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Anomalies in the stress tensor of a chiral fermion in a gauge background

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Abstract

In this thesis we study the anomalies of a chiral fermion in a gauge background, using a different regularization from those already present in literature. The aim is to study all the anomalies involving the stress tensor. The final motivation is to eventually focus on the trace anomaly, which has been of some interest recently.

Thus, after a brief introduction to the issue of anomalies in QFT, we proceed by studying the symmetries of a massless left-handed Weyl fermion coupled to an abelian gauge background and gravity as well (used as an external source for the stress tensor). The regularization of the corresponding QFT is then implemented through Pauli-Villars (PV) fields having a Dirac mass. Particular emphasis is put on the unusual mass term used at this stage, consisting of a customary Dirac mass multiplied by the vierbein determinant e raised to the generic power α.

After devoting a chapter to the mathematical tool of the heat kernel, we restrict ourselves to flat space, present the regulators of the model, the "jacobians" associated to each of its symmetries, and all the useful heat kernel coefficients needed for the anomaly calculations.

Finally, we evaluate all the anomalies of our model: the usual chiral anomaly and the anomalies in the stress tensor, namely the trace anomaly and the anomalies in the symmetry of the stress tensor and in its conservation (i.e. local Lorentz and gravitational anomalies). The latter is the most demanding task, as it requires the use of particular heat kernel coefficients which have been rarely treated in the literature.

The computation of these anomalies is the leading task accomplished in this thesis. Of course, one does not expect all of these anomalies to be genuine, as some are expected to be canceled by the variation of local counterterms, leaving at the end only the chiral and trace anomalies with their known expressions. That this is the case is left for future research.

Abstract

Nel corso della presente tesi, studiamo le anomalie di un fermione chirale posto in un background di gauge, usando una regolarizzazione differente da quelle finora usate in letteratura. Il nostro obbiettivo consiste nello studio di tutte le anomalie coinvolgenti il tensore energia impulso. Tuttavia, le motivazioni ultime sono da ricercare, eventualmente, nell'approfondimento dell'anomalia di traccia, che recentemente è stata oggetto di grande interesse.

Quindi, dopo una breve introduzione al problema delle anomalie in QFT, procediamo a studiare le simmetrie di un fermione sinstrorso di Weyl, privo di massa, accoppiato ad un background di gauge abeliano ed alla gravità (usata come sorgente del tensore energia impulso). Si implementa poi la regolarizzazione della corrispondente teoria quantistica tramite campi di Pauli-Villars dotati di massa di Dirac. Enfasi particolare è posta sull'insolito termine di massa utilizzato, che consiste in un'usuale massa di Dirac moltiplicata per l'$\alpha-$esima potenza del determinante del vierbein e.

Dopo un intero capitolo dedicato allo strumento matematico dell'heat kernel, restringendoci su uno spazio-tempo piatto, presentiamo i regolatori del modello, gli "jacobiani" associati ad ogni sua simmetria, e tutti i principali coefficienti di heat kernel, necessari al calcolo delle anomalie.

Infine valutiamo tutte le anomalie del nostro modello, tra cui l'anomalia chirale e quelle che affliggono il tensore energia impulso, ovvero l'anomalia di traccia e quelle nella simmetria e conservazione del tensore energia impulso (anomalia di Lorentz locale e anomalia gravitazionale). L' anomalia gravitazionale rappresenta il compito più arduo, poiché richiede l'uso di coefficienti di heat kernel raramente trattati in letteratura.

Il computo di tali anomalie è dunque la principale conquista portata a termine nel corso dell'elaborato. Naturalmente, ci si aspetta che, tramite la variazione di opportuni controtermini locali, soltanto le anomalie di traccia e chirale, con le loro solite espressioni, sopravvivano. Che questo sia o meno il caso è però lasciato a ricerche future.

Contents

Chapter 1

Introducing the anomalies

Quantum Field Theories (QFTs) constitute the main tool to study the fundamental interactions, in particular when applied to gauge theories and gravity. One particular aspect of QFTs deals with the implementation of symmetries and their preservation at the quantum level. It is well-known that the quantization procedure, which often requires the introduction of a regularization scheme, brings in a conflict between different symmetries, so that they cannot be maintained all together (at least if certain cancellation criteria are not met). For instance, it may happen that the regulators employed to cure the divergencies in the Feynman diagrams, such as the ones encountered in 1-loop processes, do not respect the original classical symmetries, causing their breaking at the quantum level. Likewise, the functional measure that arises in a path integral quantization often does not exhibit the same invariance properties of the classical action, so that some symmetries can get lost. In plain words, not all the symmetries survive the second quantization: in these cases, as it is customary to say, we are in the presence of a quantum anomaly.

The first anomaly was discovered in the late '60s, the so-called chiral (or Adler-Bell-Jackiw) anomaly [1, 2], and falls within the category of the global anomalies. It describes the non-conservation of the axial symmetry

$$\psi \longrightarrow \psi' = e^{i\gamma_5\theta}\psi \tag{1.1}$$

present in massless QED, and is expressed through the following non zero expectation value

$$\partial_\mu \langle J_5^\mu \rangle = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}.$$

Indeed, one of the ways we have to infer whether an anomaly is present is in verifying the quantum non-conservation of a Noether current. Phenomenologically this chiral anomaly has been very important. If the quantized theory were invariant under (1.1), with ψ considered now as the field of a quark, the decay of a (massless) neutral pion into two photons (see fig. 1.1) would be forbidden. In fact, the amplitude describing the transition would vanish classically because of the Ward identity associated to the chiral symmetry, not allowing the pion at rest to undergo this decay. In the quantum world, though, the width associated with this process is no longer negligible (as the anomaly itself can now balance the Ward identity), and the phenomenon can therefore be observed in actual experiments.

Of course, there can be anomalies ensuing from local symmetries as well, like the ones associated to gauge theories. In these cases, since chiral gauge symmetries are at the cornerstones of the

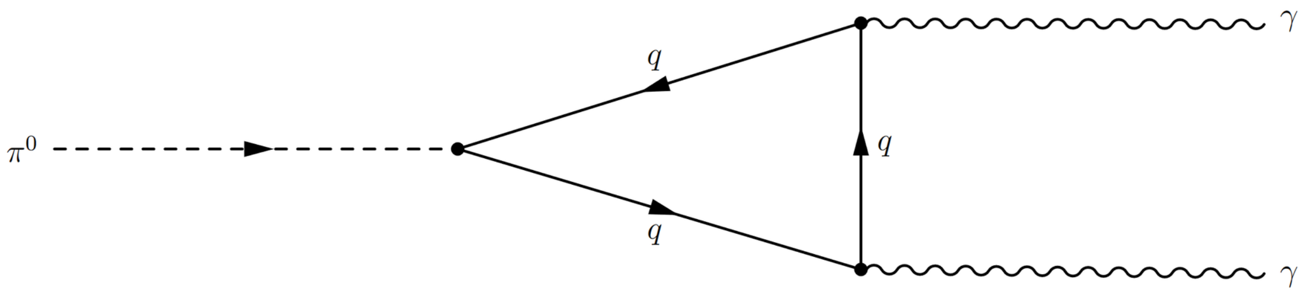

Figure 1.1: $\pi^0 \to \gamma\gamma$

electroweak sector of the Standard Model with gauge group

$$G_{EW} = SU(2) \times U(1),$$

disastrous consequences may come out fatal for the overall consistency of the theory, and one must check that these potential anomalies are canceled by the field content of the Standard Model [3]. The four conditions listed below

$$\begin{aligned} \mathrm{Tr}\left[\{\tau_a, \tau_b\}\tau_c\right] &= 0, \\ \mathrm{Tr}\left[\tau_a Y Y\right] &= 0, \\ \mathrm{Tr}\left[\{\tau_a, \tau_b\} Y\right] &= 0, \\ \mathrm{Tr}\left[Y Y Y\right] &= 0, \end{aligned} \tag{1.2}$$

where τ_a $(a = 1, 2, 3)$ and Y are the generators of $SU(2)$ and $U(1)$ respectively, have to be fulfilled for the standard model to be anomaly-free. Similar anomalies appear in theories coupled to gravity, and the corresponding gravitational anomalies have been instrumental to construct consistent string theories and their effective actions.

Among the anomalies, also the so-called trace anomalies [4] have been found to be of paramount importance. They appear in conformal theories, renowned for their scale invariance, which often have classically a traceless stress tensor. Nevertheless, at the quantum level, an anomaly may appear, especially if the theory is coupled to suitable backgrounds, e.g. by putting it in a curved space. In this respect, there have been recent claims that a particular CP-violating contribution [5, 6, 7, 8], proportional through an imaginary coupling to the Pontryagin density

$$P = \frac{1}{2}\epsilon^{abcd} R_{abmn} R_{cd}{}^{mn}, \tag{1.3}$$

appears in the trace of the stress tensor of chiral theories coupled to gravity:

$$\langle T^a{}_a \rangle = \frac{1}{180(4\pi)^2}\Big(\frac{11}{4}E - \frac{9}{2}\mathcal{W}^2 - i\frac{15}{4}P\Big), \tag{1.4}$$

where $\mathcal{W}^2$ is the Weyl density

$$\mathcal{W}^2 = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{1}{3}R^2$$

and E the Euler one

$$E = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

(with R_{abcd}, R_{ab} and R standing for the Riemann tensor, Ricci tensor and scalar curvature respectively). By any means, because of its accordance with the Sakharov conditions, (1.4) might provide a so far unexplored and new mechanism for baryogenesis. Furthermore, this imaginary contribution, being potentially fatal for unitarity, could be used as a new requirement for anomaly cancellation, introducing an additional consistency criterion for a theory [6]. However, these assertions have not found confirmation by other groups [9, 10], which instead finds for a chiral fermion just half the trace anomaly of a Dirac spinor, i.e.

$$\langle T^a{}_a \rangle = \frac{1}{180(4\pi)^2}\Big(\frac{11}{2}E - 9\mathcal{W}^2\Big) \tag{1.5}$$

with no appearence of the Pontryagin density.

A similar situation arises also in the coupling of a chiral fermion to background gauge fields. Also in this case a conjectured contribution to a topological density, now depending on the background gauge fields, has been put forward [5]. However, explict calculations again have not found its presence [11, 12]. The method used there consisted in employing a Pauli-Villars (PV) regularization with PV fields with Majorana mass. The calculation is then recasted as a standard Fujikawa "jacobian" calculation, as shown in [13], which is then implemented with heat kernel techinques.

Given this state of affairs, we wish to study again the case of a chiral fermion coupled to an abelian background gauge field, but regulating it with PV fields with a Dirac mass term. This mass term is much more difficult to treat, as it ruins also general coordinate and local Lorentz invariance once the model is put in curved space. Thus more anomalies are expected, though one expects that variation of local counterterms should relate them to those obtained with the PV Majorana mass, getting in particular a vanishing contribution of the Chern-Pontryagin topological density to the trace anomaly. In this thesis we aim to compute the anomalies with the PV Dirac mass regularization, leaving the issue of possible counterterms to future work.

1.1 The model

We consider a chiral (left-handed) spinor λ coupled to an abelian gauge field A_a, whose lagrangian takes the form

$$\mathcal{L}_W = -\overline{\lambda}\gamma^a(\partial_a - iA_a)\lambda = -\overline{\lambda}\gamma^a D_a(A)\lambda = -\overline{\lambda}\not{D}(A)\lambda \tag{1.6}$$

(see notations in appendix A and B). Analogously to the coupling to gravity, also this abelian gauge coupling may give rise to a trace anomaly with an imaginary contribution proportional to the odd-parity Chern-Pontryagin topological density $F\widetilde{F}$, as suggested in [5]. Since the Chern-Pontryagin density $F\widetilde{F}$ satisfies the consistency conditions [14] for trace anomalies, this is indeed not exclued. On the other hand its trace anomaly has been computed only recently in [11], through a Pauli-Villars regularization involving Pauli-Villars field with Majorana mass. The calculation reproduced the usual gauge U(1) anomaly on top of an explicit gauge invariant form of the trace anomaly, in contrast to the conjectures made in [5] about the existence of $F\widetilde{F}$ in the structure of the latter. In this paper, we wish to check again the results of [11] using Pauli-Villars fields with Dirac mass as alternative regulators. The Dirac mass breaks additional symmetries with respect to the Majorana mass, so it may induce anomalies in the conservation of the stress tensor (gravitational anomalies) and on its antisymmetric part (local Lorentz anomalies), in addition to the expected trace and chiral anomalies. We shall compute all these anomalies, which is a necessary step before studying eventual counterterms whose symmetry variation should reinstate both conservation and symmetry of the stress tensor. In the present case calculations are more demanding then those of ref. [11], since the gravitational anomaly requires more general heat kernel formulae. As a matter of fact, the computation of the gravitational anomaly constitutes the main technical task carried out in this paper. Moreover, for the occasion we improved the notebook that was developed in [15] so that, after every calculus, we could check the ensuing result.

In any case, before starting our analyses of the anomalies, we first need to review the general setup we are going to use, which includes the Fujikawa's method and the scheme of ref. [13] for extracting consistent regulators.

1.2 Fujikawa's method and consistent regulators

The setup we are going to review was devised to insure that the anomalies indeed satisfy the integrability conditions reported in [14]. It relates the calculation of the anomalies done with the Fujikawa's method to that of some Feynman graphs regulated via Pauli-Villars fields.

1.2.1 Fujikawa's method

Fujikawa [16, 17] approached the problem of the anomalies by considering the path integral

$$Z = \int D\upsilon \; e^{iS[\upsilon]} \tag{1.7}$$

and recognizing that the functional measure $D\upsilon$ can be regarded as responsible for their appearance. In fact, its lack of invariance under a certain symmetry of the classical action $S[\upsilon]$ causes the full path integral Z to be non invariant. For instance, let us suppose the aforementioned symmetry is a Lie symmetry that depends on a constant parameter θ, acting on the field υ and the coordinates x through the following laws

$$x^\nu \longrightarrow x'^\nu = q^\nu(x, \theta) \tag{1.8}$$

$$\upsilon(x) \longrightarrow \upsilon'(x') = Q(x, \upsilon(x), \partial_\nu \upsilon(x), \theta) \;, \tag{1.9}$$

which restrict to the identity transformations whenever $\theta = 0$. This suggests that the infinitesimal form of (1.9), valid if $\theta \ll 1$, can be stated as

$$\delta_\theta \upsilon(x) = \upsilon'(x) - \upsilon(x) = \theta \; I(x, \upsilon(x), \partial_\nu \upsilon(x)) \;, \tag{1.10}$$

where I is a generic function descending directly from Q and q.

