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THERMODYNAMIC BEHAVIOUR OF GRAVITY

Attempts at a statistical derivation of the field equations

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Introduzione

Lo scopo di questa tesi è quello di presentare una curiosa interpretazione della gravità che deriva da considerazioni statistiche ed evidenzia punti cruciali che potrebbero essere ulteriori argomenti importanti su cui poter fare ricerca scientifica. Nel tentativo di derivazione delle equazioni di campo di Einstein attraverso un approccio statistico, fenomeni termodinamici conducono alla teoria classica della Relatività Generale, tuttavia con informazioni aggiuntive che usualmente non compaiono quando si procede invece per la Teoria dei Campi.

Ciò nonostante, questo implica che sia necessario rivedere e accettare nuovi presupposti a livello fondamentale per quanto riguarda la natura dello spaziotempo. Tali assunzioni sono motivate e giustificate dall'osservazione del modello statistico dal punto di vista di un sistema di riferimento privilegiato, nel momento in cui si compara il comportamento gravitazionale con la termodinamica.

I primi tentativi di esplorare questa analogia risalgono al 1974, dopo la intrigante scoperta teorica della radiazione di buco nero del fisico Stephen Hawking, attraverso la quale l'interpretazione termodinamica della gravita è diventata sempre più evidente.

Di conseguenza, quando la gravità è considerata come un fenomeno che avviene a livello macroscopico nel modo descritto dalla Relativià Generale Classica, inevitabilmente esiste una statistica a livello microscopico. Dunque, esistono dei gradi di liberta di natura ancora sconosciuta associati allo spaziotempo, cosicché si dice che la gravità come la conosciamo "emerge". Questo collegamento tra macro e micro è simile alla derivazione di altre quantità termodinamiche, come temperatura o entropia di un gas atomico, risultato di uno stato particolare dei gradi di libertà di tutte le particelle considerate: un funzionale per tali gradi di libertà dello spaziotempo può essere trovato consentendo alla gravità di essere l'espressione a larga scale di questi, ogni qualvolta la procedura di estremizzazione viene effettutata.

Tali gradi di libertà vengono chiamati in gergo "atomi di spaziotempo", la cui dinamica avviene su scala microscopica; l'ordine di grandezza delle interazioni è quella di Plank e tali atomi suppostamente possiedono un certo numero di gradi di libertà, la cui natura ultima è considerata irrilevante sulla scala macroscopica della teoria: l'obiettivo è quello di descrivere la gravità in maniera statistica in funzione del numero di tali granuli microscopici.

Qui sotto, verrà descritto il percorso eseguito con la descrizione degli argomenti trattati in ciascun capitolo.

Nel primo capitolo si deriveranno le equazioni del moto con il metodo usuale, estremizzando la Lagrangiana di Einstein-Hilbert rispetto alla metrica. Facendo ciò, evidenzieremo l'esistenza di una corrente di Noether conservata *off-shell*, la cui natura è puramente geometrica: questo discende semplicemente dal fatto che la Lagrangiana è uno scalare.

Nel secondo capitolo, verranno fatte considerazioni particolari grazie alle quali sarà possibile fornire una interpretazione fisica alla corrente ricavata in precedenza. L'utilizzo delle equazioni del moto rivela che la carica di Noether associata può essere vista come entropia dello spaziotempo; in questo modo, le equazioni del moto sono l'espressione di un bilanciamento fra entropie: da un lato, la variazione dell'entropia associata alla materia che oltrepassa un orizzonte, dall'altro la variazione dell'entropia di natura geometrica.
Tale interpretazione, infatti, non è una mera formalità. Il fatto che a ogni orizzonte si possa associare della radiazione di buco nero, a una certa temperatura, sostiene fortemente l'idea che gli orizzonti siano sistemi termodinamici a tutti gli effetti, anche se definiti come entità puramente geometriche.

La derivazione convenzionale delle equazioni di campo non tiene in considerazione alcuna caratteristica termodinamica dello spaziotempo. Questo suggerisce di provare a cambiare visione e cercare un metodo di derivazione differente, in cui aspetti termodinamici siano presi come punti iniziali. Ecco il contenuto del capitolo tre: considerando la gravità come un fenomeno statistico a livello fondamentale, si può provare una derivazione delle equazioni di campo a partire dall'estremizzazione di un funzionale di entropia opportuno.

Il capitolo quattro esplora la possibile esistenza di una micro-struttura dello spaziotempo, o "atomi", in altre parole: risulta quindi un ingrediente piuttosto naturale per una descrizione statistica della gravità. Tale granulosità dello spaziotempo verrà descritta e implementata con l'introduzione di un particolare oggetto matematico in grado di fornire una distanza finita tra due eventi, nel limite di coincidenza tra essi; questa nuova metrica effettiva, che descrive la micro-dinamica in maniera assai curiosa, trasmuta la Lagrangiana canonica di Einstein-Hilbert in quel funzionale di entropia utilizzato nella derivazione statistica.

In conclusione, questo lavoro di tesi si riduce a una introduzione degli argomenti trattati, ma la teoria, appena descritta in breve, apre a nuove e intriganti frontiere e questioni ancora da approfondire e spiegare. Infatti, le assunzioni fondamentali sono applicabili anche su teorie a ordini derivativi maggiori a due e in dimensioni superiori di quelle usuali. Inoltre, funziona anche per teorie che non necessariamente sono caratterizzate da una proporzionalità fra entropia e area della superficie: questa è una ragione in più che la rende affascinante e interessante.

Introduction

The purpose of this thesis is to introduce an interesting interpretation for gravity that comes from statistical considerations and highlights possible milestones that might be topics of research themselves. In the attempt of deriving Einstein's field equations through a statistical approach, thermodynamic behaviours lead to the classical General Relativity Theory plus some extra information that is not visible under the usual Field Theory derivation.

Nevertheless, this implies that is necessary to review and accept new assumptions at fundamental level for nature of spacetime. These are motivated and justified by observing the statistical model under a particular system of reference when comparing gravity's behaviour with thermodynamics.

The beginning of exploration of this analogy can be traced back in 1974, thanks to the intriguing theoretical discovery of black hole radiation by the physicist Stephen Hawking, through which a thermodynamic interpretation of gravity has become more evident.

Consequently, when gravity is considered as a phenomenon that occurs at a macroscopic level in the way it is known in Classical General Relativity, a microscopic statistic exists, inevitably. Thus, some still unknown degrees of freedom associated to spacetime play an important role, so that the gravity we are familiar with it said to ”emerge”. This link between macro and micro is as similar as any derivation of thermodynamic quantities, in example temperature or entropy of an atomic gas, resulting from a particular status of the degrees of freedom of all particles involved: a functional for spacetime degrees of freedom therefore can be found allowing gravity to be their large scale expression, whenever the extremisation procedure is applied.

They way to call them in jargon is “atoms of spacetime” whose dynamics is suggested to be considered at the smallest scale; the order of magnitude of interactions is the Planck length and such atoms are expected to carry a certain number of degrees of freedom, whose ultimate nature is considered as irrelevant at the scale of the macroscopic theory: the main goal is to describe gravity in a statistical manner in terms of the number of these not-better-defined microscopic degrees of freedom.

The full path taken, with all considerations and highlights are in the following chapters, as described below.

In the first chapter, the equations of motion are derived by using the usual route, *i.e.* by extremisation of the Einstein-Hilbert Lagrangian with respect to the metric. In doing this, we will stress that a Noether current exists, of strictly geometric nature; although this is simply as a consequence of the fact that the Lagrangian is a scalar, what is relevant to notice is that this quantity is conserved already off-shell and deserves definitely particular attention, as long as it carries curious information when observed at specific frame of reference.

In the second chapter, the physical interpretation to the associated current found previously is described, as well as the conditions under which it is possible. Using the field equations, it turns out that to the Noether charge can be given the interpretation of entropy of spacetime; in this way, the equations of motion are the expression of a balancing between entropies: on one side the variation of entropy associated to matter crossing an horizon, on the other side the variation of entropy of geometric origin.
This interpretation, indeed, is not a mere formality. The fact that to any horizon is associated black body radiation at a certain temperature strongly substantiates the idea that horizons, even if defined as purely geometric entities, are thermodynamic systems in every respect.

The conventional derivation of field equations does not take into account any thermodynamic feature of spacetime. Then, the considered observation suggests to try to change perspective and search for a different derivation, with thermodynamic aspects of spacetime taken as starting points. This is the content of chapter three. By considering gravity as a statistical phenomenon at fundamental level, one can attempt to derive the equations of motion by extremisation of a certain suitable entropy functional.

Chapter four explores the possible existence of a micro-structure of spacetime, or "atoms", in other words: this appears quite a natural ingredient of a statistical description of gravity. Such granularity of spacetime will be described and implemented through the introduction of a peculiar mathematical object, which acts as a very bizarre metric, able to give a finite distance between two events even in the limit in which the two events do coincide; this new effective metric describing this micro-dynamics curiously makes the canonical Einstein-Hilbert Lagrangian transmute into the entropy functional used in the statistical derivation.

All things considered, this thesis wants to introduce the values to the reader beneath such a new intriguing approach for gravity studies. All those features qualitatively described above are open doors and arise more questions to be deepened and explained accurately.

Contents

1 An Action Principle for Gravity

Generally, in mechanics and field theory, equations of motion are derived through an action $\mathcal{A}$, which is a functional defined in terms of the integral of a Lagrangian $\mathcal{L}$. In mechanics, the Lagrangian is a function depending on the dynamical variables $q(t)$ and their first order derivatives $\dot{q}(t)$, both functions of time

$$\mathcal{A}[q(t)] = \int_{t_1}^{t_2} \mathcal{L}[q(t), \dot{q}(t), t]dt$$

where t_1 and t_2 set an interval of time, whilst $q(t_1) = q_1$ and $q(t_2) = q_2$ are the fixed endpoints of the evolution process of the studied system. When the action is stationary, the result of above integral are second order field equation of motion.

This formalism naturally extends to field theories, and is used largely in Quantum Field Theory to study the forces of nature, that is electromagnetic interaction, strong and weak force. Unfortunately, several difficulties arise when gravity is involved.

In the standard theory of gravity, i.e. general relativity, one takes the spacetime metric $g_{ab}(x)$ as the fundamental dynamical variable. The effects of gravity on matter can be included, under the so-called minimal coupling (*i.e.* without direct coupling to curvature), changing the measure of the action from d^4x to $\sqrt{-g}d^4x$ and using covariant derivatives. Thus, the matter Lagrangian depends on certain variables collectively denoted by ϕ, their first order derivatives and the metric; in such a way its action is defined as

$$\mathcal{A}_m = \int d^4x\sqrt{-g}\mathcal{L}_m(\phi, D\phi; g_{ab}) \tag{1.1}$$

As for the dynamics of gravity itself, we can follow this scheme choosing as dynamical field the metric. One thing happens, however, which makes the gravitational case unique among all fields: if we try to construct a Lagrangian which depends on the metric and its first derivatives alone, we fail; the reason being that any such Lagrangian turns out to be a trivial constant.

To see this, consider a point P of a geodesic curve $\gamma(t)$, where t is the proper time along it. We can introduce a particular set of local coordinates adapted to a geodesics, also known as

Fermi normal coordinates; precisely, we choose $x^a = (t, x^1, x^2, x^3)$ such that, in a neighbourhood of P, when t is small, the set $(t, 0, 0, 0)$ represents the geodesics on P; moreover, the metric $g_{ab} \simeq \eta_{ab}$ and any Christoffel symbol vanishes. Consequently, around the whole line of γ a frame of reference in which the metric is flat can always been found.
It follows that if we build a scalar function in the form $f(g_{ab}, \partial_c g_{ab})$ and involving Fermi coordinates, it results that near P the function reduces to $f(\eta_{ab}, 0)$, so that it is a constant along any geodesics of spacetime, trivially.
Therefore if f is a constant, its value remains the same in any other coordinate system.

A way out is to add in the gravitational Lagrangian a dependence on second derivatives of the metric. It turns out that, if this dependence is linear, the terms with second derivatives, which are those potentially giving derivatives higher than two in the equations of motion, are confined in a total derivative term, which can be managed some way or another.

Therefore, we build the action

$$\mathcal{A}_g = \int d^4x \sqrt{-g} \mathcal{L}_g(g_{ab}, \partial_c g_{ab}, \partial^2 g_{ab}) \tag{1.2}$$

where we demand for linearity on second order derivatives.

The Ricci scalar is the only scalar built from the curvature tensor satisfying these requirements. Using this as Lagrangian, the action consists in a bulk part, which leads to the equations of motion, and a surface part. A way to eliminate the effects of the latter is to add to the Lagrangian a surface term (Gibbons-Hawking-York counterterm) involving a scalar function which compensates such a boundary integral and after the variation it takes that away.
Despite this procedure actually works, it appears quite peculiar when contrasted with the characteristic clean structure of Field Theory. In the following, we prefer to leave the original surface part in (1.2) as it is.

This will enable us to highlight that this surface part, which is, from the above, fundamentally thrown away in the conventional derivation of field equations, still turns out to be equivalent to the bulk part in its information content.

