\right)^2 + \left(V_{a_y}^B\right)^2}}\right) \tag{3.36}$$

  If the air relative speed is projected on the $x_b - y_b$ plane of the body reference frame, this is the angle that is present between the projection and the x_B axis.

where V_a^B is the following column vector:

$$V_a^B = \begin{bmatrix} V_{a_x}^B \\ V_{a_y}^B \\ V_{a_y}^B \end{bmatrix} \tag{3.37}$$

At this point it is possible to describe the relationship that link the aerodynamic forces, F_a^B expressed in the body reference frame $\mathcal{F}_B$.

The forces are function of the *impact pressure*, that is computed in the following way:

$$P = \frac{1}{2}\rho V_a^2 \tag{3.38}$$

This parameter has to be multiplied by the reference surface of the vehicle S and the non-dimensional coefficients C_X, C_Y and C_Z.

The formula describing this behaviour is:

$$F_a^B = \frac{1}{2}\rho V_a^2 \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} \tag{3.39}$$

where:

- $V_a = ||V_a^B|| = ||V_a^I||$ is the modulus of the air relative speed;
- ρ is the density of the air;
- S is the cross section of the vehicle on the plane defined in the body reference frame by O_B, y_B and z_B.

Figure 3.19: Drag force on a vehicle

Furthermore it is necessary to describe with parameters influences the aerodynamic coefficients, C_X, C_Y and C_Z. These are highly non-linear coefficient that depends on the following parameters:

- α: *aerodynamic attack angle*
- β: *aerodynamic side-slip angle*
- d: distance from the centre of gravity O_B to its projection to re road
- μ_R: road inclination along the x axis with respect to the direction of motion
- γ_R: road inclination along the y axis with respect to the direction of motion
- χ: road parameter of the z inclination

Given these parameters the coefficients are:

$$\begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} = \begin{bmatrix} C_X(\alpha, \beta, d, \mu_R, \gamma_R, \chi) \\ C_Y(\alpha, \beta, d, \mu_R, \gamma_R, \chi) \\ C_Z(\alpha, \beta, d, \mu_R, \gamma_R, \chi) \end{bmatrix} \tag{3.40}$$

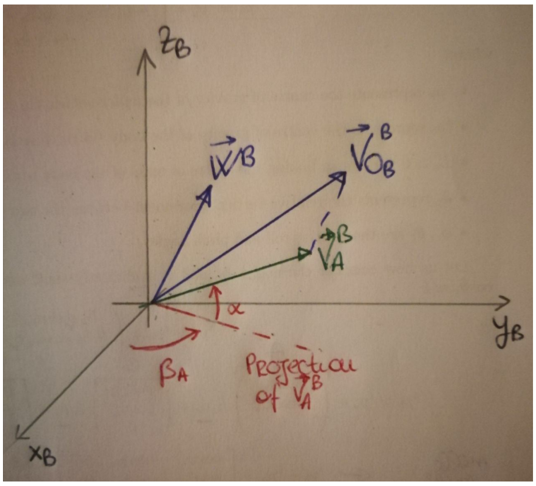


Figure 3.20: Graphical representation of the angles and speeds previously described

3.3.6 Eulero-Lagrange Equation

First of all, it is necessary to define the **Lagrangian function** L, as function of the kinetic and potential energy of the system. This is done according to the definition of the solution of the problem that can be found in the mathematical introduction at page 14.

$$L = K - T \tag{3.41}$$

$$\begin{gathered}
L = \frac{1}{2}k_z d_z^2 + \frac{1}{2}k_{\phi_r}\phi_r^2 + \frac{1}{2}k_{\theta_r}\theta_r^2 + m_b g(L_z + d_z) - \\
- \left\{ \frac{1}{2}m_u\, {}^i\dot{p}_u^\top\, {}^i\dot{p}_u + \frac{1}{2}m_b\, {}^i\dot{p}_b^\top\, {}^i\dot{p}_b + \frac{1}{2}\begin{bmatrix} \dot{\phi}_u & \dot{\theta}_u & \dot{\psi}_u \end{bmatrix} \begin{bmatrix} J_{x_u} & J_{xy_u} & J_{xz_u} \\ J_{xy_u} & J_{y_u} & J_{yz_u} \\ J_{xz_u} & J_{yz_u} & J_{z_u} \end{bmatrix} \begin{bmatrix} \dot{\phi}_u \\ \dot{\theta}_u \\ \dot{\psi}_u \end{bmatrix} + \right. \\
+ \frac{1}{2}\begin{bmatrix} \dot{\phi}_r & \dot{\theta}_r & \dot{\psi}_u \end{bmatrix} \begin{bmatrix} J_{x_b} & J_{xy_b} & J_{xz_b} \\ J_{xy_b} & J_{y_b} & J_{yz_b} \\ J_{xz_b} & J_{yz_b} & J_{z_b} \end{bmatrix} \begin{bmatrix} \dot{\phi}_r \\ \dot{\theta}_r \\ \dot{\psi}_u \end{bmatrix}
\end{gathered} \tag{3.42}$$

With the substitution of and all the specific terms that can be found at paragraph 3.2.2 and in appendix A.8:

$$
\begin{aligned}
{}^{i}\dot{p}_b \; = \; & {}^{i}\dot{p}_u + \dot{A}\,(d_z - L_x\,\theta_r + L_z) + A\left(\dot{d}_z - L_x\,\dot{\theta}_r\right) + \\
& + \dot{B}\,(L_x + L_z\,\theta_r) + B\dot{\theta}_r + \dot{C}L\,(L_z\,\phi_r) + C\dot{\phi}_r
\end{aligned}
\tag{3.43}
$$

Below, it is possible to find all the Eulero-Lagrange equations one for each variable the appears in the *lagrangian function L*. The external forces related to each variable can be found considering zero all the other variables, with except of the considered one, and taking into account all the forces and torques that influence that variable.

- ${}^{i}p_u$:

$$
\frac{\partial L}{\partial\,{}^{i}p_u} - \frac{\partial}{\partial t}\frac{\partial L}{\partial\,{}^{i}\dot{p}_u} = (m_u + m_b)\,{}^{i}\ddot{p}_u +
$$

$$
+ m_b\left[AL_z\ddot{\phi}_r + BL_z\ddot{\theta}_r + C\left(\ddot{d}_z - L_x\ddot{\theta}_r\right) + DL_x\ddot{\psi}_u\right] = \sum {}^{i}F_w + {}^{i}F_a \tag{3.44}
$$

$$
({}^{i}\ddot{p}_u(3) \equiv 0)
$$

- d_z:

$$
\begin{aligned}
\frac{\partial L}{\partial d_z} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{d}_z} & = k_z d_z + m_b g + m_b\,{}^{i}\ddot{p}_u^{\top} C + m_b\ddot{d}_z = \\
& = e_z^{\top} R_2(-\theta_r) R_1(-\phi_r)\,{}^{b}F_a - \beta_z\dot{d}_z
\end{aligned}
\tag{3.45}
$$

$$
({}^{i}\ddot{p}_u^{\top} C \equiv 0)
$$

- ϕ_r:

$$
\frac{\partial L}{\partial \phi_r} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\phi}_r} = k_{\phi_r}\phi_r + m_b\,{}^{i}\ddot{p}_u^{\top} L_z A + \begin{bmatrix} \ddot{\phi}_r & \ddot{\theta}_r & \ddot{\psi}_u \end{bmatrix} \begin{bmatrix} m_b L_{z_b}^2 + J_{x_b} \\ J_{xy_b} \\ J_{xz_b} \end{bmatrix} =
$$

$$
= -\beta_{\phi_r}\dot{\phi}_r + e_x^{\top}({}^{b}F_a \times {}^{b}L) \tag{3.46}
$$

- θ_r:

$$\frac{\partial L}{\partial \theta_r} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}_r} = k_{\theta_r}\theta_r + m_b\, {}^i\ddot{p}_u^\top L_z B - m_b\, {}^i\ddot{p}_u^\top L_x C + \frac{1}{2} m_b\, A^\top D L_{x_b} L_{z_b} \ddot{\psi}_u +$$

$$+ \begin{bmatrix} \ddot{\phi}_r & \ddot{\theta}_r & \ddot{\psi}_u \end{bmatrix} \begin{bmatrix} J_{xy_b} \\ m_b(L_{z_b}^2 + L_{x_b}^2) + J_{y_b} \\ J_{yz_b} \end{bmatrix} = -\beta_{\theta_r}\dot{\theta}_r + e_y^\top R_1(-\phi_r)({}^bF_a \times {}^bL)$$

$$({}^i\ddot{p}_u^\top C \equiv 0\; A^\top D \equiv -1) \tag{3.47}$$

- ψ_u:

$$\frac{\partial L}{\partial \psi_u} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\psi}_u} = m_b\, {}^i\ddot{p}_u^\top L_x D + \frac{1}{2}\, m_b\, A^\top D L_{x_b} L_{z_b} \ddot{\theta}_r +$$

$$+ \begin{bmatrix} \ddot{\phi}_r & \ddot{\theta}_r & \ddot{\psi}_u \end{bmatrix} \begin{bmatrix} J_{xz_b} \\ J_{yz_b} \\ J_{z_b} + J_{z_u} + m_b L_{x_b}^2 \end{bmatrix} = \tag{3.48}$$

$$e_z^\top \left[\sum {}^u\tau_w + R_2(-\theta_r)R_1(-\phi_r)({}^bF_a \times {}^bL)\right]$$

$$(A^\top D \equiv -1)$$

- ϕ_u:

$$\frac{\partial L}{\partial \phi_u} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\phi}_u} = J_{xz_b}\, \ddot{\psi}_u + {L_z}^2\, m_b\, \ddot{\phi}_r - m_b\, \cos\psi_u\, d_z\, \ddot{p}_{u_y} + L_z\, m_b\, d_z\, \ddot{\phi}_r -$$

$$+ L_x\, m_b\, d_z\, \ddot{\psi}_u + + m_b\, \sin\psi_u\, d_z\, \ddot{p}_{u_x} - L_z\, m_b\, \cos\psi_u\, \ddot{p}_{u_y} - L_x\, L_z\, m_b\, \ddot{\psi}_u +$$

$$+ L_z\, m_b\, \sin\psi_u\, \ddot{p}_{u_x} - 2{L_z}^2\, m_b\, \dot{\theta}_r\, \dot{\psi}_u - 2L_z\, m_b\, d_z\, \dot{\theta}_r\, \dot{\psi}_u = 0 \tag{3.49}$$

