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Path Integrals and Heat Kernels

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Abstract

Heat kernel methods are a useful tool for studying propagators and one-loop effective actions in quantum field theory. One particular application arises in the calculation of anomalies, which are often cast as heat kernel traces with insertion of suitable operators. Such traces can be computed using a quantum mechanical path integral.

In this thesis we present a path integral representation for the general expression needed to compute anomalies affecting the conservation of the stress tensor (the so-called gravitational anomalies) in flat spacetime. In particular, we consider a flat spacetime with a non-abelian gauge background, and discuss the traces needed for evaluating the gravitational anomalies in theories defined in two, four and six dimensions. The peculiar property of these traces is that they contain the insertion of a first order differential operator, making their evaluation much more demanding than the usual traces containing insertion of terms without differential operators.

Before focusing on the path integral representation of the above heat kernel traces, in chapter 1 we review the two mostly used boundary conditions one can adopt to compute path integral traces, namely, the Dirichlet boundary conditions (DBC) and the string inspired boundary conditions (SI). We study the perturbative expansion of the heat kernel with insertion of a scalar function by presenting two different looking path integral representations, and compute some related heat kernel coefficients (also known as Seeley-DeWitt coefficients) in DBC and SI. Then we prove their equivalence.

Next in chapter 2 we build up a path integral representation for the heat kernel associated to a scalar particle coupled to an arbitrary non-abelian gauge field and potential. We see that a way to write it consists in using a time ordering prescription on the matrix valued worldline action. Time ordering reveals itself crucial to reproduce the correct gauge transformation of the transition amplitude. After this preparation, we show how to generate a path integral representation for the generalized heat kernel traces, as needed for gravitational anomalies, starting from the one obtained for the above kernel, and compute the generalized heat kernel coefficient needed in two dimensions, already identified in [1].

In chapter 3 we present a different approach to realize a path integral representation for the generalized heat kernel traces, which makes use of new bosonic variables that upon quantization implement colour degrees of freedom. Once written a path integral for our expression we test it by reproducing [2], which contains the coefficient needed for anomalies in four dimensions.

Then, in chapter 4 we go beyond by computing the coefficient for the gravitational anomaly in six dimensions, which is the main novel task of the thesis.

Sommario

I metodi del nucleo del calore (heat kernel) rappresentano strumenti utili per studiare propagatori e azioni effettive ad un loop in teoria quantistica di campo. Una loro particolare applicazione riguarda il calcolo delle anomalie, che spesso possono essere riscritte come tracce del nucleo di calore con l'inserimento di appropriati operatori. Tali tracce possono essere calcolate utilizzando un path integral di prima quantizzazione.

In questa tesi viene presentata una rappresentazione in path integral per l'espressione generale utilizzata per calcolare anomalie associate alla conservazione del tensore energia-impulso (chiamate anomalie gravitazionali) in spazio-tempo piatto. In particolare, viene considerato uno spazio-tempo piatto con un campo di gauge non-abeliano, e vengono discusse tracce che servono per il calcolo di anomalie gravitazionali in teorie due, quattro e sei dimensionali. La caratteristica peculiare di queste tracce risiede nel fatto che al loro interno è stato inserito un operatore differenziale del primo ordine, ciò rende il calcolo molto più complesso rispetto a quello di usuali tracce senza l'inserimento di operatori differenziali.

Prima di concentrarci su una rappresentazione in path integral per l'anomalia, nel Capitolo 1 analizziamo le condizioni al contorno più utilizzate in letteratura per calcolare perturbativamente il path integral, ovvero le condizioni di Dirichlet (DBC) e le string inspired (SI). Viene inoltre studiata l'espansione perturbativa della traccia di un nucleo di calore con l'inserimento di una funzione scalare, presentando due differenti rappresentazioni in path integral e calcolando qualche coefficiente del nucleo di calore (meglio conosciuti come coefficienti di *Seeley-DeWitt*) in DBC e SI. In seguito, viene mostrata l'equivalenza tra le due rappresentazioni.

Nel Capitolo 2 viene ottenuta una rappresentazione in path integral per il nucleo di calore associato ad una particella scalare che interagisce lungo la sua linea di mondo, con un campo di gauge e un potenziale non-abeliani. Vediamo che un modo per scrivere il path integral in questo caso, è quello di usare il time ordering sull'azione matriciale della particella. Il time ordering infatti si rivela cruciale per produrre la corretta trasformazione di gauge per l'ampiezza di transizione. Finalmente, dopo questa preparazione, mostriamo come generare una rappresentazione in path integral per le tracce del nucleo di calore generalizzato, che servono per il calcolo delle anomalie gravitazionali, partendo da quella costruita sopra, e calcoliamo il coefficiente bidimensionale, già ottenuto da [1].

Nel Capitolo 3 viene proposto un approccio differente per realizzare col path integral tracce del nucleo di calore generalizzato; questo nuovo metodo si basa sull'introduzione di nuove variabili bosoniche che in prima quantizzazione realizzano i gradi di libertà di colore della particella. Ottenuto un path integral per la nostra espressione testiamo la rappresentazione trovata, controllando che riproduca correttamente [2] in cui è presente il coefficiente relativo al calcolo dell'anomalia in quattro dimensioni.

In seguito, nel Capitolo 4 procediamo oltre calcolando il coefficiente relativo all'anomalia in sei dimensioni; quest'ultimo risultato non è presente in letteratura ed è l'obbiettivo principale del lavoro di tesi.

Contents

Introduction

The heat kernel finds many applications in physics. Some of them will be briefly reviewed in this introduction to make clear its crucial role in quantum field theory and what will be analysed later on in this thesis.

The heat kernel was first introduced in QFT by J. Schwinger, who proposed in [3] that the Green's functions could be related to the dynamical properties of a fictitious particle with spacetime coordinates depending upon a proper time parameter. That relation lies in the fact that Green's function of a charged Dirac field in flat spacetime can be rewritten using what now is known as the *Schwinger's proper time representation*, from which the heat kernel naturally arises. Later B. DeWitt extended this procedure to curved spacetime [4], and found recurrence relations between the heat kernel coefficients.

To briefly review Schwinger's procedure, let us consider for instance an elliptic second order differential operator of the form

$$\hat{H} = -\partial^2 + m^2 + V(x) = \hat{H}_0 + V(x) \tag{1}$$

where ∂^2 indicates the laplacian in cartesian coordinates. The corresponding Green function (the Feynman propagator upon a Wick rotation) can be written in the Schwinger's representation as

$$\begin{aligned} D_F(x,y) &= \langle y|\frac{1}{-\partial^2 + m^2 + V}|x\rangle \\ &= \int_0^\infty d\beta \, \langle y|e^{-\beta(-\partial^2+m^2+V)}|x\rangle = \int_0^\infty d\beta \, K(\beta; x, y; \hat{H}) \end{aligned} \tag{2}$$

where β is the Schwinger's proper time. In this way the heat kernel

$$K(\beta; x, y; \hat{H}) = \langle y|e^{-\beta(-\partial^2+m^2+V)}|x\rangle \tag{3}$$

naturally arises in QFT. It satisfies the heat equation

$$-\frac{\partial}{\partial\beta}K(\beta; x, y; \hat{H}) = \hat{H}K(\beta; x, y; \hat{H}) \tag{4}$$

with boundary condition $K(0; x, y; \hat{H}) = \langle y|x\rangle = \delta^{(D)}(x - y)$.

For arbitrary potentials $V(x)$ one is not able to evaluate the heat kernel exactly. One useful way to evaluate it is to consider a perturbative expansion in the proper time β

$$K(\beta; x, y; \hat{H}) = K(\beta; x, y; \hat{H}_0)\left(1 + \beta a_2(x, y) + \beta^2 a_4(x, y) + \cdots\right) \tag{5}$$

where $K(\beta; x, y; \hat{H}_0)$ is explicitly known (the free heat kernel). The series coefficients $a_n(x, y)$ are fixed by plugging the above expansion in the heat equation, as done in [4].

The coincident points coefficients $a_n(x) \equiv a_n(x,x)$ are often called *Seeley-DeWitt* coefficients. Their main property lies in the fact that they deliver informations in terms of few geometric invariants (gauge curvature, Riemanian curvature etc.). These coefficients allow to obtain short distance behaviour of the Feynman propagator, and can be used to obtain the UV divergent part of the one-loop effective action.

Another useful application of heat kernel coefficients in QFT consists in the evaluation of anomalies, which may be identified as the Fujikawa Jacobian [5], appearing when deriving Ward identities for particular symmetries by performing a change of path integration variables. For change of variables given by infinitesimal symmetry transformations, the Jacobian differs from unity by a trace. The infinitesimal variation of the regulated Fujikawa Jacobian consists of a heat kernel trace with insertion of scalar functions or differential operators, the latter insertion describing some gravitational anomalies, and are the object of study of this thesis. So, again, one is interested in computing generalized heat kernel expansion and related Seeley-DeWitt coefficients.

In the above applications there are several ways to compute the coefficients, but one of the most elegant, which goes back to Schwinger [3], consists in representing the operator we need to compute the trace of by a quantum mechanical hamiltonian acting on a fictitious (non-physical) Hilbert space, as done above. Then, using Feynman's reformulation of quantum mechanics [6], the trace computation is recast as a worldline path integral with specific boundary conditions and evaluated. Such a program was beautifully carried out in the computation of chiral anomalies by E. Witten and L. Gaumè [7], where supersymmetric quantum mechanics was used, and later extended to the case of trace anomalies in [8, 9].

This procedure may be seen as part of the so-called *worldline formalism*, which defines a first quantized approach to quantum field theory and allows for a very elegant and efficient way to compute second quantized objects such as one-loop Feynman diagrams, propagators, anomalies etc.

In this work, we would like to implement a worldline path integral for computing perturbatively generalized heat kernel traces of the form

$$\mathrm{Tr}\left[\left(\sigma(x) + \xi^\mu(x)\nabla_\mu\right) e^{-\beta \hat{H}} \right] \tag{6}$$

where $\sigma(x)$ and $\xi_\mu(x)$ are respectively a scalar field and a vector field, while

$$\hat{H} = -\frac{1}{2}\nabla^2 + V \ , \qquad \nabla_\mu = \partial_\mu + A_\mu \tag{7}$$

is the hamiltonian operator (here written as normalized for a particle of unit mass) of a scalar particle interacting with a Yang-Mills field A_μ and a non-abelian Lorentz scalar potential V.

Such formulae appear in the study of anomalies generated by local translations and, in particular (6), is the most general expression needed in computing possible anomalies affecting the conservation of the stress tensor for a flat QFT defined in some fixed spacetime dimension. Their computation is of great interest. It may be used in verifying, possibly with different regularizations, statements about the structure of anomalies in various QFT, and in particular in the case of Weyl fermions as recently treated in [10, 11].

Chapter 1

Path integral and kernel traces

We start by introducing the heat kernel as the solution, with a particular boundary condition, of the Wick rotated Schrödinger equation, which properly speaking is a heat equation. Then, we consider the trace of the heat kernel with the insertion of a scalar function, and study perturbatively two different looking representation of the trace, using DBC and SI boundary conditions on the worldline variables. The equivalence between two different looking formulae for the insertion of an operator in the trace will be shown, as well as the worldline boundary independence of the whole result. This last step is first shown pertubatively, by computing Seeley-DeWitt coefficients up to third order in proper time both in SI and DBC finding the same result, then, we see that this fact is a consequence of a residual shift symmetry that can be turned in a BRST one, as shown in [12]. In the above study we use, as leading example, a simple hamiltonian describing a non-relativistic particle in D dimensions coupled to a scalar potential, but the results obtained in this chapter are of general validity.

1.1 Heat kernel

Starting from non relativistic quantum mechanics and doing a Wick rotation in the time parameter, one obtains a heat equation

$$-\frac{\partial}{\partial\beta}K(\beta,x,y) = \hat{H}\,K(\beta,x,y) \tag{1.1}$$

where the hamiltonian $\hat{H}$ is a second order elliptic differential operator. The formal solution with boundary condition

$$\lim_{\beta\to 0} K(\beta,x,y) = \delta(x-y) \tag{1.2}$$

is the *heat kernel* and, in the operatorial formalism, can be written as

$$K(\beta,x,y) = \langle y|e^{-\beta\hat{H}}|x\rangle \tag{1.3}$$

which corresponds, after undoing a Wick rotation, to the time evolution operator associated to a time independent quantum hamiltonian in the Schrödinger's picture. To explain what we are interested in (to start with) consider for instance a simple hamiltonian of the form

$$\hat{H} = \frac{\hat{p}^2}{2} + \hat{V}(\hat{x}) = -\frac{1}{2}\partial^2 + V(x) \tag{1.4}$$

where we indicate the laplacian by $\partial^2 = \partial^\mu \partial_\mu$. For an arbitrary potential an exact solution of (1.1) is not known, but one can perturbatively evaluate the kernel for small values of β, and the ansatz for this expansion is written as

$$\langle x|e^{-\beta\hat{H}}|y\rangle = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} e^{-\frac{(x-y)^2}{2\beta}} \Big(a_0(x,y) + a_1(x,y)\beta + a_2(x,y)\beta^2 + \cdots \Big) . \tag{1.5}$$

The coefficients at coincident points ($y \to x$) (which are called Seeley-DeWitt coefficients) are of great importance in computing for instance anomalies in QFT, terms in the one loop effective action, etc. These coefficients appears when one evaluates the functional trace of kernel operator

$$\mathrm{Tr}\,[e^{-\beta\hat{H}}] = \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \langle x|e^{-\beta\hat{H}}|x\rangle = \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \sum_{n=0}^{\infty} a_n(x)\beta^n \tag{1.6}$$

where $a_n(x) \equiv \lim_{y\to x} a_n(x,y)$ and some of them are listed below

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= -V \\
a_2 &= \frac{1}{2}V^2 - \frac{1}{12}\partial^2 V \\
a_3 &= -\frac{1}{6}V^3 + \frac{1}{12}V\partial^2 V + \frac{1}{24}\partial_\mu V \partial^\mu V - \frac{1}{240}\partial^4 V \, .
\end{aligned} \tag{1.7}$$

There exist several ways to compute the coefficients with different hamiltonian, but in this thesis we would like to use the path integral approach, which is a very physical and elegant way to study and compute objects like (1.6), even with the insertion of differential operators in that trace.

1.2 Kernel trace with insertion of a scalar function

From quantum mechanics we have a well-known path integral representation of partition function for hamiltonian (1.4), namely

$$\mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \int_{PBC} Dx\, e^{-S[x]} \tag{1.8}$$

where in this case one has the simple action

$$S[x] = \frac{1}{\beta}\int_0^1 d\tau \left(\frac{1}{2}\dot{x}^2 + \beta^2 V(x)\right) \tag{1.9}$$

obtained by Wick rotating in the action the time parameter, and rescaling the resulting euclidean time such that $\tau \in [0,1]$. The indication of PBC (periodic boundary conditions) in the path integral means a sum over periodic functions of unit period in the rescaled time, i.e. functions $x(\tau)$ such that $x(0) = x(1)$, so that in practice one makes a path integration on loops.

One can be more general and consider an insertion inside the trace in (1.8). In particular one can insert an operator $\hat{\sigma}(\hat{x})$, function of the position operator $\hat{x}$ only

$$\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] . \tag{1.10}$$

To recognize how to translate this expression into a functional integral, one can just write the trace using position eigenstates, as in (1.6), and then express the transition amplitude as a path integral

$$\begin{aligned}\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] &= \int d^D x\, \langle x|\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}|x\rangle = \int d^D x\, \sigma(x)\langle x|e^{-\beta\hat{H}}|x\rangle \\ &= \int_{PBC} Dx\, \sigma(x(0))\, e^{-S[x]} = \int_{PBC} Dx\, \sigma(x(\tau))e^{-S[x]}\end{aligned} \tag{1.11}$$

where $x(0) = x(1) = x$ is the starting point of the loop, that can be placed anywhere on the loop by using time translational invariance.

Alternatively one can exponentiate the operator $\hat{\sigma}(\hat{x})$ multiplied by a parameter λ, which may then be thought as a coupling constant that modifies the hamiltonian. Then one can take a derivative in λ and set $\lambda = 0$ to produce the insertion

$$\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] = \frac{\partial}{\partial\lambda}\mathrm{Tr}\left[e^{-\beta\hat{H}+\lambda\hat{\sigma}(\hat{x})}\right]\Big|_{\lambda=0} . \tag{1.12}$$

The trace guarantees that the insertion can be put on the left of the exponential. This relation is now translated into the path integral language by considering the action associated to the new hamiltonian. It is obtained by making the substitution $V(x) \mapsto V(x) - \frac{\lambda}{\beta}\sigma(x)$ which gives

$$S[x] \mapsto S[x,\lambda] = \int_0^1 d\tau\, \left(\frac{1}{2\beta}\dot{x}^2 + \beta V(x) - \lambda\sigma(x)\right) \tag{1.13}$$

so that the trace one the right hand side of (1.12) is rewritten as

$$\mathrm{Tr}\left[e^{-\beta\hat{H}+\lambda\sigma}\right] = \int_{PBC} Dx\, e^{-S[x,\lambda]} . \tag{1.14}$$

Now, taking a derivative respect to the real parameter λ and evaluating it at zero produces the alternative formula

$$\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] = \int_{PBC} Dx\, \left(\int_0^1 d\tau\, \sigma(x(\tau))\right) e^{-S[x]} . \tag{1.15}$$

This last formula must be then equivalent to the one obtained in (1.11).

The equivalence between these two expressions is understood by invoking the time translational invariance of the one-point function of the operator $\hat{\sigma}(\hat{x})$ appearing in (1.11), which may then be substituted by its time average in (1.15). In the following we would like to verify explicitly this equivalence.

Before that, we also need to discuss how to compute the above PBC path integral, which may be also done in two equivalent ways. First, we recall that the functional measure, formally defined in configuration space by

$$Dx = \prod_{\tau} \prod_{\mu=0}^{D} dx^{\mu}(\tau) \tag{1.16}$$

is translational invariant. Then we consider the fact that the complete set of loops, described by the PBC, can be created by first fixing a base point x, and then considering the loops with such a base point, $x(\tau) = x + q(\tau)$ where $q(0) = q(1) = 0$ define the Dirichlet boundary conditions (DBC). A final integration over all base points produces all the loops in space

$$\mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \int_{PBC} Dx\, e^{-S[x]} = \int d^D x \int_{DBC} Dq\, e^{-S[x+q(\tau)]} = \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \langle 1 \rangle_{DBC} \tag{1.17}$$

where $\langle \cdots \rangle_{DBC}$ indicates the normalized correlation functions in DBC. Thus, thanks to the translation invariance, the path integral measure can be split in the form

$$Dx = d^D x Dq \tag{1.18}$$

The factorization of the "zero mode" allows to invert the kinetic term in the action, which otherwise would present a null eigenvalue in PBC. This is precisely what was obtained from the operatorial formalism in (1.11). Using the above representation one can recast it as

$$\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] = \int_{PBC} Dx\, \sigma(x(\tau))e^{-S[x]} = \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \langle \sigma(x + q(\tau)) \rangle_{DBC} \tag{1.19}$$

while (1.15) can be rewritten as follows

$$\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] = \int_{PBC} Dx \int_0^1 d\tau\, \sigma(x(\tau))e^{-S[x]} = \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \left\langle \int_0^1 d\tau \sigma(x + q(\tau)) \right\rangle_{DBC} . \tag{1.20}$$

Similarly one can use SI boundary conditions (String Inspired), defined doing the same change of variable as before but with the condition

$$x = \int_0^1 d\tau\, x(\tau) \quad \Rightarrow \quad \int_0^1 q(\tau) = 0 \tag{1.21}$$

in which the zero mode is now the center of mass of the loop. We do not rewrite the above formulae since they look the same as before but with different boundary conditions. It should also be noted the normalization $(2\pi\beta)^{D/2}$ has been factorized in the above path integrals, so that $\langle 1 \rangle = 1$ on the free action in DBC or SI.

1.3 Equivalence between different formulae in DBC

In the last section we wrote different ways to represent a heat kernel trace with path integrals, in which we inserted a scalar function defined on the loop. Here we wish to check the equivalence between different looking formulae by using Dirichlet boundary conditions. The equivalence can be checked by a perturbative calculation of PBC path integral using Wick's theorem. This last step is carried out by computing correlators of worldline fields on the free theory, so first one needs the worldline propagator whose structure depends on the boundary chosen. The propagators in DBC and SI are shown in Appendix A, where we used the path integral to carry on the derivation.

Consider now (1.15)

$$
\begin{aligned}
\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] &= \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}} \left\langle \int_0^1 d\tau\, \sigma(x+q(\tau))\right\rangle_{DBC} \\
&= \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}} \sigma(x)\langle 1\rangle_{DBC} \\
&+ \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}} \left\langle \int_0^1 d\tau \left(q^\mu(\tau)\partial_\mu\sigma + \frac{1}{2}q^\mu(\tau)q^\nu(\tau)\partial_\mu\partial_\nu\sigma + \cdots\right)\right\rangle_{DBC}
\end{aligned} \tag{1.22}
$$

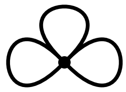
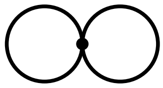
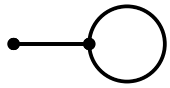
while taking (1.11) one has

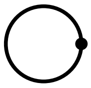
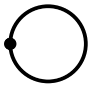
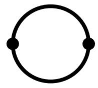
$$
\begin{aligned}
\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] &= \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}} \langle \sigma(x+q(\tau))\rangle_{DBC} \\
&= \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}} \sigma(x)\langle 1\rangle_{DBC} \\
&+ \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}} \left\langle \left(q^\mu(\tau)\partial_\mu\sigma + \frac{1}{2}q^\mu(\tau)q^\nu(\tau)\partial_\mu\partial_\nu\sigma + \cdots\right)\right\rangle_{DBC} .
\end{aligned} \tag{1.23}
$$

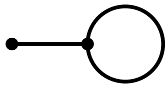
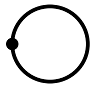
Thus equivalence is proved by showing the last line in (1.22) and (1.23) vanishes. We will use DBC to carry out the calculations, but we report the value of the correlation functions also in SI for later use. So starting form (1.22) one can compute the last line to third order and show that it produces total derivatives, which vanish when x- integrated.

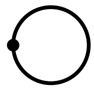
In facts up to third order one has

$$\int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \Big\langle \int_0^1 d\tau \, \Big(q^\mu(\tau)\partial_\mu\sigma + \frac{1}{2} q^\mu(\tau) q^\nu(\tau) \partial_\mu\partial_\nu\sigma + \cdots \Big) \Big\rangle_{DBC} =$$

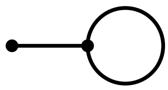
$$\int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \Big[-\frac{\beta^3}{6!} 15 \partial^6\sigma$$  $$-\frac{\beta^3}{4!} 3\partial^4\sigma$$  $$-\frac{\beta^3}{3!} 3\partial_\mu\sigma\partial^\mu\partial^2 V$$ 

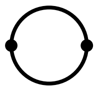
$$-\frac{\beta^3}{4}\partial^2\sigma\partial^2 V$$   $$-\frac{\beta^3}{4} 2\partial_\mu\partial_\nu\sigma\partial^\mu\partial^\nu V$$  (1.24)

$$-\frac{\beta^3}{3!} 3\partial^\mu V \partial_\mu \partial^2\sigma$$  $$-\frac{\beta^3}{4} V^2\partial^2\sigma$$ 

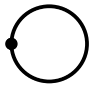
$$+ \big(\beta^2 V \partial_\mu V \partial^\mu\sigma - \beta\partial_\mu\sigma\partial^\mu V\big)$$ $$-\frac{\beta}{2}\partial^2\sigma\,(1-\beta V)$$  $$\Big]$$

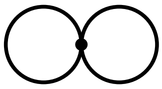
where Feynman diagrams are easily evaluated by

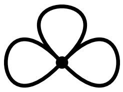
 $$= \int_0^1 d\sigma \int_0^1 d\tau\, \Delta(\tau,\sigma)\Delta(\sigma,\sigma) = \begin{cases} \frac{1}{60} & \text{DBC} \\ 0 & \text{SI} \end{cases}$$

 $$= \int_0^1 d\sigma \int_0^1 d\tau\, \Delta^2(\tau,\sigma) = \begin{cases} \frac{1}{90} & \text{DBC} \\ \frac{1}{720} & \text{SI} \end{cases}$$ (1.25)

 $$= \int_0^1 d\sigma \int_0^1 d\tau\, \Delta(\tau,\sigma) = \begin{cases} -\frac{1}{12} & \text{DBC} \\ 0 & \text{SI} \end{cases}$$

 $= \int_0^1 d\tau\, \Delta(\tau,\tau) = \begin{cases} -\dfrac{1}{6} & \text{DBC} \\ -\dfrac{1}{12} & \text{SI} \end{cases}$

 $= \int_0^1 d\tau\, \Delta^2(\tau,\tau) = \begin{cases} \dfrac{1}{30} & \text{DBC} \\ \dfrac{1}{144} & \text{SI} \end{cases}$

 $= \int_0^1 d\tau\, \Delta^3(\tau,\tau) = \begin{cases} -\dfrac{1}{140} & \text{DBC} \\ -\dfrac{1}{1728} & \text{SI} \end{cases}$

The terms in (1.24) produces a total derivative that vanishes when x- integrated, in fact

$$\begin{aligned} &\beta^3 \int d^D x\Big(+ \frac{1}{240} V\partial^4\sigma + \frac{1}{120}\sigma\partial^4 V - \frac{1}{180}\sigma\partial^4 V - \frac{1}{144}\sigma\partial^4 V\Big) \\ &= \frac{\beta^3}{80}\int d^D x \left(\sigma\partial^4 V - \sigma\partial^4 V\right) = 0 \end{aligned} \tag{1.26}$$

while considering the last line of (1.23) one has up to third order

$$\begin{aligned} &\int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \Big\langle \Big(\partial_\mu\sigma q^\mu(\tau) + \frac{1}{2}\partial_\mu\partial_\nu\sigma q^\mu(\tau) q^\nu(\tau) + \cdots\Big)\Big\rangle_{DBC} \\ &= \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \Big[\beta^2\sigma\partial^2 V \left(- \int_0^1 ds\, \Delta(s,\tau) + \frac{1}{2}\Delta(\tau,\tau)\right) \\ &+ \beta^3\sigma\partial^4 V\Big(\frac{1}{2}\int_0^1 ds\, \Delta(\tau,s)\Delta(s,s) + \frac{1}{24}\Delta(\tau,\tau) - \frac{1}{2}\Delta^2(s,\tau) \\ &+ \int_0^1 ds\, \frac{1}{2}\Delta(\tau,\tau)\Delta(s,\tau) - \frac{1}{8}\Delta^2(\tau,\tau)\Big) + \cdots\Big] \end{aligned} \tag{1.27}$$

which vanishes before x-integration, using the following results

$$\begin{aligned} \int_0^1 ds\, \Delta(\tau,s)\Delta(s,s) &= \frac{1}{12}\Delta^2(\tau,\tau) - \frac{1}{12}\Delta(\tau,\tau) \\ \int_0^1 ds\, \Delta(\tau,s) &= \frac{1}{2}\Delta(\tau,\tau) \\ \int_0^1 ds\, \Delta^2(\tau,s) &= \frac{1}{3}\Delta^2(\tau,\tau)\ . \end{aligned} \tag{1.28}$$

So this perturbative calculation checks the equivalence in DBC between the two different looking formulae. However, to prove it to all orders, we recall that one may use the time translational (TT) symmetry, which holds because the PBC action is TT invariant. In fact, under a time translation with infinitesimal constant parameter ϵ, one finds

$$\delta S = \int_0^1 d\tau\, \epsilon \frac{d}{d\tau}\left(\frac{1}{2\beta}\dot{x}^2 + \beta V(x(\tau))\right) \tag{1.29}$$

which vanishes because of periodic boundary conditions. So, using TT, in DBC, one only has to evaluate the expectation value of one on the loop

$$\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] = \int_{PBC} Dx\,\sigma(x(0))e^{-S[x]} = \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\sigma(x)\langle 1\rangle_{DBC} \tag{1.30}$$

which shows the equivalence between the two formulae (1.11) and (1.15) to all order in perturbation theory when using the DBC procedure in calculating the PBC path integral.