1.1 The Einstein-Hilbert Lagrangian

The Einstein-Hilbert action consists in the integral of the described Lagrangian, with a suitable normalization constant:

$$16\pi\kappa\, \mathcal{A}_{EH} = \int_V d^4x \sqrt{-g} \mathcal{L}_{EH} = \int_V d^4x \sqrt{-g} R \tag{1.3}$$

Explicitly,

$$\mathcal{L}_{EH} \equiv P_a^{\ bcd} R^a_{\ bcd} \tag{1.4}$$

where $P_a{}^{bcd}$ is a four-rank tensor constructed with only the metric, carrying all the same symmetries of the Riemann tensor

$$P_a{}^{bcd} = \frac{1}{2}(\delta^c_a g^{bd} - \delta^d_a g^{bc}) \tag{1.5}$$

Let us use now these symmetries properties of $P_a{}^{bcd}$ and express the Riemann tensor in terms of Christoffel symbols as

$$R^a{}_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^a_{ci}\Gamma^i_{db} - \Gamma^a_{di}\Gamma^i_{cb} \tag{1.6}$$

where the Christoffel symbols are written in terms of the metric derivatives in the form

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left[\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right] \tag{1.7}$$

Thus, the integrand in the considered action becomes

$$\sqrt{-g} P_a{}^{bcd} R^a{}_{bcd} = 2\sqrt{-g} P_a{}^{bcd} \Gamma^a_{ci}\Gamma^i_{db} + 2\partial_c \left[\sqrt{-g} P_a{}^{bcd} \Gamma^a_{db}\right] - 2\Gamma^a_{db}\partial_c \left[\sqrt{-g} P_a{}^{bcd}\right] \tag{1.8}$$

where we applied the product property of derivatives and use the anti-symmetry of the Riemann tensor in index exchange $R^a{}_{bcd} = -R^a{}_{bdc}$.
At this point, consider the partial derivative in the very last addend which can be expressed as

$$\partial_c[\sqrt{-g} P_a{}^{bcd}] = \sqrt{-g}[\partial_c P_a{}^{bcd} + \Gamma^i_{ci} P_a{}^{bcd}] \tag{1.9}$$

for which, with $g \equiv \det g_{ab}$, we have used

$$\Gamma^a_{ca} = \frac{1}{2} g^{ad} \partial_c g_{ad} = -\frac{1}{2g}\partial_c(-g) = \partial_c(\ln\sqrt{-g}) = \frac{1}{\sqrt{-g}}\partial_c(\sqrt{-g}) \tag{1.10}$$

The tensor $P_a{}^{abc}$ was said to be divergence-free, so from its covariant derivative, which is null, we can make explicit the partial derivative in terms of Christoffel symbols. Thus, writing

$$D_c P_a{}^{bcd} = \partial_c P_a{}^{bcd} + \Gamma^b_{ic} P_a{}^{icd} + \Gamma^c_{ic} P_a{}^{bid} - \Gamma^i_{ac} P_i{}^{bcd} = 0 \tag{1.11}$$

we find

$$\partial_c P_a{}^{bcd} = -\Gamma^b_{ic} P_a{}^{icd} - \Gamma^c_{ic} P_a{}^{bid} + \Gamma^i_{ac} P_i{}^{bcd} \tag{1.12}$$

Therefore, substituting it in (1.9) and multiplying both sides by Γ^a_{db}, only two terms remain:

$$\Gamma^a_{db}\partial_c[\sqrt{-g} P_a{}^{bcd}] = \sqrt{-g}[\Gamma^a_{db}\Gamma^i_{ac} P_i{}^{bcd} - \Gamma^a_{db}\Gamma^b_{ic} P_a{}^{icd}] = \sqrt{-g}[\Gamma^i_{db}\Gamma^a_{ic} P_a{}^{bcd} - \Gamma^a_{di}\Gamma^i_{bc} P_a{}^{bcd}] \tag{1.13}$$

Finally putting this into equation (1.8), we obtain the expression for the Lagrangian to be

$$\sqrt{-g}P_a^{\ bcd}R^a_{\ bcd} = 2\sqrt{-g}P_a^{\ bcd}\Gamma^a_{di}\Gamma^i_{bc} + 2\partial_c\left[\sqrt{-g}P_a^{\ bcd}\Gamma^a_{db}\right] \tag{1.14}$$

As a consequence, the only surviving Lagrangian terms defining the gravitational action are

$$\sqrt{-g}P_a^{\ bcd}R^a_{\ bcd} = \sqrt{-g}\mathcal{L}_{quad} + \mathcal{L}_{sur} \tag{1.15}$$

where we have separated the specific terms as we wanted:

$$\mathcal{L}_{quad} \equiv 2P_a^{\ bcd}\Gamma^a_{dk}\Gamma^k_{bc} \qquad \mathcal{L}_{sur} \equiv 2\partial_c\left[\sqrt{-g}P_a^{\ bcd}\Gamma^a_{bd}\right] \tag{1.16}$$

In front of this result, it is remarkable to highlight that the $\mathcal{L}_{quad}$ term is the bulk part of Einstein-Hilbert Lagrangian, the one giving back the equations of motion, and the term related to the surface enclosing it is $\mathcal{L}_{sur}$, that appears in form of a total derivative which can be treated in several ways.
Thus, if we rewrite $\sqrt{-g}\mathcal{L}_{quad} = \sqrt{-g}R - \mathcal{L}_{sur}$, the complete action to be varied at the end is in the form

$$\mathcal{A}_{tot} = \mathcal{A}_{quad} + \mathcal{A}_m = \frac{1}{16\pi\kappa}\int_V d^4x\sqrt{-g}\mathcal{L}_{quad}(g,\partial g) + \int_V d^4x\sqrt{-g}\mathcal{L}_m(\phi,\nabla\phi;g) \tag{1.17}$$

Nevertheless, although the surface part does not participate in furnishing the final field equation, the bulk and surface Lagrangians are two quantities that share a very deep relationship beneath that will be clearer later. In the following subsection it will be performed the variation of the gravitational action respect to the metric, whilst it will be shown later an equivalent method of variation involving a new set of coordinates, through which such a link between $\mathcal{L}_{quad}$ and $\mathcal{L}_{sur}$ gets evident.

1.1.1 Variation of Gravitational Action

As shown in the previous calculations, it is possible to split the Einstein-Hilbert Lagrangian putting on evidence the quadratic term represented by $\mathcal{L}_{quad}$. This piece of action is actually the part which gives the equations of motion when the total action is varied. In this section we calculate the variation with respect to the metric g_{ab} of the quadratic part first, later the surface term, which will be set to vanish opportunely at the boundary, that is asking for $\delta g_{ab} = 0$ on it.

Gravity Term Let us start with the Einstein-Hilbert Lagrangian term initially, thus

$$\delta(\sqrt{-g}R) = \delta(\sqrt{-g})R + \sqrt{-g}\delta R \tag{1.18}$$

The Riemann Tensor variation takes the form in covariant derivatives

$$\delta R^a_{bcd} = D_c(\delta\Gamma^a_{db}) - D_d(\delta\Gamma^a_{cb}) \tag{1.19}$$

noticing that

$$D_b(\delta\Gamma^a_{cd}) = \partial_b(\delta\Gamma^a_{cd}) + \Gamma^a_{mc}\delta\Gamma^m_{cd} - \Gamma^m_{cb}\delta\Gamma^a_{md} - \Gamma^m_{db}\delta\Gamma^a_{cm} \tag{1.20}$$

Therefore the Ricci tensor is found contracting two indexes in the Riemann tensor $\delta R_{ab} = \delta R^c_{acb}$, thus the Ricci scalar variation is

$$\delta R = R_{ab}\delta g^{ab} + D_a(g^{bc}\delta\Gamma^a_{bc} - g^{ba}\delta\Gamma^d_{db}) \tag{1.21}$$

where the last four-divergence does not vanish in the action integral, even invoking the Stokes Theorem.
Fortunately in our case, this is not needed because it naturally cancels out with a term appearing in the variation of the surface Lagrangian. For convenience, such a term can be rewritten using the relation

$$g^{ab}\delta R_{ab} = \frac{1}{\sqrt{-g}}\partial_a\left[\sqrt{-g}(g^{bc}\delta\Gamma^a_{bc} - g^{ba}\delta\Gamma^d_{db})\right] = \frac{2}{\sqrt{-g}}\partial_c(\sqrt{-g}g^{bk}P_{ak}{}^{cd}\delta\Gamma^a_{bd}) \tag{1.22}$$

The metric determinant variation, instead, is calculated as follows. First,

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}(g^{ab}\delta g_{ab}) \tag{1.23}$$

where $\delta g = g g^{ab}\delta g_{ab}$ due to the Jacobi formula. Hence,

$$\frac{1}{\sqrt{-g}}\delta\sqrt{-g} = -\frac{1}{2}g_{ab}\delta g^{ab} \tag{1.24}$$

Putting the last results into the action integral defined in (1.2), the expression for it becomes

$$\delta A_g = \frac{1}{16\pi\kappa}\int_V d^4x\ \sqrt{-g}\left[\left(R_{ab} - \frac{1}{2}g_{ab}R\right)\delta g^{ab} + 2\partial_c\left(\sqrt{-g}g^{bk}P_{ak}{}^{cd}\delta\Gamma^a_{bd}\right)\right] \tag{1.25}$$

As expected, the geometry part of equation of motion is contained in this term in the form of the Einstein tensor $2G_{ab} = 2R_{ab} - g_{ab}R = 0$. The second addend, instead, will be subtracted by a term appearing in the variation of the surface Lagrangian.

Surface Term The surface term variation can be computed starting from

$$\delta\mathcal{L}_{sur} = \partial_c \left[2P_{ak}{}^{cd}\Gamma^a_{bd}\delta(\sqrt{-g}g^{bk}) + \sqrt{-g}g^{bk}\delta\Gamma^a_{bd})\right] \tag{1.26}$$

As anticipated before, the second addend is the one that cancels out the term in the variation of the geometric part. The first addend, instead, involves the variation

$$\delta(\sqrt{-g}g^{bk}) = \sqrt{-g}\ \delta g^{ij}\left(\delta^b_i\delta^k_j - \frac{1}{2}g^{bk}g_{ij}\right) \tag{1.27}$$

Thus

$$\delta\mathcal{L}_{sur} = \partial_c \left[2\sqrt{-g}P_{ak}{}^{cd}\Gamma^a_{bd}\left(\delta^b_i\delta^k_j - \frac{1}{2}g^{bk}g_{ij}\right)\delta g^{ij}\right] + 2\partial_c\left(\sqrt{-g}g^{bk}P_{ak}{}^{cd}\delta\Gamma^a_{bd}\right) \tag{1.28}$$

We are ready now to put all pieces together and find the variation of the quadratic action, according to

$$\delta\mathcal{A}_{quad} = \delta\mathcal{A}_g - \delta\mathcal{A}_{sur} \tag{1.29}$$

Now, (1.28) is integrated and subtracted from (1.25) in order to have

$$\delta\mathcal{A}_{quad} = \frac{1}{16\pi\kappa}\int_V d^4x\ \sqrt{-g}\left[\left(R_{ab} - \frac{1}{2}g_{ab}R\right)\delta g^{ab} - \partial_c\left(\sqrt{-g}M^c_{ij}\delta g^{ij}\right)\right] \tag{1.30}$$

in which a new three-rank object M^c_{ij} collecting all the residue terms was defined:

$$M^c_{ij} \equiv 2P_{ak}{}^{cd}\Gamma^a_{bd}\left(\delta^b_i\delta^k_j - \frac{1}{2}g^{bk}g_{ij}\right) \tag{1.31}$$

Therefore, the Gauss theorem can be applied in order to obtain a surface integral. It gives the result

$$\frac{1}{16\pi\kappa}\int_{\partial V} d^3x\ \sqrt{h}n_c\ M^c_{ij}\delta g^{ij} \tag{1.32}$$

where n_c is a unit vector normal to the surface ∂V. This integral vanishes demanding that $\delta g^{ij} = 0$ at the boundary. Finally, we have obtained the expression for the quadratic action returning the geometric part of the classical Einstein's field equations.

1.1.2 Discovery of a Holographic Principle

We present in here an equivalent — but more meaningful — procedure in varying the gravitational action, in which it emerges an interesting principle that ties intrinsically the bulk with the surface term.

In analogy to the fact that a generic Lagrangian $L_q(\dot{q}, q)$ can give back the Euler-Lagrange equations by varying the coordinate q in the action and demanding $\delta q = 0$ on the boundary, a new Lagrangian $L_p(\ddot{q}, \dot{q}, q)$ can be defined as

$$L_p(\ddot{q}, \dot{q}, q) = L_q(\dot{q}, q) - \frac{d}{dt}\left(q\frac{\partial L_q}{\partial \dot{q}}\right) \tag{1.33}$$

However, we demand in here that the variation of the momentum $p = \partial L_q/\partial \dot{q}$ vanishes at the boundary, instead of δq. As we will show, the equations of motion arise even under this condition, thanks to linearity on second derivatives.

To see this, let us consider the action integral of such a defined Lagrangian and vary it respect to q

$$\delta A_p = \delta \int dt L_q(\dot{q}, q) - \delta \int dt \frac{d}{dt}\ q\ p(\dot{q}, q) \tag{1.34}$$

The first addend leads to the canonical Euler-Lagrangian field equations of motion, whilst the other one

$$\delta A_p = \int dt \left[\frac{\delta L_q}{\delta q} - \frac{d}{dt}\frac{\delta L_q}{\delta \dot{q}}\right]\delta q - \left.(q\ \delta p)\right|_{t_1}^{t_2} \tag{1.35}$$

Thus, on top of the bulk Euler-Lagrangian equations of motion one must also impose $\delta A_p = 0$, which is obtained by asking that $\delta p = 0$.

In a similar way, the Lagrangian for gravity can be rewritten as

$$\sqrt{-g}R = \sqrt{-g}\mathcal{L}_{quad} - \partial_c\left[g_{ab}\frac{\partial\sqrt{-g}\mathcal{L}_{quad}}{\partial(\partial_c g_{ab})}\right] \tag{1.36}$$

and comparing it with (1.15), it is evident that

$$\mathcal{L}_{sur} = -\partial_c\left[g_{ab}\frac{\partial\sqrt{-g}\mathcal{L}_{quad}}{\partial(\partial_c g_{ab})}\right] \tag{1.37}$$

The above solution is not just a casual quantity, but it consists in the so-called *holographic principle*. Actually, the relationship between surface and bulk is of holographic nature, which means that the two entities share information and, in this way, they are tightly bound to each

other.
In fact, considering physical quantities like entropy, in example, it is possible to show that such information is translated into "degrees of freedom": the entropy of a three-dimensional region of bulk express the same number of degrees, when the system is at equilibrium, of the two-dimensional surface enclosing it, on which one has a definition of "horizon entropy". [5]
In fact, this rescaling in information quantity between the term containing the equations of motion and the surface part, which is disregarded always, appears naturally for gravity in the canonical derivation, so that it deserves a proper path of research around which it might be possible to discover and develop new understandings of gravity.

Turning back our calculation, we can identify a sort of "gravitational momentum" inside the square brackets that collects all the terms led to vanish at the boundary, once the variation is carried, as argued in the following section.
We try now to rewrite the (1.16) by defining a proper pair of coordinates.

