Alma Mater Studiorum
Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

COHOMOLOGY OF HYPERPLANE AND TORIC ARRANGEMENTS

Master Thesis in Mathematics

Supervisor:
Prof. Luca Moci
Co-supervisor:
Dr. Roberto Pagaria

Author:
Claudia Agosti

Fifth Session
Accademic Year: 2018-2019

A papà,
che, se non nominato per primo, si offende.
A mamma,
che invece ci ride su.

Contents

Introduzione

Un *arrangiamento di iperpiani* è una collezione finita di sottospazi affini di codimensione uno (chiamati *iperpiani*) in uno spazio vettoriale complesso di dimensione finita.
L'algebra di coomologia del complementare di un arrangiamento di iperpiani ammette un'elegante presentazione dovuta a Orlik e Solomon, i quali hanno dimostrato che questa algebra è isomorfa ad un'algebra graduata, generata in grado uno, che soddisfa relazioni di tipo combinatorio, che dipendendono solo dal reticolo delle intersezioni ([OT92]). In particolare, se tutti gli iperpiani dell'arrangiamento passano per l'origine, l'algebra di coomologia dipende soltanto dal *matroide* associato all'arrangiamento.

Negli ultimi anni, vari autori hanno studiato gli *arrangiamenti torici*, collezioni finite di sottotori di codimensione uno (chiamati *ipertori*), in un toro complesso.
L'algebra di coomologia del complementare di un arrangiamento torico è piú complicata in quanto il toro complesso ha già di per sé una coomologia non banale e perchè l'intersezione di due sottotori in generale non è connessa.

Nel 2005, De Concini e Procesi si sono concentrati sullo studio dell'algebra di coomologia del complementare degli arrangiamenti torici nel quale le intersezioni di sottotori sono sempre connesse (arrangiamenti torici *unimodulari*) ottenendone una presentazione sullo stile di quella data da Orlik e Solomon per gli arrangiamenti di iperpiani ([DCP05]). In questo caso, l'algebra di coomologia del complementare è sempre generata in grado uno.

Nel 2018, Callegaro, D'Adderio, Delucchi, Migliorini e Pagaria hanno generalizzato il lavoro di De Concini e Procesi fornendo una presentazione, sempre sullo stile di quella data da Orlik e Solomon, dell'algebra di coomologia di un generico arrangiamento torico ([CDD+19]). Studiando il caso generale, hanno visto che questa algebra non è necessariamente generata in grado uno.
Inoltre, perfino per calcolare i numeri di Betti del complementare di un arrangiamento torico, non è sufficiente considerare le dipendenze lineari tra gli ipertori. Infatti, è necessario considerare un matroide *aritmetico* che tenga conto del numero di componenti connesse delle intersezioni di ipertori dell'arrangiamento ([DM13], [Moc12]).
A differenza del caso degli arrangiamenti di iperpiani, la coomologia del complementare di un arrangiamento torico non dipende unicamente dal matroide aritmetico dell'arrangiamento. Pagaria ha infatti costruito esplicitamente due arrangiamenti torici con matroidi isomorfi ma anelli di coomologia non isomorfi ([Pag19b]).

Piano della tesi. Nelle tre appendici richiamiamo concetti di base riguardanti i moduli, i matroidi e la coomologia di de Rham. Nel primo capitolo trattiamo gli arrangiamenti di iperpiani. Dopo alcune definizioni di base ed alcuni esempi, definiamo l'algebra di Orlik e Solomon dell'arrangiamento e, grazie alla costruzione di una particolare base di quest'algebra (chiamata base dei circuiti non rotti), dimostriamo l'esistenza di un isomorfismo tra quest'algebra e l'algebra di coomologia del complementare dell'arrangiamento.

Nel secondo capitolo affrontiamo alcuni concetti relativi al toro complesso, come il gruppo fondamentale, l'anello della coomologia di de Rham ed alcune proprietà dei rivestimenti finiti.

Nel terzo capitolo analizziamo gli arrangiamenti torici. Prima di tutto introduciamo, per un generico arrangiamento torico, il reticolo delle componenti connesse delle intersezioni di ipertori, il matroide aritmetico associato e una scelta di forme logaritmiche differenziali. Successivamente trattiamo il caso unimodulare, ottenendo una presentazione, sullo stile di quella data da Orlik e Solomon, dell'algebra di coomologia del complementare dell'arrangiamento. Infine generalizziamo questo risultato per un arrangiamento torico arbitrario, attraverso un suo rivestimento dato da un arrangiamento unimodulare.

Introduction

An *arrangement of hyperplanes* is a finite collection of codimension one affine subspaces (called *hyperplanes*) in a finite dimensional complex vector space.
The cohomology algebra of the complement of an arrangement of hyperplanes admits an elegant presentation due to Orlik and Solomon. They proved that this algebra is isomorphic to a graded algebra generated in degree one, that satisfies some combinatorially determined relations ([OT92]).
Moreover, the cohomology algebra depends only on the intersection poset. In fact, if all hyperplanes pass through the origin, it depends only on the *matroid* associated with the arrangement, which is a combinatorial abstractation of the way these hyperplanes intersect each other.

In the last years, many authors studied the *toric arrangements*, that are finite collections of subtori of codimension one (called *hypertori*), in a *complex torus.*
The cohomology algebra of the complement of a toric arrangement is more complicated because the ambient torus has its own cohomology algebra. Moreover the intersection of two subtori in general is not connected making the combinatorial data more intricated.

In 2005, De Concini and Procesi provided an Orlik-Solomon-type presentation for the cohomology algebra of the complement of *unimodular* toric arrangements ([DCP05]), that are toric arrangements in which the intersections of subtori are always connected. In this case, the cohomology algebra of the complement is always generated in degree one.

In 2018, Callegaro, D'Adderio, Delucchi, Migliorini and Pagaria generalised De Concini and Procesi's work, providing an Orlik-Solomon-type presentation for the cohomology algebra of the complement of a general toric arrangement ([CDD$^+$19]). In the general case, this algebra is not necessarily generated in degree one.
Moreover, even to encode basic topological data such as the Betti numbers of the complement of a toric arrangement, it is not enough to consider the linear dependencies among the hypertori. In fact, it is necessary to consider an *arithmetic* matroid that keeps track of the number of connected components of the intersections of subtori ([DM13], [Moc12]).
Unlike the case of hyperplane arrangements, the cohomology of the complement of a toric arrangement does not depend only on the arrangement's arithmetic matroid. In fact, Pagaria explicitly constructed two toric arrangements with isomorphic matroids and with non-isomorphic cohomology rings ([Pag19b]).

Plan of the thesis. Some background material on modules, matroid and de Rham cohomology is recalled in three appendices. In the first chapter, we treat the hyperplane arrangements. After some basic definitions and examples, we define the Orlik-Solomon algebra of an arrangement, and, thanks to the construction of a particular basis of this algebra (called non-broken circuit basis), we prove the existence of an isomorphism between this algebra and the cohomology algebra of the complement of the arrangement.

In the second chapter, we recall some basic fact of the complex torus such as its fundamental group, its de Rham cohomology ring and some properties of its finite coverings.

In the third chapter, we analyse the toric arrangements. First we introduce, for a general toric arrangement, the poset of layers, the arithmetic matroid and a choice of logarithmic forms, then we treat the unimodular case and, after some formal identities, we provide an Orlik-Solomon-type presentation for the cohomology algebra of the complement of the arrangement. Thus, finally, we consider a general toric arrangement, and, by covering it with unimodular arrangements, we obtain an Orlik-Solomon-type presentation of the cohomology algebra of the complement of an arbitrary toric arrangement.

Chapter 1

Arrangements of hyperplanes and Orlik-Solomon algebras

1.1 The poset of intersections $L(\mathcal{A})$

Definition 1.1.1. Let $\mathbb{K}$ be a field and $V_{\mathbb{K}}$ be an ℓ-dimensional $\mathbb{K}$-vector space. An **arrangement of hyperplanes** $\mathcal{A} = (\mathcal{A}_{\mathbb{K}}, V_{\mathbb{K}}) = \{H_1, \dots, H_n\}$ is a finite set of affine subspaces of dimension $(\ell - 1)$. We call every H_i in $\mathcal{A}$ a **hyperplane** in $V_{\mathbb{K}}$. If we want to emphasize the dimension of V, we say that $\mathcal{A}$ is an **ℓ-arrangement**.

From now on we will denote by n the number of hyperplanes in the arrangement, and by ℓ the dimension of the ambient space V.

We say that an arrangement $\mathcal{A}$ is **central** if $\bigcap_{H\in\mathcal{A}} H \neq \emptyset$, and we call $T(\mathcal{A}) := \bigcap_{H\in\mathcal{A}} H$ the **center** of $\mathcal{A}$.

Let V^* be the dual space of V, the space of linear forms on V. Choose a basis $\{e_1, ..., e_\ell\}$ in V and let $\{x_1, ...x_\ell\}$ be the dual basis in V^*, i.e. $x_i(e_j) = \delta_{i,j} \quad \forall i, j$. Note that if $\mathcal{A}$ is a central arrangement, the coordinates may be chosen so that each hyperplane contains the origin, then, without loss of generality, we can say that the center of $\mathcal{A}$ contains the origin.

Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial α_H of degree one. This polynomial is defined up to a non-zero scalar multiple, in fact, two distinct polynomials $p, q \in \mathbb{K}[x_1, \dots, x_l]$ define the same hyperplane H if $p = cq$ for some $c \in \mathbb{K}^*$. In this case we write $p \doteq q$. Then we are allowed to give the following definition.

Definition 1.1.2. The product

$$Q(\mathcal{A}) \doteq \prod_{H \in \mathcal{A}} \alpha_H$$

is called a **defining polynomial** of $\mathcal{A}$.

Note that if $\mathcal{A}$ is a central arrangement, then each α_H is a linear form and $Q(\mathcal{A})$ is a homogeneous polynomial of degree n.

Definition 1.1.3. Let $\mathcal{A}$ be an arrangement. Define $L(\mathcal{A})$ the poset of all non-empty intersection of elements of $\mathcal{A}$ with the partial order given by the **reverse inclusion**

$$X \leq Y \Longleftrightarrow Y \subseteq X.$$

We agree that $L(\mathcal{A})$ includes the ambient space V as the intersection of the empty collection of hyperplanes. Note that V is the unique minimal element since $X \subseteq V$ for every $X \in L(\mathcal{A})$.

Definition 1.1.4. Define a **rank function** on $L(\mathcal{A})$ by

$$r : L(\mathcal{A}) \to \mathbb{N}$$

$$X \longmapsto codim(X).$$

Note that $r(V) = 0$ and $r(H) = 1 \quad \forall H \in \mathcal{A}$. For this reason we say that every hyperplane H is an **atom** of $L(\mathcal{A})$.

Definition 1.1.5. Denote $L_p(\mathcal{A}) = \{X \in L(\mathcal{A}); r(X) = p\}$.
The **Hasse diagram** of $L(\mathcal{A})$ has vertices labeled by the elements of $L(\mathcal{A})$ and arranged on levels $L_p(\mathcal{A})$ for $p \geq 0$. If $X \in L_p(\mathcal{A})$ and $Y \in L_{p+1}(\mathcal{A})$, an edge in the Hasse diagram connects X with Y if and only if $X < Y$.

Definition 1.1.6. Let X be a maximal element of $L(\mathcal{A})$.
Define $rank(\mathcal{A}) = r(X)$. We say that an ℓ-arrangement $\mathcal{A}$ is **essential** if $rank(\mathcal{A}) = \ell$.

The rank of an arrangement is well defined since it can proved that maximal elements of $L(\mathcal{A})$ have the same rank.

Remark 1.1.7. An ℓ-arrangement $\mathcal{A}$ is essential if and only if it contains ℓ linearly independent hyperplanes. If $\mathcal{A}$ is a central essential ℓ-arrangement, then $rank(\mathcal{A}) = r(T(\mathcal{A})) = \ell$. In particular the coordinates may be chosen so that $T(\mathcal{A}) = \{0\}$.

Example 1.1.8. Let $\mathcal{A}$ be a ℓ-arrangement defined by $Q(\mathcal{A}) = x_1x_2 \dots x_\ell$, i.e. $\mathcal{A} = \{H_1, \dots, H_\ell\}$, with $H_i = \{x_i = 0\} \quad \forall i = \{1, \dots, \ell\}$, is a central essential arrangement since $\bigcap_{H\in\mathcal{A}} H = \{(0, \dots, 0)\} =: T(\mathcal{A})$ and $rank(\mathcal{A}) = r(\{(0, \dots, 0)\}) = \ell$. This arrangement $\mathcal{A}$ is called **Boolean** arrangement and it is the arrangement of the coordinate hyperplanes in $\mathbb{K}^\ell$.

Example 1.1.9. Let $\mathcal{A}$ be a 2-arrangement defined by $Q(\mathcal{A}) = xy(x+y-1)$, i.e. $\mathcal{A} = \{H_1, H_2, H_3\}$ with $H_1 = \{(x, y) \in V; x = 0\}$, $H_2 = \{(x, y) \in V; y = 0\}$ and $H_3 = \{(x, y) \in V; x + y = 1\}$.

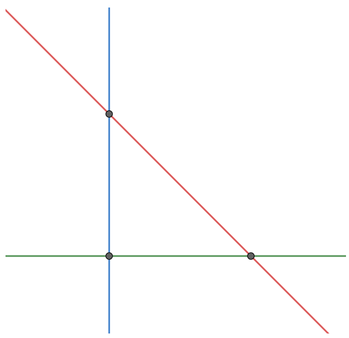
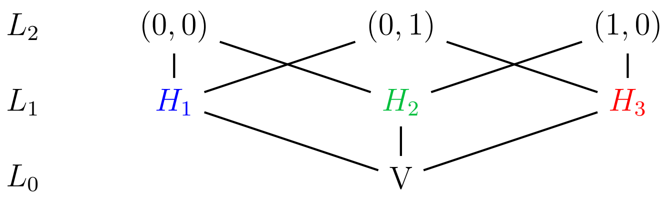


Figure 1.1: $\mathcal{A}$

The Hasse diagram of $L(\mathcal{A})$ is:



It is an essential arrangement since $rank(\mathcal{A}) = r(\{(0,0)\} = r(\{(0,1)\}) = r(\{(1,0)\}) = 2$ but it is not central since $H_1 \cap H_2 \cap H_3 = \emptyset$.

Example 1.1.10. Let $\mathcal{A}$ be a ℓ-arrangement defined by

$$Q(\mathcal{A}) = \prod_{1 \leq i < j \leq \ell} (x_i - x_j).$$

Since $T(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H = \{x_1 = \cdots = x_\ell\}$, $rank(\mathcal{A}) = \ell - 1$. It follows that $\mathcal{A}$ is a central but not essential arrangement. This arrangement $\mathcal{A}$ is called **braid** arrangement.

For $\ell = 3$, $\mathcal{A} = \{H_1, H_2, H_3\}$ with $H_1 = \{(x, y, z) \in V; x = y\}$, $H_2 = \{(x, y, z) \in V; x = z\}$ and $H_3 = \{(x, y, z) \in V; y = z\}$.

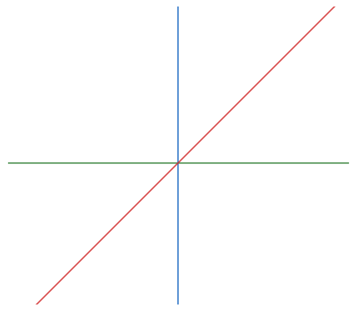


Figure 1.2: Projection of $\mathcal{A}$ in the plane $0xy$

1.2 The Möbius function

Definition 1.2.1. Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$.
Define the **Möbius function**

$$\mu_{\mathcal{A}} = \mu : L \times L \longrightarrow \mathbb{Z}$$

as follows:

- $$\mu(X, X) = 1 \qquad \text{if } X \in L,$$
- $$\sum_{\substack{Z \in L, \\ X \leq Z \leq Y}} \mu(X, Z) = 0 \qquad \text{if } X, Y \in L \text{ and } X < Y,$$
- $$\mu(X, Y) = 0 \qquad \text{otherwise.}$$

Note that for fixed X the values of $\mu(X, Y)$ may be computed recursively. It follows that if ν is any other function which satisfies the defining properties of μ, then $\nu = \mu$.

Theorem 1.2.2. *If $X \leq Y$, then $\mu(X, Y) \neq 0$ and sgn $\mu(X, Y) = (-1)^{r(X)-r(Y)}$.*

Definition 1.2.3. For $X \in L$, define $\mu(X) = \mu(V, X)$.

Remark 1.2.4. In general is not possible to give a formula for $\mu(X)$, but we know that

1. $\mu(V) = 1$;
2. $\mu(H) = -1 \quad \forall H \in L$;
3. if $r(X) = 2$, then $\mu(X) = |\mathcal{A}_X| - 1$; where $\mathcal{A}_X = \{H \in \mathcal{A}; H \leq X\}$.

Example 1.2.5. Let $\mathcal{A}$ be the Boolean ℓ-arrangement.
For $X \in L$,

$$\mu(X) = (-1)^{r(X)}.$$

Proof. Define $\nu(X) = (-1)^{r(X)}$. It suffices to show that ν satisfies $\nu(V) = 1$ and $\sum_{\substack{Z \in L, \\ Z \leq Y}} \nu(Z) = 0 \ \forall Y \in L$. Clearly $\nu(V) := (-1)^0 = 1$. If $Y \in L$ and $Y \neq V$, there exist $H_{i_1}, \ldots, H_{i_p} \in \mathcal{A}$ such that $Y = H_{i_1} \cap \cdots \cap H_{i_p}$. For every $Z \in L$ such that $Z \leq Y$, there exist $H_{j_1}, \ldots, H_{j_q} \in \mathcal{A}$ such that $Z = H_{j_1} \cap \cdots \cap H_{j_q}$ with $\{j_1, \ldots, j_q\} \subseteq \{i_1, \ldots, i_p\}$ and $q = r(X) \leq r(Y) = p$. Then

$$\sum_{\substack{Z \in L, \\ Z \leq Y}} \nu(Z) = \sum_{\substack{Z \in L, \\ Z \leq Y}} (-1)^{r(Z)} = \sum_{q=0}^{p} \binom{p}{q} (-1)^q = 0.$$

□

Definition 1.2.6. Let $\mathcal{A}$ be an arrangement with intersection poset L and Möbius function μ. Let t be an indeterminate.
Define the **Poincaré polynomial** of $\mathcal{A}$ by

$$\pi(\mathcal{A}, t) = \sum_{X \in L} \mu(X)(-t)^{r(X)}.$$

The reason for this terminology is that, as we will show in Theorem 1.5.13, if $\mathcal{A}$ is a complex arrangement, π equals the Poincaré polynomial of the cohomology ring of the complement of $\mathcal{A}$ viewed as a complex manifold. In particular, the Betti numbers of the complement of a complex arrangement are just the coefficients of the polynomial π.

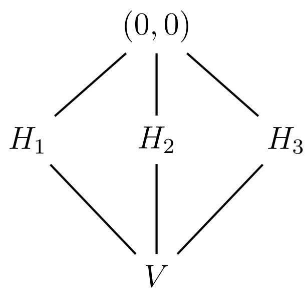
It follows from Theorem 1.2.2 that $\pi(\mathcal{A}, t)$ has nonnegative coefficients. In fact,

$$sgn\, \mu(X)(-1)^{r(X)} = sgn\, (-1)^{r(V)-r(X)}(-1)^{r(X)} = +1.$$

Example 1.2.7. Let $\mathcal{A}$ be the Boolean ℓ-arrangement.
We have already proved in Example 1.2.5 that $\mu(X) = (-1)^{r(X)}$. Then

$$\pi(\mathcal{A},t) = 1 + \binom{\ell}{1}(-1)(-t) + \binom{\ell}{2}(+1)(-t)^2 + \binom{\ell}{3}(-1)(-t)^3 + \cdots =$$
$$= \sum_{p=0}^{\ell} \binom{\ell}{p} t^p = (1+t)^p.$$

Example 1.2.8. Let $\mathcal{A}$ be the arrangement defined by $Q(\mathcal{A}) = xy(x+y)$, i.e. $\mathcal{A} = \{H_1, H_2, H_3\}$ with $H_1 = \{(x,y) \in V; x = 0\}$, $H_2 = \{(x,y) \in V; y = 0\}$ and $H_3 = \{(x,y) \in V; x = -y\}$. The Hasse diagram of $L(\mathcal{A})$ is:



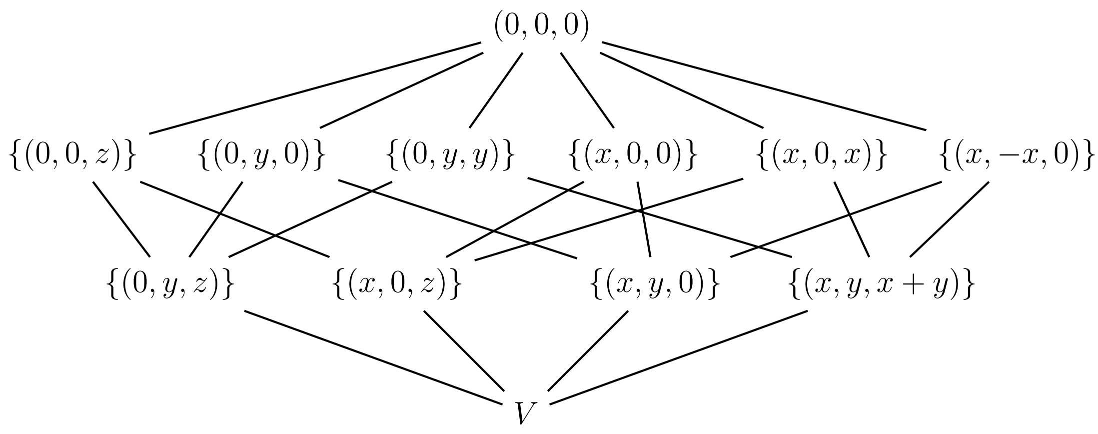
Then

$$\pi(\mathcal{A},t) = \mu(V)(-t)^0 + \mu(H_1)(-t)^1 + \mu(H_2)(-t)^1 + \mu(H_3)(-t)^1 + \mu((0,0))(-t)^2 =$$
$$= 1 + 3t + 2t^2.$$

Even if it seems like every Poincaré polynomial of a central arrangement is a product of linear terms $(1+bt) \in \mathbb{Z}[t]$, we give an example that proves that this is a false impression.

Example 1.2.9. Let $\mathcal{A}$ be the central and essential arrangement defined by $Q(\mathcal{A}) = xyz(x+y-z)$.

The Hasse diagram of $L(\mathcal{A})$ is:



and then the Poincaré polynomial of $\mathcal{A}$ is

$$\begin{aligned}\pi(\mathcal{A},t) &= 1+4(-1)(-t)+6(+1)(-t)^2-[6(+1)+4(-1)+1](-t)^3 = \\ &= (1+t)(1+3t+3t^2).\end{aligned}$$

1.3 The Orlik-Solomon algebra

In this section we assume $\mathcal{A}$ is a central arrangement.

Definition 1.3.1. Let $\mathcal{A}$ be an ℓ-arrangement over a field $\mathbb{K}$ and $\mathcal{K}$ be a commutative subring of $\mathbb{K}$.
Let $\{e_H; H \in \mathcal{A}\}$ be a set of symbols in one-to-one correspondance with the hyperplanes of $\mathcal{A}$. Define

$$E_1 = \bigoplus_{H\in\mathcal{A}} \mathcal{K}e_H$$

the free $\mathcal{K}$-module of all $\mathcal{K}$-linear combination of these symbols and

$$E = E(\mathcal{A}) = \Lambda(E_1) = \bigoplus_{j=0}^{|\mathcal{A}|} \Lambda^j E_1$$

the exterior algebra of E_1, that is a graded $\mathcal{K}$-algebra.

Write $E_j := \Lambda^j E_1$ and $uv = u \wedge v$. Note that $E_1 := \Lambda^1 E_1$ agrees with its earlier definition.

Recall that:

- $E_0 := \Lambda^0 E_1 = \mathcal{K}$;
- $e_H^2 = 0$ and $e_H e_K = -e_K e_H \quad \forall H, K \in \mathcal{A}$;
- E_p is the $\mathcal{K}$-module generated by the products of p distinct e_i with $H_i \in \mathcal{A}$.

Definition 1.3.2. Define a $\mathcal{K}$-linear map

$$\partial := \partial_E : E \longrightarrow E$$

by $\partial 1 = 0$, $\partial e_H = 1$ and

$$\partial(e_{H_1} \dots e_{H_p}) := \sum_{k=1}^{p} (-1)^{k-1} e_{H_1} \dots \widehat{e_{H_k}} \dots e_{H_p} \qquad \forall p \geq 2, H_1, ..., H_p \in \mathcal{A},$$

where the notation $\widehat{e_{H_k}}$ indicates that the factor e_{H_k} is omitted from the product.

Remark 1.3.3. Since $\partial_E \circ \partial_E = 0$ and ∂_E is homogeneous of degree -1, (E, ∂_E) is a chain complex. Moreover, since ∂_E is a derivation of the exterior algebra, it gives E the structure of a differential graded algebra.

Definition 1.3.4. Given $S = (H_1, ..., H_p)$ a p-tuple of hyperplanes of $\mathcal{A}$, we define

$$|S| := p, \qquad \cap S := H_1 \cap ... \cap H_p \quad \text{and} \quad e_S := e_{H_1} \dots e_{H_p} \in E.$$

For $S = ()$ the empty tuple, we agree that $|S| = 0$, $\cap S = V$ and $e_S = 1$. Let $\mathbf{S}_p$ denote the set of all p-tuples $(H_1, ..., H_p)$ and let $\mathbf{S} = \cup_{p \geq 0} \mathbf{S}_p$.

Since at the beginning of this chapter we have supposed $\mathcal{A}$ central, for every tuple S, we have $\cap S \neq \emptyset$, then $\cap S \in L$ with $r(\cap S) \leq |S|$.

Definition 1.3.5. We say that a tuple S is **independent** if $r(\cap S) = |S|$ and **dependent** if $r(\cap S) < |S|$.

Remark 1.3.6. The terminology has a geometric significance: the tuple $S = (H_1, \ldots, H_p)$ is independent if and only if the corresponding linear form $\alpha_{H_1}, ..., \alpha_{H_p}$ are linearly independent. Equivalently, if and only if the hyperplanes of S are in general position.

Definition 1.3.7. Let $\prec$ be a fixed total order on $\mathcal{A}$.
A p-tuple $S = (H_1, \ldots, H_p)$ is **standard** if $H_1 \prec \cdots \prec H_p$.
A p-tuple $S = (H_1, \ldots, H_p)$ is a **circuit** if it is minimally dependent, i.e. $(H_1, \ldots, H_p)$ is dependent but $(H_1, \ldots, \widehat{H_k}, \ldots, H_p)$ is independent for every $1 \leq k \leq p$.

Definition 1.3.8. Let $\mathcal{A}$ be an arrangement.
Define $I = I(\mathcal{A})$ the ideal of the $\mathcal{K}$-algebra E generated by ∂e_S for all dependent $S \in \mathbf{S}$, i.e.

$$I := \{a_{S_1}\partial e_{S_1} + \cdots + a_{S_p}\partial e_{S_p};\ p > 0, a_{S_i} \in E \text{ and } S_i \in \mathbf{S}, S_i \text{ dependent} \, \forall i\}.$$

Lemma 1.3.9. *If S is dependent, then $e_S \in I$.*

Proof. For $s \in S$, $0 = \partial(e_s e_S) = (\partial e_s)e_S \pm e_s(\partial e_S) = e_S \pm e_s(\partial e_S)$. It follows that $e_S = \pm e_s(\partial e_S) \in I$. □

Proposition 1.3.10. *E is generated by the elements e_S where S is standard and I is generated by elements of the form ∂e_S where S is a circuit.*

Proof. We need to prove only the second statement. Let S a dependent p-tuple. There exist $T, U \subseteq S$ with T circuit, such that $S = T \cup U$. For every $t \in T$, we have that

$$\partial e_S = \partial(e_T e_U) = \pm(\partial e_T)e_U \pm e_T \partial e_U = \pm(\partial e_T)e_U \pm e_t \partial e_T \partial e_U = (\pm e_U \pm e_t \partial e_U)\partial e_T.$$

□

Definition 1.3.11. Let $\varphi : E \longrightarrow E/I$ be the natural projection on the quotient $\mathcal{K}$-algebra.
Define $A = A(\mathcal{A}) := E/I$ the **Orlik-Solomon algebra** of $\mathcal{A}$ and denote $A_p := \varphi(E_p)$, $a_H := \varphi(e_H)$ for $H \in \mathcal{A}$, and $a_S := \varphi(e_S)$ for $S \in \mathbf{S}$.

Since E is a graded algebra and I is a homogeneous ideal of E, A is a graded algebra

$$A = \bigoplus_{p=0}^{|\mathcal{A}|} \frac{E_p}{I \cap E_p} = \bigoplus_{p=0}^{|\mathcal{A}|} A_p.$$

Note that, for $p > \ell$, $A_p = 0$ since every element of $\mathbf{S}_p$ is dependent.

Proposition 1.3.12. *$\partial_E I \subset I$.*

Proof. If S is dependent, by Leibniz rule, $\partial(x\partial e_S) = (\partial x)(\partial e_S) \in I$ for any $x \in E$. □

Remark 1.3.13. It follows that:

- ∂_E induces a $\mathcal{K}$-linear map $\partial_A : A \longrightarrow A$ defined by, for $x \in E$,

$$\partial_A(\varphi(x)) := \varphi(\partial_E(x)).$$

- A inherits the structure of a differential graded algebra.
- (A, ∂_A) is a chain complex.

Lemma 1.3.14. *(A, ∂_A) is acyclic complex of chains.*

Proof. We have to prove that $Ker\partial_A \subseteq Im\partial_A$. For every $a \in A$ there exists $u \in E$ such that $a = \varphi(u)$. Let $v := e_H$ with $H \in \mathcal{A}$ and $b := \varphi(v)$. We have that $\partial_A(ba) = \partial_A(\varphi(v)\varphi(u)) = \partial_A(\varphi(vu)) := \varphi(\partial_E(vu)) = \varphi(u - v\partial_E u) = \varphi(u) - \varphi(v)\varphi(\partial_E(u)) = \varphi(u) - \varphi(v)\partial_A(\varphi(u)) = a - b\partial_A(a) = a$. Then there exists $x \in A$ such that $a = \partial_A(x) \in Im\partial_A$. □

Proposition 1.3.15. *Let $\mathcal{A}$ be an ℓ-arrangement, then*

$$A = \bigoplus_{p=0}^{\ell} A_p.$$

In particular

$$A = \mathcal{K} \oplus \bigoplus_{H\in\mathcal{A}} \mathcal{K}a_H \oplus \bigoplus_{p=2}^{\ell} A_p.$$

Proof. From now on, for every p, denote $I_p := I \cap E_p$.
$I_0 = \{x\partial e_S | x \in E_r, S \in \mathbf{S}_q$ dependent and $r + (q-1) = 0\} = \{x\partial e_S | x \in E_0, S \in \mathbf{S}_1$ dependent$\} \cup \{x\partial e_S | x \in E_1, S \in \mathbf{S}_0$ dependent$\}$. Since every element of $\mathbf{S}_0$ and $\mathbf{S}_1$ is independent, we have that $I_0 = 0$ and hence $A_0 = \mathcal{K}$.
$I_1 = \{x\partial e_S | x \in E_r, S \in \mathbf{S}_q$ dependent and $r + (q-1) = 1\}$, since the only dependent elements of $\mathbf{S}_2$ are of the form $S = (H, H)$, and since $e_S = e_H^2 = 0$, we have $I_1 = 0$ and then the elements a_H are linearly independent over $\mathcal{K}$ and $A_1 = \bigoplus_{H\in\mathcal{A}} \mathcal{K}a_H$. □

Example 1.3.16. Let $\mathcal{A}$ be the Boolean ℓ-arrangement.
Since $S = (H_1, \ldots, H_p)$ is independent if and only if $H_1, \ldots, H_p$ are distinct hyperplanes. It immediately follows that if S is dependent, than $e_S = 0$, $I = 0$ and than $A = E$.

1.3.1 The grading by $L(\mathcal{A})$

In this subsection we want to discuss a grading on $A(\mathcal{A})$ finer than the standard grading inherited from E.

Definition 1.3.17. Let $X \in L(\mathcal{A})$. Denote

$$\mathbf{S}_X = \{S \in \mathbf{S} | \cap S = X\}$$

and

$$E_X = \sum_{S\in\mathbf{S}_X} \mathcal{K}e_S.$$

Note that, since $\mathbf{S} = \bigcup_{X \in L} \mathbf{S}_X$ is a disjoint union and $E_X E_Y \subseteq E_{X \cap Y}$, E is an algebra graded by $L(\mathcal{A})$, with

$$E = \bigoplus_{X \in L} E_X.$$

Proposition 1.3.18. *I is homogeneous with respect to the grading of E by $L(\mathcal{A})$, with*

$$I = \bigoplus_{X \in L} I \cap E_X.$$

Proof. If S is a circuit with $\cap S = X$, then, for every $s \in S$, we have that $\cap(S \backslash \{s\}) = X$ otherwise $S \backslash \{s\}$ would be a proper dependent set in S. Thus every homogeneous component of $\partial e_S = \sum_{s_i \in S} (-1)^{i-1} e_{S \backslash \{s_i\}} \in E_X$ belongs to the same E_X. □

Remark 1.3.19. Since E is a graded algebra with respect to $L(\mathcal{A})$ and I is a homogeneous ideal of E with respect to the same grading, A is a graded algebra with respect to $L(\mathcal{A})$, i.e.

$$A = \bigoplus_{X \in L} A_X,$$

where, for every $X \in L(\mathcal{A})$, $A_X := \frac{E_X}{I \cap E_X}$.

