Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

SEMICLASSICAL ANALYSIS OF SYSTEMS OF SCHRÖDINGER EQUATIONS.

Tesi di Laurea in Fisica Matematica

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Volevo che tu vedessi che cosa è il vero coraggio, tu che credi che sia rappresentato da un uomo col fucile in mano.
Aver coraggio significa sapere di essere sconfitti prima ancora di cominciare, e cominciare ugualmente e arrivare sino in fondo, qualsiasi cosa succeda.
È raro vincere, in questi casi, ma qualche volta succede.
Atticus Finch.

Introduzione

L'equazione di Schrödinger indipendente dal tempo

$$-h^2\Delta\psi(x) + V(x)\psi(x) = E\psi(x),$$

gioca un ruolo fondamentale nella meccanica quantistica poiché la sua soluzione modellizza la distribuzione di probabilità di trovare una particella in un punto $x \in \mathbb{R}^n$.
Nel caso monodimensionale consideriamo un potenziale continuo $V(x)$ che forma una "buca" rispetto all'energia totale $E \in \mathbb{R}$ del sistema. Nella meccanica classica le regioni in cui $E < V(x)$ sono proibite, intendendo con ciò che la probabilità di trovarvi un punto materiale è identicamente nulla. In meccanica quantistica, invece, elettroni (generalmente particelle) possono oltrepassare le barriere di potenziale e possono essere trovate in regioni dove classicamente non dovrebbero essere; matematicamente, la distribuzione di probabilità della particella è esponenzialmente piccola ma non nulla in questi domini. Ciò è detto "effetto tunnel" ed è largamente usato nelle applicazioni tecnologiche.

L'analisi semiclassica che useremo in questa tesi è basata sul cosiddetto *parametro semiclassico* $h > 0$. Esso dà una profonda caratterizzazione del modello usato: quando h è la famosa costante di Planck stiamo lavorando in un regime di meccanica quantistica, mentre quando $h = 0$ ci troviamo in regime di meccanica classica. Considereremo sempre il suo limite in 0 che ci permetterà di collegare questi due ambiti tramte il *principio di corrispondenza di Bohr.*

In questa tesi considereremo un sistema 2×2 P di operatori di Schrödinger interagenti tale che

$$Pu = Eu \tag{1}$$

dove

$$P := \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix}, \quad u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad E \in \mathbb{R},$$

e

$$P_j = -h^2 \frac{d^2}{dx^2} + V_j(x), \quad j = 1, 2$$

e W, W^* sono un operatore di interazione del primo ordine e il suo aggiunto formale, i quali verranno discussi in dettaglio durante la tesi.
Tale modello proviene dallo studio di molecole biatomiche tramite approssimazione di Born-Oppenheimer. In questo caso V_1 e V_2 sono potenziali effettivi (livelli elettronici) dovuti all'azione degli elettroni sui nuclei e h^2 rappresenta il rapporto tra la massa degli elettroni e quella dei nuclei.

Considereremo diversi casi in cui due funzioni potenziali, entrambe che formano una buca, possono interagire. In ognuno di essi costruiremo soluzioni di (1) ognuna definita nel proprio intervallo. Successivamente le estenderemo all'intera retta reale usando la loro dipendenza lineare nelle intersezioni dei loro intervalli di definizione trovando infine un'unica soluzione globale. Tutte queste funzioni dipenderanno dal parametro semiclassico h.
Lo scopo finale di questa tesi è di trovare una condizione su $E \in [E_0 - \delta, E_0 + \delta]$, dove E_0 è fissata e $\delta > 0$ abbastanza piccola, tale che l'unica soluzione globale trovata sia in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ e rappresenti una distribuzione di probabilità per due particelle in presenza di due potenziali. In meccanica quantistica questa condizione è nota come *condizione di quantizzazione di Bohr-Sommerfeld.*

Nel primo capitolo, seguendo il lavoro fondante [Y], studieremo il caso monodimensionale con un solo potenziale. Costruiremo soluzioni dell'equazione di Schrödinger e preveremo che se E è un autovalore, allora

$$\phi(E) := \int_{a(E)}^{b(E)} \big(E - V(y)\big)^{1/2} \, dy = \pi\Big(n + \frac{1}{2}\Big)h + O(h^2) \tag{2}$$

doe $n \in \mathbb{Z}$ e $a(E)$ e $b(E)$ sono i punti per cui $V(a) = V(b) = E$. E si dice *quantizzata* perché può solamente assumere valori che dipendono da numeri interi.

Successivamente considereremo due potenziali e studieremo la condizione di quantizzazione per E in base a come interagiscono.

Il primo caso considerato è quello in cui la loro intersezione si trova sopra il livello fissato di energia E_0:

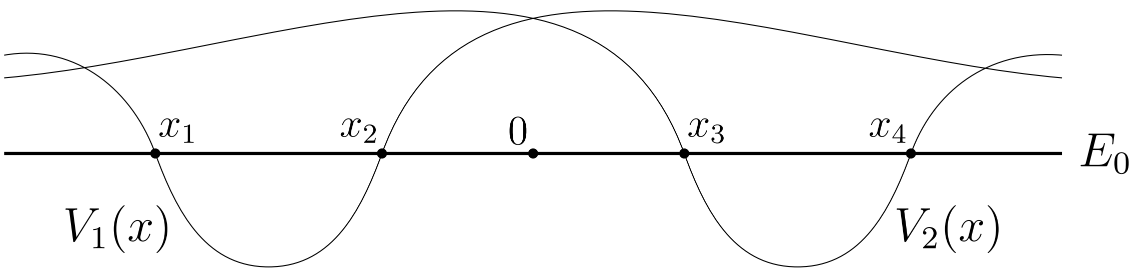


Figure 1: Intersezione sopra il livello di energia.

Seguendo il lavoro [A] tratteremo quattro intervalli L, R, I, J e in ognuno di essi costruiremo soluzioni del sistema. Successivamente, imponendo la loro dipendenza lineare nelle intersezioni degli intervalli si ottiene la condizione per cui $E \in [E_0 - \delta, E_0 + \delta]$, $\delta > 0$, è un autovalore implica

$$\boxed{\cos\left(h^{-1}\phi_1(E)\right)\cos\left(h^{-1}\phi_2(E)\right) = O(h^{-1/3}).} \tag{3}$$

Questa condizione sarà discussa in dettaglio nella Sezione 2.8 e condurrà a due condizioni su E simili a (2) ma riferite ai due potenziali.

Nel terzo capitolo consideriamo il caso in cui l'intersezione tra le funzioni potenziali è sotto il livello dell'energia E_0, seguendo il lavoro [FMW3].

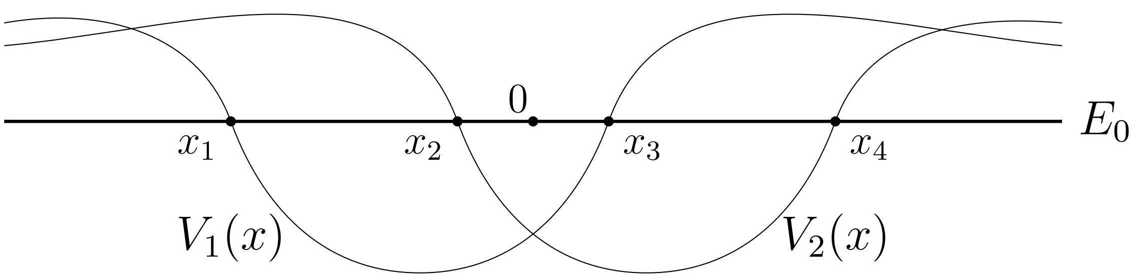


Figure 2: Intersezione sotto il livello dell'energia.

Il metodo è di ancora di trovare soluzioni di (1) e usare la loro dipendenza lineare per ottenere la condizione di quantizzazione (3) per E con $O(h^{1/6})$ invece che $O(h^{-1/3})$ nel secondo membro. Ad eccezione di questa differenza sarà discussa con lo stesso metodo del caso precedente.

Il quarto capitolo rappresenta un'eccezione poiché i potenziali non si intersecano e la buca di V_2 è completamente inclusa nella buca di V_1.

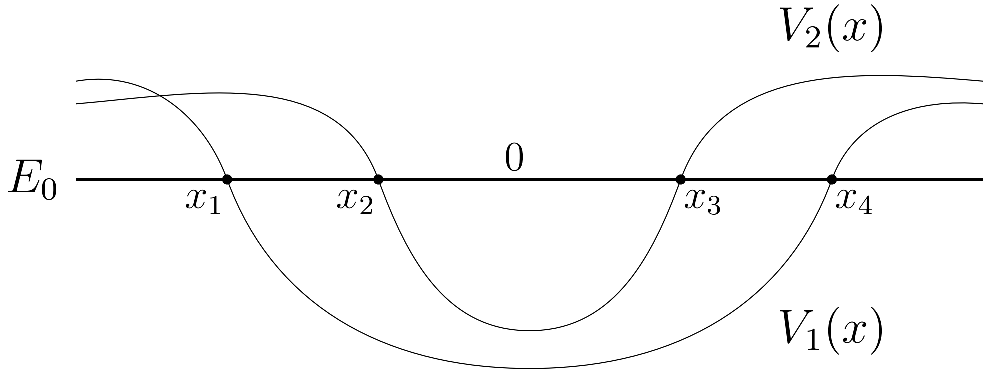


Figure 3: Potenziali che non si intersecano.

Tuttavia la discussione non ha differenze significative: useremo gli stessi operatori integrali del secondo e quinto capitolo per trovare delle soluzioni globali del sistema e per ottenre la stessa condizione di quantizzazione per ϕ_j e di conseguenza per E.

Per l'ultimo caso ci atterremo a [FMW1] e [FMW2]. L'intersezione delle funzioni potenziali questa volta coincide con l'energia E_0.

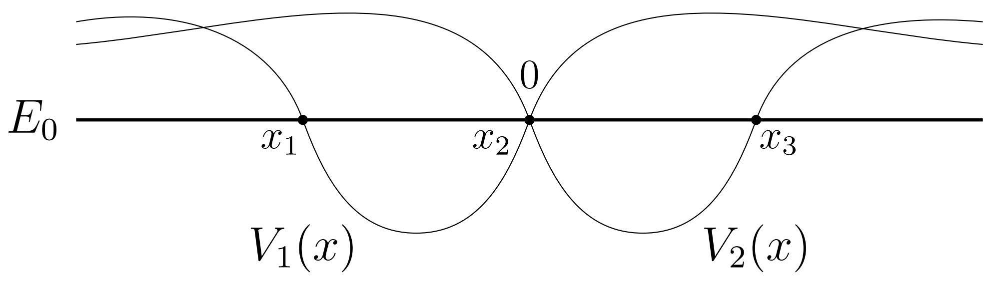


Figure 4: Intersezione dei potenziali al livello dell'energia.

Gli intervalli considerati saranno $(-\infty, 0]$ e $[0, +\infty)$ quali hanno un trattamento speculare e arriveremo ancora alla condizione (3).

Introduction

The time-independent Schrödinger equation

$$-h^2\Delta\psi(x) + V(x)\psi(x) = E\psi(x),$$

plays a fundamental role in quantum mechanics since its solution models the probability distribution of finding a particle in a point $x \in \mathbb{R}^n$.
In the one-dimensional case we consider a continuous potential $V(x)$ forming a "well" with respect to the total energy $E \in \mathbb{R}$ of the system. In classical mechanics the regions where $E < V(x)$ are prohibited, meaning that the probability to find any material point inside them is identically null. In quantum mechanics, instead, electrons (more generally particles) can overcome the potential barriers and be found in regions where they classically should not be; mathematically the distribution of probability of the particle is exponentially small but not null in these domains. This is called "tunnel effect" and is extensively used in technological applications.

The semiclassical analysis we are using during this thesis is based on the so-called *semiclassical parameter* $h > 0$. It gives a deep characterization of the model used: when h is the famous Planck constant we are working in the quantum mechanics regime, while when $h=0$ we are in the classical mechanics regime. We will always consider its limit approaching 0 permitting us two link these two frameworks via the *Bohr correspondence principle.*

In this thesis we will mainly consider a 2×2 system P of interacting Schrödinger operators such that

$$Pu = Eu \tag{4}$$

where

$$P := \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix}, \quad u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad E \in \mathbb{R},$$

and

$$P_j = -h^2\frac{d^2}{dx^2} + V_j(x), \quad j = 1, 2$$

and W, W^* are an interaction first-order operator and its formal adjoint, which will be discussed in details during the thesis.
Such a model comes from the study of diatomic molecules in the Born-Oppenheimer approximation. In this case V_1 and V_2 are the effective potentials (electronic levels) due to the action of the electrons on the nuclei and h^2 represents the quotient between the mass of the electron and that of the nuclei.

We are considering several possible cases in which two potential functions, both forming a well, can interact. In each of them we will construct solutions of (4) each one defined in its own interval. We will then extend them to the whole real line using their linear dependence in the intersections of their intervals of definition having finally a unique global solution. All these functions will depend from the semiclassical parameter h.
The final aim of this thesis is to find a condition on $E \in [E_0 - \delta, E_0 + \delta]$, where E_0 is fixed and $\delta > 0$ small enough, such that the unique global solution found is in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and represents a probability distribution for two particles in presence of two potentials. In quantum mechanics this condition is known as the *Bohr-Sommerfeld quantization condition*.

In the first chapter, following the foundational work [Y], we will study the one dimensional case with only one potential. We will build solutions of the Schrödinger equation and we will prove that if E is an eigenvalue, then

$$\phi(E) := \int_{a(E)}^{b(E)} \big(E - V(y)\big)^{1/2} \, dy = \pi\Big(n + \frac{1}{2}\Big)h + O(h^2) \tag{5}$$

where $n \in \mathbb{Z}$ and $a(E)$ and $b(E)$ are the points where $V(a) = V(b) = E$. E is said to be *quantized* because it can take only values that depend on integer numbers.

From this point on we will consider two potentials and study the quantization condition on E depending on how they interact.

The first case considered is when their intersection point is above a fixed energy level E_0:

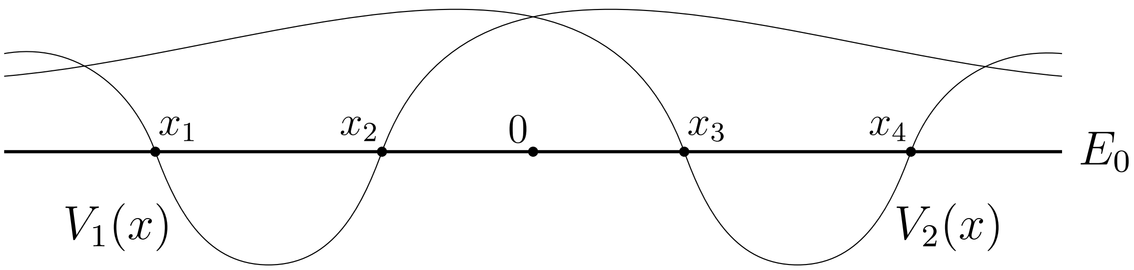


Figure 5: Intersection above energy level

Following the work [A] we are considering four intervals L, R, I, J and in each of them construct solutions of the system. Then, imposing their linear dependence in the intersection of the intervals we arrive to the condition that $E \in [E_0 - \delta, E_0 + \delta]$, $\delta > 0$, is an eigenvalue implies

$$\boxed{\cos\left(h^{-1}\phi_1(E)\right)\cos\left(h^{-1}\phi_2(E)\right) = O(h^{-1/3}).} \tag{6}$$

This condition will be discussed in details in Section 2.8 and will lead to two conditions on E similar to (5) but referred to the two potentials .

In the third chapter we consider the case in which the intersection of the potential functions is below the energy level E_0, following the work [FMW3].

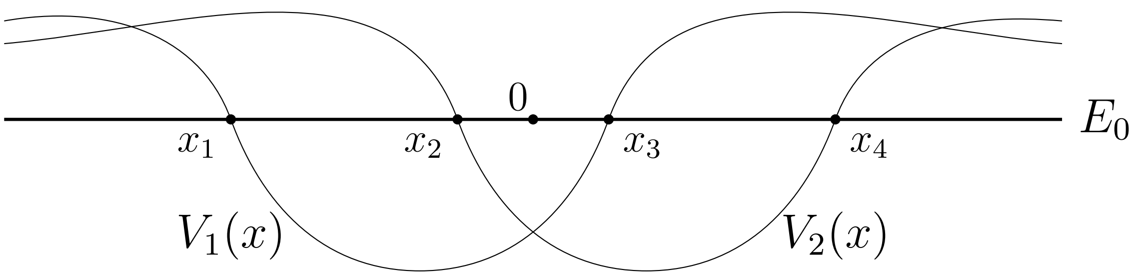


Figure 6: Intersection below the energy level

The method is again to find solutions of (4) and to use their linear dependence to obtain the quantization condition (6) on E with $O(h^{1/6})$ instead of $O(h^{-1/3})$ in the right-hand side. Except for this difference it will be discussed with the same method of the previous case.

The fourth chapter is an exception since the potentials do not intersect and the well of V_2 is completely included in the well of V_1.

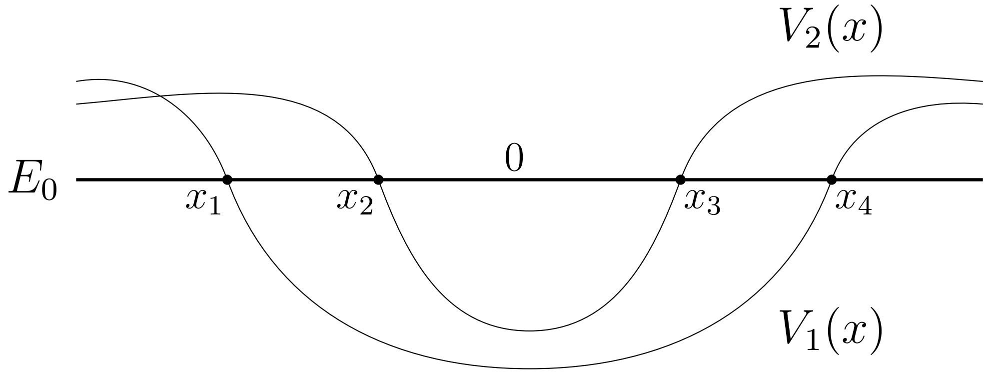


Figure 7: Non intersecting potentials

Nevertheless the discussion has not significant differences: we will use similar integral operators as the second and fifth chapter to find global solutions of the system and to reach the same quantization conditions on the ϕ_j and consequently on E.

For the last case we stick to [FMW1] and [FMW2]. The intersection of the potential functions is this time coincident with the energy E_0.

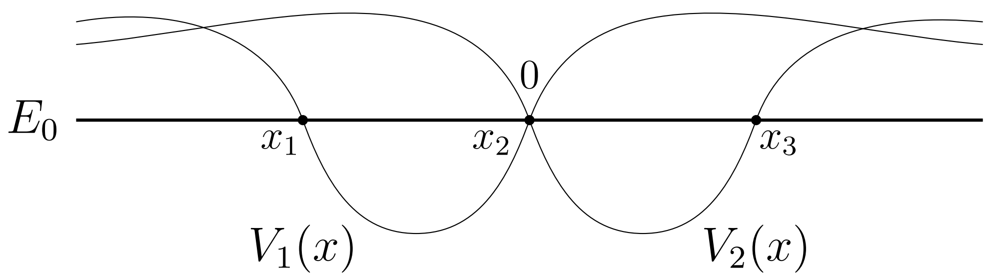


Figure 8: Intersection of potentials at energy level

The interval considered are $(-\infty, 0]$ and $[0, +\infty)$ which have a specular treatment and we then will arrive again to the condition (6).

Contents

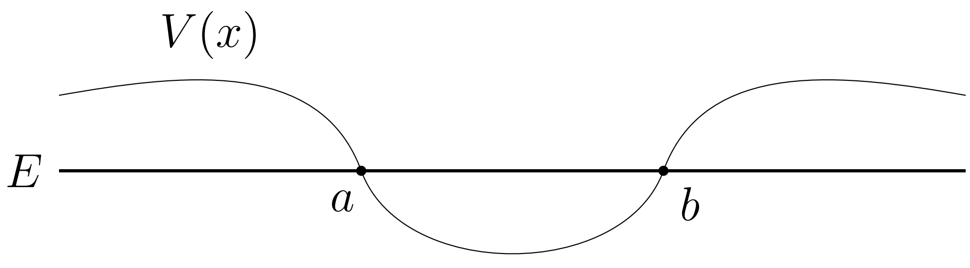
Chapter 1

Asymptotic solutions of scalar Schrödinger equation.

In this chapter we will describe the solution $\psi(x) = \psi(x; h, E)$ of the Schrödinger equation

$$-h^2\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{1.1}$$

assuming $V(x)$ to be a real scalar potential and E to be a scalar having the role of the total energy of the system.
We are following the methods in [Y] and [FMW1], Appendix A.1 and A.2.
The potential $V(x)$ and E are such that they form a potential well, that is the equation $V(x) = E$ has two distict solutions, denoted by $a = a(E)$ and $b = b(E)$ such that for $x \in (a, b)$ one has $V(x) < E$.



Moreover we assume

$$\liminf_{|x|\to+\infty} V(x) > E,$$

$V(x) \in C^\infty$ in some neighborhoods of a and b, $|V'(a)|, |V'(b)| \neq 0$ and,

$$\forall k \in \mathbb{N}, \quad V^{(k)}(x) = O(\langle x\rangle^{-k}) \tag{1.2}$$

where in general

$$\langle x\rangle^m := (1+|x|^2)^{m/2}, \quad m \in \mathbb{R}. \tag{1.3}$$

We start by setting

$$u(x) := h^2\psi(x) \tag{1.4}$$

such that (1.1) becomes

$$-u''(x) + h^{-2}(V(x)-E)u(x) = 0. \tag{1.5}$$

Now it has the form of an Airy equation

$$-w''(t) + tw(t) = 0 \tag{1.6}$$

which has two linearly independent solutions $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$. For negative arguments they have an oscillatory and slowly decaying behaviour with a phase shift. For positive t, $\mathrm{Ai}(t)$ has an exponential decay while, on the contrary, $\mathrm{Bi}(t)$ grows exponentially. Further details can be found in [O]. Explicitly:

$$\mathrm{Ai}(t) = \frac{1}{2\sqrt{\pi}}t^{-1/4}\exp\Big(-\frac{2}{3}t^{3/2}\Big)\Big(1+O(t^{-1})\Big), \qquad t\to+\infty \tag{1.7}$$

$$\mathrm{Ai}(t) = \frac{1}{\sqrt{\pi}}(-t)^{-1/4}\sin\Big(\frac{2}{3}(-t)^{3/2}+\frac{\pi}{4}\Big)+O\big(|t|^{-7/4}\big), \quad t\to-\infty \tag{1.8}$$

$$\mathrm{Bi}(t) = \frac{1}{\sqrt{\pi}}t^{-1/4}\exp\Big(\frac{2}{3}t^{3/2}\Big)\Big(1+O(t^{-1})\Big), \quad t\to+\infty \tag{1.9}$$

$$\mathrm{Bi}(t) = -\frac{1}{\sqrt{\pi}}(-t)^{-1/4}\sin\Big(\frac{2}{3}(-t)^{3/2}-\frac{\pi}{4}\Big)+O\big(|t|^{-7/4}\big), \quad t\to-\infty \tag{1.10}$$

These functions are linearly independent as we can observe that their Wronskian is

$$\mathcal{W}(\mathrm{Ai}(t),\mathrm{Bi}(t)) := \mathrm{Ai}(t)\mathrm{Bi}'(t) - \mathrm{Ai}'(t)\mathrm{Bi}(t) = \pi^{-1} \quad \forall t\in\mathbb{R}. \tag{1.11}$$

From the definition of $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$ and their Wronskian we can see that

$$\frac{d}{d\tau}\Big(\frac{\mathrm{Bi}(\tau)}{\mathrm{Ai}(\tau)}\Big) = \frac{\mathrm{Bi}'(\tau)\mathrm{Ai}(\tau)-\mathrm{Bi}(\tau)\mathrm{Ai}'(\tau)}{\mathrm{Ai}^2(\tau)} = \frac{1}{\pi\mathrm{Ai}^2(\tau)} \tag{1.12}$$

and, integrating between two values s, t yields

$$\mathrm{Bi}(s)\mathrm{Ai}^{-1}(s) - \mathrm{Bi}(t)\mathrm{Ai}^{-1}(t) = \frac{1}{\pi}\int_t^s \mathrm{Ai}^{-2}(\tau)d\tau. \tag{1.13}$$

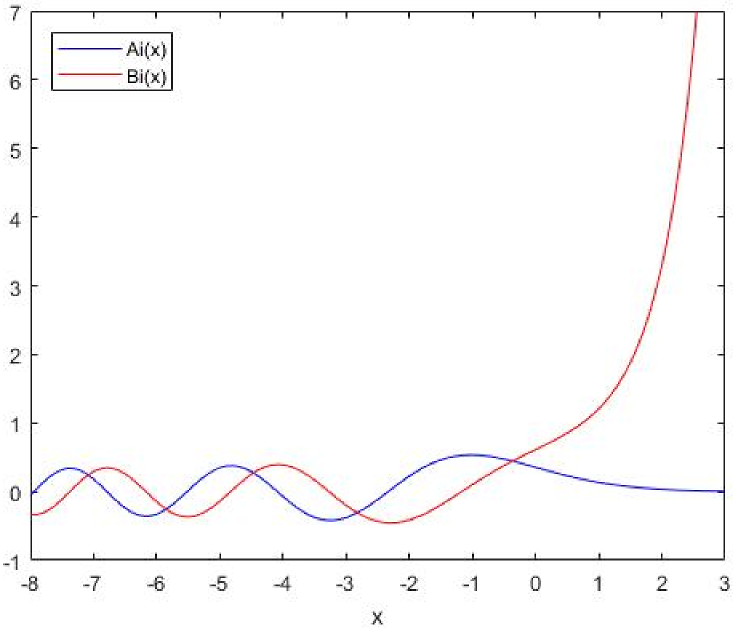


Figure 1.1: Airy functions

Definition 1.1. Now we define the following auxiliary functions $\xi_a(x;E)$ and $\xi_b(x;E)$.
Let $x_1 \in (a,b)$:

$$\xi_a(x) = \Big(\frac{3}{2}\int_x^a \sqrt{V(y)-E}\,dy\Big)^{2/3}, \quad x \le a \tag{1.14}$$

$$\xi_a(x) = -\Big(\frac{3}{2}\int_a^x \sqrt{E-V(y)}\,dy\Big)^{2/3}, \quad a \le x \le x_1. \tag{1.15}$$

Similarly let $x_0 \in (a,b)$ and $x_0 < x_1$:

$$\xi_b(x) = \Big(\frac{3}{2}\int_b^x \sqrt{V(y)-E}\,dy\Big)^{2/3}, \quad x \ge b \tag{1.16}$$

$$\xi_b(x) = -\Big(\frac{3}{2}\int_x^b \sqrt{E-V(y)}\,dy\Big)^{2/3}, \quad x_0 \le x \le b. \tag{1.17}$$

Please note that both $\xi_a(x)$ and $\xi_b(x)$ are positive when x is outside the potential well and negative otherwise. Moreover the intersection of their domains is the interval $[x_0, x_1] \subseteq (a,b)$.

These functions have the following properties:

Lemma 1.0.1. *$\xi_a(x) \in C^3(-\infty, x_1)$, $\xi_b(x) \in C^3(x_0, +\infty)$ and $\xi_{a,b}(x) \to +\infty$ as $x \to \pm\infty$.*
The derivatives are such that

$$\xi_a'(x) < 0, \quad \xi_b'(x) > 0 \tag{1.18}$$

for every x in their respective domain and

$$\xi_a'(a) = V'(a)^{1/3} \tag{1.19}$$
$$\xi_b'(b) = V'(b)^{1/3}. \tag{1.20}$$

Furthermore the functions $\xi_{a,b}(x)$ satisfy the equation

$$\xi_{a,b}'(x)^2 \xi_{a,b}(x) = V(x) - E.$$

Proof. By the Taylor theorem we can expand $V(x) - E$ near b:

$$V(x) - E = \alpha x + O(x^2) \tag{1.21}$$

where obviously $\alpha = V'(b)$. Then

$$\xi_b(x) = \Big(\frac{3}{2}\int_b^x \sqrt{\alpha t + O(t^2)}\,dt\Big)^{2/3} \sim \Big(\Big[\sqrt{\alpha} t^{3/2}\Big]_b^x\Big)^{2/3} \sim \alpha^{1/3} x. \tag{1.22}$$

Analogously the same holds in a neighbourhood of a.
The last identity can be easily proved noting that

$$\xi_{a,b}'(x) = \Big(\pm\frac{3}{2}\int_{a,b}^x \sqrt{|V(x) - E| dy}\Big)^{-1/3} \sqrt{|V(x) - E|}. \tag{1.23}$$

□

Remark 1. Moreover, since $\xi_{a,b}'(x) \neq 0$ and $\xi_{a,b} \in C^3$, the functions

$$p_{a,b}(x) = \big(|\xi_{a,b}'(x)|^{-1/2}\big)'' |\xi_{a,b}'(x)|^{-3/2}$$

are continuous.

