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Quantum description of Inflation

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Indice

Abstract

In this thesis I analyzed what happens when an inflationary space-time is described as a collection of quanta. Classically the inflationary universe can be described by the equation of moto for the metric tensor field and for the inflaton field. My purpose is to consider the two fields as systems of particles, called respectively gravitons and inflatons. Thus, the fundamental quantum quantities that are taken into account to describe these systems, are the particle occupation numbers and the coupling constants between them. Therefore, in addition to the equations of moto for the classical quantities, the temporal evolution for the number of quanta of the two systems had to be deduced. It will be shown that, the equation for the classical and quantum models give the same interpretation for the space-time, untill the microscopic interactions between quanta become important. The quantum corrections that arise, lead to a new interpretation of the inflationary universe. Briefly, the standard inflationary model admits a limit in which the well kown de Sitter metric is recovered, this is no more true when the quantum interactions between quanta of the fields are taken into account.
Before analyzing the quantum model of inflation, I will show in the second chapter, the corpuscolar interpretation of black holes. Indeed, it constitutes an instructive model in which only the gravitational field is present, and the dynamics of the associated quanta, is very similar to that of the inflationary model. The reason can be searched in the statistichal features of the respectively systems. In both models, I will show that they are a Bose-Einstein condensates, constitued by many very weakly interacting quanta, at critical phase point.
To conclude, in the last chapter I will show an alternative theory of gravity that inside itself, encodes the inflationary universe, and I will try to quantizy it.

Introduction

Gravitational systems are a main topics of physics. They can be described by different theories of gravitation, each of which explains some pieces of a big puzzle of this kind of physical phenomena. A large part of this puzzle, is put togheter by the General Theory of Relativity. The wonderfull feature of this theory is that gravitational systems, their sources and their motion, like with the structure of the space-time. The next step, one would to do, is to employ in some way, the framework of the Quantum Theory, with all its consequences, and to analyze the quantum features of the gravitational field. This latter had to be understood as a collection of particles, called gravitons. This is indeed the dual interpretation of the fields. On the other hand, a corpuscolar vision of the gravitational field involves, assuming the Einstein theory, a quantum discretization of the space-time, with very interesting conceptual consequences. Gravitational systems are composed of a source and the associated field. Thus, the quanta one could consider, neglecting the non gravitational interactions, are those of the source and the gravitons.
Following the above idea, in this thesis I will try to analyze two very important physical systems: black holes and cosmological space-time, studyng in depth the so called Inflationary model. All the universe is in some way a gravitational system, where is present, in a simplified model, a homogeneus fluid, whose density and pressure generate different kind of dynamical space-time. Thus, the hystory of the Cosmo could be analyzed as a sequence of different kind of evolution of the space-time. This description, that follows the standard Einstein Theory of Gravitation, not reproduces all the truly evolution of the cosmological parameters. Thus, it is necessary to improve the theory taking into account the General Theory of Relativity in a more deep form, and introduce the corpuscolar interpretation. The theory of inflation born to solves two important cosmological problems and to describe the early stage of the universe. It is provided by various theoretical frameworks:

- Einstein's equation with dark matter;
- Einstein's equation without source, but with the cosmological constant;
- Theory of gravity with high order term in the lagrangian;

Quantizing these theories, the fundamental physical quantities and relative equations grow, bringing more general formulas and slightly different results. Inflation is roughly defined as a very fast increase of the relative distances between the points of the space. Classically, it can be studied using the Hubble function and the slow-roll parameters. Instead, the fundamental physical quantum quantities are the number of gravitons and the gravitational coupling constant. This latter

quantifies the microscopic interactions between the corspuscles of the space-time. An important concept that will rise, is the so called quantum break time, that signs the passage between the classical description to a pure quantum gravitational system. To understand this matter, it will be necessary to study the procedure of quantization of the gravitational field, that is similar to the procedure of the quantum field theory for bosons and fermions fields. In this way, it can be built the Fock space of multiparticles states of gravitons. Then, my purpose is to understand how, from the quantum framework, it can be recovered some classical results of the Einstein theory, as the redshift, or semiclassical results as the particle evaporation. In the full quantum theory I shall show how the microscopic interaction come into account and had to be studied in the framework of the S-matrix.
At first, I shall show the quantization of the Schwarzchild space-time. As soon as the gravitational field of the black hole is described as a graviton's collection with a critical statistical behaviour, it can be recovered, for an infinite number of gravitons, the Hawking effect. The fundamental novelty is that this particle production is not an effect of the vacuum, but of the particles itself. The quantum correction in the classical formula are of the type $\frac{1}{N_g}$, where N_g is the number of gravitons.
For the quantum inflation theory, the elements are the collection of gravitons and of the fluid's quanta, called inflatons. The fundamental formulas describe the evolution of the occupation numbers of these particles. In particular, I will show that, whenever the dynamics is dominated by quantum depletion, the Hubble patch enters into a black hole-type regime, with a very similar description. In the cosmological space the link between the classical and quantum view is possible writing $\frac{N_g}{N_\phi} = \sqrt{\epsilon}$ where N_ϕ is the number of the particles associated to the fluid and ϵ is the slow-roll parameter. In the last chapter I will show a different quantization of the inflation theory, that rises without the use of the inflaton particles. For this purpose I shall adopt a more general $f(R)$ gravity function, that I will prove is equivalent to the so called Starobinsky model. Therefore, it can be obtained an interesting corpuscolar theory that uses only interacting gravitons.
In this thesis I will often use the fundamental constants of physics: $\hbar, c, G$ and the derived natural system m_p, l_p.

$$m_p = \sqrt{\frac{\hbar c}{G}}; \quad l_p = \sqrt{\frac{\hbar G}{c^3}}$$

and setting $c = 1$:

$$G = \frac{l_p}{m_p}; \quad \hbar = m_p l_p$$

Capitolo 1

General relativity and cosmology

1.1 Gravitational field equation

Classically, the forces of the nature, through which the motion of objects can be justified, are conceptually and mathematically relatad with their source by a field equation. This latter shows what kind of field is generated by a determined distribution of charged matter. Quantically, the forces act at little scale as an exchange of a mediator particle, with the effect of modify the traictories of the particles considered. In this latter sense, the coupling costant quantifies the interactions and what kind of fields are coupled. Very similar considerations can be done for the electro-magnetic force, which source owns an electrically charged of matter distribution. In this thesis, I will treat the main features of the classical and quantum gravitational field and I shall show how the passage between the two theories is quite similar to the procedure for the electro-magnetic field.
Let me focus on the gravitational field in the context of the Einstein theory. In this latter, the differential geometry is the language to describe the relation between source and the field associated. In this framework the fundamental equation is a tensorial, non linear differential equation, that must recover the Newton theory in some limit, for which the classical field equation assumes the well know Poisson form:

$$\triangle\phi = 4\pi G\rho \tag{1.1}$$

where $\phi = \phi(x,t)$ is a scalar function depending on the space-time points (x,t) and $\rho = \rho(x,t)$ is the energy density of the source. Since that, the Laplace operator doesen't contain derivatives respect to the time, the above equation will describe instantaneous forces, just as in the Newton theory of gravitation is expected. To take into account also the special theory of relativity, it is necessary to pass to a more general form of the field equation, in which the scalar field will be substituted by the metric tensor field $g_{\mu\nu} : (x,t) \to g_{\mu\nu}(x,t)$ and the energy density by the stress-energy tensor $T_{\mu\nu} : (x,t) \to T_{\mu\nu}$. Thus, the main novelty is that the gravitational field is described by the metric of the space-time! This means that, assigned a particular fluid, the real distances, area and volume in the manifold can be different from the standard one. The field equation is:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor and R the scalar curvature. Both of them can be written as depending on the metric tensor field. The stress-energy tensor on the right-hand side, is built as depending on the energy density and on the pressure of the fluid, that moves in a space with a geometry affected by the fluid itself. In particular for a perfect fluid it can be written:

$$T^{\mu\nu} = -\rho g^{\mu\nu} + (p+\rho)u^\mu u^\nu \tag{1.3}$$

where $u^\mu = \frac{dx^\mu}{dt}$ and $p = p(x,t)$ are respectively the 4-velocity and the pressure of the fluid. This latter is equal in all directions of the space for a perfect fluid. The starting Newton equation (1.1) can be recovered in the weak field limit, that means to considere small speeds and a weak gravitational field.
Then, the General Theory of Relativity provides an equation of motion for point-like objects, whose solution is a geodesic:

$$u^\mu \nabla_\mu u^\gamma = \frac{d^2x^\gamma}{d\lambda} + \Gamma^\gamma_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{1.4}$$

where ∇_μ indicates the covariant derivative along the trajectory, determined by the object itself. In the second equality $\Gamma^\gamma_{\alpha\beta}$ is the so called Levi-Civita connection. It can be interpreted as a generalized gravitational force and it can be written as depending on the metric tensor field:

$$\Gamma^l_{jk} = \frac{1}{2}g^{li}(g_{ki,j} + g_{ij,k} - g_{jk,i}) \tag{1.5}$$

where $g_{\mu\nu}$ can be viewed as a generalization of the gravitational potential. An important quantity of the differential geometry is the Riemann tensor field, that can be written as a particular combination of the second covariant derivative of $g_{\mu\nu}$:

$$R_{ijkl} = \frac{1}{2}\left(g_{il;jk} - g_{ik;jl} + g_{jk;il} - g_{jl;ik}\right) \tag{1.6}$$

From this formula, the Ricci tensor and the scalar curvature can be obtained, that are the fundamental quantities in the Einstein equation. They can be written using (1.5), as a combination of the first covariant derivatives of $\Gamma^\lambda_{\mu\nu}$:

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\lambda\nu,\mu} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\rho,\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda} \tag{1.7}$$

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}\left(\Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}\right) \tag{1.8}$$

1.2 Cosmology

The framework of the General Theory of Relativity provides the tools to investigate the nature of the space-time of all the universe. From the Einstein point of view, the space and time distances are affected by dilation and contraction, described by the metric of the manifold. This is determined by the gravitational equation (1.2), where the source is all the matter contained in the universe. To begin, the assumption that from all the points of the universe, the space appears in all directions very similar, must be considered. This is the statement of the Copernican principle combined with the assumption of the isotropy, that can be considere true for portions of space enough larger than galaxies. Saied otherwise, if a congruence of timelike free falling observers see an isotropic fluid, and assuming that the isotropy holds for every space point, then the space-time will be spatially homogeneous and isotropic. These informations, transcribed in mathematical language, allow to write the metric for the cosmo, in the following very general form:

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 d\phi^2) \right) \tag{1.9}$$

that is called Friedmann-Lemaitre-Robertson-Walker metric. To understand the implications of this formula, it is necessary to specify the meaning of the coordinates. In particular t is the proper time of a comoving observer with the cosmic fluid, whereas r is a coordinate that doesen't represent the real distance between the origin of the frame and some physical system. Indeed, considering a small radial distance $\Delta r = r_2 - r_1$, and assuming $k = 0$, the true physical distance between two radial points coordinates will be:

$$\Delta_{phy} r \approx a(t)\Delta r$$

Thus, the standard Euclidean line element is multiplied by the so called cosmic factor scale $a(t)$, that describes the distance deformations in the space, releved by some observer. Note that, the numerical factor k in the denominator, can always be rewritten in such a way that is equal to $-1, 0, +1$. It allows to distinguish three kind of spaces. Indeed, with the appropriate transformations on the coordinates, it can be seen that the line element assumes the form:

$$dl^2 = \begin{cases} dX^2 + \sin^2(X)d\Omega^2 & k = +1 \\ dr^2 + r^2 d\Omega^2 & k = 0 \\ d\chi^2 + \sinh^2(\chi)d\Omega^2 & k = -1 \end{cases} \tag{1.10}$$

that respectively describe a closed, a flat and an open space.
A fundamental question in cosmology is about the evolution of the cosmic scale factor, because it reveals the main physical informations for the present, the past and the future of the space. To obtain these informations, it had to be plugged the metric (1.9) in the gravitational field equation (1.2) and to explicit in some way, the right-hand side. For example, in absence of matter the energy-impulse tensor is null and the space is euclidian. Whereas, in presence of matter there are different physical scenarios. For a perfect and homogeneus fluid, the energy-impulse tensor will be described by (1.3), therefore for comoving observers with the fluid, it will assume the following diagonal form:

$$T_{\mu\nu} = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & p(t) & 0 & 0 \\ 0 & 0 & p(t) & 0 \\ 0 & 0 & 0 & p(t) \end{pmatrix} \tag{1.11}$$

Now, substituing this matrix in the Einstein equation (1.2) and analyzing the 00 and ii components of the Einstein tensor $G_{\mu\nu}$, the equations of motion for the function $a(t)$, called Friedmann equations, can be obtained:

$$\frac{\dot{a}^2}{a^2} = \frac{\rho}{3} - \frac{k}{a^2} \tag{1.12}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) \tag{1.13}$$

where the dots indicate the derivatives respect to the cosmic time. The first one equation is a constraint wich selects the possibile combinations for the intial conditions $a(t_0) = a_0$ and $\dot{a}(t_0) = a_0'$. Instead, the second equation is the real dynamical equation, since that it is a second order equation. At first look it can be noted that, if the second member is negative, that is the case for ordinary matter, the cosmic scale factor is decelerating. Thus, since $\ddot{a} < 0$, it can be considered as initial condition $a(t = 0) = 0$. This argument constitutes a roughly way to prove the Big Bang existence, i.e. a moment in the universe, when all the distances between physical systems are null. It is useful to write the above expressions (1.12-13), introducing the so called Hubble function $H = \frac{\dot{a}}{a} = \frac{d}{dt}\ln(a)$:

$$H^2 = \frac{1}{3}\rho - \frac{k}{a^2} \tag{1.14}$$

$$\dot{H} + H^2 = -\frac{1}{6}(\rho + 3p) \tag{1.15}$$

Hand by hand, I will show the deep physical meaning of H and its recurrence in many equations. Using (1.11) and the relation $\nabla^\mu T_{\mu\nu} = 0$, it can be written the continuity equation for the cosmic fluid:

$$\dot{\rho} + 3H(\rho + p) = 0$$

that explicitally shows the relation between the energy density of the fluid and the cosmic scale factor during time. Now, let me introduce another fundamental cosmological physical quantity, called slow-roll parameter:

$$\epsilon = -\frac{\dot{H}}{H^2} \tag{1.16}$$

Later, the value of ϵ will be used to distinguish the inflatonary space-time from other kind of cosmological spaces. Rewriting the Hubble function as depending on $a(t)$, the following identity can be derived:

$$\frac{\ddot{a}}{a} = H^2(1 - \epsilon) \tag{1.17}$$

At first look, this formula explicitally shows that for $\epsilon < 1 \rightarrow \ddot{a} > 0$.
Developing the Friedmann equations, the formula (1.12) will assume the interesting form:

$$\Omega - 1 = \frac{\rho}{\rho_{critical}} - 1 = \frac{k}{a^2 H^2} \tag{1.18}$$

where the critical density is:

$$\rho_{critical} = \frac{3H^2}{8\pi G}$$

From the equation (1.18), it can be noted that, for a flat universe ($k = 0$), the solutions of the Friedmann equations will be:

$$a \sim t^{\frac{2}{3}}; \quad Dust \quad p = 0 \tag{1.19}$$

$$a \sim t^{\frac{1}{2}}; \quad Radiation \quad p = \frac{\rho}{3} \tag{1.20}$$

$$a \sim e^{Ht}; \quad D.Matter \quad p = -\rho \tag{1.21}$$

where the three equations on the right represent the state equations of the fluid taken into account. Thus, there are three main kind of space-time. The third cosmic scale function is determinated by an exotic kind of fluid, that differently to other normal systems, has a negative pressure! Later, I will focus on this kind of evolution, applying the inflaton field model. Moreover, this topic will be generalized by the quantum framework, imaging the space-time as a collection of particles, whose dynamics has its counterpart in the evolution of the cosmological parameter H. In particular I will show that with the corpuscolar interpretation, the cosmic evolution is slightly different from (1.21) and there are no limits to reduce the new expression to it. Moreover, an important and very predictive consequence of this model is that the inflation had to stop after sometimes, in such a way that there is no necessary of an external input in the theory to stop this stage of the universe. Analyzing the formula (1.18), it can be said that during the matter or radiation dominated universe, Ω had to diverge with time. But actually in the universe, this parameter is nowday ~ 1, i.e. it is an unstable fixed point that can be obtained only by very fine initial conditions. This is the so called flatness problem, so namely because the parameter Ω can be viewed as an average between the potential energy of the matter and the characteristic energy of the space, given by ρ_{crit}. Inflation is a good candidate to solve this problem, because during this period Ω decreases toward one!
Another classical way to study the exponential evolution of the space, is making use of the so called cosmological constant Λ. Indeed, modifyng the equation (1.2) adding a costant term in the left-hand side or in the right-hand side, paying attention on the sign, it can be written:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(R - \Lambda\right) = 8\pi G T_{\mu\nu} \tag{1.22}$$

Solving this equation without source, the cosmic factor will be of the kind (1.21). In particular, to match this theory with the previous one, the following relations must be respected:

$$\rho = -p = \frac{\Lambda}{8\pi G} \tag{1.23}$$

so that:

$$a \sim e^{Ht} = e^{\sqrt{\frac{\Lambda}{3}}t} \tag{1.24}$$

Now, to understand the physical meaning of inflation, there is necessity to investigate the meaning of the so called Hubble radius. To this task, I shall show the following example. Imagine a photon that begin to travel from the past to the present time and reaches our position, that for convenience is defined by the coordinates $r = 0$ and $t = 0$. The equation of motion for this light-quanta is encoded by the formula $ds^2 = 0$. Developing this expression, it can be derived that:

$$dr = \frac{dt}{a(t)} \tag{1.25}$$

Assuming that the emission of the photon occurred at time $-t_e$, the comoving radial coordinate will be:

$$r_e = \int_0^{r_s} dr \sim \int_{-t_e}^{0} \frac{dt}{a(t)} \tag{1.26}$$

If the cosmic scale factor is a monomyal of the type $a(t) = t^{\alpha}$, where $0 < \alpha < 1$, that is the case for the matter and radiation universes, the radial coordinate traveled by the photon will be:

$$r_e = t_e^{-\alpha+1} \tag{1.27}$$

This radial coordinate doesen't say anything about the distance traveled by the photon, measured by an observer. Thus, the proper distance must be introduced:

$$R \sim a(-t_e) r_e = t_e \tag{1.28}$$

for which the distance is only a function of the time. Now, considering the cosmic scale factor for an universe filled by dark matter, it can be seen that the coordinate travelled and the proper distance are:

$$r_e = \frac{e^{-H_0 t}}{H_0} \tag{1.29}$$

$$R = \frac{1}{H_0} \tag{1.30}$$

where the proper distance doesen't depend on r_e and on the time of the emission! By a simple check, it can be noted that the Hubble function is related to the matter and to the radiation universe in the same way of the dark matter. Thus, the inverse of the Hubble function, called Hubble radius, has a very interesting meaning. It is a physical quantity that allows to build the causal horizon of a particle, i.e the maximum distance that a light ray can travel in a range of time:

$$\tau = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da}{Ha^2} = \int_0^a d\ln a \left(\frac{1}{Ha}\right) \tag{1.31}$$

where in the last equation, $\frac{1}{Ha}$ is the so called comoving Hubble radius. In particular, substituing the expressions (1.19-1.20) of the matter and of the radiation dominated universe:

$$\tau \sim a \sim \sqrt{t}$$

$$\tau \sim \sqrt{a} \sim t^{\frac{1}{3}}$$

These are monotonically growing functions of the time. Note that, particles separeted by distances greater than τ never could communicate with one another.