Proposition 1.3.20. *The grading of A by $L(A)$ is finer than the standard grading $A = \bigoplus_{p=0}^{\ell} A_p$.*

Proof. In order to prove that

$$A_p = \bigoplus_{\substack{X \in L(\mathcal{A}), \\ r(X) = p}} A_X,$$

it suffices to prove

$$A_p = \sum_{\substack{X \in L(\mathcal{A}), \\ r(X) = p}} A_X.$$

$\subseteq$: For every $a \in A_p$, there exists $u = \sum_{s \in \mathbf{S}_p} c_S e_S \in E_p$, with $c_S \in \mathcal{K}$, such that $a = \varphi(u)$. Note that, for every S dependent, $e_S \in I$ and, for every S independent with $X = \cap S$, $e_S \in E_X$ (because $r(X) = |S| = p$), it follows that $a = \sum_{\substack{S \in \mathbf{S}_p, \\ S \text{ independent}}} c_S \varphi(e_S) \in A_X$.

$\supseteq$: Let $X \in L_p$. For every $a \in A_X$, there exists $u = \sum_{S \in \mathbf{S}_X} c_S e_S \in E_X$, with $c_S \in \mathcal{K}$, such that $a = \varphi(u)$. Since $e_S \in I$ for every S dependent and $e_S \in E_p$ for every S independent (because $r(\cap S) = r(X) = p$), It follows that $a = \sum_{\substack{S \in \mathbf{S}_X, \\ S \text{ independent}}} c_S \varphi(e_S) \in A_p$.

□

1.3.2 The broken circuit basis

The aim of this subsection is to show that the $\mathcal{K}$-algebra $A(\mathcal{A})$ is a free $\mathcal{K}$-module by constructing a standard $\mathcal{K}$-basis for $A(\mathcal{A})$.

Definition 1.3.21. Let $\prec$ be a fixed total order on $A(\mathcal{A})$ and denote $\mathbf{min}S$ the minimal element of S with respect to $\prec$.
A standard p-tuple $S = (H_1, \ldots, H_p)$ is a **broken circuit** if $\exists\, K \in \mathcal{A}$ such that $K \prec minS$ and $(K, H_1, \ldots, H_p)$ is a circuit.
A standard p-tuple $S = (H_1, \ldots, H_p)$ is a **nbc-set** if it does not contain any broken circuit. If S ia a nbc-set, we say that e_S is a **nbc-monomial**.

Remark 1.3.22. Every broken circuit is a dependent tuple obtained by deleting the minimal element in a standard circuit.
Every nbc-set is an independent tuple.

Definition 1.3.23. Define

$$nbc_p(\mathcal{A}) := \{S \in \mathbf{S}_p \mid \text{S is nbc-set}\} \qquad nbc(\mathcal{A}) := \bigcup_{p \geq 0} nbc_p(\mathcal{A});$$

$C_0 := \mathcal{K} \qquad C_p$ the free $\mathcal{K}$-module with basis $\{e_S \in E \mid S \in nbc_p(\mathcal{A})\}$ for $p \geq 1$;

$$C := C(\mathcal{A}) := \bigoplus_{p \geq 0} C_p \text{ the \textbf{broken circuit module}.}$$

Note that $C(\mathcal{A})$ is a free graded $\mathcal{K}$-module and a sub-$\mathcal{K}$-module of $E(\mathcal{A})$. In general C is not closed under multiplication in E, so C is not a sub-$\mathcal{K}$-algebra of E.

Remark 1.3.24. If S is a nbc-set, for every $s \in S$, $S \backslash \{s\}$ is a nbc-set. Then $\partial_E(C) \subseteq C$ and the restriction $\partial_C : C \longrightarrow C$ of $\partial_E : E \longrightarrow E$ is well defined. It follows that (C, ∂_C) is a chain complex.

Lemma 1.3.25. *(C, ∂_C) is an acyclic complex of chains.*

Proof. First we prove that $e_{H_1}C \subseteq C$ with H_1 the minimal element of $\mathcal{A}$ with respect to $\prec$. Let S be a nbc-set. If (H_1, S) contains a broken circuit, then there exists $S' \subseteq S$ such that (H_1, S') is a broken circuit. Since every broken circuit is obtained by deleting the maximal element in a circuit, there exists a circuit T such that $(H_1, S') = T \backslash minT$. It implies that $minT \prec H_1$, that is absurd for hypothesis.
In order to prove that (C, ∂_C) is an acyclic complex of chains, we have to demonstrate that $Ker(\partial_C) \subseteq Im(\partial_C)$. Let $c \in C$ with $\partial_C(c) = 0$ and H_1 the minimal element of $\mathcal{A}$ with respect to $\prec$. Thanks to the previous statement, we know that $e_{H_1}c \in C$, then $c = c - e_{H_1}(\partial_C(c)) = \partial_C(e_{H_1}c) \in Im(\partial_C)$. $\square$

Remark 1.3.26. Since E is a graded algebra with respect to $L(\mathcal{A})$, C is a graded algebra with respect to the same grading, i.e.

$$C = \bigoplus_{X \in L} C_X,$$

where, for every $X \in L(\mathcal{A})$, $C_X := C \cap E_X$.

Lemma 1.3.27. *The grading of C by $L(\mathcal{A})$ is finer than the standard grading $C = \bigoplus_{p=0}^{\ell} C_p$.*

Proof. It suffices to prove that for every $p \geq 0$,

$$C_p = \bigoplus_{X \in L_p} C \cap E_X =: \bigoplus_{X \in L_p} C_X.$$

$\subseteq$: Let $e_S \in C_p$. Since S is a nbc-set with $|S| = p$, it is an independent tuple with $r(\cap S) = p$. It immediately follows that $e_S \in \bigoplus_{X \in L_p} C_X$.
$\supseteq$: Let $e_S \in C_X$ with $X \in L_p$. Since every $S \in nbc(\mathcal{A})$ is independent, $|S| = r(\cap S) = r(X) = p$, then $e_S \in C_p$. □

We introduce now two statements that will be very usefull in the next theorem (see Proposition 3.31 and Lemma 3.40, [OT92] for the proofs).

Proposition 1.3.28. *Let $X \in L(\mathcal{A})$ and $\mathcal{A}_X := \{H \in \mathcal{A} | H \leq X\}$.*

1. $A_X(\mathcal{A}_X) \simeq A_X(\mathcal{A})$.

2. $C_X(\mathcal{A}_X) = C_X(\mathcal{A})$.

Theorem 1.3.29. *Let $\varphi : E \longrightarrow A(\mathcal{A})$ be the natural projection such that $\varphi(e_S) = a_S$. The $\mathcal{K}$-linear homomorphism $\varphi|_{C(\mathcal{A})} : C(\mathcal{A}) \longrightarrow A(\mathcal{A})$ is an isomorphism of graded $\mathcal{K}$-module.*

Proof. It suffices to prove that, for every $X \in L(\mathcal{A})$, $\varphi|_{C_X} : C_X \longrightarrow A_X$ is an isomorphism. We use induction on $rank(\mathcal{A})$.
If $rank(\mathcal{A}) = 0$, we have that $L = \{V\}$, then $C = C_V := C \cap E_V = C \cap \mathcal{K} = \mathcal{K} = A_V = A$. It follows that $\varphi|_{C_V}$ is the identity map.
Suppose the theorem holds for $rank(\mathcal{A}) < r$. We need to prove it for $rank(\mathcal{A}) = r$. Let $X \in L(\mathcal{A})$.

- If $r(X) < r$, then $r(\mathcal{A}_X) < r$, so by induction hypothesis $\tilde{\varphi} : C_X(\mathcal{A}_X) \longrightarrow A_X(\mathcal{A}_X)$ is an isomorphism. Considering Proposition 1.3.28, it follows that $\varphi|_{C_X} : C_X(\mathcal{A}) = C_X(\mathcal{A}_X) \xrightarrow{\tilde{\varphi}} A_X(\mathcal{A}_X) \simeq A_X$ is also an isomorphism.

- If $r(X) = r$, since $\mathcal{A}$ is a central arrangement, $X = T$ where T is the center of $\mathcal{A}$. Now consider the following diagram

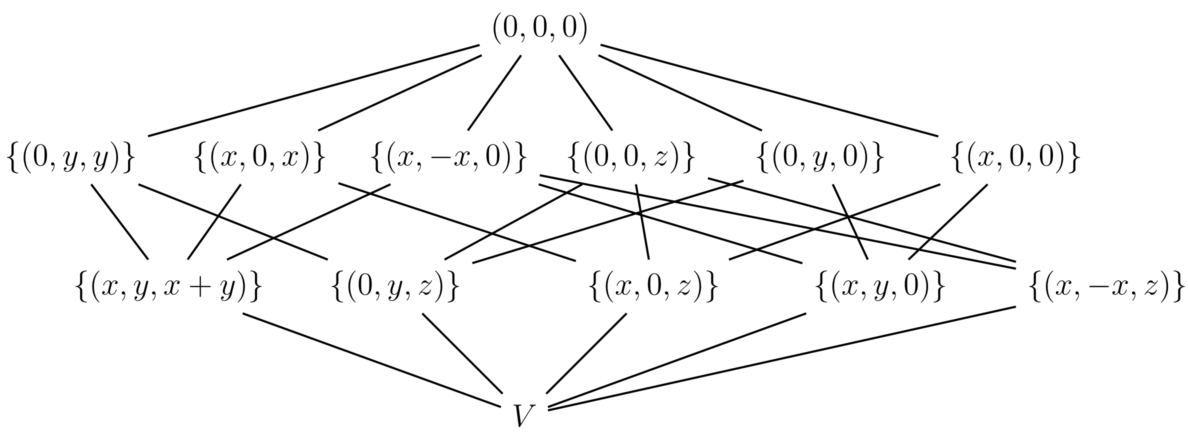
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C_r & \xrightarrow{\partial_C} & C_{r-1} & \xrightarrow{\partial_C} & \dots & \xrightarrow{\partial_C} & C_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi|_{C_r}} & & \downarrow{\scriptstyle \varphi|_{C_{r-1}}} & & & & \downarrow{\scriptstyle \varphi|_{C_0}} & & \\ 0 & \longrightarrow & A_r & \xrightarrow{\partial_A} & A_{r-1} & \xrightarrow{\partial_A} & \dots & \xrightarrow{\partial_A} & A_0 & \longrightarrow & 0 \end{array}$$

  Since $A_p = \oplus_{X \in L_p} A_X$ and $C_p = \oplus_{X \in L_p} C_X$, it follows that $\varphi|_{C_p}$ is an isomorphism for every $p < r$. For the five lemma we have that also $\varphi|_{C_r}$ is an isomorphism.

□

Corollary 1.3.30. *$\{a_S \in A(\mathcal{A}); S$ is nbc-set$\}$ is a basis, called* ***the broken circuit basis****, for $A(\mathcal{A})$ as a graded $\mathcal{K}$-module.*

Example 1.3.31. Let $\mathcal{A}$ be the central and essential arrangement defined by $Q(\mathcal{A}) = xyz(x+y)(x+y-z)$.
Define $H_0 = \{x + y = z\}, H_1 = \{x = 0\}, H_2 = \{y = 0\}, H_3 = \{z = 0\}$ and $H_4 = \{x = -y\}$. The Hasse diagram of L($\mathcal{A}$) is:



The circuits of $\mathcal{A}$ are: $\{(H_0, H_3, H_4), (H_1, H_2, H_4), (H_0, H_1, H_2, H_3)\}$, then

$$\begin{aligned} nbc(\mathcal{A}) = \emptyset \cup \mathbf{S}_1 \cup \mathbf{S}_2 \backslash \{(H_2, H_4), (H_3, H_4)\} \cup \\ \cup \{(H_0, H_1, H_2), (H_0, H_1, H_3), (H_0, H_1, H_4), (H_0, H_2, H_3)\}. \end{aligned}$$

Then $\{a_S; S \in nbc(\mathcal{A})\}$ is the broken circuit basis of $A(\mathcal{A})$. Note that in order to compute this basis we needed only the poset $L(\mathcal{A})$.

1.3.3 The deletion-restriction exact sequence

Definition 1.3.32. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a non-empty arrangement and let H' a fixed hyperplane in $\mathcal{A}$. We define two new arrangements:
$\mathcal{A}' := \mathcal{A} \backslash \{H'\}$ obtained by the **deletion** of H' and
$\mathcal{A}'' := \{H \cap H' | H \in \mathcal{A}\}$ obtained by the **restriction** of $\mathcal{A}$ to H'.

Remark 1.3.33. $\mathcal{A}'$ is an ℓ-arrangement with $|\mathcal{A}'| = n - 1$, and $\mathcal{A}''$ is an $(\ell - 1)$-arrangement in the space H' with $|\mathcal{A}''| \leq n - 1$.

For any object O associated with arrangements, we write $O = O(\mathcal{A}), O' = (\mathcal{A}')$ and $O'' = O(\mathcal{A}'')$.

Without loss of generality, from now on we will always choose H' as the last hyperplane H_n.

Definition 1.3.34. Let $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ as defined above. Define the map

$$\lambda : \mathcal{A}' \longrightarrow \mathcal{A}''$$

$$H \longmapsto H \cap H_n,$$

and, for every $S = (H_{i_1}, \ldots, H_{i_p})$, the q-tuple

$$\lambda(S) := (H_{j_i}, \ldots, H_{j_q})$$

such that $\lambda(H_{i_k}) = H_{j_k}$ for every $k \in \{1, \ldots, p\}$ and $e_{\lambda(S)} := e_{H_{j_i}} \wedge \cdots \wedge e_{H_{j_q}}$.

Note that, since distinct hyperplanes in $\mathcal{A}$ may have identical intersection with H_n, the indices j_k need not to be distinct and then $|\mathcal{A}''| \leq n - 1$.

From now on, we choose linear order on $\mathcal{A}'$ and $\mathcal{A}''$ such that $\mathcal{A}'$ inherite the order from $\mathcal{A}$, i.e. $\mathcal{A}' = \{H_1, \dots, H_{n-1}\}$, and in $\mathcal{A}''$, for every $i, j \in \{1, \dots, n-1\}$, $H_i \cap H_n$ precedes $H_j \cap H_n$ if and only if H_i precedes H_j in $\mathcal{A}'$.

Let's consider the sequence

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ j\ } E'' \longrightarrow 0,$$

where i is the natural inclusion

$$i(e_S) = e_S$$

and

$$j(e_S) = \begin{cases} e_{\lambda(S \setminus \{H_n\})} & \text{if } H_n \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Note that i is an injective map, j is surjective, $Im(i) \subseteq Ker(j)$ but $Ker(j) \nsubseteq Im(i)$. In fact, if $S = (H_1, H_2, H_n)$ where $H_1 \cap H_n = H_2 \cap H_n$, then $j(e_S) = e_{H_1 \cap H_n} e_{H_2 \cap H_n} = 0$, but S is not a tuple of hyperplanes of $\mathcal{A}'$. It follows that this sequence need not to be exact at E.

Now we prove a lemma that will be very usefull in Proposition 1.3.36 and in Theorem 1.3.37.

Lemma 1.3.35. *1. $i(I') \subset I$;*

2. $j(I) \subset I''$;

3. $i(C') \subset C$;

4. $j(C) = C''$.

Proof. 1. It immediately follows from the fact that if S is dependent in $\mathcal{A}'$ then it is dependent also in $\mathcal{A}$.

2. Let $e_S \in E$. If $H_n \notin S$, then $j(e_S) = 0$. If $S = (H_{i_1}, \dots, H_{i_q}, H_n)$, we have that $\cap\lambda(S\backslash\{H_n\}) = (H_{i_1} \cap H_n, \dots, H_{i_q} \cap H_n) = \cap S$. If S is dependent, then $\alpha_{H_{i_1}}, \dots \alpha_{H_{i_q}}, \alpha_{H_n}$ are dependent functionals, and then also $\alpha_{H_{i_1}}|_{H_n}, \dots \alpha_{H_{i_q}}|_{H_n}$ are dependent functionals, that is $\lambda\left(S\backslash\{H_n\}\right)$ is dependent too. Now

$$\begin{aligned} j(\partial e_S) &= j\left(\partial(e_{S\backslash\{H_n\}})e_{H_n} \pm e_{S\backslash\{H_n\}}\partial(e_{H_n})\right) = j\left(\partial(e_{S\backslash\{H_n\}})e_{H_n}\right) = \\ &= j\left(\sum_{j=i_1}^{i_q}(-1)^{j-1}e_{S\backslash\{H_j,H_n\}}e_{H_n}\right) = \\ &= \sum_{j=i_1}^{i_q}(-1)^{j-1}e_{H_{i_1}\cap H_n}\dots\widehat{e_{H_j\cap H_n}}\dots e_{H_{i_q}\cap H_n} = \\ &= \partial(e_{H_{i_1}\cap H_n}\dots e_{H_{i_q}\cap H_n}) = \partial(e_{\lambda(S\backslash\{H_n\})}) \in I''. \end{aligned}$$

3. Let $e_S \in C'$. If $e_S \notin C$, then there exists $S' \subseteq S$ and $H \in \mathcal{A}$ such that $H \prec minS'$ and (H, S') circuit. Since S is a nbc-set for $\mathcal{A}'$, it follows that H has to be equal to H_n but $H_i \prec H_n$ for every $i \in \{1, \dots, n-1\}$ and this contradicts the hypothesis.

4. First note that if $\tilde{S} \in \{H_1 \cap H_n, \dots, H_{n-1} \cap H_n\}$, there exists $S \in \{H_1, \dots, H_n\}$ such that $\lambda(S\backslash\{H_n\}) = \tilde{S}$. The thesis immediately follows from the fact that S is a nbc-set if and only if $\lambda(S\backslash\{H_n\})$ is a nbc-set.

□

Proposition 1.3.36. *Let $\tilde{i}$ and $\tilde{j}$ be the restriction of i and j respectively to C' and C. The short sequence*

$$0 \longrightarrow C' \xrightarrow{\tilde{i}} C \xrightarrow{\tilde{j}} C'' \longrightarrow 0$$

is exact.

Proof. Thanks to 3) and 4) of Lemma 1.3.35, the sequence is well defined. Moreover, since i is injective, j is surjective and $j \circ i = 0$, it suffices to prove that $Ker(\tilde{j}) \subseteq Im(\tilde{i})$.
Let $N \subseteq C$ be the sub-$\mathcal{K}$-module generated by $\{e_S | H_n \in S, S \text{ nbc-set}\}$, since $C = N \oplus \tilde{i}(C')$ and since $\tilde{i}(C') \subseteq Ker(\tilde{j})$, we have that $Ker(\tilde{j}) \subseteq Im(\tilde{i}) \Leftrightarrow Ker\tilde{j} \cap N = \{0\}$. Then it is enough to check $\tilde{j}$ is injective over N. Let S,T nbc-sets such that $H_n \in S, H_n \in T$ and suppose $S \neq T$. It follows that if $e_{\lambda(S\backslash\{H_n\})} = e_{\lambda(T\backslash\{H_n\})}$, then $\lambda(S\backslash\{H_n\}) = \lambda(T\backslash\{H_n\})$, so $\exists H_s \in S, H_t \in T$ with $H_s \neq H_t$, such that $H_s \cap H_n = H_t \cap H_n$. This implies that, if we suppose $H_s \prec H_t$, $\{H_s, H_t, H_n\}$ is a dependent set and then $\{H_t, H_n\}$ is a broken circuit. This contradicts the fact that T is a nbc-set. It follows that $S = T$ and this completes the proof. □

Theorem 1.3.37. *The not exact sequence*

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ j\ } E'' \longrightarrow 0$$

descends to the exact sequence

$$0 \longrightarrow A' \xrightarrow{\ \widehat{i}\ } A \xrightarrow{\ \widehat{j}\ } A'' \longrightarrow 0.$$

Proof. Thanks to 1) and 2) of Lemma 1.3.35, factorising $0 \to E(\mathcal{A}')_k \xrightarrow{i_k} E(\mathcal{A})_k \xrightarrow{j_k} E(\mathcal{A}'')_{k-1} \to 0$ we obtain the well define sequence $0 \longrightarrow A(\mathcal{A}')_k \xrightarrow{\widehat{i_k}} A(\mathcal{A})_k \xrightarrow{\widehat{j_k}} A(\mathcal{A}'')_{k-1} \longrightarrow 0$. Theorem 1.3.29 and Proposition 1.3.36 suffice to conclude. □

1.4 The matroid $\mathcal{M}(\mathcal{A})$

In the previous section we have construct the Orlik-Solomon algebra A of a central arrangement $\mathcal{A}$ and it is clear by its definition that it depends only on the poset $L(\mathcal{A})$, i.e. on the linear dependencies of the hyperplanes. We now want to show that the Orlik-Solomon algebra depends only on a particular matroid associated with the arrangement.

The next statement immediately follows from Remark 1.3.6.

Proposition 1.4.1. *The pair $(\mathcal{A}, \{S \in \boldsymbol{S}; S \text{ independent}\})$ is a matroid, called the **matroid of the arrangement** $\mathcal{M} = \mathcal{M}(\mathcal{A})$.*

Remark 1.4.2.

- $\{S \in \mathbf{S}|; S \text{ is a circuit}\} = \mathcal{C}(\mathcal{M})$ where $\mathcal{C}(\mathcal{M})$ is the set of circuits of $\mathcal{M}$;
- If r is the rank function on $\mathrm{L}(\mathcal{A}) = \{\cap X; X \subseteq \mathcal{A}\}$ and $r_{\mathcal{M}}$ is the rank function of the matroid of the arrangement, then $r(\cap X) = r_{\mathcal{M}}(X)$;
- $cl(X) = \{K \in \mathcal{A}; \alpha_K \in span\{\alpha_H; H \in X\}\}$.

Theorem 1.4.3. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement and let $L(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$ be, respectively, the poset of intersections of the arrangement and the poset of flats of the matroid of the arrangement. Then*

$$L(\mathcal{A}) \cong \mathcal{L}(\mathcal{M}).$$

Proof. Let $S, T \subseteq \{H_1, \dots, H_n\}$. Then

$$\begin{aligned} \cap S = \cap T \quad &\Leftrightarrow \quad span\{\alpha_H | H \in S\} = span\{\alpha_H | H \in T\} \\ &\Leftrightarrow \quad cl(S) = cl(T). \end{aligned}$$

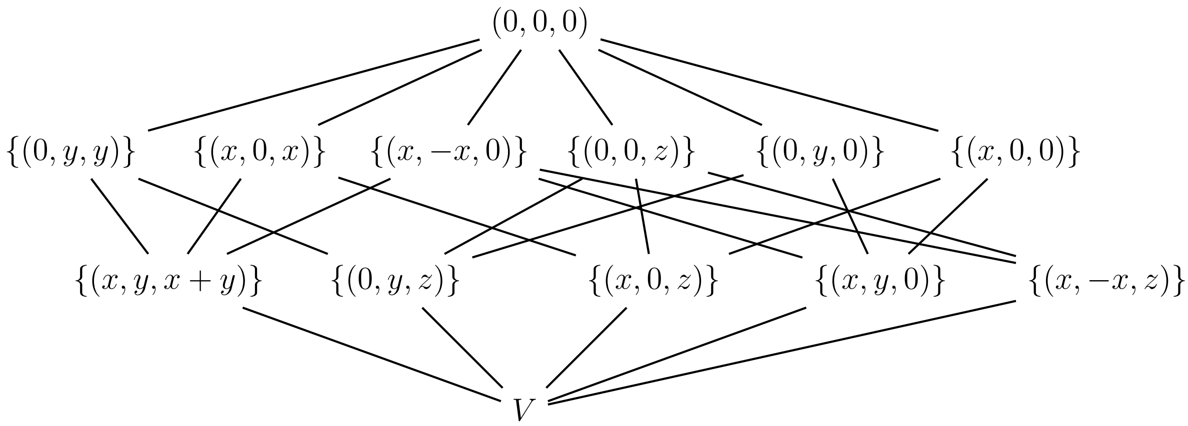
Since $\mathcal{L}(\mathcal{M}) = \{cl(X) | X \subseteq E\}$, it follows that the map ϕ that sends $\cap S$ to $cl(S)$ is an isomorphism between $L(\mathcal{A})$ and $\mathcal{L}(\mathcal{M})$.

□

It immediately follows that

Corollary 1.4.4. *Let $\mathcal{A}$ be a central arrangement. Then the poset L($\mathcal{A}$) of the intersections of the hyperplanes of $\mathcal{A}$ depends only on the matroid of the arrangement $\mathcal{M}(\mathcal{A})$. Thus the matroid $\mathcal{M}(\mathcal{A})$ encapsulates the essential information about $\mathcal{A}$ needed to define L($\mathcal{A}$) and the Orlik-Solomon algebra of the arrangement.*

Example 1.4.5. Let $\mathcal{A}$ be the central and essential arrangement of Example 1.3.31 with Hasse diagram of L($\mathcal{A}$)



Note that:
$\mathcal{I}(\mathcal{M}) := \{S \in \mathbf{S} | S \text{ is independent}\} = \{\emptyset\} \cup \mathbf{S}_1 \cup \mathbf{S}_2 \cup (\mathbf{S}_3 \backslash \{(H_0, H_3, H_4), (H_1, H_2, H_4)\})$;
$\mathcal{C}(\mathcal{M}) = \{(H_0, H_3, H_4), (H_1, H_2, H_4), (H_0, H_1, H_2, H_3)\}$
and $\mathcal{B}(\mathcal{M}) = \mathbf{S}_3 \backslash \{(H_0, H_3, H_4), (H_1, H_2, H_4)\}$.
In order to construct the lattice of the flats of $\mathcal{M}(\mathcal{A})$, first we have to compute the rank and closure function of $\mathcal{M}(\mathcal{A})$.

$$\begin{aligned} r(\emptyset) &= 0 \\ r((H_i)) &= 1 && \forall i; \\ r((H_i, H_j)) &= r((H_0, H_3, H_4)) = r((H_1, H_2, H_4)) = 2 && \forall i \neq j \\ r((H_i, H_j, H_k)) &= r(X) = 3 && \forall (i, j, k) \notin \{(0, 3, 4), (1, 2, 4)\}, \\ & && \forall X \in \mathbf{S}_4 \cup \mathbf{S}_5. \end{aligned}$$

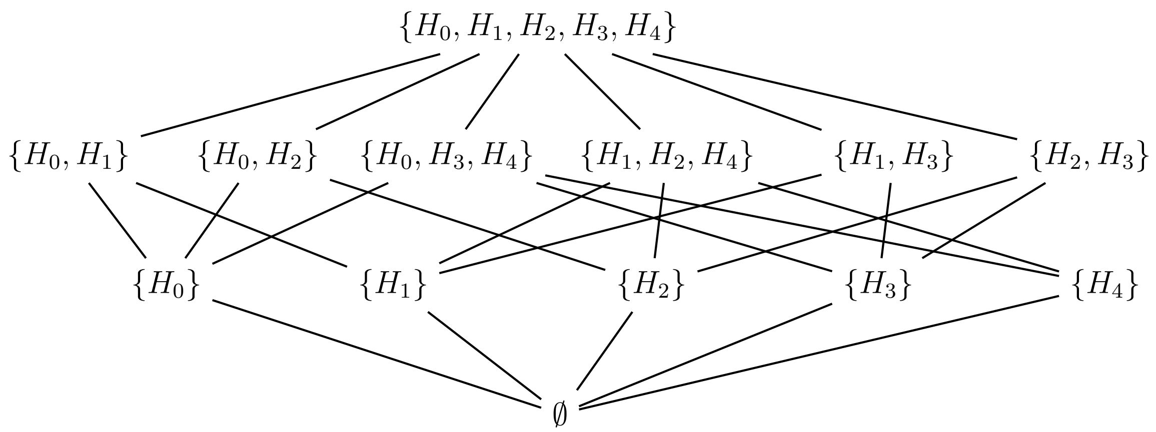
Then

$$
\begin{aligned}
cl(\emptyset) &= \emptyset \\
cl(\{H_i\}) &= \{H_i\} && \forall i \\
cl(\{H_i, H_j\}) &= \{H_i, H_j\} && \forall \{i,j\} \notin \{\{0,3\},\{0,4\},\{3,4\},\{1,2\},\{1,4\},\{2,4\}\} \\
cl(\{H_i, H_j\}) &= \{H_0, H_3, H_4\} && \forall \{i,j\} \in \{\{0,3\},\{3,4\},\{0,4\}\} \\
cl(\{H_i, H_j\}) &= \{H_1, H_2, H_4\} && \forall \{i,j\} \in \{\{1,2\},\{1,4\},\{2,4\}\} \\
cl(\{H_0, H_3, H_4\}) &= \{H_0, H_3, H_4\} \\
cl(\{H_1, H_2, H_4\}) &= \{H_1, H_2, H_4\} \\
cl(\{H_i, H_j, H_k\}) &= cl(X) = \mathcal{A} && \forall \{i,j,k\} \notin \{\{0,3,4\},\{1,2,4\}\} \\
& && \forall X \in \mathbf{S}_4 \cup \mathbf{S}_5.
\end{aligned}
$$

Finally we have:

$$
\begin{aligned}
\mathcal{L}(\mathcal{M}) =& \{\emptyset\} \cup \mathbf{S}_1 \cup \Big\{ \{H_i, H_j\} \quad \forall \{i,j\} \notin \{\{0,3\},\{0,4\},\{3,4\},\{1,2\},\{1,4\},\{2,4\}\} \Big\} \cup \\
& \cup \{H_0, H_3, H_4\} \cup \{H_1, H_2, H_4\} \cup \mathbf{S}_5.
\end{aligned}
$$

The Hasse diagram of $\mathcal{L}(\mathcal{A})$ is



which is isomorphic to the Hasse diagram of of $L(\mathcal{A})$.

1.5 Cohomology of the complement $M(\mathcal{A})$

Definition 1.5.1. Let $V_{\mathbb{C}} \backsimeq \mathbb{C}^{\ell}$ be a complex vector space and $\mathcal{A}$ be an arrangement in $V_{\mathbb{C}}$. Denote the complement of H as

$$M_H := V \backslash H$$

and the complement of $\mathcal{A}$ as

$$M = M(\mathcal{A}) := V \backslash \bigcup_{H \in \mathcal{A}} H.$$

Note that, since V is an open smooth manifold of real dimension 2ℓ and M, M_H are open in V, M, M_H are open smooth manifolds of real dimension 2ℓ. It follows that we can study theirs de Rham cohomology $H^*(M;\mathbb{Z})$ and $H^*(M_H;\mathbb{Z})$. All cohomology rings to follow have integer coefficients, and we henceforth omit the $\mathbb{Z}$ from our notation.

Remark 1.5.2. The canonical generator of $H^1(\mathbb{C}^*) \simeq \mathbb{Z}$ in the de Rham cohomology can be represented by the form $\frac{1}{2\pi i} dlog(z)$. It follows that, since M_H is homotopy equivalent to $\mathbb{C}^*$ via projection onto a complex line meeting H transversely, $H^1(M_H) \simeq H^1(\mathbb{C}^*) \simeq \mathbb{Z}$. Note that, for every $H \in \mathcal{A}$, we have a well-defined map

$$\alpha_H|_{M_H} : M_H \longrightarrow \mathbb{C}^*.$$

It follows that we are allowed to give the next definitions.

Definition 1.5.3. For every $H \in \mathcal{A}$, define on V the following logarithmic form

$$\varpi_H = \frac{1}{2\pi i} dlog(\alpha_H).$$

We will denote ϖ_i, instead of ϖ_{H_i}, the logarithmic form of $H_i \in \mathcal{A}$.
Let i and j be the inclusion maps $i : M_H \longrightarrow V$ and $j : M \longrightarrow V$.
Denote $< \varpi_H >$ the cohomology class of ϖ_H in $H^1(M_H)$ and $[\varpi_H]$ the cohomology class of ϖ_H in $H^1(M)$.

Remark 1.5.4. Let k be the inclusion map $k : M \longrightarrow M_H$, then

$$< \varpi_H >= H^1\left(\alpha_H|^*_{M_H}\right)\left(\frac{1}{2\pi i}dlog(z)\right) \in H^1(M_H);$$

$$[\varpi_H] = H^1(k^*)(< \varpi_H >) \in H^1(M).$$

Note that the cohomology classes $[\varpi_H] \in H^1(M)$ determine a homomorphism of graded algebras $\vartheta : E \to H^*(M)$ sending $e_H \longmapsto [\varpi_H]$.