The quadratic Lagrangian can be expressed in terms of derivatives of the metric by substituting the Christoffel symbols with their definition, simply

$$\mathcal{L}_{quad} = 2P_a^{\ bcd}\Gamma^a_{dk}\Gamma^k_{bc} \equiv \frac{1}{4}M^{abcijk}\partial_a g_{bc}\partial_i g_{jk} \tag{1.38}$$

where

$$M^{abcijk} = g^{ai}g^{bc}g^{jk} - g^{ai}g^{bj}g^{ck} + 2g^{ak}g^{bj}g^{ci} - 2g^{ak}g^{bc}g^{ij} \tag{1.39}$$

The surface Lagrangian, instead, can be redefined through a vector V^c as

$$\mathcal{L}_{sur} = 2\partial_c\left[\sqrt{-g}g^{bk}P_{ak}^{\ \ cd}\Gamma^a_{bd}\right] \equiv \partial_c[\sqrt{-g}V^c] \tag{1.40}$$

At this point, in the same spirit of the previous calculation, we vary each term and sum them according to

$$\delta\mathcal{A}_g = \delta\mathcal{A}_{quad} + \delta\mathcal{A}_{sur} \tag{1.41}$$

so that one obtains

$$\delta(\sqrt{-g}R) = \delta(\sqrt{-g}L_{quad}) + \delta\left[\partial_c(\sqrt{-g}V^c)\right] = -\sqrt{-g}G_{ab}\delta g^{ab} - \partial_c[g_{ik}\delta(\sqrt{-g}M^{cij})] \tag{1.42}$$

The quadratic part remains the same, but the variation of the surface one, instead, leads to consider a new object M^{cij} such that $M^c_{ik}g^{ik} = -V^c$, thus

$$\delta\left[\partial_c(\sqrt{-g}V^c)\right] = -\partial_c[g_{ik}\delta(\sqrt{-g}M^{cij})] \tag{1.43}$$

With more attention, M^c_{ij} is actually the same object introduced previously in definition (1.31) that now takes the form

$$M^{cik} = g^{il}g^{km}\Gamma^{c}_{lm} - g^{il}g^{ck}\Gamma^{d}_{ld} - \frac{1}{2}g^{ik}V^{c} \tag{1.44}$$

However, if before it was introduced as a cumulative expression of all the multiplicative terms of δg^{ik}, it appears more evident here that it consists in the *canonical momentum* for gravity associated to the field coordinate g_{ab}.
Indeed, one has that the object M^{cik} is defined through the functional derivative of the quadratic Lagrangian, respect to the derivative of the field coordinate g_{ab}, as it appears in (1.37):

$$M^{cik} \equiv \frac{\partial\sqrt{-g}\mathcal{L}_{quad}}{\partial(\partial_c g_{ik})} = \frac{1}{2}M^{cikpqr}\partial_p g_{qr} \tag{1.45}$$

Finally, the integration can be done and, invoking the Gauss theorem, the surface integral becomes

$$\frac{1}{16\pi\kappa}\int_{\partial V} d^3x \; g_{ab}\delta(\sqrt{-g}M^{0ab}) \tag{1.46}$$

where it runs over the the surface ∂V at $t = const.$ The integral vanishes by asking $\sqrt{-g}M^{0ab}$ to be fixed at the boundary.

All this was discussed to put on evidence that, despite a Lagrangian depending on second derivatives of the metric was used, a conjugate momentum related to g_{ab} exists and it carries all the other remaining terms: thanks to linearity on second order, such annoying terms are summed up into the definition of a canonical momentum that can be easily confined at the boundary, thereof. This procedure does not affect the quadratic part and the bulk term remains as the only source of the geometric part of field equations, still in terms of G_{ab}.

In addition, the canonical procedure shows the natural presence of holography between the two terms of the Einstein-Hilbert Lagrangian and it is worth to cross this door, since new features and properties of physical relevance might be exemplary in letting know more about gravitational force at fundamental level.

1.2 The Noether Current

Before proceeding further in deriving explicitly the equations of motion, we want to show first that a gravitational Noether current exists whose nature is purely geometric: this relevant aspect descends from the fact that R is a scalar. In addition, this result is worth specific attention because it is obtained without considering any equation of motion, then such an object is conserved upstream.

A fast way to calculate it can be pursued when the action is varied with respect to coordinate transformations of the kind $x^a \to x^a - \xi^a$. The starting point is a generic scalar action (1.2), from which we write below:

$$\delta(\sqrt{-g}\mathcal{L}_g) = \sqrt{-g}\left[E_{ab}\delta g^{ab} + D_a(g^{bc}\delta\Gamma^a_{bc} - g^{ba}\delta\Gamma^d_{db})\right] \tag{1.47}$$

In the left-hand side, a divergence-free behaviour, which is intrinsically satisfied, can be put under evidence whenever ξ^a vanishes at the boundary. The local variation of the metric under the infinitesimal coordinate transformation $x^a \to x^a - \xi^a$ is given by $\delta g_{ab} = D_a\xi_b + D_b\xi_a$. Similarly, the variation of a scalar object, like $\mathcal{L}_g$ is given by just the transport term, $\delta\mathcal{L}_g = \xi^a\partial_a\mathcal{L}_g$. Substituting them into the action one finds

$$\delta\mathcal{A}_g = \int d^4x\ \delta(\sqrt{-g}\mathcal{L}_g) = \int d^4x\ [\frac{\sqrt{-g}}{2}g_{ab}(D^a\xi^b + D^b\xi^a)\mathcal{L}_g + \sqrt{-g}\xi^a\partial_a\mathcal{L}_g] \tag{1.48}$$

Contracting the indexes one finds

$$\delta A_g = \int d^4x\sqrt{-g}\left((D_a\xi^a)\mathcal{L}_g + \xi^a\partial_a\mathcal{L}_g\right) = \int d^4x\sqrt{-g}D_a(\xi^a\mathcal{L}_g) \tag{1.49}$$

However, if one demands that ξ^a vanishes at the boundary of the integration, this full variation vanishes, $\delta A_g = 0$. Considering now the arbitrariness of the infinitesimal vector field ξ^a one finally gets

$$D_a(\mathcal{L}_g\xi^a) = 0 \tag{1.50}$$

and the vector $P^a \equiv \mathcal{L}_g\xi^a$ is conserved, without the use of any equation of motions. This is true for any scalar. Hence, $\delta(\sqrt{-g}\mathcal{L}_g) = \sqrt{-g}D_a(\mathcal{L}_g\xi^a) = \partial_a(\sqrt{-g}\mathcal{L}_g\xi^a)$.

Returning to the explicit expression (1.47), about the side on the right, and using the expression of δg^{ab} as before, the first addend becomes $2E_{ab}D^a\xi^b$, so $D^a(2E_{ab}\xi^b)$. In the second addend, instead, just call $\delta v^a = g^{bc}\delta\Gamma^a_{bc} - g^{ba}\delta\Gamma^d_{db}$. The equation becomes

$$D_a(\mathcal{L}_g\xi^a + 2E^a_b\xi^b + \delta_\xi v^a) := D_aJ^a = 0 \tag{1.51}$$

where δ_ξ was written to remark that is the boundary term which arises due to such a changing of coordinates. The current J^a is actually a Noether current, it satisfies a continuity equation and it is conserved *off-shell*.

1.3 Variational Principle for Matter

In general, a suitable Lagrangian for matter can be defined and coupled to gravity with the method of the minimal coupling mentioned earlier. From it a generic second rank tensor F_{ab}, which is symmetric and possesses specific properties, can be derived. We will prove that physically, it is identified with the energy-momentum tensor carrying the density of energy and

momentum T_{ab} of the matter in the studied system. Moreover, it consists in the source of the field equation in gravity and it is required to be divergence-free $D_a T^{ab} = 0$.
Recall that we are searching for something in the form

$$\mathcal{A}_m = \int d^4x \sqrt{-g} \mathcal{L}_m(\phi, D_a\phi; g_{ab}) \tag{1.52}$$

where ϕ is a generic field. The action integral variation respect to the metric is easily computed as

$$\delta\mathcal{A}_m = \int_V d^4x \, \sqrt{-g} \left(\frac{\delta\mathcal{L}_m}{\delta g^{ab}} - \frac{1}{2} g_{ab}\mathcal{L}_m \right) \delta g^{ab} \tag{1.53}$$

Now, using the expression for the total action (1.17) and setting it to be null, putting on evidence a factor 1/2, the resulting equations of motion arise:

$$\frac{1}{8\pi\kappa} \left(R_{ab} - \frac{1}{2} g_{ab} R \right) = 2\frac{\delta\mathcal{L}_m}{\delta g^{ab}} - g_{ab}\mathcal{L}_m := F_{ab} \tag{1.54}$$

First of all, it can be shown that the tensor F_{ab} actually acts as a energy-momentum tensor: *i.e.* considering the Lagrangian of a generic scalar field ϕ

$$\mathcal{L}_\phi = \frac{1}{2} g^{ab} \partial_a\phi \partial_b\phi - \frac{1}{2} m^2 \phi^2$$

and calculating the tensor defined above it results $F_{ab} \equiv T_{ab}[\phi]$, with

$$T_{ab}[\phi] = \partial_a\phi\partial_b\phi - g_{ab}\mathcal{L}_\phi$$

In analogy to (1.49), the divergence-free property, instead, must be intrinsically satisfied, according to the invariance of the field equations under generic transformations of coordinates such as $x^a \to x^a - \xi^a$, with $\delta g^{ik} = D^i\xi^k + D^k\xi^i$ and ξ^a null at the boundary. It remains

$$\delta A_m = \int d^4x \sqrt{-g} \; D_a(\mathcal{L}_m \xi^a) = 0 \tag{1.55}$$

if the equation of motion of the matter fields are used (their variations needed to verify that $\mathcal{L}_m$ is a scalar do not contribute as they just the equations of motion). The four-vector $P^a \equiv \mathcal{L}_m \xi^a$ is conserved and it is possible to build $P^a \equiv T^{ab}\xi_b$, which is conserved along the geodesic identified by the killing vector.

In few words, the integral (1.53) under such a coordinate transformation can be rewritten as

$$\delta\mathcal{A}_m = \frac{1}{2} \int d^4x \, \sqrt{-g} \; T_{ab}(D^a\xi^b + D^b\xi^a) = \int d^4x \, \sqrt{-g}(D^a T_{ab})\xi^b \tag{1.56}$$

where the total derivative term vanishes at the boundary requiring ξ^a to be null in there. When $\delta\mathcal{A}_m = 0$, the energy-momentum tensors is conserved: $D_a T^{ab} = 0$.

1.4 Gravitational Field Equations

The total action now has all the ingredients both for matter both for gravity to be varied and give back the well-known Einstein's equations of motion, indeed,

$$\delta\mathcal{A}_{tot} = \int_V d^4x\sqrt{-g}\left\{\frac{1}{16\pi\kappa}G_{ik} - \frac{1}{2}T_{ik}\right\}\delta g^{ik} = 0 \tag{1.57}$$

where T_{ik} is actually the source of the gravitational field. Consequently, when matter is not present, by demanding that δg^{ik} is null on the boundary ∂V, the above integral vanishes and the only surviving term is the one that gives the geometry part of the equation of motion, when $\delta\mathcal{A}_{quad} = 0$.
The inclusion of a matter term as in (1.17) implies the classical Einstein field equation are found.

Restoring the definition of G_{ab} in terms of Riemann and Ricci tensors and integrating it, we get

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi\kappa T_{ab} \tag{1.58}$$

Notice that the above solution was chosen with a null cosmological constant $\Lambda = 0$.
Alternatively, we could have consider it into the initial Lagrangian without affecting the equations of motion solution; indeed, this is equivalent to add a term on the right hand side, interpreted as a negative pressure $p = -\rho$.

All things considered, these expected field equations are derived only from a part of the Hilbert action, whilst the rest needs careful considerations in order to make it vanish. Of course, it is a quantity of physical relevance as well, because it was seen that it shares information content with the bulk counterpart according to a holographic principle.
However, in the previous section it was discovered thus such an unwanted surface term appears even in the expression of the Noether current as $\delta_\xi v$, so that the integration of its four-divergence in the 4-dimensional space, makes it assume an interesting signification on-shell. Indeed,

$$\int_V d^4x\ \sqrt{-g}D_a J^a = \int_{\partial V} d^3x\ \sqrt{h}\ n_a J^a \tag{1.59}$$

The surface term $\delta_\xi v^a$ vanishes if ξ^a is a Killing vector. Therefore, it is the element $n_a J^a$ which carries a conserved quantity on the surface, with particularly interesting interpretations, as a energy, then an entropy, described in details in the following chapter.

2 Gravity as a Thermodynamic Entity

The previous chapter introduced a mathematical description of gravity, formally based on an action principle formalism. We saw that a Lagrangian can be properly found in order to have the exact equations of motion of the General Relativity Theory.
The cost of it was the presence of a surface integral term which needed precise considerations in order to let it vanish. The associated Noether current, however, reminds interesting features once the dynamics is studied on the boundary, implications that now we propose to discuss in here.

2.1 Heat Densities at Equilibrium

As primary step, let us start exploring deeper the terms appearing in the current J^a, in order to give a precise meaning, of physical relevance, to each one and the contest those are realised.

Consider the Noether current J^a, satisfying the continuity equation (1.51). It is important to recall that its form

$$J^a[\xi] = \mathcal{L}_g \xi^a + 2E^a_b \xi^b + \delta_\xi v^a \tag{2.1}$$

contains terms which were residues of the appearing surface part in the gravity Lagrangian of Einstein-Hilbert. Moreover, each addend shows a four-vector field ξ^a of deep relevance when chosen properly, as we will notice soon. This quantity is *off-shell* conserved; consequently, we want to evaluate what happens at equilibrium, that is when equations of motion are put into the above expression.

Firstly, as already anticipated in section (1.2), the term $\delta_\xi v^a$ disappears if ξ^a is a Killing vector, in example, with positive norm.
This is true because J^a can be computed for any kind of Lagrangian and any vector field ξ^a, so that the term δv^a takes the general form

$$\delta v^a = \frac{1}{2} A^{a(bc)} \delta g_{bc} + \frac{1}{2} B_d^{a(bc)} \delta \Gamma^d_{bc} \tag{2.2}$$

Thus, if the variation is carried respect to a Killing vector, $\delta \to \delta_\xi$ and the first addend is null obviously because $\delta_\xi g_{ab} = D_a \xi_b + D_b \xi_a = 0$; the last one is also null for the same reason, due to the fact that the affine connection is nothing but the sum of terms involving derivatives of the metric.

A second observation follows straightforwardly. Indeed, in absence of matter, when equations of motion $E^b_a \equiv G^b_a = 0$ hold, the Noether current (2.1) reduces to the form $J^a = 2R_{ab}\xi^b$, whose physical interpretation is still obscure. However, it can be pointed out that its nature is purely geometrical and such a characteristic descends from the expression of the Hilbert Lagrangian, that is built itself through geometrical mathematical considerations.

Therefore, we introduce in here a further vector field n_a and the condition $n_a \xi^a = 0$, valid in a certain region around an event of the spacetime. By contraction of it with the Noether current, the term involving the Ricci scalar disappears under the mentioned condition and it remains

$$n_a J^a = 2G^a_b \; n_a \xi^b \tag{2.3}$$

Considering now the Einstein's equations including the matter counterpart $2G_{ab} = 2R_{ab} - g_{ab}R = T_{ab}$, instead, the expression of the above result leads to

$$n_a J^a = T^a_b \; n_a \xi^b \tag{2.4}$$

If one pays attention to the very last solution, he notices that it was discussed in section (1.3) how $T^a_b \xi^b$ was a conserved quantity along the geodesics reproduced by the Killing vector ξ^a.
Actually, such a term is a known quantity in physics and it really represents a heat density of massive origin, so, even for dimensional reasons, we are allowed to start thinking that the object $R_{ab}\xi^a$ is a kind of heat density of geometrical nature, which is conserved as well as the one of matter.