Instead, if they are separated by distances greater than the comoving Hubble radius, they cannot enter in causal contact nowday. Thus, if $\tau > \frac{1}{aH}$ the particles discussed above, could enter still in contact in the future. For this reason, the better way to define inflation is by the condition:

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \tag{1.32}$$

By this definition two important questions of cosmology can be resolved: the flatness and the horizon problems. Indeed, if it is assumed an inflation stage of the universe near the Big Bang, with condition (1.32), there could be a time whereby $\frac{1}{aH} < \tau$ and only later, with another kind of space, the condition could be changed! Resuming, the conditions for inflation are:

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \rightarrow \quad \ddot{a} > 0 \rightarrow \quad \rho + 3p < 0 \tag{1.33}$$

1.3 de Sitter space

The FLRW metric (1.9) describes an homegeneus and isotropic space-time. For this reason the inflationary solution can be mathematically defined as the maximally symmetric space-time with constant Hubble function H or equivalently, with constant positive curvature Λ, and constitutes the so called de Sitter space-time. It has been seen that, to solve the two cosmological problems, a space-time of this kind must be existed in the early time of the universe. In particular, this inflationary de Sitter phase, the cosmological constant was large and correspondingly the curvature's radius very tiny.
In the framework of the differential geometry the de Sitter solution can be defined as a four dimensional hypersurface embedded in a five dimensional space-time. Altough this "conteiner" manifold doesen't exist, the definiton that make use of this escamotage, has still a meaning. Otherwise, in the previous section, I defined this manifold and its curvature "intriniscally", without using the idea of the embedding. Note that, following this new procedure, the de Sitter space-time can be built for arbitrary dimensions. Geometrically the de Sitter space-time can be viewed as a four dimensional timelike hyperbola, embedded in a 4+1-dimensional Minkowski space-time. Note that, it is a maximally simmetric manifold. This means that the isometries in the manifold, i.e. the fields along wich the covariant derivative of the metric are null $\nabla_\mu g_{\mu\nu} = 0$, i.e. the transformations that preserve the hyperboloid in the emebbed version, are the maximum one. Since that the space is spherically simmetric, the isometries will be the rotations and the boosts. They will be encoded by the isometry group $SO(3,1)$. Now, the constrained equation that defines this hypersurface is:

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 = -R^2 \tag{1.34}$$

where R is called de Sitter radius and coincides with the Hubble radius. The metric on the embedded hyperbola will be the reduced Minkowsky metric:

$$ds^2 = -dy_0^2 + dy_1^2 + +dy_2^2 + +dy_3^2 \tag{1.35}$$

The hyperbolae can be parametrized by a new set of four coordinates (t, x_1, x_2, x_3) by means of the following transformations law:

$$
\begin{aligned}
y^0 &= R\sinh\left(\frac{t}{R}\right) + exp\left\{\frac{t}{R}\right\}\frac{x_1^2+x_2^2+x_3^2}{2R} \\
y_1 &= exp\left\{\frac{t}{R}\right\}x_1 \\
y_2 &= exp\left\{\frac{t}{R}\right\}x_2 \\
y_3 &= exp\left\{\frac{t}{R}\right\}x_3 \\
y_4 &= R\cosh\left(\frac{t}{R}\right) - exp\left\{\frac{t}{R}\right\}\frac{x_1^2+x_2^2+x_3^2}{2R}
\end{aligned}
$$

Substituing these expressions in the differential terms in (1.35), the well know de Sitter metric for a flate space can be obtained:

$$ds^2 = dt^2 - exp\left\{\frac{2t}{R}\right\}(dx_1 + dx_2 + dx_3) \tag{1.36}$$

To recover the de Sitter metric in an open and closed space, it must be slightly changed the above transformations. Without enter in mathematichal accounts, the resulting line elements will be respectively:

$$dl^2 = \begin{cases} R\cosh\left(\frac{t}{R}\right)\left(dX^2 + \sin^2(X)d\Omega^2\right) & k=+1 \\ R\sinh\left(\frac{t}{R}\right)\left(d\chi^2 + \sinh^2(\chi)d\Omega^2\right) & k=-1 \end{cases} \tag{1.37}$$

To conclude, the Ricci tensor and the scalar curvature in the flat space are:

$$R_{\mu\nu} = \frac{3}{R^2}g_{\mu\nu} \tag{1.38}$$

$$R = \frac{12}{R^2} \sim \Lambda \tag{1.39}$$

1.4 Inflaton field model

The classical theory of inflation shown in the above sections, studies this kind of universe starting from the Friedmann equations and considering a fluid, without analyze its dynamics. Therefore, it is important to study the field associated to the right-hand side of the gravitational equation (1.2), because its evolution can determine more general conclusions. One could think that the cause of inflation is given by some kind of matter, described by arbitrary boson or fermion fields, or by a combination of them. But, actually the main results can be obtained by use of a simple real scalar field $\phi : \mathcal{R}^{1,3} \to \mathcal{R}$. Excluding other kind of fields in the universe, the interactions that come into play, are described by two terms:

$$g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi \qquad V(\phi)$$

where the first describes the coupling between the metric tensor field and ϕ, whereas $V(\phi)$ is the autointeraction potential. Then, I will show that the form of this latter, allows to classify different kind of inflationary models. Considering a perfect and homogeneus fluid $\phi(x,t) = \phi(t)$, the components of the associated energy-impulse tensor $T_{\mu\nu}$ for a comoving observer, can be written as:

$$p_\phi, \rho_\phi = \frac{1}{2}\dot{\phi}^2 \pm V(\phi) \tag{1.40}$$

Plugging the above expressions in the Friedmann equations (1.14-15), it can be obtained:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0; \tag{1.41}$$

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{1.42}$$

These equations show new important informations about the evolution of the field ϕ. Indeed, the transformation of ϕ during time can change the conditions for the inflation, so that the de Sitter phase, becomes only a particular phase of the universe, in which the field satisfies some requirements. The slow-roll parameter can be written as:

$$\epsilon = \frac{3}{2}\left(\frac{\rho_\phi}{p_\phi} + 1\right) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \tag{1.43}$$

Thus the de Sitter limit $p_\phi \to \rho_\phi$ corresponds to $\epsilon \to 0$, with the consequence that the potential energy term $V(\phi)$ dominates on the kinetical term. Therefore the equations (1.41-42) simplifies to:

$$\dot{\phi} \simeq -\frac{V_{,\phi}}{3H} \tag{1.44}$$

$$m_p^2 H^2 \simeq V(\phi) \tag{1.45}$$

It is very important remark that this regime can change with time and therefore the inflation can stop. This means that the classical de Sitter metric solution, describing an eternal exponantial space, cannot be valid for arbitrarly long times. Moreover, from the equation (1.41), it can be deduced that the inflationary phase occures for enough long times if:

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V_{,\phi}| \tag{1.46}$$

To resume this last feature, a second parameter η can be introduced:

$$\eta = -\frac{\ddot{\phi}}{H\phi} = \epsilon - \frac{1}{2\epsilon}\frac{d}{Hdt}\epsilon \ll 1 \tag{1.47}$$

where the disequality is consistent with (1.46). It is important to note that ϵ and η depend on the form of the potential energy of the inflaton field.

$$\epsilon = \frac{m_p^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \tag{1.48}$$

$$\eta = m_p^2 \frac{V_{,\phi\phi}}{V} \tag{1.49}$$

It is called slow-roll regime the stage of the model for which the conditions $\epsilon, \eta \ll 1$ are respected. The key idea behind the scalar field inflation is that, if the plateau of the potential is sufficiently flat, the phase of exponential expansion lasts long enough to solve the cosmological problems mentioned above, providing also a mechanism for the universe to gracefully leave this highly accelerated phase. Considere a quadratic potential $V(\phi) = m^2\phi^2$, the approximated equations (1.44-45) can be rewritten as:

$$H^2 \sim \frac{1}{2}m^2\phi^2 \qquad \dot{\phi} \sim -\frac{m^2\phi}{3H}$$

These equations show that the limits $m \to 0$ and $\phi \to \infty$, mainteining $V(\phi)$ constant, allow to recover an eternal de Sitter universe. Then, I will show that in the corpuscolar description, if the space-time is treated as a collection of finite gravitons, a regime in which this evolution can be recovered, doesen't exist!
Since inflation has a finite life time, it is interesting to calculate its duration by means of the so called number of e-folds N_e. For example, during the de Sitter inflation, with Hubble function constant, the number of e-folds between two fixed time t_1 and t_2 will be:

$$\ln\left(\frac{a(t_2)}{a(t_1})\right) = \ln e^{H(t_2-t_1)} = H(t_2 - t_1)$$

More generally, the number of e-folds, before the inflation ends are:

$$N_e(\phi) = \int_t^{t_{end}} H dt = \int_\phi^{\phi_{end}} H\frac{d\phi}{\dot{\phi}} \sim \int_{\phi_{end}}^{\phi} d\phi\frac{V}{V_{,\phi}} = \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\epsilon}} \tag{1.50}$$

where:

$$\epsilon(\phi_{end}) \sim 1 \tag{1.51}$$

Note that to solve the horizon and flatness problemes, it must be required that $N_e > 60$. For the quadratic potential the number of e-folding is:

$$N_e = \frac{1}{\epsilon} \tag{1.52}$$

When the inflaton field rolls down to the minimum of the potential, inflation stops and the radiation dominated phase begins. The process goes under the name of reheating and the value of the field starts to oscillate. In this case the equation for the Hubble function becomes:

$$\frac{d\rho_\phi}{dt} + 3H\rho_\phi = 0$$

Moreover, if during this phase, ϕ is coupled with other fields, there will be a decay of the inflaton energy:

$$\frac{d\rho_\phi}{dt} + (3H + \Gamma_\phi)\,\rho_\phi = 0$$

where Γ_ϕ is a coupling parameter between the inflaton and another generic field. The set up shown in this section, seems to be very rich of features to describe an

inflationary space-time and to solve the horizon and flatness problems, but it fails in the description of gravitational perturbations, at early stage of the universe. Instead the full quantum description, provides new possible interpretations for this kind of effects.
An important consequence of the inflaton model is that, from different $V(\phi)$ one could provides many possible inflationary space-time and one could understand what kind of these allows the inflationary phase. During the slow-roll regime the inflaton field can move over a large or small distance $\Delta\phi = \phi_{in} - \phi_{end}$. In an important class of inflationary models, their dynamics is driven by a single monomial term in the potential $V(\phi) = \lambda_p \phi^p$. In this case $\Delta\phi > m_p$ and the gravitational waves produced by inflation, should be observed in the near future. Note that the slow-roll conditions are independent from the coupling constant λ_p. Instead, if $\Delta\phi < m_p$, the amplitude of the gravitational waves produced during inflation is too small to be detected. An example of this last model is the Higgs-like potential:

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu} \right)^2 \right]^2$$

Other kind of inflation models are theoretically possible. Then, I will focus on the so called $f(R)$ gravity theory, that provides an inflationary phase without the use of the inflaton field, but make use of a more general Einstein field equation. In this case the inflationary cosmo is a consequence of the gravity itself and not of an external fluid! Another one possibility that I will not consider, make use of a different coupling term between the metric and the inflaton field, in such a way to modify the kinetical term:

$$g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi \to F(\phi, g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi)$$

where the functional F describes an high energy kinetic term. To conclude, another model could be contain more fields than ϕ. But, from a theoretical point of view, this model should go to add the ingredients of the theory.

Capitolo 2

Quantum black hole

The task of this thesis is to develope a simple corpuscolar model for the inflationary space-time. Before to see this thing, I will begin to consider the quantum features of the black hole configuration, since that the inflationary quantum model is in some way a generalization of this latter. Whenever source of energy is localized in the space, it modifies the structure of the space-time, generating a non causally connected region, bounded by a well defined distance. This latter will contain the black hole, that can be studied as a composition of ordinary matter. But at the same time, the particles of the gravitational field called gravitons, had to be introduced. To understand the physical meaning of this topic, I will focus on the process of creation of a black hole by a collapsing matter. Without entering in too much details, I also want to see the particle production effect from the vacuum in the semiclassical approach (Hawking effect), that doesen't consider the existence of gravitons and I will compare this result with the quantum one.
Black holes are predicted by the General Theory of Relativity, since that they are perfectly consistent with this theory. However, this doesen't mean that they necessarily exist in nature. Indeed what the theory describes is a black hole as already existing, but it is necessary to describe these physical entities analyzing the whole creation process. I am talking about the study of a gravitational collapse, which science says to have various destinies, including the creation of a region of the space, that cannot communicate with all the rest.

2.1 Quantum field in curved space-time

Now, I will show the semiclassical approch and broadly I will analyze the procedure to obtain the Hawking effect. Remark that the semiclassical theory has the sake of study a first form of a quantum theory in the Einstein Gravitation theory, going to plug the quantum fields formalism of ordinary matter and strenghts, in a space-time described by a metric tensor fields more complex of the Minkowsky metric, but without considering the microscopic interactions of the matter with the background. Thus the fundamental question is about the behaviour of a quantized field in a curved space-time. What happens to a quantum field defined in a space where is occuring a gravitational collapse? Imagine that the particles of this probe field enter in the area of collapse starting from far distances and time. The amazing result is that a part of the particles that enter in the black hole can exit from it, so that an observer can see a spectrum of emission particles with an associated temperature. The key idea behind this model arises from the strange nature of the vacuum in a curved space-time. This vacuum has proprerties very different from the standard of the Minkowsky space-time. To simplify the problem, I want to consider a collapsing massive shell with spherical symmetry. For the classical properties of the gravitational field, the interior of the source is described in every point by the Minkowsky metric $\eta_{\mu\nu} = (1, -1, -1, -1)$. The outside region of the nutshell is instead described by the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 d\phi^2\right) \tag{2.1}$$

This metric is the solution of the vacuum Einstein equation. It has many important physical and matematichal consequences, that I will not treat. Only I say that the metric is doesen't defined for the value $r = R_S \equiv 2GM$, that is named Schwarzchild radius. This bound is just the horizon of the black hole! To understand how to approach to the problem, I take into account the interesting analysis concerning the basis that can be used to write a field as a decomposition. Indeed in the Quantum Field Theory, as soon as is assigned a field equation, one tries to write its general solution as finite or infinite linear combination of some kind of basis. If this latter can be found in some special reference frame, and therefore in all, it becomes possible to proceed with the quantization. The fundamental difference with the QFT approch is that there are more possible reference frames. This is a consequence of the relativity principle about the equality of the physical laws for all the reference frames. Thus, since also non linear transformations can be used, passing between two basis, new informations can be fish out by the different quantizations that arise. Considere a "free" scalar field, the equation of motion will be:

$$(\nabla_\mu \nabla^\mu + m^2)\phi = 0 \tag{2.2}$$

where $\nabla_\mu \nabla^\mu = g^{\mu\nu}\partial_\mu\partial_\nu$ substitutes the standard d'Alebertian operator. ϕ isen't really free because it is coupled with the metric field, therefore it is affected by the gravitational field. To build a global basis for all time and space points, one should analyze separately the different regions of the manifold. For long time ago and far distances from the collapse, one could describe ϕ as in the usual Minkowsky space-time and the (2.2) reduces to the Klein-Gordon equation, with solutions:

$$t \to \infty; \quad r \to \infty \quad b_\omega(x) = e^{-i(\omega t + kx)}$$

where ω and k are the energy and the vector number of the free particles, that in some way scatter with the collapsing region. For large time and opposite far distances, the resulting basis is still formed by plane waves, with a different coefficient that multiplies them. Instead, near the black hole, the problem is different, because the covariant derivative cannot be substituted by the partial derivative. Without analyzing the resulting form of the basis near this region, I want only say that the first step to find it, is demanding the continuity of the metric between the inside and the outside of the shell. This matching can be written as:

$$ds^2_M \mid_\Sigma = ds^2_S \mid_\Sigma \tag{2.3}$$

where Σ is a three dimensional surface, that describes the area horizon for all time. In this way, some features of the basis near the black hole can be understood. As soon as the basis for the various part of the manifold is obtained, it can be quantized and the black hole quantum features arise.
Considere a generic manifold, to obtain the quantization of the field ϕ, let me admit the existence of a basis b_i such that:

$$\phi(x) = \sum_i (c_i b_i(x) + c_i^* b_i^*(x)) \tag{2.4}$$

with the normalization conditions:

$$(b_i, b_j) = \delta_{ij} \tag{2.5}$$
$$(b_i^*, b_j^*) = -\delta_{ij} \tag{2.6}$$

where the scalar product is slightly different from the standard one. It is defined by:

$$(b_i, b_j) = \frac{i}{\hbar} \int_\Sigma d^3x \sqrt{-g} n^\mu \left(b_j(x)^* \partial_\mu b_i(x) - b_i(x)^* \partial_\mu b_j(x) \right) \tag{2.7}$$

where Σ is a space-like surface with the normal vector n^μ for every point. Wheres g indicates the determinant of the metric tensor. Remember that, the fundamental assumption of the Einstein theory is that the pysical phenomena must be described in the same way in all reference systems. This requirement can be mathematically expressed, in the field formalism, by the equation:

$$\phi(x) = \phi'(x) = \sum_i c'_i b'_i(x) + c_i^* b_i^*(x) \tag{2.8}$$

where c'_i are the new components of the new basis b'_i. Not all the basis's transformations admitted in the Einstein theory can be made. In order to save the concept of probability that arises in the quantum framework, it must be required that the new basis respects the same normalization equations (2.5-2.6) of the initial basis. The transformations among two basis are encoded by linear applications, i.e. matrices with indices belonging to the natural set $\mathcal{N}$, or to a subset of $\mathcal{R}$:

$$\begin{cases} b'_i(x) = \alpha_{ij} b_j(x) + \beta_{ij} b_j^*(x) \\ b_i(x) = A_{ij} b'_j(x) + B_{ij} b'^*_j(x) \end{cases}$$

where α_{ij}, β_{ij}, A_{ij} and B_{ij} are the linear applications introduced before. In the second quantization the components c_i and c_i^* become respectively the destruction $\widehat{c}_i$ and construction $\widehat{c}_i^+$ operators on a Fock space $\mathcal{F}$, containing all the physical states of the "free" multiparticles-ϕ. Note that, the states built with these operators contain quanta with dispersion relation different from the standard: $\omega_k = \sqrt{k^2 + m^2}$! Let me show how these operators transform:

$$\begin{cases} \widehat{c}'_i = \alpha_{ij} \widehat{c}_j + \beta_{ij} \widehat{c}_j^+ \\ \widehat{c}'^+_i = A_{ij} \widehat{c}_j + B_{ij} \widehat{c}_j^+ \end{cases}$$

The key point is that passing from a basis to another one, the construction and destruction operators change their form. In particular, analyzing the number operator in a determined frame $\widehat{N} = \sum_i \widehat{c}_i^+ \widehat{c}_i$, the number of particles in the vacuum state $|0\rangle$ is:

$$\langle 0| \sum_i \widehat{c}_i^+ \widehat{c}_i |0\rangle = 0 \tag{2.9}$$

Passing to another generic frame, using a transformation with non null coefficients β_{ij}, the number of particles in the vacuum state will be no more null!

$$\langle 0| \sum_i \widehat{c}_i^+ \widehat{c}_i |0\rangle = \sum_i |\beta_{ij}| \neq 0 \tag{2.10}$$

This means that a QFT in a curved space-time has the vacuum state not well defined. For the model seen before, of a collapsing body that creates a black hole, I said that the choise of the basis is dictated by the matching of the metrics on the nutshell. Without analyzing the complete procedure, I want to report the final result for spectrum of the emitted particles, for each frequency, called Hawking formula:

$$\Delta(N_\omega) = \frac{1}{e^{\omega T} - 1} \tag{2.11}$$

This is the spectrum of the black hole. Thus, this latter has non zero temperature altough classically nothing can be exit from it! To conclude, the quantum effects cause a leakage of the particles associated to the probe scalar field.

2.2 Quantum gravitational collapse

Now I will show a first simple full quantum model for gravitational systems, going to quantize the gravitational field. My first task is to study the gravitational collapse, interpreting the potential energy of the Newton Gravitational Theory, as a manifestation of a collection of gravitons quanta. In this way, for the moment, the quantization of the metric tensor field can be avoided and some interesting results can be obtained by the more simple quantization of the classical gravitational field.
Consider gravitational collapse model with spherical symmetry. Imagine a compact star of radius R made of ordinary matter, and therefore constituted of a number of baryons N_b with mass m_b. This kind of matter is auto interacting and N_b doesen't depend from the radius R. Assuming that processes of radiation or absorption of energy don't take place, the total energy E of the system is conserved, in such a way that $E = M_{ADM}$, where M_{ADM} is the Arnowitt–Deser–Misner mass. Energy conservation has different interpretations for the classical theory of gravitation and for the Einstein theory. In the first case the energy conservation is closely related to the symmetry of the physical laws respect to the time, whereas in the second case, it is related to the freedom of re-parametrizing the time coordinate.
Now, the energy of the system is given by the matter's composition of the system and by the forces between the particles involved. The baryonic energy is composed by a kinetical and a potential components. Moreover, this corpuscolar model takes into account the quanta of the gravitational field, whose energy is described in a similar way, by a kinetical and a potential energy terms. Image that at

the beginning of the collapse the baryons are fixed very far from each other, the hamiltonian constraint will be:

$$H = H_B + H_G = M_{ADM} \tag{2.12}$$

This form of hamiltonian takes into account the baryonic matter and the curvature of the space-time, described by the graviton's energy term H_G. Moreover, excluding at first the energy of the gravitons, the initial energy of the system is only given by the total baryonic mass:

$$H = E_b = M_b \simeq m_b N_b \tag{2.13}$$

Hand by hand that the system evolves, the baryons will be closer to the centre of the collapse. Gradually that the distance R of the collapsing body decreases, the kinetical energy grows, whereas the potential energy decreases (but increases in modulo). Thus the form of E_b will contain other terms:

$$E_b = K_b(R) + U_{bg}(R) + U_{bb}(R) + M_b \tag{2.14}$$

The term $U_{bb}(R)$ describes the repulsion among the baryons and therefore it is always positive. It gives the pressure that allows to the system to be in a static configuration. A stable star has a fixed radius R_s, so that $K_b = 0$ and $U_{bg}(R_s) = -U_{bb}(R_s)$. This is true because the total energy must remain equal to the ADM mass. Now, the term $U_{bg}(R)$ is given by the product of the potential generated by the collection of baryons and the total mass of the baryons themselves:

$$U_{bg}(R) \simeq M_b \phi(R) \tag{2.15}$$

where $\phi(R)$ is the classical potential, that satisfies the Poisson equation:

$$\triangle\phi(R) = G M_b j(R) \tag{2.16}$$

In the right-hand side $j(R)$ represents the energy density function of a static source, that satisfies $\int_0^R r^2 dr j(r) = 1$. The formula above is like the (1.1), rewritten using the radial coordinate. As just seen, this equation is provided by the limit of the G_{00} component of the Einstein tensor in the weak field approximation. The solution of this equation is the well known Newton potential:

$$\phi(R) = -\frac{G M_b}{R} \tag{2.17}$$

thus the potential energy takes the form:

$$U_{bg}(R) \simeq -G\frac{M_b^2}{R} \tag{2.18}$$

Now, I want to see how to write the potential field in the framework of the quantum theory. I will show how quantize ϕ and how build the Fock space of the multiparticles gravitons states associated. This is the fundamental step of the QFT. For this purpose, I will use the Fourier tansformed of the Poisson equation. In this way the functional equation will become a polynomial equation:

$$k^2\phi(k) = -\frac{M_b}{m_p} j(k) \tag{2.19}$$

This latter encodes the dispersion relation for graviton, i.e. the relation between energy, momentum and mass for the particle. Starting from (2.17), ϕ can be written as a composition of plane waves with vector number k, that is a dimensionless quantity related with the classical momentum of the quanta by the De Broglie relation $\overline{p} = \hbar\overline{k}$. Expanding the graviton field operator in the corresponding radial modes, it can be written as:

$$\widehat{\phi}_k \simeq \left(\frac{\widehat{g}_k + \widehat{g}^+_{-k}}{\sqrt{k}} \right) \tag{2.20}$$