Remark 2. By means of a Taylor expansion of the potential function near the inversion point b we can give a Taylor expansion of the function $\xi_b(x)$

with the same holding in a neighbourhood of a for the function $\xi_a(x)$:

$$\begin{aligned}
&E - V(x) = c|x-b| + O\big((x-b)^2\big) \Rightarrow \\
&\sqrt{E-V(x)} = \sqrt{c|x-b|}\big(1+O(|x-b|)\big) \Rightarrow \\
&\Big(\int_b^x \sqrt{c|t-b|}\big(1+O(|t-b|)\big)\,dt\Big)^{2/3} = \\
&= \Big(\Big[c_1|t-b|^{3/2}\Big]_b^x + O(|x-b|^{5/2})\Big)^{2/3} = \\
&= \Big(c_2|x-b|^{3/2} + O(|x-b|^{5/2})\Big)^{2/3} \Rightarrow \\
&|\xi_b(x)| = c_3|x-b| + O(|x-b|^{5/3}).
\end{aligned} \tag{1.24}$$

The scalar Schrödinger equation is a second order ordinary differential equation and every solution is a linear combination of two linearly independent functions. We will build a basis of solutions separately on the intervals $[x_0,+\infty)$ and $(-\infty,x_1]$ and will name them $u_R^-(x)$ and $u_R^+(x)$ in $[x_0,+\infty)$ and $u_L^-(x)$ and $u_L^+(x)$ on $(-\infty,x_1]$. The subscripts "R" and "L" stands for "right" and "left". This construction implies that in the interval $[x_0,x_1]\subseteq[a,b]$ all these four functions are defined and, in particular, $u_L^\pm(x)$ are linear combination of $u_R^\pm(x)$ and vice versa since, as already stated, both these couples of functions are a basis of solutions.

Theorem 1.0.2. (See [Y], Thm. 2.5 and [FMW1], Prop. A.2.) *The equation (1.5) admits two linearly independent solutions $u_R^\pm(x)$ on $[x_0,+\infty)$ such that, for $x\to+\infty$ and uniformly with respect to $h>0$:*

$$u_R^\pm(x) = \frac{h^{1/6}}{\sqrt{\pi}}(V(x)-E)^{-1/4}\exp\Big(\pm h^{-1}\int_b^x \sqrt{V(y)-E}\,dy\Big)\big(1+O(h)\big). \tag{1.25}$$

while on $[x_0,b)$ they are

$$\begin{aligned}
u_R^\pm(x) =& \frac{h^{1/6}}{\sqrt{\pi}}(E-V(x))^{-1/4}\sin\Big(h^{-1}\int_x^b \sqrt{E-V(y)}\,dy \mp \frac{\pi}{4}\Big) \\
&+ O(h^{7/6}).
\end{aligned} \tag{1.26}$$

Asymptotically as $h\to 0$ we have

$$\begin{aligned}
u_R^-(x) =& 2(\xi_b'(x))^{-1/2}\,\mathrm{Ai}(h^{-2/3}\xi_b(x))\big(1+O(h)\big), \\
u_R^+(x) =& 2(\xi_b'(x))^{-1/2}\,\mathrm{Bi}(h^{-2/3}\xi_b(x))\big(1+O(h)\big).
\end{aligned} \tag{1.27}$$

Conversely, on $(-\infty, x_1]$, the equation (1.5) admits two linearly independent solutions $u_L^{\pm}(x)$ such that, for $x \to -\infty$ and uniformly with respect to $h > 0$:

$$u_L^{\pm}(x) = \frac{h^{1/6}}{\sqrt{\pi}}(V(x)-E)^{-1/4} \exp\Big(\mp h^{-1} \int_a^x \sqrt{V(y)-E}\, dy\Big)\big(1+O(h)\big) \tag{1.28}$$

while on $(a, x_1]$ they are

$$\begin{aligned} u_L^{\pm}(x) =& \frac{h^{1/6}}{\sqrt{\pi}}(E-V(x))^{-1/4} \sin\Big(h^{-1}\int_a^x \sqrt{E-V(y)}\, dy \mp \frac{\pi}{4}\Big) \\ &+ O(h^{7/6}). \end{aligned} \tag{1.29}$$

Asymptotically as $h \to 0$:

$$\begin{aligned} u_L^{-}(x) =& 2(\xi_a'(x))^{-1/2} \operatorname{Ai}(h^{-2/3}\xi_a(x))\big(1+O(h)\big), \\ u_L^{+}(x) =& 2(\xi_a'(x))^{-1/2} \operatorname{Bi}(h^{-2/3}\xi_a(x))\big(1+O(h)\big). \end{aligned} \tag{1.30}$$

1.1 Norm of the solutions $u_{R,L}^{-}(x)$

Before proceeding to the proof of the theorem we are going to estimate the L^2-norm of the function $u_R^-(x)$ since for $u_L^-(x)$ the same reasoning holds. The functions $u_{R,L}^+(x)$ do not belong to L^2 in their domain because they have an exponential growing behaviour.

Proposition 1.1.1.

$$||u_{R,L}^{-}(x)||_{L^2} = O(h^{1/6}). \tag{1.31}$$

Proof. In order to do so we are splitting the interval $[x_0, +\infty)$ into the union of $[x_0, b-ch^{2/3}]$, $[b-ch^{2/3}, b+ch^{2/3}]$ and $[b+ch^{2/3}, +\infty)$ for some fixed $c > 0$ so that the norm is

$$\begin{aligned} ||u_R^-(x)||^2_{L^2([x_0,+\infty))} =& \int_{x_0}^{b-ch^{2/3}} u_R^-(x)^2\, dx + \int_{|x-b|\le ch^{2/3}} u_R^-(x)^2\, dx + \\ &+ \int_{b+ch^{2/3}}^{+\infty} u_R^-(x)^2\, dx. \end{aligned} \tag{1.32}$$

The function in the first integral is explicitly written as

$$u_R^-(x)^2 = \pi^{-1}h^{1/3}\big(E-V(x)\big)^{-1/2} \sin^2\Big(h^{-1}\int_x^b \sqrt{E-V(y)}\, dy + \frac{\pi}{4}\Big) + O(h^{4/3}) \tag{1.33}$$

and, since $E - V(x)$ and the sine are bounded and the former non-vanishing in that interval, all the integral is $O(h^{1/3})$.
The second integral, since we can consider $\xi_b(x) \sim x$, is

$$\int_{|x-b|\leq ch^{2/3}} |\mathrm{Ai}(h^{-2/3}x)|^2\, dx = \int_{|y|\leq c} |\mathrm{Ai}(y)|^2 h^{2/3}\, dy = O(h^{2/3}) \tag{1.34}$$

because $\mathrm{Ai}(y)$ is in L^2_{loc} after having set $y = h^{-2/3}x$.
The last integral is again $O(h^{2/3})$ using the same change of variables as above and noting that $\mathrm{Ai}^2(y)$ converges in a neighbourhood of $+\infty$ since is $O\big((V(x) - E)^{-1/4} \exp\big(-\frac{2}{3}y^{3/2}\big)\big)$.
Gathering the integrals yields:

$$||u^-_{R,L}(x)||_{L^2} = \big(O(h^{1/3}) + O(h^{2/3})\big)^{1/2} = O(h^{1/6}). \tag{1.35}$$

□

1.2 Proof of Theorem 1.0.2

We will first treat the cases $u^\pm_R(x)$ referring to the interval $[x_0, +\infty)$.

Let us make the change of variables

$$t := h^{-2/3}\xi_b(x) \tag{1.36}$$

$$f(t) := \xi_b'(x)^{1/2}u(x) \tag{1.37}$$

so that the Schrödinger equation becomes

$$-f''(t) + tf(t) = R(t)f(t) \tag{1.38}$$

where

$$R(t) = h^{4/3}p_b(h^{2/3}t) \tag{1.39}$$

and

$$p_b(x) = \big(|\xi_b'(x)|^{-1/2}\big)''|\xi_b'(x)|^{-3/2} = O\big((1 + |\xi_b(x)|)^{-2}\big) \tag{1.40}$$

already defined soon after Lemma 1.0.1.
The last estimate can be proved remembering that $\xi_b(x) = O(\xi_b(x)^{-1/2}|V(x) - E|^{1/2})$. Substituting into the definition of $p_b(x)$ yields

$$p(x) = -\frac{5}{16}\xi_b(x)^{-2} + \xi_b(x)\Big(\frac{5}{16}|V(x) - E|^{-3}\big(V'(x)\big)^2 - |V(x) - E|^{-2}V''(x)\Big). \tag{1.41}$$

By hypothesis both $|V(x)-E|^{-3}$ and $|V(x)-E|^{-3}$ are bounded and either $(V'(x))^2$ and $V''(x)$ are $O(|x|^{-2})$. Moreover, since $V(x)=O(1)$,

$$\xi_b(x) = \Big(\frac{3}{2}\int_b^x \sqrt{|c|}\,dt\Big)^{2/3} = O(x^{2/3}) \tag{1.42}$$

and then

$$|p(x)| \sim |\xi_b(x)|^{-2} + |\xi_b(x)||x|^{-2} \tag{1.43}$$

which is $O\big((1+|\xi_b(x)|)^{-2}\big)$ because the ratio

$$\frac{|\xi_b(x)|^{-2} + |\xi_b(x)||x|^{-2}}{|\xi_b(x)|^{-2}} = 1 + |\xi_b(x)^3||x|^{-2} = 1 + O(1). \tag{1.44}$$

Hence we can reduce this to a Volterra integral equation by setting the integral kernel

$$K(t,s) := -\pi\big(\mathrm{Ai}(t)\mathrm{Bi}(s) - \mathrm{Ai}(s)\mathrm{Bi}(t)\big) \tag{1.45}$$

Remark 3. Remember that a Volterra integral equation is in general defined as

$$F(x) = G(x) + \int_\alpha^x K(x,y)F(y)dy \tag{1.46}$$

with $\alpha \in \mathbb{R}$, $F(x)$ being the unknown function and $G(x)$ given.

Thus our equation (1.38) is equivalent to

$$f(t) = \mathrm{Ai}(t) + \int_t^{+\infty} K(t,s)R(s)f(s)ds \tag{1.47}$$

since derivating it twice and using the definition of Airy functions and their Wronskian one has:

$$\begin{aligned} f''(t) =& \mathrm{Ai}''(t) - \pi\big(\mathrm{Ai}'(t)\mathrm{Bi}(t) - \mathrm{Ai}(t)\mathrm{Bi}'(t)\big)R(t)f(t) + \\ & - \pi\int_t^\infty \big(\mathrm{Ai}''(t)\mathrm{Bi}(s) - \mathrm{Ai}(s)\mathrm{Bi}''(t)\big)R(s)f(s)\,ds = \\ =& t\mathrm{Ai}(t) - R(t)f(t) + t\int_t^\infty K(t,s)R(s)f(s)\,ds = \\ =& tf(t) - R(t)f(t). \end{aligned} \tag{1.48}$$

We are treating separately the functions $u_R^-(x)$ and $u_R^+(x)$ and for each one we will consider the cases $t \geq 0$ (that is $x \geq b$) and $t \leq 0$ (corresponding to $x \in [x_0, b]$).

Defining the new function

$$g(t) := f(t)\mathrm{Ai}^{-1}(t) \tag{1.49}$$

and inserting it into (1.47) yields

$$\begin{aligned}\mathrm{Ai}(t)g(t) &= \mathrm{Ai}(t) + \int_t^\infty K(t,s)R(s)\mathrm{Ai}(s)g(s)ds \Rightarrow \\ g(t) &= 1 + \int_t^\infty \mathrm{Ai}^{-1}(t)K(t,s)\mathrm{Ai}(s)R(s)g(s)ds\end{aligned} \tag{1.50}$$

Now, posing the new kernel of the last integral equation as

$$L(t,s) := \mathrm{Ai}^{-1}(t)K(t,s)\mathrm{Ai}(s), \tag{1.51}$$

we can expand it using the definition of $K(t,s)$ and the relation (1.13) as

$$\begin{aligned}L(t,s) &= -\pi\mathrm{Ai}^{-1}(t)\Big(\mathrm{Ai}(t)Bi(s) - \mathrm{Ai}(s)Bi(t)\Big)\mathrm{Ai}(s) = \\ &= -\pi\Big(\mathrm{Ai}(s)Bi(s) - \mathrm{Ai}^{-1}(t)Bi(t)\mathrm{Ai}^2(s)\Big) = \\ &= -\pi\Big(\mathrm{Ai}^{-1}(s)Bi(s) - \mathrm{Ai}^{-1}(t)Bi(t)\Big)Ai^2(s) = \\ &= -\int_t^s \mathrm{Ai}^{-2}(\tau)d\tau\mathrm{Ai}^2(s).\end{aligned} \tag{1.52}$$

The new kernel $L(t,s)$ corresponds to an integral operator

$$\begin{aligned}L : C_b^0(\Gamma \cap \{t \geq 0\}) &\to C_b^0(\Gamma \cap \{t \geq 0\}) \\ g(t) &\mapsto \int_t^\infty L(t,s)R(s)g(s)ds\end{aligned} \tag{1.53}$$

where $\Gamma := \{t := h^{-2/3}\xi_b(x); x \in [x_0, +\infty)\}$ and in general

$$C_b^k(I) := \left\{u : I \to \mathbb{C} \text{ of class } C^k, I \subseteq \mathbb{R}; \sum_{0\leq j\leq k} \sup_{x\in I} |u^{(j)}(x)| < +\infty\right\} \tag{1.54}$$

equipped with the norm

$$||u||_{C_b^k(I)} = \sum_{0\leq j\leq k} \sup_{x\in I} |u^{(j)}(x)|. \tag{1.55}$$

We want to give an estimate of its norm in order to solve equation (1.50) via the fixed-point theorem:

$$||L||_{\mathcal{L}(C_b^0,C_b^0)} = \sup_{t\in[0,+\infty)} \frac{|Lg(t)|}{|g(t)|} \tag{1.56}$$

where

$$\begin{aligned}|Lg(t)| &\leq \int_t^{+\infty} \left| \int_t^s \mathrm{Ai}^{-2}(\tau)d\tau \right| |\mathrm{Ai}^2(s)||R(s)||g(s)|ds \\ &\leq c \sup_{t\in[0,+\infty)} |g(t)| \int_t^{+\infty} \exp\big(\frac{4}{3}s^{\frac{3}{2}}\big)s^{-\frac{1}{2}} \exp\big(-\frac{4}{3}s^{\frac{3}{2}}\big)(1+h^{2/3}s)^{-2}h^{4/3}ds = \\ &= ch^{4/3} \sup_{t\in[0,+\infty)} |g(t)|O(h^{-1/3}) = \\ &= c \sup_{t\in[0,+\infty)} |g(t)|O(h).\end{aligned} \tag{1.57}$$

noting that the integral

$$\int_t^{+\infty} s^{-1/2}(1+h^{2/3}s)^{-2}ds = O(h^{-1/3}) \tag{1.58}$$

by the change of variables $y = 1 + h^{2/3}s$.
In this way $||L||_{\mathcal{L}(C_b^0,C_b^0)} = O(h)$ and the equation

$$g(t) = 1 + Lg(t) \tag{1.59}$$

can be solved by iterations:

$$\begin{aligned} g = 1 + Lg \quad &\Rightarrow \quad (1-L)g = 1 \quad \Rightarrow \\ & g = \sum_{k=0}^{+\infty} L^k(1) \end{aligned} \tag{1.60}$$

where each terms has norm

$$||L^k(1)||_{C_b^0} \leq ||L||^k_{C_b^0}||1||_{C_b^0} = O(h^k) \tag{1.61}$$

so that we can write $g(t)$ as

$$g(t) = 1 + L(1) + L^2(1) + O(L^3(1)) = 1 + O(h), \tag{1.62}$$

that is $f(t) = \mathrm{Ai}(t)(1 + O(h))$.

Please remember that all this costruction has been made in the case $t \geq 0$. We will now assume $t \leq 0$ that is, as already stated, $x \in [x_0, b]$.

Into the Volterra equation

$$f(t) = \mathrm{Ai}(t) + \int_t^{+\infty} K(t,s)R(s)f(s)ds$$

already defined we can split the integral into two parts having

$$f(t) = \mathrm{Ai}(t) + \int_0^{+\infty} K(t,s)R(s)f(s)ds + \int_t^0 K(t,s)R(s)f(s)ds. \quad (1.63)$$

and rename

$$\begin{aligned} f^{(1)}(t) &:= \int_0^{+\infty} K(t,s)R(s)f(s)ds \\ f^{(0)}(t) &:= \mathrm{Ai}(t) + f^{(1)}(t). \end{aligned} \quad (1.64)$$

Using the definition of $K(t,s)$ and (1.62) we have the estimate on $f^{(1)}(t)$

$$\begin{aligned} &\int_0^{+\infty} K(t,s)R(s)f(s)ds \le \\ &\le C\Big(|\mathrm{Ai}(t)| \int_0^{+\infty} \mathrm{Ai}(s)Bi(s)|R(s)|ds + |Bi(t)| \int_0^{+\infty} \mathrm{Ai}^2(s)|R(s)|ds\Big) \end{aligned} \quad (1.65)$$

and, taking into account that, from the definition of Airy functions for positive arguments one has

$$\begin{aligned} \mathrm{Ai}^2(t) + \mathrm{Ai}(t)\mathrm{Bi}(t) &\le c\Big(t^{-1/2}\exp\Big(-\frac{4}{3}t^{3/2}\Big) + t^{-1/2}\Big) \le \\ &\le c\big(1+t\big)^{-1/2} \end{aligned} \quad (1.66)$$

the estimate is

$$\begin{aligned} \int_0^{+\infty} K(t,s)R(s)f(s)ds \le &C_1(1+|t|)^{-1/4}h^{4/3}\int_0^{+\infty}(1+s)^{-1/2}(1+h^{2/3}s)^{-2}ds \le \\ \le &C_2(1+|t|)^{-1/4}h. \end{aligned} \quad (1.67)$$

having used the same calculation as in (1.58).
We are now solving the integral equation

$$f(t) = \mathrm{Ai}(t) + \int_t^0 K(t,s)R(s)f(s)ds \quad (1.68)$$

setting

$$g(t) := f(t)\mathrm{Ai}^{-1}(t) \quad (1.69)$$

in order solve the new integral equation by iterations

$$g(t) = 1 + \int_t^0 \mathrm{Ai}^{-1}(t)K(t,s)R(s)\mathrm{Ai}(s)g(s)\,ds. \quad (1.70)$$

The new kernel is bounded by

$$c(1+|t|)^{1/4}(1+|t|)^{-1/4}(1+|s|)^{-1/4}h^{4/3}(1+|s|)^{-1/4} = ch^{4/3}(1+|s|)^{-1/2}.$$

where $s \in [x_0, 0]$. Repeating the same calculations done when considering $t \geq 0$ yields

$$\begin{aligned} g(t) &= 1 + O\big(h^{4/3}(1+|s|)^{-1/2}\big) \Rightarrow \\ f(t) &= \mathrm{Ai}(t)\big(1 + O(h^{4/3}(1+|s|)^{-1/2})\big). \end{aligned} \tag{1.71}$$

In particular we can give another estimate on the remainder

$$\begin{aligned} |f(t) - f^{(0)}(t)| &\leq \int_t^0 |K(t,s)||R(s)||f(s)|ds \leq \\ &\leq C_1 \int_t^0 |K(t,s)||R(s)|(1+|s|)^{-1/4}ds \\ &\leq C_2 h^{4/3} \int_t^0 (1+|s|)^{-3/4}ds \leq \\ &\leq C_3 h^{4/3}(1+|t|)^{1/4}. \end{aligned} \tag{1.72}$$

where we have used that $f(s) \sim \mathrm{Ai}(s) \sim (1+|s|)^{-1/4}$ and that $|t| \geq |s| \Rightarrow (1+|t|)^{-1/4} \leq (1+|s|)^{-1/4}$.
In conclusion the first integral in the right-hand side of (1.63) tends to 0 with an order $(1+|t|)^{-1/4}h$ which dominates over the second integral having an order $(1+|t|)^{1/4}h^{4/3}$ leading to the solution

$$f(t) = \mathrm{Ai}(t) + O\big(h(1+|t|)^{-1/4}\big). \tag{1.73}$$

A very similar construction can be done to prove the representation formulae referred to the function $u^+(x)$ again defined only in the interval $[x_0, +\infty)$.
The only difference is that now we start from the integral equation

$$f(t) = \mathrm{Bi}(t) + \int_t^0 K(t,s)R(s)f(s)ds \tag{1.74}$$

that is itself equivalent to (1.38) using the definition of $\mathrm{Bi}(t)$ as in (1.48).

Let us first treat the case $t \geq 0$; in this case we define the new function as

$$g(t) = f(t)\mathrm{Bi}^{-1}(t) \tag{1.75}$$

and this time we obtain the integral equation

$$\begin{aligned}\mathrm{Bi}(t)g(t) &= \mathrm{Bi}(t) + \int_t^0 K(t,s)R(s)\mathrm{Bi}(s)g(s)ds \Rightarrow \\ g(t) &= 1 + \int_t^0 \mathrm{Bi}^{-1}(t)K(t,s)\mathrm{Bi}(s)R(s)g(s)ds = \\ &= 1 + \pi\int_0^t \Big(\frac{\mathrm{Ai}(t)}{\mathrm{Bi}(t)}\mathrm{Bi}^2(s) - \mathrm{Ai}(s)\mathrm{Bi}(s)\Big)R(s)g(s)ds = \\ &= 1 + Lg(t).\end{aligned} \tag{1.76}$$

The estimate on $||L||_{\mathcal{L}(C_b^0,C_b^0)}$ is obtained from

$$\begin{aligned}|Lg(t)| &\le \pi\int_0^t \left|\frac{\mathrm{Ai}(t)}{\mathrm{Bi}(t)}\mathrm{Bi}^2(s) - \mathrm{Ai}(s)\mathrm{Bi}(s)\right||R(s)||g(s)|ds \le \\ &\le c\pi h^{4/3}\sup_{t\in[0,+\infty)}|g(t)|\cdot \\ &\cdot\int_0^t \left(\left|\frac{\mathrm{Ai}(t)}{\mathrm{Bi}(t)}\mathrm{Bi}^2(s)\right| + \left|\mathrm{Ai}(s)\mathrm{Bi}(s)\right|\right)(1+h^{2/3}s)^{-2}ds.\end{aligned} \tag{1.77}$$

Now we fix a value T_0 and split the integral into the sum of two integrals referred to $t\in[0,T_0]$ and $t\in[T_0,+\infty)$ respectively.
The one in $[0,T_0]$ converges because all the functions involved are bounded in a bounded interval, while for the other one we have

$$\begin{aligned}&\int_{T_0}^t \left(\left|\frac{\mathrm{Ai}(t)}{\mathrm{Bi}(t)}\mathrm{Bi}^2(s)\right| + \left|\mathrm{Ai}(s)\mathrm{Bi}(s)\right|\right)(1+h^{2/3}s)^{-2}ds \le \\ &\le c\int_{T_0}^t \big(s^{-1/2}\exp\big(-\frac{4}{3}t^{3/2} - \frac{4}{3}s^{3/2}\big) + s^{-1/2}\big)(1+h^{2/3}s)^{-2}ds \le \\ &\le c_1\Big(\int_{T_0}^t s^{-1/2}\exp\big(-\frac{4}{3}s^{3/2}\big)(1+h^{2/3}s)^{-2}ds + \int_{T_0}^t s^{-1/2}(1+h^{2/3}s)^{-2}ds\Big).\end{aligned} \tag{1.78}$$

The first one converges, the second one is $O(h^{-1/3})$ as in (1.57); so that globally means

$$\begin{aligned}|Lg(t)| &\le ch^{4/3}\big(O(1)+O(1)+O(h^{-1/3})\big)\sup_{t\in[0,+\infty)}|g(t)| = \\ &= O(h)\sup_{t\in[0,+\infty)}|g(t)|,\end{aligned} \tag{1.79}$$

that is, trivially,

$$||L||_{\mathcal{L}(C_b^0,C_b^0)} = O(h). \tag{1.80}$$

The proof ends by iterations in the same way as the case $u^-(x)$ leading to

$$g(t) = 1 + O(h), \tag{1.81}$$

that is $f(t) = \mathrm{Bi}(t)(1 + O(h))$.

The only case missing is when $t \leq 0$ for which, as the previous case we write the integral equation with the new function $g(t) := f(t)\mathrm{Bi}^{-1}(t)$:

$$\begin{aligned} g(t) =& 1 + \pi \mathrm{Bi}^{-1}(t) \int_0^t \big(\mathrm{Ai}(t)\mathrm{Bi}(s) - \mathrm{Ai}(s)\mathrm{Bi}(t)\big) R(s)\mathrm{Bi}(s) g(s)\, ds = \\ =& 1 + \mathrm{Bi}^{-1}(t) L g(t) \end{aligned} \tag{1.82}$$

having the estimate

$$\begin{aligned} \mathrm{Bi}^{-1}(t)|Lg(t)| \leq& (1+|t|)^{1/4} h^{4/3} \sup_{s\in[t,0]} |g(s)| \int_0^t (1+|s|)^{-3/4}\, ds \leq \\ \leq& (1+|t|)^{1/2} h^{4/3} \sup_{s\in[t,0]} |g(s)| \end{aligned} \tag{1.83}$$

and since $t := h^{-2/3}\xi_b(x) = O(h^{-2/3})$ as $\xi_b(x)$ is bounded for $x \in [x_0, b]$,

$$(1+|t|)^{1/4}|Lg(t)| = O(h) \sup_{s\in[t,0]} |g(s)| \tag{1.84}$$

that allows us to solve (1.74) by iterations obtaining

$$f(t) = \mathrm{Bi}(t) + O(h(1+|t|)^{-1/4}). \tag{1.85}$$

By replacing the interval $[x_0, +\infty)$ with $(-\infty, x_1]$ so that one has to consider the function $\xi_a(x)$ instead of $\xi_b(x)$ and by making exactly the same calculations we can obtain the formulae (1.28) and (1.30) referred to $u_L^\pm(x)$. This concludes the proof of the theorem.

□

1.3 Quantization condition.

As already stated right before Thm 1.0.2, in the interval $[x_0, x_1]$ both the couples $u_L^\pm(x)$ and $u_R^\pm(x)$ are a basis of solutions for equation (1.5). It means that there exist four numbers $\alpha_\pm, \beta_\pm \in \mathbb{R}$ such that the two equations hold:

$$u_L^\pm(x) = \alpha_\pm u_R^-(x) + \beta_\pm u_R^+(x) \quad \forall x \in [x_0, x_1]. \tag{1.86}$$

Actually, since $u_R^{\pm}(x)$ are defined on $[x_0, +\infty)$ these relations let us extend the domain of $u_L^{\pm}(x)$ from $(-\infty, x_1]$ up to the whole set of real numbers. Naturally, the same holds considering $u_L^{+}(x)$ and $u_L^{-}(x)$ as a basis of solutions so that we can expand the domain of $u_R^{\pm}(x)$.

The scope of this section is to find values of the energy E for which there exist two proportional global solutions in L^2. This theorem refers to [Y], Theorem 4.1.