- θ_u:

$$\frac{\partial L}{\partial \theta_u} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}_u} = J_{yz_b}\,\ddot{\psi}_u - L_x\,m_b\,\ddot{d}_z + {L_x}^2\,m_b\,\ddot{\theta}_r + {L_z}^2\,m_b\,\ddot{\theta}_r - L_x\,m_b\,d_z\,\dot{\psi}_u^2$$

$$+m_b\,\cos\psi_u\,d_z\,\ddot{p}_{u_x} + L_z\,m_b\,d_z\,\ddot{\theta}_r + m_b\,\sin\psi_u\,d_z\,\ddot{p}_{u_y} +$$

$$-L_x\,L_z\,m_b\,\dot{\psi}_u^2 + L_z\,m_b\,\cos\psi_u\,\ddot{p}_{u_x} + L_z\,m_b\,\sin\psi_u\,\ddot{p}_{u_y} + 2\,{L_z}^2\,m_b\,\dot{\phi}_r\,\dot{\psi}_u +$$

$$+2\,L_z\,m_b\,d_z\,\dot{\phi}_r\,\dot{\psi}_u + L_x\,L_z\,m_b\,\sin\psi_u\,\ddot{\phi}_r + 2L_x\,L_z\,m_b\,\cos\psi_u\,\dot{\psi}_u\,\dot{\phi}_r = 0 \tag{3.50}$$

It is necessary to specify that for the aerodynamic forces, some de-rotations are applied to be coherent with the main reference system. As it is possible to see from the equation 3.39, the aerodynamic forces, as a consequence also the torques, are expressed in the *body reference frame*, while all the Euler-Lagrange equations have been calculated in the *under-vehicle reference frame*. It is also necessary to perform a selection of the forces on a specific axis, because the specific *Lagrangian variables* are not influenced by the forces along the three axis.

It is possible to find the the definition of these de-rotation and selections in the appendix A.9.

In order to solve this complex system of second order differential equations, it is useful to write it in the matrix form. At this point, all the symbolic matrix that have been substituted, to have easier computation, are going to be written in the original form.

$$\begin{bmatrix} m_b + m_u & 0 & L_z m_b \sin\psi_u & L_z m_b \cos\psi_u & -L_x m_b \sin\psi_u & 0 \\ 0 & m_b + m_u & -L_z m_b \cos\psi_u & L_z m_b \sin\psi_u & L_x m_b \cos\psi_u & 0 \\ 0 & 0 & 0 & 0 & 0 & m_b \\ L_z m_b \sin\psi_u & -L_z m_b \cos\psi_u & m_b L_z^2 + J_{x_b} & J_{xy_b} & J_{xz_b} & 0 \\ L_z m_b \cos\psi_u & L_z m_b \sin\psi_u & J_{xy_b} & m_b(L_{z_b}^2 + L_{x_b}^2) + J_{y_b} & J_{yz_b} - \frac{1}{2} m_b L_{x_b} L_{z_b} & 0 \\ -L_x m_b \sin\psi_u & L_x m_b \cos\psi_u & J_{xz_b} & J_{yz_b} - \frac{1}{2} m_b L_{x_b} L_{z_b} & J_{z_b} + J_{z_u} + m_b L_{x_b}^2 & 0 \\ d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & -d_z m_b \cos\psi_u - L_z m_b \cos\psi_u & m_b L_z^2 + m_b L_z d_z & 0 & J_{xz_b} - L_x L_z d_z & 0 \\ d_z m_b \cos\psi_u + L_z m_b \cos\psi_u & d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & L_x L_z m_b \sin\psi_u & m_b(L_{z_b}^2 + L_{x_b}^2) + L_z m_b d_z & J_{yz_b} & -L_x m_b \end{bmatrix}$$

$$\begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \\ \ddot{\phi}_r \\ \ddot{\theta}_r \\ \ddot{\psi}_u \\ \ddot{d}_z \end{bmatrix} = \begin{bmatrix} \sum {}^iF_{w_x} + {}^iF_{a_x} \\ \sum {}^iF_{w_y} + {}^iF_{a_y} \\ {}^uF_{a_z} - \beta_z\,\dot{d}_z - k_z\,d_z - m_b\,g \\ e_x^\top ({}^bF_a \times {}^bL) - \beta_{\phi_r}\,\dot{\phi}_r - k_{\phi_r}\phi_r \\ e_y^\top R_1(-\phi_r)({}^bF_a \times {}^bL) - \beta_{\theta_r}\,\dot{\theta}_r - k_{\theta_r}\,\theta_r \\ e_z^\top \left[R_2(-\theta_r)R_1(-\phi_r)({}^bF_a \times {}^bL) + \sum^u \tau_w\right] \\ 2{L_z}^2\,m_b\,\dot{\theta}_r\,\dot{\psi}_u + 2L_z\,m_b\,d_z\,\dot{\theta}_r\,\dot{\psi}_u \\ L_x\,m_b\,d_z\,\dot{\psi}_u^2 + L_x\,L_z\,m_b\,\dot{\psi}_u^2 - 2\,{L_z}^2\,m_b\,\dot{\phi}_r\,\dot{\psi}_u - 2\,L_z\,m_b\,d_z\,\dot{\phi}_r\,\dot{\psi}_u - 2L_x\,L_z\,m_b\,\cos\psi_u\,\dot{\psi}_u\,\dot{\phi}_r \end{bmatrix} \tag{3.51}$$

Considering that the equation in $\ddot{d}_z$ is independent from the others, it is possible to take it out from the system and it solution is:

$$\ddot{d}_z = \frac{{}^{u}F_{a_z} - \beta_z\,\dot{d}_z - k_z\,d_z - g\,m_b}{m_b} \tag{3.52}$$

This equation gives the translational vertical dynamics of the vehicle.

In this phase, it is possible to assume that the vehicle that under study is symmetrical along the x axes, so the left and the right side are going to be considered identical. So the terms J_{xy_b} and J_{xy_u} are null.

- $J_{xy_b} = 0$
- $J_{xy_u} = 0$

$$\begin{bmatrix} m_b + m_u & 0 & L_z m_b \sin\psi_u & L_z m_b \cos\psi_u & -L_x m_b \sin\psi_u & 0 \\ 0 & m_b + m_u & -L_z m_b \cos\psi_u & L_z m_b \sin\psi_u & L_x m_b \cos\psi_u & 0 \\ 0 & 0 & 0 & 0 & 0 & m_b \\ L_z m_b \sin\psi_u & -L_z m_b \cos\psi_u & m_b L_z^2 + J_{x_b} & 0 & J_{xz_b} & 0 \\ L_z m_b \cos\psi_u & L_z m_b \sin\psi_u & 0 & m_b(L_{z_b}^2 + L_{x_b}^2) + J_{y_b} & J_{yz_b} - \frac{1}{2} m_b L_{x_b} L_{z_b} & 0 \\ -L_x m_b \sin\psi_u & L_x m_b \cos\psi_u & J_{xz_b} & J_{yz_b} - \frac{1}{2} m_b L_{x_b} L_{z_b} & J_{z_b} + J_{z_u} + m_b L_{x_b}^2 & 0 \\ d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & -d_z m_b \cos\psi_u - L_z m_b \cos\psi_u & m_b L_z^2 + m_b L_z d_z & 0 & J_{xz_b} - L_x L_z d_z & 0 \\ d_z m_b \cos\psi_u + L_z m_b \cos\psi_u & d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & L_x L_z m_b \sin\psi_u & m_b(L_{z_b}^2 + L_{x_b}^2) + L_z m_b d_z & J_{yz_b} & -L_x m_b \end{bmatrix}$$

$$\begin{bmatrix} i\ddot{p}_u(1) \\ {}^{i}\ddot{p}_u(2) \\ \ddot{\phi}_r \\ \ddot{\theta}_r \\ \ddot{\psi}_u \\ \ddot{d}_z \end{bmatrix} = \begin{bmatrix} \sum {}^{i}F_{w_x} + {}^{i}F_{a_x} \\ \sum {}^{i}F_{w_y} + {}^{i}F_{a_y} \\ {}^{u}F_{a_z} - \beta_z\,\dot{d}_z - k_z\,d_z - m_b\,g \\ e_x^\top({}^{b}F_a \times {}^{b}L) - \beta_{\phi_r}\,\dot{\phi}_r - k_{\phi_r}\phi_r \\ e_y^\top R_1(-\phi_r)({}^{b}F_a \times {}^{b}L) - \beta_{\theta_r}\,\dot{\theta}_r - k_{\theta_r}\,\theta_r \\ e_z^\top\left[R_2(-\theta_r)R_1(-\phi_r)({}^{b}F_a \times {}^{b}L) + \sum^{u}\tau_w\right] \\ 2{L_z}^2\,m_b\,\dot{\theta}_r\,\dot{\psi}_u + 2L_z\,m_b\,d_z\,\dot{\theta}_r\,\dot{\psi}_u \\ L_x\,m_b\,d_z\,\dot{\psi}_u^2 + L_x\,L_z\,m_b\,\dot{\psi}_u^2 - 2\,{L_z}^2\,m_b\,\dot{\phi}_r\,\dot{\psi}_u - 2\,L_z\,m_b\,d_z\,\dot{\phi}_r\,\dot{\psi}_u - 2L_x\,L_z\,m_b\,\cos\psi_u\,\dot{\psi}_u\,\dot{\phi}_r \end{bmatrix} \tag{3.53}$$

This is the complete model that will be studied the next chapter for the equilibrium conditions. All these symbolic expressions, for each terms, are going to be substituted with the numerical values considering the real characteristics of the vehicle on which it will be necessary to perform some dynamics simulations.

Chapter 4

Equilibrium Analysis

In this chapter, it will be analyzed the behaviour of the proposed non-linear system under the equilibrium conditions. In order to do so, it is necessary to consider some simplifying assumption to describe in which kind of situation the system is going to be analyzed.

4.1 Definition of the equilibrium condition

First of all, it is necessary to describe the trajectory along which the analysis will be performed. It has been assumed that the car is on a point of equilibrium while performing an ideal turn, this means that the trajectory is perfectly round and the speed of the vehicle is constant during the entire route.