Around at any spacetime event point it can be always identified a surface that surrounds it and this can be mathematically done keeping a coordinate as constant. In this contest, it is obvious to select hypersurfaces at constant time $t = const$, so that, if we chose a time-like Killing vector along the time coordinate, the condition $\xi^a \xi_a = 0$ holds onto the hypersurface.
Consequently, the presence of a second arbitrary vector field suggests using it to identify clearly the surface region around the event, thus $n_a = N\xi_a$ can be thought as the normal vector of the surface at t constant; observing the dynamics of the system onto it, the term $R\xi^a \xi_a$ is null and

$$n_a J^a = 2NR_{ab}\xi^a \xi^b = NT_{ab}\xi^a \xi^b \tag{2.5}$$

Under this spark, there is no reason to wonder if also the geometric term is a energy density anymore. In addition, it seems that the system, at equilibrium, obeys at a certain balance law around the surface, between the energy of matter and the one carried by the geometry of spacetime.

Nevertheless, we remark that such an interpretation was possible only by the presence of matter in the field equations.

Hence, if J^a carries a energy density, the quantity $R_{ab}\xi^a\xi^b$ should represent a formal density of gravitational heat, per unit time, crossing the surface $t = const.$
That can be seen even in terms of a flux, as we will study later, in the integral form

$$\int_{\partial V} d^3x \ \sqrt{h} \ 2NR_{ab}\xi^a\xi^b = \int_{\partial V} d^3x \ \sqrt{h} \ NT_{ab}\xi^a\xi^b \tag{2.6}$$

where $n^a = N\xi^a$ represents the unit normal to the surface ∂V, defined parallel to a time-like Killing vector ξ^a.

In order to verify whether this mathematical result might have a possible relevant physical interpretation, it is mandatory to find suitable conditions in which matter goes missing beyond a horizon.
The next section will remind the physics of accelerating frames of reference and it will focus on the main quantities that Rindler observers are able to measure.

2.2 Rindler Horizons

The choice of a Rindler frame of reference is very helpful in giving an elegant and straight thermodynamic interpretation for the gravitational field equations [4]. The power of it lays in the fact that such a type of frame can be found at any point in space time, even in flat spacetime, regardless of the presence of gravity. It is a key example that could let understand better the physics behind all this dissertation, given by a particular class of observers described in a Rindler metric. In here thermodynamics emerges since a horizon, that is a boundary of a certain region of spacetime, appears hot according to some accelerated observers [7].

In 1995, it was showed by Jacobson that the Einstein's equation could be derived from a thermodynamic approach.
One considers a generic spacetime horizon as a null hypersurface which acts as a causality barrier, separating the outside world from the inside one containing the degrees of freedom of the system. The main curiosity is that such horizons hide information from the outside observers, thus it is natural to think that the lack of information is a kind of entropy. Such a quantity is related to an "entanglement entropy" whose scale is infinite in Quantum Field Theory, but a finite cut-off length L_c, in case it exists, makes the entanglement entropy be proportionally to it quadratically L_c^2.

Around any spacetime event enclosed by a 2D surface, it is possible to find a local flat region: we identify in this neighbourhood a *local Rindler frame* (LRF), moving with acceleration a, and the bi-dimensional surface as the *local Rindler horizon*; the system is said in "local equilibrium" inside the shell region. In here, Poincaré symmetries are preserved, so a vector killing representing generic boosts ξ^a exists, vanishing onto the horizon. A energy density crossing the horizon is equivalent to a heat flow δQ and the entropy is proportional to this by a certain temperature $T = a\eta/2\pi$, associated to the surface, in the usual form $\delta Q = TdS$.
The LRF interprets all the matter energy as heat, once that has crossed the horizon. However, a free falling observer do not experience all this.

For a LRF, with ξ^a pointing to the selected event, the current for matter $J^a_m = T^{ab}\xi_b$ crossing the horizon generates the energy density flux $T_{ab}\xi^a\xi^b$, from which the heat flow is the integral

$$\delta Q_m = \int_H T_{ab}\xi^a d\Sigma^b = -a\int_H \lambda T_{ab}k^a k^b d\lambda dA \tag{2.7}$$

The choice is $\xi^a = -a\lambda k^a$, with k^a tangent to the horizon H and λ a parameter being null on it, where dA is the area element. Recall that $T_{ab}\xi^a$ is a conserved quantity, as it was shown in section (1.3). Hence, the entropy exchange related to some matter entering the horizon is trivially $T_{loc}\delta S_m = \delta Q_m$, with $T_{loc} = a\eta_R/2\pi$ and η_R is the scale length at which the observer is situated.

Recall now what stated before about gravity; analogously to the matter quantity, if the current for gravity $J^a[\xi] = \mathcal{L}_{EH}\xi^a + 2G^a_b\xi^b$ carries a formal energy density which is conserved on-shell according to Noether theorem, it is possible to write

$$\delta Q_g = -aN\int_H \lambda\left(\mathcal{L}_{EH}k^a k_a + 2G_{ab}k^a k^b\right)d\lambda dA \equiv -aN\int_H \lambda 2R_{ab}k^a k^b d\lambda dA \tag{2.8}$$

when $G_{ab} = 0$. Hence, the associated entropy results to be $T\delta S_g = \delta Q_g$ with $T = a\eta_c/2\pi$.

At this point, if an amount Q_m has crossed the horizon, this is enlarged by the same quantity counted as Q_g, thus the Rindler observer has to impose that the two entropies must be equal, so

$$\left(\frac{2\pi}{\eta_c}\right)N\int_H \lambda 2R_{ab}k^a k^b d\lambda dA = \left(\frac{2\pi}{\eta_R}\right)\int_H \lambda T_{ab}k^a k^b d\lambda dA \tag{2.9}$$

In the coincidence limit $N\eta_R \to \eta_c \sim \eta$, in other words, when the two length scales become comparable at the horizon — *viz.* the Planck scale —, one prepares the integrand in the form

$$(2R_{ab} - T_{ab})k^a k^b = 0 \tag{2.10}$$

which holds for any arbitrary null vector k^a at any point in spacetime, so the result is independent from the specific notion carried by our specific observer. After integration the Einstein field equations emerge

$$2G_{ab} - T_{ab} = \Lambda g_{ab} \tag{2.11}$$

with the integration constant Λ. Inevitably, the equations of motion arose clearly for our Rindler observer.
At any point in spacetime, a local Rindler observer can be found always, then it can be used to study physically how matter couples with the gravitational field. However, under this view, it is relevant to remark how this link is strictly observer dependent, that is there are other frames of reference in which thermodynamics is not perceived.

At this point it is mandatory to notice that the entire above dissertation is not solely a theoretical powerful description. Certainly, a physical application can be treated when considering

black holes. Their horizons are perfectly causal barriers and, when some matter enters it, the associated horizon entropy increases.
The temperature associated to the horizon is in this case the particular Hawking temperature $T_H = \kappa/2\pi$, proportional to κ, the *superficial gravity* of the black hole: the last one is the measures of the acceleration of a near-horizon observer, so it can be identified with $\kappa \equiv a\eta$.
Now, from $T = a\eta/2\pi$, if $\eta \sim \hbar/ck_B$, one recovers exactly the expression of the real Hawking temperature

$$T_H = \frac{\kappa}{2\pi} = \frac{\hbar c^3}{8\pi GMk_B} \tag{2.12}$$

where $a = c^4/4GM$ and M is the black hole's mass, according to the Schwarzshild solution, k_B is the Boltzmann constant and G the gravitational constant: what it was just found is therefore a magnificent reality which lets state that the element $R_{ab}\xi^a\xi^b$ is definitely a heat energy density rate of gravitational contribution.

All things considered, the presence of matter, which possesses all the characteristics of thermodynamic observables, allows us to start thinking that the formal expression of the gravity entropy itself alone assumes a physical thermodynamic reality onto horizons. The existence of special observers constitutes a proof. However, in this contest the interpretation presented in (2.9) is possible if, and only if, the field equations $G_{ab} = T_{ab}$ are valid, exclusively because geometric quantities equate matter ones.

2.3 A Different Perspective

Geometry of spacetime has at this point a new face of deep relevance which deserves all the attention. The entropy associated to horizons is not just a formality anymore because, in a certain way, it compensates the flux of matter crossing this, so that it must be an existent physical thermodynamic quantity.
The proof is not only theoretical when considering generic horizons, but it is also physical if we put an observer under special frame of reference, *viz.* a local Rindler frame.

Around at any event, null surfaces can always be found – *i.e.* considering a light-like killing vector ξ^a– so that even observers, who see them heated up by crossing matter, exist. However, in our theory, the new equations of motion in the form

$$G_{ab}\xi^a\xi^b = kT_{ab}\xi^a\xi^b \tag{2.13}$$

fulfil a "zero-dissipation" principle such that the respective heat quantities satisfy $\mathcal{H}_g = \mathcal{H}_m$ (or $\delta Q_g = \delta Q_m$ equivalently) on the boundary, then it follows that the heating contribute of spacetime micro-structure contrasts the one due to matter.
Certainly, the appearing vector fields ξ^a need more care in giving a justification about their existence.

On the contrary, classically, the well known elegant field equations appear in the form

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = k T_{ab} \tag{2.14}$$

On the left side, the geometry description with the Ricci tensor and the scalar representing the curvature of spacetime; on the right one, instead, the energy-momentum tensor of matter as the source of gravity.

Firstly, the presence of the Ricci scalar in the Lagrangian is a requirement following from the equivalence principle, according to which it is always possible to find, at any point of spacetime, a reference frame which is locally Minkowskian, so that the Special Relativity laws are applicable to it. Indeed, it was noticed that the gravitational effects could be mimicked by an opportune accelerated frame, so that such an observer would not have experimented locally the presence of a gravitational force.
Beneath this amazing feature, there is the astonishing suggestion that gravity has geometrical properties, that is its effects can be described by the curvature of the spacetime background. It follows a mathematical formalism, based on differential geometry, that leads to an expression of the action involving the Ricci scalar, with all the discussions carried in the previous chapter.
As far as we are concerned, this argumentation is still present even in the derivation of (2.13) since the Noether current J^a has shown to be itself of geometrical nature intrinsically.

Secondly, the source of curvature is found to be the matter; in the Newtonian limit, the 00-component of the momentum-energy tensor T_{ab} satisfies the Poisson equation for a gravitational scalar field ϕ

$$\bigtriangledown^2 \phi \propto T_{00} = k\rho$$

Hence, in accordance with the equivalence principle, gravity is tied with a geometric description in both cases it is represented in a scalar or a tensor form. Thus geometric characteristics are preserved even in the field equations in scalar form as in (2.13).
After one hundred years, these powerful characteristics make Einstein's General Relativity acknowledged to be the "master" theory which describes gravity, enforced by the large amount of experimental proves that have been collected during all these years, day by day.

Nevertheless, observing the ordinary form (2.14), on the one hand the interpretation is related to the topology of spacetime, to the energy density of matter on the other hand: no one can state that these two entities are the same thing apparently. In the same way, Padmanabhan writes, in one of his articles [2], the tensor Einstein's result equates "apples and oranges". Thus one can accept that these two fruits are equal for the sake of mathematics and discard this paper, otherwise one can show that Einstein could have written his equation mathematically correct, in another way, avoiding such a misleading question, even keeping all the requirements cited previously.
Nowadays, finding an answer to the manner which curvature of spacetime arises is still an open field in gravity research: this is the context such a thesis concerns itself with.

Anyway, the (2.14) tell how matter deforms spacetime and the massive objects take their dynamics on it at the same time, but not the reason such different entities are equated: this is the guiding aspect of gravity which this thesis aims to dig into.

To help us, the results obtained in the 70's concerning black hole thermodynamics have set a path along to which one can appreciate a thermodynamic behaviour of gravity. This appears evident every time field solutions involving horizons are considered.
It is worth to spend effort and exploring a possible different nature for spacetime, thereof.

Only considering the scalar equations the nature of this equality is evident, therefore this suggests inevitably that the equation of motion has to be derived by statistical considerations.

The subsequent move is considering the functional Q_{tot} as the total heat density, thus one can use it to calculate the total energy crossing into the null surface generated by the ξ^a vector fields. Formally, by summing $Q_{tot} = Q_g + Q_m$, one has

$$Q_{tot} = \int \sqrt{\gamma}\; d\lambda\; d^2x \left(\mathcal{H}_m + \mathcal{H}_g\right) \tag{2.15}$$

and Q_g can be interpreted as the heat content for gravity. Reversing this relationship, then the $\mathcal{H}$s can be interpreted as the heating rates per unit area of the null surface by the respective components.
Such a idea was already introduced by Jacobson, independently from all the path covered until this point of the thesis. He easily showed, in the similar idea presented before, how the Einstein's equations of motion could have been derived from the proportionality $\delta Q = TdS$, letting them being recalled more properly as "Einstein's equation of state". [7]
Thereof, if also gravity can be hot, the Boltzmann principle guarantees that it inevitably possesses a micro-structure. We can identify hence the above quantity as a functional suitable to extremisation onto a null surface, which returns back the equations of motion in the scalar form.

Under the light of such an interpretation of the projected equations of motion, we want to find a proper statistical formalism in order to derive the gravitational field equations. The next chapter will describe this attempt.

3

A Statistical Formalism

In the chapter before, it was discussed how to interpret the geometric term appearing in the Noether current when the Einstein's equations of motion hold. It resulted that a special class of observers, accelerating onto a certain horizon, attributes a thermodynamic behaviour to it, since it measures an exchange of heat densities between matter and spacetime. Under these conditions, the equations of motion emerged together with a set of four-vector fields $q^a q^b$ with null norm.
These features suggest therefore to attempt a derivation of the Einstein equations through a proper entropy functional depending on such vector fields.

In this chapter it will be presented the form of the entropy functional which yields to the field equation by varying it with respect to the vectors q^a. We will require $q^2 = 0$ and that $F[q]$ is invariant under any transformation that leaves the four-vectors light-like.
This will be discovered to furnish not only the equations of motion we are used to, but also a cosmological constant which arises naturally attached to them.

3.1 The Entropy Functional

We search now for an expression of a functional being aware of the presence of certain geometric microscopic degrees of freedom, using the expression of the current conserved on shell.
It is obvious that such degrees need a more accurate study, but for the moment we only suppose their existence.
Hence, this discussion starts by assuming the statistical nature of gravity since the beginning, independently from the field equations.