Theorem 1.3.1. *E is an eigenvalue of the operator $P := -h^2 \frac{d^2}{dx^2} + V(x)$ if and only if the wronskian $\mathcal{W}(u_L^-, u_R^-) = 0$*

Before proceeding to the proof we calculate the wronskian for every $x \in [x_0, x_1]$ and in order to do so we introduce the new functions

$$\varphi_a(x; E) := \int_{a(E)}^{x} \big(E - V(y)\big)^{1/2} dy \tag{1.87}$$

$$\varphi_b(x; E) := \int_{x}^{b(E)} \big(E - V(y)\big)^{1/2} dy \tag{1.88}$$

$$\phi(E) = \varphi_a(x; E) + \varphi_b(x; E) = \int_{a(E)}^{b(E)} \big(E - V(y)\big)^{1/2} dy. \tag{1.89}$$

The Wronskian is

$$\begin{aligned}
\mathcal{W}(u_L^-, u_R^-) =& u_L^-(x) \frac{d}{dx}\big(u_R^-(x)\big) - \frac{d}{dx}\big(u_L^-(x)\big) u_R^-(x) = \\
=& -\frac{h^{-2/3}}{\pi} \sin\Big(h^{-1}\varphi_a(x; E) + \frac{\pi}{4}\Big) \cos\Big(h^{-1}\varphi_b(x; E) + \frac{\pi}{4}\Big) + \\
& -\frac{h^{-2/3}}{\pi} \cos\Big(h^{-1}\varphi_a(x; E) + \frac{\pi}{4}\Big) \sin\Big(h^{-1}\varphi_b(x; E) + \frac{\pi}{4}\Big) + O(h^{1/3}) = \\
=& -\frac{h^{-2/3}}{\pi} \cos\big(h^{-1}\phi(E)\big) + O(h^{1/3})
\end{aligned} \tag{1.90}$$

and the relation $\mathcal{W}(u_L^-, u_R^-) = 0$ is equivalent to

$$\cos\big(h^{-1}\phi(E)\big) = O(h) \tag{1.91}$$

which is satisfied for any $E \in \mathbb{R}$ such that

$$\phi(E) = \int_{a(E)}^{b(E)} \big(E - V(y)\big)^{1/2} dy = \pi\Big(n + \frac{1}{2}\Big)h + O(h^2) \tag{1.92}$$

for some $n \in \mathbb{Z}$. This relation is famous in quantum mechanics as the *Bohr-Sommerfeld quantization condition.*

Thus, the following corollary holds:

Corollary 1.3.2. *E is an eigenvalue of the operator if and only if*

$$\phi(E) = \pi\Big(n + \frac{1}{2}\Big)h + O(h^2)$$

Proof of Theorem 1.3.1. Let us prove that for every number $\pi\big(n + \frac{1}{2}\big)h$ exists an eigenvalue E_n sufficiently near, that is the estimate

$$\big|\phi(E_n) - \pi\big(n + \frac{1}{2}\big)h\big| \leq Ch^2 \tag{1.93}$$

holds.
We can see that $\phi(E)$ is a one-to-one correspondence via the inverse function theorem:

$$\phi'(E) = \frac{1}{2}\int_{a(E)}^{b(E)} \big(E - V(y)\big)^{-1/2} dy > 0 \tag{1.94}$$

since by costruction $V(a) = V(b) = E$. Now we can set $\mu = \phi(E)$ and

$$\epsilon(\mu, h) := \pi h^{2/3}\mathcal{W}(u_L^-, u_R^-) - \sin\big(h^{-1}\mu + \frac{\pi}{2}\big) \tag{1.95}$$

which has the role of an error and, from the calculation of the wronskian, one has

$$|\epsilon(\mu, h)| \leq Ch. \tag{1.96}$$

We have to show that the equation

$$\pi^{-1}h^{2/3}w(\phi^{-1}(\mu), h) = \sin\big(h^{-1}\mu + \frac{\pi}{2}\big) + \epsilon(\mu, h) = 0 \tag{1.97}$$

has solutions $\mu_n := \phi(E_n)$ satisfying the estimate in (1.93) and in order to do it we make the change of variables

$$s := h^{-1}\mu + \frac{\pi}{2} \quad \iff \quad \mu = h\big(s - \frac{\pi}{2}\big) \tag{1.98}$$

such that the new solution $s_n(h)$ of the new equation

$$\sin s + \epsilon\Big(h\big(s - \frac{\pi}{2}\big), h\Big) = 0 \tag{1.99}$$

satisfy the estimate

$$
\begin{aligned}
&|h\big(s_n - \frac{\pi}{2}\big) - h\pi\big(n + \frac{1}{2}\big)| \leq Ch^2 \implies \\
&|s_n - \pi(n+1)| \leq Ch.
\end{aligned}
\tag{1.100}
$$

This is true because, by means of (1.96), equation (1.99) is equivalent to

$$\sin s = O(h), \tag{1.101}$$

that is the solution of $\sin s = 0$ differ from the points $\pi(n+1)$ of an order $O(h)$ and this is actually equation (1.100).

On the contrary, we have to show that for any $n \in \mathbb{Z}$ exists only one eigenvalue E_n of H close enough to it as stated in (1.93). That means that the solution of equation (1.99) satisfying (1.100) is unique for any n.
Let us suppose the contrary and observe that, since there exist infinite solutions of (1.99), there exist also a point $\tilde{s}$ different from any of them in which the derivative of the left hand side of (1.99) is null, that is

$$
\begin{aligned}
&h^{-1}\cos\tilde{s} + \frac{\partial\epsilon}{\partial\mu}\big(h(\tilde{s} + \frac{\pi}{2}), h\big) = 0 \implies \\
&\cos\tilde{s} = -h\frac{\partial\epsilon}{\partial\mu}\big(h(\tilde{s} + \frac{\pi}{2}), h\big).
\end{aligned}
\tag{1.102}
$$

Deriving the formula for the Wronskian in μ yields

$$\frac{\partial}{\partial\mu}\mathcal{W}(u_L^-, u_R^-) = -\frac{h^{-5/3}}{\pi}\cos\left(h^{-1}\mu + \frac{\pi}{2}\right) + O(h^{-2/3}). \tag{1.103}$$

Please note that the last term $O(h^{-2/3})$ appears because in (1.90) the terms $O(h^{1/3})$ actually depend on $h^{-1}\phi(E)$ and therefore on μ.
Hence we have, from (1.95),

$$\frac{\partial\epsilon}{\partial\mu}(\mu, h) = h^{-1}\cos\left(h^{-1}\mu + \frac{\pi}{2}\right) - h^{-1}\cos\left(h^{-1}\mu + \frac{\pi}{2}\right) + O(1). \tag{1.104}$$

The left-hand side of (1.102) is, given the previous estimate,

$$\cos\tilde{s} = \cos\big(\pi(n+1) + O(h)\big) = (-1)^{n+1} + O(h) \xrightarrow{h\to 0} (-1)^{n+1} \tag{1.105}$$

while the right-hand side is $O(h)$, so we have a contradiction.

□

Now, taking into account (1.86),

$$0 = \mathcal{W}(u_L^-, u_R^-) = \mathcal{W}(\alpha_- u_R^- + \beta_- u_R^+, u_R^-) = \beta_- \mathcal{W}(u_R^+, u_R^-) \tag{1.106}$$

and, being $u_R^+(x)$ and $u_R^-(x)$ linearly independent, that implies $\beta_- = 0$, that is, if E is an eigenvalue of P,

$$u_L^-(x) = \alpha_- u_R^-(x) \tag{1.107}$$

meaning that $u_L^-(x)$ and $u_R^-(x)$ are proportional solutions of the Schrödinger equation in $L^2(\mathbb{R})$.

Chapter 2

Intersection of potential functions above the energy level

We are now adding another potential function and treat the following 2×2 Schrödinger operator

$$P = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix} \tag{2.1}$$

in order to solve the system

$$Pu = Eu \qquad \Longleftrightarrow \qquad \begin{cases} (P_1 - E)u_1 = -hWu_2 \\ (P_2 - E)u_2 = -hW^*u_1. \end{cases} \tag{2.2}$$

with E close to some fixed value E_0.
In order to do so we will refer to [A] and extensively use the results in the previous chapter.
In particular P_j are the scalar Schrödinger operators

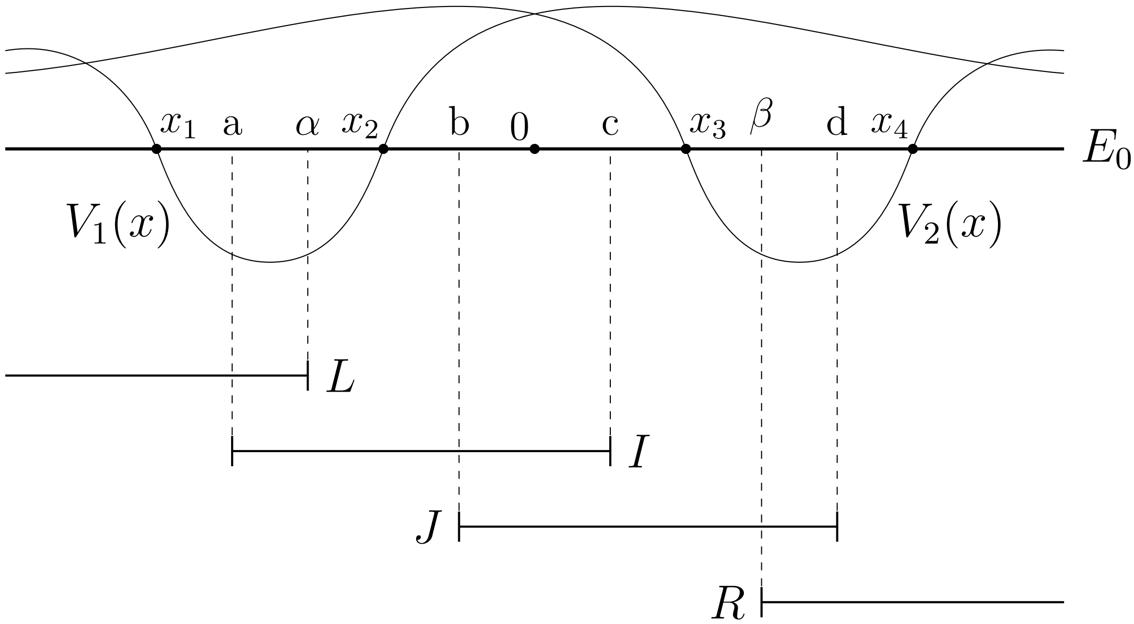
$$P_j = -h^2 \frac{d^2}{dx^2} + V_j(x) \tag{2.3}$$

and the off-diagonal part is made by a first order perturbation operator and its formal adjoint

$$W = W(x, h\partial_x) = r_0(x) + r_1(x)h\frac{d}{dx} \tag{2.4}$$

$$W^* = W^*(x, h\partial_x) = \overline{r_0(x)} - \overline{r_1(x)}h\frac{d}{dx} - \overline{r_1'(x)} \tag{2.5}$$

where $r_0(x)$ and $r_1(x)$ are bounded and analytic functions. In the case considered now the two potential wells are disjoint. It leads us to consider different intervals between them as illustrated in figure.



The intervals L and R (respectively for "left" and "right") span from an arbitrary point inside the wells, excluded x_2 for L and x_3 for R, towards $-\infty$ or $+\infty$. The intervals I and J have one extreme inside one well and the other in the interval (x_2, x_3) between them. We will also assume I and J to be closed.
The crucial features for these four intervals are that each of them must contain only one inversion point for the potential and must intersect two-by-two along an interval.

From the costruction made in Chapter 1 in the interval L only the functions $u^-_{1,L}(x)$ and $u^-_{2,L}(x)$ are well defined and in L^2. Simmetrically in the interval R we must consider only the functions $u^-_{1,R}(x)$ and $u^-_{2,R}(x)$.
In I and J, since they are bounded, all the four functions $u^-_{1,R}(x)$,$u^+_{1,R}(x)$,$u^-_{2,L}(x)$ and $u^+_{2,L}(x)$ must be taken in consideration as they are all in L^2.

In order to find solutions to the system we have to introduce new functions from the solutions already calculated in the first chapter:

$$v^+_{2,b} := e^{S_2/h} u^-_{2,L}, \quad v^-_{2,b} := e^{-S_2/h} u^+_{2,L} \tag{2.6}$$

$$v^+_{1,c} := e^{S_1/h} u^-_{1,R}, \quad v^-_{1,c} := e^{-S_1/h} u^+_{1,R} \tag{2.7}$$

with $S_1 := \int_{x_2}^{x_3} (V_1(t) - E)^{1/2}\, dt$ and $S_2 := \int_{x_2}^{x_3} (V_2(t) - E)^{1/2}\, dt$.

2.1 Fundamental operators on I

We are defining fundamental solutions of the operators $P_1 - E$ and $P_2 - E$ on the interval I.
The operators are

$$K_{1,I}, K'_{1,I}, K''_{1,I} : C(I) \to C^2(I) \tag{2.8}$$

for $P_1 - E$ and

$$K_{2,I} : C(I) \to C^2(I) \tag{2.9}$$

for $P_2 - E$ defined as, $\forall v \in C(I)$:

$$K_{1,I}(v)(x) := \\ = \frac{1}{h^2 \mathcal{W}(u^-_{1,R}, u^+_{1,R})} \Big(u^-_{1,R}(x) \int_a^x u^+_{1,R}(t) v(t)\, dt + u^+_{1,R}(x) \int_x^c u^-_{1,R}(t) v(t)\, dt \Big); \tag{2.10}$$

$$K'_{1,I}(v)(x) := \\ = \frac{1}{h^2 \mathcal{W}(u^-_{1,R}, u^+_{1,R})} \Big(u^-_{1,R}(x) \int_a^x u^+_{1,R}(t) v(t)\, dt - u^+_{1,R}(x) \int_a^x u^-_{1,R}(t) v(t)\, dt \Big); \tag{2.11}$$

$$K''_{1,I}(v)(x) := \\ = \frac{1}{h^2 \mathcal{W}(u^-_{1,R}, u^+_{1,R})} \Big(- u^-_{1,R}(x) \int_x^c u^+_{1,R}(t) v(t)\, dt + u^+_{1,R}(x) \int_x^c u^-_{1,R}(t) v(t)\, dt \Big); \tag{2.12}$$

$$K_{2,I}(v)(x) := \\ = \frac{1}{h^2 \mathcal{W}(v^-_{2,b}, v^+_{2,b})} \Big(v^-_{2,b}(x) \int_a^x v^+_{2,b}(t) v(t)\, dt + v^+_{2,b}(x) \int_x^c v^-_{2,b}(t) v(t)\, dt \Big). \tag{2.13}$$

Lemma 2.1.1. *These operators satisfy*

$$(P_1 - E) K_{1,I} = Id, \quad (P_1 - E) K'_{1,I} = Id, \quad (P_1 - E) K''_{1,I} = Id, \tag{2.14}$$

and

$$(P_2 - E) K_{2,I} = Id \tag{2.15}$$

where Id is the identity operator.

Proof. We are proving only the relations for $K_{1,I}$, the others being similar. Writing explicitly $(P_1-E)K_{1,I}f(x)$ one has

$$\begin{aligned}\big(h^2\frac{d^2}{dx^2}+V_1(x)-E\big)\Big(&\frac{u^-_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\int_a^x u^+_{1,R}(t)f(t)dt+\\&+\frac{u^+_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\int_x^c u^-_{1,R}(t)f(t)dt\Big)=\end{aligned}\tag{2.16}$$

$$\begin{aligned}=&h^2\frac{d^2}{dx^2}\Big(\frac{u^-_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\Big)\int_a^x u^+_{1,R}(t)f(t)dt+\\&+\Big(V_1(x)-E\Big)\frac{u^-_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\int_a^x u^+_{1,R}(t)f(t)dt+\\&+2h^2\frac{d}{dx}\Big(\frac{u^-_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\Big)u^+_{1,R}(x)f(x)+\\&+h^2\frac{u^-_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\Big(\frac{d}{dx}u^+_{1,R}(x)f(x)+u^+_{1,R}(x)f'(x)\Big)+\\&+h^2\frac{d^2}{dx^2}\Big(\frac{u^+_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\Big)\int_x^c u^-_{1,R}(t)f(t)dt+\\&+\Big(V_1(x)-E\Big)\frac{u^+_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\int_x^c u^-_{1,R}(t)f(t)dt+\\&-2h^2\frac{d}{dx}\Big(\frac{u^+_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\Big)u^-_{1,R}(x)f(x)+\\&+h^2\frac{u^+_{1,R}(x)}{h^2\mathcal{W}(u^-_{1,R},u^+_{1,R})}\Big(-\frac{d}{dx}u^-_{1,R}(x)f(x)-u^-_{1,R}(x)f'(x)\Big)=\\=&\frac{2}{\mathcal{W}(u^-_{1,R},u^+_{1,R})}\frac{d}{dx}\big(u^-_{1,R}(x)\big)u^+_{1,R}(x)f(x)+\\&+\frac{1}{\mathcal{W}(u^-_{1,R},u^+_{1,R})}u^-_{1,R}(x)\frac{d}{dx}\big(u^+_{1,R}(x)\big)f(x)+\\&-\frac{2}{\mathcal{W}(u^-_{1,R},u^+_{1,R})}\frac{d}{dx}\big(u^+_{1,R}(x)\big)u^-_{1,R}(x)f(x)+\\&-\frac{1}{\mathcal{W}(u^-_{1,R},u^+_{1,R})}u^+_{1,R}(x)\frac{d}{dx}\big(u^-_{1,R}(x)\big)f(x)=\\=&f(x)\frac{1}{\mathcal{W}(u^-_{1,R},u^+_{1,R})}\Big(\frac{d}{dx}\Big(u^+_{1,R}(x)\Big)u^-_{1,R}(x)-u^+_{1,R}(x)\frac{d}{dx}u^-_{1,R}(x)\Big)=f(x).\end{aligned}$$

having used that $u^{\pm}_{1,R}(x)$ are solutions of $P_1u=Eu$. $\square$

We are defining a new function space where to work: let $v(x,h)$ be a continuous function on I such that $v(x,h) \neq 0\, \forall x \in I$, we set

$$C(v,h) := \{u : I \to \mathbb{R}\, \text{continuous}\}$$

equipped with the norm

$$||u||_{C(v,h)} = \sup_{x\in I} \frac{|u(x)|}{|v(x,h)|}.$$

In the following lemma we are giving an estimate of the norm of these operators.

Lemma 2.1.2. (See [A], Lemma 4.1.)
In the limit as $h \to 0^+$ the following estimates hold:

$$||hK_{1,I}W||_{\mathcal{L}(C(u^{\pm}_{1,R},h))} = O(1), \tag{2.17}$$

$$||hK_{2,I}W^*||_{\mathcal{L}(C(u^{\pm}_{1,R},h))} = O(h^{5/6}), \tag{2.18}$$

$$||hK'_{1,I}W||_{\mathcal{L}(C(v^{+}_{2,b},h))} = O(h^{2/3}), \tag{2.19}$$

$$||hK''_{1,I}W||_{\mathcal{L}(C(v^{-}_{2,b},h))} = O(h^{1/2}), \tag{2.20}$$

$$||hK_{2,I}W^*||_{\mathcal{L}(C(v^{\pm}_{2,b},h))} = O(1). \tag{2.21}$$

Proof. We are adapting the proof in [A], Lemma 4.1, with the difference that here the variables x and t are intended to be $x-c$ and $t-c$. The estimates (2.18) and (2.19) are better than the corresponding ones in [A] because the interval I does not have 0 as extreme point.

In order to prove (2.17) we define the new functions

$$U_1(x,t) := |u^+_{1,R}(x)u^-_{1,R}(t)|\mathbf{1}_{\{t<x\}} + |u^-_{1,R}(x)u^+_{1,R}(t)|\mathbf{1}_{\{t>x\}} = U_1(t,x) \tag{2.22}$$

$$U'_1(x,t) := |u^+_{1,R}(x)h\partial_t u^-_{1,R}(t)|\mathbf{1}_{\{t<x\}} + |u^-_{1,R}(x)h\partial_t u^+_{1,R}(t)|\mathbf{1}_{\{t>x\}} \tag{2.23}$$

$$\tilde{U}_1(x,t) = U_1(x,t) + U'_1(x,t). \tag{2.24}$$

so that, from the definition of the operators and integrating by parts,

$$\begin{aligned}|hK_{1,I}Wv(x)| =& \Big|\frac{u^-_{1,R}(x)}{h\mathcal{W}(u^-_{1,R},u^+_{1,R})}\int_a^x u^+_{1,R}(t)\big(r_0(t)+hr_1(t)\partial_t\big)v(t)dt+\\ &+\frac{u^+_{1,R}(x)}{h\mathcal{W}(u^-_{1,R},u^+_{1,R})}\int_x^c u^-_{1,R}(t)\big(r_0(t)+hr_1(t)\partial_t\big)v(t)dt\Big| =\end{aligned}$$

$$\begin{aligned}
=&O(h^{-1/3})\Big|\int_a^x u^-_{1,R}(x)u^+_{1,R}(t)v(t)\,dt+\\
&-\int_a^x u^-_{1,R}(x)h\partial_t\big(u^+_{1,R}(t)\big)v(t)\,dt+hu^-_{1,R}(x)u^+_{1,R}(x)v(x)+\\
&-hu^-_{1,R}(x)u^+_{1,R}(a)v(a)+\int_x^c u^+_{1,R}(x)u^-_{1,R}(t)v(t)\,dt+\\
&-\int_x^c u^+_{1,R}(x)h\partial_t\big(u^-_{1,R}(t)\big)v(t)\,dt+hu^+_{1,R}(x)u^-_{1,R}(c)v(c)+\\
&-hu^+_{1,R}(x)u^-_{1,R}(x)v(x)\Big|=\\
=&O(h^{-1/3})\Big(\int_a^c \tilde{U}_1(x,t)|v(t)|\,dt+hU_1(x,a)|v(a)|+\\
&+hU_1(x,c)|v(c)|\Big).
\end{aligned}\tag{2.25}$$

Estimating (2.17) is equivalent to give evaluations for

$$\int_a^c \tilde{U}_1(x,t)|v(t)||u^\pm_{1,R}(x)|^{-1}\,dt,\tag{2.26}$$

$$hU_1(x,a)|v(a)||u^\pm_{1,R}(x)|^{-1}\quad\text{and}\quad hU_1(x,c)|v(c)||u^\pm_{1,R}(x)|^{-1}\tag{2.27}$$

as a consequence of the definition of the norm in $C(u^\pm_{1,R},h)$.
Let us remember the asymptotics of $u^+_{1,R}(x)$ and its derivative as found in the first chapter:

$$u^\pm_{1,R}(x)=\frac{h^{1/6}}{\sqrt{\pi}}\big(V_1(x)-E\big)^{-1/4}\exp\Big(\pm h^{-1}\int_a^x\sqrt{V_1(t)-E}\,dt\Big)\big(1+O(h)\big)\tag{2.28}$$

$$\begin{aligned}
h\partial_x u^\pm_{1,R}(x)=&\exp\Big(\pm h^{-1}\int_a^x\sqrt{V_1(t)-E}\,dt\Big)\cdot\\
&\cdot\Big(\pm\big(V_1(x)-E\big)^{1/4}\frac{h^{1/6}}{\sqrt{\pi}}+O(h^{7/6})\Big)\big(1+O(h)\big).
\end{aligned}\tag{2.29}$$

Considering the different possible values taken by x and t we have, having fixed a positive constant C_1:

- If $a\le x,t\le a+C_1h^{2/3}$:

$$\tilde{U}_1(x,t)|u^\pm_{1,R}(t)||u^\pm_{1,R}(x)|^{-1}=O(1)\tag{2.30}$$

because all the functions involved are bounded inside the well.

- If $a \le x \le a + C_1 h^{2/3} \le t \le c$:
$$\begin{aligned} &\tilde{U}_1(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1} = \\ &= O(h^{1/3})|V_1(t)-E|^{-1/2} \exp\Big(-\frac{1}{h}\int_x^t (V_1(s)-E)^{1/2} ds + \\ &\pm \frac{1}{h}\int_x^t (V_1(s)-E)^{1/2} ds\Big), \end{aligned} \tag{2.31}$$
and the same holds simmetrically exchanging x and t.
- If $a + C_1 h^{2/3} \le x, t \le c$:
$$\begin{aligned} &\tilde{U}_1(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1} = \\ &= O(h^{1/3})|V_1(t)-E|^{-1/2} \exp\Big(-\frac{1}{h}\int_x^t (V_1(s)-E)^{1/2} ds + \\ &\pm \frac{1}{h}\int_x^t (V_1(s)-E)^{1/2} ds\Big). \end{aligned} \tag{2.32}$$

These estimates show that uniformly with respect to $x, t \in [a, c]$,
$$\tilde{U}_1(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1} = O(1) \tag{2.33}$$
as $h \to 0^+$ and, in particular,
$$\big|\big|\tilde{U}_1(x,a)|v(a)|\big|\big|_{C(u^{\pm}_{1,R})} \le C\big|\big|v\big|\big|_{C(u^{\pm}_{1,R})} \tag{2.34}$$
$$\big|\big|\tilde{U}_1(x,c)|v(c)|\big|\big|_{C(u^{\pm}_{1,R})} \le C\big|\big|v\big|\big|_{C(u^{\pm}_{1,R})} \tag{2.35}$$
and this gives estimates of the two functions in (2.27) as $O(h)$ in $||\cdot||_{C(u^{\pm}_{1,R})}$. The integral in (2.26) is estimated as
$$\begin{aligned} \int_a^c \tilde{U}_1(x,t)|v(t)||u^{\pm}_{1,R}(x)|^{-1}\,dt =& O(1)\int_a^{a+C_1h^{2/3}} dt + \\ &+ O(h^{1/3})\int_{a+C_1h^{2/3}}^c (t-a)^{-1/2}\,dt = \\ =& O(h^{1/3}) \end{aligned} \tag{2.36}$$
where we used that in a bounded interval $\big(V_1(t)-E\big)^{-1/2} \sim (t-a)^{-1/2}$ and the integrals are:
$$\int_a^{a+C_1h^{2/3}} dt = O(h^{2/3}),$$
$$\int_{a+C_1h^{2/3}}^c (t-a)^{-1/2}\,dt = \Big[2(t-a)^{1/2}\Big]_{a+C_1h^{2/3}}^c = 2|c|^{1/2} - 2\sqrt{C_1}h^{1/3}.$$

Finally, from the last step in (2.25),

$$\big|\big|hK_{1,I}W\big|\big|_{\mathcal{L}(C(u^{\pm}_{1,R},h))} = O(h^{-1/3})\Big(O(h^{1/3}) + O(h) + O(h)\Big) = O(1). \quad (2.37)$$

Proceeding, we are proving (2.18). Please note that we are going to use the new functions in (2.6).
We are defining the functions

$$V_{2,b}(x,t) := |v^{-}_{2,b}(x)v^{+}_{2,b}(t)|\mathbf{1}_{\{t<x\}} + |v^{+}_{2,b}(x)v^{-}_{2,b}(t)|\mathbf{1}_{\{t>x\}} = V_{2,b}(t,x), \quad (2.38)$$

$$V'_{2,b}(x,t) := |v^{-}_{2,b}(x)h\partial_t v^{+}_{2,b}(t)|\mathbf{1}_{\{t<x\}} + |v^{+}_{2,b}(x)h\partial_t v^{-}_{2,b}(t)|\mathbf{1}_{\{t>x\}}, \quad (2.39)$$

$$\tilde{V}_{2,b}(x,t) = V_{2,b}(x,t) + V'_{2,b}(x,t) \quad (2.40)$$

and, from an integration by parts similar to the previous case one obtains

$$|hK_{2,I}W^{*}v(x)| = \\ = O(h^{-1/3})\Big(\int_a^c \tilde{V}_{2,b}(x,t)|v(t)|dt + h\tilde{V}_{2,b}(x,a)|v(a)| + h\tilde{V}_{2,b}(x,c)|v(c)|\Big). \quad (2.41)$$

Similarly as above we are giving different estimates of the functions involved based on the values of x and t from the solutions already found in the first chapter

$$u^{\pm}_{2,L}(x) = \frac{h^{1/6}}{\sqrt{\pi}}\big(V_2(x) - E\big)^{-1/4}\exp\big(\mp h^{-1}\int_0^x \sqrt{V_2(t) - E}\,dt\big)\big(1 + O(h)\big), \quad (2.42)$$

$$h\partial_x u^{\pm}_{2,L}(x) = \exp\big(\mp h^{-1}\int_0^x \sqrt{V_2(t) - E}\,dt\big)\cdot \\ \cdot\Big(\mp\big(V_2(x) - E\big)^{1/4}\frac{h^{1/6}}{\sqrt{\pi}} + O(h^{7/6})\Big)\big(1 + O(h)\big). \quad (2.43)$$

- If $a \le x, t \le a + C_1 h^{2/3}$:

$$\tilde{V}_{2,b}(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1} = \\ = O(h^{1/3})|V_2(x) - E|^{-1/4}|V_2(t) - E|^{-1/4}\exp\Big(-\frac{1}{h}\int_t^x (V_2(s) - E)^{1/2}ds\Big)\cdot \\ \cdot\frac{|V_1(x) - E|^{1/4}}{|V_1(t) - E|^{1/4}}\exp\Big(\mp\frac{1}{h}\int_t^x (V_1(s) - E)^{1/2}ds\Big) = \\ = O(h^{1/3})\exp\Big(-\frac{1}{h}\int_t^x (V_2(s) - E)^{1/2}ds\Big). \quad (2.44)$$

- If $a \le x \le a + C_1 h^{2/3} \le t \le c$:

$$\tilde{V}_{2,b}(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1} =$$
$$= O(h^{1/3})|V_2(x)-E|^{-1/4}|V_2(t)-E|^{-1/4}\exp\Big(-\frac{1}{h}\int_x^t (V_2(s)-E)^{1/2}ds\Big)\cdot$$
$$\cdot\frac{|V_1(x)-E|^{1/4}}{|V_1(t)-E|^{1/4}}\exp\Big(\mp\frac{1}{h}\int_t^x (V_1(s)-E)^{1/2}ds\Big) =$$
$$= O(h^{1/3})\frac{\exp\big(-\frac{1}{h}\int_x^t (V_2(s)-E)^{1/2}ds \mp \frac{1}{h}\int_t^x (V_1(s)-E)^{1/2}ds\big)}{|V_1(t)-E|^{1/4}}. \tag{2.45}$$

- If $a \le t \le a + C_1 h^{2/3} \le x \le c$:

$$\tilde{V}_{2,b}(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1} =$$
$$= O(h^{1/6})\exp\Big(-\frac{1}{h}\int_x^t (V_2(s)-E)^{1/2}ds \mp \frac{1}{h}\int_t^x (V_1(s)-E)^{1/2}ds\Big) \tag{2.46}$$

  making the same calculations as above and considering $v^{+}_{2,b}(t) = e^{S_2/h}u^{-}_{2,L}(t) = O(1)$.