- $R = const$
- $V_{O_B} = const$

There is also an additional kinematic condition deriving from this ideal situation that is preferable to achieve. he β_U angle is constant, this is the angle between the bow of the vehicle and the tangent to the curve, as it has been defined at paragraph 3.1. As a consequence, of this kinematic condition, the first order dynamic of ψ_u is constant and it depends only from the tangential speed V_{tg} and the radius of the turn R.

Considering the fact that the analysis is done in static conditions, this means, together with the previous consideration, that the relative angles $[\phi_r \, \theta_r \, \beta]$, belonging to the system, are constant. Furthermore, in this situation it is considered that the suspension are in rest position, thus lead actually to null relative angles, $[\phi_r \, \theta_r]$.

Taking into account all these considerations, it is possible to gather that:

1. $\dot{\psi}_u$:

$$\dot{\psi}_u = const(K) \Rightarrow \ddot{\psi}_u = 0 \tag{4.1}$$

2. ϕ_r:

$$\phi_r \Rightarrow \dot{\phi}_r = 0 \Rightarrow \ddot{\phi}_r = 0 \tag{4.2}$$

3. θ_r:

$$\theta_r = 0 \Rightarrow \dot{\theta}_r = 0 \Rightarrow \ddot{\theta}_r = 0 \tag{4.3}$$

Meanwhile, the formal definition of the side-slip angle as function of the vehicle speed is:

$$\beta = \frac{{}^{u}V_{o_x}}{{}^{u}V_{o_y}} \tag{4.4}$$

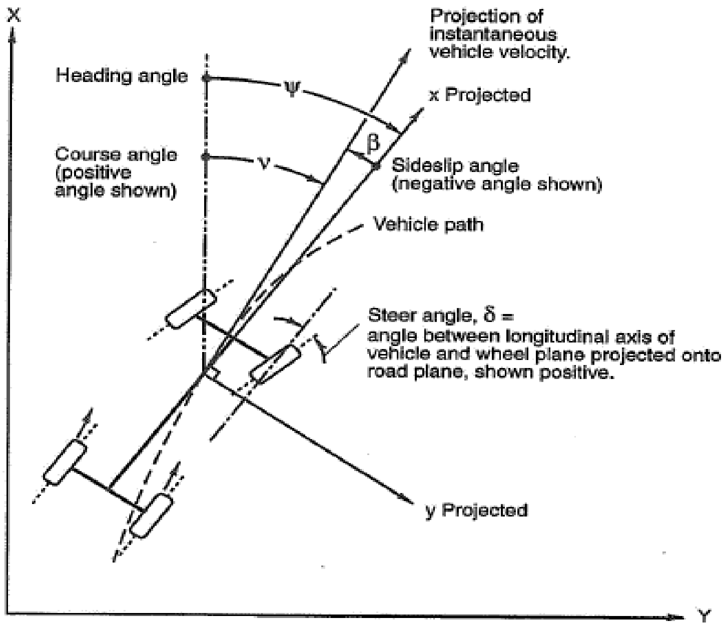


Figure 4.1: Graphical representation of the side-slip angles

It is also necessary to consider the degree of freedom along the vertical axis. Indeed, in static condition, since it has been assumed that the wheels don't leave the ground floor, it means that the dynamic along this axis is null.

Which means:

$$d_z = const(D) \Rightarrow \dot{d}_z = 0 \Rightarrow \ddot{d}_z = 0 \tag{4.5}$$

At this point it is necessary to describe the acceleration in inertial axis, ${}^{i}\ddot{p}_u$, such that the vehicle keeps the trajectory and it remains on the circle. It is important to know the modulus of the acceleration because actually the direction is known and it is along the radius that sweeps the turn.

In this conditions, also the modulus of the acceleration depends only on the tangential speed V_{tg} and the radius of the turn R, according to the following formula:

$$|{}^{i}\ddot{p}_{u}| = \frac{V_t^2}{R} \tag{4.6}$$

This is valid only in case of constant speed and radius.

In inertial axis, the two components are:

$${}^{i}\ddot{p}_{u_x} = -\frac{V_t^2}{R}\cos\alpha \tag{4.7}$$

$${}^{i}\ddot{p}_{u_y} = -\frac{V_t^2}{R}\sin\alpha \tag{4.8}$$

Where α is a generic angle that indicates at which point of the curvilinear trajectory the vehicle is arrived. It is a generic angle considered during the rotation taking into account the position of the vehicle with respect to the centre of the circle, so it is function of time. While considering this representation β is the angle of projection of the acceleration.

While in tangential and radial axes, they are:

$$\ddot{p}_{u_t} = 0 \tag{4.9}$$

$$\ddot{p}_{u_r} = \frac{V_t^2}{R} \tag{4.10}$$

While considering this representation, in the tangential-radial plane, the side-slip angle β is the angle of projection of the acceleration in the inertial reference frame.

It is necessary now to express the relationship between the acceleration decomposed along the inertial axis and it expression along the tangential and radial axes. To do so, it is necessary first to pass through the representation in the under-vehicle reference frame. It is possible to perform two consecutive rotations first of angle β and than of angle ψ_u, according to the following matrices:

$${}^{u}R_t(\beta) = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.11}$$

$$
{}^{i}R_u\left(\psi_u\right) = \begin{bmatrix} \cos\psi_u & -\sin\psi_u & 0 \\ \sin\psi_u & \cos\psi_u & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.12}
$$

In conclusion, the relationship between the acceleration expressed in the inertial reference frame and the one expressed in the radial-tangential plane is:

$$
{}^{i}\ddot{p}_{u_{xy}} = {}^{i}R_u\left(\psi_u\right)\,{}^{u}R_t\left(\beta\right)\,\ddot{p}_{u_{tr}} \tag{4.13}
$$

A graphical representation of all these consideration can be found in figure 5.1.

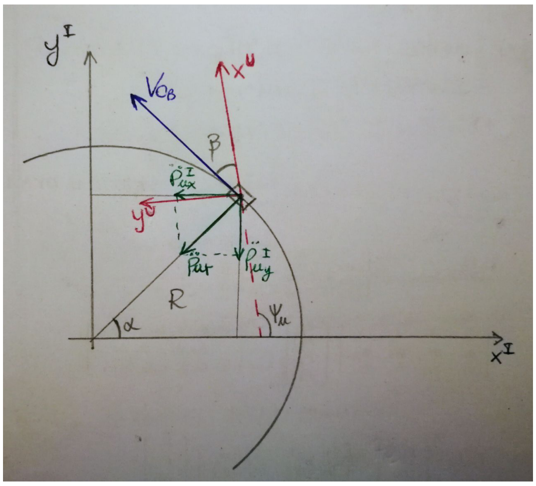


Figure 4.2: Graphical representation of the vehicle accelerations according to two different reference frame

4.2 Complete model at the equilibrium conditions

Now it is necessary to go back and consider again the complete model, that can be found in the equation 3.3.6 and gather together all the equilibrium assumptions:

- $\dot{\psi}_u = const(K) \Rightarrow \ddot{\psi}_u = 0$
- $\phi_r = \theta_r = 0 \Rightarrow \dot{\phi}_r = \dot{\theta}_r = 0 \Rightarrow \ddot{\phi}_r = \ddot{\theta}_r = 0$
- $d_z = const(D) \Rightarrow \dot{d}_z = 0 \Rightarrow \ddot{d}_z = 0$

Thus leading to the following simplified set of equations:

$$\begin{bmatrix} m_b + m_u & 0 & L_z m_b \sin\psi_u & L_z m_b \cos\psi_u & -L_x m_b \sin\psi_u & 0 \\ 0 & m_b + m_u & -L_z m_b \cos\psi_u & L_z m_b \sin\psi_u & L_x m_b \cos\psi_u & 0 \\ 0 & 0 & 0 & 0 & 0 & m_b \\ L_z m_b \sin\psi_u & -L_z m_b \cos\psi_u & m_b L_z^2 + J_{x_b} & 0 & J_{xz_b} & 0 \\ L_z m_b \cos\psi_u & L_z m_b \sin\psi_u & 0 & m_b(L_{z_b}^2 + L_{x_b}^2) + J_{y_b} & J_{yz_b} - \frac{1}{2} m_b L_{x_b} L_{z_b} & 0 \\ -L_x m_b \sin\psi_u & L_x m_b \cos\psi_u & J_{xz_b} & J_{yz_b} - \frac{1}{2} m_b L_{x_b} L_{z_b} & J_{z_b} + J_{z_u} + m_b L_{x_b}^2 & 0 \\ d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & -d_z m_b \cos\psi_u - L_z m_b \cos\psi_u & m_b L_z^2 + m_b L_z d_z & 0 & J_{xz_b} - L_x L_z d_z & 0 \\ d_z m_b \cos\psi_u + L_z m_b \cos\psi_u & d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & L_x L_z m_b \sin\psi_u & m_b(L_{z_b}^2 + L_{x_b}^2) + L_z m_b d_z & J_{yz_b} & -L_x m_b \end{bmatrix}$$

$$\begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sum {}^iF_{w_x} + {}^iF_{a_x} \\ \sum {}^iF_{w_y} + {}^iF_{a_y} \\ {}^uF_{a_z} - k_z\, d_z - m_b\, g \\ e_x^\top ({}^bF_a \times {}^bL) \\ e_y^\top R_1(-\phi_r)({}^bF_a \times {}^bL) \\ e_z^\top \left[R_2(-\theta_r) R_1(-\phi_r)({}^bF_a \times {}^bL) + \sum {}^u\tau_w\right] \\ 0 \\ L_x\, m_b\, d_z\, \dot{\psi}_u^2 + L_x\, L_z\, m_b\, \dot{\psi}_u^2 \end{bmatrix} \tag{4.14}$$

Furthermore, eliminating from the first matrix all the terms which are multiplied by zero, it is possible to obtain the following simplified system:

$$\begin{bmatrix} m_b + m_u & 0 \\ 0 & m_b + m_u \\ 0 & 0 \\ L_z m_b \sin\psi_u & -L_z m_b \cos\psi_u \\ L_z m_b \cos\psi_u & L_z m_b \sin\psi_u \\ -L_x m_b \sin\psi_u & L_x m_b \cos\psi_u \\ d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & -d_z m_b \cos\psi_u - L_z m_b \cos\psi_u \\ d_z m_b \cos\psi_u + L_z m_b \cos\psi_u & d_z m_b \sin\psi_u + L_z m_b \sin\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} = \begin{bmatrix} \sum {}^iF_{w_x} + {}^iF_{a_x} \\ \sum {}^iF_{w_y} + {}^iF_{a_y} \\ {}^uF_{a_z} - k_z\, d_z - m_b\, g \\ e_x^\top ({}^bF_a \times {}^bL) \\ e_y^\top R_1(-\phi_r)({}^bF_a \times {}^bL) \\ e_z^\top \left[R_2(-\theta_r) R_1(-\phi_r)({}^bF_a \times {}^bL) + \sum {}^u\tau_w\right] \\ 0 \\ L_x\, m_b\, d_z\, \dot{\psi}_u^2 + L_x\, L_z\, m_b\, \dot{\psi}_u^2 \end{bmatrix} \tag{4.15}$$

Considering this simplified version of the set of equation necessary to describe the behaviour of the system, it is possible to divide it, in order to easily study the single contributions to the overall vehicle dynamics.