Now, we bestow a space-time dependence upon θ, so that it gets promoted from constant to real valued function $\theta(x)$, and (1.10) becomes

$$\delta_{\theta(x)} \upsilon(x) = \theta(x) \; I(x, \upsilon(x), \partial_\nu \upsilon(x)) \;. \tag{1.11}$$

Even though in the very beginning, by definition of symmetry, we had

$$\delta_\theta S[\upsilon] = 0 \ ,$$

as soon as (1.11) is involved, the status of symmetry is generally lost and the new variation of the action has to be necessarily expressed as

$$\delta_{\theta(x)} S[\upsilon] = \int d^n x \ \partial_\nu\big(\theta(x)\big) C^\nu(x) \ , \tag{1.12}$$

which guarantees the restoring of the symmetry, $\delta_{\theta(x)} S[\upsilon] = 0$, every time that $\theta(x) = const.$ As concerns $C^\nu(x)$, it represents the classically conserved Noether current of the model: if (1.12) is evaluated on-shell, then not only the least action principle ensures a null result, but since an integration by parts produces

$$\delta_{\theta(x)} S[\upsilon] = -\int d^n x \ \theta(x) \partial_\nu\big(C^\nu(x)\big) = 0 \ , \tag{1.13}$$

the continuity equation

$$\partial_\nu\big(C^\nu(x)\big) = 0$$

follows as expected.

Anyway, getting back to the problem at hand, we subject (1.7) to a series of algebraic manipulations: first, we rename the dummy integration variable

$$\int D\upsilon \ e^{iS[\upsilon]} = \int D\upsilon' \ e^{iS[\upsilon']} \ , \tag{1.14}$$

and, by identifying υ' with the old variable υ increased by the infinitesimal space-time-depending variation (1.11), we execute the following change of the functional integration variable in the RHS of (1.14)

$$\upsilon' = \upsilon + \delta_{\theta(x)} \upsilon \ . \tag{1.15}$$

Thus, in doing so, we have to recall that the measure changes according to

$$\begin{aligned} D\upsilon' &= D\upsilon \ \mathrm{Det} \left| \frac{\partial \upsilon'}{\partial \upsilon} \right| = \\ &= D\upsilon \ \mathrm{Det} \left| \frac{\partial \upsilon}{\partial \upsilon} + \frac{\partial\big(\delta_{\theta(x)}\upsilon\big)}{\partial \upsilon} \right| = \\ &= D\upsilon \ \mathrm{Det} \left| \mathbb{1} + \frac{\partial\big(\delta_{\theta(x)}\upsilon\big)}{\partial \upsilon} \right| = \\ &\approx D\upsilon \left(1 + \mathrm{Tr} \left[\frac{\partial\big(\delta_{\theta(x)}\upsilon\big)}{\partial \upsilon} \right] \right) \ , \end{aligned}$$

where we were able to exploit the fact that $\mathrm{Det}|\mathbb{1}+A| \approx 1+\mathrm{Tr}[A]$, if $A \ll \mathbb{1}$. Meanwhile, the action undergoes to

$$
\begin{aligned}
S[\upsilon'] = \underbrace{S[\upsilon+\delta_{\theta(x)}\upsilon]}_{\text{functional Taylor expand}} = \\
&\approx S[\upsilon] + \frac{\delta_{\theta(x)}S[\upsilon]}{\delta_{\theta(x)}\upsilon}\ \delta_{\theta(x)}\upsilon = \\
&= S[\upsilon] + \delta_{\theta(x)}S[\upsilon] \ ,
\end{aligned}
$$

causing its exponential to be Taylor expanded as

$$
\begin{aligned}
e^{iS[\upsilon']} &= e^{iS[\upsilon+\delta_{\theta(x)}\upsilon]} = \\
&\approx e^{iS[\upsilon]+i\delta_{\theta(x)}S[\upsilon]} = \\
&\approx e^{iS[\upsilon]}\left(1+i\delta_{\theta(x)}S[\upsilon]\right) \ .
\end{aligned}
$$

Therefore, putting all together in (1.14), we recognize (at first order)

$$
\begin{aligned}
\int D\upsilon\ e^{iS[\upsilon]} = \int D\upsilon'\ e^{iS[\upsilon']} = \\
&\approx \underbrace{\int D\upsilon \left(1+\mathrm{Tr}\left[\frac{\partial\big(\delta_{\theta(x)}\upsilon\big)}{\partial\upsilon}\right]\right) e^{iS[\upsilon]}\left(1+i\delta_{\theta(x)}S[\upsilon]\right)}_{\text{keep the } 1^{\text{st}} \text{ order}} = \\
&\approx \int D\upsilon\ e^{iS[\upsilon]} + \int D\upsilon\ \mathrm{Tr}\left[\frac{\partial\big(\delta_{\theta(x)}\upsilon\big)}{\partial\upsilon}\right] e^{iS[\upsilon]} + i\int D\upsilon\ \delta_{\theta(x)}S[\upsilon]\ e^{iS[\upsilon]} \ ,
\end{aligned}
$$

which in turn could be inverted to find that

$$
\int D\upsilon\ \mathrm{Tr}\left[\frac{\partial\big(\delta_{\theta(x)}\upsilon\big)}{\partial\upsilon}\right] e^{iS[\upsilon]} + i\int D\upsilon\ \delta_{\theta(x)}S[\upsilon]\ e^{iS[\upsilon]} = 0 \ . \tag{1.16}
$$

For the sake of convenience, we then proceed by relabeling the infinitesimal part of the jacobian[1] as

$$
J = \frac{\partial\big(\delta_{\theta(x)}\upsilon\big)}{\partial\upsilon} \ ,
$$

[1]Albeit not reported, the jacobian itself carries a space-time dependence that, when made explicit, should read as

$$
\frac{\partial\big(\delta_{\theta(x)}\upsilon\big)}{\partial\upsilon} \equiv \frac{\delta\big(\delta_{\theta(x)}\upsilon(x)\big)}{\delta\upsilon(y)} .
$$

Of course, this calls for an expression of its trace of the form

$$
\mathrm{Tr}\left[\frac{\partial\delta_{\theta(x)}\upsilon}{\partial\upsilon}\right] = \int d^4x\, d^4y\ \frac{\delta\big(\delta_{\theta(x)}\upsilon(x)\big)}{\delta\upsilon(y)}\delta^4(x-y) \ ,
$$

where the Dirac distribution is needed to ensure locality.

after which we divide (1.16) by the original path integral Z. Operating in this way, we can provide a legitimate expression for the quantum anomaly in terms of normalized correlation functions

$$\frac{1}{Z}\int D\upsilon \ \mathrm{Tr}\,[J]\, e^{iS[\upsilon]} + \frac{i}{Z}\int D\upsilon \ \delta_{\theta(x)} S[\upsilon]\ e^{iS[\upsilon]} =$$

$$\equiv \langle\, \mathrm{Tr}[J]\,\rangle + i\langle\, \delta_{\theta(x)} S[\upsilon]\,\rangle =$$
$$= \ \ \mathrm{Tr}[J] + i\langle\, \delta_{\theta(x)} S[\upsilon]\,\rangle \ \ = 0\ ,$$

which presumed $\mathrm{Tr}[J]$ to be independent of any quantum field. In the very end, this last equation gets tidied in

$$i\langle\, \delta_{\theta(x)} S[\upsilon]\,\rangle = -\mathrm{Tr}[J]\ , \tag{1.17}$$

and further developed in

$$i\underbrace{\langle \delta_{\theta(x)} S[\upsilon]\rangle}_{(1.13)} = -i\int d^n x\ \theta(x)\partial_\nu\langle\, C^\nu(x)\,\rangle = -\mathrm{Tr}[J]\ ,$$

revealing us that the true origin of the anomaly is to be found in the non-triviality

$$J \neq 0 \tag{1.18}$$

of the jacobian we employed in the change of variable. In addition, we see that the Noether current $C^\nu(x)$ cannot be quantum preserved,

$$\partial_\nu\langle\, C^\nu(x)\,\rangle \neq 0\ ,$$

as long as condition (1.18) on J is met.

We're almost there. In fact, a well-defined prescription on how to compute the trace in (1.17), without incurring in any infinity, is the only thing missing. Fujikawa managed to figure out how to do that. Using a suitable negative-definite[2] operator R, the so-called regulator, weighted by the inverse of a squared mass, M^2, he first put the exponential of the ratio $-\frac{R}{M^2}$ in the above-mentioned trace, alongside with J, and then took the infinite mass limit, thus obtaining the one-loop regulated anomaly

$$i\langle\, \delta_{\theta(x)} S[\upsilon]\,\rangle = -\lim_{M\to\infty} \mathrm{Tr}[J\, e^{-\frac{R}{M^2}}] \tag{1.19}$$

with eventual divergent terms (for $M \to \infty$) in the expression assumed to be canceled by renormalization. Despite this achieves a fair regularization, one should bear in mind that an anomaly is eventually recognized as such only if it cannot be erased by the variation of a local counterterm. In this set-up, the main problem is how to identify those regulators that indeed produce the consistent anomalies, i.e. those that arise from the symmetry variation of an effective action.

The next section will help us to clarify how the extraction of an adequate regulator may be obtained.

[2]More correctly, Fujikawa envisaged R to be negative-definite only after a Wick rotation.

1.2.2 Consistent regulators

A procedure for identifying consistent regulators, i.e. those regulators that indeed produce consistent anomalies in the Fujikawa scheme, has been put forward in [13], by considering a Pauli-Villars regularization of the quantum theory.

Being mostly interested in describing the physics of a massless field or set of fields, collectively denoted by φ, we rewrite the lagrangian of the corresponding model as

$$\mathcal{L} = \frac{1}{2}\varphi^T T\mathcal{O}\varphi \ , \tag{1.20}$$

to maintain the same notation of [13]. Moreover, we are going to assume that (1.20) is invariant under the linear transformation

$$\delta\varphi = K\varphi \ . \tag{1.21}$$

To keep the argument as general as possible, one assumes that the kinetic operator $T\mathcal{O}$ might depend on some background fields as well.

Furthermore, since we intend to cancel all the divergences that this theory possesses at one-loop, we must introduce a Pauli-Villars (PV) field, say ϕ, endowed with a mass M. This comes along with a lagrangian density of the following kind

$$\mathcal{L}_{PV} = \frac{1}{2}\phi^T T\mathcal{O}\phi + \frac{1}{2}M\phi^T T\phi \ , \tag{1.22}$$

here written in the same way as (1.20), and again T is allowed to depend on the eventual background fields. After that, we subtract the massive PV one-loop to the original one, and, as is the custom with this type of regularization, we implement the limit $M \to \infty$, which should decouple the massive PV fields from the theory. To this end, the symmetry (1.21) has to be extended to the kinetic part of (1.22), too,

$$\delta\phi = K\phi,$$

meaning that the symmetry can be broken solely by the massive part of $\mathcal{L}_{PV}$, i.e. that marked by the matrix T, assumed to be invertible. In that case, we have

$$\begin{aligned} \delta\mathcal{L}_{PV} &= \frac{1}{2}M(\phi^T K^T T\phi + \phi^T \delta T\phi + \phi^T TK\phi) = \\ &= \frac{1}{2}M\phi^T(K^T T + \delta T + TK)\phi \ . \end{aligned} \tag{1.23}$$

It is worth to notice that if we could spot a PV field such that $\delta\mathcal{L}_{PV} = 0$, then no quantum anomaly would show at all, and the ensuing theory would be anomaly free. In view of the intended use of the mechanism we are developing, from now on ϕ and ϕ^T will be considered as spinor. This implies that, by dropping the hypercondensed notation[3] quietly adopted until now, it would be easy to see that only the antisymmetric part of $T\mathcal{O}$ and T will contribute in (1.22). Said differently, $T\mathcal{O}$ and T are

[3]In a hypercondensed notation all the indices outlining the multi-component structure of the fields (together with any space-time dependence) are omitted.

antisymmetric matrices (i.e. operators). This fact is used to manipulate the implicit index structure of (1.23) into

$$\delta\mathcal{L}_{PV} = \frac{1}{2}M\phi^T(\delta T + 2TK)\phi \ . \tag{1.24}$$

At this stage, the regularization procedure can at last be implemented. This is easily accomplished by modifying the path integral defining the one-loop effective action Γ, that is

$$Z = e^{i\Gamma} = \int D\varphi \ e^{iS[\varphi]} \ , \tag{1.25}$$

by adding the PV action $S_{PV}[\phi]$ to the one already displayed in (1.25). This entails that the infinite mass limit and the further functional integration upon ϕ have to be evaluated:

$$Z_{reg} = e^{i\Gamma_{reg}} = \int D\phi D\varphi \ e^{iS_{reg}[\varphi,\phi]} \equiv \lim_{M\to\infty}\int D\phi D\varphi \ e^{i(S[\varphi]+S_{PV}[\phi])} \ . \tag{1.26}$$

Now, it is the variation of (1.26) with respect to the extended version of (1.21) that allows for the anomaly to emerge. Moving in this direction, we get that

$$i\,\delta\Gamma_{reg}\,e^{i\Gamma_{reg}} = \int D\phi D\varphi \ (i\,\delta S_{reg})\,e^{iS_{reg}[\varphi,\phi]} = \lim_{M\to\infty}\int D\phi D\varphi \ (i\,\delta S_{PV})e^{i(S[\varphi]+S_{PV}[\phi])} \ , \tag{1.27}$$

where, based on the above, only the variation of S_{PV}, or rather its massive part, will be present in (1.27). Now, reintroducing indices and space-time dependence, we have

$$\begin{aligned}\delta S_{PV} &= \int d^4x\,d^4y\ \delta\mathcal{L}_{PV}(x,y) = \\ &= \frac{M}{2}\int d^4x\,d^4y\ \phi_i^T(x)\Big(\delta T^{\,ij}(x,y) + 2\big(TK\big)^{ij}(x,y)\Big)\phi_j(y) \ .\end{aligned} \tag{1.28}$$

Thus, the quotient of (1.27) by (1.26) brings in the new equation

$$i\,\delta\Gamma_{reg} = \underbrace{\frac{1}{Z_{reg}}\int D\phi D\varphi \ (i\,\delta S_{reg})\,e^{iS_{reg}[\varphi,\phi]}}_{i\langle\delta S_{reg}\rangle} = \frac{1}{Z_{reg}}\lim_{M\to\infty}\int D\phi D\varphi \ (i\,\underbrace{\delta S_{PV}}_{(1.28)})e^{i(S[\varphi]+S_{PV}[\phi])}$$

$$\Downarrow$$

$$i\,\delta\Gamma_{reg} = i\langle\delta S_{reg}\rangle = \frac{i}{Z_{reg}}\lim_{M\to\infty}\int D\phi D\varphi\ \frac{M}{2}\int d^4x\,d^4y\ \phi_i^T(x)\,X^{\,ij}(x,y)\,\phi_j(y)e^{i(S[\varphi]+S_{PV}[\phi])} \ , \tag{1.29}$$

where we identified

$$X^{\,ij}(x,y) = \Big(\big(\delta T^{\,ij}(x,y) + 2\big(TK\big)^{ij}(x,y)\Big) \ .^{4} \tag{1.30}$$

[4] We cannot help but notice that in writing (1.28) down we employed two different event dependences for ϕ and its transposed ϕ^T. Despite being the most general course of action, this is clearly contrary to the requirement of locality. This implies that $X^{\,ij}(x,y)$ and the matrices contained therein should be accordingly amended, if necessary, to include at least a delta distribution $\delta^4(x-y)$.