Thus, the above computation proves the equivalence in DBC between different looking path integral representations of the heat kernel, in which we made the insertion of a scalar function defined on the loop and shows how the worldline expansion is quite efficient, among other methods, in computing perturbatively the kernel.

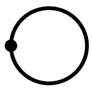
A final remark should be done if one uses SI to compute the path integral. In SI quantum fluctuations do not vanish, but they sum up to $\langle 1\rangle_{SI}$ to produce the correct final result (because this time x is not a point on the loop so they interfere with the results on the loop). Thus equivalence to all order in SI is more demanding then DBC, but it can be shown as explained in the next section of this chapter.

1.4 Equivalence between SI and DBC

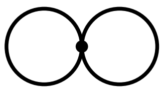
We showed equivalence between different looking formulae in DBC even if as a final remark we said that quantum fluctuations in SI are different from zero. Here we would like to compute Seeley-De Witt coefficients listed in (1.7) both using DBC and SI. This will make clear the remark about non vanishing quantum fluctuations in the string inspired boundary and also, it will verify perturbatively the equivalence between the two boundary conditions.

Let us start by considering DBC in which there is not any contribution by quantum fluctuations, as showed above by using TT. Taylor expanding the action in the exponential and using for instance (1.15) one has

$$\mathrm{Tr}\left[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}\right] = \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\sigma(x)\Big[1 - \beta V + \frac{1}{2}\beta^2V^2 + \frac{\beta^2}{2}\partial^2V$$

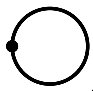


$$-\frac{1}{6}\beta^3V^3 - \frac{\beta^3}{4!}3\partial^4V$$



$$-\frac{\beta^3}{2}\partial_\mu V\partial^\mu V$$

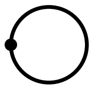
$$-\frac{\beta^3}{4}V\partial^2V$$

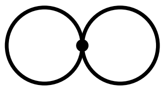
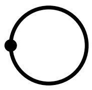
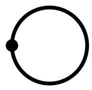


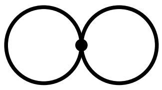
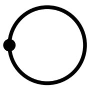
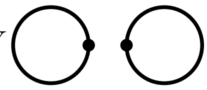
$$\Big] \tag{1.31}$$

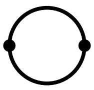
which correctly generates the coefficients in (1.7) that are multiplied by the scalar function.

Switching now to SI, one has to include some quantum fluctuations to the above terms (evaluated in SI), so we have

$$\mathrm{Tr}\,[\hat{\sigma}(\hat{x})e^{-\beta\hat{H}}] = \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}}\,\sigma(x)\Big[1 - \beta V + \frac{1}{2}\beta^2 V^2 + \frac{\beta^2}{2}\partial^2 V$$

$$- \frac{1}{6}\beta^3 V^3$$

$$-\frac{\beta^3}{4!}3\partial^4 V$$

$$-\frac{\beta^3}{2}V\partial^2 V$$

$$\Big] + \frac{\beta^2}{2}V\partial^2\sigma$$


$$-\frac{\beta^3}{4!}3V\partial^4\sigma$$

$$-\frac{\beta^3}{4}V^2\partial^2\sigma$$

$$-\frac{\beta^3}{4}\partial^2\sigma\partial^2 V$$


$$-\frac{\beta^3}{4}2\partial_\mu\partial_\nu\sigma\partial^\mu\partial^\nu V$$

(1.32)

where the terms containing derivatives of σ are the non vanishing contributions from quantum fluctuations. One can easily see that (after some part integration) the two results coincide and correctly reproduce the coefficients in (1.7). So the above computation proves perturbatively the equivalence between DBC and SI, by showing they produce the same result, further shows that there is a deeper link between them.

In fact one can show to all order that results do not depend upon the boundary chosen, using BRST symmetry. When splitting the zero mode, by using translational invariance of functional measure, one is left with a residual shift symmetry

$$\begin{aligned} \delta x^\mu &= \epsilon^\mu \\ \delta q^\mu(\tau) &= -\epsilon^\mu \end{aligned} \tag{1.33}$$

which behaves as a gauge symmetry, so one can rewrite that introducing new ghost fields such that

$$\begin{aligned} \delta x^\mu &= \eta^\mu \Lambda \\ \delta q^\mu(\tau) &= -\eta^\mu \Lambda \\ \delta \eta^\mu &= 0 \\ \delta \bar{\eta}^\mu &= i\pi^\mu \Lambda \\ \delta \pi_\mu &= 0 \end{aligned} \tag{1.34}$$

where Λ is the BRST grassman odd parameter, η and $\bar{\eta}$ are constant anticommuting fields while π_μ is a bosonic field. To fix the gauge (which in this case means choosing boundary conditions on the worldline field $q^\mu(\tau)$) one introduces a "gauge fixing fermion"

$$\psi[\rho] = \bar{\eta}_\mu \int_0^1 d\tau\, \rho(\tau) q^\mu(\tau) \tag{1.35}$$

where $\rho(\tau)$ is a distribution defined on the loop such that $\int_0^1 d\tau \rho(\tau) = 1$ and one adds, in the action (1.9), the following gauge fixing term

$$\frac{\delta_L \psi}{\delta \Lambda} = -\bar{\eta}^\mu \eta_\mu + i\pi_\mu \int_0^1 d\tau\, \rho(\tau) q^\mu(\tau) \tag{1.36}$$

this last term preserves BRST symmetry of the action since nilpotence of the BRST variation (the operator $\delta_L/\delta\Lambda$ means the left Λ derivative of $\delta_{BRST}\psi$). So the action rewrites as

$$S[x,q,\pi,\eta,\bar{\eta}] = S[x+q] - \bar{\eta}^\mu\eta_\mu + i\pi_\mu \int_0^1 d\tau\, \rho(\tau) q^\mu(\tau) \tag{1.37}$$

where $S[x+q]$ is the action (1.9). Now, the path integration over anti-commuting fields produces one, while the integration over the bosonic field π_μ produces the constraint

$$\int_0^1 d\tau\, \rho(\tau) q^\mu(\tau) = 0 \quad \Rightarrow \quad \int_0^1 d\tau\, \rho(\tau) x^\mu(\tau) = x^\mu \tag{1.38}$$

since BRST invariance (note that gauge fixed action is BRST invariant but we broke shift symmetry by fixing the gauge) of the action the result should be ρ independent so, for instance, one can choose $\rho(\tau) = 1$ to produce SI boundary or $\rho(\tau) = \delta(\tau)$ for having DBC.

This proves how the two boundaries are simply different gauge fixings so the whole computation will not depend upon the fixed gauge because of the BRST invariance of the action. It also checks equivalence between (1.11) and (1.15) to all order in SI. In fact, since DBC and SI arise just as different gauge fixing conditions, once proved the equivalence between the two formulae in DBC it should also hold for a different gauge fixing condition.

As a final remark one should note that the advantage of SI lies in the fact that the free theory propagator is TT invariant, so this should simplify calculations. This indeed happens in many situations, but in our case, as showed above, one has non vanishing fluctuations (while they vanish to all orders in DBC, as proved in the previous section with the DBC propagator which is not TT invariant), and thus fluctuations must be added to $\langle 1 \rangle_{SI}$ when evaluating kernels.

Chapter 2

Worldline path integral for coloured particles

We start this chapter by considering a scalar particle on the worldline coupled to a non-abelian gauge field and a non-abelian Lorentz scalar potential, then we study gauge transformation property of gauge fields, potential and quantum states of the theory, which are crucial to build up the transition amplitude for the model. In fact we show that the path integral for this kind of transition amplitude needs a time ordering prescription in front of the exponential of the action. This is crucial to produce the same gauge variation of transition amplitude one would obtain by using operator formalism. Finally, starting with the above path integral, we build up a path integral representation for the generalized heat kernel trace given in (6). We do that by additionally coupling the particle to an abelian gauge field (i.e. the vector field $\xi_\mu(x)$ appearing in (6)), using a parameter as coupling constant. Then, taking a derivative respect to the coupling and setting it to zero, allows to obtain the generalized heat kernel trace.

2.1 Coloured scalar particle on the worldline

We start by writing down the hamiltonian of a worldline scalar particle interacting with a non-abelian gauge field A_μ and a non-abelian Lorentz scalar potential V. We expand these potentials on anti-hermitian Lie algebra generators T^a, with Lie algebra $[T^a, T^b] = f^{abc}T^c$, as follow $A_\mu(x) = A^a_\mu(x)T^a$ and $V(x) = iV^a(x)T^a$.

Using the standard covariant derivative one can write down the following hamiltonian

$$\hat{H} = \frac{1}{2}(\hat{p} - iA)^2 + V = -\frac{1}{2}\nabla^2 + V \qquad \nabla_\mu = \partial_\mu + A_\mu \,. \tag{2.1}$$

Let us now rederive how A_μ transforms under a gauge symmetry generated by an element $g(x) = e^{\omega(x)} = e^{i\omega^a(x)T^a}$ of the gauge group, whose generators are indicated by T^a while $\omega^a(x)$ are the local gauge group parameters. We can start by considering the action of the gauge group on wave functions in configuration space, which are assumed to transform covariantly together with their derivatives,

$$\begin{aligned} \Psi(x) &\longmapsto \Psi^{'}(x) = g(x)\Psi(x) \\ \nabla_\mu\Psi &\longmapsto \nabla^{'}_\mu\Psi' = g(x)\nabla_\mu\Psi(x) \end{aligned} \tag{2.2}$$

the last equation can be further evaluated

$$\begin{aligned}\nabla'_\mu \Psi' = \partial_\mu \Psi' + A'_\mu \Psi' &= \partial_\mu g\Psi + g\partial_\mu \Psi + A'_\mu g\Psi \\ &= g\partial_\mu \Psi + gA_\mu \Psi\end{aligned}$$

so that one gets the transformation rule for the Yang-Mills field

$$A'_\mu(x) = g(x)A_\mu(x)g^\dagger(x) - \partial_\mu g(x)\, g^\dagger(x) \tag{2.3}$$

where we used the fact that $g(x)$ is unitary. Further, requiring the gauge invariance of the Schrödinger equation fixes transformation rule for the scalar potential

$$V'(x) = g(x)\, V(x)\, g^\dagger(x)\ . \tag{2.4}$$

Thus we see that both the gauge field and scalar potential belongs to the gauge algebra since they transform according to the adjoint representation of the gauge group (note that A_μ is a connection on the gauge bundle, so it presents an extra piece in the transformation).

Starting with the gauge field one has the background curvature

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{2.5}$$

which lies in gauge algebra and can be obtained considering the commutator of covariant derivatives acting on a wave function on which covariant derivative reduces to the one defined in (2.1). We are interested in making explicit how covariant derivative acts on gauge algebra elements, since in chapter 4 we are going to manipulate this kind of objects. Requiring a transformation rule like (2.4) for the curvature derivative gives

$$\nabla_\mu F_{\nu\lambda} = \partial_\mu F_{\nu\lambda} + [A_\mu, F_{\nu\lambda}]\ . \tag{2.6}$$

which is a way of writing the covariant derivative for objects that transform in the adjoint representation of the gauge group. To conclude this brief description we list some useful geometrical properties used to simplify calculations

$$\begin{aligned}[]
[\nabla_\mu, \nabla_\nu]F_{\rho\lambda} &= [F_{\mu\nu}, F_{\rho\lambda}] \\
\mathrm{tr}_{\,YM}(\nabla_\rho \nabla_\lambda F_{\mu\nu}) &= \mathrm{tr}_{\,YM}(\nabla_\lambda \nabla_\rho F_{\mu\nu}) \\
\nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \nabla_\mu F_{\nu\lambda} &= 0\ .
\end{aligned} \tag{2.7}$$

The first identity could even be written in a much more general form (making implicit the generator of the adjoint representation on the r.h.s) but we specialize it to our case, the second one practically says that under traces of that kind covariant derivatives commute, while the third one is the Bianchi identity on a gauge background. We stress that the first two properties in (2.7) apply to a generic gauge algebra element.

2.2 Gauge symmetry and coloured kernel path integral

In order to build up the transition amplitude one needs to set up the quantum states of the system (i.e. the Hilbert space) and, particularly, position eigenstates. The latter are defined as a tensor product of colourless position eigenstates and a basis of colour states in some fixed representation of the gauge group. This generates the whole particle Hilbert space

$$|x, j\rangle = |x\rangle \otimes |j\rangle \; . \tag{2.8}$$

Thus a generic wave function can be written in the basis of position eigenstates as

$$\Psi_j(x) = \langle x, j|\Psi\rangle \tag{2.9}$$

with the usual identity resolution and scalar product of states (the difference in writing them is that now one has colour indices in states)

$$\hat{\mathbb{1}} = \sum_j \int d^D x \, |x, j\rangle\langle x, j| \tag{2.10}$$

such that

$$\langle x, i|y, j\rangle = \delta(x - y)\, \delta_{ij} \; . \tag{2.11}$$

Expanding momentum and position operator using the identity resolution produces the usual quantum correspondence, but with a colour Kronecker delta

$$\begin{aligned} \langle x, i|\hat{x}^\mu|y, j\rangle &= x^\mu \, \delta(x - y)\delta_{ij} \\ \langle x, i|\hat{p}^\mu|y, j\rangle &= -i\partial_\mu \, \delta(x - y)\delta_{ij} \; . \end{aligned} \tag{2.12}$$

We would like to build up now a path integral representation for the transition amplitude associated to the coloured scalar particle. This can be done by using gauge symmetry, which allows to identify the correspondence between operator formalism and path integral. Let us now explain this strategy in details. First we need to know how position eigenstates transform under gauge group action. It is easily done by considering the action of gauge group on wave functions in configuration space, indeed one has

$$\Psi'_i(x) = {}_g\langle x, i|\Psi\rangle = \langle x, i|g(\hat{x})|\Psi\rangle = g_i{}^j(x)\, \langle x, j|\Psi\rangle = g_i{}^j(x)\, \Psi_j(x) \tag{2.13}$$

from which one reads the following gauge transformation for position eigenstates

$$|x, i\rangle_g = g^\dagger{}_j{}^i(x)\, |x, j\rangle \; . \tag{2.14}$$

This implies the following gauge transformation for the transition amplitude

$$\langle x_f, t_f|x_i, t_i\rangle_g = g(x_f)\langle x_f, t_f|x_i, t_i\rangle g^\dagger(x_i) \tag{2.15}$$

where colour indices have been suppressed and unitarity of the gauge operator $g(x)$ has been used. The infinitesimal version of this gauge transformation, with $g(\omega) = 1 + \omega$, reads

$$\begin{aligned} \delta_g\langle x_f, t_f|x_i, t_i\rangle_g &= \langle x_f, t_f|\omega(x_f) - \omega(x_i)|x_i, t_i\rangle \\ &= \omega(x_f)\langle x_f, t_f|x_i, t_i\rangle - \langle x_f, t_f|x_i, t_i\rangle\omega(x_i) \; . \end{aligned} \tag{2.16}$$

We show now that the path integral representation for the above transition amplitude is given by

$$\langle x_f, t_f|x_i, t_i\rangle = \int_{x(t_i)=x_i}^{x(t_f)=x_f} Dx \, \mathrm{T}\, e^{iS[x;A]} \tag{2.17}$$

where T denotes time ordering prescription while the minkowskian action is defined as

$$S[x;A]=\int_{t_i}^{t_f}dt\left(\frac{1}{2}\dot{x}^2+i\dot{x}^\mu A_\mu-V\right)\ . \tag{2.18}$$

It should be noted that the action is matrix valued, since A_μ and V are Lie algebra valued. To show that (2.17) is the correct path integral representation for the transition amplitude associated to the coloured particle, we have to evaluate its infinitesimal gauge variation and check that it exactly reproduces (2.16), where we used operator formalism. First one considers the infinitesimal variation of the action under a gauge group transformation

$$\begin{aligned}\delta_g S[x;A]&=\int_{t_i}^{t_f}dt\ (i\dot{x}^\mu\delta_g A_\mu-\delta_g V)\\&=\int_{t_i}^{t_f}dt\,(-i\dot{x}^\mu[A_\mu,\omega(x)]-i\dot{x}^\mu\partial_\mu\omega(x)-[\omega(x),V])\\&=\int_{t_i}^{t_f}dt\,(-i\dot{x}^\mu\nabla_\mu\omega(x)-[\omega(x),V])\\&=\int_{t_i}^{t_f}dt\left(-i\frac{d\omega(x)}{d\tau}-[\omega(x),V]\right)\\&=-i\omega(x_f)+i\omega(x_i)-\int_{t_i}^{t_f}dt\,[\omega(x),V]\end{aligned} \tag{2.19}$$

then evaluating the gauge variation of path integral, using the above variation of the action, one has

$$\begin{aligned}\delta_g\int_{x(t_i)=x_i}^{x(t_f)=x_f}Dx\,\mathrm{T}\,e^{iS[x;A]}&=\int_{x(t_i)=x_i}^{x(t_f)=x_f}Dx\,\mathrm{T}\ \delta_g S\,\frac{\delta}{\delta S}\,e^{iS[x;A]}\\&=\int_{x(t_i)=x_i}^{x(t_f)=x_f}Dx\,\mathrm{T}\left[\omega(x_f)e^{iS[x;A]}-\omega(x_i)e^{iS[x;A]}-i\int_{t_i}^{t_f}[\omega,V]e^{iS[x;A]}\right]\\&=\omega(x_f)\int_{x(t_i)=x_i}^{x(t_f)=x_f}Dx\,\mathrm{T}\,e^{iS[x;A]}-\int_{x(t_i)=x_i}^{x(t_f)=x_f}Dx\,\mathrm{T}\,e^{iS[x;A]}\,\omega(x_i)\\&=\omega(x_f)\langle x_f,t_f|x_i,t_i\rangle-\langle x_f,t_f|x_i,t_i\rangle\omega(x_i)\\&=\delta_g\langle x_f,t_f|x_i,t_i\rangle\end{aligned} \tag{2.20}$$

which exactly reproduces the variation (2.16) and shows how time ordering prescription is crucial to obtain the same gauge transformation of transition amplitude written in the operator formalism.

Now, taking PBC in (2.17) after Wick rotating in time parameter and rescaling the resulting euclidean time such that $\tau\in[0,1]$, one obtains the coloured heat kernel trace

$$\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]=\int_{PBC}Dx\,\mathrm{T}\,e^{-S[x;A]} \tag{2.21}$$

where the euclidean action is defined as

$$S\,[x;A]=\frac{1}{\beta}\int_0^1 d\tau\left(\frac{1}{2}\dot{x}^2+\beta\dot{x}^\mu A_\mu+\beta^2 V\right) \tag{2.22}$$

Further one can consider traces as (1.15) by using results of chapter 1 (note that time ordering acts on matrices so objects with no colour indices are not affected). The evaluation of the above traces can be carried out as usual (factorizing the zero mode and using DBC or SI boundary conditions) but this time one also traces over colour indices in the fundamental representation of gauge group. Some final remark is needed about the amplitude (2.17).

We proved that the two representations (operator and path integral one) transform under gauge action the same way however this doesn't fully complete the proof, in fact one should also check that the action produces hamiltonian (2.1) in first quantization and that the amplitudes transform the same way under global symmetries of action, but all this is true in our case.

2.3 Generalized heat kernel and path integral

Once written a path integral representation for the Yang-Mills kernel trace we would like to consider more general coloured kernel traces in which we insert a first order differential operator inside the trace

$$\mathrm{Tr}\left[\left(\sigma(x)+\xi^{\mu}(x)\nabla_{\mu}\right)e^{-\beta\hat{H}}\right] \tag{2.23}$$

where $\xi^{\mu}(x)$ is a vector field and $\sigma(x)$ a scalar field while the hamiltonian and the covariant derivative have been defined in (2.1). We refer to (2.23) as generalized heat kernel trace while we will refer to the associated kernel coefficients as generalized coefficients.

Formula (2.23) is the general expression we find in computing possible anomalies affecting the conservation of the stress tensor. Indeed, under an arbitrary infinitesimal change of coordinates, $x_{\mu}\to x_{\mu}-\xi_{\mu}(x)$, a generic scalar field $\phi(x)$ transforms as $\delta\phi=\xi^{\mu}\partial_{\mu}\phi$, and in the Fujikawa's approach to the anomalies one considers the infinitesimal part of the jacobian dictated by a symmetry transformation on ϕ, given by the functional trace of

$$\frac{\partial\delta\phi(x)}{\partial\phi(y)}=\xi^{\mu}(x)\partial_{\mu}\delta^{D}(x-y)\ . \tag{2.24}$$

Sometimes, one finds more convenient to use the Hawking-Fujikawa variables $\tilde{\phi}=g^{\frac{1}{4}}\phi$, where g is the determinant of the spacetime metric, which produces instead the functional trace of

$$\frac{\partial\delta\tilde{\phi}(x)}{\partial\tilde{\phi}(y)}=\left(\xi^{\mu}(x)\partial_{\mu}+\frac{1}{2}(\partial_{\mu}\xi^{\mu}(x))\right)\delta^{D}(x-y)\ . \tag{2.25}$$

All of these traces are regulated by the heat kernel with a suitable $\hat{H}$, as in (2.23).

In literature there are results as needed for the anomalies in two and four dimensions, corresponding to the generalized coefficients of first and second order in proper time, obtained by using operator methods. In this section, starting from (2.21), we implement a worldline path integral to represent (2.23) and then, in chapter 4, using perturbation theory on the worldline, we compute the generalized coefficient at third order in proper time.

Now, in order to insert a first order differential operator inside the trace (2.21), we couple the coloured particle to ξ_{μ}, which we now consider as an abelian gauge field, by making the substitution $A_\mu \to A_\mu + \lambda\xi_\mu$ in which λ is a parameter thought as a coupling constant. This produces the following change in the worldline euclidean action

$$S[x;A] \to S[x,\lambda;A] = \int_0^1 d\tau \left(\frac{\dot{x}^2}{2\beta} + \dot{x}^\mu A_\mu + \lambda \dot{x}^\mu \xi_\mu + \beta V\right) \tag{2.26}$$

which in turn modifies the hamiltonian, which acquires additional pieces, so that (2.21) is rewritten as

$$\mathrm{Tr}\left[e^{-\beta(\hat{H}-\lambda\xi^\mu\nabla_\mu - \frac{\lambda}{2}\partial_\mu\xi^\mu - \frac{\lambda^2}{2}\xi^2)}\right] = \int_{PBC} Dx\, \mathrm{T}\, e^{-S[x,\lambda;A]} \,. \tag{2.27}$$

Indeed, the new coupling produces the following change in the particle quantum hamiltonian

$$\hat{H} \to \hat{H} - \lambda\xi^\mu\nabla_\mu - \frac{\lambda}{2}\partial_\mu\xi^\mu - \frac{\lambda^2}{2}\xi^2 \tag{2.28}$$

in which we used the fact that ξ_μ behaves as a scalar under covariant differentiation. Taking a derivative respect to λ in (2.27) and evaluating it at zero produces

$$\mathrm{Tr}\left[\left(\xi^\mu\nabla_\mu + \frac{1}{2}(\partial_\mu\xi^\mu)\right)e^{-\beta\hat{H}}\right] = \int_{PBC} Dx \left(-\frac{1}{\beta}\int_0^1 d\tau\, \xi_\mu(x)\dot{x}^\mu\right) \mathrm{T}\, e^{-S[x;A]} \tag{2.29}$$

further, the trace guarantees that the operator insertion can be placed on the left of the exponential so no ordering ambiguity arises from this procedure.

Let us now rewrite (2.29) in a more comfortable way to make computation. Splitting the zero mode integration by using DBC on path integral one has

$$\mathrm{Tr}\left[\left(\xi^\mu\nabla_\mu + \frac{1}{2}(\partial_\mu\xi^\mu)\right)e^{-\beta\hat{H}}\right] = \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}} \Big\langle -\frac{1}{\beta}\int_0^1 d\tau\, \xi_\mu(x+q(\tau))\dot{q}^\mu(\tau)\Big\rangle_{DBC} \tag{2.30}$$

where the expectation value in the last line is defined as

$$\Big\langle -\frac{1}{\beta}\int_0^1 ds\, \xi_\mu(x+q(\tau))\,\dot{q}^\mu(\tau)\Big\rangle_{DBC} = \int_{DBC} Dq\left(-\frac{1}{\beta}\int_0^1 d\tau\, \dot{q}^\mu(\tau)\,\xi_\mu(x+q(\tau))\right) \mathrm{T}\, e^{-S[q;A]} \tag{2.31}$$

with the following euclidean action

$$S[q;A] = \int_0^1 d\tau \left(\frac{1}{2\beta}\dot{q}^2 + \dot{q}^\mu(\tau)\,A_\mu(x+q(\tau)) + \beta V(x+q(\tau))\right). \tag{2.32}$$

In the above lines we pass ξ_μ through time ordering prescription since it is abelian. As a remark we would like to say that the above derivation also works in SI, because of the results of chapter 1, even if we decided to use DBC for the present calculations.

Now, in order to make calculations easier we need to fix a gauge and, as once said by prof. Bastianelli, it is an art. In our case, the artful choice is Fock-Schwinger (FS) gauge, defined by the condition

$$q^\mu(\tau)A_\mu(x^\nu + q^\nu(\tau)) = 0\,. \tag{2.33}$$

FS gauge directly allows to expand gauge field in terms of background curvature as

$$A_\mu(x+q) = A_\mu(x) + \frac{1}{2}q^\nu F_{\nu\mu}(x) + \frac{1}{3}q^\alpha q^\beta \nabla_\beta F_{\alpha\mu}(x) + \frac{1}{8}q^\lambda q^\sigma q^\rho \nabla_\rho\nabla_\sigma F_{\lambda\mu}(x) + \cdots \tag{2.34}$$

the same expansion holds for ξ^μ replacing covariant derivative with partial ones, since it is an abelian gauge field, further we defined abelian curvature as

$$G_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) - \partial_\nu \xi_\mu(x) \; . \tag{2.35}$$

Now, in order to build up some notation, we rewrite (2.30) by making explicit its dependence by generalized coefficients

$$\begin{aligned} \mathrm{Tr}\left[\Big(\xi^\mu \nabla_\mu + \frac{1}{2}(\partial_\mu \xi^\mu)\Big) e^{-\beta \hat{H}}\right] &= \mathrm{tr}_{\,YM} \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \sum_{n=0}^{\infty} \tilde{b}_n \, \beta^n \\ &= \mathrm{tr}_{\,YM} \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \Big(\tilde{b}_0 + \tilde{b}_1 \beta + \tilde{b}_2 \beta^2 + \tilde{b}_3 \beta^3 + \cdots\Big) \end{aligned} \tag{2.36}$$

where the generalized coefficients up to third order in proper time are listed below

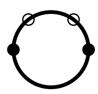
$$\begin{aligned} \tilde{b}_0 &= 0 \\ \tilde{b}_1 &= \frac{1}{24} G_{\mu\nu} F^{\mu\nu} \\ \tilde{b}_2 &= \frac{1}{720}\Big[-2\nabla_\lambda F_{\mu\nu} \partial^\lambda G^{\mu\nu} + \nabla^\nu F_{\nu\mu} \partial_\lambda G^{\lambda\mu} - 30 V F_{\mu\nu} G^{\mu\nu} \Big] \\ \tilde{b}_3 &= \frac{1}{1440}\Big[-6 G_{\mu\nu} V \nabla^2 F^{\mu\nu} - 6 F^{\mu\nu} V \partial^2 G_{\mu\nu} - \partial_\alpha G_{\mu\nu} F^{\mu\nu} \nabla^\alpha V - 2\partial^\lambda G_{\lambda\mu} F^{\mu\alpha} \nabla_\alpha V \\ &\quad + G_{\mu\nu} \nabla^\alpha F^{\mu\nu} \nabla_\alpha V + 2 G^{\mu\nu} \nabla^\lambda F_{\lambda\mu} \nabla_\nu V - 8 \partial_\lambda G_{\mu\nu} \nabla^\lambda F^{\mu\nu} V - 2\partial^\lambda G_{\lambda\alpha} \nabla_\mu F^{\mu\alpha} V \\ &\quad + 30 G_{\mu\nu} F^{\mu\nu} V^2 + 10 G_{\mu\lambda} F^{\lambda\alpha} F_\alpha{}^\mu V + \frac{5}{4} G_{\mu\nu} F^{\mu\nu} F^2 + \frac{3}{28} G_{\mu\nu} \nabla^4 F^{\mu\nu} + G_{\mu\nu} F^{\nu\rho} F_{\rho\lambda} F^{\lambda\mu} \Big] \; . \end{aligned} \tag{2.37}$$

The above coefficients are up to total derivatives and valid only under Yang-Mills trace. Further, they are written in terms of Lorentz and gauge invariant objects. Their computation involves the evaluation of vacuum expectation value of fields which produce a tensor part coupled to some Feynman diagram generating the numerical coefficient. These generalized heat kernel coefficients at coinciding points are exact, and do not receive additional contributions.

