Alma Mater Studiorum · Università di Bologna

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CANONICAL FORMALISM FOR COMPACT SOURCES

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Anno Accademico 2018/2019

To W.W.

My Star, My Perfect Silence.

Abstract

This thesis aims to describe the ADM formalism of General Relativity and to use the latter to describe a spherical compact source consisting of a perfect fluid. With two different choices for three-dimensional metric on hypersurfaces, we analyze the constraints of the system in the non-static case and the resulting equations of motion, both for canonical gravitational variables and those of matter. After examining some special cases, we also show that it is possible, in the case of static nature, to obtain the value of the Misner-Sharp mass from the Hamiltonian constraint, while near the trapping surfaces we obtain a relationship between the density of matter and the dynamic variables of the metric. Finally we propose a possible method for quantizing the constraints using the procedure that in the vacuum leads to the Wheeler-DeWitt equations.

Contents

Introduction

The ADM formalism is an approach to General Relativity and to Einstein equations that relies on the slicing of the four-dimensional spacetime by three-dimensional hypersurfaces. These latters have to be spacelike, so that the metric induced on them by the Lorentzian spacetime metric is Riemannian.
From the mathematical point of view, the ADM formalism allows to formulate the problem of resolution of Einstein equations as a Cauchy problem with constraints, while from the physical point of view it amounts to a decomposition of spacetime into "space" + "time", so it is also known as "3+1 formalism". As a matter of fact, one manipulates only time-varying tensor fields in the three-dimensional space, where the standard scalar product is Riemannian.
The ADM formalism originates from works by Georges Darmois in the 1920's, Andrè Lichnerowicz in the 1930-40's and Yvonne Choquet-Bruhat in the 1950's. Notably, in 1952, Yvonne Choquet-Bruhat was able to show that the Cauchy problem arising from this decomposition has locally a unique solution. In the late 1950's and early 1960's, the formalism received a considerable impulse, serving as foundation of Hamiltonian formulations of general relativity by Paul A.M. Dirac, Richard Arnowitt, Stanley Deser and Charles W. Misner (ADM). In the 1970's, the ADM formalism became the basic tool for the nascent numerical relativity. Today, most numerical codes for solving Einstein equations are based on the 3+1 formalism.
More recently, in order to describe the gravitational radius of spherically symmetric compact sources and determining the existence of a horizon in a quantum mechanical fashion, the Horizon Quantum Mechanics [32] has been introduced. The reason we are interested in compact spherical sources is that in a classical spherically symmet-

ric system, the gravitational radius defined in terms of the (quasi-)local Misner-Sharp mass, uniquely determines the location of the trapping surfaces [30,31], where the null geodesic expansion vanishes. In a non-spherical space-time, such as the one generated by an axially-symmetric rotating source, although there are candidates for the quasi-local mass function that should replace the Misner-Sharp mass, the locations of trapping surfaces, and horizons, remain to be determined separately.

The first two Chapters of this thesis are devoted to the description of the essential elements of the General Relativity theory and the concept of gravitational collapse in the latter. Chapter 3 is devoted to the description of constrained systems, while Chapter 4 illustrates the initial value formulation in Einstein's theory. Chapter 5 is dedicated to the description of the ADM formalism and in Chapter 6 this formalism is used in order to describe a compact spherical source of perfect fluid, also introducing a local mass in the diagonal radial component of the metric. Finally, in Chapter 7 we briefly mention the problem of quantization of the General Relativity and we apply the "Wheeler-DeWitt quantization" to the constraints thus obtained.

Chapter 1

Einstein's field equations and Schwarzschild solution

The purpose of this Chapter, in which we use conventions of [1], is to briefly illustrate the field equations of General Relativity and assumptions that Einstein made in the early twentieth century in order to develop the latter, to make gravity consistent with Special Relativity. Then, we describe the Schwarzschild solution, which according to Birkhoff's theorem [5] is the most general spherically symmetric vacuum solution of the Einstein field equations: it describes the gravitational field outside a spherical mass, on the assumption that the electric charge and the angular momentum of the latter (toghether with universal cosmological constant) are all zero.

1.1 Einstein's equations

The gravitational field equations were obtained by Einstein in 1915, after about ten years of research, and they can be written as:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.1}$$

Where $R_{\mu\nu}$ is the *Ricci tensor*, defined as the only possible non trivial contraction of the *Riemann tensor*:

$$R_{\mu\nu} = g^{\lambda\rho}R_{\lambda\mu\rho\nu} = R^{\rho}{}_{\mu\rho\nu} \tag{1.2}$$

$$R^{\lambda}{}_{\mu\rho\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\lambda}_{\mu\rho}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\sigma\rho}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho} \tag{1.3}$$

In which $\Gamma^{\sigma}_{\lambda\mu}$ is the *affine connection.* It turns out that this latter can be related to the partial derivatives of the metric tensor as:

$$\Gamma^{\nu}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\sigma\lambda}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\lambda}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\sigma}} \right\} \tag{1.4}$$

R is the *curvature scalar*, defined as the contraction of the Ricci tensor:

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu} \tag{1.5}$$

Finally, G is the gravitational constant and $T_{\mu\nu}$ the energy-momentum tensor of the system, which is the source of gravitational field. We have also written the term $\Lambda g_{\mu\nu}$, that corresponds to the cosmological constant, on the left side of the equation as Einstein did in the beginning of the 20th century (1917). In modern cosmology it is convenction to write it on the right side (interpreting it as "dark energy" due to the matter). On the four coordinates x^{μ}, it is possible to operate an arbitrary transformation, which allows to choose four between the ten components of $g_{\mu\nu}$, leaving six of the latter to be determined from the field equations. Furthermore, the four components of four-vector velocity $U^{\mu} \equiv dx^{\mu}/ds$ which enter in the definition of $T_{\mu\nu}$, are subject to the relation $U^{\mu}U_{\mu} = 1$, thus only three of them are independent.

So, as expected, there are ten partial differential equations (1.1) for ten functions to be determined: six components of $g_{\mu\nu}$, three components of U^{μ} and the density of the matter ω/c^2 (or its pressure p). These equations were originally derived by assuming the following requirements:

- Since the choice of the reference system is arbitrary, laws of nature must be formally the same for any coordinate system (x^0, x^1, x^2, x^3), thus field equations should be tensor equations in order to exhibit this covariance property.

- They must be (like all the other field equations of physics) partial differential equations of, at most, second order in time for the components of the metric tensor $g_{\mu\nu}$, which are the functions to be determined in the present context.

- We assume that spatial derivatives are of, at most, second order and the equations must be also linear in the highest derivatives.

- In the appropriate limit, they should go over to the Poisson equation of the Newtonian theory:

$$\nabla^2\phi = 4\pi G\rho \tag{1.6}$$

  In which ϕ is the potential of the gravitational field and ρ is the mass density of the source. The so called "Newtonian limit" is obtained by assuming a weak and static field and that the velocities of the sources of the latter are very small compared to the velocity of light c

- The energy-momentum tensor $T_{\mu\nu}$ should be the source of the gravitational field (Einstein was guided by the analogy with special relativity, in which $T_{\mu\nu}$ is the analogue of the mass density)

It is important to emphasize that, since the geometric properties of the space-time (i.e. its metric) are determined from physics phenomena in General Relativity and they are not invariable properties of space and time, the exact definition of "reference system" in General Relativity turns out to be an infinite continue system of bodies to which, in every point, a clock is bound that marks an arbitrary time. In Special Relativity the same notion consists simply in a set of bodies at rest with respect to each other and rigidly bound. We also note that the formal invariance of the laws of nature does not mean that all the reference systems are physically equivalent (as for the equivalence of the latters in Special Relativity).

1.2 Schwarzschild external solution

We are, at first, intersted in the solution of the Einstein's field equation for a system composed by a statical spherical source in the vacuum, which means with no other sources of gravitational field, characterized by an energy-momentum tensor $T_{\mu\nu}$.
In simple terms, a solution is said *stationary* if it is time-independent [3]. This doesn't mean that the solution is in no way evolutionary but simply that time does not enter into it explicitly. On the other hand, the stronger requirement that a solution is *static* means that it can't be evolutionary: in such a case, nothing would change if at any time we ran time backward, thus "static" means time-symmetric about any origin of time.

The field equations in the vacuum become:

$$R_{\mu\nu} = 0 \tag{1.7}$$

and they must be solved in order to find the metric components $g_{\mu\nu}$, "deformed" by the field, outside the spherical source that generates the gravitational field itself. The field equations (1.7) are subject to the four geometrical relations:

$$R^{l}{}_{m;\,l} \equiv \frac{\partial R^{l}{}_{m}}{\partial x^{l}} + \Gamma^{l}_{l\sigma} R^{\sigma}{}_{m} - \Gamma^{\sigma}_{ml} R^{l}{}_{\sigma} = \frac{1}{2}\frac{\partial R}{\partial x^{m}} \tag{1.8}$$

that are obtained from the contraction of the Bianchi identities, thus as was anticipated there are six independent field equations for six corresponding unknown functions $g_{\mu\nu}$.
The gravitational field must exhibit central simmetry as the source that produces it, so it is convenient to use a coordinate system which is adapted to the sperichal simmetry of the latter. The central simmetry of the field means that the space-time metric $g_{\mu\nu}$ or, equivalently, the interval

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{1.9}$$

which is the analogue of the interval $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ in a non-inertial reference system, must have the same value in all the points with the same distance from the center of the coordinate system, that is where the source of the field is supposed to be located. In doing so, we can write the most general line element in cartesian coordinates which turns out to be invariant for rotations as:

$$ds^2 = I(r,t)dt^2 + B(r,t)(\vec{x}\cdot d\vec{x})dt + D(r,t)(\vec{x}\cdot d\vec{x})^2 + C(r,t)(d\vec{x})^2 \tag{1.10}$$

In fact, the line element must be quadratic in the differentials, and $(\vec{x}\cdot\vec{x})$, $(\vec{x}\cdot d\vec{x})$, $(d\vec{x}\cdot d\vec{x})$ are the only possible non trivial terms that turns out to be invariant for rotations, while their coefficients in (1.10) are arbitrary functions of r and t. It turns out that in General Relativity the choice of the reference system is not subject to any kind of limitations: the role of the three spatial coordinates (x^1, x^2, x^3) can be assigned to whichever quantities which define the position of the objects in the space. Similarly, the temporal coordinate x^0 can be determined by a clock that signs the "time" t in an arbitrary way.
By writing the line element (1.10), i.e. only supposing central simmetry, we have reduced

the number of functions from 6 to 4 to solve the problem. Writing the same line element in spatial "spherical" $(x^1, x^2, x^3) = (r, \theta, \phi)$ coordinates we have

$$ds^2 = I(r,t)dt^2 + B(r,t)rdrdt + D(r,t)r^2dr^2 + C(r,t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) \quad (1.11)$$

and, for convenience, we re-write the the last expression by introducing other arbirary functions, which are defined in terms of the previous ones, as:

$$ds^2 = I(r,t)dt^2 + A(r,t)drdt + H(r,t)dr^2 + K(r,t)(\sin^2\theta d\varphi^2 + d\theta^2) \quad (1.12)$$

in which

$$B(r,t)r = A(r,t), \quad D(r,t)r^2 + C(r,t) = H(r,t), \quad C(r,t)r^2 = K(r,t) \quad (1.13)$$

The field equations must be invariant with respect to general coordinate transformations, thus we can now transform the coordinates r and t in order to preserve the central simmetry of ds^2 as:

$$\begin{cases} r = f(r', t') \\ t = g(r', t') \end{cases} \quad (1.14)$$

where $f(r', t')$, $g(r', t')$ are arbitrary functions of the new coordinates (r', t') . We choose r and t in order to have

$$\begin{cases} A(r,t) = 0 \\ K(r,t) = -r^2 \end{cases} \quad (1.15)$$

By making this assumption, it turns out that r is determined in order to have the lenght of the circonference with center in the origin of the coordinate system equal to $2\pi r$ (in analogy with an euclidean space), while it is still possible to transform the temporal coordinate t as $t \to f(t')$. This feature will be soon used in order to simplify the expression of ds^2. We now set:

$$\begin{cases} H(r,t) = -e^{\lambda(r,t)} \\ I(r,t) = c^2e^{\nu(r,t)} \end{cases} \quad (1.16)$$

In order to write the line element as:

$$ds^2 = c^2e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (1.17)$$

If we suppose that $(x^0, x^1, x^2, x^3) \equiv (ct, r, \theta, \varphi)$, we have the following form for the metric tensor:

$$g_{\mu\nu} = diag(e^{\nu}, -e^{\lambda}, -r^2, -r^2 \sin^2\theta) \tag{1.18}$$

Its inverse can be found from $g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu}$ and it simply results as:

$$g^{\mu\nu} = diag(e^{-\nu}, -e^{-\lambda}, -r^{-2}, -r^{-2}\sin^{-2}\theta) \tag{1.19}$$

Now we are able to determine the affine connection by using (1.18) and (1.4), the calculation is straightforward:

$$\Gamma^0_{\mu\nu} = \begin{pmatrix} \frac{\dot{\nu}}{2} & \frac{\nu'}{2} & 0 & 0 \\ \frac{\nu'}{2} & \frac{\dot{\lambda}}{2}e^{\lambda-\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^1_{\mu\nu} = \begin{pmatrix} \frac{\nu'}{2}e^{\nu-\lambda} & \frac{\dot{\lambda}}{2} & 0 & 0 \\ \frac{\dot{\lambda}}{2} & \frac{\lambda'}{2} & 0 & 0 \\ 0 & 0 & -re^{-\lambda} & 0 \\ 0 & 0 & 0 & -re^{-\lambda}\sin^2\theta \end{pmatrix}$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \qquad \Gamma^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{1}{\tan\theta} \\ 0 & \frac{1}{r} & \frac{1}{\tan\theta} & 0 \end{pmatrix} \tag{1.20}$$

where $a' \equiv da/dr$, $\dot{a} \equiv da/d(ct)$.

We can express the Ricci tensor, according to (1.2) and (1.3), as:

$$R_{\mu\nu} = \frac{\partial\Gamma^{\sigma}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial\Gamma^{\sigma}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} \tag{1.21}$$

Now, calculating the Ricci's tensor components with (1.21) and after replacing the results in the field equations (1.1), we obtain the following system:

$$\begin{cases} \frac{8\pi G}{c^4}T^1_1 = -e^{-\lambda}(\frac{\nu'}{r} + \frac{1}{r^2}) + \frac{1}{r^2} \\ \frac{8\pi G}{c^4}T^2_2 = -\frac{1}{2}e^{-\lambda}(\nu'' + \frac{\nu'^2}{2} + \frac{\nu'-\lambda'}{r} - \frac{\nu'\lambda'}{2}) + \frac{1}{2}e^{-\nu}(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}) \\ \frac{8\pi G}{c^4}T^3_3 = \frac{8\pi G}{c^4}T^2_2 \\ \frac{8\pi G}{c^4}T^0_0 = -e^{-\lambda}(\frac{1}{r^2} - \frac{\lambda'}{r}) + \frac{1}{r^2} \\ \frac{8\pi G}{c^4}T^1_0 = -e^{-\lambda}\frac{\dot{\lambda}}{r} \end{cases} \tag{1.22}$$

This equations are integrable in the configuration that we have considered, namely a gravitational field with central simmetry in the vacumm and out of the source that produces it. As a matter of fact setting $T_{\mu\nu} = 0$ in (1.22) we obtatin the following equations:

$$\begin{cases} e^{-\lambda}(\frac{\nu'}{r} + \frac{1}{r^2}) - \frac{1}{r^2} = 0 \\ e^{-\lambda}(\frac{\lambda'}{r} - \frac{1}{r^2}) + \frac{1}{r^2} = 0 \\ \dot{\lambda} = 0 \end{cases} \tag{1.23}$$

From the third equation we see that λ does not depend on t, while from the first and the second ones we see that $\lambda' + \nu' = 0$, which implies that:

$$\lambda + \nu = f(t) \tag{1.24}$$

So we use the possibility to operate a transformation on the variable t of the form $t = f'(t')$ in order to set $f(t) = 0$ in the last expression. We note that a gravitational field with central simmetry in the vacuum automatically becomes static. The second equation in (1.23) gives:

$$e^{-\lambda} = e^{\nu} = 1 + \frac{constant}{r} \tag{1.25}$$

As we espected, for $r \to \infty$ we recover the galileian metric, that is $g_{00} = 1$. To fix the *constant* we impose that, at large distancies (where the gravitationl field is weak), we must find the Newtonian law:

$$g_{00} = 1 + \frac{2\varphi}{c^2} \tag{1.26}$$

where

$$\varphi = -GM/r \tag{1.27}$$

is the Newtonian classical potential and M is the mass of the spherical source of the field, so we conclude that:

$$-costant \equiv r_g = \frac{2GM}{c^2} \tag{1.28}$$

This quantity is usually called *gravitational radius*, *Schwarzschild radius* or *gravitational radius* of the source. For normal stars or planets r_g is very small in relation to the geometrical radius: the Schwarzschild radius of the Sun, for example, has the value

$r_g = 2.96\,km$, while for the Earth one finds $r_g = 8.8\,mm$.
Therefore, it is possible to derive the following form of the space-time metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1.29}$$

This solution of the Einstein's equation, which completely determines the gravitational field in the vacuum produced by any distribution of the masses with central simmetry, was found in 1916 from K. Schwarzschild and it depends only on the total mass M of the gravitational source of the field (as for the Newtonian theory). This solution is valid not only for masses at rest but also for masses with central simmetry motion. Since the Schwarzschild metric describes only the gravitational field outside the matter distribution, whilst the Schwarzschild radius mostly lies far in the interior, we shall suppose that $r \gg r_g$.
It is possible to show [1] that, for a spherical source with radius a that generates a gravitational field in the vacuum, the total mass M of the latter is given by:

$$M = \frac{4\pi}{c^2} \int_0^a T_0^0 r^2 dr \tag{1.30}$$

In particular, for the static distribution of the matter one has $T_0^0 = \omega$, thus

$$M = \frac{4\pi}{c^2} \int_0^a \omega r^2 dr \tag{1.31}$$

where the integration is done in $4\pi r^2 dr$, while the spatial volume element in the metric (1.17) is $dV = 4\pi e^{\lambda/2}/dr$. Since one can show that $e^{\lambda/2} > 1$ we see that this difference expresses the "gravitational mass defect" of the body.

1.2.1 The interior Schwarzschild solution

We are now intersted in determine the spherically simmetric gravitational field inside the matter and using a comoving coordinate system. A model for the matter is required, thus is necessary to say something about its energy-momentum tensor. Ignoring thermodynamic effects, such as heat conduction and viscosity, a useful approximation is the ideal fluid medium, which in comoving coordinates has the following form:

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \tag{1.32}$$

Applications in Riemannian spaces often deal with a velocity field $U^{\mu} = dx^{\mu}/d\sigma$ (a flux of bodies, or of observers). Since σ is a coordinate-independent parameter, the components of this velocity transform like the coordinate differential

$$U^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} U^{\mu'} \tag{1.33}$$

By means of a coordinate transformation it is always possible to make the three spatial components u^i of the velocity zero, since the differential equations

$$\begin{aligned} U^1 &= \frac{\partial x^1}{\partial x^{0'}} U^{0'} + \frac{\partial x^1}{\partial x^{1'}} U^{1'} + \frac{\partial x^1}{\partial x^{2'}} U^{2'} + \frac{\partial x^1}{\partial x^{3'}} U^{3'} = 0 \\ U^2 &= \frac{\partial x^2}{\partial x^{0'}} U^{0'} + \frac{\partial x^2}{\partial x^{1'}} U^{1'} + \frac{\partial x^2}{\partial x^{2'}} U^{2'} + \frac{\partial x^2}{\partial x^{3'}} U^{3'} = 0 \\ U^3 &= \frac{\partial x^3}{\partial x^{0'}} U^{0'} + \frac{\partial x^3}{\partial x^{1'}} U^{1'} + \frac{\partial x^3}{\partial x^{2'}} U^{2'} + \frac{\partial x^3}{\partial x^{3'}} U^{3'} = 0 \end{aligned} \tag{1.34}$$

always have a solution $x^i(x^{\mu'})$. In the resulting coordinate system, the particles do not change their position and the coordinates move with the particles (one can visualize the coordinate values attached to the particles as names). Although the coordinate difference of two particles never alters, their separation can vary because of the time-dependence of the metric.

We seek a spherically symmetric gravitational field and thus, in the line element (1.12), we use the two possible transformations of the coordinates r and t in order to set $A(r,t)$ and, according to the discussion above, the radial component of the velocity equal to zero in every point of the space-time.

Defining the following coordinates for the comoving system:

$$(x^0, x^1, x^2, x^3) \equiv (c\tau, \rho, \theta, \varphi) \tag{1.35}$$

and setting:

$$I(r,t) \equiv e^{\nu(\tau,\rho)} \quad H(r,t) \equiv -e^{\lambda(\tau,\rho)} \quad K(r,t) \equiv -e^{\mu(\tau,\rho)} \tag{1.36}$$

we are driven to the following expression for the line element:

$$ds^2 = c^2 e^\nu d\tau^2 - e^\mu(\sin^2\theta d\varphi^2 + d\theta^2) - e^\lambda d\rho^2 \tag{1.37}$$

The field equations turn out to be (setting $' \equiv \frac{\partial}{\partial\rho}$ and $\cdot \equiv \frac{\partial}{\partial(c\tau)}$):

$$\begin{cases} -\frac{8\pi G}{c^4}T^1_1 = \frac{8\pi G}{c^4}p = \frac{1}{2}e^{-\lambda}(\frac{\mu'^2}{2} + \mu'\nu') - e^{-\nu}(\dot{\mu} - \frac{1}{2}\dot{\mu}\dot{\nu} + \frac{3}{4}\dot{\mu}^2) - e^{-\mu} \\ -\frac{8\pi G}{c^4}T^2_2 = \frac{8\pi G}{c^4}p = \frac{1}{4}e^{-\lambda}(2\nu'' + \nu'^2 + 2\mu'' + \mu'^2 - \mu'\lambda' - \nu'\lambda' + \mu'\nu') \\ \qquad\qquad + \frac{1}{4}e^{-\nu}(\dot{\lambda}\dot{\nu} + \dot{\mu}\dot{\nu} - \dot{\lambda}\dot{\mu} - 2\ddot{\lambda} - \dot{\lambda}^2 - \dot{\mu} - \dot{\mu}^2) \\ \frac{8\pi G}{c^4}T^0_0 = \frac{8\pi G}{c^4}\omega = -e^{-\lambda}(\mu'' + \frac{3}{4}\mu'^2 - \frac{\mu'\lambda'}{2}) + \frac{1}{2}e^{-\nu}(\dot{\lambda}\dot{\mu} + \frac{\dot{\mu}^2}{2}) + e^{-\mu} \\ \frac{8\pi G}{c^4}T^1_0 = 0 = \frac{1}{2}e^{-\lambda}(2\dot{\mu}' + \dot{\mu}\mu' - \dot{\lambda}\mu' - \nu'\dot{\mu}) \end{cases} \tag{1.38}$$

These are only integrable if the balance equations of energy and momentum (implicitly contained in the field equations):

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{1.39}$$

are satisfied. These conservation laws often give an important indication of how to solve the field equations, in fact using them one can obtain some general equations for λ, μ, ν:

$$\dot{\lambda} + 2\dot{\mu} = -\frac{2\dot{\omega}}{p+\omega} \qquad \nu' = -\frac{2p'}{p+\omega} \tag{1.40}$$

If an equation of state $p = p(\omega)$ is kown, it is possible to integrate them in order to obtain:

$$\lambda + 2\mu = -2\int\frac{d\omega}{p+\omega} + f_1(\rho) \qquad \nu = -\int\frac{dp}{p+\omega} + f_2(\tau) \tag{1.41}$$

where $f_1(\rho)$ and $f_2(\tau)$ can be chosen arbitrarily since one still has the possibility to perform transformations of the type $\rho = \rho(\rho')$ and $\tau = \tau(\tau')$.

Chapter 2

Gravitational Collapse of a Spherically Symmetric Source

In this Chapter, in which again we use conventions of [1], we will briefly describe the phenomenon of gravitational collapse in General Relativity, analyzing the latter in a reference system solidary with matter. In particular, the exact solution of the field equations for gravitational collapse for a source with spherical symmetry and consisting of dust (originally found by Tolman in 1934) is described.

2.1 Collapse in a comoving reference system

If, during its evolution, a massive, spherically symmetric star does not succeed in ejecting or radiating away suffcient mass to become a neutron star, then there is no stable, final state available to it. At some time or other it will reach a state in which the pressure gradient can no longer balance the gravitational attraction and, as a consequence, it will continue to contract further and its radius will pass the Schwarzschild radius (1.29) and tend to $r = 0$: in such a case, the star suffers a *gravitational collapse*.
A quick glance at the Schwarzschild metric (1.29) shows that g_{00} goes to zero and a singularity of the metric tensor's component g_{11} is present at $r = r_g$. In our earlier discussion of the Schwarzschild metric we had set this problem aside with the remark that the radius r_g lies far inside a celestial body, where the vacuum solution is of course

no longer appropriate. Now, however, we shall turn to the question of whether and in what sense there is a singularity of the metric at $r = r_g$ and what the physical aspects of this are.

A singular coordinate system can give a false indication of a singularity of the space. If the metric is singular at a point, one investigates whether this singularity can be removed by introducing a new coordinate system. Or, appealing more to physical intuition, one asks whether a freely falling observer can reach this point and can use a local Minkowski system there. If both are possible, then the observer notices no peculiarities of the physical laws and phenomena locally, and hence there is no singularity present.

Soon after the Schwarzschild metric had been obtained as a solution of the field equations, it was recognized that both the determinant of the metric $-detg_{\mu\nu} = r^4 \sin^2\theta$ and also the *Kretschmann scalar* $K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48m^2}{r^6}$ associated with the curvature tensor are regular on the "singular" surface $r = r_g$. This suggests that no genuine singularity is present there, but rather that only the coordinate system becomes singular.

In order to clarify the real meaning of the metric in this region of the space-time, it is possible to make the following coordinate transformation (in this Chapter for simplicity we set $c = 1$):

$$\tau = \pm\, t \pm \int \frac{f(r)dr}{1 - \frac{r_g}{g}}, \qquad R = t \pm \int \frac{dr}{\left(1 - \frac{r_g}{g}\right) f(r)} \tag{2.1}$$

Choosing $f(r) = \sqrt{r_g/r}$ and the upper sign in (2.1) one obtains:

$$r = \left[\frac{3}{2}(R - \tau)\right]^{2/3} r_g^{1/3} \tag{2.2}$$

and inserting this result in the Schwarszchild metric (1.29) we find the non-stationary *Lemaitre metric*:

$$ds^2 = d\tau^2 - \frac{dR^2}{\left[\frac{3}{2r_g}(R - \tau)\right]^{2/3}} - \left[\frac{3}{2}(R - \tau)\right]^{4/3} r_g^{2/3}(d\theta^2 + \sin^2\theta d\varphi^2) \tag{2.3}$$

The singularity on the Schwarzschild sphere ($r_g = \frac{3}{2}(R - \tau)$) in these coordinates is absent. R and τ are everywhere spatial and temporal coordinates, respectively. The time lines in this metric are geodetics, so the test particles in a state of rest relative to this system are in free motion in the given gravitational field. The lines of Universe

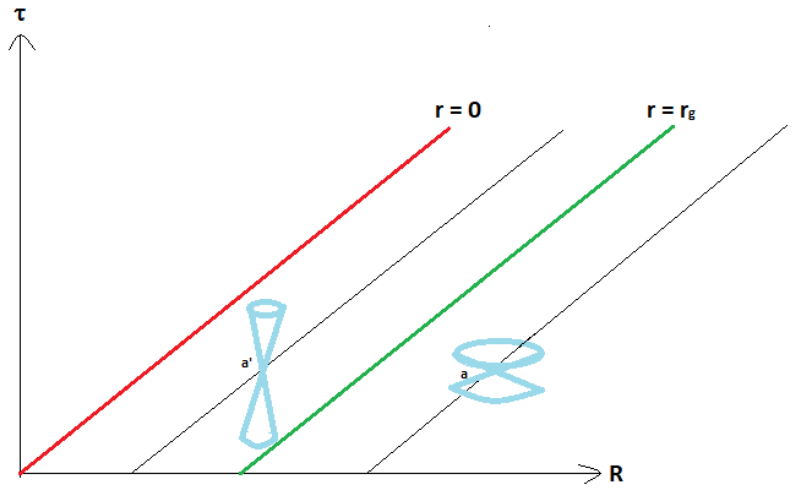

Figure 2.1: Lines of Universe $R - \tau = const.$ for the Lemaitre metric.

$R - \tau = const.$ correspond to certain values of r represented by the inclined lines in the diagram of figure (2.1). The particles at rest are clearly represented by vertical lines, which moving along these vertical lines fall (in a finite interval of their proper time) in the center of the field $r = 0$, that represents a true singular point of the metric.
The equation $ds^2 = 0$, which corresponds to the propagation of radial light signals (for constant angular coordinates), gives the variation of τ with respect to R along the radius:

$$\frac{d\tau}{dR} = \pm\sqrt{\frac{r_g}{r}} \tag{2.4}$$

in which the two signs correspond at the two frontiers of the vertex light cone at the given universe point. For $r > r_g$, which corresponds to the point a in the diagram of figure (2.1), the inclination of these frontiers is less than one, so that the line $r = const.$ (along which the inclination is equal to one) ends up inside the cone. For $r < r_g$ (the point a' in the very same diagram) we have the opposite situation.
Therefore, in the region $r < r_g$, all interactions and signals can't be at rest and they must fall toward the center in a finite proper time τ.

2.2 Tolman solution

One would like to confirm these plausible intuitive ideas by making exact calculations on a stellar model with an equation of state (with a physically reasonable relation between pressure and mass density). The only model for which this is possible without great mathematical complexity is that of *dust* or *incoherent matter*, defined by the condition $p = 0$ (R. Tolman, 1934). Since the pressure vanishes, it is to be expected here that once a star started to contract it would "fall in" to a point. Although it is unrealistic to neglect the pressure, this is example is not trivial since it yields an exact solution of the Einstein equations which is valid inside and outside the collapsing star, and which in a certain sense can serve as a model for all collapsing stars. In Chapter 6 we will also examine the case of non-zero pressure, within the *ADM formalism* for General Relativity which has been briefly described in the introduction.
As the starting point for treating this collapsing stellar dust we do not take the canonical form (1.17) of the line element used in the Schwarzchild solution, but a system which is both synchronous (which means $g_{00} = 1$, $g_{oi} = 0$) and comoving with the dust. It can be shown that in incoherent matter and supposing a radial motion for the latter one can always choose such a system. We obtain it by considering again (1.35) and defining:

$$A(\rho,\tau) \equiv 0 \quad I(\rho,\tau) \equiv 1 \quad H(\rho,\tau) \equiv -e^{\lambda(\tau,\rho)} \quad K(\rho,\tau) \equiv -r^2(\tau,\rho) \tag{2.5}$$

in (1.12), hence bringing the metric into the form:

$$ds^2 = d\tau^2 - e^{\lambda(\tau,\rho)}d\rho^2 - r^2(\tau,\rho)(d\theta^2 + \sin^2\theta d\varphi^2) \tag{2.6}$$

The coordinate τ, which is univocally fixed by (2.6), is the proper time of a particle at rest in the comoving coordinate system, while the curves $\rho = const.$, $\theta = const.$, $\varphi = const.$ are geodesics. The function $r(\tau,\rho)$ represents a "radius" detemined in order to have the lenght of the circonference (whose center in the coordinate's origin) equal to $2\pi r$. It is still possible to transform the radial coordinate as $\rho = \rho(\rho')$. We note that because of

$$U^{\mu}{}_{;i}U^i = 0$$

$$(\omega U^{\mu})_{;\mu} = 0$$

where ω is the rest mass of the dust, the latter always moves along geodesics.
The field equations can be obtained from (1.38) setting $\nu = 0$, $e^{\mu} = r^2$, $p = 0$:

$$\begin{cases} 0 = -e^{-\lambda} r'^2 + 2r\ddot{r} + \dot{r}^2 + 1 = 0 \\ 0 = -\frac{e^{-\lambda}}{r}(2r'' - r'\lambda') + \frac{\dot{r}\dot{\lambda}}{r} + \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} + \frac{2\ddot{r}}{r} \\ 8\pi G\omega = -\frac{e^{-\lambda}}{r^2}(2rr'' + r'^2 - rr'\lambda') + \frac{1}{r^2}(r\dot{r}\dot{\lambda} + \dot{r}^2 + 1) \\ 0 = 2\dot{r}' - \dot{\lambda} r' \end{cases} \tag{2.7}$$

First integrals of these equations can be obtained very easily. The first step is to integrate the last equation of (2.7) to give

$$e^{\lambda} = \frac{r'^2}{1 + f(\rho)} \tag{2.8}$$

with $f(\rho)$ as an arbitrary function which satisfy $1 + f(\rho) > 0$. Substitution into the first equation of (2.7) leads to

$$2\ddot{r}r + \dot{r}^2 - f(\rho) = 0 \tag{2.9}$$

If one now chooses r as the independent variable, then one obtains the first integral of the previous equation as

$$\dot{r}^2 = f(\rho) + \frac{F(\rho)}{r} \tag{2.10}$$

where $F(\rho)$ is another arbitrary function which satisfy $F > 0$.
Thus, one obtains:

$$\tau = \pm \int \frac{dr}{\sqrt{f + \frac{F}{r}}} \tag{2.11}$$

The function $r(\tau, \rho)$, which can be obtained by integration, can be written in a parametric form:

$$\begin{cases} r = \frac{F}{2f}(\cosh\eta - 1), \quad \tau_0(\rho) - \tau = \frac{F}{2f^{3/2}}(\sinh\eta - \eta) \quad if \;\; f > 0 \\ r = \left(\frac{9F}{4}\right)^{1/3} [\tau_0(\rho) - \tau]^{2/3} \quad if \;\; f = 0 \\ r = -\frac{F}{2f}(1 - \cos\eta), \quad \tau_0(\rho) - \tau = \frac{F}{2(-f)^{3/2}}(\eta - \sin\eta) \quad if \;\; f < 0 \end{cases} \tag{2.12}$$

where $\tau_0(\rho)$ is another arbitrary function.
If one next eliminates f in (2.8) with the aid of (2.10), then one finds that the second

equation of (2.7) is satisfied identically and that the fourth equation of (2.7) leads to the following expression for the matter density:

$$8\pi G\omega = \frac{F'}{r'r^2} \tag{2.13}$$

Of the three free functions $F(\rho)$, $f(\rho)$ and $\tau_0(\rho)$ (whose must satisfy only the conditions that ensure the positivity of e^λ, r, and ω), at most two have a physical significance, since the coordinate ρ is defined only up to scale transformations $\rho = \rho(\rho')$.
Unfortunately one cannot simply specify the matter distribution $\omega(\tau, \rho)$ and then determine the metric, but rather through a suitable specification of f, F and τ_0 one can produce meaningful matter distributions. Since layers of matter which move radially with different velocities can overtake and cross one another, one must expect the occurrence of coordinate singularities in the comoving coordinates used here. To each particle of the matter corresponds a determined value of ρ, the function r determines the law of motion of the particle for this value of ρ, while the derivative $\dot{r}$ is its radial velocity.
We now want to apply the Tolman solution to the problem of a star of finite dimensions. An important property of the solution is that arbitrary functions, given in the range from 0 to ρ_0, completely determine the behavior of the sphere of this radius, which therefore does not depend on the properties of these functions for $\rho > \rho_0$. As a result, the internal problem is automatically solved for any finite sphere.
The total mass of this sphere is given, in accordance with (1.30), by:

$$m(\tau, \rho_0) = 4\pi \int_0^{r(\tau,\rho_0)} \omega r^2 dr = 4\pi \int_0^{\rho_0} \omega r^2 r' d\rho \tag{2.14}$$

which is usually called *Misner-Sharp mass.* Taking into account the spherically simmetry of the system, we suppose it depends on the "temporal" coordinate $x^0 = \tau$ and the "radial" coordinate $x^1 = \rho$. The original definition introduced by Misner and Sharp for the mass $m(\tau, \rho)$, assumed to be contained inside a spherically symmetric shell of radius $r(\tau, \rho)$, is:

$$\frac{\partial m}{\partial \rho} = 4\pi r^2 \omega \frac{\partial r}{\partial \rho} \tag{2.15}$$

This definition follows actually from the very intuitive requirement that within a spherical layer of infinitesimal thickness dr, one finds the element of mass $dm = 4\pi r^2 \omega dr$.