We manipulate the outer RHS of (1.29), by taking the reciprocal of the regulated path integral Z_{reg} inside the limit and reversing the order of space-time and functional integrations. In so doing, we are allowed to write

$$
\begin{aligned}
i\,\delta\Gamma_{reg} = i\langle \delta S_{reg}\rangle &= \frac{i}{2}\lim_{M\to\infty} M \int d^4x\, d^4y\; X^{ij}(x,y)\, \underbrace{\frac{1}{Z_{reg}}\int D\phi D\varphi\; \phi_i^T(x)\,\phi_j(y) e^{i(S[\varphi]+S_{PV}[\phi])}}_{\langle\phi(y)\phi^T(x)\rangle_{ji}} = \\
&= \frac{i}{2}\lim_{M\to\infty} M \int d^4x\, d^4y\; X^{ij}(x,y)\; \langle\phi(y)\phi^T(x)\rangle_{ji} = \\
&= \frac{i}{2}\lim_{M\to\infty} M \int d^4x\, d^4y\; X^{ij}(x,y)\; \Big(\frac{i}{T\mathcal{O}+TM}\Big)_{ji}(y,x) = \\
&= -\frac{1}{2}\lim_{M\to\infty} M \int d^4x\, d^4y\; X^{ij}(x,y)\; \big(T\mathcal{O}+TM\big)^{-1}{}_{ji}(y,x)\;,
\end{aligned} \tag{1.31}
$$

in which the exact definition of the 2-point Green's function

$$
\langle\phi(x_a)\,\phi^T(x_b)\rangle_{mn} = -\frac{1}{Z_{reg}}\int D\phi D\varphi\; \phi_m(x_a)\,\phi_n(x_b) e^{i(S[\varphi]+S_{PV}[\phi])}\;,
$$

with the opposite sign due to the Pauli-Villars nature of ϕ, and the corresponding fermionic propagator[5]

$$
\langle\phi\,\phi^T\rangle = \Big(\frac{i}{T\mathcal{O}+TM}\Big)
$$

have been used.

Finally, the ultimate form of (1.31) can be now explored. All it takes is for us to acknowledge that the complete index contraction and the two space-time integrals there performed might be equally expressed as the trace of the operator product they affect: $X(T\mathcal{O}+TM)^{-1}$. In fact, we have:

$$
\begin{aligned}
i\langle\delta S_{reg}\rangle &= -\frac{1}{2}\lim_{M\to\infty} M \int d^4x\, d^4y\; X^{ij}(x,y)\; \big(T\mathcal{O}+TM\big)^{-1}{}_{ji}(y,x) = \\
&= -\frac{1}{2}\lim_{M\to\infty} M\,\mathrm{Tr}\big[\underbrace{X}_{(1.30)}\big(T\mathcal{O}+TM\big)^{-1}\big] = \\
&= -\frac{1}{2}\lim_{M\to\infty} M\,\mathrm{Tr}\big[\big(\delta T+2TK\big)\big(T\mathcal{O}+TM\big)^{-1}\big] = \\
&= -\frac{1}{2}\lim_{M\to\infty} M\,\mathrm{Tr}\Big[\big(\delta T+2TK\big)\Big(\frac{\mathcal{O}}{M}+\mathbb{1}\Big)^{-1}\big(TM\big)^{-1}\Big] = \\
&\overset{(B.16)}{=} -\lim_{M\to\infty}\mathrm{Tr}\Big[\Big(\frac{1}{2}T^{-1}\delta T+K\Big)\Big(\frac{\mathcal{O}}{M}+\mathbb{1}\Big)^{-1}\Big] = \\
&= -\lim_{M\to\infty}\mathrm{Tr}\Big[\Big(\frac{1}{2}T^{-1}\delta T+K\Big)\underbrace{\Big(\mathbb{1}-\frac{\mathcal{O}}{M}\Big)\Big(\mathbb{1}-\frac{\mathcal{O}}{M}\Big)^{-1}}_{\mathbb{1}}\Big(\frac{\mathcal{O}}{M}+\mathbb{1}\Big)^{-1}\Big] = \\
&= -\lim_{M\to\infty}\mathrm{Tr}\Big[\Big(\frac{1}{2}T^{-1}\delta T+K+\frac{\delta\mathcal{O}}{2M}\Big)\Big(\mathbb{1}-\frac{\mathcal{O}^2}{M^2}\Big)^{-1}\Big].
\end{aligned} \tag{1.32}
$$

[5]As usual, the fermionic propagator is deduced from the inverse of the operator specified in the PV lagrangian (1.22).

The last equality follows from a result derived in [18], where the well-known invariance of (1.20) under (1.21)

$$\delta\mathcal{L} = \frac{1}{2}\varphi^T(\delta T\mathcal{O} + T\delta\mathcal{O} + 2T\mathcal{O}K)\varphi = 0$$

has been employed. Next, by approximating

$$\Big(\mathbb{1} - \frac{\mathcal{O}^2}{M^2}\Big)^{-1} \approx e^{\frac{\mathcal{O}^2}{M^2}} \ ,$$

eq. (1.32) becomes

$$i\langle\delta S_{reg}\rangle = -\lim_{M\to\infty} \mathrm{Tr}\Big[\Big(K + \frac{1}{2}T^{-1}\delta T + \frac{\delta\mathcal{O}}{2M}\Big)e^{\frac{\mathcal{O}^2}{M^2}}\Big] \ , \tag{1.33}$$

whose comparison with (1.19) results in a precise and consistent expression for the infinitesimal jacobian

$$J = \Big(K + \frac{1}{2}T^{-1}\delta T + \frac{\delta\mathcal{O}}{2M}\Big) \ , \tag{1.34}$$

and regulator

$$R = -\mathcal{O}^2 \ , \tag{1.35}$$

formerly discussed in section 1.2.1 (we are implicitly requesting for the operator $-\mathcal{O}^2$ to be negative-definite after a Wick rotation).

In conclusion, this improved Fujikawa method permits one to regulate the jacobian that produces the consistent anomaly by cutting off all the ultraviolet frequencies, because of the mass M, and recognize the anomaly as the finite mass-independent terms in (1.33) (as possible diverging terms are canceled by renormalization).

Chapter 2

Symmetries and the Dirac mass

To study our model, and in particular the structure of its stress tensor, it is useful to couple it to background gravity, so that one can use the vierbein as a source of the stress tensor. After insertion of one stress tensor into correlation functions, we restrict ourselves to flat space, as we are interested in the contribution to the anomalies of the U(1) gauge field only.

The coupling to gravity of the lagrangian in (1.6) reads

$$\mathcal{L}_W = -e\,\overline{\lambda}\gamma^\mu\nabla_\mu\lambda = -e\,\overline{\lambda}\not\nabla\lambda \tag{2.1}$$

where $\gamma^\mu = e^\mu{}_a\gamma^a$ are the gamma matrices with curved indices[1], $e^\mu{}_a$ is the inverse of the vierbein $e_\mu{}^a$, and ∇_μ is the covariant derivative

$$\nabla_\mu = \partial_\mu - iA_\mu + \tfrac{1}{4}\omega_{\mu ab}(e)\gamma^a\gamma^b \,, \tag{2.2}$$

containing the $U(1)$ gauge potential A_μ and the spin connection $\omega_{\mu ab}(e_\mu{}^a)$, which is another function of the vierbein. The action $S_W = \int d^4x\,\mathcal{L}_W$ is gauge invariant and invariant under general coordinate, local Lorentz, and Weyl transformations. The energy-momentum tensor, or stress tensor, is defined by

$$T^{\mu a}(x) = \frac{1}{e}\frac{\delta S_W}{\delta e_{\mu a}(x)} \tag{2.3}$$

where e is the determinant of the vierbein. It is covariantly conserved (up to some classical breaking term), symmetric, and traceless on-shell, as consequence of diffeomorphisms, local Lorentz invariance, and Weyl symmetry, respectively. In flat space, and simplified with the equations of motion, it reads

$$T_{ab} = \tfrac{1}{4}\overline{\lambda}\left(\gamma_a\overset{\leftrightarrow}{D}_b + \gamma_b\overset{\leftrightarrow}{D}_a\right)\lambda \,, \tag{2.4}$$

with $\overset{\leftrightarrow}{D}_a = D_a - \overset{\leftarrow}{D}_a$, and the conservation law it satisfies has the form

$$D_a T^{ab} = -i\overline{\lambda}\gamma_a\lambda\,F^{ab} \tag{2.5}$$

[1]Tensors portrayed with greek, "curved" (or "coordinate") indices correspond to those that are being described by observers living in a general Einstein reference frame. Here an index gets lowered or raised by the generic curved metric $g_{\mu\nu}$ and its inverse. On the contrary, latin, "flat" (or "frame") indices characterize tensor quantities as they appear in the eyes of an observer residing in a locally flat Lorentz frame, where the metric is seen to be the Minkowskian one ($g_{ab} \equiv \eta_{ab}$).

The vierbein $e_\mu{}^a$ acts on curved indices by making them flat; its inverse does the opposite.

being

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{2.6}$$

the usual U(1) field strenght.

For later purposes, it is useful to report the infinitesimal form of the background local symmetries in curved space. They take the form

$$\begin{aligned}
\delta {e_\mu}^a &= \xi^\nu \partial_\nu {e_\mu}^a + (\partial_\mu \xi^\nu) {e_\nu}^a + {\omega^a}_b {e_\mu}^b + \sigma {e_\mu}^a \\
\delta A_\mu &= \xi^\nu \partial_\nu A_\mu + (\partial_\mu \xi^\nu) A_\nu \\
\delta \lambda &= \xi^\mu \partial_\mu \lambda + \tfrac{1}{4} \omega_{ab} \gamma^{ab} \lambda - \tfrac{3}{2} \sigma \lambda \\
\delta \rho &= 0
\end{aligned} \tag{2.7}$$

where ξ^μ, ω_{ab}, and σ are the infinitesimal local parameters of the Einstein, Lorentz, and Weyl symmetries, respectively. We have included a right handed PV field ρ which remains inert under these symmetries, as it is uncoupled to the curved background. In addition, there is the U(1) gauge symmetry that acts non-trivially only on λ and A_μ with infinitesimal local parameter ζ

$$\begin{aligned}
\delta A_\mu &= \partial_\mu \zeta \\
\delta \lambda &= i \zeta \lambda \ .
\end{aligned} \tag{2.8}$$

Let us also review for completeness how the background invariance of the action produces the described properties of the stress tensor. Under the chiral and Weyl symmetry, with local parameter $\zeta(x)$ and $\sigma(x)$ respectively, using the invariance of the action one finds

$$\begin{aligned}
\delta_{\zeta(x)} S &= \int d^4x \left(\frac{\delta S}{\delta {e_\mu}^a(x)} \delta_{\zeta(x)} {e_\mu}^a(x) + \frac{\delta S}{\delta A_\mu(x)} \delta_{\zeta(x)} A_\mu(x) + \frac{\delta_R S}{\delta \lambda(x)} \delta_{\zeta(x)} \lambda(x) + \delta_{\zeta(x)} \overline{\lambda}(x) \frac{\delta_L S}{\delta \overline{\lambda}(x)} \right) = \\
&= i \int d^4x e \, \overline{\lambda}(x) \gamma^\mu \lambda(x) \, \big(\delta_{\zeta(x)} A_\mu(x) \big) = i \underbrace{\int d^4x e \, \overline{\lambda}(x) \gamma^\mu \lambda(x) \, \partial_\mu \big(\zeta(x) \big)}_{\text{integrate by parts}} = \\
&= - \int d^4x e \, \partial_\mu \big(i \overline{\lambda}(x) \gamma^\mu \lambda(x) \big) \zeta(x) = 0
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\delta_{\sigma(x)} S &= \int d^4x \left(\frac{\delta S}{\delta {e_\mu}^a(x)} \delta_{\sigma(x)} {e_\mu}^a(x) + \frac{\delta_R S}{\delta \lambda(x)} \delta_{\sigma(x)} \lambda(x) + \delta_{\sigma(x)} \overline{\lambda}(x) \frac{\delta_L S}{\delta \overline{\lambda}(x)} \right) = \\
&= \int d^4x e \, {T^\mu}_a(x) \delta_{\sigma(x)} {e_\mu}^a(x) = \int d^4x e \, {T^\mu}_a(x) \sigma(x) {e_\mu}^a(x) = \\
&= \int d^4x e \, {T^a}_a(x) \sigma(x) = 0 \ ,
\end{aligned} \tag{2.10}$$

where the equations of motion of the spinor field have been implemented (we used left and right derivatives for the Grassmann valued fields), and the fact that the inert fields under a given transformation do not contribute to the corresponding variation has been used. Because of the arbitrariness of both $\zeta(x)$ and $\sigma(x)$, we end up with the on-shell preservation of the $U(1)$ current

$$\partial_\mu \big(i \overline{\lambda}(x) \gamma^\mu \lambda(x) \big) = 0 \ , \tag{2.11}$$

and the tracelessness (always on-shell) of the stress tensor

$$T^a{}_a(x) = 0 \ . \tag{2.12}$$

Similarly, the Lorentz symmetry with local parameters $\omega_{ab}(x)$ implies

$$\begin{aligned}
\delta_{\omega(x)} S &= \int d^4x \left(\frac{\delta S}{\delta e_\mu{}^a(x)} \delta_{\omega(x)} e_\mu{}^a(x) + \frac{\delta_R S}{\delta \lambda(x)} \delta_{\omega(x)} \lambda(x) + \delta_{\omega(x)} \overline{\lambda}(x) \frac{\delta_L S}{\delta \overline{\lambda}(x)} \right) = \\
&= \int d^4x e\, T^\mu{}_a(x) \delta_{\omega(x)} e_\mu{}^a(x) = \int d^4x e\, T^\mu{}_a(x) \omega^a{}_b(x) e_\mu{}^b(x) = \\
&= \int d^4x e\, T^{ba}(x) \omega_{ab}(x) = 0
\end{aligned} \tag{2.13}$$