As mentioned above, the final result is expressed in terms of gauge invariants objects and further, it could be possible to reduce it to a minimal set of invariants using all available simplifications. A useful one is cyclic invariance under colour trace which reveals crucial when expanding time ordering of gauge fields, since it allows to couple different time ordered strings by getting rid of step functions which increase complexity of worldgraphs. Another one is the Bianchi identity, crucial to simplify tensor part of correlators allowing to couple, after some variables renaming, different looking worldline correlators.

After this digression, as a hint of calculation, we show in the following lines how to produce the two dimensional generalized coefficient $\tilde{b}_1$ already obtained in [1] and [2] (in curved background). First we Taylor expand to first order the exponential of interacting action in path integral, next we use FS gauge on both gauge fields by taking the first term in FS expansion, finally one is left with the following

$$\begin{aligned} \frac{1}{4\beta} \int_{01} \langle \dot{q}_0^\mu q_0^\alpha \dot{q}_1^\nu q_1^\rho \rangle \, G_{\alpha\mu} \, F_{\rho\nu} &= \frac{\beta}{4} \int_{01} \big({}^\bullet\Delta^\bullet_{01} \Delta_{01} G_{\alpha\mu} F^{\alpha\mu} + {}^\bullet\Delta_{01} \Delta^\bullet_{01} G_{\alpha\mu} F^{\mu\alpha} \big) \\ &= \frac{\beta}{2} G_{\mu\nu} F^{\mu\nu} \end{aligned} \tag{2.38}$$



$$= \frac{\beta}{24} G_{\mu\nu} F^{\mu\nu}$$

where an empty ring close to a vertex denotes a derivative of DBC propagator with respect to the vertex argument. Below we list all non vanishing terms which produce the four dimensional generalized coefficient $\tilde{b}_2$

$$\begin{aligned}
\frac{1}{9\beta}\int_{01}\langle\dot{q}_0^\mu\dot{q}_1^\lambda q_0^\nu q_0^\alpha q_1^\rho q_1^\sigma\rangle\nabla_\alpha F_{\nu\mu}\partial_\rho G_{\sigma\lambda} &= \frac{\beta^2}{720}\left(4\nabla_\lambda F_{\mu\nu}\partial^\lambda G^{\mu\nu}+\nabla^\nu F_{\nu\mu}\partial_\lambda G^{\lambda\mu}\right)\\
\frac{1}{16\beta}\int_{01}\langle\dot{q}_0^\lambda q_0^\sigma q_0^\rho q_0^\gamma\dot{q}_1^\mu q_1^\alpha\rangle\partial_\gamma\partial_\rho G_{\sigma\lambda}F_{\alpha\mu} &= -\frac{\beta^2}{240}\partial_\rho G_{\mu\alpha}\nabla^\rho F^{\mu\alpha}\\
\frac{1}{16\beta}\int_{01}\langle\dot{q}_0^\mu q_0^\nu\dot{q}_1^\rho q_1^\alpha q_1^\sigma q_1^\lambda\rangle\nabla_\sigma\nabla_\lambda F_{\alpha\rho}G_{\nu\mu} &= -\frac{\beta^2}{240}\partial^\rho G^{\mu\nu}\nabla_\rho F_{\mu\nu}\\
-\frac{1}{4}\int_{01}\langle\dot{q}_1^\mu q_1^\nu\dot{q}_0^\alpha q_0^\sigma\rangle G_{\sigma\alpha}F_{\nu\mu}V &= \frac{\beta^2}{24}G_{\mu\nu}F^{\mu\nu}V\ .
\end{aligned}\tag{2.39}$$

The computation of four dimensional coefficient $\tilde{b}_2$ has been worked out using (2.30) even if in chapter 3 we present a different path integral representation for generalized heat kernel trace and use it to recompute the two and four dimensional coefficient for checking correctness of this novel representation.

The generalized heat kernel coefficients in two and four dimensions have been used as a test for our worldline path integral representations, and, the results correctly reproduce [2] which represents the most extensive reference concerning generalized heat kernel traces. Turning back to our result, they have already been expressed in a minimal set of invariants by using the above mentioned symmetries and part integrating covariant derivatives.

Next we have the novel six dimensional generalized kernel coefficient $\tilde{b}_3$, obtainded by adding up the following terms

$$\begin{aligned}
&-\frac{1}{8}\int_{012}\langle\dot{q}_1^\mu q_1^\nu\dot{q}_0^\lambda q_0^\alpha q_2^\rho q_2^\sigma\rangle G_{\lambda\rho}F_{\nu\mu}\nabla_\sigma\nabla_\alpha V\\
&\quad=-\frac{\beta^3}{720}\Big(10G_{\mu\nu}F^{\mu\nu}\nabla^2V-4G_{\mu\nu}F^{\alpha\mu}\nabla^\nu\nabla_\alpha V-4G_{\mu\nu}F^{\alpha\mu}\nabla_\alpha\nabla^\nu V\Big)
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{6}\int_{012}\langle\dot{q}_0^\mu\dot{q}_1^\lambda q_0^\alpha q_1^\sigma q_1^\rho q_2^\gamma\rangle G_{\alpha\mu}\nabla_\sigma F_{\rho\lambda}\nabla_\gamma V\\
&\quad=\frac{\beta^3}{720}\Big(G^{\alpha\mu}\nabla^\lambda F_{\lambda\alpha}\nabla_\mu V-3G^{\mu\alpha}\nabla_\lambda F_{\mu\alpha}\nabla^\lambda V\Big)
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{90\beta}\int_{01}\langle\dot{q}_0^\mu q_0^\alpha q_0^\sigma\dot{q}_1^\nu q_1^\rho q_1^\lambda q_1^\eta q_1^\Omega\rangle\partial_\sigma G_{\alpha\mu}\nabla_\Omega\nabla_\eta\nabla\lambda F_{\rho\nu}\\
&\quad=\frac{\beta^3}{10080}\Big(6\partial_\sigma G_{\mu\alpha}\nabla^2\nabla^\sigma F^{\mu\alpha}+\partial^\mu G_{\mu\alpha}\nabla^2\nabla_\lambda F^{\lambda\alpha}\Big)
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{16}\int_{01}\langle\dot{q}_1^\mu q_1^\alpha q_1^\rho q_1^\sigma\dot{q}_0^\lambda q_0^\nu\rangle G_{\nu\lambda}\nabla_\rho\nabla_\sigma F_{\alpha\mu}V\\
&\quad=\frac{\beta^3}{480}\Big(G_{\mu\lambda}\nabla^\lambda\nabla_\alpha F^{\alpha\mu}V+G_{\mu\lambda}\nabla_\alpha\nabla^\lambda F^{\alpha\mu}V-G^{\mu\lambda}\nabla^2F_{\mu\lambda}V\Big)
\end{aligned}$$

$$+\frac{1}{16\beta}\int_{0123}\langle \dot q_0^\eta q_0^\omega \dot q_1^\mu q_1^\sigma \dot q_2^\nu q_2^\rho \dot q_3^\lambda q_3^\Omega\rangle\theta_{12}\theta_{23}G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F_{\Omega\lambda}$$
$$=\frac{\beta^3}{16}\left(\frac{1}{720}\right)\left(10G_{\omega\eta}F^{\omega\eta}F_{\lambda\nu}F^{\lambda\nu}+8G_{\omega\eta}F^{\eta\rho}F_{\rho\lambda}F^{\lambda\omega}\right)$$

$$+\frac{1}{90\beta}\int_{01}\langle \dot q_0^\mu q_0^\alpha q_0^\sigma q_0^\rho q_0^\lambda \dot q_1^\nu q_1^\Omega q_1^\eta\rangle\partial_\lambda\partial_\rho\partial_\sigma G_{\alpha\mu}\nabla_\eta F_{\Omega\nu}$$
$$=\frac{\beta^3}{10080}\left(6\partial^2\partial^\lambda G^{\mu\nu}\nabla_\lambda F_{\mu\nu}+\partial^2\partial^\mu G_{\mu\alpha}\nabla_\nu F^{\nu\alpha}\right)$$

$$\frac{\beta}{4}\int_{0123}\langle \dot q_1^\mu \dot q_0^\lambda q_1^\alpha q_0^\rho\rangle\left(\theta_{12}\theta_{23}+\theta_{31}\theta_{12}+\theta_{32}\theta_{21}\right)G_{\rho\lambda}F_{\alpha\mu}V^2=\frac{\beta^3}{48}G_{\mu\nu}F^{\nu\mu}V^2$$
$$-\frac{1}{9}\int_{012}\langle \dot q_0^\mu \dot q_1^\lambda q_0^\nu q_0^\alpha q_1^\rho q_1^\sigma\rangle\nabla_\alpha F_{\nu\mu}V\partial_\sigma G_{\rho\lambda}=-\frac{\beta^3}{720}\left(\nabla^\mu F_{\mu\alpha}V\partial_\rho G^{\rho\alpha}+4\nabla_\rho F_{\mu\alpha}V\partial^\rho G^{\mu\alpha}\right)$$
$$-\frac{1}{16}\int_{012}\langle \dot q_1^\mu q_1^\alpha \dot q_0^\lambda q_0^\sigma q_0^\rho q_0^\gamma\rangle\partial_\gamma\partial_\rho G_{\sigma\lambda}F_{\alpha\mu}V=-\frac{\beta^3}{240}\partial^2 G_{\mu\nu}F^{\mu\nu}V$$
$$-\frac{1}{6}\int_{012}\langle \dot q_0^\mu q_0^\nu q_0^\alpha \dot q_1^\lambda q_1^\sigma q_2^\rho\rangle\partial_\alpha G_{\nu\mu}F_{\sigma\lambda}\nabla_\rho V=\frac{\beta^3}{720}\left(\partial^\lambda G_{\lambda\alpha}F^{\alpha\mu}\nabla_\mu V-3\partial_\lambda G_{\mu\alpha}F^{\mu\alpha}\nabla^\lambda V\right)$$
$$+\frac{1}{288\beta}\int_{01}\langle \dot q_0^\mu q_0^\alpha \dot q_1^\nu q_1^\lambda q_1^\rho q_1^\sigma q_1^\Omega q_1^\eta\rangle G_{\alpha\mu}\nabla_\eta\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\nu}=\frac{\beta^3}{4480}G^{\mu\nu}\nabla^4 F_{\mu\nu}$$
$$+\frac{1}{288\beta}\int_{01}\langle \dot q_0^\mu q_0^\alpha q_0^\sigma q_0^\rho q_0^\lambda q_0^\Omega \dot q_1^\nu q_1^\eta\rangle\partial_\Omega\partial_\lambda\partial_\rho\partial_\sigma G_{\alpha\mu}F_{\eta\nu}=\frac{\beta^3}{4480}\partial^4 G_{\mu\nu}F^{\mu\nu}$$
$$\frac{1}{64\beta}\int_{01}\langle \dot q_0^\lambda \dot q_1^\nu q_0^\rho q_0^\sigma q_0^\gamma q_1^\mu q_1^\alpha q_1^\Omega\rangle\partial_\Omega\partial_\alpha G_{\mu\nu}\nabla_\sigma\nabla_\rho F_{\gamma\lambda}$$
$$=\frac{\beta^3}{8}\left(\frac{1}{5040}\right)\left(4\partial^\lambda\partial^\alpha G_{\alpha\mu}\nabla_\lambda\nabla_\nu F^{\nu\mu}+17\partial^2 G_{\mu\nu}\nabla^\sigma\nabla_\sigma F^{\mu\nu}+18\partial^\rho\partial^\alpha G^{\mu\nu}\nabla_\rho\nabla_\alpha F_{\mu\nu}\right).$$

We stress that the above results are valid under colour trace, which is implicit in the above terms, further we refer to chapter 4 for the computation of third order terms.

As a last remark one should note that switching off non-abelian gauge field in (2.37) let all coefficients vanish (this is true even for higher dimensional ones), reproducing the well known result that there is not anomaly if one only deals with a scalar potential; so in searching for terms one can easily avoid writing and computing the ones containing only abelian field and scalar potential, since they vanish or add up to produce exactly zero.

Chapter 3

Alternative approach for the coupling to the Yang-Mills field

In this chapter we first review an alternative approach to the computation of the heat kernel trace in the presence of a coupling to a background Yang-Mills fields. This alternative method was introduced in [16] and [15], where the time ordering prescription is replaced by introducing new bosonic variables. The latter, upon canonical quantization, spans the coloured inner part of the Hilbert space. However the bosonic wave function of the scalar particle corresponds to a reducible representation of gauge group, which can be decomposed in a direct sum of symmetric tensor products of the fundamental representation, so one has to use some procedure to select a desired representation, which in our calculations is the fundamental one. As a novelty, starting from the above approach to coloured particles we derive a worldline path integral representation for (2.23) which reveals quite useful, according to us, for computing low dimensional generalized coefficients.

3.1 Coherent state path integral for coloured particles

In order to build up a path integral for the worldline coloured particle, using auxiliary bosonic fields, one has to introduce them in the action such that upon canonical quantization they become creation and annihilation operators. This allows to regard them as conjugate variables, which is crucial to make them generate a suitable coupling with Yang-Mills field and, most important, to work out the mechanism selecting the fundamental representation of the gauge group.

To start with, consider the following coupling

$$A^a_\mu T^a \;\;\rightarrow\;\; A^a_\mu\, \bar{c}^\alpha (T^a)_\alpha{}^\beta\, c_\beta = A^a_\mu \rho^a \tag{3.1}$$

where c and $\bar{c}$ are the bosonic worldline variables while T^a are the gauge group generators. Assuming for instance that they are conjugate variables, so that their Poisson bracket in Minkowski space has the form

$$\{c_\alpha, \bar{c}^\beta\}_{PB} = -i\delta_\alpha{}^\beta \tag{3.2}$$

one can check that the quantity ρ^a form a representation of gauge group algebra on the phase space generated by bosonic fields, indeed

$$
\begin{aligned}
\{\rho^a, \rho^b\}_{PB} &= (T^a)_\alpha{}^\beta (T^b)_\rho{}^\sigma \{\bar{c}^\alpha c_\beta, \bar{c}^\rho c_\sigma\}_{PB} \\
&= (T^a)_\alpha{}^\beta (T^b)_\rho{}^\sigma \left[\frac{\partial (\bar{c}^\alpha c_\beta)}{\partial \bar{c}^\mu} \frac{\partial (\bar{c}^\rho c_\sigma)}{\partial c_\mu} - \frac{\partial (\bar{c}^\alpha c_\beta)}{\partial c^\mu} \frac{\partial (\bar{c}^\rho c_\sigma)}{\partial \bar{c}_\mu} \right] \\
&= [(T^a)_\alpha{}^\beta, (T^b)_\beta{}^\sigma]\, \bar{c}^\alpha c_\sigma = f^{abc} \rho^c \ .
\end{aligned}
\tag{3.3}
$$

This explains why we have such coupling, and so, to fulfil (3.2), one needs to rewrite the worldline euclidean action as

$$
S[x, \bar{c}, c; A] = \int_0^1 d\tau \left(\frac{\dot{x}^2}{2\beta} + \bar{c}^\alpha \dot{c}_\alpha + \dot{x}^\mu A^a_\mu \, \bar{c}^\alpha (T^a)_\alpha{}^\beta c_\beta \right) \ . \tag{3.4}
$$

Switching to quantum mechanics, the wave functions for coloured particle can be Taylor expanded as follows

$$
\Phi(x, \bar{c}^\alpha) = (\langle x| \otimes \langle \bar{c}^\alpha|)|\Phi\rangle = \Phi(x) + \Phi_\alpha(x)\bar{c}^\alpha + \frac{1}{2}\Phi_{\alpha\beta}(x)\bar{c}^\alpha \bar{c}^\beta + \cdots \tag{3.5}
$$

note that this time, since bosonic fields span the colour inner space, position eigenstates are written as tensor product of colourless one and coherent states, further, given (3.2), creation and annihilation operators act on the full wave function as

$$
\begin{aligned}
\langle x, \bar{c}_\alpha | \hat{c}^\dagger_\alpha | \Phi \rangle &= \bar{c}_\alpha \Phi(x, \bar{c}_\alpha) \\
\langle x, \bar{c}_\alpha | \hat{c}_\alpha | \Phi \rangle &= \frac{\partial}{\partial \bar{c}^\alpha} \Phi(x, \bar{c}_\alpha) \ .
\end{aligned}
\tag{3.6}
$$

So, the full wave function contains single pieces transforming according to the symmetric tensor products of gauge group fundamental representation. In order to use it in our calculations we need to project the full wave function on the Hilbert space sector whose occupation number n is equal to one, which corresponds to the first level of harmonic oscillator occupied by one group index objects (sitting in fundamental representation of the gauge group).

This is achieved by coupling bosonic fields to a U(1) worldline gauge field $a(\tau)$ and introducing further a Chern-Simons term such that bosonic action is rewritten as

$$
S_B[c, \bar{c}, a] = \int_0^1 d\tau \ (\bar{c}^\alpha (\partial_\tau + ia) c_\alpha - ias) \ . \tag{3.7}
$$

To check the above procedure works one sees that upon quantization the equation of motion for the gauge field $a(\tau)$ turns into an operator constraint that selects only wave function piece transforming in some fixed symmetric representation of gauge group. Indeed one has

$$
\frac{\delta S}{\delta a} = \frac{i}{2} (\bar{c}^\alpha c_\alpha - 2s) \sim \left(\bar{c}^\alpha \frac{\partial}{\partial \bar{c}^\alpha} - n \right) \Phi(x, \bar{c}^\lambda) = 0 \tag{3.8}
$$

where symmetrization has been used to solve ordering ambiguity in the position of creation and annihilation operators and the Chern-Simons term has been fixed as $s = n + \frac{N}{2}$ to generate the proper constraint. Finally the coloured scalar particle action takes the following form

$$
S[x, c, \bar{c}, a; A] = \int_0^1 d\tau \left(\frac{\dot{x}^2}{2\beta} + \bar{c} \left(\partial_\tau + ia\right) c_\alpha - ias + \dot{x}^\mu A^a_\mu \, \bar{c}^\alpha (T^a)_\alpha{}^\beta c_\beta \right) \ . \tag{3.9}
$$

Now, in order to build up a path integral representation for partition function of the model one has to fix the worldline U(1) gauge field. Intuitively this redundancy describes the fact that all equivalent physical configurations lies on a circle of unit radius, so to make computations one has select a point on the circle; this makes us think that $a(\tau)$ behaves as an angle and, indeed one can fix it such that $a(\tau) = \phi$. But different values of the angle ϕ are gauge inequivalent, and when computing the path integral one has to integrate over all of their possible values.

Finally, using the above intuition, one can compute heat kernel trace as

$$\mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{is\phi} \int_{PBC} Dx \int_{PBC} D\bar{c}Dc\, e^{-S_{gf}[x,c,\bar{c},\phi;A]} \tag{3.10}$$

where the local gauge fixed action is

$$S_{gf}[x,c,\bar{c},\phi;A] = \int_0^1 d\tau \left(\frac{\dot{x}^2}{2\beta} + \bar{c}^\alpha \left(\partial_\tau + i\phi\right) c_\alpha + \dot{x}^\mu A^a_\mu\, \bar{c}^\alpha (T^a)_\alpha{}^\beta\, c_\beta\right) \tag{3.11}$$

and, since bosonic, we used PBC even on auxiliary fields c_α and $\bar{c}^\alpha$.

However given the gauge symmetry one can rotate the bosonic fields ($c(\tau) \to e^{-i\phi\tau}c(\tau)$, $\bar{c}(\tau) \to e^{i\phi\tau}\bar{c}(\tau)$) in order to get rid of the U(1) coupling in the action, but this switches the PBC integration in TBC (twisted boundary conditions) integration, defined as ($c(0) = e^{-i\phi}c(1)$, $\bar{c}(0) = e^{i\phi}\bar{c}(1)$). This finally allows to rewrite the kernel trace as

$$\mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{is\phi} \int_{PBC} Dx \int_{TBC} D\bar{c}Dc\, e^{-S_{gf}[x,c,\bar{c};A]} \tag{3.12}$$

where finally, the gauge fixed TBC action reads as

$$S_{gf}[x,c,\bar{c};A] = \int_0^1 d\tau \left(\frac{\dot{x}^2}{2\beta} + \bar{c}^\alpha \dot{c}_\alpha + \dot{x}^\mu A^a_\mu\, \bar{c}^\alpha (T^a)_\alpha{}^\beta\, c_\beta\right) . \tag{3.13}$$

For our purposes, we further couple our coloured particle to a non-abelian Lorentz scalar potential. This is easily done by inserting $V^a\, \bar{c}^\alpha (T^a)_\alpha{}^\beta\, c_\beta$ in the above action, since the non-abelian potential lies in colour algebra.

3.2 Generalized heat kernel trace with auxiliary bosonic variables

Once obtained a path integral representation for the kernel trace one can write down, using the same mechanism as in chapter 2, a trace for the kernel with the insertion of a differential first order operator inside the trace, which we referred to as generalized heat kernel trace.

Since the external gauge field we introduced to produce (2.29) is abelian, one has not to deal (just for the abelian field) with annihilation and creation operators, such that the generalized kernel trace writes down as

$$\mathrm{Tr}\left[\left(\frac{1}{2}\partial_\mu\xi^\mu + \xi^\mu\nabla_\mu\right) e^{-\beta\hat{H}}\right] = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{is\phi} \Big\langle -\frac{1}{\beta}\int_0^1 d\tau\, \dot{x}^\mu(\tau)\xi_\mu(x(\tau))\Big\rangle \tag{3.14}$$

where non subscript expectation value is defined as

$$\Big\langle -\frac{1}{\beta}\int_0^1 d\tau\,\dot{x}^\mu \xi_\mu(x(\tau))\Big\rangle = \int_{PBC} Dx \int_{TBC} D\bar{c}Dc \left(-\frac{1}{\beta}\int_0^1 d\tau\,\dot{x}^\mu \xi_\mu(x(\tau))\right) e^{-S_{gf}[x,c,\bar{c};A]} \tag{3.15}$$

To proceed in computing to some fixed order (3.14), as usual one splits out the zero mode from position integration and fixes DBC that we chose to invert the kinetic operator. Next one factorizes out normalization constants arising from position and coherent states integration, whose computation is worked out in Appendix A; further, one has to Taylor expand the exponential of interacting action and fix FS gauge for both gauge fields, finally, one is left with bosonic and configuration fields correlators.

These last correlators produce worldgraphs with suitable contractions of curvatures indices as already seen in the previous approach, while bosonic field correlators produce Heaviside step functions implementing time ordering prescription of the previous approach and contract group generators colour indices. At the end one is left with the angle integration that can be rewritten as a complex one on the unit circle centred in the origin of complex plane. Then, using the residue theorem, one can easily evaluate the integral. However one has to regularize the complex integration since poles lies on the contour of the circle, even if it is not a huge problem in this case we will explain later on how to do it. Those integrals select exactly the correct terms produced by using time ordering prescription.

After this description we would like to illustrate the above procedure by computing the two dimensional generalized coefficient listed in (2.37). Starting from (3.14) we have

$$\begin{aligned}
&\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\Big\langle -\frac{1}{\beta}\int_0^1 d\tau\,\dot{x}^\mu \xi_\mu(x(\tau))\Big\rangle\\
&= \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\int_{DBC} Dq\int_{TBC} D\bar{c}Dc\,e^{-S_0[\bar{c},c]}\Big(-\frac{1}{\beta}\int_0^1 d\tau\,\dot{q}^\mu\xi_\mu(x+q)\Big)e^{-S_{gf}}\\
&\sim \frac{1}{4\beta}\int\frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\int_{01}\langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\lambda q_1^\rho\rangle\langle c_\gamma^1\,\bar{c}_1^\alpha\rangle\,G_{\nu\mu}F^a_{\rho\lambda}(T^a)_\alpha{}^\gamma\\
&\sim \int\frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\,\frac{\beta}{24}G_{\mu\nu}F^{\mu\nu}\underbrace{\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\frac{e^{-\frac{\phi}{2}}}{2i\sin\frac{\phi}{2}}}_{I=1}\sim\int\frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\,\frac{\beta}{24}G_{\mu\nu}F^{\mu\nu}
\end{aligned}$$

where, setting Chern-Simons coupling $s = 1+\frac{N}{2}$ since in the fundamental representation of gauge group, the angle integral can be evaluated as follows

$$\begin{aligned}
I &= \oint\frac{dz}{2\pi i z}\,z^{\frac{N}{2}+1}\,(z-1)^{-N-1}\,z^{\frac{N}{2}}\\
&= \oint\frac{dz}{2\pi i}\,\frac{z^N}{(z-1)^{N+1}} = \frac{1}{N!}\frac{d^N}{dz^N}\,z^N\Big|_{z=1} = 1
\end{aligned} \tag{3.16}$$

where we used the bosonic propagator, whose evaluation has been carried out in Appendix A. Further, for regularizing the contour integral, we pushed the pole inside the unit circle by giving it a small real part ϵ which has been set to zero at the end of computation.

This should test the correctness of (3.14) and give an alternative way to compute Yang-Mills kernel traces with insertion of a first order differential operator in flat spacetime. So, one can choose to proceed in the computation of higher order generalized coefficients this way or adopting the time ordering prescription.

In the thesis we decided to use time ordering prescription for the computation of the third order generalized coefficient, in order to avoid bosonic propagators and contour integrals, which may let calculations become heavier then they already are. Indeed, the computation of third order generalized coefficient involves configuration space correlators made up of eight worldline fields, even if we adopted some simplifications to reduced this correlators.

However as shown in next section the computation of bosonic correlators drastically simplifies at second order in proper time, since all contributing terms contain the same correlator whose contour integral produces one such to generate the corresponding term one obtains using time ordering approach, and so, showing second order equivalence between the two approaches, as it should be.