- If $a + C_1 h^{2/3} \le x, t \le c$:

$$\tilde{V}_{2,b}(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1} =$$
$$= O(h^{1/3})\frac{\exp\big(-\frac{1}{h}\int_x^t (V_2(s)-E)^{1/2}ds \mp \frac{1}{h}\int_t^x (V_1(s)-E)^{1/2}ds\big)}{|V_1(t)-E|^{1/4}} \tag{2.47}$$

  like in the second case.

Every integral appearing in these cases converges and so, $\forall x,t \in [a,c]$

$$\tilde{V}_{2,b}(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1} = O(h^{1/6}) \tag{2.48}$$

and in particular

$$\big|\big|h\tilde{V}_{2,b}(x,a)|v(a)|\big|\big|_{C(u^{\pm}_{1,R},h)} = O(h^{7/6}), \tag{2.49}$$
$$\big|\big|h\tilde{V}_{2,b}(x,c)|v(c)|\big|\big|_{C(u^{\pm}_{1,R},h)} = O(h^{7/6}). \tag{2.50}$$

Now we are going to study $\int_a^c \tilde{V}_{2,b}(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1}dt$ dividing into three possibile cases for the value of x.

If $a \leq x \leq a + C_1 h^{2/3}$ there exist a constant $\alpha > 0$ and $\delta > 0$ such that

$$\begin{aligned}\int_a^c \tilde{V}_{2,b}(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1}dt =& O(h^{1/3})\int_a^{a+\delta} e^{-\alpha|x-t|/h}dt + O(e^{-\alpha/h}) = \\ =& O(h^{4/3})\end{aligned} \tag{2.51}$$

having used equations (2.44) and (2.45) and noting that the integral is

$$\int_a^{a+\delta} e^{-\alpha|x-t|/h}dt = \Big[-\frac{h}{\alpha}e^{-\alpha|x-t|/h}\Big]_a^{a+\delta} = O(h).$$

When $a + C_1 h^{2/3} < x \leq c - \delta'$ for some $\delta' > 0$ there exists a constant $\alpha > 0$ such that

$$\begin{aligned}\int_a^c \tilde{V}_{2,b}(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1}dt =& O(h^{1/6})\int_a^{\delta'/2} e^{-\alpha|x-t|/h}dt + O(e^{-\alpha/h}) = \\ =& O(h^{7/6})\end{aligned} \tag{2.52}$$

from equations (2.46) and (2.47).

When $c - \delta' < x \leq c$ there exist two constant $\alpha, \beta > 0$ such that:

$$\begin{aligned}\int_a^c \tilde{V}_{2,b}(x,t)|u^{\pm}_{1,R}(t)||u^{\pm}_{1,R}(x)|^{-1}dt =& O(h^{1/3})\int_{c-2\delta'}^{c} e^{-\beta|x-t|/h}dt + O(e^{-\alpha/h}) = \\ =& O(h^{4/3})\end{aligned} \tag{2.53}$$

because $\int_{c-2\delta'}^{c} e^{-\beta|x-t|/h}dt = O(h)$ simply calculating its antiderivative.
Returning back to (2.51), (2.52) and (2.53) we have

$$\Big|\Big|\int_a^c \tilde{V}_{2,b}(x,t)|v(t)|dt\Big|\Big|_{C(u^{\pm}_{1,R},h)} \leq Ch^{5/6}||v||_{C(u^{\pm}_{1,R},h)} \tag{2.54}$$

and globally, from (2.41):

$$||hK_{2,I}W^*||_{C(u^{\pm}_{1,R},h)} = O(h^{-1/3})\big(O(h^{5/6}) + O(h^{7/6})\big) = O(h^{1/2}). \tag{2.55}$$

The proof of (2.19) is similar to the previous one, with the important difference that the norm is referred to $v^{+}_{2,b} = e^{S_2/h}u^{-}_{2,L}$.
The usual integration by parts yields

$$\begin{aligned}&|hK'_{1,I}Wv(x)| = \\ &= O(h^{-1/3})\Big(\int_a^x \tilde{U}_1(x,t)|v(t)|dt + h\tilde{U}_1(x,x)|v(x)| + h\tilde{U}_1(x,a)|v(a)|\Big)\end{aligned} \tag{2.56}$$

where, analogously as the estimate for (2.17) and (2.18),

- If $a \le t \le x \le a + C_1 h^{2/3}$:

$$\tilde{U}_1(x,t)|v_{2,b}^+(t)||v_{2,b}^+(x)|^{-1} = O(1)\exp\Big(-\frac{1}{h}\int_t^x (V_2(s)-E)^{1/2}ds\Big). \tag{2.57}$$

- If $a \le t \le a + C_1 h^{2/3} \le x \le c$:

$$\tilde{U}_1(x,t)|v_{2,b}^+(t)||v_{2,b}^+(x)|^{-1} = \\ = O(h^{1/6})\frac{\exp\big(-\int_t^x (V_2(s)-E)^{1/2}ds/h - \int_a^x (V_1(s)-E)^{1/2}ds/h\big)}{|V_1(x)-E|^{1/4}} \tag{2.58}$$

where we considered $u_{1,L}^-(t)$ in $\tilde{U}_1(x,t)$ as $O(1)$, as well as $\frac{|V_2(x)-E|^{1/4}}{|V_2(t)-E|^{1/4}}$ from $|v_{2,b}^+(t)||v_{2,b}^+(x)|^{-1}$.

- If $a + C_1 h^{2/3} \le t \le x \le c$:

$$\tilde{U}_1(x,t)|v_{2,b}^+(t)||v_{2,b}^+(x)|^{-1} = \\ = O(h^{1/3})\frac{\exp\big(-\int_t^x (V_2(s)-E)^{1/2}ds/h - \int_t^x (V_1(s)-E)^{1/2}ds/h\big)}{|V_1(t)-E|^{1/4}|V_1(x)-E|^{1/4}}. \tag{2.59}$$

Globally that means that $\tilde{U}_1(x,t)|v_{2,b}^+(t)||v_{2,b}^+(x)|^{-1} = O(1)$ $\forall x \in [a,c]$, $\forall t \in [a,x]$ as $h \to 0^+$, that is

$$\big|\big|\tilde{U}_1(x,x)|v(x)|\big|\big|_{C(v_{2,b}^+,h)} \le C||v||_{C(v_{2,b}^+,h)} \tag{2.60}$$

$$\big|\big|\tilde{U}_1(x,a)|v(a)|\big|\big|_{C(v_{2,b}^+,h)} \le C||v||_{C(v_{2,b}^+,h)}. \tag{2.61}$$

For what it concerns $\int_a^x \tilde{U}_1(x,t)|v(t)|dt$, when $a \le x \le a + C_1 h^{2/3}$ or $a + C_1 h^{2/3} < x \le c - \delta'$ for some constant $\delta' > 0$, there exist a constant $\alpha > 0$ such that

$$\int_a^x \tilde{U}_1(x,t)|v_{2,b}^+(t)||v_{2,b}^+(x)|^{-1}dt \le O(1)\int_a^x e^{-\alpha(t-x)/h}dt = O(h) \tag{2.62}$$

from the first two cases above, while when $c - \delta' < x \le c$, there exists a positive constant β such that

$$\int_a^x \tilde{U}_1(x,t)|v_{2,b}^+(t)||v_{2,b}^+(x)|^{-1}dt = O(h^{1/3})\int_{c-2\delta'}^x e^{-\beta|x-t|/h}dt + O(e^{\alpha/h}) = \\ = O(h^{4/3}). \tag{2.63}$$

Hence we have

$$\Big|\Big|\int_a^x \tilde{U}_1(x,t)|v(t)|dt\Big|\Big|_{C(v^+_{2,b},h)} \leq Ch||v||_{C(v^+_{2,b},h)} \tag{2.64}$$

and, from (2.56)

$$||hK'_{1,I}W||_{C(v^+_{2,b},h)} = O(h^{2/3}). \tag{2.65}$$

We are now proving (2.20). Please note that this time the norm is referred to $v^-_{2,b} = e^{-S_2/h}u^+_{2,L}$. Once again we have

$$\begin{aligned} &|hK''_{1,I}Wv(x)| = \\ &= O(h^{-1/3})\Big(\int_x^c \tilde{U}_1(x,t)|v(t)|dt + h\tilde{U}_1(x,x)|v(x)| + h\tilde{U}_1(x,c)|v(c)|\Big) \end{aligned} \tag{2.66}$$

and the various estimate for the functions appearing are

- If $a \leq x \leq t \leq a + C_1h^{2/3}$:
$$\tilde{U}_1(x,t)|v^-_{2,b}(t)||v^-_{2,b}(x)|^{-1} = O(1)\exp\Big(-\frac{1}{h}\int_x^t (V_2(s)-E)^{1/2}ds\Big) \tag{2.67}$$
- If $a \leq x \leq a + C_1h^{2/3} \leq t \leq c$:
$$\begin{aligned} &\tilde{U}_1(x,t)|v^-_{2,b}(t)||v^-_{2,b}(x)|^{-1} = \\ &= O(h^{1/6})\frac{\exp\big(-\int_x^t (V_2(s)-E)^{1/2}ds/h - \int_a^t (V_1(s)-E)^{1/2}ds/h\big)}{|V_1(t)-E|^{1/4}} \end{aligned} \tag{2.68}$$
where $u^-_{1,L}(x) = O(1)$ in $\tilde{U}_1(x,t)$, similarly to $\frac{|V_2(x)-E|^{1/4}}{|V_2(t)-E|^{1/4}}$ from $|v^-_{2,b}(t)||v^-_{2,b}(x)|^{-1}$.
- If $a + C_1h^{2/3} \leq x \leq t \leq c$:
$$\begin{aligned} &\tilde{U}_1(x,t)|v^-_{2,b}(t)||v^-_{2,b}(x)|^{-1} = \\ &= O(h^{1/3})\frac{\exp\big(-\int_x^t (V_2(s)-E)^{1/2}ds/h - \int_x^t (V_1(s)-E)^{1/2}ds/h\big)}{|V_1(t)-E|^{1/4}|V_1(x)-E|^{1/4}}. \end{aligned} \tag{2.69}$$

having $\tilde{U}_1(x,t)|v^-_{2,b}(t)||v^-_{2,b}(x)|^{-1} = O(1)\ \forall t \in [a,c],\ \forall x \in [a,t]$ as $h \to 0^+$. That means

$$\Big|\Big|\tilde{U}_1(x,x)|v(x)|\Big|\Big|_{C(v^-_{2,b},h)} \leq C||v||_{C(v^-_{2,b},h)} \tag{2.70}$$

$$\Big|\Big|\tilde{U}_1(x,c)|v(c)|\Big|\Big|_{C(v^-_{2,b},h)} \leq C||v||_{C(v^-_{2,b},h)}. \tag{2.71}$$

For what it concerns $\int_x^c \tilde{U}_1(x,t)|v(t)|dt$, when $a \le x \le a + C_1 h^{2/3}$ there exists a constant $\delta > 0$ and $\alpha > 0$ such that

$$\int_x^c \tilde{U}_1(x,t)|v^-_{2,b}(t)||v^-_{2,b}(x)|^{-1}dt \le O(1)\int_x^{a+\delta} e^{-\alpha(t-x)/h}dt + O(e^{\alpha/h}) = O(h) \tag{2.72}$$

from the first two cases above.
Instead, when $a + C_1 h^{2/3} < x \le c - \delta'$, for some constant $\delta' > 0$ small enough there exists a positive constant α (possibly different from the previous one) such that

$$\int_x^c \tilde{U}_1(x,t)|v^-_{2,b}(t)||v^-_{2,b}(x)|^{-1}dt = O(1)\int_x^{c-\delta'/2} e^{-\alpha(t-x)/h}dt + O(e^{\alpha/h}) = O(h). \tag{2.73}$$

Finally, if $c - \delta' < x \le c$, there exist two positive constants α, β such that

$$\begin{aligned}\int_x^c \tilde{U}_1(x,t)|v^-_{2,b}(t)||v^-_{2,b}(x)|^{-1}dt =& O(h^{1/3})\int_x^c e^{-\beta|x^2-t^2|/h}dt + O(e^{\alpha/h}) = \\ =& O(h^{5/6}).\end{aligned} \tag{2.74}$$

Hence we have

$$\Big|\Big|\int_x^c \tilde{U}_1(x,t)|v(t)|dt\Big|\Big|_{C(v^-_{2,b},h)} \le Ch^{5/6}||v||_{C(v^-_{2,b},h)} \tag{2.75}$$

and, from (2.66)

$$||hK''_{1,I}W||_{C(v^-_{2,b},h)} = O(h^{-1/3})\big(O(h^{5/6}) + O(h)\big) = O(h^{1/2}). \tag{2.76}$$

At last, let us prove (2.21). In order to do so we have to estimate (2.41)

$$\begin{aligned}&|hK_{2,I}W^*v(x)| = \\ &= O(h^{-1/3})\Big(\int_a^c \tilde{V}_{2,b}(x,t)|v(t)|dt + h\tilde{V}_{2,b}(x,a)|v(a)| + h\tilde{V}_{2,b}(x,c)|v(c)|\Big).\end{aligned} \tag{2.77}$$

but referring to the norm centered in $v^{\pm}_{2,b}$.
It is sufficient to observe that, for every $x, t \in [a,c]$:

$$\begin{aligned}&\tilde{V}_{2,b}(x,t)|v^{\pm}_{2,b}(t)||v^{\pm}_{2,b}(x)|^{-1} = \\ &= O(h^{1/3})\exp\Big(-\frac{1}{h}\int_t^x \big(V_2(s)-E\big)^{1/2}ds \pm \frac{1}{h}\int_t^x \big(V_2(s)-E\big)^{1/2}ds\Big)\end{aligned} \tag{2.78}$$

so that

$$\left|\left|\tilde{V}_{2,b}(x,a)|v(a)|\right|\right|_{C(v^{\pm}_{2,b},h)} \le Ch^{1/3}||v||_{C(v^{\pm}_{2,b},h)} \tag{2.79}$$

$$\left|\left|\tilde{V}_{2,b}(x,c)|v(c)|\right|\right|_{C(v^{\pm}_{2,b},h)} \le Ch^{1/3}||v||_{C(v^{\pm}_{2,b},h)}. \tag{2.80}$$

The integral, instead, is calculated as, for every $a \le x \le c$,

$$\int_a^c \tilde{V}_{2,b}(x,t)|v^{\pm}_{2,b}(t)||v^{\pm}_{2,b}(x)|^{-1}dt = O(h^{1/3})\int_a^c dt = O(h^{1/3}). \tag{2.81}$$

The estimate of (2.21) is

$$||hK_{2,I}W^*||_{\mathcal{L}(C(v^{\pm}_{2,b},h))} = O(h^{-1/3})O(h^{1/3}) = O(1). \tag{2.82}$$

□

2.2 Fundamental operators on J

For what it concerns the interval $J = [b,d]$ there are little adjustments to be done with respect to the interval I.
The operators to be considered are this time:

$$K_{1,J}(v)(x) := \\ = \frac{1}{h^2\mathcal{W}(v^+_{1,c},v^-_{1,c})}\Big(v^+_{1,c}(x)\int_b^x v^-_{1,c}(t)v(t)\,dt + v^-_{1,c}(x)\int_x^d v^+_{1,c}(t)v(t)\,dt\Big) \tag{2.83}$$

$$K_{2,J}(v)(x) := \\ = \frac{1}{h^2\mathcal{W}(u^+_{2,L},u^-_{2,L})}\Big(u^+_{2,L}(x)\int_b^x u^-_{2,L}(t)v(t)\,dt + u^-_{2,L}(x)\int_x^d u^+_{2,L}(t)v(t)\,dt\Big) \tag{2.84}$$

$$K'_{2,J}(v)(x) := \\ = \frac{1}{h^2\mathcal{W}(u^+_{2,L},u^-_{2,L})}\Big(-u^+_{2,L}(x)\int_x^d u^-_{2,L}(t)v(t)\,dt + u^-_{2,L}(x)\int_x^d u^-_{2,L}(t)v(t)\,dt\Big) \tag{2.85}$$

$$K''_{2,J}(v)(x) := \\ = \frac{1}{h^2\mathcal{W}(u^+_{2,L},u^-_{2,L})}\Big(u^+_{2,L}(x)\int_b^x u^-_{2,L}(t)v(t)\,dt - u^-_{2,L}(x)\int_b^x u^+_{2,L}(t)v(t)\,dt\Big) \tag{2.86}$$

and, like before they map $C(J)$ in $C^2(J)$ and the analogous of lemma (2.1.1) holds:

Lemma 2.2.1.

$$(P_1 - E)K_{1,J} = Id, \quad (P_2 - E)K_{2,J} = Id,$$
$$(P_2 - E)K'_{2,J} = Id, \quad (P_2 - E)K''_{2,J} = Id.$$

and the norms of the operators that will be used to build solutions are

Lemma 2.2.2. (See [A], Lemma 4.2.)

$$||hK_{2,J}W^*||_{\mathcal{L}(C(u^{\pm}_{2,L},h))} = O(1), \tag{2.87}$$
$$||hK_{1,J}W||_{\mathcal{L}(C(u^{\pm}_{2,L},h))} = O(h^{5/6}), \tag{2.88}$$
$$||hK'_{2,J}W^*||_{\mathcal{L}(C(v^{\pm}_{1,c},h))} = O(h^{2/3}), \tag{2.89}$$
$$||hK''_{2,J}W^*||_{\mathcal{L}(C(v^{-}_{1,c},h))} = O(h^{1/2}), \tag{2.90}$$
$$||hK_{1,J}W||_{\mathcal{L}(C(v^{\pm}_{1,c},h))} = O(1) \tag{2.91}$$

as $h \to 0^+$.

2.3 Fundamental operators on L and R.

We are now defining integral opertors acting on functions defined on each interval. Starting from this operators we will be then able to build solutions of the system.
For this section we refer to [A], Section 4.3.
Let us define the new function spaces, $\forall\, k \in \mathbb{N}$:

$$C^k_b(L) = \Big\{u : L \to \mathbb{R} \text{ of class } C^k \big| \sum_{0\le j\le k} \sup_{x\in L} |u^{(j)}(x)| \le \infty\Big\};$$
$$C^k_b(R) = \Big\{u : R \to \mathbb{R} \text{ of class } C^k \big| \sum_{0\le j\le k} \sup_{x\in R} |u^{(j)}(x)| \le \infty\Big\}.$$

with the norms

$$||u||_{C^0_b(L,R)} = \sum_{0\le j\le k} \sup_{x\in L,R} |u^{(j)}(x)|.$$

The four operators to be considered are

$$K_{j,L} : C^0_b(L) \to C^2_b(L), \quad K_{j,L} : C^0_b(R) \to C^2_b(R),$$

$$K_{j,L}(v)(x) := \\ = \frac{1}{h^2\mathcal{W}(u^+_{j,L}, u^-_{j,L})}\Big(u^+_{j,L}(x)\int_{-\infty}^{x} u^-_{j,L}(t)v(t)\,dt + u^-_{j,L}(x)\int_{x}^{\alpha} u^+_{j,L}(t)v(t)\,dt\Big); \tag{2.92}$$

for $v \in C^0_b(L)$;

$$K_{j,R}(v)(x) := \\ = \frac{1}{h^2\mathcal{W}(u^-_{j,R}, u^+_{j,R})}\Big(u^-_{j,R}(x)\int_{\beta}^{x} u^+_{j,R}(t)v(t)\,dt + u^+_{j,R}(x)\int_{x}^{+\infty} u^-_{j,R}(t)v(t)\,dt\Big); \tag{2.93}$$

for $v \in C^0_b(R)$.

The following lemma holds:(see [A], Lemma 4.3)

Lemma 2.3.1. *For $h \to 0^+$ we have*

$$\big|\big|hK_{2,L}W^*\big|\big|_{\mathcal{L}(C^0_b(L))} = O(h); \tag{2.94}$$

$$\big|\big|hK_{1,L}W\big|\big|_{\mathcal{L}(C^0_b(L))} = O(h^{-1/6}); \tag{2.95}$$

and, for R: (see [A], Lemma 4.4)

Lemma 2.3.2. *For $h \to 0^+$ we have*

$$\big|\big|hK_{1,R}W\big|\big|_{\mathcal{L}(C^0_b(R)} = O(h); \tag{2.96}$$

$$\big|\big|hK_{2,R}W^*\big|\big|_{\mathcal{L}(C^0_b(R))} = O(h^{-1/6}); \tag{2.97}$$

Proof. We will only prove Lemma (2.3.1), the other being similar.
Setting

$$\tilde{u}^{\pm}_{j,L} := \max\{|u^{\pm}_{j,L}|, |h\partial_x u^{\pm}_{j,L}|\}$$
$$\mathcal{U}_{j,L}(x,t) := \tilde{u}^+_{j,L}(x)\tilde{u}^-_{j,L}(t)\mathbf{1}_{\{t<x\}} + \tilde{u}^-_{j,L}(x)\tilde{u}^+_{j,L}(t)\mathbf{1}_{\{t>x\}}$$

and integrating by parts we obtain

$$|hK_{1,L}Wv(x)| = O(h^{-1/3})\Big(\int_{-\infty}^{\alpha}\mathcal{U}_{1,L}(x,t)|v(t)\,dt + h\mathcal{U}_{1,L}(x,\alpha)|v(\alpha)|\Big) \tag{2.98}$$

and

$$|hK_{2,L}W^*v(x)| = O(h^{-1/3})\Big(\int_{-\infty}^{\alpha}\mathcal{U}_{2,L}(x,t)|v(t)\,dt + h\mathcal{U}_{2,L}(x,\alpha)|v(\alpha)|\Big). \tag{2.99}$$

To estimate $\mathcal{U}_{2,L}(x,t)$ we observe that for any $x, t < \alpha$,

$$\mathcal{U}_{2,L}(x,t) = O(h^{1/3})e^{-|\int_t^x (V_2(s)-E)^{1/2}ds|/h} = O(h^{1/3})$$

and, moreover there exists a constant $\gamma > 0$ such that

$$\int_{-\infty}^{\alpha} \mathcal{U}_{2,L}(x,t)|v(t)\,dt = O(h^{1/3})\int_{-\infty}^{\alpha} e^{-\gamma|x-t|/h}\,dt = O(h^{4/3}). \qquad (2.100)$$

That is,

$$\big|\big|hK_{2,L}W^*\big|\big|_{\mathcal{L}(C_b^0(L))} = O(h^{-1/3})O(h^{4/3}) = O(h). \qquad (2.101)$$

To estimate $\mathcal{U}_{1,L}(x,t)$, instead, we see that for any $\delta > 0$ small enough there exist (a different) $\gamma > 0$ such that

- If $|t - x_1| \le C_1 h^{2/3}$ or $\alpha - C_1 h^{2/3} \le t \le \alpha$, then for any $-\infty \le x \le \alpha$:
$$\mathcal{U}_{1,L}(x,t) = O(1);$$
- If $x_1 + C_1 h^{2/3} \le t \le \alpha - C_1 h^{2/3}$ then for any $-\infty \le x \le \alpha$:
$$\mathcal{U}_{1,L}(x,t) = O(h^{1/6})|t-\alpha|^{-1/4};$$
- If $x_1 - 2\delta \le t \le x_1 - C_1 h^{2/3}$ then for any $-\infty \le x \le \alpha$:
$$\mathcal{U}_{1,L}(x,t) = O(h^{1/6})|t-x_1|^{-1/4};$$
- If $t \le x_1 - 2\delta$ and $x \le x_1 - \delta$ then
$$\mathcal{U}_{1,L}(x,t) = O(h^{1/3})e^{-\gamma|t-x|/h};$$
- If $t \le x_1 - 2\delta$ and $x_1 - \delta \le x \le \alpha$ then
$$\mathcal{U}_{1,L}(x,t) = O(h^{1/6})e^{-\gamma|t-x_1+\delta|/h}.$$

Hence $\mathcal{U}_{1,L}(x,t) = O(1)$. For what it concerns the integral, when $x \le x_1 - \delta$:

$$\begin{aligned}\int_{-\infty}^{\alpha} \mathcal{U}_{1,L}(x,t)\,dt =& O(h^{1/3})\int_{-\infty}^{x_1-2\delta} e^{-\gamma|t-x|/h}\,dt + O(1)\int_{x_1-C_1h^{2/3}}^{x_1+C_1h^{2/3}} dt + \\ &+ O(1)\int_{\alpha-C_1h^{2/3}}^{\alpha} dt + O(h^{1/6})\int_{x_1+C_1h^{2/3}}^{\alpha-C_1h^{2/3}} |t-\alpha|^{-1/4}\,dt + \\ &+ O(h^{1/6})\int_{x_1-2\delta}^{x_1-C_1h^{2/3}} |t-x_1|^{-1/4}\,dt = \\ =& O(h^{1/3}) + O(h^{2/3}) + O(h^{2/3}) + O(h^{2/3}) + O(h^{1/6}) = \\ =& O(h^{1/6}).\end{aligned} \qquad (2.102)$$

When $x_1 - \delta \leq x \leq \alpha$ we have

$$\begin{aligned}\int_{-\infty}^{\alpha} \mathcal{U}_{1,L}(x,t)\,dt =& O(h^{1/6}) \int_{-\infty}^{x_1-2\delta} e^{-\gamma|t-x_1+\delta|/h}\,dt + O(1)\int_{x_1-C_1h^{2/3}}^{x_1+C_1h^{2/3}} dt + \\ & + O(1)\int_{\alpha-C_1h^{2/3}}^{\alpha} dt + O(h^{1/6})\int_{x_1+C_1h^{2/3}}^{\alpha-C_1h^{2/3}} |t-\alpha|^{-1/4}\,dt + \\ & + O(h^{1/6})\int_{x_1-2\delta}^{x_1-C_1h^{2/3}} |t-x_1|^{-1/4}\,dt = \\ =& O(h^{1/6}),\end{aligned} \tag{2.103}$$

so, from (2.99),

$$\big|\big|hK_{1,L}W\big|\big|_{\mathcal{L}(C_b^0(L))} = O(h^{-1/3})\big(O(h^{1/6}) + O(h)\big) = O(h^{-1/6}). \tag{2.104}$$

□

2.4 Solutions of the system on I

The purpose of the costruction done in the previous sections is to build solutions in each interval considered and to give conditions in order to have global solutions in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

We make use of new operators on the interval I

$$M_I := h^2 K_{2,I} W^* K_{1,I} W, \tag{2.105}$$
$$M_I' := h^2 K_{2,I} W^* K_{1,I}' W, \tag{2.106}$$
$$M_I'' := h^2 K_{2,I} W^* K_{1,I}'' W \tag{2.107}$$

which are respectively $O(h^{5/6})$, $O(h^{2/3})$ and $O(h^{1/2})$ when acting on $C(u_{1,R}^{\pm}, h)$, $C(v_{2,b}^{+}, h)$ and $C(v_{2,b}^{-}, h)$. The key fact is that they all tend to 0 as $h \to 0^+$.