We recall the heat densities rates in the form

$$A\ R_{ab} q^a q^b \tag{3.1}$$

for gravity, whilst for matter

$$B\ T_{ab} q^a q^b \tag{3.2}$$

with A, B two arbitrary constants. Therefore, the functional $F[q]$ is built starting from these quantities in the spirit of what stated earlier:

$$F[q] = L_P^2 \; R_{ab}q^a q^b - 8\pi L_P^4 \; T_{ab}q^a q^b \tag{3.3}$$

Such a definition involves the dependence of the functional only on the vectors q^a, which is actually the variable respect to which the variation will be done.
In an analogy to statistical mechanics procedure, at equilibrium, the extremisation of (3.3) furnishes exactly the classical equations of motion of gravity, for which we demand to be valid for all vectors q^a in spacetime.
Quantitatively, the imposition $\delta F[q] = 0$ resumes the balancing explored earlier of a heat equilibrium between the two contributes of matter and geometry at the horizon, which arise when equations of motion hold. In this way, the second addend is the heat content of matter crossing it whilst the first one is a heat content of geometry nature, each one with q^a as generator.

3.2 Extremisation Procedure

The entropy functional (3.3) is now ready to be studied through the extremisation formalism. We already mentioned that the variable is the vector field q^a and $F[q]$ is invariant for any displacement of it.

In section (1.3), the matter Lagrangian was discussed to be in the form of the energy-momentum tensor. It results relevant at this stage to point out another aspect which takes into consideration a symmetry under an additive constant.
Indeed, classically, the equations of motion remain valid and unvaried for any constant added to L_m, in example $-\lambda$, which is equivalent to sum λ to the Hamiltonian term: one interprets it as an arbitrary shift in the zero-level energy point.
However, in the case of gravity, such a circumstance affects the source of gravity so that it becomes $T_{ab} + \lambda g_{ab}$ and an ulterior condition is necessary in order to let it vanish, because we want reasonably to not break this symmetry. Moreover, it is obvious that any extremum principle in this form can not have the metric as the dynamical variable.

It follows in our procedure, instead, that we demand $F[q]$ to not change under any transformation that keeps valid the condition $g_{ab}q^a q^b = 0$, at any point in spacetime.
The variation of our functional can now be taken, with the condition reflected by the null vectors $q_a \delta q^a = 0$, therefore

$$\delta F = 2\left(L_P^2 \; R_{ab}q^a - 8\pi L_P^4 \; T_{ab}q^a + \lambda(x) g_{ab}q^a\right)\delta q^b = 0 \tag{3.4}$$

with the constant $\lambda(x)$ that does not depend on q^a, but it can depend, in general, on the coordinates. Thus,

$$\left(L_P^2 \; R_{ab} - 8\pi L_P^4 \; T_{ab} + \lambda(x) g_{ab}\right) q^a = 0 \tag{3.5}$$

is an expression valid for all arbitrary q^a, in this way

$$L_P^2\ R_{ab} - 8\pi L_P^4\ T_{ab} = -\lambda(x)g_{ab} \tag{3.6}$$

Notice that any derivative of the field q^a is not appearing in the equation we found, thus the power of the method described leads to the wanted solution just through the variation of this interesting variable.

At this point, the free-divergence property of both G_{ab} and T_{ab} returns a very important result. In fact, applying a covariant derivative to both sides one has $D_a T^a_b = 0$ and

$$L_P^2\ D_a R^a_b = \frac{L_P^2}{2}\partial_a(R\delta^a_b) = -\partial_a(\lambda(x)\delta^a_b) \tag{3.7}$$

so that

$$\partial_a \left[\lambda(x)\delta^a_b + \frac{L_P^2}{2}R\delta^a_b\right] = 0 \tag{3.8}$$

This means obviously that the quantity inside brackets is a constant in the form

$$\lambda(x) + \frac{L_P^2}{2}R = constant \tag{3.9}$$

This solution, if put into equation (3.6) leads to the field equations for gravity

$$G^a_b = R^a_b - \frac{1}{2}R\delta^a_b = 8\pi L_P^2 T^a_b + \Lambda\delta^a_b \tag{3.10}$$

where the integration constant called Λ acts as a natural *cosmological constant*.

Nevertheless, this solution contains a very new deep meaning because of the presence of the element Λ. Behind this, it might be hidden some natural physical interpretations about the cosmological constant problem. At this level we will not get deep into such a wide contest, but it is enough just to highlight that such a theory, that is without a precise zero-level of energy density, is equivalent to any other theory containing a cosmological constant, with observables consequences. Hence, Λ is physically an undetermined cosmological constant to be fixed with further considerations .
Hence, under no circumstances one can introduce in such a formalism a cosmological constant manually, since it appears by itself as an integration constant. In this way, one has to work on that in order to use it in the cosmological contest.

For the above reasons, field equations obtained involving light-like vector q^s are conceptually very different from a theory in which $q^2 > 0$, in example, representing the four-velocity of an observer. In this case, the equations of motion are the equality of quantity measured in the geometric and the matter sector respectively, as we explored in section (2.2): indeed, only considering horizons and $q^2 = 0$ the field equations arose, holding for all q^a.

All the path followed until this moment let us formulate a new principle in which the field equations are obtained with a cosmological constant which arises without having put it into consideration from the beginning. It seems that the conventional field theory approach is blind to its existence and the only way to introduce it in the final equations is artificially, thus adding it to the Lagrangian initially, in example. Of course, this leads to the open problems physicist are trying to solve nowadays on cosmology, so that the alternative approach presented in the thesis might be very helpful in this area, too.

In conclusion, a statistical point of view, conversely, might be able to avoid some misleading doubts from a fundamental level. Indeed, alternatively to a variation principle, one can ask that at fundamental level the equations of motion are in a projected form onto surfaces with $q^2 = 0$. More words will be spent on this feature in the last section.

3.3 A new understanding

Under the light switched on before, an equivalent form to the Einstein equation is defined by the introduction of a null vector q^a such that we obtain a scalar equation demanding that it holds for all q^as:

$$G_{ab}q^a q^b = kT_{ab}q^a q^b \tag{3.11}$$

Geometry described by $R_{ab}q^a q^b$ and matter depicted as $T_{ab}q^a q^b$ can be seen balancing each other making the product of the respective densities $\rho_g \rho_m$ being equal to unity on shell, when equations of motion hold at equilibrium. Without this feature, we would not have any guiding principle which tell us how matter influences the evolution of spacetime: the canonical equations of motion, again, limit only to the kinematic of gravity rather than its dynamic framework [2][8].

Only under this new form of the field equations for gravity there is no doubt that both sides are the same thing. It results they can be both interpreted independently and acting as a balancing between matter and spacetime: no more "apples" are said to be equal to "oranges" as it occurs in the canonical form [2]. Nevertheless, we just ask for a new revisit, so there is no point of hesitation on this feature, since the Classical General Relativity Theory remains unchanged.
Therefore, we are dealing with a new derivation of the equation of motion which merely changes the perspective of spacetime at fundamental level, leaving untouched all the useful and spectacular applications of Einstein work.

Moreover, the form in (3.11) tells that they constitute the dynamics at fundamental level: this is intriguing because it was shown that a cosmological constant emerges coupled with the other terms, as an effect of microscopic natural dynamics between degrees of freedom of matter and the spacetime.
The subsequent chapter will present a form of a metric which is able to relate the usual metric on the macroscopic scale, with the minimum finite distance of the atoms of spacetime: in this way, a real link between the two dynamics of macro and micro scales will be shown to exist.

4 Measure of Minimal Length

In the last chapter, a entropy functional returning the projected equations of motion was introduced, following the idea that geometry of spacetime behaves thermodynamically. This perspective implies the assumption of the existence of a micro-structure of spacetime, *i.e.* the existence of discrete degrees of freedom at microscopic scale.

Such a granular structure of the spacetime is a very new concept respect to the one we are used to, that is the ordinary view of it as a continuum, aspect that might be sometimes very difficult to accept rather than trying to accomplish it. This granularity may be thought in terms of the existence of a minimal finite distance between events.

It has recently risen the idea to implement this by building a mathematical tool which can take into consideration the spacetime graining. More precisely, what we would like to have is an object q_{ab} through which we measure the distance between two events recovering a minimal finite length, instead of zero, in the coincidence limit between two events. On the contrary, when the distance is large enough, the macroscopic scale is kept described by the usual metric g_{ab} we are accustomed to. This consists in a real challenge because a metric giving back a no-null minimal distance can not exist.

We present in this chapter the expression and the characteristics of the "*quasi*-metric".

4.1 The quasi-metric

We want, in this section, to discuss the existence and the characteristics of a suitable metric through which, in the limit it tends to zero, it is possible to recover a non-vanishing measure; conversely, it results that it is finite and the length scale corresponds to the Plank unit.
The derivation of this special metric is quite complex and might deserve many chapters dedicated to. Thus only the main features will be enlightened [9] [12] [13].

Suppose to have an event point P called "base" which is the center of a normal convex neighbourhood $\mathcal{N}(P)$ and another point $p \in \mathcal{N}(P)$ named "field". This means that a unique geodesics connecting any two points exists always in $\mathcal{N}(P)$. If $\sigma^2(p, P|g_{ab})$ is the geodesic distance squared calculated with the usual background metric, the aim is to find a modified metric q_{ab} so that we can use it as basilar parameter measuring the distance of the two events. Furthermore, the distance between two points is naturally better represented by a quadratic quantity rather than the metric, naturally.

We define the geodesic distance as

$$\sigma^2(p,P) = 2\Omega(p,P) \tag{4.1}$$

through the so-called *Synge's world function.* This is a *biscalar* in that it is a scalar depending on two points.
The Synge's world function is written as

$$\Omega(p,P) = \frac{1}{2}[\lambda(p) - \lambda(P)] \int_{\lambda(P)}^{\lambda(p)} [g_{ab}t^a t^b](x(\lambda))d\lambda \tag{4.2}$$

where λ is an affine parameter, whilst t^a is the tangent vector to the geodesic connecting the points p and P, satisfying $t^2 = \epsilon = \pm 1$. It is evident that we are considering space-like or time-like vectors for which it is possible to take the affine parameter as the distance $\lambda = \sqrt{\epsilon\sigma^2}$. In this case,

$$t_a = \frac{\partial_a \sigma^2}{2\sqrt{\epsilon\sigma^2}} \tag{4.3}$$

Notice that all the main geometric properties of the spacetime, generally described by the background metric, can be expressed using the Synge's function as well: this choice do entail no loss of information about the metric characteristics of the manifold [10]. Indeed, all information is already encoded in the calculation of the geodesic distance in the expression $\sigma^2 = g_{ab}(x^a - X^a)(x^b - X^b)$, where x, X are coordinates of the two points p, P respectively. Hence, one has that the usual metric is recovered in the coincidence limit

$$g_{ab} = \lim_{p\to P}\; D_a D_b \sigma^2 \simeq \lim_{p\to P}\; D_a D_b \Omega \tag{4.4}$$

What said previously can be now generalised by considering a function S of the biscalar σ^2, under certain conditions. Thus, we set a generic functional acting from the space of the geodesic distances related to g_{ab} to the ones concerning the q_{ab}, as $S(\sigma^2) : 2\Omega \mapsto 2\widetilde{\Omega}$, and take into account three main requirements:

i. The first one is that a zero-point Lorentz invariant length L_0 exists as the minimum distance between two events p, P. Thus, we set the modification to the most general form to be a function $\sigma^2 \to S_{L_0}(\sigma^2)$, such that

$$S_{L_0}(0) = L_0^2$$

ii. The second condition is the trivial identity

$$S_0(\sigma^2) = \sigma^2$$

iii. The last one implies the convergence

$$\frac{|S_{L_0}|}{S'^2_{L_0}}(0) < \infty$$

All these ingredients will be necessary to our purposes after having found the most general form for q_{ab}, which is presented starting from the following section.

4.1.1 Expression of the q-metric

The expression of q_{ab} is expected to assume the generic form [8]

$$q_{ab} = Ag_{ab} - \epsilon B t_a t_b \qquad B \equiv \frac{QA}{A^{-1} + Q} = \frac{AQ}{\alpha} \tag{4.5}$$

whose controvariant version is

$$q^{ab} = A^{-1} g^{ab} + \epsilon Q t^a t^b \tag{4.6}$$

with A, Q bi-scalar functions of the spacetime event points to be determined opportunely. Hence, two conditions are required to fix the above quantities definitely.

I) The first requirement concerns the existence of a minimal length such that the geodesic distance gets modified as

$$\sigma^2 \to \sigma^2 + L_0^2 \tag{4.7}$$

This can be imposed by introducing the modified *Hamilton-Jacobi* equation in the q-metric; if satisfied by σ^2, it is expressed as:

$$g^{ab} \partial_a \sigma^2 \partial_b \sigma^2 = 4\sigma^2 \tag{4.8}$$

Hence, for a generic functional $S_{L_0}(\sigma^2)$, we require that

$$q^{ab} \partial_a S_{L_0} \partial_b S_{L_0} = 4 S_{L_0} \tag{4.9}$$

Putting trivially (4.6) into the last equation and using the chain rule for derivatives ones finds

$$\left(A^{-1}g^{ab}+\epsilon Q t^a t^b\right)\partial_a\sigma^2\partial_b\sigma^2\left(\frac{dS_{L_0}}{d\sigma^2}\right)^2=4S_{L_0} \tag{4.10}$$

Let us call $S'_{L_0}=dS/d\sigma^2$. By adding algebraically a quantity $\epsilon A^{-1}t^a t^b$ inside the brackets on the left side, we get

$$\left[A^{-1}(g^{ab}-\epsilon t^a t^b)+\epsilon\alpha t^a t^b\right]\partial_a\sigma^2\partial_b\sigma^2=4\frac{S_{L_0}}{S'^2_{L_0}} \tag{4.11}$$

with $\alpha=A^{-1}+Q$. However, this means that equation (4.9) fixes the *quasi*-metric only on the hypersurface generated by the four-vectors $t^a t^b$, that is, introducing the induced metric on such a surface as $h^{ab}=g^{ab}-\epsilon t^a t^b$ and making use of (4.8), one consequently finds

$$\alpha=A^{-1}+Q=\frac{1}{\sigma^2}\frac{S_{L_0}(\sigma^2)}{S'^2_{L_0}(\sigma^2)} \tag{4.12}$$

II) The second condition, instead, fixes the q-metric definitely and it permits to find the remaining biscalar A. We make use of the modified *d'Alembertian* operator and we require that the corresponding two-points Green functions satisfy

$$\widetilde{G}[\sigma^2]=G[S_{L_0}(\sigma^2)] \tag{4.13}$$

in all maximally symmetric spacetime, that is $\widetilde{\Box}\widetilde{G}[\sigma^2]=0$, when $\Box G[\sigma^2]=0$, at any $p\neq P$. By definition, a *maximally symmetric spacetime* is the manifold which contains locally the maximum number of Killing vectors.