The connection between the classical and the quantum field can be done with the same procedure that allows to pass from a classical eletromagnetic field to the photon vector quantum field. Indeed, to describe the electromagnetic field, the best way is to use the mean field, that is directly associated with the electric and the magnetic fields that solve the Maxwell equations. The key point to recover the mean field associated to a systems of photons, is using the so called N-coherent states in the Fock space of the photons. They haven't a well defined number of photon, but their energy distribution has very important proprerties. Let me focus on the gravitons Fock space built starting from the vacuum state, the construction and the destruction operator $(F, \widehat{g}_k, \widehat{g}^+, |0\rangle)$. I will show that the bra-operator-ket, using the coherent states in the Fock space, can make recover the standard potential form for a spherical symmetric source of matter. For the moment I define a coherent state $\mid g >$ as the eigenstate of the annihilation operator:

$$\widehat{g}_k \, |g\rangle = g(k) \, |g\rangle \tag{2.21}$$

In particular it can be chosen:

$$g(k) \simeq -\frac{M}{m_p} \frac{j(k)}{k^{\frac{3}{2}}}) \tag{2.22}$$

in such a way that:

$$\phi(k) \simeq \langle g|\widehat{\phi}_k|g\rangle \simeq -\frac{M}{m_p} \frac{j(k)}{k^2} \tag{2.23}$$

that represents the mean field in the momentum space. Thus, the number of gravitons in the coherent state $|g\rangle$ will be:

$$N_g = \langle g| \, \widehat{N} \, |g\rangle = \int d^2k k^2 \, \langle g| \, g^+_k g_k \, |g\rangle \simeq \frac{M_p^2}{m_p^2} \int dk \frac{j^2(k)}{k} \simeq \left(\frac{R_S}{l_p} \right)^2 \tag{2.24}$$

The last equality shows explicitly the relation between the quantum quantity N_g and the classical one R_S. Following the idea of the quantum interpretation, the potential energy can be interpreted as consequence of a sum of the graviton's energies:

$$U_{bg}(R) \simeq N_g \epsilon_g(R) \tag{2.25}$$

where ϵ_g can be calculated by means of the De Broglie relation:

$$\epsilon_g \simeq -\frac{\hbar}{\lambda} \tag{2.26}$$

where λ is the wavelenght of the quanta. Note that gravitons has negative energies since that gravitational strenght is only attractive. Comparing the above expression with the (2.17), it can be deduce that the graviton's energy scales with the radius of the matter:

$$\epsilon_g \simeq -\frac{\hbar}{R} \tag{2.27}$$

Now, the auto-interacting term for gravitons U_{gg} must be considered. It constitutes a complete novelty. It is the so called post-newtonian term of the potential function. It must be positive since that it describes the repulsive component of the gravitational strenght due to the interaction between each graviton with the gravitational field:

$$U_{gg}(R) \approx E_g(R)\phi(R) \simeq N_g\epsilon_g(R)\phi(R) \simeq N_g M_b \frac{l_p^2}{R^2} \tag{2.28}$$

Its behavior is of the kind $\sim \frac{1}{R^2}$. In particular:

$$\begin{aligned} U_{bg}(R) \gg U_{gg}(R) \qquad & R \gg R_S \\ U_{bg}(R) \approx -U_{gg}(R) \qquad & R \approx R_S \end{aligned}$$

The final complete resulting hamiltonian will be:

$$M = E_b + U_{gg}(R) = M + K_b(R) + U_{bb}(R) + U_{bg}(R) + U_{gg}(R) \tag{2.29}$$

At this point let me try to describe the black hole configuration. This can be viewed as a very massive, compact and "stable" astrophysical star, with dimension of the size of the Schwarzschild radius. This configuration can be static if the kinetical energy and the repulsive strenght of the baryons are null: $K_B(R_S) = U_{bb}(R_S) = 0$. By the assumption of the energy conservation during the collapse, it can be written:

$$U_{bg}(R_S) \approx -U_{gg}(R_S) \approx M \tag{2.30}$$

Thus, the only possible condition for the black hole formation, by a collapsing matter untill to R_S is:

$$K_B(R_H) + U_{BB}(R_H) \simeq 0 \tag{2.31}$$

Untill now, I haven't defined the so called dimensionaless coupling costant that quantifies, as in all QFT, the interacting strenght between quanta. In graviton's theory, it represents the microscopic interaction between gravitons themselves. It changes with the size of the source, following the formula:

$$\alpha = \frac{\epsilon_g^2 R_S}{m_p^2} \tag{2.32}$$

From this relation it can be said that:

$$\alpha N_G \simeq 1 \tag{2.33}$$

i.e. the graviton's system, seen as a statistichal ensamble, is critical. It is on a verge of a phase transition between the liquid and gas phases. Moreover, it can be said that the black hole radius, the energy of the gravitons and the density energy scale with the graviton's number:

$$R_S \simeq l_p \sqrt{N_g} \tag{2.34}$$

$$\epsilon_G \simeq \frac{M}{N_g} \tag{2.35}$$

$$\rho \simeq \frac{1}{N_g} \tag{2.36}$$

To large N_g correspond a small density and a more classical black hole.
To conclude, let me note that in the model seen above the repulsive potential U_{bb} is not null. But, even it is vanished, the source will end its process to the Schwarschild radius, because of the existence of the new term U_{gg}! Indeed, the kinetical energy is, also in this case, null for $R = R_S$.

2.3 Black hole as N-bound state

Now I want to resume the concepts about the quantum nature of the Schwarzschild space-time, going to study in more depth the quantum statistichal features of the graviton's system, considered as a quantum state. The main features of a quantum system of gravitons are the number of constituents N_g and the coupling constant α. Manipulating these parameters of the theory, very different descriptions can be obtained. Moreover, I want introduce the so called collective parameter $C = N\alpha$, that can be interpreted as a sum of all the little quantum interactions. It measures the global effect of the microscpic dynamics. Note that this collective parameter cannot always be defined. For example, a generic classical object is understood as a state with high occupation number, but with α ill-defined. Instead in this sense, the black hole is a very interesting object. It has a big occupation number $N \sim \frac{1}{\alpha}$ and $C \sim 1$, so that his quantum nature is manifest at large scale. In particular, among all the possible sources of some physical size R, N is maximed by the black hole! I shown in the model of the previous section that N is given for a graviting source of mass M by:

$$N_g = N_g(M) = \frac{R_S M}{\hbar} = \frac{M^2}{m_p^2} \tag{2.37}$$

that is a function of the mass of the source. Thus, the number of gravitons depends only on the mass of the source, but not on the effective radius of the object! Many features can be derived by the quantum number N, also semiclassical quantities as the entropy or the spectrum thermality. Not only, also features given by the classical interpretation of gravity, as the horizon radius or the geometry of the space-time can be recovered by the mathematical manipulation of this parameter. In some way this parameter substitutes the classical physical quantities, that are the gravitational radius and the mass. Rewriting the formula (2.37):

$$N_g = \frac{l_p^2}{R_S^2} \tag{2.38}$$

The number of gravitons can be interpreteted as the number of cell of size l_p that fill the area of the horizon.

In this description the black hole gravitational field is a bound state of gravitons. They are massless, spin 2 particles very weakly interagent among them. As seen before, their energies is very small, because their wavelenghts is of order of the size of the Schwarschild radius. For this reason, combined with the very large occupation number N_g, the graviton's state is leaky bound. The black hole is so full of gravitons that evaporates! The self-coupling (2.32) can be rewritten as:

$$\alpha = \frac{l_p^2}{\lambda^2(R)} \simeq \frac{l_p}{R_S^2} \tag{2.39}$$

Since that $\alpha \ll 1$, it is possible a perturbative study of the quantum gravity. The wavelenght of the gravitons can be rewritten as a function of the occupation number:

$$\lambda = l_p\sqrt{N} \tag{2.40}$$

Each individual graviton is also subjected to a collective binding potential (2.18), created by all the N gravitons:

$$U_{bg} = -\frac{\hbar}{l_p\sqrt{N}} \tag{2.41}$$

Resuming, black hole is a bound N-state. Classically it has a null temperature, but since that in the quantum description, it is leaky bound, it emits particles with a charachteristic temperature. All the physical quantities seen before, can be described and scale with N, so that during the evaporation, all these quantities scale with N. I shown that the potential energy is a sum of the graviton's energies ϵ_g (2.26), determined by the their wavelenghts, that are of the order of the size of the Schwarzschild radius. Actually, they are described by a density function N_λ that quantifies the number of gravitons with wavelenght λ. Thus the (2.26) can be generalized by

$$E \sim \sum_\lambda \hbar \frac{N_\lambda}{\lambda} \tag{2.42}$$

The peak of N_λ must be for $\lambda = R_S$. The contributions given by short wavelenghts or equivalently large energies are exponentially suppressed. Thus, the resulting graviton's system is charachterized by long wavelenghts and weakly interactions. Note that, for a bodies with $R \gg R_S$, the interaction between each gravitons with the collective potential becomes negligible and the quantum effects are excluded. Instead, the interaction between the gravitons are negligible also for object of radius R_S, but cannot more be trascured for $R_S \simeq l_p$.
Note that, the semiclassical theory, i.e. the quantum field theory in a curved space-time, can be recovered with the limit:

$$N_g \to \infty$$

This means that in this limit, the dynamics of the graviton's condensate matches with the metric tensor field description, affected by quantum fluctuations.
The gravitons in the condensate are described by a distribuiton of wavelenght λ around the size R_S that allows the existence of particles with energies above

the thresold and that could come out from the condensate. This evaporation is described by the formula:

$$\dot{N}_g \sim -\frac{\hbar}{\epsilon_g} \sim -\frac{\hbar}{\lambda(N_g)} \sim -\frac{\hbar}{l_p\sqrt{N_g}} \tag{2.43}$$

where in the first equation is clear that the evaporation is related with the characteristic energy of the gravitons. By integration, the form of the function $N(t)$ can obtained:

$$N_g(t) = \left(\frac{t}{l_p}\right)^{\frac{2}{3}} \tag{2.44}$$

This function describes the density distribution of the evaporating gravitons. This evaporation law can be applied to the gravitational collapse model analyzed before, because it is only due by the scattering between gravitons:

$$g + g \rightarrow g + g \tag{2.45}$$

In this process one of the final gravitons aquires the necessary energy to escape from the collective binding. During the evaporation, the loss of one graviton corresponds to the emission of a quanta with wavelenght $\sqrt{N_g}l_p$. Now, there is a fundamental difference with the evaporation of a system like a liquid. For this latter the energy redistribution makes slow down the rate of the evaporation. For the black hole this process doesen't happen, because the gravitational radius decreases and the graviton's energy consequently grows! Said otherwise, since the potential energy (2.41) of the gravitational system is conserved, the wavelenghts of gravitons must decreases. The rate of the process (2.45) must be as in (2.43). To prove this fact, three terms must be considered in the account:

$$N_g^2 \qquad \alpha^2 = \frac{1}{N_g^2} \qquad E = \frac{\hbar}{\sqrt{N_g}l_p}$$

where the first indicates the number of possible scattering couples $g-g$, the second is the strenght of the quantum interaction and the third is the characteristic energy of the process. Finally, from the product of these three terms the formula (2.43) can be recovered. Therefore, a graviton of wavelenght λ is emitted during the time scale $\Delta t = \frac{\hbar}{\Gamma}$. The variation of the mass of the black hole is:

$$\dot{M} = -\frac{E}{\Delta t} = -\frac{\hbar}{Nl_p^2} \tag{2.46}$$

The evaporation causes a continuous transformation of the black hole to black holes of smaller size. It can be calculate also the half life time of the black hole as:

$$\tau \sim E \sim \frac{\hbar}{\sqrt{N}l_p} \tag{2.47}$$

Moreover, the mass decay rate can be rewritten as a function of the temperature:

$$\dot{M} = -\frac{T^2}{\hbar} \tag{2.48}$$

Thus the half life time can be written as a function of the temperature:

$$\tau = \frac{\hbar^2}{T^3 G_N} \tag{2.49}$$

An interesting consequence of the evaporation process is that the characteristic strenght between two gravitons increases with time. This is a novelty respect to the standard quantum fields theory, for which the coupling constants are fixed. Resuming, black holes are maximally packed N-bound state states. Thus, if a graviton is added to it, the size unavoidably increases. Moreover N doesen't depend on the physical composition of the system and doesen't change also if $R \to \infty$. The proprerties of the black holes are the same for all of them; this is expected in this framework since the universality of N. Note that for other bosons fields, for example the photon field, this kind of study cannot be done, because the number of quanta is of the order of the coupling constant, therefore less than one. Then, I will show that the N-bound state model can be applied to different kind of space-time, where for example matteris not localized and the quantum interpretation of the gravitational field is little more complicated. In particular, I will apply all this considerations on the inflationary cosmological space-time.
Now, I want to generalize the evaporation formula (2.43) adding to the model a generic number of different species of particles N_s. Since that the gravitational coupling is universal, these particles are equally coupled with the quanta of the gravitational field. In this kind of universe, a new process of scattering must be taken into account. In particular, to recover the collapsing model of the previous section, I want to foucs on one specie of baryon, that is the costituent of the matter collapsed. The scattering process will be:

$$g + b \to g + b \tag{2.50}$$

Thus, it is necessary to add to the formula (2.43) another term, making the same considerations done before, with the only difference that the number of scattering couples will be $N_g N_b$, so that the new term can be written as:

$$-\frac{N_g N_b}{N_g^2} \frac{1}{l_p \sqrt{N_g}} = -\frac{N_b}{N_g} \frac{1}{l_p \sqrt{N_g}} \tag{2.51}$$

It describes the graviton depletion given by the scattering with baryons. The complete depletion formula will be:

$$\dot{N}_g \simeq -\frac{1}{l_p \sqrt{N_g}} \left(1 + \frac{N_b}{N_g}\right) \tag{2.52}$$

Note that the graviton depletion is more intense than the baryon depletion, but the energy flows have the same magnitude. For a generic number of species, the rate of evaporation will contain an extra term:

$$\Gamma = \frac{\hbar}{\sqrt{N_g} l_p} N_s \tag{2.53}$$

From the observation that $\Gamma < M$ to have a self-sustained black hole, the "strange" unexpected bound on the number of gravitons can be obtained:

$$N_g > N_s \tag{2.54}$$

Writing N_g as function of the wavelenght λ of gravitons, also a lower bound for the size of black hole can be obtained:

$$\lambda > \sqrt{N_s} l_p \tag{2.55}$$

If this relation is not respected, the black hole cannot be self-sustained. This means that the particles in the universe affect the size of the black holes! This result couldn't be derived in the classical picture. Taking the time derivative of the equation (2.48) and using (2.43), it can be written:

$$\frac{\dot{T}}{T} = \frac{N_s}{N} T \tag{2.56}$$

This equation shows that, as soon as the number of gravitons in the condensate depletes and becomes comparable to N_s, the quantity T cannot even approximately be interpreted as a temperature, since the rate of its change exceeds its own value. Let me note that at the quantum critical point small subsystems as small black hole are maximally entangled. I will show later in depth, for the quantum inflationary universe, the meaning of this sentence.
To conclude this section, I want to discuss briefly the concept of entropy of a black hole, that arises in the QFT in curved space-time, introduced by Hawking. This semiclassical quantity must be derived in some way by the occupation number of the graviton' state. In this full quantum approch, the notion of entropy can emerge as the number of "dispositions" in which these N_g gravitons can exist. Without enter in the details, the final result is that:

$$S \sim N_g \tag{2.57}$$

The equality becomes exact in the limit $N \to \infty$, for which the pure quantum interpretation matches the semiclassical theory. For finite N_g, small corrections due to the interaction term $\sim \frac{1}{N_g}$, must be considered in the formula.

2.4 Black hole as Bose-Einstein condensate

For ordinary objects, with size much larger than the de- Broglie wave-lengths of the constituents, the coupling α_{ij} between a pair of constituents i and j, strongly depends on the relative positions of the constituents and cannot be defined universally. In contrast, an universal coupling can be defined in the systems in which every element "talks" with everyone with an equal strength. Such is the fundamental property of Bose-Einstein condensates, where all the constituents are in a common quantum state. In particular, the black holes represent such kind of system because, as seen before, the collective parameter C can be defined simply as a product of N_g and α. This parameter plays a central role since that, it determines how close the system is from quantum criticality. For ordinary objects the quantum coupling cannot be defined in this way. For generic Bose-Einstein condesates it can be defiied, but the occupation number is far from criticality, so that $C \ll 1$.
Now, I want to derive again the evaporation law of a black hole, studying this

latter as a Bose-Einstein condensate with the condition of a maximally packed system $C = \alpha N_g = 1$ and using the analysis of the Bogoliubov transformations. The conditions for the black hole are equivalent to have a BEC's at the critical point of a phase transition. As result black holes can never be treated classically. Consider that the system is inside a finite region of size L. In the formalism of the second quantization the hamiltonian is:

$$H = \sum_k k^2 \widehat{g}_k^+ \widehat{g}_k - \frac{1}{4}\alpha \sum_k \widehat{g}_{k+p}^+ \widehat{g}_{k'-p}^+ \widehat{g}_k \widehat{g}_k \tag{2.58}$$

The first sum is the kinetical energy of the system, whereas the second term represents the interaction vertex between gravitons: two gravitons of wave number k and k' scatter producing two new gravitons of wave number $k + p$ and $k' - p$. Starting from (2.57), I want to calculate the spectrum of low lying excitations of the uniform BEC, in such a way to obtain the graviton's energy necessary to escape from it. Since that the temperature of the condensate is null, it can be assumed that most of the particles occupy the $\omega_{k=0}$ energy level, with some small quantum fluctuation around this state. In first approximation, it can be used the Bogoliubov replacement:

$$\widehat{g}_0^+ = \widehat{g}_0 = \sqrt{N_0} \sim \sqrt{N_g} \tag{2.59}$$

where N_0 is the graviton occupation number in the fundamental state. In order to diagonalize the hamiltonian (2.57), I perform a Bogoliubov transformation where the coefficients of the transformations will be:

$$u_k, v_k = \pm\frac{1}{2}\left(\frac{k^2 - \frac{\alpha N_g}{2}}{\epsilon(k)} \pm 1\right) \tag{2.60}$$

The new hamiltonian refers to "new" particles states. With the new basis, the energy spectrum can be written in a simple form:

$$\epsilon(k) = \sqrt{k^2(k^2 - \alpha N_g)} \tag{2.61}$$

To study the quantum jump between the first two levels, it must be noted that if $N_g = \frac{1}{\alpha} \rightarrow \epsilon(k = 1) = 0$, i.e. the system must undergo a phase transition. Now, the number density of the depleted starting real particles is given by the modulo quadro of the component of the Bogoliubov transformations v_k. Moreover, the energy gap between the first two energy levels is:

$$\epsilon_1 = \frac{\hbar}{l_p\sqrt{N_g}} \tag{2.62}$$

and just coincides with the characteristich energy seen in (2.43). Thus in this model, the resulting depletion formula, can be derived from the analysis of the Bogoliubov transformations. This remebers the approch to obtain the Hawking effect in the semiclassical theory.

Without enter in too much details, I want to conclude saying that, the analogy between a curved space-time and a BEC can be revealed starting from the mean-field description of the condensates, encoded in the Gross-Pitaevskii equation:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\triangle+V_{ext}(x)+U|\psi(x,t)|^2\right]\psi(x,t) \tag{2.63}$$

where $\psi(x,t)=\langle\psi(x,t)\rangle$ is a mean-field term, and represents the wave function of the condensate, whereas $V_{ext(x)}$ is any external potential that confines the systeme in a finite region and U is the two bodies potential.

2.5 Redshift effect

The theory of gravitation can be studied in different ways. The classical one is resolving the field equation (1.2) for some kind of fluid and analyzing the structure of the space-time generated from it. The second one, is to considere the same equation, but interpreting the source as generated by quantum fields. This is the semiclassical theory of gravity. The third way, is what has been done before, i.e. to build a quantum theory, where the space-time can be interpreted as an N-bound quantum state. These three approaches can be discriminated by use of $\hbar$ and N, as follows:

$$\begin{array}{c}\text{Classical:}\quad \hbar=0;\quad \frac{1}{N}=0\\ \text{Semiclassical:}\quad \hbar\neq0;\quad \frac{1}{N}=0\\ \text{Quantum:}\quad \hbar\neq0;\quad \frac{1}{N}\neq0\end{array}$$

In some way, the first and the second approach, must be recovered from the full quantum theory in the limits $\hbar\to0$ and $N\to\infty$. In this section I want to study a particular phenomenon of the Einstein theory, i.e. the redshift effect in the Schwarzchild space-time, with the classical approach. Then, I want to study the same effect in the quantum framework and recover the classical result.