The set of the divided equations are:

$$\begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} = \frac{1}{m_b + m_u} \left(\begin{bmatrix} \sum {}^iF_{w_x} + {}^iF_{a_x} \\ \sum {}^iF_{w_y} + {}^iF_{a_y} \end{bmatrix} \right) \tag{4.16}$$

$${}^iF_{a_z} - k_z\, d_z - g\, m_b = 0 \tag{4.17}$$

$$\begin{bmatrix} L_z m_b \sin\psi_u & -L_z m_b \cos\psi_u \\ L_z m_b \cos\psi_u & L_z m_b \sin\psi_u \\ -L_x m_b \sin\psi_u & L_x m_b \cos\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} = \begin{bmatrix} e_x^\top ({}^bF_a \times {}^bL) \\ e_y^\top R_1(-\phi_r)({}^bF_a \times {}^bL) \\ e_z^\top R_2(-\theta_r) R_1(-\phi_r)({}^bF_a \times {}^bL) + \sum {}^u\tau_w \end{bmatrix} \tag{4.18}$$

$$\begin{bmatrix} d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & -d_z m_b \cos\psi_u - L_z m_b \cos\psi_u \\ d_z m_b \cos\psi_u + L_z m_b \cos\psi_u & d_z m_b \sin\psi_u + L_z m_b \sin\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ L_x\, m_b\, d_z\, \dot{\psi}_u^2 + L_x\, L_z\, m_b\, \dot{\psi}_u^2 \end{bmatrix} \tag{4.19}$$

Now it is possible to study individually the behaviour of the different equation, taking into account all the necessary substitutions and simplifications.

Starting from the equations 4.16:

$$\begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} = \frac{1}{m_b + m_u} \left(\begin{bmatrix} \sum {}^iF_{w_x} + {}^iF_{a_x} \\ \sum {}^iF_{w_y} + {}^iF_{a_y} \end{bmatrix} \right) \tag{4.20}$$

Where the forces of the wheels and the aerodynamic forces needs to be expressed in the inertial reference frame, while they have been defined in the paragraph 3.3.5 in a different one, in the wheel reference frame and the body reference frame respectively. So they need to be rotated according to the following formulas:

$${}^iF_a = {}^iR_u\, {}^uR_b\, {}^bF_a \tag{4.21}$$

$${}^iF_w = {}^iR_u\, {}^uR_{w_i}\, {}^{w_i}F_{w_i} \tag{4.22}$$

It is also necessary to substitute in the previous equation, 4.16, the relationship coming from the formula 4.13.

$${}^iR_u\,(\psi_u)\ {}^uR_t\,(\beta)\ \ddot{p}_{utr} = \frac{1}{m_b + m_u} \left(\begin{bmatrix} \sum {}^iF_{w_x} + {}^iF_{a_x} \\ \sum {}^iF_{w_y} + {}^iF_{a_y} \end{bmatrix} \right) \tag{4.23}$$

It is possible to see that there is the same rotation matrix on both side, between the inertial reference frame and the under-vehicle, that it means that it is possible to simplify it. As a consequence it remains only the the one in β that is the one of interest.

$${}^uR_t\,(\beta)\ \ddot{p}_{utr} = \frac{1}{m_b + m_u} \left(\begin{bmatrix} \sum {}^uF_{w_x} + {}^uF_{a_x} \\ \sum {}^uF_{w_y} + {}^uF_{a_y} \end{bmatrix} \right) \tag{4.24}$$

where all the terms are expressed in the under-vehicle reference frame.

At this point it is necessary to isolate the variable of interest, that are the wheel forces, arriving to the following formula:

$$\sum {}^{u}F_{w_i} = (m_b + m_u)\, {}^{u}R_t\,(\beta)\; \ddot{p}_{u_{tr}} - {}^{u}F_a \tag{4.25}$$

This equation tells how the forces on the wheel varies as function of the side slip angle. These are centripetal forces that have the same direction of the acceleration of the vehicle projected with β.

Afterwards, it is necessary to analyze the third set of equations, 4.26:

$$\begin{bmatrix} L_z m_b \sin\psi_u & -L_z m_b \cos\psi_u \\ L_z m_b \cos\psi_u & L_z m_b \sin\psi_u \\ -L_x m_b \sin\psi_u & L_x m_b \cos\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^{i}\ddot{p}_u(2) \end{bmatrix} = \begin{bmatrix} e_x^{\top}({}^{b}F_a \times {}^{b}L) \\ e_y^{\top} R_1(-\phi_r)({}^{b}F_a \times {}^{b}L) \\ e_z^{\top} R_2(-\theta_r) R_1(-\phi_r)({}^{b}F_a \times {}^{b}L) + \sum {}^{u}\tau_w \end{bmatrix} \tag{4.26}$$

In this case, it is necessary to substitute the accelerations with the same relationship used before, the one coming from the equation 4.13. Considering the matrix that pre-multiplies the acceleration now is multiplied by the rotational matrix ${}^{i}R_u\,(\psi_u)$, a lot of simplifications come as a consequence.

It is also necessary to take into account, on the right side of the equation, which are the cross product between the aerodynamic forces and there own arms, together with all the de-rotation matrices and the selection vectors, that have been defined in the appendix A.9. All these consideration together leads to the following simplified relationship:

$$\begin{bmatrix} 0 & -L_z m_b & 0 \\ L_z m_b & 0 & 0 \\ 0 & L_x m_b & 0 \end{bmatrix} {}^{u}R_t\,(\beta)\; \ddot{p}_{u_{tr}} = \begin{bmatrix} {}^{b}F_{a_y}\,{}^{b}L_z \\ {}^{b}F_{a_z}\,{}^{b}L_x - {}^{b}F_{a_x}\,{}^{b}L_z \\ -{}^{b}F_{a_z}\,{}^{b}L_x + \sum {}^{u}\tau_w \end{bmatrix} \tag{4.27}$$

The last equation gives the yaw dynamics of the system.

Where the definition of the torques forces is the following one:

$${}^{u}\tau_w = ({}^{u}R_w(\delta_w)\,{}^{w}F_w) \times {}^{u}L_w = {}^{u}F_w \times {}^{u}L_w = -S({}^{u}L_w)\,{}^{u}F_w \tag{4.28}$$

$$S({}^{u}L_{w}) = \begin{bmatrix} 0 & -{}^{b}L_{z} & 0 \\ {}^{b}L_{z} & 0 & -{}^{b}L_{x} \\ 0 & {}^{b}L_{x} & 0 \end{bmatrix} \tag{4.29}$$

$$\begin{aligned} {}^{u}\tau_{w} = -S({}^{u}L_{w})\,{}^{u}F_{w} = -\begin{bmatrix} 0 & -{}^{b}L_{z} & 0 \\ {}^{b}L_{z} & 0 & -{}^{b}L_{x} \\ 0 & {}^{b}L_{x} & 0 \end{bmatrix} \begin{bmatrix} {}^{u}F_{w_x} \\ {}^{u}F_{w_y} \\ {}^{u}F_{w_z} \end{bmatrix} = \\ = -\begin{bmatrix} -{}^{b}L_{z}\,{}^{u}F_{w_y} \\ {}^{b}L_{z}\,{}^{u}F_{w_x} - {}^{b}L_{x}\,{}^{u}F_{w_z} \\ {}^{b}L_{x}\,{}^{u}F_{w_z} \end{bmatrix} = \begin{bmatrix} {}^{b}L_{z}\,{}^{u}F_{w_y} \\ {}^{b}L_{x}\,{}^{u}F_{w_z} - {}^{b}L_{z}\,{}^{u}F_{w_x} \\ -{}^{b}L_{x}\,{}^{u}F_{w_z} \end{bmatrix} \end{aligned} \tag{4.30}$$

All the calculation done to consider the de-rotation matrices and selection of the specific torques con be found in the appendix A.10.

The last set of equation is given by the equation, 4.19:

$$\begin{aligned} \begin{bmatrix} d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & -d_z m_b \cos\psi_u - L_z m_b \cos\psi_u \\ d_z m_b \cos\psi_u + L_z m_b \cos\psi_u & d_z m_b \sin\psi_u + L_z m_b \sin\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^{i}\ddot{p}_u(2) \end{bmatrix} = \\ \begin{bmatrix} 0 \\ L_x\, m_b\, d_z\, \dot{\psi}_u^2 + L_x\, L_z\, m_b\, \dot{\psi}_u^2 \end{bmatrix} \end{aligned} \tag{4.31}$$

As in the previous cases, the substitution of the acceleration expressed in the tangential reference frame, according to the relationship 4.13, leads to some simplification:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} {}^{u}R_t\,(\beta)\; \ddot{p}_{u_{tr}} = \begin{bmatrix} 0 \\ L_x\, \dot{\psi}_u^2 \end{bmatrix} \tag{4.32}$$

The calculation on how the terms have been simplified can be found in the appendix A.11.