Proposition 1.5.5. *The map ϑ defined above, descends to a homomorphism of graded algebras $\theta : A \to H^*(M)$.*

Proof. It suffice to prove that $\vartheta(I) = 0$. Let $S = \{H_{i_1}, \dots, H_{i_k}\}$ be a dependent set and consider $\{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ the set of the correspondent linear dependent functionals. Without loss of generality, we can suppose $\alpha_{i_k} = \sum_{j=1}^{k-1} c_j\alpha_{i_j}$ where $c_j \in \mathbb{C}$ for every j, then $\varpi_{i_k} := \frac{1}{2\pi i}\frac{d\alpha_{i_k}}{\alpha_{i_k}} = \frac{1}{\alpha_{i_k}}\sum_{j=1}^{k-1} c_j\alpha_{i_j}\varpi_{i_j}$.
Thus

$$\begin{aligned}
\vartheta(\partial e_S) &:= \sum_{j=1}^{k}(-1)^{j-1}\vartheta(e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_k}) = \\
&= \sum_{j=1}^{k-1}(-1)^{j-1}(\varpi_{i_1} \wedge \cdots \wedge \widehat{\varpi_{i_j}} \wedge \cdots \wedge \frac{1}{\alpha_{i_k}}c_j\alpha_{i_j}\varpi_{i_j})+ \\
&+ (-1)^{k-1}[\varpi_{i_1} \wedge \cdots \wedge \varpi_{i_{k-1}}] = \\
&= \sum_{j=1}^{k-1}(-1)^{j-1}(-1)^{k-j-1}\frac{1}{\alpha_{i_k}}c_j\alpha_{i_j}[\varpi_{i_1} \wedge \cdots \wedge \varpi_{i_{k-1}}]+ \\
&+ (-1)^{k-1}[\varpi_{i_1} \wedge \cdots \wedge \varpi_{i_{k-1}}] = \\
&= \Big((-1)^{k-2}\sum_{j=1}^{k-1}\frac{1}{\alpha_{i_k}}c_j\alpha_{i_j} + (-1)^{k-1}\Big)[\varpi_{i_1} \wedge \cdots \wedge \varpi_{i_{k-1}}] = \\
&= \Big((-1)^{k-1}\big(-\frac{1}{\alpha_{i_k}}\alpha_{i_k} + 1\big)\Big)[\varpi_{i_1} \wedge \cdots \wedge \varpi_{i_{k-1}}] = \\
&= 0.
\end{aligned}$$

□

1.5.1 Isomorphism between $A(\mathcal{A})$ and $H^*(M(\mathcal{A}))$

Our aim is to prove that the map θ of Proposition 1.5.5 is an isomorphism of graded algebras. In order to do this we construct an exact sequence in cohomology analogous to the one that we have construct with Orlik-Solomon algebras in Theorem 1.3.37.

Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be the deletion restriction triple determined by the hyperplane $H_n \in \mathcal{A}$. We denote the complements of these arrangements by M', M and M'' respectively.

Remark 1.5.6. $M'' = H_n \backslash \bigcup_{i \neq n}(H_i \cap H_n) = M' \cap H_n$ and $M = M' \backslash M''$.

Lemma 1.5.7. *There exist two maps ϕ and ψ such that*

$$... \longrightarrow H^k(M') \xrightarrow{H^k(i_M^*)} H^k(M) \xrightarrow{\phi} H^{k-1}(M'') \xrightarrow{\psi} H^{k+1}(M') \longrightarrow ...$$

is a long exact sequence in cohomology, where $i_M : M \longrightarrow M'$ is the inclusion map.

Proof. By the long exact sequence of the pair (M', M),

$$... \longrightarrow H^k(M') \xrightarrow{H^k(i_M^*)} H^k(M) \xrightarrow{H^k i} H^{k+1}(M', M) \xrightarrow{H^{k+1} j} H^{k+1}(M') \longrightarrow ...$$

it is enough to find an isomorphism $H^{k-1}(M'') \simeq H^{k+1}(M', M)$.

Since $M'' = M' \cap H_n$, we can consider a tubular neighborhood N of M'' in M' and $N_0 := N \backslash M''$. For example, in the arrangement of Example 1.1.9, we would have:

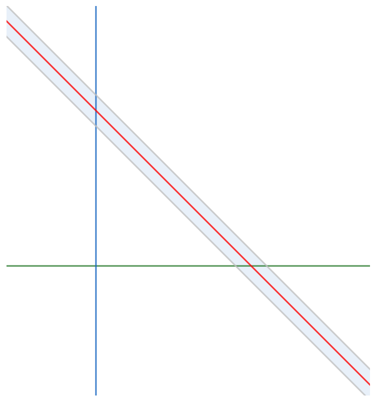


Figure 1.3: Tubular neighborhood of $M'' = M' \cap H_3$.

It follows that there exists a vector bundle $\xi = (N, \pi, M'')$ of rank 1 such that the restriction on N_0 is a bundle with fibre homeomorphic to $\mathbb{C}^*$. Since both bundles are restriction of trivial bundles over H_n, it follows that they are trivial and $(N, N_0) = M'' \times (\mathbb{C}, \mathbb{C}^*)$ up to homeomorphism. Thus

$$H^*(N, N_0) = H^*(M'') \otimes H^*(\mathbb{C}, \mathbb{C}^*).$$

For every k there is an isomorphism $\tau : H^k(M'') \longrightarrow H^{k+2}(N, N_0)$ called **Thom isomorphism** (for further information see [BT95, Chapter 6]).
Since $M' \backslash N \subseteq M \subseteq M'$, by excision we have

$$H^k(M', M) \simeq H^k(M' \backslash (M' \backslash N), M \backslash (M' \backslash N))$$

and since $M \backslash (M' \backslash N) = N \backslash M'' = N_0$, we have an isomorphism

$$r : H^k(M', M) \longrightarrow H^k(N, N_0).$$

Thus if suffice to define $\phi := \tau^{-1} \circ r \circ H^k i$ and $\psi = H^{k+1} j \circ r^{-1} \circ \tau$. $\square$

Now we can finally conclude that

Theorem 1.5.8. *The map $\theta : A \to H^*(M)$ is an isomorphism of graded algebras.*

Proof. We use induction on $|\mathcal{A}|$. If $\mathcal{A} = \emptyset$, $A \simeq H^*(M) = \mathbb{Z}$ and $\theta : A \longrightarrow H^*(M)$ is the identity map. If $\mathcal{A} \neq \emptyset$, we can consider the deletion restriction triple $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ determined by the last hyperplane H_n and we have already proved in Theorem 1.3.37 that, for every k, the sequence

$$0 \longrightarrow A'_k \xrightarrow{\widehat{i}} A_k \xrightarrow{\widehat{j}} A''_k \longrightarrow 0$$

is exact. Let's consider the following commutative diagram for every k.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A'_k & \xrightarrow{\widehat{i}} & A_k & \xrightarrow{\widehat{j}} & A''_k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta''} & & \\ 0 & \longrightarrow & H^k(M') & \xrightarrow{H^k(i_M^*)} & H^k(M) & \xrightarrow{\phi} & H^{k-1}(M'') & \longrightarrow & 0 \end{array}$$

By induction hypothesis, θ' and θ'' are isomorphisms. If we prove that the second row is exact, then, thanks to the five lemma, we can conclude that θ is an isomorphism too. From Lemma 1.5.7 we know that the second row is exact in $H^k(M)$. Since the first row is exact in A''_k, $\theta'' \circ \widehat{j}$ is surjective. It follows from the commutativity of the diagram that $\theta \circ \psi$ is surjective, thus ϕ is too and the second row is exact in $H^{k-1}(M'')$. Again from Lemma 1.5.7, $Ker\psi = Im\phi = H^{k-1}(M'')$ i.e. $\psi = 0$. It immediately follows that $KerH^k(i_M^*) = Im\psi = 0$ and then the second row is exact in $H^k(M')$.

□

Corollary 1.5.9. *The cohomology classes $[\varpi_H]$ generate $H^*(M)$.*

Remark 1.5.10. In other words, the cohomology algebra $H^*(M(\mathcal{A}))$ is isomorphic to the algebra A with

- A generated by $\{a_S; S \text{ independent set}\}$. The degree of a_S is $|S|$.

- The following types of relations

  1. For any two generators a_S and $a_{S'}$,

  $$a_S a_{S'} = \begin{cases} 0 & \text{if } S \sqcup S' \text{ is a dependent set} \\ (-1)^{\ell(S,S')} a_{S \sqcup S'} & \text{otherwise} \end{cases}$$

  where, for $C = \{c_1 < \cdots < c_l\}$ and $D = \{d_1 < \cdots < d_h\}$, $\ell(C, D)$ denote the lenght of the permutation that takes $(C, D) = \{c_1, \ldots, c_l, d_1, \ldots, d_h\}$ into $C \cup D$.

  2. For every circuit C,

  $$\partial(a_C) = \sum_{H_j \in C} (-1)^{j-1} a_{C \setminus H_j} = 0.$$

1.5.2 Consequences

In Section 1.4 we have seen that the Orlik-Solomon algebra $A(\mathcal{A})$ depends only on the matroid $\mathcal{M}(\mathcal{A})$. Thanks to Theorem 1.5.8 we can say that also the cohomology of $M(\mathcal{A})$ is completely determined by the matroid $\mathcal{M}(\mathcal{A})$.

Since the complement of a hyperplane ℓ-arrangement in $V \simeq \mathbb{C}^\ell$ is a manifold of real dimension 2ℓ, we know that $H^k(M(\mathcal{A})) = 0$, for every $k > 2\ell$. Moreover, thanks to Theorem 1.5.8, we can also state that

Corollary 1.5.11.

$$H^k(M(\mathcal{A})) = 0, \text{ for every } k > \ell.$$

Definition 1.5.12. The **Poincaré polynomial of the complement** is defined as

$$Poin(M(\mathcal{A}), t) = \sum_{p \geq 0} b_p(M) t^p,$$

where, for every p, $b_p(M) = rank(H^p(M))$ is the p-th **Betti number** of M.

Theorem 1.5.13. *Let $\mathcal{A}$ be a complex arrangement. Then*

$$Poin(M(\mathcal{A}), t) = \pi(\mathcal{A}, t).$$

where $\pi(\mathcal{A}, t)$ is the Poincaré polynomial of $\mathcal{A}$.
In particular

$$b_p(M) = \sum_{X \in L_p} (-1)^p \mu(X).$$

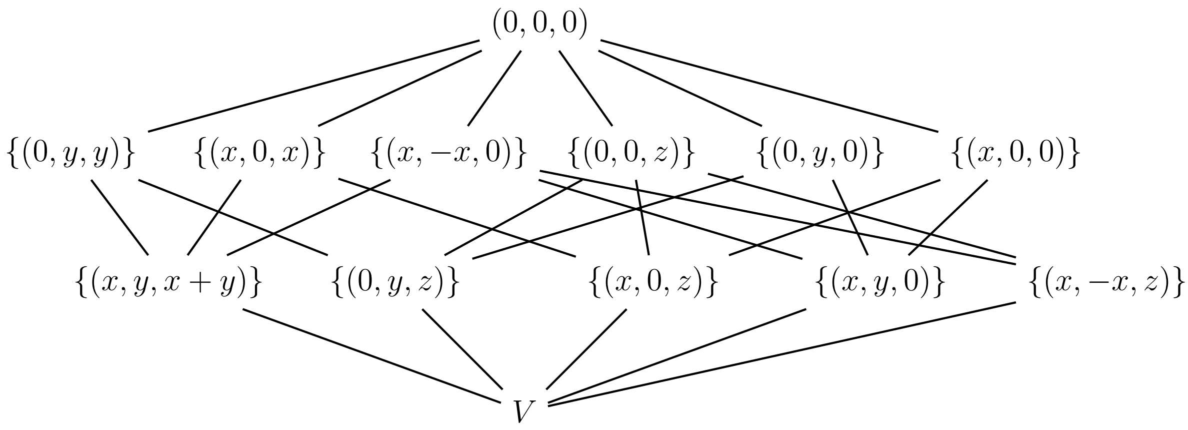
Example 1.5.14. Let $\mathcal{A}$ be the arrangement of Example 1.3.31, in which we have already computed $\{a_S; S \in nbc(\mathcal{A})\}$ the broken circuit basis of $A(\mathcal{A})$, with

$$\begin{aligned} nbc(\mathcal{A}) = \emptyset \cup \mathbf{S}_1 \cup \mathbf{S}_2 \backslash \{(H_2, H_4), (H_3, H_4)\} \cup \\ \cup \{(H_0, H_1, H_2), (H_0, H_1, H_3), (H_0, H_1, H_4), (H_0, H_2, H_3)\}. \end{aligned}$$

Since $A(\mathcal{A})$ is a graded algebra isomorphic to $H^*(M(\mathcal{A}))$,

$$b_0 = 1 \qquad b_1 = 5 \qquad b_2 = \binom{5}{2} - 2 = 8 \qquad b_3 = 4.$$

If we compute b_p using Theorem 1.5.13 we obtain the same results. In fact



$\sum_{X \in L_0} (-1)^0 \mu(X) = (-1)^0(1) = 1;$
$\sum_{X \in L_1} (-1)\mu(X) = 5(-1)(-1) = 5;$
$\sum_{X \in L_2} (-1)^2 \mu(X) = 4(-1)^2(1) + 2(-1)^2(2) = 8;$
$\sum_{X \in L_3} (-1)^3 \mu(X) = (-1)^3 \mu((0, 0, 0)) = \sum_{Z < (0,0,0)} \mu(Z) = 1 - 5 + 8 = 4.$

Chapter 2

Complex tori

Definition 2.0.1. The d-dimensional **complex torus** is the set $T := (\mathbb{C}^*)^d$ of d-tuples of non-zero complex numbers. This is a group under coordinate-wise multiplication.

Let's now consider the surjective group homomorphism $f : \mathbb{C} \longrightarrow \mathbb{C}^*$ such that $f(z) = exp(2\pi iz)$. Since $Ker(f) = \mathbb{Z}$, $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$. It immediately follows that exists a group isomorphism $\tilde{f}$ such that $\mathbb{C}^d/\mathbb{Z}^d \simeq (\mathbb{C}^*)^d$.

2.1 Characters of the torus

Definition 2.1.1. A **character** of $(\mathbb{C}^*)^d$ is a group homomorphism
$f : (\mathbb{C}^*)^d \longrightarrow \mathbb{C}^*$.
Denote Λ the set of characters of $(\mathbb{C}^*)^d$.

Definition 2.1.2. Define $\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ the ring consisting of **Laurent polynomials**

$$f(t_1, \dots, t_d) := \sum_{(a_1,\dots,a_d) \in \mathbb{Z}^d} \lambda_{(a_1,\dots,a_d)} t_1^{a_1} \dots t_d^{a_d}$$

with finitely many $\lambda_{(a_1,\dots,a_d)} \neq 0$.

Proposition 2.1.3. *Let $f : (\mathbb{C}^*)^d \longrightarrow \mathbb{C}^*$ be a Laurent polynomial.*
f is a character of $(\mathbb{C}^*)^d$ if and only if it is a Laurent monomal, i.e. for every $\underline{t} \in (\mathbb{C}^*)^d$,

$$f(t_1, \dots, t_d) = \prod_{i=1}^{d} t_i^{a_i}$$

with $\underline{a} \in \mathbb{Z}^d$.

Proof. $\Leftarrow$: It is immediate to verify since, for every $\underline{t}, \underline{s} \in (\mathbb{C}^*)^d$,
$f(\underline{ts}) = (t_1 s_1)^{a_1} \dots (t_d s_d)^{a_d} = t_1^{a_1} \dots t_d^{a_d} s_1^{a_1} \dots s_d^{a_d} = f(\underline{t}) f(\underline{s})$.
$\Rightarrow$: Let $\underline{s} \in (\mathbb{C}^*)^d$ and define $m_{\underline{s}}$ the multiplication operator

$$m_{\underline{s}} : \Lambda \longrightarrow \Lambda$$

$$f \longmapsto \tilde{f}$$

such that $\tilde{f}(\underline{t}) := f(\underline{st})$. If f is a character of $(\mathbb{C}^*)^d$, it is an eigenvector for the operator $m_{\underline{s}}$ of eigenvalue $f(\underline{s})$, in fact $m_{\underline{s}}(f)(\underline{t}) = f(\underline{s}) f(\underline{t})$.
In particular, if $\underline{a}, \underline{b} \in \mathbb{Z}^d$ with $\underline{a} \neq \underline{b}$, then the characters $\prod_{i=1}^d t_i^{a_i}$ and $\prod_{i=1}^d t_i^{b_i}$ are eigenvectors for $m_{\underline{s}}$ of distinct eigenvalues.
It follows that there exist $\lambda \in \mathbb{C}^*$, $\underline{a} \in \mathbb{Z}^d$ such that

$$f = \lambda \prod_{i=1}^{d} t_i^{a_i}.$$

In fact, let $f(\underline{t}) := \lambda_{\underline{a}} \prod t_i^{a_i} + \lambda_{\underline{b}} \prod t_i^{b_i}$ and suppose $\underline{a} \neq \underline{b}$. Since f is a eigenvector of $m_{\underline{s}}$, there exists $c \in \mathbb{C}^*$ such that $m_{\underline{s}}(f) = cf$. Since every monomial of the form $\prod_{i=1}^d t_i^{a_i}$ is an eigenvector of $m_{\underline{s}}$, if follows that there exist $c_1, c_2 \in \mathbb{C}^*$ such that $m_{\underline{s}}(f)(\underline{t}) = \lambda_{\underline{a}} m_{\underline{s}}(\prod t_i^{a_i}) + \lambda_{\underline{b}} m_{\underline{s}}(\prod t_i^{b_i}) = \lambda_{\underline{a}} c_1 \prod t_i^{a_i} + \lambda_{\underline{b}} c_2 \prod t_i^{b_i}$. Thus $c_1 = c = c_2$ that is absurd since $\underline{a} \neq \underline{b}$.
We conclude noting that, since f is a non-zero group homomorphism, $f(1) = 1$, then $\lambda = 1$.

□

Looking only at characters that lie in $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$, so we can say that

Corollary 2.1.4. *The set* Λ *of characters of* $(\mathbb{C}^*)^d$ *is an abelian group under coordinatewise multiplication, naturally isomorphic to* $\mathbb{Z}^d$*.*

Proof. It immediately follows from Proposition 2.1.3 that the mapping

$$(a_1, \ldots, a_d) \longmapsto e^{(a_1, \ldots, a_d)} := \prod_{i=1}^{d} t_i^{a_i}$$

is a group isomorphism between the additive group $\mathbb{Z}^d$ and the multiplicative group Λ.

□

Example 2.1.5. Consider two characters on $(\mathbb{C}^*)^2 \longrightarrow \mathbb{C}^*$:

$$(t_1, t_2) \mapsto (t_2^3) \qquad (t_1, t_2) \mapsto (t_1^{-1} t_2^2).$$

The first is identified with $(0, 3) \in \mathbb{Z}^2$ and the second one
with $(-1, 2) \in \mathbb{Z}^2$. Their sum is the character $(t_1, t_2) \mapsto (t_1^{-1} t_2^5)$ which is identified with $(-1, 5) = (0, 3) + (-1, 2) \in \mathbb{Z}^2$.

Definition 2.1.6. A character χ of $(\mathbb{C}^*)^d$ viewed as an element $(a_1, \ldots, a_d)$ of $\mathbb{Z}^d$, is called **primitive** if $GCD\{a_1, \ldots, a_d\} = 1$.

Example 2.1.7. Consider again the characters on $(\mathbb{C}^*)^2 \longrightarrow \mathbb{C}^*$:

$$(t_1, t_2) \mapsto (t_2^3) \qquad (t_1, t_2) \mapsto (t_1^{-1} t_2^2).$$

It is obvious that the first one isn't primitive ($GCD\{0, 3\} = 3$) while the second is.

2.2 Fundamental group and de Rham cohomology ring

Since $(\mathbb{C}^*)^d$ is a d-dimensional complex manifold, it is a $2d$-dimensional real manifold and as such we can compute its de Rham cohomology.

First we introduce a very important statement that will be very useful in the computation of the fundamental group and the coomology groups of $(\mathbb{C}^*)^d$ (see Appendix C for the definition of *strong deformation retract*).

Proposition 2.2.1. *S^1 is a strong deformation retract of $\mathbb{C}^*$. In particular S^1 and $\mathbb{C}^*$ are homotopy equivalent.*

Proof. If we take

$$r : \mathbb{C}^* \longrightarrow S^1$$

$$z \longmapsto \frac{z}{||z||},$$

we immediately see that $r(z) = z$ for every $z \in S^1$, and then r is the retraction wanted.
We can conclude considering

$$F : \mathbb{C}^* \times [0,1] \longrightarrow \mathbb{C}^*$$

$$(z,t) \quad \longmapsto tz + (1-t)\frac{z}{||z||}.$$

In fact, for every $z \in \mathbb{C}^*$ and for every $a \in S^1$, we have that

- $F(z,0) = \frac{z}{||z||} = (i \circ r)(z)$;
- $F(z,1) = z = Id_{\mathbb{C}^*}(z)$;
- $F(a,t) = a \quad \forall t \in [0,1]$.

Moreover, through $F|_{S^1 \times [0,1]}$, S^1 and $\mathbb{C}^*$ are homotopic equivalent.

□

Corollary 2.2.2.

$$\pi_1(S^1) \simeq \pi_1(\mathbb{C}^*),$$
$$H^*_{dR}(S^1) \simeq H^*_{dR}(\mathbb{C}^*).$$

Recall that $\pi_1(S^1) \simeq \mathbb{Z}, H^k_{dR}(S^1) \simeq \mathbb{Z}\ \forall k = 0, 1$ and $H^k_{dR}(S^1) \simeq 0$ $\forall k \neq 0, 1$. We can now easily compute the fundamental group and the de Rham cohomology groups of the torus $T = (\mathbb{C}^*)^d$, using basic properties of algebraic topology. In fact,

Proposition 2.2.3.

$$\pi_1(T) = \mathbb{Z}^d.$$

Proof. For every $\underline{z} = (z_1, \ldots, z_d) \in T = (\mathbb{C}^*)^d$:

$$\begin{aligned}
\pi_1(T, \underline{z}) &\simeq \pi_1(\mathbb{C}^*, z_1) \times \cdots \times \pi_1(\mathbb{C}^*, z_d) \simeq \\
&\simeq \pi_1\left(S^1, \frac{z_1}{||z_1||}\right) \times \cdots \times \pi_1(S^1, \frac{z_d}{||z_d||}) \simeq \\
&\simeq \mathbb{Z} \times \cdots \times \mathbb{Z} \simeq \\
&\simeq \mathbb{Z}^d.
\end{aligned}$$

□

Proposition 2.2.4.

$$H^k_{dR}(T) \simeq \mathbb{Z}^{\binom{d}{k}} \qquad \forall k \leq d.$$

Proof. Thanks to Corollary 2.2.2 and Kunneth formula, we obtain, for every k:

$$\begin{aligned}
H^k_{dR}(T) &= \bigoplus_{i_1+\cdots+i_d=k} H^{i_1}_{dR}(\mathbb{C}^*) \otimes \cdots \otimes H^{i_d}_{dR}(\mathbb{C}^*) \simeq \\
&\simeq \bigoplus_{i_1+\cdots+i_d=k} H^{i_1}_{dR}(S^1) \otimes \cdots \otimes H^{i_d}_{dR}(S^1) \simeq \\
&\simeq \bigoplus_{i_1+\cdots+i_d=k} \mathbb{Z} = \\
&= \mathbb{Z}^{\binom{d}{k}}.
\end{aligned}$$

□

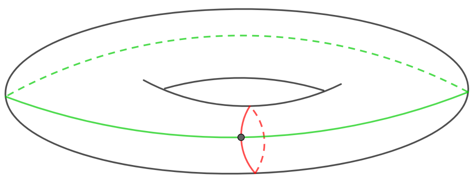
Remark 2.2.5. In order to study the ring structure of $H^*_{dR}(T)$, we must recall that $H^*_{dR}(S^1) \simeq \Lambda(x)$ with $deg(x) = 1$. Thanks to Kunneth formula, we have that $H^*_{dR}((\mathbb{C}^*)^d) \simeq H^*_{dR}(\mathbb{C}^*) \otimes H^*_{dR}((\mathbb{C}^*)^{d-1})$, then, by induction we have that $H^*_{dR}((\mathbb{C}^*)^d) \simeq \Lambda(x_1, \ldots, x_d)$ with $deg(x_i) = 1 \quad \forall i \in \{1, \ldots, d\}$. Thus

$$H^k_{dR}(T) \simeq \Lambda^k(x_1, \ldots, x_d) =< \{x_{i_1} \wedge \cdots \wedge x_{i_k}, \text{ with } i_\ell \neq i_m \quad \forall \ell \neq m\} > .$$

This means that every k-form of the torus can be written as the wedge of k different 1-forms. Then $H^*_{dR}(T) = \Lambda(H^1_{dR}(T))$ is generated in degree one.

Example 2.2.6. Let $T = \{(t, s) \in (\mathbb{C}^*)^2\}$ be the 2 dimensional complex torus.

$$H^0_{dR}(T) \simeq < [1] >= \mathbb{Z},$$
$$H^1_{dR}(T) =< \left[\frac{1}{2\pi i}\frac{dt}{t}\right], \left[\frac{1}{2\pi i}\frac{ds}{s}\right] >=:< [\psi_1], [\psi_2] >,$$
$$H^2_{dR}(T) =< \left[\frac{1}{2\pi i}\frac{dt}{t}\right] \wedge \left[\frac{1}{2\pi i}\frac{ds}{s}\right] >=:< [\psi_1 \wedge \psi_2] > .$$



2.3 Coverings

In order to study the coverings of $T = (\mathbb{C}^*)^d$, we have to recall a fundamental theorem in algebraic topology.

Theorem 2.3.1. *Let X be a topological space path connected and locally path connected such that there exists a universal covering $q : \tilde{X} \longrightarrow X$. Then, for every $x \in X$ and for every subgroup $H < \pi_1(X, x)$, there exists an unique covering $p : E \longrightarrow X$ and a point $e \in p^{-1}(x)$ such that $H = p_*\pi_1(E, e)$. Moreover*

$$deg(p) = [\pi_1(X, x) : H].$$

Corollary 2.3.2. *If $p : U \longrightarrow T$ is a finite covering, then $U = (\mathbb{C}^*)^d$.*

Proof. We want to prove that, for every $H < \pi_1(T)$ of finite index, there exists a finite covering $p : (\mathbb{C}^*)^d \longrightarrow T$ such that $p_*\pi_1((\mathbb{C}^*)^d) = H$, then, thanks to the uniqueness of the finite covering we conclude.
Since $\pi_1(T) \simeq \mathbb{Z}^d$, it follows that every subgroup of finite index H of $\pi_1(T)$ is isomorphic to $\mathbb{Z}^d$, then if $\{v_1, \ldots, v_d\}$ is a basis of H and $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{Z}$, we have that

$$\begin{array}{ccccc} \mathbb{Z}^d & \simeq & H & \hookrightarrow & \mathbb{Z}^d \\ & & v_i & \longmapsto & a_{i,1}e_1 + \cdots + a_{i,d}e_d. \end{array}$$

This implies that for every $H < \mathbb{Z}^d$ there exists $A \in GL_d(\mathbb{Z})$ such that $H = Im(A)$. Now we conclude considering

$$\begin{array}{cccc} p: & (\mathbb{C}^*)^d & \longrightarrow & T \\ & (u_1, \ldots, u_d) & \longmapsto & \left(u_1^{a_{1,1}} \ldots u_d^{a_{1,d}}, \ldots, u_1^{a_{d,1}} \ldots u_d^{a_{d,d}}\right) \end{array}$$

which is a finite covering of T with $p_*(\pi_1(U)) = H$.

$\square$

Chapter 3

Toric arrangements

3.1 The poset of layers $L(\mathcal{A})$

Definition 3.1.1. Let $T = (\mathbb{C}^*)^d$ be a complex torus, and let Λ be the set of characters of T.
Consider a list $\vec{\chi} = (\chi_1, \dots, \chi_n) \subseteq \Lambda^d$ and $\vec{b} = (b_1, \dots, b_n) \subseteq (\mathbb{C}^*)^d$. The **toric arrangement** defined by $\vec{\chi}$ and $\vec{b}$ is

$$\mathcal{A} = \{H_i; i \in \{1, \dots, n\}\},$$

where $H_i := \chi_i^{-1}(b_i)$ is the levet set of χ_i at level b_i. We call every H_i a **hypertorus**. We will always denote by n the number of hypertori in the arrangement and d the dimension of the complex torus.

Remark 3.1.2. Every toric arrangement can be described by primitive characters. In fact, if $\chi = (a_1, \dots, a_d)$ is a non-primitive character, there exists a primitive character χ' such that $\chi = m\chi'$ with $m := GCD\{a_1, \dots, a_d\}$, then $\chi^{-1}(b) = \{(\chi')^{-1}(c); c^m = b\}$.

Example 3.1.3. Consider the character χ such that $(t,s) \mapsto s^3$. It is non-primitive since, viewed as an element of $\mathbb{Z}^2$, $\chi = (0,3) = 3(0,1) = 3\chi'$.

$$\begin{aligned}\chi^{-1}(1) &= \{(t,s) \in (\mathbb{C}^*)^2; s^3 = 1\} = \\ &= \{(t,s) \in (\mathbb{C}^*)^2; s = e^{\frac{2}{3}\pi i}\} \cup \{(t,s) \in (\mathbb{C}^*)^2; s = e^{\frac{4}{3}\pi i}\} \cup \{(t,s) \in (\mathbb{C}^*)^2; s = 1\} = \\ &= (\chi')^{-1}(e^{\frac{2}{3}\pi i}) \cup (\chi')^{-1}(e^{\frac{4}{3}\pi i}) \cup (\chi')^{-1}(1).\end{aligned}$$

Definition 3.1.4. A toric arrangement is **central** if $\vec{b} = (1,\dots,1)$ i.e. if H_i is the kernel of χ_i for all i.

Definition 3.1.5. Let $\mathcal{A}$ be a central toric arrangement defined by $\vec{\chi} = (\chi_1,\dots,\chi_n)$ with $\chi_i(t_1,\dots,t_d) = t_1^{a_{i1}}\dots t_d^{a_{id}}$. Define, in $\mathbb{C}^d$, the hyperplane arrangement

$$\tilde{\mathcal{A}} := \{\tilde{H}_i^k; k \in \mathbb{Z}, i = 1\dots,n\}$$

with

$$\tilde{H}_i^k := \{(x_1,\dots,x_d) \in \mathbb{C}^d; a_{i1}x_1 + \dots + a_{id}x_d = k\}.$$

Such hyperplane arrangement is called **periodic**.

Remark 3.1.6. The family of hyperplanes $\{\tilde{H}^k; k \in \mathbb{Z}\}$ is the preimage of H in the universal cover of $(\mathbb{C}^*)^d$. In fact, in the previous section, we have seen that there is a group isomorphism $\tilde{f}$ such that $\mathbb{C}^d/\mathbb{Z}^d \simeq (\mathbb{C}^*)^d$. It follows that

$$\begin{aligned}\tilde{f}^{-1}(H) &= \{(x_1,\dots,x_d) \in \mathbb{C}^d | (e^{2\pi i x_1})^{a_1}\dots(e^{2\pi i x_d})^{a_d} = 1\} = \\ &= \{(x_1,\dots,x_d) \in \mathbb{C}^d | a_1x_1 + \dots + a_dx_d = k \quad \forall k \in \mathbb{Z}\} = \\ &= \bigcup_{k\in\mathbb{Z}} (\tilde{H}^k).\end{aligned}$$

Example 3.1.7. Let $T = (\mathbb{C}^*)^2$ and consider the central toric arrangement $\mathcal{A}$ defined by $\vec{\chi} = (\chi_1,\chi_2,\chi_3)$ with $\chi_1(t,s) = t$, $\chi_2(t,s) = s$ and $\chi_3(t,s) = t^2s^{-1}$.

We can then consider the hyperplane arrangement $\tilde{\mathcal{A}}$ given by the hyperplanes, for every $k \in \mathbb{Z}$, $\tilde{H_1}^k := \{(x, y) \in \mathbb{C}^2 | x = k\}$, $\tilde{H_2}^k := \{(x, y) \in \mathbb{C}^2 | y = k\}$ and $\tilde{H_3}^k := \{(x, y) \in \mathbb{C}^2 | 2x - y = k\}$.

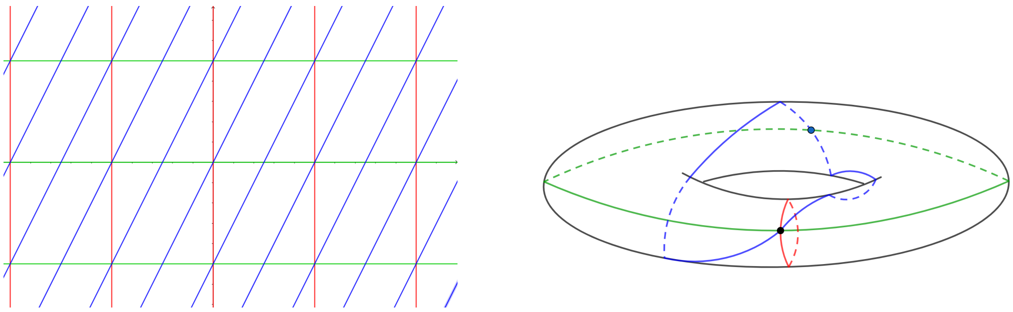


Figure 3.1: $\tilde{\mathcal{A}}$ $\xrightarrow{/\mathbb{Z}^2}$ $\mathcal{A}$

Definition 3.1.8. A toric arrangement $\mathcal{A}$ is **unimodular** if, for every $A \subseteq \{H_1, \ldots H_n\}$, $\cap_{H_i \in A} H_i$ is either connected or empty.

Example 3.1.9. Let $T = (\mathbb{C}^*)^2$ be the 2-dimensional torus and $\mathcal{A} = \{H_1, H_2\}$ where $H_1 := \{(t, s) \in (\mathbb{C}^*)^2 | t = 1\}$, $H_2 := \{(t, s) \in (\mathbb{C}^*)^2 | s = 1\}$. It is obviously unimodular since $H_1 \cap H_2 = \{(1, 1)\}$ is connected.