2.5.1 Classical redshift

The redshift is a physical phenomenon for which the frequency of a free, undisturbed photon becomes smaller or larger during time, when it moves in a non trivial space-time. Indeed, this kind of effect couldn't happen in a Minkowsky space-time. The redshift is expected by the General Theory of Relativity and it was verified the first time with the Pound-Rebka experiment. Roughly speaking, if a photon moves away from a region where a gravitational field is strong, it loses its kinetical energy, that results in a decrease of its frequency. The redshift so defined is called gravitational, to distinguish it from the redshift of the Special Theory of Relativity, that arises from very different assumptions. The Pound-Rebka experiment is described theorically in the framework of the Schwarzschild space-time, that is my intent to explain. To show quantitative the gravitational redshift, I want to define the so called killing energy:

$$E=-K_\mu u_\mu \tag{2.64}$$

where u_μ is the 4-velocity of the photon and $K_\mu=\partial_t$ is a killing time vector field of the Schwarzchild space-time. It is a vector field that satisfies

$$\nabla_{K_\mu}g_{\mu\nu}=0$$

For construction, the derivative of E respect to a parameter λ that describes the photon's traiectory, is null. Therefore E is conserved along the photon's trajectory. Now, to express mathematically the redshift, I want to calculate the value of the ratio between the energy of the emitted photon, from a point of the space, and the energy of the same photon measured by an observer who meets it, in an another point of the space, closer or farther from the gravitational source. Considering the normalized killing vector $\frac{K_\mu}{\sqrt{K_\mu K^\mu}}$, it can be deduced that:

$$\frac{E_o}{E_e} = \frac{\sqrt{K_\mu K^\mu}|_{x=x_o}}{\sqrt{K_\mu K^\mu}|_{x=x_e}} = \sqrt{\frac{g_{tt}|_o}{g_{tt}|_e}} \tag{2.65}$$

Therefore:

$$\frac{E_o}{E_e} = \sqrt{\frac{1-\frac{R_S}{r_o}}{1-\frac{R_S}{r_e}}} \tag{2.66}$$

2.5.2 Quantum redshift

Now, I want to analyze the same process making use of the corpuscolar interpretation of the gravitational system. The classical and the quantum language must coincide since that the quantity $G_N = \frac{l_p^2}{\hbar}$ is non-zero in the limit $\hbar \to 0$. In some way, the full quantum gravity can explain this classical effect of the Einstein theory, without using the notion of the metric tensor field, but only by use of the microscopic interactions, between the gravitons of the background and the photon. In particular, the emergence of the classical background is just the effect of a coherent quantum state of a large number of gravitons.
Again, I want to see what happens to a free falling photon probe. The solution of the equation of motion (1.4) will be a geodesic, that from a quantum point of view, is the consequence of the coupling between the energy-impulse tensor $\tau_{\mu\nu}$ of the probe, with the energy-impulse tensor associeted to all the space-time $T_{\mu\nu}$. At small distances, the process that affects the traiectory of the photon particle is due by an exchange of a graviton between $\tau_{\mu\nu}$ and $T_{\mu\nu}$. Thus, it is necessary to calculate the amplitude of this process, by the well know formula of QFT:

$$s_1 = \frac{l_p^2}{\hbar} \int dy \int dx \tau_{\mu\nu} \Delta_{\mu\nu,\alpha\beta}(x,y) T_{\mu\nu} \tag{2.67}$$

where s_1 is the first dominant term of this kind of process, whereas $\Delta_{\mu\nu}$ is the graviton's propagator. To build this object there is necessity to quantize the first term of the weak field expansion of the metric tensor field, named $h_{\mu\nu}$. In the next chapter I will analyze in depth this topic. For the moment I want to exprime this amplitude term in the operatorial formalism, as an ordined chronological product of the quantized perturbation operator field with itself:

$$\Delta_{\mu\nu,\alpha\beta} =< 0 \mid T\widehat{h}_{\mu\nu}(x)\widehat{h}_{\alpha\beta}(y) \mid 0 > \tag{2.68}$$

Remember that the chronological product serves to maintain a causal relationship between events. In QFT one would to study the different processes, among the quanta considered, and sum all the probabilities associated. For the above process, one could consider also the loop diagrams and other more complex processes. For example, one could calculate the second order term of the amplitude. This can be obtained considering the dependence of $T_{\mu\nu}$ on the perturbation operator field $h_{\mu\nu}$:

$$s_2 = \frac{l_p^2}{\hbar} \int \int dx dy \tau_{\mu\nu} \Delta_{\mu\nu,\alpha\beta}(x,y) T_{\mu\nu}(\widehat{h}_{\mu\nu}) \tag{2.69}$$

and so on. To recover the redshift effect, the transition amplitude had to be calculated:

$$\langle f|s_n|i\rangle \tag{2.70}$$

for the final $\langle f|$ and initial $|i\rangle$ states associated to the free photon. In particular, the classical and quantum description of the lost energy of the probe light-quanta must be the same for very large N. If N is small, purely quantum effects come into account and modify in some way the classical redishift result.

2.6 Break times

It is important to say that the quantum model for a finite N graviton's state matches with the classical, but after sometimes, it necessary departes from it. Indeed, for finite N the bra-operator-ket representation of the mean field (2.23) becomes wrong after sometimes, because the state $|g\rangle$ is not more coherent. Then, I will show that this breaking time will be important also for the quantum de Sitter space, for wich the quantum correction on the evaporation law, breaks the bra-operator-ket representation of the mean field. For black holes an interior and an exterior region can be distinguisched after the half-life time. But actually, since that the $\frac{1}{N}$ quantum corrections become important, the interior of the Schwarschild radius changes his description.
In particular, there are two kind of corrections: the classical and the quantum one. The first are due from the non linearity of the Einstein equation, therefore they are quantified by the collective parameter C. This break time can be more large if, the gravitational field equation can be solved more accurately. Instead the second break time depends on the graviton's occupation number and it is intrinsic of the quantum description.

$$\begin{array}{lll} \sim \alpha N; & \text{classical} & \text{correction} \\ \sim \frac{1}{N}; & \text{quantum} & \text{correction} \end{array}$$

The first correction there is also in the limit:

$$\hbar \to 0 \longleftrightarrow \alpha \to 0 \tag{2.71}$$

therefore it is not null also in the absence of the quantum interactions between particles. Indeed, also if $N \to \infty$ the collective parameter $C \sim 1$. Instead, the quantum break time is the time after that the mean field representation falls down. The corrections that arises, lead the gravitons to have wavelenghts getting closer to the planck lenght, so that also the quantum description is not more possible, because the system enters in a non perturbative regime.

Without enter in details, I want to show another kind of break time, that describes the passage between a non entangled black hole state to a one-particle entangled black hole state. To this sake, it must be said that the initial wavefunction of the black hole, can be written as a tensor product of single quanta wave functions:

$$\psi = \prod_i \phi_i \qquad i = 1, ..N_g \tag{2.72}$$

where the single ones wavefunctions ϕ_i solve the Schrodinger equation with the collective binding potential energy. After sometimes ψ cannot longer be written as the product of a single wavefunction particle and a state describing all the other particles. The minimal time scale for the generation of this effect is:

$$t = \frac{\ln N}{\Gamma} = l_p \sqrt{N} \ln N = R_S \ln N \tag{2.73}$$

Moreover, since the black hole depletion rate is sensitive to the number of extra particles species, the time to generate entaglement must be smaller:

$$t = \frac{l_p \sqrt{N}}{N_{species}} \ln N = \frac{R_S}{N_{species}} \ln N \tag{2.74}$$

Note that, when this effect becomes important, the background associated to the graviton's condensate is not more well defined.

Capitolo 3

Quantum inflation

The main topic of this thesis is about the consequences that can be derived quantizing a cosmological space, filled by dark matter. There are different kind of inflationary models, but in this chapter I will consider the inflaton field model. I have just shown something about what happens considering the gravitational field or equivalently the potential energy, as a collection of gravitons. The wave functions of these quanta describe the quantum features of the space-time. They are characterized by parameters like the frequency or the wavelenght, that must be related with the cosmological physical quantities, like for example the horizon radius R_H. Starting from the results obtained from the black hole analysis, it can be deduced that the wavelenght of these gravitons, are of the same size of the Hubble horizon:

$$\lambda_{quanta} \approx R_H \tag{3.1}$$

The novelty respect to the Schwarzschild model is that the energy-impulse tensor is not null in the universe, since that it is filled by the inflaton field. Thus, the inflationary universe must be thought as a reservoir of two kind of particles: gravitons and inflatons. To pass from a classical to a quantum description of inflation, a relation that relates the classical and quantum quantities must be found. It can be assumed the form:

$$\frac{N_g}{N_\phi} = f(\epsilon) \tag{3.2}$$

where, besides the just seen N_g, there is the fundamental parameter N_ϕ that describes the number of inflatons, whereas $f(\epsilon)$ is a functon of the slow roll parameter. Note that, the existence of gravitons and inflatons is very difficult to prove by direct relevations but, by theoretical considerations, one could prove more accurately equations, that match with cosmological relevations. The main result I will show is that, as soon as the quantum framework is taken into account, the well know de Sitter space-time must be substituted by a quasi-de Sitter space-time, that in no limits can be reduced to the previous one, differently from the standard inflaton model. This is due by the evolution of the systems of inflatons and gravitons, that in some way affects the Hubble function. Let me begin with the study of the solutions of the gravitational field equation (1.2), in absence of any source. To introduce formally the graviton states, I had to quantize these solutions, if they could be written in a explicit form. Note that, in the black hole analysis, I did a very similar thing, but considering the more simple Newton field equation. The next step will be to quantize the field solutions in presence of a source, described by dark matter that fills homogeneously the space or equivalently by the cosmological constant Λ. The main problem that will arise is that the Einstein equation in the vacuum is a non linear tensorial equation, for which doesen't exist a true free solution, that can be written with an explicit mathematical expression. From a quantum point of view this means that the particles associated to the gravitational field are unavoidably auto-interacting. Therefore, before starting with the Einstein equation, I want to present a very similar non linear scalar model and see how quantize it. The procedure for the gravittational theory will be the same, but applied to more degrees of freedom.

3.1 Non linear scalar theory

The so called toy scalar model is described by a lagrangian for a single scalar field, with an auto-interagent term:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\omega^2\phi^2 - \frac{1}{4}\lambda^2\phi^4 \tag{3.3}$$

where ω replaces the mass and λ is the coupling of the field with itself. The equation of motion can be obtained by the Euler-Lagrange equations:

$$\frac{\delta\mathcal{L}}{\delta\phi} = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial^\mu\phi} \tag{3.4}$$

so that the equation becomes:

$$\Box\phi - \omega^2\phi - \lambda^2\phi^3 = 0 \tag{3.5}$$

If the non linear term could be excluded, the solution of the equation can be written as an harmonic function $\phi(t,x) = f(x)A\cos(\omega t)$. This approximation can be made for small values of the coupling constant λ. But, also for very small λ, exists a time after that the non linear term cannot more be excluded:

$$t \equiv \frac{\omega}{\lambda^2 A^2} \tag{3.6}$$

This break time is a consequence of the fact that the solution is only an approximation of the equation (3.5). Now, the quantization of the approximate field can be made. Then the interaction term can be reintroduced, calculating its influence on the amplitude of the microscopic interactions processes. The general approximed quantum field is:

$$\widehat{\phi}(x,t) = \int \frac{d^3k}{2\sqrt{\omega_k}}(\widehat{a}_k e^{ikx} + \widehat{a}_k{}^+ e^{-ikx}) \tag{3.7}$$

where $\widehat{a}_k$ and $\widehat{a}_k^+$ are respectively the destruction and construction operators of the field particles with vector number k, of the Fock space of the field. As just seen in the collapsing gravitational model, the correspondence between the classical mean field and the quantum operator field can be obtained by:

$$\phi = \langle N|\widehat{\phi}|N\rangle \tag{3.8}$$

where $|N\rangle$ is a coherent state with mean occupation number N. It can be obtained by a superposition of the eigenstates $|n\rangle$ of the number operator $\widehat{N} = \sum_k \widehat{a}_k^+ \widehat{a}_k$:

$$|N\rangle = e^{-\frac{N}{2}} \sum_{n=0} \frac{N^{\frac{n}{2}}}{\sqrt{n!}} |n\rangle \tag{3.9}$$

$$\langle N\rangle = \sqrt{\langle N|\widehat{N}\,|N\rangle} = N \tag{3.10}$$

$$\langle N^2\rangle = \sqrt{\langle N|\,|\widehat{N}^2\,|N\rangle\, N} = N^2 + N \tag{3.11}$$

for which $\Delta N = \sqrt{\langle N^2\rangle - \langle N\rangle^2} = \sqrt{N}$, that is a feature of the Poisson distribution. Indeed the probability of measuring n gravitons in the state $|N\rangle$ is:

$$P_N = |\langle n|N\rangle|^2 = e^{-N} \frac{|N|^n}{n!} \tag{3.12}$$

Plugging the relations (3.7-3.9) in (3.8), it can be obtained:

$$\phi = \langle N| \int \frac{d^3k}{2\sqrt{\omega_k}} (\widehat{a}_k e^{ikx} + \widehat{a}_k^+ e^{-ikx})\,|N\rangle = \sqrt{\frac{\hbar N}{\omega_k V}} \cos(\omega_k t) \tag{3.13}$$

where V is the volume of the region where the system is confined. In this way the amplitude A of the mean field ϕ, can be identified with $\sqrt{\frac{\hbar N}{\omega_k V}}$ and the number density with:

$$n \equiv \frac{N}{V} = \frac{\omega_k A^2}{\hbar} \tag{3.14}$$

The representation (3.8) cannot be valid always, because the coherent state $|N\rangle$ loses its coherence after some time. This latter can be interpreted as the time after that a fraction of order one of the particles experiences scattering. The process taken into account is: $\phi + \phi \to \phi + \phi$, with rate $\Gamma \sim \frac{\omega^2}{\alpha^2}$. Thus this time will be:

$$t_{quantum} \sim N N^{-2} \Gamma = \frac{1}{\omega \alpha^2 N} \tag{3.15}$$

where the subscrit "quantum" is used to distinguish this time from the classical break time, that can be obtained comparing this equation with (3.6):

$$t_{quantum} \sim \frac{t_{classical}}{\alpha} \tag{3.16}$$

This relation shows that for $\alpha \to 0 \Rightarrow t_q \to \infty$, the full quantum behaviour will never appear. To conclude, it is interesting to note that adding another particle in the system, the modified decay rate doesen't change the value of the quantum break time, because the coherence of the state $|N\rangle$ is not affected by it.

3.2 Graviton quantum field

I will show that the gravitational counterpart of the cosmological classical background is a degenerate state of off-shell longitudinal gravitons, with frequencies of order of the inverse of the Hubble horizon. To this end, I must find an approximate solution of the Einstein's equation, applying the results of the toy scalar model to a reduced form of the metric tensor field. The first step is to linearize the field equation (1.2). For this task it is necessary to consider the linearized metric field $h_{\mu\nu}$ on top of the Minkowski metric $\eta_{\mu\nu}$, in such a way to exclude higher order terms. These latter are negligible in the weak field approximation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \tag{3.17}$$

where $h_{\mu\nu}$ is a small pertubation of the metric. This is the physical object that had to be analyzed and no more $g_{\mu\nu}$. Using this strategy, the small perturbation must be quantized, rather thant the total background. Remark that the decomposition (3.17) is not unique, because one can applies on it a diffeomorphism that preserves the condition $|h_{\mu\nu}| \ll 1$. Now, the resulting linearized field equation, in absence of a source, will be given by the application of some operators on the first order perturbation field.

$$G_{\mu\nu} \simeq \Box h_{\mu\nu} - \eta_{\mu\nu}\Box h - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\alpha h_{\alpha\mu} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} = 0 \tag{3.18}$$

This expression can be obtained inserting the (3.17) in the Einstein tensor $G_{\mu\nu}$ and excluding the higher order terms respect to $h_{\mu\nu}$. Another way to obtain the (3.18) is analyzing the Einstein-Hilbert action, that "generates" the Einstein's field equation:

$$S_{EH} = -\frac{1}{16\pi G_N}\int d^4x\sqrt{-g}R \tag{3.19}$$

where g is the determinant of the metric. Inserting the (3.17) in the formula (1.8) for R and excluding higher order terms in the expansion (3.17), the action (3.19) can be well approximated by:

$$S_{EH} \simeq -\frac{1}{32\pi G_N}\int d^4x\left(\frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\mu - \partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \partial_\mu h\partial_\rho h^{\mu\rho} - \frac{1}{2}\partial_\mu h\partial^\mu h\right) \tag{3.20}$$

where the expression in the integral is the so called Fierz-Pauli lagrangian. In a very similar way with the vector field for the photon, one can choose a gauge constraint on the small perturbation function, without affecting the physical dynamics. In particular, the so called De Donder gauge can be choose:

$$\partial^\mu(h_{\mu\nu} - \partial_\nu h) = 0 \tag{3.21}$$

where h is the trace of $h_{\mu\nu}$. The equation (3.18) simplifies as:

$$\Box h_{\mu\nu} = 0 \tag{3.22}$$

This is a free equation very similar to the free photon equation field. Since that this equation can be solved exactly, its quantized general solution, can be interpreted as a superposition of gravitons states. The more simple solutions are plane travelling waves, representing free propagating gravitons with four-momenta p^μ and a well defined dispersion relation $p^\mu p_\mu = 0$:

$$h_{\mu\nu} \sim \epsilon_{\mu\nu}(\overline{p}) e^{\frac{i}{\hbar}k^\mu x_\mu} \tag{3.23}$$

where $\epsilon_{\mu\nu}(\overline{p})$ is the polarization tensor. Moreover, a coordinate transformation can be made $x^\mu \to x^\mu + \lambda^\mu$, with $\Box\lambda^\mu = 0$, in such a way that:

$$h^{0\mu} = 0 \tag{3.24}$$

$$h^i_i = 0 \tag{3.25}$$

$$\partial^i h_{ij} = 0 \tag{3.26}$$

Thus, some restrictions can be got also on the polarization tensor: $\epsilon_{0\mu}$,$\epsilon^i_i = 0$ and $p^i\epsilon_{ij} = 0$. Now, to make contact with the quantum theory, $h_{\mu\nu}$ had to be promote to a quantum field:

$$h_{\mu\nu} \to \frac{l_p}{\sqrt{\hbar}}\widehat{h}_{\mu\nu} \tag{3.27}$$

The operatorial field expansion can be written as:

$$\widehat{h}_{\mu\nu} = \sum_{k=\pm 1} \int d\mu(\overline{p}) \left[\widehat{a}(\overline{p}, k)\epsilon_{\mu\nu}(\overline{p}, k)e^{\frac{i}{\hbar}p^\mu x_\mu} + h.c.\right] \tag{3.28}$$

The components $\widehat{a}(\overline{p}, k)$ and $\widehat{a}(\overline{p}, k)^+$ are respectively the destruction and construction operators for the gravitons states. They had to satisfied the following canonical commutations relations:

$$\left[\widehat{a}(\overline{p}, k), \widehat{a}(\overline{p}', k')^+\right] = \delta^3(\overline{p} - \overline{p}')\delta_{kk'} \tag{3.29}$$

The graviton Feynamn propagator in the momentum space is:

$$iD_{\alpha\beta\mu\nu}(p) = \frac{iP_{\alpha\beta\mu\nu}}{p^2 - i\epsilon} = \frac{1}{2}\frac{i(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\beta}\eta_{\mu\nu})}{p^2 - i\epsilon} \tag{3.30}$$

where $P(\alpha\beta\mu\nu)$ is the projector operator. The propagator constitutes the solution of the equation (3.18), with a delta function on the right-hand side as impulsive source. The propagator is a very fundamental element of the QFT because allows to build the interaction diagrams.
Now, I want to study the linearized Einstein action in presence of the cosmlogical constant as source:

$$\mathcal{L}_E = \frac{1}{16\pi}\left(\frac{1}{2}\overline{h}^{\mu\nu}\epsilon^{\alpha\beta}_{\mu\nu}\overline{h}_{\alpha\beta} + \frac{2}{\sqrt{G_N}}h\Lambda\right) \tag{3.31}$$

where $\epsilon^{\alpha\beta}_{\mu\nu}\overline{h}^{\mu\nu}$ can be written as the right-hand side in the formula (3.18), where $h^{\mu\nu}$ is substituted with $\overline{h}^{\mu\nu}$, that represents a small perturbation on top of the Minkowsky metric, caused by the cosmological constant. As seen before, using the De Donder gauge on $\overline{h}^{\mu\nu}$, the equation of motion with the cosmological source can be written as:

$$\Box(\overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h}) = -2\Lambda\eta_{\mu\nu} \tag{3.32}$$

The solutions are expressed by the following components:

$$\overline{h}_{00} = \Lambda t^2 \tag{3.33}$$

$$\overline{h}_{0i} = -\frac{2}{3} t x_i \tag{3.34}$$

$$\overline{h}_{ij} = -\Lambda t^2 \delta_{ij} - \frac{\Lambda}{3} x_i x_j \tag{3.35}$$

Thus the perturbation metric tensor assumes the compact form:

$$\overline{h}_{\mu\nu} = \begin{pmatrix} \Lambda t^2 & -\frac{2}{3}tx & -\frac{2}{3}ty & -\frac{2}{3}tz \\ -\frac{2}{3}tx & -\Lambda t^2 - \frac{\Lambda}{3}x^2 & -\frac{\Lambda}{3}xy & \frac{\Lambda}{3}xz \\ -\frac{2}{3}ty & \frac{\Lambda}{3}yx & -\Lambda t^2 - \frac{\Lambda}{3}y^2 & \frac{\Lambda}{3}yz \\ -\frac{2}{3}tz & -\frac{\Lambda}{3}zx & \frac{\Lambda}{3}zy & -\Lambda t^2 - \frac{\Lambda}{3}z^2 \end{pmatrix} \tag{3.36}$$

This matrix can be rewritten in a more simple form applying a diffeomorphism:

$$\overline{h}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(1+\frac{\Lambda t^2}{3}) & -\frac{\Lambda}{3}xy & -\frac{\Lambda}{3}xz \\ 0 & -\frac{\Lambda}{3}yx & -(1+\frac{\Lambda t^2}{3}) & -\frac{\Lambda}{3}yz \\ 0 & -\frac{\Lambda}{3}zx & -\frac{\Lambda}{3}zy & -(1+\frac{\Lambda t^2}{3}) \end{pmatrix} \tag{3.37}$$