and constrains the antisymmetric part of the stress tensor to vanish on-shell. Again, the U(1) gauge field is inert under a local Lorentz transformation and does not contribute to the variation of the action. Thus

$$T^{ab}(x) = T^{ba}(x) \ . \tag{2.14}$$

Finally, a conservation law for the stress tensor arises as a consequence of infinitesimal diffeomorphism invariance and takes the following form

$$\begin{aligned}
\delta_{\xi(x)} S &= \int d^4x \left(\frac{\delta S}{\delta e_\mu{}^a(x)} \delta_{\xi(x)} e_\mu{}^a(x) + \frac{\delta S}{\delta A_\mu(x)} \delta_{\xi(x)} A_\mu(x) + \frac{\delta_R S}{\delta \lambda(x)} \delta_{\xi(x)} \lambda(x) + \delta_{\xi(x)} \overline{\lambda}(x) \frac{\delta_L S}{\delta \overline{\lambda}(x)} \right) = \\
&= \int d^4x e \Big(T^\mu{}_a(x) \underbrace{\mathcal{L}_{\xi(x)} e_\mu{}^a(x)}_{\xi^\nu \partial_\nu e_\mu{}^a + \partial_\mu \xi^\nu e_\nu{}^a} + \frac{1}{e} \frac{\delta S}{\delta A_\mu(x)} \mathcal{L}_{\xi(x)} A_\mu(x) \Big) = \\
&= \int d^4x e \, \left(T^\mu{}_a(x) \nabla_\mu \xi^a(x) + i \overline{\lambda} \gamma^\mu \lambda \xi^\nu(x) F_{\nu\mu} \right) = \\
&= \int d^4x e\, \xi_a(x) \left(-\nabla_\mu T^{\mu a}(x) + i \overline{\lambda} \gamma_b \lambda F^{ab}(x) \right) = 0 \ .
\end{aligned} \tag{2.15}$$

An explanation is in order: in the second line we used the fermion equation of motions, while in the third, after replacing the Lie derivative of the vierbein, we employed the property

$$\nabla_\nu e_\mu{}^a = \partial_\nu e_\mu{}^a - \Gamma_{\nu\mu}{}^\lambda e_\lambda{}^a + \omega_\nu{}^a{}_b e_\mu{}^b = 0$$

to write

$$\partial_\nu e_\mu{}^a = \Gamma_{\nu\mu}{}^\lambda e_\lambda{}^a - \omega_\nu{}^a{}_b e_\mu{}^b \ . \tag{2.16}$$

We then added for free a spin connection term

$$\xi^\nu \omega_\nu{}^a{}_b e_\mu{}^b \ ,$$

as it drops out once the stress tensor is symmetric, recognized the covariant derivative $\nabla_\mu \xi^a(x)$ and integrated by parts.

2.1 Pauli-Villars fields with Dirac mass

To regulate the quantum theory we consider massive Pauli-Villars (PV) fields, whose mass must be sent to infinity. In particular, we consider a left handed PV field with the same coupling of the original chiral fermion, plus a free right handed PV field which is needed to write down a Dirac mass term. We name λ and ρ these PV fields (calling λ one of the PV field should not cause any confusion in the following). Besides, since we are free to choose any arbitrary mass term, we adopt the one acquired from a customary Dirac mass term $M(\overline{\lambda}\rho + \overline{\rho}\lambda)$ by attaching to it a generic power of the vierbein determinant e^{α}. Therefore, the Pauli-Villars lagrangian we are going to employ in the regularization procedure is

$$\mathcal{L}_{PV} = -e\overline{\lambda}\nabla\!\!\!/\lambda - \overline{\rho}\partial\!\!\!/\rho - M(\overline{\lambda}\rho + \overline{\rho}\lambda)e^{\alpha} \; . \tag{2.17}$$

A few remarks about this choice are now in order: i) the massless part of the PV field λ should have precisely the same couplings to the gauge field and gravity as the original chiral spinor that we want to regulate, ii) the massless part of the PV field ρ should have no couplings at all, not to spoil the regularization, iii) the Dirac mass term is arbitrary, in the sense that it could contain also coupling to the background fields, possibly chosen in order to manifestly preserve some symmetries. By inspection one recognizes that no symmetry can be manifestly preserved, so we have decided to be as general as possible by introducing a coupling to gravity even in the mass term. In fact, we expect that such an arbitrariness should allow one to scan a one parameter family of distinct regularizations, and eventually check the independence of the final anomalies from the regularization scheme adopted. Nevertheless, we will not explore this captivating chance, as it lies outside our purpose, leaving the analysis for future debates.

As anticipated, being not symmetric, this mass term may produce anomalies on all possible symmetries (gauge, Einstein, local Lorentz and Weyl), which we are going to compute. Again, one expects that only a trace anomaly (on top of the chiral one) will survive at the end, as already obtained with the Majorana mass in [11], though we will not face here this issue that requires the study of counterterms.

2.2 Regulators and jacobians

As explained in sec. 1.2.2, the PV lagrangian presented earlier can be cast in the form (1.22)

$$\mathcal{L}_{PV} = \frac{1}{2}\phi^T T\mathcal{O}\phi + \frac{1}{2}M\phi^T T\phi \; ,$$

where ϕ is now a 16−dimensional column vector made up of 4 PV spinors

$$\phi = \begin{pmatrix} \lambda \\ \lambda_c \\ \rho \\ \rho_c \end{pmatrix} \; . \tag{2.18}$$

Hence, we rewrite (2.17) solely in terms of the the charge conjugate fields λ_c and ρ_c, rather then making use of the Dirac conjugates $\overline{\lambda}$ and $\overline{\rho}$. In fact, since the above lagrangian and the one that encodes the dynamics of the corresponding antiparticle should both lead to the exact same physics, we are allowed to write

$$
\begin{aligned}
\mathcal{L}_{PV} &= -e\overline{\lambda}\nabla\!\!\!/\lambda - \overline{\rho}\partial\!\!\!/\rho - M(\overline{\lambda}\rho + \overline{\rho}\lambda)e^{\alpha} = \\
&= \frac{e}{2}\left(\lambda_c^T C \nabla\!\!\!/(A)\lambda + \lambda^T C \nabla\!\!\!/(-A)\lambda_c\right) + \frac{1}{2}\left(\rho_c^T C \partial\!\!\!/\rho + \rho^T C \partial\!\!\!/\rho_c\right) + \\
&\quad + \frac{M}{2}e^{\alpha}\left(\lambda_c^T C\rho + \lambda^T C\rho_c + \rho_c^T C\lambda + \rho^T C\lambda_c\right),
\end{aligned}
$$

where (see appendix A and B)

$$
\begin{aligned}
\lambda_c &= C^{-1}\overline{\lambda}^T = -C\overline{\lambda}^T, \\
\rho &= C^{-1}\overline{\rho}^T = -C\overline{\rho}^T, \\
\nabla\!\!\!/(A) &= \gamma^{\mu}(\partial_\mu - iA_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b), \\
\nabla\!\!\!/(-A) &= \gamma^{\mu}(\partial_\mu + iA_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b).
\end{aligned}
$$

This allows to recognize the matrix T and the regulator $\mathcal{O}^2$ to be used later on:

$$
T\mathcal{O} = \begin{pmatrix} 0 & eC\nabla\!\!\!/(-A)P_R & 0 & 0 \\ eC\nabla\!\!\!/(A)P_L & 0 & 0 & 0 \\ 0 & 0 & 0 & C\partial\!\!\!/P_L \\ 0 & 0 & C\partial\!\!\!/P_R & 0 \end{pmatrix} \tag{2.19}
$$

$$
T = \begin{pmatrix} 0 & 0 & 0 & e^{\alpha}CP_L \\ 0 & 0 & e^{\alpha}CP_R & 0 \\ 0 & e^{\alpha}CP_R & 0 & 0 \\ e^{\alpha}CP_L & 0 & 0 & 0 \end{pmatrix} \tag{2.20}
$$

$$
\mathcal{O} = \begin{pmatrix} 0 & 0 & e^{-\alpha}\partial\!\!\!/P_R & 0 \\ 0 & 0 & 0 & e^{-\alpha}\partial\!\!\!/P_L \\ e^{1-\alpha}\nabla\!\!\!/(A)P_L & 0 & 0 & 0 \\ 0 & e^{1-\alpha}\nabla\!\!\!/(-A)P_R & 0 & 0 \end{pmatrix} \tag{2.21}
$$

$$
\mathcal{O}^2 = \begin{pmatrix} e^{(1-2\alpha)}\partial\!\!\!/\nabla\!\!\!/(A)P_L & 0 & 0 & 0 \\ 0 & e^{(1-2\alpha)}\partial\!\!\!/\nabla\!\!\!/(-A)P_R & 0 & 0 \\ 0 & 0 & e^{(1-2\alpha)}\nabla\!\!\!/(A)\partial\!\!\!/P_R & 0 \\ 0 & 0 & 0 & e^{(1-2\alpha)}\nabla\!\!\!/(-A)\partial\!\!\!/P_L \end{pmatrix}. \tag{2.22}
$$

Note that the projectors

$$
P_L = \frac{\mathbb{1} + \gamma^5}{2} \qquad \text{and} \qquad P_R = \frac{\mathbb{1} - \gamma^5}{2} \tag{2.23}
$$

appear because λ and ρ are chiral spinors.

Now, to apply the general scheme of refs. [13, 18] to compute the anomalies, we saw from (1.34) and (1.35) that the infinitesimal jacobian and regulator taking part in the Fujikawa set-up, read (in the same notation as [11])

$$J = K + \frac{1}{2}T^{-1}\delta T + \frac{1}{2}\frac{\delta \mathcal{O}}{M}\,, \qquad R = -\mathcal{O}^2\,.$$

While it's true that $\delta\mathcal{O}$ is off-diagonal and won't contribute to $\mathrm{Tr}[J]$, as long as we consider the gravity covariant extension of the PV lagrangian (2.17), it's possible that the term $\frac{1}{2}T^{-1}\delta T$ may actually bring some contributions. Hence, we are forced to ascertain whether it vanishes or not, when restricting ourselves to flat space. In particular, from (2.20) it's easy to verify that

$$T^{-1} = e^{-\alpha}\begin{pmatrix} 0 & 0 & 0 & C^{-1}P_L \\ 0 & 0 & C^{-1}P_R & 0 \\ 0 & C^{-1}P_R & 0 & 0 \\ C^{-1}P_L & 0 & 0 & 0 \end{pmatrix}, \tag{2.24}$$

$$\delta T = \alpha e^{\alpha-1}\delta e\begin{pmatrix} 0 & 0 & 0 & CP_L \\ 0 & 0 & CP_R & 0 \\ 0 & CP_R & 0 & 0 \\ CP_L & 0 & 0 & 0 \end{pmatrix}. \tag{2.25}$$

Moreover, we notice that by taking the variation of both sides of the following identity

$$\ln(\det B) = \mathrm{tr}(\ln B) \qquad \longrightarrow \qquad \frac{\delta(\det B)}{\det B} = \mathrm{tr}\big(B^{-1}\delta B\big)$$

we can assert that a valid expression for the vierbein determinant variation is

$$\delta e = e e^{\mu a}\delta e_{\mu a}. \tag{2.26}$$

Then, after replacing (2.26) in (2.25), we are permitted to inspect the subsequent three cases.

- Local chiral transformation: $e_{\mu a}(x) \to e'_{\mu a}(x) = e_{\mu a}(x)$. As the vierbein remains untouched

  $$\delta e_{\mu a} = 0\,, \tag{2.27}$$

  δe also vanishes

  $$\delta e = 0.$$

  The same fate is suffered by the mass matrix variation (2.25), and thus

  $$\frac{1}{2}T^{-1}\delta T = 0 \tag{2.28}$$

  is seen to hold.

- Weyl transformation: $e_{\mu a}(x) \to e'_{\mu a}(x) = e^{\sigma(x)} e_{\mu a}(x)$. Its infinitesimal form

$$e_{\mu a}(x) \to e'_{\mu a}(x) = (1 + \sigma(x)) e_{\mu a}(x) \; ,$$

showing up when $\sigma(x) << 1$, allows us to write

$$\delta e_{\mu a}(x) = \sigma(x) e_{\mu a}(x) \tag{2.29}$$

and

$$\delta e(x) = e(x) e^{\mu a}(x)\, \delta e_{\mu a}(x) = e(x) e^{\mu a}(x) \sigma(x) e_{\mu a}(x) = 4 e(x) \sigma(x).$$

Under these circumstances, it should be pretty straightforward to show that

$$\frac{1}{2} T^{-1} \delta T = 2\alpha\sigma(x) \begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & P_R & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix} . \tag{2.30}$$

- Local Lorentz transformation: $e_{\mu a}(x) \to e'_{\mu a}(x) = \Lambda_a{}^b(x) e_{\mu b}(x)$. As a consequence to this law, we get the ensuing infinitesimal one

$$e_{\mu a}(x) \to e'_{\mu a}(x) = (\delta_a{}^b + \omega_a{}^b(x)) e_{\mu b}(x),$$

where ω^{ab}, with $|\omega^{ab}| << 1$, is the antisymmetric matrix that gathers all the infenitesimal parameters of the Lorentz Lie group. Thus, we have

$$\delta e_{\mu a}(x) = \omega_a{}^b(x) e_{\mu b}(x) \; , \tag{2.31}$$

$$\delta e(x) = e(x) e^{\mu a}(x)\, \delta e_{\mu a}(x) = e(x) e^{\mu a}(x) \omega_a{}^b(x) e_{\mu b}(x) = e(x) \omega_a{}^a(x) = 0$$

and more importantly:

$$\frac{1}{2} T^{-1} \delta T = 0 \tag{2.32}$$

- General Einstein transformation: $e_{\mu a}(x) \to e'_{\mu a}(x') = \dfrac{\partial x^\nu}{\partial x'^\mu} e_{\nu a}(x)$. For a small change in the coordinates