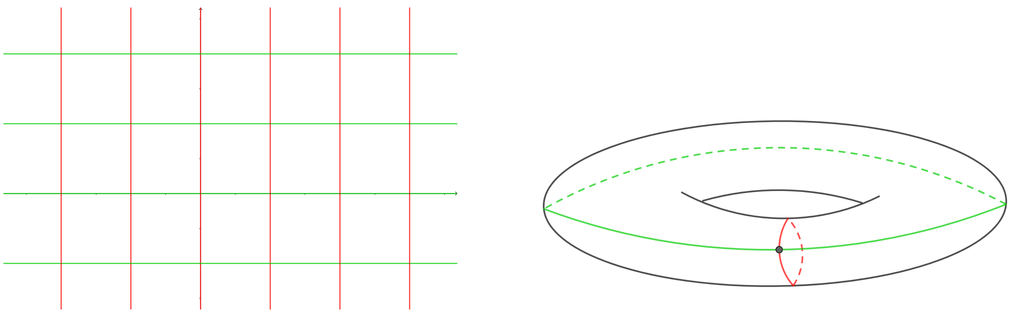


Figure 3.2: $\tilde{\mathcal{A}}$ $\xrightarrow{/\mathbb{Z}^2}$ $\mathcal{A}$

Take, instead, $\mathcal{A}' = \{H_1, H_2\}$ where $H_1 = \{(t,s) \in (\mathbb{C}^*)^2 | ts = 1\}$, $H_2 = \{(t,s) \in (\mathbb{C}^*)^2 | ts^{-1} = 1\}$. $\mathcal{A}'$ is not unimodular since $H_1 \cap H_2 = \{(1,1), (-1,-1)\}$.

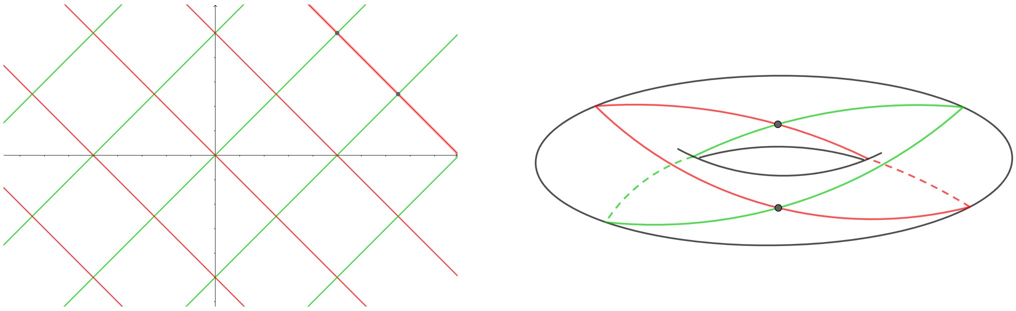


Figure 3.3: $\tilde{\mathcal{A}}'$ $\xrightarrow{/\mathbb{Z}^2}$ $\mathcal{A}'$

Definition 3.1.10. Let $\mathcal{A}$ be a toric arrangement on $T = (\mathbb{C}^*)^d$ and let $L(\mathcal{A})$ be the set of all connected components of non-empty intersection of hypertori of $\mathcal{A}$. We agree that $L(\mathcal{A})$ includes T as the intersection of the empty collection of hypertori.
The elements of $L(\mathcal{A})$ are called **layers** of the arrangement.
Define a partial order on $L(\mathcal{A})$ by the reverse inclusion:

$$X \leq Y \Leftrightarrow Y \subseteq X.$$

Define also a **rank function** on $L(\mathcal{A})$ such that, for every $X \in L(\mathcal{A})$, $r(X)$ is the lenght of any chain $T < \cdots < X$.

Definition 3.1.11. A toric arrangement $\mathcal{A}$ is **essential** if the maximal elements in $L(\mathcal{A})$ are points.

From now on we will consider only essential arrangement with $\prec$ a fixed total order on it.

3.2 The arithmetic matroid $\mathcal{M}(\mathcal{A})$

Since $\Lambda \simeq \mathbb{Z}^d$, we can identify every character of $T = (\mathbb{C}^*)^d$ with an element of $\mathbb{Z}^d$. We say that a subset of hypertori is **independent** if the defining characters of these hypertori are linearly independent over $\mathbb{Q}$.
For every toric arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ in $T = (\mathbb{C}^*)^d$ we can consider the pair $(\mathcal{A}, \{S \subseteq \{H_1, \dots, H_n\}$ with S independent$\})$ that is a matroid called **matroid of the toric arrangement** $\mathcal{M} := \mathcal{M}(\mathcal{A})$.

Definition 3.2.1. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a toric arrangement and S a p-tuple of hypertori of $\mathcal{A}$. Define $[S]$ the integer $d \times |S|$-matrix whose columns are the characters in S viewed as elements of $\mathbb{Z}^d$. The columns appear in the total order $\prec$ chosen on $\mathcal{A}$.
We say that S is a **circuit** if it is minimally dependent, that is, S is linearly dependent but any proper subset of S is not.
Let $min(S)$ be the minimal element of S with respect to $\prec$. S is a **broken circuit** if $\exists K \in \mathcal{A}; K \prec min(S)$ and (K, S) is a circuit.
S is a **no-broken-circuit set (nbc set)** if it does not contain any broken circuit. Denote **nbc($\mathcal{A}$)** the set of all nbc sets.

Remark 3.2.2.

- $S \subseteq \{H_1, \dots, H_n\}$ is an independent set if and only if $\operatorname{rank}([S]) = |S|$;
- $\{S \subseteq \{H_1, \dots, H_n\}$ with S circuit$\} = \mathcal{C}(\mathcal{M})$ the set of circuits of the matroid $\mathcal{M}(\mathcal{A})$;
- every nbc-set is an independent set.

Example 3.2.3. Let $T = (\mathbb{C}^*)^2$ be the 2-dimensional torus and $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$ where $H_1 := \{(t,s) \in (\mathbb{C}^*)^2 | t = 1\}$, $H_2 := \{(t,s) \in (\mathbb{C}^*)^2 | s = 1\}$,
$H_3 := \{(t,s) \in (\mathbb{C}^*)^2 | ts = 1\}$ and $H_4 := \{(t,s) \in (\mathbb{C}^*)^2 | ts^{-1} = 1\}$.

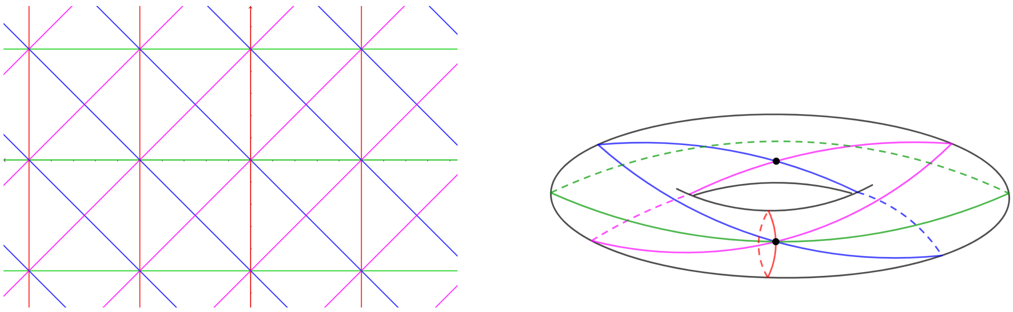
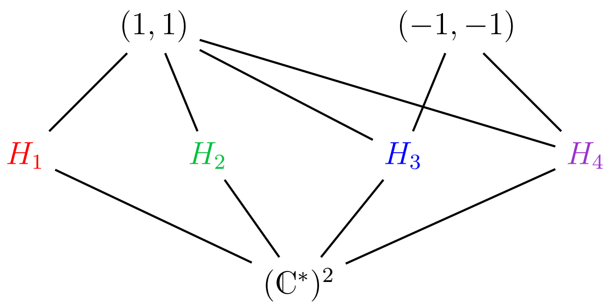


Figure 3.4: $\tilde{\mathcal{A}}' \xrightarrow{/\mathbb{Z}^2} \mathcal{A}'$

The Hasse diagram of $L(\mathcal{A})$ is:



Note that every hypertorus and every pair of hypertori is independent. Moreover $\mathcal{C}(\mathcal{M}) = \{\{H_1, H_2, H_3\}, \{H_1, H_2, H_4\}, \{H_1, H_3, H_4\}, \{H_2, H_3, H_4\}\}$, the broken circuits are: $\{H_2, H_3\}$ $\{H_2, H_4\}$ $\{H_3, H_4\}$ and
$nbc(\mathcal{A}) = \{\emptyset, \{H_1\}, \{H_2\}, \{H_3\}, \{H_4\}, \{H_1, H_2\}, \{H_1, H_3\}, \{H_1, H_4\}\}$.

Definition 3.2.4. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ a toric arrangement in $(\mathbb{C}^*)^d$. For every subset S of hypertori of $\mathcal{A}$, viewed as a subset of Λ, define

$$\Lambda_S := < S > \subseteq \Lambda,$$
$$\Lambda^S := (\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_S) \cap \Lambda,$$
$$m(S) := [\Lambda^A : \Lambda_A] \quad \text{multiplicity of } S,$$

where $< S >$ is the $\mathbb{Z}-$module generated by S.

For $S = ()$ we agree that $m(S) = 1$.

Remark 3.2.5. $m(S)$ is the greatest common divisor of all minors of $[S]$ with size equal to the rank of $[S]$. In particular if $[S]$ is a non singular square matrix, $m(S) = |det([S])|$.

Example 3.2.6. Take $S = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{Z}^2$.

$$\Lambda_S = \left\{ m_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + m_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}; m_1, m_2 \in \mathbb{Z} \right\} = \left\{ \begin{pmatrix} 3m_2 \\ m_1 + m_2 \end{pmatrix}; m_1, m_2 \in \mathbb{Z} \right\} \subseteq \mathbb{Z}^2,$$

$$\Lambda^S = \left\{ q_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + q_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}; q_1, q_2 \in \mathbb{Q} \right\} \cap \mathbb{Z}^2 = \left\{ \begin{pmatrix} 3q_2 \\ q_1 + q_2 \end{pmatrix}; q_1, q_2 \in \mathbb{Q} \right\} \cap \mathbb{Z}^2.$$

Since $\mathbb{Z} \subseteq \mathbb{Q}$, obviously $\Lambda_S \subseteq \Lambda^S$, but $\Lambda_S \subsetneq \Lambda^S$ since, for example
$\frac{2}{3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \in \Lambda^S \backslash \Lambda_S$.
Since $\Lambda_S \simeq 3\mathbb{Z} \oplus \mathbb{Z}$ and $\Lambda^S \simeq \mathbb{Z}^2$, we have that

$$[\Lambda^S : \Lambda_S] = \frac{[\Lambda^S : \Lambda]}{[\Lambda_S : \Lambda]} = 3,$$

that equals

$$\left| det \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \right|.$$

We now introduce the definition of arithmetic matroid.

Definition 3.2.7. A triple $\mathcal{M} = (E, \mathcal{I}, M)$ is an *arithmetic matroid* if $(E, \mathcal{I})$ is a matroid and $m : \mathcal{P}(E) \longrightarrow \mathbb{N}$ is a function such that:

- if $A \subseteq E$, $v \in E$ is dependent on A (i.e. $r_{\mathcal{M}}(A \cup \{v\}) = r_{\mathcal{M}}(A)$), then $m(A \cup \{v\})$ divides $m(A)$.
- if $A \subseteq E$, $v \in E$ is independent on A (i.e. $r_{\mathcal{M}}(A \cup \{v\}) = r_{\mathcal{M}}(A) + 1$), then $m(A)$ divides $m(A \cup \{v\})$.
- if $A \subseteq B \subseteq E$ with $B = A \sqcup F \sqcup T$ and, for every $A \subseteq C \subseteq B$, $r_{\mathcal{M}}(C) = r_{\mathcal{M}}(A) + |C \cap F|$, then
$$m(A)m(B) = m(A \cup F)m(A \cup T).$$
- if $A \subseteq B \subseteq E$ with $r_{\mathcal{M}}(A) = r_{\mathcal{M}}(B)$, then
$$\sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(T) \geq 0.$$
- if $A \subseteq B \subseteq E$ with $r^*_{\mathcal{M}}(A) = r^*_{\mathcal{M}}(B)$, then
$$\sum_{A \subseteq T \subseteq B} (-1)^{|T|-|A|} m(E \backslash T) \geq 0$$
with $r^*_{\mathcal{M}}$ the rank function of the matroid $\mathcal{M}^*$ on E with bases the complement of the bases of $\mathcal{M}$ ($\mathcal{M}^*$ is called **dual** of $\mathcal{M}$).

We say that an arithmetic matroid $\mathcal{M} = (E, \mathcal{I}, m)$ is **GCD** if, for every $A \subseteq E$, $m(A) = GCD(\{m(B);\, B \subseteq A \text{ and } |B| = r_{\mathcal{M}}(B) = r_{\mathcal{M}}(A)\})$.

Proposition 3.2.8. *Let $\mathcal{A}$ be a toric arrangement on $(\mathbb{C}^*)^d$. The matroid of the arrangement $\mathcal{M}(\mathcal{A})$ together with the function m of Definition 3.2.4, determines a arithmetic matroid.*
In particular, thanks to Remark 3.2.5, it is a GCD arithmetic matroid.

Remark 3.2.9. It follows from the properties of the arithmetic matroids (see [DM13]), that $m(S)$ is the number of connected components of $\bigcap_{H_i \in S} H_i$ when this intersection is non-empty. In particular, if $\mathcal{A}$ is unimodular, then $m(S) = 1$ for all S.

Remark 3.2.10. The poset of layers $L(\mathcal{A})$ determines the arithmetic matroid $\mathcal{M}(\mathcal{A}) = (\mathcal{A}, \{S \subseteq \mathcal{A} | S \text{ independent}\}, m)$. In fact, for every S, if X_S is the set of minimal upper-bounds in $L(\mathcal{A})$, we have that

- S is independent if and only if $r(x) = |S|$ for every $x \in X_S$;
- $m(S) = |X_S|$.

Example 3.2.11. Let $T = (\mathbb{C}^*)^2$ be the 2-dimensional torus and
$\mathcal{A} = \{H_1, H_2, H_3\}$ the central not unimodular toric arrangement with
$H_1 := \{(t, s) \in (\mathbb{C}^*)^2 | t = 1\}$, $H_2 := \{(t, s) \in (\mathbb{C}^*)^2 | s = 1\}$ and
$H_3 := \{(t, s) \in (\mathbb{C}^*)^2 | t^3 s = 1\}$.

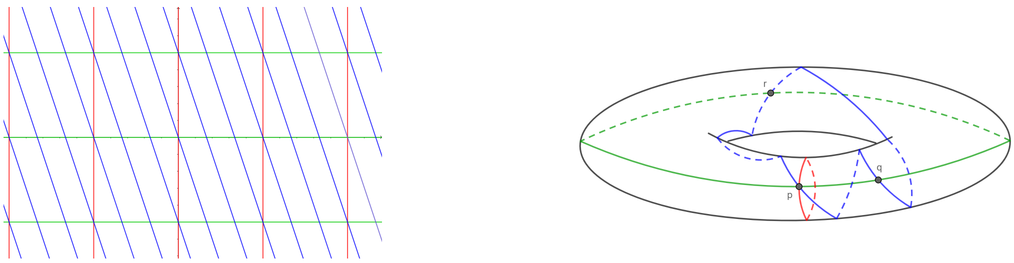
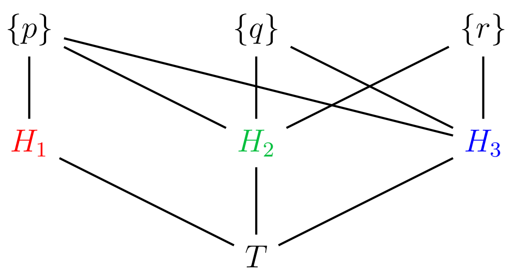


Figure 3.5: $\tilde{\mathcal{A}}$ $\xrightarrow{/\mathbb{Z}^2}$ $\mathcal{A}$

We want to compute the arithmetic matroid $\mathcal{M}(\mathcal{A})$. Thanks to the Hasse diagram of $L(\mathcal{A})$,



we can immediately say that $\{H_1, H_2, H_3\}$ is the only dependet set.
In order to compute the multiplicity function, thanks to Remark 3.2.5, we need to construct the matrices of every tuple of hypertori

$$[\{H_1, H_2, H_3\}] = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}, \qquad [\{H_1, H_2\}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$[\{H_1, H_3\}] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \qquad [\{H_2, H_3\}] = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}.$$

Then we obtain
$m(\{H_1\}) = m(\{H_2\}) = m(\{H_3\}) = 1$, indeed H_1, H_2, H_3 are connected;
$m(\{H_1, H_2\}) = |det[\{H_1, H_2\}]| = 1$ as well as $m(\{H_1, H_3\}) = 1,$ indeed $H_1 \cap H_2 = H_1 \cap H_3 = (1, 1) =: p$ connected;
$m(\{H_2, H_3\}) = |det[\{H_2, H_3\}]| = 3$, indeed $H_2 \cap H_3 = \{p, q := (e^{2/3\pi i}, 1), r := (e^{4/3\pi i, 1)})\}$ that has 3 connected components;
$m(\{H_1, H_2, H_3\}) = GCD\{1, 2\} = 1$, indeed $H_1 \cap H_2 \cap H_3 = \{p\}$ connected.

Thanks to Remark 3.2.10 we can also construct the arithmetic matroid $\mathcal{M}(\mathcal{A})$ looking only the poset $L(\mathcal{A})$. In fact,
$X_{\{H_1\}} = \{H_1\}$, $r(H_1) = 1 = |\{H_1\}|$ and $m(H_1) = |X_{\{H_1\}}|$. The same argument for H_2 and H_3.
For $j \in \{2, 3\}$, $X_{\{H_1, H_j\}} = \{p\}$, $r(p) = 2 = |\{H_1, H_j\}|$ and $m(\{H_1, H_j\}) = 1 = |\{p\}|$. $X_{\{H_2, H_3\}} = \{p, q, r\}$, $r(p) = r(q) = r(r) = 2 = |\{H_2, H_3\}|$ and $m(\{H_2, H_3\}) = 3 = |\{p, q, r\}|$.
$X_{\{H_1, H_2, H_3\}} = \{p\}$, $r(p) = 2 \neq 3 = |\{H_1, H_2, H_3\}|$ and $m(\{H_1, H_2, H_3\}) = 1 = |\{p\}|$.

Lemma 3.2.12. *If $C = \{H_{i_1}, \dots, H_{i_p}\}$ is a circuit, then there exist $\{c_1, \dots, c_p\}$ with $c_j \in \{-1, +1\} \quad \forall j \in \{1, \dots, p\}$ such that*

$$\sum_{j=1}^{p} c_j m(C \backslash \{H_{i_j}\}) \chi_{i_j} = 0$$

where every χ_{i_j} is viewed as element of $\mathbb{Z}^d$.

It can be proved that the coefficients c_j depend only on $L(\mathcal{A})$ (see [Pag19a]).

Corollary 3.2.13. *If* $\mathcal{A}$ *is unimodular, then every circuit can be realized by a minimal linear dependency*

$$\sum_{j=1}^{p} c_j \chi_{i_j} = 0$$

where $c_j \in \{-1, +1\}$ *for all* $j \in \{1, \dots, p\}$.

Proof. It immediately follows from Lemma 3.2.12 and Remark 3.2.9. □

Example 3.2.14. Let $\mathcal{A}$ be the non-unimodular toric arrangement of Example 3.2.11. We have that $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\chi_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Since $C = \{H_1, H_2, H_3\}$ is the unique circuit of $\mathcal{A}$, we have that, for $c_1 = +1$, $c_2 = +1, c_3 = -1$,

$$\begin{aligned} c_1 m(C \backslash \{H_1\}) \chi_1 + c_2 m(C \backslash \{H_2\}) \chi_2 + c_3 m(C \backslash \{H_3\}) \chi_3 = \\ = 3c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 0. \end{aligned}$$

3.3 Cohomology of the complement $M(\mathcal{A})$

Definition 3.3.1. Let $\mathcal{A}$ be a toric arrangement in $T = (\mathbb{C}^*)^d$. Define the **complement** of $\mathcal{A}$ by

$$M(\mathcal{A}) = T \backslash \bigcup_{H \in \mathcal{A}} H$$

and the **Poincaré polynomial of the complement** by

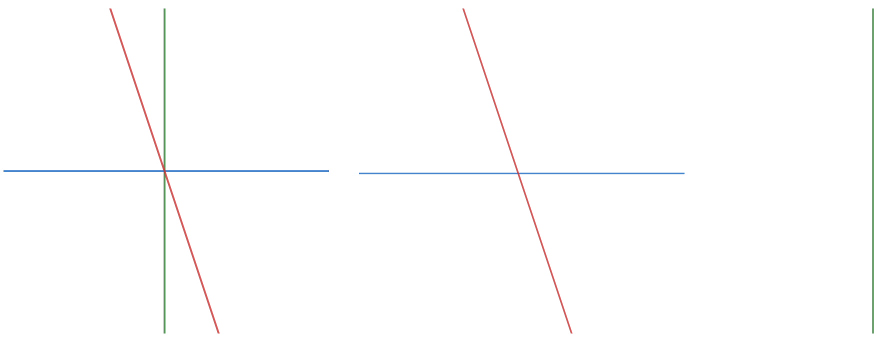
$$Poin(M(\mathcal{A}), t) = \sum_{j=0}^{d} b_j(M) t^j$$

where, for every j, $b_j(M) := rank(H^j_{dR}(M))$ is the j-th Betti number of $M(\mathcal{A})$.

Definition 3.3.2. Let $\mathcal{A}$ be a toric arrangement in $(\mathbb{C}^*)^d$.
Given a point $p \in T$, we define $\mathcal{A}[p]$ the linear arrangement in the tangent space $T_p(T)$ as the arrangement given by the hyperplane $T_p(H)$ for all $H \in \mathcal{A}$ such that $p \in H$.
For $W \in L(\mathcal{A})$, a point $p \in W$ is called **generic** if, for any $H \in \mathcal{A}$ such that $W \not\subseteq H$, we have that $p \notin H$. In other words, $p \in W$ is generic if it does not lie in any other hypertorus.
We define $\mathcal{A}[W]$ as the hyperlane arrangement $\mathcal{A}[p]$ for a generic point $p \in W$.

Note that $\mathcal{A}[W]$ is well defined since it does not depend on the choice of the generic point p.

Example 3.3.3. Let $\mathcal{A}$ be the toric arrangement on $T = (\mathbb{C}^*)^2$ of Example 3.2.11. For every point p_0 in any hyperplane H in $\mathcal{A}$, we identify $T_{p_0}(T)$ with $\mathbb{C}^2$ with coordinates $(\bar{t}, \bar{s})$.
Let $\bar{H}_1 := \{(\bar{t}, \bar{s}) \in \mathbb{C}^2 | \bar{t} = 0\}$, $\bar{H}_2 := \{(\bar{t}, \bar{s}) \in \mathbb{C}^2 | \bar{s} = 0\}$ and $\bar{H}_3 := \{(\bar{t}, \bar{s}) \in \mathbb{C}^2 | 3\bar{t} + \bar{s} = 0\}$.
We have that $\mathcal{A}[p] = \{\bar{H}_1, \bar{H}_2, \bar{H}_3\}$, $\mathcal{A}[q] = \mathcal{A}[r] = \{\bar{H}_2, \bar{H}_3\}$.
For $W = H_1$, every point $p_0 \neq p = (1,1)$ is generic and we have that $\mathcal{A}[H_1] = \mathcal{A}[p_0] = \{\bar{H}_1\}$.



$\mathcal{A}[p]$ $\mathcal{A}[q] = \mathcal{A}[r]$ $\mathcal{A}[H_1]$

The following lemma is essentially proved in Theorem 4.2 of [DCP05].

Lemma 3.3.4. *If $\mathcal{A}$ is a toric arrangement in $T = (\mathbb{C}^*)^d$,*

$$Poin(M(\mathcal{A}, t)) = \sum_{j=0}^{d} N_j (t+1)^{d-j} t^j,$$

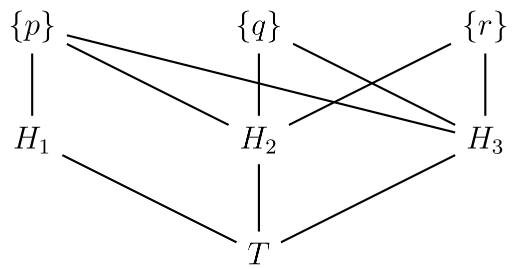
where, for $j \in \{0, \dots, d\}$,

$$N_j := \sum_{\substack{L \text{ layer} \\ |L|=j}} |nbc_j(\mathcal{A}[L])|$$

and nbc_j is the set of nbc of cardinality j in the arrangement $\mathcal{A}[L]$.
In particular,

$$b_j = \sum_{i=0}^{j} N_i \binom{d-i}{j-i}.$$

Example 3.3.5. Let $\mathcal{A}$ be the toric arrangement on $T = (\mathbb{C}^*)^2$ of Example 3.2.11. We want to compute the Poincaré polynomial of the complement. First we have to compute $\mathcal{A}[W]$ for every $W \in L(\mathcal{A})$. The Hasse diagram of $L(\mathcal{A})$ is:



$$\mathcal{A}[T] = \emptyset$$

$$\mathcal{A}[H_1] = \{\bar{H}_1\} \qquad \mathcal{A}[H_2] = \{\bar{H}_2\} \qquad \mathcal{A}[H_3] = \{\bar{H}_3\}$$

$$\mathcal{A}[p] = \{\bar{H}_1, \bar{H}_2, \bar{H}_3\} \qquad \mathcal{A}[q] = \{\bar{H}_2, \bar{H}_3\} \qquad \mathcal{A}[r] = \{\bar{H}_2, \bar{H}_3\}.$$

Then

$$N_0 = |nbc_0(\mathcal{A}[T])| = |\mathbb{C}^2| = 1$$

$$N_1 = |nbc_1(\mathcal{A}[H_1])| + |nbc_1(\mathcal{A}[H_2])| + |nbc_1(\mathcal{A}[H_3])| =$$
$$= |\{\bar{H}_1\}| + |\{\bar{H}_2\}| + |\{\bar{H}_3\}| = 3$$

$$N_2 = |nbc_2(\mathcal{A}[p])| + |nbc_2(\mathcal{A}[q])| + |nbc_2(\mathcal{A}[r])| =$$
$$= |\{\bar{H}_1, \bar{H}_2\}, \{\bar{H}_1, \bar{H}_3\}| + |\{\bar{H}_2, \bar{H}_3\}| + |\{\bar{H}_2, \bar{H}_3\}| = 4.$$

Thus

$$b_0 = N_0 = 1 \qquad b_1 = 2N_0 + N_1 = 5 \qquad b_2 = N_0 + N_1 + N_2 = 8.$$

and

$$Poin(M(\mathcal{A}, t)) = 1 + 5t + 8t^2.$$

Logarithmic forms

Definition 3.3.6. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ a toric arrangement in $(\mathbb{C}^*)^d$. For every $i \in \{1, \ldots, n\}$, define the following logarithmic forms:

$$\omega_i := \frac{1}{2\pi i} dlog(1 - e^{\chi_i}), \qquad \psi_i := \frac{1}{2\pi i} dlog(e^{\chi_i}),$$
$$\bar{\omega}_i := \frac{1}{2\pi i} dlog(1 - e^{\chi_i}) + \frac{1}{2\pi i} dlog(1 - e^{-\chi_i}).$$

For every $S = \{H_{i_1}, \ldots, H_{i_p}\} \subseteq \{H_1, \ldots, H_n\}$,

$$\omega_S := \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \qquad \psi_S := \psi_{i_1} \wedge \cdots \wedge \psi_{i_p} \qquad \bar{\omega}_S := \bar{\omega}_{i_1} \wedge \cdots \wedge \bar{\omega}_{i_p}.$$

Lemma 3.3.7. *For every $i \in \{1, \ldots, n\}$,*

$$\bar{\omega}_i = 2\omega_i - \psi_i.$$

Proof. If suffices to prove that $\omega_i - \psi_i = \bar{\omega}_i - \omega_i$. Note that

$$\omega_i := \frac{1}{2\pi i} \frac{e^{\chi_i}}{e^{\chi_i} - 1} dlog(e^{\chi_i}) = \frac{e^{\chi_i}}{e^{\chi_i} - 1} \psi_i,$$

then

$$\omega_i - \psi_i = \left(\frac{e^{\chi_i}}{e^{\chi_i} - 1} - 1 \right) \psi_i = \frac{1}{2\pi i} \frac{1}{e^{\chi_i} - 1} dlog(e^{\chi_i}) = \frac{1}{2\pi i} dlog(1 - e^{-\chi_i}).$$

□

3.3.1 Unimodular case

Let $\mathcal{A}$ be an unimodular toric arrangement in $T = (\mathbb{C}^*)^d$ and $C = \{H_1, \ldots, H_k\}$ a circuit of $\mathcal{A}$. Thanks to Corollary 3.2.13, we have that

$$\sum_{j=1}^{k} c_j \chi_j = 0$$

where $c_j \in \{-1, +1\}$. The aim of this subsection is to prove that:

$$\prod_{j=2}^{k} (\bar{\omega}_j + c_j \psi_j - \bar{\omega}_{j-1} + c_{j-1}\psi_{j-1}) = 0.$$

In order to do it, we need to introduce two lemmas that deal with two particular cases.

Lemma 3.3.8. *If $\chi_1 = \sum_{i=2}^{k} \chi_i$, then*

$$\omega_2 \ldots \omega_k = \omega_1 \prod_{i=2}^{k-1} (\omega_{i+1} - \omega_i + \psi_i)$$

Proof. For every fixed $\bar{j} \in \{2, \ldots, k\}$, we consider the product $\prod_{i=2}^{k-1}(\omega_{i+1} - \omega_i + \psi_i)$ without the factors $\omega_{\bar{j}}$ and $\psi_{\bar{j}}$, i.e.

$$\prod_{i=2}^{\bar{j}-2} (\omega_{i+1} - \omega_i + \psi_i)(\widehat{\omega_{\bar{j}}} - \omega_{\bar{j}-1} + \psi_{\bar{j}-1})(\omega_{\bar{j}+1} - \widehat{\omega_{\bar{j}}} + \widehat{\psi_{\bar{j}}}) \prod_{i=\bar{j}+1}^{k-1} (\omega_{i+1} - \omega_i + \psi_i),$$

where the notation$\widehat{\omega_{\bar{j}}}$ and $\widehat{\psi_{\bar{j}}}$ indicate that $\omega_{\bar{j}}$ and $\psi_{\bar{j}}$ are omitted.
Since, for every i, $\omega_i \wedge \omega_i = 0$ and $\omega_i \wedge \psi_i = 0$, we obtain

$$\prod_{i=2}^{\bar{j}-1} (-\omega_i + \psi_i) \prod_{i=\bar{j}}^{k-1} \omega_{i+1},$$

then

$$\omega_1 \prod_{i=2}^{k-1} (\omega_{i+1} - \omega_i + \psi_i) = \sum_{j=2}^{k} \omega_1 \left(\prod_{i=2}^{j-1} (-\omega_i + \psi_i) \prod_{i=j}^{k-1} \omega_{i+1} \right).$$

Now, for every $A \subsetneq C = \{H_1, \dots, H_k\}$, denote

$$\eta_A = \eta_1 \dots \widehat{\eta_{max(C\backslash A)}} \dots \eta_k \qquad \text{with} \qquad \eta_i = \eta_i^A = \begin{cases} \omega_i & \text{if } H_i \in A \\ \psi_i & \text{if } H_i \notin A \end{cases}$$

where $H_{max(C\backslash A)} := max(C\backslash A)$.