Therefore, the total mass would simply be given by (2.14). Another (geometric) definition of the Misner-Sharp mass $m(\tau,\rho)$ mostly used in the literature is the following:

$$m(\tau,\rho) = \frac{R}{2G}(1 - g^{\mu\nu}\partial_\mu R \partial_\nu R) \tag{2.16}$$

where R is the areal radius of the spherical region of space-time under study.
Inserting (2.13) in (2.14) (noting that $\rho = 0$ implies $r = 0$ and $F(0) = 0$), we find:

$$m = \frac{F(\rho_0)}{2G}, \quad r_g = F(\rho_0) \tag{2.17}$$

We obtain the simplest interior solution [2] when ω does not depend upon position (upon ρ) and r has (for a suitable scale) the form $r = K(\tau)\rho$. These restrictions lead to an interior $\rho \leq \rho_0$ of the star which is a three-dimensional space of constant curvature, whose radius K depends on time (in the language of cosmological models, it is a section of a *Friedmann universe*). A great circle on the surface of the star has the radius $\rho_0 K(c\tau)$ and, because of the time-dependence of K, the star either expands or contracts. It can be shown that, depending of the value of η, one can have models which correspond to stars whose radius decreases continuously from arbitrarily large values until at the time $\tau = 0$ (in which a collapse occurs), and models which represent stars which first expand to a maximum radius and then contract.
The solution in the exterior space to the star is clearly a spherically symmetric vacuum solution, and because of the Birkhoff theorem it can only be the Schwarzschild solution. Since the Tolman solution holds for arbitrary mass density ω, it must contain the exterior

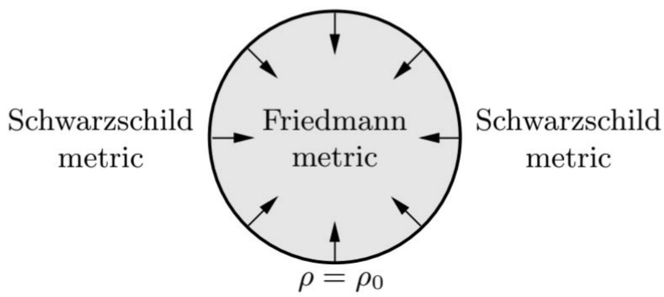

Figure 2.2: Snapshot of a collapsing star. From [2]

Schwarzschild solution as a special case ($F = const.$, while $F = 0$ corresponds to have no gravitational field). For example, setting $F = r_g$, $f = 0$, $\tau_0 = \rho$ we recover the Lemaitre metric (2.3).

In the Tolman solution the coordinates in the exterior space are chosen so that the surface of the star is at rest. In the usual Schwarzschild metric, on the other hand, the stellar surface is in motion, but in both cases the motion of a particle of the surface takes place on a geodesic. Since scale transformations $\rho = \rho(\rho')$ are still possible, $f(\rho)$ cannot be uniquely determined here and, the following, we shall not need $f(\rho)$.

The solutions are written so as to have the compression (which corresponds to the real physical problem of the collaapse of an unstable body) when τ tends to τ_0. At the time $\tau = \tau_0(\rho)$ corresponds the fact that matter with radial coordinate ρ reaches the center (one must have also $\tau_0' > 0$). The limit behaviour of the metric in the interior of the sphere is the same in the three cases of (2.12) for $\tau \to \tau_0(\rho)$:

$$r \sim \left(\frac{9F}{4}\right)^{1/3} (\tau_0 - \tau)^{2/3}, \quad e^{\lambda/2} \sim \left(\frac{2F}{3}\right)^{1/3} \frac{\tau_0'}{\sqrt{1+f}} (\tau_0 - \tau)^{-1/3} \tag{2.18}$$

This means that all the radial distances (in the comoving system, in motion with the dust) tend to infinity, while the transversal ones tend to zero, all volumes equally tend to zero as $(\tau - \tau_0)^2$. For this reason the density matter increases indefinitely for every value of the mass (this is because we have neglected the pressure):

$$8\pi G\omega \sim \frac{2F'}{3F\tau_0'(\tau_0 - \tau)} \tag{2.19}$$

So, one has the collapse of all the distribution the matter to the center.

We conclude this section by saying that in all cases, the instant of the passage of a collapsing sphere inside the Schwarzchild sphere ($r(\tau, \rho) = r_g$) is not significant for the purposes of internal dynamics in the comoving coordinate system.

Chapter 3

Constraints and Reparametrization Invariance

A brief recapitulation of some of the basic properties of constrained systems is necessary beacuse of their importance in the canonical formalism and quantization of General Relativity, as it has been briefly explained in the introduction.
In this Chapter, in which we use conventions of [7], we describe (in addition to the general formalism for dealing with first class constrained systems and references to the methods used up to now for the quantization of the latters) three application examples: the classical point particle, the relativistic particle and the parametrized field theories.
The most important aspect for our purpose is the concept of *reparametrization invariance* and the ensuing existence of constraints. Such invariance properties are often named as "general covariance" of the system, because they refer to an invariance with respect to a relabelling of the underlying space–time manifold.
In General Relativity, the full metric is dynamical and the invariance group coincides with the covariance group, which is the group of all diffeomorphisms. Absolute elements can also appear in "disguised form" if a theory has been reparametrized artificially: this is the case in the model of parametrized field theories and the non-relativistic particle to be discussed here, but not in General Relativity.

3.1 Constrained systems

Constrained Hamiltonian systems tipically emerge when one tries to set up an Hamiltonian formulation of *singular* Lagrangians. With the term "singular" one refers to Lagrangians $L(q^i, \dot{q}^i)$ for which is verified the following relation:

$$det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0 \tag{3.1}$$

In such a case, when one tries to pass to the Hamiltonian formalism by introducing the momenta p_i as independent variables, it turns out that the relations between momenta and velocities are not invertible. As a matter of fact, the momenta are related to the velocities $\dot{q}^i$ by the equations

$$p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i} \tag{3.2}$$

These equations are not invertible in terms of the velocities precisely when eq. (3.1) holds. This condition gives rise to "constraints" between the canonical coordinates (q^i, p_i) of phase space. The latter define a hypersurface (embedded in the phase space) on which the dynamics of the physical system takes place. Additional constraints may also arise by requiring that the time evolution of the constraints themselves vanishes. Having found all constraints for a physical system, it is possible to show that they can be classified in two main classes: *first-class constraints* and *second-class constraints.*

- **FIRST CLASS CONSTRAINTS**
  Assume that $\phi_a(q, p)$, where $a = 1, 2...n$, is a set of first-class constraints. Then the latters satisfie, by definition:

  $$\phi_a(q, p) \approx 0 \tag{3.3}$$

  where q and p represent positions and momenta for N particles, while the symbol $\approx$ (originally introduced by Dirac) means that the constraints $\phi_a(q, p)$, as functions on the whole phase space, do not vanish identically and thus should not be set to zero before computing the Poisson brackets, which are defined for functions of the whole phase space. This kind of constraints is related to gauge symmetries, defining a constraint hypersurface in phase space as in figure (3.1), on which the

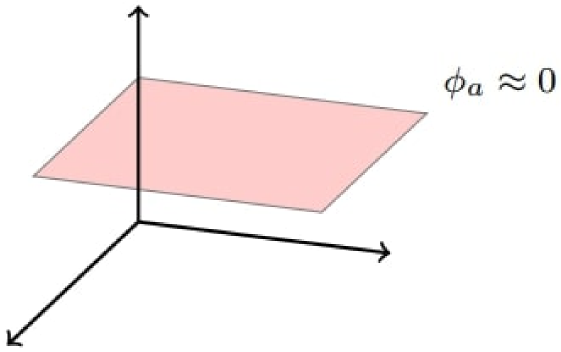

Figure 3.1: Constraint surface identified by $\phi_a \approx 0$ in phase space. From [8]

dynamics must take place (when the constraints are set to vanish as in (3.3)). In addition, they are the generators of gauge transformations which relate different points of the surface describing the same physics configurations: all points related this way make up a "gauge orbit" and different gauge orbits correspond to different physical configurations. First class constraints obey, by definition, the Poisson-bracket relations:

$$\{\phi_a, \phi_b\} = f_{ab}{}^c \phi_c \tag{3.4}$$

where $f_{ab}{}^c$ are called "structure functions", as they may depend on the phase space coordinates, thus we should properly write $f_{ab}{}^c(p, q)$. It turns out that these constraints are compatible with the following time evolution:

$$\{\phi_a, H\} = d_a{}^b \phi_b \tag{3.5}$$

for suitable functions $d_a{}^b$, while H is a gauge invariant Hamiltonian that generates time evolution via previous equation.

- **SECOND CLASS CONSTRAINTS**
  We consider a set of constraints $\psi_a(z) \approx 0$, with $a = 1, ..., n$ (where we identify with z the canonical coordinates of the phase space) that identify an hypersurface in phase space on which the dynamics takes place. These constraints are called

second-class if they satisfy:

$$det\{\psi_a, \psi_b\} \not\approx 0 \tag{3.6}$$

In such a case, the above condition is sufficient to guarantee that the restriction of the the symplectic structure of the original phase space to the constraint surface is still symplectic, which is used to identify the Poisson brackets on the constraint surface, the latter is then called *reduced phase space.* At this stage, one can simply work on the reduced phase space, defined by the constraints $\psi_a = 0$, using the reduced symplectic structure. The formula defining this structure in terms of the variables of the original phase space is given by the so-called *Dirac brackets*, given for any two functions f, g of phase space by:

$$\{f, g\}_D = \{f, g\} - \{f, \psi_a\}(M^{-1})^{ab}\{\psi_b, g\} \tag{3.7}$$

where $M_{ab} = \{\psi_a, \psi_b\}$.

Canonical quantization may then proceed as usual, setting up commutation relations defined by the Dirac brackets. This kind of constraints is not related to gauge symmetries and they arise, essentially, because one tries to set up an Hamiltonian formulation of a system that is already in an Hamiltonian form. Dirac brackets play the role of the Poisson brackets on the constraint surface, the latter make it consistent to solve the second class constraints for a set of independent coordinates of the constraint surface. Thus, the constraint surface must be considered as the appropriate phase space on which the Hamiltonian dynamics takes place.

The methods developed nowadays in order to treat first class constraints and construct canonical quantization of gauge systems can be grouped into three main classes:

1) Reduced phase space method

2) Dirac method

3) BRST method

In the following, we assume the first class constraints to be independent of each other, otherwise certain reducibility relations must be taken into account.

- **REDUCED PHASE SPACE METHOD**
  Since the constraint surface is made up by gauge orbits generated by the first class constraints ϕ_a, the idea is to pick a representative from each gauge orbit by using suitable gauge fixing functions $F^a = 0$. In order to verify that the gauge is properly chosen, one has to check if the set of constraints (ϕ_a, F^a) form a system of second class constraints, which identify a "reduced phase space" embedded in original phase space, as in figure (3.2). Then, one can use the corresponding

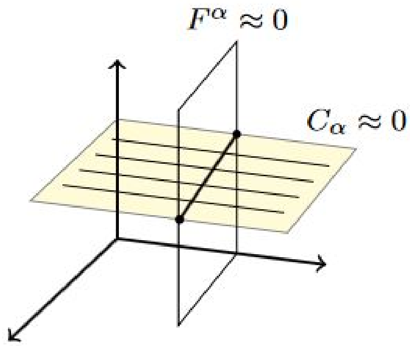

Figure 3.2: The reduced phase space is the intersection of the two sufaces. From [8]

  Dirac brackets and solve the constraints explicitly to find a set of independent coordinates of the reduced phase space. Canonical quantization now normally proceeds by finding linear operators with commutation relations specified by the Dirac brackets. This last step may be not so obvious because Dirac brackets in the chosen coordinates of the reduced phase space might be complicated, anyway Darboux's theorem guarantees that canonical coordinates always exist locally.

- **BRST METHOD**
  This is the most general method that allows for much flexibility in selecting gauge fixing conditions. It encodes the use of Faddeev-Popov ghosts and the ensuing BRST symmetry, originally found in the path integral quantization of Lagrangian gauge theories: this method consists essentially in enlarging even further the phase space by introducing ghosts degrees of freedom. In the full phase space one finds

a symmetry, which is called "BRST symmetry", that encodes the complete information about the first class gauge algebra. The key property of this construction is the nilpotency of a "BRST charge", which generates the BRST symmetry, and the associated concept of cohomology, used to select physical states and physical operators.

- **DIRAC METHOD**
In this method, the space in which one works is the full phase space, thus it is possible to take advantage of the canonical sympletic structure. Nevertheless, physical configurations must lie on the constraints surface $\phi_a = 0$. As Poisson brackets are well-defined on the full phase space, it is possible to proceed with canonical quantization, so one constructs operators acting on a Hilbert space for all of the phase space variables. However, not all states of the Hilbert space are going to be physical, for this reason the Hilbert space of all the states has no, generally, positive norm. Classical first-class constraints ϕ_a turn into operators $\hat{\phi}_a$, which generate gauge transformations at the quantum level and are used to select the vectors $|\psi_{ph}\rangle$ of the Hilbert space that describe physical (i.e. gauge invariant) configurations by requiring that a physical state $|\psi_{ph}\rangle$ satisfies

$$\hat{\phi}_a|\psi_{ph}\rangle = 0 \qquad \forall a = 1, 2...n \tag{3.8}$$

It may happen that this requirement is too strong for certain theories, so that it might be necessary to step back and require the weaker condition

$$\langle\psi'_{ph}|\hat{\phi}_a|\psi_{ph}\rangle = 0 \qquad \forall a = 1, 2...n \tag{3.9}$$

for arbitrary physical states belonging to the Hilbert space. These subsidiary conditions has been used by Gupta and Bleuler in describing quantum electrodynamics in the Lorenz gauge, for this reason this method is sometimes called the "Dirac-Gupta-Bleuler" method. Having found the subspace of physical states in the Hilbert space, one should be careful to define a proper scalar product between them (this usually requires the use of some gauge fixing functions), so that the subspace of physical states forms a "true" Hilbert space, ie with positive norm.

3.2 Classical point particle

In this section, it will be analyzed a model that exhibits some features of General Relativity's canonical approach and quantization, but which is much easier to discuss, which is the classical point particle.
The action for a point particle in classical mechanics can be written as:

$$S[q(t)] = \int_{t_1}^{t_2} dt L\left(q, \frac{dq}{dt}\right) \tag{3.10}$$

Where $q(t)$ is the position function parametrized by the Newton's "absolute time" t, for simplicity it will be considered one single particle described by a time-independent Lagrangian. We now formally elevate t to the rank of a dynamical variable by introducing a time parameter τ that will be named "label time": this is an example for an absolute structure in disguise, as mentioned in the description of this Chapter.
The action (3.10) can then be rewritten in terms of $q(\tau)$ and $t(\tau)$ as:

$$S[q(\tau), t(\tau)] = \int_{\tau_1}^{\tau_2} d\tau\, \dot{t} L\left(q, \frac{\dot{q}}{\dot{t}}\right) \equiv \int_{\tau_1}^{\tau_2} d\tau \tilde{L}\left(q, \dot{q}, \dot{t}\right) \tag{3.11}$$

where derivatives with respect to τ are denoted by $^{\cdot}$, and restriction to $\dot{t} > 0$ is made.
The Lagrangian $\tilde{L}$ possesses the important property that is homogeneous of degree one in the velocities:

$$\tilde{L}\left(q, \lambda\dot{q}, \lambda\dot{t}\right) = \lambda\tilde{L}\left(q, \dot{q}, \dot{t}\right) \tag{3.12}$$

where $\lambda \neq 0$ can be an arbitrary function of τ, thus it is more appropriate to write $\lambda(\tau)$. For this reason, homogeneous Lagrangians lead to actions that are invariant under time reparametrizations $\tau \to \tilde{\tau} \equiv f(\tau)$, which means that they can be written as a $\tilde{\tau}$-integral over the same Lagrangian depending, now, on $dq/d\tau$. Thus, it turns out that homogeneity is equivalent to reparametrization invariance, assuming $\dot{f} > 0$, we express the action in the following way:

$$S = \int_{\tau_1}^{\tau_2} d\tau L(q, \dot{q}) = \int_{\tilde{\tau}_1}^{\tilde{\tau}_2} \frac{d\tilde{\tau}}{\dot{f}} L\left(q, \frac{dq}{d\tilde{\tau}}\dot{f}\right) = \int_{\tilde{\tau}_1}^{\tilde{\tau}_2} d\tilde{\tau} L\left(q, \frac{dq}{d\tilde{\tau}}\right) \tag{3.13}$$

where the property (3.12) has been used. The canonical momentum for q is found from (3.11), which coincides with the momentum corresponding to (3.10), to read:

$$\tilde{p}_q = \frac{\partial\tilde{L}}{\partial\dot{q}} = \dot{t}\frac{\partial L}{\partial\left(\frac{\dot{q}}{\dot{t}}\right)}\frac{1}{\dot{t}} = \frac{\partial L}{\partial(\frac{\dot{q}}{\dot{t}})} = p_q \tag{3.14}$$

The main difference between (3.10) and (3.11) lies in the fact that associated to the latter there's also a momentum canonically conjugated to t, which has been elevated to the rank of dynamical variable:

$$p_t = \frac{\partial \tilde{L}}{\partial \dot{t}} = L\left(q, \frac{\dot{q}}{\dot{t}}\right) + \dot{t}\frac{\partial L\left(q, \frac{\dot{q}}{\dot{t}}\right)}{\partial \dot{t}} = L\left(q, \frac{dq}{dt}\right) - \frac{dq}{dt}\frac{\partial L(q, dq/dt)}{\partial(dq/dt)} = -H \tag{3.15}$$

where H is the Hamiltonian corresponding to the action (3.10) that can be obtainend, as usually, with a Legendre transformation and it is assumed that the latter has the usual form $H = p_q^2/2m + V(q)$.
The Hamiltonian belonging to $\tilde{L}$ is found again by a Legendre transformation as:

$$\tilde{H} = p_q\dot{q} + p_t\dot{t} - \tilde{L} = \dot{t}(H + p_t) \tag{3.16}$$

This Hamiltonian is constrained to vanish because of (3.15), which establishes that that t and $-H$ are canonically conjugate pairs.
We now introduce a new quantity called *Super-Hamiltonian*, defined as:

$$H_S \equiv H + p_t \tag{3.17}$$

for which we find the following constraint, whose existence is a consequence of the reparametrization invariance (i.e. homogeneity in the velocities) with respect to τ:

$$H_S \approx 0 \tag{3.18}$$

As in the previous section of this Chapter, the symbol $\approx$ defines a subspace in phase space and can be set to zero only after all Poisson brackets have been evaluated. As will be examinated in Chapter 5, there's an analogue to (3.17) and (3.18) in General Relativity, but in contrast to here all momenta will occur quadratically. One can now use, instead of (3.10), the new action principle:

$$S = \int_{\tau_1}^{\tau_2} d\tau (p_q\dot{q} + p_t\dot{t} - NH_S) \tag{3.19}$$

where all quantities have to be varied, and N has been formally introduced as Lagrange multiplier, for which the variation of S with respect to the latter just yelds the constraint (3.18). From Hamilton's equations and (3.17), one finds

$$\dot{t} = \frac{\partial(NH_S)}{\partial p_t} = N \tag{3.20}$$

N is called *lapse function* because it gives the rate of change of Newton's time t with respect to label time τ. The concept of lapse function will be very useful for the parametrized field theories, which will be analyzed later.
Given an homogeneous Lagrangian in the velocities $L(q,\dot{q})$ for which one has $L(q,\lambda\dot{q}) = \lambda L(q,\dot{q})$, it turns out that the canonical Hamiltonian reads:

$$H_c = \frac{\partial L(q,\dot{q})}{\partial \dot{q}}\dot{q} - L = \lambda^{-1}\left(\frac{\partial L(q,\lambda\dot{q})}{\partial(\lambda\dot{q})}\lambda\dot{q} - L(q,\lambda\dot{q})\right) = \lambda^{-1}H_c \tag{3.21}$$

and H_c vanishes (as we have seen for (3.16)), since λ is an arbitrary function of τ.
It is very important to discuss the concept of *deparametrization*, that is, the identification of a distinguished time-like variable when it is possible. In this example, although Newton's time has been mixed amongst the other dynamical variables, it can easily be recovered, for its momentum p_t enters *linearly* into the super Hamiltonian definition (3.17). Therefore, one can trivially solve (3.18) to find $p_t = -H$, choose ("fixing the gauge") the label $\tau = t$ and find from (3.19):

$$S = \int dt \left(p_q \frac{dq}{dt} - H\right) \tag{3.22}$$

which is the Hamiltonian form of the standard action (3.10). As previously mentioned, in General Relativity all momenta occur quadratically, this leads to the interesting question whether a deparametrization for General Relativity is possible.
We now add first-class constraints ϕ_a to the action (3.22) with Lagrange multipliers λ_a:

$$S = \int dt \left(p_q \frac{dq}{dt} - H - \lambda_a\phi_a\right) \tag{3.23}$$

Therefore, the time evolution of an arbitrary function of the position and momentum $A(q,p)$ reads

$$\dot{A}(q,p) = \{A,H\} + \lambda_a\{A,\phi_a\} \tag{3.24}$$

As first-class constraints generate gauge transformations, it turns out that the latter introduce an arbitrariness into the time evolution. It is possible to obtain the infinitesimal "gauge transformation":

$$\delta A = \Delta\tau(\lambda_a^{(1)}(0) - \lambda_a^{(2)}(0))\{A,\phi_a\} \equiv \omega_a\{A,\phi_a\} \tag{3.25}$$

and so we can definitely write:

$$\delta A = \omega_a \{A, \phi_a\} \tag{3.26}$$

The previous equation represents the gauge transformations on the "constraint hypersurface" Γ_c in phase space, generated by the constraints, discussed in the last section. Functions $A(q,p)$ for which $\{A, \phi_a\} \approx 0$ holds are sometimes called "observables", in fact they they do not change under a gauge transformation, but it is important to emphasize that there is no a priori relation between these observables and observables in an operational sense. This "observables" notion was historically introduced by Bergmann, in the hope that these quantities might play the role of the standard observables in quantum theory. 0n order to select one physical representative amongst all equivalent gauge configurations, one frequently employs "gauge conditions", which should be generally chosen in such a way that there is no further gauge freedom left and that any system's configuration can be transformed in one satisfying the gauge.

QUANTIZATION

In order to quantize a system given by a constraint such as (3.18), we can use the Dirac method (1964) previously described, which essentially consists in implementing a classical constraint in the quantum theory as a restriction on physically allowed wave functions. Using the coordinate representation, obtained by considering the eigenstates $|x\rangle$ of the position operator $\hat{x}$, that satisfy $\hat{x}|x\rangle = |x\rangle x$ with x a real number, and projecting the various states of the Hilbert space onto them to identify the wave functions, one finds the familiar way of realizing quantum mechanics as wave mechanics. Thus, using the wave functions in the position representation (instead of the Dirac notation, which was used to discuss the Dirac method), (3.18) is translated into

$$\hat{H}_S \psi = 0 \tag{3.27}$$

The $\hat{q}$ are represented by multiplication with q and the momenta $\hat{p}$ by derivatives $-i\hbar\partial/\partial q$, for the parametrized particle, this includes also $\hat{p}_t = -i\hbar\partial/\partial t$ since we treated t as dynamical variable. Therefore the quantum version of the constraint (3.18) is, remembering (3.17),

$$\left(\hat{H} - i\hbar\frac{\partial}{\partial t}\right)\psi(q,t) = 0 \tag{3.28}$$

which is the Schroedinger equation.
It is important to emphasize that t is not a dynamical variable in quantum mechanics: time cannot be represented by an operator because it would be in contradiction with the boundedness of energy. This is the consequence of having an absolute structure in disguise: in quantum theory framework, in spite of its formal appearance as a quantum variable, it remains an absolute structure.

3.3 Relativistic particle

Let us now study the action for a free relativistic particle, in natural units, with mass $m \neq 0$. By definition it must be consistent with Lorentz invariance and, more generally, with transformations of the Poincarè group, thus its action can be write as proportional to the total proper time (which is a relativistic invariant, as it measures the invarianth lenght of the wordline) along its wordline:

$$S = -m \int_{s_1}^{s_2} ds \tag{3.29}$$

In order the express this action in a different way, one could use different approaches according to which one is more convenient for the actual purpose. We will use the four dynamical variables x^μ, which form a four-vector. Of course one of latters (or more generally one combination of them) will have to be redundant, so to guarantee equivalence with the same action [7] expressed in the inertial frame with coordinates $x^\mu = (x^0, x^i) \equiv (t, x^i)$, in which the position $x^i(t)$ and time t are considered as "distincted" variables and only the first are dynamical:

$$S[x^i(t)] = -m \int dT_0 = -m \int \sqrt{1 - \dot{x}^i(t)\dot{x}^i(t)} \tag{3.30}$$

This is possible, but according with the previous example local symmetries (gauge symmetries) occur. It is achieved in the following way: we indicate by $x^\mu(\tau)$ the dynamical variables which describe the worldline traveled by the particle in terms of an arbitrary parameter τ, thus the action is geometrically the same as before (proportional to the proper time), but now it takes the form of a functional of four variables

$$S = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\dot{x}^2} \equiv \int_{\tau_1}^{\tau_2} d\tau L \tag{3.31}$$

where $\dot{x}^\mu \equiv dx^\mu/d\tau$ and tangent vector to the wordline is time-like, so $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu < 0$. The action is clearly homogeneous in the velocities and therefore (as it has been established in the discussion of the classical point particle) there's invariance under the reparametrization $\tau \to f(\tau)$.
The canonical momenta can be obtained as

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \tag{3.32}$$

From the previous equation it turns out that the momenta satisfie the "mass-shell condition":

$$p^2 + m^2 = 0 \tag{3.33}$$

This is a constraint in the phase space and thus should be more properly written, according to the notation introduced by Dirac and previously discussed, as

$$p^2 + m^2 \approx 0 \tag{3.34}$$

Because of reparametrization invariance, the canonical Hamiltonian vanishes

$$H_c = p_\mu \dot{x}^\mu - L = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}\dot{x}^\mu + m\sqrt{-\dot{x}^2} = 0 \tag{3.35}$$

Because of the relation (3.33), it is possible to show that:

$$H_c(x,p) = \dot{x}^0(-p^0 + \sqrt{\vec{p}^2 + m^2}) \approx 0 \tag{3.36}$$

where the positive square root $p^0 = \sqrt{\vec{p}^2 + m^2}$ must be chosen in order to render the energy positive. We thus find again the the canonical Hamiltonian vanishes because of the reparametrization invariance of the action, which arises having elevated t to the rank of dynamical variable by the introduction of τ.
In this example we can define the Super-Hamiltonian to be:

$$H_S \equiv \eta^{\mu\nu} p_\mu p_\nu + m^2 \approx 0 \tag{3.37}$$

which is constrained to vanish. One can now transform the action into Hamiltonian form:

$$S = \int_{\tau_1}^{\tau_2} d\tau (p_\mu \dot{x}^\mu - N H_S) \tag{3.38}$$

To interpret the Lagrange multipler N one can use the Hamilton's equations

$$\dot{x}^0 = \frac{\partial(NH_S)}{\partial p_0} = -2Np_0 \tag{3.39}$$

to give

$$N = \frac{\dot{x}^0}{2\sqrt{\vec{p}^2 + m^2}} = \frac{\dot{x}^0}{2m\gamma} = \frac{1}{2m}\frac{ds}{d\tau} \tag{3.40}$$

where γ is the relativistic factor.

So for the relativistic particle, in contrast to the classical case (3.20), the lapse function N is proportional to the rate of change of proper time s, instead of x^0, with respect to parameter time τ (this ultimately depends on the choice of the super Hamiltonian we have made). With regard to the Hamilton action (3.38), how x, p, and N must transform under time reparametrizations in order to leave the action invariant?

Since the first class constraint H_S generates gauge transformations in the sense of (3.26), if we neglet the space-time indices and shall keep the formalism general, we find:

$$\delta x(\tau) = \epsilon(\tau)\{x, H_S\} = \epsilon\frac{\partial H_S}{\partial p} \tag{3.41}$$

$$\delta p(\tau) = \epsilon(\tau)\{p, H_S\} = -\epsilon\frac{\partial H_S}{\partial x} \tag{3.42}$$

To show how does N transform, we calculate

$$\delta S = \int_{\tau_1}^{\tau_2} d\tau(\dot{x}\delta p + p\delta\dot{x} - H_S\delta N - N\delta H_S)$$

The last term is zero, and partial integration of the second term leads to

$$\delta S = \int_{\tau_1}^{\tau_2} d\tau\left(-\epsilon\frac{\partial H_S}{\partial x}\dot{x} - \epsilon\frac{\partial H_S}{\partial p}\dot{p} - H_S\delta N\right) + \left[p\epsilon\frac{\partial H_S}{\partial p}\right]_{\tau_1}^{\tau_2}$$

In order that only a surface term remains, one has to choose

$$\delta N(\tau) = \dot{\epsilon}(\tau) \tag{3.43}$$

Because of the Hamilton equations this leads to

$$\delta S = \left[\epsilon(\tau)\left(p\frac{\partial H_S}{\partial p} - H_S\right)\right]_{\tau_1}^{\tau_2}$$

Since the term in brackets is $p^2 - m^2 \neq 0$, one must demand

$$\epsilon(\tau_1) = 0 = \epsilon(\tau_2) \tag{3.44}$$

thus, boundaries must not be transformed in order to have $\delta S = 0$. It is important to underline that the restriction (3.44) only holds if the action is an integral over the Lagrangian without additional boundary terms (if appropriate boundary terms are present in the action principle, one can relax the condition on ϵ). Also for the relativistic particle is possible to fix the gauge, if the latter is independent of the lapse function N, it is called *canonical gauge*, otherwise it is called *non-canonical.*

- **CANONICAL GAUGE**

  We first consider a canonical gauge. It can be written as:

$$\chi(x, p, \tau) \approx 0 \tag{3.45}$$

An example of this kind of gauge is the same used in the deparametrization of the non-relativistic particle, which was $x^0 - \tau \approx 0$. A potential problem is that (3.45) holds at all times, including the endpoints, and may be in conflict with (3.44). Since there is no gauge freedom at the endpoints, $\chi \approx 0$ could restrict physically relevant degrees of freedom. For reparametrization-invariant systems, a canonical gauge must depend explicitly on τ. From the condition that (3.45) be invariant under time evolution:

$$0 \approx \frac{d\chi}{d\tau} = \frac{\partial\chi}{\partial\tau} + N\{\chi, H_S\}$$

we see from the last equation that a τ-independent gauge χ would lead to the unacceptable value $N = 0$, freezing the motion, while for the gauge to break the reparametrization invariance generated by H_S, $\{\chi, H_S\}$ must be non vanishing. In the case of relativistic particle, this yelds

$$0 \approx \frac{\partial\chi}{\partial\tau} + N\frac{\partial\chi}{\partial x^\mu}\frac{\partial H_S}{\partial p_\mu} = \frac{\partial\chi}{\partial\tau} + 2Np^\mu\frac{\partial\chi}{\partial x^\mu}$$

for the example $x^0 - \tau \approx 0$, one has $N = \frac{1}{2p^0}$ in accordance with (3.40). One can look for an equation of second order in ϵ (since there are two conditions (3.44)) in order to avoid potential problems with the boundary. Since x and p transform proportional to ϵ, it would be necessary to involve $\ddot{x}$ or $\ddot{p}$, which would render the action functional unnecessarily complicated.

- **NON-CANONICAL GAUGE**
  Since (3.43) is valid and, as seen for the canonical gauge, looking for equation of second order in ϵ would complicate the expression of S, it is possible to choose the "non-canonical gauge":
  $$\dot{N} = \chi(p, x, N) \tag{3.46}$$
  In Electrodynamics, for example, A^0 plays the role of N and thus the Lorentz gauge $\partial_\mu A^\mu = 0$ is a non-canonical gauge, whereas the Coulomb gauge $\partial_k A^k = 0$ would be an example of a canonical gauge.

As final remark, we emphasize that if boundary terms (which can be determine once the equation of motion are solved) are present in the action, one can even choose τ-independent caonical gauges (an extreme choice would be $x^0(\tau) = 0$ for all τ).

QUANTIZATION
If we apply Dirac's quantization rule on the classical constraint (3.37) we get

$$\hat{H}_S \psi(x^\mu) \equiv (-\hbar^2 \partial^\mu \partial_\mu + m^2)\psi(x^\mu) = 0 \tag{3.47}$$

This is the Klein-Gordon equation for one relativistic spinless particle in quantum mechanics, therefore we conclude that the Klein-Gordon equation is obtained by quantizing canonically the relativistic partice (this is known as the "first quantization" of the relativistic particle). It is important to emphasize that the classical parameter τ has totally disappeared since particle trajectories, because of the Heisenberg uncertainty principle, do not exist in quantum theory.

3.4 Parametrized field theories

Parametrized field theories can be considered as a possible generalization of the parametrized non-relativistic particle previously discussed. We introduce this class of theories by defining standard inertial coordinates $X^\mu \equiv (T, X^a)$ and a real scalar field in Minkowski space $\phi(X^\mu)$. Now it is convenient to introduce arbitrary coordinates $x^\mu \equiv (t, x^a)$ and let the X^μ depend parametrically on x^μ, the latter are in general curved coordinates. The relation between X^μ and x^μ is analogue, rispectively, to the relation between t and τ discussed in the example of the classical point particle. The functions $X^\mu(x^\nu)$ describe a family of hypersurfaces in Minkowski space parametrized by $x^0 \equiv t$ (we shall restrict ourselves to the space-like case).