3.3 Generalized coefficient at order β^2

In this section we use (3.14) to reproduce second order β^2 terms corresponding to the generalized coefficient $\tilde{b}_2$ in (2.37). In order to find terms at second order in proper time, one has to Taylor expand the exponential of interacting action in path integral to first order then, using Fock-Schwinger gauge on Yang-Mills field and applying from the left the first order FS expansion for the abelian field ξ_μ, the full contribution is produced.

So, adopting the above procedure one obtains the following second order terms

$$\frac{1}{16\beta}\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\int_{01}\langle\dot{q}_0^\rho q_0^\lambda \dot{q}_1^\mu q_1^\alpha q_1^\sigma q_1^\gamma\rangle\langle c_j^1\,\bar{c}_1^i\rangle G_{\lambda\rho}(\nabla_\gamma\nabla_\sigma F_{\alpha\mu})^a(T^a)_i{}^j \tag{3.17}$$

$$\frac{1}{9\beta}\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\int_{01}\langle\dot{q}_0^\rho q_0^\lambda q_0^\alpha \dot{q}_1^\mu q_1^\sigma q_1^\gamma\rangle\langle c_j^1\,\bar{c}_1^i\rangle\partial_\alpha G_{\lambda\rho}(\nabla_\gamma F_{\sigma\mu})^a(T^a)_i{}^j \tag{3.18}$$

$$\frac{1}{16\beta}\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\int_{01}\langle\dot{q}_0^\rho q_0^\lambda q_0^\alpha q_0^\sigma \dot{q}_1^\mu q_1^\nu\rangle\langle c_j^1\,\bar{c}_1^i\rangle\partial_\sigma\partial_\alpha G_{\lambda\rho}(F_{\nu\mu})^a(T^a)_i{}^j\;. \tag{3.19}$$

One immediately notes that the bosonic field correlator is the same for all the above terms, it can be easily computed using property (A.39) so that one is left with a contour integral that can be evaluated as follows

$$\begin{aligned}\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\langle c_1\,\bar{c}_1\rangle &= \int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N-1}e^{-i\frac{\phi}{2}}\\ &= \oint\frac{dz}{2\pi i}\,\frac{z^N}{(z-1)^{N+1}} = \frac{1}{N!}\frac{d^N}{dz^N}z^N\Big|_{z=1} = 1\end{aligned} \tag{3.20}$$

where we set the Chern-Simons coupling $s = 1+\frac{N}{2}$ to extract the fundamental representation. This has to be expected since the above terms must coincide with the ones obtained by using time ordering prescription, which have the same form but no angle integration, so the latter must produce one.

Coming back to the evaluation of terms, since we have six fields correlator we first reduce the calculation to four fields correlators (adopting some simplifications) then write the final result in terms of worldline Feynman diagrams, collecting the tensor parts. We stress that the whole calculation has been carried out using DBC. The final result has to be compared with [2] in order to check its correctness.

Let us start by considering (3.17)

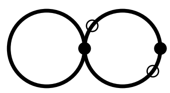
$$
\begin{aligned}
&+\frac{1}{16\beta}\int_{01}\langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\rho q_1^\alpha q_1^\sigma q_1^\lambda\rangle \nabla_\sigma\nabla_\lambda F_{\alpha\rho}G_{\nu\mu} = \\
&-\frac{1}{16}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\nu \dot{q}_1^\rho q_1^\alpha q_1^\lambda\rangle \Big(2G_{\nu\mu}\nabla_\lambda\nabla^\mu F_{\alpha\rho} + G_{\nu\mu}\nabla_\lambda\nabla_\alpha F^\mu{}_\rho\Big) \\
&-\frac{1}{16}\int_{01} {}^\bullet\Delta^\bullet_{01}\langle q_0^\nu q_1^\alpha q_1^\sigma q_1^\lambda\rangle G_{\nu\mu}\nabla_\lambda\nabla_\sigma F_\alpha{}^\mu = \\
&-\frac{1}{16}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\nu \dot{q}_1^\rho q_1^\alpha q_1^\lambda\rangle \left(-2G_\nu{}^\mu\nabla_\rho\nabla_\lambda F_{\alpha\mu} - G_\nu{}^\mu\nabla_\lambda\nabla_\alpha F_{\rho\mu} + 2G_\nu{}^\mu\nabla_\lambda\nabla_\mu F_{\alpha\rho} + G_\nu{}^\mu\nabla_\lambda\nabla_\alpha F_{\mu\rho}\right)
\end{aligned}
\tag{3.21}
$$

where the last correlator in the second line has been part integrated. One can use Bianchi identity on last line

$$
-2G_\nu{}^\mu\nabla_\rho\nabla_\lambda F_{\alpha\mu} + 2G_\nu{}^\mu\nabla_\lambda\nabla_\mu F_{\alpha\rho} = 2G_\nu{}^\mu\nabla_\lambda\nabla_\rho F_{\mu\alpha} + 2G_\nu{}^\mu\nabla_\lambda\nabla_\mu F_{\alpha\rho} = 2G_\nu{}^\mu\nabla_\lambda\nabla_\alpha F_{\mu\rho}
$$

So, recasting (3.21) one has

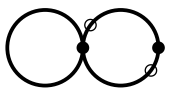
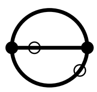
$$
\begin{aligned}
&\frac{1}{16\beta}\int_{01}\langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\rho q_1^\alpha q_1^\sigma q_1^\lambda\rangle \nabla_\sigma\nabla_\lambda F_{\alpha\rho}G_{\nu\mu} = \\
&-\frac{1}{4}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\nu \dot{q}_1^\rho q_1^\alpha q_1^\lambda\rangle G_\nu{}^\mu\nabla_\lambda\nabla_\alpha F_{\mu\rho} = \frac{\beta^2}{4}G^{\mu\nu}\nabla^2 F_{\mu\nu}
\end{aligned}
$$

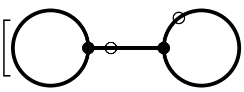
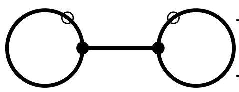


$$
= \frac{\beta^2}{240}G_{\mu\nu}\nabla^2 F^{\mu\nu}\,. \tag{3.22}
$$

Consider now (3.18)

$$
\begin{aligned}
&+\frac{1}{9\beta}\int_{01}\langle \dot{q}_0^\nu q_0^\sigma q_0^\rho \dot{q}_1^\mu q_1^\alpha q_1^\lambda\rangle \partial_\rho G_{\sigma\nu}\nabla_\lambda F_{\alpha\mu} = \\
&-\frac{1}{9}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\sigma q_0^\rho \dot{q}_1^\mu q_1^\alpha\rangle \left(-\partial_\rho G_\sigma{}^\nu\nabla_\alpha F_{\mu\nu} - \partial_\rho G_\sigma{}^\nu\nabla_\mu F_{\alpha\nu} + \partial_\rho G_\sigma{}^\nu\nabla_\alpha F_{\nu\mu} + \partial_\rho G_\sigma{}^\nu\nabla_\nu F_{\alpha\mu}\right) \\
&-\frac{1}{9}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\sigma \dot{q}_1^\mu q_1^\alpha q_1^\lambda\rangle \partial^\nu G_{\sigma\nu}\nabla_\lambda F_{\alpha\mu} = \\
&-\frac{1}{3}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\sigma q_0^\rho \dot{q}_1^\mu q_1^\alpha\rangle \partial_\rho G_\sigma{}^\nu\nabla_\alpha F_{\nu\mu} - \frac{1}{9}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\sigma \dot{q}_1^\mu q_1^\alpha q_1^\lambda\rangle \partial^\nu G_{\sigma\nu}\nabla_\lambda F_{\alpha\mu} =
\end{aligned}
$$

$+\frac{\beta^2}{3}\partial_\alpha G^{\alpha\nu}\nabla^\lambda F_{\lambda\nu}$  $+ \frac{\beta^2}{2}\partial^\alpha G^{\mu\nu}\nabla_\alpha F_{\mu\nu}$ 

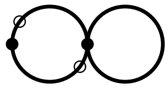
$+\frac{\beta^2}{9}\partial^\nu G_{\nu\mu}\nabla_\lambda F^{\lambda\mu}\Big[$  $-$  $\Big] =$

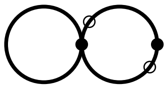
$$
+\frac{\beta^2}{720}\left(4\partial^\alpha G^{\mu\nu}\nabla_\alpha F_{\mu\nu} + \partial_\lambda G^{\lambda\nu}\nabla^\rho F_{\rho\nu}\right)\,. \tag{3.23}
$$

Next one has (3.19)

$$+\frac{1}{16\beta}\int_{01}\langle \dot{q}_0^\lambda q_0^\sigma q_0^\rho q_0^\gamma \dot{q}_1^\mu q_1^\alpha\rangle \partial_\gamma\partial_\rho G_{\sigma\lambda}F_{\alpha\mu} =$$

$$\underbrace{-\frac{1}{8}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\sigma q_0^\gamma \dot{q}_1^\mu q_1^\alpha\rangle \partial_\gamma\partial^\lambda G_{\sigma\lambda}F_{\alpha\mu}}_{=0} - \frac{1}{8}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\sigma q_0^\rho q_0^\gamma \dot{q}_1^\alpha\rangle \partial_\gamma\partial_\rho G_{\sigma\mu}F^\mu{}_\alpha = \qquad (3.24)$$

$$-\frac{\beta^2}{8}\left[2\partial_\alpha\partial G_{\rho\mu}F^\mu{}_\alpha - F^{\mu\nu}\partial^2 G_{\mu\nu}\right]$$  $=$

$$+\frac{\beta^2}{4}F^{\mu\nu}\partial^2 G_{\mu\nu}$$  $$= \frac{\beta^2}{240}F^{\mu\nu}\partial^2 G_{\mu\nu}\,.$$

We even have another second order term obtained by Taylor expanding the exponential of interacting action to second order and considering products of A_μ and V, namely

$$-\frac{1}{4}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\int_{012}\langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\sigma\rangle\langle \bar{c}_1^\rho\, c_\eta^1\, \bar{c}_2^\omega\, c_\gamma^2\rangle G_{\sigma\alpha}(F_{\nu\mu})^a V^b\,(T^a)_\rho{}^\eta\,(T^b)_\omega{}^\gamma\,. \qquad (3.25)$$

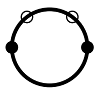
To compute it one can proceed considering contractions produced by bosonic correlator as follows

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\langle \bar{c}_1^\rho\, c_\eta^1\, \bar{c}_2^\omega\, c_\gamma^2\rangle(T^a)_\rho{}^\eta\,(T^b)_\omega{}^\gamma$$

$$=\int_0^{2\pi}\frac{d\phi}{2\pi}e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}(\Delta_B^2(\tau-\tau,\phi)(T^a)_\rho{}^\rho\,(T^b)_\lambda{}^\lambda + \Delta_B(\sigma-\tau,\phi)\Delta_B(\tau-\sigma,\phi)(T^a)_\alpha{}^\rho(T^b)_\rho{}^\alpha)$$

$$=\underbrace{\oint\frac{dz}{2\pi i}\frac{z^N}{(z-1)^{N+2}}}_{=0}(T^a)_\rho{}^\rho\,(T^b)_\lambda{}^\lambda + \underbrace{\oint\frac{dz}{2\pi i}\frac{z^{N+1}}{(z-1)^{N+2}}}_{=1}(T^a)_\alpha{}^\rho(T^b)_\rho{}^\alpha$$

where we used (A.39) and (A.40) while Δ_B is the bosonic worldline propagator. Finally (3.25) can be rewritten as

$$-\frac{1}{4}\int_{01}\langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\sigma\rangle G_{\sigma\alpha}F_{\nu\mu}V = -\frac{\beta^2}{2}G_{\mu\nu}F^{\mu\nu}V$$  $$= -\frac{\beta^2}{24}G_{\mu\nu}F^{\mu\nu}V\,. \qquad (3.26)$$

This completes the computation of the above terms, but there still is a last term that can be found by Taylor expanding to second order the exponential of interacting action in path integral, namely

$$\frac{1}{16\beta}\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N}\int_{012}\langle \dot{q}_1^\mu q_1^\alpha \dot{q}_2^\nu q_2^\sigma \dot{q}_0^\rho q_0^\lambda\rangle\langle c_j^1\,\bar{c}_1^i\,c_l^2\,\bar{c}_2^k\rangle G_{\lambda\rho}F^a_{\alpha\mu}F^b_{\sigma\nu}(T^a)_i{}^j(T^b)_k{}^l\,. \qquad (3.27)$$

Let us compute first contour integrals which arise when contracting bosonic fields. One has two integrals, namely

$$\begin{aligned}\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{i(s-1)\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N-2} &= \oint\frac{dz}{2\pi i}\frac{z^N}{(z-1)^{N+2}} = 0\\ \int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{is\phi}\left(2i\sin\frac{\phi}{2}\right)^{-N-2} &= \oint\frac{dz}{2\pi i}\frac{z^{N+1}}{(z-1)^{N+2}} = 1\end{aligned} \qquad (3.28)$$

thus (3.27) can be recast as

$$
\begin{aligned}
&\frac{1}{16\beta}\int_{012}\langle \dot q_1^\mu q_1^\alpha \dot q_2^\nu q_2^\sigma \dot q_0^\rho q_0^\lambda\rangle G_{\lambda\rho}F_{\alpha\mu}F_{\sigma\nu}\\
&=\frac{1}{8}\int_{012} {}^\bullet\Delta_{01}\,\langle \dot q_2^\rho q_2^\lambda \dot q_1^\nu q_0^\alpha\rangle G_{\alpha\mu}F_\nu{}^\mu F_{\lambda\rho}+\frac{1}{8}\int_{012} {}^\bullet\Delta_{02}\,\langle \dot q_1^\nu q_1^\sigma \dot q_2^\lambda q_0^\alpha\rangle G_{\alpha\mu}F_{\sigma\nu}F_\lambda{}^\mu\\
&=\frac{1}{4}\int_{012} {}^\bullet\Delta_{01}\,\langle \dot q_2^\rho q_2^\lambda \dot q_1^\nu q_0^\alpha\rangle G_{\alpha\mu}F_\nu{}^\mu F_{\lambda\rho}=0
\end{aligned}
\tag{3.29}
$$

since all proper time integrals vanish.

In order to add together all the terms one can show that total derivatives vanishes when integrated, assuming a suitable behaviour of fields at the boundary. Consider for instance terms like this

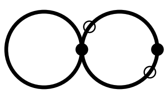
$$
\begin{aligned}
&\int d^Dx\,\mathrm{tr}_{YM}\Big(\nabla^\rho\left(G^{\mu\nu}\nabla_\nu F_{\rho\mu}\right)\Big)=\\
&\int d^Dx\,\mathrm{tr}_{YM}\Big(\partial^\rho\left(G^{\mu\nu}\nabla_\nu F_{\rho\mu}\right)\Big)+\int d^Dx\,G^{\mu\nu}\,\mathrm{tr}_{YM}\Big([A^\rho,\nabla_\nu F_{\rho\mu}]\Big)=0
\end{aligned}
$$

the first one vanishes because we're integrating a total derivative, while the second one is zero because of cyclicity of the trace.

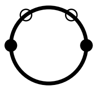
Finally collecting terms the result can be written as

$$
\frac{\beta^2}{720}\mathrm{tr}_{YM}\Big(-2\nabla^\lambda F^{\mu\nu}\partial_\lambda G_{\mu\nu}+\nabla^\mu F_{\mu\alpha}\partial_\lambda G^{\lambda\alpha}-30G_{\mu\nu}F^{\mu\nu}V\Big) \tag{3.30}
$$

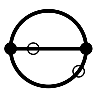
this completes the second order test and correctly reproduces [2]. For completeness we write down the evaluation of the above Feynman diagrams



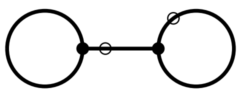
$$= \int_0^1 d\tau\int_0^1 d\sigma\,{}^\bullet\Delta(\tau,\sigma)\Delta^\bullet(\tau,\sigma)\Delta(\sigma,\sigma)=\frac{1}{60}$$



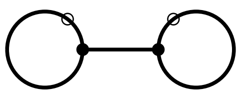
$$= \int_0^1 d\tau\int_0^1 d\sigma\,{}^\bullet\Delta^\bullet(\tau,\sigma)\Delta(\tau,\sigma)=\frac{1}{12}$$



$$= \int_0^1 d\tau\int_0^1 d\sigma\,\Delta(\tau,\sigma){}^\bullet\Delta(\tau,\sigma)\Delta^\bullet(\tau,\sigma)=\frac{1}{90} \tag{3.31}$$



$$= \int_0^1 d\tau\int_0^1 d\sigma\,\Delta(\tau,\tau){}^\bullet\Delta(\tau,\sigma)\Delta^\bullet(\sigma,\sigma)=\frac{1}{360}$$



$$= \int_0^1 d\tau\int_0^1 d\sigma\,{}^\bullet\Delta(\tau,\tau)\Delta(\tau,\sigma)\Delta^\bullet(\sigma,\sigma)=-\frac{1}{720}$$

Chapter 4

Generalized heat kernel trace at order β^3

Here we get deep into the computation of the coefficient $\tilde{b}_3$ in (2.37) by using the time ordering prescription approach. Starting by (2.30) we first search for non vanishing contributions in the Taylor expansion of the exponential in path integral, by adopting some useful rules allowing to write down the correlators we need to evaluate. In this procedure we first use cyclicity of colour trace to couple different time ordered strings and use the FS gauge. Next we start its computation by reducing correlators through integration by parts on the worldline and by simplifying the tensor part of the result by using some properties of covariant derivative and background curvature. The numerical coefficient is written down in terms of Feynman worldline diagrams whose evaluation is carried out in DBC.

4.1 Searching for terms in Taylor expansion

The searching for terms at some order in β is carried out first by Taylor expanding the exponential of interacting action in path integral, then one has to deal with the time ordering prescription on non-abelian fields, trying to simplify the results as much as possible. As a guideline one can use dimensional analysis to recover the structure of terms, then going back to the correct Taylor order in expansion, one writes down the complete term. Let us explain in details this procedure.

Since hamiltonian has dimensions L^{-2} in units of length, this gives that gauge fields and curvatures are respectively L^{-1} and L^{-2}, so even scalar potential. Now, generalized coefficient at order β^n is obtained by considering correlators producing β^{n+1} (since we have a β in the denominator of abelian insertion from path integral) so it will be obtained by a $2(n+1)$ fields correlator so the dimension of tensor part is $L^{2(n+1)}$.

This allows to guess the structure of a term and particularly, it allows to count the number of non-abelian fields of which it is made up, such that we can guess until which order in Taylor expansion we need to look for. For six dimensional generalized coefficient we are interested in, the most general terms would have dimensions L^{-8} corresponding to an eight field worldline correlator and can be produced Taylor expanding action till third order, this even suggests that two and four dimensional coefficients are reached at least to second order.

So, these reasons suggest that we should look to first, second and third order Taylor expansion of exponential to write down the correct correlator and tensor part we wish to compute. However, it is not the whole story, since once made the Taylor expansion we need to take care of time ordering prescription such that the final result is as much as possible simplified, then we use FS gauge on non-abelian field to produce curvatures and finally coupling by the left with FS expansion of abelian field we pass to evaluation.

As a starting point considering first order Taylor expansion of exponential one can easily write down all terms contributing to third order when coupled with abelian gauge field expansion, since he has not to deal with time ordering prescription. It is enough to expand Yang-Mills field in FS and reach fourth order terms, then all the couplings with abelian field naturally arise.

Next considering second order Taylor expansion of the exponential of interacting action we have

$$\begin{aligned}
\mathrm{T}\,\frac{1}{2}\int_{01}&\left(\dot{q}_0^\mu A_\mu(x+q_0)+\beta V(x+q_0)\right)\left(\dot{q}_1^\mu A_\mu(x+q_1)+\beta V(x+q_1)\right)=\\
&\frac{1}{2}\int_{01}\dot{q}_0^\mu\dot{q}_1^\nu\left(A_\mu(x+q_0)A_\nu(x+q_1)\theta_{01}+A_\nu(x+q_1)A_\mu(x+q_0)\theta_{10}\right)\\
&+\beta\int_{01}\dot{q}_0^\mu\left(A_\mu(x+q_0)V(x+q_1)\theta_{01}+V(x+q_1)A_\mu(x+q_0)\theta_{10}\right)\\
&+\frac{\beta^2}{2}\int_{01}\left(V(x+q_0)V(x+q_1)\theta_{01}+V(x+q_1)V(x+q_0)\theta_{10}\right)\ .
\end{aligned}$$

The above lines can be drastically simplified by using cyclicity of colour trace such that

$$\begin{aligned}
&\frac{1}{2}\int_{01}\dot{q}_0^\mu\dot{q}_1^\nu\left(A_\mu(x+q_0)A_\nu(x+q_1)\theta_{01}+A_\nu(x+q_1)A_\mu(x+q_0)\theta_{10}\right)=\\
&\frac{1}{2}\int_{01}\dot{q}_0^\mu\dot{q}_1^\nu A_\mu(x+q_0)A_\nu(x+q_1)\left(\theta_{01}+\theta_{10}\right)=\frac{1}{2}\int_{01}\dot{q}_0^\mu\dot{q}_1^\nu A_\mu(x+q_0)A_\nu(x+q_1)=\\
&\frac{1}{2}\int_{01}\dot{q}_0^\mu\dot{q}_1^\nu\left(A_\mu+\frac{1}{2}q_0^\alpha F_{\alpha\mu}++\frac{1}{3}q_0^\alpha q_0^\lambda\nabla_\alpha F_{\lambda\mu}\cdots\right)\left(A_\nu+\frac{1}{2}q_1^\alpha F_{\alpha\nu}+\frac{1}{3}q_1^\alpha q_1^\lambda\nabla_\alpha F_{\lambda\nu}\cdots\right)\\
\\
&\beta\int_{01}\dot{q}_0^\mu\left(A_\mu(x+q_0)V(x+q_1)\theta_{01}+V(x+q_1)A_\mu(x+q_0)\theta_{10}\right)=\\
&\beta\int_{01}\dot{q}_0^\mu A_\mu(x+q_0)V(x+q_1)\left(\theta_{01}+\theta_{10}\right)=\beta\int_{01}\dot{q}_0^\mu A_\mu(x+q_0)V(x+q_1)=\\
&\beta\int_{01}\dot{q}_0^\mu\left(A_\mu+\frac{1}{2}q_0^\alpha F_{\alpha\mu}++\frac{1}{3}q_0^\alpha q_0^\lambda\nabla_\alpha F_{\lambda\mu}\cdots\right)\left(V(x)+q_1^\mu\nabla_\mu V+\frac{1}{2}q_1^\mu q_1^\nu\nabla_\mu\nabla_\nu V+\cdots\right)\\
\\
&\frac{\beta^2}{2}\int_{01}\left(V(x+q_0)V(x+q_1)\theta_{01}+V(x+q_1)V(x+q_0)\theta_{10}\right)=\\
&\frac{\beta^2}{2}\int_{01}V(x+q_0)V(x+q_1)\left(\theta_{01}+\theta_{10}\right)=\frac{\beta^2}{2}\int_{01}V(x+q_0)V(x+q_1)=\\
&\frac{\beta^2}{2}\int_{01}\left(V(x)+q_0^\mu\nabla_\mu V+\frac{1}{2}q_0^\mu q_0^\nu\nabla_\mu\nabla_\nu V+\cdots\right)\left(V(x)+q_1^\mu\nabla_\mu V+\frac{1}{2}q_1^\mu q_1^\nu\nabla_\mu\nabla_\nu V+\cdots\right)\ .
\end{aligned}\tag{4.1}$$

Further we consider third order Taylor expansion of action, in this case we anticipate that not all the time ordered terms contained in third order expansion give contributions to our computation, the only ones are

$$\begin{aligned} &-\frac{\beta}{6}\mathrm{T}\int_{012}\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha A_\mu(x+q_0)A_\nu(x+q_1)A_\alpha(x+q_2) \\ &-\frac{\beta}{2}\mathrm{T}\int_{012}\dot{q}_0^\mu A_\mu(x+q_0)V(x+q_1)V(x+q_2) \end{aligned} \tag{4.2}$$

others produce vanishing contributions or higher dimensional terms which we are not interested in now.

Expanding time ordering in the first one we have

$$\begin{aligned} -\frac{\beta}{6}\mathrm{T}\int_{012}&\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha A_\mu(x+q_0)A_\nu(x+q_1)A_\alpha(x+q_2) = \\ &-\frac{\beta}{6}\int_{012}\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha A_\mu(x+q_0)A_\nu(x+q_1)A_\alpha(x+q_2)\theta_{01}\theta_{12} \\ &-\frac{\beta}{6}\int_{012}\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha A_\mu(x+q_0)A_\nu(x+q_2)A_\alpha(x+q_1)\theta_{02}\theta_{21} \\ &-\frac{\beta}{6}\int_{012}\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha A_\mu(x+q_1)A_\nu(x+q_0)A_\alpha(x+q_2)\theta_{10}\theta_{02} \\ &-\frac{\beta}{6}\int_{012}\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha A_\mu(x+q_1)A_\nu(x+q_2)A_\alpha(x+q_0)\theta_{12}\theta_{20} \\ &-\frac{\beta}{6}\int_{012}\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha A_\mu(x+q_2)A_\nu(x+q_1)A_\alpha(x+q_0)\theta_{21}\theta_{10} \\ &-\frac{\beta}{6}\int_{012}\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha A_\mu(x+q_2)A_\nu(x+q_0)A_\alpha(x+q_1)\theta_{20}\theta_{01} \end{aligned} \tag{4.3}$$

renaming dummy variables in the integrals and using cyclicity of trace such to produce in all string the order of the first line, one obtains

$$\begin{aligned} -\frac{\beta}{6}\mathrm{T}\int_{012}&\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha A_\mu(x+q_0)A_\nu(x+q_1)A_\alpha(x+q_2) = \\ &-\beta\int_{012}\dot{q}_0^\mu\dot{q}_1^\nu\dot{q}_2^\alpha\,\theta_{01}\theta_{12}A_\mu(x+q_0)A_\nu(x+q_1)A_\alpha(x+q_2) \sim \\ &-\frac{\beta}{8}\int_{012}\dot{q}_0^\mu q_0^\nu\dot{q}_1^\alpha q_1^\rho\dot{q}_2^\lambda q_2^\gamma\,\theta_{01}\theta_{12}\,F_{\nu\mu}F_{\rho\alpha}F_{\gamma\lambda} \end{aligned} \tag{4.4}$$

where $\sim$ indicates that other terms, when coupled with the abelian field expansion, generate higher dimensional contributions which we are not interested in for our perturbative calculations.

Next, using the same algorithm for second one in (4.2) we have

$$
\begin{aligned}
-\frac{\beta}{2}\,\mathrm{T}\int_{012} \dot{q}_0^{\mu} A_\mu(x+q_0)V(x+q_1)V(x+q_2) = \\
-\beta\int_{012} \dot{q}_0^{\mu} A_\mu(x+q_0)V(x+q_1)V(x+q_2)\theta_{01}\theta_{12} \\
-\beta\int_{012} \dot{q}_0^{\mu} A_\mu(x+q_0)V(x+q_1)V(x+q_2)\theta_{20}\theta_{01} \\
-\beta\int_{012} \dot{q}_0^{\mu} A_\mu(x+q_0)V(x+q_1)V(x+q_2)\theta_{12}\theta_{20} \sim \\
-\frac{\beta}{2}\int_{012} \dot{q}_0^{\mu} q_0^{\nu} F_{\nu\mu} V^2 \theta_{01}\theta_{12} - \frac{\beta}{2}\int_{012} \dot{q}_0^{\mu} q_0^{\nu} F_{\nu\mu} V^2 \theta_{20}\theta_{01} - \frac{\beta}{2}\int_{012} \dot{q}_0^{\mu} q_0^{\nu} F_{\nu\mu} V^2 \theta_{12}\theta_{20}\ .
\end{aligned}
\tag{4.5}
$$

This completes the searching for terms, now we pass to computation in next section.