The vector-valued functions

$$w_{1,I}^{\pm} := \begin{pmatrix} u_{1,R}^{\pm} + hK_{1,I}W \sum_{j\geq 0} M_I^j \big(hK_{2,I}W^* u_{1,R}^{\pm}\big) \\ -\sum_{j\geq 0} M_I^j \big(hK_{2,I}W^* u_{1,R}^{\pm}\big) \end{pmatrix}, \tag{2.108}$$

$$w_{2,I}^{+} := \begin{pmatrix} -hK_{1,I}'W \sum_{j\geq 0} (M_I')^j v_{2,b}^{+} \\ \sum_{j\geq 0} (M_I')^j v_{2,b}^{+} \end{pmatrix}, \tag{2.109}$$

$$w_{2,I}^{-} := \begin{pmatrix} -hK''_{1,I}W\sum_{j\geq 0}(M''_I)^j v_{2,b}^{-} \\ \sum_{j\geq 0}(M''_I)^j v_{2,b}^{-} \end{pmatrix}, \tag{2.110}$$

are solution of the system (2.2)

$$\begin{cases} (P_1 - E)u_1 = -hWu_2 \\ (P_2 - E)u_2 = -hW^*u_1. \end{cases}$$

From (2.14) the second equation is equivalent to

$$u_2 = (P_2 - E)^{-1}(-hW^*u_1) = -hK_{2,I}W^*u_1 \tag{2.111}$$

and so the first equation becomes

$$(P_1 - E)u_1 = h^2WK_{2,I}W^*u_1 \tag{2.112}$$

equivalent to

$$u_1 = (P_1 - E)^{-1}h^2WK_{2,I}W^*u_1 \quad \Longleftrightarrow \quad u_1 = h^2K_{1,I}WK_{2,I}W^*u_1 \tag{2.113}$$

which has solution of the type

$$u_1 = u_{1,R}^{\pm} + h^2K_{1,I}WK_{2,I}W^*u_1 = u_{1,R}^{\pm} + M_Iu_1. \tag{2.114}$$

The function (2.108) satisfies this relation since

$$\begin{aligned} u_{1,R}^{\pm} + M_Iu_1 =& u_{1,R}^{\pm} + M_I\big(u_{1,R}^{\pm} + hK_{1,I}W\sum_{j\geq 0}M_I^j\big(hK_{2,I}W^*u_{1,R}^{\pm}\big)\big) = \\ =& u_{1,R}^{\pm} + M_Iu_{1,R}^{\pm} + hK_{1,I}W\sum_{j\geq 1}M_I^j\big(hK_{2,I}W^*u_{1,R}^{\pm}\big)\big) = u_1 \end{aligned} \tag{2.115}$$

where u_1 is the first component of $w_{1,I}^{\pm}$.
For what it concerns $w_{2,I}^{+}$ (and $w_{2,I}^{-}$ changing $K'_{1,I}$ into $K''_{1,I}$ and M'_I into M''_I) we make a substitution from the other equation in order to have

$$u_1 = -hK'_{1,I}Wu_2 \quad \Rightarrow \quad (P_2 - E)u_2 = h^2W^*K'_{1,I}Wu_2 \tag{2.116}$$

where we used again the relations in (2.14), (2.15).
This last equation has solutions of the type

$$u_2 = v_{2,b}^{+} + M'_Iu_2 \tag{2.117}$$

and, posing $u_2 = \sum_{j\geq 0}(M'_I)^j v^+_{2,b}$, this satisfies

$$u_2 = \sum_{j\geq 0}(M'_I)^j v^+_{2,b} = v^+_{2,b} + M'_I \sum_{j\geq 0}(M'_I)^j v^+_{2,b} = v^+_{2,b} + M'_I u_2. \tag{2.118}$$

These solutions, thanks to the estimates given in the previous lemmas, tends to, as $h \to 0^+$,

$$w^{\pm}_{1,I} = \begin{pmatrix} u^{\pm}_{1,L} + O(h^{1/2}u^{\pm}_{1,R}) \\ O(h^{1/2}u^{\pm}_{1,L}) \end{pmatrix} \longrightarrow \begin{pmatrix} u^{\pm}_{1,R} \\ 0 \end{pmatrix}, \tag{2.119}$$

$$w^{\pm}_{2,I} = \begin{pmatrix} O(h^{1/2}v^{\pm}_{2,b}) \\ v^{\pm}_{2,b} + O(h^{1/2}v^{\pm}_{2,b}) \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ v^{\pm}_{2,b} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{\pm S_2/h}u^{\mp}_{2,L} \end{pmatrix}. \tag{2.120}$$

2.5 Solutions of the system on J

The same holds again for the interval J with the operators

$$M_J := h^2 K_{1,J} W K_{2,J} W^*, \tag{2.121}$$

$$M'_J := h^2 K_{1,J} W K'_{2,J} W^*, \tag{2.122}$$

$$M''_J := h^2 K_{1,J} W K''_{2,J} W^* \tag{2.123}$$

which are respectively $O(h^{5/6})$, $O(h^{2/3})$ and $O(h^{1/2})$ when acting on $C(u^{\pm}_{2,L}, h)$, $C(v^+_{1,c}, h)$ and $C(v^-_{1,c}, h)$. Once again, they all vanish for $h \to 0^+$.

Having defined these operators the solutions of the system (2.2) are

$$w^+_{1,J} := \begin{pmatrix} \sum_{j\geq 0}(M'_J)^j v^+_{1,c} \\ -hK'_{2,J}W^* \sum_{j\geq 0}(M'_J)^j v^+_{1,c} \end{pmatrix}, \tag{2.124}$$

$$w^-_{1,J} := \begin{pmatrix} \sum_{j\geq 0}(M''_J)^j v^-_{1,c} \\ -hK''_{2,J}W \sum_{j\geq 0}(M''_J)^j v^-_{1,c} \end{pmatrix}, \tag{2.125}$$

$$w^{\pm}_{2,J} := \begin{pmatrix} -\sum_{j\geq 0} M^j_J\big(hK_{1,J}Wu^{\pm}_{2,L}\big) \\ u^{\pm}_{2,L} + hK_{2,J}W \sum_{j\geq 0} M^j_J\big(hK_{1,J}Wu^{\pm}_{2,L}\big) \end{pmatrix}. \tag{2.126}$$

which, thanks to the several estimates done,

$$w^{\pm}_{1,J} \longrightarrow \begin{pmatrix} v^{\pm}_{1,c} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{\pm S_1/h}u^{\mp}_{1,R} \\ 0 \end{pmatrix}, \tag{2.127}$$

$$w^{\pm}_{2,J} \longrightarrow \begin{pmatrix} 0 \\ u^{\pm}_{2,L} \end{pmatrix}. \tag{2.128}$$

2.6 Solutions of the system in L, R

Similarly as the intervals I and J, the vector functions on L

$$w_{1,L} := \begin{pmatrix} \sum_{j\geq 0}(M_L)^j u^-_{1,L} \\ -hK_{2,L}W^* \sum_{j\geq 0}(M_J)^j u^-_{1,L} \end{pmatrix}, \tag{2.129}$$

$$w_{2,L} := \begin{pmatrix} -\sum_{j\geq 0} M_L^j \big(hK_{1,L}Wu^-_{2,L}\big) \\ u^-_{2,L} + hK_{2,L}W^* \sum_{j\geq 0} M_L^j \big(hK_{1,L}Wu^-_{2,L}\big) \end{pmatrix} \tag{2.130}$$

and on R

$$w_{1,R} := \begin{pmatrix} u^-_{1,R} + hK_{1,R}W \sum_{j\geq 0} M_R^j \big(hK_{2,R}W^* u^-_{1,R}\big) \\ -\sum_{j\geq 0} M_R^j \big(hK_{2,R}W^* u^-_{1,R}\big) \end{pmatrix}, \tag{2.131}$$

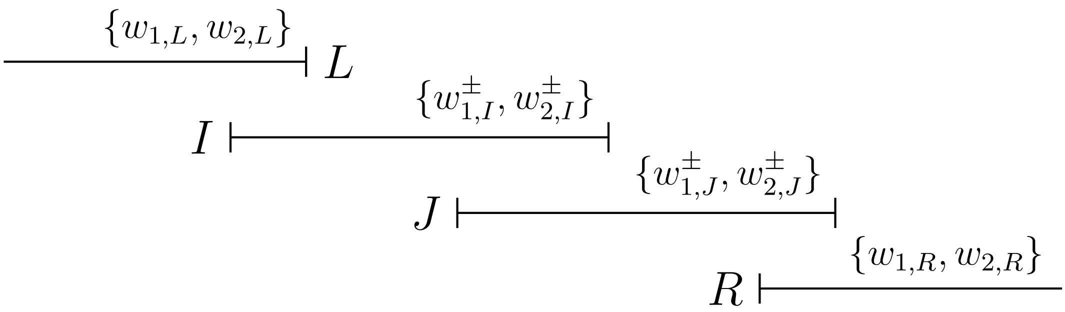
$$w_{2,R} := \begin{pmatrix} -hK_{1,R}W \sum_{j\geq 0}(M_R)^j u^-_{2,R} \\ \sum_{j\geq 0}(M_R)^j u^-_{2,R} \end{pmatrix}, \tag{2.132}$$

are solutions of the Schrödinger system (2.2).
Since the norm of the operators involved vanish for $h \to 0^+$, these solutions tend to

$$w_{1,L} \to \begin{pmatrix} u^-_{1,L} \\ 0 \end{pmatrix}; \quad w_{2,L} \to \begin{pmatrix} 0 \\ u^-_{2,L} \end{pmatrix};$$
$$w_{1,R} \to \begin{pmatrix} u^-_{1,R} \\ 0 \end{pmatrix}; \quad w_{2,R} \to \begin{pmatrix} 0 \\ u^-_{2,R} \end{pmatrix}.$$

2.7 Connection of the solutions

Let us come back to the beginning of the chapter. As stated there we have now two solutions in the intervals L, R respectively in $\big(L^2(L)\big)^2$ and $\big(L^2(R)\big)^2$, while in the middle intervals I, J the space of solutions of the system is four-dimensional, since these intervals are bounded.



Taken any of the functions $w_{k,L}$, $k = 1, 2$, it can be written, in the intersection $L \cap I$, as a linear combination of the four functions $w^{\pm}_{k,I}$ but, thanks to these linear combinations, $w_{k,L}$ is extended to the whole interval I. Now, analogously, any of the functions $w^{\pm}_{k,I}$ is written as a linear combination on $I \cap J$ of the function $w^{\pm}_{k,J}$ and we can extend $w^{\pm}_{k,I}$ and, as a fundamental consequence, $w_{k,L}$ to the whole interval J. At last, using that $w_{k,R}$ is a basis of solutions in R, we prolong $w^{\pm}_{k,J}$ from $J \cap R$ to R and so, transitively, we have defined $w_{k,L}$ not only on L but on $L \cup I \cup J \cup R = \mathbb{R}$.
The same can be done reversely starting from $w_{k,R}$ on R up to the interval L in order to have a global definition of $w_{k,R}$.

The condition to have a global solution in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is that the intersection between the space generated by $w_{k,L}$ and the space generated by $w_{k,R}$ must not be reduced only to the null function, that is there must exist four real numbers $\alpha, \beta, \gamma, \delta$, with $(\alpha, \beta, \gamma, \delta) \neq (0, 0, 0, 0)$, such that

$$\alpha w_{1,L} + \beta w_{2,L} = \gamma w_{1,R} + \delta w_{2,R}. \tag{2.133}$$

Obviously α and β are different from the extreme points of the intervals L and R.

2.8 Quantization condition

Proposition 2.8.1. *E is an eigenvalue of the operator P if and only if*

$$\cos\left(h^{-1}\phi_1(E)\right)\cos\left(h^{-1}\phi_2(E)\right) = O(h^{-1/3}). \tag{2.134}$$

Proof. The condition (2.133) is equivalent to ask the Wronskian of the solutions to be vanishing. It is defined as

$$\mathcal{W}(w_{1,L}, w_{2,L}, w_{1,R}, w_{1,R}) = \det\begin{pmatrix} w_{1,L} & w_{2,L} & w_{1,R} & w_{2,R} \\ w'_{1,L} & w'_{2,L} & w'_{1,R} & w'_{2,R} \end{pmatrix}. \tag{2.135}$$

This matrix is well defined and is 4×4. We will use the limit of the vector functions for $h \to 0$ and omit the vanishing parts. As it is explained in [FMW3], Section 4, derivating the operators reduce the exponent in their norm by 1. Namely, for what it concerns the operators in L (in R being similar):

$$||\partial_x M_L||_{\mathcal{L}(C_b^0(L))} = O(h^{-1/6}), \quad ||h\partial_x K_{2,L}W^*||_{\mathcal{L}(C_b^0(L))} = O(1), \tag{2.136}$$

$$||h\partial_x K_{1,L}W||_{\mathcal{L}(C_b^0(L))} = O(h^{-7/6}), \quad ||h^2\partial_x K_{2,L}W^*K_{1,L}W||_{\mathcal{L}(C_b^0(L))} = O(h^{-1/6}) \tag{2.137}$$

and so the Wronskian becomes

$$\mathcal{W}(w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R}) = \\ = \begin{vmatrix} u_{1,L}^- & 0 & u_{1,R}^- & 0 \\ 0 & u_{2,L}^- & 0 & u_{2,R}^- \\ (u_{1,L}^-)' + O(1) & O(h^{-1}) & (u_{1,R}^-)' + O(1) & O(h^{1/6}) \\ O(h^{1/6}) & (u_{2,L}^-)' + O(1) & O(h^{-1}) & (u_{2,R}^-)' + O(1) \end{vmatrix} \tag{2.138}$$

and by developing by Laplace rule with respect to the first row and considering $u_{j,S}^- = O(h^{1/6})$ and $(u_{j,S}^-)' = O(h^{-5/6})$ in 0, $j = 1,2$, $S = L, R$ we obtain

$$\mathcal{W}(w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R}) = \mathcal{W}\big(u_{1,L}^-, u_{1,R}^-\big)\mathcal{W}\big(u_{2,L}^-, u_{2,R}^-\big) + O(h^{-5/3}). \tag{2.139}$$

Both the Wronskians in the right-hand side have been calculated in the previous chapter being , $j = 1, 2$:

$$\mathcal{W}\big(u_{j,L}^-, u_{j,R}^-\big) = -\frac{h^{-2/3}}{\pi}\cos\big(h^{-1}\phi_j(E)\big) + O(h^{1/3}), \tag{2.140}$$

where

$$\phi_1(E) = \int_{x_1(E)}^{x_2(E)} \big(E - V_1(y)\big)^{1/2} dy, \qquad \phi_2(E) = \int_{x_3(E)}^{x_4(E)} \big(E - V_2(y)\big)^{1/2} dy.$$

Hence it becomes:

$$\mathcal{W}(w_{1,L}, w_{2,L}, w_{1,R}, w_{1,R}) = \\ = \pi^{-2}h^{-4/3}\cos\big(h^{-1}\phi_1(E)\big)\cos\big(h^{-1}\phi_2(E)\big) + O(h^{-1/3}) + O(h^{-5/3}) \tag{2.141}$$

Finally, posing it equal to 0 gives the equation (2.134).

□

Now, let us discuss the condition (2.134). Using the formulae for the sum of cosine functions the condition is equivalent to

$$\frac{1}{2}\left(\cos\left(\frac{\phi_1(E)+\phi_2(E)}{h}\right)+\cos\left(\frac{\phi_1(E)-\phi_2(E)}{h}\right)\right)=O(h^{-1/3}). \quad (2.142)$$

which is equivalent to

$$\cos\left(\frac{\phi_1(E)-\phi_2(E)}{h}\right)=-\cos\left(\frac{\phi_1(E)+\phi_2(E)}{h}\right)+O(h^{-1/3}). \quad (2.143)$$

When $\frac{\phi_1(E)-\phi_2(E)}{h}\neq 0,\pi \mod 2\pi$ the cosine is invertible, and so

$$\frac{\phi_1(E)-\phi_2(E)}{h}=\pm\left(\frac{\phi_1(E)+\phi_2(E)}{h}\right)+\pi+O(h^{-1/3}) \mod 2\pi \quad (2.144)$$

and when we consider the "$-$" sign the solution is

$$2\frac{\phi_1(E)}{h}=\pi+O(h^{-1/3})+2n\pi \implies$$

$$\phi_1(E)=\pi\left(\frac{1}{2}+n\right)h+O(h^{2/3}) \quad (2.145)$$

for $n\in\mathbb{Z}$ and analogously with the "+" sign:

$$\phi_2(E)=\pi\left(\frac{1}{2}+m\right)h+O(h^{2/3}) \quad (2.146)$$

with $m\in\mathbb{Z}$.
On the contrary, when $\frac{\phi_1(E)-\phi_2(E)}{h}$ is near 0 or π modulus 2π, we use the Taylor expansion of the cosine (renaiming $x=\frac{\phi_1(E)+\phi_2(E)}{h}$, $y=\frac{\phi_1(E)-\phi_2(E)}{h}$ and considering $O(h^{-1/3})=\alpha h^{-1/3}$, $\alpha\in\mathbb{R}$):

$$\cos x=\cos y+O(h^{-1/3}) \implies$$

$$1-\frac{x^2}{2}\sim 1-\frac{y^2}{2}+O(h^{-1/3}) \implies$$

$$x^2\sim y^2+\alpha h^{-1/3} \implies$$

$$\frac{\phi_1(E)-\phi_2(E)}{h}=\pm\left(\frac{\phi_1(E)+\phi_2(E)}{h}\right)+\pi+O(h^{-1/6}) \mod 2\pi \quad (2.147)$$

which is discussed as above giving the results

$$\phi_1(E)=\pi\left(\frac{1}{2}+n\right)h+O(h^{5/6}) \quad (2.148)$$

for the "−" sign and

$$\phi_2(E) = \pi\Big(\frac{1}{2} + m\Big)h + O(h^{5/6}) \tag{2.149}$$

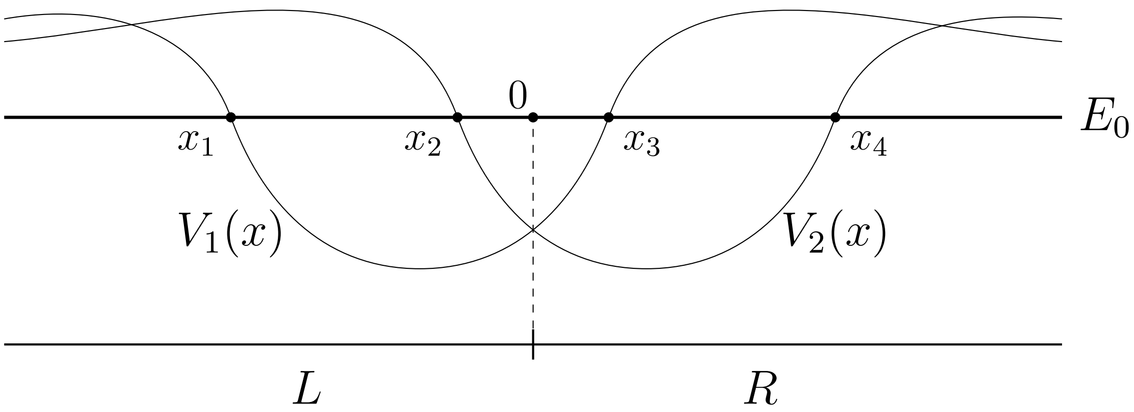
when considering the "+" sign.
Please note that the only difference from the previous case when the cosine was invertible is in the bigger exponent of h.

These are the *Bohr-Sommerfeld conditions* on the energy in order to be an eigenvalue of the system. As stated in the introduction the energy is *quantized* because its admissible values depend on integer numbers.

Chapter 3

Intersection of potential functions below the energy level

In the case treated in this chapter we are assuming the two potentials to have an intersection below the value of the energy. We are mainly following the work [FMW3].



We are considering only the two intervals $L := (-\infty, 0]$ and $R := [0, +\infty)$. Their intersection point 0 will be crucial in linking the solutions built separately in the two intervals.

The bidimensional Schrödinger operator is again

$$P = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix} \tag{3.1}$$

bringing to the 2×2 system

$$Pu = Eu \quad \Longleftrightarrow \quad \begin{cases} (P_1 - E)u_1 = -hWu_2 \\ (P_2 - E)u_2 = -hW^*u_1. \end{cases} \tag{3.2}$$

P_j are the scalar Schrödinger operators previously treated

$$P_j = -h^2 \frac{d^2}{dx^2} + V_j(x) \tag{3.3}$$

and the off-diagonal part is made by a first order perturbation operator and its formal adjoint

$$W = W(x, h\partial_x) = r_0(x) + r_1(x) h \frac{d}{dx} \tag{3.4}$$

$$W^* = W^*(x, h\partial_x) = \overline{r_0(x)} - \overline{r_1(x)} h \frac{d}{dx} - h\overline{r_1'(x)} \tag{3.5}$$

where $r_0(x)$ and $r_1(x)$ are bounded and analytic functions.

3.1 Fundamental operators on L

Given the spaces

$$C_b^k(L) = \left\{ u : (-\infty, 0] \to \mathbb{R}; \sum_{j=0}^{k} \sup_{x \le 0} |u^{(j)}(x)| \right\} \tag{3.6}$$

equipped with the norm

$$||u||_{C_b^k} = \sum_{j=0}^{k} \sup_{x \le 0} |u^{(j)}(x)|. \tag{3.7}$$

we can define two fundamental operators $K_{1,L}$, $K_{2,L}$ on $L = (-\infty, 0]$ as

$$K_{j,L} : C_b^0(L) \to C_b^2(L) \tag{3.8}$$

$$K_{j,L}[v](x) = \frac{u_{j,L}^+(x)}{h^2 \mathcal{W}_{j,L}} \int_{-\infty}^{x} u_{j,L}^-(t) v(t) dt + \frac{u_{j,L}^-(x)}{h^2 \mathcal{W}_{j,L}} \int_{x}^{0} u_{j,L}^+(t) v(t) dt, \tag{3.9}$$

where by $\mathcal{W}_{j,L}$ we have denoted $\mathcal{W}(u^-_{j,L}, u^+_{j,L})$ which is

$$\mathcal{W}(u^-_{j,L}, u^+_{j,L}) = \frac{-2}{\pi h^{-2/3}}(1 + O(h)) \tag{3.10}$$

already calculated in the first chapter.

The core of the chapter is in the following theorem

Theorem 3.1.1. *Given the previous operators, one has*

$$\big|\big|h^2 K_{1,L} W K_{2,L} W^*\big|\big|_{\mathcal{L}(C^0_b(L))} + \big|\big|h^2 K_{2,L} W^* K_{1,L} W\big|\big|_{\mathcal{L}(C^0_b(L))} = O(h^{1/3}), \tag{3.11}$$

$$\big|hK_{2,L} W^* v(0)\big| + \big|hK_{1,L} W v(0)\big| = O(\sup_L |v|). \tag{3.12}$$

The proof of this theorem is rather involved and makes use of several lemmas inside.

Proof. We are denoting by $U_1(x,t)$ and $U_2(x,t)$ respectively the distributional kernels of $h^2\mathcal{W}_{1,L} K_{1,L} W$ and $h^2\mathcal{W}_{2,L} K_{2,L} W^*$. Explicitly we have

$$\begin{aligned} h^2\mathcal{W}_{1,L} K_{1,L} W[v](x) =& u^+_{1,L}(x) \int_{-\infty}^x u^-_{1,L}(t)\big(r_0(t) + r_1(t) h\frac{\partial}{\partial t}\big) v(t) dt + \\ &+ u^-_{1,L}(x) \int_x^0 u^+_{1,L}(t)\big(r_0(t) + r_1(t) h\frac{\partial}{\partial t}\big) v(t) dt \end{aligned} \tag{3.13}$$

and integrating by parts

$$\begin{aligned} &h^2\mathcal{W}_{1,L} K_{1,L} W[v](x) = \\ =& u^+_{1,L}(x) \int_{-\infty}^x W(u^-_{1,L}(t) v(t)) dt + u^+_{1,L}(x) \int_{-\infty}^x {}^tW(u^-_{1,L}(t)) v(t)) dt + \\ &+ u^-_{1,L}(x) \int_x^0 W(u^+_{1,L}(t) v(t)) dt + u^-_{1,L}(x) \int_x^0 {}^tW(u^-_{1,L}(t)) v(t)) dt = \\ =& u^+_{1,L}(x) h r_1(x) u^-_{1,L}(x) v(x) + u^+_{1,L}(x) \int_{-\infty}^x {}^tW(u^-_{1,L}(t)) v(t)) dt + \\ &+ u^-_{1,L}(x) h r_1(0) u^+_{1,L}(0) v(0) - u^-_{1,L}(x) h r_1(x) u^+_{1,L}(x) v(x) + \\ &+ u^-_{1,L}(x) \int_x^0 {}^tW(u^-_{1,L}(t)) v(t)) dt, \end{aligned} \tag{3.14}$$

that is

$$U_1(x,t) = \tilde{U}_1(x,t) + h r_1(0) u^-_{1,L}(x) u^+_{1,L}(0) \delta_{t=0} \tag{3.15}$$

where $\delta_{t=0}$ is the Dirac delta in $t=0$,

$$\tilde{U}_1(x,t) := u^+_{1,L}(x)\big(W_1 u^-_{1,L}\big)(t)\mathbf{1}_{\{t<x\}} + u^-_{1,L}(x)\big(W_1 u^+_{1,L}\big)(t)\mathbf{1}_{\{x<t<0\}} \tag{3.16}$$

and

$$W_1 = {}^tW = r_0 - hr_1\frac{\partial}{\partial t}. \tag{3.17}$$

In the same way we can calculate

$$U_2(x,t) = \tilde{U}_2(x,t) + hr_1(0)u^-_{2,L}(x)u^+_{2,L}(0)\delta_{t=0}, \tag{3.18}$$

$$\tilde{U}_2(x,t) = u^+_{2,L}(x)\big(W_2 u^-_{2,L}\big)(t)\mathbf{1}_{\{t<x\}} + u^-_{2,L}(x)\big(W_2 u^+_{2,L}\big)(t)\mathbf{1}_{\{x<t<0\}}. \tag{3.19}$$

The theorem requires an estimate of the norm of $h^2K_{1,L}WK_{2,L}W^*$, which, in our new notation reads

$$\begin{aligned} h^2K_{1,L}WK_{2,L}W^*[v](x) =& \frac{1}{h^2\mathcal{W}_{1,L}\mathcal{W}_{2,L}}\int_{-\infty}^{0}\int_{-\infty}^{0} U_1(x,t)U_2(t,s)v(s)\,ds\,dt = \\ =& O(h^{-2/3})\int_{-\infty}^{0}\int_{-\infty}^{0} U_1(x,t)U_2(t,s)v(s)\,ds\,dt \end{aligned} \tag{3.20}$$

Referring to the definitions of U_1 and U_2 above this last integral can be decomposed into the sum of four terms:

$$\int_{-\infty}^{0}\int_{-\infty}^{0} U_1(x,t)U_2(t,s)v(s)\,ds\,dt = A_1(x)+A_2(x)+A_3(x)+A_4(x) \tag{3.21}$$

where the various $A_i(x)$ are

$$A_1(x) = \int_{-\infty}^{0}\int_{-\infty}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)v(s)\,ds\,dt, \tag{3.22}$$

$$A_2(x) = hr_1(0)u^+_{2,L}(0)v(0)\int_{-\infty}^{0}\tilde{U}_1(x,t)u^-_{2,L}(t)dt, \tag{3.23}$$

$$A_3(x) = hr_1(0)u^-_{1,L}(x)u^+_{1,L}(0)\int_{-\infty}^{0}\tilde{U}_2(0,s)v(s)ds, \tag{3.24}$$

$$A_4(x) = h^2r_1(0)^2u^-_{1,L}(x)u^+_{1,L}(0)u^+_{2,L}(0)v(0) \tag{3.25}$$

and from now on we are estimating them with respect to the parameter h.