The standard action for a scalar field can be rewritten in terms of the arbitrary coordinates x^μ as:

$$S = \int d^4X \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial X^\mu}\right) \equiv \int d^4x \tilde{\mathcal{L}} \tag{3.48}$$

where

$$\tilde{\mathcal{L}}(\phi, \phi_{,a}\dot{\phi}; X^\mu_{,a}, \dot{X}^\mu) = J\mathcal{L}\left(\phi, \phi_{,\nu}\frac{\partial x^\nu}{\partial X^\mu}\right) \tag{3.49}$$

and J denotes the Jacobi determinant of the X with respect to the x:

$$J = \epsilon_{\rho\nu\lambda\sigma}\frac{\partial X^\rho}{\partial x^0}\frac{\partial X^\nu}{\partial x^1}\frac{\partial X^\lambda}{\partial x^2}\frac{\partial X^\sigma}{\partial x^3} \tag{3.50}$$

The notation used in (3.49) is $\dot{\phi} \equiv d\phi/dt$, $\phi_{,a} \equiv d\phi/dx^a$. Before calculating directly the momentum canonically conjugate to X^μ, it is more convenient to consider first the Hamiltonian density $\tilde{\mathcal{H}}$ corresponding to $\tilde{\mathcal{L}}$ with respect to ϕ, which is:

$$\begin{aligned} \tilde{\mathcal{H}} &= \tilde{p}_\phi\dot{\phi} - \tilde{\mathcal{L}} = J\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - J\mathcal{L} = J\frac{\partial x^0}{\partial X^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial X^\mu)}\frac{\partial\phi}{\partial X^\nu} - \delta^\mu_\nu\mathcal{L}\right)\dot{X}^\nu \\ &\equiv J\frac{\partial x^0}{\partial X^\mu}T^\mu{}_\nu\dot{X}^\nu \end{aligned} \tag{3.51}$$

Where the notation is again: $\dot{X}^\nu = dX^\nu/dt = \partial X^\nu/\partial t = \partial X^\nu/\partial x^0$.

To show that both J and $T^\mu{}_\nu$ do not depend on the "kinematical velocities" $\dot{X}^\mu$, one can obtain from (3.50) the following relation

$$J\frac{\partial x^0}{\partial X^\mu} = \epsilon_{\mu\nu\lambda\sigma}\frac{\partial X^\nu}{\partial x^1}\frac{\partial X^\lambda}{\partial x^2}\frac{\partial X^\sigma}{\partial x^3}$$

which is just the vectorial surface element on $t = \textit{constant}$, that does not depend on the $\dot{X}^\mu$ and, for the same reason, the energy-momentum tensor does not depend on these velocities. As a generalization of (3.16) and (3.18), it is possible to introduce the kinematical momenta Π_ν via the constraint

$$\mathcal{H}_\nu \equiv \Pi_\nu + J\frac{\partial x^0}{\partial X^\mu}T^\mu{}_\nu \approx 0 \tag{3.52}$$

Taking the action

$$S = \int d^4x(\tilde{p}_\phi\dot{\phi} - \tilde{\mathcal{H}}) \tag{3.53}$$

Inserting the expression (3.51) and adding the constraints (3.52) with Lagrange multipliers N^ν, one gets the action principle

$$S = \int d^4x(\tilde{p}_\phi\dot{\phi} + \Pi_\nu\dot{X}^\nu - N^\nu\mathcal{H}_\nu) \tag{3.54}$$

This result, which is the analogue of (3.19), can be also obtained by defining the kinematical momenta directly from (3.49)
We can decompose (3.52) into components orthogonal and parallel to the hypersurfaces $x^0 = constant$, for this purpose we introduce the normal vector n^μ and the tangential vectors $X^\nu_{,a}$ to the latters. We have the following relations

$$\eta_{\mu\nu}n^\mu n^\nu = -1 \tag{3.55}$$

$$n_\nu X^\nu_{,a} = 0 \tag{3.56}$$

Furthermore, with the previous decomposition, we now see that (3.52) is turned into the constraints:

$$\mathcal{H}_\perp \equiv \mathcal{H}_\nu n^\nu \approx 0 \tag{3.57}$$

$$\mathcal{H}_a \equiv \mathcal{H}_\nu X^\nu_{,a} \approx 0 \tag{3.58}$$

Which are called, respectively, *Hamiltonian constraint* and *momentum constraint* (of *diffeomorphism constraint*). Decomposing in this way (3.52), we insert the constraints (3.57) and (3.58) in the action (3.54), therefore we get

$$S = \int d^4x(\tilde{p}_\phi\dot{\phi} + \Pi_\nu\dot{X}^\nu - N\mathcal{H}_\perp - N^a\mathcal{H}_a) \tag{3.59}$$

To interpret the Lagrange multipliers N and N^a one can vary this action with respect to Π_μ and from the relations (remembering (3.52)):

$$\frac{\delta\mathcal{H}_\perp}{\delta\Pi_\mu} = \frac{\delta\left(\mathcal{H}_\nu n^\nu\right)}{\delta\Pi_\mu} = n^\mu$$

$$\frac{\delta\mathcal{H}_a}{\delta\Pi_\mu} = \frac{\delta\left(\mathcal{H}_\nu X^\nu_{,a}\right)}{\delta\Pi_\mu} = X^\mu_{,a}$$

it is easy to obtain the following condition:

$$\dot{X}^\mu \equiv t^\mu = N n^\mu + N^a X^\mu{}_{,a} \tag{3.60}$$

The relation between neighboured hypersurfaces identified by $x^0 \equiv t = constant$ is the same as shown in figure (3.3), in which (3.60) is a vector that points from a point with spatial coordinates x^a on $t = constant$ to a point with the same coordinates on a neighbouring hypersurface $t + dt = constant$. N represents the pure temporal distance between the hypersurfaces and it is called lapse function: this the same terminology used for the classical point particle (because of the analogy between the physical meaning, remembering the analogy between X^μ and x^μ with t and τ). Similarly, N^a is called *shift vector*, it points from the point with coordinates x^a on $t = constant$ to the point on the same hypersurface from which the normal is erected to reach the point with the same coordinates x^a on $t + dt = constant$. One can choose an arbitrary curved back-ground

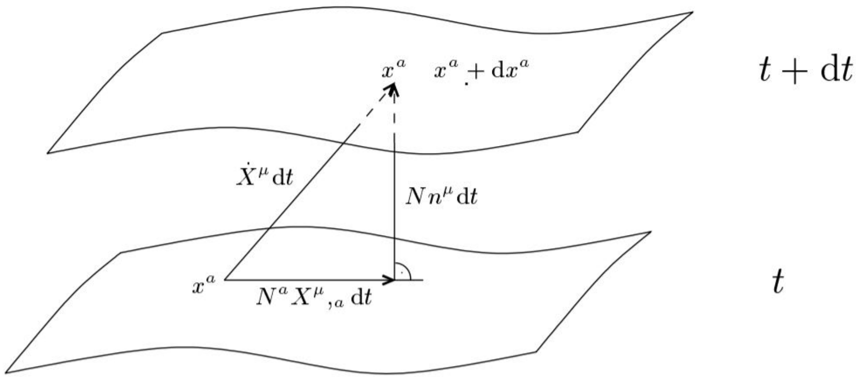

Figure 3.3: The geometric interpretation of the lapse function and the shift vector for the parametrized field theories. From [4]

for the embedding instead of Minkowski space (and this will be the General Relativity case), denoting by h_{ab} the spatial metric obtained according to

$$h_{ab} = g_{\mu\nu} \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\nu}{\partial x^b} \tag{3.61}$$

it is possible to decompose the 4-dimensional line element as follows

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= -N^2dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \\ &= (h_{ab}N^aN^b - N^2)dt^2 + 2h_{ab}N^a dx^b dt + h_{ab}dx^a dx^b \end{aligned} \tag{3.62}$$

As one can verify by direct inspection, the action (3.59) is invariant under the following reparametrizations

$$x^0 \to x^{0'} = x^0 + f(x^a) \tag{3.63}$$

$$x^a \to x^{a'} = g(x^b) \tag{3.64}$$

with arbitrary functions f and g which obey standard differentiability conditions. This is not equivalent to the full set of space-time diffeomorphisms.

Chapter 4

Causal Structure and Initial Value Formulation

In General Relativity, the causal structure of space-time is *locally* of the same qualitative nature as in the flat Minkowski space-time, which is the framework of the Special Relativity theory. Otherwise, significant differences can occur globally because of non-trivial topology, space-time singularities or "twisting of the directions" of the light cones. The purpose of the first part of this Chapter is to give an account of the definitions and basic results concerning the causal structure of space-times in General Relativity.
The second part aims to establish the precise criteria and mathematical meaning of *well posed* initial value formulation. This concept depends, generally, on the type of theory considered, thus this discussion only aims to establish the conceptual bases that will allow us to show that General Relativity admits a well posed inital value formulation (the latter will be briefly analyzed in the next Chapter).
The discussion throughout this Chapter, in which the conventions of [6] are used, will concern arbitrary space-times $(\mathcal{M}, g_{\mu\nu})$ and only in the next one (after the discussion of the 3+1 decomposition) we will impose Einstein's equations on $g_{\mu\nu}$.

4.1 Futures, Pasts and Causality Conditions

Let $(\mathcal{M}, g_{\mu\nu})$ be a space-time. At each event $p \in \mathcal{M}$, we define the *light cone* of p as the light cone passing through the origin of the tangent space V_p of p, which is isomorphic to Minkowski space-time. In a non-simply connected manifold it may be not possible to make a continuous designation of "future" and "past" as p varies over $\mathcal{M}$. If such a continuous choice can be made then $(\mathcal{M}, g_{\mu\nu})$ is said to be *time orientable*. Thus, in a time orientable space-time one has the physical essential property to consistently distinguish between the notions of going "forward in time" as opposed to "backward in time", in the following we will consider only time orientable space-times and will assume that a continuous designation has made of the "future" and "past" halves of the light cones at each point.
It is possible to show that for time orientable space-times always exists a smooth non vanishing timelike vector field t^μ on $\mathcal{M}$, which is not unique in general and, conversely, if a continuous time-like vector field can be chosen, then $(\mathcal{M}, g_{\mu\nu})$ is time orientable.
A differentiable curve $\lambda(t)$ in a time orientable space-time is said to be a *future directed timelike curve* if, at each $p \in \lambda$, the tangent t^μ is a future directed (which means lying in the "future half" of the cone) time-like vector.
The *chronological future* of $p \in \mathcal{M}$, denoted by $I^+(p)$, is defined as the set of events that can be reached by a future directed timelike curve starting from p, $I^+(p)$ is always an open subset of $\mathcal{M}$. We now define, for any subset $S \subset \mathcal{M}$, $I^+(S)$ as

$$I^+(S) = \bigcup_{p \in S} I^+(p) \tag{4.1}$$

and, since an arbitrary union of open sets is open, it follows $I^+(S)$ always is an open set. Totally analogous definitions and properties apply to the *chronological pasts* $I^-(p)$ and $I^-(S)$. In Minkowski space-time $I^+(p)$ consists precisely of the points that can be reached by future directed time-like geodesics starting from p.
A differential curve $\lambda(t)$ is said to be a *future directed causal curve* if, at each $p \in \lambda$, t^μ is either a future directed time-like or null vector. The *causal future* of p, denoted by $J^+(p)$, is defined in the same way as $I^+(p)$ except that "causal curve" replace future "directed time-like curve" in its definition.
Thus, we always have $p \in J^+(p)$. In flat space-times $J^+(p)$ is a closed set but in general

space-times this may not be the case. It is possible to show that $J^+(p)$ must be closed in any globally hyperbolic space-time. Again we define

$$J^+(S) = \bigcup_{p \in S} J^+(p) \tag{4.2}$$

and define the causal pasts $J^-(p)$ and $J^-(S)$ analogously.
In order to discuss the global hyperbolicity of the space-time (that will be a fundamental request in order to discuss the 3+1 decomposition of Geeneral Relativity), we now introduce the definition of *achronal set*: a subset $S \subset \mathcal{M}$ is said to be achronal if there do not exist $p, q \in S$ such that $q \in I^+(p)$.
The following theorem, whose first half proof can be found in Hicks (1965) and the second part in [29], establishes that *locally* in arbitrary space-times $I^+(p)$ consists of the points that can be reached by future directed time-like geodesics starting from p, while the boundary $\dot{I}^+(p)$ is generated by the future directed null geodesics starting from p.

Theorem 4.1.1. *Let $(\mathcal{M}, g_{\mu\nu})$ be an arbitrary space-time and let $p \in \mathcal{M}$. Then there exists an open set U with $p \in U$ such that for all $q, p \in U$ there exists a unique geodesic γ connecting q and r that stays entirely within U.*
Furthermore, for any such U, the chronological future $I^+(p)|_U$ of p in the space-time $(U, g_{\mu\nu})$ consists of all points reached by future directed time-like geodesics starting from p and contained within U, in addition $\dot{I}^+(p)|_U$ is generated by the future directed null geodesics in U emanating from p.

According to theorem (4.1.1), all space-times in General Relativity have locally the same qualitative causal structure as in Special Relativity, but globally very significant differences can occur, but this discussion is beyond our purposes.
The characterization that allows to formulate precise conditions on these space-times is the *strong causality condition*: a space-time $(\mathcal{M}, g_{\mu\nu})$ is said to be strongly causal if for all $p \in \mathcal{M}$ and every neighborhood O of p, there exists a neighborhood V of p contained in O such that no causal curve intersects V more than one. It is possible to construct more sophisticated examples where strong causality is satisfied but a modification of $g_{\mu\nu}$ in an arbitrarily small neighborhood of, at least, two points produces closed causal curves.

Thus, strong causality does not fully express the condition that one is not on the verge of producing causality violation, this latter is expressed by the stronger notion of *stable causality*, defined as follows.
Let t^μ be a time-like vector at point $p \in \mathcal{M}$ and define $\tilde{g}_{\mu\nu}$ at p by

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - t_\mu t_\nu \tag{4.3}$$

where $g_{\mu\nu}$ is the space-time metric.
We define a space-time $(\mathcal{M}, g_{\mu\nu})$ to be *stable causal* if there exists a continuous nonvanishing time-like vector field t^μ such that the space-time $(\mathcal{M}, \tilde{g}_{\mu\nu})$ possesses no closed time-like curves. The light cone of $\tilde{g}_{\mu\nu}$ is strictly larger than that of $g_{\mu\nu}$ or, in other terms, every time-like and null vector of $g_{\mu\nu}$ is a time-like vector $\tilde{g}_{\mu\nu}$. The following theorem shows that stable causality is equivalent to the existence of a *global time function* on the space-time:

Theorem 4.1.2. *A space-time $(\mathcal{M}, g_{\mu\nu})$ is stably causal if and only if there exists a differentiable function f on $\mathcal{M}$ such that $\nabla^\mu f$ is a past directed time-like vector field.*

This greatly strenghtens the above suggestion that the requirement of stable causality should suffice to rule out any causal pathologies, in addition we mention that a corollary of the previous theorem establishes that stable causality implies strong causality.
In conclusion, stable causality appears to be the appropriate notion which expresses the idea that a space-time is not “on the verge” of displaying bad causal behaviour.

4.2 Global Hyperbolicity

In this section, instead of focusing the attention on the collection of events $I^+(S)$ or $J^+(S)$ that could be influenced by a set S of events, we shall be concerned with the collection of events which are “entirely determined” by a closed and achronal set of events S. There will be also explored some properties of space-times in which all events are “determined” by an appropriate S.
Let S be a closed and achronal set, then we define the *future domain of dependence* of S, denoted by $D^+(S)$, as the set of every $p \in \mathcal{M}$ for which every past inextendible causal

curve through p intersects S. An example illustrating the nature of this set is given in figure (4.1), in which the *future Cauchy horizon* of the achronal set S is defined as:

$$H^+(S) = \overline{D^+(S)} - I^-[D^+(S)] \tag{4.4}$$

Since from a physical point of view nothing can travel faster than light, if we are given

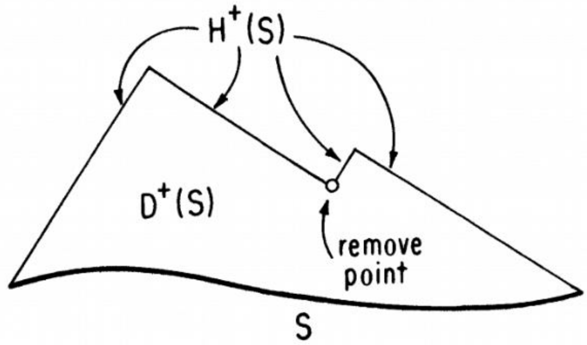

Figure 4.1: Space-time diagram showing the future domain of dependence $D^+(S)$ and Cauchy horizon $H^+(S)$ of a particular closed achronal set S in Minkowski space-time with a point removed. From [6]

appropriate information about "initial conditions" on S, we should be able to predict what happens at $p \in D^+(S)$, thus the latter is of interest because any signal sent to $p \in D^+(S)$ must have "registered" on S.

The *domain of dependence* of S, which represents the complete set of events for which all conditions should be determined by a knowledge of conditions on S, is denoted by $D(S)$ and turns out to be defined as:

$$D(S) = D^+(S) \bigcup D^-(S) \tag{4.5}$$

where $D^-(S)$ is the past domain of dependence of S and is defined analogously to $D^+(S)$ (simply replacing "future" with "past").

A closed achronal set Σ for which $D(\Sigma) = \mathcal{M}$ is called a *Cauchy surface.* We use the term "surface" because it can be shown that every Cauchy surface is an embedded C^0 submanifold of $\mathcal{M}$ and, since Σ is achronal, we may think of Σ as representing an "instant of time" throughout the universe (this analogy it is useful in order to discuss the

Hamiltonian formulation of General Relativity).
A space-time $(\mathcal{M}, g_{\mu\nu})$ which possesses a Cauchy surface Σ is said to be *globally hyperbolic*. In such a space-time, the entire future and past history of the universe can be predicted or retrodicted from conditions at the instant of time represented by Σ. Conversely, in a non-globally hyperbolic space-time there's a breakdown of predictability: a complete knowledge of conditions at a single "instant of time" can never suffice to determine the entire history of the universe. There are good reasons (see Penrose 1979) for believing that all physically realistic space-times must be globally hyperbolic.
It is possible to show that if Σ is a Cauchy surface, then every inextendible causal curve intersects Σ, furthermore, in a global hyperbolic space-time $(\mathcal{M}, g_{\mu\nu})$ with a Cauchy surface Σ, no closed time-like curves can exist. In fact, it turns out that such a space-time is *always* stable causal (of course, as a consequence, strong causality holds), as established by the following theorem, which summarize the main results that will be essentially in order to discuss the Hamiltonian formulation of General Relativity.

Theorem 4.2.1. *Let $(\mathcal{M}, g_{\mu\nu})$ be a globally hyperbolic space-time. Then, $(\mathcal{M}, g_{\mu\nu})$ is stably causal. Furthermore, a global time function f can be chosen such that each surface of constant f is a Cauchy surface, therefore $\mathcal{M}$ can be foliated by Cauchy surfaces and the topology of $\mathcal{M}$ is $\mathbb{R} \times \Sigma$, where Σ denotes any Cauchy surface*

4.3 Initial Value Formulation

General Relativity asserts that space-time structure and gravitation are described by a space-time $(\mathcal{M}, g_{\mu\nu})$, where $\mathcal{M}$ is a four-dimensional manifold and $g_{\mu\nu}$ is a metric with Lorentzian signature satisfying Einstein's equations.
In classical physics, one has quite generally a great deal of physical control over initial conditions of the system and, if the latter is allowed to evolve freely, then its behaviour is completely determined by initial conditions. Although our pratical ability to control initial conditions in gravitational problems is far more limited, it seems natural to believe that we should, in principle, be able to control the initial conditions of the gravitational field and matter distribution (at least over regions much smaller than cosmological scales), perhaps subject to some constraints as in EM.

Thus, unless General Relativity differs drastically from other theories of classical physics, it should permit a physically reasonable specification of initial data, through which the Einstein's equations should determine the subsequent evolution (eventuallly supplemented by additional equations for the matter).
A theory is said to possess an "Initial value formulation" if it can be formulated in order to specify "appropriate initial data", possibly subject to constraints, such that the subsequent dynamical evolution of the system is uniquely determined. Anyway, even if such a formulation exists, there remain further properties that a physically viable theory should satisfy in order to exhibit a *well posed* initial value formulation. First, in order to preserve the predictive power, "small changes" in initial data of the theory should produce only corresponding "small changes" in the solution over any fixed compact region of space-time. Second, to preserve the framework of the theory and do not propagate signals faster than light, changes in the initial data region S of the initial data surface should not produce any changes in the solution outside the causal future $J^+(S)$ of this region.

4.3.1 Initial value formulation of Particles and Fields

The Newton's second law of motion in ordinary and non-relativistic particle mechanics possesses the fondamental feature to relate the second time derivatives of spatial position of particles to the force, which is usually a known function of the position and velocity of the particles in the physical system.
Therefore, for a system of particles interacting with themselves and/or external potentials with forces dependent on positions and velocities (but no on higher time derivatives of the particle positions), the laws of mechanics take the form:

$$\frac{d^2 q_i}{dt^2} = F_i\left(q_1, ...q_n; \frac{dq_1}{dt}, ...\frac{dq_n}{dt}\right) \tag{4.6}$$

where $i = 1, ...n$ and n is the number of degrees of freedom of the system. From the theory of differential equations (see Coddington and Levinson 1955), given arbitrary initial values for the particle positions $q_{10}...q_{n0}$ and velocities $(dq_1/dt)_0...(dq_n/dt)_0$ at $t = t_0$, the system of n ordinary second order differential equations (4.6) for the n quantities $q_1(t), ...q_n(t)$ always possesses a unique solution, over a finite time interval about t_0, with these initial values. Therefore, ordinary particle mechanics possess an initial value formulation and it turns out that the latter is also well posed, since at fixed time t the positions $q_1(t)...q_n(t)$ are continuous functions of initial positions and velocities of the particles and, in non-relativistic mechanics, the causal propagation of changes in the initial data is not an issue.

If we consider the massive Klein-Gordon field ϕ, propagating in a flat Minkowski spacetime, which obeys the standard Klein-Gordon equation:

$$(\Box - m^2)\phi = 0 \tag{4.7}$$

and we choose global intertial coordinates $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$, it is possible to write this equation in the form

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - m^2 \phi \tag{4.8}$$

The mathematical structure of equation (4.8) is markedly different from that of (4.6): the first one is a single partial derivatives differential equation, while the second one turns out to be a system of ordinary differential equation. Nevertheless, the essential content of (4.6) and (4.8) is quite similar. In fact, they both tell us how to compute the second time derivative of the unknown quantity (or quantities) at an istant of time, given the value and first time derivative of the quantity (or quantities) at that time. Indeed one can heuristically view equation (4.8) as arising from the limit as $N \to \infty$ of a system of N particles coupled by nearest neighbor harmonic oscillator interactions: in this limit the index i goes over to the continuous label $\vec{x}$ and the finite set of variables $q_i(t)$ satisfying equation (4.6) goes over to the field variable $\phi(\vec{x}, t)$ which satisfy (4.8).

From the previous mathematical and physical analogy between (4.8) and (4.6), we may conclude that Klein-Gordon theory should have the initial value formulation which consists in specifying the values of ϕ and $\partial\phi/\partial t$ on a spatial hypersurface Σ_0 of constant (inertial) time $t = t_0$, then there should exist a unique solution of (4.8) having this initial data. Actually, if one considers only analytic initial data (i.e. when ϕ and $\partial\phi/\partial t$ are analytic functions on Σ_0), by making use of the Cauchy-Kowaleski theorem [6] it is possible to show that such a formulation exists for the Klein-Gordon theory.

In analogy to particle mechanics, the initial value of ϕ and its time derivative may be

specified arbitrarily and these initial values determine the subsequent evolution of ϕ.
It is even possible to show, by using the linearity and other properties of the Klein-Gordon equation, that the theory of the massive scalar Klein-Gordon field exhibits an initial value formulation which is also well posed. The demonstration is quite sophisticated and it requires to deeply analyze the structure of the Klein-Gordon equation, in addition to using the flatness of the Minkowski space-time on which the theory is defined [6].
In order to generalyze the above discussion for the Klein-Gordon field to our purpose, we replace the Klein-Gordon equation in $\mathbb{R}^4$ by any equation on a generic manifold $\mathcal{M}$ of the form:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + A^\mu\nabla_\mu\phi + B\phi + C = 0 \tag{4.9}$$

where A^μ is an arbirary smooth vector field, B and C are arbitrary smooth functions, ∇_μ is any derivative operator and $g_{\mu\nu}$ is an arbitrary smooth metric with Lorentz signature such that the space-time $(\mathcal{M}, g_{\mu\nu})$ is globally hyperbolic, that is, it possesses a Cauchy surface Σ on which initial data can be described to determine uniquely the whole space-time.
A second order linear partial differential equation is said to by "hyperbolic" if and only if it can be expressed in the form (4.9), such an equation will have a well posed initial vale formulation for initial data $(\phi, n^\mu\nabla_\mu\phi)$ on any smooth, space-like Cauchy surface Σ, where n^μ is the unit normal to Σ. We will not give a proof of this result which, however, it is not difficult to demonstrate after one has given a well posed initial value formulation of the Klein-Gordon theory.
These results can be further generalized to system of equations, resulting in the following theorem:

Theorem 4.3.1. *Let $(\mathcal{M}, g_{\mu\nu})$ be a globally hyperbolic space-time, ∇_μ be any derivative operator and Σ be a smooth and space-like Cauchy surface. Consider the linear, diagonal and second order hyperbolic system of n equations for n unknown functions $\phi_1 ... \phi_n$ of the form:*

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi_i + \sum_j (A_{ij})^\mu\nabla_\mu\phi_j + \sum_j B_{ij}\phi_j + C_i = 0 \tag{4.10}$$

Then, given arbitrary smooth initial data $(\phi_i, n^\mu\nabla_\mu\phi_i)$ for $i = 1...n$ on Σ, there exists a

unique solution of the above system through $\mathcal{M}$.
Furthermore, the solutions depend continuously on the initial data and a variation of the latters outside of a closed substet S of Σ does not affect the solution in $D(S)$.

A complete proof of this theorem, which tells us that the system of equations (4.10) has a well posed initial value formulation, can be found in [29].
The last step that it is important to discuss, at this level and for our purpose, is the generalization of (4.10) to some general and nonlinear systems of equations, for which very few results about the initial value formulation has been successfully developed.
For a diagonal and hyperbolic system of n second order partial differential and *quasilinear* (i.e. linear in the highest derivative term) equations

$$g^{\mu\nu}(x;\phi_j;\nabla_\rho\phi_j)\nabla_\mu\nabla_\nu\phi_i = F_i(x;\phi_j;\nabla_\rho\phi_j) \tag{4.11}$$

where each F_i is a smooth function of its variables, there's an important theorem due to Leray (1952), which we enounce (note that in (4.11) $g^{\mu\nu}$ is now permitted to depend on the unknown variables and their first derivatives nd F_i now may have nonlinear dependence on these variables).

Theorem 4.3.2. *Let $(\phi_0)_1...(\phi_0)_n$ be any solution of the quasilinear hyperbolic system* (4.11) *on a manifold $\mathcal{M}$ and let $(g_0)^{\mu\nu} = g^{\mu\nu}(x;(\phi_0)_j;\nabla_\rho(\phi_0)_j)$. Suppose $(\mathcal{M},(g_0)_{\mu\nu})$ is globally hyperbolic and let Σ be a smooth space-like Cauchy surface for $(\mathcal{M},(g_0)_{\mu\nu})$. Then for initial data on Σ sufficiently close to the initial data for $(\phi_0)_1...(\phi_0)_n$ there exists an open neighborhood O of Σ such that equation* (4.10) *has a solution $\phi_1...\phi_n$ in O and $(O, g_{\mu\nu}(x;\phi_j;\nabla_\rho\phi_j))$ is globally hyperbolic. The solution is unique in O, it propagates causally and depends continuously on the initial data.*

Thus the initial value formulation of (4.11) exists and it is also well posed on Σ, the complete proof of this theorem can be found in Leray (1952) and [29].
After illustrating the 3+1 decomposition process of the space-time in the next Chapter, starting from these results we will show that even General Relativity exhibits a well-posed initial value formulation by casting the Einstein's equations in the form (4.11).

Chapter 5

ADM Formalism

The purpose of this Chapter is to illustrate the ADM formalism, which essentially is a Hamiltonian formulation of General Relativity (as it has been explained in the introduction). While a Lagrangian formulation of a field theory is "space-time covariant" and it consists in specifing an action functional of the field on the space-time manifold (whose extremization yields the field equations), a Hamiltonian formulation of a field theory requires a breakup of space-time into space and time.

The Hamiltonian formalism starts from the definition of a momentum variabile related to a choice of a configuration variable:

$$p = \frac{\partial L}{\partial \dot{q}} \tag{5.1}$$

Since the latter requires a time coordinate, it is necessary to cast General Relativity in a form where it exhibits a "distinguished" time. This step is achieved in the ADM formalism by foliating the space-time, described by $(\mathcal{M}, g)$, into a set of three-dimensional space-like hypersurfaces Σ_t, with t denoting the global time function and with coordinates on each slice given by x^i.

The dynamic variables of this formalism are taken to be the metric tensor of three dimensional spatial slices h_{ab} and their conjugate momenta p^{ab}. Using these variables it is possible to define a Hamiltonian, and thereby write the equations of motion for General Relativity in the form of Hamilton's equations. This approach is usually called *3+1 decomposition*, while the covariance of the theory is preserved by allowing for the possibility to consider all possible foliations of this type. We want to demand, as a

necessary condition, that $(\mathcal{M}, g)$ be globally hyperbolic, in such a way it possesses a Cauchy surface Σ on which initial data can be described to determine uniquely the whole space-time (see Chapter 5). In this case, we'll show (by setting the Einstein equations in the form (4.11)) that classical initial value formulation make sense and the Hamiltonian form of General Relativity can be constructed.
In this Chapter we will also analyze the Hamiltonian constraints that emerge in the ADM formulation of General Relativity, with particular regard to the choice of configuration space and the problem of determining the physical degrees of freedom of this theory. We use conventions of [19].

5.1 The 3+1 Decomposition of General Relativity

As we have shown, for such a globally hyperbolic space-time $(\mathcal{M}, g)$ there exists a global time function f such that each surface $f = constant$ is a Cauchy surface, therefore $(\mathcal{M}, g)$ can be foliated into Cauchy hypersurfaces and its topology (which is fixed) results as a direct product:

$$\mathcal{M} \cong \mathbb{R} \times \Sigma \tag{5.2}$$

It is worthwhile to emphasize that if the manifold Σ is supposed to be compact, the previous topology is the same of a closed universe. In quantum theory, topology change may be a viable option, thus its absence in the formalism could be a possible weakness of the canonical approach, nevertheless the resulting quantum theory is general enough to cope with many of the intersting situations. A more general formulation allowing topology change to occur, in principle, is the path integral approach [4].
The vector field "flow of time" corresponding to the global time function t is denoted by t^μ and it is chosen in order to satisfie $t^\mu t_{,\mu} = 1$, this vector field can be interpreted as describing the "gloabl" flow of time and can be used to identify each Σ_t with the initial surface Σ. In order to view dynamical evolution as the change of fields on the fixed manifold Σ it is convenient to choose a time-independent volume element on Σ_t [6].
So, we have introduced a time function t and a vector field t^μ on a space-time such that the surfaces Σ_t of constant t are space-like Cauchy surfaces. It is important to underline that one cannot interpret t and t^μ in terms of physical measurements using clocks until

one knows the space-time metric, which turns out to be the unknown field variable in Einstein's equations.

Introducing n^μ and $X^\nu{}_{,a}$ as, rispectively, the normal vector and tangential vectors to the space-like hypersurfaces Σ_t characterized by $x^0 \equiv t = constant$, the relation between neighboured Cauchy hypersurfaces Σ_t into which space-time is foliated is the same as shown in figure (3.3) seen in the description of the parametrized field theories, were t^μ was written as $\dot{X}^\mu$. The space-time metric $g_{\mu\nu}$ induces a three-dimensional Riemannian spatial metric on each Σ_t according to:

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{5.3}$$

where the unit normal n_μ satisfies:

$$n^\mu n_\mu = -1 \tag{5.4}$$

$$X^\mu{}_{,a} n_\mu = 0 \tag{5.5}$$

Multipliyng (5.3) with $X^\mu_{,a} X^\nu_{,b}$ and application of $X^\mu_{,a} n_\mu = 0$ leads to:

$$h_{ab} = g_{\mu\nu} \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\nu}{\partial x^b}$$

which is nothing but (3.61).

In fact, $h_{\mu\nu}$ is a three-dimensional object only, since it acts as a projector on Σ_t and we shall write for it h_{ab}, since there is an isomorphism between tensor fields on $\mathcal{M}$ that are orthogonal to n_μ in each index and tensor fields on Σ_t. The three-dimensional metric satisfies:

$$h_{\mu\nu} n^\nu = 0 \tag{5.6}$$

$$h_{\mu\nu} h^{\nu\rho} = h^\rho_\mu \tag{5.7}$$

The last equation can be also expressed as $h^{ab} h_{bc} = \delta^a_c$.