Te most general form of the box operator corresponding to q_{ab} takes the form

$$\begin{aligned}\widetilde{\Box}=&A^{-1}\left\{\Box_g+g^{ij}\partial_i\ln A\,\partial_j+\epsilon t^i\partial_i\ln A\; t^i\partial_i\right\}+\\&+\epsilon Q\left\{\left[\nabla_i t^i+\frac{3}{2}t^i\partial_i\ln A\right]t^i\partial_i+(t^i\partial_i)^2\right\}+\\&+\sqrt{\epsilon\sigma^2}\frac{d\alpha}{d\sigma^2}t^i\partial_i\end{aligned} \tag{4.14}$$

In the case of a maximally symmetric spacetime, the two d'Alemebert operators assume the form

$$\begin{aligned}\widetilde{\Box}&=\alpha\Box+2\alpha\sigma^2\frac{\partial}{\partial\sigma^2}[\ln(\alpha A^3)]\frac{\partial}{\partial\sigma^2}\\\Box&=\frac{\partial^2}{\partial\sigma^2}+\left(\frac{\partial}{\partial\sigma^2}\ln\Delta^{-1}+\frac{3}{\sigma}\right)\frac{\partial}{\partial\sigma}\end{aligned} \tag{4.15}$$

where a new object Δ, called *van Vleck determinant*, was introduced and whose general definition is the following:

$$\Delta(p, P) = \frac{1}{\sqrt{g(p)g(P)}} \det\left(\partial_a^{(p)}\partial_b^{(P)}\frac{\sigma^2}{2}\right) \tag{4.16}$$

with the superscripts that identify the respective coordinates points at which the derivative is taken. Its role is analogous to the one of the Jacobian determinant, so it consists in a powerful instrument in any contest in which curvatures are present. Actually, we can notice that in flat spacetimes $\Delta(p, P) = 1$, thus in arbitrary curved spacetimes $\Delta(p, P) \to 1$, when $p \to P$.

In equation (4.15), that is for maximally symmetric spacetime, the van Vleck determinant becomes

$$\Delta^{-1/3} = \left\{\frac{\sin y}{y}, 1, \frac{\sinh y}{y}\right\} \tag{4.17}$$

with $y = |\sigma|/a$ and a is the curvature radius related to the geodesics.
Consequently, considering the condition (4.13), the solution of the obtained differential equation reads

$$A = \frac{S_{L_0}}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{2/3} \tag{4.18}$$

When $S_{L_0} = \sigma^2$, $A = 1$ fixes the constant of integration. The subscripts S tells that it is calculated respect to the function S, instead of σ^2.

Putting all pieces together the final expression of the q-metric assumes the form

$$q^{ab} = \frac{\sigma^2}{S_{L_0}}\left(\frac{\Delta}{\Delta_S}\right)^{-2/3} g^{ab} + \epsilon\left\{\frac{S_{L_0}}{\sigma^2 S'^2_{L_0}} - \frac{\sigma^2}{S_{L_0}}\left(\frac{\Delta}{\Delta_S}\right)^{-2/3}\right\} t^a t^b \tag{4.19}$$

Equivalently in terms of the surface matrix, we have

$$q^{ab} = \frac{\sigma^2}{S_{L_0}}\left(\frac{\Delta}{\Delta_S}\right)^{-2/3} h^{ab} + \epsilon\left(\frac{S_{L_0}}{\sigma^2 S'^2_{L_0}}\right) t^a t^b \tag{4.20}$$

The whole argument pursued in this section carries a very deep geometrical significance in such a contest. Thanks to the arbitrariness in the choice of $S_{L_0}(\sigma^2)$, we can now discuss some cases using the three conditions presented earlier.

The invocation of requirement **i.** introduces a metric whose geodesic distance is actually $S_{L_0}(\sigma^2) = \sigma^2 + L_0^2$, that takes into account a minimal finite length L_0. Hence,

$$q^{ab} = \frac{\sigma^2}{\sigma^2 + L_0^2} \left(\frac{\Delta}{\Delta_S}\right)^{-2/3} h^{ab} + \epsilon \left(\frac{1 + L_0^2/\sigma^2}{1 + \sigma^2}\right) t^a t^b \tag{4.21}$$

In the limit of large distances between events, in which $\sigma^2/L_0 \gg 1$, we are left with the trivial identity inherent to condition **ii.**, that is $S_{L_0} \to S_0(\sigma^2) = \sigma^2$ which makes $\Delta_S \to \Delta$ and $A \to 1$; as a consequence, the modified metric tends to $q^{ab} \simeq g^{ab}$: this is the result that one expects under such constraints, naturally, even when g_{ab} describes a flat spacetime.
In general, this does not happen for any $S(\sigma^2)$ and the q-metric might lead to a non-zero curvature also in presence of a flat spacetime.

Moreover, observing (4.19), we highlight the fact that the appearing ratio of the van Vleck determinants express, in a certain way, the discrepancy between the curvatures measured with the usual geodesic distance σ^2 and its mapping $S_{L_0}(\sigma^2)$: these two becomes coincident once in the surface Σ generated by $t^a t^b$ and the quasi-metric assumes the exact form of h^{ab}.
The role of this quantity will be relevant in the calculation of the Ricci scalar later, as we will see.

In the chapter below, such a result will be used in order to derive the expression of the Ricci biscalar and all the main features related to it will be enlightened and discussed properly.

4.2 Ricci bi-scalar

Our interest now is focused on finding the exact expression of the Ricci scalar for the q-metric, because, in our dissertation, it consists in the Lagrangian functional for gravity. The generic expression for it will be derived using the so-called *Gauss-Codazzi* relation [11]. Each term of it, which is calculated respect to q_{ab}, will be rewritten in terms of the elements related to the usual background metric g_{ab}, trough the introduction of a "conformal transformation" on the induced geometries, trick that simplifies the computation notably.
In addition, we will notice that the final result is composed by a conformal part, which depends only on A, and another part, where $B \neq 0$, that does not play any role in conformal transformations: this is a relevant discriminant in the solution we want to find.
The notation with a "tilde" refers always to quantities calculated using the q-metric.

The *Gauss-Codazzi* formula for the Ricci scalar $\widetilde{R}$ is the following:

$$\widetilde{R} = \widetilde{R}_\Sigma - \epsilon \left[\widetilde{K}^2 + \widetilde{K}_{ab}^2 + 2\widetilde{T}^i \widetilde{D}_i \widetilde{K}\right] + 2\epsilon \widetilde{D}_i \widetilde{a}^i \tag{4.22}$$

where $\widetilde{T}^a = q^{ab}\widetilde{T}_b$, in which we have introduced such a vector defined as

$$\widetilde{T}_a = \sqrt{A - B}\, t_a \equiv \sqrt{\alpha}\, t_a$$

thus $\widetilde{a}^i = \widetilde{T}^b \widetilde{D}_b \widetilde{T}_a$ is the acceleration vector associated to $\widetilde{T}_a$.
Moreover, it were defined the objects

$$\begin{aligned}\widetilde{K}_{ab} &= \widetilde{D}_a \widetilde{T}_b - \epsilon \widetilde{a}_a \widetilde{T}_b \\ \widetilde{K} &= q^{ab} \widetilde{K}_{ab}\end{aligned} \tag{4.23}$$

so that $\widetilde{K}^2_{ab} = q^{ia} q^{jb} \widetilde{K}_{ab} \widetilde{K}_{ij}$. The Ricci scalar R_Σ is obtained through the induced metric $\widetilde{h}_{ab}$. Covariant derivatives, instead, are defined through the Christoffel symbols as

$$\widetilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{1}{2} q^{ak} \left(D_b q_{ck} + D_c q_{bk} - D_k q_{ab} \right) \tag{4.24}$$

in this way $\widetilde{D}_i \widetilde{T}_k = \partial_i \widetilde{T}_k - \widetilde{\Gamma}^a_{ik} \widetilde{T}_a$, obviously.

At this point, as primary step, in order to rewrite the surface addends in terms of elements corresponding to g_{ab}, we introduce the definition of *conformal transformation*, which is defined as a transformation that preserves angles locally. This gives the relationship between only the terms of the induced geometries, that is related to Σ; if the metrics on the hypersurface are respectively $\widetilde{h}_{ab}$ and h_{ab}, one finds that

$$\begin{aligned}\widetilde{h}_{ab} &= A h_{ab} \\ \widetilde{R}_\Sigma &= A^{-1} R_\Sigma\end{aligned} \tag{4.25}$$

Notice that these two quantities are simply rescaled by a factor A, so no dependence on B is present.

The remaining terms in square brackets of (4.22), instead, can be manipulated with some algebra by calculating the affine connection explicitly and restoring the definition given earlier of the vectors $\widetilde{T}^a$ in terms of the t^a. By doing this, one obtains the covariant derivative as

$$\widetilde{D}_b \widetilde{T}_c = \sqrt{\alpha} \left(D_b t_c + \frac{1}{2} \frac{t^j D_j A}{\alpha} h_{bc} + \frac{B}{\alpha} [K_{bc} - K_{cb}] \right) \tag{4.26}$$

where the expressions of K and K_{ab} related to g_{ab} are defined as above:

$$\begin{aligned}K_{ab} &= D_a t_b - \epsilon a_b t_a \\ K &= g^{ai} g^{bj} K_{ab} K_{ij}\end{aligned} \tag{4.27}$$

Thus, the acceleration is found trivially to be $\widetilde{a}_i = a_i = t^a D_a t_i$.

The final form of searched addends results therefore

$$\begin{aligned}\widetilde{K}_{ab} &= \frac{A}{\sqrt{A-B}} \left[K_{ab} + \frac{1}{2} (t^a D_a \ln A) h_{ab} \right] \\ \widetilde{K} &= \frac{1}{\sqrt{A-B}} \left[K + \frac{3}{2} (t^a D_a \ln A) \right]\end{aligned} \tag{4.28}$$

Now, substituting the ingredients (4.25) and (4.28) into the Gauss-Codazzi formula, we get the final result for $\widetilde{R}$ to be [12]

$$\widetilde{R}(p, P|q_{ab}) = A^{-1}R(p, P|g_{ab}) + \epsilon(\alpha - A^{-1})\mathcal{J}_d - \epsilon\alpha\mathcal{J}_c \tag{4.29}$$

in which some terms have been renamed as

$$\mathcal{J}_c = \epsilon\frac{6}{\sqrt{A}}\Box A + \left(K + \frac{3}{2}t^a D_a \ln A\right)(t^a D_a \ln \alpha A) \tag{4.30}$$

$$\mathcal{J}_d = 2R_{ab}t^a t^b + K_{ab}^2 - K^2 - 2D_i a^i = \epsilon(R - R_\Sigma - 2D_i a^i) \tag{4.31}$$

In here it results clearer which terms play a role in conformal transformations. Actually, when $B = 0$, this case corresponds to conformal metrics for which the $\mathcal{J}_d$ term does not contribute and the second addend above becomes null. Such a coincidence is not trivial because the structure of this term reveals itself to be special because it contains the element $R_{ab}t^a t^b$ of wide interest in our contest.
In the following section we are actually discussing in details these features and the way in which this term is a crucial pillar in the whole emergent gravity paradigm treated in this paper.

4.2.1 Equigeodesic Surfaces

In this section we propose to recover explicitly the exact expression of the gravitational Lagrangian, which is nothing but the Ricci scalar. The starting point is to see what happens to the general expression (4.29) when two event points do coincide, $p \to P$, and whether it is comparable with the general Ricci scalar $R(g_{ab})$ of the background metric.
It will be consequently found that a leading term exists and it astonishingly will be the exact term contained inside the definition of the entropy functional $F[q]$, rather than $R(g_{ab})$ as well as one might expect.

For the sake of simplicity, we now consider the spacetime as foliated by equi-geodesic surfaces Σ_G and call the one of the event P as $\Sigma_{G,P}$. The related normal vector to it is the vector tangent to the geodesics connecting P to p, that is affinely parametrised as 4.3.
By doing this, we make "disappear" the acceleration a^i terms in all the definitions of the previous section, thus we are left with modified terms such as

$$\begin{aligned} K_{ab} \quad &\Longrightarrow \quad K_{ab} = D_a t_b = \frac{\partial_a\partial_b(\sigma^2/2) - \epsilon t_a t_b}{\sqrt{\epsilon\sigma^2}} \\ \mathcal{J}_d \quad &\Longrightarrow \quad \mathcal{J}_d = \epsilon(R - R_{\Sigma_{G,P}}) \end{aligned}$$

At this point, it is possible to insert in (4.29) the expressions (4.12) and (4.18) for both α and A. In the computation of such an expression two equalities for the van Vleck determinant are used. Indeed, the van Vleck determinant satisfies two main differential identities which connect them with K_{ab} and K related, this time, to the surface $\Sigma_{G,P}$:

$$t^i D_i \ln \Delta = \frac{3}{\sqrt{\epsilon\sigma^2}} - K \tag{4.32}$$

$$t^i D_i (t^j D_j \ln \Delta) = -\frac{3}{\epsilon\sigma^2} - K_{ab}^2 + R_{ab} t^a t^b \tag{4.33}$$

Finally, the result obtained is

$$\begin{aligned} \tilde{R}(p, P|q_{ab}) = &\left[\frac{\sigma^2}{S_{L_0}} \left(\frac{\Delta}{\Delta_S} \right)^{-2/3} \mathcal{R}_\Sigma - \frac{6}{S_{L_0}} + 20 \frac{d}{dS_{L_0}} \ln \Delta_S \right] - \\ &- \frac{S_{L_0}}{\lambda^2 S'^2_{L_0}} \left(K_{ab} K^{ab} - \frac{1}{3} K^2 \right) + \\ &+ 4 S_{L_0} \left[2 \frac{d^2}{dS^2_{L_0}} \ln \Delta_S - \frac{3}{2} \left(\frac{d}{dS_{L_0}} \ln \Delta_S \right)^2 \right] \end{aligned} \tag{4.34}$$

Firstly, three main relevant contributions emerged, named as Q_0, Q_K, Q_Δ corresponding to the first, second and third addend, respectively: *intrinsic curvature*, *extrinsic curvature* and *van Vleck determinant*. Each of these will be defined later piece by piece.
Thus, Ricci scalar result is mathematically described by the geodesic structure of spacetime, completely.

Moreover, notice that, as far as $S(\sigma^2)$ represents effects of quantum gravity, only its first order derivatives appears in the Ricci formula because all the other higher order terms cancels out; this luck of dependence suggests that semi-classical effects of quantum gravity can be grasped only from precise details of quantum gravity itself, rather than its any perturbative contribution.

Finally, the extrinsic curvature of the surface $\Sigma_{G,P}$ related to the event base point P does not imply any coincidence limit divergence.

4.2.2 Coincidence Limit

We are about to take the limit of two events coinciding, $L_0 \to 0$. The peculiar aspect of the previous result for the Ricci bi-scalar tensor is that it can be decomposed essentially into three additive terms, the ones that sum up the main characteristics of the geometry of spacetime.