Since $u^-_{1,L}(x)$ is bounded for negative x the estimate on the last is

$$\sup_L |A_4(x)| = O(h^2) \sup_L |v|. \tag{3.26}$$

For the same reason

$$\sup_L |A_3(x)| = O(h) \Big| \int_{-\infty}^{0} \tilde{U}_2(0,s)v(s)dt \Big| \tag{3.27}$$

and

$$\int_{-\infty}^{0} \tilde{U}_2(0,s)v(s)dt = \int_{-\infty}^{x_2(E)} \tilde{U}_2(0,s)v(s)dt + \int_{x_2(E)}^{0} \tilde{U}_2(0,s)v(s)dt. \tag{3.28}$$

The first one is

$$\int_{-\infty}^{x_2(E)} \tilde{U}_2(0,s)v(s)dt = O(h^{2/3}) \sup_L |v| \tag{3.29}$$

Looking at the functions in $\tilde{U}_2(x,t)$ we note that $u^\pm_{2,L}(x)$ are oscillating in $[b,0]$, that is inside the well of V_2 and so, from their definition,

$$|u^\pm_{2,L}(s)| + |h\partial_s u^\pm_{2,L}(s)| = O(h^{1/6}|s-b(E)|^{-1/4}), \tag{3.30}$$

that is

$$\int_{x_2(E)}^{0} \tilde{U}_2(0,s)v(s)dt = O(h^{1/3}) \sup_L |v| \tag{3.31}$$

which dominates over the first integral leading to an estimate

$$\sup_L |A_3(x)| = O(h)O(h^{1/3}) \sup_L |v| = O(h^{4/3}) \sup_L |v|. \tag{3.32}$$

Now we are going to estimate $A_2(x)$. $u^+_{2,L}(0)$ is bounded but in particular is $O(h^{1/6})$ one has

$$\sup_L |A_2(x)| = O(h^{7/6}) \Big| \int_{-\infty}^{0} \tilde{U}_1(x,t)u^-_{2,L}(t)dt \Big| \sup_L |v| \tag{3.33}$$

In order to estimate this integral we need the following

Lemma 3.1.2.

$$\int_{-\infty}^{0} \tilde{U}_1(x,t)u^-_{2,L}(t)dt = O(h^{1/3}).$$

Proof of lemma 3.1.2. Since $\tilde{U}_1(x,t) = O(1)$ on $(-\infty, x_3(E)]^2$ because, as it is defined, the parts with $u^-_{1,L}$ compensate the exponentially big parts with $u^+_{1,L}$ and $\forall\delta > 0\, \exists\alpha > 0$ constant such that

- $u^-_{2,L}(t) = O(h^{1/6}e^{-\alpha|t|/h})$ on $(-\infty, x_2(E) - \delta]$, that is outside the well of V_2;
- $u^-_{2,L}(t) = O(h^{1/6}|x_2(E) - t|^{-1/4}e^{-\alpha|x_2(E)-t|^{3/2}/h})$ on $[x_2(E) - \delta, x_2(E) - h^{2/3}]$ because we have to take into account the behaviour near the inversion point;
- $u^-_{2,L}(t) = O(1)$ on $(-\infty, x_3(E)]$

one obtains

$$\int_{-\infty}^{x_2(E)} \tilde{U}_1(x,t)u^-_{2,L}(t)dt = O(h^{2/3}) \tag{3.34}$$

Otherwise, when $t \in [x_2(E), 0]$ one has $\tilde{U}_1(x,t) = O(h^{1/6})$ and $u^-_{2,L}(t) = O(h^{1/6}|t - x_2(E)|^{-1/4})$ because obviously the sine function is bounded. That implies

$$\int_{x_2(E)}^{0} \tilde{U}_1(x,t)u^-_{2,L}(t)dt = O(h^{1/3}) \tag{3.35}$$

that brings to

$$\int_{-\infty}^{0} \tilde{U}_1(x,t)u^-_{2,L}(t)dt = O(h^{2/3}) + O(h^{1/3}) = O(h^{1/3}). \tag{3.36}$$

End of proof of lemma 3.1.2

Inserting this result into (3.33) we obtain

$$\sup_L |A_2(x)| = O(h^{3/2}) \sup_L |v| \tag{3.37}$$

The only estimate missing in (3.21) is the one for

$$A_1(x) = \int_{-\infty}^{0}\int_{-\infty}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)v(s)\, ds\, dt. \tag{3.38}$$

In order to do so we observe that the set $(-\infty, 0]\times(-\infty, 0]$ can be decomposed into four set by dividing the interval $(-\infty, 0]$ into the union of $(-\infty, x_2(E)]$ and $[x_2(E), 0]$.
For the first term, the estimate is

$$\int_{-\infty}^{x_2(E)}\int_{-\infty}^{x_2(E)} \tilde{U}_1(x,t)\tilde{U}_2(t,s)v(s)\, ds\, dt = O(h^{4/3}) \sup_L |v| \tag{3.39}$$

as can be seen in [FMW1], Section 3.1, eq. (3.15) and in equation (5.46) of this thesis.
The other three quantities to be evaluated are

$$A_{1,1}(x) = \int_{x_2(E)}^{0} \int_{x_2(E)}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)v(s)\,ds\,dt, \tag{3.40}$$

$$A_{1,2}(x) = \int_{-\infty}^{x_2(E)} \int_{x_2(E)}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)v(s)\,ds\,dt, \tag{3.41}$$

$$A_{1,3}(x) = \int_{x_2(E)}^{0} \int_{-\infty}^{x_2(E)} \tilde{U}_1(x,t)\tilde{U}_2(t,s)v(s)\,ds\,dt. \tag{3.42}$$

In $A_{1,2}(x)$, since $t \leq s$, from the definition of $\tilde{U}_2$ we can rewrite, separating the integrals

$$A_{1,2}(x) = \int_{-\infty}^{x_2(E)} \tilde{U}_1(x,t)u_{2,L}^{-}(t)dt \int_{x_2(E)}^{0} \big(W_2 u_{2,L}^{+}\big)(s)ds \tag{3.43}$$

and, since on $[x_2(E), 0]$,

$$\big(W_2 u_{2,L}^{+}\big)(s) = O\big(h^{1/6}|s - x_2(E)|^{-1/4}\big), \tag{3.44}$$

it becomes

$$A_{1,2}(x) = O(h^{1/6}) \sup_L |v| \int_{-\infty}^{x_2(E)} \tilde{U}_1(x,t)u_{2,L}^{-}(t)dt. \tag{3.45}$$

As above, $\tilde{U}_1(x,t)$ is exponentially decading for $t \in (-\infty, x_2(E)]$, that is $\forall\delta > 0\ \exists\alpha(\delta) > 0$ such that

$$\int_{-\infty}^{x_2(E)-\delta} \tilde{U}_1(x,t)u_{2,L}^{-}(t)dt = O(e^{-\alpha/h})$$

and otherwise, when $t \in [x_2(E) - \delta, x_2(E)]$,

$$\tilde{U}_1(x,t) = O(h^{1/6})$$

from the features of the functions involved inside the well of V_1 and

$$u_{2,L}^{-}(t) = O(h^{1/6}|t - x_2(E)|^{-1/4}e^{-\beta(x_2-t)^{3/2}/h})$$

for some constant $\beta > 0$ and then,

$$\begin{aligned}\int_{x_2(E)-\delta}^{x_2(E)} \tilde{U}_1(x,t)u_{2,L}^{-}(t)dt =& O(h^{1/3}) \int_{-\delta}^{\delta} t^{-1/4}e^{-\beta t^{3/2}/h}dt \underset{y=th^{-2/3}}{=} \\ =& O(h^{1/3}) \int_{-\delta h^{-2/3}}^{\delta h^{2/3}} h^{-1/6}y^{-1/4}e^{-\beta y^{3/2}}h^{2/3}dt = \\ =& O(h^{5/6})\end{aligned} \tag{3.46}$$

from which,

$$\sup_L |A_{1,2}| = O(h^{1/6})\big(O(h^{5/6}) + O(e^{-\alpha/h})\big) \sup_L |v| = O(h) \sup_L |v|. \quad (3.47)$$

About $A_{1,3}(x)$, like $A_{1,2}(x)$, it can be rewritten as

$$A_{1,3}(x) = \int_{x_2(E)}^{0} \tilde{U}_1(x,t) u^+_{2,L}(t) dt \int_{-\infty}^{x_2(E)} \big(W_2 u^-_{2,L}\big)(s) ds. \quad (3.48)$$

From [FMW1], Section 3.1, equation (3.11) the second integral has the value

$$\int_{-\infty}^{x_2(E)} \big(W_2 u^-_{2,L}\big)(s) ds = O(h^{2/3}) \quad (3.49)$$

leading to

$$A_{1,3}(x) = O(h^{2/3}) \sup_L |v| \int_{b(E)}^{0} \tilde{U}_1(x,t) u^+_{2,L}(t) dt \quad (3.50)$$

and for $t \in [0, x_2(E)]$, the integrand function is $O(h^{1/3}|t - x_2(E)|^{-1/4})$ bringing to the final estimate

$$\sup_L |A_{1,3}(x)| = O(h) \sup_L |v| \quad (3.51)$$

The estimation on the part $A_{1,1}(x)$ in (3.40) makes use of several auxiliary lemmas. First of all we split

$$A_{1,1}(x) = A^+_{1,1}(x) + A^-_{1,1}(x) \quad (3.52)$$

having set

$$A^\pm_{1,1}(x) = \int_{x_2(E)}^{0} \tilde{U}_1(x,t) u^\pm_{2,L}(t) w_\pm(t) dt \quad (3.53)$$

$$w_+(t) = \int_{x_2(E)}^{t} \big(W_2 u^-_{2,L}\big)(s) v(s) ds \quad (3.54)$$

$$w_-(t) = \int_{t}^{0} \big(W_2 u^+_{2,L}\big)(s) v(s) ds \quad (3.55)$$

Since $|u^\pm_{2,L}| + |W_2 u^\pm_{2,L}| = O(h^{1/6}|E - V_2|^{-1/4})$ on $[0, b(E)]$ because both the sine and the cosine are trivially bounded and there $\big(E - V_2(s)\big)^{-1/4} \sim s^{1/4}$ which is integrable we obtain

$$w_\pm(t) = O(h^{1/6}) \sup_L |v| \quad (3.56)$$

and

$$w'_{\pm}(t) = \big(W_2 u^{\mp}_{2,L}\big)(t)v(t) = O(\sup_L |v|). \tag{3.57}$$

Now we can fix $\lambda \geq 1$ so we can write

$$A^{\pm}_{1,1}(x) = \int_{b(E)}^{b(E)+\lambda h^{2/3}} \tilde{U}_1(x,t)u^{\pm}_{2,L}(t)w_{\pm}(t)dt + \int_{b(E)+\lambda h^{2/3}}^{0} \tilde{U}_1(x,t)u^{\pm}_{2,L}(t)w_{\pm}(t)dt \tag{3.58}$$

Using the last estimates on $w_{\pm}$, $\tilde{U}_1(x,t) = O(h^{1/6})$ and $u^{\pm}_{2,L}(t) = O(h^{1/6}|t - b(E)|^{-1/4})$ when $t \in [b(E), b(E) + \lambda h^{2/3}]$, the first integral becomes

$$\begin{aligned} &\int_{b(E)}^{b(E)+\lambda h^{2/3}} O(h^{1/2})|t-b(E)|^{-1/4} \sup_L |v| dt = \\ &= O(h^{1/2}) \sup_L |v| \big[|t-b(E)|^{3/4}\big]_{b(E)}^{b(E)+\lambda h^{2/3}} = O(h) \sup_L |v|. \end{aligned} \tag{3.59}$$

Assuming $x \leq b(E)$, hence $x \leq t$, we have

$$\begin{aligned} A^{\pm}_{1,1}(x) &= \int_{b(E)+\lambda h^{2/3}}^{0} \tilde{U}_1(x,t)u^{\pm}_{2,L}(t)w_{\pm}(t)dt + O(h) \sup_L |v| = \\ &= \int_{b(E)+\lambda h^{2/3}}^{0} W_1(x,t)u^{+}_{1,L}(t)u^{\pm}_{2,L}(t)w_{\pm}(t)\, dt\, u^{-}_{1,L}(x) + O(h) \sup_L |v| \end{aligned} \tag{3.60}$$

from the definition of $\tilde{U}_1(x,t)$ and rename

$$C(v) = \int_{b(E)+\lambda h^{2/3}}^{0} W_1 u^{+}_{1,L}(t)u^{\pm}_{2,L}(t)w_{\pm}(t)\, dt \tag{3.61}$$

Since the auxiliary function $\xi_2(E)$ used in the first chapter for scalar solutions behaves like $|t - b(E)|$ in the interval $[b(E) + \lambda h^{2/3}, 0]$ we can write

$$C(v) = C_{+}(v) + C_{-}(v) + R(v) \tag{3.62}$$

where

$$C_{\pm}(v) = \int_{b(E)+\lambda h^{2/3}}^{0} h^{1/3} \frac{a_{\pm}(t)\exp\big(\pm i\nu_1(t)/h\big)}{\big(t-b(E)\big)^{1/4}} \sin\Big(\frac{2\xi_2(t)^{3/2}}{3h} + \frac{\pi}{4}\Big) w_{\pm}(t)\, dt \tag{3.63}$$

and

$$R(v) = \int_{b(E)+\lambda h^{2/3}}^{0} \left(\frac{O(h^{\frac{7}{6}+\frac{1}{6}})}{(t-b(E))^{1/4}} + \frac{O(h^{\frac{1}{6}+\frac{7}{6}})}{(t-b(E))^{\frac{1}{4}+\frac{3}{2}}} + O(h^{7/6}) \right) |w_\pm(t)|\, dt \tag{3.64}$$

where $a_\pm(t)$ is a smooth function and $\nu_1(t) = \int_{a(E)}^{t} \big(E - V_1(s)\big)^{1/2} ds$. The estimate on $R(v)$ is

$$\begin{aligned} R(v) =& \Big(O(h^{3/2}) \big[|t-b(E)|^{3/4} \big]_{b(E)+\lambda h^{2/3}}^{0} + O(h^{3/2}) \big[|t-b(E)|^{-3/4} \big]_{b(E)+\lambda h^{2/3}}^{0} + \\ &+ O(h^{4/3}) \big[t \big]_{b(E)+\lambda h^{2/3}}^{0} \Big) \sup_L |v| = \\ =& O\big(h^2 + h + h^{4/3}\big) \sup_L |v| = O(h) \sup_L |v| \end{aligned} \tag{3.65}$$

Posing

$$\nu_2(t) = \int_{b(E)}^{t} \big(E - V_2(s)\big)^{1/2} ds \tag{3.66}$$

we see that $C_+(v)$ and $C_-(v)$ are sums of terms of the type

$$B_+(v) = \int_{b(E)+\lambda h^{2/3}}^{0} h^{1/3} \frac{a_\pm(t) \exp\big(\pm i(\nu_1(t) + \nu_2(t))/h \big)}{(t-b)^{1/4}} w_\pm(t)\, dt \tag{3.67}$$

or

$$B_-(v) = \int_{b(E)+\lambda h^{2/3}}^{0} h^{1/3} \frac{a_\pm(t) \exp\big(\pm i(\nu_1(t) - \nu_2(t))/h \big)}{(t-b)^{1/4}} w_\pm(t)\, dt \tag{3.68}$$

where $a_\pm(t)$ is the same as the definition of $C_\pm$ and the signs in the exponent are not related to the signs in $w_\pm$.
When $t \in [b(E) + \lambda h^{2/3}, 0]$, that is strictly inside both the wells, one has

- $$\nu_1'(t) + \nu_2'(t) = \sqrt{E - V_1(t)} + \sqrt{E - V_2(t)} \geq \frac{1}{C} \tag{3.69}$$

  for some $C \in \mathbb{R}^+$.

- $$\nu_1'(t) - \nu_2'(t) = \sqrt{E - V_1(t)} - \sqrt{E - V_2(t)} \tag{3.70}$$

  is null only for $t = 0$, which is the only point where $V_1(t) = V_2(t)$ by construction.

- $$|\nu_1''(t)-\nu_2''(t)| = \Big| -\frac{V_1'(t)}{2\sqrt{E-V_1(t)}} + \frac{V_2'(t)}{2\sqrt{E-V_2(t)}}\Big| \geq \frac{1}{C} \tag{3.71}$$

  for some $C \in \mathbb{R}^+$.

We observe that

$$e^{\pm i(\nu_1(t)+\nu_2(t))/h} = \frac{\pm h}{i(\nu_1(t)+\nu_2(t))}\frac{d}{dt}e^{\pm i(\nu_1(t)+\nu_2(t))/h} \tag{3.72}$$

and integrating by parts on $B_+(v)$ and using the notation $\varphi(t) := \pm(\nu_1(t)+\nu_2(t))$:

$$\begin{aligned}
B_+(v) =& \Big[\pm h^{4/3}\frac{a_\pm(t)}{(t-b)^{1/4} i\varphi'(t)}\exp\big(i\varphi(t)/h\big)w_\pm(t)\Big]_{b(E)+\lambda h^{2/3}}^{0} + \\
& - \int_{b(E)+\lambda h^{2/3}}^{0} \frac{d}{dt}\Big(\pm h^{4/3}\frac{a_\pm(t)}{(t-b)^{1/4} i\varphi'(t)}w_\pm(t)\Big)\exp\big(i\varphi(t)/h\big)\,dt = \\
=& O(h^{4/3})\big(O(1)+O(h^{-1/6})\big)\sup_L|w_\pm| + \\
& + ih^{4/3}\int_{b(E)+\lambda h^{2/3}}^{0} e^{i\varphi(t)/h}\frac{d}{dt}\Big(\frac{a_\pm(t)w_\pm(t)}{(t-b(E))^{1/4}\varphi'(t)}\Big)\,dt = \\
=& O(h^{4/3})\sup_L|v| + O(h^{4/3})\int_{b(E)+\lambda h^{2/3}}^{0} |e^{i\varphi(t)/h}| O\big(|w_\pm'| + (t-b(E))^{-5/4}|w_\pm|\big)\,dt \\
=& O(h^{4/3})\sup_L|v| + O(h^{4/3})\sup_L|v| + O(h^{\frac{4}{3}+\frac{1}{6}-\frac{1}{6}})\sup_L|v| = \\
=& O(h^{4/3})\sup_L|v|.
\end{aligned} \tag{3.73}$$

where we have used that $|w_\pm'| = O(\sup|v|)$ near $b(E)$ and $|w_\pm| = O(h^{1/6}\sup|v|)$.
For the term $B_-(v)$ we are using the stationary phase method and two auxiliary lemmas.
Let $\chi \in C_0^\infty([b(E),0])$ be a cut-off function which is $\chi = 1$ in a neighbourhood of 0; we can hence write $B_-(v) = B_{-,1}(v) + B_{-,2}(v)$ with

$$B_{-,1}(v) = \int_{b(E)+\lambda h^{2/3}}^{0} h^{1/3}\big(1-\chi(t)\big)\frac{a(t)\exp\big(\pm i(\nu_1(t)-\nu_2(t))/h\big)}{(t-b)^{1/4}}w_\pm(t)\,dt \tag{3.74}$$

and

$$B_{-,2}(v) = \int\limits_{b(E)+\lambda h^{2/3}}^{0} h^{1/3}\chi(t)\frac{a(t)\exp\big(\pm i(\nu_1(t)-\nu_2(t))/h\big)}{(t-b)^{1/4}}w_{\pm}(t)\,dt. \quad (3.75)$$

The estimate on $B_{-,1}$ is identical to the one for B_+ giving

$$B_{-,1}(v) = O(h^{4/3})\sup_L |v| \quad (3.76)$$

As said slightly above we are using a lemma related to the stationary phase theorem.

Lemma 3.1.3. *Let $\chi_0 \in C_0^\infty(\mathbb{R},[0,1])$, $\chi_0 = 1$ near 0, $\psi \in C^\infty(\mathbb{R})$ admitting 0 as the unique stationary point in Suppχ_0 with $\psi''(0) \neq 0$. Then, denoting by K the convex hull of Suppχ_0 one has, for $f \in C^2(\mathbb{R})$,*

$$\int e^{i\psi(t)/h}\chi_0(t)f(t)\,dt = f(0)e^{i\frac{\pi}{4}sgn\psi''(0)}\sqrt{\frac{2\pi h}{|\phi''(0)|}} + O(h)\sup_K\big(|f'|+|f''|\big). \quad (3.77)$$

uniformly with respect to $h > 0$ small enough.

Proof of Lemma 3.1.3. We can make a smooth change of variables for ψ setting

$$\psi = \pm\mu t^2/2$$

with $\mu > 0$ constant, since ψ has only one stationary point. Then we can write f as

$$f(t) = f(0) + tg(t),$$

with

$$g(t) := \int_0^1 f'(\theta t)\,d\theta$$

obtaining thus

$$\begin{aligned}&\int e^{i\psi(t)/h}\chi_0(t)f(t)\,dt = \\ &= f(0)\int e^{\pm i\mu t^2/2h}\chi_0(t)\,dt \pm \frac{h}{i\mu}\int\frac{d}{dt}(e^{\pm i\mu t^2/2h})\chi_0(t)g(t)\,dt.\end{aligned} \quad (3.78)$$

From the stationary-phase theorem (in [Ma], Theorem 2.6.1) we have

$$\int e^{\pm i\mu t^2/2h}\chi_0(t)\,dt = \mu^{-1/2}e^{\pm i\frac{\pi}{4}}\sqrt{2\pi h} + O(h^\infty) \quad (3.79)$$

and from an integration by parts:

$$\begin{aligned}\int e^{i\psi(t)/h}\chi_0(t)f(t)\,dt =& f(0)\big(\mu^{-1/2}e^{\pm i\frac{\pi}{4}}\sqrt{2\pi h}+O(h^\infty)\big)+\\ &\pm\frac{ih}{\mu}\int e^{\pm i\mu t^2/2h}\frac{d}{dt}\big(\chi_0(t)g(t)\big)\,dt\end{aligned} \tag{3.80}$$

Taking into account that $\sup_{\mathrm{Supp}\chi_0}(|g|+|g'|)\leq \sup_K(|f|+|f'|)$ the thesis follows.
End of proof of lemma 3.1.3.

From this lemma another result descends:

Lemma 3.1.4. *Let $I\subset\mathbb{R}$ an open interval containig 0 and $\psi\in C^\infty(\mathbb{R})$ admitting 0 as a unique stationary point in $\bar{I}$ with $\psi''(0)\neq 0$. Then, for $u\in C^1_0(I)$,*

$$\int e^{i\psi(t)/h}u(t)\,dt = O(h^{1/2})\sup(|u|+|u'|) \tag{3.81}$$

uniformly with respect to $h>0$ small enough.

Proof of lemma 3.1.4. Setting

$$f(t)=\frac{1}{\sqrt{2\pi h}}\int e^{-(t-s)^2/2h}u(s)\,ds$$

we can write

$$f(t)-u(t)=\frac{1}{\sqrt{2\pi h}}\int e^{-(t-s)^2/2h}\big(u(s)-u(t)\big)\,ds$$

From the continuity of u, $u(s)-u(t)\leq|s-t|\sup|u'|$ and $\int e^{-(s-t)^2/2h}|s-t|\,ds=2h$, so we have

$$|f(t)-u(t)|=\sqrt{\frac{2h}{\pi}}\sup|u'|.$$

Fixing $\chi_0\in C^\infty_0(\mathbb{R},[0,1])$ with $\chi_0=1$ on I, such that $\chi u=u$, we obtain

$$\Big|\int e^{i\psi(t)/h}u(t)\,dt-\int e^{i\psi(t)/h}\chi_0(t)f(t)\,dt\Big|=|I|\sqrt{\frac{2h}{\pi}}\sup|u'|.$$

Applying lemma 3.1.3 to f,

$$\int e^{i\psi(t)/h}\chi_0(t)f(t)\,dt=f(0)\sqrt{\frac{2\pi h}{|\psi''(0)|}}+O(h)\sup\big(|f'|+|f''|\big) \tag{3.82}$$

Now we calculate the derivatives of f:

$$f(t) \underset{y=s-t}{=} \frac{1}{\sqrt{2\pi h}} \int e^{-y^2/2h} u(y+t)\, dy \implies$$

$$f'(t) = \frac{1}{\sqrt{2\pi h}} \int e^{-y^2/2h} \frac{d}{dt} u(y+t)\, dy = \frac{1}{\sqrt{2\pi h}} \int e^{-(t-s)^2/2h} \frac{d}{ds} u(s)\, ds \quad (3.83)$$

and

$$f''(t) = -\frac{1}{h\sqrt{2\pi h}} \int e^{-(t-s)^2/2h} (t-s) u'(s)\, ds \quad (3.84)$$

It gives

$$|f'(t)| = O(1) \sup |u'| \quad (3.85)$$

making the change of variables $z = (t-s)/\sqrt{2h}$ in the integral and

$$|f'(t)| = O(h^{-1/2}) \sup |u'| \quad (3.86)$$

again with the same change.
Since $f(0) = O(\sup |u|)$, substituting in (3.82) we obtain the thesis.
End of proof of lemma 3.1.4.

Remark 4. This lemma remains valid when the integration is restricted to a half line $\mathbb{R}_+$ or $\mathbb{R}_-$.

If we apply the previous lemma to the functions

$$\psi(t) = \pm(\nu_1(t) - \nu_2(t)), \quad u(t) = h^{1/3} \frac{a(t)}{(t-b(E))^{1/4}} w_\pm(t) \quad (3.87)$$

we obtain the estimate on $B_{-,2}(v)$:

$$B_{-,2} = O(h^{5/6}) \sup_{\mathrm{Supp}\chi} (|w_\pm| + |w'_\pm|)$$

because from the definition (3.75) and the lemma:

$$B_{-,2}(v) = h^{1/3} O(h^{1/2}) \sup_{\mathrm{Supp}\chi} (|w_\pm| + |w'_\pm|)$$

but, from (3.56) and $w'_\pm = O(h^{1/6}) \sup_L |v|$:

$$B_{-,2}(v) = O(h) \sup_L |v| \quad (3.88)$$

We are now concluding the estimate on $A^{\pm}_{1,1}(t)$ using (3.60), (3.62), (3.65), (3.73), (3.76) and (3.88):

$$\sup_L |A^{\pm}_{1,1}| = O(h) \sup_L |v|. \tag{3.89}$$

Here is a recap of the various functions and estimates we have used:

$$\begin{aligned}|A^{\pm}_{1,1}| =& C(v) u^-_{1,L}(x) + O(h) \sup_L |v| = \\ =& \big(C_+(v) + C_-(v) + R(v)\big) u^-_{1,L}(x) + O(h) \sup_L |v| = \\ =& \big(B_+(v) + B_-(v) + R(v)\big) u^-_{1,L}(x) + O(h) \sup_L |v| = \\ =& \big(B_+(v) + B_{-,1}(v) + B_{-,2}(v) + R(v)\big) u^-_{1,L}(x) + O(h) \sup_L |v| = \\ =& \big(O(h^{4/3}) + O(h^{4/3}) + O(h) + O(h)\big) \sup_L |v| O(1) + O(h) \sup_L |v| = \\ =& O(h) \sup_L |v|.\end{aligned} \tag{3.90}$$

Please note that right before (3.60) we have assumed $x \leq b(E)$. Now, when $x \in [b(E), 0]$, $A^{\pm}_{1,1}$ must be written as

$$\begin{aligned}A^{\pm}_{1,1}(x) =& u^+_{1,L}(x) \int_{b(E)}^{x} \big(W_1 u^-_{1,L}\big)(t) u^{\pm}_{2,L}(t) w_{\pm}(t)\, dt + \\ &+ u^-_{1,L}(x) \int_x^0 \big(W_1 u^+_{1,L}\big)(t) u^{\pm}_{2,L}(t) w_{\pm}(t)\, dt\end{aligned} \tag{3.91}$$

in consequence of the definition of $\tilde{U}_1(x,t)$. The proof remains essentially similar as long as x is away from the point 0 critical for $\nu_1 - \nu_2$. Before, x was less or equal $b(E)$ and we have taken χ to be 1 in a neighbourhood of the origin. This time, whenever $x \neq 0$, it suffices to choose χ such that its support does not contain x.

In the case $x \approx 0$, in the proof of lemma 3.1.3, when integrating by parts in eq. (3.80) the boundary term

$$\mp \frac{ih}{\mu} e^{\pm i\mu\tilde{x}^2/2h}$$

appears, because χ is not vanishing anymore. $\tilde{x}$ is the transformed of x after the change of variables. This term is $O(h)$ because trivially all parts other from h are bounded.

The other difference is the integral

$$\int_{\pm t \geq \pm\tilde{x}} e^{\pm i\mu t^2/2h} \chi_0(t)\, dt \tag{3.92}$$

to be added in the right-hand side of (3.79) and to be estimated. In order to do so we prove

Lemma 3.1.5. *For any $\mu > 0$ independent from h and for any $a \in \mathbb{R}$ (which may depend from h), and χ_0 as in lemma 3.1.3, one has,*

$$\int_{\pm t \geq \pm a} e^{\pm i\mu t^2/2h} \chi_0(t)\, dt = O(h^{1/2}),$$

uniformily with respect to h small enough.