As in (3.60), it is convenient to decompose the vector field t^μ into its normal and tangential components with respect to the surfaces Σ_t:

$$t^\mu = N n^\mu + N^\mu \tag{5.8}$$

where N is the lapse function and N^μ is the shift vector, the latter was previously written as $N^a X^\mu_{,a}$, as for the metric induced on each Σ_t, N^μ is a three dimensional object and it can be identified with N^a. The lapse function can be written as $N = -t^\mu n_\mu$, from which one can infer:

$$N = -t^\mu n_\mu = \frac{1}{n^\mu t_{,\mu}} \tag{5.9}$$

As for the relativistic particle example, one can interpret the previous equation as the ratio between proper time, given by $t^\mu t_{,\mu} = 1$ and coordinate time $n^\mu t_{,\mu}$. Thus, N measures the rate of flow of proper time with respect to coordinate time as one moves normally to Σ_t, whereas N^μ measures the amount of tangential "shift" to Σ_t contained in the time flow vector field (5.8):

$$N_\mu = h_{\mu\nu} t^\nu \tag{5.10}$$

The four-metric $g_{\mu\nu}$ can be decomposed, from (5.3) and (3.62), into spatial and temporal components as:

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_c & h_{ab} \end{pmatrix} \tag{5.11}$$

Its inverse can be found as:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^b}{N^2} \\ \frac{N^c}{N^2} & h^{ab} - \frac{N^a N^b}{N^2} \end{pmatrix} \tag{5.12}$$

Here, h^{ab} is the inverse of the three-dimensional metric, and one recognizes that the spatial part of $g^{\mu\nu}$ is not identical with h^{ab} but contains an additional term that involves the shift vector N^a. The components of the normal vector can be also expressed, by using (5.12) and the one-from $n_\mu dx^\mu = -N dt$, as:

$$n^\mu = \left(\frac{1}{N}, -\frac{\vec{N}}{N}\right) \tag{5.13}$$

$$n_\mu = (-N, 0, 0, 0) \tag{5.14}$$

The various hypersurfaces Σ_t can be identified by a diffeomorphism that is generated by the integral curves of t^μ, which can be interpreted as the "flow of time" throughout space-time. If we identify the hypersurfaces Σ, Σ_t by the diffeomorphism resulting from following integral curves of t^μ, we may view the effect of moving forward in time as that of changing the spatial metric on an abstract three-dimensional manifold Σ from $h_{ab}(0)$

to $h_{ab}(t)$, therefore we may view a globally hyperbolic space-time $(\mathcal{M}, g)$ as representing the time development of a Riemannian metric on a fixed three-dimensional manifold. This suggests the use of the spatial three-dimensional metric h_{ab} on a three-dimensional hypersurface as the appropriate dynamical variables for the canonical formalism in General Relativity (the lapse function N and the covariant form of the shift vector $N_a = h_{ab}N^b$ are not dynamical, since they only prescribe how to "move forward in time" and strictly speaking they are Lagrange multipliers, as we shall see, thus they are arbitrary). One can otherwise consider, as we shall do in order to discuss the Lagrangian formulation of General Relativity, the inverse metric $g^{\mu\nu}$ as field variable.
From (5.12) one can see that information contained in (h_{ab}, N, N_a) is equivalent to that one contained in $g^{\mu\nu}$ (further motivation for viewing the spatial metric as the dynamical variable in General Relativity will arise in the Hamiltonian formulation): space–time then becomes nothing but a "trajectory of spaces".
There is even no need to assume from the beginning that Σ is embedded in some space–time, only after solving the equations of motion we can interpret $h_{ab}(t)$ as being brought about by "wafting" through $\mathcal{M}$ via a one-parameter family of embeddings. Thus, we would expect appropriate initial data to consist of the Riemannian metric h_{ab} and it is "time derivative" or "velocity" on the three-dimensional manifold Σ. In order to

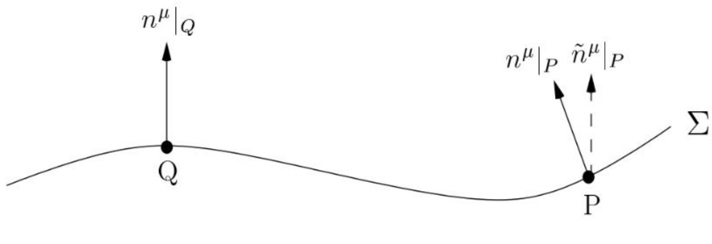

Figure 5.1: A space-time diagram illustrating the notion of the extrinsic curvature of a hypersurface Σ. From [4]

introduce the corresponding "velocity" for h_{ab}, we first consider the *Extrinsic curvature* or *Second fundamental form* tensor field:

$$K_{\mu\nu} \equiv h_\mu{}^\rho \nabla_\rho n_\nu \tag{5.15}$$

where $\nabla^\mu = g^{\mu\nu}\nabla_\nu$ is the covariant derivative, that acts on a generic tensor $T^{\mu\sigma}{}_\lambda$ defined in every point of $(\mathcal{M}, g)$ as:

$$\nabla_\rho T^{\mu\sigma}{}_\lambda = \frac{\partial T^{\mu\sigma}{}_\lambda}{\partial x^\rho} + \Gamma^\mu_{\rho\nu} T^{\nu\sigma}{}_\lambda + \Gamma^\sigma_{\rho\nu} T^{\mu\nu}{}_\lambda - \Gamma^k_{\lambda\rho} T^{\mu\sigma}{}_k \tag{5.16}$$

Since $K_{\mu\nu} n^\mu = 0 = K_{\mu\nu} n^\nu$, the tensor field $K_{\mu\nu}$ is a purely spatial quantity and can be mapped to its spatial version K_{ab} (with indices being moved by the three-metric h_{ab}).
The extrinsic curvature $K_{\mu\nu}$ of the C^2 space-like hypersurface Σ represents a well-defined notion of the "time derivative" of the spatial metric on a hypersurface Σ embedded in space-time. One can prove, using Frobenius' theorem for the hypersurface orthogonal vector field n^μ, that this tensor field is symmetric, $K_{\mu\nu} = K_{\nu\mu}$. Its geometric interpretation in terms of the "bending" of Σ in space-time can be inferred from figure (5.1). Consider the normal vectors at two different points P and Q of a hypersurface Σ. Be $\tilde{n}^\mu$ the vector at P resulting from parallel transporting n^μ along a geodesic from Q to P. The difference between n^μ and $\tilde{n}^\mu$ is a measure for the embedding curvature of Σ into $\mathcal{M}$ at P. One therefore recognizes that the tensor field $K_{\mu\nu}$ can be used in order to describe this embedding curvature, since it vanishes for $\tilde{n}^\mu = n^\mu$. $K_{\mu\nu}$ or, equivalently, K_{ab} can be interpreted as the "velocity" associated with h_{ab} while its trace $K \equiv K_a{}^a = h^{ab} K_{ab} \equiv \theta$ can be interpreted as the "expansion" of a geodesic congruence orthogonal to Σ (in cosmology, for a Friedmann universe K is three times the Hubble parameter).
These considerations suggest that in General Relativity, appropriate initial data should consist of a triple (Σ, h_{ab}, K_{ab}), where Σ is a three-dimensional manifold, h_{ab} is the Riemannian metric on Σ and K_{ab} is a symmetric tensor field on the same hypersurface.
In the hyperbolic space-time $(\mathcal{M}, g_{\mu\nu})$ in which is defined a smooth space-like hypersurface Σ on which the induced metric is h_{ab}, let D_a denote the derivative operator associated whith h_{ab} (as ∇_a is the derivative operator associated with g_{ab}), an important theorem [6] states that h_{ab} uniquely determines this "natural" derivative operator on the hypersurfaces which foliate the space-time. Furthermore, the derivative operator D_a on Σ gives rise to a curvature three-dimensional tensor ${}^{(3)}R_{abc}{}^d$ on Σ, which is defined analogously to $R_{\mu\nu\rho}{}^\sigma$ in $(\mathcal{M}, g_{\mu\nu})$, with ∇_μ replaced by D_a and $g_{\mu\nu}$ replaced by h_{ab}.
Now we will establish some useful relations between the space-time metric, derivative operator, curvature and the corresponding quantities they induce on a space-like hypersurface Σ embedded in $(\mathcal{M}, g_{\mu\nu})$: we obtain now formulas relating D_a and ${}^{(3)}R_{abc}{}^d$ to

four-dimensional quantities. This step will be fundamental in order to show that General Relativity exhibits a well-posed initial value formulation, because it will make possible to use as dynamical variables the quantities (Σ, h_{ab}, K_{ab}) on the hypersurfaces parametrized by t.
Let v^μ be a (space-time) vector at a point $p \in \Sigma$. We may uniquely decompose v^μ into components tangent to and perpendicular to Σ via

$$v^\mu = v_\perp n^\mu + v_\parallel^\mu \tag{5.17}$$

where n^μ is the unit normal to Σ, so it turns out that $v_\parallel^\mu n_\mu = 0$.
If $v_\perp = 0$ so that $v^\mu = v_\parallel^\mu$, we may view v^μ as a vector lying in the tangent space to Σ at p. The condition that $v_\perp = 0$ is equivalent to

$$v^\mu = h^\mu{}_\nu v^\nu \tag{5.18}$$

where $h_{\mu\nu}$ is given by (5.3) and the first index of h_{ab} is raised by g^{ab}.
More generally, we may view a space-time tensor $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ at $p \in \Sigma$ as a tensor over the tangent space to Σ at p if

$$T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = h^{a_1}{}_{c_1} \dots h^{a_k}{}_{c_k} h_{b_1}{}^{d_1} \dots h_{b_l}{}^{d_l} T^{c_1 \dots c_k}{}_{d_1 \dots d_l} \tag{5.19}$$

Conversely, any tensor defined at point p on the manifold Σ uniquely gives rise to a space-time tensor at p (i.e. a tensor over the tangent space to $\mathcal{M}$ at p) which satisfies previous equation. Thus, as it has been already emphsized, $h^a{}_b$ plays the role of a projection operator from the tangent space to $\mathcal{M}$ at p to the tangent space to Σ at p.
Let $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ be a tensor field on the manifold Σ: if we view $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ as a space-time tensor satisfying (5.19), we still cannot define $\nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ since, in order to calculate this quantity, we would need to know how $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ varies as we move off of Σ. However, $h_d{}^c \nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ is well defined since, for this quantity, no derivatives in directions pointing out of Σ are taken. This latter tensor doesn't need to satisfy equation (5.19), but we can project its indices using $h^a{}_b$ to obtain a tensor field on Σ.
It can be shown [6] that D_a acts on $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ as:

$$D_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = h^{a_1}{}_{d_1} \dots h_{b_l}{}^{e_l} h_c{}^f \nabla_f T^{d_1 \dots d_k}{}_{e_1 \dots e_l} \tag{5.20}$$

where ∇_a is the derivative operator associated with g_{ab}. Furthermore, since $\nabla_d g_{ef} = 0$ and $h_{ab}n^b = 0$, we have

$$D_a h_{bc} = h_a{}^d h_b{}^e h_c{}^f \nabla_d (g_{ef} + n_e n_f) = 0 \tag{5.21}$$

From (5.20) one can derive that the Gauss equation generalized to higher dimensions is:

$${}^{(3)}R_{\mu\nu\lambda}{}^{\rho} = h_\mu{}^{\mu'} h_\nu{}^{\nu'} h_\lambda{}^{\lambda'} h^\rho{}_{\rho'} R_{\mu'\nu'\lambda'}{}^{\rho'} - K_{\mu\lambda} K_\nu{}^\rho + K_{\nu\lambda} K_\mu{}^\rho \tag{5.22}$$

and a similar calculation leads to the so called *generalized Codazzi equation*:

$$D_\mu K_{\nu\lambda} - D_\nu K_{\mu\lambda} = h_\mu{}^{\mu'} h_\nu{}^{\nu'} h_\lambda{}^{\lambda'} R_{\mu'\nu'\lambda'\rho} n^\rho \tag{5.23}$$

In the much simpler case of a two-dimensional hypersurface embedded in three dimensional flat euclidean space, (5.22) is the famous "theorema egregium" of Gauss.
In 3+1 dimensions, however, the situation is more complicated. Now we calculate:

$$\begin{aligned} K_{ab} &= \frac{1}{2} \left(n^c \nabla_c h_{ab} + h_{ac} \nabla_b n^c + h_{cb} \nabla_a n^c \right) \\ &= \frac{1}{2N} \left(N n^c \nabla_c h_{ab} + h_{ac} \nabla_b (N n^c) + h_{cb} \nabla_a (N n^c) \right) \end{aligned} \tag{5.24}$$

So, in terms of lapse and shift, the extrinsic curvature can be written as:

$$K_{ab} = \frac{1}{2N} \left(\dot{h}_{ab} - D_a N_b - D_b N_a \right) \tag{5.25}$$

We shall see that the components of the canonical momenta are obtained by a linear combination of the K_{ab}: this is reason why we specified in the introduction to this Chapter that the dynamic variables of this formalism are tipically taken to be the metric tensor of three dimensional spatial slices and their conjugate momenta.

5.2 Lagrangian formulation of General Relativity

We have introduced the mathematical structures that allow us to illustrate the Lagrangian formulation of the General Relativity in order to derive the field equations (1.1), whose can be determined by an action principle. The action for a gravitational field must be expressed by a scalar integral extended to all the space with respect to the

spatial coordinate (x^1, x^2, x^3) and collocated between two given values with respect to the time coordinate x^0, furthermore the integrand of the action must not contain second or higher order derivatives of $g_{\mu\nu}$ in order to satisfie the general requirements, that we have mentioned in the first Chapter, for the field equations.
The formalism of General Relativity can thus be defined by the *Einstein - Hilbert action*:

$$S_{E-H} = \frac{c^4}{16\pi G}\int_M d^4x\sqrt{-g}(R-2\Lambda) - \frac{c^4}{8\pi G}\int_{\partial M} d^3x\sqrt{h}K \tag{5.26}$$

The integration in the first integral of (5.26) covers a region M of the space-time manifold, while the second integral is defined on its space-like boundary ∂M, in both the integrals the invariant measure is understood as $d^4x\sqrt{-g} = d^4x\sqrt{-det(g_{\mu\nu})}$
As we have seen, the space-like nature of the boundary ∂M implies that what happens on a point on this surface is determined by the initial conditions on the surface itself in the past and in the future. We have also specified that a curved space which is not globally hyperbolic does not admit space-like (i.e. Cauchy) surfaces. The second term in (5.26), in which h is the determinant of the three dimensional metric on the boundary and K is the trace of the second fundamental form, is necessary in order to obtain a consistent variational principle as noticed by Einstein in 1916.
To show how the Einstein equations in the vacuum are recovered from the Einstein - Hilbert action, we can calculate the variation of the terms that appear in (5.26):

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} \tag{5.27}$$

where we used the fact that the differential dg of the metric tensor's determinant can be written as $dg = gg^{\mu\nu}dg_{\mu\nu} = -gg_{\mu\nu}dg^{\mu\nu}$ (the last equality is due to the fact that $g_{\mu\nu}g^{\mu\nu} = \delta^\mu_\mu = 4$). Also, it can be shown that:

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla^\mu v_\mu = \nabla^\mu\left(\nabla^\nu(\delta g_{\mu\nu}) - g^{\rho\sigma}\nabla_\mu(\delta g_{\rho\sigma})\right) \tag{5.28}$$

Moreover we can write:

$$\int_M \nabla_\mu v^\mu\sqrt{-g}\, d^4x = \int_{\partial M} v_\mu n^\mu\sqrt{h}\, d^3x \tag{5.29}$$

where $\sqrt{h}\, d^3x$ is the natural volume element on ∂M and n^μ is the unit normal to the very same boundary surface. Using (5.28) we have on ∂M:

$$\begin{aligned} v_\mu n^\mu &= n^\mu g^{\rho\sigma} \left[\nabla_\sigma (\delta g_{\mu\rho}) - \nabla_\mu (\delta g_{\rho\sigma}) \right] \\ &= n^\mu h^{\rho\sigma} \left[\nabla_\sigma (\delta g_{\mu\rho}) - \nabla_\mu (\delta g_{\rho\sigma}) \right] \\ &= -n^\mu h^{\rho\sigma} \nabla_\mu (\delta g_{\rho\sigma}) \end{aligned} \tag{5.30}$$

where $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ is the metric induced on ∂M and we have $\delta g_{\mu\sigma} = 0$ on ∂M. Now we can relate the very last expression to the variation of the trace of the extrinsic curvature of the boundary. Remembering (5.15) we define:

$$K \equiv K^\mu{}_\mu = h^\mu{}_\nu \nabla_\mu n^\nu \tag{5.31}$$

it can be shown that:

$$\delta K = \frac{1}{2} n^\mu h^{\sigma\rho} \nabla_\mu (\delta g_{\sigma\rho}) \tag{5.32}$$

So we can write, using (5.29), (5.30) and (5.32) :

$$\int_M \nabla_\mu v^\mu \sqrt{-g}\, d^4x = -2 \int_{\partial M} \delta K \sqrt{h}\, d^3x \tag{5.33}$$

where the variation of $g_{\mu\nu}$ for which $\delta g_{\mu\nu} = 0$ on ∂M requires to satisfy $\delta h_{\mu\nu} = 0$.
So, using (5.33), (5.28), (5.27) we can definitely write:

$$\frac{\delta S_{E-H}}{\delta g^{\mu\nu}} = 0 \quad \Rightarrow \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{5.34}$$

We have chosen for convenience $g^{\mu\nu}$ instead of $g_{\mu\nu}$ as field variable.
It is important to underline that, by Stokes theorem, (5.29) does not vanish for general variations where $g^{\mu\nu}$ is held fixed on the boundary but not the first derivatives of the latter, and this is the reason why the boundary term in (5.26) does appear. It is also important to emphasize that, since not all variations of $g_{\mu\nu}$ correspond to a variation of the space-time metric (that is the real variation of the gravitational field), one cannot deduce that in a real gravitational field the action has a minimun (instead of a more general extreme) with respect to all possible variations of $g_{\mu\nu}$.
Every transformation of the coordinates between two reference system to another one in the same space-time represents, in general, a set of four (because the latter is the number

of the x^μ) independent transformations.
Thus, in General Relativity, it turns out that the minimum principle action only establishes that the $g_{\mu\nu}$ can be vincolated to restrictive conditions in order for the action to have a minimum when $g_{\mu\nu}$ are varied. However the field equations can be obtained only requiring that the action exhibits an extreme (not necessarily a minimum), then in order to determine them one can vary all the $g_{\mu\nu}$ independently.
To recover the Einstein equations in presence of a gravitational source characterized by an energy-momentum tensor $T_{\mu\nu}$, we can add a "matter action" S_m to (5.26) in order to have:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \tag{5.35}$$

In this way we recover the most general form of the Einstein equations (1.1) by requiring that the variation of $S_{E-H} + S_m$ does vanish.

5.3 Initial Value Formulation for General Relativity

In order to show that General Relativity has a well posed initial value formulation, we will cast the Einstein's equation into the form (4.11). The analysis of Einstein's equation differs from that of the Klein-Gordon field in that there are initial value constraints and in that it is necessary to make a "gauge choice", i.e., a choice of coordinates, so that Einstein's equation takes the desired form.
The first issue to consider is the nature of the initial value formulation in the theory of General Relativity, because in such a theory we are solving for the space-time metric itself, while in other theories of classical physics we are given the space-time background and the task is to determine the time evolution of the quantities (in the background) from their initial values and time derivatives. In order to choose the quantity or quantities to prescribe initially in General Relativity in order to determine the space-time structure, it is necessary to view this theory as describing the time evolution of some quantity.
The initial value formulation should be relevant only in the case of an hyperbolic space-time, thus, let $(\mathcal{M}, g_{\mu\nu})$ be a globally hyperbolic space-time. As we have proven, we can foliate $(\mathcal{M}, g_{\mu\nu})$ by Cauchy surfaces, Σ_t, parametrized by a global time function t.
We have also shown that appropriate initial data should consist in the triple (Σ, h_{ab}, K_{ab}).

Once we have chosen the time function t and the vector field t^μ previously introduced, the next step is to reformulate the Einstein–Hilbert action in terms of the three dimensional variables h_{ab} and K_{ab}. For this purpose one needs the relationship between the four-dimensional and the three-dimensional curvatures: this is given by the Gauss equation (5.22) and the generalized Codazzi equation (5.23). Contraction of (5.23) with $g^{\mu\lambda}$ gives, using (1.2) and the simmetry of $K_{\mu\nu}$ and $h_{\mu\nu}$:

$$D_\mu K^\mu{}_\nu - D_\nu K = R_{\rho\lambda} n^\lambda h^\rho{}_\nu \tag{5.36}$$

We turn, now, to the analysis of the vacuum Einstein's equations.
We give initial data (h_{ab}, K_{ab}) on a three-dimensional manifold Σ and write down Einstein's equations for the metric components $g_{\mu\nu}$ in a local coordinate system $\{y^\mu\}$ with the time coordinate t chosen in order to have the $t = 0$ surface which corresponds to Σ. By casting the equations in the form (4.11) we will prove local existence of a solution with the desired properties. Finally, we shall mention how to "globalize" our global results in order to obtain the final conclusion.
Einstein's equations in vacuum yield a system of 10 second-order partial differential equations for the ten unknown metric components. Furthermore these equations are linear in the second derivatives of the metric, thus they exhibit what we have defind as a "quasilinear" form.
Addressing the vacuum Einstein equations without the cosmological constant:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \equiv G_{\mu\nu} = 0 \tag{5.37}$$

Ir's possible to show that the equations

$$G_{\mu\nu} n^\nu = 0 \tag{5.38}$$

Where n^μ is the unit normal to the $t = constant$ surfaces, contain no second time derivatives of any of the metric components, this means that these components of $G_{ab} = 0$ at $t = 0$ depend only on the initial data.
Thus, these equations provide initial value constraints, we can express them in coordinate invariant form by using the Gauss and Codazzi equations (5.22), (5.23). Referring to the equations (5.37), one finds for the "space–time component", which turns out to be

the initial value constraint (obtained by multipliyng (5.37) for $h^{\mu}{}_{\rho}n^{\nu}$), remembering that (5.6) holds:

$$0 = h^{\mu}{}_{\rho}G_{\mu\nu}n^{\nu} = h^{\mu}{}_{\rho}R_{\mu\nu}n^{\nu} \tag{5.39}$$

which can be rewritten by using (5.36) as

$$D_b K^b{}_a - D_a K = 0 \tag{5.40}$$

For the "time–time component", that is an additional constraint, by multipliyng (5.37) for $n^{\mu}n^{\nu}$ one has:

$$0 = R_{\mu\nu}n^{\mu}n^{\nu} + \frac{R}{2} \tag{5.41}$$

in which the condition (5.4) has been used.
From (5.22) one easily finds upon contraction of indices and remembering the definition (1.5):

$$R + K_{\mu}{}^{\mu}K_{\nu}{}^{\nu} - K_{\mu\nu}K^{\mu\nu} = h^{\mu\mu'}h_{\nu}{}^{\nu'}h_{\mu}{}^{\lambda'}h^{\nu}{}_{\rho'}R_{\mu'\nu'\lambda'}{}^{\rho'} \tag{5.42}$$

Using (5.3), is possible to express the right-hand side as

$$^{(3)}R + 2R_{\mu\nu}n^{\mu}n^{\nu} = 2G_{\mu\nu}n^{\mu}n^{\nu} \tag{5.43}$$

and so, with (5.42) and (5.43), we come to the "time–time component" of Einstein's equations (5.41) written as:

$$K^2 - K_{ab}K^{ab} +^{(3)} R = 0 \tag{5.44}$$

This is the (3+1)-dimensional version of the theorema egregium.
Both (5.40) and (5.44) are the initial value constraint equations of General Relativity, in fact they only contain first-order time derivatives, these constraints play a crucial role in the initial value formulation of classical General Relativity, see for example Choquet-Bruhat and York (1980) for details. If the constraints (5.40) and (5.44) are satisfied initially and the spatial components of Einstein's equation are satisfied everywhere, then the constraints also are satisfied (this can be easily demonstrated by using the Bianchi identity $\nabla^{\mu}G_{\mu\nu} = 0$). We will return on this point at the end of this section.
Thus (5.37) is an undetermined system of equations for the metric components $g_{\mu\nu}$: we have only six evolution equations (the pure spatial components of $G_{\mu\nu} = 0$) for 10

unknown metric components. However this undetermination is not physical and it results from the redundancy in the description of space-time geometry by metric components $g_{\mu\nu}$. As will be later discussed, if $\phi : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, then $(\mathcal{M}, g_{\mu\nu})$ and $(\mathcal{M}, \phi^* g_{\mu\nu})$ represent the same physical space-time. Since the coordinate basis metric components of $g_{\mu\nu}$ and $\phi^* g_{\mu\nu}$ are related by the coordinate transformation associated with ϕ, any two solutions of Einstein's equation (whose coordinate basis metric components are related by the tensor transformation law) represent the same physical solution. Since four arbitrary functions appear in the transformation law, roughly speaking there should be only six "nongauge functions" in the 10 metric components $g_{\mu\nu}$.
Therefore, it is plausible that Einstein's equation contains the correct number of evolution equations and that a well posed initial value formulation exists. It is possible to show that this precisely is the case: there exists one globally hyperbolic space–time obeying Einstein's equations (i.e. a unique solution for the four-metric up to diffeomorphisms), which has a Cauchy surface on which the induced metric and the extrinsic curvature are just h_{ab} and K_{cd}, respectively. The demonstration is quite sophisticated [6], the main idea is to introduce a convenient choice of "gauge" (i.e., coordinates), for which Einstein's equation has the form (4.11), which are the *harmonic coordinates* x^μ that satisfie by definition:

$$H^\mu \equiv \nabla_\nu \nabla^\nu x^\mu = 0 \tag{5.45}$$

The vacuum Einstein equations (5.37) become, by using harmonic coordinates, the so-called *reduced Einstein equation*:

$$0 = -\frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + \hat{F}_{\mu\nu}(g, \partial g) \tag{5.46}$$

where $\hat{F}_{\mu\nu}(g, \partial g)$ is a nonlinear function of the metric components $g_{\alpha\beta}$ and their first derivatives.
The fundamental point of the latter is that it has the same form of (4.11), for which we have seen the theorem (4.3.2) that establishes that the initial value formulation is well posed in a particular sense (see last section of the previous Chapter). In order not to make long and sophisticated digressions, we only report final conclusions of the discussion about the initial value formulation of the General Relativity, which are summarized in the following theorem, whose proof can be found in [6]:

Theorem 5.3.1. *Let Σ be a three-dimensional C^∞ manifold, let h_{ab} be a smooth Riemannian metric on Σ and let K_{ab} be a smooth symmetric tensor field on Σ. Suppose h_{ab} and K_{ab} satisfy the constraint equations* (5.40) *and* (5.44)*. Then there exists a unique C^∞ space-time $(\mathcal{M}, g_{\mu\nu})$, called the "maximal Cauchy development" of (Σ, h_{ab}, K_{ab}), satisfying the following four properties:*
(i) $(\mathcal{M}, g_{\mu\nu})$ is a solution of Einstein's equation
(ii) $(\mathcal{M}, g_{\mu\nu})$ is a globally hyperbolic with Cauchy surface Σ
(iii) The induced metric and extrinsic curvature of Σ are, respectively, h_{ab} and K_{ab}
(iv) Every other space-time satisfying (i)-(iii) can be mapped isometrically into a subset of $(\mathcal{M}, g_{\mu\nu})$
Furthermore, $(\mathcal{M}, g_{\mu\nu})$ satisfies the desired domain of dependence property in the following sense. Suppose (Σ, h_{ab}, K_{ab}) amd $(\Sigma', h'_{ab}, K'_{ab})$ are initial data sets with maximal developments $(\mathcal{M}, g_{\mu\nu})$ and $(\mathcal{M}', g'_{\mu\nu})$. Suppose there is a diffeomorphism between $S \subset \Sigma$ and $S' \subset \Sigma'$ which carries (h_{ab}, K_{ab}) on S into (h'_{ab}, K'_{ab}) on S'. Then $D(S)$ in the space-time $(\mathcal{M}, g_{\mu\nu})$ is isometric to $D(S')$ in the space-time $(\mathcal{M}', g'_{\mu\nu})$ and the solution $g_{\mu\nu}$ on $\mathcal{M}$ depends continuously on the initial data (h_{ab}, K_{ab}) on Σ.

Aside from showing that General Relativity has the physically desirable property of possessing a well-posed initial value formulation, this theorem puts globally hyperbolic space-time $(\mathcal{M}, g_{\mu\nu})$ satisfying the constraint equation into correspondence with initial data sets (Σ, h_{ab}, K_{ab}) satisfying the constraint equations (5.40) and (5.44) (this association is, however, not 1-1 because of the freedom of choosing a space-like Cauchy surface in $\mathcal{M}$). It usually is far easier to solve the constraint equations on Σ than to solve Einstein's equation on $\mathcal{M}$. In electrodynamics, for comparison, one has to specify $\vec{A}$ and $\vec{E}$ on Σ satisfying the constraint $\nabla\vec{E} = 0$, which is the Gauss law. One then gets in space–time a solution of Maxwell's equations that is unique up to gauge transformation. The important point is that the space–time is fixed in Maxwell's theory, whereas in the gravitational case it is part of the solution.
That the dynamical laws follow from the laws of the instant can be inferred from the validity of the following "interconnection theorems", to which we have already referred [15]:
1) If the constraints are valid on an initial hypersurface and if $G_{ab} = 0$ (pure spatial

components of the vacuum Einstein equations) on space–time, the constraints hold on every hypersurface.
2) If the constraints hold on every hypersurface, the equations $G_{ab} = 0$ hold on space–time. Similar properties hold in Maxwell's theory, in the presence of non-gravitational fields, $\nabla_\mu T^{\mu\nu} = 0$ is needed as an integrability condition (analogously to $\partial_\mu j^\mu = 0$ for Maxwell's equations).

5.4 ADM Action

In order to reformulate the Einstein–Hilbert action, it is necessary to express the volume element and the Ricci scalar in terms of the new dynamical variables (on the 3-dimensional hypersurfaces) h_{ab} and K_{cd}. For the volume element one finds

$$\sqrt{-g} = N\sqrt{h} \tag{5.47}$$

Equation (5.47) can be easily found calculating the determinant of $g_{\mu\nu}$ from (5.11). Nevertheless, it also can be seen by defining the three-dimensional volume element [6]:

$$^{(3)}e_{\mu\nu\lambda} = e_{\rho\mu\nu\lambda}t^\rho \tag{5.48}$$

with t^ρ decomposed as in (5.8) and $e_{\rho\mu\nu\lambda}$ denoting the time-independent four-dimensional volume element, one has by using $\epsilon_{\rho\mu\nu\lambda} = \sqrt{-g}\; e_{\rho\mu\nu\lambda}$:

$$\epsilon_{\rho\mu\nu\lambda}t^\rho = \sqrt{-g}\; e_{\mu\nu\lambda} = \sqrt{-\frac{g}{h}}\epsilon_{\mu\nu\lambda}$$

from which (5.47) follows after using (5.8) and taking purely spatial components.
We shall now assume in the following that Σ is compact without boundary (the boundary terms for the non-compact case will be discussed separately). In order to rewrite the curvature scalar, we use first (5.41) written as follows:

$$R =^{(3)} R + K^2 - K_{ab}K^{ab} - 2R_{\mu\nu}n^\mu n^\nu \tag{5.49}$$

where $^{(3)}R$ is the three-dimensional Ricci scalar, obtained as

$$^{(3)}R =^{(3)} R^i{}_i \tag{5.50}$$

Using the definition of the Riemann tensor in terms of second covariant derivatives,

$$R^{\rho}{}_{\mu\rho\nu}n^{\mu} = R_{\mu\nu}n^{\mu} = \nabla_{\rho}\nabla_{\nu}n^{\rho} - \nabla_{\nu}\nabla_{\rho}n^{\rho} \tag{5.51}$$

The second term on the right-hand side becomes:

$$\begin{aligned} -2R_{\mu\nu}n^{\mu}n^{\nu} = {} & 2(\nabla_{\rho}n^{\nu})(\nabla_{\nu}n^{\rho}) - 2(\nabla_{\nu}n^{\nu})(\nabla_{\rho}n^{\rho}) \\ & + 2\nabla_{\nu}(n^{\nu}\nabla_{\rho}n^{\rho}) - 2\nabla_{\rho}(n^{\nu}\nabla_{\nu}n^{\rho}) \end{aligned} \tag{5.52}$$

The third and fourth term are total divergences, as consequence they can be cast into surface terms at the temporal boundaries. The last surface term yields $-2(n^{\nu}\nabla_{\nu}n^{\rho})n_{\rho} = 0$, while the first one gives, recalling the definition of $K_{\mu\nu}$, $2\nabla_{\mu}n^{\mu} = -2K$. The two remaining terms in the last expression can be written as $2K_{ab}K^{ab}$ and $-2K^2$, respectively. We now introduce the *DeWitt's metric* [20]:

$$G^{abcd} = \frac{\sqrt{h}}{2}(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd}) \tag{5.53}$$

which plays [4] the role of a metric in the space of all metrics.
Inspecting the Einstein–Hilbert action (5.26), one recognizes that the temporal surface term is cancelled, and that the action now reads, using (5.47), (5.49), (5.51), (5.52):

$$\begin{aligned} 16\pi G S_{E-H} &= \int_{\mathcal{M}} dt d^3x N\sqrt{h}(K_{ab}K^{ab} - K^2 +^{(3)} R - 2\Lambda) \\ &\equiv \int_{\mathcal{M}} dt d^3x N(G^{abcd}K_{ab}K_{cd} + \sqrt{h}(^{(3)}R - 2\Lambda)) \end{aligned} \tag{5.54}$$

where, in the second line, DeWitt's metric was used.
The gravitational action (5.54) has the classic form of kinetic energy minus potential energy, since the extrinsic curvature (as one can see from (5.25)) contains the "velocities" $\dot{h}_{ab}$. It is also called the *ADM action* in recognition of the work by Arnowitt, Deser, and Misner.

5.5 Hamiltonian formulation of General Relativity

Before describing the Hamiltonian formulation for General Relativity, one should properly define a configuration space for the field by specifying what tensor fields q

on Σ_t physically describs the instantaneous configuration of the field, but this subject will be deeply analized in the last section of this Chapter. Since the set of possible configurations of the field is infinite-dimensional, we shall not attemp here to give a precise definition of the cotangent space, which is the space of possible momenta of the field at a given configuration q. The final and most non trivial step required for a Hamiltonian formulation of a field theory is the specification of a functional $H[q,p]$ on Σ_t, called the Hamiltonian, which is of the form [9,10]:

$$H = \int_{\Sigma_t} \mathcal{H} \tag{5.55}$$

where the Hamiltonian density $\mathcal{H}$ is the local function of q and p (and of their spatial derivatives up to a finite order) such that the pair of equations:

$$\dot{q} = \frac{\delta H}{\delta p} \tag{5.56}$$

$$\dot{p} = -\frac{\delta H}{\delta q} \tag{5.57}$$

is equivalent to the field equation satisfied by the field.
Introducing now the ADM Lagrangian by writing:

$$S_{E-H} \equiv \int_{\mathcal{M}} dt d^3x \mathcal{L}^g \tag{5.58}$$

since $\mathcal{L}^g$ does not contain any time derivatives of N or N_a, one gets for the momenta canonically coniugated to the latters the following expressions

$$p_N \equiv \frac{\partial \mathcal{L}^g}{\partial \dot{N}} = 0 \qquad p^g_a \equiv \frac{\partial \mathcal{L}^g}{\partial \dot{N}^a} = 0 \tag{5.59}$$

Because lapse function and shift vector are only Lagrange multipliers, as for A^0 in electrodynamics, thus they should not be viewed as dynamical variables, in fact these are primary constraints since they do not involve the dynamical equations. Second, from (5.25) and (5.53) it turns out that the momentum canonically conjugate to h_{ab} is

$$p^{ab} \equiv \frac{\partial \mathcal{L}^g}{\partial \dot{h}_{ab}} = \frac{1}{16\pi G} G^{abcd} K_{cd} = \frac{\sqrt{h}}{16\pi G}(K^{ab} - K h^{ab}) \tag{5.60}$$

It is important to underline that, according to our conventions, the gravitational constant G appears here explicitly, although no coupling to matter is involved (this is the reason why it appears also in vacuum quantum gravity, this topic will be briefly treated in the last Chapter). Since p^{ab} is canonically conjugated to h_{ab}, one therefore has the following Poisson-bracket relation (in which we don't take into account that $\sqrt{h} > 0$, so this is formal at this stage):

$$\{h_{ab}(x), p^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\delta(x,y) \tag{5.61}$$

Recalling the expression of the extrinsic curvature (5.25) and taking the trace of (5.60), one can express the velocities in terms of the momenta:

$$\dot{h}_{ab} = \frac{32\pi GN}{\sqrt{h}}(p_{ab} - \frac{1}{2}ph_{ab}) + D_aN_b + D_bN_a \tag{5.62}$$

where $p \equiv p^a{}_a = p^{ab}h_{ab}$.
The canonical Hamiltonian density reads:

$$\mathcal{H}^g = p^{ab}\dot{h}_{ab} - \mathcal{L}^g \tag{5.63}$$

for which, using the inverse of DeWitt metric:

$$G_{abcd} = \frac{1}{2\sqrt{h}}(h_{ac}h_{bd} + h_{ad}h_{bc} + h_{ab}h_{cd}) \tag{5.64}$$

one gets the following expression, which holds modulo a total divergence which does not contribute in the integral of Hamiltonian (because we are supposing Σ is compact):

$$\mathcal{H}^g = 16\pi GNG_{abcd}p^{ab}p^{cd} - N\frac{\sqrt{h}({}^{(3)}R - 2\Lambda)}{16\pi G} - 2N_b(D_ap^{ab}) \tag{5.65}$$

The full Hamiltonian is found, as usual, by integration over the space coordinates:

$$H^g \equiv \int d^3x\mathcal{H}^g \equiv \int d^3x(N\mathcal{H}^g_\perp + N^a\mathcal{H}^g_a) \tag{5.66}$$

which has been decomposed as previously discussed.
The ADM Action (5.54) can be written in the form

$$16\pi GS_{E-H} = \int_{\mathcal{M}} dtd^3x(p^{ab}\dot{h}_{ab} - N\mathcal{H}^g_\perp - N^a\mathcal{H}^g_a) \tag{5.67}$$

Variation with respect to the Lagrange multipliers N and N^a yields the same constraints that can be found from the preservation of the primary constraints:

$$\{p_N, H^g\} = 0 \tag{5.68}$$

$$\{p_a^g, H^g\} = 0 \tag{5.69}$$

which are:

$$\mathcal{H}_\perp^g = 16\pi G G_{abcd} p^{ab} p^{cd} - \frac{\sqrt{h}}{16\pi G}({}^{(3)}R - 2\Lambda) \approx 0 \tag{5.70}$$

$$\mathcal{H}_a^g = -2D_b {p_a}^b \approx 0 \tag{5.71}$$

In fact, the constraint (5.70) is equivalent to the time-time component of Einstein's equations, and (5.71) is equivalent to their space-time component, they are called Hamiltonian constraint and diffeomorphism (or momentum) constraint, respectively. From its structure, (5.70) has some similarity to the "mass-shell constraint" for the relativistic particle (3.33), while (5.71) is similar to the Gauss law of electrodynamics. The total Hamiltonian is thus constrained to vanish, a result that is in accordance with our general discussion of reparametrization invariance (see second Chapter). In the case of non-compact space, boundary terms are present in the Hamiltonian, as will be later discussed.
Of course we have, in addition to the constraints, the six dynamical equations which are the Hamiltonian equations of motion. The first half, $\dot{h}_{ab} = \{h_{ab}, H^g\}$, just gives the "velocities" (5.62), the second half $\dot{p}_{ab} = \{p_{ab}, H^g\}$, yields a cumbersome expression [6] that is not needed for canonical quantization (it is needed for applications of the classical canonical formalism such as gravitational-wave emission from compact binary objects). If non-gravitational fields are coupled, as will be the case in the next Chapter, the constraints acquire extra terms. In the time-time component of Einstein's equations one has to use that:

$$2G_{\mu\nu} n^\mu n^\nu = 16\pi G T_{\mu\nu} n^\mu n^\nu \equiv 16\pi G \rho \tag{5.72}$$

Instead of (5.70) one now has the following expression for the Hamiltonian constraint:

$$\mathcal{H}_\perp = 16\pi G G_{abcd} p^{ab} p^{cd} - \frac{\sqrt{h}}{16\pi G}({}^{(3)}R - 2\Lambda) + \sqrt{h}\rho \approx 0 \tag{5.73}$$

Similarly, one has instead of (5.71) for the diffeomorphism constraints:

$$\mathcal{H}_a = -2D_b {p_a}^b + \sqrt{h} J_a \approx 0 \tag{5.74}$$

where $J_a \equiv {h_a}^\mu T_{\mu\nu} n^\nu$ is the "Poynting vector".
The classical canonical formalism for the gravitational field as discussed up to now was

pioneered by Bergmann, Dirac, “ADM” and other in 1950s. The canonical quantization of higher-derivative theories such as R^2- gravity can also be performed: the formalism is then more complicated since one has to introduce additional configuration-space variables and momenta.