If $max(C\backslash A) = H_j$, then $\{H_{j+1}, \dots, H_k\} \subseteq A$, thus

$$\eta_A = \eta_1 \dots \eta_{j-1}\omega_{j+1} \dots \omega_k = \eta_1 \dots \eta_{j-1} \prod_{i=j}^{k-1} \omega_{i+1}.$$

It easily can be proved that

$$\prod_{i=2}^{j-1}(-\omega_i + \psi_i) = \sum_{\substack{A \subseteq C,\\ max(C\backslash A)=H_j}} (-1)^{|A_{\leq j}|}\eta_1 \dots \eta_{j-1},$$

where $A_{\leq j} := A \cap \{H_1, \dots, H_j\}$. Then it follows that

$$\omega_1 \prod_{i=2}^{k-1}(\omega_{i+1} - \omega_i + \psi_i) = \sum_{j=2}^{k} \sum_{\substack{H_1 \in A \subsetneq C,\\ max(C\backslash A)=H_j}} (-1)^{|A_{\leq j}|-1}\eta_A = \sum_{H_1 \in A \subsetneq C} (-1)^{|A_{\leq max(C\backslash A)}|-1}\eta_A.$$

Define

$$\vartheta^{(1)} := \frac{1}{2\pi i} dlog(1 - \prod_{j=2}^{k} e^{\chi_j})$$

and, for every $B \subseteq C' := \{H_2, \dots, H_k\}$,

$$\Phi_B^{(1)} := (-1)^{\ell(B, C\backslash B)} \prod_{H_i \in B} \omega_i \prod_{\substack{H_j \in C'\backslash B,\\ H_j \neq max(C'\backslash B)}} \psi_j \vartheta^{(1)}.$$

Since $\chi_1 = \sum_{i=2}^{k} \chi_i$, we have that $\vartheta^{(1)} = \omega_1$ and

$$\Phi_B^{(1)} = (-1)^{max(C\backslash B)-1}\eta_{B\cup\{H_1\}}.$$

In [DP11, eq. (15.3)] it has been proved that

$$\sum_{B \subsetneq C} (-1)^{|B|+k}\Phi_B^{(1)} = \omega_2 \dots \omega_k,$$

then, if we take $A = B \cup \{H_1\}$, then $max(C\backslash B) = max(C\backslash A)$ and $|B| = |A| - 1$. Since $k - 1 - max(C\backslash A) = |A| - |A_{\leq max(C\backslash A)}|$ we obtain the claimed equality. $\square$

Lemma 3.3.9. *If* $\sum_{i=1}^{k} \chi_i = 0$*, then*

$$\prod_{i=1}^{k-1}(\omega_{i+1} - \omega_i + \psi_i) = 0$$

equivalently

$$\prod_{i=1}^{k-1}(\bar{\omega}_{i+1} + \psi_{i+1} - \bar{\omega}_i + \psi_i) = 0$$

Proof. First we prove by induction on k that

$$\omega_2 \dots \omega_k = \omega_2 \prod_{i=2}^{k-1}(\omega_{i+1} - \omega_i + \psi_i).$$

For $k = 2$ it is immediate. If it is true for $k = \bar{k}$, then it is true also for $k = \bar{k} + 1$. In fact,

$$\omega_2 \prod_{i=2}^{\bar{k}}(\omega_{i+1} - \omega_i + \psi_i) = \omega_2 \dots \omega_{\bar{k}}(\omega_{\bar{k}+1} - \omega_{\bar{k}} + \psi_{\bar{k}}) = \omega_2 \dots \omega_{\bar{k}}\omega_{\bar{k}+1}.$$

Now, if we denote $\chi_1' = -\chi_1$, we have that $\chi_1' = \sum_{i=2}^{k} \chi_i$ and then we can rewrite the left-hand side of the last equation using Lemma 3.3.8 and we obtain:

$$\omega_2 \prod_{i=2}^{k-1}(\omega_{i+1} - \omega_i + \psi_i) = \omega_2 \dots \omega_k = \omega_1' \prod_{i=2}^{k-1}(\omega_{i+1} - \omega_i + \psi_i).$$

Collecting terms we have $(\omega_2 - \omega_1') \prod_{i=2}^{k-1}(\omega_{i+1} - \omega_i + \psi_i) = 0$. It follows that, thanks to Lemma 3.3.7

$$\begin{aligned}
0 &= (\omega_2 - \omega_1') \prod_{i=2}^{k-1}(\omega_{i+1} - \omega_i + \psi_i) \\
&= (\omega_2 - \frac{1}{2\pi i} dlog(1 - e^{-\chi_1})) \prod_{i=2}^{k-1}(\omega_{i+1} - \omega_i + \psi_i) \\
&= (\omega_2 - \omega_1 + \psi_1) \prod_{i=2}^{k-1}(\omega_{i+1} - \omega_i + \psi_i) \\
&= \prod_{i=1}^{k-1}(\omega_{i+1} - \omega_i + \psi_i).
\end{aligned}$$

□

Now we can easily prove:

Proposition 3.3.10. *If* $\sum_{j=1}^{k} c_j \chi_j = 0$ *where* $c_j \in \{-1, +1\}$, *then*

$$\prod_{j=2}^{k} (\bar{\omega}_j + c_j \psi_j - \bar{\omega}_{j-1} + c_{j-1} \psi_{j-1}) = 0.$$

Proof. If we denote $\chi_i' = c_i \chi_i$ for every $i \in \{1, \dots, k\}$, we obtain $\sum_{i=1}^{k} \chi_i' = 0$, then, since $\bar{\omega}_i' = \bar{\omega}_i$ and $\psi_i' = c_i \psi_i$, thanks to Lemma 3.3.9 we conclude. □

Example 3.3.11. Let $T = (\mathbb{C}^*)^2$ be the 2-dimensional torus and $\mathcal{A}'$ the unimodular arrangement defined by $H_1 := \{(t, s) \in (\mathbb{C}^*)^2 | t = 1\}$, $H_2 := \{(t, s) \in (\mathbb{C}^*)^2 | s = 1\}$ and $H_3 := \{(t, s) \in (\mathbb{C}^*)^2 | ts = 1\}$.

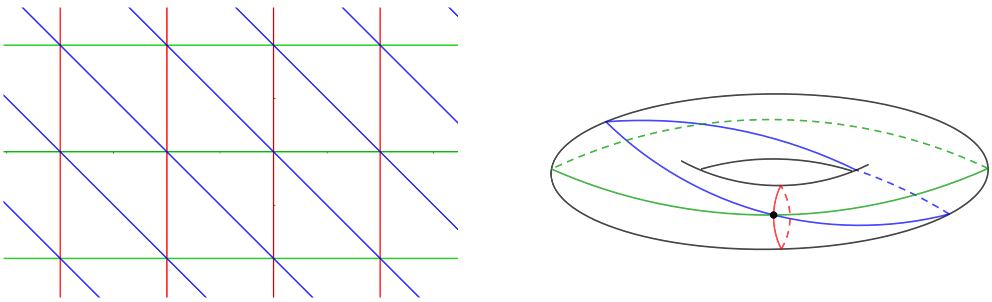


Figure 3.6: $\tilde{\mathcal{A}}'$ $\xrightarrow{/\mathbb{Z}^2}$ $\mathcal{A}'$

Following the calculus of Example 3.2.3, we have that $\{H_1, H_2, H_3\}$ is the only circuit in $\mathcal{A}'$. In this case we have that $\chi_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi_1 + \chi_2$. Since

$$\omega_3 = \frac{1}{2\pi i} dlog(1 - st) = \frac{1}{2\pi i} \frac{d(st)}{st - 1} = \frac{1}{2\pi i} \frac{sd(t) + td(s)}{st - 1},$$

and

$$\begin{aligned}\omega_2 - \omega_1 + \psi_1 &= \frac{1}{2\pi i} dlog(1-s) - \frac{1}{2\pi i} dlog(1-t) + \frac{1}{2\pi i} dlog(t) \\ &= \frac{1}{2\pi i}\frac{d(s)}{s-1} - \frac{1}{2\pi i}\frac{d(t)}{t-1} + \frac{1}{2\pi i}\frac{d(t)}{t} \\ &= \frac{1}{2\pi i}\frac{d(s)}{s-1} - \frac{1}{2\pi i}\frac{d(t)}{t(t-1)},\end{aligned}$$

it follows that, exactly as Lemma 3.3.8 states,

$$\begin{aligned}\omega_3(\omega_2 - \omega_1 + \psi_1) &= \frac{-1}{4\pi^2}\left(\frac{sd(t)}{st-1}\frac{d(t)}{s-1} + \frac{td(s)}{st-1}\frac{d(t)}{t(t-1)}\right) = \\ &= \frac{-1}{4\pi^2}\left(\frac{(st-1)d(t)d(s)}{(st-1)(s-1)(t-1)}\right) = \frac{-1}{4\pi^2}\frac{d(t)d(s)}{(s-1)(t-1)} = \\ &= \frac{-1}{4\pi^2} dlog(1-t)dlog(1-s) = \\ &= \omega_1\omega_2,\end{aligned}$$

If we denote $\chi_3' := -\chi_3$ we have that $\chi_1 + \chi_2 + \chi_3' = 0$.
Note that, since $\psi_3' = -\psi_3$ and $\bar{\omega}_3' = \bar{\omega}_3$, we have that

$$\begin{aligned}\omega_3' &= \frac{1}{2}(\bar{\omega}_3' - \psi_3') = \frac{1}{2}(\bar{\omega}_3 + \psi_3) = \frac{1}{2}(\bar{\omega}_3 - \psi_3) + \psi_3 = \omega_3 + \psi_3 = \\ &= \omega_3 - \psi_3'.\end{aligned}$$

Thus

$$\begin{aligned}0 &= \omega_1\omega_2 - \omega_3(\omega_2 - \omega_1 + \psi_1) = \omega_1\omega_2 - (\omega_3' - \psi_3')(\omega_2 - \omega_1 + \psi_1) = \\ &= (\omega_1 - \omega_3' + \psi_3')(\omega_2 - \omega_1 + \psi_1),\end{aligned}$$

that is exactly what Lemma 3.3.9 states.

Definition 3.3.12. Let $C = \{H_1, \ldots, H_k\}$ a circuit of $\mathcal{A}$.
For every $A, B \subseteq C$ such that $A \cap B = \emptyset$, denote

$$\bar{\eta}_{A,B} := \prod_{H_i \in A \cup B} \bar{\eta}_i^A \qquad \text{with} \qquad \bar{\eta}_i^A = \begin{cases} \bar{\omega}_i & \text{if } H_i \in A \\ \psi_i & \text{if } H_i \in B \end{cases}$$

Proposition 3.3.13. *Let $C = \{H_1, \dots, H_k\}$ a circuit of $\mathcal{A}$ such that $\sum_{H_j \in C} c_i\chi_i = 0$ where $c_i \in \{-1, +1\}$ for all i. Then*

$$\sum_{j=1}^{k} \sum_{\substack{A,B \subset C\\ C = A \sqcup B \sqcup \{H_j\}}} (-1)^{|A_{\leq j}|} c_B \bar{\eta}_{A,B} = 0$$

where $c_B := \prod_{H_i \in B} c_i$ and $A_{\leq j} = A \cap \{H_1, \dots, H_j\}$.

In particular

$$\sum_{H_j \in C} \sum_{\substack{A,B \subset C\\ C = A \sqcup B \sqcup \{H_j\}\\ |B| even}} (-1)^{|A_{\leq j}|} c_B \bar{\eta}_{A,B} = 0.$$

Proof. From Proposition 3.3.10, we have that $\prod_{i=2}^{k}(\bar{\omega}_i + c_i\psi_i - \bar{\omega}_{i-1} + c_{i-1}\psi_{i-1}) = 0$. Exactly like we have done in the proof of Lemma 3.3.8, we can rewrite this equation as

$$\sum_{j=1}^{k} \prod_{i=1}^{j-1} (-\bar{\omega}_i + c_i\psi_i) \prod_{i=j+1}^{k} (\bar{\omega}_i + c_i\psi_i) = 0.$$

Expanding all the products we obtain exactly

$$\sum_{j=1}^{k} \sum_{\substack{A,B \subset C\\ C = A \sqcup B \sqcup \{H_j\}}} (-1)^{|A_{\leq j}|} c_B \bar{\eta}_{A,B} = 0. \tag{3.1}$$

Moreover, for $\chi_i' := -\chi_i$, $\sum_{i=1}^{k} c_i\chi_i' = 0$, $\psi' = -\psi$ and $\bar{\omega}_i' = \bar{\omega}_i$. Then

$$\sum_{j=1}^{k} \prod_{i=1}^{j-1} (-\bar{\omega}_i - c_i\psi_i) \prod_{i=j+1}^{k} (\bar{\omega}_i - c_i\psi_i) = 0,$$

thus

$$0 = \sum_{j=1}^{k} \sum_{\substack{A,B \subset C\\ C = A \sqcup B \sqcup \{H_j\}}} (-1)^{|A_{\leq j}|} c_B \bar{\eta}'_{A,B} = \sum_{j=1}^{k} \sum_{\substack{A,B \subset C\\ C = A \sqcup B \sqcup \{H_j\}}} (-1)^{|A_{\leq j}|} c_B (-1)^{|B|} \bar{\eta}_{A,B}.$$

Adding this to (3.1), we obtain

$$0 = \sum_{j=1}^{k} \sum_{\substack{A,B \subset C\\ C = A \sqcup B \sqcup \{H_j\}}} (-1)^{|A_{\leq j}|} c_B \bar{\eta}_{A,B} \left(1 + (-1)^{|B|}\right).$$

It immediately follows that

$$\sum_{H_j \in C} \sum_{\substack{A,B \subset C \\ C = A \sqcup B \sqcup \{H_j\} \\ |B| even}} (-1)^{|A_{\leq j}|} c_B \bar{\eta}_{A,B} = 0.$$

□

Theorem 3.3.14. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential unimodular toric arrangement.*
The rational cohomology algebra $H^*(M(\mathcal{A}), \mathbb{Q})$ is isomorphic to the algebra $\mathcal{E}$ with

- *$\mathcal{E}$ generated by $\{e_{A;B} | A \cap B = \emptyset, A \sqcup B$ is a independent set$\}$. The degree of $e_{A;B}$ is $|A \sqcup B|$.*
- *The following types of relations*
  1. *For any two generators $e_{A;B}$ and $e_{A';B'}$,*

     $$e_{A;B} e_{A';B'} = \begin{cases} 0 & \text{if } A \sqcup B \sqcup A' \sqcup B' \text{ dependent} \\ (-1)^{\ell(A \cup B, A' \cup B')} e_{A \cup A'; B \cup B'} & \text{otherwise} \end{cases}$$

     where, for $C = \{c_1 < \cdots < c_l\}$ and $D = \{d_1 < \cdots < d_h\}$, $\ell(C, D)$ denote the lenght of the permutation that takes $(C, D) = \{c_1, \dots, c_l, d_1, \dots, d_h\}$ into $C \cup D$.
  2. *If $\sum_{i=1}^n n_i \chi_i = 0$ where $n_i \in \mathbb{Z}$, then $\sum_{i=1}^n n_i e_{\emptyset;\{H_i\}} = 0$.*
  3. *For every circuit $C \subseteq \{H_1, \dots, H_n\}$, with associated linear dependency $\sum_{H_i \in C} n_i \chi_i = 0$ with $n_i \in \mathbb{Z}$, then*

     $$\sum_{H_j \in C} \sum_{\substack{A,B \subset C \\ C = A \sqcup B \sqcup \{H_j\} \\ |B| \, even}} (-1)^{|A_{\leq j}|} c_B e_{A;B} = 0$$

     where $c_B := \prod_{H_i \in B} sgn\ n_i$.

Proof. We want to prove that the map Φ given by $e_{A;B} \longmapsto [\bar{\eta}_{A,B}]$ is well defined.

- if $A \sqcup B$ is an independent set, A and B are independent sets too.
- The elements $[\bar{\eta}_{W,A,B}]$ satisfy the same relations. In fact

  1. For $\bar{\eta}_{A,B}$ and $\bar{\eta}_{A',B'}$, with $A \sqcup B \sqcup A' \sqcup B'$ independent set, we have

$$\begin{aligned}\bar{\eta}_{A,B}\bar{\eta}_{A',B'} = \prod_{H_i \in A\cup B} \bar{\eta}_i^A \prod_{H_i \in A'\cup B'} \bar{\eta}_i^{A'} = \\ = (-1)^{\ell(A\cup B, A'\cup B')} \prod_{H_i \in A\cup B\cup A'\cup B'} \bar{\eta}_i^A \bar{\eta}_i^{A'} = \\ = (-1)^{\ell(A\cup B, A'\cup B')} \bar{\eta}_{A\cup A', B\cup B'}.\end{aligned}$$

  2. If $\sum_{i=1}^n n_i \chi_i = 0$, then

$$\begin{aligned}\sum_{i=1}^n n_i \bar{\eta}_{\emptyset,H_i} = \sum_{i=1}^n n_i \psi_i = \sum_{i=1}^n n_i dlog(e^{\chi_i}) = d\left(\sum_{i=0}^n log(e^{n_i\chi_i})\right) = \\ = dlog\left(\prod_{i=1}^n e^{n_i\chi_i}\right) = dlog\left(e^{\sum_{i=1}^n n_i\chi_i}\right) = 0.\end{aligned}$$

  3. it immediately follows from Proposition 3.3.13.

We will prove that Φ is bijective at the end of this chapter, in Theorem 3.3.55 that is a generalization of this theorem. □

Remark 3.3.15.
- Every generator of degree k can be written as the product of k generators of degree one. In fact, if $|A \sqcup B| = k$, by definition

$$\bar{\eta}_{A,B} = (-1)^{\ell(A,B)} \prod_{H_i \in A} \bar{\omega}_i \prod_{Hj \in B} \psi_j = (-1)^{\ell(A,B)} \prod_{H_i \in A} \bar{\eta}_{H_i,\emptyset} \prod_{Hj \in B} \bar{\eta}_{\emptyset,H_j}.$$

  This implies that every form in $H^*(M(\mathcal{A}), \mathbb{Q})$ is the product of 1-forms.

- Among the generators of $H^*(M(\mathcal{A}),\mathbb{Q})$ there are the generators of $H^*(T,\mathbb{Q})$, in fact $\{\bar{\eta}_{\emptyset,i}; i=1,\dots,n\}=\{\psi_1,\dots,\psi_n\}$ generates $H^*(T,\mathbb{Q})$ since $\mathcal{A}$ is essential.

- This theorem gives a Orlik-Solomon-type presentation for the cohomology algebra of an unimodular toric arrangement.
  By Remark 1.5.10, the reason for this name is clear.

Example 3.3.16. Let $\mathcal{A}$ be the toric arrangement of Example 3.3.11.

- The generators of $H^*(M(\mathcal{A}),\mathbb{Q})$ can be represented by $\{\bar{\omega}_1,\bar{\omega}_2,\bar{\omega}_3,\psi_1,\psi_2,\psi_3\}$, with

$$\omega_1=\frac{1}{2\pi i}dlog(1-t)=\frac{1}{2\pi i}\frac{d(t)}{t-1} \qquad \psi_1=\frac{1}{2\pi i}dlog(t)=\frac{1}{2\pi i}\frac{d(t)}{t}$$

$$\omega_2=\frac{1}{2\pi i}dlog(1-s)=\frac{1}{2\pi i}\frac{d(s)}{s-1} \qquad \psi_2=\frac{1}{2\pi i}dlog(s)=\frac{1}{2\pi i}\frac{d(s)}{s}$$

$$\omega_3=\frac{1}{2\pi i}dlog(1-ts)=\frac{1}{2\pi i}\frac{sd(t)+td(s)}{ts-1} \qquad \psi_3=\frac{1}{2\pi i}dlog(ts)=\frac{1}{2\pi i}\frac{sd(t)+td(s)}{ts}$$

and

$$\bar{\omega}_1=2\omega_1-\psi_1=\frac{1}{2\pi i}\left(\frac{t+1}{t(t-1)}d(t)\right)$$
$$\bar{\omega}_2=2\omega_2-\psi_2=\frac{1}{2\pi i}\left(\frac{s+1}{s(s-1)}d(s)\right)$$
$$\bar{\omega}_3=2\omega_3-\psi_3=\frac{1}{2\pi i}\left(\frac{s(ts+1)d(t)+t(ts+1)d(s)}{ts(ts-1)}\right).$$

- The relations are

  1. For $A=H_1, B=H_3$, $A'=H_2$ and $B'=\{H_1,H_3\}$, we have that

$$\bar{\eta}_{A,B}\bar{\eta}_{A',B'}=\bar{\omega}_1\psi_3\bar{\omega}_2\psi_1\psi_3=0.$$

  If we take instead $B'=\emptyset$, we obtain

$$\bar{\eta}_{A,B}\bar{\eta}_{A',B'}=\bar{\omega}_1\psi_3\bar{\omega}_2=-\bar{\omega}_1\bar{\omega}_2\psi_3=(-1)^{\ell(A\cup B,A'\cup B')}\bar{\eta}_{A\cup A'}.$$

2. Since $\chi_1 + \chi_2 - \chi_3 = 0$,

$$\bar{\eta}_{\emptyset,\{1\}}+\bar{\eta}_{\emptyset,\{2\}}-\bar{\eta}_{\emptyset,\{3\}} = \psi_1+\psi_2-\psi_3 = \frac{1}{2\pi i}\left(\frac{d(t)}{t} + \frac{d(s)}{s} - \frac{sd(t)+td(s)}{ts}\right) = 0.$$

3. If $|B|$ even, then we have

$$(A,B) \in \{(\emptyset,\{H_1,H_2\}),(\emptyset,\{H_1,H_3\}),(\emptyset,\{H_2,H_3\}),(\{H_1,H_2\},\emptyset),(\{H_1,H_3\},\emptyset),(\{H_2,H_3\},\emptyset)\},$$

then

$$\begin{aligned}
&(-1)^0 c_1c_2\psi_1\psi_2 + (-1)^0 c_1c_3\psi_1\psi_3 + (-1)^0 c_2c_3\psi_2\psi_3 + (-1)^2\bar{\omega}_1\bar{\omega}_2 + (-1)^1\bar{\omega}_1\bar{\omega}_3 + \\
&\qquad + (-1)^0\bar{\omega}_2\bar{\omega}_3 = \\
&= \psi_1\psi_2 - \psi_1\psi_3 - \psi_2\psi_3 + \bar{\omega}_1\bar{\omega}_2 - \bar{\omega}_1\bar{\omega}_3 + \bar{\omega}_2\bar{\omega}_3 = \\
&= \psi_1\psi_2 - \psi_3(\psi_1+\psi_2) + \bar{\omega}_1\bar{\omega}_2 - \bar{\omega}_1\bar{\omega}_3 + \bar{\omega}_2\bar{\omega}_3 = \\
&= \psi_1\psi_2 - \psi_3\psi_3 + \bar{\omega}_1\bar{\omega}_2 - \bar{\omega}_1\bar{\omega}_3 + \bar{\omega}_2\bar{\omega}_3 = \\
&= \frac{-1}{4\pi^2}\left(\frac{d(t)d(s)}{ts} + \frac{(t+1)(s+1)d(s)}{ts(t-1)(s-1)} - \frac{(t+1)(ts+1)d(t)d(s)}{ts(t-1)(ts-1)}\right) + \\
&\qquad - \frac{-1}{4\pi^2}\left(\frac{(s+1)(ts+1)d(t)d(s)}{ts(s-1)(ts-1)}\right) = \\
&= \frac{-1}{4\pi^2}\left(\frac{(t-1)(s-1)(ts-1)+(t+1)(s+1)(ts-1)-(t+1)(s-1)(ts+1)}{ts(t-1)(s-1)(ts-1)}\right) + \\
&\qquad - \frac{-1}{4\pi^2}\left(\frac{(t-1)(s+1)(ts+1)}{ts(t-1)(s-1)(ts-1)}\right) = \\
&= \frac{-1}{4\pi^2}\left(\frac{(ts-1)[(t-1)(s-1)+(t+1)(s+1)]}{ts(t-1)(s-1)(ts-1)}\right) + \\
&\qquad - \frac{-1}{4\pi^2}\left(\frac{(ts+1)[(t+1)(s-1)+(t-1)(s+1)]}{ts(t-1)(s-1)(ts-1)}\right) = \\
&= \frac{-1}{4\pi^2}\left(\frac{(ts-1)[2ts+2]-(ts+1)[2ts-2]}{ts(t-1)(s-1)(ts-1)}\right) = \\
&= \frac{-1}{4\pi^2}\left(\frac{2((ts)^2-1)-2((ts)^2-1)}{ts(t-1)(s-1)(ts-1)}\right) = \\
&= 0.
\end{aligned}$$

3.3.2 General case

Coverings

Definition 3.3.17. Let $\mathcal{A}$ be a primitive arrangement in $T = (\mathbb{C}^*)^d$ and $f : U \longrightarrow (\mathbb{C}^*)^d$ be a finite covering. Define

$$a_i := |\pi_0(f^{-1}(H_i))| \quad \text{for every } H_i \in \mathcal{A},$$

$$\mathcal{A}_U := \bigcup_{H \in \mathcal{A}} \pi_0(f^{-1}(H)) = \bigcup_{H_i \in \mathcal{A}} \bigcup_{j=1}^{a_i} H^U_{i,j},$$

where $H^U_{i,j}$ is the j-th connected component of $f^{-1}(H_i)$, and for every $q \in f^{-1}(H_i)$,

$$H^U_i(q) \text{ the connected component of } f^{-1}(H_i) \text{ containing } q.$$

Note that $\mathcal{A}_U$ is a primitive arrangement, since every hypertorus is connected.

Proposition 3.3.18. *The connected components $f^{-1}(H_i)$ are associated with the primitive character $\frac{\widehat{\chi}_i}{a_i}$, where $\widehat{\chi}_i := \chi_i \circ f$.*
In particular, every $L \in \pi_0(f^{-1}(H_i))$ has equation

$$\frac{\widehat{\chi}_i}{a_i} = \frac{\widehat{\chi}_i}{a_i}(q)$$

with q any point in L.

Example 3.3.19. Let $\mathcal{A} = \{H_1, H_2, H_3\}$ be the toric arrangement in $T := (\mathbb{C}^*)^2$ of Example 3.2.11, with $H_1 := \{(t,s) \in (\mathbb{C}^*)^2 | t = 1\}$, $H_2 := \{(t,s) \in (\mathbb{C}^*)^2 | s = 1\}$ and $H_3 := \{(t,s) \in (\mathbb{C}^*)^2 | t^3 s = 1\}$.

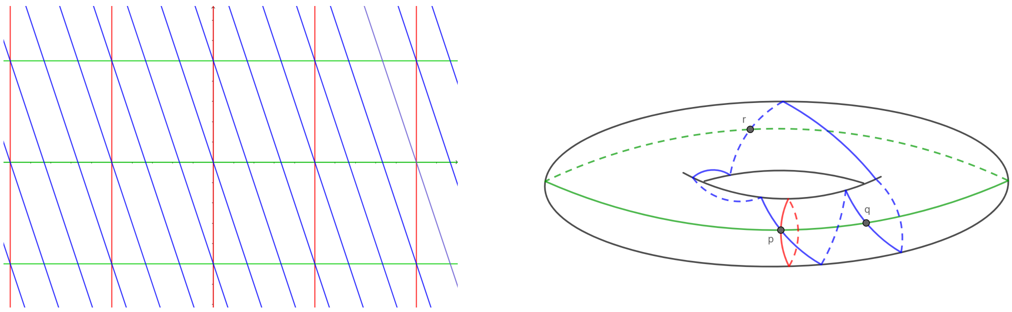


Figure 3.7: $\tilde{\mathcal{A}} \xrightarrow{/\mathbb{Z}^2} \mathcal{A}$

Let $f : (\mathbb{C}^*)^2 \longrightarrow T$ the covering given by $f(u,v) \mapsto (u, v^3)$. For $p = (1,1) \in T$, $f^{-1}(p) = \{p_1 = (1, e^{\frac{2}{3}\pi i}), p_2 = (1, e^{\frac{4}{3}\pi i}), p_3 = (1,1)\}$. In $\mathcal{A}_U$ we have

$$f^{-1}(H_1) = \{(u,v) \in (\mathbb{C}^*)^2 | u = 1\} = H_1^{(\mathbb{C}^*)^2}, \quad \text{with} \quad \widehat{\chi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad a_1 = 1,$$

$$\begin{aligned} f^{-1}(H_2) &= \{(u,v) \in (\mathbb{C}^*)^2 | v^3 = 1\} = \bigcup_{k \in \mathbb{Z}} \{(u,v) \in (\mathbb{C}^*)^2 | v = e^{\frac{2}{3}k\pi i}\} = \\ &= \{(u, e^{\frac{2}{3}\pi i}) \in (\mathbb{C}^*)^2\} \cup \{(u, e^{\frac{4}{3}\pi i}) \in (\mathbb{C}^*)^2\} \cup \{(u,1) \in (\mathbb{C}^*)^2\} = \\ &= {H_{2,1}}^{(\mathbb{C}^*)^2} \cup {H_{2,2}}^{(\mathbb{C}^*)^2} \cup {H_{2,3}}^{(\mathbb{C}^*)^2} = \\ &= H_2^{(\mathbb{C}^*)^2}(p_1) \cup H_2^{(\mathbb{C}^*)^2}(p_2) \cup H_2^{(\mathbb{C}^*)^2}(p_3) \end{aligned}$$

$$\text{with} \quad \widehat{\chi}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad \text{and} \quad a_2 = 3$$

$$f^{-1}(H_3) = \{(u,v) \in (\mathbb{C}^*)^2 | u^3v^3 = 1\} = \bigcup_{k\in\mathbb{Z}} \{(u,v) \in (\mathbb{C}^*)^2 | uv = e^{\frac{2}{3}k\pi i}\} =$$
$$= \{(u,v) \in (\mathbb{C}^*)^2 | uv = e^{\frac{2}{3}\pi i}\} \cup \{(u,v) \in (\mathbb{C}^*)^2 | uv = e^{\frac{4}{3}\pi i}\} \cup$$
$$\cup \{(u,v) \in (\mathbb{C}^*)^2 | uv = 1\} =$$
$$= {H_{3,1}}^{(\mathbb{C}^*)^2} \cup {H_{3,2}}^{(\mathbb{C}^*)^2} \cup {H_{3,3}}^{(\mathbb{C}^*)^2} =$$
$$= H_3^{(\mathbb{C}^*)^2}(p_1) \cup H_3^{(\mathbb{C}^*)^2}(p_2) \cup H_3^{(\mathbb{C}^*)^2}(p_3)$$

$$\text{with} \quad \widehat{\chi}_3 = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{and} \quad a_3 = 3.$$

In order to visualize this 7 hypertori in $\mathcal{A}_{(\mathbb{C}^*)^2}$, we consider the associated periodic hyperplane arrangement $\tilde{\mathcal{A}}_{(\mathbb{C}^*)^2}$, defined by

$$\begin{aligned}
{\tilde{H}_1}^{(\mathbb{C}^*)^2} &= \{(x,y) \in \mathbb{C}^2 | x = 0\}, \\
\tilde{H}_{2,1}^{(\mathbb{C}^*)^2} &= \{(x,y) \in \mathbb{C}^2 | 3y = 1\} \\
\tilde{H}_{2,2}^{(\mathbb{C}^*)^2} &= \{(x,y) \in \mathbb{C}^2 | 3y = 2\}, \\
\tilde{H}_{2,3}^{(\mathbb{C}^*)^2} &= \{(x,y) \in \mathbb{C}^2 | 3y = 3\}, \\
\tilde{H}_{3,1}^{(\mathbb{C}^*)^2} &= \{(x,y) \in \mathbb{C}^2 | 3x + 3y = 1\}, \\
\tilde{H}_{3,2}^{(\mathbb{C}^*)^2} &= \{(x,y) \in \mathbb{C}^2 | 3x + 3y = 2\}, \\
\tilde{H}_{3,3}^{(\mathbb{C}^*)^2} &= \{(x,y) \in \mathbb{C}^2 | 3x + 3y = 3\}.
\end{aligned}$$

Note that, in this example, $\mathcal{A}_{(\mathbb{C}^*)^2}$ is unimodular.

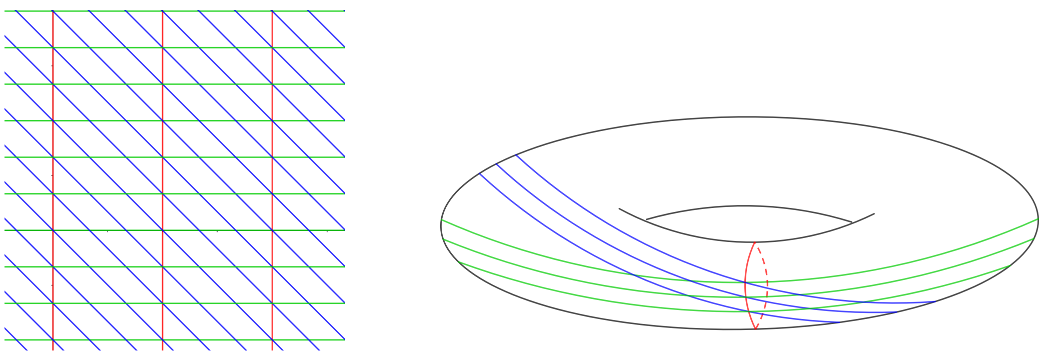


Figure 3.8: $\tilde{\mathcal{A}}_{(\mathbb{C}^*)^2} \xrightarrow{/\mathbb{Z}^2} \mathcal{A}_{(\mathbb{C}^*)^2}$

Logarithmic forms of coverings

The aim of this subsection is to describe the logarithmic forms on $M(\mathcal{A}_U)$.

Definition 3.3.20. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $T = (\mathbb{C}^*)^d$ and $f : U \longrightarrow T$ be a finite covering.
For every $i \in \{1, \ldots, n\}$ and every $L \in \pi_0(f^{-1}(H_i))$, we define the following logarithmic forms:

$$\omega_L = \omega_i^U(q) := \frac{1}{2\pi i} dlog(1 - e^{\frac{\widehat{\chi}_i}{a_i} - \frac{\widehat{\chi}_i}{a_i}(q)}) \qquad \psi_L = \psi_i^U := \frac{f^*(\psi_i)}{a_i} = \frac{1}{2\pi i} dlog(e^{\frac{\widehat{\chi_i}}{a_i}})$$

$$\bar{\omega}_L = \bar{\omega}_i^U(q) := 2\omega_i^U(q) - \psi_i^U$$

with q any point in L, and f^* the pull-back of f.

For $\pi_0(f^{-1}(H_i)) = \{H_{i,1}^U, \ldots, H_{i,a_i}^U\}$, we also denote $\bar{\omega}_{H_{i,j}^U}$ as $\bar{\omega}_{i,j}^U$.

For every $S \subseteq \{H_1, \ldots, H_n\}$, define

$$\omega_S^U(q) := \prod_{H_i \in S} \omega_i^U(q) \qquad \psi_S^U := \prod_{H_i \in S} \psi_i^U \qquad \bar{\omega}_S^U(q) := \prod_{H_i \in S} \bar{\omega}_i^U(q),$$

where $q \in \cap_{H_i \in S} H_i^U(q)$.

For every $A, B \subseteq \{H_1, \ldots, H_n\}$ with $A \cap B = \emptyset$, denote

$$\bar{\eta}_{A,B}^U(q) = \prod_{H_i \in A \cup B} \bar{\eta}_i^U(q) \quad \text{with} \quad \bar{\eta}_i^U(q) = \begin{cases} \bar{\omega}_i^U(q) & \text{if } H_i \in A \\ \psi_i^U & \text{if } H_i \in B, \end{cases}$$

where $q \in \cap_{H_i \in A} H_i^U(q)$.

Remark 3.3.21. Note that $\omega_i^U(q)$ does not depend on the choice of the point $q \in L$.