Proof of lemma 3.1.5. We are assuming $t \geq a$ and $a \in Supp\chi$.
If $a \geq -h^{1/2}$, we have,

$$\begin{aligned}\int_{t\geq a} e^{\pm i\mu t^2/2h}\chi_0(t)\,dt = &\int_a^{a+2h^{1/2}} e^{\pm i\mu t^2/2h}\chi_0(t)\,dt + \int_{t\geq a+2h^{1/2}} e^{\pm i\mu t^2/2h}\chi_0(t)\,dt = \\ =&O(h^{1/2}) \pm \frac{h}{i\mu}\int_{t\geq a+2h^{1/2}} \frac{d}{dt}\Big(e^{\pm i\mu t^2/2h}\Big)\frac{\chi_0(t)}{t}\,dt = \\ =&O(h^{1/2}+h^{1/2}) \pm \frac{ih}{\mu}\int_{t\geq a+2h^{1/2}} e^{\pm i\mu t^2/2h}\frac{d}{dt}\Big(\frac{\chi_0(t)}{t}\Big)\,dt = \\ =&O(h^{1/2}) + O(h)\int_{t\geq a+2h^{1/2}}\frac{dt}{t^2} = O(h^{1/2}).\end{aligned} \tag{3.93}$$

If $a \leq -h^{1/2}$

$$\begin{aligned}\int_{t\geq a} e^{\pm i\mu t^2/2h}\chi_0(t)\,dt = &\int_a^{-h^{1/2}} e^{\pm i\mu t^2/2h}\chi_0(t)\,dt + \int_{t\geq -h^{1/2}} e^{\pm i\mu t^2/2h}\chi_0(t)\,dt = \\ =&\int_a^{-h^{1/2}} e^{\pm i\mu t^2/2h}\chi_0(t)\,dt + O(h^{1/2}+h^{1/2}) = \\ =&\pm\frac{h}{i\mu}\int_a^{-h^{1/2}} \frac{d}{dt}\Big(e^{\pm i\mu t^2/2h}\Big)\frac{\chi_0(t)}{t}\,dt + O(h^{1/2}) = \\ =&O(h^{1/2})\end{aligned} \tag{3.94}$$

from an integration by parts on the last step.
End of proof of lemma 3.1.5

Thus, the estimate of $|A_{1,1}(x)|$ is the same as for $x \leq b(E)$ giving

$$\sup_L |A_{1,1}| = O(h)\sup_L |v| \tag{3.95}$$

The estimate of the operator $h^2 K_{1,L} W K_{2,L} W^*$ comes from (3.20), (3.21), (3.26), (3.32), (3.37), (3.38), (3.47), (3.51), (3.95):

$$\begin{aligned}
&\big|\big|h^2 K_{1,L} W K_{2,L} W^*\big|\big|_{\mathcal{L}(C_b^0(L))} = \\
&= O(h^{-2/3})\big(|A_{1,1}| + |A_{1,2}| + |A_{1,3}| + O(h^{4/3}) + |A_2| + |A_3| + |A_4|\big) \sup_L |v| = \\
&= O(h^{-2/3})\big(O(h) + O(h) + O(h) + O(h^{4/3}) + \\
&\quad + O(h^{3/2}) + O(h^{4/3}) + O(h^2)\big) \sup_L |v| = \\
&= O(h^{1/3}) \sup_L |v|.
\end{aligned} \tag{3.96}$$

The estimate on $h^2 K_{2,L} W^* K_{1,L} W$ is made in the same way and it proves the first estimate in Theorem 3.1.1.

For the estimate of $hK_{2,L} W^* v(0)$, we observe

$$hK_{2,L} W^* v(0) = O(h^{-1/3}) \int_{-\infty}^0 \tilde{U}_2(0,t) v(t)\, dt + O(h^{2/3}) r_1(0) u_{2,L}^-(0) u_{2,L}^+(0) v(0) \tag{3.97}$$

and using that $u_{2,L}^{\pm} = O(h^{1/6})$,

$$\begin{aligned}
hK_{2,L} W^* v(0) =& O(h^{-1/3}) \int_{-\infty}^0 \tilde{U}_2(0,t) v(t)\, dt + O(h) \sup_L |v| = \\
=& O(h^{-1/6}) \int_{-\infty}^0 W_2 u_{2,L}^-(t) v(t)\, dt + O(h) \sup_L |v| = \\
=& O(\sup_L |v|).
\end{aligned} \tag{3.98}$$

The estimate of $hK_{1,L} W v(0)$ is the same and Theorem 3.1.1 follows. □

3.2 Fundamental operators on R

Concerning to the interval $R = [0, +\infty)$, there is not any difference from the interval L. We only have to consider the two integral operators

$$K_{j,R} : C_b^0(R) \to C_b^2(R),$$

$$\begin{aligned}
&K_{j,R}(v)(x) := \\
&= \frac{u_{j,R}^-(x)}{h^2 \mathcal{W}(u_{j,R}^-, u_{j,R}^+)} \int_0^x u_{j,R}^+(t) v(t) dt + \frac{u_{j,R}^+(x)}{h^2 \mathcal{W}(u_{j,R}^-, u_{j,R}^+)} \int_x^{+\infty} u_{j,R}^-(t) v(t) dt.
\end{aligned} \tag{3.99}$$

In the same way as Theorem 3.1.1 we can prove

Theorem 3.2.1. *Given the previous operators, one has*

$$\left|\left|h^2 K_{2,R} W^* K_{1,R} W\right|\right|_{\mathcal{L}(C_b^0(R))} + \left|\left|h^2 K_{1,R} W K_{2,R} W^*\right|\right|_{\mathcal{L}(C_b^0(R))} = O(h^{1/3}), \tag{3.100}$$

$$\left|hK_{1,R} W v(0)\right| + \left|hK_{2,R} W^* v(0)\right| = O(\sup_R |v|). \tag{3.101}$$

3.3 Solutions of the system on L and R and quantization condition.

We are renaming the operators in Theorems 3.1.1 and 3.2.1

$$M_L = h^2 K_{1,L} W K_{2,L} W^*, \quad \tilde{M}_L = h^2 K_{2,L} W^* K_{1,L} W$$
$$M_R = h^2 K_{2,R} W^* K_{1,R} W, \quad \tilde{M}_R = h^2 K_{1,R} W K_{2,R} W^*$$

where the index "L" and "R" trivially refers to the interval in which they are defined. They permit us to define the solutions of the system (3.2)

$$w_{1,L} = \begin{pmatrix} \sum_{j\geq 0} M_L^j u_{1,L}^- \\ -hK_{2,L} W^* \sum_{j\geq 0} M_L^j u_{1,L}^- \end{pmatrix}$$

$$w_{2,L} = \begin{pmatrix} -hK_{1,L} W \sum_{j\geq 0} \tilde{M}_L^j u_{2,L}^- \\ \sum_{j\geq 0} \tilde{M}_L^j u_{2,L}^- \end{pmatrix}$$

on the interval L and

$$w_{1,R} = \begin{pmatrix} \sum_{j\geq 0} \tilde{M}_R^j u_{1,R}^- \\ -hK_{2,R} W^* \sum_{j\geq 0} \tilde{M}_R^j u_{1,R}^- \end{pmatrix}$$

$$w_{2,R} = \begin{pmatrix} -hK_{1,R} W \sum_{j\geq 0} M_R^j u_{2,R}^- \\ \sum_{j\geq 0} M_R^j u_{2,R}^- \end{pmatrix}$$

on the interval R.
As for the solutions in the previous chapter each couple $\{w_{1,L}, w_{2,L}\}$ and $\{w_{1,R}, w_{2,R}\}$ is a basis for the space of solutions in its own interval and using that in 0 $\{w_{1,L}, w_{2,L}\}$ are linearly dependent from $\{w_{1,R}, w_{2,R}\}$ and viceversa, we can extend all these four functions to $\mathbb{R}$. That brings to the condition on their Wronskian

$$\mathcal{W}(w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R}) = 0 \tag{3.102}$$

in order to have solutions in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.
We are calculating the Wronskian in 0 since is constant with respect to the point x for Abel's identity. In order to do so we are using the values of the solutions and their derivative in 0, setting $S = L$ or R (see [FMW3], Proposition 4.1):

$$w_{1,S}(0) = \begin{pmatrix} u^-_{1,S}(0) \\ 0 \end{pmatrix} + O(h^{1/3}); \quad w'_{1,S}(0) = \begin{pmatrix} (u^-_{1,S})'(0) \\ 0 \end{pmatrix} + O(h^{-2/3});$$
$$w_{2,S}(0) = \begin{pmatrix} 0 \\ u^-_{2,S}(0) \end{pmatrix} + O(h^{1/3}); \quad w'_{2,S}(0) = \begin{pmatrix} 0 \\ (u^-_{2,S})'(0) \end{pmatrix} + O(h^{-2/3}).$$

The wronskian is, from [FMW3], Proposition 4.3 and, with the same calculation as (2.138) using these functions above

$$\mathcal{W}(E) = \frac{16}{\pi^2} h^{-4/3} \cos(h^{-1}\phi_1(E)) \cos(h^{-1}\phi_2(E)) + O(h^{-7/6}). \tag{3.103}$$

Imposing it to be null we have the equation

$$\cos(h^{-1}\phi_1(E)) \cos(h^{-1}\phi_2(E)) = O(h^{1/6}). \tag{3.104}$$

and following the resolution on Section 2.8, when $\phi_1(E) \neq \phi_2(E)$ the conditions are

$$\phi_1(E) = \pi\Big(\frac{1}{2} + n\Big)h + O(h^{7/6}) \tag{3.105}$$

for $n \in \mathbb{Z}$ or

$$\phi_2(E) = \pi\Big(\frac{1}{2} + m\Big)h + O(h^{7/6}) \tag{3.106}$$

with $m \in \mathbb{Z}$, while when $\phi_1(E) \approx \phi_2(E)$:

$$\phi_1(E) = \pi\Big(\frac{1}{2} + n\Big)h + O(h^{13/12}) \tag{3.107}$$

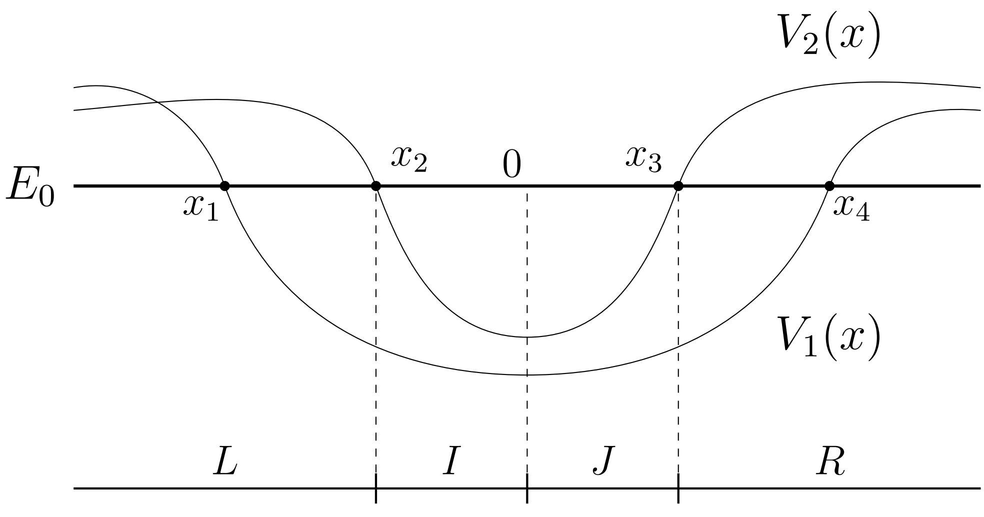
or

$$\phi_2(E) = \pi\Big(\frac{1}{2} + m\Big)h + O(h^{13/12}). \tag{3.108}$$

Chapter 4

Non intersecting potential functions

We are dividing again $\mathbb{R}$ into four interval having a point in common that can be either one of the inversion point for $V_2(x)$ or 0. In this way we are able to use construction already studied in other chapters and refer to them.



4.1 Fundamental operators on L and R

On the intervals $L := (-\infty, x_2]$ and $R := [x_3, +\infty)$ the integral operators are the same as in the fifth chapter since again the interval ends on an inversion point for a potential and the other potential goes from negative values to positive ones.

For L they are:

$$K_{j,L} : C^0_b(L) \to C^2_b(L)$$

$$K_{j,L}[v](x) = \\ = \frac{u^+_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L}, u^+_{j,L})} \int_{-\infty}^{x} u^-_{j,L}(t)v(t)dt + \frac{u^-_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L}, u^+_{j,L})} \int_{x}^{x_2} u^+_{j,L}(t)v(t)dt. \tag{4.1}$$

and in R

$$K_{j,R} : C^0_b(R) \to C^2_b(R)$$

$$K_{j,R}[v](x) = \\ = \frac{u^-_{j,R}(x)}{h^2\mathcal{W}(u^-_{j,R}, u^+_{j,R})} \int_{x_2}^{x} u^+_{j,R}(t)v(t)dt + \frac{u^+_{j,R}(x)}{h^2\mathcal{W}(u^-_{j,R}, u^+_{j,R})} \int_{x}^{+\infty} u^-_{j,R}(t)v(t)dt. \tag{4.2}$$

and for them the relations

$$(P_j - E)K_{j,L} = Id, \quad (P_j - E)K_{j,R} = Id \tag{4.3}$$

hold (see 5.1.1 in Chapter 5 for the proof).
The estimates on the norms of these operators are, from Theorem 5.1.2, Theorem 5.2.1, [FMW1], Prop. 3.1 and Prop. 3.2.

Theorem 4.1.1.

$$||hK_{2,L}W^*||_{\mathcal{L}(C^0_b)} = O(h^{1/3}); \tag{4.4}$$

$$||h^2K_{1,L}WK_{2,L}W^*||_{\mathcal{L}(C^0_b)} = O(h^{2/3}); \tag{4.5}$$

$$||hK_{1,R}W||_{\mathcal{L}(C^0_b(R)} = O(h^{1/3}); \tag{4.6}$$

$$||h^2K_{2,R}W^*K_{1,R}W||_{\mathcal{L}(C^0_b(R)} = O(h^{2/3}). \tag{4.7}$$

4.2 Fundamental operators on I and J.

For what it concerns the operators to use in the intervals $I := [x_2, 0]$ and $J := [0, x_3]$ we use the ones in Section 3.1 and 3.2 with $-\infty$ in the integral of $K_{j,L}$ replaced by x_2 and $+\infty$ in the integral of $K_{j,R}$ replaced by x_3. One extreme of the interval is an inversion point for one potential and the other potential is below the energy through all the interval I or J considered.
The operators are

$$K_{j,I} : C^0_b(I) \to C^2_b(I)$$

$$K_{j,I}[v](x) = \\ = \frac{u^+_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L}, u^+_{j,L})}\int_{x_2}^{x} u^-_{j,L}(t)v(t)dt + \frac{u^-_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L}, u^+_{j,L})}\int_{x}^{0} u^+_{j,L}(t)v(t)dt, \tag{4.8}$$

and

$$K_{j,J} : C^0_b(J) \to C^2_b(J)$$

$$K_{j,J}(v)(x) := \\ = \frac{u^-_{j,R}(x)}{h^2\mathcal{W}(u^-_{j,R}, u^+_{j,R})}\int_{0}^{x} u^+_{j,R}(t)v(t)dt + \frac{u^+_{j,R}(x)}{h^2\mathcal{W}(u^-_{j,R}, u^+_{j,R})}\int_{x}^{x_3} u^-_{j,R}(t)v(t)dt. \tag{4.9}$$

The estimate on the norms of the compositions of these operators are the same of Theorem 3.1.1 and Theorem 3.2.1 in Sections 3.1 and 3.2.

Theorem 4.2.1. *Given the previous operators, one has*

$$\left|\left|h^2K_{1,I}WK_{2,I}W^*\right|\right|_{\mathcal{L}(C^0_b(I))} + \left|\left|h^2K_{2,I}W^*K_{1,I}W\right|\right|_{\mathcal{L}(C^0_b(I))} = O(h^{1/3}); \tag{4.10}$$

$$\left|hK_{2,I}W^*v(0)\right| + \left|hK_{1,I}Wv(0)\right| = O(\sup_I |v|); \tag{4.11}$$

$$\left|\left|h^2K_{2,J}W^*K_{1,J}W\right|\right|_{\mathcal{L}(C^0_b(J))} + \left|\left|h^2K_{1,J}WK_{2,J}W^*\right|\right|_{\mathcal{L}(C^0_b(J))} = O(h^{1/3}); \tag{4.12}$$

$$\left|hK_{1,J}Wv(0)\right| + \left|hK_{2,J}W^*v(0)\right| = O(\sup_J |v|). \tag{4.13}$$

Proof. The proof is identical to the proof of Theorem 3.1.1 and we are not repeating it. The only difference is that, now, neither $\nu_1 + \nu_2$ nor $\nu_1 - \nu_2$ have a stationary point. For this reason B_+ and B_- can be estimated in the same way and we obtain

$$B_+(v) + B_-(v) = O(h^{4/3}) \sup_I |v|. \tag{4.14}$$

Inserting the last estimate into (3.90) we observe that the estimate for $A_{1,1}$ is again $O(h)\sup_I |v|$ and continuing the proof the result follows. □

4.3 Quantization condition

We are now giving solutions of the system in each interval, using for L and R the ones in Sections 5.1 and 5.2, while in I and J the ones already defined in Section 3.3. Namely, after having defined the new operators

$$M_L = h^2 K_{1,L} W K_{2,L} W^*, \quad M_R = h^2 K_{2,R} W^* K_{1,L} W, \tag{4.15}$$

$$M_I = h^2 K_{1,I} W K_{2,I} W^*, \quad \tilde{M}_I = h^2 K_{2,I} W^* K_{1,I} W, \tag{4.16}$$

$$M_J = h^2 K_{2,J} W^* K_{1,J} W, \quad \tilde{M}_J = h^2 K_{1,J} W K_{2,J} W^*, \tag{4.17}$$

they are

$$w_{1,L} = \begin{pmatrix} \sum_{j\geq 0} M_L^j u_{1,L}^- \\ -hK_{2,L} W^* \sum_{j\geq 0} M_L^j u_{1,L}^- \end{pmatrix};$$

$$w_{2,L} = \begin{pmatrix} -\sum_{j\geq 0} M_L^j (hK_{1,L} W u_{2,L}^-) \\ u_{2,L}^- + hK_{2,L} W^* \sum_{j\geq 0} M_L^j (hK_{1,L} W u_{2,L}^-) \end{pmatrix};$$

$$w_{1,R} := \begin{pmatrix} u_{1,R}^- + hK_{1,R} W \sum_{j\geq 0} M_R^j (hK_{2,R} W^* u_{1,R}^-) \\ -\sum_{j\geq 0} M_R^j (hK_{2,R} W^* u_{1,R}^-) \end{pmatrix}$$

$$w_{2,R} := \begin{pmatrix} -hK_{1,R} W \sum_{j\geq 0} M_R^j u_{2,R}^- \\ \sum_{j\geq 0} M_R^j u_{2,R}^- \end{pmatrix};$$

$$w_{1,I} = \begin{pmatrix} \sum_{j\geq 0} M_I^j u_{1,L}^- \\ -hK_{2,I} W^* \sum_{j\geq 0} M_I^j u_{1,L}^- \end{pmatrix}; \quad w_{2,I} = \begin{pmatrix} -hK_{1,I} W \sum_{j\geq 0} \tilde{M}_I^j u_{2,L}^- \\ \sum_{j\geq 0} \tilde{M}_I^j u_{2,L}^- \end{pmatrix};$$

$$w_{1,J} = \begin{pmatrix} \sum_{j\geq 0} \tilde{M}_J^j u_{1,R}^- \\ -hK_{2,J} W^* \sum_{j\geq 0} \tilde{M}_J^j u_{1,R}^- \end{pmatrix}; \quad w_{2,J} = \begin{pmatrix} -hK_{1,J} W \sum_{j\geq 0} M_J^j u_{2,R}^- \\ \sum_{j\geq 0} M_J^j u_{2,R}^- \end{pmatrix}.$$

As in Sections 2.7 and 2.8 we can extend $w_{j,L}$ and $w_{j,R}$ to $\mathbb{R}$ and come to the condition on the Wronskian

$$\mathcal{W}(w_{1,L},\, w_{2,L},\, w_{1,R},\, w_{2,R}) = 0 \tag{4.18}$$

which leads to the equation

$$\cos\left(h^{-1}\phi_1(E)\right)\cos\left(h^{-1}\phi_2(E)\right) = O(h^{-1/3}) \tag{4.19}$$

solved, for $\frac{\phi_1(E)-\phi_2(E)}{h} \neq 0, \pi \mod 2\pi$ with the condition on E

$$\phi_1(E) = \pi\Big(\frac{1}{2} + n\Big)h + O(h^{2/3}) \tag{4.20}$$

for $n \in \mathbb{Z}$ or

$$\phi_2(E) = \pi\Big(\frac{1}{2} + m\Big)h + O(h^{2/3}) \tag{4.21}$$

with $m \in \mathbb{Z}$ and, when $\frac{\phi_1(E)-\phi_2(E)}{h} \approx 0, \pi \mod 2\pi$ the conditions become

$$\phi_1(E) = \pi\Big(\frac{1}{2} + n\Big)h + O(h^{5/6}) \tag{4.22}$$

for $n \in \mathbb{Z}$ or

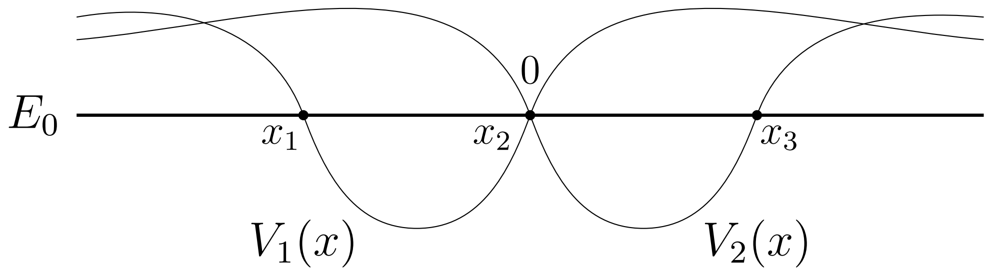
$$\phi_2(E) = \pi\Big(\frac{1}{2} + m\Big)h + O(h^{5/6}) \tag{4.23}$$

for $m \in \mathbb{Z}$.

Chapter 5

Intersection of potential functions at energy level

This Chapter follows the works [FMW1] and [FMW2].
In the last case the potentials $V_1(x)$ and $V_2(x)$ have a point of intersection x_2 where $V_1(x_2) = V_2(x_2) = E_0$. We are considering possible values of the energy E in the interval $[E_0 - C_0 h^{2/3}, E_0 + C_0 h^{2/3}]$ for some fixed constant C_0 arbitrarily large. Since the point x_2 does not depend by E we will assume $x_2 = 0$.



We are treating the problem separately in the two intervals $(-\infty, 0]$ and $[0, +\infty)$.

5.1 Solutions on $(-\infty, 0]$.

For what it concerns $L := (-\infty, 0]$ we observe that in the first chapter we have built a couple of solutions $u^{\pm}_{j,L}(x)$ related to the potential $V_j(x)$ so

we have globally four linearly independent solutions.
In particular the Wronskians are

$$\mathcal{W}(u_{j,L}^{-}, u_{j,L}^{+}) = \frac{-2}{\pi h^{2/3}}\big(1 + O(h)\big). \tag{5.1}$$

Defining the space of functions as in the previous chapter

$$C_b^k\big((-\infty, 0]\big) = \Big\{u : (-\infty, 0] \to \mathbb{R};\ \sum_{j=0}^{k} \sup_{x\leq 0} |u^{(j)}(x)|\Big\} \tag{5.2}$$

equipped with the norm

$$||u||_{C_b^k} = \sum_{j=0}^{k} \sup_{x\leq 0} |u^{(j)}(x)|, \tag{5.3}$$

we can set an operator

$$K_{j,L} : C_b^0\big((-\infty, 0]\big) \to C_b^2\big((-\infty, 0]\big) \tag{5.4}$$

acting on $v \in C_b^0\big((-\infty, 0]\big)$ as

$$\begin{aligned} K_{j,L}[v](x) =& \frac{u_{j,L}^{+}(x)}{h^2\mathcal{W}(u_{j,L}^{-}, u_{j,L}^{+})} \int_{-\infty}^{x} u_{j,L}^{-}(t)v(t)dt + \\ &+ \frac{u_{j,L}^{-}(x)}{h^2\mathcal{W}(u_{j,L}^{-}, u_{j,L}^{+})} \int_{x}^{0} u_{j,L}^{+}(t)v(t)dt. \end{aligned} \tag{5.5}$$

Lemma 5.1.1. *$K_{j,L}$ is a fundamental solution for $P_j - E$, that is*

$$(P_j - E)K_{j,L}f(x) = f(x) \quad \forall f \in C_b^0\big((-\infty, 0]\big). \tag{5.6}$$

Proof. Writing explicitly $(P_j - E)K_{j,L}f(x)$ one has

$$\begin{aligned} \big(h^2\frac{d^2}{dx^2} + V_j(x) - E\big)\Big(&\frac{u_{j,L}^{+}(x)}{h^2\mathcal{W}(u_{j,L}^{-}, u_{j,L}^{+})} \int_{-\infty}^{x} u_{j,L}^{-}(t)f(t)dt + \\ &+ \frac{u_{j,L}^{-}(x)}{h^2\mathcal{W}(u_{j,L}^{-}, u_{j,L}^{+})} \int_{x}^{0} u_{j,L}^{+}(t)f(t)dt\Big) = \end{aligned} \tag{5.7}$$

$$
\begin{aligned}
=&h^2\frac{d^2}{dx^2}\Big(\frac{u^+_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\Big)\int_{-\infty}^{x}u^-_{j,L}(t)f(t)dt+\\
&+\Big(V_j(x)-E\Big)\frac{u^+_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\int_{-\infty}^{x}u^-_{j,L}(t)f(t)dt+\\
&+2h^2\frac{d}{dx}\Big(\frac{u^+_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\Big)u^-_{j,L}(x)f(x)+\\
&+h^2\frac{u^+_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\big(\frac{d}{dx}u^-_{j,L}(x)f(x)+u^-_{j,L}(x)f'(x)\big)+\\
&+h^2\frac{d^2}{dx^2}\Big(\frac{u^-_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\Big)\int_{x}^{0}u^+_{j,L}(t)f(t)dt+\\
&+\Big(V_j(x)-E\Big)\frac{u^-_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\int_{x}^{0}u^+_{j,L}(t)f(t)dt+\\
&-2h^2\frac{d}{dx}\Big(\frac{u^-_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\Big)u^+_{j,L}(x)f(x)+\\
&+h^2\frac{u^-_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\big(-\frac{d}{dx}u^+_{j,L}(x)f(x)-u^+_{j,L}(x)f'(x)\big)=
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{2}{\mathcal{W}(u^-_{j,L},u^+_{j,L})}\frac{d}{dx}u^+_{j,L}(x)u^-_{j,L}(x)f(x)+\\
&+\frac{1}{\mathcal{W}(u^-_{j,L},u^+_{j,L})}u^+_{j,L}(x)\frac{d}{dx}u^-_{j,L}(x)f(x)+\\
&-\frac{2}{\mathcal{W}(u^-_{j,L},u^+_{j,L})}\frac{d}{dx}u^-_{j,L}(x)u^+_{j,L}(x)f(x)+\\
&-\frac{1}{\mathcal{W}(u^-_{j,L},u^+_{j,L})}u^-_{j,L}(x)\frac{d}{dx}u^+_{j,L}(x)f(x)=\\
=&f(x)\frac{1}{\mathcal{W}(u^-_{j,L},u^+_{j,L})}\big(\frac{d}{dx}u^-_{j,L}(x)u^+_{j,L}(x)-u^-_{j,L}(x)\frac{d}{dx}u^+_{j,L}(x)\big)=f(x).
\end{aligned}
$$

having used that $u^{\pm}_{j,L}(x)$ are solutions of $P_ju=Eu$. □

Moreover we can give a characterization of the operators $K_{j,L}W$ and

$K_{j,L}W^*$ by an integration by parts obtaining

$$\begin{aligned}
K_{j,L}W[v](x) =& \frac{u^+_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\int_{-\infty}^{x} u^-_{j,L}(t)\big(r_0(t)+hr_1(t)\partial_t\big)v(t)dt+ \\
&+\frac{u^-_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\int_{x}^{0} u^+_{j,L}(t)\big(r_0(t)+hr_1(t)\partial_t\big)v(t)dt = \\
=& \frac{u^+_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\int_{-\infty}^{x} u^-_{j,L}(t)r_0(t)v(t)dt+ \\
&+\frac{u^+_{j,L}(x)}{h\mathcal{W}(u^-_{j,L},u^+_{j,L})}u^-_{j,L}(x)r_1(x)v(x)+ \\
&-\frac{u^+_{j,L}(x)}{h\mathcal{W}(u^-_{j,L},u^+_{j,L})}\int_{-\infty}^{x}\partial_t\big(u^-_{j,L}(t)r_1(t)\big)v(t)dt+ \\
&+\frac{u^-_{j,L}(x)}{h^2\mathcal{W}(u^-_{j,L},u^+_{j,L})}\int_{x}^{0} u^+_{j,L}(t)r_0(t)v(t)dt+ \\
&-\frac{u^-_{j,L}(x)}{h\mathcal{W}(u^-_{j,L},u^+_{j,L})}u^+_{j,L}(x)r_1(x)v(x)+ \\
&+\frac{u^-_{j,L}(x)}{h\mathcal{W}(u^-_{j,L},u^+_{j,L})}u^+_{j,L}(x_2)r_1(x_2)v(x_2)+ \\
&-\frac{u^-_{j,L}(x)}{h\mathcal{W}(u^-_{j,L},u^+_{j,L})}\int_{x}^{0}\partial_t\big(u^+_{j,L}(t)r_1(t)\big)v(t)dt,
\end{aligned} \tag{5.8}$$

that is

$$K_{j,L}W,\ K_{j,L}W^* : C^0_b \to C^0_b \tag{5.9}$$

since the integrals converge and the multiplication $u^+_{j,L}(x)u^-_{j,L}(x)$ is bounded on $(-\infty,0]$.