5.5.1 Open spaces

Up to now we have neglected the presence of possible spatial boundary terms in the Hamiltonian. In this subsection, we shall briefly discuss the necessary modifications for the case of open spaces, where “open” means asymptotically flat (the necessary details can be found, for example, in [18]).

- **CLOSED UNIVERSE**
  Consider, first, the case of a closed universe, i.e. $\mathcal{M} = \mathbb{R} \times \Sigma$, where Σ is compact. If one considers a region U of $\mathcal{M}$ bounded by two constant time hypersurfaces Σ_1 and Σ_2, it is possible to show [6] that the modification of the gravitational action due to the boundary contributions from Σ_1 and Σ_2 are such to leave $\mathcal{H}^g$ unchanged in a closed universe. Furthermore, the numerical value of the latter vanishes for every solution. This suggests that we should define the total energy of a closed universe to be zero, i.e. there does not exist a nontrivial notion of total energy in a closed universe. However, this argument is not conclusive since it is always possible to “deparametrize” a theory in the manner mentioned above so as to make its Hamiltonian vanish: if General Relativity could be “deparametrized”, a notion of total energy in a closed universe could well emerge.

- **FLAT SPACE-TIME**
  Consider, now, the case of asymptotically flat space-times. If we take a region of $\mathcal{M}$ bounded by two hypersurfaces Σ_1 and Σ_2 and we wish to consider variations of the metric for which h_{ab} is held fixed on Σ_1 and Σ_2, it results that the most natural spatial boundary condition is that the variations preserve asymptotic flatness rather than that the induced metric be held fixed on a distant spatial boundary. This new boundary condition requires the addition of further boundary terms into the gravitational action and H^g is modified. Instead of keeping careful account of all

these terms, we first shall proceed by calculating the boundary terms arising from variations of H^g and then modifying the latter to get rid off these terms.

Firstly, we consider the case where t^μ asymptotically becomes a time translation at spatial infinity, which corresponds to take $N \to 1$ ad $N_a \to 0$ as $r \to \infty$. In order to get a Hamiltonian whose variation produces no boundary terms from spatial infinity we definire a new gravitational Hamiltonian H'^g by:

$$H'^g = H^g + \alpha \tag{5.75}$$

where α is given, in cartesian coordinates, by [28]:

$$\alpha = \lim_{r\to\infty} \sum_{\nu=1}^{3} \int_S \left(\frac{\partial h_{\mu\nu}}{\partial x^\nu} - \frac{\partial h_{\nu\nu}}{\partial x^\mu} \right) \tag{5.76}$$

in which S denotes a coordinate sphere of radius r.

The numerical value of H'^g for a solution of Einstein's equation is just α: this suggests that α should be interpreted as proportional to the total energy of an asymptotically flat space-time. The constant of proportionality between α and energy can be determined, for example, by evaluating α for the Schwarzchild solution.

Secondary, the definition of a total momentum can be motivated by examining the boundary terms in H^g which occur when we take $N \to 0$ and require N_a to go a translation as $r \to \infty$. Indeed, a notion of angular momentum arises from consideration of more general asymptotic behaviour of the lapse and shift [18].

In order to see the same things in more detail, we write the variation of the full Hamiltonian H^g with respect to the canonical variables h_{ab} and p^{cd} yields:

$$\delta H^g = \int d^3x (A^{ab}\delta h_{ab} + B_{ab}\delta p^{ab}) - \delta C$$

where δC denotes surface terms.

Because H^g must be a differentiable function with respect to h_{ab} and p^{cd} (otherwise Hamilton's equations of motion would not make sense), δC must be cancelled by introducing explicit surface terms to H^g. For the derivation of such surface terms, one must impose fall-off conditions for the canonical variables. For the three-metric they read

$$h_{ab} \sim \delta_{ab} + \mathcal{O}\left(\frac{1}{r}\right), \qquad h_{ab,c} \sim \delta_{ab} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{5.77}$$

and analogously for the momenta, where the asymptothical behaviour is understood for $r \to \infty$. The lapse and shift, if again combined to the four-vector N^μ, are supposed to obey

$$N^\mu \sim \alpha^\mu + \beta^\mu_a x^a \tag{5.78}$$

again for $r \to \infty$, where α^μ describe space–time translations, $\beta_{ab} = -\beta_{ba}$ spatial rotations, and $\beta^\perp_a$ a boosts. Together, they form the Poincare group of Minkowski space–time, which is a symmetry in the asymptotic sense.

The procedure mentioned above then leads to the following expression for the total Hamiltonian:

$$H^g = \int d^3x(N\mathcal{H}^g_\perp + N^a\mathcal{H}^g_a) + \alpha E_{ADM} - \alpha^a P_a + \frac{1}{2}\beta_{\mu\nu}J^{\mu\nu} \tag{5.79}$$

where E_{ADM} (also called “ADM energy”, see Arnowitt et al. 1962), P_a, and $J^{\mu\nu}$ are the total energy, the total momentum, and the total angular momentum plus the generators of boosts, respectively. Together they form the generators of the Poincare group at infinity. For the ADM energy, in particular, one finds the expression

$$E_{ADM} = \frac{1}{16\pi G}\oint_{r\to\infty} d^2\sigma_a(h_{ab,b} - h_{bb,a}) \tag{5.80}$$

Note that the total energy is defined by a surface integral over a sphere for $r \to \infty$ not by a volume integral. One can prove that $E_{ADM} \geq 0$.

Because of the Hamiltonian and diffeomorphism constraints, H^g is numerically equal to the surface terms. For vanishing asymptotic shift and lapse equal to one, it is just given by the ADM energy. We emphasize that the asymptotic Poincare transformations must not be interpreted as gauge transformations (otherwise E_{ADM}, P^a, and $J^{\mu\nu}$ would be constrained to vanish).

Making a brief summary, we have seen that ADM energy is a peculiar way to define the energy in General Relativity, it is only applicable to some special geometries of space-time that asymptotically approach a well-defined metric tensor at infinitye. The ADM energy in these cases is defined as a function of the “deviation” of the metric tensor from its prescribed asymptotic form: the ADM energy is computed as the strength of the gravitational field at infinity. If the required asymptotic form is time-independent (such as the Minkowski space), then it respects the time-translational symmetry and Noether's theorem applies, which implies that the ADM energy is conserved.

5.6 Discussion of the constraints

We have given a constrained Hamiltonian formulation of Einstein's equation, but we have not isolated the "true dynamical degrees of freedom" in our choice of configuration space, because we see the presence of constraints in the Hamiltonian formulations of Einstein's equations. Even though we already have eliminated N and N_a as dynamical variables, the constraints tell us that our phase space is still "too large", this is directly related to the gauge freedom present in our configuration variables h_{ab}.
Therefore, the presence of the constraints derived in the last subsection means that only part of the variables constitute "physical degrees of freedom" (i.e. how many distinct solutions of the equations exist). In order to count the physical degrees of freedom of the gravitational field in General Relativity, we can use the two fully equivalent following arguments.

- The three-metric $h_{ab}(x)$, which expression is given by (5.3) is characterized by six numbers per space point, which can be symbolically denoted as $6\times\infty^3$. The diffeomorphism constraints (5.71) generate coordinate transformations on three-space, which are characterized by three numbers, thus $6-3=3$ numbers per space point remain. The constraint (5.70) corresponds, in a sense, to "time" because it corresponds to one variable per space point describing the location of Σ in space–time, since the latter changes under normal deformations. Therefore $2\times\infty^3$ degrees of freedom remain.

- One can alternatively perform the following counting in phase space: the canonical variables $(h_{ab}(x), p_{cd}(y))$ are $12\times\infty^3$ variables because they correspond to twelve numbers per space point. Due to the presence of the four constraints in phase space, $4\times\infty^3$ variables have to be subtracted, and remaining $8\times\infty^3$ variables define the constraint hypersurface Γ_c on which the dynamic takes place. Since the constraints generate a four-parameter set of gauge transformations on Γ_c according to the discussion of the third Chapter, $4\times\infty^3$ degrees of freedom must be subtracted in order to "fix the gauge". The remaining $4\times\infty^3$ variables define the reduced phase space Γ_r and correspond to $2\times\infty^3$ degrees of freedom in configuration space, in accordance with the counting above.

The gravitational field thus seems to be characterized by $2\times\infty^3$ intrinsic degrees of freedom, i.e. two degrees of freedom per point of space. It is important to emphasized that this counting always holds modulo a finite number of degrees of freedom.
Does a three-dimensional geometry indeed contain information about time? As ca be shown, two configurations (e.g. of a clock) in classical mechanics do not suffice to determine the motion, in fact one needs in addition the two times of the clock configurations or its speed.
The situation in the gravitational case is related to the so called "sandwich conjecture" [19], which states that two three-geometries do (in the generic case) determine the temporal separation (the proper times) along each time-like worldline connecting them in the resulting space-time. Whereas still not much is known about the finite version of this conjecture, results are available for the infinitesimal case, but we will not dwell on this argument here as this is beyond our purpose.
The somewhat ambiguous nature of the Hamiltonian constraint (5.73) leads to the question whether it really generates gauge transformations. The answer should be 'yes' in view of the general fact that first-class constraints have this property. On the other hand, $H^g_\perp$ is also responsible for the time evolution, mediating between different hypersurfaces. Can this time evolution be interpreted as the "unfolding" of a gauge transformation? This is indeed possible because the presence of the constraint $H^g_\perp \approx 0$ expresses the fact that evolutions along different foliations are equivalent.
A related issue concerns the notion of an "observable": this can be defined as a variable that weakly commutes with the constraints. In the present situation, an observable $\mathcal{O}$ should therefore satisfy

$$\{\mathcal{O}, \mathcal{H}^g_a\} \approx 0 \tag{5.81}$$

$$\{\mathcal{O}, \mathcal{H}^g_\perp\} \approx 0 \tag{5.82}$$

While the first condition is certainly reasonable (observables should not depend on the chosen coordinates of Σ), the situation is not so clear for the second condition: Kuchar

[21] refers (in order to emphasize the difference between both equations) to quantities obeying (5.81) already as observables, while to variables that obey in addition (5.82) as "perennials", which are constants of motion, since they weakly commute with the full Hamiltonian. It is possible to say more about these interpretational issues in the context of the quantum theory, but this is beyond our purpose.

We have already seen that the transformations generated by the constraints (5.70) and (5.71) are different from the original space–time diffeomorphisms of General Relativity: the formal reason is that $H^g_\perp$ is non-linear in the momenta, so the transformations in the phase space Γ spanned by (h_{ab}, p^{cd}) cannot be reduced to space-time transformations.

In order to understand the relation between both types of transformations, we can reason as follows.

Let $(\mathcal{M}, g)$ be a globally hyperbolic space–time, we shall denote by Riem $\mathcal{M}$ the space of all (pseudo-) Riemannian metrics on $\mathcal{M}$. Since the group of space-time diffeomorphisms, Diff $\mathcal{M}$, does not act transitively, there exist non-trivial orbits in Riem $\mathcal{M}$. One can make a projection down to the space of all four-geometries, which we formally denote as Riem $\mathcal{M}/$ Diff $\mathcal{M}$. By considering a particular section σ : Riem $\mathcal{M}/$ Diff $\mathcal{M} \to \mathcal{M}$, one can choose a particular representative metric on $\mathcal{M}$ for each geometry. In this way one can define formal points of the "background manifold" $\mathcal{M}$, which a priori have no meaning (in General Relativity, points cannot be disentangled from the metric fields).

5.6.1 Superspace

An important issue, both for the quantum theory and the Hamiltonian formulation, is an investigation into the structure of the configuration space, because this turns out to be the space on which the wave functional is defined. The gauge arbitrariness in the configuration field h_{ab} lies in the fact that if φ is any diffeomorphism of Σ_t, then h_{ab} and $\varphi^* h_{ab}$ represent the same physical configuration. This suggests that we should take the configuration space of General Relativity to be the set equivalence classes of Riemannian metrics on Σ_t where two metrics are considered equivalent if they can be carried into each other by a diffeomorphism. This configuration space is known as *Superspace*.

We have seen that the canonical formalism deals with the set of all three metrics on a given manifold Σ. We call this space Riem Σ (not to be confused with Riem $\mathcal{M}$

considered above). As configuration space of the theory we want to address the quotient space in which all metrics corresponding to the same three geometry are identified, which is precisely the superspace cited above and originally introduced by Wheeler.
The latter is defined by:

$$\mathcal{S} \equiv \text{Riem } \Sigma/\text{Diff } \Sigma \tag{5.83}$$

Choosing superspace as configuration space, the momentum constraints (5.71) are automatically fulfilled (thus they turns out to be eliminated by the choice of superspace), but the constraint (5.70) remains. The latter can be viewed as resulting from the gauge arbitrariness involved in the choice of how to "slice" space-time into space and time. It is very closely analogous to the constraint which arises when one "parametrizes" an originally unconstrained theory in a fixed, background space-time, that is when one introduces into the Lagrangian a time function (which defines the choice of hypersurfaces Σ_t with respect to a reference surface Σ) and treats this time function as a dynamical variable. In the case of such parametrized theories, as we have seen in the corrispective section, the constraint analogous to (5.70) is linear in the momentum conjugate to the time function and one can "deparametrize" the theory by solving the constraint for this momentum.
In the case of Einstein's equation, the constraint is quadratic in the momentum, so a similar deparametrization does not appear to be possible and, as a consequence, it does not appear to be possible to find a choice of configuration space for General Relativity such that only the "true dyamical degrees of freedom" are present in its phase space. The presence of constraint (5.70) appears to be unavoidable feature of the Hamiltonian formulation of General Relativity, this provides a serious obstacle to the formulation of a quantum theory of gravitation by the canonical quantization approach.
Whereas Riem Σ has a simple topological structure (it is a cone in the vector space of all symmetric second-rank tensor fields), the topological structure of $\mathcal{S}(\Sigma)$ is very complicated because it inherits (through Diff Σ) some of the topological information contained in Σ. In general, Diff Σ can be divided into a "symmetry part" and a "redundancy part" [17]. Symmetries arise typically in the case of asymptotically flat spaces and describe, for example, rotations with respect to the remaining part of the universe ("fixed stars"). Since they have physical significance, they should not be factored out, and Diff Σ is then

understood to contain only the "true" diffeomorphisms (redundancies). In the closed case (relevant in particular for cosmology) only the redundancy part is present. For closed Σ, $\mathcal{S}(\Sigma)$ has a non-trivial singularity structure due to the occurrence of metrics with isometries (Fischer 1970), at such singular points, superspace is not a manifold (a situation like e.g. at the tip of a cone).

In the open case is possible to perform a so called "one-point compactification", which will be denoted as $\bar{\Sigma} \equiv \Sigma \cup \{\infty\}$, then the corresponding superspace is defined as

$$\mathcal{S}(\Sigma) \equiv \text{Riem } \bar{\Sigma}/\mathcal{D}_F(\bar{\Sigma}) \tag{5.84}$$

where $\mathcal{D}_F(\bar{\Sigma})$ represents all diffeomorphisms that fix the frames at infinity.

The DeWitt metric G^{abcd} plays the role of metric on Riem Σ in the following sense:

$$G(l,k) \equiv \int_\Sigma d^3x G^{abcd} l_{ab} k_{cd} \tag{5.85}$$

where l ad k denote tangent vectors at $h \in$ Riem Σ. Due to its symmetry properties, it can formally be considered as a symmetric 6×6-matrix at each space point [20] and, thanks to the spectral theorem, at each point this matrix can therefore be diagonalized, and the signature turns out to read

$$diag(-,+,+,+,+,+)$$

It must be emphasized that the negative sign in DeWitt's metric has nothing to do with the Lorentzian signature of space–time. In the Euclidean case, the minus sign stays and only the relative sign between potential and kinetic term will change and due to the presence of this minus sign, the kinetic term for the gravitational field is indefinite.

In Riem Σ, one can distinguish between "vertical" and "horizontal" directions. The vertical directions are the directions along the orbits generated by the three-dimensional diffeomorphisms. Metrics along a given orbit describe the same geometry, while orizontal directions are defined as being orthogonal to the orbits, where orthogonality holds with respect to the DeWitt metric. Since the latter is indefinite, the horizontal directions may also contain vertical directions (this happens in the "light-like" case for zero norm). Calling $V_h(H_h)$ the vertical (horizontal) subspace with respect to a given metric h_{ab}, one can show that:

- if $V_h \cap H_h = \{0\}$, then G^{abcd} can be projected to the horizontal subspace where it defines a metric.
- if $V_h \cap H_h \neq \{0\}$ then there exist critical points in $S(\Sigma)$ where the projected metric changes signature.

The situation is illustrated in figure (5.2). The task is then to classify the regions in

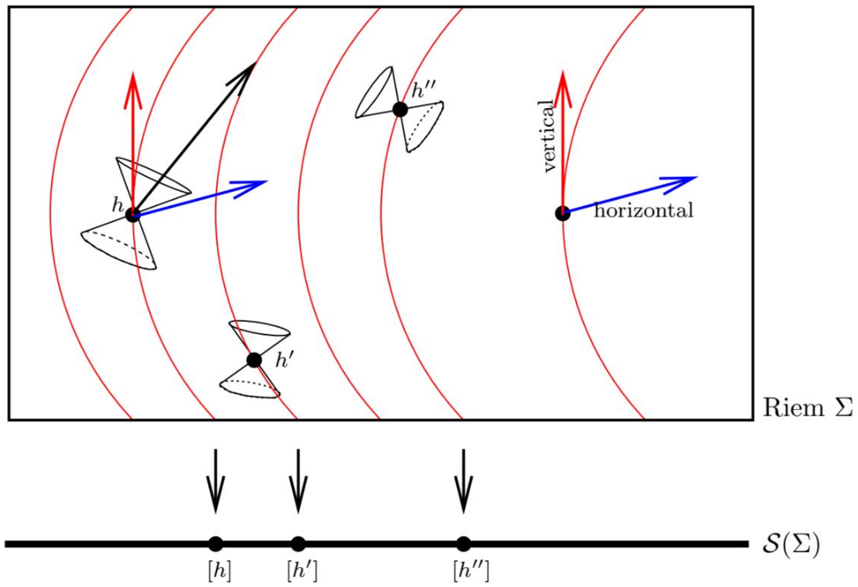

Figure 5.2: The space Riem Σ, fibred by the orbits of Diff Σ (curved vertical lines). Tangent directions to these orbits are called "vertical", the G-orthogonal directions "horizontal". Horizontal and vertical directions intersect whenever the "hyper-lightcone" touches the vertical directions, as in point h'. At h, h', and h'' the vertical direction is depicted as time-, light-, and space-like respectively. Hence $[h']$ corresponds to a transition point where the signature of the metric in superspace changes. From [4]

Riem Σ according to these two cases. There exist some partial results [17], on which we will not dwell here as they are beyond our purposes.

Chapter 6

ADM Formalism for a Spherical Source

In this Chapter we use the ADM formalism, introduced and analyzed in the previous Chapter, in order to describe a matter's source (consisting of a perfect fluid) with spherical symmetry and analyze the equations for the Hamiltonian and diffeomorphism constraints.
Besides using the metric form already employed for the Lagrangian formalism in Tolman's solution, we also perform a change of variable by introducing a local mass function as dynamic variable and analyze the resulting equations for the constraints, also showing how the value of the Misner-Sharp mass can be recovered by requiring that the Hamiltonian costraint weakly vanishes.

6.1 Coordinates

Suppose we have a mass source with spherical symmetry, consisting of a perfect fluid, which generates a gravitational field. We use the previously introduced coordinates (1.35), which are $(x^0, x^1, x^2, x^3) \equiv (\tau, \rho, \theta, \varphi)$, while in this Chapter we use the convenctions of tables (6.1)-(6.3). We again emphasize that this coordinate system is in motion with the matter which constitutes the spherically symmetric source, so the particles do not change their position in this system and the metric is time-dependent.

6.2 Metric tensor and Units of measure

We choose $N(\tau,\rho)$ and $N^1(\tau,\rho) \equiv N^\rho(\tau,\rho)$ as lapse function and shift vector, assuming that these last quantities depend on τ and ρ. Because of the spherically simmetry, we set $N^\theta = N^\varphi = 0$. The lapse function N encodes the possibility to perform arbitrary reparametrizations of the time parameter, while the shift function N^ρ is responsible for reparametrizations of the radial coordinate (this is the only freedom in performing spatial coordinate transformations that is left after spherical symmetry is imposed).
According to (5.11), we write the metric tensor as:

$$g_{\mu\nu} = \begin{pmatrix} N_\rho N^\rho - N^2 & N_\rho & 0 & 0 \\ N_\rho & e^{\lambda(\tau,\rho)} & 0 & 0 \\ 0 & 0 & r^2(\rho,\tau) & 0 \\ 0 & 0 & 0 & r^2(\tau,\rho)\sin^2\theta \end{pmatrix} \tag{6.1}$$

where $\lambda(\tau,\rho)$ and $r(\tau,\rho)$ are functions that we use as dynamical variables, while the three-dimensional metric on the hypersurfaces h_{ab} is given by:

$$h_{ab} = \begin{pmatrix} e^{\lambda(\tau,\rho)} & 0 & 0 \\ 0 & r^2(\rho,\tau) & 0 \\ 0 & 0 & r^2(\tau,\rho)\sin^2\theta \end{pmatrix} \tag{6.2}$$

The inverse of the metric tensor is given by (5.12):

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^\rho}{N^2} & 0 & 0 \\ \frac{N^\rho}{N^2} & h^{\rho\rho} - \frac{N^\rho N^\rho}{N^2} & 0 & 0 \\ 0 & 0 & h^{\theta\theta} & 0 \\ 0 & 0 & 0 & h^{\varphi\varphi} \end{pmatrix} \tag{6.3}$$

where the three-dimensional inverse metric is:

$$h^{ab} = \begin{pmatrix} e^{-\lambda(\rho,\tau)} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{6.4}$$

We clearly use the indexes with the following correspondence: $\tau = 0, \rho = 1, \theta = 2, \varphi = 3$.
In this way, according to (3.62), we write the metric as:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (N_\rho N^\rho - N^2)d\tau^2 + 2d\tau d\rho N_\rho + e^{\lambda(\tau,\rho)}d\rho^2 + r^2(\rho,\tau)d\Omega^2 \tag{6.5}$$

Table 6.1: Units of measure of constants

Physical constant	Dimensions
c	1
G	[Lenght]/[Mass]
$\hbar$	[Lenght] $\cdot$ [Mass]

Table 6.2: Units of measure of coordinates

Physical coordinates	Dimensions
τ	[Lenght]
ρ	[Lenght]
θ	1
φ	1

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$.
In order to treat gravitational and quantum systems, the conventions for the units of measure of table (6.1) is usually adopted, while for the coordinates we choose conventions of table (6.2) and dimensions of other quantities involved can be read in table (6.3). We conclude this section by saying that the field equations for the metric tensor (6.1) are calculated in the appendix.

6.3 ADM Action and Canonical Momenta

For a metric tensor with form (6.1), in which we suppose that r and λ are real functions of the coordinates τ and ρ, one has:

$$\sqrt{h} = \sqrt{det(h_{ab})} = r^2 \sin\theta\, e^{\lambda/2} \tag{6.6}$$

Table 6.3: Units of measure of quantities

Physical quantities	Dimensions
Energy	[Mass]
ADM Action	[Lenght] · [Mass]
ADM Lagrangian	[Mass]/[Lenght]
$\lambda(\rho,\tau)$	1
$r(\tau,\rho)$	[Lenght]
$m(\tau,\rho)$	[Mass]
N	1
N^1	1

Regarding the expression of the second fundamental form, we can use (5.25):

$$K_{11} \equiv K_{\rho\rho} = \frac{1}{2N}\left(\frac{d}{d\tau}\left(e^{\lambda}\right) - 2D_\rho N_\rho\right) \tag{6.7}$$

$$K_{22} \equiv K_{\theta\theta} = \frac{1}{2N}\left(\frac{d(r^2)}{d\tau} - 2D_\theta N_\theta\right) \tag{6.8}$$

$$K_{33} \equiv K_{\varphi\varphi} = \sin^2\theta K_{22} \tag{6.9}$$

We want to express the term $D_a N_b$ in a more explicit way, remembering that we have only one radial non vanishing component of the shift vector, which is $N_1 \equiv N_\rho$.
In this case we have that D_1 is acting on N_1, so we write:

$$D_a N_b = N_{b,a} - \Gamma^d_{ba} N_d \tag{6.10}$$

in such a way that we have:

$$D_\rho N_\rho \equiv D_1 N_1 = N_{1,1} - \Gamma^1_{11} N_1 \tag{6.11}$$

$$D_\theta N_\theta \equiv D_2 N_2 = -\Gamma^1_{22} N_1 \tag{6.12}$$

$$D_\varphi N_\varphi \equiv D_3 N_3 = -\Gamma^1_{33} N_1 \tag{6.13}$$

The metric on the hypersurfaces is h_{ab}, so (in the calculation of the three-dimensional covariant derivative) we have:

$$\Gamma^1_{11} = \frac{1}{2}\frac{d\lambda}{d\rho} \qquad D_1 N_1 = \frac{d(N_1)}{d\rho} - \frac{N_1}{2}\frac{d\lambda}{d\rho} \tag{6.14}$$

$$\Gamma^1_{22} = -e^{-\lambda} r \frac{dr}{d\rho} \qquad D_2 N_2 = N_1 e^{-\lambda} r \frac{dr}{d\rho} \tag{6.15}$$

$$\Gamma^1_{33} = \sin^2\theta\,\Gamma^1_{22} \qquad D_3 N_3 = \sin^2\theta\, D_2 N_2 \tag{6.16}$$

We choose to work with the contravariant radial component of the shift vector N^1, so we have:

$$N_1 = e^{\lambda} N^1 \tag{6.17}$$

from which:

$$D_1 N_1 = \frac{e^{\lambda} N^1}{2}\frac{d\lambda}{d\rho} + e^{\lambda}\frac{d(N^1)}{d\rho} \qquad etc. \tag{6.18}$$

Finally, in order to avoid confusion between indices, derivatives and powers, setting $N^1 \equiv \beta$ and $N \equiv \alpha$ we can finally write the expressions for the diagonal elements of the second fundamental form:

$$K_{11} \equiv K_{\rho\rho} = \frac{e^{\lambda}}{\alpha}\left(\frac{1}{2}\frac{d\lambda}{d\tau} - \frac{\beta}{2}\frac{d\lambda}{d\rho} - \frac{d\beta}{d\rho}\right) \tag{6.19}$$

$$K_{22} \equiv K_{\theta\theta} = \frac{r}{\alpha}\left(\frac{dr}{d\tau} - \beta\frac{dr}{d\rho}\right) \tag{6.20}$$

$$K_{33} \equiv K_{\varphi\varphi} = \sin^2\theta K_{22} \tag{6.21}$$

The ADM Lagrangian $\mathcal{L}^g$, as can be seen from (5.54) and (5.58), is given by (we neglet the cosmological constant Λ):

$$\mathcal{L}^g = \frac{\alpha\sqrt{h}}{16\pi G}\left({}^{(3)}R - K^2 + K_{ab}K^{ab}\right) \tag{6.22}$$

The AMD Action is written, according to the previous description of the 3+1 decomposition of General Relativity, as:

$$S_{E-H} = \int d\tau \int d\rho \int d\theta \int d\varphi \;\; \mathcal{L}^g \tag{6.23}$$

One can thus calculate from the metric (6.2) the following curvature scalar:

$${}^{(3)}R = \frac{2e^{-\lambda}}{r}\left(-2r'' + \lambda' r' - \frac{r'^2}{r}\right) + \frac{2}{r^2} \tag{6.24}$$

The canonical momenta coniugated to the three-dimensional metric elements h_{ab} can be written, according to (5.60), as:

$$p^{ab} = \frac{\sqrt{h}}{16\pi G}(K^{ab} - Kh^{ab}) \tag{6.25}$$

where the trace of K^{ab} is given by:

$$K \equiv h^{ab}K_{ab} = h^{\rho\rho}K_{\rho\rho} + h^{\theta\theta}K_{\theta\theta} + h^{\varphi\varphi}K_{\varphi\varphi} \tag{6.26}$$

The last expression can be calculated as:

$$K = \frac{1}{2\alpha}\left(\frac{d\lambda}{d\tau} - \beta\frac{d\lambda}{d\rho} - 2\frac{d\beta}{d\rho} + \frac{4}{r}\frac{dr}{d\tau} - \frac{4\beta}{r}\frac{dr}{d\rho}\right) \tag{6.27}$$

Using the three-dimensional metric in order to raise the indicies, settings $\dot{} \equiv \frac{\partial}{\partial\tau}$ and $' \equiv \frac{\partial}{\partial\rho}$, we can calculate the following expressions for the momenta:

$$p^{11} \equiv p^{\rho\rho} = \frac{\sin\theta\, e^{-\lambda/2}}{16\pi G\alpha}(-2r\dot{r}) \tag{6.28}$$

$$p^{22} \equiv p^{\theta\theta} = \frac{\sin\theta\, e^{\lambda/2}}{16\pi G\alpha}\left(-\frac{\dot{r}}{r} - \frac{\dot{\lambda}}{2} + \frac{\beta\lambda'}{2} + \beta' + \frac{\beta r'}{r}\right) \tag{6.29}$$

$$p^{33} \equiv p^{\varphi\varphi} = \frac{p^{22}}{\sin^2\theta} \tag{6.30}$$

Now we write the full expression of the ADM Action integrated over the angular coordinates according to (6.23) and (6.22), which turns out to be:

$$\begin{aligned} S_{E-H} = \int d\tau \int d\rho \Bigg[\left(\frac{e^{\lambda/2}}{2G\alpha}\right) &\left(-r\dot{r}\dot{\lambda} + \dot{r}\beta r\lambda' + 2\dot{r}\beta' r - \dot{r}^2 + \dot{\lambda}\beta r r' - \beta^2\lambda' r r' - 2\beta\beta' r r' \right. \\ &\left. + 2\beta\dot{r}r' - \beta^2 r'^2\right) + \left(\frac{\alpha}{2G}\right)\left(-2r'' r e^{-\lambda/2} + \lambda' r' r e^{-\lambda/2} - r'^2 e^{-\lambda/2} + e^{\lambda/2}\right)\Bigg] \\ \equiv \int d\tau \int d\rho\, \tilde{\mathcal{L}}^g \end{aligned} \tag{6.31}$$

We used the tilde notation in order to emphasize that integration over the angular coordinates has been used in (6.31). The 4π factor doesn't appear explicitly beacuse we have divided the definition of $\mathcal{L}^g$ for $16\pi G$ (as one can see from (6.22)), so only the term

$2G$ remains in the denominator of (6.31) and the ADM Lagrangian has the dimensions of a mass divided by a lenght, according to our conventions.
One can derive, from the ADM Lagrangian, all the momenta coniugated to quantities that appear in the latter: for example, the momenta coniugated to λ and to r, can be obtained from (6.31) as:

$$p_\lambda = \frac{e^{\lambda/2} r}{2G\alpha} \left(\beta r' - \dot{r}\right) \tag{6.32}$$

$$p_r = \frac{e^{\lambda/2}}{2G\alpha} \left(2\beta r' - 2\dot{r} + 2\beta' r + \beta\lambda' r - \dot{\lambda} r\right) \tag{6.33}$$

It is important to underline that p_λ and p_r are calculated by an ADM Lagrangian that was integrated over the angular coordinates. We note that the momenta coniugated to λ and r are regular fields of the variables τ and ρ for every choice of the lapse function $\alpha(\tau, \rho)$ different from zero and every "sufficiently regular" shift vector $\beta(\tau, \rho)$.
Now we want to invert the relations (6.32) and (6.33) in order to have the velocities $\dot{r}$ and $\dot{\lambda}$ written in terms of the momenta coniugated to the latters:

$$\dot{\lambda} = 2\beta' + \beta\lambda' + \frac{2G\alpha e^{-\frac{\lambda}{2}} \left(\frac{2p_\lambda}{r} - p_r\right)}{r} \tag{6.34}$$

$$\dot{r} = \frac{2G\alpha e^{-\frac{\lambda}{2}} p_\lambda}{r} + \beta r' \tag{6.35}$$

then we replace the results in (6.31) in order to introduce the Hamiltonian density and the constraints, according with the previous discussion of the 3+1 decomposition of General Relativity. In doing so, we are driven to the following expression for the ADM Lagrangian:

$$\begin{aligned} \tilde{\mathcal{L}}^g =& \frac{2Ge^{-\lambda/2}}{r} p_\lambda \left(\frac{p_\lambda}{r} - p_r\right) \\ & + \text{potential terms} \end{aligned} \tag{6.36}$$

6.3.1 ADM Hamiltonian and Constraints

We introduce the ADM Hamiltonian density according to (5.63), using λ and r as dynamical variables:

$$\begin{aligned} \tilde{\mathcal{H}}^g &= p_\lambda \dot{\lambda} + p_r \dot{r} - \tilde{\mathcal{L}}^g \\ &\equiv \alpha \tilde{\mathcal{H}}^g_\perp + \beta \tilde{\chi}^g \end{aligned} \tag{6.37}$$

where in the second passage the Hamiltonian and the diffeomorphism constraints have been introduced, according to (5.66).
The full Hamiltonian is found by integration:

$$\tilde{H}^g = \int d\rho \left(\alpha\tilde{\mathcal{H}}^g_\perp + \beta\tilde{\chi}^g\right) \tag{6.38}$$

where the only integration left is on the radial coordinate, because of the spherical simmetry.
The full expression of the ADM Hamiltonian density turns out to be:

$$\begin{aligned}\tilde{\mathcal{H}}^g = &-\frac{\alpha e^{\frac{\lambda}{2}}}{2G} - \frac{2G\alpha e^{-\frac{\lambda}{2}} p_\lambda p_r}{r} + \frac{2G\alpha e^{-\frac{\lambda}{2}} p_\lambda^2}{r^2} - \frac{\alpha e^{-\frac{\lambda}{2}}\lambda' r r'}{2G} + \frac{\alpha e^{-\frac{\lambda}{2}} r'^2}{2G} \\ &+ \frac{\alpha e^{-\frac{\lambda}{2}} r r''}{G} + 2\beta' p_\lambda + \beta\lambda' p_\lambda + \beta p_r r'\end{aligned} \tag{6.39}$$

The expressions for the Hamiltonian and the diffeomorphism constraints are:

$$\tilde{\mathcal{H}}^g_\perp = \frac{2Ge^{-\lambda/2}}{r} p_\lambda \left(\frac{p_\lambda}{r} - p_r\right) + \frac{e^{-\lambda/2}}{2G}\left(2r''r - r'\lambda' r + r'^2\right) - \frac{e^{\lambda/2}}{2G} \tag{6.40}$$

$$\tilde{\chi}^g = -2p'_\lambda + \lambda' p_\lambda + p_r r' \tag{6.41}$$

where an integration by part has been performed in order to obtain (6.41).