Definition 3.3.22. Let $S \subseteq \{H_1, \ldots, H_n\}$ be an independent set, W a connected component of $\cap_{H_i \in S} H_i$ and p any point in W. Since the pullback f^* is injective, we can define the forms $\omega^f_{W,S}$ and $\bar{\omega}^f_{W,S}$ as the unique forms on $M(\mathcal{A})$ such that

$$f^*(\bar{\omega}^f_{W,S}) = \frac{1}{|\cap_{H_i \in S} H_i^U(q_0) \cap f^{-1}(p)|} \sum_{q \in f^{-1}(p)} \bar{\omega}^U_S(q)$$

and

$$f^*(\omega^f_{W,S}) = \frac{1}{|\cap_{H_i \in S} H_i^U(q_0) \cap f^{-1}(p)|} \sum_{q \in f^{-1}(p)} \omega^U_S(q)$$

where q_0 is any point in $f^{-1}(p)$.

Lemma 3.3.23. *If $\mathcal{A}_U$ is unimodular, then for any $S \subseteq \{H_1, \ldots, H_n\}$ independent, $W \in \pi_0(\cap_{H_i \in S} H_i)$ and $p \in W$,*

$$\pi_0(f^{-1}(W)) = \left\{ \bigcap_{H_i \in S} H_i^U(q_0) | q_0 \in f^{-1}(p) \right\}.$$

Proof. $\supseteq$: If $\mathcal{A}_U$ is unimodular, $\cap_{H_i \in S} H_i^U(q_0)$ is a connected component of $\cap_{H_i \in S} f^{-1}(H_i)$, then $f(\cap_{H_i \in S} H_i^U(q_0))$ is a connected component of $\cap_{H_i \in S} H_i$. Since $q_0 \in \cap_{H_i \in S} H_i^U(q_0)$, $f(\cap_{H_i \in S} H_i^U(q_0))$ has to contain $f(q_0) = p$, thus $f(\cap_{H_i \in S} H_i^U(q_0)) = W$ and then $\cap_{H_i \in S} H_i^U(q_0)$ is a connected component of $f^{-1}(W)$.
$\subseteq$: Take $L \in \pi_0(f^{-1}(W))$ and take a point $q \in L$. Since $L \subseteq \cap_{H_i \in S} f^{-1}(H_i)$ and $q \in L$, $L \subseteq \cap_{H_i \in S} H_i^U(q)$, but, since $\cap_{H_i \in S} H_i^U(q)$ is connected, we have that $L = \cap_{H_i \in S} H_i^U(q)$.

□

It immediately follows that:

Remark 3.3.24. If $\mathcal{A}_U$ is unimodular, then

$$f^*(\bar{\omega}^f_{W,S}) = \frac{1}{|L \cap f^{-1}(p)|} \sum_{q \in f^{-1}(p)} \bar{\omega}^U_S(q)$$

with L any connected component of $f^{-1}(W)$.

Example 3.3.25. Let $\mathcal{A}$ be the toric arrangement in $T := (\mathbb{C}^*)^2$ of Example 3.2.11 and $f : U = (\mathbb{C}^*)^2 \longrightarrow T$ such that $f(u,v) = (u, v^3)$. As we have already computed in Example 3.3.19,

$$\begin{aligned}
H_2^U(p_1) &= \{(u,v) \in (\mathbb{C}^*)^2; v = e^{\frac{2}{3}\pi i}\} = \{(u,v) \in (\mathbb{C}^*)^2; e^{\frac{4}{3}\pi i}v = 1\}\\
H_2^U(p_2) &= \{(u,v) \in (\mathbb{C}^*)^2; v = e^{\frac{4}{3}\pi i}\} = \{(u,v) \in (\mathbb{C}^*)^2; e^{\frac{2}{3}\pi i}v = 1\}\\
H_2^U(p_3) &= \{(u,v) \in (\mathbb{C}^*)^2; v = 1\}\\
H_3^U(p_1) &= \{(u,v) \in (\mathbb{C}^*)^2; uv = e^{\frac{2}{3}\pi i}\} = \{(u,v) \in (\mathbb{C}^*)^2; e^{\frac{4}{3}\pi i}uv = 1\}\\
H_3^U(p_2) &= \{(u,v) \in (\mathbb{C}^*)^2; uv = e^{\frac{4}{3}\pi i}\} = \{(u,v) \in (\mathbb{C}^*)^2; e^{\frac{2}{3}\pi i}uv = 1\}\\
H_3^U(p_3) &= \{(u,v) \in (\mathbb{C}^*)^2; uv = 1\}.
\end{aligned}$$

Denoting $\xi = e^{\frac{2}{3}\pi i}$, we obtain

$$\begin{aligned}
&\bar{\omega}_2^U(p_1) = \frac{-1-\xi^2 v}{1-\xi^2 v}\frac{dv}{v} && \bar{\omega}_2^U(p_2) = \frac{-1-\xi v}{1-\xi v}\frac{dv}{v} && \bar{\omega}_2^U(p_3) = \frac{-1-v}{1-v}\frac{dv}{v}\\
&\bar{\omega}_3^U(p_1) = \frac{-1-\xi^2 uv}{1-\xi^2 uv}\frac{d(uv)}{uv} && \bar{\omega}_3^U(p_2) = \frac{-1-\xi uv}{1-\xi uv}\frac{d(uv)}{v} && \bar{\omega}_3^U(p_3) = \frac{-1-uv}{1-uv}\frac{d(uv)}{uv}.
\end{aligned}$$

For $W = p$, $S = \{H_2, H_3\}$, we obtain

$$\begin{aligned}
f^*(\bar{\omega}_{W,S}^f) &= \bar{\omega}_{2,3}(p_1) + \bar{\omega}_{2,3}(p_2) + \bar{\omega}_{2,3}(p_3) =\\
&= \frac{-1}{4\pi}\left(\frac{1+\xi^2 v}{1-\xi^2 v}\frac{dv}{v}\frac{1+\xi^2 uv}{1-\xi^2 uv}\frac{d(uv)}{uv} + \frac{1+\xi uv}{1-\xi uv}\frac{dv}{v}\frac{1+\xi uv}{1-\xi uv}\frac{d(uv)}{uv}\right) +\\
&\qquad + \frac{-1}{4\pi}\left(\frac{1+v}{1-v}\frac{dv}{v}\frac{1+uv}{1-uv}\frac{d(uv)}{uv}\right) =\\
&= \frac{-3}{4\pi}\frac{u^3v^6 + u^3v^3 + 4u^2v^3 + 4uv^3 + v^3 + 1}{uv(v^3-1)(u^3v^3-1)}dudv,
\end{aligned}$$

thus

$$\bar{\omega}_{W,S}^f = \frac{-1}{4\pi}\frac{t^3s^2 + t^3s + 4t^2s + 4ts + s + 1}{ts(s-1)(t^3s-1)}dtds.$$

Separating coverings

Definition 3.3.26. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement in $T = (\mathbb{C}^*)^d$ and $S \subseteq \{H_1, \dots, H_n\}$ be an independent set. We say that a covering $f : U \longrightarrow T$ **separates** S if, for any connected component W of $\cap_{H_i \in S} H_i$ and for every $H_i \in S$, there exist $q_i \in f^{-1}(H_i)$ such that $f(\cap_{H_i \in S} H_i^U(q_i)) = W$.

Lemma 3.3.27. *If f is a covering such that $\mathcal{A}_U$ is unimodular, then f separates every independent set $S \subseteq \{H_1, \dots, H_n\}$.*

Proof. Thanks to Lemma 3.3.23, there exists $q \in f^{-1}(H_i)$ such that $W = f(f^{-1}(W)) = f(\cap_{H_i \in S} H_i^U(q))$. □

Example 3.3.28. Let $\mathcal{A}$ be the toric arrangement in $T := (\mathbb{C}^*)^2$ of Example 3.2.11 and $f : U \longrightarrow T$ such that $f(u, v) = (u, v^3)$.

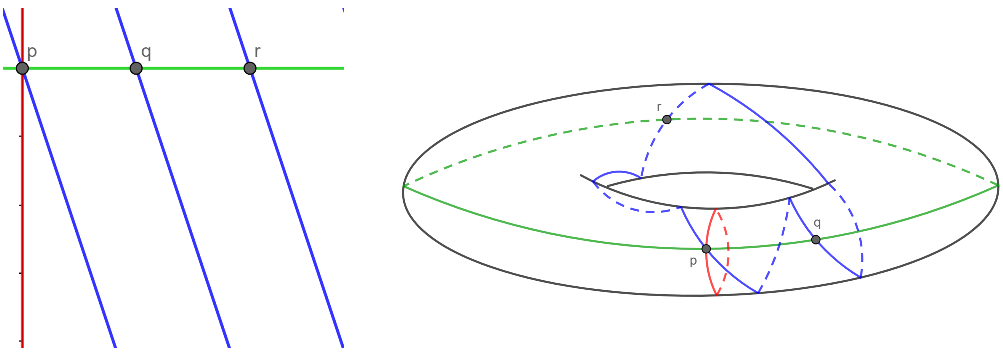


Figure 3.9: $\tilde{\mathcal{A}} \xrightarrow{/\mathbb{Z}^2} \mathcal{A}$

Since $\mathcal{A}_U$ is unimodular, f separates every independent set. In fact for $S = \{H_2, H_3\}$, $\pi_0(H_2 \cap H_3) = \{p, q, r\}$ and

$$f(H_2^U(p_1) \cap H_3^U(p_1)) = f(p_1) = p,$$

$$f(H_2^U(q_1) \cap H_3^U(q_1)) = f(q_1) = q,$$

$$f(H_2^U(r_1) \cap H_3^U(r_1)) = f(r_1) = r.$$

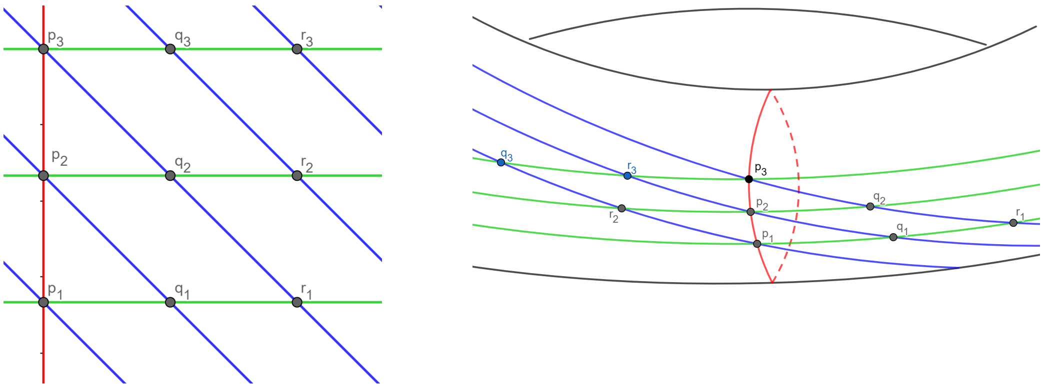


Figure 3.10: $\tilde{\mathcal{A}}_{(\mathbb{C}^*)^2} \xrightarrow{/\mathbb{Z}^2} \mathcal{A}_{(\mathbb{C}^*)^2}$

Here we state an important result of [CDD+19, Proposition 5.3].

Proposition 3.3.29. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement in $T = (\mathbb{C}^*)^d$ and $S \subseteq \{H_1, \dots, H_n\}$ an independent set. There exist a covering $f : U \longrightarrow T$ that separates S.*

Lemma 3.3.30. *If f is a covering that separates $S = \{H_1, \dots, H_k\}$, then*

$$f^*(\bar{\omega}^f_{W,S}) = \sum_{\substack{\vec{1} \leq \vec{j} \leq \vec{a}, \\ \cap_i H^U_{i,j_i} \subseteq f^{-1}(W)}} \prod_{H_i \in S} \bar{\omega}^U_{i,j_i},$$

where $\pi_0(f^{-1}(H_i)) = \{H^U_{i,1}, \dots, H^U_{i,a_i}\}$ and the $k-$tuple $\vec{1}, \vec{j}, \vec{a}$ are defined by $\vec{1} = (1, \dots, 1), \vec{j} = (j_1, \dots, j_k), \vec{a} = (a_1, \dots, a_k)$.

Theorem 3.3.31. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement in $T = (\mathbb{C}^*)^d$ and $S \subseteq \{H_1, \dots, H_n\}$ an independent set. If $f : U \longrightarrow T$,$g : V \longrightarrow T$ are coverings that separate S, then*

$$\bar{\omega}^f_{W,S} = \bar{\omega}^g_{W,S}.$$

Analogously $\omega^f_{W,S} = \omega^g_{W,S}$.

Proof. First we suppose there exists a finite covering $h : V \longrightarrow U$ such that $g = f \circ h$.

$$\begin{array}{ccc} U & \xrightarrow{f} & T \\ & \nwarrow^{h} & \uparrow g \\ & & V \end{array}$$

In this case, thanks to Lemma 3.3.30, we have

$$g^*(\omega^f_{W,S}) = h^*(f^*(\omega^f_{W,S})) = h^*\left(\sum_{\substack{\vec{1} \le \vec{j} \le \vec{a}, \\ \cap_i H^U_{i,j_i} \subseteq f^{-1}(W)}} \prod_{H_i \in S} \bar{\omega}^U_{i,j_i} \right),$$

$$g^*(\omega^g_{W,S}) = \sum_{\substack{\vec{1} \le \vec{k} \le \vec{b}, \\ \cap_i H^V_{i,k_i} \subseteq g^{-1}(W)}} \prod_{H_i \in S} \bar{\omega}^V_{i,k_i},$$

where $\pi_0(f^{-1}(H_i)) = \{H^U_{i,1}, \dots, H^U_{i,a_i}\}$ and $\pi_0(g^{-1}(H_i)) = \{H^V_{i,1}, \dots, H^V_{i,b_i}\}$. Since

$$h^*(\bar{\omega}^U_{i,j_i}) = \sum_{h(H^V_{i,k_i}) = H^U_{i,j_i}} \bar{\omega}^V_{i,k_i},$$

we obtain

$$\begin{aligned} g^*(\omega^f_{W,S}) &= \sum_{\substack{\vec{1} \le \vec{j} \le \vec{a}, \\ \cap_i H^U_{i,j_i} \subseteq f^{-1}(W)}} \prod_{H_i \in S} h^*(\bar{\omega}^U_{i,j_i}) = \\ &= \sum_{\substack{\vec{1} \le \vec{j} \le \vec{a}, \\ \cap_i H^U_{i,j_i} \subseteq f^{-1}(W)}} \prod_{H_i \in S} \sum_{h(H^V_{i,k_i}) = H^U_{i,j_i}} \bar{\omega}^V_{i,k_i} = \\ &= \sum_{\substack{\vec{1} \le \vec{j} \le \vec{a}, \\ \cap_i H^U_{i,j_i} \subseteq f^{-1}(W)}} \sum_{\substack{\vec{1} \le \vec{k} \le \vec{b}, \\ h(H^V_{i,k_i}) = H^U_{i,j_i}}} \prod_{H_i \in S} \bar{\omega}^V_{i,k_i} = \\ &= \sum_{\substack{\vec{1} \le \vec{k} \le \vec{b}, \\ \cap_{H_i \in S} H^V_{i,k_i} \subseteq g^{-1}(W)}} \prod_{H_i \in S} \bar{\omega}^V_{i,k_i} = \\ &= g^*(\omega^g_{W,S}), \end{aligned}$$

and then

$$\bar{\omega}^{f}_{W,S} = \bar{\omega}^{g}_{W,S}.$$

Finally, in the general case, we have two coverings $f : U \longrightarrow T$ and $g : V \longrightarrow T$. Consider $h : V' \longrightarrow U$ the pullback of g by f, $g' := f \circ h$ and note that g' separates S since f does.

$$\begin{array}{ccc} U & \xrightarrow{f} & T \\ & \nwarrow^{h} & \uparrow{\scriptstyle g'} \\ & & V' \end{array}$$

Now, applying the previous part of the proof, we can say that

$$g'^{*}(\bar{\omega}^{f}_{W,S}) = g'^{*}(\bar{\omega}^{g'}_{W,S}) \quad \Rightarrow \quad \bar{\omega}^{f}_{W,S} = \bar{\omega}^{g'}_{W,S}.$$

Analogously, considering the diagram

$$\begin{array}{ccc} V & \xrightarrow{g} & T \\ & \nwarrow^{h} & \uparrow{\scriptstyle g'} \\ & & V' \end{array}$$

we can say that

$$g'^{*}(\bar{\omega}^{g}_{W,S}) = g'^{*}(\bar{\omega}^{g'}_{W,S}) \quad \Rightarrow \quad \bar{\omega}^{g}_{W,S} = \bar{\omega}^{g'}_{W,S},$$

then we conclude. □

It immediately follows that we are allowed to give the following definitions:

Definition 3.3.32. Let S be an independent set of $\mathcal{A} = \{H_1, \ldots, H_n\}$ and W a connected component of $\cap_{H_i \in S} H_i$. Define

$$\bar{\omega}_{W,S} := \bar{\omega}^{f}_{W,S} \qquad\qquad \omega_{W,S} := \omega^{f}_{W,S}\,,$$

where $f : U \longrightarrow T$ is any covering that separates S.

Lemma 3.3.33. *Let S, S' be two sets of $\mathcal{A} = \{H_1, \ldots, H_n\}$ such that $S \sqcup S'$ is an independent set and W,W' be, respectively, connected component of $\cap_{H_i \in S} H_i$, $\cap_{H_i \in S'} H_i$. Then*

$$\bar{\omega}_{W,S}\bar{\omega}_{W',S'} = (-1)^{\ell(S,S')} \sum_{L \in \pi_0(W \cap W')} \bar{\omega}_{L,S \sqcup S'}.$$

Proof. First notice that the left-hand side is well defined since, if $S \sqcup S'$ is independent, then S and S' are too. Take a covering $f : U \longrightarrow T$ that separates $S \sqcup S'$ (remember that f exists thanks to Proposition 3.3.29). Obviously f separates also S and S', then we have

$$\begin{aligned}
f^* \left(\bar{\omega}_{W,S}\bar{\omega}_{W',S'}\right) &= f^*(\bar{\omega}_{W,S}) f^*(\bar{\omega}_{W',S'}) = \\
&= \left(\sum_{\substack{\vec{1} \le \vec{j} \le \vec{a}, \\ \bigcap_{H_i \in S} H^U_{i,j_i} \subseteq f^{-1}(W)}} \prod_{H_i \in S} \bar{\omega}^U_{i,j_i} \right) \left(\sum_{\substack{\vec{1} \le \vec{k} \le \vec{b}, \\ \bigcap_{H_i \in S'} H^U_{i,k_i} \subseteq f^{-1}(W')}} \prod_{H_i \in S'} \bar{\omega}^U_{i,k_i} \right) = \\
&= \sum_{\substack{\vec{1} \le \vec{j} \le \vec{a}, \quad \vec{1} \le \vec{k} \le \vec{b}, \\ \left(\bigcap_{H_i \in S} H^U_{i,j_i}\right) \cap \left(\bigcap_{H_i \in S'} H^U_{i,k_i}\right) \subseteq f^{-1}(W \cap W')}} \prod_{H_i \in S} \bar{\omega}^U_{i,j_i} \prod_{H_i \in S'} \bar{\omega}^U_{i,k_i} = \\
&= \sum_{\substack{\vec{1} \le \vec{p} \le (\vec{a}, \vec{b}), \\ \left(\bigcap_{H_i \in S \sqcup S'} H^U_{i,p_i}\right) \subseteq f^{-1}(W \cap W')}} (-1)^{\ell(S,S')} \prod_{H_i \in S \sqcup S'} \bar{\omega}^U_{i,p_i} = \\
&= (-1)^{\ell(S,S')} \sum_{L \in \pi_0(W \cap W')} \sum_{\substack{\vec{1} \le \vec{p} \le (\vec{a}, \vec{b}), \\ \left(\bigcap_{H_i \in S \sqcup S'} H^U_{i,p_i}\right) \subseteq f^{-1}(L)}} \prod_{H_i \in S \sqcup S'} \bar{\omega}^U_{i,p_i} = \\
&= (-1)^{\ell(S,S')} \sum_{L \in \pi_0(W \cap W')} f^*(\bar{\omega}_{L,S \cup S'}) = \\
&= f^* \left((-1)^{\ell(S,S')} \sum_{L \in \pi_0(W \cap W')} \bar{\omega}_{L,S \cup S'} \right)
\end{aligned}$$

where $p_i := (j_i, k_i)$.

□

Lemma 3.3.34. *If* $\cap_{H_i \in S} H_i$ *is connected and* $\cap_{H_i \in S} H_i = W$*, then*

$$\bar{\omega}_{W,S} = \bar{\omega}_S.$$

Proof. The identity map $id : T \longrightarrow T$ separates S, in fact:

$$id(\cap_{H_i \in S}(H_i^T(q_i))) = \cap_{H_i \in S} H_i = W.$$

Then

$$\bar{\omega}_{W,S} = \bar{\omega}_{W,S}^{id} = \sum_{1 \leq j \leq 1} \prod_{H_i \in S} \bar{\omega}_{i,j}^T = \prod_{H_i \in S} \bar{\omega}_i = \bar{\omega}_S.$$

□

Definition 3.3.35. Define

$$\bar{\eta}_{W,A,B} := (-1)^{\ell(A,B)} \bar{\omega}_{W,A} \psi_B.$$

Lemma 3.3.36. *If* $\cap_{H_i \in S} H_i$ *is connected and* $\cap_{H_i \in S} H_i = W$*, then*

$$\bar{\eta}_{W,A,B} = \bar{\eta}_{A,B}$$

where $\bar{\eta}_{A,B}$ *of Definition 3.3.12.*

Proof. Thanks to Lemma 3.3.34,

$$\bar{\eta}_{W,A,B} = (-1)^{\ell(A,B)} \bar{\omega}_{W,A} \psi_B = (-1)^{\ell(A,B)} \bar{\omega}_A \psi_B = \prod_{H_i \in A \cup B} \bar{\eta}_i^A = \bar{\eta}_{A,B}.$$

□

From Lemma 3.3.33, we have

Lemma 3.3.37. *Let* A, A', B, B' *be four sets of* $\mathcal{A} = \{H_1, \ldots, H_n\}$ *such that* $A \sqcup A' \sqcup B \sqcup B'$ *is an independent set and* W*,*W' *be, respectively, connected component of* $\cap_{H_i \in A} H_i$*,* $\cap_{H_i \in A'} H_i$*. Then*

$$\bar{\eta}_{W,A,B} \bar{\eta}_{W',A',B'} = (-1)^{\ell(A \cup B, A' \cup B')} \sum_{L \in \pi_0(W \cap W')} \bar{\eta}_{L, A \sqcup A', B \sqcup B'}.$$

Definition 3.3.38. Define $\mathcal{F}_* = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ the increasing filtration of $H^*(M(\mathcal{A}),\mathbb{Z})$ defined by

$$\mathcal{F}_{-1} = 0 \qquad\qquad \mathcal{F}_0 = H^*(T,\mathbb{Z})$$

and

$$\mathcal{F}_i = \bigoplus_{j\leq i} H^j(M(\mathcal{A}),\mathbb{Z}) \otimes H^*(T,\mathbb{Z}),$$

identifying $H^*(T,\mathbb{Z})$ with its image $j^*(H^*(T,\mathbb{Z}))$ in $H^*(M(\mathcal{A}),\mathbb{Z})$ where $j : M(\mathcal{A}) \longrightarrow T$ is the natural inclusion.
The associated graded module is defined by

$$gr_*(H^*(M(\mathcal{A}))) := \bigoplus_{i\geq 0} \frac{\mathcal{F}_i}{\mathcal{F}_{i-1}}.$$

It can be proved that there exist an isomorphism of graded modules such that

$$gr_k(H^*(M(\mathcal{A}))) \simeq \bigoplus_{\substack{W\in L(\mathcal{A})\\ rk(W)=k}} H^*(W) \otimes H^k(M(\mathcal{A}[W])),$$

where $\mathcal{A}[W]$ is the hyperplanes arrangement of Definition 3.3.2. Recall that $\mathcal{A}[W]$ is a $rk(W)$-arrangement.

Lemma 3.3.39. *For* $A, B \subseteq \{H_1,\dots,H_n\}$ *with* $A \sqcup B$ *independent and for any* $W \in \pi_0(\cap_{H_i\in A} H_i)$*, we have that the image of* $\bar{\eta}_{W,A,B}$ *in* $gr_{|A|}(H^{|B|}(M(\mathcal{A})))$ *equals*

$$(-1)^{\ell(B,A)} 2^{|A|} \psi_B \otimes \varpi_A \in H^{|B|}(W) \otimes H^{|A|}(M(\mathcal{A}[W])),$$

where ϖ_A *is the canonical generator in the top degree of the Orlik Solomon algebra of the hyperplane arrangement* $\mathcal{A}[W]$ *associated with the hyperplanes indexed by* A*.*

Proof. See [CDD+19, Lemma 5.14]. □

Example 3.3.40. Consider $\mathcal{A}$ the arrangement of Example 3.2.11, with $H_1 := \{(t,s) \in (\mathbb{C}^*)^2 | t = 1\}$, $H_2 := \{(t,s) \in (\mathbb{C}^*)^2 | s = 1\}$ and $H_3 := \{(t,s) \in (\mathbb{C}^*)^2 | t^3 s = 1\}$.

We have already computer in Example 3.3.3 that $\mathcal{A}[p] = \{\bar{H}_1, \bar{H}_2, \bar{H}_3\}$ with $\bar{H}_1 := \{(\bar{t}, \bar{s}) \in \mathbb{C}^2 | \bar{t} = 0\}$, $\bar{H}_2 := \{(\bar{t}, \bar{s}) \in \mathbb{C}^2 | \bar{s} = 0\}$ and
$\bar{H}_3 := \{(\bar{t}, \bar{s}) \in \mathbb{C}^2 | 3\bar{t} + \bar{s} = 0\}$.

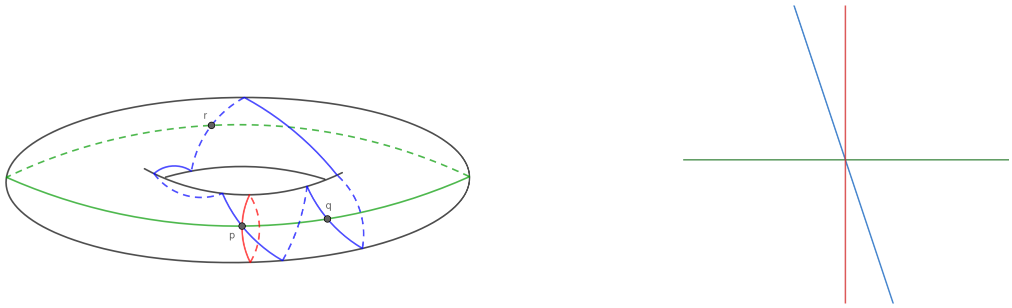


Figure 3.11: $\mathcal{A}$ $\mathcal{A}[p]$

For $A = \{H_2, H_3\}$ and $W = p$, the image of $\bar{\omega}_{p,\{H_2,H_3\}}$ in $gr_2(H^*(M(\mathcal{A})))$ equals

$$4 \otimes e_{\{\bar{H}_2,\bar{H}_3\}} \qquad \text{with} \qquad e_{\bar{H}_2} = \frac{1}{2\pi i}\frac{d(\bar{s})}{\bar{s}} \qquad e_{\bar{H}_3} = \frac{1}{2\pi i}\frac{d(3\bar{s}+\bar{t})}{3\bar{s}+\bar{t}}.$$

Unimodular coverings

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a primitive, central and essential arrangement in $T = (\mathbb{C}^*)^d$ and denote E the ground set of the arithmetic matroid $\mathcal{M}(\mathcal{A})$ associated with $\mathcal{A}$. Suppose the arrangement contains exactly one circuit $C = \{H_1, \dots, H_k\}$, hence $rk(E) = n - 1$. Denote $F = \{H_{k+1}, \dots, H_n\}$. Thanks to Lemma 3.2.12, $\sum_{i=1}^{k} c_i m(C \backslash \{H_i\}) \chi_i = 0$, with $c_1 \in \{-1, +1\}$.

Definition 3.3.41. For every $i \in \{1, \dots, n\}$, define

$$a_i := \begin{cases} \frac{m(E)^2}{m(C)} \prod_{\substack{H_j \in C \\ H_j \neq H_i}} m(C \backslash \{H_j\}) & \text{for } i = 1, \dots, k \\ \frac{m(E)^2}{m(C \cup F_{\leq i})} & \text{for } i = k+1, \dots, n. \end{cases}$$

where $F_{\leq i} := F \cap \{H_1, \dots, H_i\}$.
Denote $\Lambda(E)$ the set in $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ generated by $\{\frac{\chi_1}{a_1}, \dots, \frac{\chi_n}{a_n}\}$.

Remark 3.3.42.

- $\Lambda = \Lambda^E$. In particular $m(E) = [\Lambda : \Lambda_E]$.
- $[\Lambda(E) : \Lambda_{E \setminus H_i}] = \prod_{j \neq i} a_j \quad \forall i.$

Example 3.3.43. Let $\mathcal{A}$ be the toric arrangement in $T := (\mathbb{C}^*)^2$ of Example 3.2.11. In this case, $C = \{H_1, \dots, H_n\}$, then

$$\begin{aligned}\Lambda^C &= \left\{ q_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + q_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + q_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} ; q_1, q_2, q_3 \in \mathbb{Q} \right\} \cap \mathbb{Z}^2 = \\ &= \left\{ q_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + q_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; q_1, q_2 \in \mathbb{Q} \right\} \cap \mathbb{Z}^2 = \mathbb{Z}^2.\end{aligned}$$

We have already computed computed in Example 3.2.11

$$m(\{H_1, H_2\}) = m(\{H_1, H_3\}) = 1 \qquad \text{and} \qquad m(\{H_2, H_3\}) = 3,$$

then it follows that

$$a_1 = 1 \qquad a_2 = 3 \qquad a_3 = 3,$$

hence

$$\begin{aligned}\Lambda(C) &= \left\{ m_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_2 \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} + m_3 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} ; m_1, m_2, m_3 \in \mathbb{Z} \right\} = \\ &= \left\{ m_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_2 \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} ; m_1, m_2 \in \mathbb{Z} \right\} \simeq \mathbb{Z} \oplus \frac{1}{3}\mathbb{Z} \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^2\end{aligned}$$

Now consider $C \setminus \{H_1\}$, then

$$\Lambda_{C \setminus \{H_1\}} = \left\{ m_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + m_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} ; m_1, m_2 \in \mathbb{Z} \right\} \simeq 3\mathbb{Z} \oplus \mathbb{Z},$$

thus

$$[\Lambda(C) : \Lambda_{C \setminus \{H_1\}}] = 9 = a_2 a_3.$$

Lemma 3.3.44.
$$\sum_{i=1}^{k} c_i \frac{\chi_i}{a_i} = 0.$$

Proof. Note that $a_i m(C\backslash\{H_i\}) = \frac{m(E)^2}{m(C)} \prod_{j=1}^{k} m(C\backslash\{H_j\})$, then we have

$$0 = \sum_{i=1}^{k} c_i m(C\backslash\{H_i\})\chi_i = \frac{m(E)^2}{m(C)} \prod_{j=1}^{k} m(C\backslash\{H_j\}) \sum_{i=1}^{k} c_i \frac{\chi_i}{a_i},$$

and since $\frac{m(E)^2}{m(C)} \prod_{j=1}^{k} m(C\backslash\{H_j\}) \neq 0$, we conclude. □

Proposition 3.3.45. $\Lambda \subseteq \Lambda(E)$ *with*

$$[\Lambda(E) : \Lambda] = \frac{\prod_{j=1}^{n} a_j}{a_i m(\mathcal{A}\backslash H_i)} =: g,$$

where H_i is any element of C.
In particular there exists a special covering of T of degree g.

Definition 3.3.46. Let $\pi_U : U \longrightarrow T$ the covering of Proposition 3.3.45. Denote $\mathcal{A}_U$ the arrangement in the torus U induced by the characters $\frac{\chi_i}{a_i}$ in $\Lambda(E)$, i.e. $\mathcal{A}_U = \left\{ \left(e^{\frac{\chi_1}{a_1}}\right)^{-1} \left(e^{\frac{2k\pi i}{a_1}}\right), \ldots, \left(e^{\frac{\chi_n}{a_n}}\right)^{-1} \left(e^{\frac{2k\pi i}{a_n}}\right); k \in \mathbb{Z} \right\}$.

Lemma 3.3.47. $\mathcal{A}_U$ *is primitive and unimodular.*

Proof. For every $H_j \in C$, $\{\frac{\chi_i}{a_i}\}_{i \neq j}$ is a basis for $\Lambda(E)$. It follows that $\mathcal{A}_U$ is primitive.
In order to prove the unimodularity of $\mathcal{A}_U$, we can use Proposition 15.7 of [DCP11] which state that an arrangement is unimodular if and only if, for every $A = \{H_1, \ldots, H_m\}$ independent and H_0 dependent on A, χ_0 can be written as linear combination of $\{\chi_1, \ldots, \chi_m\}$ with coefficients in $\{-1, 0, +1\}$.
Now, since C is the unique circuit of $\{H_1, \ldots, H_n\}$, then, for every $H_j \in C$, $\{\frac{\chi_1}{a_1}, \ldots, \widehat{\frac{\chi_j}{a_j}}, \ldots, \frac{\chi_n}{a_n}\}$ is an independent set. Moreover, by Lemma 3.3.44, $\sum_{i=1}^{k} c_i \frac{\chi_i}{a_i} = 0$, then, every $\frac{\chi_j}{a_j}$ can be written as linear combination, with integer coefficients, of the characters $\{\frac{\chi_1}{a_1}, \ldots, \widehat{\frac{\chi_j}{a_j}}, \ldots, \frac{\chi_n}{a_n}\}$. Thus, if
$A \subseteq \{H_1, \ldots, H_n\}$ is an independent set, there exists $H_j \in C\backslash A$, then $\{\frac{\chi_i}{a_i}\}_{H_i \in A}$ can be completed to a basis $\{\frac{\chi_1}{a_1}, \ldots, \widehat{\frac{\chi_j}{a_j}}, \ldots, \frac{\chi_n}{a_n}\}$ of $\Lambda(E)$. □

Remark 3.3.48. For every i, $\chi_i = a_i \frac{\chi_i}{a_i}$, then it can be written as linear combination of the elements of the basis $\{\frac{\chi_j}{a_j}\}_{j\neq i}$. Thanks to Remark 3.1.2, we have that $\left|\pi_0\left(\pi_U^{-1}(H_i)\right)\right| = a_i$.