The (3.37) is an approximation of the de Sitter metric in closed FRW slicing (1.10), valid for short time-scales and distances:

$$t \ll \frac{1}{\sqrt{\Lambda}} \tag{3.38}$$

$$r \ll \frac{1}{\sqrt{\Lambda}} \tag{3.39}$$

These conditions show clearly that, for an high value of the cosmological constant, the effects of non linearity begin to be important very early. In analogy with the scalar toy model, I want to investigate the three fundamental parameters of the field λ, ω, A. For short times and distances they must be functions of the two constant of physics G_N and Λ, so that the three parameters are not independent. From the last considerations it can be said that the classical break time is $t_{cl} = \frac{1}{\Lambda}$. Thus:

$$\omega_{grav} = \frac{1}{t_{cl}} = \sqrt{\Lambda}; \quad A_{grav} = \frac{1}{\sqrt{G_N}}; \quad \lambda^2_{grav} = \Lambda G_N \tag{3.40}$$

A very interesting matter is that the above Λ-theory is for short times completelly equivalent and physcally indistinguishable for an observer, from a theory without the cosmological constant, but with a massive graviton with mass m. This equivalence is possible by the following association:

$$\Lambda \to m^2 \tag{3.41}$$

This is a map from the interacting off-shell massless gravitons onto free massive ones. The graviton mass is merely a computational device which replaces the effects of non linearities. Thus, the de Sitter metric in closed FRW slicing, can be also described by an oscillating massive spin 2 field, instead of its massless counterpart. The field equation for the massive field $h_{\mu\nu}$ will become:

$$\epsilon_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} + m^2(h_{\mu\nu} - \eta_{\mu\nu}) = 0 \tag{3.42}$$

To proceed, I want to decompose the metric perturbation field, following the irriducible representations of the Poincarè group:

$$h_{\mu\nu} = \widetilde{h}_{\mu\nu} + \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{1}{6}\eta_{\mu\nu}\chi - \frac{1}{3}\frac{\partial_\mu \partial_\nu}{m^2}\chi \tag{3.43}$$

where $\widetilde{h}_{\mu\nu}$, A_μ, χ are respectively the helicity 2, helicity 1 and helicity 0 components of the fields. The lagrangian (3.31) can be rewritten in the massive theory as follows:

$$L_{FP} = \frac{1}{16\pi}\left(\frac{1}{2}\widetilde{h}_{\mu\nu}\epsilon_{\mu\nu}^{\alpha\beta}\widetilde{h}_{\mu\nu} + \frac{m^2}{2}(\widetilde{h}_{\mu\nu}\widetilde{h}^{\mu\nu} - \widetilde{h}^2)\right) \tag{3.44}$$

where only the helicity-2 component of $h_{\mu\nu}$ is considered, without to lose the fundamental physics behind the problem. Using the constrain seen before, the equation of motion can be written as:

$$(\Box^2 + m^2)(\widetilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widetilde{h}) = 0 \tag{3.45}$$

One possible solution is expressed by the components:

$$\widetilde{h}_{00} = -\frac{2\Lambda}{m^2}\cos(mt) \tag{3.46}$$

$$\widetilde{h}_{0i} = -\frac{2\Lambda}{3m}\sin(mt)x_i \tag{3.47}$$

$$\widetilde{h}_{ij} = \frac{2\Lambda}{m^2}\cos(mt)\delta_{ij} - \frac{\Lambda}{3}\cos(mt)x_i x_j \tag{3.48}$$

These solutions reproduce for $t \ll \frac{1}{m}$ the closed de Sitter solution of the field equation whe there is the cosmological constant. Now, the massless Einstein theory and the Fierz-Pauli massive theory are among them consistents, if the relatives classical break time coincide. This is possible if the mass of the graviton is set as:

$$m \approx \sqrt{\Lambda} \tag{3.49}$$

so that also in the massive theory the classical break time will be $\frac{1}{\sqrt{\Lambda}}$. Come back to the massless theory, the trace of the helicity-2 component $\overline{h_{\mu\nu}}$ will be:

$$\overline{h^s_{\mu\nu}} = -\sqrt{16\pi}\Phi\eta_{\mu\nu} \tag{3.50}$$

where now, Φ is the unique dynamical physical degree:

$$\Phi = \frac{m_p}{\sqrt{4\pi}}\cos(mt) \tag{3.51}$$

In this way the linearized de Sitter solution can be studied in terms of only one scalar degree of freedom and the procedure of the toy scalar model can be applyied.

Resuming, the fundamental ideas to formally study the graviton quantum field are based on the weak field limit (3.17), that allows to write an approximate solution of the field equation, and the (3.43), that allows to reduce the degrees of freedom of the field. Ideally, to describe the quantum nature of the inflationary space-time generated by the inflaton field or, as seen before, by Λ, the creation $\widehat{a}^{+}$ and annihilation $\widehat{a}$ operators that generate the multi-gravitons states in the Fock space must be find. Since that the theory is non linear, the quanta associated with a^{+} are not free. These quanta interact among them, so that the dispersion relation $E = E(p^{\mu})$ is slightly different from the standard relation, seen before, for the approximate field. Since that, it is impossible writing explicitly this kind of operators, $\widehat{a}^{+}_{free}$ and $\widehat{a}_{free}$ must be used. Indeed, they describe approximately free gravitons in the weak field limit:

$$\widehat{a}^{+} \simeq \widehat{a}^{+}_{free}; \qquad \widehat{a} \simeq \widehat{a}_{free} \tag{3.52}$$

The "corrections" on these operators can be interpreted as given by the interactions among quanta and they can be calculated analyzing the vertices of the theory.

3.3 Quasi de Sitter space-time

The eternal accelerated universe, described by the de Sitter metric, doesen't represent accurately the history at the early stage of the universe. One could assume this kind of space-time for finite time life and then consider the other type of fluid domination. Anyway, this fixed sequence of space-time, is a very simplified model and doesen't reproduce differnt cosmological features of the universe. Instead, the inflaton field model is more predictive, but doesen't take into account the fundamental quantum nature of the gravitational systems. The corpuscolar inflationary model, I want to analyze, proves the existence of a finite life-time quasi de Sitter space-time. This result is conceptually very interesting, because it is not a generalization of the classical theories, but a departure from them. The reason of this difference, can be found in the quantum corrections on the equation of motion for the Hubble function H and for the cosmic scale factor a.
The inflaton field model, seen in the first chapter, arises from the action:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - V(\phi)\right) \tag{3.53}$$

The integral is a combination of the metric tensor field and of the inflaton field hamiltonians. It can be written as:

$$H = H_{EH} + H_{\phi} \tag{3.54}$$

Quantizing the above action (3.53), means to introduce the gravitons collection and the quanta associated with the inflaton field, called inflatons. The fundamental difference with the quantum black hole is that, the behaviour of the gravitons is in some way affected by the inflatons. This other collection of particles, represents a Bose-Einstein condensate of off-shell inflatons degrees of freedom, with mass

screened by the gravitational binding energy. Thus, the global quantum system is a mixture of two Bose-Einstein gases. Both of them are at critical point of a phase transition.
Now, going to analyze the Friedmann equation with the scalar field, different situations can be distinguished, with different quantum interpretations. Thus, to estimate the number of inflatons and gravitons it is necessary to distinguish the different kind of regimes. I want to remember that analyzing the dynamical equations (1.41-1.42) for the fluid field in the inflaton model, the system exibits the inflation regime when:

$$\phi \gg \frac{\sqrt{\hbar}}{l_p}$$

If this condition is not satisfied the universe is matter dominated. In this latter case the system is oscillating around the minimum of the potential and the number of gravitons for Hubble volume, can be estimated through the formula:

$$N_g = \left(\frac{R_H}{l_p}\right)^2 \tag{3.55}$$

This is a consequence of the wavelenght of the gravitons $\sim R_H$. Note that, the notion of the graviton occupation number can be defined within the region of the curvature radius. The cupling constant is:

$$\alpha_g = \frac{1}{N_g} \tag{3.56}$$

Inverting the relation (3.55), the scaling relations for the for the Hubble radius and for the coupling constant can obtained:

$$R_H \sim l_p \sqrt{N_g}; \quad \alpha_g = \frac{1}{N_g} \tag{3.57}$$

In a generic D dimensions space-time, the above relations can be written as:

$$N = \frac{R^{D-2}}{l_p^{D-2}} \tag{3.58}$$

and:

$$\lambda = N^{\frac{1}{D-2}} l_p; \quad \alpha_g = \frac{1}{N} \tag{3.59}$$

During the reheating era, the scalar background can be described by the asymptotic degrees of freedom. These latter can be obtained by the usual creation operators $\widehat{a}_k^+$, which create a ϕ-quanta with the dispersion relation $\omega_k = \sqrt{k^2 + m_\phi^2}$. Thus, the description of these massive particles, could be made considering states built with these asymptotic creation operators. Note that, during the rehating era inflation undergoes damped oscillations. Therefore, in this regime, the scalar background is a standard BEC of ϕ-quanta with $k = 0$ and energy $\omega = m_\phi$.
Now, I want to consider the inflation regime. Likewise for the graviton's gas, the inflaton's system is characterized by a coupling constant, that describes the attraction strenght between quanta:

$$\alpha_\phi = \left(\frac{m_\phi}{l_p \hbar}\right)^2 \tag{3.60}$$

To estimate the number of inflatons, I had to analyze the features of the inflaton field and of the potential energy. Since the inflationary background is a homogeneous field configuration, it can be viewed as a condensate of finite number of ϕ-quanta in $k = 0$ state. Therefore, their wavefunctions are completely delocalized throughout the full universe. This means that the number of ϕ-quanta, within a given Hubble patch, is not fixed. Ignoring the interactions, one can shows that the number of quanta in a given region of the universe, is given by the Poisson distribution. The question is about what kind of condensate the inflationary stage correspond to. Let me assume, for the moment that during inflation the scalar background can still be thought as a collection of $k = 0$ and $\omega = m_\phi$ weakly interacting quanta. In order to make arise the horizon radius R_H, the number of ϕ-quanta within one Hubble patch, had to be:

$$N_\phi = \frac{m_p^2}{m_\phi H} \tag{3.61}$$

During reheating, the background undergoes oscillations and its coeherent state have costituents with frequencies m_ϕ, but when the state enters in the inflationary era, the scalar field undergoes a slow-roll, with rate much smaller than m_ϕ:

$$\dot{\phi} = -\frac{m_\phi^2}{H}\phi \tag{3.62}$$

For this reason, appears to be more natural to think the inflaton background as a condensate of off-shell ϕ-quanta in $k = 0$ state, with a mass replaced by an effective mass:

$$m_{eff} = \frac{m_\phi^2}{H} \tag{3.63}$$

Thus, the background quantum geometry in the inflationary phase, corresponds to a degenerate state of off-shell longitudinal gravitons with $\omega = H$ and $N_\phi = \frac{m_p^2}{H^2}$ for Hubble patch. Therefore, the interior of the horizon can be described by the following number eigenstate:

$$|N_g^{\omega=H}, N_\phi^{\omega=\frac{m_\phi^2}{H}}\rangle \tag{3.64}$$

that is an eigenstate of the graviton and of the inflaton number operators. It can be noted that, the constituents of inflation are not the same as the asymptotic degrees of freedom, seen in the reheating era. Thus, it is necessary to define a new set of construction operators, which create quanta with the mass gap (3.63). On the other hand, assuming the universality of the gravitational couplings, the interaction vertices can be considered very similar to the ones of the asymptotic theory.
For a homogeneus field, in a quadratic potential, the number density is:

$$n_\phi = \frac{m_\phi \phi^2}{\hbar^2} \tag{3.65}$$

Thus the number of inflatons in one Hubble volume R_H^3 will be:

$$N_\phi = \frac{\hbar H}{m_\phi}\frac{R_H^2}{l_p^2} \tag{3.66}$$

Going to calculate the ratio between N_g and N_ϕ it can be seen that:

$$\frac{N_g}{N_\phi} = \frac{m_\phi^2}{(\hbar H)^2} = \sqrt{\epsilon} \tag{3.67}$$

where ϵ is the slow-roll parameter for the quadratic potential. This is a very important relation because combines in a single formula, the two fundamental quantum parameters with the classical one. Analyzing the (3.67), it can be said that inflation drives when the inflaton quanta dominates on the graviton's number, so that $\epsilon \ll 1$. In the classical inflaton model, ϵ separates the two regime of inflation and at the same time, in the quantum framework, it shows what kind of particle is dominant. Using the relations (3.60) and (3.66), the collective parameter for the inflaton collection, that measures the global effect on the inflaton's gas, will be:

$$\alpha_\phi N_\phi = \sqrt{\epsilon} \tag{3.68}$$

This means that for the value $\epsilon = 1$, the system is between the two regime and it is on a verge of a phase transition. As soon as the deplation for inflatons becomes strong and $N = N_\phi$, inflation stops. Moreover, in matter dominated phase $\epsilon \gg 1$ the inflaton is overcritical, unstable and had to collapse. Note that, the energy density for gravitons and inflatons are respectively:

$$\rho \simeq \frac{m_p}{l_p^3 N_g}; \quad \rho_\phi \simeq N_\phi m H^3$$

so that:

$$\rho \simeq \rho_\phi \simeq \frac{3 m_p H^2}{8\pi l_p}$$

This latter equation shows that the energy densities of gravitons and inflatons remain equal. This fact reminds that the effective Friedmann equation is just a statement of the total energy conservation, altough in the slow-roll regime $N_\phi \gg N_g$. The history of the gravitons can be resumed by the fundamental equation:

$$\frac{\dot{N}_g}{N_g} = H\epsilon - \frac{H}{\sqrt{\epsilon} N_g} \tag{3.69}$$

This is the equation of motion for the graviton's system. The solution for the occupation number of gravitons, can be explicitly written as:

$$N_g = exp\left\{\int dt \left(H\epsilon - \frac{H}{\sqrt{\epsilon} N_g}\right)\right\} \tag{3.70}$$

The evolution of gravitons is given by a classical and a quantum term, that are respectively, the first and the second term in the right-hand side of (3.69). Excluding the second term, in the limit for $\epsilon \to 0$, the de Sitter metric can be recovered. Note that the classical evolution can be also written as depending on the energy density:

$$\frac{\dot{N}_g}{N_g} \simeq \epsilon H \simeq \frac{\dot{\rho}}{\rho} \tag{3.71}$$

Now, if the quantum term is considered, there is no limit in which one can recover the eternal de Sitter metric, indeed for $\epsilon \to 0$, the depletion equation becomes:

$$\frac{\dot{N}_g}{N_g} \simeq -\frac{H}{\sqrt{\epsilon} N_g} \tag{3.72}$$

and using the relation $\epsilon = -\frac{\dot{H}}{H^2}$, it can also be written as:

$$\dot{N}_g \simeq -\frac{H^2}{\sqrt{-\dot{H}}} \tag{3.73}$$

In particular, a minimum "distance" between the de Sitter and quasi de Sitter universe exists. Indeed, rewriting the (3.69) as:

$$\frac{\dot{N}_g}{N_g} = H\epsilon \left(1 - \frac{1}{\epsilon^{\frac{3}{2}} N_g}\right) \tag{3.74}$$

it can be noted that, the right-hand side is not negative if:

$$\epsilon* \geq \frac{1}{N_g^{\frac{3}{2}}} \sim \left(\frac{l_p}{R_H}\right)^3 \tag{3.75}$$

The evolution equation (3.69) can also be interpreted as an equation for the Hubble function and equivalently for the Hubble radius:

$$R_H = \frac{N}{\dot{N}}\epsilon - \frac{1}{\sqrt{\epsilon} N} \tag{3.76}$$

in such a way that, also the fundamental cosmic scale factor is affected by the quantum dynamics. Moreover, rewriting the slow-roll parameter as in (3.67) the deplation equation will be:

$$\frac{\dot{N}_g}{N_g} = H\left(\frac{N_g^2}{N_\phi^2} - \frac{N_\phi}{N_g^2}\right) \tag{3.77}$$

Instead, the evolution of the inflaton occupation number N_ϕ for a generic potential, is determined by the equation:

$$\frac{\dot{N}_\phi}{N_\phi} = H\eta - \frac{H}{N_g} \tag{3.78}$$

where η is the second slow-roll parameter in (1.47), that measures if the inflation is enough long. Also in this case, the classical and the quantum evolution terms are given by the first and the second term in the right-hand side of the equation. The particles of the inflaton field for Hubble volume increases, but slower than gravitons, so that the graviton occupation number catches up. It can be noted that the quantum effects, that arise in this framework, are encoded by the terms of the kind $\frac{1}{N}$ in the equations (3.69-77), so that the depletion of gravitons and inflatons is a cumulative effect. When the quantum terms are negligible, the classical evoultion of the cosmic scale factor can be recovered, due by the Friedmann equations in the inflaton field model. Indeed the classical terms are not null, also in the limit $\hbar \to 0$. Analyzing the depletion equations (3.69-77), it can be said that, N_g and N_ϕ can be described as functions of the classical quantities ϕ, ϵ and η, as long as, the quantum terms in the evolution equations are sub-dominant. Over the time scale of the classical change $\frac{\delta H}{H} \sim 1$, the quantum contribution can be negligible, i.e. the variation of N due to the quantum term, is less than the classical variation. If this condition is not satisfied, the full quantum regime must be taken into account. Now, I want to prove how the quantum corrections on the depletions equations can be obtained. As seen in the quantum description of black holes, the main physical novelty in the quantum framework, are the quantum interaction vertices. In the cosmological inflationary space-time model there are two particles systems, with gravitons and inflatons, thus there are three kind of scattering processes:

$$g + g \to g + g; \quad g + \phi \to g + \phi; \quad g + g \to \phi + \phi \tag{3.79}$$

Since that, the system starts at the beginning of the inflationary phase with $N_\phi \gg N_g$, the depletion is mainly given by the process: $g + \phi \to g + \phi$. The rate of this process can be calculated by the product of the characteristic energy of the process, the strenght and the number of combinations of the possible scattering pairs:

$$\Gamma_{\phi g} = H\alpha^2 N_g N_\phi = H\frac{N_\phi}{N_g} = \frac{1}{l_p\sqrt{N_g}}\frac{N_\phi}{N_g} \tag{3.80}$$

where in the last two equalities, I used the criticality condition for the graviton's gas: $\alpha_g N_g = 1$. The last term is just the depletion term written in the depletion equation (3.69). Instead, the rate of the gravitons depletion, due to the graviton-graviton scattering $g + g \to g + g$ is:

$$\Gamma_{gg} = H = \frac{1}{l_p\sqrt{N_g}} \tag{3.81}$$

that is like the (3.74), but with the term $\frac{N_\phi}{N_g}$ suppressed. Note that, it is the same seen in the black hole evaporation law. Another interaction channel is $g+g \to \phi+\phi$, for which the rate is also of order H. For these two latter processes, I should expect to produce approximately few ϕ-quanta with energy H.
In the standard analysis of inflation, there is a single classical clock set by a slow-rolling scalar field, which leads to a classical decrease of H. This clock countinues to be present also in the quantum picture and corresponds to the classical evolution term, that increases the occupation numbers N_g and N_ϕ. Now, a second clock can be introduced, that borns from quantum considerations. This clock is different from the other one because it decreases both N_g and N_ϕ. The classical clock tries to fill up the reservoir, whereas the quantum clock depletes it. Whenever it can be tried to make the classical clock run slower than a certain critical rate, the quantum clock speeds up and the reservoir is emptied before that the classical clock has any chance to refills it! For this reason the system becomes inconsistent.

$$\dot{H} \quad \text{c. clock}; \quad \dot{N} \quad \text{q. clock}$$

The first one evolves as consequence of the General Theory of Relativity, applied on the inflationary universe. $\dot{H} = 0$ is inconsistent if N is finite and $\dot{N} = 0$.

The variation of the gravitons and inflatons in a range of time $[t_{in}, t_{fin}]$, given by the quantum term is:

$$\Delta N_g = \Delta N_\phi = \int_{t_{in}}^{t_{fin}} dt \frac{H}{\sqrt{\epsilon}}$$

Let me focus on the quadratic potential for the inflaton field, the above variation can be expressed as a function of the number of e-folds and is equal to $-\Delta N_e^{\frac{3}{2}}$. Therefore the following bound had to be satisfied:

$$N_e < N^{\frac{2}{3}} \tag{3.82}$$

The annihilation of the constituent gravitons causes the gradual depletion of the graviton condensate, so that an upper bound on the lifetime of the quasi de Sitter space arises. This upper bound excludes any potential energy that slopes to a positive cosmological constant and puts a restriction on how slow the potential energy can decrease. Moreover, this bound can be recast as a constraint on the inflaton potential:

$$m_p^2 \left(\frac{V_{,\phi}}{V}\right)^2 < \frac{m_p^{\frac{8}{3}}}{\hbar} \frac{1}{V^{\frac{2}{3}}} \tag{3.83}$$

Thus, the potential derivative must to satisfie the following relation:

$$V_{,\phi} < \frac{m_p{}^{\frac{2}{3}}}{\hbar} V^{\frac{4}{3}} \tag{3.84}$$

To conclude this section I want to say that, the quantum description of the universe as a graviton bound-state with a very high occupation number $N \gg 1$ will decay after the time:

$$t_{dec} = N^{\frac{3}{2}} l_p = \frac{1}{l_p^2 H_0} = \frac{N}{H_0} \tag{3.85}$$

where in the formula is assumed that the gravitons energies are $\epsilon \sim \hbar k \sim \hbar H_0$.