The complete set of equation in which it has been explicit the terms ${}^{u}R_t\,(\beta)$ and $\ddot{p}_{u_{tr}}$, according to the definitions given at page 49, are the following:

$$\begin{bmatrix} \sum {}^uF_{w_{x_i}} \\ \sum {}^uF_{w_{y_i}} \end{bmatrix} = (m_b + m_u) \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} 0 \\ \frac{V_t^2}{R} \end{bmatrix} - \begin{bmatrix} {}^uF_{a_x} \\ {}^uF_{a_y} \end{bmatrix} \tag{4.33}$$

$${}^iF_{a_z} - k_z\, d_z - g\, m_b = 0 \tag{4.34}$$

$$\begin{bmatrix} 0 & -L_z m_b & 0 \\ L_z m_b & 0 & 0 \\ 0 & L_x m_b & 0 \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{V_t^2}{R} \\ 0 \end{bmatrix} = \begin{bmatrix} {}^bF_{a_y}\, {}^bL_z \\ {}^bF_{a_z}\, {}^bL_x - {}^bF_{a_x}\, {}^bL_z \\ -{}^bF_{a_z}\, {}^bL_x - \sum {}^bL_x\, {}^uF_{w_z} \end{bmatrix} \tag{4.35}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{V_t^2}{R} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ L_x\, \dot{\psi}_u^2 \\ 0 \end{bmatrix} \tag{4.36}$$

The equation 4.46 gives the vertical equilibrium of the system. An additional constrain needs to be taken into account, i.e. the equilibrium of the torques of the system. Since the system is in static conditions, the equilibrium is give for fixed roll and pitch angle. This is given according to the following definitions:

$$\sum {}^u\tau = \sum {}^u\tau_{w_i} + {}^u\tau_a + {}^u\tau_g = 0 \tag{4.37}$$

$$\sum {}^u\tau_{w_i} = \sum ({}^uR_{w_i}(\delta_{w_i})\, {}^wF_{w_i}) \times {}^uL_{w_i} \tag{4.38}$$

$${}^u\tau_a = ({}^uR_b\, {}^bF_a) \times {}^uL_a \tag{4.39}$$

$${}^u\tau_g = ({}^uR_i\, {}^igm_b) \times {}^uL_m \tag{4.40}$$

With the arm ${}^uL_m$ obtained from the difference of the position of the under-vehicle and the body parts:

$${}^uL_m = {}^up_b - {}^up_u = {}^u \begin{bmatrix} 0 \\ 0 \\ d_z \end{bmatrix} + {}^uR_b(\phi_r, \theta_r)^b \begin{bmatrix} L_x \\ 0 \\ L_z \end{bmatrix} \tag{4.41}$$

It is necessary to take into account only the equations that gives the roll and pitch dynamics, taking into account the cross-product:

$$\sum {}^bL_z\, {}^uF_{w_y} + {}^uL_{a_z}\, {}^uF_{a_y} + {}^uL_{m_z}\, {}^igm_b = 0 \tag{4.42}$$

$$\sum ({}^bL_x\, {}^uF_{w_z} - {}^bL_z\, {}^uF_{w_x}) + {}^bL_{a_x}\, {}^uF_{a_z} - {}^bL_{a_z}\, {}^uF_{a_x} + {}^bL_{m_x}\, {}^igm_b - {}^bL_{m_z}\, {}^igm_b = 0 \tag{4.43}$$

The final list of equations is:

$${}^uF_{w_{x_1}} + {}^uF_{w_{x_2}} + {}^uF_{w_{x_3}} + {}^uF_{w_{x_4}} = -(m_b + m_u)\sin\beta\frac{V_t^2}{R} - {}^uF_{a_x} \tag{4.44}$$

$${}^uF_{w_{y_1}} + {}^uF_{w_{y_2}} + {}^uF_{w_{y_3}} + {}^uF_{w_{y_4}} = (m_b + m_u)\cos\beta\frac{V_t^2}{R} - {}^uF_{a_y} \tag{4.45}$$

$${}^iF_{a_z} - k_z\, d_z - g\, m_b = 0 \tag{4.46}$$

$$-L_z m_b \cos\beta\frac{V_t^2}{R} = {}^bF_{a_y}\, {}^bL_z \tag{4.47}$$

$$-L_z m_b \sin\beta\frac{V_t^2}{R} = {}^bF_{a_z}\, {}^bL_x - {}^bF_{a_x}\, {}^bL_z \tag{4.48}$$

$$\sum {}^bL_x\, {}^uF_{w_z} = -{}^bF_{a_z}\, {}^bL_x - L_x m_b \cos\beta\frac{V_t^2}{R} \tag{4.49}$$

$$-\cos\beta\frac{V_t^2}{R} = 0 \tag{4.50}$$

$$-\sin\beta\frac{V_t^2}{R} = L_x\, \dot{\psi}_u^2 \tag{4.51}$$

$$\sum {}^bL_z\, {}^uF_{w_y} = -{}^uL_{a_z}\, {}^uF_{a_y} - {}^uL_{m_z}\, {}^igm_b \tag{4.52}$$

$$\sum ({}^bL_x\, {}^uF_{w_z} - {}^bL_z\, {}^uF_{w_x}) = -{}^bL_{a_x}\, {}^uF_{a_z} + {}^bL_{a_z}\, {}^uF_{a_x} - {}^bL_{m_x}\, {}^igm_b + {}^bL_{m_z}\, {}^igm_b \tag{4.53}$$

Chapter 5

Conclusion

At this point, with all the gathered equations, it is possible to describe the feed-forward law, that given the forces and torques on the vehicle, that gives the wheel force that are applied to the ground.

The unknown of the problem, since the beginning, are forces acting on the wheels. From the complete system, given at the equation 3.3.6, it is possible to see that there are 8 equations with 12 unknown. This means that the problem has an infinite number of solution. Indeed this is an hyperstatic problem because it comes from an undersized system.

In order to solve the problem, it is necessary to select one of the infinite solutions and in this case it has been decided to exploit the one that minimize the norm. To do so, it is necessary to calculate the **Moore-Penrose pseudo inverse**, that in case of a matrix that has linearly independent rows, is defined in the following way:

$$A^+ = A^*(A \cdot A^*)^{-1} \tag{5.1}$$

This is a right pseudoinverse, as $A \cdot A^+ = 1$ for a non injective problem. This is a non injective matrix because the A matrix belongs to the space $\mathbf{K}^{m \times n}$ where $n > m$.

For this specific problem, the A is the following one:

$$\begin{bmatrix} \cos\delta_{w_{x_1}} & -\sin\delta_{w_{x_1}} & 0 & \cos\delta_{w_{x_2}} & -\sin\delta_{w_{x_2}} & 0 & \cos\delta_{w_{x_3}} & -\sin\delta_{w_{x_3}} & 0 & \cos\delta_{w_{x_4}} \\ \sin\delta_{w_{x_1}} & \cos\delta_{w_{x_1}} & 0 & \sin\delta_{w_{x_2}} & \cos\delta_{w_{x_2}} & 0 & \sin\delta_{w_{x_3}} & \cos\delta_{w_{x_3}} & 0 & \sin\delta_{w_{x_4}} \\ 0 & 0 & {}^uL_{x_1} & 0 & 0 & {}^uL_{x_2} & 0 & 0 & {}^uL_{x_3} & 0 \\ {}^uL_{z_1}\sin\delta_{w_{x_1}} & {}^uL_{z_1}\cos\delta_{w_{x_1}} & 0 & {}^uL_{z_2}\sin\delta_{w_{x_2}} & {}^uL_{z_2}\cos\delta_{w_{x_2}} & 0 & {}^uL_{z_3}\sin\delta_{w_{x_3}} & {}^uL_{z_3}\cos\delta_{w_{x_3}} & 0 & {}^uL_{z_4}\sin\delta_{w} \\ -{}^uL_{z_1}\cos\delta_{w_{x_1}} & {}^uL_{z_1}\sin\delta_{w_{x_1}} & {}^uL_{x_1} & -{}^uL_{z_2}\cos\delta_{w_{x_2}} & {}^uL_{z_2}\sin\delta_{w_{x_2}} & {}^uL_{x_2} & -{}^uL_{z_3}\cos\delta_{w_{x_3}} & {}^uL_{z_3}\sin\delta_{w_{x_3}} & {}^uL_{x_3} & -{}^uL_{z_4}\cos\delta \end{bmatrix} \tag{5.2}$$

This is obtained from the final system of chapter 4, substituting all the specific rotation matrix of each wheel to have all the terms in the under-vehicle reference frame. While the vector ${}^{b}L$ is coincident with ${}^{u}L$ if the relative angles ϕ_r and θ_r are null.

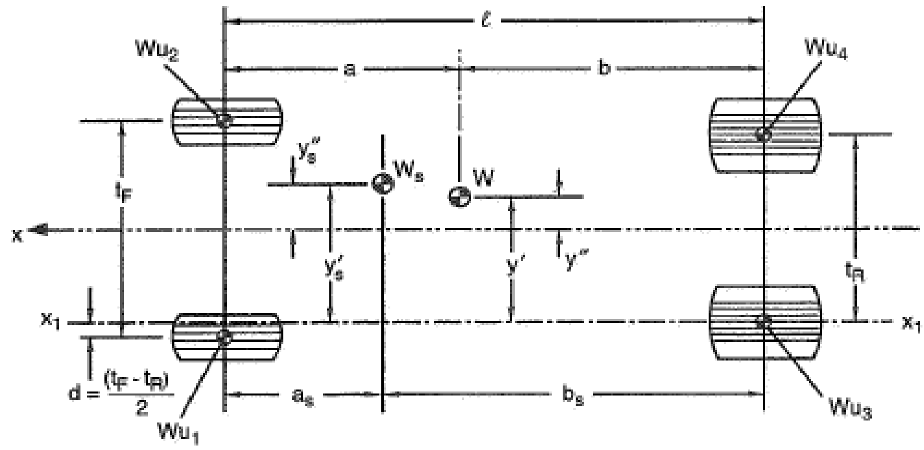


Figure 5.1: Definition of the wheel numeration

The previous matrix can be simplified considering the fact that only the front wheel are steering wheels. This consideration gives:

- $\delta_{w_{x_3}} = 0$
- $\delta_{w_{x_4}} = 0$

The simplified matrix is:

$$\begin{bmatrix}
\cos\delta_{w_{x_1}} & -\sin\delta_{w_{x_1}} & 0 & \cos\delta_{w_{x_2}} & -\sin\delta_{w_{x_2}} & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\sin\delta_{w_{x_1}} & \cos\delta_{w_{x_1}} & 0 & \sin\delta_{w_{x_2}} & \cos\delta_{w_{x_2}} & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & {}^{u}L_{x_1} & 0 & 0 & {}^{u}L_{x_2} & 0 & 0 & {}^{u}L_{x_3} & 0 & 0 & {}^{u}L_{x_4} \\
{}^{u}L_{z_1}\sin\delta_{w_{x_1}} & {}^{u}L_{z_1}\cos\delta_{w_{x_1}} & 0 & {}^{u}L_{z_2}\sin\delta_{w_{x_2}} & {}^{u}L_{z_2}\cos\delta_{w_{x_2}} & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
-{}^{u}L_{z_1}\cos\delta_{w_{x_1}} & {}^{u}L_{z_1}\sin\delta_{w_{x_1}} & {}^{u}L_{x_1} & -{}^{u}L_{z_2}\cos\delta_{w_{x_2}} & {}^{u}L_{z_2}\sin\delta_{w_{x_2}} & {}^{u}L_{x_2} & -1 & 0 & {}^{u}L_{x_3} & -1 & 0 & {}^{u}L_{x_4}
\end{bmatrix} \tag{5.3}$$

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Appendix A

Calculations

A.1 Complete relative rotational matrix

$$^{u}R_b(\Theta_r) = R_1(\phi_r)R_2(\theta_r)R_3(\psi_r) =$$

$$= \begin{pmatrix} \cos\psi_r\cos\theta_r & \cos\theta_r\sin\psi_r & -\sin\theta_r \\ \cos\psi_r\sin\phi_r\sin\theta_r - \cos\phi_r\sin\psi_r & \cos\phi_r\cos\psi_r + \sin\phi_r\sin\psi_r\sin\theta_r & \cos\theta_r\sin\phi_r \\ \sin\phi_r\sin\psi_r + \cos\phi_r\cos\psi_r\sin\theta_r & \cos\phi_r\sin\psi_r\sin\theta_r - \cos\psi_r\sin\phi_r & \cos\phi_r\cos\theta_r \end{pmatrix} \tag{A.1}$$

A.2 Complete under-vehicle rotational matrix

$$^{i}R_u(\Theta_u) = R_3^{\top}(\psi_u)R_2^{\top}(\theta_u)R_1^{\top}(\phi_u) =$$

$$= \begin{pmatrix} \cos\psi_u\cos\theta_u & \cos\psi_u\sin\phi_u\sin\theta_u - \cos\phi_u\sin\psi_u & \sin\phi_u\sin\psi_u + \cos\phi_u\cos\psi_u\sin\theta_u \\ \cos\theta_u\sin\psi_u & \cos\phi_u\cos\psi_u + \sin\phi_u\sin\psi_u\sin\theta_u & \cos\phi_u\sin\psi_u\sin\theta_u - \cos\psi_u\sin\phi_u \\ -\sin\theta_u & \cos\theta_u\sin\phi_u & \cos\phi_u\cos\theta_u \end{pmatrix} \tag{A.2}$$

A.3 Complete navigation rotational matrix

$$
{}^{U}R_{N}\left(\gamma_U,\beta_U\right)=\begin{pmatrix}\cos\beta_U\cos\gamma_U & -\cos\gamma_U\sin\beta_U & \sin\gamma_U\\ \sin\beta_U & \cos\beta_U & 0\\ -\cos\beta_U\sin\gamma_U & \sin\beta_U\sin\gamma_U & \cos\gamma_U\end{pmatrix} \tag{A.3}
$$

A.4 Complete i-th wheel reference frame

$$
{}^{U}R_{W_i}\left(\phi_{W_i},\delta_{W_i}\right)=\begin{pmatrix}\cos\delta_{W_i} & -\sin\delta_{W_i} & 0\\ \cos\phi_{W_i}\sin\delta_{W_i} & \cos\delta_{W_i}\cos\phi_{W_i} & -\sin\phi_{W_i}\\ \sin\delta_{W_i}\sin\phi_{W_i} & \cos\delta_{W_i}\sin\phi_{W_i} & \cos\phi_{W_i}\end{pmatrix} \tag{A.4}
$$

A.5 Vehicle position

$$
{}^{i}p_b = {}^{i}p_u + \begin{pmatrix}
\left(\sin\phi_u\,\sin\psi_u+\cos\phi_u\,\cos\psi_u\,\sin\theta_u\right)\left(d_z-L_x\,\sin\theta_r+L_z\,\cos\phi_r\,\cos\theta_r\right)+\\
+\cos\psi_u\,\cos\theta_u\,\left(L_x\,\cos\theta_r+L_z\,\cos\phi_r\,\sin\theta_r\right)+L_z\,\sin\phi_r\,\left(\cos\phi_u\,\sin\psi_u-\cos\psi_u\,\sin\phi_u\,\sin\theta_u\right)\\
\\
-\left(\cos\psi_u\,\sin\phi_u-\cos\phi_u\,\sin\psi_u\,\sin\theta_u\right)\left(d_z-L_x\,\sin\theta_r+L_z\,\cos\phi_r\,\cos\theta_r\right)+\\
+\cos\theta_u\,\sin\psi_u\,\left(L_x\,\cos\theta_r+L_z\,\cos\phi_r\,\sin\theta_r\right)-L_z\,\sin\phi_r\,\left(\cos\phi_u\,\cos\psi_u+\sin\varphi_u\,\sin\psi_u\,\sin\theta_u\right)\\
\\
\cos\phi_u\,\cos\theta_u\,\left(d_z-L_x\,\sin\theta_r+L_z\,\cos\phi_r\,\cos\theta_r\right)-\\
+\sin\theta_u\,\left(L_x\,\cos\theta_r+L_z\,\cos\phi_r\,\sin\theta_r\right)-L_z\,\cos\theta_u\,\sin\phi_r\,\sin\phi_u
\end{pmatrix} \tag{A.5}
$$

A.6 Small angle approximation

$$ {}^{i}p_b \approx^{i} p_u + \begin{pmatrix} (\sin\phi_u \, \sin\psi_u + \cos\phi_u \, \cos\psi_u \, \sin\theta_u)\,(d_z - L_x\,\theta_r + L_z) + \\ + \cos\psi_u \, \cos\theta_u \, (L_x \, + L_z\,\theta_r) + L_z\,\phi_r \, (\cos\phi_u \, \sin\psi_u - \cos\psi_u \, \sin\phi_u \, \sin\theta_u) \\ \\ -(\cos\psi_u \, \sin\phi_u - \cos\phi_u \, \sin\psi_u \, \sin\theta_u)\,(d_z - L_x\,\theta_r + L_z) + \\ + \cos\theta_u \, \sin\psi_u \, (L_x + L_z\,\theta_r) - L_z\,\phi_r \, (\cos\phi_u \, \cos\psi_u + \sin\phi_u \, \sin\psi_u \, \sin\theta_u) \\ \\ \cos\phi_u \, \cos\theta_u \, (d_z - L_x\,\theta_r + L_z) - \\ + \sin\theta_u \, (L_x + L_z\,\theta_r) - L_z\,\phi_r \cos\theta_u \, \sin\phi_u \end{pmatrix} \tag{A.6} $$

A.7 Simplified position equation

$$ {}^{i}p_b \approx {}^{i}p_u + \begin{pmatrix} (\sin\phi_u \, \sin\psi_u + \cos\phi_u \, \cos\psi_u \, \sin\theta_u) \\ -(\cos\psi_u \, \sin\phi_u - \cos\phi_u \, \sin\psi_u \, \sin\theta_u) \\ \cos\phi_u \, \cos\theta_u \end{pmatrix} (d_z - L_x\,\theta_r + L_z) + $$

$$ + \begin{pmatrix} \cos\psi_u \, \cos\theta_u \\ \cos\theta_u \, \sin\psi_u \\ -\sin\theta_u \end{pmatrix} (L_x + L_z\,\theta_r) + \begin{pmatrix} (\cos\phi_u \, \sin\psi_u - \cos\psi_u \, \sin\phi_u \, \sin\theta_u) \\ -(\cos\phi_u \, \cos\psi_u + \sin\phi_u \, \sin\psi_u \, \sin\theta_u) \\ -\cos\theta_u \, \sin\phi_u \end{pmatrix} (L_z\,\phi_r) \tag{A.7} $$

A.8 Partial derivatives used in the position computation

$$ \frac{\partial A}{\partial \phi_u} = \begin{pmatrix} \sin\psi_u \cos\phi_u - \cos\psi_u t \sin\theta_u \sin\phi_u \\ -\cos\psi_u \cos\phi_u - \sin\psi_u \sin\theta_u \sin\phi_u \\ -\cos\theta_u \sin\phi_u \end{pmatrix} \tag{A.8} $$

$$ \frac{\partial A}{\partial \theta_u} = \begin{pmatrix} \cos\phi_u \cos\psi_u \cos\theta_u \\ \cos\phi_u \sin\psi_u \cos\theta_u \\ -\cos\phi_u \sin\theta_u \end{pmatrix} \tag{A.9} $$

$$\frac{\partial A}{\partial \psi_u} = \begin{pmatrix} \sin\phi_u \cos\psi_u - \cos\phi_u \sin\theta_u \sin\psi_u \\ \sin\phi_u \sin\psi_u + \cos\phi_u \sin\theta_u \cos\psi_u \\ 0 \end{pmatrix} \tag{A.10}$$

$$\frac{\partial B}{\partial \phi_u} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{A.11}$$

$$\frac{\partial B}{\partial \theta_u} = \begin{pmatrix} -\cos\psi_u \sin\theta_u \\ -\sin\psi_u \sin\theta_u \\ -\cos\theta_u \end{pmatrix} \tag{A.12}$$

$$\frac{\partial B}{\partial \psi_u} = \begin{pmatrix} -\cos\theta_u \sin\psi_u \\ \cos\theta_u \cos\psi_u \\ 0 \end{pmatrix} \tag{A.13}$$

$$\frac{\partial C}{\partial \phi_u} = \begin{pmatrix} -\sin\psi_u \sin\phi_u - \cos\psi_u \sin\theta_u \cos\phi_u \\ \cos\psi_u \sin\phi_u + \sin\psi_u \sin\theta_u \cos\phi_u \\ 0 \end{pmatrix} \tag{A.14}$$

$$\frac{\partial C}{\partial \theta_u} = \begin{pmatrix} -\cos\psi_u \sin\phi_u \cos\theta_u \\ \sin\phi_u \sin\psi_u \cos\theta_u \\ \sin\psi_u \sin\theta_u \end{pmatrix} \tag{A.15}$$

$$\frac{\partial C}{\partial \psi_u} = \begin{pmatrix} \sin\phi_u \cos\psi_u - \cos\phi_u \sin\theta_u \sin\psi_u \\ \sin\phi_u \sin\psi_u + \cos\varphi_u \sin\theta_u \cos\psi_u \\ 0 \end{pmatrix} \tag{A.16}$$