4.2 Computation of third order terms

Here we compute third order terms in proper time obtained by coupling with abelian field the terms in the previous section.

Starting with (4.5) one has

$$-\frac{\beta^2}{2}\int_{123}\dot{q}_1^\mu q_1^\nu\Big(F_{\nu\mu}V^2\theta_{12}\theta_{23}+VF_{\nu\mu}V\theta_{31}\theta_{12}+V^2F_{\nu\mu}\theta_{32}\theta_{21}\Big)$$

then, coupling to the abelian field in order to reproduce a third order term, one obtains the followings

$$+\frac{\beta}{4}\int_{0123}\langle\dot{q}_1^\mu q_1^\nu\dot{q}_0^\alpha q_0^\lambda\rangle G_{\lambda\alpha}F_{\nu\mu}V^2\theta_{12}\theta_{23}=$$
$$+\frac{\beta^3}{4}2G_{\mu\nu}F^{\mu\nu}V^2\underbrace{\int_{0123}{}^\bullet\Delta^\bullet_{01}\Delta_{01}\theta_{12}\theta_{23}}_{\frac{1}{80}}=+\frac{\beta^3}{160}G_{\mu\nu}F^{\mu\nu}V^2$$

$$+\frac{\beta}{4}\int_{0123}\langle\dot{q}_1^\mu q_1^\nu\dot{q}_0^\alpha q_0^\lambda\rangle G_{\lambda\alpha}VF_{\nu\mu}V\theta_{31}\theta_{12}=$$
$$+\frac{\beta^3}{4}2G_{\mu\nu}VF^{\mu\nu}V\underbrace{\int_{0123}{}^\bullet\Delta^\bullet_{01}\Delta_{01}\theta_{31}\theta_{12}}_{\frac{1}{60}}=+\frac{\beta^3}{120}G_{\mu\nu}VF^{\mu\nu}V$$

$$+\frac{\beta}{4}\int_{0123}\langle\dot{q}_1^\mu q_1^\nu\dot{q}_0^\alpha q_0^\lambda\rangle G_{\lambda\alpha}V^2F_{\nu\mu}\theta_{32}\theta_{21}=$$
$$+\frac{\beta^3}{4}2G_{\mu\nu}V^2\left(x\right)F^{\mu\nu}\underbrace{\int_{0123}{}^\bullet\Delta^\bullet_{01}\Delta_{01}\theta_{32}\theta_{21}}_{\frac{1}{80}}=+\frac{\beta^3}{160}G_{\mu\nu}V^2F^{\mu\nu}$$

adding terms and using cyclicity of trace we have

$$\frac{\beta^3}{48}G_{\mu\nu}\,\mathrm{tr}_{YM}\left(F^{\mu\nu}V^2\right)\ . \tag{4.6}$$

This is the only third order term that one can obtain from third order expansion of action.

Consider now second order expansion (4.1) from which one can write down the following abelian coupled terms

$$-\frac{1}{8}\int_{012}\langle\dot{q}_1^\mu q_1^\nu\dot{q}_0^\rho q_0^\lambda q_2^\alpha q_2^\sigma\rangle G_{\lambda\rho}F_{\nu\mu}\nabla_\alpha\nabla_\sigma V \tag{4.7}$$
$$-\frac{1}{16}\int_{01}\langle\dot{q}_1^\mu q_1^\alpha\dot{q}_0^\lambda q_0^\sigma q_0^\rho q_0^\gamma\rangle\partial_\gamma\partial_\rho G_{\sigma\lambda}F_{\alpha\mu}V \tag{4.8}$$
$$-\frac{1}{16}\int_{01}\langle\dot{q}_1^\mu q_1^\alpha\dot{q}_0^\lambda q_0^\nu q_1^\rho q_1^\sigma\rangle G_{\nu\lambda}\nabla_\rho\nabla_\sigma F_{\alpha\mu}V \tag{4.9}$$

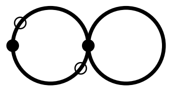
$$- \frac{1}{9}\int_{01} \langle \dot{q}_1^\mu q_1^\alpha q_1^\sigma \dot{q}_0^\nu q_0^\rho q_0^\lambda \rangle \partial_\lambda G_{\rho\nu} \nabla_\sigma F_{\alpha\mu} V \tag{4.10}$$

$$- \frac{1}{6}\int_{012} \langle \dot{q}_0^\mu q_0^\alpha \dot{q}_1^\lambda q_1^\sigma q_1^\rho q_2^\gamma \rangle G_{\alpha\mu} \nabla_\sigma F_{\rho\lambda} \nabla_\gamma V \tag{4.11}$$

$$- \frac{1}{6}\int_{012} \langle \dot{q}_0^\mu q_0^\nu q_0^\alpha \dot{q}_1^\lambda q_1^\sigma q_2^\rho \rangle \partial_\alpha G_{\nu\mu} F_{\sigma\lambda} \nabla_\rho V \ . \tag{4.12}$$

Consider (4.9)

$$\begin{aligned}
&-\frac{1}{16}\int_{01} \langle \dot{q}_1^\mu q_1^\alpha \dot{q}_0^\lambda q_0^\nu q_1^\rho q_1^\sigma \rangle G_{\nu\lambda} \nabla_\rho \nabla_\sigma F_{\alpha\mu} V = \\
&\frac{\beta}{16}\int_{01} {}^\bullet\Delta_{11} \langle q_1^\alpha \dot{q}_0^\lambda q_0^\nu q_1^\sigma \rangle \Big(G_{\nu\lambda} \nabla^\mu \nabla_\sigma F_{\alpha\mu} V + G_{\nu\lambda} \nabla_\sigma \nabla^\mu F_{\alpha\mu} V \Big) \\
&+\frac{\beta}{16}\int_{01} {}^\bullet\Delta^\bullet_{01} \langle q_1^\alpha q_0^\nu q_1^\rho q_1^\sigma \rangle G_{\nu\mu} \nabla_\rho \nabla_\sigma F_\alpha{}^\mu V + \frac{\beta}{16}\int_{01} {}^\bullet\Delta_{10} \langle q_1^\alpha \dot{q}_0^\lambda q_1^\rho q_1^\sigma \rangle G_{\mu\lambda} \nabla_\rho \nabla_\sigma F_\alpha{}^\mu V = \\
&\frac{\beta}{16}\int_{01} {}^\bullet\Delta_{11} \langle q_1^\alpha \dot{q}_0^\lambda q_0^\nu q_1^\sigma \rangle \Big(G_{\nu\lambda} \nabla^\mu \nabla_\sigma F_{\alpha\mu} V + G_{\nu\lambda} \nabla_\sigma \nabla^\mu F_{\alpha\mu} V \Big) \\
&+\frac{\beta}{8}\int_{01} {}^\bullet\Delta_{10} \langle q_1^\alpha \dot{q}_0^\lambda q_1^\rho q_1^\sigma \rangle G_{\mu\lambda} \nabla_\rho \nabla_\sigma F_\alpha{}^\mu V = \\
&\frac{\beta^3}{8}\Big(- G_{\mu\lambda} V \nabla^2 F^{\mu\lambda} + G_{\mu\lambda} V \nabla_\alpha \nabla^\lambda F^{\alpha\mu} + G_{\mu\lambda} V \nabla^\lambda \nabla_\alpha F^{\alpha\mu} \Big)
\end{aligned}$$



$$\begin{aligned}
&= \frac{\beta^3}{480}\Big(G_{\mu\lambda} V \nabla^\lambda \nabla_\alpha F^{\alpha\mu} + G_{\mu\lambda} V \nabla_\alpha \nabla^\lambda F^{\alpha\mu} - G_{\mu\lambda} V \nabla^\rho \nabla_\rho F^{\mu\lambda} \Big) \ .
\end{aligned} \tag{4.13}$$

This result can be further simplified using antisymmetry of abelian gauge field and Bianchi identity on non-abelian field strength

$$\begin{aligned}
&G_{\mu\lambda} V \nabla_\alpha \nabla^\lambda F^{\alpha\mu} = \frac{1}{2} G_{\mu\lambda} V \nabla_\alpha \left(\nabla^\lambda F^{\alpha\mu} - \nabla^\mu F^{\alpha\lambda} \right) = -\frac{1}{2} G_{\mu\lambda} V \nabla_\alpha \nabla^\alpha F^{\mu\lambda} \\
&G_{\mu\lambda} V \nabla^\lambda \nabla_\alpha F^{\alpha\mu} = G_{\mu\lambda} V \nabla_\alpha \nabla^\lambda F^{\alpha\mu} + G_{\mu\lambda} V [F^\lambda{}_\alpha, F^{\alpha\mu}] = -\frac{1}{2} G_{\mu\lambda} V \nabla_\alpha \nabla^\alpha F^{\mu\lambda} + G_{\mu\lambda} V [F^\lambda{}_\alpha, F^{\alpha\mu}] \\
&G_{\mu\lambda} V [F^\lambda{}_\alpha, F^{\alpha\mu}] = G_{\mu\lambda} V F^\lambda{}_\alpha F^{\alpha\mu} - G_{\mu\lambda} V F^{\alpha\mu} F^\lambda{}_\alpha = 2 G_{\mu\lambda} F^\lambda{}_\alpha F^{\alpha\mu} V
\end{aligned}$$

where in the last line indices has been renamed in the second term to sum up the two terms (note that this last term vanishes if all the terms commute). So the final result can be cast as

$$- \frac{1}{16}\int_{01} \langle \dot{q}_1^\mu q_1^\alpha \dot{q}_0^\lambda q_0^\nu q_1^\rho q_1^\sigma \rangle G_{\nu\lambda} \nabla_\rho \nabla_\sigma F_{\alpha\mu} V = \frac{\beta^3}{240}\Big(- G_{\mu\lambda} V \nabla_\alpha \nabla^\alpha F^{\mu\lambda} + G_{\mu\lambda} F^\lambda{}_\alpha F^{\alpha\mu} V \Big) \ . \tag{4.14}$$

Next one has (4.7)

$$
\begin{aligned}
&-\frac{1}{8}\int_{012}\langle \dot q_1^\mu q_1^\nu \dot q_0^\rho q_0^\lambda q_2^\alpha q_2^\sigma\rangle G_{\lambda\rho}F_{\nu\mu}\nabla_\alpha\nabla_\sigma V = \\
&\quad\frac{\beta}{8}\int_{012} {}^\bullet\Delta_{12}\langle \dot q_0^\rho q_1^\nu q_0^\lambda q_2^\alpha\rangle\Big(G_{\lambda\rho}F_{\nu\mu}\nabla_\alpha\nabla^\mu V + G_{\lambda\rho}F_{\nu\mu}\nabla^\mu\nabla_\alpha V\Big) \\
&+\frac{\beta}{4}\int_{012} {}^\bullet\Delta_{10}\langle \dot q_0^\lambda q_1^\nu q_2^\alpha q_2^\sigma\rangle G_{\mu\lambda}F_\nu{}^\mu\nabla_\sigma\nabla_\alpha V = \\
&\quad\frac{\beta^3}{8}\left(\frac{1}{360}\right)\Big(-\delta^{\lambda\nu}\delta^{\alpha\rho}+\delta^{\alpha\lambda}\delta^{\nu\rho}\Big)\Big(G_{\lambda\rho}F_{\nu\mu}\nabla_\alpha\nabla^\mu V + G_{\lambda\rho}F_{\nu\mu}\nabla^\mu\nabla_\alpha V\Big) \\
&+\frac{\beta^3}{4}\left(\frac{1}{360}\right)\Big(5\delta^{\lambda\nu}\delta^{\alpha\sigma}+\delta^{\alpha\nu}\delta^{\lambda\sigma}+\delta^{\alpha\lambda}\delta^{\nu\sigma}\Big)G_{\mu\lambda}F_\nu{}^\mu\nabla_\sigma\nabla_\alpha V = \\
&\quad\frac{\beta^3}{8}\left(\frac{1}{360}\right)\Big(-10G_{\mu\lambda}F^{\mu\lambda}\nabla^\alpha\nabla_\alpha V + 4G_{\mu\lambda}F^{\alpha\mu}\nabla^\lambda\nabla_\alpha V + 4G_{\mu\lambda}F^{\alpha\mu}\nabla_\alpha\nabla^\lambda V\Big)\,.
\end{aligned}
\tag{4.15}
$$

This result can be written in a different way, considering the first piece

$$
-10G_{\mu\lambda}F^{\mu\lambda}\nabla^\alpha\nabla_\alpha V = -10\nabla_\alpha\Big(G_{\mu\lambda}F^{\mu\lambda}\nabla^\alpha V\Big) + 10\partial_\alpha G_{\mu\lambda}F^{\mu\lambda}\nabla^\alpha V + 10G_{\mu\lambda}\nabla_\alpha F^{\mu\lambda}\nabla^\alpha V
$$

while for the second one

$$
\begin{aligned}
&+4G_{\mu\lambda}F^{\alpha\mu}\nabla^\lambda\nabla_\alpha V + 4G_{\mu\lambda}F^{\alpha\mu}\nabla_\alpha\nabla^\lambda V = 8G_{\mu\lambda}F^{\alpha\mu}\nabla^\lambda\nabla_\alpha V + 4G_{\mu\lambda}F^{\alpha\mu}[F_\alpha{}^\lambda, V] = \\
&+8\nabla^\lambda\Big(G_{\mu\lambda}F^{\alpha\mu}\nabla_\alpha V\Big) - 8\partial^\lambda G_{\mu\lambda}F^{\alpha\mu}\nabla_\alpha V + 8G_{\lambda\mu}\nabla^\lambda F^{\alpha\mu}\nabla_\alpha V + 8G_{\mu\lambda}F^{\alpha\mu}F_\alpha{}^\lambda V = \\
&+8\nabla^\lambda\Big(G_{\mu\lambda}F^{\alpha\mu}\nabla_\alpha V\Big) - 8\partial^\lambda G_{\mu\lambda}F^{\alpha\mu}\nabla_\alpha V + 4G_{\lambda\mu}\left(\nabla^\lambda F^{\alpha\mu} - \nabla^\mu F^{\alpha\lambda}\right)\nabla_\alpha V + 8G_{\mu\lambda}F^{\alpha\mu}F_\alpha{}^\lambda V = \\
&+8\nabla^\lambda\Big(G_{\mu\lambda}F^{\alpha\mu}\nabla_\alpha V\Big) - 8\partial^\lambda G_{\mu\lambda}F^{\alpha\mu}\nabla_\alpha V + 4G_{\mu\lambda}\nabla^\alpha F^{\mu\lambda}\nabla_\alpha V + 8G_{\mu\lambda}F^{\alpha\mu}F_\alpha{}^\lambda V
\end{aligned}
$$

thus the whole result can be recast as

$$
\begin{aligned}
&-\frac{1}{8}\int_{012}\langle \dot q_1^\mu q_1^\nu \dot q_0^\rho q_0^\lambda q_2^\alpha q_2^\sigma\rangle G_{\lambda\rho}F_{\nu\mu}\nabla_\alpha\nabla_\sigma V \\
&\quad=\frac{\beta^3}{8}\left(\frac{1}{360}\right)\Big(10\partial_\alpha G_{\mu\lambda}F^{\mu\lambda}\nabla^\alpha V + 14G_{\mu\lambda}\nabla^\alpha F^{\mu\lambda}\nabla_\alpha V - 8\partial^\lambda G_{\lambda\mu}F^{\mu\alpha}\nabla_\alpha V + 8G_\mu{}^\lambda F^{\alpha\mu}F_{\alpha\lambda}V\Big)\,.
\end{aligned}
$$

Consider now (4.11)

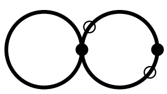
$$
\begin{aligned}
&-\frac{1}{6}\int_{012}\langle \dot q_0^\mu q_0^\alpha \dot q_1^\lambda q_1^\sigma q_1^\rho q_2^\gamma\rangle G_{\alpha\mu}\nabla_\sigma F_{\rho\lambda}\nabla_\gamma V = \\
&+\frac{\beta}{6}\int_{012} {}^\bullet\Delta^\bullet_{01}\langle q_0^\alpha q_1^\sigma q_1^\rho q_2^\gamma\rangle G_{\alpha\mu}\nabla_\sigma F_\rho{}^\mu\nabla_\gamma V \\
&+\frac{\beta}{6}\int_{012} {}^\bullet\Delta_{01}\langle q_0^\alpha \dot q_1^\lambda q_1^\rho q_2^\gamma\rangle\Big(G_{\alpha\mu}\nabla^\mu F_{\rho\lambda}\nabla_\gamma V + G_{\alpha\mu}\nabla_\rho F^\mu{}_\lambda\nabla_\gamma V\Big) \\
&+\frac{\beta}{6}\int_{012} {}^\bullet\Delta_{02}\langle q_0^\alpha \dot q_1^\lambda q_1^\sigma q_1^\rho\rangle G_{\alpha\mu}\nabla_\sigma F_{\rho\lambda}\nabla^\mu V = \\
&\frac{\beta}{6}\int_{012} {}^\bullet\Delta_{01}\langle q_0^\alpha \dot q_1^\lambda q_1^\rho q_2^\gamma\rangle\Big(-G_\alpha{}^\mu\nabla_\lambda F_{\rho\mu}\nabla_\gamma V - G_\alpha{}^\mu\nabla_\rho F_{\lambda\mu}\nabla_\gamma V \\
&+G_\alpha{}^\mu\nabla_\mu F_{\rho\lambda}\nabla_\gamma V + G_\alpha{}^\mu\nabla_\rho F_{\mu\lambda}\nabla_\gamma V\Big) + \frac{\beta}{6}\int_{012} {}^\bullet\Delta_{02}\langle q_0^\alpha \dot q_1^\lambda q_1^\sigma q_1^\rho\rangle G_{\alpha\mu}\nabla_\sigma F_{\rho\lambda}\nabla^\mu V = \\
&\frac{\beta}{2}\int_{012} {}^\bullet\Delta_{01}\langle q_0^\alpha \dot q_1^\lambda q_1^\rho q_2^\gamma\rangle G_\alpha{}^\mu\nabla_\rho F_{\mu\lambda}\nabla_\gamma V + \frac{\beta}{6}\int_{012} {}^\bullet\Delta_{02}\langle q_0^\alpha \dot q_1^\lambda q_1^\sigma q_1^\rho\rangle G_{\alpha\mu}\nabla_\sigma F_{\rho\lambda}\nabla^\mu V = \\
&\frac{\beta^3}{2}\left(\frac{1}{720}\right)\Big(6\delta^{\alpha\lambda}\delta^{\gamma\rho} + \delta^{\alpha\gamma}\delta^{\lambda\rho}\Big)G_\alpha{}^\mu\nabla_\rho F_{\mu\lambda}\nabla_\gamma V \\
&+\frac{\beta^3}{6}\left(\frac{1}{720}\right)\Big(-\delta^{\alpha\rho}\delta^{\lambda\sigma} + 2\delta^{\alpha\lambda}\delta^{\rho\sigma}\Big)G_{\alpha\mu}\nabla_\sigma F_{\rho\lambda}\nabla^\mu V = \\
&\frac{\beta^3}{720}\Big(-3G^{\mu\alpha}\nabla_\lambda F_{\mu\alpha}\nabla^\lambda V + G^{\alpha\mu}\nabla^\lambda F_{\lambda\alpha}\nabla_\mu V\Big)\,.
\end{aligned}
$$

Next we have (4.10)

$$
\begin{aligned}
&-\frac{1}{9}\int_{01}\langle \dot q_1^\mu q_1^\alpha q_1^\sigma \dot q_0^\nu q_0^\rho q_0^\lambda\rangle \partial_\lambda G_{\rho\nu}\nabla_\sigma F_{\alpha\mu}V = \\
&\frac{\beta}{3}\int_{01} {}^\bullet\Delta_{10}\langle q_1^\alpha q_1^\sigma \dot q_0^\nu q_0^\lambda\rangle \partial_\lambda G_{\mu\nu}\nabla_\sigma F_\alpha{}^\mu V + \frac{\beta}{3}\int_{01} {}^\bullet\Delta_{11}\langle q_1^\alpha \dot q_0^\nu q_0^\rho q_0^\lambda\rangle \partial_\lambda G_{\rho\nu}\nabla^\mu F_{\alpha\mu}V = \\
&+\frac{\beta^3}{3}\left(\frac{1}{360}\right)\Big(\delta^{\lambda\nu}\delta^{\alpha\sigma} + 4\delta^{\alpha\nu}\delta^{\lambda\sigma} + 4\delta^{\alpha\lambda}\delta^{\nu\sigma}\Big)\partial_\lambda G_{\mu\nu}\nabla_\sigma F_\alpha{}^\mu V \\
&+\frac{\beta^3}{9}\left(\frac{1}{720}\right)\Big(-\delta^{\lambda\nu}\delta^{\alpha\rho} + 2\delta^{\alpha\nu}\delta^{\lambda\rho}\Big)\partial_\lambda G_{\rho\nu}\nabla^\mu F_{\alpha\mu}V = \\
&-\frac{\beta^3}{720}\Big(\partial^\lambda G_{\lambda\mu}V\nabla_\alpha F^{\alpha\mu} + 4\partial_\lambda G_{\mu\nu}V\nabla^\lambda F^{\mu\nu}\Big)\,.
\end{aligned}
$$

Further one has (4.8)

$$
-\frac{1}{16}\int_{01}\langle \dot q_1^\mu q_1^\alpha \dot q_0^\lambda q_0^\sigma q_0^\rho q_0^\gamma\rangle \partial_\gamma\partial_\rho G_{\sigma\lambda}F_{\alpha\mu}V = \frac{\beta^3}{4}\int_{01} {}^\bullet\Delta_{10}\langle q_1^\alpha \dot q_0^\lambda q_0^\rho q_0^\gamma\rangle \partial_\gamma\partial_\rho G_{\mu\lambda}F_\alpha{}^\mu V =
$$

$-\dfrac{\beta^3}{4}$  $\partial^2 G_{\mu\lambda}F^{\mu\lambda}V = -\dfrac{\beta^3}{240}\partial^2 G_{\mu\lambda}F^{\mu\lambda}V\,.$

Finally we have (4.12)

$$
\begin{aligned}
&-\frac{1}{6}\int_{012}\langle \dot{q}_0^\mu q_0^\nu q_0^\alpha \dot{q}_1^\lambda q_1^\sigma q_2^\rho\rangle \partial_\alpha G_{\nu\mu}F_{\sigma\lambda}\nabla_\rho V = \\
&+\frac{\beta}{2}\int_{012} {}^\bullet\Delta_{01}\langle q_0^\alpha \dot{q}_1^\lambda q_1^\sigma q_2^\gamma\rangle \partial_\sigma G_{\mu\lambda}F_\alpha{}^\mu \nabla_\gamma V + \frac{\beta}{6}\int_{012} {}^\bullet\Delta_{02}\langle q_0^\alpha \dot{q}_1^\lambda q_1^\rho q_1^\sigma\rangle \partial_\sigma G_{\rho\lambda}F_{\alpha\mu}\nabla^\mu V = \\
&+\frac{\beta^3}{6}\left(\frac{1}{720}\right)\Big(-\delta^{\alpha\rho}\delta^{\lambda\sigma}+2\delta^{\alpha\lambda}\delta^{\rho\sigma}\Big)\partial_\sigma G_{\rho\lambda}F_{\alpha\mu}\nabla^\mu V \\
&+\frac{\beta^3}{2}\left(\frac{1}{720}\right)\Big(6\delta^{\alpha\lambda}\delta^{\gamma\sigma}+\delta^{\alpha\gamma}\delta^{\lambda\sigma}\Big)\partial_\sigma G_{\mu\lambda}F_\alpha{}^\mu \nabla_\gamma V = \\
&\frac{\beta^3}{720}\Big(\partial^\lambda G_{\lambda\alpha}F^{\alpha\mu}\nabla_\mu V - 3\partial_\lambda G_{\mu\alpha}F^{\mu\alpha}\nabla^\lambda V\Big)\,.
\end{aligned}
$$

where all the terms have been simplified using part integration on ${}^\bullet\Delta^\bullet$(since all border terms vanish) and Bianchi identity.