6.4 Matter distribution

We assume that the distribution of matter with spherical symmetry that generates the gravitational field consists of a perfect fluid. By definition, the latter has the following expression for the energy-stress tensor:

$$T^{\mu\nu} = \omega U^\mu U^\nu + p\left(g^{\mu\nu} + U^\mu U^\nu\right) \tag{6.42}$$

It must be emphasized that we are using in this Chapter a space-time metric of signature $(-1, 1, 1, 1)$, opposite to the one used for the very same perfect fluid in (1.32). We first deal with the fundamental equations in the general case, without initially adapting the formalism to a spherical source.
Let us consider a fluid element at point $p \in \Sigma_\tau$ as in figure (6.1). Let τ_e be the Eulerian

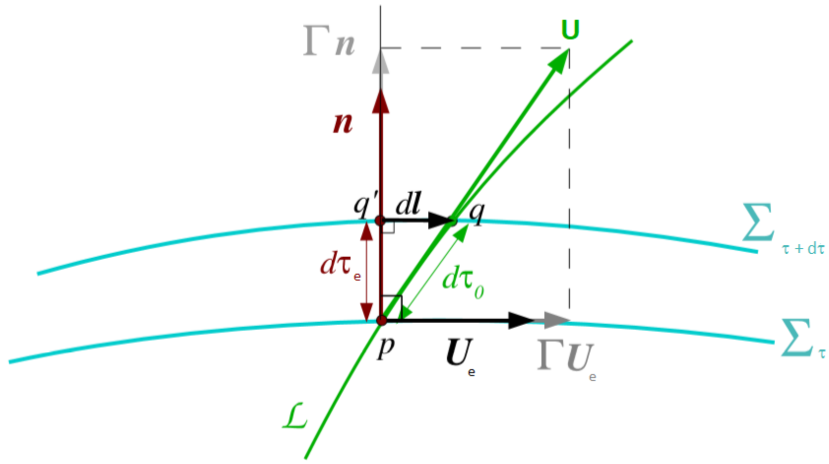

Figure 6.1: Worldline $\mathcal{L}$ of a fluid element crossing the spacetime foliation. U^{μ} is the fluid 4-velocity and $\vec{U}_e = d\vec{l}/d\tau_e$ the relative velocity of the fluid with respect to the Eulerian observer, whose 4-velocity is n^{μ}. U_e^{μ} is tangent to Σ_{τ} and enters in the orthogonal decomposition of U^{μ} with respect to Σ_{τ}, via $U^{\mu} = \Gamma(n^{\mu} + U_e^{\mu})$. Contrary to what the figure might suggest, $d\tau_e > d\tau_0$ (conflict between the figure's underlying Euclidean geometry and the actual Lorentzian geometry of spacetime).

observer's proper time at p. At the coordinate time $\tau + d\tau$, the fluid element has moved to the point $q \in \Sigma_{\tau+d\tau}$. The date $\tau_e + d\tau_e$ attributed to the event q by the Eulerian observer moving through p is given by the orthogonal projection q' of q onto the wordline of that observer. Indeed, let us recall that the space of simultaneous events (local rest frame) for the Eulerian observer is the space orthogonal to his 4-velocity U^{μ}, i.e. locally Σ_{τ}. Let $d\vec{l}$ be the infinitesimal vector connecting q' to q. Let $d\tau_0$ be the increment of the fluid proper time between the events p and q. The Lorentz factor of the fluid with respect to the Eulerian observer is defined as being the proportionality factor Γ between the proper times $d\tau_0$ and $d\tau_e$:

$$d\tau_e \equiv \Gamma d\tau_0 \tag{6.43}$$

It is very easy to show that:

$$\Gamma = -\, U^{\mu} n_{\mu} \tag{6.44}$$

Table 6.4: Units of measure for the fluid's variables

Physical quantities	Dimensions
U	1
ω	[Mass]/[Lenght]^3
p	[Mass]/[Lenght]^3

From a pure geometrical point of view, the Lorentz factor is thus nothing but minus the scalar product of the two 4-velocities, the fluid's one and the Eulerian observer's one. By using (5.14) it is immediate to show that $\Gamma = \alpha U^0$. The *fluid velocity relative to the Eulerian observer* is defined as the quotient of the displacement $d\vec{l}$ by the proper time $d\tau_e$, both quantities being relative to the Eulerian observer:

$$U_e^i = \frac{dl^i}{d\tau_e} \tag{6.45}$$

It is also very easy to show that

$$U^\mu = \Gamma \left(n^\mu + U_e^\mu\right) \tag{6.46}$$

Since U_e^μ is by construction tangent to Σ_τ, one has $U_e^\mu n_\mu = 0$ and therefore (6.23) constitutes the orthogonal ADM decomposition of the fluid 4-velocity U^μ.
The physical dimensions of the perfect fluid energy stress tensor's components are summarized in table (6.4). We note that already at this point, without having introduced any relativistic action for the fluid, we can already write, by considering (5.73) and (5.74), the constraints that the latter adds to the system we are describing. In fact, by using (6.44) and inserting (5.12) in the relation $U^\mu U_\mu = -1$, one has:

$$\Gamma = -n_\mu U^\mu = \left(1 + U_i h^{ij} U_j\right)^{1/2} \tag{6.47}$$

Using the previous equation together with (5.3), it is immediate to show that:

$$\mathcal{H}^{fluid} \equiv \sqrt{h} T^{\mu\nu} n_\mu n_\nu = \sqrt{h} \left[\omega \left(1 + U_i h^{ij} U_j\right) + p U_i h^{ij} U_j\right] \tag{6.48}$$

$$\chi_i^{fluid} \equiv \sqrt{h} T^\mu{}_i n_\mu = \sqrt{h} \left(\omega + p\right) \left(1 + U_i h^{ij} U_j\right)^{1/2} U_i \tag{6.49}$$

We can also write the constraints in terms of the velocity U_e^μ by using (6.46):

$$\mathcal{H}^{fluid} = \sqrt{h} \left[\left(\omega + p\right) \Gamma^2 - p\right] \tag{6.50}$$

$$\chi_i^{fluid} = \sqrt{h}\,(\omega + p)\,\Gamma^2 U_{e\,i} \tag{6.51}$$

We note that $\mathcal{H}^{fluid}/\sqrt{h}$ and $\chi_i^{fluid}/\sqrt{h}$ are just the fluid energy density and fluid momentum density as measured by the Eulerian observer, as figure (6.1) suggests.
However, we want to construct an action that describes the fluid, from which the relative constraints derive from the application ADM formalism, as seen in the previous Chapter. In this regard, it is necessary to introduce canonical variables that describe the fluid and whose equations of motion are somehow coupled with the gravitational canonical variables.

6.4.1 Action for a perfect fluid

Perfect fluids are described locally by various thermodynamical variables, the latter are space-time scalar fields whose values represent measurements made in the rest frame of the fluid, that is, along the fluid worldline of figure (6.1): the particle number density n, the energy density ω, the pressure p, the temperature T and the entropy per particle s. We note that the energy density in the fluid reference system ω enters its energy-momentum tensor (6.42) together with the velocity field U^μ, which is the velocity in the fluid's reference system.
As summarized by J. David Brown [14], the action functionals describing relativistic perfect fluids and the 3+1 decomposition for the latters can be formulated, but no perfect fluid action can be constructed solely from the variables (n, ω, p, T, s) and U^μ unless the variations among those variables are constrained [25]. Two of the required constraints are particle number conservation $\nabla_\mu(nU^\mu) = 0$ and the absence of entropy exchange between neighboring flow lines $\nabla_\mu(nsU^\mu) = 0$, while the remaining one is that the fluid flow lines should be fixed on the boundaries of space-time.
A method for handling the constraint that the flow lines should be fixed on the space-time boundaries consists in characterize the history of fluid by a set of space-time scalar fields $V^i, i = 1, 2, 3$ (instead of the four–velocity U^μ), that are interpreted as Lagrangian coordinates for the fluid. That is, $V^i(x)$ serve as labels for the fluid, specifying which flow line passes through a given space-time point x. A set of Lagrangian coordinates can be generated by choosing an arbitrary spacelike hypersurface and a coordinate system V^i on that surface, then each flow line is labeled by the coordinate value of the point where

it intersects the hypersurface. By building an action functional using the Lagrangian coordinates V^i, the fluid flow lines are held fixed on the space-time boundaries by simply fixing V^i on the boundaries. In spherical case we only need one non-vanishing component of the Lagrangian coordinates to describe the fluid, thus we will use only $V^1 \equiv V$. The particle number and entropy exchange constraints can be incorporated directly into the action via Lagrange multipliers which we indicate as ϕ and ϑ: they encode the particle number conservation constraint and the entropy exchange constraint, respectively [14]. We write the action for the perfect fluid as:

$$S^{fluid} = \int d^4x\sqrt{-g}\left[nU^\mu\left(\phi_{,\mu} + s\vartheta_{,\mu} + \delta V_{,\mu}\right) - \omega(n,s)\right] \tag{6.52}$$

where δ is a Lagrange multiplier for the constraints that restricts the fluid four-velocity vector to be directed along the flow lines $V = constant$. Defining the pressure as $p = n\frac{\partial\omega}{\partial n} - \omega$, the stress-energy tensor (6.42) is obtained by the perfect fluid action through (5.35).

The fields ϕ and ϑ can be brought to zero on any spacelike hypersurface by a symmetry transformation. Thus, there is no loss of generality in choosing ϕ and ϑ to be zero on the initial hypersurface. Moreover, we can choose Lagrangian coordinates V^i to coincide with the coordinates x^i on the initial surface, so that $V^j_{,i} = \delta^j_i$ and $\dot{V}^i = 0$. Nevertheless, we will write explicitly V' and $\dot{V}$ in order not to lose generality in the formalism. Considering this aspect, we note that the constraint represented by the Lagrange multiplier δ forces the four-velocity of the fluid to move along the lines $\rho = constant$, according to the physical interpretation of the spherical symmetry of the system.

The Eulerian densities are related to the Lagrangian (or comoving) densities n and ns by a kinematical or boost factor Γ that in turn is determined by the local spatial velocity of the fluid, as (6.47) establishes.

We have seen that the fluid contributions to the Hamiltonian and momentum constraints are just the appropriate projections of the perfect fluid stress–energy–momentum tensor (6.42), and involve the spatial components U_i of the fluid four–velocity. These components U_i can be explicitly expressed in terms of the canonical fluid variables and the Lagrangian particle number density n through the equations of motion that are derived from the relativistic action for the perfect fluid [19]. The Lagrangian number density n is itself a function of the canonical variables, as implicitly determined by the equation that relates the Eulerian number density $\Pi/\sqrt{h}$ of the fluid and n:

$$\Pi = \sqrt{h} n(1 + U_i h^{ij} U_j)^{1/2} \tag{6.53}$$

That is, n is the density of the number of particles measured by an observer in motion with the fluid, while $\Pi/\sqrt{h}$ is the density measured by an observer at rest on the hypersurfaces $\tau = const$, whose 4-velocity is n^μ. It is not difficult to prove, by performing the canonical analysis of the relativistic action for the fluid [19] and remembering (6.47), that:

$$\Gamma = \frac{\Pi}{\sqrt{h} n} = (1 + U_i h^{ij} U_j)^{1/2} \tag{6.54}$$

Furthermore, it is possible to show that $\Pi \equiv \Pi_\phi$ coincides with the momentum canonically conjugated to φ (this justifies the notation just employed), while $\Pi s \equiv \Pi_\vartheta$ (the product between Π and the entropy per particle s) is conjugated to ϑ and $\Pi\delta \equiv \Pi_V$ is the momentum conjugated to V.

Taking into account the spherical simmetry of the system, the final resulting expression for the Hamiltonian form of the action can be written (as for the theory without matter, we set $N \equiv \alpha$ and $N^1 \equiv \beta$) as:

$$S^{fluid} = \int d\tau \int d\rho \int d\theta \int d\varphi \left[\Pi_\phi \dot{\phi} + \Pi_\vartheta \dot{\vartheta} + \Pi_V \dot{V} - \alpha \mathcal{H}_\perp^{fluid} - \beta \chi^{fluid}\right] \tag{6.55}$$

Where the expressions for the Hamiltonian and the diffeomorphism constraints, as expected of a matter action with nonderivative coupling to gravity, are (we have only one non-vanishing component of the velocity, thus we set $\tilde{\chi}^{fluid} \equiv \tilde{\chi}_1^{fluid}$):

$$\mathcal{H}_\perp^{fluid} = \sqrt{h}\left[\omega(1 + h_{11}U^2) + p h_{11} U^2\right] \tag{6.56}$$

$$\chi^{fluid} = -\sqrt{h}\,(\omega + p)\left(1 + h_{11}U^2\right)^{1/2} h_{11} U \tag{6.57}$$

that coincide with our previous expressions (with only $U^1 \equiv U$ different from zero) and in terms of which it is possible to write the full Hamiltonian of the fluid:

$$\begin{aligned} H^{fluid} &\equiv \int d\rho \int d\theta \int d\varphi \left(\alpha \mathcal{H}_\perp^{fluid} + \beta \chi^{fluid}\right) \\ &\equiv \int d\rho \left(\alpha \tilde{\mathcal{H}}_\perp^{fluid} + \beta \tilde{\chi}^{fluid}\right) \end{aligned} \tag{6.58}$$

In order to obtain (6.56) and (6.57), we have implicitly written the space-time metric in the ADM form:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2 d\tau^2 + h_{ij}(dx^i + N^i d\tau)(dx^j + N^j d\tau) \tag{6.59}$$

where x^i are the spatial coordinates on the τ = constant hypersurfaces, which agrees with our previous definition (3.62) and (5.11).
If we describe the system in the comoving reference of the fluid, the contributions of the perfect fluid to the constraints, after an integration over the angular coordinates and using (6.2), can be written as:

$$\tilde{\mathcal{H}}_\perp^{fluid} = 4\pi r^2 e^{\lambda/2}\left[\omega + (\omega+p)\, e^\lambda U^2\right] \tag{6.60}$$

$$\tilde{\chi}^{fluid} = -4\pi r^2 e^{3\lambda/2}\,(\omega+p)\left(1+e^\lambda U^2\right)^{1/2} U \tag{6.61}$$

The fluid contribution to the Hamiltonian constraint (6.56) can be rewritten in various useful forms by using (6.53). Also the momentum constraint (6.57) can be written the form dictated by the role of $\tilde{\chi}^{fluid}$ as the generator of spatial diffeomorphisms for the scalar fields ϕ, ϑ, V and their conjugates:

$$\chi^{fluid} = \Pi_\phi \phi' + \Pi_\vartheta \vartheta' + \Pi_V V' \tag{6.62}$$

6.4.2 Initial and boundary value problems

A perfect fluid coupled to the gravitational field is described by his canonical action plus the canonical action for gravity. The Cauchy data for this system consist of the fluid variables ϕ, Π_ϕ, ϑ, Π_ϑ, V, Π_V, and the canonical gravitational variables. These initial data cannot be specified independently, but must satisfy the Hamiltonian and momentum constraints. Any set of initial data that does satisfy these constraints can be transformed into an equivalent set by the symmetries of the perfect fluid action.
If we choose $V^i = x^i$ on the initial surface and we set ϕ and ϑ equal to zero, then a complete set of initial data for a perfect fluid consists of the (Eulerian) particle number density, the (Eulerian) entropy density, ω and p along with $\phi = \vartheta = 0$ and $V^i = x^i$. These initial data are then evolved according to the Hamiltonian differential equations of motion, which we will examine in the next section.

Now consider the boundary value problem. Assume the space-time manifold admits closed spacelike hypersurfaces so the boundary data is specified only on initial and final hypersurfaces. In the most general case in which the system has no spherical symmetry, one possible set of boundary data consists in specifying the canonical coordinates ϕ, ϑ, and V^i on the initial and final hypersurfaces. These boundary data include $10 \times \infty^3$ boundary values, $5 \times \infty^3$ on the initial surface and $5 \times \infty^3$ on the final surface, where ∞^3 is the number of space points. With these boundary data, the $10 \times \infty^3$ Hamiltonian first order differential equations of motion generically determine a solution for the ten canonical field variables.
Another possible set of boundary data consists in specifying the canonical momentum Π_ϕ along with the coordinates ϑ and V^i on the initial and final hypersurfaces. This means that the independent boundary data consist of only $9 \times \infty^3$ boundary values. With these boundary data, the $10 \times \infty^3$ Hamiltonian first order differential equations of motion generically determine the ten canonical field variables to within one symmetry transformation.
The distinctions among the various types of boundary data can be clarified by a simple example that we have discussed in Chapter 3, namely, the free nonrelativistic particle. If the initial and final positions of the particle are given as boundary data, the equations of motion can be solved uniquely for the particle position as a function of time. But the initial and final momenta cannot be specified independently, because space translation invariance implies that the momentum is conserved. By specifying equal values for the intial and final momenta, the equations of motion can be solved to within a constant spatial translation of the particle.

6.5 Time evolution

As we have specified in Chapter 5, in addition to the constraints one has the dynamical equations, which are the Hamiltonian equations of motion.
By postponing the analysis for the evolution of the fluid variables, the first half are

$$\dot{\lambda} = \{\lambda, H^g + H^{fluid}\} \quad \dot{r} = \{r, H^g + H^{fluid}\} \tag{6.63}$$

while the second half are:

$$\dot{p}_\lambda = \{p_\lambda, H^g + H^{fluid}\} \quad \dot{p}_r = \{p_r, H^g + H^{fluid}\} \tag{6.64}$$

We impose, according to (5.61):

$$\{\lambda(\tau, x), p_\lambda(\tau, y)\} = \delta(x, y) \tag{6.65}$$

$$\{r(\tau, x), p_r(\tau, y)\} = \delta(x.y) \tag{6.66}$$

where x and y are radial coordinates. By using (6.38) and (6.58) with above equations, it is easy to see that (as must be expcted from the ADM formalism described in Chapter 5) (6.63) just gives the relations (6.34) and (6.35), while the equations of motion (6.64) give the following expressions:

$$\begin{aligned}\dot{p}_\lambda = &-\frac{Gp_\lambda p_r \alpha e^{-\frac{\lambda}{2}}}{r} + \frac{Gp_\lambda^2 \alpha e^{-\frac{\lambda}{2}}}{r^2} - \frac{re^{-\frac{\lambda}{2}} r' \alpha'}{2G} - \frac{\alpha e^{-\frac{\lambda}{2}} r'^2}{4G} + \frac{\alpha e^{\frac{\lambda}{2}}}{4G} + \beta p_\lambda' + p_\lambda \beta' \\ &- 6\alpha\pi r^2 e^{\lambda/2}\left(U^2 p e^\lambda + U^2 \omega e^\lambda + \frac{\omega}{3}\right) + \frac{\beta\pi r^2 e^{\frac{3\lambda}{2}} U(p+\omega)\left(7U^2 e^{\frac{\lambda}{2}} + 6\right)}{\sqrt{U^2 e^{\frac{\lambda}{2}} + 1}}\end{aligned} \tag{6.67}$$

$$\begin{aligned}\dot{p}_r = &-\frac{2Gp_\lambda p_r \alpha e^{-\frac{\lambda}{2}}}{r^2} + \frac{4Gp_\lambda^2 \alpha e^{-\frac{\lambda}{2}}}{r^3} - \frac{\alpha e^{-\frac{\lambda}{2}} r''}{G} - \frac{e^{-\frac{\lambda}{2}} r' \alpha'}{G} + \frac{\alpha e^{-\frac{\lambda}{2}} r' \lambda'}{2G} - \frac{re^{-\frac{\lambda}{2}} \alpha''}{G} \\ &+ \frac{re^{-\frac{\lambda}{2}} \alpha' \lambda'}{2G} + \beta p_r' + p_r \beta' - 8\alpha\pi e^{\lambda/2} r\left(U^2 p e^\lambda + U^2 \omega e^\lambda + \omega\right) \\ &+ 8\beta\pi U r e^{\frac{3\lambda}{2}}(p+\omega)\sqrt{U^2 e^{\frac{\lambda}{2}} + 1}\end{aligned} \tag{6.68}$$

We note that the Poisson brackets between the constraints is not equal to zero, thus the quantum version of the latter does not commute (in a weak sense):

$$\{\tilde{\mathcal{H}}_\perp^g(\tau, x) + \tilde{\mathcal{H}}_\perp^{fluid}(\tau, x), \tilde{\chi}^g(\tau, y) + \tilde{\chi}^{fluid}(\tau, y)\} \neq 0 \tag{6.69}$$

Defining $\tilde{H}_\perp^g(\tau) \equiv \int dx \tilde{\mathcal{H}}_\perp^g(\tau, x)$ and $\tilde{\Xi}^g(\tau) \equiv \int dy \tilde{\chi}^g(\tau, y)$ one finds after some calculations (which include, as the previous expressions, integrations by parts in order to eliminate the derivatives of the Dirac's delta distributions) that the following Poisson bracket between the constraints holds at equal times:

$$\begin{aligned}\{\tilde{H}_\perp^g, \tilde{\Xi}^g\} = \int d\rho \Bigg(&-\frac{2Gp_\lambda e^{-\frac{\lambda}{2}} p_r'}{r} + \frac{4Gp_\lambda p_r e^{-\frac{\lambda}{2}} r'}{r^2} + \frac{Gp_\lambda p_r e^{-\frac{\lambda}{2}} \lambda'}{r} - \frac{4Gp_\lambda^2 e^{-\frac{\lambda}{2}} r'}{r^3} \\ &- \frac{Gp_r^2 e^{-\frac{\lambda}{2}} r'}{r} - \frac{re^{-\frac{\lambda}{2}} r'' \lambda'}{2G} + \frac{re^{-\frac{\lambda}{2}} r' \lambda'^2}{4G} - \frac{e^{-\frac{\lambda}{2}} r'^2 \lambda'}{2G} + \frac{e^{-\frac{\lambda}{2}} r' r''}{G}\Bigg)\end{aligned} \tag{6.70}$$

in which we used the fact that $\tilde{\mathcal{H}}^g_\perp \approx 0$, $\tilde{\chi}^g \approx 0$, therefore this set of constraints turns out to be of second class since satisfies (3.6). For simplicity we have not considered the contributions of the fluid in the previous explicit calculation of the Poisson bracket between the constraints, but this is enough to see that the latter belong to the second class ones, which we have briefly discussed in Chapter 3, and for which the canonical quantization can take place after the Poisson brackets have been replaced with Dirac brackets.

Turning our attention now to the canonical fluid variables, the fundamental Poisson brackets among the latters are:

$$\{\phi(\tau,x),\Pi_\phi(\tau,y)\} = \delta(x,y) \tag{6.71}$$

$$\{\vartheta(\tau,x),\Pi_\vartheta(\tau,y)\} = \delta(x,y) \tag{6.72}$$

$$\{V(\tau,x),\Pi_V(\tau,y)\} = \delta(x,y) \tag{6.73}$$

We compute the canonical fluid equations of motion for the canonical fluid variables ϕ, ϑ, V by using the Hamilton equations with the generator (6.58):

$$\dot{\phi} = \{\phi, H^{fluid}\} = \alpha\left(\frac{\omega+p}{n} - Ts - U\phi'\right)\left(1+e^\lambda U^2\right)^{-1/2} + \beta\phi' \tag{6.74}$$

$$\dot{\vartheta} = \{\vartheta, H^{fluid}\} = \alpha\left(T - U\vartheta'\right)\left(1+e^\lambda U^2\right)^{-1/2} + \beta\vartheta' \tag{6.75}$$

$$\dot{V} = \{V, H^{fluid}\} = \alpha\left(-UV'\right)\left(1+e^\lambda U^2\right)^{-1/2} + \beta V' \tag{6.76}$$

While the relative momenta evolves according to:

$$\dot{\Pi}_\phi = \{\Pi_\phi, H^{fluid}\} = -\left(4\pi\alpha r^2 e^{\lambda/2}U\right)' + (\beta\Pi_\phi)' \tag{6.77}$$

$$\dot{\Pi}_\vartheta = \{\Pi_\vartheta, H^{fluid}\} = -\left(4\pi\alpha r^2 e^{\lambda/2}nsU\right)' + (\beta\Pi_\vartheta)' \tag{6.78}$$

$$\dot{\Pi}_V = \{\Pi_V, H^{fluid}\} = -\left(4\pi\alpha r^2 e^{\lambda/2}nVU\right)' + (\beta\Pi_V)' \tag{6.79}$$

It turns out that evolution of the canonical fluid variables (and the relative momenta) is not affected by the constraints (6.40) and (6.41): this justifies the fact that we have used only the fluid generator to compute the equations of motion for the fluid variables. Conversely, the constraints (6.105) and (6.106) actually involve a change in the evolution of the dynamical variables and corresponding canonical momenta. In particular, $\dot{\lambda}$ and $\dot{r}$ are not directly modified since constraints which encode the fluid contribution only

contain functions of λ and r, but actually $\dot{p}_\lambda$ and $\dot{p}_r$ explicitly change.
It must be emphasized that, nevertheless, the volution of λ and r is implicitly modified by the fluid because, as one can see from (6.34) and (6.35), the evolution of the latters depend on p_λ and p_r.

6.6 Constraints coupled with the matter

The equations for the constraints, including the contribution of the perfect fluid, can be written as:

$$\tilde{\mathcal{H}}^g_\perp + \tilde{\mathcal{H}}^{fluid}_\perp \approx 0 \tag{6.80}$$

$$\tilde{\chi}^g + \tilde{\chi}^{fluid} \approx 0 \tag{6.81}$$

Replacing previous expressions, the constraints turn out to be:

$$\begin{aligned}\tilde{\mathcal{H}}^g_\perp + \tilde{\mathcal{H}}^{fluid}_\perp =& \frac{2Ge^{-\lambda/2}}{r} p_\lambda \left(\frac{p_\lambda}{r} - p_r\right) + \frac{e^{-\lambda/2}}{2G}\left(2r''r - r'\lambda' r + r'^2\right) - \frac{e^{\lambda/2}}{2G} \\ &+ 4\pi r^2 e^{\lambda/2}\left[\omega + (\omega + p)\, e^\lambda U^2\right]\end{aligned} \tag{6.82}$$

$$\tilde{\chi}^g + \tilde{\chi}^{fluid} = -\, 2p'_\lambda + \lambda' p_\lambda + p_r r' - 4\pi r^2 e^{3\lambda/2}(\omega + p)\left(1 + e^\lambda U^2\right)^{1/2} U \tag{6.83}$$

In addition to the constraints, it is also necessary to take into account the ADM decomposition for the continuity equation satisfied by (6.42). This decomposition is slightly laborious [28], so we report only the final results, where the projection along the normal vector to the hypersurfaces is called "energy conservation equation", while the orthogonal one is the "relativistic Euler equation":

$$\begin{aligned}n^\nu \nabla_\mu T^\mu{}_\nu =& \frac{2Ge^{3\lambda/2}p_\lambda}{r^2}\left(\frac{\beta^2 e^\lambda p}{\alpha} - \alpha p - U^2\alpha e^\lambda p + 3U\alpha e^{-\lambda} p + \alpha e^{-2\lambda}\omega - U^2 \alpha e^\lambda \omega\right. \\ &\left. + 3U\alpha e^{-\lambda}\omega\right) - \frac{Ge^{3\lambda/2}p_r}{r}\left(\frac{\beta^2 e^\lambda p}{\alpha} - \alpha p - U^2\alpha e^\lambda p + U\alpha e^{-\lambda}p - \alpha e^{-2\lambda}\omega\right. \\ &\left. - U^2\alpha e^\lambda \omega + U\alpha e^{-\lambda}\omega\right) + U\alpha e^\lambda\sqrt{Ue^\lambda + 1}\left(p' + 2\alpha' p + \frac{3}{2}\lambda' p + 2\alpha'\omega\right. \\ &\left. + \frac{3}{2}\lambda'\omega + \omega'\right) + \frac{U^2\alpha e^{2\lambda}\lambda'(\omega + p)}{2\sqrt{Ue^\lambda + 1}} - U\beta e^\lambda(p' + \omega') + Ue^\lambda(\dot{p} + \dot{\omega}) \\ &+ 2U\beta' e^\lambda(p + \omega) + \dot{\omega} - \beta\omega'\end{aligned} \tag{6.84}$$

$$
\begin{aligned}
h^{\nu}{}_{1}\nabla_{\mu}T^{\mu}{}_{\nu} =& \alpha' + \frac{4GU^3\alpha e^{\lambda/2}p_\lambda - 2GU^3 r\alpha e^{\lambda/2}p_r - 2U^2r^2\alpha' - 2Ur^2\beta' - Ur^2\beta\lambda'}{2r^2\sqrt{Ue^\lambda+1}} \\
&+ \frac{-4GU^2\alpha e^{\lambda/2}p_\lambda + 2GU^2 r\alpha e^{\lambda/2}p_r - 2U^2r^2\beta' e^\lambda + U^2r^2\beta e^\lambda\lambda' + Ur^2\beta\lambda'}{2r^2(Ue^\lambda+1)^{3/2}} \\
&+ \frac{\alpha p'}{(p+\omega)\left(Ue^\lambda+1\right)} + \frac{-U\beta p' + U\dot{p}}{(p+\omega)\left(Ue^\lambda+1\right)^{3/2}} + \frac{-2U^3\alpha e^\lambda\lambda' - U^2\alpha\lambda'}{2\left(Ue^\lambda+1\right)^2}
\end{aligned}
\tag{6.85}
$$

So far in this Chapter we have applied the ADM formalism to a compact spherical source consisting of a perfect fluid using the metric (6.2).
Instead of analyzing the resulting equations for the constraints (which we have obtained in the geneal case) with this choice of dynamic variables, in the next section we will make a change of variable and repeat the analysis performed up to now.

6.7 Change of variable

We are now interested in making the following variable change:

$$e^{-\lambda(\tau,\rho)} = 1 - \frac{2Gm(\rho,\tau)}{r(\rho,\tau)} \tag{6.86}$$

where we assume that $m(\rho, \tau)$ is a function of temporal and radial coordinates. We are interested in understanding if it is possible to derive the value of this "local mass function" (in analogy with the static case of the Schwarzschild metric) from the constraints of the theory. Therefore, we use now (m, r) as dynamical variables.
After considering the change of variable (6.86), we must follow the same procedure with which we obtained the Lagrangian ADM density in the previous subsection, replacing λ by using m, in order to compute the constraints.
The three-dimensional metric on the hypersurfaces therefore becomes:

$$h_{ab} = g_{ab} = \begin{pmatrix} \left(1 - \frac{2Gm(\rho,\tau)}{r(\rho,\tau)}\right)^{-1} & 0 & 0 \\ 0 & r^2(\rho,\tau) & 0 \\ 0 & 0 & r^2(\tau,\rho)\sin^2\theta \end{pmatrix} \tag{6.87}$$

By making this sobstitution, we can write the full expression of $\mathcal{L}^g$ with the variable change (6.86) and, as in the previous section, we can integrate S_{E-H} over θ and φ, so the latter turns out to be:

$$\begin{aligned} S_{E-H} &= \int d\tau \int d\rho \int d\theta \int d\varphi \, \mathcal{L}^g \\ &\equiv \int d\tau \int d\rho \;\, \tilde{\mathcal{L}}^g \end{aligned} \tag{6.88}$$

where the tilde notation $\tilde{\mathcal{L}}^g$ has been introduced again in order to distinguish the ADM Lagrangian from the one integrated over the angular coordinates.
The full expression of the ADM Lagrangian density with the variable change (6.86) can

be calculated as:

$$\tilde{\mathcal{L}}^g = \left(2G\sqrt{1-\frac{2Gm}{r}}(r-2Gm)\alpha\right)^{-1}\Big(2\beta'\dot{r}r^2 - 2\beta r'\beta' r^2 - \beta^2 r'^2 r - \dot{r}^2 r - 2G\beta^2 m' r' r$$
$$+ 4Gm\beta r'\beta' r + 2G\beta r'\dot{m}r + 2G\beta m'\dot{r}r + 2\beta r'\dot{r}r - 4Gm\beta'\dot{r}r - 2G\dot{m}\dot{r}r + 4Gm\beta^2 r'^2$$
$$+ 4Gm\dot{r}^2 - 8Gm\beta r'\dot{r}\Big) + \left(2G\sqrt{1-\frac{2Gm}{r}}\right)^{-1}\Big(-2\alpha r'' r - \alpha r'^2 + \alpha + 2G\alpha m' r'$$
$$+ 4Gm\alpha r''\Big) \tag{6.89}$$

From which one can derive the momenta coniugated to $r(\rho,\tau)$ and $m(\rho,\tau)$:

$$p_m = \frac{1}{\alpha}\left(1-\frac{2Gm}{r}\right)^{-3/2}(\beta r' - \dot{r}) \tag{6.90}$$

$$p_r = \frac{1}{\alpha}\left(1-\frac{2Gm}{r}\right)^{-3/2}\left(\frac{2m\dot{r}}{r} - \dot{m} + \beta m' - \frac{2\beta m r'}{r}\right) + \frac{1}{\alpha G}\left(1-\frac{2Gm}{r}\right)^{-1/2}(\beta' r - \dot{r} + r'\beta) \tag{6.91}$$

Again, we emphasize that these momenta are determined by a Lagrangian integrated over the angular coordinates.