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu,$$

with $|\xi^\mu| << 1$, it's a simple matter for us to exhibit that

$$\begin{aligned} \delta e_{\mu a} = e'_{\mu a}(x') - e_{\mu a}(x') &= \frac{\partial x^\nu}{\partial x'^\mu} e_{\nu a}(x) - e_{\mu a}(x - \xi(x)) = \\ &= \big(\delta_\mu{}^\nu + \partial_\mu(\xi^\nu(x))\big) e_{\nu a}(x) - e_{\mu a}(x) + \xi^\nu \partial_\nu(e_{\mu a}(x)) = \\ &= \partial_\mu(\xi^\nu(x)) e_{\nu a}(x) + \xi^\nu(x) \partial_\nu(e_{\mu a}(x)) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} \delta e = e e^{\mu a} \delta(e_{\mu a}) &= e e^{\mu a} \left[\partial_\mu(\xi^\nu) e_{\nu a} + \partial_\nu(e_{\mu a}) \xi^\nu\right] = \\ &= e \delta^\mu{}_\nu \partial_\mu(\xi^\nu) + e e^{\mu a} \partial_\nu(e_{\mu a}) \xi^\nu = \\ &= e \partial_\nu(\xi^\nu) + \partial_\nu(e) \xi^\nu = \partial_\nu \left(e \xi^\nu\right), \end{aligned}$$

from which it follows:

$$\frac{1}{2}T^{-1}\delta T = \frac{\alpha}{2e}\partial_\nu(e\xi^\nu)\begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & P_R & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix} . \tag{2.34}$$

As for the K operator, we know it is defined by the relation

$$\delta\phi = K\phi, \tag{2.35}$$

establishing the infinitesimal transformation rules of the Pauli-Villars fields under the mappings under scrutiny. Its expression can therefore be easily inferred by probing the local variation endured by λ, λ_c, ρ and ρ_c respectively under

- local chiral transformation:

$$\begin{cases} \lambda(x) & \rightarrow & \lambda'(x) = e^{i\zeta(x)}\lambda(x) \\ \lambda_c(x) & \rightarrow & \lambda'_c(x) = e^{-i\zeta(x)}\lambda_c(x) \\ \rho(x) & \rightarrow & \rho'(x) = \rho(x) \\ \rho_c(x) & \rightarrow & \rho'_c(x) = \rho_c(x) \end{cases}$$

that, from an infinitesimal point of view, should look as

$$\begin{aligned} \delta\lambda &= i\zeta(x)\lambda \\ \delta\lambda_c &= -i\zeta(x)\lambda_c \\ \delta\rho = \delta\rho_c &= 0 \ . \end{aligned} \tag{2.36}$$

Accordingly, if we used the (2.36) to build the K matrix defined in (2.35), we would realize that

$$K_C = i\zeta(x)\begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & -P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \tag{2.37}$$

- Weyl transformation:

$$\begin{cases} \lambda(x) & \rightarrow & \lambda'(x) = e^{-\frac{3}{2}\sigma(x)}\lambda(x) \\ \lambda_c(x) & \rightarrow & \lambda'_c(x) = e^{-\frac{3}{2}\sigma(x)}\lambda_c(x) \\ \rho(x) & \rightarrow & \rho'(x) = \rho(x) \\ \rho_c(x) & \rightarrow & \rho'_c(x) = \rho_c(x) \end{cases}$$

from which it clearly emerges that

$$\begin{aligned} \delta\lambda &= -\frac{3}{2}\sigma(x)\lambda \\ \delta\lambda_c &= -\frac{3}{2}\sigma(x)\lambda_c \\ \delta\rho = \delta\rho_c &= 0 \ , \end{aligned} \tag{2.38}$$

and, through a comparison between (2.38) and (2.35),

$$K_W = -\frac{3}{2}\sigma(x)\begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.39}$$

immediatly follows;

- local Lorentz transformation:

$$\begin{cases} \lambda(x) & \rightarrow & \lambda'(x) = e^{\frac{i}{2}\omega_{ef}(x)\Sigma^{ef}}\lambda(x) \\ \lambda_c(x) & \rightarrow & \lambda'_c(x) = e^{\frac{i}{2}\omega_{ef}(x)\Sigma^{ef}}\lambda_c(x) \\ \rho(x) & \rightarrow & \rho'(x) = \rho(x) \\ \rho_c(x) & \rightarrow & \rho'_c(x) = \rho_c(x) \end{cases}$$

where the Lorentz infinitesimal generators Σ^{ef} in the spinorial $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation are, as usual,

$$\Sigma^{ef} = -\frac{i}{4}\left[\gamma^e,\gamma^f\right] \equiv -\frac{i}{2}\gamma^{ef} \ .$$

Hence, the PV fields suffer a local variation of the form

$$\begin{aligned} \delta\lambda &= \frac{1}{4}\omega_{ef}(x)\gamma^{ef}\lambda \\ \delta\lambda_c &= \frac{1}{4}\omega_{ef}(x)\gamma^{ef}\lambda_c \\ \delta\rho = \delta\rho_c &= 0 \ , \end{aligned} \tag{2.40}$$

resulting in

$$K_L = \frac{1}{4}\omega_{ef}(x)\gamma^{ef}\begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \tag{2.41}$$

- general Einstein transformation ($x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu$)

$$\begin{cases} \lambda(x) & \rightarrow & \lambda'(x') = \lambda(x) \\ \lambda_c(x) & \rightarrow & \lambda'_c(x') = \lambda_c(x) \\ \rho(x) & \rightarrow & \rho'(x) = \rho(x) \\ \rho_c(x) & \rightarrow & \rho'_c(x) = \rho_c(x) \end{cases}$$

locally affecting the fields through

$$\begin{aligned} \delta\lambda &= \xi^\nu(x)\partial_\nu(\lambda) \\ \delta\lambda_c &= \xi^\nu(x)\partial_\nu(\lambda_c) \\ \delta\rho = \delta\rho_c &= 0 \end{aligned} \tag{2.42}$$

and therefore

$$K_E = \xi^\nu(x)\partial_\nu \begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.43}$$

We point out that the trivial right-fields transformation laws have been imposed to ensure that their massless free lagrangian is invariant under the four simmetries above, which in turn provides null $\rho-$ and ρ_c-contributions to the physical Noether currents.

Finally, the time has come for us to restrict ourselves to flat space. This choice is dictated by the fact that, whenever an anomaly is present in the model, it already reveals itself on a manifold having null curvature. Thus, for the sake of simplicity, we merely replace every occurrence of the einbein determinant e with 1

$$e \to 1, \tag{2.44}$$

and the gravity-gauge covariant derivative $\nabla(A)$ with the simpler gauge covariant one $D(A)$

$$\nabla(A) \to D(A). \tag{2.45}$$

Against this background, the combination of (2.37), (2.39), (2.41) and (2.43) all together translates into[2]

$$K = [is^\pm\zeta(x) + \xi^\mu(x)\partial_\mu + \frac{1}{4}\omega_{ab}(x)\gamma^{ab} - \frac{3}{2}\sigma(x)] \begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.46}$$

Accordingly, the jacobian which we end up with is the following

$$J = \begin{pmatrix} [C(x) + D(x)]\, P_L & 0 & 0 & 0 \\ 0 & [C(x) + D(x)]\, P_R & 0 & 0 \\ 0 & 0 & D(x)P_R & 0 \\ 0 & 0 & 0 & D(x)P_L \end{pmatrix} , \tag{2.47}$$

where

$$C(x) = is^\pm\zeta(x) + \xi^\mu(x)\partial_\mu + \frac{1}{4}\omega_{ab}(x)\gamma^{ab} - \frac{3}{2}\sigma(x)$$

and

$$D(x) = \frac{\alpha}{2}\partial_\nu\left(\xi^\nu(x)\right) + 2\alpha\sigma(x).$$

So, even though K vanishes in the (ρ, ρ_c) sector, we can't simply restrict to the (λ, λ_c) one. Indeed, as a consequence of the non vanishing contributions coming from $\frac{1}{2}T^{-1}\delta T$, the regulator we are to use is

$$-R = \mathcal{O}^2 = \begin{pmatrix} \partial\!\!\!/ D\!\!\!\!/(A)P_L & 0 & 0 & 0 \\ 0 & \partial\!\!\!/ D\!\!\!\!/(-A)P_R & 0 & 0 \\ 0 & 0 & D\!\!\!\!/(A)\partial\!\!\!/ P_R & 0 \\ 0 & 0 & 0 & D\!\!\!\!/(-A)\partial\!\!\!/ P_L \end{pmatrix} . \tag{2.48}$$

[2] $s^\pm$ is a sector-depending factor, whose value coincides with $+1$ as long as it acts on the $\lambda-$section of the theory, while it is -1 otherwise.

To summarize we have to compute

$$i\langle \delta S\rangle = -\lim_{M\to\infty} \mathrm{Tr}[Je^{-\frac{R}{M^2}}] \;, \tag{2.49}$$

which we will do separately for the various symmetries and using heat kernel expansion. However, while the calculation behind is somewhat standard when J is a matrix function, the corresponding heat kernel fomulae become far too complicated whenever the infinitesimal jacobian does contain a first order differential operator, as the $\xi^\mu(x)\partial_\mu$ term due to the gravitational anomalies[3]. Chapter 3 will entirely be devoted to deepening this subject.

[3]See ref. [19] for the two dimensional case, and [20] for the four dimensional one.

Chapter 3

Heat kernel

The heat kernel originates from the heat equation

$$-\frac{\partial}{\partial\beta}\psi = H\psi \ , \tag{3.1}$$

the fundamental solution of which is precisely designed to be its definition. When represented in operatorial form, (3.1) gets solved by the following transition amplitude

$$\psi(x,y;\beta) = \langle y|\, e^{-\beta H}\, |x\rangle \ , \tag{3.2}$$

which in turn undergoes, for small β (i.e. for small propagation times), a perturbative expansion. In order to see this, we'll have to consider at most a second order differential operator H, that will be taken in the form of

$$H = -\nabla^2 + V. \tag{3.3}$$

Here V is allowed to be a matrix potential, while $\nabla^2 = \nabla_a\nabla^a$ is the d'Alembertian built from the gauge covariant derivative of our model

$$\nabla_a = \partial_a + W_a \ . \tag{3.4}$$

Depending on the circumstances, there's no guarantee that W_a is abelian, so that ∇_a might not commute with itself; namely

$$[\nabla_a, \nabla_b] = \partial_a W_b - \partial_b W_a + [W_a, W_b] \equiv \mathcal{F}_{ab} \tag{3.5}$$

is likely to play an active part in all the computations we are about to undertake. However, the well known free solution is seen to emerge by simply using path integral methods with $V = 0$, and, in a flat D$-$dimensional space time, it appears to be

$$\psi_0(x,y;\beta) = \frac{1}{(4\pi\beta)^{\frac{D}{2}}}e^{-\frac{(x-y)^2}{4\beta}}.$$

3.1 Seeley-DeWitt coefficients

If we had included an arbitrary potential $V \neq 0$, things would have gotten far more complicated, but luckily for us, as we stated before, it should still be possible to treat the overall problem in perturbation theory, when $\beta << 1$. Of course, what will result from the current discussion is something like the following

$$\psi_V(x, y; \beta) = \frac{1}{(4\pi\beta)^{\frac{D}{2}}} e^{-\frac{(x-y)^2}{4\beta}} \left[\sum_{n=0}^{\infty} a_n(x, y, H) \beta^n \right], \tag{3.6}$$

in which the heat kernel coefficients $a_n(x, y, H)$, also known as Seeley-DeWitt coefficients[1], have been introduced. As we'll see shortly, of particular importance is the case in which $x = y$. For such situations, the coefficients' values can be inferred from [20]:

$$\begin{aligned} a_0(x, x, H) &= \mathbb{1}; \\ a_1(x, x, H) &= -V; \\ a_2(x, x, H) &= \frac{1}{2}V^2 - \frac{1}{6}\nabla^2 V + \frac{1}{12}\mathcal{F}_{ab}\mathcal{F}^{ab}. \end{aligned} \tag{3.7}$$

As a matter of fact, these terms show up when dealing with the trace of certain operators, like the one reported here under

$$\begin{aligned} \mathrm{Tr}[Ge^{-\beta H}] &= \int d^D x \ \mathrm{tr}[G(x) \langle x| e^{-\beta H} |x\rangle] = \\ &= \int d^D x \ i \ \mathrm{tr}[G(x) \underbrace{\langle x| e^{-itH} |x\rangle}_{(3.6)}] = \\ &= \int d^D x \frac{i}{(4\pi i t)^{D/2}} \sum_{n=0}^{\infty} \mathrm{tr}[G(x) a_n(x, x, H)](it)^n . \end{aligned} \tag{3.8}$$

In the previous equality, a Wick rotation $\beta \to it$ allowed us to go from the first to the second line, while the symbol "tr", instead, is designed to extract the trace of the discrete matrix structure that $G(x) \langle x| e^{-itH} |x\rangle$ is endowed with. Moreover, it is important to specify that this scheme shall only apply when $G(x)$ is a generic matrix-valued function of the coordinate x not involving any differential operator (this case will be treated in some detail during the very next section).

Hence, by comparing (3.8) with the trace needed in the Fujikawa approach (2.49), a proper expression for the anomaly can now be inferred. First and foremost, we identify

$$\begin{aligned} J(x) &\equiv G(x), \\ R = -\mathcal{O}^2 &\equiv H, \\ \frac{1}{M^2} &\equiv it, \end{aligned} \tag{3.9}$$

[1] Generally, the Seeley-DeWitt coefficients $a_n(x, y, H)$ are matrix valued objects depending on two position eigenstates, $|x\rangle$ and $|y\rangle$.

and then we're allowed to rewrite (2.49) in the following form

$$i\langle\delta S\rangle = -\lim_{M\to\infty}\mathrm{Tr}[Je^{-\frac{R}{M^2}}] = -\lim_{M\to\infty}\int d^Dx\frac{iM^D}{(4\pi)^{D/2}}\sum_{n=0}^{\infty}\mathrm{tr}[J(x)a_n(x,R)](\frac{1}{M^2})^n,$$

in which, for the sake of notation, we set

$$a_n(x,x,R)\equiv a_n(x,R).$$

It's easy to see that the previous limit pulls out of the trace only the mass indipendent term, as negative powers of M of course vanish, and the positive ones can always be erased by adding new PV fields. Therefore we end up with an anomaly that, in $D=4$ space-time dimensions, would read as follows

$$i\langle\delta S\rangle = -\lim_{M\to\infty}\mathrm{Tr}[Je^{-\frac{R}{M^2}}] = -\int d^4x\frac{i}{(4\pi)^2}\mathrm{tr}[J(x)a_2(x,R)]\ , \tag{3.10}$$

thus reducing the anomaly issue to that of determining $J(x)$ and the second Seeley-DeWitt coefficient $a_2(x,R)$, an argument that will be addressed in the forthcoming chapter.