A.9 Derotation and selection

1. Selections:

$$e_x = (1;0;0) \quad e_y = (0;1;0) \quad e_z = (0;0;1) \tag{A.17}$$

2. De-rotation only along the first axis x_B:

$$R_1(-\phi_r) = R_1^\top(\phi_r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_r & -\sin\phi_r \\ 0 & \sin\phi_r & \cos\phi_r \end{pmatrix} \tag{A.18}$$

3. De-rotation only along the second axis y_U:

$$R_2(-\theta_r) = R_2^\top(\theta_r) = \begin{pmatrix} \cos\theta_r & 0 & \sin\theta_r \\ 0 & 1 & 0 \\ -\sin\theta_r & 0 & \cos\theta_r \end{pmatrix} \tag{A.19}$$

A.10 Derotation and selection computation

$$e_x^\top({}^bF_a \times {}^bL) - k_{\phi_r}\phi_r = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} {}^bF_{a_x} \\ {}^bF_{a_y} \\ {}^bF_{a_z} \end{bmatrix} \times \begin{bmatrix} {}^bL_x \\ 0 \\ {}^bL_z \end{bmatrix} \right) - k_{\phi_r}\phi_r \tag{A.20}$$

$$e_x^\top({}^bF_a \times {}^bL) - k_{\phi_r}\phi_r = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i & j & k \\ {}^bF_{a_x} & {}^bF_{a_y} & {}^bF_{a_z} \\ {}^bL_x & 0 & {}^bL_z \end{bmatrix} - k_{\phi_r}\phi_r \tag{A.21}$$

$$e_x^\top({}^bF_a \times {}^bL) - k_{\phi_r}\phi_r = {}^bF_{a_y}{}^bL_z - k_{\phi_r}\phi_r \tag{A.22}$$

$$e_y^\top R_1(-\phi_r)({}^bF_a \times {}^bL) - k_{\theta_r}\theta_r =$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_r & -\sin\phi_r \\ 0 & \sin\phi_r & \cos\phi_r \end{bmatrix} \left(\begin{bmatrix} {}^bF_{a_x} \\ {}^bF_{a_y} \\ {}^bF_{a_z} \end{bmatrix} \times \begin{bmatrix} {}^bL_x \\ 0 \\ {}^bL_z \end{bmatrix} \right) - k_{\phi_r}\phi_r \tag{A.23}$$

$$e_y^\top R_1(-\phi_r)({}^bF_a \times {}^bL) - k_{\theta_r}\theta_r =$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_r & -\sin\phi_r \\ 0 & \sin\phi_r & \cos\phi_r \end{bmatrix} \begin{bmatrix} i & j & k \\ {}^bF_{a_x} & {}^bF_{a_y} & {}^bF_{a_z} \\ {}^bL_x & 0 & {}^bL_z \end{bmatrix} - k_{\phi_r}\phi_r \tag{A.24}$$

$$e_y^\top R_1(-\phi_r)({}^bF_a \times {}^bL) - k_{\theta_r}\,\theta_r =$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\phi_r \\ 0 & \phi_r & 1 \end{bmatrix} \begin{bmatrix} {}^bL_z\,{}^uF_{a_y} \\ {}^bL_x\,{}^uF_{a_z} - {}^bL_z\,{}^uF_{a_x} \\ -{}^bL_x\,{}^uF_{a_z} \end{bmatrix} - k_{\phi_r}\phi_r \tag{A.25}$$

$$e_y^\top R_1(-\phi_r)({}^bF_a \times {}^bL) - k_{\theta_r}\,\theta_r = {}^bF_{a_z}\,{}^bL_x - {}^bL_z\,{}^uF_{a_x} + {}^bL_x\,{}^uF_{a_z}\phi_r - k_{\phi_r}\phi_r \tag{A.26}$$

$$e_z^\top R_2(-\theta_r)R_1(-\phi_r)({}^bF_a \times {}^bL) =$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_r & 0 & \sin\theta_r \\ 0 & 1 & 0 \\ -\sin\theta_r & 0 & \cos\theta_r \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_r & -\sin\phi_r \\ 0 & \sin\phi_r & \cos\phi_r \end{bmatrix} \left(\begin{bmatrix} {}^bF_{a_x} \\ {}^bF_{a_y} \\ {}^bF_{a_z} \end{bmatrix} \times \begin{bmatrix} {}^bL_x \\ 0 \\ {}^bL_z \end{bmatrix} \right) \tag{A.27}$$

$$e_z^\top R_2(-\theta_r)R_1(-\phi_r)({}^bF_a \times {}^bL) =$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_r & 0 & \sin\theta_r \\ 0 & 1 & 0 \\ -\sin\theta_r & 0 & \cos\theta_r \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_r & -\sin\phi_r \\ 0 & \sin\phi_r & \cos\phi_r \end{bmatrix} \begin{bmatrix} i & j & k \\ {}^bF_{a_x} & {}^bF_{a_y} & {}^bF_{a_z} \\ {}^bL_x & 0 & {}^bL_z \end{bmatrix} \tag{A.28}$$

$$e_z^\top R_2(-\theta_r)R_1(-\phi_r)({}^bF_a \times {}^bL) =$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \theta_r \\ 0 & 1 & 0 \\ -\theta_r & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\phi_r \\ 0 & \phi_r & 1 \end{bmatrix} \begin{bmatrix} {}^bL_z\,{}^uF_{a_y} \\ {}^bL_x\,{}^uF_{a_z} - {}^bL_z\,{}^uF_{a_x} \\ -{}^bL_x\,{}^uF_{a_z} \end{bmatrix} \tag{A.29}$$

$$e_z^\top R_2(-\theta_r)R_1(-\phi_r)({}^bF_a \times {}^bL) =$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \theta_r \\ 0 & 1 & 0 \\ -\theta_r & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^bF_{a_y}\, {}^bL_z \\ {}^bL_x\, {}^uF_{a_z} - {}^bL_z\, {}^uF_{a_x} + {}^bL_x\, {}^uF_{a_z}\phi_r \\ \left({}^bL_x\, {}^uF_{a_z} - {}^bL_z\, {}^uF_{a_x}\right)\phi_r - {}^bL_x\, {}^uF_{a_z} \end{bmatrix} \quad \text{(A.30)}$$

$$e_z^\top R_2(-\theta_r)R_1(-\phi_r)({}^bF_a \times {}^bL) = -\,{}^bF_{a_y}\, {}^bL_z\, \theta_r + \left({}^bL_x\, {}^uF_{a_z} - {}^bL_z\, {}^uF_{a_x}\right)\phi_r - {}^bL_x\, {}^uF_{a_z} \quad \text{(A.31)}$$

A.11 Simplifications

$$\begin{bmatrix} d_z m_b \sin\psi_u + L_z m_b \sin\psi_u & -d_z m_b \cos\psi_u - L_z m_b \cos\psi_u \\ d_z m_b \cos\psi_u + L_z m_b \cos\psi_u & d_z m_b \sin\psi_u + L_z m_b \sin\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ L_x\, m_b\, d_z\, \dot{\psi}_u^2 + L_x\, L_z\, m_b\, \dot{\psi}_u^2 \end{bmatrix} \quad \text{(A.32)}$$

$$\begin{bmatrix} (d_z + L_z)m_b \sin\psi_u & -(d_z + L_z)m_b \cos\psi_u \\ (d_z + L_z)m_b \cos\psi_u & (d_z + L_x)m_b \sin\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} = \begin{bmatrix} 0 \\ L_x\, m_b\, d_z\, \dot{\psi}_u^2 + L_x\, L_z\, m_b\, \dot{\psi}_u^2 \end{bmatrix} \quad \text{(A.33)}$$

$$(d_z + L_z)m_b \begin{bmatrix} \sin\psi_u & -\cos\psi_u \\ \cos\psi_u & \sin\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} = \begin{bmatrix} 0 \\ L_x\, \dot{\psi}_u^2\, m_b(d_z + L_z) \end{bmatrix} \quad \text{(A.34)}$$

$$\begin{bmatrix} \sin\psi_u & -\cos\psi_u \\ \cos\psi_u & \sin\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} = \begin{bmatrix} 0 \\ L_x\, \dot{\psi}_u^2 \end{bmatrix} \quad \text{(A.35)}$$

$$\begin{bmatrix} \sin\psi_u & -\cos\psi_u \\ \cos\psi_u & \sin\psi_u \end{bmatrix} \begin{bmatrix} i\ddot{p}_u(1) \\ {}^i\ddot{p}_u(2) \end{bmatrix} = \begin{bmatrix} 0 \\ L_x\, \dot{\psi}_u^2 \end{bmatrix} \quad \text{(A.36)}$$

$$\begin{bmatrix} \sin\psi_u & -\cos\psi_u & 0 \\ \cos\psi_u & \sin\psi_u & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\psi_u & -\sin\psi_u & 0 \\ \sin\psi_u & \cos\psi_u & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{u}R_t\,(\beta)\ \ddot{p}_{u_{tr}} = \begin{bmatrix} 0 \\ L_x\,\dot{\psi}_u^2 \end{bmatrix} \tag{A.37}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} {}^{u}R_t\,(\beta)\ \ddot{p}_{u_{tr}} = \begin{bmatrix} 0 \\ L_x\,\dot{\psi}_u^2 \end{bmatrix} \tag{A.38}$$