As in the previous subsection, we want to use relations (6.90) and (6.91) in order to write the velocities $\dot{r}$ and $\dot{m}$ in terms of the momenta coniugated to the latter quantities.

$$\dot{m} = \alpha\left(1-\frac{2Gm}{r}\right)^{5/2}\frac{p_m}{G} - \alpha\left(1-\frac{2Gm}{r}\right)^{3/2}\left(\frac{2mp_m}{r} + p_r\right) + \left(1-\frac{2Gm}{r}\right)\frac{\beta' r}{G} + \beta m' \tag{6.92}$$

$$\dot{r} = \beta r' - \alpha p_m\left(1-\frac{2Gm}{r}\right)^{3/2} \tag{6.93}$$

In doing so, we are driven to the following expression for the ADM Lagrangian:

$$\tilde{\mathcal{L}}^g = \alpha\left(1-\frac{2Gm}{r}\right)^{5/2}\frac{p_m^2}{G} - \alpha\left(1-\frac{2Gm}{r}\right)^{3/2} p_m\left(p_r + \frac{p_m}{2G}\right) + \text{potential terms} \tag{6.94}$$

We see that, with this choice of the metric, the trapping surfaces (which by definition occurs where the expansion of outgoing null geodesics vanishes) correspond [30,31] to $r = r_g$. If we analyze the momenta coniugated to m and r for $r \sim 2Gm$ (which corresponds to the gravitational radius (1.28) which has been described in the Schwarzschild solution), we see that in order to render p_m and p_r reguar we can choose the following value for the lapse function α:

$$\alpha = \left(1 - \frac{2Gm}{r}\right)^{-\gamma} \qquad \gamma \geq 3/2 \tag{6.95}$$

while the shif vector β can be any regular function of the dynamical variables differentiable with respect to the radial coordinate. For example, if we choose $\gamma = 3/2$, we find for $r \sim 2Gm$:

$$p_m \sim 2G(\beta m' - \dot{m}) \tag{6.96}$$

$$p_r \sim -\dot{m} - m'\beta \tag{6.97}$$

6.7.1 ADM Hamiltonian and Constraints

In order to introduce the ADM Hamiltonian density $\tilde{\mathcal{H}}^g$ (we maintain the "tilde" notation to underline that the only remaining integration is over the radial coordinate), we write:

$$\begin{aligned} \tilde{\mathcal{H}}^g &= p_m\dot{m} + p_r\dot{r} - \tilde{\mathcal{L}}^g \\ &\equiv \alpha\tilde{\mathcal{H}}^g_\perp + \beta\tilde{\chi}^g \end{aligned} \tag{6.98}$$

where $\tilde{\mathcal{H}}^g_\perp$ and $\tilde{\chi}^g$ are, respectively, the Hamiltonian and the momentum constraint. The full Hamiltonian is found by integration:

$$\tilde{H}^g = \int d\rho \left(\alpha\tilde{\mathcal{H}}^g_\perp + \beta\tilde{\chi}^g\right) \tag{6.99}$$

The ADM Hamiltonian density can be calculated as:

$$\begin{aligned} \tilde{\mathcal{H}}^g = &\left(2Gr^2\sqrt{1 - \frac{2Gm}{r}}\right)^{-1} \Big(- 8G^3\alpha m^2 r p_m p_r - 16G^3\alpha m^3 p_m^2 - 2G\alpha m' r^3 r' \\ &+ 8G^3\alpha m r^2 p_m p_r - 2G\alpha r^3 p_m p_r + 20G^2\alpha m^2 r p_m^2 - 8G\alpha m r^2 p_m^2 + \alpha r^3 p_m^2 \\ &- 4G\alpha m r^3 r'' + \alpha r^3 r'^2 + 2\alpha r^4 r'' - \alpha r^3\Big) + \frac{\beta' p_m r}{G} + \beta m' p_m - 2\beta' m p_m + \beta p_r r' \end{aligned} \tag{6.100}$$

As can be directly verified, the expression for the constraints turns out to be:

$$\begin{aligned}\tilde{\mathcal{H}}^g_\perp &= \left(1-\frac{2Gm}{r}\right)^{3/2} p_m\left(\frac{p_m}{2G}-\frac{2mp_m}{r}-p_r\right)\\ &\quad+\left(1-\frac{2Gm}{r}\right)^{-1/2}\left(\frac{r'^2}{2G}-2mr''+\frac{rr''}{G}-r'm'-\frac{1}{2G}\right)\end{aligned} \tag{6.101}$$

$$\tilde{\chi}^g = p_m\left(3m'-\frac{r'}{G}\right)+p'_m\left(2m-\frac{r}{G}\right)+p_r r' \tag{6.102}$$

We see that for $r \sim 2Gm$, the constraints exhibit the following behaviour (in particular, the Hamiltonian constraint is divergent):

$$\tilde{\mathcal{H}}^g_\perp \sim \left(1-\frac{2Gm}{r}\right)^{-1/2}\left(-\frac{1}{2G}\right) \tag{6.103}$$

$$\tilde{\chi}^g \sim 2Gm'p_r \tag{6.104}$$

Fluid constraints can be written, according to (6.56), (6.57) and (6.58), as:

$$\tilde{\mathcal{H}}^{fluid}_\perp = 4\pi r^2\left(1-\frac{2Gm}{r}\right)^{-1/2}\left[\omega+(\omega+p)\left(1-\frac{2Gm}{r}\right)^{-1}U^2\right] \tag{6.105}$$

$$\tilde{\chi}^{fluid} = -4\pi r^2\left(1-\frac{2Gm}{r}\right)^{-3/2}(\omega+p)\left[1+\left(1-\frac{2Gm}{r}\right)^{-1}U^2\right]^{1/2}U \tag{6.106}$$

where we see that for $r \sim 2Gm$ and $U \neq 0$ both the constraints diverge. Note that the Hamiltonian constraint for the fluid diverges with the same asymptotic behavior of (6.101).

6.8 Time evolution

The Hamiltonian equations of motion are

$$\dot{m} = \{m, H^g + H^{fluid}\} \qquad \dot{r} = \{r, H^g + H^{fluid}\} \tag{6.107}$$

$$\dot{p}_m = \{p_m, H^g + H^{fluid}\} \quad \dot{p}_r = \{p_r, H^g + H^{fluid}\} \tag{6.108}$$

We impose as usually:

$$\{m(\tau,x), p_m(\tau,y)\} = \delta(x,y) \tag{6.109}$$

$$\{r(\tau,x), p_r(\tau,y)\} = \delta(x,y) \tag{6.110}$$

where x and y are again radial coordinate.
As regards the temporal evolution for the canonical variables of the fluid, once the Poisson brackets (6.71), (6.72) and (6.73) are imposed, the resulting equations are the following:

$$\dot{\phi} = \{\phi, H^{fluid}\} = \alpha\left(\frac{\omega+p}{n} - Ts - U\phi'\right)\left[1+\left(1-\frac{2Gm}{r}\right)^{-1}U^2\right]^{-1/2} + \beta\phi' \tag{6.111}$$

$$\dot{\vartheta} = \{\vartheta, H^{fluid}\} = \alpha\left(T - U\vartheta'\right)\left[1+\left(1-\frac{2Gm}{r}\right)^{-1}U^2\right]^{-1/2} + \beta\vartheta' \tag{6.112}$$

$$\dot{V} = \{V, H^{fluid}\} = \alpha\left(-UV'\right)\left[1+\left(1-\frac{2Gm}{r}\right)^{-1}U^2\right]^{-1/2} + \beta V' \tag{6.113}$$

$$\dot{\Pi}_\phi = \{\Pi_\phi, H^{fluid}\} = -\left[4\pi\alpha r^2\left(1-\frac{2Gm}{r}\right)^{-1/2}U\right]' + (\beta\Pi_\phi)' \tag{6.114}$$

$$\dot{\Pi}_\vartheta = \{\Pi_\vartheta, H^{fluid}\} = -\left[4\pi\alpha r^2\left(1-\frac{2Gm}{r}\right)^{-1/2}nsU\right]' + (\beta\Pi_\vartheta)' \tag{6.115}$$

$$\dot{\Pi}_V = \{\Pi_V, H^{fluid}\} = -\left[4\pi\alpha r^2\left(1-\frac{2Gm}{r}\right)^{-1/2}nVU\right]' + (\beta\Pi_V)' \tag{6.116}$$

where we used only the fluid generator H^{fluid}, since H^g does not contribute to the temporal evolution of the canonical variables related to the fluid.
With the change of variable (6.86) in the metric, we see that considerations made in the previous sections remain valid: the temporal evolution of the canonical variables m and r is not *directly* influenced by the Hamiltonian relative to the perfect fluid, while the corresponding conjugate momenta acquire terms due to the latter, which in turn modify the time evolution of m and r through (6.93) and (6.92).
In fact, by using (6.99) and (6.58) with previous expressions, it turns again out that (6.107) just gives the relations (6.93) and (6.92) according to the general description of the ADM formalism, while equations (6.108) give the following expressions for the

temporal evolution of conjugate momenta:

$$\dot{p}_m = \left(2r^3\sqrt{1-\frac{2Gm}{r}}\right)^{-1}\Big(12G^2mrp_mp_r\alpha + 40G^2m^2p_m^2\alpha - 4G^2r^2\alpha - 6Gr^2p_mp_r\alpha$$
$$- 34Gmrp_m^2\alpha + 7r^2p_m^2\alpha - 2r^3r'\alpha' - \alpha r^2r'^2\Big) + 3p_m\beta' + \beta p_m'$$
$$+ \left(\frac{4\pi Gr^3\sqrt{1-\frac{2Gm}{r}}}{(r-2Gm)^4}\right)\Big(6GpU^2mr\alpha - 4G^2\omega m^2\alpha - 6GpUmr\beta + 6GU^2\omega mr\alpha$$
$$- 6GU\omega mr\beta + 4G\omega mr\alpha + 5pU^3r^2\beta - 3pU^2r^2\alpha + 3pUr^2\beta + 5U^3\omega r^2\beta - 3U^2\omega r^2\alpha$$
$$+ 3U\omega r^2\beta - \omega r^2\alpha\Big) \tag{6.117}$$

$$\dot{p}_r = \left(2Gr^4\sqrt{1-\frac{2Gm}{r}}\right)^{-1}\Big(4G^3r^2m\alpha - 12G^3m^2rp_mp_r\alpha - 40G^3m^3p_m^2\alpha + 2Gr^3m'r'\alpha'$$
$$+ 2Gr^4m'\alpha' + 6G^2mr^2p_mp_r\alpha + 34G^2m^2rp_m^2\alpha - 7Gmr^2p_m^2\alpha + 4Gmr^3r''\alpha - 2r^4r''\alpha$$
$$+ 4Gmr^3r'\alpha' - 2r^4r'\alpha' - Gmr^2r'^2\alpha + 4Gmr^4\alpha'' - 2r^5\alpha''\Big) - \frac{p_m\beta'}{G} + \beta p_r' + p_r\beta'$$
$$+ \left(\sqrt{1-\frac{2Gm}{r}}(r-2Gm)^3\right)^{-1}\Big(56G^2\pi U\omega m^2\beta r^2 + 80G^3\pi\omega m^3\alpha r - 8p\pi U^2\alpha r^4$$
$$- 8\pi U^2\omega\alpha r^4 - 8\pi\omega\alpha r^4 + 8p\pi U^3\beta r^4 + 8p\pi U\beta r^4 + 8\pi U^3\omega\beta r^4 + 8\pi U\omega\beta r^4$$
$$+ 44Gp\pi U^2m\alpha r^3 + 44G\pi U^2\omega m\alpha r^3 + 52G\pi\omega m\alpha r^3 - 36Gp\pi U^3m\beta r^3 - 44Gp\pi Um\beta r^3$$
$$- 36G\pi U^3\omega m\beta r^3 - 44G\pi U\omega m\beta r^3 - 56G^2p\pi U^2m^2\alpha r^2 - 56G^2\pi U^2\omega m^2\alpha r^2$$
$$- 112G^2\pi\omega m^2\alpha r^2 + 56G^2p\pi Um^2\beta r^2\Big) \tag{6.118}$$

As in the previous section, the Poisson brackets between the constraints is different from zero:

$$\{\tilde{\mathcal{H}}_\perp^g(\tau,x) + \mathcal{H}_\perp^{fluid}(\tau,x), \tilde{\chi}^g(\tau,y) + \tilde{\chi}^{fluid}(\tau,y)\} \neq 0 \tag{6.119}$$

Defining $\tilde{H}_\perp^g(\tau) \equiv \int dx\tilde{\mathcal{H}}_\perp^g(\tau,x)$ and $\tilde{\Xi}^g(\tau) \equiv \int dy\tilde{\chi}^g(\tau,y)$, the following Poisson bracket between the constraints (for simplicity we again disregard the fluid contribution) holds at equal times:

$$
\begin{aligned}
\{\tilde{H}^g_\perp, \tilde{\Xi}^g\} = \int d\rho \Big[& -512G^7 p_m^4 m^7 - 256G^6 p_m^3 (Gp_r - 5p_m) r m^6 \\
& + 32G^5 p_m^2 r^2 \left(-39p_m^2 + 18Gp_r p_m + 10r'^2 - 4rr''\right) m^5 \\
& - 16G^4 p_m r^3 (-35p_m^3 + 30Gp_r p_m^2 + \left(-8G^2 + 43r'^2 - 14rr''\right) p_m \\
& + 4Grp_r' r' + 2Gp_r \left(4G^2 - 7r'^2\right)) m^4 \\
& + 8G^3 r^4 (16G^4 - 8rr''G^2 + 20p_m^3 p_r G + 2p_m \left(8rp_r' r' + p_r \left(16G^2 - 25r'^2\right)\right) G \\
& - 10p_m^4 + r'^4 + 4r'^2 \left(p_r^2 G^2 - G^2 - rr''\right) - 2p_m^2 \left(16G^2 - 37r'^2 + 8rr''\right)) m^3 \\
& - 8G^2 r^5 (24G^4 - 12rr''G^2 + 3m'r'^3 G + 3p_m \left(4rp_r' r' + p_r \left(8G^2 - 11r'^2\right)\right) G \\
& + 3p_m^4 - r'^4 + 2r'^2 \left(3p_r^2 G^2 - 3G^2 + 2rm''G - 5rr''\right) - 2p_m^2 \left(12G^2 - 16r'^2 + rr''\right)) m^2 \\
& + 2Gr^6 (48G^4 - 24rr''G^2 + 12m'r'^3 G - 6p_m^3 p_r G + 2p_m \left(8rp_r' r' + p_r \left(16G^2 - 19r'^2\right)\right) G \\
& + 8rm'r'r''G + 5p_m^4 - 3r'^4 + 4r'^2 \left(3p_r^2 G^2 + 2m'^2 G^2 - 3G^2 + 4rm''G - 7rr''\right) \\
& + 4p_m^2 \left(-8G^2 + 7r'^2 + rr''\right)) m + r^7 (-16G^4 + 8rr''G^2 + 6m'r'^3 G + 2p_m^3 p_r G \\
& - 4p_m \left(rp_r' r' + 2p_r \left(G^2 - r'^2\right)\right) G - 8rm'r'r''G - p_m^4 \\
& - 4r'^2 \left(p_r^2 G^2 + 8m'^2 G^2 - G^2 + 2rm''G - 3rr''\right) \\
& + p_m^2 \left(8G^2 - 5r'^2 - 2rr''\right)) \Big] \left(4Gr^6 r' \sqrt{1 - \frac{2Gm}{r}} (r - 2Gm)^2\right)^{-1}
\end{aligned}
\tag{6.120}
$$

in which we have taken into account that $\tilde{\mathcal{H}}^g_\perp \approx 0$, $\tilde{\chi}^g \approx 0$, thus also this set is a second class constraints set, as how to be expected from the previous analysis for the variables (λ, r).

6.9 Constraints coupled with the matter

The equations for the constraints, including the contribution of the perfect fluid, can be written as:

$$\tilde{\mathcal{H}}^g_\perp + \tilde{\mathcal{H}}^{fluid}_\perp \approx 0 \tag{6.121}$$

$$\tilde{\chi}^g + \tilde{\chi}^{fluid} \approx 0 \tag{6.122}$$

Replacing previous expressions, the constraints turn out to be:

$$\begin{aligned}\tilde{\mathcal{H}}_\perp^g + \tilde{\mathcal{H}}_\perp^{fluid} &= \left(1-\frac{2Gm}{r}\right)^{3/2} p_m \left(\frac{p_m}{2G} - \frac{2mp_m}{r} - p_r\right) \\ &+ \left(1-\frac{2Gm}{r}\right)^{-1/2} \left(\frac{r'^2}{2G} - 2mr'' + \frac{rr''}{G} - r'm' - \frac{1}{2G}\right) \\ &+ 4\pi r^2 \left(1-\frac{2Gm}{r}\right)^{-1/2} \left[\omega + (\omega+p)\left(1-\frac{2Gm}{r}\right)^{-1} U^2\right]\end{aligned} \tag{6.123}$$

$$\begin{aligned}\tilde{\chi}^g + \tilde{\chi}^{fluid} &= p_m \left(3m' - \frac{r'}{G}\right) + p'_m \left(2m - \frac{r}{G}\right) + p_r r' \\ &- 4\pi r^2 \left(1-\frac{2Gm}{r}\right)^{-3/2} (\omega+p) \left[1 + \left(1-\frac{2Gm}{r}\right)^{-1} U^2\right]^{1/2} U\end{aligned} \tag{6.124}$$

It is possible to eliminate the dipendence on p_r from the Hamiltonian constraint, taking advantage of the condition $\tilde{\chi}^g + \tilde{\chi}^{fluid} \approx 0$:

$$\begin{aligned}\tilde{\mathcal{H}}_\perp^g + \tilde{\mathcal{H}}_\perp^{fluid} &= \left(1-\frac{2Gm}{r}\right)^{3/2} p_m \left(-\frac{2mp_m}{r} + \frac{3p_m m'}{r'} - \frac{p_m}{2G} + \frac{2mp'_m}{r'} - \frac{rp'_m}{r'G}\right) \\ &+ \left(1-\frac{2Gm}{r}\right)^{-1/2} \left(\frac{r'^2}{2G} - 2mr'' + \frac{rr''}{G} - r'm' - \frac{1}{2G} + 4\pi\omega r^2\right) \\ &- 4\pi p_m r^2 \frac{(\omega+p)}{r'} \left[1 + \left(1-\frac{2Gm}{r}\right)^{-1} U^2\right]^{1/2} U\end{aligned} \tag{6.125}$$

At this point, once we have obtained the equations for the system constraints in the general case, we want to examine some particular cases.

- $U = 0$

  In this case, in the co-moving reference system with the fluid, the latter is at rest. The diffeomorphism constraint relative to the fluid (as one can expect) identically vanishes, while the Hamiltonian constraint for the fluid becomes

  $$\tilde{\mathcal{H}}_\perp^{fluid} = 4\pi r^2 \left(1-\frac{2Gm}{r}\right)^{-1/2} \omega \tag{6.126}$$

  The equations of motion are considerably simplified, as can be seen directly by imposing $U = 0$ in the latters. In particular, while the time evolution of the gravitational canonical variables m and r (6.93), (6.92) is not directly modified by the condition $U = 0$ for the fluid, the evolution (6.117), (6.118) of the conjugate momenta is considerably simplified, however for arbitrary values of the lapse function and of the radial component of the shift vector, the latters are still too complicated to be explicitly solved.

For $r \sim 2Gm$ the sum of the diffeomorphism constraints gives:

$$p_r m' \approx 0 \tag{6.127}$$

while the sum of Hamiltonian constraints become:

$$\left(1 - \frac{2Gm}{r}\right)^{-1/2} \left(16\pi G^2 m^2 \omega - \frac{1}{2G}\right) \approx 0 \tag{6.128}$$

The relation that is possible to extrapolate for the mass density of the fluid (which multiplies the divergent factor in the previous equation) is

$$8\pi G\omega = \frac{1}{4G^2 m^2} \tag{6.129}$$

it is clear that the previous relation, which relates the dynamic variable m to the energy density of the fluid ω, is valid in the case in which there is a trapping surface in the region where the fluid is present.

- $U = 0$, $p = 0$

  In this case the perfect fluid becomes dust (Chapter 2). The constraints, for $U = 0$, do not change depending on whether the pressure is positive or zero. However, it is easy (although the calculations are laborious) to show that the relativistic Euler equation (6.85) written in terms of variables (m, r) in this case is reduced to:

$$\alpha' = 0 \tag{6.130}$$

  that is, we recover the known result [23] according to which the lapse function for a compact spherical source consisting of dust cannot depend on radial coordinate, thus $\alpha = \alpha(\tau)$. Conversely, if we take $\alpha = 1$, (6.85) is reduced to:

$$p' = 0 \tag{6.131}$$

  A pressure independent of the radial coordinate seems not to be a plausible physical situation unless $p = 0$, but on the other hand the lapse function (as we saw in the previous Chapter) is an arbitrary choice. The reason for this is that $\alpha = 1$ implies that the particles move along timelikes geodesics of spacetime, but this is only possible in the absence of pressure.

- $U = 0$, $r = \rho$
  This condition implies that the system becomes totally static: the radius of the spherical source does not depend on time and therefore it can neither expand nor contract. In this case, the constraints become:

$$\begin{aligned} \tilde{\mathcal{H}}_\perp^g + \tilde{\mathcal{H}}_\perp^{fluid} =& \left(1 - \frac{2Gm}{\rho}\right)^{3/2} p_m \left(\frac{p_m}{2G} - \frac{2mp_m}{\rho} - p_r\right) \\ &+ \left(1 - \frac{2Gm}{\rho}\right)^{-1/2} \left(-m' + 4\pi\omega\rho^2\right) \end{aligned} \tag{6.132}$$

$$\tilde{\chi}^g + \tilde{\chi}^{fluid} = p_m \left(3m' - \frac{1}{G}\right) + p_m' \left(2m - \frac{\rho}{G}\right) + p_r \tag{6.133}$$

where the expressions of the momenta are given by

$$p_m = \frac{\beta}{\alpha} \left(1 - \frac{2Gm}{\rho}\right)^{-3/2} \tag{6.134}$$

$$p_r = \frac{1}{\alpha} \left(1 - \frac{2Gm}{\rho}\right)^{-1/2} \left[\left(1 - \frac{2Gm}{\rho}\right)^{-1} \left(\beta m' - \frac{2\beta m}{\rho}\right) + \frac{1}{G} \left(\beta'\rho + \beta\right)\right] \tag{6.135}$$

Having used this formalism to describe this source, in this condition we are interested in understanding if it is possible to derive the value of the mass m from the Hamiltonian constraint. Imposing $\beta = 0$ and keeping α arbitrary, we see from (6.90) that $p_m = 0$, thus from the condition $\tilde{\mathcal{H}}_\perp^g + \tilde{\mathcal{H}}_\perp^{fluid} \approx 0$ we find:

$$m = 4\pi \int_0^{\rho_0} \omega\rho^2 d\rho \tag{6.136}$$

which is the Misner-Sharp mass (2.14) for $r = \rho$, while also the conjugate momentum to r vanishes, therefore the radial constraint undergoes the same fate. From (6.93) we see that $\dot{m} = 0$, condition which coincides with the result obtained by carrying out the same hypotheses in the Lagrangian formalism, as we show in the appendix. In this formalism, however, we must be able to prove that ω does not depend on the temporal coordinate. It is not difficult to show that the energy conservation equation (6.84) is reduced, under these assumptions and again using (m, r), to the condition:

$$\dot{\omega} = 0 \tag{6.137}$$

So we see that the Misner-Sharp mass turns out to be a constraint in the ADM formalism, at least in the totally static case. If these hypotheses are released on U, r and β, the constraint equations are transformed into cumbersome partial differential equations on m and p_m.

In the appendix we also show that, with these hypotheses within the Lagrangian formalism, the value of the Misner-Sharp mass m can be recovered from the time-time component of the field equations, while setting also $g_{00} = e^{\nu(\rho)}$ we obtain the so-called *Tolman-Oppenheimer-Volkoff equation*.

Finally we notice that, referring to the Tolman solution (which was obtained with a different choice of the metric) discussed in Chapter 2 with the hypotesis $r = \rho$, we would have obtained the relation (6.129) if we had imposed $F = \rho$ in (2.13) for $\rho \sim 2Gm$.

Chapter 7

Canonical Quantization For a Spherical Source

We have established in the previous Chapter a Hamiltonian formulation of General Relativity following the work of Bergmann, Dirac, "ADM" and others in 1950s. This can be considered the starting point for a canonical quantization, which requires the definition of a configuration variable and its conjugate momentum.
The dynamics of General Relativity (as in all reparametrization-invariant systems) is entirely generated by constraints: the total Hamiltonian either vanishes as a constraint (for the spatially compact case) or solely consists of surfaces terms (in the asymptotically flat case). The central difficulty is thus, both conceptually and technically, the correct treatment of the quantum constraints: given a classical theory, it is not possible to derive a unique quantum theory from the latter, one can only try to guess such a theory and to test it by experiment. For this purpose, we can try to use sets of "quantization rules" which turned out to be successful in the construction of quantum theories (such as QED). The task turns out to be, strictly speaking, to construct a quantum theory from its classical limit.

7.1 Steps for Quantization

For the sake of brevity we deal in this Chapter only with the analysis for the variables (m, r) used to describe the spherical source. Following Kuchar [15], we shall divide the programme of canonical quantization into six steps, which will shortly be presented and described here using again conventions of table (6.1), (6.2) and (6.3).

- **CONFIGURATION VARIABLES AND MOMENTA**

  The first step consists in the identification of configuration variables and their momenta. Together with the unit operator, these variables are called the *fundamental variables* V_i. The implementation of Dirac's procedure for the canonical quantization is, as usually, the translation of Poisson brackets into commutators for the fundamental variables according to the well know formula:

$$V_3 = \{V_1, V_2\} \rightarrow \hat{V_3} = \frac{1}{i\hbar}[V_1, V_2] \tag{7.1}$$

  As we have seen in the ADM formalism section, in the formulation of General Relativity the fundamental variables are, apart from the unit operator, the three-metric $h_{ab}(x)$ and its momentum $p^{cd}(x)$ (or, in the approach of reduced quantization, a subset of them). The V_i form a vector space that is closed under Poisson brackets and complete (in the sense that every dynamical variable can be expressed as a sum of products of fundamental variables).

  Application of the above Dirac's prescription to (5.61) would yield the following relation:

$$[\hat{h}_{ab}(x), \hat{p}^{cd}(y)] = i\hbar\delta^c_{(a}\delta^d_{b)}\delta(x, y) \tag{7.2}$$

  plus vanishing commutators between, respectively, the metric components and the momentum components. Since p^{cd} is linearly related to the extrinsic curvature (describing the embedding of the three-geometry into the fourth temporal dimension), the presence of the commutator (7.2) and the ensuing "uncertainty relation" between intrinsic and extrinsic geometry means that the classical space–time picture has completely dissolved in quantum gravity: this is analogous to the disappearance of particle trajectories as fundamental concepts in quantum mechanics and constitutes one of the central interpretational ingredients of quantum gravity. Equation

(7.2) does not implement the positivity requirement $deth > 0$ of the classical theory, but this could only be a problem if $\hat{p}^{ab}$ were self-adjoint and its exponentiation therefore a unitary operator, which could "shift" the metric to negative values. Applying the same canonical quantization procedure to the Poisson brackets (6.109), (6.110), (6.71), (6.72) and (6.73) one obtains:

$$[\hat{m}(x), \hat{p}_m(y)] = i\hbar\delta(x,y) \tag{7.3}$$

$$[\hat{r}(x), \hat{p}_r(y)] = i\hbar\delta(x,y) \tag{7.4}$$

$$[\hat{\phi}(x), \hat{\Pi}_\phi(y)] = i\hbar\delta(x,y) \tag{7.5}$$

$$[\hat{\vartheta}(x), \hat{\Pi}_\vartheta(y)] = i\hbar\delta(x,y) \tag{7.6}$$

$$[\hat{V}(x), \hat{\Pi}_V(y)] = i\hbar\delta(x,y) \tag{7.7}$$

Of course we must not forget that, as shown in Chapter 6, for a compact spherical source described by a perfect fluid, the two resulting constraints are of second class and, therefore, the Poisson brackets must be replaced, according to (3.7), by Dirac brackets.

- **QUANTIZATION OF VARIABLES**
  This step addresses the quantization of a general variable $F(h_{ab}, p^{cd})$, which is a function of the fundamental variables discussed in the previous step. From general theorems of quantum theory (Groenewald and van Hove), it turns out that that it is impossible to respect the transformation rule (7.2) in the general case, while assuming an irreducible representation of the commutation rules: this failure is related to the problem of "factor ordering" on which we will not dwell here.
  Therefore, additional criteria must be invoked to find the "correct" quantization.

- **REPRESENTATION SPACE**
  The third step concerns the construction of an appropriate representation space, $\mathcal{F}$, for the dynamical variables, on which the latter should act as operators. We shall usually employ the functional Schroedinger picture, in which the operators act on wave functionals defined in an appropriate functional space.
  The implementation of (7.2) woul be achieved, for example (following the well

known procedure of the quantum mechanics), by

$$\hat{h}_{ab}(x)\Psi[h_{ab}(x)] = h_{ab}(x)\Psi[h_{ab}(x)] \tag{7.8}$$

$$\hat{p}^{cd}(x)\Psi[h_{ab}(x)] = -i\hbar\frac{\delta}{\delta h_{cd}(x)}\Psi[h_{ab}(x)] \tag{7.9}$$

Riem Σ would have to be invariant under translations in function space in order to define self-adjoint operators from the previous equations, but since there is not Lebesgue measure on Riem Σ this is not the case. Thus, one would not expect the fundamental relations (7.2) to be necessarily in conflict with $deth > 0$. Before the constraints are implemented, the representation space $\mathcal{F}$ does not necessarily contain only physical states, therefore neither does it have to be a Hilbert space nor do operators acting on $\mathcal{F}$ have to be self-adjoint.

It might even be inconsistent to demand that the constraints be self-adjoint operators on an auxiliary Hilbert space $\mathcal{F}$, which can therefore be merely seen as an auxiliary space.

This step applied to the example of Chapter 6 gives:

$$\hat{m}\Psi = m\Psi \qquad \hat{p}_m\Psi = -i\hbar\frac{\delta}{\delta m}\Psi \tag{7.10}$$

$$\hat{r}\Psi = r\Psi \qquad \hat{p}_r\Psi = -i\hbar\frac{\delta}{\delta r}\Psi \tag{7.11}$$

$$\hat{\phi}\Psi = \phi\Psi \qquad \hat{\Pi}_\phi\Psi = -i\hbar\frac{\delta}{\delta \phi}\Psi \tag{7.12}$$

$$\hat{\vartheta}\Psi = \vartheta\Psi \qquad \hat{\Pi}_\vartheta\Psi = -i\hbar\frac{\delta}{\delta \vartheta}\Psi \tag{7.13}$$

$$\hat{V}\Psi = V\Psi \qquad \hat{\Pi}_V\Psi = -i\hbar\frac{\delta}{\delta V}\Psi \tag{7.14}$$

- **CONSTRAINTS**

This step consists in the implementation of the constraints. According to the Dirac's quantization method of the constraints discussed in previous Chapters, one would implement the classical constraints $H_\perp \approx 0$ and $H_a \approx 0$ as:

$$\mathcal{H}_\perp\Psi = 0 \tag{7.15}$$

$$\mathcal{H}_a\Psi = 0 \tag{7.16}$$

These are infinitely many equations, in fact we have one of them at each space point, which is usually indicated collectively as $H_\mu \Psi = 0$. Only solutions to these "quantum constraints" can be regarded as candidates for physical states, in accordance with what we saw in Chapter 3. The solution space will be indicated as $\mathcal{F}_0$. How the constraints above can be written in detail depends on one's approach to the *problem of time*, which will be investigated soon. It has to be expected that the solution space is still too large: as in quantum mechanics, one may have to impose further conditions on the wave functions such as normalizability, whose requirement is needed in quantum mechanics because of the probability interpretation (but it is far from clear whether this interpretation can be maintained in quantum gravity). The physical space on which wave functionals act, $\mathcal{F}_{phys}$, should thus form in the ideal case a genuine subspace, $\mathcal{F}_{phys} \subset \mathcal{F}_0 \subset \mathcal{F}$.

- **OBSERVABLES**
  The fifth step concerns the role of *observables* $\mathcal{O}$, which are characterized by having weakly vanishing Poisson brackets with the constraints: $\{\mathcal{O}, \mathcal{G}_a\} \approx 0$. The latters should not be confused with observables in an operationalistic sense. In quantum mechanics, observables are associated with self-adjoint operators, however in practice only few operators correspond in fact with quantities that are "measured" in laboratory. For an operator corresponding to a classical observable satisfying $\{\mathcal{O}, \mathcal{H}_\mu\} \approx 0$, one would expect that in the quantum theory the following relation holds:

$$\left[\hat{\mathcal{O}}, \hat{\mathcal{H}}_\mu\right] \Psi = 0 \tag{7.17}$$

  This is sometimes interpreted as meaning that the "measurement" of the quantity being related to this operator leads to a state that is no longer annihilated by the constraints, "throwing one out" of the solution space. This would, however, only be a problem for a "collapse" interpretation of quantum gravity, an interpretation that seems to be highly unlikely to hold in quantum gravity [4]. Since the interpretations of classical Hamiltonian and diffeomorphism constraints differ from each other, the same is exptected to happen for their quantum versions (7.15) and (7.16).

- **HILBERT SPACE**

  The last step concerns the role of the physical Hilbert space (cf. also step 3).

  Do the observables have to be represented by operators in some Hilbert space? If yes, which Hilbert space? It can't be the auxiliary space $\mathcal{F}$ (because the latter is "too large"), but it is unclear whether it is $\mathcal{F}_0$ or only $\mathcal{F}_{phys}$ (which is a subspace of $\mathcal{F}_0$). A general method to deal with the construction of a physical Hilbert space, in the quantization of constrained systems, is the "group averaging procedure". However, the situation for General Relativity, where the constraint algebra is not a Lie algebra at all, remains unclear and the "problem of Hilbert space" is intimately connected with the "problem of time" in quantum gravity, whose discussion follows.

The quantum theory and the concept of time in General Relativity differ drastically from each other physical theory, because time in quantum theory is an external parameter (an "absolute" element of the theory as we have defined it in the third Chapter), whereas in General Relativity is dynamical because of the role of $g_{\mu\nu}$ in the latter (as seen in the discussion of the Einstein's equation in Chapter 3).