3.4 Cosmological redshift

In this section I want to describe the cosmological redshift, in the framework of the General Theory of Relativity and in the full quantum picture introduced before. The redshift effect is expected in the FRW universe, for all kind of fluids. Otherwise from the Schwarzschild case, in cosmology the reason of the changing of the frequency of a propagating probe photon, isen't given by the presence of a localized source of energy, but it is a consequence of the evolution of the cosmic scale factor. This latter assumes, in the inflaton field model, different forms due by the different regimes of the field. In particular, in the inflationary regime, it is described by the formula (1.21). The novelty of the quantum description is that the cosmic scale factor is slightly different from it, with corrections that can be fished out from the equation (3.69) and that affect in some way the redshift formula.

3.4.1 Classical redshift

Imagine two photons emitted from the same position respectively at time t_e and $t_e + \delta t_1$ and detected in an another point of the space, respectively at time t_r and $t_r + t_2$. The photons travel along the same coordinate distance, but the measured distances are different, since that they are affected by the metric of the space-time. By the equation:

$$\int_{t_e}^{t_r} \frac{dt}{a(t)} = \int_{t_e+t_1}^{t_r+t_2} \frac{dt}{a(t)} = \int_{r_1}^{r_2} dr \tag{3.86}$$

it can be obtained for short times δt_1 and δt_2:

$$\int_{t_1}^{t_1+\delta t_1} \frac{dt}{a(t)} = \int_{\delta t_1}^{\delta t_2} \frac{dt}{a(t)} \Rightarrow \frac{\delta t_2}{a(t_2)} = \frac{\delta t_1}{a(t_1)} \tag{3.87}$$

From the last equality, the redshift formula can be calculated:

$$\frac{\omega(t_1)}{\omega(t_2)} = \frac{a(t_2)}{a(t_1)} = e^{H(t_2-t_1)} \tag{3.88}$$

where the last equality is true for the de Sitter universe.

3.4.2 S-matrix formalism

From a quantum point of view the redshift effect can be analyzed using the depletion formula (3.69) or by means of the S-matrix evolution operator. In this section I want to describe this latter approach. In this formalism the redshift can be interpreted as a scattering process between the quanta of the perturbed metric $\widehat{h}_{\mu\nu}$ and the probe particle. The test corpuscle interacts with the gravitons of the background $|N\rangle$ producing a new total final state:

$$|i_\Psi\rangle \otimes |N\rangle \, N \to |f_\Psi\rangle \otimes |N'\rangle \tag{3.89}$$

where $|i_\Psi\rangle = \widehat{b}_p^+ |0\rangle$ and $|f_\Psi\rangle = \widehat{b}_{p'}^+ |0\rangle \, 0$ are respectively the initial and final state for the probe Ψ-particle, representing a generic scalar field. The quantum S-matrix evolution operator is:

$$S = -i \int dx T \widehat{H}_{int}^{eff} = -i \int dx T \frac{\widehat{\Phi}\widehat{T}_\mu^\mu(\Psi)}{m_p} \tag{3.90}$$

Following this idea, the main physical number I want to obtain is the transition amplitude:

$$A = \langle f_\Psi | S^{eff} | i_\Psi \rangle = \langle f_\Psi | \otimes \langle N' | S | i_\Psi \rangle \otimes |N\rangle \tag{3.91}$$

In the semiclassical limit $N \to \infty$, it must be considered a coherent state with an infinite mean occupation number, in such a way that the generalized initial and final states contain the same number of gravitons. In this case, the quantum operator field Φ can be replaced by its classical counterpart $\Phi_{cl} = \langle N|\widehat{\Phi}|N\rangle$, that is a function representing an eternal background. Equivalently, it can be said that, in the semiclassical treatment any back reaction can be ignored and $\Lambda = m^2$ must

be considered fixed, with the consequence that the gravitons will be on-shell. Formally, to obtain the quantum evolution, the weak field limit must be used and it is necessary to expand the above expression (3.89) in the parameter $\frac{\Phi_{cl}}{m_p} \ll 1$, that corresponds to consider the limit $m_p \to \infty$.
Note that the quantum dynamics of the de Sitter space-time with the probe particle, takes into account the possibility that the mean value of the gravitons number change, with a possible deviation from an initial coherent state. Using the formula for a general coherent state (3.9), it can be observed that:

$$\langle N'| |N\rangle = \langle N + \Delta N| |N\rangle = e^{\frac{-(\Delta N)^2}{8N}} \tag{3.92}$$

that is a gaussian function. It shows the probability that the $|N\rangle$ state transforms into $|N'\rangle$. In the semiclassical limit $N \to \infty$, this scalar product will be approximately equal to 1 and $|N\rangle \approx |N'\rangle$, whereas if N is finite, the back reaction becomes important after some time. The quantum amplitude can be written as:

$$A = \langle N'| \widehat{b}_{p'} S \widehat{b}_p |N\rangle = K(p, p') \int d^4x e^{-i(p-p')} \langle N'| \widehat{\Phi} |N\rangle \tag{3.93}$$

where $K(p, p')$ is the kinetic factor associated to the probe particle:

$$K(p, p') = \frac{i\sqrt{16\pi}}{m_p} \frac{p \cdot p' - m_\Psi^2}{(2\pi)^{\frac{3}{2}} \sqrt{2p_0} (2\pi)^{\frac{3}{2}} \sqrt{2p'_0}} \tag{3.94}$$

Considering the semiclassical limit, for wich $\Phi = \Phi_{cl}$ and $N' = N$ the integral in the formula (3.92) will be:

$$\delta(p - p') \left(\delta(-p_0 + p'_0 + m) \sqrt{\frac{N}{V}} + \delta(-p_0 + p'_0 - m) \sqrt{\frac{N}{V}} \right) \tag{3.95}$$

From this formula the process of redshift can be understood. Analyzing the delta terms it can be deduced that the induced emission or absorbation of one graviton corresponds to the lose or the increment of the energy of the probe quanta (redshift, blueshift). A priori, this formula doesen't distinguis between the two processes, but going to manipulate the boundary conditions, the redshift can be imposed. The situation can be interpreted as follows: a particle with 4-momentum $(E, \overline{p})$ deposits one graviton in the background state of N gravitons, so that the final 4-momentum of the particle will be $(E - m, \overline{p})$.
Another way to deduce immediately the redshift effect is by use of the formula (3.80) for the graviton scattering. From this equation, it can be deduced that the lost energy by a particle during time is:

$$\frac{\dot{E}}{E} = -H \tag{3.96}$$

that exactly matches the rate of the energy lost by the redshift effect in a de Sitter space-time. Indeed, the (3.95) can be rewritten as:

$$\Delta E = exp \left\{ - \int_{t_1}^{t_2} dtH \right\} = e^{H(t_2 - t_1)} \tag{3.97}$$

To conclude, I want to say that the classical expansion of the cosmic scale factor $a(t) \sim e^{Ht}$ doesen't affect the instability of the condensate. Its evolution only changes the frequencies of the probe particle, but not of the gravitons of the condesate.

3.4.3 Redshift and deplation formula

The equation of motion for the graviton condensate (3.69), can be read also as an equation for the Hubble function and for the cosmic scale factor:

$$H = \frac{\dot{a}}{a} = \frac{\dot{N}}{N\left(\epsilon - \frac{1}{\sqrt{\epsilon}N}\right)} = \frac{\dot{N}}{N\epsilon - \frac{1}{\sqrt{\epsilon}}} = \frac{\dot{N}}{\epsilon\left(N - \frac{1}{\epsilon^{\frac{3}{2}}}\right)} \tag{3.98}$$

The three expressions on the right-hand side of this equation affect the evolution of the classical cosmological quantities. For this reason the redshift effect must have a different value from the standard one. At the beginning, I want to consider only the effect of the classical term of the equation (3.69) and to apply the limit $N \to \infty$. For a generic moment during the inflationary phase $1 < \epsilon < 1$, the cosmic scale factor evolves as:

$$a(t) = exp\left\{\int^t dt' H(t')\right\} = exp\left\{\int^t dt' \frac{\dot{N}}{N\epsilon}\right\} \tag{3.99}$$

To quantify the modified energy of a probe particle, given by the evolution of the cosmic scale factor, the following ratio must be calculated:

$$r_\omega = \frac{a(t_2)}{a(t_1)} \tag{3.100}$$

Thus, for two generic time $t_1 < t_2$, the (3.99) can be written as:

$$r_\omega = exp\left\{\int_{t_1}^{t_2} dt H(t)\right\} = exp\left\{\int_{t_1}^{t_2} dt' \frac{\dot{N}}{N\epsilon}\right\} \tag{3.101}$$

For a small interval of time $t_2 - t_1$, the above expression assumes the form:

$$r_\omega \simeq e^{(H(t_2)-H(t_1))(t_2-t_1)} \simeq exp\left\{\left[\frac{\dot{N}(t_2)}{N(t_2)\epsilon(t_2)} - \frac{\dot{N}(t_1)}{N(t_1)\epsilon(t_1)}\right](t_2 - t_1)\right\} \tag{3.102}$$

Since the Hubble function is decreasing during the above interval, the resulting r_ω will be slightly minor of the classical de Sitter result (3.87). In particular, for the beginning and for the end of inflation can be written respectively:

$$\epsilon \sim 0 \qquad a(t) \sim e^{Ht} \tag{3.103}$$

$$\epsilon \sim 1 \qquad a(t) \sim kN \tag{3.104}$$

For two values of time, close to the end of inflation, r_ω is given by:

$$r_\omega \simeq \frac{N(t_2)}{N(t_1)} \tag{3.105}$$

Now, considering the full quantum picture, i.e. finite N, to account the cosmic scale factor, the complete expression (3.97) must be solved, leading to the following cosmic scale factor:

$$a(t) = exp\left\{\int^{t} dt' \frac{\dot{N}}{N\epsilon - \frac{1}{\sqrt{\epsilon}}}\right\} \tag{3.106}$$

so that during the inflationary phase:

$$r_{\omega} = exp\left\{\int_{t_1}^{t_2} dt' \frac{\dot{N}}{N\epsilon - \frac{1}{\sqrt{\epsilon N}}}\right\} \tag{3.107}$$

and for a small interval of time:

$$r_{\omega} \simeq exp\left\{\left[\frac{\dot{N}(t_2)}{N(t_2)\epsilon(t_2) - \frac{1}{\sqrt{\epsilon(t_2)N(t_2)}}} - \frac{\dot{N}(t_1)}{N(t_1)\epsilon(t_1) - \frac{1}{\sqrt{\epsilon(t_1)N(t_1)}}}\right](t_2 - t_1)\right\} \tag{3.108}$$

In the previous section I shown that, in the quantum description of the graviton condensate exists a minimum distance from the de Sitter space-time, i.e. a minimum value for the slow-roll parameter (3.74). Thus, inserting this expression in the formula (3.105), the "minimum" distance between $a(t)$ and the de Sitter cosmic scale factor can be obtained:

$$a(t) = exp\left\{\int^{t} dt' \frac{\dot{N}}{\sqrt{N} - \frac{1}{N^{\frac{5}{4}}}}\right\} \tag{3.109}$$

To conclude, at the end of inflation $\epsilon \sim 1$, the cosmic scale factor assumes the following simplified form:

$$a(t) \sim exp\left\{\int^{t} dt' \frac{\dot{N}}{N - 1}\right\} \tag{3.110}$$

so that, for two values of time, close to the end of inflation, r_{ω} is given by:

$$a(t) \sim exp\left\{\int_{t_1}^{t_2} dt' \frac{\dot{N}}{N - 1}\right\} \tag{3.111}$$

3.5 Quantum particles creation

The task of this section is to study the semiclassical effect of particles production from the inflationary universe. This effect is described in the semiclassical theory as a vacuum process very similar to the black hole evaporation. It can be considered as a consequence of the gravitational field equation (1.2), with the right-hand side replaced by the mean value of the energy-impulse tensor, for some quantum field:

$$G_{\mu\nu} = \langle T_{\mu\nu} \rangle \tag{3.112}$$

In the quantum picture, the particle creation is a consequence of the quantum depletion of the particles of the graviton and inflaton Bose-Einstein condensates . This latter approach matches with the first one in the limit $N \to \infty$.

3.5.1 Gibbons-Hawking particle creation

The quantum description of the de Sitter space-time corresponds to a critical system, whose gravitons are able to produce on-shell particles in their re-scattering. The main contribution to the particles creation process, comes from the quantum depletion term in (3.69), that leads the gravitons to push out from the condensate and to become propagating. Thus, the self coupling of the gravitons and their coupling to other particle, lead to processes of scattering and decay of the constituent gravitons of de Sitter universe. The final products of such decays and scatterings will contain particles with a dispersion relations of free quanta propagating on a classical de Sitter background.
I want to obtain an explicit formula for the production rate of generic external particles, using the S-matrix formalism. Considere an initial coherent state $|N\rangle$ for the de Sitter space-time, in wich a quanta of energy m decays into two particles-Ψ with four momenta: $p = (p_0, p_i)$ and $p' = (p'_0, p'_i)$. The final state $|N'\rangle = |N + \Delta N\rangle$ can be different from the initial one. This is the fundamental novelty respect to the semiclassical approch, for wich this difference is negligible, since that N is very large. The differential transition probability is given by:

$$dw = \frac{|A|^2 d^3pV d^3p'V}{\langle N| |N\rangle \langle N'| |N\rangle \langle f_\Psi| |f_\Psi\rangle} \tag{3.113}$$

where:

$$A = (2\pi)^4 \delta(m - p_0 - p'_0)\delta(p - p')M \tag{3.114}$$

and:

$$M = \frac{K(-p, p')}{\sqrt{2m}} \sqrt{\frac{N'}{V}} \left(1 - \frac{\Delta N^2}{8N}\right) \tag{3.115}$$

Note that, in the semiclassical theory, the corrections due by ΔN are "invisible". In the previous formulas the approximation $\Delta N \ll N$ has been used. The differential decay will be:

$$\frac{d\Gamma}{d\Omega} = \frac{2\sqrt{\frac{m^2}{4} - m_{\Psi^2}}N'}{2\pi m_p^2 m^2} \left(\frac{m^2}{2} + m_\Psi\right)^2 \left(1 - \frac{\Delta N^2}{4N}\right) \tag{3.116}$$

so that the particles production rate can be obtained integrating over Ω:

$$\Gamma = \frac{2\sqrt{\frac{m^2}{4} - m_{\Psi^2}}N}{m_p^2 m^2} \left(\frac{m^2}{2} + m_\Psi\right)^2 \left(1 - \frac{\Delta N^2 - 4\Delta N}{4N}\right) \tag{3.117}$$

Remark that, only transitions whereby all initial and final constituent gravitons are off-shell, contribute to the recovery of the classical metric, without particle creation. Whereas, processes in which some of the initial off-shell constituent gravitons become on-shell, lead to particle creation. Moreover, it can be noted that, the process of gravitons production must have a similar rate, since that the gravitational coupling is universal.

3.5.2 Distribution of the produced particles

The energy distribution of the produced particles is not uniform. Indeed, the Hubble horizon radiates like a black body of temperature $T \sim \sqrt{\Lambda}$. Namely A the horizon area, the emitted power is given by the Stefan-Boltzmanm formula:

$$P \sim T^4 A \sim \Lambda \tag{3.118}$$

From the equation (3.117) it can be proved that, the power spectrum in the quantum theory is given by the (3.118) multiplied for $\hbar$. This means that in the classical limit $\hbar \to 0$, the particle production vanishes.
To first order, all the produced particles have the same energy $\frac{m}{2}$. To higher order the energy distribution, will be:

$$\Gamma \sim e^{-\frac{E}{\sqrt{\Lambda}}} \tag{3.119}$$

The particles production is an Hamiltonian process in which the background coherent state of gravitons get converted into free quanta. If it is considered the emission of particles with $E \gg m$, the annihilation of n background gravitons with $n > \frac{E}{m}$ and the creation of two Ψ-particles of total energy $E = nm$ must be taken into account. The probability of such process is highly suppressed, because there is a large partecipation of soft gravitons in it. For estimating this suppression can be analyzed the following formula:

$$\Gamma \sim \left(\frac{1}{N}\right)^n n! \binom{N}{n}$$

where $\binom{N}{n}$ accounts the number n of gravitons combination for the total number of gravitons N. For $N \gg 1$ and $N - n \gg 1$:

$$\Gamma \sim e^{-n} \left(\frac{N}{N-n}\right)^{N-n} \sim e^{-n} \tag{3.120}$$

Since that $n = \frac{E}{m}$ and $m \approx \sqrt{\Lambda}$, the above expression reproduces the exponential suppression of the Boltzmann factor for a thermal bath.
On the other hand, arbitrarily soft quanta can be produced in the processes of decay and of annihilations of the background gravitons. For example one of the gravitons can decay into two particles of energy $E \ll m$ and tranfers the energy difference $\Delta E = m - E$ to the remaining $N - 1$ gravitons. This kind of processes are not exponantially suppressed.

3.6 Entangled time and Entropy

The hystory of the de Sitter space-time is imprinted in the evolution of the quantum state associated. In particular, the history is encoded by the evolution of the condensates and by the entanglement of gravitons and inflatons. This means that, if one could measure the entanglement of the inflationary state, the absolute age of the de Sitter space-time should be possible to measure. A near critical condensate of gravitons generates entanglement very efficiently, because of the deplation of the constituents quanta. This effect is true also for the Hubble graviton's reservoir. Indeed, after a minimal time scale, the system becomes an entangled quantum system. For realistic values of the inflationary parameters, the entanglement time scale is within 30 e-folds. The quantum entanglement time for de Sitter condensate is:

$$t = \sqrt{\epsilon} R_H \ln N \tag{3.121}$$

that is very similar to the entanglement time for the balck hole gravitons quantum state, except for the term $\sqrt{\epsilon}$, due to the presence of the inflatons quanta. After this time, the de Sitter space-time cannot longer be viewed as a well defined classical background.
The initial N graviton's state is not entangled. This means that in the mean

field approximation, it is well described by a single wave function. Thus, the initial state can be represented as a N quanta tensor product:

$$\Psi = \psi_1 \times \psi_2 ... \times \psi_N \tag{3.122}$$

Hand by hand that the condensate emits gravitons, the system generates entaglement. After the time scale (3.120), the system becomes one-particle entangled, i.e. Ψ cannot longer be expressed as:

$$\Psi = \psi_j \times \psi' \tag{3.123}$$

for any j, where ψ' is an arbitrary $N-1$ particle state. Moreover, another kind of entaglement time scale exists. It is defined as:

$$t = \sqrt{\epsilon} R_H N \tag{3.124}$$

After this time, the graviton's system becomes maximally entangled. i.e. it cannot longer be represented as the tensor product (3.121).
In analogy with the black hole case, the entropy of the de Sitter universe is:

$$S_{dS} \sim R_H^2 \sim N_{dS} \tag{3.125}$$

This relation is very similar to the Bekenstein-Hawking entropy law for black holes. Since that the de Sitter graviton's condensate is critical $\alpha N_{dS} \sim 1$, the equation (3.124) can be written as:

$$S_{dS} \sim \frac{1}{\alpha} \tag{3.126}$$

so that the entropy scales like the inverse of the coupling.
During every e-fold of the de Sitter expansion $\Delta t = \frac{1}{H}$, two soft gravitons are emitted at the de Sitter horizon. Thus, the variation of the entropy for e-fold satisfies:

$$\frac{dS_e}{dN_e} \gtrsim 1 \tag{3.127}$$

After N e-folds, $N_{dS} \sim N_e$ indipendent quanta are emitted making up $2^{N_{dS}}$ possibles microstates, which exeed N when:

$$S_{dS} \sim N_{dS} \tag{3.128}$$

The perturbation theory will break down when $S_{dS} \gtrsim N_e$, that corresponds to the time scale:

$$t_{ds} \sim R_{dS} S_{dS} \sim \frac{m_p^2}{H^3} \tag{3.129}$$

This quantity is dimensionally equivalent to the black hole evaporation time. To conclude, I want to say that the amplitude of the process $2 \to N_g$ can be rewritten as depending on the entropy:

$$A \sim e^{-S_{dS}} \sim e^{-N} \tag{3.130}$$

3.7 Density quantum perturbation

An important consequence of the quantum description of the inflationary universe, is the creation of small perturbations on the energy density $\rho + \delta\rho$. In the slow-roll regime there are two origins of the density perturbations. The first is the uncertainty in the position of the inflatons, the second is the scattering of the gravitons. I want to start calculating the density perturbations due to the uncertainty principle. Remembering that the distribution of the inflaton's number is the Poisson distribution, if the expected number of the scalar quanta in a given region, let me say the Hubble patch, is N_ϕ, the fluctuation will be of the order $\sqrt{N_\phi}$. Thus, the density perturbation due to the uncertainty principle will be:

$$\delta\rho_u = \sqrt{N_\phi} m H^3 = m_\phi H m_p \delta\phi \tag{3.131}$$

so that, the perturbation on ϕ can be calculated as:

$$\delta\phi = H\sqrt{\frac{H}{m}} \tag{3.132}$$

This result is not consistent with the standard one obtained in the inflaton field theory, that is $\delta\phi = H$. To resolve this problem, I must consider the effective mass of the inflatons, so that the typical density perturbation in (3.130) will be replaced by:

$$\delta\rho_u = \sqrt{N_\phi} m_{eff} H^3 = m_p m H^2 \tag{3.133}$$

This value corresponds to the correct amplitude of the scalar fluctuation on the horizon.