Example 3.3.49. Let $\mathcal{A}$ be the toric arrangement in $T := (\mathbb{C}^*)^2$ of Example 3.2.11. In Example 3.3.43 we have proved that $\Lambda(E) = \Lambda(C)$ is generated by $\{e_1, \frac{e_2}{3}, e_1 + \frac{e_2}{3}\}$ where $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$.
$\mathcal{A} = \{H_1, H_2, H_3\}$ with

$$\begin{aligned}
H_1 &= (e^{\chi_1})^{-1}(1) = \{(t,s); t = 1\}\\
H_2 &= (e^{\chi_2})^{-1}(1) = \{(t,s); s = 1\}\\
H_3 &= (e^{\chi_3})^{-1}(1) = \left\{(t,s); t^3 s = 1\right\}
\end{aligned}$$

that are primitive in

$$\Lambda =< \left\{e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix}\right\} > .$$

$\mathcal{A}_U = \{H^U_{1,1}, H^U_{2,1}, H^U_{2,2}, H^U_{2,3}, H^U_{3,1}, H^U_{3,2}, H^U_{3,3}\}$ with

$$\begin{aligned}
H^U_{1,1} &= \left(e^{\frac{\chi_1}{a_1}}\right)^{-1}(1) = \{(u,v) \in U; u = 1\}\\
H^U_{2,1} &= \left(e^{\frac{\chi_2}{a_2}}\right)^{-1}\left(e^{\frac{2}{3}\pi i}\right) = \{(u,v) \in U; v = e^{\frac{2}{3}\pi i}\}\\
H^U_{2,2} &= \left(e^{\frac{\chi_2}{a_2}}\right)^{-1}\left(e^{\frac{4}{3}\pi i}\right) = \{(u,v) \in U; v = e^{\frac{4}{3}\pi i}\}\\
H^U_{2,3} &= \left(e^{\frac{\chi_2}{a_2}}\right)^{-1}(1) = \{(u,v) \in U; v = 1\}\\
H^U_{3,1} &= \left(e^{\frac{\chi_3}{a_3}}\right)^{-1}\left(e^{\frac{2}{3}\pi i}\right) = \{(u,v) \in U; uv = e^{\frac{2}{3}\pi i}\}\\
H^U_{3,2} &= \left(e^{\frac{\chi_3}{a_3}}\right)^{-1}\left(e^{\frac{4}{3}\pi i}\right) = \{(u,v) \in U; uv = e^{\frac{4}{3}\pi i}\}\\
H^U_{3,3} &= \left(e^{\frac{\chi_3}{a_3}}\right)^{-1}(1) = \{(u,v) \in U; uv = 1\}
\end{aligned}$$

that are primitive in

$$\Lambda(C) =< \left\{e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2' = \begin{pmatrix}0\\\frac{1}{3}\end{pmatrix}\right\} > .$$

Then we have that $\mathcal{A}$ is represented by

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix},$$

while $\mathcal{A}_U$ is represented by

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 3 \end{pmatrix}.$$

Note that the covering π_U is exactly the covering f of Example 3.3.19.

Lemma 3.3.50. *Let $S \subsetneq \{H_1, \ldots, H_n\}$, W a connected component of $\cap_{H_i \in S} H_i$ and $p \in W$.*
For every layer L of $\mathcal{A}_U$ with $\pi_U(L) = W$, the number of preimages of p contained in L is

$$\left|L \cap \pi_U^{-1}(p)\right| = \frac{m(S)}{m(E\backslash H_j)} \prod_{\substack{H_i \in E\backslash S), \\ i \neq j}} a_i,$$

where H_j is any element of $C\backslash S$.

Proof. We know that $\left|\pi_U^{-1}(p)\right| = \frac{\prod_{j=1}^n a_j}{a_i m(E\backslash H_i)}$. On the other hand

$$|\pi_0(\pi_U^{-1}(W))| = \frac{\left|\pi_0(\pi_U^{-1}(\cap_{H_i \in S} H_i))\right|}{|\pi_0\left(\cap_{H_i \in S} H_i\right))|} = \frac{\prod_{H_i \in S} a_i}{m(S)}.$$

Then we obtain, for any $H_j \in C\backslash S$,

$$\begin{aligned}\left|L \cap \pi_U^{-1}(p)\right| &= \frac{\prod_{i=1}^n a_i}{a_j m(E\backslash H_j)} \frac{m(S)}{\prod_{H_i \in S} a_i} = \\ &= \frac{m(S)}{m(E\backslash H_j)} \prod_{\substack{H_i \in (E\backslash S), \\ i \neq j}} a_i.\end{aligned}$$

□

Example 3.3.51. Let $\mathcal{A}$ be the toric arrangement in $T := (\mathbb{C}^*)^2$ of Example 3.2.11.

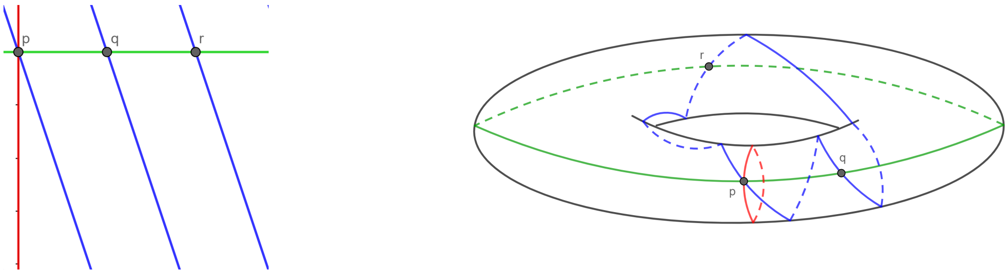


Figure 3.12: $\tilde{\mathcal{A}} \xrightarrow{/\mathbb{Z}^2} \mathcal{A}$

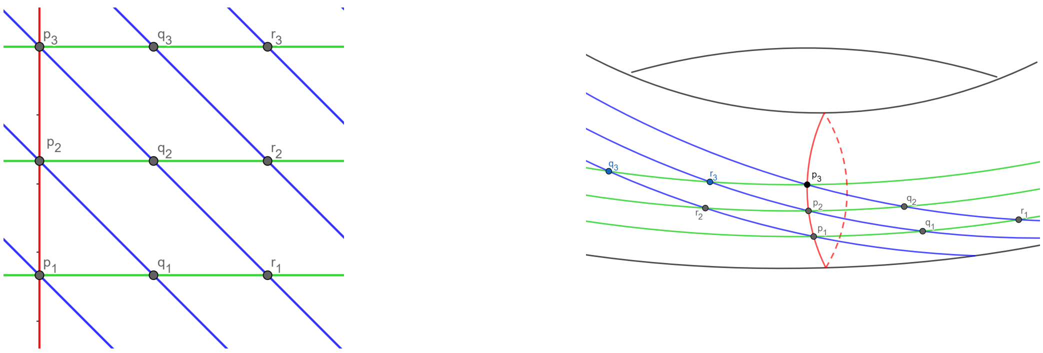


Figure 3.13: $\tilde{\mathcal{A}}_{(\mathbb{C}^*)^2} \xrightarrow{/\mathbb{Z}^2} \mathcal{A}_{(\mathbb{C}^*)^2}$

Take $S = \{H_2\}$, $W = H_2$, $q = (e^{\frac{2}{3}\pi i}, 1) \in H_2$ and $L = H_{2,1}^U$. We have $\pi_U^{-1}(q) = \{q_1 := (e^{\frac{2}{3}\pi i}, e^{\frac{2}{3}\pi i}), q_2 := (e^{\frac{2}{3}\pi i}, e^{\frac{4}{3}\pi i}), q_3 := (e^{\frac{2}{3}\pi i}, 1)\}$, then

$$|H_{2,1}^U \cap \pi_U^{-1}(q)| = |\{q_1\}| = 1 = \frac{1}{3} \cdot 3 = \frac{m(H_2)}{m(H_2, H_3)} a_3.$$

If we take instead $S = \{H_1\}$, $W = H_1$, $p = (1, 1) \in H_1$ and $L = H_1^U$. We have $\pi_U^{-1}(p) = \{p_1 := (1, e^{\frac{2}{3}\pi i}), p_2 := (1, e^{\frac{4}{3}\pi i}), p_3 := (1, 1)\}$, then

$$|H_1^U \cap \pi_U^{-1}(p)| = |\{p_1, p_2, p_3\}| = 3 = \frac{1}{1} \cdot 3 = \frac{m(H_1)}{m(H_1, H_3)} a_3.$$

Definition 3.3.52. For any $A, B \subseteq E$ with $A \sqcup B$ independent set and for every $q \in \pi_U^{-1}(\cap_{H_i \in A} H_i)$, we define

$$\bar{\eta}^U_{A,B}(q) := (-1)^{\ell(A,B)} \bar{\omega}^U_A(q) \psi^U_B.$$

Lemma 3.3.53. *Let $A, B \subseteq \{H_1, \dots, H_n\}$ with $A \sqcup B$ is a maximal independent subset of $\{H_1, \dots, H_n\}$. For every connected component W of $\cap_{H_i \in A} H_i$, we have*

$$\pi_U^*(\bar{\eta}_{W,A,B}) = \frac{m(A \sqcup B)}{m(A)} \sum_{q \in \pi_U^{-1}(p_0)} \bar{\eta}^U_{A,B}(q),$$

with p_0 any point in W.

Proof. Recall that, from Definition 3.3.35, $\bar{\eta}_{W,A,B} := (-1)^{\ell(A,B)} \bar{\omega}_{W,A} \psi_B$. Thanks to Definition 3.3.20, we have that

$$\pi_U^*(\psi_B) = \prod_{H_i \in B} \pi_U^*(\psi_i) = \prod_{H_i \in B} a_i \psi_i^U = \left[\prod_{H_i \in B} a_i \right] \psi^U_B.$$

It follows that, with Remark 3.3.24, we obtain:

$$\pi_U^*(\bar{\omega}_{W,A}) = \pi_U^*(\bar{\omega}^{\pi_U}_{W,A}) = \frac{1}{|L \cap \pi_U^{-1}(p)|} \sum_{q \in \pi_U^{-1}(p)} \bar{\omega}^U_A(q).$$

Since $A \sqcup B$ is a maximal independent set, $|A| + |B| = rk(A \sqcup B) = rk(E) = |E| - 1$, then there exists a unique H_j such that $E = A \sqcup B \sqcup \{H_j\}$. Thus, by Lemma 3.3.50,

$$\begin{aligned}
\pi_U^*(\bar{\eta}_{W,A,B}) &= (-1)^{\ell(A,B)} \pi_U^*(\bar{\omega}_{W,A}) \pi_U^*(\psi_B) = \\
&= (-1)^{\ell(A,B)} \frac{\prod_{H_i \in B} a_i}{|L \cap \pi_U^{-1}(p)|} \sum_{q \in \pi_U^{-1}(p)} \bar{\omega}^U_A(q) \psi^U_B = \\
&= (-1)^{\ell(A,B)} \frac{m(E \backslash H_j)}{m(A)} \frac{\prod_{H_i \in B} a_i}{\prod_{H_i \in (E \backslash (A \cup H_j))} a_i} \sum_{q \in \pi_U^{-1}(p)} \bar{\omega}^U_A(q) \psi^U_B = \\
&= (-1)^{\ell(A,B)} \frac{m(A \cup B)}{m(A)} \sum_{q \in \pi_U^{-1}(p)} \bar{\omega}^U_A(q) \psi^U_B = \\
&= \frac{m(A \sqcup B)}{m(A)} \sum_{q \in \pi_U^{-1}(p_0)} \bar{\eta}^U_{A,B}(q).
\end{aligned}$$

□

Proposition 3.3.54. *For every L connected component of $\cap_{H_i \in X} H_i$,*

$$\sum_{H_j \in C} \sum_{\substack{X = A \sqcup B \sqcup \{H_j\} \\ F \subseteq A \\ |B| \text{ even} \\ W \supseteq L}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A \cup B)} \bar{\eta}_{W,A,B} = 0,$$

with $A_{\leq j} = A \cap \{H_1, \ldots, H_j\}$, $c_B := \prod_{H_i \in B} c_i$.

Proof. Since $\mathcal{A}_U$ is an unimodular arrangement, thanks to Proposition 3.3.13, for any $q \in \pi_U^{-1}(p)$ with $p \in \cap_{H_i \in C} H_i$, we have

$$\sum_{H_j \in C} \sum_{\substack{A', B \subset C \\ C = A' \cup B \cup H_j \\ |B| \text{ even}}} (-1)^{|A'_{\leq j}|} c_B \bar{\eta}^U_{A',B}(q) = 0,$$

then, since $X = C \sqcup F$,

$$0 = \sum_{H_j \in C} \sum_{\substack{A', B \subset C \\ C = A' \cup B \cup H_j \\ |B| \text{ even}}} (-1)^{|A'_{\leq j}|} c_B \bar{\eta}^U_{A',B}(q) \bar{\eta}^U_{F,\emptyset}(q) = \sum_{H_j \in C} \sum_{\substack{X = A \cup B \cup H_j \\ F \subseteq A \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \bar{\eta}^U_{A,B}(q).$$

Thus, thanks to Lemma 3.3.53,

$$\begin{aligned}
0 &= \sum_{q \in \pi_U^{-1}(p)} \sum_{H_j \in C} \sum_{\substack{X = A \cup B \cup H_j \\ F \subseteq A \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \bar{\eta}^U_{A,B}(q) = \\
&= \sum_{H_j \in C} \sum_{\substack{X = A \cup B \cup H_j \\ F \subseteq A \\ |B| \text{ even}}} (-1)^{|A_{\leq j}|} c_B \sum_{q \in \pi_U^{-1}(p)} \bar{\eta}^U_{A,B}(q) = \\
&= \sum_{H_j \in C} \sum_{\substack{X = A \cup B \cup H_j \\ F \subseteq A \\ |B| \text{ even} \\ W \supseteq L}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A \cup B)} \pi_U^*(\bar{\eta}_{W,A,B}) = \\
&= \pi_U^* \left(\sum_{H_j \in C} \sum_{\substack{X = A \cup B \cup H_j \\ F \subseteq A \\ |B| \text{ even} \\ W \supseteq L}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A \cup B)} \bar{\eta}_{W,A,B} \right).
\end{aligned}$$

□

Now, dropping the assumption that $\mathcal{A}$ has an unique circuit, we can finally give an Orlik-Solomon-type presentation for the cohomology algebra of a general toric arrangement.

Theorem 3.3.55. *Let $\mathcal{A}$ be an essential toric arrangement. The rational cohomology algebra $H^*(M(\mathcal{A}), \mathbb{Q})$ is isomorphic to the algebra $\mathcal{E}$ with:*

- *$\mathcal{E}$ generated by $\{e_{W,A;B} | W$ ranges over all layers of $\mathcal{A}$, A is a set generating W, $A \cap B = \emptyset, A \sqcup B$ is an independent set$\}$. The degree of $e_{W,A;B}$ is $|A \sqcup B|$.*
- *The following types of relations*
  1. *For any two generators $e_{W,A;B}$ and $e_{W',A';B'}$, if $A \sqcup B \sqcup A' \sqcup B'$ dependent,*
  $$e_{W,A;B}e_{W',A';B'} = 0, \tag{3.2}$$
  otherwise
  $$e_{W,A;B}e_{W',A';B'} = (-1)^{\ell(A\cup B, A'\cup B')} \sum_{L\in\pi_0(W\cap W')} e_{L,A\cup A';B\cup B'}. \tag{3.3}$$
  2. *If $\sum_{i=1}^n n_i\chi_i = 0$ where $n_i \in \mathbb{Z}$, then*
  $$\sum_{i=1}^n n_i e_{T,\emptyset;\{i\}} = 0. \tag{3.4}$$
  3. *For every $X \subseteq \{H_1, \ldots, H_n\}$ with $rk(X) = |X| - 1$ and every $L \in \pi_0(\cap_{H_i\in X} H_i)$, then*
  $$\sum_{H_j\in C} \sum_{\substack{A,B\subset X \\ X=A\sqcup B\sqcup\{H_j\} \\ F\subseteq A \\ |B| \text{ even} \\ W\supseteq L}} (-1)^{|A_{\leq j}|} c_B \frac{m(A)}{m(A\cup B)} e_{W.A;B} = 0 \tag{3.5}$$
  where $X = C \sqcup F$ with C the unique circuit in X with associated linear dependency $\sum_{H_i\in C} n_i\chi_i$ with $n_i \in \mathbb{Z}$ and $c_B := \prod_{H_i\in B} sgn\ n_i$.

Proof. First we prove that the map Φ given by $e_{W,A;B} \longmapsto [\bar{\eta}_{W,A;B}]$ is well defined (see Definition 3.3.35).

- If $A \sqcup B$ is independent, then A is independent too and, if $W \in L(\mathcal{A})$ and A is generating W, then $W \in \pi_0(\cap_{H_i \in A} H_i)$, thus $\bar{\eta}_{W,A;B}$ is well defined and has degree $|A| + |B| = |A \sqcup B|$.

- The elements $[\bar{\eta}_{W,A,B}]$ satisfy the same relations. In fact,

  1. It immediately follows from Lemma 3.3.37,
  2. Already proved in Theorem 3.3.14.
  3. It immediately follows from Proposition 3.3.54. Note that, since $F \subseteq A$, then $B \subseteq C$ and c_B is well defined.

It remains to prove the bijectivity of Φ.

It is surjective since $gr\,(Im(\Phi)) = gr(H^*(M(\mathcal{A}), \mathbb{Q}))$. In fact, from Lemma 3.3.39, we have that, for every $e_{W,A;B}$,

$$gr(\Phi(e_{W,A;B})) = (-1)^{\ell(A,B)} 2^{|A|} \psi_B \otimes \varpi_A \in gr_{|A|}(H^*(M(A))).$$

On the other hand, for every generators $\psi_B \otimes \varpi_A$ of $gr(H^*(M(\mathcal{A})))$ there exists a preimage in $\mathcal{E}$ defined by $(-1)^{\ell(A,B)} 2^{-|A|} e_{W,A;B}$.

Now, let $A \subseteq \{H_1, \ldots, H_n\}$ an independent set and take $D(A) \subseteq \{H_1, \ldots, H_n\}$ such that $A \cup D(A)$ is a maximal independent set.
In order to prove the injectivity of Φ, we prove that $\mathcal{E}$ is generated by $e_{W,A;B}$ with A nbc set in the arrangement $\mathcal{A}[W]$ and $B \subseteq D(A)$. Thanks to equation 3.3, $e_{W,A;B} = (-1)^{\ell(A,B)} e_{W,A;\emptyset} e_{T,\emptyset,B}$, than it suffices to prove that each factor of the rhs can be written as linear combination of $e_{W,S;R}$ with S nbc set such that $W \in \pi_0(\cap_{H_i \in S} H_i)$ and $R \subseteq D(S)$. First we prove that every $e_{W,A;\emptyset}$ satisfies the thesis.

Define on $\mathcal{P}(E)$ a total order given by

$$A < A' \iff \begin{array}{c} |A| < |A'| \\ \text{or} \\ |A| = |A'| \text{ with } A \text{ greater than } A' \text{ wrt the lexicographic order.} \end{array}$$

We prove the thesis by induction on $\mathcal{P}(E)$. For the base case we take $A = max(E)$ that is a nbc set, then obviously the thesis is satisfied.
Let $A = \{H_1, \ldots, H_m\}$ be an independent non-nbc set and suppose that for every $A' < A$ the thesis is true. Since A is a non-nbc set, there exists $A_1 \subseteq A$ broken circuit i.e. there exists $K \in E$ such that (K, A_1) is a circuit. For $X = (K, A) = \{K, H_1, \ldots, H_m\}$, $rk(X) = rk(A) = |A| = |X| - 1$, then by equation 3.5, we have that $e_{W,A;\emptyset}$ can be written as linear combination of the ones $e_{W,S;R}$ with $R \neq \emptyset$ and $|S| < |A|$ and $e_{W,S;\emptyset}$ with $S = X \backslash H_i$. It follows that $e_{W,A,\emptyset}$ can be written as linear combination of $e_{W,S,R}$ with $S < A$, then by inductive hypothesis we can conclude.
From definition and equation 3.4, $e_{T,\emptyset,B} = \prod_{b\in B} e_{T,\emptyset,b}$ where for every $b \in B$, $e_{T,\emptyset,b} = \sum_{a\in A} \lambda_{b,a} e_{T,\emptyset,a} + \sum_{d\in D(A)} \mu_{b,d} e_{T,\emptyset,d}$, and, for equation 3.2, we have that $e_{T,\emptyset;B}$ can be written as linear combination of $e_{T,\emptyset;B'}$ with $B' \subseteq D(A)$.

Since $\mathcal{A}$ is essential, for every $W \in \pi_0(\cap_{H_i\in A} H_i)$, $rk(W) = |A|$, then

$$\begin{aligned} dim(\mathcal{E}) &\leq \sum_{W\in L(\mathcal{A})} 2^{d-rk(W)} |nbc_{rk(W)}(\mathcal{A}[W])| = \\ &= \sum_{W\in L(\mathcal{A})} 2^{dim(W)} |nbc_{rk(W)}(\mathcal{A}[W])| = \\ &= Poin(M(\mathcal{A}), 1) = \\ &= dim(H^*(M(\mathcal{A}))), \end{aligned}$$

thus Φ is also injective. $\square$

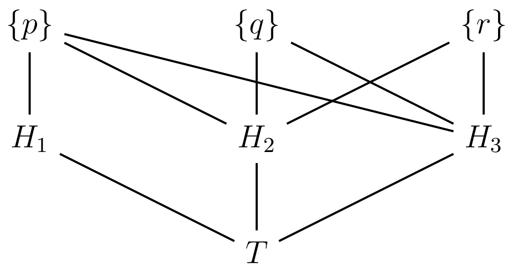
Remark 3.3.56. • This theorem generalized the one for the unimodular arrangement, in fact, by Lemma 3.3.36, if $\cap_{H_i \in A} H_i$ is connected, we have that $\bar{\eta}_{W,A,B} = \bar{\eta}_{A,B}$ with $W = \cap_{H_i \in A} H_i$. Thus, in the unimodular case, the relations of this theorem are exactly the same of Theorem 3.3.14.

- There are examples of non-unimodular arrangements whose cohomology algebra is not generated in degree one. Anyway, as we have already seen in Remark 3.3.15, if $\mathcal{A}$ is unimodular, the cohomology algebra is generated in degree one.

- Φ determines an isomorphism that cannot be restricted to $H^*(M(\mathcal{A}), \mathbb{Z})$ due to the $2^{|A|}$ factor in Lemma 3.3.39. However it is possible to give a presentation also for $H^*(M(\mathcal{A}), \mathbb{Z})$: see Theorem 7.4 [CDD+19].

- Since the equation 3.2, 3.3 and 3.5 can be determined by the poset of layers $L(\mathcal{A})$, and it can be proved that also 3.4 can be recovered by the poset [Pag19a], this presentation of the cohomology algebra of the complement of a toric arrangement is completely determined by $L(\mathcal{A})$. We have also seen, in Remark 3.2.10, that the poset of layers $L(\mathcal{A})$ determines uniquely the arithmetic matroid of the arrangement, but unlike the case of hyperplane arrangements (Corollary 1.4.4), $L(\mathcal{A})$ does not depends only on arithmetic matroids. In fact, in [Pag19b], Pagaria constructed two toric arrangements with isomorphic matroids but non-isomorphic cohomology rings.

Example 3.3.57. Let $\mathcal{A}$ be the toric arrangement in $T := (\mathbb{C}^*)^2$ of Example 3.2.11. In order to compute the generators, we take the Hasse diagram of the poset of layers $L(\mathcal{A})$, that is:



- For $W = T$,

$$\bar{\eta}_{T,\emptyset;H_i} = \psi_i \quad \forall i, \qquad \bar{\eta}_{T,\emptyset;\{H_i,H_j\}} = \psi_i\psi_j \quad \forall i \neq j.$$

For $W = H_i$,

$$\bar{\eta}_{H_i,H_i;\emptyset} = \bar{\eta}_{H_i;\emptyset} = \bar{\omega}_i \qquad \bar{\eta}_{H_i,H_i;H_j} = \bar{\eta}_{H_i;H_j} = \bar{\omega}_i\psi_j \quad \forall i \neq j.$$

For $W \in \{p, q, r\}$,

$$\bar{\eta}_{p,\{H_1,H_2\};\emptyset} = \bar{\omega}_{p,\{H_1,H_2\}} \qquad \bar{\eta}_{p,\{H_1,H_3\};\emptyset} = \bar{\omega}_{p,\{H_1,H_3\}} \qquad \bar{\eta}_{p,\{H_2,H_3\};\emptyset} = \bar{\omega}_{p,\{H_2,H_3\}}$$

$$\bar{\eta}_{q,\{H_2,H_3\};\emptyset} = \bar{\omega}_{q,\{H_2,H_3\}} \qquad \bar{\eta}_{r,\{H_2,H_3\};\emptyset} = \bar{\omega}_{r,\{H_2,H_3\}}.$$

Note that thanks to relation 3.3, we have that

$$\bar{\omega}_1\bar{\omega}_2 = \bar{\omega}_{p,\{H_1,H_2\}} \qquad \bar{\omega}_1\bar{\omega}_3 = \bar{\omega}_{p,\{H_1,H_3\}}$$

Then we conclude that the generators of $H^*(M(\mathcal{A}), \mathbb{Q})$ can be represented by

$$\begin{array}{ccc} \bar{\omega}_1 & \bar{\omega}_2 & \bar{\omega}_3 \\ \psi_1 & \psi_2 & \psi_3 \\ \bar{\omega}_{p,\{H_2,H_3\}} & \bar{\omega}_{q,\{H_2,H_3\}} & \bar{\omega}_{r,\{H_2,H_3\}} \end{array}$$

with, as computed in Example 3.3.19,

$$\bar{\omega}_{p,\{H_2,H_3\}} = \frac{-1}{4\pi}\frac{t^3s^2+t^3s+4t^2s+4ts+s+1}{ts(s-1)(t^3s-1)}dtds,$$

similarly,

$$\bar{\omega}_{q,\{H_2,H_3\}} = \frac{-1}{4\pi}\frac{t^3s^2+t^3s+4\xi t^2s+4\xi ts+s+1}{ts(s-1)(t^3s-1)}dtds,$$

$$\bar{\omega}_{r,\{H_2,H_3\}} = \frac{-1}{4\pi}\frac{t^3s^2+t^3s+4\xi^2t^2s+4\xi^2ts+s+1}{ts(s-1)(t^3s-1)}dtds,$$

where $\xi = e^{\frac{2}{3}\pi i}$.

- The relations are

  1.
  $$\bar{\omega}_i\psi_i = 0 \quad \forall i;$$
  $$\bar{\omega}_i\bar{\omega}_{s,\{H_2,H_3\}} = 0 \qquad \forall i = 2,3 \quad \forall s \in \{p,q,r\};$$
  $$\bar{\omega}_1\bar{\omega}_2 = \bar{\omega}_{p,\{H_1,H_2\}} \qquad \bar{\omega}_1\bar{\omega}_3 = \bar{\omega}_{p,\{H_1,H_3\}};$$
  $$\bar{\omega}_2\bar{\omega}_3 = \bar{\omega}_{p,\{H_2,H_3\}} + \bar{\omega}_{q,\{H_2,H_3\}} + \bar{\omega}_{r,\{H_2,H_3\}}.$$

  2. Since $3\chi_1 + \chi_2 - \chi_3 = 0$,
  $$3\psi_1 + \psi_2 - \psi_3 = 0.$$

  3. $C = \{H_1, H_2, H_3\}$ is the unique circuit with $\{p\} = H_1 \cap H_2 \cap H_3$, then
  $$0 = -\frac{1}{3}\bar{\eta}_{T,\emptyset,\{H_2,H_3\}} + \bar{\eta}_{p,\{H_2,H_3\}\emptyset} + \bar{\eta}_{T,\emptyset,\{H_1,H_2\}} + \bar{\eta}_{p,\{H_1,H_2\},\emptyset} +$$
  $$- \bar{\eta}_{T,\emptyset,\{H_1,H_3\}} - \bar{\eta}_{p,\{H_1,H_3\},\emptyset} =$$
  $$= -\frac{1}{3}\psi_2\psi_3 + \bar{\omega}_{p,\{H_2,H_3\}} + \psi_1\psi_2 + \bar{\omega}_{p,\{H_1,H_2\}} - \psi_1\psi_3 - \bar{\omega}_{p,\{H_1,H_3\}} =$$
  $$= -\frac{1}{3}\psi_2\psi_3 + \bar{\omega}_{p,\{H_2,H_3\}} + \psi_1\psi_2 + \bar{\omega}_1\bar{\omega}_2 - \psi_1\psi_3 - \bar{\omega}_1\bar{\omega}_3.$$

Considering the relations above, we have that

- a basis for $H^1(M(\mathcal{A}))$ is represented by

$$\{\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \psi_1, \psi_2\},$$

  then $dim(H^1(M(\mathcal{A}))) = 5$;

- a basis for $H^2(M(\mathcal{A}))$ is represented by

$$\{\bar{\omega}_1\bar{\omega}_2, \bar{\omega}_1\bar{\omega}_3, \bar{\omega}_2\bar{\omega}_3, \psi_1\psi_2, \bar{\omega}_1\psi_2, \bar{\omega}_2\psi_1, \bar{\omega}_3\psi_1, \bar{\omega}_{r,\{H_2,H_3\}}\},$$

  then $dim(H^2(M(\mathcal{A}))) = 8$.

Note that these results are the same of Example 3.3.5 in which we computed $b_1 = 5$ and $b_2 = 8$.

Appendix A

Modules and algebras

A.1 $\mathcal{K}$-modules

Definition A.1.1. Let $(\mathcal{K}, +, ^*)$ be an abelian ring.
$(M, \perp)$ is a $\mathcal{K}$**-module** if $(M, \perp)$ is an abelian group and exists an operation $* : \mathcal{K} \times M \longrightarrow M$ such that:

- $(a + b) * m = a * m \perp b * m$;
- $a * (m \perp n) = a \ * \ m \perp a * n$;
- $(a^* b) * m = a * (b * m)$;
- $1_R * m = m$

$\forall\, a, b \in \mathcal{K}$ and $\forall m, n \in M$.

A $\mathbb{K}$-module where $\mathbb{K}$ is a field, is called $\mathbb{K}$**-vector space**.

Definition A.1.2. Let M be a $\mathcal{K}$-module and N a subgroup of $(M, \perp)$.
N is a **sub-**$\mathcal{K}$**-module** of M if, for every $a \in \mathcal{K}, n \in N$, $a * n \in N$.

If X is a non-empty subset of the $\mathcal{K}$-module M, then we define

$$< X >:= \bigcap_{\substack{N \text{ sub-}\mathcal{K}\text{-module of } M \\ X \subseteq N}} N$$

the **sub-$\mathcal{K}$-module generated by X**, that is the smallest sub-$\mathcal{K}$-module of M that contains X.
If $X = \{x_1, \ldots, x_n\}$ is a finite set, then $< X >= \{\sum_{i=1}^{n} a_i * x_i | a_i \in \mathcal{K}\}$ is said to be **finitely generated**.
If $X = \{x\}$, we write $\mathcal{K}x :=< x >= \{a * x | a \in \mathcal{K}\}$.
If $M =< X >$ we say that X **generates** M.

Definition A.1.3. Let M be a $\mathcal{K}$-module and N be a subset of M. N is **linearly independent** if for any distinct $x_1, \ldots, x_n \in N$ and $a_1, \ldots, a_n \in \mathcal{K}$ we have

$$\sum_{i=1}^{n} a_i * x_i = 0 \quad \Rightarrow \quad a_i = 0 \quad \forall i.$$

A set that is not linearly independent is **linearly dependent**.

Definition A.1.4. Let M be a $\mathcal{K}$-module. A subset $\mathcal{B}$ of M is a **basis** for M if $\mathcal{B}$ is linearly independent and generates M.
M is said to be a **free $\mathcal{K}$-module** if $M = \{0\}$ or if M admits a basis.

Theorem A.1.5. *A subset $\mathcal{B}$ of a $\mathcal{K}$-module M is a basis if and only if every nonzero $x \in M$ is an essentially unique combination of the vectors in $\mathcal{B}$.*

Definition A.1.6. Let M be a $\mathcal{K}$-module and N be a sub-$\mathcal{K}$-module of M. The binary relation

$$x \mathcal{R} y \Leftrightarrow x - y \in N$$

is an equivalence relation on M, whose equivalence classes are the cosets

$$[x] = \{x + n | n \in N\}.$$

The set M/N of all cosets of N in M is a $\mathcal{K}$-module under the well defined operations $\tilde{\perp} : M/N \times M/N \longrightarrow M/N$ and $\tilde{*} : M/N \times M/N \longrightarrow M/N$ such that

$$[x]\tilde{\perp}[y] := [x \perp y] \qquad \text{and} \qquad a\tilde{*}[x] := [a * x].$$

This $\mathcal{K}$-module is called **quotient $\mathcal{K}$-module**.