Therefore, a consistent theory of quantum gravity should exhibit a "novel concept" of time: the history of physics has shown that new theories often entail a new concept of space and time, thus the same should happen again with quantum gravity.

The absolute nature of time in quantum mechanics is crucial for its interpretation in the General Relativity framework.

- **TIME IN QUANTUM MECHANICS**

  Matrix elements are usually evaluated at fixed t, and the scalar product is conserved in time (this is the *unitarity* requirement): this express, in quantum mechanics, the conservation of the total probability. "Time" is part of the classical background which, according to the Copenhagen interpretation, is needed for the interpretation of measurements. As we have remarked at the end of discussion of the classical point particle, the introduction of a time operator in quantum mechanics is problematic. The time parameter t appears explicitly in the Schroedinger equation (3.28), it comes together with the imaginary unit i, a fact that finds an explanation in the semiclassical approximation to quantum geometrodynamics [4].

- **TIME IN GENERAL RELATIVITY**
  Space–time is dynamical and, therefore, there is no absolute time: space–time influences material clocks in order to allow them to show proper time. The clocks react on the metric and change the geometry, in this sense, the metric itself can be seen as a clock. A quantization of the metric $g_{\mu\nu}$ can, according to this point of view, be interpreted as a quantization of the concept of time. Since the nature of time in quantum gravity is not yet clear (because the classical constraints do not contain any time parameter), one speaks of the "problem of time".
  As reviewed by Isham and Kuchar [21], one can distinguish essentially three possible solutions of this problem:
  - The choice of a concept of time before quantization
  - "Timeless" options
  - The identification of a concept of time after quantization

  In this context we intend to examine and apply only the third solution to this (still open) problem. For further details we refer again to the monograph [4].

7.2 Time after quantization

Using directly the commutation rules (7.2) and their formal implementation (7.8) and (7.9), it is possible to determine wave functionals $\Psi[h_{ab}(\vec{x})]$ defined on Riem Σ, i.e. the space of all three-metrics which is the central kinematical quantity, while the "dynamics" must be implemented through the quantization of the constraints (5.73) and (5.74). It turns out that this is all that remains in the quantum theory.
One then gets the following equations for the wave functional:

$$\hat{\mathcal{H}}^g_\perp \Psi \equiv \left(-16\pi G\hbar^2 G_{abcd} \frac{\delta^2}{\delta h_{ab}\delta_{cd}} - \frac{\sqrt{h}}{16\pi G}({}^{(3)}R - 2\Lambda) \right) \Psi = 0 \tag{7.18}$$

$$\hat{\mathcal{H}}^g_a \Psi \equiv 2i\hbar D_b h_{ac} \frac{\delta \Psi}{\delta h_{bc}} = 0 \tag{7.19}$$

The first one is called the "Wheeler–DeWitt equation" in honour of the work by DeWitt and Wheeler [20,22]: these are again infinitely many equations. The constraints (7.19) are called the "quantum diffeomorphism (or momentum) constraints" but, sometimes in the literature, both (7.18) and (7.19) are referred to as Wheeler–DeWitt equations. Applying this quantization procedure to the example of Chapter 6, the following equations are obtained:

$$
\begin{aligned}
(\hat{\tilde{\mathcal{H}}}^g_\perp + \hat{\tilde{\mathcal{H}}}^{fluid}_\perp)\Psi = \Bigg[& \left(1-\frac{2Gm}{r}\right)^{3/2}\left(\frac{2m\hbar^2}{r}\frac{\delta^2}{\delta m^2}+\frac{\hbar^2\delta^2}{\delta m\delta r}-\frac{\hbar^2}{2G}\frac{\delta^2}{\delta m^2}\right)\\
& + \left(1-\frac{2Gm}{r}\right)^{-1/2}\left(\frac{r'^2}{2G}-2mr''+\frac{rr''}{G}-r'm'-\frac{1}{2G}\right)\\
& - \left(1-\frac{2Gm}{r}\right)^{1/2}\left(\frac{\omega+p}{4\pi r^2 n^2}\frac{\hbar^2\delta^2}{\delta\phi^2}\right)-4\pi r^2\left(1-\frac{2Gm}{r}\right)^{-1/2}p\Bigg]\Psi = 0
\end{aligned}
\tag{7.20}
$$

$$
\begin{aligned}
(\hat{\tilde{\chi}}^g + \hat{\tilde{\chi}}^{fluid})\Psi = (-i\hbar)\Bigg(& 3m'\frac{\delta}{\delta m}-\frac{r'}{G}\frac{\delta}{\delta m}+2m\frac{\delta}{\delta m'}-\frac{r}{G}\frac{\delta}{\delta m'}+r'\frac{\delta}{\delta r'}+\phi'\frac{\delta}{\delta\phi}\\
& +\vartheta'\frac{\delta}{\delta\vartheta}+V'\frac{\delta}{\delta V}\Bigg)\Psi
\end{aligned}
\tag{7.21}
$$

Nevertheless, there are many problems associated with these equations.
The first obvious problem is the "factor-ordering problem": the precise form of the kinetic term is open, thus there could be additional terms proportional to $\hbar$ containing, at most first derivatives in the $g_{\mu\nu}$. Since second functional derivatives at the same space point usually lead to undefined expressions (such as $\delta(0)$), a regularization and, perhaps, renormalization scheme has to be employed, connected with this is the potential presence of anomalies.
Continuing with the discussion of the "time problem" and the related Hilbert-space problem, since (7.18) does not have the structure of a local Schroedinger equation, the choice of Hilbert space is not clear a priori.
The first option to define an Hilbert space appropriated to the theory is related to the use of a Schroedinger-type inner product, that is, the standard quantum-mechanical inner product as generalized to quantum field theory:

$$
\langle\Psi_1|\Psi_2\rangle = \int_{Riem\Sigma}\mathcal{D}\mu[h]\ \Psi_1^*[h]\Psi_2[h] \tag{7.22}
$$

where h is here a shorthand for h_{ab} as argument of the measure and the functionals. It is known that such a construction is at best formal, since the measure $D\mu[h]$ cannot be rigorously defined, that is, there is no Lebesgue measure in the functional case as in QFT. The elementary operators $\hat{h}_{ab}$ and $\hat{p}^{cd}$ are (formally) self-adjoint with respect to this inner product. Besides the lack of mathematical rigour, this inner product (7.22) has other problems, which we do not dwell on as they are beyond our purposes [4].
Since the Wheeler–DeWitt equation is (unlike the Schroedinger equation) a real equation, one would expects that real solutions should possess some significance.
For the standard Klein–Gordon equation in Minkowski space, it is always possible to make a separation between "positive" frequencies and "negative" frequencies. As long as one can stay within the one-particle picture, it is consistent to make a restriction to the positive-frequency sector and, for such solutions, the inner product is positive.
Nevertheless Kuchar showed that such a separation into positive and negative frequencies cannot be made for the Wheeler-DeWitt equation [15].
As known, for the standard Klein–Gordon equation the failure of the one-particle picture leads to "second quantization" and QFT. The Wheeler– DeWitt equation, however, corresponds already to a field-theoretic situation. Kuchar has, therefore, suggested to proceed with a "third quantization" and to turn the wave function $\Psi[h]$ into an operator [15]. No final progress, however, has been achieved with such attempts nowadays.
Finally, let us make a few other specific considerations concerning the case of a compact spherical source, which we discussed in the previous Chapter. As shown in [24], it is possible to introduce a a global gravitational radius operator for a static and spherically symmetric quantum mechanical matter state: this can be done by lifting the classical Hamiltonian constraint that relates the gravitational radius to the ADM mass, thus giving rise to a "horizon wave-function" [32].
For a static and spherically symmetric self-gravitating source, the Misner-Sharp mass (6.136) approaches the ADM mass M of the source [28] for $\rho \to \infty$, where also the gravitational radius becomes the Schwarzschild radius (1.28). If we want to describe the source by quantum physics, the Misner-Sharp mass should be described by corresponding quantum variables, thus one expects the gravitational radius will undergo the same treatment. In order to describe the "fuzzy" gravitational radius of a localised (but like-

wise fuzzy) quantum source, the *Horizon Quantum Mechanics* has been introduced [26]. This theory differs from most previous attempts in which the gravitational degrees of freedom are quantised independently of the state of the source and becomes particularly relevant for sources of the Planck size, for which quantum effects may not be neglected. The argument that grants the Planck mass m_p and Planck length l_p a remarkable role in the search for a quantum theory of gravity is the following. The Heisenberg principle introduces an uncertainty in the particle's spatial localisation of the order of the Compton–de Broglie length λ:

$$\lambda \simeq \frac{l_p m_p}{M} \tag{7.23}$$

and the Schwarzschild radius only makes sense if the latter is greater than λ, which means

$$M \gtrsim m_p \tag{7.24}$$

As shown in [24], analyzing the quantum constraint that relates the gravitational radius of a spherically symmetric source to its spectral decomposition, the same quantum state can be employed in order to describe both the global radius associated with the ADM mass and the local radius associated with the Misner–Sharp mass function.

Nevertheless, while the local gravitational radius at the quantum level requires the spectral decomposition in terms of localised energy eigenmodes, global radius can be defined in any case. The global gravitational radius should be, from a strictly physical point of view, rather insensitive to the details of its internal structure (in fact it is an asymptotic property of a self-gravitating system), whereas the local radius should be determined by the precise internal structure of the source. It therefore appears consistent that the local gravitational radius can be defined only provided the inner structure of the source is properly characterised as well. Finally, since any realistic astrophysical sources should have very finely-spaced energy levels, the fact the spectral decomposition must be discrete does not constitute a real limitation in most practical situations.

Conclusions

We have obtained the equations of motion and constraints for a compact spherical source of perfect fluid coupled with gravity. We first wrote the three-dimensional metric on hypersurfaces in accordance with the “usual form” (that is, by using the same metric which describes the Tolman’s solution), while secondly we made a change of variable by introducing a local mass in the radial component of the latter. Repeating the analysis with this metric we have seen that, in the case of a static system and if the shift vector’s radial component is equal to zero, the Misner-Sharp mass is a solution of the Hamiltonian constraint, while in the non-static case this constraint is reduced to a differential equation to partial derivatives on the local mass and its conjugate momentum. We have also shown how, by requiring that the particles of the fluid move along geodesics, the condition of constant pressure is obtained. In a similar way, imposing that the pressure is zero, the theory predicts that the lapse function cannot depend on the radial coordinate. Regarding the analysis of trapping surfaces, from the Hamiltonian constraint we have obtained a relationship between the local mass and the density of matter that seems to exist near these surfaces. Finally, as regards the quantum description, by applying the “Wheeler-DeWitt quantization procedure”, we have obtained the corresponding quantum version of the constraints described above.
The discussion could be extended by further examining some other particular cases and by making hypotheses on the lapse function and on the radial component of the shift vector, trying to understand if it is possible to draw some information from the constraints for the system in the non-static case in which gravitational collapse can occur. We could also try to make a more specific and restrictive choice on the variables to be quantized. All these possible extensions are left for future development.

Appendices

Einstein tensor

For a metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} \beta(\tau,\rho)^2 e^{\lambda(\tau,\rho)} - \alpha(\tau,\rho)^2 & \beta(\tau,\rho)e^{\lambda(\tau,\rho)} & 0 & 0 \\ \beta(\tau,\rho)e^{\lambda(\tau,\rho)} & e^{\lambda(\tau,\rho)} & 0 & 0 \\ 0 & 0 & r(\tau,\rho)^2 & 0 \\ 0 & 0 & 0 & \sin^2(\theta) r(\tau,\rho)^2 \end{pmatrix} \tag{25}$$

with inverse given by:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha(\tau,\rho)^2} & \frac{\beta(\tau,\rho)}{\alpha(\tau,\rho)^2} & 0 & 0 \\ \frac{\beta(\tau,\rho)}{\alpha(\tau,\rho)^2} & e^{-\lambda(\tau,\rho)} - \frac{\beta(\tau,\rho)^2}{\alpha(\tau,\rho)^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r(\tau,\rho)^2} & 0 \\ 0 & 0 & 0 & \frac{\csc^2(\theta)}{r(\tau,\rho)^2} \end{pmatrix} \tag{26}$$

and coordinates $(x^0, x^1, x^2, x^3) = (\tau, \rho, \theta, \varphi)$, it turns out that the only non-vanishing components of the Einstein tensor, defined according to (5.37) (we neglet the cosmological constant) as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{27}$$

are G_{00}, G_{01}, G_{10}, G_{11} , G_{22} , G_{22}.

One can calculate the following expressions for the non-vanishing components of the

Einstein tensor:

$$
\begin{aligned}
G_{00} =& \frac{2\alpha'\beta^4 e^\lambda r'}{\alpha^3 r} - \frac{2\alpha'\beta^3 e^\lambda \dot{r}}{\alpha^3 r} - \frac{2\dot{\alpha}\beta^3 e^\lambda r'}{\alpha^3 r} + \frac{2\dot{\alpha}\beta^2 e^\lambda \dot{r}}{\alpha^3 r} - \frac{2\alpha'\beta^2 r'}{\alpha r} + \frac{4\alpha'\beta\dot{r}}{\alpha r} \\
&- \frac{2\beta'\beta^3 e^\lambda r'}{\alpha^2 r} + \frac{2\dot{\beta}\beta^2 e^\lambda r'}{\alpha^2 r} - \frac{\beta^4 e^\lambda r'^2}{\alpha^2 r^2} - \frac{2\beta^4 e^\lambda r''}{\alpha^2 r} + \frac{2\beta^3 e^\lambda r'\dot{r}}{\alpha^2 r^2} + \frac{4\beta^3 e^\lambda \dot{r}'}{\alpha^2 r} \\
&- \frac{\beta^2 e^\lambda \dot{r}^2}{\alpha^2 r^2} - \frac{2\beta^2 e^\lambda \ddot{r}}{\alpha^2 r} + \frac{\alpha^2 e^{-\lambda}\lambda' r'}{r} - \frac{\alpha^2 e^{-\lambda} r'^2}{r^2} - \frac{2\alpha^2 e^{-\lambda} r''}{r} + \frac{2\beta'\beta r'}{r} \\
&- \frac{2\beta'\dot{r}}{r} - \frac{\beta^2\lambda' r'}{r} - \frac{\beta\lambda'\dot{r}}{r} + \frac{\beta\dot{\lambda} r'}{r} + \frac{2\beta^2 r'^2}{r^2} + \frac{4\beta^2 r''}{r} \\
&- \frac{2\beta r'\dot{r}}{r^2} - \frac{4\beta\dot{r}'}{r} + \frac{\dot{\lambda}\dot{r}}{r} + \frac{\dot{r}^2}{r^2} + \frac{\alpha^2}{r^2} - \frac{\beta^2 e^\lambda}{r^2}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
G_{10} =& \frac{2\alpha'\beta^3 e^\lambda r'}{\alpha^3 r} - \frac{2\alpha'\beta^2 e^\lambda \dot{r}}{\alpha^3 r} - \frac{2\dot{\alpha}\beta^2 e^\lambda r'}{\alpha^3 r} + \frac{2\dot{\alpha}\beta e^\lambda \dot{r}}{\alpha^3 r} + \frac{2\alpha'\dot{r}}{\alpha r} - \frac{2\beta'\beta^2 e^\lambda r'}{\alpha^2 r} \\
&+ \frac{2\dot{\beta}\beta e^\lambda r'}{\alpha^2 r} - \frac{\beta^3 e^\lambda r'^2}{\alpha^2 r^2} - \frac{2\beta^3 e^\lambda r''}{\alpha^2 r} + \frac{2\beta^2 e^\lambda r'\dot{r}}{\alpha^2 r^2} + \frac{4\beta^2 e^\lambda \dot{r}'}{\alpha^2 r} - \frac{\beta e^\lambda \dot{r}^2}{\alpha^2 r^2} \\
&- \frac{2\beta e^\lambda \ddot{r}}{\alpha^2 r} - \frac{\beta\lambda' r'}{r} + \frac{\beta r'^2}{r^2} + \frac{2\beta r''}{r} + \frac{\dot{\lambda} r'}{r} - \frac{2\dot{r}'}{r} - \frac{\beta e^\lambda}{r^2}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
G_{11} =& \frac{2\alpha'\beta^2 e^\lambda r'}{\alpha^3 r} - \frac{2\alpha'\beta e^\lambda \dot{r}}{\alpha^3 r} - \frac{2\dot{\alpha}\beta e^\lambda r'}{\alpha^3 r} + \frac{2\dot{\alpha} e^\lambda \dot{r}}{\alpha^3 r} + \frac{2\alpha' r'}{\alpha r} - \frac{2\beta'\beta e^\lambda r'}{\alpha^2 r} \\
&+ \frac{2\dot{\beta} e^\lambda r'}{\alpha^2 r} - \frac{\beta^2 e^\lambda r'^2}{\alpha^2 r^2} - \frac{2\beta^2 e^\lambda r''}{\alpha^2 r} + \frac{2\beta e^\lambda r'\dot{r}}{\alpha^2 r^2} + \frac{4\beta e^\lambda \dot{r}'}{\alpha^2 r} - \frac{e^\lambda \dot{r}^2}{\alpha^2 r^2} - \frac{2e^\lambda \ddot{r}}{\alpha^2 r} \\
&+ \frac{r'^2}{r^2} - \frac{e^\lambda}{r^2}
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
G_{22} =& -\frac{\beta'^2 r^2}{\alpha^2} - \frac{\beta^2\lambda'^2 r^2}{4\alpha^2} - \frac{\dot{\lambda}^2 r^2}{4\alpha^2} + \frac{\beta\alpha'\beta' r^2}{\alpha^3} + \frac{\beta^2\alpha'\lambda' r^2}{2\alpha^3} - \frac{e^{-\lambda}\alpha'\lambda' r^2}{2\alpha} \\
&- \frac{3\beta\beta'\lambda' r^2}{2\alpha^2} + \frac{e^{-\lambda}\alpha'' r^2}{\alpha} - \frac{\beta\beta'' r^2}{\alpha^2} - \frac{\beta^2\lambda'' r^2}{2\alpha^2} - \frac{\beta'\dot{\alpha} r^2}{\alpha^3} - \frac{\beta\lambda'\dot{\alpha} r^2}{2\alpha^3} \\
&+ \frac{\lambda'\dot{\beta} r^2}{2\alpha^2} - \frac{\beta\alpha'\dot{\lambda} r^2}{2\alpha^3} + \frac{\beta'\dot{\lambda} r^2}{\alpha^2} + \frac{\beta\lambda'\dot{\lambda} r^2}{2\alpha^2} + \frac{\dot{\alpha}\dot{\lambda} r^2}{2\alpha^3} + \frac{\dot{\beta}' r^2}{\alpha^2} \\
&+ \frac{\beta\dot{\lambda}' r^2}{\alpha^2} - \frac{\ddot{\lambda} r^2}{2\alpha^2} + \frac{\beta^2 r'\alpha' r}{\alpha^3} + \frac{e^{-\lambda} r'\alpha' r}{\alpha} - \frac{2\beta r'\beta' r}{\alpha^2} - \frac{1}{2} e^{-\lambda} r'\lambda' r \\
&- \frac{\beta^2 r'\lambda' r}{2\alpha^2} + e^{-\lambda} r'' r - \frac{\beta^2 r'' r}{\alpha^2} - \frac{\beta\alpha'\dot{r} r}{\alpha^3} + \frac{\beta'\dot{r} r}{\alpha^2} + \frac{\beta\lambda'\dot{r} r}{2\alpha^2} \\
&- \frac{\beta r'\dot{\alpha} r}{\alpha^3} + \frac{\dot{r}\dot{\alpha} r}{\alpha^3} + \frac{r'\dot{\beta} r}{\alpha^2} + \frac{\beta r'\dot{\lambda} r}{2\alpha^2} - \frac{\dot{r}\dot{\lambda} r}{2\alpha^2} + \frac{2\beta\dot{r}' r}{\alpha^2} - \frac{\ddot{r} r}{\alpha^2}
\end{aligned} \tag{31}
$$

$$
G_{33} = \sin^2\theta\, G_{22} \tag{32}
$$

It is straightforward to verify that placing $\alpha = 1$ and $\beta = 0$ and adding the contribution of the incoherent matter, one ricovers (note that in Chapters 1 and 2 we used a metric with signature (1,-1-1-1)) the field equations (2.7) and the result Tolman solution.

Change of variable and TOV equations

If we introduce the variable change (6.86), the metric tensor turns out to be:

$$g_{\mu\nu} = \begin{pmatrix} \frac{\beta(\tau,\rho)^2}{1-\frac{2Gm(\tau,\rho)}{r(\tau,\rho)}} - \alpha(\tau,\rho)^2 & \frac{\beta(\tau,\rho)}{1-\frac{2Gm(\tau,\rho)}{r(\tau,\rho)}} & 0 & 0 \\ \frac{\beta(\tau,\rho)}{1-\frac{2Gm(\tau,\rho)}{r(\tau,\rho)}} & \frac{1}{1-\frac{2Gm(\tau,\rho)}{r(\tau,\rho)}} & 0 & 0 \\ 0 & 0 & r(\tau,\rho)^2 & 0 \\ 0 & 0 & 0 & \sin^2\theta \cdot r(\tau,\rho)^2 \end{pmatrix} \tag{33}$$

with inverse given by:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha(\tau,\rho)^2} & \frac{\beta(\tau,\rho)}{\alpha(\tau,\rho)^2} & 0 & 0 \\ \frac{\beta(\tau,\rho)}{\alpha(\tau,\rho)^2} & -\frac{\beta(\tau,\rho)^2}{\alpha(\tau,\rho)^2} - \frac{2Gm(\tau,\rho)}{r(\tau,\rho)} + 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r(\tau,\rho)^2} & 0 \\ 0 & 0 & 0 & \frac{\csc^2\theta}{r(\tau,\rho)^2} \end{pmatrix} \tag{34}$$

The expressions for the non-vanishing components of the Einstein tensor become:

$$
\begin{aligned}
G_{00} ={}& \frac{r'^2\beta^4}{(2Gm-r)r\alpha^2} - \frac{2r'\alpha'\beta^4}{(2Gm-r)\alpha^3} + \frac{2r''\beta^4}{(2Gm-r)\alpha^2} + \frac{2r'\beta'\beta^3}{(2Gm-r)\alpha^2} - \frac{2r'\dot{r}\beta^3}{(2Gm-r)r\alpha^2} \\
&+ \frac{2\alpha'\dot{r}\beta^3}{(2Gm-r)\alpha^3} + \frac{2r'\dot{\alpha}\beta^3}{(2Gm-r)\alpha^3} - \frac{4\dot{r}'\beta^3}{(2Gm-r)\alpha^2} - \frac{2r'^2\beta^2}{(2Gm-r)r} + \frac{2Gmr'^2\beta^2}{(2Gm-r)r^2} \\
&+ \frac{\dot{r}^2\beta^2}{(2Gm-r)r\alpha^2} + \frac{2Gm'r'\beta^2}{(2Gm-r)r} + \frac{2r'\alpha'\beta^2}{(2Gm-r)\alpha} - \frac{4Gmr'\alpha'\beta^2}{(2Gm-r)r\alpha} - \frac{4r''\beta^2}{2Gm-r} \\
&+ \frac{8Gmr''\beta^2}{(2Gm-r)r} - \frac{2\dot{r}\dot{\alpha}\beta^2}{(2Gm-r)\alpha^3} - \frac{2r'\dot{\beta}\beta^2}{(2Gm-r)\alpha^2} + \frac{2\ddot{r}\beta^2}{(2Gm-r)\alpha^2} + \frac{\beta^2}{(2Gm-r)r} \\
&- \frac{2r'\beta'\beta}{2Gm-r} + \frac{4Gmr'\beta'\beta}{(2Gm-r)r} - \frac{2Gr'\dot{m}\beta}{(2Gm-r)r} + \frac{2Gm'\dot{r}\beta}{(2Gm-r)r} + \frac{2r'\dot{r}\beta}{(2Gm-r)r} \\
&- \frac{4Gmr'\dot{r}\beta}{(2Gm-r)r^2} - \frac{4\alpha'\dot{r}\beta}{(2Gm-r)\alpha} + \frac{8Gm\alpha'\dot{r}\beta}{(2Gm-r)r\alpha} + \frac{4\dot{r}'\beta}{2Gm-r} - \frac{8Gm\dot{r}'\beta}{(2Gm-r)r} \\
&- \frac{\alpha^2}{(2Gm-r)r} + \frac{2Gm\alpha^2}{(2Gm-r)r^2} + \frac{\alpha^2r'^2}{(2Gm-r)r} - \frac{2Gm\alpha^2r'^2}{(2Gm-r)r^2} - \frac{\dot{r}^2}{(2Gm-r)r} \\
&+ \frac{4Gm\dot{r}^2}{(2Gm-r)r^2} - \frac{2G\alpha^2m'r'}{(2Gm-r)r} + \frac{4G^2m\alpha^2m'r'}{(2Gm-r)r^2} + \frac{2\alpha^2r''}{2Gm-r} - \frac{8Gm\alpha^2r''}{(2Gm-r)r} \\
&+ \frac{8G^2m^2\alpha^2r''}{(2Gm-r)r^2} + \frac{2\beta'\dot{r}}{2Gm-r} - \frac{4Gm\beta'\dot{r}}{(2Gm-r)r} - \frac{2G\dot{m}\dot{r}}{(2Gm-r)r}
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
G_{01} ={}& -\frac{2G\beta m'r'}{r(r-2Gm)} + \frac{2G\dot{m}r'}{r(r-2Gm)} + \frac{2\alpha'\beta^3r'}{\alpha^3(r-2Gm)} - \frac{2\alpha'\beta^2\dot{r}}{\alpha^3(r-2Gm)} - \frac{2\dot{\alpha}\beta^2r'}{\alpha^3(r-2Gm)} \\
&+ \frac{2\dot{\alpha}\beta\dot{r}}{\alpha^3(r-2Gm)} - \frac{4G\alpha'm\dot{r}}{\alpha r(r-2Gm)} + \frac{2\alpha'\dot{r}}{\alpha(r-2Gm)} - \frac{2\beta'\beta^2r'}{\alpha^2(r-2Gm)} + \frac{2\dot{\beta}\beta r'}{\alpha^2(r-2Gm)} \\
&- \frac{\beta^3r'^2}{\alpha^2r(r-2Gm)} - \frac{2\beta^3r''}{\alpha^2(r-2Gm)} + \frac{2\beta^2r'\dot{r}}{\alpha^2r(r-2Gm)} + \frac{4\beta^2\dot{r}'}{\alpha^2(r-2Gm)} - \frac{\beta\dot{r}^2}{\alpha^2r(r-2Gm)} \\
&- \frac{2\beta\ddot{r}}{\alpha^2(r-2Gm)} + \frac{\beta r'^2}{r(r-2Gm)} - \frac{4G\beta mr''}{r(r-2Gm)} + \frac{2\beta r''}{r-2Gm} - \frac{2Gmr'\dot{r}}{r^2(r-2Gm)} \\
&+ \frac{4Gm\dot{r}'}{r(r-2Gm)} - \frac{2\dot{r}'}{r-2Gm} - \frac{\beta}{r(r-2Gm)}
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
G_{11} ={}& \frac{2\alpha'\beta^2r'}{\alpha^3(r-2Gm)} - \frac{2\alpha'\beta\dot{r}}{\alpha^3(r-2Gm)} - \frac{2\dot{\alpha}\beta r'}{\alpha^3(r-2Gm)} - \frac{4G\alpha'mr'}{\alpha r(r-2Gm)} + \frac{2\alpha'r'}{\alpha(r-2Gm)} \\
&+ \frac{2\dot{\alpha}\dot{r}}{\alpha^3(r-2Gm)} - \frac{2\beta'\beta r'}{\alpha^2(r-2Gm)} + \frac{2\dot{\beta}r'}{\alpha^2(r-2Gm)} - \frac{\beta^2r'^2}{\alpha^2r(r-2Gm)} - \frac{2\beta^2r''}{\alpha^2(r-2Gm)} \\
&+ \frac{2\beta r'\dot{r}}{\alpha^2r(r-2Gm)} + \frac{4\beta\dot{r}'}{\alpha^2(r-2Gm)} - \frac{\dot{r}^2}{\alpha^2r(r-2Gm)} - \frac{2\ddot{r}}{\alpha^2(r-2Gm)} + \frac{r'^2}{r(r-2Gm)} \\
&- \frac{2Gmr'^2}{r^2(r-2Gm)} - \frac{1}{r(r-2Gm)}
\end{aligned} \tag{37}
$$

The expression of G_{22} turns out to be very long and it is not useful for our purpose, so we don't write it explicitly, while for simmetry we can easily write $G_{33} = \sin^2\theta\, G_{22}$. By direct inspection it turns out that, for general values of α and β, the non trivial Einstein's equations become too complex to be solved.

If we suppose $r(\tau,\rho) = \rho$, $g_{00} = e^{\nu(\rho)}$, $\beta(\tau,\rho) = 0$ and we use the energy-tensor of the perfect fluid (6.42) for the matter diistribution (supposing that the spatial components of the 4-velocity is equal to zero and that ω and p only depend on the radial coordinate), we find:

$$8\pi G e^{\nu(\rho)}\omega(\rho) = \frac{2Ge^{\nu(\rho)}m'(\tau,\rho)}{\rho^2} \tag{38}$$

$$0 = \frac{2G\dot{m}(\tau,\rho)}{\rho^2 - 2G\rho m(\tau,\rho)} \tag{39}$$

$$\frac{8\pi G\rho p(\rho)}{(1-2Gm(\tau,\rho))} = \frac{\rho^2\nu'(\rho) - 2m(\tau,\rho)\left(G\rho\nu'(\rho)+G\right)}{\rho^2(\rho - 2Gm(\tau,\rho))} \tag{40}$$

$$\begin{aligned}
8\pi G\rho^2 p(\rho) = &- \frac{2G^3\rho\nu'(\rho)m(\tau,\rho)^2m'(\tau,\rho)}{(\rho-2Gm(\tau,\rho))^2} - \frac{4G^3m(\tau,\rho)^2m'(\tau,\rho)}{(\rho-2Gm(\tau,\rho))^2} \\
&- \frac{2G^3\rho\nu'(\rho)^2m(\tau,\rho)^3}{(\rho-2Gm(\tau,\rho))^2} - \frac{2G^3\nu'(\rho)m(\tau,\rho)^3}{(\rho-2Gm(\tau,\rho))^2} + \frac{4G^3m(\tau,\rho)^3}{\rho(\rho-2Gm(\tau,\rho))^2} \\
&+ \frac{2G^2\rho^2\nu'(\rho)m(\tau,\rho)m'(\tau,\rho)}{(\rho-2Gm(\tau,\rho))^2} - \frac{3G^2\rho^2e^{-\nu(\rho)}\dot{m}(\tau,\rho)^2}{(\rho-2Gm(\tau,\rho))^2} \\
&+ \frac{4G^2\rho m(\tau,\rho)m'(\tau,\rho)}{(\rho-2Gm(\tau,\rho))^2} + \frac{6G^2\rho^2\nu''(\rho)m(\tau,\rho)^2}{(\rho-2Gm(\tau,\rho))^2} + \frac{3G^2\rho^2\nu'(\rho)^2m(\tau,\rho)^2}{(\rho-2Gm(\tau,\rho))^2} \\
&+ \frac{4G^2\rho\nu'(\rho)m(\tau,\rho)^2}{(\rho-2Gm(\tau,\rho))^2} - \frac{4G^2m(\tau,\rho)^2}{(\rho-2Gm(\tau,\rho))^2} - \frac{G\rho^3\nu'(\rho)m'(\tau,\rho)}{2(\rho-2Gm(\tau,\rho))^2} \\
&- \frac{G\rho^3e^{-\nu(\rho)}\ddot{m}(\tau,\rho)}{(\rho-2Gm(\tau,\rho))^2} - \frac{G\rho^2m'(\tau,\rho)}{(\rho-2Gm(\tau,\rho))^2} + \frac{\rho^4\nu''(\rho)}{2(\rho-2Gm(\tau,\rho))^2} \\
&- \frac{3G\rho^3\nu''(\rho)m(\tau,\rho)}{(\rho-2Gm(\tau,\rho))^2} + \frac{\rho^4\nu'(\rho)^2}{4(\rho-2Gm(\tau,\rho))^2} - \frac{3G\rho^3\nu'(\rho)^2m(\tau,\rho)}{2(\rho-2Gm(\tau,\rho))^2} \\
&+ \frac{\rho^3\nu'(\rho)}{2(\rho-2Gm(\tau,\rho))^2} - \frac{5G\rho^2\nu'(\rho)m(\tau,\rho)}{2(\rho-2Gm(\tau,\rho))^2} + \frac{G\rho m(\tau,\rho)}{(\rho-2Gm(\tau,\rho))^2} \\
&+ \frac{2G^2\rho^2e^{-\nu(\rho)}m(\tau,\rho)\ddot{m}(\tau,\rho)}{(\rho-2Gm(\tau,\rho))^2} - \frac{4G^3\rho\nu''(\rho)m(\tau,\rho)^3}{(\rho-2Gm(\tau,\rho))^2}
\end{aligned} \tag{41}$$

From the first equation we again find

$$m = 4\pi \int_0^{\rho_0} \omega(\rho)\rho^2 d\rho \tag{42}$$

which is (2.14), while from the second equation we see that $\dot{m} = 0$. From third equation we find:

$$\frac{d\nu(\rho)}{d\rho} = \left(1 - \frac{2Gm(\rho)}{\rho}\right)^{-1} \left(8\pi G\rho p(\rho) + \frac{2Gm(\rho)}{\rho^2}\right) \tag{43}$$

By using this result, after some calculus we find that fourth equation becomes

$$\begin{aligned} 8\pi G\rho^2 p(\rho) =& \frac{4\pi G^2\rho^3 p(\rho)m'(\rho)}{2Gm(\rho)-\rho} + \frac{G^2 m(\rho)m'(\rho)}{2Gm(\rho)-\rho} - \frac{8\pi G^2\rho^3 p'(\rho)m(\rho)}{2Gm(\rho)-\rho} + \frac{16\pi^2 G^2\rho^5 p(\rho)^2}{2Gm(\rho)-\rho} \\ &- \frac{12\pi G^2\rho^2 p(\rho)m(\rho)}{2Gm(\rho)-\rho} + \frac{4\pi G\rho^4 p'(\rho)}{2Gm(\rho)-\rho} + \frac{8\pi G\rho^3 p(\rho)}{2Gm(\rho)-\rho} \end{aligned} \tag{44}$$

We could have obtained the same result by considering the balance equation $\nabla_\mu T^{\mu\nu} = 0$. The latter can be written as

$$-\frac{dp(\rho)}{d\rho} = \frac{G}{\rho^2}\left(1 - \frac{2Gm(\rho)}{\rho}\right)^{-1} (\omega(\rho) + p(\rho)) \big(m(\rho) + 4\pi\rho^3 p(\rho)\big) \tag{45}$$

and it is called *Tolman-Oppenheimer-Volkoff equation*, which constrains the structure of a spherically symmetric body of isotropic material which is in static gravitational equilibrium. In fact, when supplemented with an equation of state which relates density to pressure, the Tolman–Oppenheimer–Volkoff equation completely determines the structure of a spherically symmetric body of isotropic material in equilibrium.

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