Next we have terms obtained by Taylor expanding the exponential of interacting action to first order. First consider the below one, obtained by FS expanding non-abelian gauge field taking four fields terms and coupling with abelian field

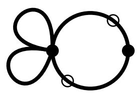
$$
\begin{aligned}
&+\frac{1}{288\beta}\int_{01}\langle \dot{q}_0^\mu q_0^\alpha \dot{q}_1^\nu q_1^\lambda q_1^\rho q_1^\sigma q_1^\Omega q_1^\eta\rangle G_{\alpha\mu}\nabla_\eta\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\nu} = \\
&-\frac{1}{288}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\lambda q_0^\sigma q_0^\Omega q_0^\alpha \dot{q}_1^\nu q_1^\eta\rangle G_{\eta\nu}\nabla_\alpha\nabla_\Omega\nabla_\sigma\nabla^\mu F_{\lambda\mu} \\
&-\frac{1}{288}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\lambda q_0^\rho q_0^\Omega q_0^\alpha \dot{q}_1^\nu q_1^\eta\rangle G_{\eta\nu}\nabla_\alpha\nabla_\Omega\nabla^\mu\nabla_\rho F_{\lambda\mu} \\
&-\frac{1}{288}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\lambda q_0^\rho q_0^\sigma q_0^\alpha \dot{q}_1^\nu q_1^\eta\rangle G_{\eta\nu}\nabla_\alpha\nabla^\mu\nabla_\sigma\nabla_\rho F_{\lambda\mu} \\
&-\frac{1}{288}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\lambda q_0^\rho q_0^\Omega q_0^\sigma \dot{q}_1^\nu q_1^\eta\rangle G_{\eta\nu}\nabla^\mu\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\mu} \\
&-\frac{1}{288}\int_{01} {}^\bullet\Delta^\bullet_{01}\langle q_0^\lambda q_0^\rho q_0^\sigma q_0^\Omega q_0^\alpha q_1^\eta\rangle G_\eta{}^\mu\,\nabla_\alpha\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\mu} \\
&-\frac{1}{288}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\lambda q_0^\rho q_0^\sigma q_0^\Omega q_0^\alpha \dot{q}_1^\nu\rangle G^\mu{}_\nu\,\nabla_\alpha\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\mu}\,.
\end{aligned}
$$

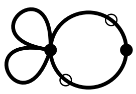
Consider the double derivative contraction, it can be part integrated since the border term vanishes

$$
\begin{aligned}
&-\frac{1}{288}\int_{01} {}^\bullet\Delta^\bullet_{01}\langle q_0^\lambda q_0^\rho q_0^\sigma q_0^\Omega q_0^\alpha q_1^\eta\rangle G_\eta{}^\mu\,\nabla_\alpha\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\mu} = \\
&+\frac{1}{288}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\lambda q_0^\rho q_0^\sigma q_0^\Omega q_0^\alpha \dot{q}_1^\eta\rangle G_\eta{}^\mu\,\nabla_\alpha\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\mu}\,.
\end{aligned}
$$

Collecting contractions, renaming indices and using the second property in (2.7) one has

$$+\frac{1}{288\beta}\int_{01}\langle \dot{q}_0^\mu q_0^\alpha \dot{q}_1^\nu q_1^\lambda q_1^\rho q_1^\sigma q_1^\Omega q_1^\eta\rangle G_{\alpha\mu}\nabla_\eta\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\nu} =$$
$$-\frac{1}{144}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\lambda q_0^\rho q_0^\sigma q_0^\Omega q_0^\alpha \dot{q}_1^\nu\rangle G^\mu{}_\nu\nabla_\alpha\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\mu}$$
$$-\frac{1}{72}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\lambda q_0^\rho q_0^\Omega q_0^\alpha \dot{q}_1^\nu q_1^\eta\rangle G_{\eta\nu}\nabla_\alpha\nabla_\Omega\nabla_\rho\nabla^\mu F_{\lambda\mu} =$$
$$+\frac{\beta}{36}\int_{01} {}^\bullet\Delta_{01}\Delta_{00}\langle q_0^\rho q_0^\Omega q_0^\alpha \dot{q}_1^\nu\rangle G^\mu{}_\nu\nabla_\alpha\nabla_\Omega\nabla_\rho\nabla^\lambda F_{\lambda\mu}$$
$$+\frac{\beta}{144}\int_{01} {}^\bullet\Delta_{01}{}^\bullet\Delta_{10}\langle q_0^\rho q_0^\sigma q_0^\Omega q_0^\alpha\rangle G^{\mu\lambda}\nabla_\alpha\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\mu}$$
$$+\frac{\beta}{72}\int_{01} {}^\bullet\Delta_{00}{}^\bullet\Delta_{10}\langle q_1^\eta q_0^\lambda q_0^\rho q_0^\alpha\rangle \Big(G_{\eta\nu}\nabla_\alpha\nabla_\lambda\nabla_\rho\nabla_\mu F^{\nu\mu} + 3G_{\eta\nu}\nabla_\alpha\nabla_\rho\nabla^\nu\nabla^\mu F_{\lambda\mu}\Big) =$$

$+\frac{\beta^3}{12}$  $\left(G^{\mu\alpha}\nabla^2\nabla_\alpha\nabla^\lambda F_{\lambda\mu} - \frac{1}{4}G^{\mu\nu}\nabla^4 F_{\mu\nu}\right) =$

$-3\frac{\beta^3}{48}$  $G^{\mu\nu}\nabla^4 F_{\mu\nu} = \frac{\beta^3}{4480}G_{\mu\nu}\nabla^4 F^{\mu\nu}$

where we use the following procedure

$$\begin{aligned} G^{\mu\alpha}\nabla^2\nabla_\alpha\nabla^\lambda F_{\lambda\mu} &= G^{\alpha\mu}\nabla^2\nabla^\lambda\nabla_\alpha F_{\mu\lambda} \\ &= \frac{1}{2}G^{\alpha\mu}\nabla^2\nabla^\lambda\left(\nabla_\alpha F_{\mu\lambda} + \nabla_\mu F_{\lambda\alpha}\right) \\ &= -\frac{1}{2}G^{\mu\nu}\nabla^4 F_{\mu\nu}\ . \end{aligned} \tag{4.16}$$

Thus, rewriting the result one has

$$\frac{1}{288\beta}\int_{01}\langle \dot{q}_0^\mu q_0^\alpha \dot{q}_1^\nu q_1^\lambda q_1^\rho q_1^\sigma q_1^\Omega q_1^\eta\rangle G_{\alpha\mu}\nabla_\eta\nabla_\Omega\nabla_\sigma\nabla_\rho F_{\lambda\nu} = \frac{\beta^3}{4480}G_{\mu\nu}\nabla^4 F^{\mu\nu}\ . \tag{4.17}$$

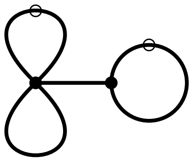
Consider now this term

$$+\frac{1}{90\beta}\int_{01}\langle \dot{q}_0^\mu q_0^\alpha q_0^\sigma q_0^\rho q_0^\lambda \dot{q}_1^\nu q_1^\Omega q_1^\eta\rangle \partial_\lambda\partial_\rho\partial_\sigma G_{\alpha\mu}\nabla_\eta F_{\Omega\nu} =$$
$$-\frac{1}{30}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\alpha q_0^\rho q_0^\lambda \dot{q}_1^\nu q_1^\Omega q_1^\eta\rangle \partial_\lambda\partial_\rho\partial^\mu G_{\alpha\mu}\nabla_\eta F_{\Omega\nu}$$
$$-\frac{1}{30}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\alpha q_0^\sigma q_0^\rho q_0^\lambda \dot{q}_1^\nu q_1^\eta\rangle \partial_\lambda\partial_\rho\partial_\sigma G_\alpha{}^\mu\nabla_\eta F_{\mu\nu}\ .$$

obtained by applying Wick's theorem as before, renaming indices and using Bianchi identity.

Consider the first six fields contraction

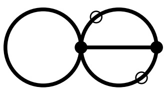
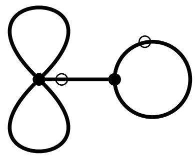
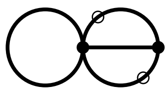
$$-\frac{1}{30}\int_{01} {}^{\bullet}\Delta_{00}\langle q_0^\alpha q_0^\rho q_0^\lambda \dot q_1^\nu q_1^\Omega q_1^\eta\rangle \partial_\lambda\partial_\rho\partial^\mu G_{\alpha\mu}\nabla_\eta F_{\Omega\nu} =$$
$$+\frac{\beta}{30}\int_{01} {}^{\bullet}\Delta_{00}{}^{\bullet}\Delta_{11}\langle q_1^\Omega q_0^\alpha q_0^\rho q_0^\lambda\rangle \partial_\lambda\partial_\rho\partial^\mu G_{\alpha\mu}\nabla^\nu F_{\Omega\nu}$$
$$+\frac{\beta}{30}\int_{01} {}^{\bullet}\Delta_{00}{}^{\bullet}\Delta_{10}\langle q_1^\Omega q_1^\eta q_0^\rho q_0^\lambda\rangle \Big(\partial_\lambda\partial_\rho\partial^\mu G_{\alpha\mu}\nabla_\eta F_\Omega{}^\alpha + \partial_\lambda\partial_\nu\partial^\mu G_{\rho\mu}\nabla_\eta F_\Omega{}^\nu + \partial_\nu\partial_\rho\partial^\mu G_{\lambda\mu}\nabla_\eta F_\Omega{}^\nu\Big) =$$

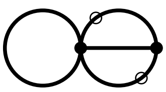
$-\frac{\beta^3}{30}$  $\big(\partial^2\partial^\mu G_{\alpha\mu}\nabla_\nu F^{\nu\alpha} + 2\partial^2\partial^\mu G_{\alpha\mu}\nabla_\lambda F^{\lambda\alpha}\big) =$

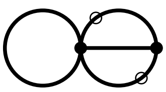
$$-\frac{\beta^3}{50400}\partial^2\partial^\mu G_{\alpha\mu}\nabla_\lambda F^{\lambda\alpha}$$

while for the second six fields contraction we have

$$-\frac{1}{30}\int_{01} {}^{\bullet}\Delta_{01}\langle q_0^\alpha q_0^\sigma q_0^\rho q_0^\lambda \dot q_1^\nu q_1^\eta\rangle \partial_\lambda\partial_\rho\partial_\sigma G_\alpha{}^\mu\nabla_\eta F_{\mu\nu} =$$
$$+\frac{\beta}{10}\int_{01} {}^{\bullet}\Delta_{01}\Delta_{00}\langle q_0^\lambda q_0^\rho \dot q_1^\nu q_1^\eta\rangle \partial^\alpha\partial_\rho\partial_\lambda G_\alpha{}^\mu\nabla_\eta F_{\mu\nu} + \frac{\beta}{30}\int_{01} {}^{\bullet}\Delta_{01}\Delta^{\bullet}_{01}\langle q_0^\sigma q_0^\rho q_0^\lambda q_1^\eta\rangle \partial_\lambda\partial_\rho\partial_\sigma G_\alpha{}^\mu\nabla_\eta F_\mu{}^\alpha$$
$$+\frac{\beta}{30}\int_{01} {}^{\bullet}\Delta_{01}\Delta_{01}\langle q_0^\sigma q_0^\rho q_0^\lambda \dot q_1^\nu\rangle \partial_\lambda\partial_\rho\partial_\sigma G_\alpha{}^\mu\nabla^\alpha F_{\mu\nu} =$$

$+\frac{\beta^3}{10}\Big[$  $\delta^{\lambda\nu}\delta^{\eta\rho} +$  $\delta^{\eta\nu}\delta^{\lambda\rho} +$  $\delta^{\eta\lambda}\delta^{\nu\rho}\Big]\partial_\lambda\partial_\rho\partial^\alpha G_\alpha{}^\mu\nabla_\eta F_{\mu\nu}$

$+\frac{\beta^3}{30}$  $\Big(\delta^{\lambda\rho}\delta^{\eta\sigma} + \delta^{\eta\rho}\delta^{\lambda\sigma} + \delta^{\eta\lambda}\delta^{\rho\sigma}\Big)\partial_\lambda\partial_\rho\partial_\sigma G_\alpha{}^\mu\nabla_\eta F_\mu{}^\alpha$

$+\frac{\beta^3}{30}$  $\Big(\delta^{\nu\rho}\delta^{\lambda\sigma} + \delta^{\lambda\rho}\delta^{\nu\sigma} + \delta^{\lambda\nu}\delta^{\rho\sigma}\Big)\partial_\lambda\partial_\rho\partial_\sigma G_\alpha{}^\mu\nabla^\alpha F_{\mu\nu} =$

$$+\frac{\beta^3}{1680}\partial^2\partial^\rho G^{\mu\nu}\nabla_\rho F_{\mu\nu} + \frac{\beta^3}{12600}\partial^2\partial^\alpha G_{\alpha\mu}\nabla_\nu F^{\nu\mu}\ ,$$

adding up the two results one has

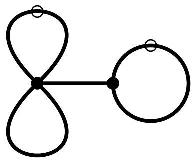
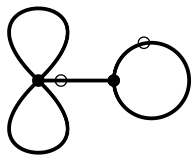
$$\frac{1}{90\beta}\int_{01}\langle \dot q_0^\mu q_0^\alpha q_0^\sigma q_0^\rho q_0^\lambda \dot q_1^\nu q_1^\Omega q_1^\eta\rangle \partial_\lambda\partial_\rho\partial_\sigma G_{\alpha\mu}\nabla_\eta F_{\Omega\nu}$$
$$= \frac{\beta^3}{10080}\Big(6\partial^2\partial^\rho G^{\mu\nu}\nabla_\rho F_{\mu\nu} + \partial^2\partial^\alpha G_{\alpha\mu}\nabla_\nu F^{\nu\mu}\Big)\ . \tag{4.18}$$

Consider next the following term

$$+\frac{1}{90\beta}\int_{01}\langle \dot q_0^\mu q_0^\alpha q_0^\sigma \dot q_1^\nu q_1^\rho q_1^\lambda q_1^\eta q_1^\Omega\rangle \partial_\sigma G_{\alpha\mu}\nabla_\Omega\nabla_\eta\nabla_\lambda F_{\rho\nu} =$$
$$-\frac{1}{30}\int_{012} {}^\bullet\Delta_{11}\langle \dot q_0^\mu q_0^\alpha q_0^\sigma q_1^\rho q_1^\eta q_1^\Omega\rangle \partial_\sigma G_{\alpha\mu}\nabla_\Omega\nabla_\eta\nabla^\nu F_{\rho\nu}$$
$$+\frac{1}{30}\int_{01} {}^\bullet\Delta_{10}\langle \dot q_0^\mu q_0^\sigma q_1^\rho q_1^\lambda q_1^\eta q_1^\Omega\rangle \partial_\sigma G_{\mu\nu}\nabla_\Omega\nabla_\eta\nabla_\lambda F_\rho{}^\nu\ .$$

Consider the first six fields contraction

$$-\frac{1}{30}\int_{012} {}^\bullet\Delta_{11}\langle \dot q_0^\mu q_0^\alpha q_0^\sigma q_1^\rho q_1^\eta q_1^\Omega\rangle \partial_\sigma G_{\alpha\mu}\nabla_\Omega\nabla_\eta\nabla^\nu F_{\rho\nu} =$$
$$+\frac{\beta}{30}\int_{01} {}^\bullet\Delta_{11}{}^\bullet\Delta_{00}\langle q_0^\alpha q_1^\rho q_1^\eta q_1^\Omega\rangle \partial^\mu G_{\alpha\mu}\nabla_\Omega\nabla_\eta\nabla^\nu F_{\rho\nu}$$
$$+\frac{\beta}{30}\int_{01} {}^\bullet\Delta_{11}{}^\bullet\Delta_{01}\langle q_0^\alpha q_0^\sigma q_1^\eta q_1^\Omega\rangle \partial_\sigma G_{\alpha\mu}\nabla_\Omega\nabla_\eta\nabla_\nu F^{\mu\nu}$$
$$+\frac{\beta}{15}\int_{01} {}^\bullet\Delta_{11}{}^\bullet\Delta_{01}\langle q_0^\alpha q_0^\sigma q_1^\rho q_1^\Omega\rangle \partial_\sigma G_{\alpha\mu}\nabla_\Omega\nabla^\mu\nabla^\nu F_{\rho\nu} =$$

$\frac{\beta^3}{30}\Big[$  $-$  $\Big]\ \partial^\mu G_{\mu\alpha}\nabla^2\nabla_\nu F^{\nu\alpha} =$

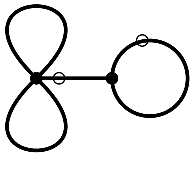
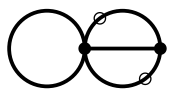
$$\frac{\beta^3}{10}\left(\frac{1}{5040}\right)\partial^\mu G_{\mu\alpha}\nabla^2\nabla_\nu F^{\nu\alpha}$$

where we used the following simplification

$$2\partial_\sigma G_{\alpha\mu}\nabla^\sigma\nabla_\nu\nabla^\alpha F^{\mu\nu} = \partial_\sigma G_{\alpha\mu}\nabla^\sigma\nabla_\nu\underbrace{\left(\nabla^\alpha F^{\mu\nu} + \nabla^\mu F^{\nu\alpha}\right)}_{=-\nabla^\nu F^{\alpha\mu}} = -\partial_\sigma G_{\alpha\mu}\nabla^2\nabla^\sigma F^{\alpha\mu}$$

while for the second one

$$+\frac{1}{30}\int_{01} {}^\bullet\Delta_{10}\langle \dot q_0^\mu q_0^\sigma q_1^\rho q_1^\lambda q_1^\eta q_1^\Omega\rangle \partial_\sigma G_{\mu\nu}\nabla_\Omega\nabla_\eta\nabla_\lambda F_\rho{}^\nu =$$
$$-\frac{\beta}{30}\int_{01} {}^\bullet\Delta_{10}{}^\bullet\Delta_{00}\langle q_1^\rho q_1^\lambda q_1^\eta q_1^\Omega\rangle \partial^\mu G_{\mu\nu}\nabla_\Omega\nabla_\eta\nabla_\lambda F_\rho{}^\nu$$
$$-\frac{\beta}{30}\int_{01} {}^\bullet\Delta_{10}\Delta^\bullet_{10}\langle q_0^\sigma q_1^\lambda q_1^\eta q_1^\Omega\rangle \partial_\sigma G_{\mu\nu}\nabla_\Omega\nabla_\eta\nabla_\lambda F^{\mu\nu}$$
$$-\frac{\beta}{10}\int_{01} {}^\bullet\Delta_{10}\Delta^\bullet_{10}\langle q_0^\sigma q_1^\rho q_1^\lambda q_1^\Omega\rangle \partial_\sigma G_{\mu\nu}\nabla_\lambda\nabla_\Omega\nabla^\mu F_\rho{}^\nu =$$

$-\frac{\beta^3}{10}$  $\partial^\mu G_{\mu\nu}\nabla^2\nabla_\lambda F^{\lambda\nu} - \frac{\beta^3}{4}$  $\partial_\sigma G_{\mu\nu}\nabla^2\nabla^\sigma F^{\mu\nu} =$

$$\frac{\beta^3}{12600}\partial^\mu G_{\mu\nu}\nabla^2\nabla_\lambda F^{\lambda\nu} + \frac{\beta^3}{1680}\partial_\sigma G_{\mu\nu}\nabla^2\nabla^\sigma F^{\mu\nu}\ ,$$

adding up results one has

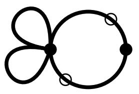
$$\begin{aligned}&\frac{1}{90\beta}\int_{01}\langle \dot{q}_0^{\mu} q_0^{\alpha} q_0^{\sigma} \dot{q}_1^{\nu} q_1^{\rho} q_1^{\lambda} q_1^{\eta} q_1^{\Omega}\rangle \partial_\sigma G_{\alpha\mu}\nabla_\Omega\nabla_\eta\nabla_\lambda F_{\rho\nu} \\ &= \frac{\beta^3}{10080}\Big(6\partial_\sigma G_{\mu\alpha}\nabla^2\nabla^\sigma F^{\mu\alpha} + \partial^\mu G_{\mu\alpha}\nabla^2\nabla_\lambda F^{\lambda\alpha}\Big)\ .\end{aligned} \tag{4.19}$$

Consider next this one

$$+\frac{1}{288\beta}\int_{01}\langle \dot{q}_0^{\mu} q_0^{\alpha} q_0^{\sigma} q_0^{\lambda} q_0^{\rho} q_0^{\Omega} \dot{q}_1^{\nu} q_1^{\eta}\rangle \partial_\Omega\partial_\lambda\partial_\rho\partial_\sigma G_{\alpha\mu}F_{\eta\nu} =$$

$$-\frac{1}{144}\int_{01} {}^\bullet\Delta_{01}\langle q_0^{\alpha} q_0^{\sigma} q_0^{\rho} q_0^{\lambda} q_0^{\Omega} \dot{q}_1^{\nu}\rangle \partial_\Omega\partial_\lambda\partial_\rho\partial_\sigma G_{\alpha\mu}F^{\mu}{}_{\nu} =$$

$$+\frac{\beta}{36}\int_{01} {}^\bullet\Delta_{01}\Delta_{00}\langle q_0^{\rho} q_0^{\lambda} q_0^{\Omega} \dot{q}_1^{\nu}\rangle \partial_\Omega\partial_\lambda\partial_\rho\partial^\alpha G_{\alpha\mu}F^{\mu}{}_{\nu} + \frac{\beta}{144}\int_{01} {}^\bullet\Delta_{01}{}^\bullet\Delta_{10}\langle q_0^{\sigma} q_0^{\rho} q_0^{\lambda} q_0^{\Omega}\rangle \partial_\Omega\partial_\lambda\partial_\rho\partial_\sigma G_{\alpha\mu}F^{\mu\alpha} =$$

$$-\frac{\beta^3}{16}$$  $$\partial^4 G_{\mu\nu}F^{\mu\nu} = \frac{\beta^3}{4480}\partial^4 G_{\mu\nu}F^{\mu\nu}$$

rewriting we have

$$\frac{1}{288\beta}\int_{01}\langle \dot{q}_0^{\mu} q_0^{\alpha} q_0^{\sigma} q_0^{\lambda} q_0^{\rho} q_0^{\Omega} \dot{q}_1^{\nu} q_1^{\eta}\rangle \partial_\Omega\partial_\lambda\partial_\rho\partial_\sigma G_{\alpha\mu}F_{\eta\nu} = \frac{\beta^3}{4480}\partial^4 G_{\mu\nu}F^{\mu\nu}\ . \tag{4.20}$$

Consider this one

$$+\frac{1}{64\beta}\int_{01}\langle \dot{q}_0^{\mu} q_0^{\alpha} q_0^{\sigma} q_0^{\Omega} \dot{q}_1^{\nu} q_1^{\lambda} q_1^{\eta} q_1^{\rho}\rangle \partial_\Omega\partial_\sigma G_{\alpha\mu}\nabla_\rho\nabla_\eta F_{\lambda\nu} =$$

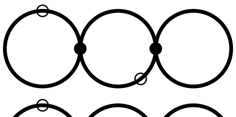
$$-\frac{1}{32}\int_{01} {}^\bullet\Delta_{00}\langle q_0^{\alpha} q_0^{\Omega} \dot{q}_1^{\nu} q_1^{\lambda} q_1^{\eta} q_1^{\rho}\rangle \partial_\Omega\partial^\mu G_{\alpha\mu}\nabla_\rho\nabla_\eta F_{\lambda\nu} - \frac{1}{16}\int_{01} {}^\bullet\Delta_{01}\langle q_0^{\alpha} q_0^{\sigma} q_0^{\Omega} \dot{q}_1^{\nu} q_1^{\lambda} q_1^{\rho}\rangle \partial_\Omega\partial_\sigma G_{\alpha}{}^{\mu}\nabla_\rho\nabla_\lambda F_{\mu\nu}\ .$$

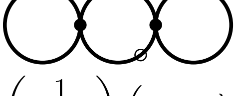
Considering the first six field contractions

$$-\frac{1}{32}\int_{01} {}^\bullet\Delta_{00}\langle q_0^{\alpha} q_0^{\Omega} \dot{q}_1^{\nu} q_1^{\lambda} q_1^{\eta} q_1^{\rho}\rangle \partial_\Omega\partial^\mu G_{\alpha\mu}\nabla_\rho\nabla_\eta F_{\lambda\nu} =$$

$$+\frac{\beta}{32}\int_{01} {}^\bullet\Delta_{00}{}^\bullet\Delta_{10}\langle q_1^{\lambda} q_1^{\eta} q_1^{\rho} q_0^{\Omega}\rangle\Big(\partial_\Omega\partial_\mu G^{\nu\mu}\nabla_\rho\nabla_\eta F_{\lambda\nu} + \partial^\mu\partial^\nu G_{\Omega\mu}\nabla_\rho\nabla_\eta F_{\lambda\nu}\Big)$$

$$+\frac{\beta}{16}\int_{01} {}^\bullet\Delta_{00}{}^\bullet\Delta_{11}\langle q_1^{\lambda} q_1^{\rho} q_0^{\alpha} q_0^{\Omega}\rangle \partial_\Omega\partial^\mu G_{\alpha\mu}\nabla_\rho\nabla^\nu F_{\lambda\nu} =$$

$$+\frac{\beta^3}{32}$$  $$\big(\delta^{\lambda\rho}\delta^{\eta\Omega} + \delta^{\eta\rho}\delta^{\lambda\Omega} + \delta^{\eta\lambda}\delta^{\rho\Omega}\big)\Big(\partial_\Omega\partial_\mu G^{\nu\mu}\nabla_\rho\nabla_\eta F_{\lambda\nu} + \partial^\mu\partial^\nu G_{\Omega\mu}\nabla_\rho\nabla_\eta F_{\lambda\nu}\Big)$$

$$-\frac{\beta^3}{16}$$  $$\big(\delta^{\alpha\rho}\delta^{\lambda\Omega} + \delta^{\alpha\lambda}\delta^{\rho\Omega}\big)\,\partial_\Omega\partial^\mu G_{\alpha\mu}\nabla_\rho\nabla^\nu F_{\lambda\nu} =$$

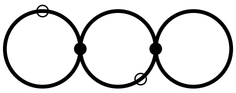
$$-\frac{\beta^3}{32}\left(\frac{1}{5040}\right)\Big(-4\partial^\lambda\partial^\mu G_{\alpha\mu}\nabla^\alpha\nabla_\nu F^{\lambda\nu} - 4\partial^\lambda\partial^\mu G_{\alpha\mu}\nabla_\nu\nabla_\lambda F^{\alpha\nu}\Big)$$

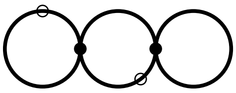
while for the second six field contraction

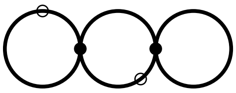
$$-\frac{1}{16}\int_{01} {}^{\bullet}\Delta_{01}\langle q_0^\alpha q_0^\sigma q_0^\Omega \dot q_1^\nu q_1^\lambda q_1^\rho\rangle \partial_\Omega\partial_\sigma G_\alpha{}^\mu \nabla_\rho\nabla_\lambda F_{\mu\nu} =$$

$$+\frac{\beta}{8}\int_{01} {}^{\bullet}\Delta_{01}\Delta_{00}\langle q_0^\Omega \dot q_1^\nu q_1^\lambda q_1^\rho\rangle \partial_\Omega\partial_\alpha G^{\alpha\mu}\nabla_\rho\nabla_\lambda F_{\mu\nu} + \frac{\beta}{16}\int_{01} {}^{\bullet}\Delta_{01}{}^{\bullet}\Delta_{10}\langle q_0^\sigma q_0^\Omega q_1^\lambda q_1^\rho\rangle \partial_\Omega\partial_\sigma G^{\alpha\mu}\nabla_\rho\nabla_\lambda F_{\mu\alpha}$$

$$+\frac{\beta}{8}\int_{01} {}^{\bullet}\Delta_{01}\Delta_{01}\langle q_0^\sigma q_0^\Omega \dot q_1^\nu q_1^\rho\rangle \partial_\Omega\partial_\sigma G^{\alpha\mu}\nabla_\rho\nabla_\alpha F_{\mu\nu} =$$

$+\frac{\beta^3}{8}$  $\left(\delta^{\nu\rho}\delta^{\lambda\Omega} + 17\delta^{\lambda\rho}\delta^{\nu\Omega} + \delta^{\lambda\nu}\delta^{\rho\Omega}\right)\partial_\Omega\partial_\alpha G^{\alpha\mu}\nabla_\rho\nabla_\lambda F_{\mu\nu}$

$+\frac{\beta^3}{8}$  $\left(9\delta^{\rho\sigma}\delta^{\nu\Omega} + 9\delta^{\nu\sigma}\delta^{\rho\Omega} + \delta^{\nu\rho}\delta^{\sigma\Omega}\right)\partial_\Omega\partial_\sigma G^{\alpha\mu}\nabla_\rho\nabla_\alpha F_{\mu\nu}$

$+\frac{\beta^3}{16}$  $\left(9\delta^{\rho\sigma}\delta^{\lambda\Omega} + 9\delta^{\lambda\sigma}\delta^{\rho\Omega} + 17\delta^{\lambda\rho}\delta^{\sigma\Omega}\right)\partial_\Omega\partial_\sigma G^{\alpha\mu}\nabla_\rho\nabla_\lambda F_{\mu\alpha} =$

$$-\frac{\beta^3}{16}\left(\frac{1}{5040}\right)\left(4\partial^\lambda\partial^\alpha G_{\alpha\mu}\nabla_\nu\nabla_\lambda F^{\mu\nu} - 35\partial^2 G^{\mu\nu}\nabla^2 F_{\mu\nu} - 36\partial^\rho\partial^\alpha G^{\mu\nu}\nabla_\rho\nabla_\alpha F_{\mu\nu}\right),$$

adding up the two results one has

$$-\frac{\beta^3}{16}\left(\frac{1}{5040}\right)\Big(6\partial^\lambda\partial^\alpha G_{\alpha\mu}\nabla_\nu\nabla_\lambda F^{\mu\nu} + 2\partial_\lambda\partial^\alpha G_{\alpha\mu}\nabla^\mu\nabla_\nu F^{\lambda\nu} - 35\partial^2 G^{\mu\nu}\nabla^2 F_{\mu\nu} - 36\partial^\rho\partial^\alpha G^{\mu\nu}\nabla_\rho\nabla_\alpha F_{\mu\nu}\Big).$$

we can use Bianchi identity to rewrite the following term

$$\begin{aligned} 2\partial_\lambda\partial^\alpha G_{\alpha\mu}\nabla_\nu\nabla^\mu F^{\lambda\nu} &= 2\partial_\lambda\partial^\alpha G_{\alpha\mu}\nabla_\nu\left(\nabla^\lambda F^{\mu\nu} + \nabla^\nu F^{\lambda\mu}\right) \\ &= 2\partial^\lambda\partial^\alpha G_{\alpha\mu}\nabla_\nu\nabla_\lambda F^{\mu\nu} + 2\partial_\lambda\partial^\alpha G_{\alpha\mu}\nabla_\nu\nabla^\nu F^{\lambda\mu} \\ &= 2\partial^\lambda\partial^\alpha G_{\alpha\mu}\nabla_\nu\nabla_\lambda F^{\mu\nu} + \partial^2 G_{\mu\nu}\nabla^2 F^{\mu\nu} \end{aligned}$$

thus the whole result can be cast as

$$\begin{aligned} &\frac{1}{64\beta}\int_{01}\langle \dot q_0^\mu q_0^\alpha q_0^\sigma q_0^\Omega \dot q_1^\nu q_1^\lambda q_1^\eta q_1^\rho\rangle \partial_\Omega\partial_\sigma G_{\alpha\mu}\nabla_\rho\nabla_\eta F_{\lambda\nu} \\ &= \frac{\beta^3}{8}\left(\frac{1}{5040}\right)\left(4\partial^\lambda\partial^\alpha G_{\alpha\mu}\nabla_\lambda\nabla_\nu F^{\nu\mu} + 17\partial^2 G_{\mu\nu}\nabla^2 F^{\mu\nu} + 18\partial^\lambda\partial^\alpha G^{\mu\nu}\nabla_\lambda\nabla_\alpha F_{\mu\nu}\right). \end{aligned} \tag{4.21}$$