The second source of scalar perturbations is the scattering of the gravitons. In particular, for the scattering process $g + \phi \to g + \phi$, the number of excited quanta within one Hubble time is $\delta N = \delta N_\phi = 1$. The density perturbation given by this process will be:

$$\delta\rho_s = \delta N H H^3 = H^4 \tag{3.134}$$

The ratio between the two kind of density perturbations will be:

$$\frac{\delta\rho_u}{\delta\rho_s} = \frac{m_p m}{H^2} \tag{3.135}$$

To distinguish what kind of perturbation is dominant, it can be defined the so called crossover curvature scale:

$$H_* \equiv H\sqrt{\frac{\delta\rho_u}{\delta\rho_s}} = \sqrt{m_p m} \tag{3.136}$$

In the low curvature regime $H < H_*$, the dominant source of the density perturbations is the uncertainty principle. Instead for $H > H_*$, the perturbation given by the gravitons scattering becomes dominant. In order to find the scalar amplitude generated by the uncertanty principle, it is necessary to relate the density perturbation to $\delta\phi$. The formula (3.130) is true as long as the non linear terms in $\delta\phi$ is negligible. Thus considering the largest non linear correction, the (3.130) becomes:

$$\delta\rho_u = m_\phi^2 \phi \delta\phi + \delta\dot{\phi}^2 = m_\phi H m_p \delta\phi + H^2 \delta\phi^2 \tag{3.137}$$

where the second equality is true for perturbations of order H. Thus, the non linear term becomes important when $H > H_*$ and $\delta\phi = H_*$, otherwise from H. Another interaction channel which could result in the scalar amplitude perturbations is $g + g \rightarrow \phi + \phi$. The corresponding density perturbation amplitude is:

$$\delta\rho = H^4 \tag{3.138}$$

In this case, for $H \gg H_*$ the perturbation will be $\delta\phi = H$, otherwise from $\frac{H^3}{H_*^2}$. Now, I want to rewrite the background energy density in terms of the graviton occupation number:

$$\rho = E_\lambda \frac{N}{V} = \frac{\hbar}{R_H} \frac{N}{R_H^3} = \hbar \frac{N}{R_H^4} \tag{3.139}$$

Then, the perturbation on ρ due by the depletion of gravitons, can be calculated by the variation on (3.138) respect to the variation of the number of gravitons and inflatons N:

$$\delta\rho = \frac{\hbar}{\lambda R_H^3} \delta N_\lambda \tag{3.140}$$

The above equation can be rewritten as:

$$\delta\rho = \frac{\hbar H}{R_H^3} \frac{N_\phi}{N} \tag{3.141}$$

Since neither inflatons nor the longitudinal gravitons carry tensor helicity, the expression (3.140) contributes only to the scalar mode of density perturbations.

$$\delta_\Phi = l_p R_H^2 \delta\rho = l_p H \frac{N_\phi}{N} = l_p \frac{H}{\sqrt{\epsilon}} \tag{3.142}$$

Instead, the power-spectrum of the tensor modes due to the graviton-graviton scattering is:

$$\delta_T = l_p H \tag{3.143}$$

The ratio between the tensor and scalar perturbations will be:

$$r = \frac{\delta_T^2}{\delta_\Phi^2} = \frac{N^2}{N_\phi^2} = \epsilon \tag{3.144}$$

Considering the potential energy $V(\phi) = m^2\phi^2$, the above expressions (3.142-144) can be rewritten as depending on the number of e-folds:

$$\delta_\Phi = l_p H \sqrt{N_e}$$
$$r = \frac{1}{N_e}$$

Let me define the tilt:

$$n_s - 1 = \frac{d \ln \delta_\phi^2}{d \ln n_k} \tag{3.145}$$

It must be calculated on the value of the Hubble horizon:

$$1 - n_s = -\frac{\dot{\Gamma}}{H\Gamma} = \frac{3}{2}\left(\frac{\dot{N}}{N}\right) - \left(\frac{\dot{N}_\phi}{N}\right) \tag{3.146}$$

Considering only the purely classical evolution of the inflaton and graviton condensates, the (3.145) will be:

$$1 - n_s = \frac{3}{2}\epsilon - \eta \simeq \frac{2}{N_e} \tag{3.147}$$

where the last line is true for the quadratic potential. In this approximation, the purely quantum depleton has been neglected, so that the classical result is recovered. This approximation is consistent with treating the system as an infinite capacity source of particles, i.e $N \to \infty$, but keeping the ratio $\frac{N_\phi}{N}$ finite.
Resuming, in the quantum framework the perturbations in the inflationary universe are a consequence of the quantum processes, quantified by the two quantum contributions in the depletion formulas for inflatons and gravitons:

$$-\frac{H}{\sqrt{\epsilon} N}; \qquad -\frac{H}{N} \tag{3.148}$$

The quantum variation on the tilt is:

$$\Delta(1 - n_s) = -\frac{N_\phi}{N^2} \tag{3.149}$$

Navely, using the quadratic potential, the new expression can be rewritten as a function of the number of e-folds:

$$1 - n_s = \frac{1}{N_e} - \frac{\sqrt{N_e}}{N} \tag{3.150}$$

where the new contribute is more important at the earlier times and reinforces the upper bound on the number of e-folds. Moreover the quantum change on the r parameter is:

$$\Delta r = 2\frac{N}{N_\phi^2}\Delta N \tag{3.151}$$

so that the new ratio will be:

$$r = \frac{N^2}{N_\phi^2}\left(1 - \frac{\Delta N}{N}\right) = \frac{1}{N_e}\left(1 - \frac{\Delta N_e^{\frac{3}{2}}}{N}\right) \tag{3.152}$$

where the second equality is valid for the quadratic potetnial.

3.8 And the cosmological constant?

Vacuum energy can be loosely defined as the energy of the fundamental state of the universe. It is a measurable quantity only in the context of gravity, indeed in a non gravitational context, only differences of energies can be measured. But in the framework of the Einstein's theory, any form of energy that participates to the expansion of the Universe, can be measured absolutely, therefore also the energy of the vacuum.
Theoretically a gravitational theory can be built with eternal inflation given by a positive cosmological constant. Actually, if also the corpuscolar interpretation of inflation is taken into account, the gravitational attraction between the particles will slow down this eternal acceleration. Thus, any theory that in the classical limit would flow to a state with a constant positive curvature is inconsistent. It has been shown that the Hubble radius is related with the cosmological constant by the relation:

$$R_\Lambda = cT_\Lambda = \sqrt{\frac{3}{\Lambda}} \tag{3.153}$$

With this relation (3.152), a lenght and a time scale inside the General Theory of Relativity can be introduced. The cosmological constant strongly affects the space-time dynamics at all scales larger than R_Λ and T_Λ. Now, the smallness of Λ can be deduced just from the fact that, the universe is large compared to the Planck length, and old compared to the Planck time, that are the scales inside the quantum theory of gravity. In particular, considering an empty space $T_{\mu\nu} = 0$ and a positive cosmological constant, the only cosmological isotropic solution of the Einstein's equation, is the de Sitter universe with $R_\Lambda \gg l_p$ and therefore with $\Lambda \ll 1$. Instead with a negative Λ, the universe had to recollapse indipendently from the spatial curvature, in a time scale T_Λ.

$$-\frac{3}{T^2} < \Lambda < -\frac{3}{R^2} \tag{3.154}$$

where T and R are the time and the lenght observable cosmological scales. Subsituing the lenght and the time scale of the universe it can be said that $\Lambda \ll 1$.
The quantum compositeness of gravity is incompatible with the definition of a constant Λ. Space-times with positive cosmological constant exibits a finite quantum break time $t_q = \frac{1}{H\alpha} = \frac{1}{H^3}$. The quantum break-time puts a consistency constraint on the classically describable duration of the universe. Over this time the model becomes inconsistent. This thing can be mathematically expressed also as a lower bound condition on the potential of the inflaton field:

$$|\nabla V|^2 > \frac{\sqrt{V}}{t_q} = V\alpha \tag{3.155}$$

and, if $\alpha = V$:

$$|\nabla V|^2 > V^2 \tag{3.156}$$

Moreover, if $\alpha = \ln^{-1}(V^{-1})$:

$$|\nabla V|^2 > V \ln^{-1}(V^{-1}) \tag{3.157}$$

Capitolo 4

Corpuscolar f(R)-theory for inflation

I shown that to describe the quantum inflationary universe, the particles associated to the perturbation on the metric tensor field and that's one associated to the inflaton field, must be introduced. Now, I want to show that to study the inflationary space-time, an alternately lagrangian can be used, in the so called framework of the $f(R)$ theories of gravity. In this way, it is not necessary to use the scalar field ϕ, but only the scalar curvature R, in a different functional form of the integrand of the action. Thus, one could think that going to quantize this theory the inflatons particles not will appear, but will appear only the particles associated to the perturbed metric field $h_{\mu\nu}$, i.e. gravitons. The result is that inflation is allowed inside the gravitational theory itself, but in a more complicate form. Indeed, using a conformal transformation on the metric tensor field, the following passage is allowed:

$$\mathcal{L}(R) \to \mathcal{L}(R, \phi)$$

In particular there is a $f(R)$ theory with the same number of degrees of freedom and an equivalent dynamics of an inflaton field model, with a well defined potential field.
To conclude, I want to analyze the corpuscolar interpretation of this new lagrangian functional, going to associate different graviton's hamiltonian functions to each term in $f(R)$.

4.1 Lagrangian of general relativity

The canonical treatment of GR starts building an action that yields the Einstein's equations as expression of the Euler-Lagrange equations. To develope this formalism are necessary an action and a set of variables that can be varied. This latter are the components of the metric tensor field $g_{\mu\nu}(x)$, that encode the geometric evolution of the space-time. The proceed, it must be required that the the action is local and gauge invariant. In particular, the functional in the action cannot be written as an integral, because the causality principle should be lost. The action assumes the well know form:

$$S = \int dV \mathcal{L} \tag{4.1}$$

where $\mathcal{L}$ is the lagrangian density and dV is the volume element in the four dimensional space-time. The integral is extended on the whole space and between two values of the time coordinate. To understand the meaning of the volume element, it must be noted that in a Minkowsky space-time dV is:

$$dV = dx_0 dx_1 dx_2 dx_3 \tag{4.2}$$

where x_μ are the coordinates of the system in some reference frame. Applying a coordinate transformation $x'_\mu = x'_\mu(x^\nu)$, in the integral the modulo of the determinat of the Jacobian of the transformation must be added. Since that, for the equivalence principle, the relation between the metric in a general frame and the Minkowsky metric must be:

$$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\eta_{\alpha\beta} \tag{4.3}$$

and the determinat of the members of the equation must be equal, the volume element can be rewritten as:

$$dV = \sqrt{-g}dx_0 dx_1 dx_2 dx_3 \tag{4.4}$$

Since that $g_{\mu\nu}$ is symmetric, only the symmetric part of $\frac{\delta S}{g_{\mu\nu}}$ is defined, with the antisymmetric undetermined.
Now, the action must be invariant for coordinate transformations of GR, thus the most simple lagrangian is a constant, for example the cosmological constant Λ. A more complex lagrangian can be built by use of the single scalar curvatur R. It depends on the metric components $g_{\mu\nu}$, their derivatives $g_{\mu\nu,\alpha}$ and their second derivatives $g_{\mu\nu,\alpha\beta}$. Thus the most simple, but non trivial action will be:

$$S \sim \int d^4x\sqrt{-g}R \tag{4.5}$$

4.2 f(R) theories of gravity

The formula (4.5) generates the Einstein equations in a vacuum space. To obatin the energy-impulse term, a lagrangian described by a matter field must be added. Now, my sake is to investigate alternative theories of gravity, because what I will show later, about the inflationary cosmological universe, is just a consequence of a particular choise of the $f(R)$ function. A priori one could modify the classical Einstein-Hilbert action (4.5), adding higher order invariants terms with respect to the Ricci scalar. I don't want to consider higher order invariant as $R_{\mu\nu}R^{\mu\nu}$, because these kind of terms are renormalizable, but not unitary. It can be thought for example an action of the form:

$$S = \frac{1}{8\pi G}\int d^4x\sqrt{-g}\left(\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right) \tag{4.6}$$

that contains the term $R_{\mu\nu}R^{\mu\nu}$, that I don't want to consider. But, let me say only that for this action the cosmic no-hair theorem, valid in GR, it's not more true, because this lagrangian provides an anisotropy inflation.

The relevance of high order terms in the action must be considered restricted to very strong gravity regimes. Therefore, corrections to GR are considered to be important only at scales close to the Planck scale, for example in the early universe or near a black hole singularity. A "good" generalization of the action (4.5) is:

$$S = \frac{1}{8\pi G}\int d^4x\sqrt{-g}f(R) \tag{4.7}$$

where $f(R)$ is a function of the only scalar curvature. It can be thought as a polynomial:

$$f(R) = ...\frac{a_{-2}}{R^2} + \frac{a_{-1}}{R} - 2\Lambda + a_1 R + a_2 R^2... \tag{4.8}$$

where Λ appears as the monomyal of zero degree. An interesting model arises extracting from the (4.8) the sum $R - \frac{\mu^{2(n+1)}}{R^n}$, where μ is a suitably chosen parameter. This function can be used, with some precautions, to study the exit from the radiation era.

At this point, to the above action (4.7) must be added a term S_M to describe the energy density of the matter, that it would be taken into account. Going to calculate the variation respect to the metric tensor field, the following generalization of the Einstein field equation can be obtained:

$$(R_{\mu\nu} - \nabla_\mu \nabla^\nu + \Box g_{\mu\nu})f'(R) - \frac{1}{2}g_{\mu\nu}f(R) = kT_{\mu\nu} \tag{4.9}$$

where the derivative $f'(R)$ is calculated respect to R, whereas $k = \frac{8\pi G}{c^4}$. The energy-impulse tensor $T_{\mu\nu}$ is obtained by the variation of S_M respect to the metric tensor field:

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}} \tag{4.10}$$

The equations (4.9) are fourth order partial differential equations respect to the metric, since that R already includes second order derivatives. Taking the trace, the equations become:

$$f'(R)R - 2f(R) + 3\Box f' = kT \tag{4.11}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. This equation is differential and not algebrica as in GR, where $R = -kT$. For $T = 0$ the equation (4.11) no longer implies $R = 0$. For R costant and $T_{\mu\nu} = 0$, the equation (4.11) becomes:

$$f'(R)R - 2f(R) = 0 \tag{4.12}$$

Thus, if $R = 0$ the generalized Einstein equation becomes $R_{\mu\nu} = 0$, and the Minkowsky space-time is a possible solution. Whereas, if $R = C$ the equation will be $R_{\mu\nu} = \frac{C}{4}g_{\mu\nu}$ so that, depending on the sign of C, the universe will be a de Sitter or an anti de Sitter metric. This is an indication that from a particular choice of the $f(R)$ function, something like a cosmological space-time can be reproduced. Indeed, also without the precence of the matter, by the left-hand side of the equation (4.9) can be extracted a mathematical object that assumes the form of a energy-impulse tensor. I will show in depth this topic in the next section. Now, I want only say that, following the previous considerations, the cosmic scale factor

can be introduced. In particular, one could choise a particular form for $a(t)$ and tries to obtain the $f(R)$ function that reproduces it. But, the prescribed evolution for $a(t)$ doesen't determine uniquely the form of $f(R)$ but, at best, only a class of $f(R)$ models. Therefore, the observed cosmological data, that provide the history of $a(t)$, are not sufficient to find $f(R)$: one needs of additional informations, which can come out for example, from the cosmological density perturbations.
Note that a $f(R)$ theory must satisfy some requirements:

- it must have the correct weak-field limit at both the Newtonian and post-Newtonian levels;
- it must be stable at the classical and semiclassical level;
- it must not contain ghost fields, because they should increase the degrees of freedom;
- it must admit a well-posed Cauchy problem;

4.3 Brans–Dicke theory and conformal frames

I want to introduce the so called Brans-Dicke theories, that are prototypes of scalar-tensor models. These latter can be viewed as generalizations of the inflaton field model. To build the Brans-Dicke action, it can be start from the action (4.7), rewritten using a scalar field χ:

$$S = \frac{1}{2k} \int d^4x \sqrt{-g} \left[f(\chi) + f'(\chi)(R - \chi) \right] + S_M \tag{4.13}$$

where the last term describes a generic matter field. Introducing $\phi = f'(\chi)$ and:

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)) \tag{4.14}$$

the action (4.13) assumes the form:

$$S = \frac{1}{2k} \int d^4x \sqrt{-g} \left[\phi R - V(\phi) \right] + S_M \tag{4.15}$$

where the scalar function ϕ is a dynamical degree of freedom. The (4.15) is the Jordan frame representation of the action of a Brans–Dicke theory with parameter $\omega = 0$. The generalization of (4.15) can be written in the form:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[f(\phi)R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + \Lambda(\phi) \right] + \int d^4x \sqrt{-g} L_{matter} \tag{4.16}$$

The functional in the integral can be viewed as a generalization of the lagrangian of the inflaton model. Note that the kinetic term for the scalar field is not positive definite, therefore this theory hasen't a stable ground state. The system decays toward lower and lower energy states, without a lower bound. For simplicity, I want to consider the function $\omega(\phi)$ constant and $\Lambda = 0$, so that the field equations will be:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) \quad (4.17)$$

$$\Box\phi + \frac{1}{2\omega}\phi R = 0 \quad (4.18)$$

Then, applying the following conformal transformations and scale redefinitions:

$$g_{\mu\nu} \to g'_{\mu\nu} = f'(R)g_{\mu\nu} = \phi g_{\mu\nu} \quad (4.19)$$

$$\phi \to \phi' = \int d\phi \frac{\sqrt{(2\omega+3)}}{\phi} \quad (4.20)$$

the theory with the Einstein-Hilbert action can be recovered, but with the tranformed fields ϕ' and $g'_{\mu\nu}$. This is the theory in the so called Einstein frame. The field equations will assume the usual form:

$$R'_{\mu\nu} - \frac{1}{2}R'g'_{\mu\nu} = 8\pi\left(\nabla'_\mu\phi'\nabla'_\nu\phi' - \frac{1}{2}g'_{\mu\nu}\nabla'^\alpha\phi'\nabla'_\mu\phi'\right) \quad (4.21)$$

$$\Box\phi' = 0 \quad (4.22)$$

Thus, two formulations of scalar–tensor theories are possibles: the version in the so–called Jordan conformal frame and that in the Einstein conformal frame. Both of them can be built for generic Brans-Dicke parameter ω. Note that the theory (4.17-18) has the desidered stability property. The formulations of a scalar–tensor theory in the two conformal frames are physically equivalent. Note that, if it is performed a conformal transformation also for the lagrangian density matter, I must consider, in the Einstein frame, this term multiplied by a factor $exp\{-\alpha\phi'\}$. This anomalous coupling is responsible of a violation of the equivalence principle in the Einstein frame.

4.4 Starobinsky model

My task is explaining the cosmic accelaration at the early stage of the universe, without the need of the dark matter and of the associated field, as an external input of the theory. Thus, I would to reproduce the cosmological effects of dark matter, by use of an energy-impulse tensor provided by a $f(R)$ function itself. The fundamental point is that, defining an effective energy-impulse tensor as:

$$T^{eff}_{\mu\nu} = \frac{1}{k}\left[\frac{f(R) - Rf'(R)}{2}g_{\mu\nu} + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\right] \quad (4.23)$$

the generalized Einstein equation, without matter, can be rewritten in the very instructive form:

$$G_{\mu\nu} = \frac{k}{f'(R)}T^{eff}_{\mu\nu} \quad (4.24)$$

Note that the expression (4.23) hasen't the canonical quadratic form in the first derivative of the field $f'(R)$, but contains terms linear in the second derivatives.