3.2 $G(x)$ contains a differential operator: gravitational anomaly

Despite the level of technical refinement reached by Seeley and DeWitt, who originally proposed (3.10), their work does not cover all the possible cases. Among all, the one where $G(x)$ comprises a differential operator, although being of the utmost importance for the gravitational anomaly's derivation (dealt with in section 5.4), still remains not addressed. It is precisely for this reason that we must depend on the paper by Branson, Gilkey and Vassilevich [20], that provides us with a recipe to follow along in order to obtain the previous trace (3.10) when a second order differential operator takes the place of J. Actually, the formula they developed applies only to those operators, from now on denoted as Q, that possess the following features:

1. Q must consist of a purely second (Q_2), first (Q_1) and zero order (Q_0) differential operator, and thus be expressible through their sum

$$Q = Q_2 + Q_1 + Q_0\ ; \tag{3.11}$$

2. while Q_0 is not constrained by any limitation at all, Q_2 and Q_1 are endowed with a specific operator form, that is

$$Q_2(\bullet) = r^{ij}\nabla_i\nabla_j(\bullet) + \frac{1}{2}\big(2\nabla_j(r^{ij}) - \nabla^i(r^j{}_j)\big)\nabla_i(\bullet)\ , \tag{3.12}$$

$$Q_1(\bullet) = -2q^i\nabla_i(\bullet) - \nabla^i(q_i)(\bullet)\ , \tag{3.13}$$

where r^{ij} is a symmetric 2−tensor and q^i is an endomorphism valued 1−tensor.

However, since the gravitational anomaly, which arises from the quantization of the diffeomorphism invariance possessed by (2.1), presents itself such an operator form, we decided to describe the expansion procedure devised in [20] by its direct application to this particular case.

3.2.1 Generalized heat kernel expansion in the gravitational anomaly case

As we shall see in 5.4, the anomaly under scrutiny requires for something like

$$\mathrm{Tr}[\xi^i(x)\partial_i e^{-\beta H}] \tag{3.14}$$

to be evaluated, so, first things first, we have to make sure that (3.14) does conform to (3.11)-(3.13). Through a comparison, it's quite instant for us to come up with the ensuing identifications

$$Q = (\mathbb{1})\xi^i\partial_i \tag{3.15}$$

$$\text{with} \begin{cases} r^{ij} & = 0 \\ q_i & = -\frac{\xi_i}{2}(\mathbb{1}) \\ Q_2 & = 0 \\ Q_1 & = \xi^i(\mathbb{1})\nabla_i + \frac{1}{2}\nabla^i\big(\xi_i\big)(\mathbb{1}) \\ Q_0 & = -\frac{1}{2}\nabla^i\big(\xi_i\big)(\mathbb{1}) - \xi^i w_i \end{cases} , \tag{3.16}$$

where $\mathbb{1}$ is the spinor identity matrix, omitted from now on for the sake of simplicity (unless strictly necessary), and w_i represents the covariant derivative's total connection

$$\nabla_i = \partial_i + w_i \ . \tag{3.17}$$

Thus, not only can we surmise that (3.14) meets the demanded requirements, but, as we are interested in the flat space-time case, (3.17) retains just the same gauge connection of (3.4)

$$w_i = W_i \ ,$$

in turn projecting (3.16) onto a new set of equalities

$$\begin{cases} r^{ij} = 0 \\ q_i = -\frac{\xi_i}{2} \\ Q_2 = 0 \\ Q_1 = \xi^i(\partial_i + W_i) + \frac{1}{2}\nabla^i\big(\xi_i\big) \ = \ \xi^i(\partial_i + W_i) + \frac{1}{2}\partial^i\big(\xi_i\big) + \frac{1}{2}\underbrace{[W^i, \xi_i(\mathbb{1})]}_{0} \ = \ \xi^i(\partial_i + W_i) + \frac{1}{2}\partial^i\big(\xi_i\big) \\ Q_0 = -\frac{1}{2}\nabla^i\big(\xi_i\big) - \xi^i W_i \ = \ -\frac{1}{2}\partial^i\big(\xi_i\big) - \frac{1}{2}\underbrace{[W^i, \xi_i(\mathbb{1})]}_{0} - \xi^i W_i \ = \ -\frac{1}{2}\partial^i\big(\xi_i\big) - \xi^i W_i \end{cases} . \tag{3.18}$$

We observe that, in drawing (3.18) up, we made use of the matrix covariant differentiation rule

$$\nabla_a(\xi_b) = \partial_a(\xi_b) + [W_a, \xi_b(\mathbb{1})] \ ,$$

together with the fact that $\xi^a(x)$ is a simple abelian vector field

$$[W_a, \xi_b(\mathbb{1})] = 0.$$

In essence, the trace we are interested in is

$$\mathrm{Tr}[Qe^{-\beta H}] \;=\; \mathrm{Tr}[(Q_1+Q_0)e^{-\beta H}] \;=\; \mathrm{Tr}[Q_1 e^{-\beta H}] + \mathrm{Tr}[Q_0 e^{-\beta H}]\;, \tag{3.19}$$

and even though we'll always need the Seeley-DeWitt coefficients (3.7) to deal with the part of (3.19) that comes from Q_0, i.e.

$$\mathrm{Tr}[Q_0 e^{-\beta H}],$$

if we are now able to solve the remaining portion, that is

$$\mathrm{Tr}[Q_1 e^{-\beta H}], \tag{3.20}$$

we owe it to [20]. There, generalizing the model to a $D-$dimensional flat space-time, we are taught that the insertion of the first order differential operator

$$Q_1 = \xi^i(\partial_i + W_i) + \frac{1}{2}\partial_i(\xi^i)$$

causes (3.20) to take the form

$$\begin{aligned}
\mathrm{Tr}\left[Q_1 e^{-\beta H}\right] \;&=\int d^D x \,\mathrm{tr}\,\left[Q_1(x)\langle x|e^{-\beta H}|x\rangle\right] = \\
&\overset{\beta\to it}{=} \int d^D x\, \frac{i}{(4\pi)^{\frac{D}{2}}}\sum_{n=0}^{\infty}\mathrm{tr}[b_n(x,H)](it)^{n-\frac{D}{2}} = \\
&= \int d^D x\,\frac{i}{(4\pi it)^{\frac{D}{2}}}\,\mathrm{tr}\left[b_0(x,H) + b_1(x,H)it + b_2(x,H)(it)^2 + ...\right)]
\end{aligned} \tag{3.21}$$

where, after introducing the tensor

$$G_{ij} \equiv \nabla_i q_j - \nabla_j q_i = -\frac{1}{2}(\partial_i\xi_j - \partial_j\xi_i)(\mathbb{1})\;, \tag{3.22}$$

the first few generalizations of the heat kernel coefficients, as they can be extracted from [20], are

$$\begin{aligned}
&b_0(x,H) = 0;\\
&b_1(x,H) = -\frac{1}{6}\mathcal{F}_{ij}G^{ij};\\
&b_2(x,H) = \frac{1}{45}\nabla_k(\mathcal{F}_{ij})\nabla^k(G^{ij}) - \frac{1}{90}\nabla^j(\mathcal{F}_{ij})\nabla_k(G^{ik}) + \frac{1}{6}V\mathcal{F}_{ij}G^{ij}.
\end{aligned} \tag{3.23}$$

Before we continue, we have to adjust our notation to the one used in Fujikawa's method, imposing the following equalities

$$\begin{aligned}
R = -\mathcal{O}^2 &\equiv H,\\
\frac{1}{M^2} &\equiv it.
\end{aligned} \tag{3.24}$$

Moreover, for reasons that will be clear only in section 5.4, it's far more convenient to rewrite (3.21) by collecting ξ^a outside of the trace

$$\begin{aligned}\mathrm{Tr}\left[Q_1 e^{-\frac{R}{M^2}}\right] &= \int d^D x\, \frac{iM^D}{(4\pi)^{\frac{D}{2}}}\, \mathrm{tr}\,[b_0(x,H) + b_1(x,H)\frac{1}{M^2} + b_2(x,H)\Big(\frac{1}{M^2}\Big)^2 + ...)] = \\ &= \int d^D x\, \frac{iM^D}{(4\pi)^{\frac{D}{2}}}\, \xi^a(x)\,\mathrm{tr}\,[b_{a,0}(x,H) + b_{a,1}(x,H)\frac{1}{M^2} + b_{a,2}(x,H)\Big(\frac{1}{M^2}\Big)^2 + ...)].\end{aligned} \tag{3.25}$$

This leads to a redefinition of the whole set of coefficients (3.23), achieved by repeatedly resorting to integrations by parts (performed in detail in section C.1):

$$\begin{aligned} b_0(x,H) &= 0 = \xi^a(x) b_{a,0}(x,H)\ , \\ b_1(x,H) &= -\frac{1}{6}\xi^a(x)\nabla^i \mathcal{F}_{ia}(x) = \xi^a(x) b_{a,1}(x,H)\ , \\ b_2(x,H) &= \xi^a(x)\Big(-\frac{1}{45}\nabla^i\nabla_k\nabla^k\big(\mathcal{F}_{ia}(x)\big) + \frac{1}{180}\nabla^i\nabla_a\nabla^j\big(\mathcal{F}_{ij}(x)\big) + \\ &\qquad - \frac{1}{180}\nabla^k\nabla_k\nabla^j\big(\mathcal{F}_{aj}(x)\big) + \frac{1}{6}\nabla^i\big(V(x)\mathcal{F}_{ia}(x)\big)\Big) = \\ &= \xi^a(x) b_{a,2}(x,H)\ . \end{aligned} \tag{3.26}$$

from which we perceive that

$$\begin{aligned} b_{a,0}(x,H) &= 0\ ; \\ b_{a,1}(x,H) &= -\frac{1}{6}\nabla^i\mathcal{F}_{ia}\ ; \\ b_{a,2}(x,H) &= -\frac{1}{45}\nabla^i\nabla_k\nabla^k\big(\mathcal{F}_{ia}(x)\big) + \frac{1}{180}\nabla^i\nabla_a\nabla^j\big(\mathcal{F}_{ij}(x)\big) - \frac{1}{180}\nabla^k\nabla_k\nabla^j\big(\mathcal{F}_{aj}(x)\big) + \\ &\qquad + \frac{1}{6}\nabla^i\big(V(x)\mathcal{F}_{ia}(x)\big)\ . \end{aligned} \tag{3.27}$$

Actually, because of the actual expression we'll deduce for the involved fields ($W_i(x)$, $V(x)$, $\mathcal{F}_{ij}(x)$), it will be possible for us to show that the (3.27) are subject to a remarkable simplification. In fact, as we will soon confirm in section 4.5, the majority of terms surfacing from the covariant derivatives in (3.27) will vanish when the trace operation within equation (3.25) is carried out.

In the end, since the method conceived by Fujikawa requests the limit procedure (2.49) to be attached to (3.19), we are allowed to complete the current section by presenting the ultimate form of an anomaly, like the gravitational one, which happens to be equipped with the differential operator (3.15) in a $D = 4-$dimensional flat space-time:

$$\begin{aligned} i\langle\delta S\rangle &= -\lim_{M\to\infty}\mathrm{Tr}[Q(x)e^{-\frac{R}{M^2}}] = \\ &= -\lim_{M\to\infty}\mathrm{Tr}[Q_1(x)e^{-\frac{R}{M^2}}] \underbrace{-\lim_{M\to\infty}\mathrm{Tr}[Q_0(x)e^{-\frac{R}{M^2}}]}_{(3.10)} = \\ &= -\Big(\int d^4x \frac{i}{(4\pi)^2}\xi^a(x)\mathrm{tr}[b_{a,2}(x,R)] + \int d^4x\frac{i}{(4\pi)^2}\mathrm{tr}[Q_0(x)a_2(x,R)]\Big)\ . \end{aligned} \tag{3.28}$$

Chapter 4

Heat kernel coefficients

In order to implement the expansions of chapter 3, some sort of relation have to be first established between the hamiltonian H (3.3) and the regularizator R (2.48) of the model. To this end, we write down the components of R separately, i.e.

$$\begin{aligned} R_\lambda &= -\partial\!\!\!/ D\!\!\!\!/(A) P_L &(4.1)\\ R_{\lambda_c} &= -\partial\!\!\!/ D\!\!\!\!/(-A) P_R &(4.2)\\ R_\rho &= -D\!\!\!\!/(A)\partial\!\!\!/ P_R &(4.3)\\ R_{\rho_c} &= -D\!\!\!\!/(-A)\partial\!\!\!/ P_L \ . &(4.4)\end{aligned}$$

Moreover, at this stage we'll neglect the two projectors P_L and P_R, serving mostly as a reminder of which sector the regulators act on. However, it's important to reinsert them while dealing with the actual trace computations of chapter 5, where $a_2(x,R)$ and $b_{a,2}(x,R)$ are in the matrix form

$$a_2(x,R) = \begin{pmatrix} a_2(x,R_\lambda)P_L & 0 & 0 & 0 \\ 0 & a_2(x,R_{\lambda_c})P_R & 0 & 0 \\ 0 & 0 & a_2(x,R_\rho)P_R & 0 \\ 0 & 0 & 0 & a_2(x,R_{\rho_c})P_L \end{pmatrix} , \tag{4.5}$$

$$b_{a,2}(x,R) = \begin{pmatrix} b_{a,2}(x,R_\lambda)P_L & 0 & 0 & 0 \\ 0 & b_{a,2}(x,R_{\lambda_c})P_R & 0 & 0 \\ 0 & 0 & b_{a,2}(x,R_\rho)P_R & 0 \\ 0 & 0 & 0 & b_{a,2}(x,R_{\rho_c})P_L \end{pmatrix} , \tag{4.6}$$

in which every sector-restriceted coefficient, $a_2(x,R_i)$ or $b_{a,2}(x,R_i)$, appears alongside the proper projector.

Then, by expanding the hamiltonian (3.3) into

$$\begin{aligned} H &= -\nabla^2 + V = -(\partial_a + W_a)(\partial^a + W^a) + V = \\ &= -\partial_a\partial^a - \partial_a(W^a) - 2W_a\partial^a - W^aW_a + V \ , \end{aligned} \tag{4.7}$$

and comparing it with the (4.1)$-$(4.4), the whole set of heat kernel coefficients can be seen to emerge. We'll start with the evaluation of the $a_2(x,R_i)$, saving the $b_{a,2}(x,R_i)$ for last.