Let us now make some simplifications. Consider (4.17), (4.18), (4.19), (4.20), (4.21). We can rewrite each one of them up to total derivatives and commutators. Considering for instance (4.19) one has

$$\frac{1}{10080}\Big(6\partial_\sigma G_{\mu\alpha}\nabla^2\nabla^\sigma F^{\mu\alpha} + \partial^\mu G_{\mu\alpha}\nabla^2\nabla_\lambda F^{\lambda\alpha}\Big) =$$

$$\frac{1}{10080}\Big(6\nabla_\sigma(G_{\mu\alpha}\nabla^2\nabla^\sigma F^{\mu\alpha}) - 6G_{\mu\alpha}\nabla^4 F^{\mu\alpha} + \nabla^\mu(G_{\mu\alpha}\nabla^2\nabla_\lambda F^{\lambda\alpha}) - G_{\mu\alpha}\nabla^2\nabla_\lambda\nabla^\mu F^{\lambda\alpha}\Big) =$$

$$\frac{1}{10080}\Big(-\frac{13}{2}G_{\mu\nu}\nabla^4 F^{\mu\nu}\Big)$$

where total derivatives and commutators have been dropped out.
Proceeding similarly for the others one can rewrite and add up them, obtaining

$$(4.17) + (4.18) + (4.19) + (4.20) + (4.21) = \left(\frac{\beta^3}{10080}\right)\frac{3}{4}G_{\mu\nu}\nabla^4 F^{\mu\nu} \tag{4.22}$$

Further one has just a term from the third order Taylor expansion of the exponential, namely

$$\begin{aligned}
&+\frac{1}{16\beta}\int_{0123}\langle \dot q_0^\eta q_0^\omega \dot q_1^\mu q_1^\sigma \dot q_2^\nu q_2^\rho \dot q_3^\lambda q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F_{\Omega\lambda} = \\
&-\frac{1}{16}\int_{0123} {}^\bullet\Delta^\bullet_{01}\langle q_0^\omega q_1^\sigma \dot q_2^\nu q_2^\rho \dot q_3^\lambda q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F_\sigma{}^\eta F_{\rho\nu}F_{\Omega\lambda} \\
&-\frac{1}{16}\int_{0123} {}^\bullet\Delta_{01}\langle q_0^\omega \dot q_1^\mu \dot q_2^\nu q_2^\rho \dot q_3^\lambda q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F^\eta{}_\mu F_{\rho\nu}F_{\Omega\lambda} \\
&-\frac{1}{16}\int_{0123} {}^\bullet\Delta^\bullet_{02}\langle q_0^\omega \dot q_1^\mu q_1^\sigma q_2^\rho \dot q_3^\lambda q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F_{\sigma\mu}F_\rho{}^\eta F_{\Omega\lambda} \\
&-\frac{1}{16}\int_{0123} {}^\bullet\Delta_{02}\langle q_0^\omega \dot q_1^\mu q_1^\sigma \dot q_2^\nu \dot q_3^\lambda q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F_{\sigma\mu}F^\eta{}_\nu F_{\Omega\lambda} \\
&-\frac{1}{16}\int_{0123} {}^\bullet\Delta^\bullet_{03}\langle q_0^\omega \dot q_1^\mu q_1^\sigma \dot q_2^\nu q_2^\rho q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F_\Omega{}^\eta \\
&-\frac{1}{16}\int_{0123} {}^\bullet\Delta_{03}\langle q_0^\omega \dot q_1^\mu q_1^\sigma \dot q_2^\nu q_2^\rho \dot q_3^\lambda\rangle \theta_{12}\theta_{23} G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F^\eta{}_\lambda
\end{aligned}$$

we evaluated separately the correlators applying Wick's theorem and renaming variables in order to produce the same order of contractions that have been used for the first above.

$$\begin{aligned}
&-\frac{1}{16}\int_{0123} {}^\bullet\Delta^\bullet_{01}\langle q_0^\omega q_1^\sigma \dot q_2^\nu q_2^\rho \dot q_3^\lambda q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F_\sigma{}^\eta F_{\rho\nu}F_{\Omega\lambda} \\
&\quad = +\frac{\beta^3}{16}\left(\frac{1}{10080}\right)\left(35G_{\omega\eta}F^{\omega\eta}F_{\lambda\nu}F^{\lambda\nu} - 7G_{\omega\eta}F^{\rho\eta}F^{\omega\lambda}F_{\lambda\rho} - 21G_{\omega\eta}F^{\rho\eta}F_{\lambda\rho}F^{\omega\lambda}\right)
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{16}\int_{0123} {}^\bullet\Delta_{01}\langle q_0^\omega \dot q_1^\mu \dot q_2^\nu q_2^\rho \dot q_3^\lambda q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F^\eta{}_\mu F_{\rho\nu}F_{\Omega\lambda} \\
&\quad = +\frac{\beta^3}{16}\left(\frac{1}{10080}\right)\left(35G_{\omega\eta}F^{\omega\eta}F_{\lambda\nu}F^{\lambda\nu} + 7G_{\omega\eta}F^{\eta\lambda}F^{\omega\rho}F_{\rho\lambda} + 7G_{\omega\eta}F^{\eta\lambda}F_{\rho\lambda}F^{\omega\rho}\right)
\end{aligned}$$

$$-\frac{1}{16}\int_{0123} {}^\bullet\Delta^\bullet_{02}\langle q_0^\omega \dot q_1^\mu q_1^\sigma q_2^\rho \dot q_3^\lambda q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F_{\sigma\mu}F_\rho{}^\eta F_{\Omega\lambda} = 0$$

$$\begin{aligned}
&-\frac{1}{16}\int_{0123} {}^\bullet\Delta_{02}\langle q_0^\omega \dot q_1^\mu q_1^\sigma \dot q_2^\nu \dot q_3^\lambda q_3^\Omega\rangle \theta_{12}\theta_{23} G_{\omega\eta}F_{\sigma\mu}F^\eta{}_\nu F_{\Omega\lambda} \\
&\quad = +\frac{\beta^3}{16}\left(\frac{1}{10080}\right)\left(14G_{\omega\eta}F^{\omega\rho}F^{\eta\lambda}F_{\rho\lambda} - 14G_{\omega\eta}F_{\rho\lambda}F^{\eta\lambda}F^{\rho\omega}\right)
\end{aligned}$$

$$-\frac{1}{16}\int_{0123} {}^\bullet\Delta_{03}^\bullet\langle q_0^\omega \dot{q}_1^\mu q_1^\sigma \dot{q}_2^\nu q_2^\rho q_3^\Omega\rangle\theta_{12}\theta_{23}G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F_\Omega{}^\eta$$
$$= +\frac{\beta^3}{16}\left(\frac{1}{10080}\right)\Big(35G_{\omega\eta}F^{\omega\eta}F_{\lambda\nu}F^{\lambda\nu} - 7G_{\omega\eta}F^{\lambda\eta}F_{\rho\lambda}F^{\omega\rho} + 21G_{\omega\eta}F^{\lambda\eta}F^{\rho\omega}F_{\rho\lambda}$$

$$-\frac{1}{16}\int_{0123} {}^\bullet\Delta_{03}\langle q_0^\omega \dot{q}_1^\mu q_1^\sigma \dot{q}_2^\nu q_2^\rho \dot{q}_3^\lambda\rangle\theta_{12}\theta_{23}G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F^\eta{}_\lambda$$
$$= +\frac{\beta^3}{16}\left(\frac{1}{10080}\right)\Big(35G_{\omega\eta}F^{\omega\eta}F_{\lambda\nu}F^{\lambda\nu} + 7G_{\omega\eta}F^{\eta\lambda}F_{\rho\lambda}F^{\omega\rho} - 7G_{\omega\eta}F^{\eta\lambda}F^{\rho\omega}F_{\rho\lambda}\Big)\,.$$

Adding up results from contractions we have

$$\frac{1}{16\beta}\int_{0123}\langle \dot{q}_0^\eta q_0^\omega \dot{q}_1^\mu q_1^\sigma \dot{q}_2^\nu q_2^\rho \dot{q}_3^\lambda q_3^\Omega\rangle\theta_{12}\theta_{23}G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F_{\Omega\lambda}$$
$$= +\frac{\beta^3}{16}\left(\frac{1}{10080}\right)\Big(140G_{\omega\eta}F^{\omega\eta}F_{\lambda\nu}F^{\lambda\nu}) + 112G_{\omega\eta}F^{\eta\lambda}F^{\omega\rho}F_{\rho\lambda}\Big)$$
$$= +\frac{\beta^3}{16}\left(\frac{1}{720}\right)\Big(10G_{\omega\eta}F^{\omega\eta}F_{\lambda\nu}F^{\lambda\nu}) + 8G_{\omega\eta}F^{\eta\lambda}F^{\omega\rho}F_{\rho\lambda}\Big)\,.$$

Consider the last piece in the above result making explicit colour trace

$$\begin{aligned}8G_{\omega\eta}\mathrm{tr}_{\,YM}(F^{\eta\lambda}F^{\omega\rho}F_{\rho\lambda}) &= 8G_{\omega\eta}\mathrm{tr}_{\,YM}(F^{\omega\rho}F_{\rho\lambda}F^{\eta\lambda})\\ &= 8G_{\eta\omega}\mathrm{tr}_{\,YM}(F^{\omega\rho}F_{\rho\lambda}F^{\lambda\eta})\\ &= 8G_{\omega\eta}\mathrm{tr}_{\,YM}(F^{\eta\rho}F_{\rho\lambda}F^{\lambda\omega})\end{aligned}$$

where in the first line we used cyclicity of trace, in the second we switch two pairs of indices and in the third line we renamed indices. So recasting one has

$$+\frac{1}{16\beta}\int_{0123}\langle \dot{q}_0^\eta q_0^\omega \dot{q}_1^\mu q_1^\sigma \dot{q}_2^\nu q_2^\rho \dot{q}_3^\lambda q_3^\Omega\rangle\theta_{12}\theta_{23}G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F_{\Omega\lambda}$$
$$= +\frac{\beta^3}{16}\left(\frac{1}{720}\right)\Big(10G_{\omega\eta}F^{\omega\eta}F_{\lambda\nu}F^{\lambda\nu} + 8G_{\omega\eta}F^{\eta\rho}F_{\rho\lambda}F^{\lambda\omega}\Big)\,.$$

In order to check the correctness of the above result, since its computation was really involved, one can compute the abelian partner and confront results. The abelian partner is written as

$$+\frac{1}{96\beta}\int_{0123}\langle \dot{q}_0^\eta q_0^\omega \dot{q}_1^\mu q_1^\sigma \dot{q}_2^\nu q_2^\rho \dot{q}_3^\lambda q_3^\Omega\rangle G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F_{\Omega\lambda}\,.$$

where since no time ordering prescription (this allows to get rid of Heaviside step functions), one only has to Taylor expand the usual action to third order, use FS gauge and couple with the abelian curvature. Proceeding with the computation we have

$$+\frac{1}{96\beta}\int_{0123}\langle \dot q_0^\eta q_0^\omega \dot q_1^\mu q_1^\sigma \dot q_2^\nu q_2^\rho \dot q_3^\lambda q_3^\Omega\rangle G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F_{\Omega\lambda} =$$

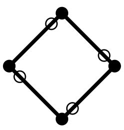
$$-\frac{1}{48}\int_{0123} {}^\bullet\Delta_{01}\langle \dot q_2^\nu q_2^\rho \dot q_3^\lambda q_3^\Omega q_0^\omega \dot q_1^\sigma\rangle G_{\omega\eta}F^\eta{}_\sigma F_{\rho\nu}F_{\Omega\lambda} - \frac{1}{48}\int_{0123} {}^\bullet\Delta_{02}\langle \dot q_3^\lambda q_3^\Omega q_0^\omega \dot q_1^\mu q_1^\sigma \dot q_2^\nu\rangle G_{\omega\eta}F_{\sigma\mu}F^\eta{}_\nu F_{\Omega\lambda} =$$

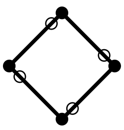
$$-\frac{1}{48}\int_{0123} {}^\bullet\Delta_{03}\langle \dot q_1^\mu q_1^\sigma q_0^\omega \dot q_2^\nu q_2^\rho \dot q_3^\lambda\rangle G_{\omega\eta}F_{\sigma\mu}F_{\rho\nu}F^\eta{}_\lambda =$$

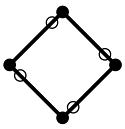
$$-\frac{1}{16}\int_{0123} {}^\bullet\Delta_{01}\langle \dot q_2^\nu q_2^\rho \dot q_3^\lambda q_3^\Omega q_0^\omega \dot q_1^\sigma\rangle G_{\omega\eta}F^\eta{}_\sigma F_{\rho\nu}F_{\Omega\lambda} =$$

$$+\frac{3}{4}\beta\int_{0123} {}^\bullet\Delta_{01}{}^\bullet\Delta_{23}\langle \dot q_3^\lambda \dot q_1^\sigma q_2^\rho q_0^\omega\rangle G_{\omega\eta}F^\eta{}_\sigma F_{\rho\nu}F^\nu{}_\lambda + \frac{3}{8}\beta\int_{0123} {}^\bullet\Delta_{01}{}^\bullet\Delta_{21}\langle \dot q_3^\lambda q_3^\Omega q_2^\rho \dot q_1^\sigma\rangle G_{\nu\eta}F^\eta{}_\sigma F_\rho{}^\nu F_{\Omega\lambda}$$

$$+\frac{3}{8}\beta\int_{0123} {}^\bullet\Delta_{01}{}^\bullet\Delta^\bullet_{21}\langle \dot q_3^\lambda q_3^\Omega q_2^\rho q_0^\omega\rangle G_{\omega\eta}F^{\eta\nu}F_{\rho\nu}F_{\Omega\lambda} =$$

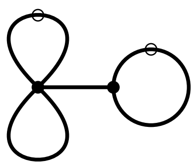
$+\frac{3}{4}\beta^3$  $\left(\delta^{\rho\sigma}\delta^{\lambda\omega} + \delta^{\lambda\sigma}\delta^{\rho\omega} + 5\delta^{\lambda\rho}\delta^{\sigma\omega}\right) G_{\omega\eta}F^\eta{}_\sigma F_{\rho\nu}F^\nu{}_\lambda$

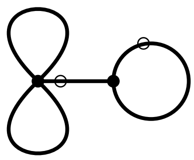
$+\frac{3}{8}\beta^3$  $\left(-\delta^{\lambda\sigma}\delta^{\rho\Omega} + \delta^{\lambda\rho}\delta^{\sigma\Omega}\right) G_{\nu\eta}F^\eta{}_\sigma F_\rho{}^\nu F_{\Omega\lambda}$

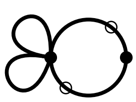
$+\frac{3}{8}\beta$  $\left(-\delta^{\lambda\omega}\delta^{\rho\Omega} + \delta^{\lambda\rho}\delta^{\Omega\omega}\right) G_{\omega\eta}F^{\eta\nu}F_{\rho\nu}F_{\Omega\lambda} =$

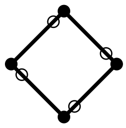
$$+\frac{\beta^3}{16}\left(\frac{1}{720}\right)\left(10G_{\omega\eta}F^{\omega\eta}F_{\rho\nu}F^{\rho\nu} + 8G_{\omega\eta}F^{\eta\rho}F_{\rho\nu}F^{\nu\omega}\right)$$

which is exactly the abelian limit of the previous result. For completeness we list below the worldgraph values

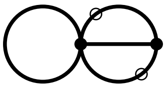
 $= \int_0^1 d\tau \int_0^1 d\sigma\ {}^\bullet\Delta(\sigma,\sigma)\Delta(\sigma,\sigma)\Delta(\sigma,\tau){}^\bullet\Delta(\tau,\tau) = \frac{1}{5040}$

 $= \int_0^1 d\tau \int_0^1 d\sigma\ \Delta^2(\sigma,\sigma){}^\bullet\Delta(\sigma,\tau){}^\bullet\Delta(\tau,\tau) = -\frac{1}{1260}$

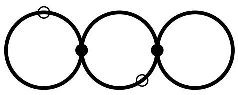
 $= \int_0^1 d\tau \int_0^1 d\sigma\ \Delta^2(\sigma,\sigma){}^\bullet\Delta(\sigma,\tau)\Delta^\bullet(\sigma,\tau) = -\frac{1}{280}$



$$= \int_0^1 d\tau \int_0^1 d\sigma \int_0^1 d\omega \int_0^1 d\alpha\, {}^\bullet\Delta(\sigma,\tau)\Delta^\bullet(\sigma,\omega)\Delta^\bullet(\omega,\alpha){}^\bullet\Delta(\tau,\alpha) = \frac{1}{720}$$



$$= \int_0^1 d\tau \int_0^1 d\sigma\, \Delta(\sigma,\sigma){}^\bullet\Delta(\sigma,\tau)\Delta^\bullet(\sigma,\tau)\Delta(\sigma,\tau) = -\frac{1}{420}$$



$$= \int_0^1 d\tau \int_0^1 d\sigma\, {}^\bullet\Delta(\sigma,\sigma)\Delta^\bullet(\sigma,\tau)\Delta(\sigma,\tau)\Delta(\tau,\tau) = -\frac{1}{5040}\,.$$

Finally we rewrite the generalized heat kernel trace expansion as

$$\begin{aligned}\mathrm{Tr}\left[\left(\xi^\mu\nabla_\mu + \frac{1}{2}(\partial_\mu\xi^\mu)\right)e^{-\beta\hat{H}}\right] &= \mathrm{tr}_{YM}\int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\sum_{n=0}^{\infty}\tilde{b}_n\beta^n \\ &= \mathrm{tr}_{YM}\int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\left(\tilde{b}_0 + \tilde{b}_1\beta + \tilde{b}_2\beta^2 + \tilde{b}_3\beta^3 + \cdots\right)\end{aligned} \tag{4.23}$$

where, for completeness, we rewrite the generalized coefficients up to third order in proper time

$$\begin{aligned}\tilde{b}_0 &= 0 \\ \tilde{b}_1 &= \frac{1}{24}G_{\mu\nu}F^{\mu\nu} \\ \tilde{b}_2 &= \frac{1}{720}\Big[-2\nabla_\lambda F_{\mu\nu}\partial^\lambda G^{\mu\nu} + \nabla^\nu F_{\nu\mu}\partial_\lambda G^{\lambda\mu} - 30VF_{\mu\nu}G^{\mu\nu}\Big] \\ \tilde{b}_3 &= \frac{1}{1440}\Big[-6G_{\mu\nu}V\nabla^2F^{\mu\nu} - 6F^{\mu\nu}V\partial^2G_{\mu\nu} - \partial_\alpha G_{\mu\nu}F^{\mu\nu}\nabla^\alpha V - 2\partial^\lambda G_{\lambda\mu}F^{\mu\alpha}\nabla_\alpha V \\ &\quad + G_{\mu\nu}\nabla^\alpha F^{\mu\nu}\nabla_\alpha V + 2G^{\mu\nu}\nabla^\lambda F_{\lambda\mu}\nabla_\nu V - 8\partial_\lambda G_{\mu\nu}\nabla^\lambda F^{\mu\nu}V - 2\partial^\lambda G_{\lambda\alpha}\nabla_\mu F^{\mu\alpha}V \\ &\quad + 30G_{\mu\nu}F^{\mu\nu}V^2 + 10G_{\mu\lambda}F^{\lambda\alpha}F_\alpha{}^\mu V + \frac{5}{4}G_{\mu\nu}F^{\mu\nu}F^2 + \frac{3}{28}G_{\mu\nu}\nabla^4F^{\mu\nu} + G_{\mu\nu}F^{\nu\rho}F_{\rho\lambda}F^{\lambda\mu}\Big]\,.\end{aligned} \tag{4.24}$$

This ends up the perturbative computation of the generalized coefficients up to order β^3. As a last remark we would like to say that one may rewrite them in the following form, perhaps more useful for the application to anomalies,

$$\begin{aligned}\mathrm{Tr}\left[\xi^\mu\nabla_\mu e^{-\beta\hat{H}}\right] &= \mathrm{tr}_{YM}\int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\sum_{n=0}^{\infty}\xi^\mu b_{\mu,n}\beta^n \\ &= \mathrm{tr}_{YM}\int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\left(\xi^\mu b_0 + \xi^\mu b_{\mu,1}\beta + \xi^\mu b_{\mu,2}\beta^2 + \xi^\mu b_{\mu,3}\beta^3 + \cdots\right)\end{aligned} \tag{4.25}$$

indeed starting by (4.23) one has

$$\begin{aligned}
\mathrm{Tr}\left[\xi^{\mu}\nabla_{\mu}e^{-\beta\hat{H}}\right] &= \mathrm{tr}_{YM}\int\frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\left[\left(-\frac{1}{2}\partial_{\mu}\xi^{\mu}\right)\sum_{n=0}^{\infty}\beta^n\tilde{a}_n+\sum_{n=0}^{\infty}\beta^n\tilde{b}_n\right] \\
&= \mathrm{tr}_{YM}\int\frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\left[\xi^{\mu}\sum_{n=0}^{\infty}\beta^n\frac{1}{2}\nabla_{\mu}\tilde{a}_n+\sum_{n=0}^{\infty}\beta^n\tilde{b}_n\right] \\
&= \mathrm{tr}_{YM}\int\frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\,\xi^{\mu}\sum_{n=0}^{\infty}\beta^n\underbrace{\left(\frac{1}{2}\nabla_{\mu}\tilde{a}_n+\tilde{b}_{\mu,n}\right)}_{b_{\mu,n}}
\end{aligned} \tag{4.26}$$

where $\tilde{b}_{\mu,n}$ correspond to the coefficients obtained by partial integrating $\tilde{b}_n$ in (4.24) such to factorize out the abelian field ξ_{μ}, while $\tilde{a}_n$ are the heat kernel trace coefficients whose computation up to third order in proper time has been worked out in Appendix B and, to be found directly in (B.11).

Conclusions

The purpose of this work was to implement a quantum mechanical path integral to represent and compute traces of the heat kernel with insertion of first order differential operator, so to identify the explicit expression of the related heat kernel coefficients (which we called generalized heat kernel coefficients). The latter may be used to calculate the stress tensor anomalies (i.e. gravitational anomalies) in various QFT. Such anomalies are expected to cancel in $4D$, but they may arise in specific regularizations. Only after their identification one may study the local counterterms that added to the effective action should cancel them. On the other hand, genuine gravitational anomalies are expected in $2D$, $6D$ and, more generally, in $D = 2 + 4k$ dimensions with k an integer number.

Here we studied the problem of computing the generalized coefficients in flat space, but with an arbitrary non-abelian gauge field. The implementation of a path integral for the coloured particle, i.e. a particle that interacts with a non-abelian gauge field, is guided by the obvious requirement of gauge symmetry. This leads to the insertion of a time ordering prescription in the path integral. In addition, a further coupling of the particle to an abelian gauge field has allowed to obtain a path integral formula for the generalized heat kernel coefficients of our interest.

We considered also an alternative approach for treating the color degrees of freedom of the particle, so to eliminate the time ordering. This consists in introducing auxiliary bosonic variables on the worldline. They allow to derive a path integral which does not require the time ordering. For computing the heat kernel traces one has to path integrate over loops, defined by using PBC for the particle coordinates, and TBC (twisted boundary conditions) for the auxiliary bosonic variables. The latter have the effect of implementing the time ordering when computing vacuum expectation values. This last representation perturbatively produces the same result as the time ordered one, but eventually we chose the latter to compute the third order kernel coefficients since it is much more direct, and because the higher order computation is more involved, so one would not like to deal with further correlators in their evaluation.

The path integral representations of the generalized coefficients is one of the main results of this thesis, and we first checked their correctness by reproducing the coefficients needed for the anomalies in two and four dimensions. These were known in the literature, though computed with different methods [1, 2]. The check was successful, so we performed also the computation of the coefficient needed for anomalies in six dimensions, which is also a new result not present in the literature, as far as we know. One of the advantage of using the path integral, among other methods employed for heat kernel calculations, is its directness and efficiency in computations.

Symmetries of our generalized kernel trace could in principle be studied (we think of discrete ones, in particular) as a future line of investigation. They may allow to simplify computations by forbidding the existence of some terms which, for the calculation of higher order coefficients, would be really welcome.

Currently heat kernel coefficients have a great variety of applications, and the use of the worldline formalism provides according to us a great tool beside the standard operator ones.

Acknowledgments

I wish to particularly thank prof. Bastianelli for having introduced me to this beautiful and really interesting topic in my opinion, and further for helpful supervisions and teachings which revealed crucial to let me understand some aspects of this work and in particular the use of path integral in physics.

Appendix A

Free theory propagators and normalizations

In the following appendix we first compute the normalization constant for configuration space path integral that we decided to factorize out in our computation, since arising when evaluating vacuum expectation values of worldline fields. We fix it by requiring that free particle kernel is a properly normalized solution of heat equation. Next we use path integral to derive the worldline free theory propagators in DBC or SI, that are crucial to compute heat kernel coefficients. Furthermore we compute coherent path integral normalization constant by using the harmonic oscillator hamiltonian produced upon canonical quantization of auxiliary bosonic fields, then we use path integral to evaluate TBC (or PBC) bosonic propagator which has been used to compute two and four dimensional generalized coefficients.

A.1 Free particle kernel

Here we show how to properly fix the free laplacian DBC determinant, obtained when one has to DBC path integrate the free kinetic term.
To start with, consider the scalar particle kernel

$$\langle y|e^{-\beta\hat{H}}|x\rangle = \int Dx\ \exp\left(-\int_0^1 d\tau\ \frac{1}{2\beta}\dot{x}^2\right) \tag{A.1}$$

where the path integration is evaluated on trajectories with fixed initial and ending point given by $x^\mu(0) = x^\mu$ and $x^\mu(1) = y^\mu$.

To proceed further one needs to split the zero mode integration by writing the worldline field as a classical solution to Euler-Lagrange equation plus quantum fluctuations, namely

$$x^\mu(\tau) = x^\mu_{cl}(\tau) + q^\mu(\tau) \tag{A.2}$$

where the classical field is the solution to the free problem

$$\ddot{x}^\mu_{cl}(\tau) = 0, \quad x^\mu(0) = x^\mu, \quad x^\mu(1) = y^\mu \tag{A.3}$$

and is written as

$$x^\mu_{cl}(\tau) = x^\mu + (y^\mu - x^\mu)\tau\ . \tag{A.4}$$

Inserting it in (A.1) one recast the kernel as

$$K(\beta, x, y) = \langle y|e^{-\beta\hat{H}}|x\rangle = e^{-\frac{(y-x)^2}{2\beta}} \underbrace{\int_{DBC} Dq \,\exp\left(-\frac{1}{\beta}\int_0^1 d\tau\, \frac{\dot{q}^2}{2}\right)}_{A(\beta)} . \tag{A.5}$$

The last DBC path integral produces function of the proper time (namely an integration constant) which corresponds to the DBC laplacian determinant in D dimensions (to the minus a half) where the zero mode eigenvalue has been dropped out. This constant can be fixed by requiring that the free kernel is a solution to the heat equation with the boundary condition (1.2).