The main assumption for the purposes of our calculations is to consider regions of spacetime as smooth, so that we are allowed to expand the terms in a covariant Taylor series around L_0. Each term expansion is indicated below:

$$\begin{aligned} K_{ab} &= \frac{1}{L_0} h_{ab} - \frac{1}{3} L_0 \mathcal{E}_{ab} + \frac{1}{12} L_0^2 t^i D_i \mathcal{E}_{ab} - \frac{1}{60} L_0^3 \left(t^i t^j D_i D_j \mathcal{E}_{ab} + \frac{4}{3} \mathcal{E}_{ak} \mathcal{E}_b^k \right) + O(L_0^4) \\ K &= \frac{3}{L_0} - \frac{1}{3} L_0 \mathcal{E} + \frac{1}{12} L_0^2 t^i D_i \mathcal{E} - \frac{1}{60} L_0^3 g^{ab} \left(t^i t^j D_i D_j \mathcal{E}_{ab} + \frac{4}{3} \mathcal{E}_{ak} \mathcal{E}_b^k \right) + O(L_0^4) \end{aligned} \tag{4.35}$$

In addition, we use the covariant Taylor expansion of the van Vleck determinant

$$\Delta^{1/2}(p,P)[L_0] = 1 + \frac{1}{12}L_0^2 R_{ab}t^a t^b + O(L_0^3) \tag{4.36}$$

together with the expansion of the Ricci surface element

$$R_{\Sigma_{G,P}} = R - \epsilon\left[\frac{10}{3}\mathcal{E} - \frac{6}{L_0^2}\right] + O(\lambda^3) \tag{4.37}$$

Precisely, K_{ab} describes the local geodesic structure of any kind of spacetime, and each term results completely depending on the tidal tensor $\mathcal{E}_{ab} = R_{akbi}t^k t^i$ and $\mathcal{E} = g^{ab}\mathcal{E}_{ab}$.

Notice that the contribution of L_0 is contained only in the intrinsic curvature part Q_0, while it appears at higher order in the others because $S_{L_0}(0) = L_0^2$. Let us therefore examine each piece separately.

Extrinsic Curvature The extrinsic curvature term Q_K can be discussed using the condition **iii.**, that is asking for convergence of the ratio $|S_{L_0}|/S'^2_{L_0}(0) \ll \infty$. It depends on L_0 quadratically, actually

$$Q_K = \frac{S_{L_0}}{S'^2_{L_0}}\frac{1}{L_0^2}\left[\frac{L_0^2}{9}\left(\mathcal{E}^2_{ab} - \frac{1}{3}\mathcal{E}^2\right) + O(L_0^3)\right]$$

in such a way this term does not contribute in the limit and no divergent objects appear. Such a powerful peculiarity is direct consequence of the presence of the van Vleck determinant without whom we would have had the problem to face divergent terms.

Van Vleck Determinant The term Q_Δ vanishes trivially as well as before because it furnishes further dependences on L_0^2, as it can be noticed observing the Taylor series (4.36) and considering that

$$\lim_{L_0\to 0}\lim_{\sigma^2\to 0}\left(\frac{d}{dS_{L_0}}\ln\Delta_S\right) = \frac{1}{6}\epsilon[R_{ab}t^a t^b](P) \tag{4.38}$$

Intrinsic Curvature We are left with the only surviving Q_0, the intrinsic curvature that, at this point, constitutes the main contribution to $\widetilde{R}$ in the coincidence limit, as anticipated before. Using equation (4.37) and $\Delta(0) = 1$ one gets, together with (4.38),

$$\lim_{L_0\to 0}\lim_{\sigma^2\to 0} Q_0 = \frac{2}{3}\epsilon[R_{ab}t^a t^b](P) + 20\frac{1}{6}\epsilon[R_{ab}t^a t^b](P) = \epsilon 4[R_{ab}t^a t^b](P) \tag{4.39}$$

As a consequence we have found the most important result for the expression of the Ricci scalar in the *quasi*-metric and in the coincidence limit, which gets proportional to the tidal element $\mathcal{E}$:

$$\lim_{L_0 \to 0} \widetilde{R}(P) = \epsilon \; 4[R_{ab}t^a t^b](P) \tag{4.40}$$

4.2.3 Observations

It was shown how the Ricci scalar in the q-metric becomes proportional to the element $R_{ab}q^a q^b$, in the limit the distance of two points of spacetime tends to zero, a quantity having a key role in the emergent gravity contests.
The first time we encountered it was in section (1.2), when the existence of a Noether current for gravity was discovered: it is not a casualty that these two different topics lead to a similar result.

The form of the Noether charge pushed to consider a new Lagrangian for gravity in the form of a entropy functional, as in (3.3). The resulting equations of motion validated such a choice and this entropy functional is now even more accredited under the result obtained in this chapter: actually, a metric carrying the existence of a microscopic structure of spacetime suggests that, at fundamental level, the leading functional term has to be in the tidal form $\mathcal{E}_{ab}$. This magnificent solution would not have never been found with the usual metric, due to the fact that the resulting geodesic interval vanishes in the coincidence limit. Afterwards, it follows that the theory concerning an arbitrary functional $S(\sigma^2)$, with all the characteristic of the geometric spacetime left unaltered, permits to insert a no-null finite length at small scale and understand better some new basilar physic aspects on gravity.
Thereof, the necessity of a statistical derivation for it, with all the considerations argued until now, is now definitely clear.

Moreover, in spite of the doubts about the presence of the vector field q^a, as argued in section (3.3), their existence is now not ambiguous anymore because it emerges naturally just considering a discrete structure for spacetime. Indeed, these vector fields are intrinsically participating in the guiding curvature term Q_0.

4.3 Statistical Formalism

We want to conclude the dissertation explaining how such atoms of spacetime can be treated microscopically through statistical considerations. Therefore, the aim is to link these degrees of freedom with the macroscopic behaviour of gravity, emerging in the form of thermodynamic quantities such as heat densities of energy and entropy, as argued since the beginning.
In this way, it is more evident how the functional $F[q]$ plays a perfect role in giving back the gravitational equations of motion as counting the granules of spacetime, together with the matter degrees of freedom.

As for any fluid, one can define a fundamental function called generally *distribution density function* $f(x^i, p_j)$ which gives the amount of atoms $dN = f(x^i, p_j)d^3x d^3p$ contained in the unit volume of phase space $d^3x d^3p$, each with the proper space coordinates x^i and momenta p_j. In our case of the spacetime, the idea is the same, thus we define an object which counts the number

of degrees of freedom of its "atoms": doing this, we are already stating that we suppose the existence of these degrees and that they are equivalent to each other, so their nature does not affect the theory we are trying to build. For our purpose, we define a function $\rho(x, \phi_A)$ whose dependence are the coordinate x and a sort of momentum ϕ_A; the last one takes an index A being a natural number ($A = 1, 2, 3...$) representing the possible internal degrees of freedom.
Consequently, the matter contribution is introduced through the stress energy tensor T^a_b. Then, in such a space, there are both atoms of spacetime both atoms of matter, hence the most generic distribution density function assumes the dependence

$$\rho[\mathcal{G}_N(x), \phi_A, T^a_b(x)] \tag{4.41}$$

Notice that the coordinates dependence was put into the definition of $\mathcal{G}_N$ which represents the geometrical variables related to spacetime, in example the metric, the curvature tensor and so on, each one identified by $N = 1, 2, 3....$

Therefore, the total number of degrees of freedom for a certain spacetime configuration is counted through a product all over the phase space and linked to a configuration entropy value S:

$$\Omega_{tot} = \prod_{\phi_A} \prod_x \rho[\mathcal{G}_N(x), \phi_A, T^a_b(x)] \equiv \exp S \tag{4.42}$$

At the equilibrium, it is possible to separate the total density function into the product of the two contribution respectively of matter and spacetime, hence exponentiating those quantities, Ω_{tot} becomes

$$\Omega_{tot} = \prod_{\phi_A} \exp \sum_x \left(\ln \rho_g + \ln \rho_m\right) \tag{4.43}$$

in which the gravity part only depends on $\mathcal{G}_N$ and ϕ_A, whilst the matter density depends on T_{ab} and ϕ_A.
The point at which the functional Ω_{tot} takes its maximum value, respect to ϕ_A, corresponds to the state of equilibrium of the system, hence it reproduces the Einstein gravity equations.

Now, consider the expression of the entropy functional (3.3) integrated in a four-dimensional space in the covariant volume dV, under a certain function $\Gamma(F[x; \xi^a])$ of it, having chosen the arbitrary $q^a \to \xi^a$ as a light-like killing vector:

$$S[\xi^a(x)] = \int_V d^4x \Gamma(F[x; \xi^a]) \tag{4.44}$$

Hence,

$$F[x; \xi^a] \equiv \left[B\ T^a_b(x) - A\ R^a_b(x)\right] \xi_a \xi^b \equiv F(x) \delta^a_b\ \xi_a \xi^b \tag{4.45}$$

Indeed, varying $\delta S = 0$ respect to the null vector one gets (3.10) as expected. Under this glance, we interpreted the variables x and ξ^a as a set of internal parameters describing the atoms of spacetime; at any event a set of null vectors ξ^a is defined with the condition $\xi^2 = 0$ at each point.

4.3.1 Heat Density for Gravity

By considering (4.43), it can be shown that the proper form for the entropy density for gravity ρ_g is related to the tidal term $\mathcal{E}_{ab}$. Intuitively, we saw the association to heat rate content to be

$$\mathcal{H}_g[q] \equiv \frac{dQ_g}{\sqrt{\sigma}d^2xd\lambda} = R_{ab}q^a q^b \tag{4.46}$$

where we have set the approximation $\xi^a \to q^a$ at the null surface, with $q^a = dx^a/d\lambda$ null vector normal to it.

By basic differential geometry knowledge, area and volume are related by a flat part contribution plus a term which takes into consideration the curvature measure. In our case we want to relate $\sqrt{h}d^3x$ and $\sqrt{g}d^4x$, which both vanish in classical spacetime if the size of the region goes to zero. This should not be surprising even if we rescale the zero value to be a finite number, because we expect that at this value both measures of area and volume vanish naturally as well.
However, the area element does not vanish in the definition of density we are about to find. Indeed, a non null vector field q^a is present and the density $\rho_g(x^i, q^a)$ depends on its choice.

We propose now to calculate $\sqrt{h}$ using the q-metric described earlier. The metric can be thought in the general form in the Euclidean sector

$$ds_E^2 = d\sigma^2 + h_{ab}dx^a dx^b \tag{4.47}$$

A typical set of coordinates can be identified through three angles and the coordinate length σ, along the geodesics connecting to points p and P; thereafter, taking a limit $\sigma \to 0$, the non-zero value should emerge, evidently if and only if the spacetime is not threaten as a continuum, but in a discrete way.

The respective determinants are related in this way:

$$\sqrt{q} = \sqrt{\alpha^{-1}}A^{3/2}\sqrt{g} \equiv \sqrt{\alpha^{-1}}\sqrt{h} \tag{4.48}$$

Hence, setting $S_{L_0}(\sigma^2) = \sigma^2 + L_0^2$, it is possible to recover the volume element

$$\sqrt{q} = \sigma(\sigma^2 + L_0^2)\left[1 - \frac{1}{6}\mathcal{E}(\sigma^2 + L_0^2) + O(\mathcal{E}^2)\right]\sqrt{h_\Omega} \tag{4.49}$$

and the surface element

$$\sqrt{h} = (\sigma^2 + L_0^2)^{3/2} \left[1 - \frac{1}{6}\mathcal{E}(\sigma^2 + L_0^2) + O(\mathcal{E}^2) \right] \sqrt{h_\Omega} \tag{4.50}$$

where $\mathcal{E} \equiv R_{ab} q^a q^b$ and $q_a = \nabla_a \sigma$ is the vector along the direction of σ, normal to the surface Σ; the element $\sqrt{h_\Omega}$ is related to the other coordinates, *i.e.* the angular part. It follows that the second addend in brackets appears as the curvature correction to the area at the considered event-point.
Thereof, when $\sigma \to 0$ and $L_0 \sim L_P$ is a typical length of Plank order, the volume element vanishes but the above solution becomes

$$\sqrt{h} = L_P^3 \left[1 - \frac{1}{6}\mathcal{E} L_0^2 \right] \sqrt{h_\Omega} \tag{4.51}$$

At this point, a simple algebra manipulation leads to the definition of a density function as

$$\rho_g(x^i, q_a) \equiv \frac{\sqrt{h}}{L_P^3 \sqrt{h_\Omega}} = 1 - \frac{L_0^2}{6} R_{ab} q^a q^b \tag{4.52}$$

where ρ_g can be assumed to be the number of atoms of spacetime. In here, it appears natural and evident the dependence on the vectors q^a.
Nevertheless, in order to interpret the above density function as the number of atoms of spacetime per unit volume at a given event P, we imagine them as a solid made of ordinary matter, in which the particles are not point-like anymore.

Consider therefore q^a as a sort of momentum of the spacetime atoms and the density function represents the numbers of microscopic degrees of freedom of such particles.
Hence, if one sets the identification $\mathcal{H}_g \equiv L_P^{-4} \rho_g(x, q)$, the definition for the gravity heat density follows similarly to the one for matter:

$$\mathcal{H}_g[q] \equiv \frac{dQ_g}{\sqrt{h} d^3 x} = \frac{1}{L_P^4} \left[1 - \frac{L_0^2}{6} R_{ab} q^a q^b \right] \tag{4.53}$$

Thus, it is possible to count now the discrete number of degrees of freedom of the atoms of spacetime by exponentiation according to $S_g = \ln \rho_g$, hence

$$\Omega_g = \exp \ln \rho_g = \exp \left[\mu \int \frac{\sqrt{\sigma} d^2 x d\lambda}{L_P^3} \left(L_P^4 \mathcal{H}_g \right) \right] \tag{4.54}$$

where $\mu = 1/L_P T$ and T has dimensions of a temperature.

In conclusion, the total heat content including matter carries actually the form of the functional (3.3), that is

$$Q_{tot} = \int \sqrt{\sigma} d^2x d\lambda \left[\frac{1}{L_P^4} + \left(T_{ab} - \frac{1}{8\pi L_P^2} R_{ab} \right) q^a q^b \right] \tag{4.55}$$

where we have put the right value of the constant $L_0^2 = (3/4\pi)L_P^2$. The analogous expression to the above one includes the entropy definitions, by dividing for a temperature.