4.1 $\lambda-$sector

Hence, let's focus on the $\lambda-$sector for the time being, in which we get for (4.1)

$$\begin{aligned} R_\lambda &= -\not\partial \not D(A) = -\underbrace{\gamma^a\gamma^b}_{\text{(B.8)}}\partial_a(\partial_b - iA_b) = \\ &= -\big(g^{ab}+\gamma^{ab}\big)(\partial_a\partial_b - i\partial_a A_b - iA_b\partial_a) = \\ &= -\partial^a\partial_a - \underbrace{\gamma^{ab}\partial_a\partial_b}_{0} + i\partial^a A_a + iA^a\partial_a + i\gamma^{ab}\underbrace{\partial_a A_b}_{\text{only its antisymmetric part will contribute}} + i\gamma^{ab}A_b\partial_a = \\ &= -\partial^a\partial_a + i\partial^a(A_a) + \frac{i}{2}\gamma^{ab}F_{ab} + (iA^a + i\gamma^{ab}A_b)\partial_a \ , \end{aligned} \tag{4.8}$$

where we used the decomposition (B.8) and the vanishing of the full contraction of an antisymmetric tensor with a symmetric one ($\gamma^{ab}\partial_a\partial_b = 0$). We also adopted the customary field strenght tensor's definition (2.6). Now, by comparing (4.8) with (4.7), it's pretty straightforward to infer that

$$W^a = -\frac{i}{2}A^a - \frac{i}{2}\gamma^{ab}A_b \, , \tag{4.9}$$

from whose divergence

$$\begin{aligned} \partial_a W^a &= -\frac{i}{2}\partial_a A^a - \frac{i}{2}\gamma^{ab}\underbrace{\partial_a A_b}_{\text{only its antisym. part will contribute}} = \\ &= -\frac{i}{2}\partial_a A^a - \frac{i}{4}\gamma^{ab}F_{ab} \end{aligned}$$

and square contraction

$$W^a W_a \overset{\text{(C.2)}}{=} \frac{1}{2}A^a A_a$$

we notice that, by adding and subtracting $\frac{1}{2}A^a A_a$ to (4.8), V follows:

$$V = \frac{i}{2}\partial_a A^a + \frac{i}{4}\gamma^{ab}F_{ab} + \frac{1}{2}A^a A_a \ . \tag{4.10}$$

Now, in order to get the coefficient $a_2(x, R_\lambda)$'s value, we simply have to follow the recipe provided in (3.7):

$$a_2(x, R_\lambda) = \frac{1}{2}V^2 - \frac{1}{6}\nabla^2 V + \frac{1}{12}\mathcal{F}_{ab}\mathcal{F}^{ab}.$$

Let's start by determining what form V^2 does have:

$$\begin{aligned} V^2 &= \left(\frac{i}{2}\partial_a A^a + \frac{i}{4}\gamma^{ab}F_{ab} + \frac{1}{2}A^a A_a\right)\left(\frac{i}{2}\partial_c A^c + \frac{i}{4}\gamma^{cd}F_{cd} + \frac{1}{2}A^c A_c\right) = \\ &= -\frac{1}{4}(\partial_a A^a)^2 - \frac{1}{4}\gamma^{ab}\partial_c(A^c)F_{ab} + \frac{i}{2}A^2\partial_c(A^c) + \frac{i}{4}\gamma^{ab}A^2 F_{ab} - \frac{1}{16}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} + \frac{1}{4}A^4 \ . \end{aligned} \tag{4.11}$$

Later, we need the covariant d'Alembertian of the potential, $\nabla^2 V$, but we must be careful. In fact, since V happens to be a matrix, its gauge covariant derivative is $\nabla_a V = \partial_a V + [W_a, V]$, as its nature demands. Therefore we have

$$\begin{aligned}\nabla_a V &= \partial_a V + [W_a, V] = \\ &\overset{(C.3)}{=} \frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) + \frac{1}{2}\gamma_{bc}A^b F^c{}_a + \frac{1}{2}\gamma_{ca}A_d F^{cd} \ ,\end{aligned} \tag{4.12}$$

and, by taking one further covariant derivative, we find:

$$\begin{aligned}\nabla^a\nabla_a V &= \partial^a\big(\nabla_a V\big) + \Big[W^a, \nabla_a V\Big] = \\ &\overset{(C.5)}{=} \frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) + \frac{1}{2}\gamma^{ca}\partial^b(A_c)F_{ab} + \gamma^{bc}A_b\partial^a(F_{ca}) + \\ &\quad + \frac{1}{2}\gamma^{ca}\partial_a(A^d)F_{cd} - \gamma^{ac}A^d\partial_a(F_{cd}) + \frac{i}{2}\gamma^{ac}A^2 F_{ac} \ .\end{aligned} \tag{4.13}$$

What is left to compute is $\mathcal{F}_{ab}$, as well as its complete contraction with itself $\mathcal{F}_{ab}\mathcal{F}^{ab}$. Let's proceed by degrees. By definition (3.5), we have

$$\begin{aligned}\mathcal{F}_{ab} &= \partial_a(W_b) - \partial_b(W_a) + \Big[W_a, W_b\Big] = \\ &\overset{(C.7)}{=} -\frac{i}{2}F_{ab} - \frac{i}{2}\gamma_{bc}\partial_a(A^c) + \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_b A^c + \frac{1}{2}\gamma_{bc}A_a A^c + \frac{1}{2}\gamma_{ab}A^c A_c \ ,\end{aligned} \tag{4.14}$$

from which the evaluation of $\mathcal{F}_{ab}\mathcal{F}^{ab}$ is just a matter of algebra

$$\begin{aligned}\mathcal{F}_{ab}\mathcal{F}^{ab} &= \Big(-\frac{i}{2}F_{ab} - \frac{i}{2}\gamma_{bc}\partial_a(A^c) + \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_b A^c + \frac{1}{2}\gamma_{bc}A_a A^c + \frac{1}{2}\gamma_{ab}A^2\Big)\cdot \\ &\quad \cdot\Big(-\frac{i}{2}F^{ab} - \frac{i}{2}\gamma^{bd}\partial^a(A_d) + \frac{i}{2}\gamma^{ad}\partial^b(A_d) - \frac{1}{2}\gamma^{ad}A^b A_d + \frac{1}{2}\gamma^{bd}A^a A_d + \frac{1}{2}\gamma^{ab}A^2\Big) = \\ &= -\frac{1}{4}F_{ab}F^{ab} + \gamma^{ac}\partial^b(A_c)F_{ab} + i\gamma^{cb}A^a A_c F_{ab} - \frac{i}{2}\gamma^{ab}A^2 F_{ab} + \frac{3}{2}\partial_b(A_c)\partial^b(A^c) + \\ &\quad + \frac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) + iA_a A_c\partial_b(A_d) + iA_b A_d\partial_a(A_c)\Big] - \frac{3}{2}A^4 + \\ &\quad - 3i\Big[A^2\partial^d(A_d) - A_a A_d\partial^a(A^d)\Big]\end{aligned} \tag{4.15}$$

(see section C.2.5 of appendix C for the detailed calculation). Ultimately, from (3.7) we infer the precise structure of the second Seeley-DeWitt coefficient, that is

$$\begin{aligned}a_2(x, R_\lambda) &= \frac{1}{2}V^2 - \frac{1}{6}\nabla^2 V + \frac{1}{12}\mathcal{F}_{ab}\mathcal{F}^{ab} = \\ &\overset{(C.11)}{=} -\frac{1}{8}(\partial_a A^a)^2 - \frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab} - \frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} - \frac{i}{12}\Box(\partial_s A^s) - \frac{i}{24}\gamma^{cd}\Box(F_{cd}) + \\ &\quad - \frac{1}{12}\Box(A^2) + \frac{1}{6}\gamma^{ac}\partial^b(A_c)F_{ab} - \frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca}) - \frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd} + \\ &\quad + \frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd}) - \frac{1}{48}F_{ab}F^{ab} + \frac{i}{12}\gamma^{cb}A^a A_c F_{ab} + \frac{1}{8}\partial_b(A_c)\partial^b(A^c) + \\ &\quad + \frac{1}{24}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) + iA_a A_c\partial_b(A_d) + iA_b A_d\partial_a(A_c)\Big] + \frac{i}{4}A_a A_d\partial^a(A^d) \ .\end{aligned} \tag{4.16}$$

4.2 λ_c-sector and $c-$prescription

Unfortunately, as we saw, the anomalies we are striving for require the knowledge of the coefficient $a_2(x, R_{\lambda_c})$, too, whose expression still follows (3.7):

$$a_2(x, R_{\lambda_c}) = \frac{1}{2}\widetilde{V}^2 - \frac{1}{6}\nabla^2\widetilde{V} + \frac{1}{12}\widetilde{\mathcal{F}}_{ab}\widetilde{\mathcal{F}}^{ab}.$$

Nevertheless, $\widetilde{V}^2$, $\nabla^2\widetilde{V}$ and $\widetilde{\mathcal{F}}_{ab}\widetilde{\mathcal{F}}^{ab}$ can now be deduced from formulae (4.11), (4.13) and (4.15) determined beforehand, instead of being computed through. This is made possible by stating what we'll refer to as the "$c-$prescription", namely a rule concerning the porting of a well defined quantity on the $\lambda-$sector to the λ_c-one. All it takes is for us to notice that the only difference occurring in the definition of these two sectors' regulators is the charge conjugation process, easily accomplished by mapping the gauge vector A^a into its opposite

$$A^a \longrightarrow -A^a \ . \tag{4.17}$$

Clearly, we are now allowed to achieve every λ_c-sector operator by acting on the corresponding $\lambda-$one with (4.17). For instance, it is quite immediate to ascertain that

$$\begin{aligned} W^a &= -\frac{i}{2}A^a - \frac{i}{2}\gamma^{ab}A_b \\ &\Downarrow \\ \widetilde{W}^a &= \frac{i}{2}A^a + \frac{i}{2}\gamma^{ab}A_b \ , \end{aligned} \tag{4.18}$$

or similarly that

$$\begin{aligned} V &= \frac{i}{2}\partial_a A^a + \frac{i}{4}\gamma^{ab}F_{ab} + \frac{1}{2}A^a A_a \\ &\Downarrow \\ \widetilde{V} &= -\frac{i}{2}\partial_a A^a - \frac{i}{4}\gamma^{ab}F_{ab} + \frac{1}{2}A^a A_a \ , \end{aligned} \tag{4.19}$$

where (4.17) has been further extended to include the transformation law for $F_{ab} = \partial_a A_b - \partial_b A_a$

$$F_{ab} \longrightarrow -F_{ab} \ . \tag{4.20}$$

Therefore, we could get everything we need just by acting via (4.17) and (4.20) upon (4.11), (4.13), (4.14), (4.15) and (4.16):

$$\widetilde{V}^2 = -\frac{1}{4}(\partial_a A^a)^2 - \frac{1}{4}\gamma^{ab}\partial_c(A^c)F_{ab} - \frac{i}{2}A^2\partial_c(A^c) - \frac{i}{4}\gamma^{ab}A^2F_{ab} - \frac{1}{16}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} + \frac{1}{4}A^4 \ , \tag{4.21}$$

$$\begin{aligned} \nabla^2\widetilde{V} = -\frac{i}{2}\Box(\partial_s A^s) - \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) + \frac{1}{2}\gamma^{ca}\partial^b(A_c)F_{ab} + \gamma^{bc}A_b\partial^a(F_{ca}) + \\ + \frac{1}{2}\gamma^{ca}\partial_a(A^d)F_{cd} - \gamma^{ac}A^d\partial_a(F_{cd}) - \frac{i}{2}\gamma^{ac}A^2F_{ac} \ , \end{aligned} \tag{4.22}$$

$$\widetilde{\mathcal{F}}_{ab} = \frac{i}{2}F_{ab} + \frac{i}{2}\gamma_{bc}\partial_a(A^c) - \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_bA^c + \frac{1}{2}\gamma_{bc}A_aA^c + \frac{1}{2}\gamma_{ab}A^cA_c \qquad (4.23)$$

$$\begin{aligned}\widetilde{\mathcal{F}}_{ab}\widetilde{\mathcal{F}}^{ab} = &-\frac{1}{4}F_{ab}F^{ab} + \gamma^{ac}\partial^b(A_c)F_{ab} - i\gamma^{cb}A^aA_cF_{ab} + \frac{i}{2}\gamma^{ab}A^2F_{ab} + \frac{3}{2}\partial_b(A_c)\partial^b(A^c) + \\ &+ \frac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) - iA_aA_c\partial_b(A_d) - iA_bA_d\partial_a(A_c)\Big] - \frac{3}{2}A^4 + \\ &+ 3i\Big[A^2\partial^d(A_d) - A_aA_d\partial^a(A^d)\Big] \qquad (4.24)\end{aligned}$$

and finally

$$\begin{aligned}a_2(x, R_{\lambda_c}) = &-\frac{1}{8}(\partial_aA^a)^2 - \frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab} - \frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} + \frac{i}{12}\Box(\partial_sA^s) + \frac{i}{24}\gamma^{cd}\Box(F_{cd}) + \\ &- \frac{1}{12}\Box(A^2) + \frac{1}{6}\gamma^{ac}\partial^b(A_c)F_{ab} - \frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca}) - \frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd} + \\ &+ \frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd}) - \frac{1}{48}F_{ab}F^{ab} - \frac{i}{12}\gamma^{cb}A^aA_cF_{ab} + \frac{1}{8}\partial_b(A_c)\partial^b(A^c) + \\ &+ \frac{1}{24}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) - iA_aA_c\partial_b(A_d) - iA_bA_d\partial_a(A_c)\Big] - \frac{i}{4}A_aA_d\partial^a(A^d)\ . \qquad (4.25)\end{aligned}$$