Appendix B

Matlab™Code

B.1 Variables definition

```
clear all
close all
clc

syms t real %time
syms x y z real %generic variables
syms m_u m_b %masses
syms L_x L_z %arm
syms J_xu J_yu J_zu J_xyu J_xzu J_yzu %under-vehicle inertia
syms J_xb J_yb J_zb J_xyb J_xzb J_yzb %body inertias
syms F_ax F_ay F_az %aerodynamic forces
syms F_wx F_wy F_wz %wheel forces
syms tau_wx tau_wy tau_wz %wheel momenta
syms k_z k_pr k_tr k_x k_y %elastic constants
syms B_pr B_tr B_z %damping constants
syms d_z(t) %vertical movement
syms d %vertical movement derivative
syms g %gravitational constant
syms tau_w %wheel momenta
syms de %a determinant

syms phi_r(t) theta_r(t) real %relative angles
syms dphi_r(t) dtheta_r(t) real %rotational speed of the body
syms phi_u(t) theta_u(t) psi_u(t) real %undervehicle angle
syms dphi_u(t) dtheta_u(t) dpsi_u(t) real %rotational speed of the undervehicle
syms p_x(t) p_y(t) real %undervehicle positions
syms dp_x(t) dp_y(t) real %undervehicle speeds
```

```
syms R_1(x) R_2(x) R_3(x) real
syms R_bu real
syms gamma(t) beta(t) %navigations angles
syms phi_W(t) delta_W(t) %wheel angles
```

B.2 Matrix definition for position, angles and intertias

```
p_u = [p_x(t); p_y(t); 0];

d = [0; 0; d_z(t)];

L = [L_x; 0; L_z];

R_1(x) = [1 0 0; 0 cos(x) sin(x); 0 -sin(x) cos(x)]; %to undervehicle/body

R_2(x) = [cos(x) 0 -sin(x); 0 1 0; sin(x) 0 cos(x)]; %to undervehicle/body

R_3(x) = [cos(x) sin(x) 0; -sin(x) cos(x) 0; 0 0 1]; %to undervehicle/body

R_ui = R_3(psi_u).'*R_2(theta_u).'*R_1(phi_u).'; %to interial

R_bu = R_1(phi_r)*R_2(theta_r); %to body from undervehicle

p_b = p_u + R_ui*(d+R_bu.'*L);
```

B.3 Position first time derivative

```
A(x,y,z) = [(sin(x)*sin(z)+cos(x)*sin(y)*cos(z)); -(cos(z)*sin(x)-cos(x)*sin(z)*sin(y)); (cos(x)*cos(y))];

B(x,y,z) = [(cos(z)*cos(y)); (cos(y)*sin(z)); -sin(y) ];

C(x,y,z) = [(cos(x)*sin(z)-cos(z)*sin(x)*sin(y)); -(cos(x)*cos(z)-sin(x)*sin(z)*sin(y)); -cos(y)*sin(z)];

p_b = p_u + A(phi_u,theta_u,psi_u)*(d_z+L_z-(L_x*theta_r)) + B(phi_u,theta_u,psi_u)*
((L_z*theta_r)+L_x) + C(phi_u,theta_u,psi_u)*L_z*phi_r

A_x = subs(A(phi_u,theta_u,psi_u), phi_u, x);

dA_phi = diff(A_x, x);
```

```
A_y = subs(A(phi_u,theta_u,psi_u), theta_u, x);
dA_theta = diff(A_y, x);
A_z = subs(A(phi_u,theta_u,psi_u), psi_u, x);
dA_psi = diff(A_z, x);
B_x = subs(B(phi_u,theta_u,psi_u), phi_u, x);
dB_phi = diff(B_x, x);
B_y = subs(B(phi_u,theta_u,psi_u), theta_u, x);
dB_theta = diff(B_y, x);
B_z = subs(B(phi_u,theta_u,psi_u), psi_u, x);
dB_psi = diff(B_z, x);
C_x = subs(C(phi_u,theta_u,psi_u), phi_u, x);
dC_phi = diff(C_x, x);
C_y = subs(C(phi_u,theta_u,psi_u), theta_u, x);
dC_theta = diff(C_y, x);
C_z = subs(A(phi_u,theta_u,psi_u), psi_u, x);
dC_psi = diff(C_z, x);
dp_b = diff(p_b);
dp_u = diff(p_u);
dd_z = diff(d);
```

B.4 Angular speed definition

```
a_b = [phi_r(t); theta_r(t); psi_u(t)];
```

```
a_u = [phi_u(t); theta_u(t); psi_u(t)];

w_b = diff(a_b);

w_u = diff(a_u);
```

B.5 Kinetic and potential energy

```
J_b = [J_xb J_xyb J_xzb; J_xyb J_yb J_yzb; J_xzb J_yzb J_zb];

J_u = [J_xu J_xyu J_xzu; J_xyu J_yu J_yzu; J_xzu J_yzu J_zu];

T = ((1/2)*m_u*(dp_u.'*dp_u)) + ((1/2)*m_b*(dp_b.'*dp_b)) +
((1/2)*w_b.'*J_b*w_b) + ((1/2)*w_u.'*J_b*w_u);

K = ((1/2)*k_z*d_z^2) + ((1/2)*k_pr*phi_r^2) + ((1/2)*k_tr*theta_r^2) + m_b*g*(L_z+d_z);

L = K - T;
```

B.6 Euler-Lagrange equations

```
L_pux = subs(L, p_u(1), x);
dL_pux = diff(L_pux,x);
dL_pux = subs(dL_pux, x, p_u(1));

L_puy = subs(L, p_u(2), x);
dL_puy = diff(L_puy,x);
dL_puy = subs(dL_puy, x, p_u(2));

L_dpux = subs(L, dp_u(1), x);
dL_dpux = diff(L_dpux, x);
dL_dpux = subs(dL_dpux, x, dp_u(1));
ddL_dpux = diff(dL_dpux);

L_dpuy = subs(L, dp_u(2), x);
dL_dpuy = diff(L_dpuy, x);
dL_dpuy = subs(dL_dpuy, x, dp_u(2));
ddL_dpuy = diff(dL_dpuy);
```

```
L_dz = subs(L, d_z, x);
dL_dz = diff(L_dz, x);
dL_dz = subs(dL_dz, x, d_z);

L_ddz = subs(L, dd_z(3), x);
dL_ddz = diff(L_ddz, x);
dL_ddz = subs(dL_ddz, x, dd_z(3));
ddL_ddz = diff(dL_dz);

L_phr = subs(L, phi_r, x);
dL_phr = diff(L_phr, x);
dL_phr = subs(dL_phr, x, phi_r);

L_dphr = subs(L, w_b(1), x);
dL_dphr = diff(L_dphr, x);
dL_dphr = subs(dL_dphr, x, w_b(1));
ddL_dphr = diff(dL_dphr);

L_thr = subs(L, theta_r, x);
dL_thr = diff(L_thr, x);
dL_thr = subs(dL_thr, x, theta_r);

L_dthr = subs(L, w_b(2), x);
dL_dthr = diff(L_dthr, x);
dL_dthr = subs(dL_dthr, x, w_b(2));
ddL_dthr = diff(dL_dthr);

L_phu = subs(L, phi_u, x);
dL_phu = diff(L_phu, x);
dL_phu = subs(dL_phu, x, phi_u);

L_dphu = subs(L, w_u(1), x);
dL_dphu = diff(L_dphu, x);
dL_dphu = subs(dL_dphu, x, w_u(1));
ddL_dphu = diff(dL_dphu);

L_thu = subs(L, theta_u, x);
dL_thu = diff(L_thu, x);
dL_thu = subs(dL_thu, x, theta_u);

L_dthu = subs(L, w_u(2), x);
dL_dthu = diff(L_dthu, x);
dL_dthu = subs(dL_dthu, x, w_u(2));
ddL_dthu = diff(dL_dthu);
```

```
L_psu = subs(L, psi_u, x);
dL_psu = diff(L_psu, x);
dL_psu = subs(dL_psu, x, psi_u);

L_dpsu = subs(L, w_u(3), x);
dL_dpsu = diff(L_dpsu, x);
dL_dpsu = subs(dL_dpsu, x, w_u(3));
ddL_dpsu = diff(dL_dpsu);
```

B.7 Symplified Euler-Lagrange equations

```
phi_u = 0;
theta_u = 0;

NdL_pux = subs(dL_pux);

NdL_puy = subs(dL_puy);

NddL_dpux = subs(ddL_dpux);

NddL_dpuy = subs(ddL_dpuy);

NdL_dz = subs(dL_dz);

NddL_ddz = subs(ddL_ddz);

NdL_phr = subs(dL_phr);

NddL_dphr = subs(ddL_dphr);

NdL_thr = subs(dL_thr);

NddL_dthr = subs(ddL_dthr);

NdL_phu = subs(dL_phu);

NddL_dphu = subs(ddL_dphu);

NdL_thu = subs(dL_thu);

NddL_dthu = subs(ddL_dthu);
```

```
%NdL_psu = subs(dL_psu);

%NddL_dpsu = subs(ddL_dpsu);
```

B.8 Additional complete rotational matrices

```
R_ti = R_3(psi_u).'*R_3(beta).';

R_nu = R_2(gamma).'*R_3(beta).';

R_Wu = R_1(phi_W).'*R_3(delta_W).';
```

Bibliography

Euler–Lagrange equation. URL https://en.wikipedia.org/wiki/Euler%E2%80%93Lagrange_equation.

Lagrangian mechanics. URL https://en.wikipedia.org/wiki/Lagrangian_mechanics.

Inverted pendulum. URL https://en.wikipedia.org/wiki/Inverted_pendulum.

Frame of reference. URL https://en.wikipedia.org/wiki/Frame_of_reference.

Rigid body. URL https://en.wikipedia.org/wiki/Rigid_body.

J. P. Alberto Parra, Asier Zubizarreta and M. Dendaluce. *Intelligent Torque Vectoring Approach for Electric Vehicles with Per-Wheel Motors*. Complexity Volume 2018, Article ID 7030184, 14 pages. URL https://www.hindawi.com/journals/complexity/2018/7030184/.

D. Morin. *The Lagrangian Method*. URL http://www.people.fas.harvard.edu/~djmorin/chap6.pdf.

D. L. M. William F. Milliken. *Race cars vehicle dynamics*. SAE International.

Sorted in alphabetic order

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