An important requirement to use the $f(R)$ theory to describe the cosmo, is that it must be associated to an universe with smooth transitions between consecutive eras, how it happens in the inflaton field model. Now, the generalized Friedmann equations can be obtained for a generic $f(R)$:

$$H^2 = \frac{8\pi G}{3f'(R)}\frac{\rho}{2}(Rf'(R) - f(R)) - 3H\dot{R}f''(R) \tag{4.25}$$

$$\dot{H} + H^2 = -\frac{8\pi G}{f'}\left[p + \ddot{R}f''' + 2H\dot{R}f''\ddot{R}f'' + \frac{1}{2}(f' - Rf')\right] \tag{4.26}$$

Moreover, the general form of the slow-roll parameters are:

$$\epsilon = \frac{m^2}{2}\left(\frac{d\ln(V)}{\phi}\right)^2 = \frac{(f'(R)R - 2f)^2}{3\,(f'(R)R - f)^2} \tag{4.27}$$

$$\eta = \frac{m^2}{V}\frac{d^2V}{d\phi^2} = \frac{2\,(f'(R)R - 4f)}{3\,(f'(R)R - f)} + \frac{f'(R)^2}{3\,(f'(R)R - f)\,f''(R)} \tag{4.28}$$

The number of e-folds can be calculated as function of ϕ and R:

$$N(\phi) = \frac{1}{m^2}\int d\phi V\left(\frac{dV}{d\phi}\right)^{-1} = \frac{3}{2}\int dR\frac{V}{V'}\frac{f''(R)^2}{f'(R)^2} \tag{4.29}$$

Note that all these formulas are valid for any $f(R)$ theory and hold whenever $f'(R)$ is positive definite and invertible.
Now, the very interesting thing is that, analyzing the general Friedmann equations (4.25-4.26) one could think to define the density $\rho_{eff} = \rho_{eff}(H, f)$ and the $p_{eff} = p_{eff}(H, f)$, associated to the tensor $T^{eff}_{\mu\nu}$, as functions of the Hubble function and of f. In some way the non linear terms depending on R, generate an effective pressure and a effective density.

$$\rho_{eff} = \frac{Rf' - f}{2f'} - \frac{3H\dot{R}f''}{f'} \tag{4.30}$$

$$p_{eff} = \frac{\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')}{f'} \tag{4.31}$$

Now rewriting the equation of state: $\frac{p_{eff}}{\rho_{eff}} = w_{eff}$ and imposing the condition for the dark matter, that is $w_{eff} = -1$, the following relation had to be satisfied:

$$\frac{f'''}{f''} = \frac{\dot{R}H - \ddot{R}}{(\dot{R})^2} \tag{4.32}$$

Substituing the monomyal R^n in $f(R)$ and a power dependence $a_0\left(\frac{t}{t_0}\right)^{\alpha}$ for the cosmic scale factor $a(t)$, an equation for the power n can be obtained:

$$w_{eff} = \frac{6n^2 - 7n - 1}{6n^2 - 9n + 3} \tag{4.33}$$

that for $w_{eff} = -1$ is solved by $n = 2$. Instead the equation for the exponent α is:

$$\alpha = \frac{-2n^2 + 3n - 1}{n - 2} \tag{4.34}$$

that shows that $n \to \infty \Rightarrow \alpha \to \infty$. This last result suggests that a quadratic term in the scalar curvatur is related in some way with a very fastly expansion of the universe. Indeed, I will show later that, it exactly reproduces a de Sitter space-time.
Now, I want to focus on the so called Starobinsky model that contains a linear and a quadratic term in R. It is described by the following action:

$$S = \int d^4x\sqrt{-g}\left(-\alpha m_p^2 R + \beta\frac{m_p^2}{12m^2}R^2\right) \tag{4.35}$$

where m is a dimensional massive parameter, whereas α and β are two constants. If $m \gg m_p$ the Hilbert-Einstein action can be recovered. The general equations of motion for this model are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \gamma l_p^2 R\left(2R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + 2\gamma l_p^2(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) = 0 \tag{4.36}$$

Now, the fundamental point is that performing a conformal trasformation $g'_{\mu\nu} = f'(R)g_{\mu\nu}$, the action (4.35) will assume the well known form of the inflaton field model:

$$S = \int d^4x\sqrt{-g'}\left[\frac{R'}{16\pi G} - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{4.37}$$

where the scalar field ϕ is related with the scalar curvature R by the relation:

$$\phi = \sqrt{\frac{3}{16\pi G}}\ln f'(R) \tag{4.38}$$

and the potential is:

$$V(\phi) = \frac{f'(R(\phi))R(\phi) - f(R(\phi))}{16\pi G f'(R(\phi)^2)} \tag{4.39}$$

Note that, if it is considered only the Einstein-Hilbert term αR, only the 2-helicity component of the perturbed metric tensor field must be taken into account. Instead, for the term $\alpha R + \beta R^2$, also the 0-helicity component of $h_{\mu\nu}$ had to be considered. Thus, the lost degree of freedom due by the scalar field is conserved. Developing the formula (4.38), the relations between R and ϕ can be expressed as:

$$\phi = \sqrt{\frac{3m_p}{16\pi l_p}}\ln(\alpha + 2\gamma l_p^2 R) \tag{4.40}$$

$$R(\phi) = \frac{1}{2\gamma l_p^2}e^{\sqrt{\frac{16\pi l_p}{3m_p}}\phi} - \alpha \tag{4.41}$$

Moreover the potential (4.39) becomes:

$$V(\phi) = \frac{m_p^2}{64\pi l_p^3\gamma}\left(1 - \alpha e^{-\sqrt{\frac{16\pi l_p}{3m_p}}\phi}\right)^2 \tag{4.42}$$

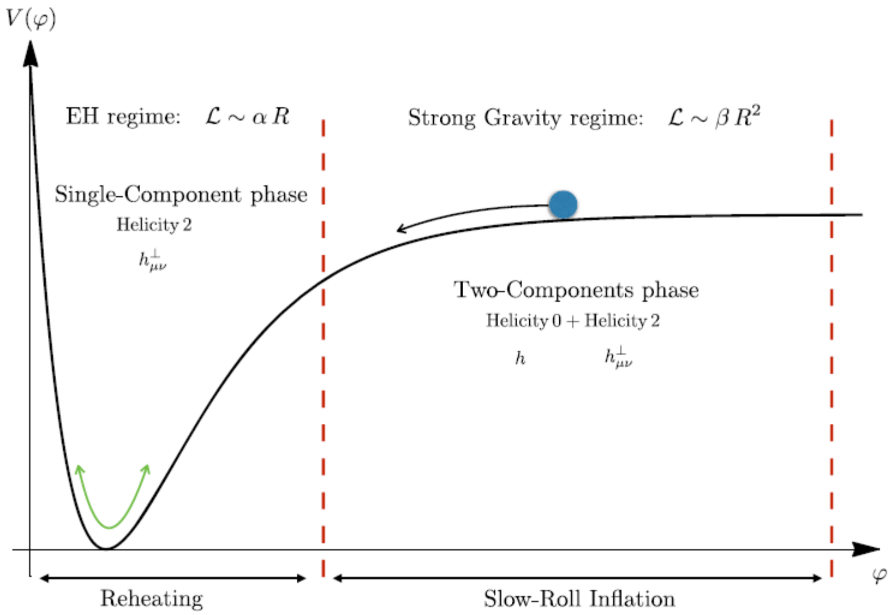

Figura 4.1: Starobinsky's potential

that is the so called Starobinsky's potential (4.1). From the figure it is clear that:

$$f(R) \sim \beta R^2 \quad \alpha \ll 1, \beta \sim 1$$
$$f(R) \sim \alpha R + \beta R^2 \quad \alpha \sim \beta \sim 1$$

Thus, the inflation phase can be recovered by the limits:

$$V(\phi)_{\alpha\to 0} = V(\phi)_{\phi\to 0} = \frac{m_p}{64\pi l_p \gamma} \tag{4.43}$$

that correspond to the $\sim R^2$ regime. Inflation stops as soon as the inflaton goes to the minumum of the potential:

$$\phi_o = \frac{3m_p}{16\pi l_p} \ln \alpha \tag{4.44}$$

The general equation of motion for (4.25) can be rewritten as:

$$6f'(R)H^2 = Rf'(R) - f(R) - 6H\dot{R}f''(R) \tag{4.45}$$

that for $f(R) = \gamma l_p^2 R^2$ simplifies:

$$12RH^2 = R^2 - 12H\dot{R} \tag{4.46}$$

This equation is very important because allows to reproduce the de Sitter space-time. The equation for the scalar curvature is:

$$\frac{R}{12} = \frac{\dot{H} + 2H^2}{2} = H^2 = \frac{\Lambda}{3} \tag{4.47}$$

and inserting directly $a(t)$:

$$R = 6\left(H^2 + \frac{\ddot{a}}{a}\right) \tag{4.48}$$

$$\dot{R} = 6\left(2H\dot{H} + \frac{\dddot{a}a - \ddot{a}\dot{a}}{a^2}\right) \tag{4.49}$$

The equation for H is:

$$H^2 = \frac{\Lambda}{3} - \frac{4H^2\dot{H}}{H^2 + \frac{\Lambda}{3}} \tag{4.50}$$

If H is a constant and equal to $\frac{\Lambda}{3}$, the second term in the equation becomes null and the well know result of the de Sitter universe with $R = 4\Lambda$ is recovered.
Now, I want to study the equation of moto (4.45) for the action (4.35):

$$\frac{6\alpha}{\gamma l_p^2}H^2 + 12RH^2 = R^2 - 12H\dot{R} \tag{4.51}$$

The derivative of the Hubble function is a costant, as expected in the de Sitter inflation:

$$\dot{H} \simeq -\frac{\alpha}{l_p^2\beta} = const. \tag{4.52}$$

The Hubble function is slowly decresing for $\frac{\alpha}{\gamma} \ll 1$ and the slow-roll parameter $\epsilon \simeq -\frac{\dot{H}}{H^2}$ is very small on the plateau.
The fundamental result of the $f(R)$ theory in (4.35) is that, it can be studied by the same action of the inflaton field theory, but with an energy potential, that doesen't constitute an external input, but it is direct consequence of the conformal transformation between the two frames. When R^2 dominates, the background can be approximated by an ideal de Sitter spacetime. This fact is consistent with the scale invariance of the curvature quadratic term. The Einstein-Hilbert term acts as a perturbation to the quadratic one and it is responsible for driving the early universe out of the inflationary phase, in a smooth way as required. Note that, it is impossibile a priori to distinguish between the $f(R)$ action (4.35) and the Starobinsky model by means of the only unperturbed FLRW cosmological model, i.e. by means of only probes that are sensitive to the history of the expansion of the universe. But, analyzing other kind of physical phenomena, like the cosmological perturbation, it is possible to discriminate between the two theories and at the same time, they can be chosen among the equivalent $f(R)$ theories. At this point of the thesis, I want to show in the last chapter a possible quantization of the Starobinsky theory, going to interpret the terms in the action (4.35), as energy potential terms due to a collection of interacting quanta.

4.5 Inflation from gravitons

In the previous chapter I shown that, an inflationary space-time can be interpreted as a system of two kind of quanta's collections. Now, I want to treat this system as composed by only the ensamble of gravitons. This can be done if, to describe the universe, it is used the action (4.35). Indeed it contains only terms associated to the metric tensor field. The cosmological condensate is made of N_g gravitons, which are loosely confined in a potential of the size of their Compton wavelength λ, and interact among them with an effective gravitational coupling $\sim \frac{1}{N_g}$. The starting point to proceed is to analyze the energy of the system, researching a suitable expression for the graviton's hamiltonian $\mathcal{H}_g$. In particular, I want to assume the following association:

$$\begin{aligned} f(R) &\rightarrow \mathcal{H}_g = \mathcal{H}_g^{(1)} + \mathcal{H}_g^{(2)} \\ \alpha R &\rightarrow \mathcal{H}_g^{(1)} = \alpha U_N \\ \beta R^2 &\rightarrow \mathcal{H}_g^{(2)} = \beta(U_N + U_{PN}) \end{aligned}$$

where to the quadratic term is associated also a post-newtonian correction. The hamiltonian constraint for the universe must be:

$$\mathcal{H}_g \simeq U_N + U_{PN} = 0 \tag{4.53}$$

Otherwise from the black hole situation, here the corpuscolar hamiltonian has a matter's energy null $H_M = 0$, since that there is not a source of energy. The Newton energy potential U_N can be obtained making use of the classical description. To this sake, I introduce H_Λ as the solution of the Friedmann equation (1.14) with ρ_Λ and $k = 0$. Integrating over a volume $R_\Lambda^3 = \frac{1}{H_\Lambda^3}$, the following relation can be obtained:

$$R_\Lambda \simeq l_p \frac{M_\Lambda}{m_p} \tag{4.54}$$

where M_Λ can be viewed as generated by N_Λ gravitons with energy:

$$\epsilon_g = -\frac{m_p l_p}{R_\Lambda} \longrightarrow M_\Lambda = m_p l_p \frac{N_\Lambda}{R_\Lambda} \tag{4.55}$$

In the first formula the wavelenghts of the gravitons are assumed to be of order of the size of the Hubble horizon. For this reason the first term of the Newton potential for a coherent state description of the condensate, can be calculated as given by an expanding gravitational field ϕ_N, of the kind of that generated by a static, spherically symmetrical source:

$$\phi_N = -\sqrt{N_\Lambda} \frac{l_p}{R_\Lambda} \tag{4.56}$$

and:

$$U_N = M_\Lambda \phi_N = \epsilon_g N_\Lambda \tag{4.57}$$

where the last equality developes the decomposition of the potential energy in the universe. The total energy is negative since that the gravitational force is "mainly" attractive. Note that, despite it is used the subscript Λ, the particle in the systems are gravitons and not inflatons! The post-Newtonian correction will be positive and will describe the auto-interaction between the constituents of the space-time.

$$U_{PN} = U_N \phi_N = (N_\Lambda \epsilon_g)\phi_N = m_p N_\Lambda^{\frac{3}{2}} \frac{l_p^2}{R_\Lambda^2} \tag{4.58}$$

This term describes a repulsive force, that balances the attraction of the gravitons in the condensate. It is strong for small sizes, but decreases faster of U_N, as the distance increases. The post-post Newtonian correction can be calculated in a iterative way, multiplicand the previous post-Newton term by the Newton potential. Now, integrating over an Hubble volume the terms U_N and U_{PN} (4.57-4.58) will be:

$$U_N \simeq -R_\Lambda^3 H_\Lambda^2 = -R_\Lambda \tag{4.59}$$

$$U_{PN} \simeq +R_\Lambda^3 \left(\frac{\Lambda}{3}\right) = R_\Lambda \tag{4.60}$$

These expressions must be generalized for a generic Hubble function, that not solves the equation (1.14):

$$U_N \simeq -R^3 H^2 \tag{4.61}$$

$$U_{PN} \simeq R^3 R_\Lambda^{-2} \tag{4.62}$$

Inserting this relation in the constrained equation (4.53) and using the association for $f(R)$ seen before, an equation for H can be found:

$$R^3 \left[-(\alpha+\beta)H^2 + \frac{\beta}{R_\Lambda^2}\right] \simeq 0 \tag{4.63}$$

with solution:

$$H^2 \simeq \frac{\beta}{\alpha+\beta}\frac{1}{R_\Lambda^2} \tag{4.64}$$

From this relation it can be noted that, the usual relation between the Hubble function and the Hubble radius is slightly modified. Because of the term proportional to α, the de Sitter metric is not more a solution, but for $\alpha = 0$ it can be recovered. Moreover, the condition $\alpha > 0$ implies that $H < H_\Lambda$. This means that if the system begins with $H = H_\Lambda$, the time derivative of H is always negative. The slow roll parameter assumes the form $\epsilon(\alpha, \beta)$:

$$\epsilon \simeq -\frac{\dot{H}}{H^2} \simeq \frac{\alpha}{\alpha+\beta} \tag{4.65}$$

Also this formula shows that, for $\alpha = 0$, the condition for eternal inflation is manteined. Substing the relation (4.65) in the depletion formula (3.69) the following relation can be obtained:

$$\frac{\dot{N}_g}{N_g} = \frac{\alpha H}{\alpha+\beta} - \frac{H}{N}\sqrt{1+\frac{\beta}{\alpha}} \tag{4.66}$$

This formula shows how the gravitons depletion is a consequence of the paramters of the $f(R)$ gravity theory. With these new results the inflation can be studyied in a very similar way of the quantum inflaton field approch: at the beginning mostly gravitons are in the condensate, so that the system is in the regime R^2, with the R-term negligible. As soon as the condensate evaporates, because of the scattering processes, there will be the regime in which R is dominant, with the consequence that inflation begins to stop. The depletion's term given by quantum effects is:

$$\frac{\dot{N}\Lambda}{\sqrt{\dot{N}_\Lambda}} = l_p \dot{M}_\Lambda \simeq -\frac{m_p^2}{M_\Lambda^2} \simeq -l_p^2 H^2 \simeq -\frac{\beta}{\beta+\alpha}\frac{1}{N_\Lambda} \tag{4.67}$$

Now, it can be noted that the minimum distance (3.74) from a pure de Sitter space-time, can be expressed as depending on α. Indeed, in the inflationary phase $\alpha \sim \epsilon$, thus it is running from the minimum value $\frac{1}{R_\Lambda^3}$ to the maximum, that is one. This means that the corpuscolar model, not allows the limit to obtain the only quadratic term. Including the effects of depletion, a more precise restriction on α can be obtained:

$$\left(\frac{l_p}{R_\Lambda}\right)^{\frac{4}{3}} \lesssim \alpha \lesssim 1 \tag{4.68}$$

An important consideration is about the creation of gravitational waves during the inflation phase. These are given by depletion of the graviton's condensate! The primordial tensor perturbation P_T can be calculated in the context of the classical inflaton theory as:

$$P_T \sim l_p^2 H^2 \sim \frac{l_p^2}{R_{\Lambda^2}} \tag{4.69}$$

The relative correction will be:

$$\frac{\Delta P_T}{P_T} \simeq \frac{H\delta H}{H^2} \sim \frac{\dot{H}\delta t}{H} \sim -\epsilon\left(1 - \frac{1}{\epsilon^{\frac{3}{2}} N_\Lambda}\right)$$

where the time variation δt is of the order of the Hubble radius. The above correction vanishes at the beginning of the inflation, that is for $\epsilon \simeq \epsilon*$ and then goes on as $\frac{\dot{N}_\Lambda}{N_\Lambda}$. When the corrections become of the order of the tensor perturbation the inflation ends.

Another consideration can be done about the cosmological redshift in the framework shown in this section. This effect is a consequence of the form of the cosmic scale factor, determinated by the three expressions on the right-hand side in (3.97). Now, the evolution of the Hubble function can be see as depending on the parameters of the action (4.35) and of the graviton's hamiltonian:

$$H = \frac{\dot{a}}{a} = \frac{\dot{N}}{\frac{\alpha N}{\alpha+\beta} - \sqrt{1+\frac{\beta}{\alpha}}} \tag{4.70}$$

In particular, considering only the classical evolution term in the depletion formula, the cosmic scale factor will be:

$$a(t) = exp\left\{\int^t dt' \frac{\dot{N}}{N}\left(1+\frac{\beta}{\alpha}\right)\right\} \tag{4.71}$$

so that:

$$r_\omega = exp\left\{\int_{t_1}^{t_2} dt' \frac{\dot{N}}{N}\left(1+\frac{\beta}{\alpha}\right)\right\} \tag{4.72}$$

Instead, in the full quantum picture, the redshift formula will be:

$$r_\omega = exp\left\{\int_{t_1}^{t_2} \frac{\dot{N}}{\frac{\alpha N}{\alpha+\beta} - \sqrt{1+\frac{\beta}{\alpha}}}\right\} \tag{4.73}$$

Thus, the redshift effect can be seen as a consequence of the depletion of the graviton's condensate, described at the same time by means of a corpuscolar hamiltonian .
To conclude, I want to analyze a possible corpuscolar interpretation of a $f(R)$ function of the kind of (4.8), with N monomyal terms:

$$f(R) = \sum_{k=1}^{N} a_k l_p^{2k-2} R^k \tag{4.74}$$

Substituing this expression in the generalized Friedmann equation (4.45), it can be obtained:

$$H^2\left(a_1 + \sum_{k=2}^{N} k a_k l_p^{2k-2} R^{k-1}\right) \simeq \sum_{k=2}^{N} (k-1) a_k l_p^{2k-2} R^k \tag{4.75}$$

Using $R \simeq \Lambda$ and $n = k - 1$:

$$a_1 H^2 \simeq \sum_{n}^{N-1} (\Lambda - H^2) n a_{n+1} l_p^{2n} \Lambda^n \tag{4.76}$$

Using this formula with the relations for the Newton and post-Newton potential (4.59-60), the Hubble function and the cosmological constant can be rewritten as:

$$H^2 \simeq -\frac{l_p}{m_p}\frac{U_N}{R_\Lambda^3} \tag{4.77}$$

$$\Lambda \simeq \frac{l_p}{m_p}\frac{U_{PN}}{R_\Lambda^3} \tag{4.78}$$

Then:

$$a_1 U_N + (U_N + U_{PN}) \sum_{n=1}^{N-1} n a_{n+1} \left(\frac{l_p^3 U_{PN}}{m_p R_\Lambda^3}\right)^n \simeq 0 \tag{4.79}$$

Naively, for the Starobinsky model ($N = 2$):

$$a_1 U_N + a_2 \left(\frac{l_p^3 U_{PN}}{m_p R_\Lambda^3}\right)(U_N + U_{PN}) \simeq 0 \tag{4.80}$$

Conclusions

The universe is the arena of all the physical systems. Classically it is a simple box where physical phenomena occur. The General Theory of Relativity changes drastically this vision. In this framework the space-time becomes a dynamical entity describing the collection of all the events and interacts with the physical systems by a characteristic field equation. This means that the relative distances between events are not fixed, but depend on the systems in the universe, and viceversa affect the motion of them. Thus, the universes are not all the same, but one could study their different form and evolution. This is, roughly speaking, the object of cosmology.
My purpose in this thesis has been to analyze in depthe the so called inflationary universe, trying to combine the classical cosmological description with the quantum one. Every physical fundamental entity has a dual behaviour, as a field and as a collection of particles. So that, one can think that this is true also for the universe background, for which the field is the metric $g_{\mu\nu}$ of the space-time. Whereas, the quantum counterpart will be a quantum Fock space $\mathcal{F}$, wich contains all the possible states for the collections of particles, called gravitons, associated to the background:

$$\text{classical background} \rightarrow \text{quantum state}$$
$$g_{\mu\nu} \rightarrow |\text{ gravitons }\rangle \in \mathcal{F}$$

In particular, starting from the quantum description of the inflationary space-time, one would to recover the classical description given by the background field. This matching between the two theories is possible, if one considers the small perturbation field on top of the Minkowsky metric, rather than the complete metric $g_{\mu\nu}$ and if the number of quanta of this approximate field is very large and weakly interacting. Then, one can study separately the quantum interactions between gravitons, trying to understand how they can affect the description of the background metric and the cosmological parameters.
I have shown that the graviton'system associated to the inflationary space-time, is a Bose-Einstein condensate on a verge of a phase transition. Its constituents evaporate with a well defined depletion formula, that combines a classical and a pure quantum term. This latter is due by scattering processes, with a magnitude of the order of the inverse of the number of particles. A very interesting consequence is that, if the condensate has a finite number of gravitons, there is no limit in which one can recover the de Sitter space-time. Moreover, in the infaltionary quantum model, the field associated to the matter that causes the inflation, must be considered. It is called inflaton field and the quanta are named inflatons. I have shown that also the system of inflatons is a critical Bose-Einstein condensate,

whose equation of motion describes its evaporation. By this description, classical and semiclassical quantities can be recovered, as the Hubble horizon, the potential energy, the entropy etc. In particular, I wanted to analyze how the quantum description modifies the form of the cosmic scale factor, leading to a different formula for the redshift effect.
In the last chapter, I studied an alternative model for the inflationary universe, that arises from a more general gravity lagrangian. I showed how this model generates a very interesting classical inflatonary theory, without the inflaton field. This means that inflation could be also thought of as an intrinsic phenomenon of a more general Einstein theory. Moreover, I have quantized this model associating to the modified Lagrangian an Hamiltonian for a collection of particles and I showed how the resulting formulas are very similar to the quantum description with inflaton particles.

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