Definition A.1.7. Let M,N be $\mathcal{K}$-modules and $f : M \longrightarrow N$ a group homomorphism. f is a **$\mathcal{K}$-linear map** or **$\mathcal{K}$-homomorphism** if

$$f(a * m) = a * f(m) \quad \forall a \in \mathcal{K}, \forall m \in M.$$

A bijective $\mathcal{K}$-linear map is called $\mathcal{K}$ **isomorphism**.

Definition A.1.8. Let M, N be two $\mathcal{K}$-modules and consider the set of the pairs (m, n) with $m \in M, n \in N$.
This set is a $\mathcal{K}$-module with the operations $+$ and $*$ defined as follow:

$$(m, n) + (m', n') = (m + m', n + n')$$

and

$$a * (m, n) = (a * m, a * n)$$

$\forall m, m' \in M, n, n' \in N, a \in \mathcal{K}$.
This $\mathcal{K}$-module, denoted as $M \oplus N$, is called **direct sum** of M and N.

In the same way, it is defined the direct sum of a finite collection of $\mathcal{K}$-modules, and it will be denoted as

$$\bigoplus_{i \in \{1,\dots,n\}} M_i := M_1 \oplus \cdots \oplus M_n.$$

The direct sum of n copies of M is denoted as

$$M^n := M \oplus \cdots \oplus M.$$

Definition A.1.9. Let $(\mathcal{K}, +, ^*)$ be a graded ring with $\mathcal{K} = \oplus_{i=0}^m \mathcal{K}_i$.
A $\mathcal{K}$-module $M = \oplus_{j=0}^n M_j$ is a **graded $\mathcal{K}$-module** if $\mathcal{K}_\lambda * M_\mu \subset M_{\lambda+\mu} \quad \forall \lambda, \mu$.

Definition A.1.10. Let M,M' be graded $\mathcal{K}$-modules with $M = \oplus_{i=1}^{n} M_i$, $M' = \oplus_{i=1}^{m} M'_i$. A $\mathcal{K}$-homomorphism of $\mathcal{K}$-modules $f : M \longrightarrow M'$ is a **graded homomorphism** if $f(M_i) \subseteq M'_i \quad \forall i$. A bijective homomorphism is called **graded isomorphism**. f is called **homogeneous of degree** k if $f(M_i) \subseteq M'_{i+k} \quad \forall i$.

A.2 $\mathcal{K}$-algebras

Definition A.2.1. Let $(\mathcal{K}, +, ^*)$ be an abelian ring. A $\mathcal{K}$-module $(A, \perp)$ with scalar product $*$ is a **$\mathcal{K}$-algebra** if there exists a bilinear operation $\star : A \times A \longrightarrow A$, called **multiplication** of A.

Recall that a **bilinear operation** is an operation that satisfies the following properties:

- $(x \perp y) \star z = x \star z + y \star z$
- $x \star (y \perp z) = x \star y \perp x \star z$
- $(a * x) \star y = a * (x \star y)$
- $x \star (b * y) = b * (x \star y)$

$\forall x, y, z \in A$ and $\forall a, b \in \mathcal{K}$.

Definition A.2.2. Let A be a $\mathcal{K}$-algebra and B a sub-$\mathcal{K}$-module of A. B is a **sub-$\mathcal{K}$-algebra** of A if $x \star y \in B \quad \forall x, y \in B$.

Definition A.2.3. Let A be a $\mathcal{K}$-algebra and I a sub-$\mathcal{K}$-module of A. I is an **ideal** of A if $\quad i \star x \in I, x \star i \in I \quad \forall i \in I, x \in A$.

Definition A.2.4. Let A be a $\mathcal{K}$-algebra and I an ideal of A. The binary relation

$$x\mathcal{R}y \Leftrightarrow x - y \in I$$

is an equivalence relation on A, whose equivalence classes are the cosets

$$[x] = \{x + i | i \in I\}.$$

The set A/I of all cosets of I in A is a $\mathcal{K}$-algebra under the bilinear operation $\tilde{\star} : A/I \times A/I \longrightarrow A/I$ defined by

$$[x]\tilde{\star}[y] := [x \star y].$$

This $\mathcal{K}$-algebra is called **quotient $\mathcal{K}$-algebra**.

Note that $\tilde{\star}$ is well defined, since, if $[x] = [x']$ and $[y] = [y']$, $x \star y - x' \star y' = x \star (y - y') + (x - x') \star y' \in I$, thus $[x \star y] = [x' \star y']$.

Definition A.2.5. Let A,A' be $\mathcal{K}$-algebras and $f : A \longrightarrow A'$ a $\mathcal{K}$-linear map. f is a **homomorphism of $\mathcal{K}$-algebras** if $f(x \star y) = f(x) \star' f(y) \quad \forall x \in A, x \in A'$.
A bijective homomorphism is called $\mathcal{K}$ **isomorphism**.

Definition A.2.6. Let A be a $\mathcal{K} - algebra$. A $\mathcal{K}$-linear map $\partial : A \longrightarrow A$ is a **$\mathcal{K}$-derivation** if it satisfies the **Leibniz rule**, i.e, $\forall a, b \in A$,

$$\partial(a \star b) = (\partial a) \star b + a \star (\partial b).$$

In this case we say that A is a **differential $\mathcal{K}$-algebra**

A.2.1 Graded $\mathcal{K}$-algebras

Definition A.2.7. Let A be a $\mathcal{K}$-algebra. A is a **graded $\mathcal{K}$-algebra** if there exist a collection $\{A_i\}_{i\in\{1,\ldots,n\}}$ of sub-$\mathcal{K}$-modules of the $\mathcal{K}$-module A such that $A = \bigoplus_{i=1}^{n} A_i$ and $A_i \star A_j \subset A_{i+j} \quad \forall i, j \in \{1, \ldots, n\}$.
An element $x \in A$ is called **homogeneous** if it belongs to one of the A_i and **homogeneous of degree i** if $x \in A_i$.

Remark A.2.8.
- 0 is homogeneous of all degrees;
- if $x \neq 0$ is homogeneous, it belongs to only one of the A_i. If $x \in A_{\bar{i}}$, we say that $\bar{i}$ is the degree of x and we write $deg(x) = \bar{i}$;
- Every $x \in A$ may be written uniquely as sum of homogeneous elements with, for $x = \sum_{i=1}^{n} x_i$, $x_i \in A_i$. In this case we say that x_i is the **homogeneous component of degree** i **of** x.

Definition A.2.9. Let A be a graded $\mathcal{K}$-algebra with $A = \oplus_{i=1}^{n} A_i$ and I an ideal of A. I is an **homogeneous** or **graded** ideal of A if $I = \oplus_{i=1}^{n} I \cap A_i$.

Proposition A.2.10. *If A is a graded $\mathcal{K}$-algebra with $A = \oplus_{i=1}^{n} A_i$ and I is a homogeneous ideal of A, then $A/I = \bigoplus_{i=1}^{n} \frac{A_i}{I \cap A_i}$ is graded $\mathcal{K}$-algebra.*

Definition A.2.11. Let A,A' be $\mathcal{K}$-algebras with $A = \oplus_{i=1}^{n} A_i$, $A' = \oplus_{i=1}^{m} A'_i$. A homomorphism of $\mathcal{K}$-algebras $f : A \longrightarrow A'$ is a **graded homomorphism** of graded algebras if $f(A_i) \subseteq A'_i \quad \forall i$. A bijective homomorphism is called **graded isomorphism**.
f is **graded of degree** k if $f(A_i) \subseteq A'_{i+k} \quad \forall i$.

Definition A.2.12. Let A be a graded $\mathcal{K}$-algebra. A is a **differential graded $\mathcal{K}$-algebra** if there exists a $\mathcal{K}$-linear map $\partial : A \longrightarrow A$ of degree 1 or degree -1, such that:

- $\partial \circ \partial = 0$;
- ∂ respects the **graded Leibniz rule**, i.e.

$$\partial(a \star b) = (\partial a) \star b + (-1)^{deg(a)} a \star (\partial b)$$

  for every $a, b \in A$.

In this case ∂ is called **derivation** of A.

A.2.2 Exterior algebras of $\mathcal{K}$-modules

Definition A.2.13. Let M,N $\mathcal{K}$-modules. Define B the $\mathcal{K}$-module of formal linear combinations of elements of $M \times N$, and R the $\mathcal{K}$ submodule of B generated by the element of one of the following types:

- $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$;
- $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$;
- $(a * m, n) - (m, a * n)$

$\forall m, m_1, m_2 \in M,\ n, n_1, n_2 \in N, a \in \mathcal{K}$.

The **tensor product of** M **and** N, denoted by $M \otimes N$, is the quotient $\mathcal{K}$-module B/R. For every $m \in M, n \in N$, define $x \otimes y$ the **tensor product** of m and n the image of (m, n) through the natural projection
$\pi : M \times N \longrightarrow M \otimes N$.

Remark A.2.14. Directly from this definition, it follows that

- $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$;
- $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$;
- $(a * m) \otimes n = m \otimes (a * n)$

$\forall\, m, m_1, m_2 \in M,\ n, n_1, n_2 \in N, a \in \mathcal{K}$.

Definition A.2.15. Let M be a $\mathcal{K}$-module. Define $M^{\otimes 0} := \mathcal{K}$ and for every $n > 0$,

$$M^{\otimes n} := M \otimes \cdots \otimes M$$

the $\mathcal{K}$-module defined as the tensor product of n modules equal to M, and

$$T(M) := \bigoplus_{n \geq 0} M^{\otimes n}$$

the $\mathcal{K}$-module defined as the direct sum of $\{M^{\otimes n}\}_{n \geq 0}$.

$T(M)$ is a $\mathcal{K}$-algebra, called **tensor $\mathcal{K}$-algebra** of M, with the operation $\star : T(M) \times T(M) \longrightarrow T(M)$ defined on the generators as

$$(x_1 \otimes \cdots \otimes x_p) \star (y_1 \otimes \cdots \otimes y_q) := x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q$$

for every $(x_1 \otimes \cdots \otimes x_p) \in M^{\otimes p}, (y_1 \otimes \cdots \otimes y_q) \in M^{\otimes r}, p, q > 0$.

Clearly $T(M)$ is a graded algebra.

Definition A.2.16. Let M be a $\mathcal{K}$-module. Define I the ideal of $T(M)$ generated by the elements $x \otimes y - y \otimes x$, with $x, y \in M$. The **exterior $\mathcal{K}$-algebra** of M, denoted as $\Lambda(M)$ is the quotient $\mathcal{K}$- algebra of the tensor algebra $T(M)$ by the ideal I.

For every $x_1 \otimes \cdots \otimes x_p \in T(M)$, denote $x_1 \wedge \cdots \wedge x_p$ its class in $\Lambda(M)$.

It follows from the definition that:

- $x \wedge y = -y \wedge x \quad \forall x, y, \in M$;
- $x \wedge x = 0 \quad \forall x \in M$.

Since $T(M)$ is a graded algebra and I is a graded ideal, $\Lambda(M)$ is a graded algebra, i.e.

$$\Lambda(M) := \bigoplus_{n \geq 0} \frac{M^{\otimes n}}{I \cap M^{\otimes n}} := \bigoplus_{n \geq 0} \Lambda^n(M)$$

and $\Lambda^r E \wedge \Lambda^s E \subseteq \Lambda^{r+s} E$.

Remark A.2.17. Since $I \cap M^{\otimes 0} = I \cap M^{\otimes 1} = \{0\}$, then $\Lambda^0(M) = M^{\otimes 0} = \mathcal{K}$ and $\Lambda^1(M) = M^{\otimes 1} = M$.

It can be proved that $x \wedge y = (-1)^{pq} y \wedge x \quad \forall x \in \Lambda^p(M), y \in \Lambda^q(M)$.

For $X = \{x_1, \ldots, x_n\}$, denote $\Lambda^k(x_1, \ldots, x_n) := \Lambda^k(< X >)$ and $\Lambda(x_1, \ldots, x_n) = \Lambda(< X >)$. Since $x \wedge x = 0$ for every $x \in X$, we have that $\Lambda^k(x_1, \ldots, x_n)$ is generated by $\{x_{i_1} \wedge \cdots \wedge x_{i_k}$, with $i_\ell \neq i_m \quad \forall \ell \neq m, x_{i_j} \in X\}$, i.e. every element of $\Lambda^k(x_1, \ldots, x_n)$ can be written as the wedge of k distinct elements of $< X >$, which are elements of degree 1 since $< X >= \Lambda^1(< X >)$. For this reason we say that $\Lambda(x_1, \ldots, x_n)$ is **generated in degree one**.

Appendix B

Matroids

Definition B.0.1. A **matroid** $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a collection of subsets of E having the following properties:

(I1) $\emptyset \in \mathcal{I}$;

(I2) If $B \subseteq A$ and $A \in \mathcal{I}$, then $B \in \mathcal{I}$;

(I3) If $A, B \in \mathcal{I}$ where $|B| < |A|$, then $\exists a \in A \backslash B$ such that $B \cup \{a\} \in \mathcal{I}$

The elements of $\mathcal{I} := \mathcal{I}(\mathcal{M})$ are called **independent sets**, and E is called **ground set** of $\mathcal{M}$. A subset of E that is not in $\mathcal{I}$ is called **dependent**.

Definition B.0.2. A **circuit of** $\mathcal{M}$ is a minimal dependent set. Define $\mathcal{C}(\mathcal{M})$ the set of circuits of $\mathcal{M}$, i.e.

$$\mathcal{C}(\mathcal{M}) = \{X \notin \mathcal{I};\ \forall x \in X, X\backslash\{x\} \in \mathcal{I}\}.$$

Note that $\mathcal{I}(\mathcal{M}) = \{X \subseteq E;\ \nexists Y \subseteq X \text{ with } Y \in \mathcal{C}(\mathcal{M})\}$. It immediately follows that

Proposition B.0.3. *A matroid is uniquely determined by its set $\mathcal{C}(\mathcal{M})$ of circuits.*

Definition B.0.4. $B \subseteq E$ is a **basis** of $\mathcal{M}$ if it is a maximal independent set in $\mathcal{M}$.Denote $\mathcal{B}$ the set of basis of $\mathcal{M}$.

Lemma B.0.5. *If* B_1 *and* B_2 *are basis of a matroid* $\mathcal{M}$*, then* $|B_1| = |B_2|$.

Note that $\mathcal{B}$ is the collection of maximal subsets of E that contain no member of $\mathcal{C}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ is the collection of minimal sets that are contained in no member of $\mathcal{B}$.

Definition B.0.6. Let $\mathcal{M} = (E, \mathcal{I})$ a matroid and $X \in E$. Define

$$\mathcal{I}|_X := \{I \subseteq X | I \in \mathcal{I}\};$$

$\mathcal{M}|_X := (X, \mathcal{I}|_X)$ the **restriction matroid** of $\mathcal{M}$ to X;

$$\mathcal{C}(\mathcal{M}|_X) = \{C \subseteq X | C \in \mathcal{C}(\mathcal{M})\}$$

and the **rank function**

$$r_{\mathcal{M}} : \mathcal{P}(E) \longrightarrow \mathbb{Z}^+ \cup \{0\}$$

$$X \longmapsto |B|$$

where B basis of $\mathcal{M}|_X$. Denote $r_{\mathcal{M}}(\mathcal{M}) := r(E)$.

Note that the rank function is well defined thanks to Lemma B.0.5.

Remark B.0.7. For every $X \subseteq E$,

- $X \in \mathcal{I} \quad \Leftrightarrow \quad |X| = r_{\mathcal{M}}(X)$;
- $X \in \mathcal{B} \quad \Leftrightarrow \quad |X| = r_{\mathcal{M}}(X) = r_{\mathcal{M}}(\mathcal{M})$;
- $X \in \mathcal{C} \quad \Leftrightarrow \quad X \neq \emptyset$ and $r_{\mathcal{M}}(X) = |X| - 1 = r_{\mathcal{M}}(X \setminus \{x\}) \, \forall x \in X$.

Definition B.0.8. Define the **closure operator** of $\mathcal{M}$ as

$$cl : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$$

$$X \longmapsto \{x \in E | r_{\mathcal{M}}(X \cup x) = r_{\mathcal{M}}(X)\}$$

Remark B.0.9. For every $X \subseteq E$,

1. $r_{\mathcal{M}}(X) = r_{\mathcal{M}}(cl(X))$;
2. $X \subseteq cl(X)$;
3. $cl(cl(X)) = cl(X)$;
4. $cl(X) \subseteq cl(Y)$, for $X \subseteq Y$.

Definition B.0.10. A subset X of E is a **flat** (or **closed set**) of $\mathcal{M}$ if $cl(X) = X$. Denote $\mathcal{L}(\mathcal{M})$ the set of flats of $\mathcal{M}$ and note that it is a poset under inclusion.

Remark B.0.11. $\{X \subseteq E | X \text{ flat}\} = \{cl(X) | X \subseteq E\}$. In fact, in order to prove the right to left inclusion, we take $Y = cl(X)$ which is a flat since, by Remark B.0.9, $cl(Y) = cl(cl(x)) = cl(X) = Y$.

Definition B.0.12. Let X, Y be subsetes of E. We say that X **spans** Y if $Y \subseteq cl(X)$ and that X is a **spanning set** of $\mathcal{M}$ if $cl(X) = E$.

Proposition B.0.13. *Let X be a subset of E, then:*

- *X is a spanning set $\Leftrightarrow r_{\mathcal{M}}(X) = r_{\mathcal{M}}(\mathcal{M})$.*

- *X is a base of $\mathcal{M} \Leftrightarrow X$ is a spanning set and independent set $\Leftrightarrow$ X is a minimally spanning set.*

- *X is a circuit $\Leftrightarrow X$ is a minimal non-empty set such that $x \in cl(X - x) \quad \forall x \in X$.*

- $cl(X) = X \cup \{x \in E | \exists\ C \text{ circuit such that } x \in C \subseteq X \cup x\}$.

Appendix C

Vector bundles and de Rham cohomology

C.1 Vector bundles

Definition C.1.1. Let E, M be C^∞ manifolds and $\pi : E \longrightarrow M$ be a C^∞ map. The triple $\xi = (E, \pi, M)$ is a **vector bundle** of rank n over a field $\mathbb{K}$, if

- $\pi^{-1}(x)$ is a $\mathbb{K}$-vector space for every $x \in M$;

- there exists an open cover $\{U_\alpha\}$ of M and diffeomorphisms $\varphi_\alpha : U_\alpha \times \mathbb{K}^n \longrightarrow \pi^{-1}(U_\alpha)$ such that

  $$\pi \circ \varphi_\alpha = proj_1$$

  where $proj_1 : U_\alpha \times \mathbb{K}^n \longrightarrow U_\alpha$ is the projection on the first component, and $\forall x \in U_\alpha, \forall y \in \mathbb{K}^n$ the maps

  $$\varphi_{\alpha,x} : \mathbb{K}^n \longrightarrow \pi^{-1}(x)$$

  $$y \longmapsto \varphi_\alpha(x, y)$$

  are linear isomorphisms.

If $\mathbb{K}$ is equal to $\mathbb{R}$ or $\mathbb{C}$, ξ is called, respectively, **real** or **complex** vector bundle.

Remark C.1.2.

- π is a surjective map since for every $x \in M$, $\pi^{-1}(x)$ is a $\mathbb{K}$ vector space and then it can't be empty;
- $E = \bigsqcup_{x \in M} \pi^{-1}(x)$.

Definition C.1.3. Let M be a C^∞ manifold. $\xi = (M \times \mathbb{K}^n, proj_1, M)$ is called **trivial bundle** of rank n over $\mathbb{K}$.

Definition C.1.4. Let $\xi = (E, \pi, M)$ be a vector bundle and let O be an open submanifold of M. The vector bundle $\xi|_O := (\pi^{-1}(O), \pi|_{\pi^{-1}(O)}, O)$ is called **restriction of** ξ.

Definition C.1.5. Let $\xi = (E, \pi, M)$ be a vector bundle.
A C^∞ map $s : M \longrightarrow E$ is a **section** of ξ if $\pi \circ s$ is the identity on M.
Let U be an open set in M, a C^∞ map is a **section of** ξ **over U** if it is a section of $\xi|_U$, i.e. $\pi \circ s$ is the identity on U.

Proposition C.1.6. *Every vector bundle $\xi = (E, \pi, M)$ admits a section $s_0 : M \longrightarrow E$ such that, $\forall x \in M$, $s_0(x) = 0_x$ where 0_x is the zero element of the vector space $\pi^{-1}(x)$. s_0 is called* ***zero section of*** *ξ.*

Definition C.1.7. Let M be a n-dimensional C^∞ manifold, and S be a k-dimensional C^∞ submanifold of M. An open neighborhood T of S in M is a **tubular neighborhood** if there exist a vector bundle $\xi = (E, \pi, S)$ of rank $n-k$ and a diffeomorphism $\psi : T \longrightarrow E$ such that $\psi|_S = s_0$, where s_0 is the zero section of ξ.

Theorem C.1.8 (Tubular Neighborhood Theorem)**.** *Let M is a C^∞ manifold. Every C^∞ submanifold S in M has a tubular neighborhood T.*

Let's now recall some notions about homotopy:

Definition C.1.9. Let M,N be C^∞ manifolds and f,g be C^∞ maps between M and N. f and g are C^∞**-homotopic** if there exists a C^∞ map

$$F : M \times \mathbb{R} \longrightarrow N$$

such that

$$F|_{M\times\{0\}} = f \quad \text{and} \quad F|_{M\times\{1\}} = g.$$

In this case F is called **homotopy from** f **to** g and we write $f \simeq_\infty g$.

Definition C.1.10. Let M,N be C^∞ manifolds. We say that M is **homotopy equivalent** to N if there exist $f : M \longrightarrow N$ and $g : N \longrightarrow M$ C^∞ maps such that

$$g \circ f \simeq \mathbb{1}_M \quad \text{and} \quad f \circ g \simeq \mathbb{1}_N.$$

In this case f and g are called **homotopy equivalences** and we write $M \simeq_\infty N$.
If a manifold M is homotopy equivalent to a point, we say that M is **contractible**.

Definition C.1.11. Let S be submanifold of a manifold M, with $i : S \longrightarrow M$ the inclusion map.
A C^∞ map $r : M \longrightarrow S$ is a **retraction** from M to S if $r \circ i = \mathbb{1}_S$. In this case we say that S is a **retract** of M.
S is a **deformation retract** of M if there exists an homotopy F from $\mathbb{1}_M$ to $i \circ r$. S is a **strong deformation retract** of M if it is a deformation retract such that $F(s,t) = s$ for every $s \in S, t \in \mathbb{R}$ (i.e. F leaves S fixed for every time t).

Proposition C.1.12. *Let S be submanifold of a manifold M. If S is a deformation retract of M, then S is homotopy equivalent to M.*

Proposition C.1.13. *A vector bundle over a contractible manifold is trivial.*

C.2 De Rham cohomology

We assume the reader is familiar with basic concepts of differential topology such as C^∞ manifold, C^∞ map, tangent space, pullback of a map and smooth differential form. (See [Tu08]).

Let M be a C^∞ manifold. We denote $\Omega^k(M)$ the real vector space of smooth differential k-forms on M.
From now on we will call a smooth differential k-form, simply k-form.
Recall that

- $\Omega^0(M) = C^\infty(M) := \{f : M \longrightarrow \mathbb{R}, f \text{ is a } C^\infty \text{ map}\}$;
- $\Omega^k(M) = 0$ for every $k \geq \dim(\mathrm{M})$ the dimension of M;
- $\Omega^*(M) := \bigoplus_{k=0}^{\dim(\mathrm{M})} \Omega^k(M)$, with the wedge product $\wedge$ of forms, is a graded algebra;
- $\alpha \wedge \beta = (-1)^{k+\ell} \beta \wedge \alpha$ for every $\alpha \in \Omega^k(M), \beta \in \Omega^\ell$.

Definition C.2.1. Let M be a C^∞ manifold. An **exterior derivative** on M is a $\mathbb{R}$-linear map

$$d : \Omega^*(M) \longrightarrow \Omega^*(M)$$

of degree 1 such that:

- on 0-forms it agrees with the differential df of a function f;
- $d \circ d = 0$;
- $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$, where $\alpha \in \Omega^k(M)$.

Theorem C.2.2. *On any manifold M there exists an unique exterior derivative* $d : \Omega^*(M) \longrightarrow \Omega^*(M)$ *characterized uniquely by the three properties of Definition C.2.1.*

It follows that $\Omega^*(M)$ is a differential graded anticommutative algebra and $(\Omega^*(M), d)$ is a cochain complex, called **de Rham complex**.
Define $d^k := d\big|_{\Omega^k(M)} : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$.

Definition C.2.3. Let M be a C^∞ manifold.
A k-form ω is a **closed** k-form if $\omega \in Ker(d^k)$ and it is a **exact** k-form if $\omega \in Im(d^{k-1})$.
The cohomology of the de Rham complex is the quotient vector space

$$H^k_{dR}(M) := \frac{Ker(d^k)}{Im(d^{k-1})} = \frac{\{\text{closed k-forms on M}\}}{\{\text{exact k-forms on M}\}}.$$

It is called **de Rham cohomology** of M in degree k.
Define

$$H^*_{dR}(M) = \bigoplus_{k=0}^{\dim(M)} H^k_{dR}(M).$$

Note that, for every α, β forms:

- α, β closed $\Rightarrow \alpha \wedge \beta$ closed;
- α exact, β closed $\Rightarrow \alpha \wedge \beta$ exact;
- α closed, β exact $\Rightarrow \alpha \wedge \beta$ exact.

It follows that the wedge product of the k-forms induces a **wedge product on the classes** defined by

$$[\alpha] \wedge [\beta] := [\alpha \wedge \beta].$$

This product gives to $H^*_{dR}(M)$ the structure of an anticommutative graded $\mathbb{R}$-algebra.

Definition C.2.4. Let M,N be C^∞ manifolds and $f : M \longrightarrow N$ be a C^∞ map.
Define

$$H^k_{dR}(f^*) : H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$$

$$[\omega] \longmapsto [f^*(\omega)]$$

where $f^* : \Omega^k(N) \longrightarrow \Omega^k(M)$ is the pullback map of f.

These maps induce an homomorphism of algebras

$$H^*_{dR}(f^*) : H^*_{dR}(N) \longrightarrow H^*_{dR}(M).$$

Note that $H^k_{dR}(f^*)$ is well defined since it can be proved that the exterior derivative commutes with the pullback of every C^∞ map.

Proposition C.2.5. *Let M,N be C^∞ manifolds and $f : M \longrightarrow N$ be a C^∞ map. If f is a diffeomorphism, then $H^k_{dR}(f^*)$ is a isomorphism of vector spaces for every k.*
In particular

$$H^*_{dR}(M) \simeq H^*_{dR}(N).$$

From this proposition immediately follows that the de Rham cohomology is a diffeomorphism invariant of C^∞ manifold.

C.2.1 Homotopy axiom for the de Rham cohomology

Proposition C.2.6. *Let M,N be C^∞ manifolds and f, g be C^∞ maps between M and N. If $f \simeq_\infty g$ then $H^k_{dR}(f^*) = H^k_{dR}(g^*)$.*

Corollary C.2.7. *Let M,N be C^∞ manifolds. If M is homotopy equivalent to N, then $H^*_{dR}(M)$ is isomorphic to $H^*_{dR}(N)$.*

Corollary C.2.8. *Let S be a submanifold of a manifold M. If S is a deformation retract of M, then $H^*_{dR}(S)$ is isomorphic to $H^*_{dR}(M)$.*

C.2.2 Mayer-Vietoris sequence

Theorem C.2.9. *Let $\{U, V\}$ be an open cover of a C^∞-manifold M. Consider, for each k,*

$$i : \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V)$$

$$\alpha \longmapsto (\alpha|_U, \alpha|_V)$$

and

$$j : \Omega^k(U) \oplus \Omega^k(V) \longrightarrow \Omega^k(U \cap V)$$

$$(\beta, \gamma) \longmapsto \beta|_{U\cap V} - \gamma|_{U\cap V}.$$

The sequence

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U \cap V)$$

is exact for each k.

Remark C.2.10. If $B \subseteq A$, for every form ω in A, $\omega|_B = i_B^*(\omega)$ where $i_B : B \longrightarrow A$ is the canonical inclusion.

In particular the sequence of cochain complexes

$$0 \to \Omega^*(M) \xrightarrow{i} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j} \Omega^*(U \cap V) \to 0 \tag{C.1}$$

is exact.
Note that $\Omega^*(U) \oplus \Omega^*(V) = \Omega^*(U \sqcup V)$.

Thanks to Snake's lemma, the sequence C.1 gives rise to a long exact sequence in cohomology, called **Mayer-Vietoris sequence**

$$\cdots \to H^k_{dR}(M) \xrightarrow{H^k_{dR}(i)} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{H^k_{dR}(j)} H^k_{dR}(U \cap V) \xrightarrow{\delta^k} H^{k+1}_{dR}(M) \to \ldots$$

where δ^k is the connecting homomorphism.

C.2.3 Relative de Rham cohomology

Definition C.2.11. Let M be C^∞ manifold, S be a submanifold of M and $i_S : S \longrightarrow M$ be the inclusion map. Define a complex $\Omega^*(M,S) := \oplus_k \Omega^k(M,S)$ by

$$\Omega^k(M,S) := \Omega^k(M) \oplus \Omega^{k-1}(S)$$

and

$$d : \Omega^k(M,S) \longrightarrow \Omega^{k+1}(M,S)$$
$$(\alpha,\beta) \longmapsto (d\alpha, \alpha|_S - d\beta).$$

It is easy to verify that $d^2 = 0$. Note that a cohomology class in $\Omega^*(M,S)$ is represented by a closed form ω on M such that $\omega|_S$ is exact.

Proposition C.2.12. *Let*

$$i : \Omega^{k-1}(S) \longrightarrow \Omega^k(M,S)$$

$$\alpha \longmapsto (0,\alpha)$$

and

$$j : \Omega^k(M,S) \longrightarrow \Omega^k(M)$$

$$(\beta,\gamma) \longmapsto \beta.$$

The sequence

$$0 \to \Omega^{k-1}(S) \xrightarrow{i} \Omega^k(M,S) \xrightarrow{j} \Omega^k(M)$$

is exact for each k.

Proposition C.2.13. *Let M be C^∞ manifold, S a submanifold of M. There is an exact sequence, called **long exact sequence of the pair** (M,S)*

$$\cdots \to H^{k-1}_{dR}(S) \xrightarrow{H^{k-1}_{dR}(i)} H^k_{dR}(M,S) \xrightarrow{H^k_{dR}(j)} H^k_{dR}(M) \xrightarrow{H^k_{dR}(i_S^*)} H^k_{dR}(S) \to \dots$$

*where $H^k_{dR}(M,S)$ is called **relative de Rham cohomology**.*

In conclusion we state an important theorem called **excision theorem**

Theorem C.2.14. *Let U,A,X be three manifolds. If $U \subseteq A \subseteq X$, then the inclusion map $(X\backslash U, A\backslash U) \longrightarrow (X, A)$ induces, for every k, an isomorphism*

$$r : H^k_{dR}(X, A) \longrightarrow H^k_{dR}(X\backslash U, A\backslash U).$$

Bibliography

[Bir99] Christina Birkenhake. *Complex tori.* Progress in Mathematics 177. Springer, 1999.

[BLVS+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Gunter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of mathematics and its applications.* Cambridge University Press, 2nd edition, 1999.

[Bou98] Nicolas Bourbaki. *Algebra I*, volume Chapters 1-3. Springer, 1998.

[BT95] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology.* Graduate Texts in Mathematics 82. Springer, 1995.

[CDD+19] Filippo Callegaro, Michele D'Adderio, Emanuele Delucchi, Luca Migliorini, and Roberto Pagaria. Orlik-Solomon-type presentations for the cohomology algebra of toric arrangements. 2019. To appear in Trans. Amer. Math. Soc., arXiv:1806.02195.

[DCP05] Corrado De Concini and Claudio Procesi. On the geometry of toric arrangements. *Transformation Groups*, 10(3):387–422, 2005.

[DCP11] Corrado De Concini and Claudio Procesi. *Topics in hyperplane arrangements, polytopes and box-splines.* Universitext. Springer-Verlag New York, 1st edition, 2011.

[DM13] Michele D'Adderio and Luca Moci. Arithmetic matroids, the Tutte polynomial and toric arrangements. *Advances in Mathematics*, 232(1):335 – 367, 2013.

[Hat01] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 1st edition, 2001.

[Man08] Marco Manetti. *Topologia*. Unitext. Springer, 2008.

[Moc12] Luca Moci. A Tutte polynomial for toric arrangements. *Transactions of the American Mathematical Society*, 364(2):1067–1088, 2012.

[OS80] Peter Orlik and Louis Solomon. Combinatorics and topology of complements of hyperplanes. *Inventiones mathematicae*, 56(2):167–189, 1980.

[OT92] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin, 1992.

[Oxl11] James G. Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2nd edition, 2011.

[Pag19a] Roberto Pagaria. Combinatorics of toric arrangements. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.*, 30:317–349, 2019.

[Pag19b] Roberto Pagaria. Two examples of toric arrangements. *J. Comb. Theory, Ser. A*, 167:389–402, 2019.

[Rom07] Steven Roman. *Advanced Linear Algebra*. Graduate Texts in Mathematics. Springer, 3rd edition, 2007.

[Tu08] Loring W. Tu. *An introduction to manifolds*. Springer, 2008.

[Yuz01] Sergey Yuzvinsky. Orlik-Solomon algebras in algebra and topology. *Uspekhi Mat. Nauk*, 56(2):87–166, 2001.