Thus, considering the action of free hamiltonian on the kernel one has

$$-\frac{1}{2}\partial^2 K(\beta, x, y) = -\frac{A(\beta)}{2}\left[-\frac{D}{\beta} + \frac{1}{\beta^2}(y-x)^2\right] e^{-\frac{(y-x)^2}{2\beta}} \tag{A.6}$$

while for the left hand side of heat equation one obtains

$$-\frac{\partial}{\partial\beta}K(\beta, x, y) = \left[-\frac{dA(\beta)}{d\beta} - \frac{A(\beta)(y-x)^2}{2\beta^2}\right] e^{-\frac{(y-x)^2}{2\beta}} \tag{A.7}$$

gathering results one gets an equation for our normalisation constant

$$\frac{dA(\beta)}{d\beta} = -A(\beta)\frac{D}{2\beta} \tag{A.8}$$

whose solution is easily written as $A(\beta) = A_0\beta^{-D/2}$, where D is euclidean space dimension. The constant A_0 is fixed by using (1.2) which in practice implies that the kernel has to be normalized since its square represents the probability distribution that the particle moves between two fixed space point. So, requiring normalization one fixes $A_0 = (2\pi)^{-D/2}$. This completely fixes the DBC determinant of the free laplacian

$$\int_{DBC} Dq \,\exp\left(-\frac{1}{\beta}\int_0^1 d\tau\, \frac{\dot{q}^2}{2}\right) = (2\pi\beta)^{-\frac{D}{2}} . \tag{A.9}$$

A.2 DBC and SI worldline propagator

Here, we would like to derive DBC and SI free theory propagator by using functional integration. Starting with the DBC one, from the definition and using the normalized DBC path integral (where the factor $(2\pi\beta)^{-D/2}$ has been factorised out in the zero mode integration) we have

$$\begin{aligned}\langle q^\mu(\sigma)q^\nu(\tau)\rangle_{DBC} &= \int_{DBC} Dq\, q^\mu(\sigma)q^\nu(\tau)\exp\left(-\frac{1}{2\beta}\int_0^1 ds\, \dot{q}^2(s)\right)\\ &= \int_{DBC} Dq\, q^\mu(\sigma)q^\nu(\tau)\exp\left(\frac{1}{2\beta}\int_0^1 ds\, q^\rho(s)\frac{d^2}{ds^2}q_\rho(s)\right)\ .\end{aligned} \tag{A.10}$$

Now, we expand worldline fields in eigenstates of the DBC kinetic operator, such that one writes the following Fourier expansion

$$q^\mu(\sigma) = \sum_{n=1}^\infty y_n^\mu \sin(n\pi\sigma) \tag{A.11}$$

from the expansion one has

$$\langle q^\mu(\sigma)q^\nu(\tau)\rangle_{DBC} = \sum_{\ell=1}^\infty\sum_{m=1}^\infty \sin(\ell\pi\sigma)\sin(m\pi\tau)\langle y_\ell^\mu y_n^\nu\rangle_{DBC}\ . \tag{A.12}$$

In order to evaluate the above correlator, we first rewrite the action in Fourier space

$$S[q] = -\frac{1}{2\beta}\sum_{n=1}^\infty\sum_{m=1}^\infty y_n^\mu y_m^\mu \int_0^1 ds\ \sin(n\pi s)\sin(m\pi s) = -\frac{\pi^2}{4\beta}\sum_{n=1}^\infty n^2 y_n^2 \tag{A.13}$$

then rewrite the measure as

$$Dq(\sigma) = \prod_{n=1}^\infty\prod_{\mu=0}^D dy_n^\mu\left(\frac{\pi n^2}{4\beta}\right)^{\frac{1}{2}} \tag{A.14}$$

where the factor in the measure needs to normalise the gaussian integral below, so one recasts as

$$\sum_{\ell=1}^\infty\sum_{m=1}^\infty \sin(\ell\pi\sigma)\sin(m\pi\tau)\int_{-\infty}^{+\infty}\prod_{n=1}^\infty\prod_{\mu=1}^D dy_n^\mu\left(\frac{\pi n^2}{4\beta}\right)^{\frac{1}{2}} y_\ell^\mu y_m^\nu \prod_{k=1}^\infty \exp\left(-\frac{\pi^2k^2}{4\beta}y_k^2\right)\ . \tag{A.15}$$

To solve the above gaussian integral we consider the generating one

$$Z[J] = \int_{-\infty}^{-\infty} dy_\ell^\mu\left(\frac{\pi\ell}{4\beta}\right)^{\frac{1}{2}}\exp\left(-\frac{\pi^2\ell^2}{4\beta}y_\ell^2 + J_\ell^\mu y_\mu^\ell\right) = \exp\left(\frac{\beta}{\pi^2\ell^2}J_\ell^2\right)$$

$$\frac{\partial}{\partial J_\ell^\mu \partial J_m^\nu}Z[J]\Big|_{J=0} = \int_{-\infty}^{+\infty} dy_n^\alpha\left(\frac{\pi n^2}{4\beta}\right)^{\frac{1}{2}} y_\ell^\mu y_m^\nu \exp\left(-\frac{\pi^2n^2}{4\beta}y_n^2\right) = \frac{2\beta}{\pi^2\ell^2}\delta_{\ell,m}\delta_{\mu\nu}$$

from which one reads

$$\langle y^\mu_\ell y^\nu_m\rangle_{DBC} = \frac{2\beta}{\pi^2\ell^2}\delta_{\ell,m}\delta^{\mu\nu}\ . \tag{A.16}$$

Finally one obtains the Fourier expansion of the DBC worldline propagator

$$\begin{aligned}\langle q^\mu(\sigma)q^\nu(\tau)\rangle_{DBC} &= \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sin(\ell\pi\sigma)\sin(m\pi\tau)\langle y^\mu_\ell y^\nu_n\rangle_{DBC}\\ &= \beta\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\frac{2}{\pi^2\ell^2}\sin(\ell\pi\sigma)\sin(m\pi\tau)\delta_{\ell,m}\delta^{\mu\nu} = -\beta\Delta_{DBC}(\sigma,\tau)\delta^{\mu\nu}\end{aligned} \tag{A.17}$$

where the last equality is our definition of the two point function (since we rescaled the action to unit time).

Finally, from the last equality in A.17 one reads the DBC propagator in Fourier representation

$$\Delta_{DBC}(\sigma,\tau) = \sum_{n=1}^{\infty} -\frac{2}{\pi^2 n^2}\sin(n\pi\sigma)\sin(n\pi\tau) \tag{A.18}$$

the above Fourier expansion can be resummed and written on the worldline as

$$\Delta_{DBC}(\sigma,\tau) = (\sigma-1)\tau\,\theta(\sigma-\tau) + (\tau-1)\sigma\,\theta(\tau-\sigma) \tag{A.19}$$

where the step function is defined such that $\theta(0) = 1/2$ since we used time slicing regularization to produce path integrals.

Given the extensive use of the DBC worldline propagator in computations, it is useful to list here its derivatives and coincident points limits

$$\begin{aligned}{}^\bullet\Delta(\tau,\sigma) &= \sigma - \theta(\sigma-\tau)\\ \Delta^\bullet(\tau,\sigma) &= \tau - \theta(\tau-\sigma)\\ {}^\bullet\Delta^\bullet(\tau,\sigma) &= 1 - \delta(\tau-\sigma)\\ {}^\bullet\Delta(\sigma,\sigma) &= \sigma - \frac{1}{2}\\ \Delta(\sigma,\sigma) &= \sigma^2 - \sigma\end{aligned} \tag{A.20}$$

where a left/right dot corresponds to a DBC propagator derivative with respect to the left/right vertex argument.

Considering now SI propagator one has a different basis of kinetic operator eigenstates, so the field expansion reads

$$q^\mu(\sigma) = \sum_{n\neq 0}\left(c^\mu_n\, e^{2\pi i n\sigma} + c^{*\mu}_n e^{-2\pi i n\sigma}\right) \tag{A.21}$$

such that two point function can be written as

$$\langle q^\mu(\tau)q^\nu(\sigma)\rangle_{SI} = \sum_{n,m\neq 0} 2\,\langle c^\mu_n c^{*\nu}_m\rangle\, e^{2\pi i n\tau - 2\pi i m\sigma}\ , \tag{A.22}$$

obtained by using reality of worldline fields which implies $c_{-n} = c^*_n$. Inserting the above expansion in free rescaled action one recasts the action as

$$S[c^*,c] = \sum_{n\neq 0}\frac{8\pi^2 n^2}{\beta}\, c^{*\mu}_n c_{\mu n} \tag{A.23}$$

while functional measure is rewritten as

$$Dq = \prod_{k\neq 0}\prod_{\mu=0}^{D}\left(\frac{4\pi k^2}{\beta}\right) dc_k^{*\mu} dc_k^{\mu} \ . \tag{A.24}$$

The normalized generating functional this time has the form

$$Z[\xi,\xi^*] = \int dc_k^* dc_k \left(\frac{4\pi k^2}{\beta}\right) \exp\left(-\frac{8\pi^2 k^2}{\beta} c_k^* c_k - \xi_k c_k^* - \xi_k^* c_k\right) = \exp\left(\frac{\beta}{8\pi^2 k^2}\xi_n^*\xi_n\right) \tag{A.25}$$

taking double derivative generates two point function in Fourier space

$$\langle c_n^{\mu} c_m^{*\nu}\rangle = \frac{\partial^2}{\partial\xi_n^{*\mu}\partial\xi_m^{\nu}} Z[\xi,\xi^*]\Big|_{\xi,\xi^*=0} = \frac{\beta}{8\pi^2 n^2}\delta_{nm}\delta^{\mu\nu} \ . \tag{A.26}$$

Finally, (A.22) can be rewritten as

$$\langle q^{\mu}(\tau) q^{\nu}(\sigma)\rangle_{SI} = \delta^{\mu\nu}\sum_{n\neq 0}\frac{\beta}{(2\pi n)^2} e^{2\pi i n(\tau-\sigma)} = -\beta\delta^{\mu\nu}\Delta_{SI}(\tau,\sigma) \tag{A.27}$$

so one reads out string inspired propagator as

$$\Delta_{SI}(\tau,\sigma) = \sum_{n\neq 0} -\frac{1}{(2\pi n)^2} e^{2\pi i n(\tau-\sigma)} \ . \tag{A.28}$$

The above Fourier series can be resummed obtaining the following worldline representation

$$\Delta_{SI}(\sigma,\tau) = \frac{1}{2}|\sigma-\tau| - \frac{1}{2}(\sigma-\tau)^2 - \frac{1}{12} \tag{A.29}$$

which makes manifest time translation invariance of string inspired propagator

A.3 Coherent states path integral normalization and propagator

In chapter 3 we computed two and four dimensional generalized coefficients by introducing bosonic creation and annihilation operators to implement coloured degrees of freedom of the worldline scalar particle. Here we compute the overall normalization constant for the free coherent state path integral corresponding in one dimension to TBC inverse determinant of bosonic kinetic operator.

To start with, TBC determiant can be recast as a PBC one by redefining back bosonic fields such that

$$\begin{aligned} \mathrm{Det}^{-N}_{TBC}(\partial_\tau) &= \int_{TBC} D\bar{c}Dc\, e^{-\int_0^1 d\tau \bar{c}^\alpha \dot{c}_\alpha} \\ &= \int_{PBC} D\bar{c}Dc\, e^{-\int_0^1 d\tau \bar{c}^\alpha (\partial_\tau + i\phi) c_\alpha} = \mathrm{Det}^{-N}_{PBC}(\partial_\tau + i\phi)\ . \end{aligned} \tag{A.30}$$

Now, since bosonic fields are quantum mechanically realized as creation and annihilation operators, the above determinant can be seen as a kernel trace whose hamiltonian describes a harmonic oscillator, namely

$$\mathrm{Det}^{-1}_{PBC}(\partial_\tau + i\phi) = \mathrm{Tr}\left[e^{-i\phi(\hat{c}^\dagger \hat{c} + \frac{1}{2})}\right] = \sum_{k=0}^{\infty} e^{-i\phi(k+\frac{1}{2})} = \frac{1}{\left(2i \sin\frac{\phi}{2}\right)} \tag{A.31}$$

where one has to regularize the sum by sending $\phi \to \phi - i\epsilon$ pushing positive ϵ to zero at the end.

Considering now the bosonic propagator, we can evaluate it by making use of coherent state path integral. Starting for instance by PBC path integral, where normalisation constant has been factorized out, one can expand bosonic fields in PBC eigenstates of kinetic operator $\partial_\tau + i\phi$ such that one obtains the following

$$c^\alpha(\tau) = \sum_{n\in\mathbb{Z}} z_n^\alpha\, e^{2\pi i n\tau} \tag{A.32}$$

thus coherent free action in this basis reads as

$$S[\bar{z}, z] = \sum_{n\in\mathbb{Z}} \bar{z}_n z_n \underbrace{i\,(\phi + 2\pi n)}_{\lambda_n} \tag{A.33}$$

where λ_n corresponds to the succession of eigenvalues (note that there is not zero eigenvalue so no splitting is needed).

As in section A.2 one is interested in computing two point function of Fourier coefficients so even this time we consider the following normalized generating functional

$$Z[\xi, \bar{\xi}] = \int d\bar{z}_n dz_n \frac{\lambda_n}{2\pi} \exp\left(-\lambda \bar{z}_n z_n - \bar{\xi}_n z_n - \xi_n \bar{z}_n\right) = \exp\left(\lambda_n^{-1} \bar{z}_n z_n\right) \tag{A.34}$$

taking double derivatives one generates the two point correlator

$$\langle z_n^\mu \bar{z}_m^\nu \rangle = \frac{\partial}{\partial \xi_n^\mu \partial \bar{\xi}_m^\nu} Z[\xi, \bar{\xi}]\Big|_{\xi,\bar{\xi}=0} = -\frac{i}{\phi + 2\pi n}\delta_{nm}\delta^{\mu\nu} \tag{A.35}$$

thus two point bosonic function can be written as

$$\langle c_\mu(\tau)\bar{c}^\nu(\sigma)\rangle_{PBC/TBC} = \delta_\mu{}^\nu \sum_{n\in\mathbb{Z}} -\frac{i}{\phi+2\pi n}\, e^{2\pi i n(\tau-\sigma)} = \delta_\mu{}^\nu \Delta_B\left(\tau-\sigma,\phi\right)\ . \tag{A.36}$$

The above series can be resummed such that one writes the following worldline representation of bosonic propagator

$$\Delta_B(\tau-\sigma,\phi) = \frac{1}{2i\sin\frac{\phi}{2}}\left[e^{i\frac{\phi}{2}}\theta(\tau-\sigma)+e^{-i\frac{\phi}{2}}\theta(\sigma-\tau)\right] \tag{A.37}$$

which is manifest translational invariant.

It's useful now to derive some properties of bosonic propagator which has been used in calculations done in chapter 3, first, considering TBC one has

$$\langle c(0)\bar{c}(\sigma)\rangle = e^{-i\phi}\langle c(1)\bar{c}(\sigma)\rangle \to \Delta_B(0-\sigma,\phi) = e^{-i\phi}\Delta_B(1-\sigma,\phi)\ . \tag{A.38}$$

Setting σ to zero in (A.38) and using translational invariance of ghost propagator allows one to obtain

$$\Delta_B(0,\phi) = \Delta_B(\tau-\tau,\phi) = \frac{e^{-i\frac{\phi}{2}}}{2i\sin\frac{\phi}{2}}\ . \tag{A.39}$$

Finally one also has the following

$$\Delta_B(\tau-\sigma,\phi)\Delta_B(\sigma-\tau,\phi) = \left(2i\sin\frac{\phi}{2}\right)^{-2} \tag{A.40}$$

obtained by using Heaviside step function properties.

Appendix B

Heat kernel trace coefficients at order β^3

In the following we use the path integral representation found in chapter 2 for Yang-Mills heat kernel to compute six dimensional traces with insertion of a position dependent operator inside the trace. Results of chapter 1 can be used to make such insertion, then we pass to computation, which as always is carried out using Fock-Schwinger gauge and Taylor expanding the exponential of the interacting action in path integral. Finally one evaluates vacuum expectation values whose computation is simplified using Bianchi identity and geometrical properties of curvatures. The result we find exactly matches [2].

B.1 Heat kernel trace computation

Here we us path integral representation of Yang-Mills kernel trace (2.21) to compute, up to third order in proper time, the following trace

$$\mathrm{Tr}\left[\hat{\sigma}(\hat{x})\, e^{-\beta\hat{H}}\right] \tag{B.1}$$

where $\hat{\sigma}(\hat{x})$ is a Lorentz scalar field operator, while the hamiltonian is defined in (2.1). Using results of chapter 1 we can write the above trace as

$$\begin{aligned}\mathrm{Tr}\left[\hat{\sigma}(\hat{x})\, e^{-\beta\hat{H}}\right] &= \int_{PBC} Dx \left(\int_0^1 d\tau\, \sigma(x(\tau))\right) \mathrm{T}\, e^{-S[x;A]} \\ &= \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\, \sigma(x) \left\langle \mathrm{T}\, e^{-S_{int}[x;A]} \right\rangle_{DBC,0} \\ &= \mathrm{tr}_{\,YM} \int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}}\, \sigma(x) \sum_{n=0}^{\infty} \tilde{a}_n \beta^n\end{aligned} \tag{B.2}$$

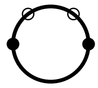
where the action is given by

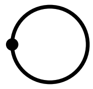
$$S[x;A] = \frac{1}{\beta}\int_0^1 d\tau \left(\frac{1}{2}\dot{x}^2 + \beta\dot{x}^\mu A_\mu(x(\tau)) + \beta^2 V(x(\tau))\right) . \tag{B.3}$$

As a remark we'd like to emphasize that this calculation is needed to compute for instance one loop effective action for scalar QCD in some space-time dimension (by setting $\hat{\sigma}(\hat{x})$

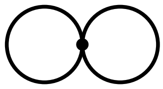
to identity operator). Indeed the higher derivative expansion of the kernel we are going to compute is relevant to identify UV divergences in one loop effective action and so counterterms to introduce in the lagrangian for getting rid of this UV divergences. Further the calculation is relevant for computing Weyl anomalies, affecting the trace of stress tensor, corresponding to the quantum breaking of conformal symmetry.

To proceed in calculations we first Taylor expand the exponential of interacting action in path integral, next we deal with time ordering prescription using cyclicity of colour trace to couple different strings and finally we fix Fock-Schwinger gauge for Yang-Mills field. Adopting this procedure one gets $-\beta V$, $\frac{1}{2}\beta^2 V^2$, which are the easy ones, further we also have

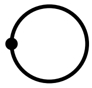
$$\frac{1}{8}\int_{01}\langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\lambda q_1^\sigma\rangle F_{\nu\mu}F_{\sigma\lambda} = \frac{\beta^2}{4}F_{\mu\nu}F^{\mu\nu}$$  $$= \frac{\beta^2}{48}F^2$$

$$-\frac{\beta}{2}\int_0 \langle q_0^\mu q_0^\nu\rangle \nabla_\mu\nabla_\nu V = \frac{\beta^2}{2}\nabla^2 V$$  $$= -\frac{\beta^2}{12}\nabla^2 V \ . \tag{B.4}$$

Next, third order terms are obtained by the followings

$$-\frac{\beta}{4!}\int \langle q_0^\mu q_0^\nu q_0^\alpha q_0^\rho\rangle \nabla_\rho\nabla_\alpha\nabla_\nu\nabla_\mu V = -\frac{\beta^3}{8}\nabla^4 V$$  $$= -\frac{\beta^3}{240}\nabla^4 V$$

$$+\frac{\beta^2}{2}\int_{01}\langle q_0^\mu q_1^\nu\rangle \nabla_\mu V\nabla_\nu V = -\frac{\beta^3}{2}\nabla_\mu V\nabla^\mu V$$ $$= \frac{\beta^3}{24}\nabla^\mu V\nabla_\mu V$$

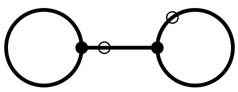
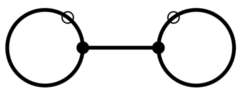
$$+\frac{\beta^2}{2}\int \langle q_0^\mu q_0^\nu\rangle V\nabla_\mu\nabla_\nu V = -\frac{\beta^3}{2}V\nabla^2 V$$  $$= \frac{\beta^3}{12}V\nabla^2 V \tag{B.5}$$

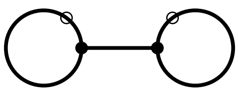
$$-\beta^3 V^3 \underbrace{\int_{012}\theta_{01}\theta_{12}}_{=\frac{1}{6}} = -\frac{\beta^3}{6}V^3 \ .$$

Further we have the following term

$$\frac{1}{18}\int_{01}\langle \dot{q}_0^\mu q_0^\alpha q_0^\sigma \dot{q}_1^\nu q_1^\lambda q_1^\rho\rangle \nabla_\sigma F_{\alpha\mu}\nabla_\rho F_{\lambda\nu} =$$

$$-\frac{\beta}{18}\int_{01} {}^\bullet\Delta_{00}\langle q_0^\alpha \dot{q}_1^\nu q_1^\lambda q_1^\rho\rangle \nabla^\mu F_{\alpha\mu}\nabla_\rho F_{\lambda\nu} - \frac{\beta}{6}\int_{01}\langle \dot{q}_1^\nu q_1^\lambda q_0^\alpha q_0^\sigma\rangle \nabla_\sigma F_\alpha{}^\mu \nabla_\lambda F_{\mu\nu}$$

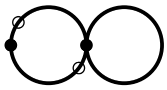
$$-\frac{\beta^3}{18}\nabla^\mu F_{\mu\alpha}\nabla_\nu F^{\nu\alpha}\Big[-4$$  $$+$$  $$\Big] \tag{B.6}$$

$$-\frac{\beta^3}{4}\nabla^\lambda F^{\mu\nu}\nabla_\lambda F_{\mu\nu}\ 8$$  $$=$$

$$\frac{\beta^3}{2}\left(\frac{1}{720}\right)\left(\nabla^\alpha F_{\alpha\mu}\nabla_\lambda F^{\lambda\mu} + 4\nabla^\lambda F^{\mu\nu}\nabla_\lambda F_{\mu\nu}\right) \ .$$

Consider this one

$$
\begin{aligned}
&\frac{1}{16}\int_{01}\langle \dot{q}_0^\mu q_0^\alpha q_0^\sigma q_0^\rho \dot{q}_1^\nu q_1^\lambda\rangle F_{\lambda\nu}\nabla_\rho\nabla_\sigma F_{\alpha\mu} = \\
&-\frac{\beta}{8}\int_{01} {}^\bullet\Delta_{01}\langle q_0^\alpha q_0^\sigma q_0^\rho \dot{q}_1^\nu\rangle F^\mu{}_\nu\nabla_\rho\nabla_\sigma F_{\alpha\mu} = \\
&-\frac{\beta^3}{8}\left(-2F^{\mu\nu}\nabla^\alpha\nabla_\alpha F_{\mu\nu} + 2F_{\nu\alpha}F^\alpha{}_\mu F^{\mu\nu}\right)
\end{aligned}
$$



$$
\begin{aligned}
&= \\
&-\frac{\beta^3}{480}\left(-2F^{\mu\nu}\nabla^\alpha\nabla_\alpha F_{\mu\nu} + 2F_{\nu\alpha}F^\alpha{}_\mu F^{\mu\nu}\right)
\end{aligned} \tag{B.7}
$$

here, in order to write the above final result we employed the following simplification

$$
\begin{aligned}
&F^{\mu\nu}\nabla_\nu\nabla^\alpha F_{\alpha\mu} + F^{\mu\nu}\nabla^\alpha\nabla_\nu F_{\alpha\mu} \\
&= 2F^{\mu\nu}\nabla^\alpha\nabla_\nu F_{\alpha\mu} + F^{\mu\nu}[F_\nu{}^\alpha, F_{\alpha\mu}] \\
&= -F^{\nu\mu}\nabla^\alpha\left(\nabla_\nu F_{\alpha\mu} - \nabla_\mu F_{\alpha\nu}\right) + 2F_{\nu\alpha}F^\alpha{}_\mu F^{\mu\nu} \\
&= -F^{\mu\nu}\nabla^2 F_{\mu\nu} + 2F_{\nu\alpha}F^\alpha{}_\mu F^{\mu\nu}
\end{aligned} \tag{B.8}
$$

where Bianchi identity and anti-symmetry of gauge curvature has been used in third line. There still is a last third order term

$$
\begin{aligned}
&-\frac{\beta}{4}\int_{012}\langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\lambda\rangle \underbrace{\left(\theta_{01}\theta_{12} + \theta_{12}\theta_{20} + \theta_{20}\theta_{01}\right)}_{g_{012}} F_{\nu\mu}F_{\lambda\alpha}V = \\
&-\frac{\beta^3}{4}\underbrace{\int_{012} {}^\bullet\Delta^\bullet_{01}\Delta_{01}\, g_{012}}_{=\frac{1}{24}}\ F_{\mu\nu}F^{\mu\nu}V - \frac{\beta^3}{4}\underbrace{\int_{012} {}^\bullet\Delta_{01}\Delta^\bullet_{01}\, g_{012}}_{=-\frac{1}{24}} F_{\nu\mu}F^{\mu\nu}V = -\frac{\beta^3}{48}F^2 V\ .
\end{aligned} \tag{B.9}
$$

Finally we can present the whole result as follows

$$
\begin{aligned}
\mathrm{Tr}\left[\hat{\sigma}(\hat{x})\, e^{-\beta\hat{H}}\right] &= \mathrm{tr}_{\,YM}\int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}}\,\sigma(x)\sum_{n=0}^{\infty}\tilde{a}_n\beta^n \\
&= \mathrm{tr}_{\,YM}\int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}}\ \sigma(x)\ \left(\tilde{a}_0 + \tilde{a}_1\beta + \tilde{a}_2\beta^2 + \tilde{a}_3\beta^3 + \cdots\right)
\end{aligned} \tag{B.10}
$$

where the coefficients are given by

$$
\begin{aligned}
\tilde{a}_0 &= 1 \\
\tilde{a}_1 &= -V \\
\tilde{a}_2 &= \frac{1}{720}\Big(360V^2 + 15F_{\mu\nu}F^{\mu\nu} - 60\nabla^2 V\Big) \\
\tilde{a}_3 &= \frac{1}{1440}\Big(\nabla^\alpha F_{\alpha\mu}\nabla_\lambda F^{\lambda\mu} + 4\nabla^\lambda F^{\mu\nu}\nabla_\lambda F_{\mu\nu} + 6F^{\mu\nu}\nabla^2 F_{\mu\nu} - 6F_{\mu\nu}F^{\nu\alpha}F_\alpha{}^\mu \\
&\qquad - 30F_{\mu\nu}F^{\mu\nu}V - 6\nabla^4 V + 60\nabla_\mu V\nabla^\mu V + 120V\nabla^2 V - 240V^3\Big)\ .
\end{aligned} \tag{B.11}
$$

We stress that even the above coefficients are up to total derivatives and valid under Yang-Mills trace (which is implicit in the above result).

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