Proposition 5.1.2. (See [FMW1], Prop. 3.1).
The following estimates hold:

$$||hK_{2,L}W^*||_{\mathcal{L}(C^0_b)} = O(h^{1/3}) \tag{5.10}$$

$$||h^2K_{1,L}WK_{2,L}W^*||_{\mathcal{L}(C^0_b)} = O(h^{2/3}) \tag{5.11}$$

Proof. Let us define the new functions

$$U_j(x,t) := |u^+_{j,L}(x)u^-_{j,L}(t)|\mathbf{1}_{\{t<x\}} + |u^-_{j,L}(x)u^+_{j,L}(t)|\mathbf{1}_{\{t>x\}} = U_j(t,x) \tag{5.12}$$

$$U'_j(x,t) := |u^+_{j,L}(x)h\partial_t u^-_{j,L}(t)|\mathbf{1}_{\{t<x\}} + |u^-_{j,L}(x)h\partial_t u^+_{j,L}(t)|\mathbf{1}_{\{t>x\}} \tag{5.13}$$

$$\tilde{U}_j(x,t) = U_j(x,t) + U'_j(x,t). \tag{5.14}$$

Integrating by parts one obtains

$$
\begin{aligned}
|hK_{1,L}Wv(x)| =& \Big| \frac{u^+_{1,L}(x)}{h\mathcal{W}(u^-_{j,L}, u^+_{j,L})} \int_{-\infty}^{x} u^-_{1,L}(t)\big(r_0(t) + hr_1(t)\partial_t\big)v(t)dt + \\
& + \frac{u^-_{1,L}(x)}{h\mathcal{W}(u^-_{j,L}, u^+_{j,L})} \int_{x}^{0} u^+_{1,L}(t)\big(r_0(t) + hr_1(t)\partial_t\big)v(t)dt\Big| = \\
=& O(h^{-1/3})\Big| \int_{-\infty}^{x} u^+_{1,L}(x)u^-_{1,L}(t)v(t)\,dt + \\
& - \int_{-\infty}^{x} u^+_{1,L}(x)h\partial_t\big(u^-_{1,L}(t)\big)v(t)\,dt + hu^+_{1,L}(x)u^-_{1,L}(x)v(x) + \\
& + \int_{x}^{0} u^-_{1,L}(x)u^+_{1,L}(t)v(t)\,dt + \\
& - \int_{x}^{0} u^-_{1,L}(x)h\partial_t\big(u^+_{1,L}(t)\big)v(t)\,dt - hu^-_{1,L}(x)u^+_{1,L}(x)v(x) + \\
& + hu^-_{1,L}(x)u^+_{1,L}(0)v(0)\Big| = \\
=& O(h^{-1/3})\Big(\int_{-\infty}^{0} \tilde{U}_1(x,t)|v(t)|\,dt + hU_1(x,0)|v(0)|\Big)
\end{aligned}
\tag{5.15}
$$

and the same calculations yields also

$$
|hK_{2,L}W^*v(x)| = O(h^{-1/3})\Big(\int_{-\infty}^{0} \tilde{U}_2(x,t)|v(t)|\,dt + hU_2(x,0)|v(0)|\Big) \tag{5.16}
$$

from which, in particular,

$$
||hK_{2,L}W^*v(x)||_{\mathcal{L}(C^0_b)} = O(h^{-1/3}) \sup_{x\le 0} \int_{-\infty}^{0} \tilde{U}_2(x,t)\,dt + O(h^{2/3}) \sup_{x\le 0} U_2(x,0). \tag{5.17}
$$

Remembering the asymptotics of $u^{\pm}_{2,L}(x)$ and $h\partial_x u^{\pm}_{2,L}(x)$ when $h \to 0^+$ already treated in the previous chapter

$$
u^{\pm}_{2,L}(x) = \frac{h^{1/6}}{\sqrt{\pi}}\big(V_2(x) - E\big)^{-1/4} \exp\Big(\mp h^{-1}\int_0^x \sqrt{V_2(t) - E}\,dt\Big)\big(1 + O(h)\big), \tag{5.18}
$$

$$
\begin{aligned}
h\partial_x u^{\pm}_{2,L}(x) =& \exp\Big(\mp h^{-1}\int_0^x \sqrt{V_2(t) - E}\,dt\Big)\cdot \\
& \cdot\Big(\mp\big(V_2(x) - E\big)^{1/4}\frac{h^{1/6}}{\sqrt{\pi}} + O(h^{7/6})\Big)\big(1 + O(h)\big)
\end{aligned}
\tag{5.19}
$$

we are going to study the different asymptotics of $\tilde{U}_2(x,t)$ depending on the values of x and t.
Fixing a constant $C_1 > 0$:

- If $x, t \le -C_1 h^{2/3}$:

$$\tilde{U}_2(x,t) = O(h^{1/3})|V_2(t) - E|^{\frac{1}{4}} \frac{\exp\big(-h^{-1}|\int_t^x \sqrt{V_2(s)-E}\,ds|\big)}{|V_2(x)-E|^{\frac{1}{4}}} \tag{5.20}$$

  because the part $h\partial_t u^{\pm}_{2,L}(t)u^{\mp}_{2,L}(x)$ dominates over $u^{\pm}_{2,L}(t)u^{\mp}_{2,L}(x)$.

- If $t \le -C_1 h^{2/3} \le x \le 0$:

$$\tilde{U}_2(x,t) = O(h^{1/6})|V_2(t) - E|^{1/4} \exp\Big(-h^{-1}\Big|\int_0^x \sqrt{V_2(s)-E}\,ds\Big|\Big) \tag{5.21}$$

  because $u^{\pm}_{2,L}(x)$ is bounded in $[-C_1h^{2/3}, 0]$. The same holds simmetrically exchanging x and t.

- If $x, t \in [-C_1 h^{2/3}, 0]$

$$\tilde{U}_2(x,t) = O(1) \tag{5.22}$$

  because all the functions involved are $O(1)$.

We can observe that actually in all these three cases $\tilde{U}_2(x,t) = O(1)$ and we can give estimates of the integral of $\tilde{U}_2(x,t)$ in (5.17) considering the position of x with respect to $-\delta$, with δ constant and $\delta > C_1h^{2/3} > 0$.

When $x \le -\delta$ there exists a constant $\alpha > 0$ such that

$$\int_{-\infty}^0 \tilde{U}_2(x,t)\,dt = O(h^{1/3})\int_{-\infty}^{-\delta/2} e^{-\alpha|x-t|/h}\,dt + O(e^{-\alpha/h}) = O(h^{4/3}) \tag{5.23}$$

noting that

$$\int_{-\infty}^{-\delta/2} e^{-\alpha|x-t|/h}\,dt = O(h). \tag{5.24}$$

When $x \in [-\delta, -C_1 h^{2/3}]$:

$$\begin{aligned}\int_{-\infty}^{0} \tilde{U}_2(x,t)\,dt = &\int_{-C_1h^{2/3}}^{0} \tilde{U}_2(x,t)\,dt + \int_{-2\delta}^{-C_1h^{2/3}} \tilde{U}_2(x,t)\,dt + \\ &+ \int_{-\infty}^{-2\delta} \tilde{U}_2(x,t)\,dt = \\ =&O(h^{\frac{2}{3}}) + O(h^{\frac{1}{3}}|x|^{-\frac{1}{4}}) \int_{-2\delta}^{-C_1h^{2/3}} t^{\frac{1}{4}} \exp\big(-\frac{\alpha}{h}\big||t|^{\frac{3}{2}} - |x|^{\frac{3}{2}}\big|\big)\,dt + \\ &+ O(h^{\frac{4}{3}})\end{aligned} \tag{5.25}$$

because in a neighbourhood of 0, $V_2(x) \sim x$ and $\xi_2(x) \sim x$ (see first chapter for details). Now setting $t = (hs)^{\frac{2}{3}}$ the integral becomes

$$O(h^{5/6}|x|^{-1/4}) \int_{C'}^{\frac{C''}{h}} \frac{\exp\big(-\alpha\big|s - h^{-1}|x|^{3/2}\big|\big)}{s^{1/2}}\,ds + O(h^{2/3}) \tag{5.26}$$

where $C' = C_1^{3/2}$ and $C'' = (2\delta)^{3/2}$ and, since this last integral converges,

$$\int_{-\infty}^{0} \tilde{U}_2(x,t)\,dt = O(h^{2/3}). \tag{5.27}$$

The last case missing is when $x \in [-C_1 h^{2/3}, 0]$:

$$\begin{aligned}\int_{-\infty}^{0} \tilde{U}_2(x,t)\,dt = &\int_{-C_1h^{2/3}}^{0} \tilde{U}_2(x,t)\,dt + \int_{-\delta}^{-C_1h^{2/3}} \tilde{U}_2(x,t)\,dt + \\ &+ \int_{-\infty}^{-\delta} \tilde{U}_2(x,t)\,dt = \\ =&O(h^{2/3}) + O(h^{1/6}) \int_{-\delta}^{-C_1h^{2/3}} |t|^{1/4} e^{-\alpha|t|^{3/2}/h}\,dt + O(h^{7/6}) = \\ =&O(h^{2/3})\end{aligned} \tag{5.28}$$

since the last integral can be estimated with the change of variables $t \mapsto h^{2/3}t$:

$$\int_{-\delta h^{-2/3}}^{-C_1} h^{1/6} t^{1/4} e^{\alpha t^{3/2}} h^{2/3}\,dt = O(h^{5/6}). \tag{5.29}$$

Finally we have treated all the possibilities when $x \leq 0$ concluding

$$\sup_{x\leq 0} \int_{-\infty}^{0} \tilde{U}_2(x,t)\, dt = O(h^{2/3}). \tag{5.30}$$

Now, since $U_2(x,t) = O(h^{1/3})|V(x) - E|^{-1/4}|V(t) - E|^{-1/4} = O(h^{1/3})$, the estimate on (5.10) is, from (5.17),

$$||hK_{2,L}W^*||_{\mathcal{L}(C_b^0)} = O(h^{-1/3})O(h^{2/3}) + O(h^{2/3})O(h^{1/3}) = O(h^{1/3}). \tag{5.31}$$

We are now treating the norm of the operator $M_L := h^2 K_{1,L} W K_{2,L} W^*$. Taking into account (5.15) and (5.16) we have

$$\begin{aligned} |M_L v(x)| =& O(h^{-2/3}) \int_{-\infty}^{0}\int_{-\infty}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)|v(s)|ds dt + \\ &+ O(h^{1/3}) \int_{-\infty}^{0} \tilde{U}_1(x,t) U_2(t,0)|v(0)|dt + \\ &+ O(h^{1/3}) U_1(x,0) \int_{-\infty}^{0} \tilde{U}_2(0,t)|v(t)|dt + \\ &+ O(h^{4/3}) U_1(x,0) U_2(0,0)|v(0)|. \end{aligned} \tag{5.32}$$

The last three terms can be automatically estimated as $O(h)\sup_{x\leq 0}|v|$ from (5.30) and from the fact that $U_j(x,t) = O(1)$.
Similarly to the estimate of the previous operator we can give estimates of $\tilde{U}_1(x,t)$ depending on x, t: chosen $\delta > 0$ exists $\alpha > 0$ such that

- When $x, t \leq x_1 - \delta$:

$$\tilde{U}_1(x,t) = O(h^{1/3}) \exp\big(-\alpha|t - x|/h\big). \tag{5.33}$$

- When $t \leq x_1 - 2\delta$ and $x_1 - \delta \leq x \leq 0$

$$\tilde{U}_1(x,t) = O(h^{1/6} e^{-\alpha/h}) \tag{5.34}$$

  because $u^+_{1,L}(x) = O(1)$ in a neighbourhood of the well.

- When $x \in [x_1 - 4\delta, 0]$ and $t \in [-\delta, -C_1 h^{2/3}]$:

$$\tilde{U}_1(x,t) = O(h^{1/6}|t|^{-1/4}) \tag{5.35}$$

  because, like above, $u^\pm_{1,L}(x)$ are bounded near the well.

- When $x \in [x_1 - 4\delta, 0]$ and $t \in [-C_1 h^{2/3}, 0] \cup [x_1 - 4\delta, -\delta]$:

$$\tilde{U}_1(x,t) = O(1). \tag{5.36}$$

Please note that the role of x and t can be exchanged.
From the definition of $\tilde{U}_2(x,t)$ every part of its integral with $|t-s| \geq \delta > 0$, with δ constant is exponentially small.
In order to give an estimate of the first integral appearing in (5.32) we split $[-\infty, 0]$ into the union of $(-\infty, x_1 - 2\delta]$ and $[x_1 - 2\delta, 0]$ having chosen a constant $\delta > 0$.

For what it concerns $x \in (-\infty, x_1 - 2\delta]$ the integral is

$$\begin{aligned}\int_{-\infty}^{0}\int_{-\infty}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)dsdt =& O(h^{2/3}) \int_{-\infty}^{x_1-\delta}\int_{-\infty}^{x_1-\delta/2} \exp\Big(-\frac{\alpha}{h}\big(|t-x|+|s-t|\big)\Big)\, ds\, dt + \\ &+ O(e^{-\alpha/h}) = \\ =& O(h^{8/3})\end{aligned} \tag{5.37}$$

thanks to the estimates (5.33) for $\tilde{U}_1(x,t)$ and (5.23) for $\tilde{U}_2(t,s)$ and to the fact that every integration of that exponential function yields $O(h)$.

Instead, for $x \in [x_1 - 2\delta, 0]$ the calculation is more involved:

$$\int_{-\infty}^{0}\int_{-\infty}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)dsdt = \int_{x_1-3\delta}^{0}\int_{x_1-4\delta}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)\, ds\, dt + O(e^{-\alpha/h}) \tag{5.38}$$

because on the left of the well for $V_1(x)$ both $\tilde{U}_1(x,t)$ and $\tilde{U}_2(t,s)$ are exponentially decaying.

This last integral can be decomposed into the sum of many parts:

$$\begin{aligned}\int_{-\infty}^{0}\int_{-\infty}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)dsdt =&O(h^{1/3}) \int_{x_1-3\delta}^{-\delta}\int_{x_1-4\delta}^{-\delta/2} e^{-\alpha|t-s|/h}\, ds\, dt + O(e^{-\alpha/h})+\\ &+O(h^{1/2}) \int_{-\delta}^{-C_1h^{2/3}}\int_{-2\delta}^{-C_1h^{2/3}} \frac{|s|^{1/4}e^{-\alpha\left||t|^{\frac{3}{2}}-|s|^{\frac{3}{2}}\right|/h}}{|t|^{1/2}}\, ds\, dt+\\ &+O(h^{1/6}) \int_{-\delta}^{-C_1h^{2/3}}\int_{-C_1h^{2/3}}^{0} \frac{e^{-\alpha|t|^{\frac{3}{2}}/h}}{|t|^{1/4}}\, ds\, dt+\\ &+O(h^{1/6}) \int_{-C_1h^{2/3}}^{0}\int_{-\delta}^{-C_1h^{2/3}} |s|^{1/4}e^{-\alpha|s|^{\frac{3}{2}}/h}\, ds\, dt+\\ &+O(h^{4/3}).\end{aligned} \tag{5.39}$$

and we are going to discuss them one by one.
For the first integral just note that $\tilde{U}_1(x,t) = O(1)$ if x, t are in a neighbourhood of the well and $\tilde{U}_2(t,s)$ is exponentially decaying outside the well of $V_2(x)$. It is $O(h^{7/3})$ because each integration of the exponential gives $O(h)$. In the second integral $\tilde{U}_1(x,t)$ behaves like (5.20) and $\tilde{U}_2(t,s) \sim h^{1/6}|t|^{-1/4}$. In the third one consider $\tilde{U}_2(t,s) = O(1)$ near the well of $V_2(x)$ and $\tilde{U}_1(x,t) \sim O(1)u^{\pm}_{1,L}(t)$ since x is in a bounded interval. It can be estimated further as

$$O(h^{5/6}) \int_{-\delta}^{-C_1h^{2/3}} \frac{e^{-\alpha|t|^{\frac{3}{2}}/h}}{|t|^{1/4}}\, dt \tag{5.40}$$

since

$$\int_{-C_1h^{2/3}}^{0} ds = O(h^{2/3}) \tag{5.41}$$

For the last integral again $\tilde{U}_1(x,t) = O(1)$ and $\tilde{U}_2(x,t)$ has estimate (5.20). The last remaining $O(h^{4/3})$ comes from (5.23).

All these observations give the estimate

$$\int\limits_{-\infty}^{0}\int\limits_{-\infty}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)dsdt = O(h^{1/2}) \int\limits_{-\delta}^{-C_1h^{2/3}}\int\limits_{-2\delta}^{-C_1h^{2/3}} \frac{|s|^{1/4}e^{-\alpha\left||t|^{\frac{3}{2}}-|s|^{\frac{3}{2}}\right|/h}}{|t|^{1/2}}\,ds\,dt + \\ + O(h^{5/6}) \int\limits_{-\delta}^{-C_1h^{2/3}} \frac{e^{-\alpha|t|^{\frac{3}{2}}/h}}{|t|^{1/4}}\,dt + O(h^{4/3}). \tag{5.42}$$

In the first integral we can make the change of variables $(t,s) \mapsto (t^{\frac{2}{3}}, s^{\frac{2}{3}})$ such that it becomes

$$O(h^{1/2}) \int_{-\delta'}^{-C_2h}\int_{-\delta''}^{-C_2h} \frac{e^{-\alpha\left||t|-|s|\right|/h}}{|t|^{1/2}|s|^{1/6}}\,ds\,dt \tag{5.43}$$

and supposing $t \le s$ it is equal to

$$O(h^{1/2}) \int_{-C_2h}^{-\delta'} \frac{e^{\alpha t/h}}{t^{\frac{5}{6}}}\,dt \int_{t}^{-\delta''} e^{-\alpha s/h}\,ds = O(h^{1/2})O(h^{1/6})O(h) = O(h^{5/3}). \tag{5.44}$$

The second integral of (5.42) can be calculated setting $y := t^{3/4}h^{-1/2}$ so that it becomes

$$O(h^{5/6})O(h^{1/2}) \int_{C'}^{O(h^{-1/2})} e^{-\alpha y^2}\,dy = O(h^{4/3}). \tag{5.45}$$

Inserting these last estimates (5.44) and (5.45) into (5.42) gives

$$\int\limits_{-\infty}^{0}\int\limits_{-\infty}^{0} \tilde{U}_1(x,t)\tilde{U}_2(t,s)dsdt = O(h^{4/3}) \tag{5.46}$$

and going back to (5.32) we have

$$||M_L||_{\mathcal{L}(C_b^0)} = ||h^2K_{1,L}WK_{2,L}W^*||_{\mathcal{L}(C_b^0)} = O(h^{2/3}). \tag{5.47}$$

□

Let us recap what we have done until this point: referring only on the interval $(-\infty, 0]$ we have the fundamental solution $K_{j,L}$ for the operator $P_j - E$ and the new operators $hK_{2,L}W^*$ and $h^2K_{1,L}WK_{2,L}W^*$ have a vanishing

norm when $h \to 0$.
Our purpose is to build solutions $u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of the 2×2 system (2.2)

$$\begin{cases} (P_1 - E)u_1 = -hWu_2 \\ (P_2 - E)u_2 = -hW^*u_1 \end{cases} \tag{5.48}$$

which tends to

$$w^0_{1,L} := \begin{pmatrix} u^-_{1,L} \\ 0 \end{pmatrix}, \qquad w^0_{2,L} := \begin{pmatrix} 0 \\ u^-_{2,L} \end{pmatrix} \tag{5.49}$$

as $h \to 0^+$.
The second equation can be written as

$$u_2 = (P_2 - E)^{-1}(-hW^*u_1) = -hK_{2,L}W^*u_1 \tag{5.50}$$

thanks to Lemma (5.1.1) so that the first equation becomes

$$(P_1 - E)u_1 = h^2 WK_{2,L}W^*u_1 \tag{5.51}$$

which has solutions of the type

$$u_1 = u^-_{1,L} + h^2 K_{1,L}WK_{2,L}W^*u_1 \tag{5.52}$$

since $u^-_{1,L}(x)$ is a solution of the homogeneous equation $(P_1 - E)u_1 = 0$ built in the previous chapter and $h^2K_{1,L}WK_{2,L}W^*u_1$ is a particular solution again from Lemma (5.1.1). Please note that these operators involved are the ones we estimated in the previous theorem.
The new vector-valued function

$$w_{1,L} = \begin{pmatrix} \sum_{j\geq 0} M^j_L u^-_{1,L} \\ -hK_{2,L}W^* \sum_{j\geq 0} M^j_L u^-_{1,L} \end{pmatrix} \in C^0_b\big((-\infty, 0]\big)^2 \tag{5.53}$$

is a solution of the system because, posing $u_1 = \sum_{j\geq 0} M^j_L u^-_{1,L}$ it satisfies (5.52) since

$$\sum_{j\geq 0} M^j_L u^-_{1,L} = u^-_{1,L} + M_L \sum_{j\geq 0} M^j_L u^-_{1,L} = M^0_L u^-_{1,L} + \sum_{j\geq 1} M^j_L u^-_{1,L}, \tag{5.54}$$

and tends to (5.49) because, using Prop. (5.1.2) its components are

$$w_{1,L} = \begin{pmatrix} u^-_{1,L} + O(h^{2/3}u^-_{1,L}) \\ O(h^{1/3}u^-_{1,L}) \end{pmatrix} \xrightarrow{h\to 0} w^0_{1,L} = \begin{pmatrix} u^-_{1,L} \\ 0 \end{pmatrix}. \tag{5.55}$$

The space of solutions of the system is bidimensional and the function

$$w_{2,L} = \begin{pmatrix} -\sum_{j\geq 0} M_L^j(hK_{1,L}Wu_{2,L}^-) \\ u_{2,L}^- + hK_{2,L}W^*\sum_{j\geq 0} M_L^j(hK_{1,L}Wu_{2,L}^-) \end{pmatrix} \in C_b^0\big((-\infty,0]\big)^2 \quad (5.56)$$

is a solution linearly independent from $w_{1,L}$.
It is a solution because, similarly as before, the first equation in equivalent to

$$u_1 = -hK_{1,L}Wu_2 \quad (5.57)$$

so that the second equation becomes

$$(P_2 - E)u_2 = h^2 W^* K_{1,L} W u_2 \quad (5.58)$$

having solutions

$$u_2 = u_{2,L}^- + h^2 K_{2,L} W^* K_{1,L} W u_2 = u_{2,L}^- + h^2 K_{2,L} W^* u_1 \quad (5.59)$$

and, renaming u_1 the first component in $w_{2,L}$, the second component u_2 is exactly

$$u_{2,L}^- + h^2 K_{2,L} W^* u_1. \quad (5.60)$$

In particular again $w_{2,L}$ is close to $\begin{pmatrix} 0 \\ u_{2,L}^- \end{pmatrix}$ when $h \to 0^+$ because it is

$$\begin{pmatrix} O(h^{1/3}u_{2,L}^-) \\ u_{2,L}^- + O(h^{2/3}u_{2,L}^-) \end{pmatrix}. \quad (5.61)$$

Please note that neither $w_{1,L}^0$ nor $w_{2,L}^0$ are solution to the system due to the presence of the perturbative terms hW and hW^*.

5.2 Solutions on $[0,+\infty)$.

In the interval $R := [0,+\infty)$ the operator to start with in our method are

$$\begin{aligned} K_{j,R}[v](x) = & \frac{u_{j,R}^-(x)}{h^2\mathcal{W}(u_{j,R}^-, u_{j,R}^+)} \int_0^x u_{j,R}^+(t)v(t)dt + \\ & + \frac{u_{j,R}^+(x)}{h^2\mathcal{W}(u_{j,R}^-, u_{j,R}^+)} \int_x^{+\infty} u_{j,R}^-(t)v(t)dt. \end{aligned} \quad (5.62)$$

. Similarly as the previous section for $(-\infty, 0]$ there hold the estimates:

Proposition 5.2.1.

$$||hK_{1,R}W||_{\mathcal{L}(C_b^0(R))} = O(h^{1/3}) \tag{5.63}$$

$$||h^2K_{2,R}W^*K_{1,R}W||_{\mathcal{L}(C_b^0(R))} = O(h^{2/3}) \tag{5.64}$$

From these operators we build

$$M_R = h^2K_{2,R}W^*K_{1,R}W$$

and we can so define solutions $u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of the 2×2 system (2.2)

$$\begin{cases} (P_1 - E)u_1 = -hWu_2 \\ (P_2 - E)u_2 = -hW^*u_1 \end{cases} \tag{5.65}$$

which tends to

$$w_{1,R}^0 := \begin{pmatrix} u_{1,R}^- \\ 0 \end{pmatrix}, \qquad w_{2,R}^0 ;= \begin{pmatrix} 0 \\ u_{2,R}^- \end{pmatrix} \tag{5.66}$$

as $h \to 0^+$.

The new vector-valued functions

$$w_{1,R} := \begin{pmatrix} u_{1,R}^- + hK_{1,R}W\sum_{j\geq 0} M_R^j(hK_{2,R}W^*u_{1,R}^-) \\ -\sum_{j\geq 0} M_R^j(hK_{2,R}W^*u_{1,R}^-) \end{pmatrix} \tag{5.67}$$

and

$$w_{2,R} := \begin{pmatrix} -hK_{1,R}W\sum_{j\geq 0} M_R^j u_{2,R}^- \\ \sum_{j\geq 0} M_R^j u_{2,R}^- \end{pmatrix} \tag{5.68}$$

using the same argument as the previous section are both solutions of the system when $x \geq 0$.
Moreover they respectively tend to $w_{1,R}^0$ and to $w_{2,R}^0$ as $h \to 0$ using the norms of the operators calculated in the last theorem.

5.3 Quantization condition

Like we have done in previous chapters, using that in 0 both the couples $\{w_{1,L}, w_{2,L}\}$ and $\{w_{1,R}, w_{2,R}\}$ are a basis for the vector space of solutions we can extend all these functions to the whole real line exploiting the coefficients

of their dependence (see section 2.7 for details).
Once again this means that the Wronskian of this two solution must be

$$\mathcal{W}(w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R}) = \mathcal{W}(w_{1,L}, w_{2,L})\mathcal{W}(w_{1,R}, w_{2,R}) + O(h^{-5/3}) = 0 \tag{5.69}$$

leading to the conditions *Bohr-Sommerfeld condition* on the energy

$$\cos\left(h^{-1}\phi_1(E)\right)\cos\left(h^{-1}\phi_2(E)\right) = O(h^{-1/3}). \tag{5.70}$$

This conditions implies, when $\phi_1(E) - \phi_2(E) \neq 0, \pi \mod 2\pi$ as explained in Section 2.8, E to be such that

$$\phi_1(E) = \pi\Big(\frac{1}{2} + n\Big)h + O(h^{2/3}) \tag{5.71}$$

for $n \in \mathbb{Z}$ or

$$\phi_2(E) = \pi\Big(\frac{1}{2} + m\Big)h + O(h^{2/3}) \tag{5.72}$$

with $m \in \mathbb{Z}$.
In the case $\phi_1(E) - \phi_2(E) \approx 0, \pi \mod 2\pi$ the conditions remains the same except for the bigger error term being

$$\phi_1(E) = \pi\Big(\frac{1}{2} + n\Big)h + O(h^{5/6}) \tag{5.73}$$

for $n \in \mathbb{Z}$ or

$$\phi_2(E) = \pi\Big(\frac{1}{2} + m\Big)h + O(h^{5/6}) \tag{5.74}$$

with $m \in \mathbb{Z}$.

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