The above expression tells the way in which the dynamics of the system is driven to equilibrium, indeed, it is evident how the respective densities act balancing so that $\rho_g \rho_m = 1$ on shell.
Straightforwardly, the statistical approach offers spacetime shells as the "place" where the gravitational system equilibrium is realised. Thereof, in the case shells are also specific physical horizons, they deserve to be explored because they might contain plenty of information about the natural dynamics of gravity: actually, this is a direct consequence of the fact that they share information with their bulk content.
Next section will show how surface and bulk share information content on shells, only through thermodynamic considerations.

4.4 Holographic Equipartition

In this last section, we want to analyse briefly some consequences of the holographic principle by counting numerically the physical degrees of freedom of a certain region of spacetime, as anticipated in section 1.1.2. In fact, it was pointed out that the bulk term of the Einstein-Hilbert Lagrangian shares information content with the related surface part.
Since we have explored the thermodynamic behaviour for the Noether current as an entropy quantity, we want to rewrite it in terms of number of degrees of freedom, contained inside the bulk N_{bulk} and in the surface part N_{sur}. [5]
The discussion will end up with an amazing equality between these numbers: this permits to understand better the spacetime dynamical evolution.

Let us start by considering again the Noether current (1.51) in the contracted form $q_a J^a$, where $q^a = dx^a/d\lambda$ and λ is the parameter along the null generator q^a. It is possible to show that this can be expressed as

$$16\pi q_a J^a[\xi] = \frac{1}{\sqrt{\sigma}} \frac{d}{d\lambda} (2a\sqrt{\sigma}) \tag{4.56}$$

where a is a positive function depending on coordinates. We suppose the current to depend on a time evolution vector ξ^a as usual, so that $\xi^a \to q^a$ at the surface.
The derivation of the above solution, consisting just in several algebraic calculations, would require a too long digression that only takes away the attention from the main purpose we want to highlight simply, therefore it is not presented in here; it can be found in [16].

It was seen that the integration over the three-dimensional space region H of the term $q_a J^a$

reproduces a physical heat density. Now, the measure of the integral for an interacting vector field over a null surface is $q_a\sqrt{\sigma}d^2xd\lambda$. Thus, observing equation (2.8), one has

$$\int_H \sqrt{\sigma}d^2xd\lambda\ q_aJ^a = \int_H d^2xd\lambda\ \frac{d}{d\lambda}(2a\sqrt{\sigma}) = \int_H \sqrt{\sigma}d^2xd\lambda\ q_a(\delta_\xi v^a + 2G^a_b\xi^b + R\xi^a) \tag{4.57}$$

so that we can rewrite it by inserting the field equations in the form $G_{ab} = 8\pi L_P^2 T_{ab} + \Lambda g_{ab}$ and recalling the null-norm condition $g_{ab}q^aq^b = 0$:

$$-\int_H \frac{d^2xd\lambda\sqrt{\sigma}}{8\pi L_P^2}\ q_a\ \delta_\xi v^a = \int_H \frac{d^2xd\lambda}{8\pi L_P^2}\ \frac{d}{d\lambda}(2a\sqrt{\sigma}) - \int_H d^2xd\lambda\sqrt{\sigma}\ (2T_{ab}q^aq^b) \tag{4.58}$$

At this point, the very last integrand can be identified with the energy density of matter crossing the surface into the bulk region.
Therefore, the whole integral constitutes the total energy and if the matter heat flux thermalizes at the average temperature of the null surface, we can count the degrees of freedom, invoking the energy equipartition theorem, as

$$E_m = \frac{1}{2}N_{bulk}k_BT_{avg} \tag{4.59}$$

where

$$T_{avg} \equiv \frac{1}{A_{S_H}}\int_{S_H} d^2x\sqrt{\sigma}\ T_{loc} \tag{4.60}$$

was introduced as an average temperature of the local temperature $T_{loc} = a/2\pi$. Obviously, A_{S_H} is the area of the surface.

The other integral, instead, can be solved over λ from a certain value of λ_1 to another one λ_2, trivially, so that one obtains a difference of heat quantities on the boundaries located at λ_1 and λ_2 respectively. In this way,

$$\int_{S_H} \frac{d^2x}{L_P^2}\left(\frac{a}{2\pi}\right)\left(\frac{\sqrt{\sigma}}{4}\right)\Bigg|_{\lambda_1}^{\lambda_2} = Q(\lambda_2) - Q(\lambda_1) \tag{4.61}$$

As done previously, $T_{loc} = a/2\pi$ and we take the surface number of degrees of freedom to be

$$N_{sur} = \frac{A_{S_H}}{L_P^2} = \int_{S_H}\frac{\sqrt{\sigma}d^2x}{L_P^2} \tag{4.62}$$

so that the final expression for (4.58) becomes

$$\int_H \frac{d^2x\sqrt{\sigma}}{8\pi L_P^2}\ q_a\ \delta_\xi v^a = \frac{1}{2}k_B T_{avg}(N_{bulk} - N_{sur}|_{\lambda_1}^{\lambda_2}) \tag{4.63}$$

Therefore, whenever ξ^a is a killing vector generating a boosts, coincident with the normal vector q^a at the null surface, the left hand side vanishes and the *holographic equipartition* arises: $N_{sur} = N_{bulk}$.
Equation (4.63), acts as guiding principle in describing the dynamical evolution of the spacetime. In fact, equilibrium is found whenever the left-hand side is zero and the discrepancy from holographic equipartition vanishes.
Conversely, the number of degrees of freedom of a certain region of spacetime tends to become equal to the ones contained in the surface that surrounds it, so that the gravitational equilibrium is realised.

Moreover, the power of this natural characteristic assumes a deep relevance since a bulk and a surface can always be found around any point of spacetime. Hence, if treated on specific physical horizons, the information accessible to it is analogous to the information content of its bulk volume, so that this approach might result quite interesting in the study of regions of spacetime which are not accessible directly, *viz.* black holes.

5 Conclusion

This dissertation is related to some fundamental questions and unsolved puzzles in physics about gravity. On one hand, formally, we face deep problems when studying it under a Quantum Field Theory approach in the same way one does with all the other three forces of nature; on the other hand, instead, it is conceptually inexplicable why in Einstein's equations, two different quantities, geometry and matter, appear to two be equated.
The reader was introduced to a new possible and valid approach to the study of gravity, by offering a possible answer to those questions.
Actually, observing the canonical Einstein's field equation under a special frame of reference, it was possible to highlight similarities between gravity behaviour and thermodynamics at macroscopic level. This discovery, let us then evaluate a different consideration about the nature of spacetime (see 3.3).

The starting point was deriving the field equations for gravity using the Canonical Quantum Field Theory approach, that is finding a suitable Lagrangian that was possible to extremise according to the action principle. The Hilbert Lagrangian

$$\mathcal{L}_{EH} = \frac{1}{2}(\delta^c_a g^{bd} - \delta^d_a g^{bc}) R^a_{bcd}$$

was shown to be right candidate, as long as it is a scalar functional, not trivially constant, however dependent on the metric and its first and second derivative, with the property of linearity on these.

The resulting equations of motion, by variation of the metric g_{ab}, are safely still at second order, whilst the extra higher order derivative terms, appearing in the action, are collected into a surface integral, which vanishes whether δg^{ab} is set to be null on it.
Indeed, the Hilbert Lagrangian is made by a quadratic part, which is the one giving back the equations of motion, and a surface part that asks for particular attentions.
Nevertheless, these two parts do share information content according to a holographic principle which arises naturally (see 1.1.2).

Afterwards, it was shown that such a surface part leads to the existence of a Noether current conserved off-shell and its related charge assumes formally the expression of a heat density quantity of geometric nature in the form (see 1.2)

$$R_{ab}q^a q^b$$

However, this physical interpretation is allowed *if, and only if*, the equations of motion with both geometry and matter are considered onto a generic null hypersurface of spacetime.
Indeed, such a formal quantity assumes a physical meaning on a special case in which the mathematical surfaces are actually horizons for a certain class of accelerated observers, for example, black holes.

Local Rindler frames of reference were a spectacular proof, rather than an example, of the fact that an entropy balancing between matter and spacetime occurs onto the horizon (see 2.2). In this way, by definition of entropy, an horizon appears to the observers thermodynamically "hot".
Now, as long as entropy is a macroscopic quantity that arises statistically from a microscopic physics of the particles of matter, one can invoke the Boltzmann principle also for the entropy of the geometric counterpart, so that we postulate inevitably the existence of a discrete micro-structure at fundamental level, called as "degrees of freedom" of spacetime.

Accepting therefore that spacetime is actually granular and possesses a certain number of degrees of freedom, of still unknown nature, one wants to build a suitable entropy functional for gravity, which gives back a set of equations of motion of statistical nature (see 3.2). The searched functional has the form

$$F[q] = L_P^2 \; R_{ab}q^a q^b - 8\pi L_P^4 \; T_{ab}q^a q^b$$

The variation is done respect to the field q^a, which furnishes a set of equations projected onto the hypersurface generated by the couple $q^a q^b$, with the condition $q^2 = 0$, that is

$$\left(R_{ab} - \frac{1}{2}R \; g_{ab}\right) q^a q^b = 8\pi L_P^2 T_{ab} q^a q^b$$

First thing to notice is that, by this procedure, it emerges that the new set of scalar field equations equates two same quantities both for geometry and matter, solving the philosophical problem that keeps affecting the classical Einstein's equations in its tensor form.
However, it is necessary to ask the functional $F[q]$ to be invariant for any transformation that keeps unchanged the norm of the vectors $q^2 = 0$: only in this way a set of equations of motion arise with an integration constant inside, whose role is the one of a *cosmological constant.* So,

$$R_{ab} - \frac{1}{2}R \; g_{ab} = 8\pi L_P^2 T_{ab} + \Lambda g_{ab}$$

It follows that the granular structure of spacetime makes, in this way, a cosmological constant arise naturally by the integration of such projected equations of motion, feature that is invisible in Quantum Field Theory and needs to be added manually, later on.
Moreover, this result is even more astonishing because it might contain answers to many of the open problems in cosmology, since the information carried by the constant is directly contained in the nature of spacetime itself, at the smallest scale.

A discrete background for the gravitational dynamics requires innovative mathematical tools to work with. Actually, a quasi-metric can be built in order to give a finite length in the coincidence limit of two events: such a finite small distance is of the order of Plank length (see 4.1).

The quasi-metric q_{ab}, at large scale, instead, reduces to the ordinary metric g_{ab}, as expected. However, the most sensational result is obtained if the Ricci scalar is constructed from the q-metric. In the coincidence limit, in which the minimum length scale $L_0 \to 0$ and the geodesic distance $\sigma^2 \to 0$, the Ricci *quasi*-scalar $\widetilde{R}(q)$ does not coincide with the expression of the Ricci scalar constructed with the usual metric g_{ab}, as one might expect. By contrast,

$$\widetilde{R}(q) \propto \; R_{ab} q^a q^b$$

which is the exact expression appearing in the Noether current when Einstein's field equations hold. This suggests the form can describe gravity at fundamental level. In such a way, a certain class of light-like q-vector fields exists intrinsically beneath, so that a functional built from these, completely with matter, constitutes a perfect generator of equations of motion with a likewise natural cosmological constant.

All things considered, this new statistical approach for gravity possesses plenty of theoretical validation, both mathematical both physical. It could be therefore a new road to follow in order to evaluate new methods to quantisation. Indeed, the attention should be moved to the quantisation of the degrees of freedom of spacetime and the fields q^a.
All this debate is enforced by the fact that the canonical approach, in which the metric is considered a variable, is not a proper choice conceptually, due to the nature of spacetime which does not behave like an external field propagating in it. Such a concept reflects disappointing consequences to any attempt in canonical quantization. Thus, if the physicists have the patience, nowadays, to show some interest into the emergent theory, a new door for quantum gravity might be opened and maybe, one day, crossed too.

As it was possible to read, several deep topics were inevitably touched and mentioned during the whole journey. Many of those are actually either wide field of researched themselves or fundamental parts for other topics. Therefore, it worth to write some extra lines about those and convince the reader how the statistical approach to gravity can contribute significantly also to other open areas of research.

Below, it will be shortly discuss how this approach can be extended to existing higher order theories of gravity, beyond the Einstein's, and some generalisations about the Holographic principle.

Higher Order Theories In addition to the whole discussion, it is mandatory to inform that this thesis has presented the emergent gravity formalism in the contest of the usual "physical gravity". The Einstein theory in a 4D spacetime, in fact, can be treated as a special case and framed in the wider mechanism of broader theories of gravity.

In fact, emergent gravity operates even in gravitational models both at dimensions higher than four, both at higher order derivatives of the metric: an example of such a class of theories are the ones named as *Lanczos-Lovelock* [1] [14].

In these models, a Lagrangian is built for generic spacetimes with dimension D, so Einstein's General Relativity is a special case for $D = 3$ and $D = 4$. Essentially, the most general Lagrangian expression is

$$\mathcal{L}_m^{(D)} = Q_a{}^{bcd} R_{bcd}{}^a \tag{5.1}$$

which contains a more compact form of the tensor

$$^{(m)}Q_a{}^{ijk} = \frac{1}{16\pi} 2^{-m} \delta^{cda_3 \dots a_{2m}}_{abb_3 \dots b_{2m}} R^{b_3 b_4}_{a_3 a_4} \dots R^{b_{2m-1} b_{2m}}_{a_{2m-1} a_{2m}} \tag{5.2}$$

where a sum over m is implicit with certain arbitrary constants c_m; m is the order in the theory. We recognise the tensor used in the thesis contest when $^{(1)}Q_a{}^{bcd} \equiv P_a{}^{bcd}$.
Thus, expanding it in power series of its derivatives, one gets (in matrix notation)

$$P(g, R) = c_1 P_1(g) + c_2 P_2(g, R) + \dots \tag{5.3}$$

General Relativity is found when only c_1 is non zero, in example $c_1 = 1/16\pi G$. Any other further element can be considered as higher order correction terms to the Einstein theory.

Holographic Principle Another interesting aspect involves the generalisation of the holographic principle. Such a powerful relationship between the quadratic and the surface part appearing in the Hilbert Lagrangian, choosing g_{ab} as dynamical variable and $\partial_c g_{ab}$ as its conjugate momentum, for $D > 2$, results

$$\mathcal{L}_{sur} = -\left[\frac{D}{2} - 1\right]^{-1} \partial_i \left(g_{ab} \frac{\partial \mathcal{L}_{quad}}{\partial(\partial_i g_{ab})} \right)$$

Such an identity is satisfied naturally, without any further requirement.

As well as it was shown, the above principle leads to the equality of the number of degrees of freedom contained in the D-dimensional volume bulk and the ones present in the horizon with dimension $D - 1$, when the system is at equilibrium [5].

Hence, Holography is a property also of Lanczos-Lovelock Lagrangian models. Moreover, it constitutes basilar aspects in other important theories such as *String Theory*, for example.

This is another reason which makes emergent gravity worthy of attention.

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