4.3 $\rho-$sector

Even though we gained access to the overall left sector of the theory, we haven't acquired the right Seeley-DeWitt coefficients, $a_2(x, R_\rho)$ and $a_2(x, R_{\rho_c})$, yet. Actually, a real prescription that grants us to know, for instance, every $\rho-$quantity starting from the $\lambda-$ones does not exist this time, and we are thus forced to repeat once again the whole procedure we followed for the $\lambda-$sector. Hence, starting with

$$\begin{aligned}R_\rho = -\not{D}(A)\not{\partial} &= -\gamma^a\gamma^b(\partial_a - iA_a)\partial_b = \\ &= -\partial^a\partial_a + (iA^a - i\gamma^{ab}A_b)\partial_a\ , \qquad (4.26)\end{aligned}$$

we'll get

$$Z^a = -\frac{i}{2}A^a + \frac{i}{2}\gamma^{ab}A_b\ , \qquad (4.27)$$

from which

$$\begin{aligned}\partial_aZ^a &= -\frac{i}{2}\partial_aA^a + \frac{i}{4}\gamma^{ab}F_{ab}, \\ Z_aZ^a &= \frac{1}{2}A^aA_a\end{aligned}$$

and of course

$$U = -\frac{i}{2}\partial_aA^a + \frac{i}{4}\gamma^{ab}F_{ab} + \frac{1}{2}A^aA_a \qquad (4.28)$$

will ensue. Here in the right sectors, we choose Z^a and U to play the role of W^a and V respectively. In the same way as before, we go ahead with the appraisal of U^2

$$\begin{aligned} U^2 &= \Big(-\frac{i}{2}\partial_a A^a + \frac{i}{4}\gamma^{ab}F_{ab} + \frac{1}{2}A^a A_a\Big)\Big(-\frac{i}{2}\partial_c A^c + \frac{i}{4}\gamma^{cd}F_{cd} + \frac{1}{2}A^c A_c\Big) = \\ &= -\frac{1}{4}(\partial_a A^a)^2 + \frac{1}{4}\gamma^{ab}\partial_c(A^c)F_{ab} - \frac{i}{2}A^2\partial_c(A^c) + \frac{i}{4}\gamma^{ab}A^2 F_{ab} - \frac{1}{16}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} + \frac{1}{4}A^4 \ , \end{aligned} \tag{4.29}$$

and then of $\nabla_a U$

$$\begin{aligned} \nabla_a U &= \partial_a U + [Z_a, U] = \\ &\overset{(C.4)}{=} -\frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) - \frac{1}{2}\gamma_{bc}A^b F^c{}_a - \frac{1}{2}\gamma_{ca}A_d F^{cd} \ , \end{aligned} \tag{4.30}$$

from which $\nabla^2 U$ can be easily extracted

$$\begin{aligned} \nabla^a\nabla_a U &= \partial^a\big(\nabla_a U\big) + \Big[Z^a, \nabla_a U\Big] = \\ &\overset{(C.6)}{=} -\frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) - \frac{1}{2}\gamma^{ca}\partial^b(A_c)F_{ab} - \gamma^{bc}A_b\partial^a(F_{ca}) + \\ &\qquad - \frac{1}{2}\gamma^{ca}\partial_a(A^d)F_{cd} + \gamma^{ac}A^d\partial_a(F_{cd}) + \frac{i}{2}\gamma^{ac}A^2 F_{ac} \ . \end{aligned} \tag{4.31}$$

Lastly, it's compulsory for us to work $\mathcal{E}_{ab}$ out, defined to be the right analogous of $\mathcal{F}_{ab}$

$$\begin{aligned} \mathcal{E}_{ab} &= \partial_a(Z_b) - \partial_b(Z_a) + \Big[Z_a, Z_b\Big] = \\ &\overset{(C.8)}{=} -\frac{i}{2}F_{ab} + \frac{i}{2}\gamma_{bc}\partial_a(A^c) - \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_b A^c + \frac{1}{2}\gamma_{bc}A_a A^c + \frac{1}{2}\gamma_{ab}A^c A_c \ ; \end{aligned} \tag{4.32}$$

then it's the turn of its full contraction $\mathcal{E}_{ab}\mathcal{E}^{ab}$ as well, to whose computation we devoted section C.2.6 of appendix C, getting

$$\begin{aligned} \mathcal{E}_{ab}\mathcal{E}^{ab} &= \Big(-\frac{i}{2}F_{ab} + \frac{i}{2}\gamma_{bc}\partial_a(A^c) - \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_b A^c + \frac{1}{2}\gamma_{bc}A_a A^c + \frac{1}{2}\gamma_{ab}A^2\Big)\cdot \\ &\quad \cdot\Big(-\frac{i}{2}F^{ab} + \frac{i}{2}\gamma^{bd}\partial^a(A_d) - \frac{i}{2}\gamma^{ad}\partial^b(A_d) - \frac{1}{2}\gamma^{ad}A^b A_d + \frac{1}{2}\gamma^{bd}A^a A_d + \frac{1}{2}\gamma^{ab}A^2\Big) = \\ &= -\tfrac{1}{4}(\mathbb{1})F_{ab}F^{ab} - \gamma^{ac}\partial^b(A_c)F_{ab} + i\gamma^{cb}A^a A_c F_{ab} - \tfrac{i}{2}\gamma^{ab}A^2 F_{ab} + \tfrac{3}{2}(\mathbb{1})\partial_b(A_c)\partial^b(A^c) + \\ &\quad + \tfrac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) - iA_a A_c\partial_b(A_d) - iA_b A_d\partial_a(A_c)\Big] - \tfrac{3}{2}(\mathbb{1})A^4 + \\ &\quad + 3i\Big[A^2\partial^d(A_d) - A_a A_d\partial^a(A^d)\Big] \ . \end{aligned} \tag{4.33}$$

Finally, the $\rho-$sector second Seeley-DeWitt coefficient, which complies once again with (3.7), can be written as

$$
\begin{aligned}
a_2(x,R_\rho) = \frac{1}{2}U^2 - \frac{1}{6}\nabla^2 U + \frac{1}{12}\mathcal{E}_{ab}\mathcal{E}^{ab} = \\
\overset{(C.12)}{=} -\frac{1}{8}(\partial_a A^a)^2 + \frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab} - \frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} + \frac{i}{12}\Box(\partial_s A^s) - \frac{i}{24}\gamma^{cd}\Box(F_{cd}) + \\
- \frac{1}{12}\Box(A^2) - \frac{1}{6}\gamma^{ac}\partial^b(A_c)F_{ab} + \frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca}) + \frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd} + \\
- \frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd}) - \frac{1}{48}F_{ab}F^{ab} + \frac{i}{12}\gamma^{cb}A^aA_cF_{ab} + \frac{1}{8}\partial_b(A_c)\partial^b(A^c) + \\
+ \frac{1}{24}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) - iA_aA_c\partial_b(A_d) - iA_bA_d\partial_a(A_c)\Big] - \frac{i}{4}A_aA_d\partial^a(A^d).
\end{aligned}
\tag{4.34}
$$

In appendix D we decided to report the mechanism we devised to quickly ascertain the validity of our $\lambda-$ and $\rho-$results.

4.4 ρ_c-sector

The last quantities left to compute are those defined in the ρ_c-section of our theory. They are easily obtained from the corresponding $\rho-$operators by means of the $c-$prescription (the same we exploited to move from the $\lambda-$sector to the λ_c-one), as the difference in the regulators

$$
R_\rho = -\not{D}(A)\not{\partial} \qquad \text{and} \qquad R_{\rho_c} = -\not{D}(-A)\not{\partial}
$$

clearly suggests. That's how we arrive at

$$
\widetilde{Z}^a = +\frac{i}{2}A^a - \frac{i}{2}\gamma^{ab}A_b \tag{4.35}
$$

$$
\widetilde{U} = +\frac{i}{2}\partial_a A^a - \frac{i}{4}\gamma^{ab}F_{ab} + \frac{1}{2}A^aA_a \tag{4.36}
$$

$$
\widetilde{U}^2 = -\frac{1}{4}(\partial_a A^a)^2 + \frac{1}{4}\gamma^{ab}\partial_c(A^c)F_{ab} + \frac{i}{2}A^2\partial_c(A^c) - \frac{i}{4}\gamma^{ab}A^2F_{ab} - \frac{1}{16}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} + \frac{1}{4}A^4 \ , \tag{4.37}
$$

$$
\begin{aligned}
\nabla^2\widetilde{U} = \frac{i}{2}\Box(\partial_s A^s) - \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) - \frac{1}{2}\gamma^{ca}\partial^b(A_c)F_{ab} - \gamma^{bc}A_b\partial^a(F_{ca}) + \\
- \frac{1}{2}\gamma^{ca}\partial_a(A^d)F_{cd} + \gamma^{ac}A^d\partial_a(F_{cd}) - \frac{i}{2}\gamma^{ac}A^2F_{ac} \ ,
\end{aligned}
\tag{4.38}
$$

$$
\widetilde{\mathcal{E}}_{ab} = \frac{i}{2}F_{ab} - \frac{i}{2}\gamma_{bc}\partial_a(A^c) + \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_bA^c + \frac{1}{2}\gamma_{bc}A_aA^c + \frac{1}{2}\gamma_{ab}A^cA_c \ , \tag{4.39}
$$

$$
\begin{aligned}
\widetilde{\mathcal{E}}_{ab}\widetilde{\mathcal{E}}^{ab} = -\frac{1}{4}F_{ab}F^{ab} - \gamma^{ac}\partial^b(A_c)F_{ab} - i\gamma^{cb}A^aA_cF_{ab} + \frac{i}{2}\gamma^{ab}A^2F_{ab} + \frac{3}{2}\partial_b(A_c)\partial^b(A^c) + \\
+ \frac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big] - \frac{3}{2}A^4 + \\
- 3i\Big[A^2\partial^d(A_d) - A_aA_d\partial^a(A^d)\Big] \ ,
\end{aligned}
\tag{4.40}
$$

$$a_2(x,R_{\rho_c}) = -\frac{1}{8}(\partial_a A^a)^2 + \frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab} - \frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} - \frac{i}{12}\Box(\partial_s A^s) + \frac{i}{24}\gamma^{cd}\Box(F_{cd}) +$$
$$- \frac{1}{12}\Box(A^2) - \frac{1}{6}\gamma^{ac}\partial^b(A_c)F_{ab} + \frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca}) + \frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd} +$$
$$- \frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd}) - \frac{1}{48}F_{ab}F^{ab} - \frac{i}{12}\gamma^{cb}A^a A_c F_{ab} + \frac{1}{8}\partial_b(A_c)\partial^b(A^c) +$$
$$+ \frac{1}{24}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big] + \frac{i}{4}A_aA_d\partial^a(A^d)\ . \qquad (4.41)$$

Nota bene: all the quantities we've obtained so far must be understood as operators acting on the spinors inhabiting the sector of interest. Thus, each one of their term not displaying at least one matrix object (such as γ^a or γ^{ab}, etc.) should be thought of as being endowed with a spinor identity matrix ($\mathbb{1}$). Strictly speaking, for example, the above $\widetilde{W}_a$ (4.18) should be written as

$$\widetilde{W}^a = \frac{i}{2}(\mathbb{1})A^a + \frac{i}{2}\gamma^{ab}A_b\ ,$$

$\widetilde{V}$ as

$$\widetilde{V} = -\frac{i}{2}(\mathbb{1})\partial_a A^a - \frac{i}{4}\gamma^{ab}F_{ab} + \frac{1}{2}(\mathbb{1})A^aA_a\ ,$$

and so forth. The same logic applies to the entire ongoing discussion.

4.5 $b_{a,2}(x,R)$−computation

At this juncture, having available all the fields we need, we can finally undertake the proper calculation of $b_{a,2}(x,R)$. We make it clear from the very beginning that the jacobian obtained by inserting (2.34) and (2.43) into (2.49), scilicet the one we'll use to treat the gravitational anomaly, does not provide for any differential operator acting on the right sectors, i.e.

$$b_{a,2}(x,R_\rho) = b_{a,2}(x,R_{\rho_c}) = 0\ , \qquad (4.42)$$

permitting us to restrict our computations to the $\lambda-$ and λ_c-sector only. That means, in turn, we can focus on $b_{a,2}(x,R_\lambda)$ alone, and then utilize (4.17) to extract $b_{a,2}(x,R_{\lambda_c})$.

As anticipated in section 3.2, the (3.27) will suffer a noteworthy improvement, as most of the traces acting upon them result to be zero (for a more accurate discussion see appendix E). In particular, this process simplifies $b_{a,2}(x,R_\lambda)$ into

$$b_{a,2}(x,R_\lambda) = -\frac{1}{60}\Box\partial^i\mathcal{F}_{ia} + \frac{1}{6}\partial^i(V\mathcal{F}_{ia})\ , \qquad (4.43)$$

defining the simplified heat kernel coefficient we were looking for. Finally, if we go ahead and substitute (4.14) and (4.10), we'll get

$$b_{a,2}(x,R_\lambda) = -\frac{1}{60}\Box\partial^i\Big(-\frac{i}{2}F_{ia} - \frac{i}{2}\gamma_{ac}\partial_i(A^c) + \frac{i}{2}\gamma_{ic}\partial_a(A^c) - \frac{1}{2}\gamma_{ic}A_aA^c + \frac{1}{2}\gamma_{ac}A_iA^c + \frac{1}{2}\gamma_{ia}A^2\Big) +$$

$$
\begin{aligned}
&+\frac{1}{6}\partial^i\Big[\Big(\frac{i}{2}\partial_b A^b+\frac{i}{4}\gamma^{db}F_{db}+\frac{1}{2}A^2\Big)\cdot\\
&\qquad\cdot\Big(-\frac{i}{2}F_{ia}-\frac{i}{2}\gamma_{ac}\partial_i(A^c)+\frac{i}{2}\gamma_{ic}\partial_a(A^c)-\frac{1}{2}\gamma_{ic}A_aA^c+\frac{1}{2}\gamma_{ac}A_iA^c+\frac{1}{2}\gamma_{ia}A^2\Big)\Big]=\\
&=-\frac{1}{60}\Box\partial^i\Big(-\frac{i}{2}F_{ia}-\frac{i}{2}\gamma_{ac}\partial_i(A^c)+\frac{i}{2}\gamma_{ic}\partial_a(A^c)-\frac{1}{2}\gamma_{ic}A_aA^c+\frac{1}{2}\gamma_{ac}A_iA^c+\frac{1}{2}\gamma_{ia}A^2\Big)+\\
&+\frac{1}{6}\partial^i\Big(\frac{1}{4}\partial_b(A^b)F_{ia}+\frac{1}{4}\gamma_{ac}\partial_b(A^b)\partial_i(A^c)-\frac{1}{4}\gamma_{ic}\partial_b(A^b)\partial_a(A^c)-\frac{i}{4}\gamma_{ic}\partial_b(A^b)A_aA^c+\\
&\qquad+\frac{i}{4}\gamma_{ac}\partial_b(A^b)A_iA^c+\frac{i}{4}\gamma_{ia}A^2\partial_b(A^b)+\frac{1}{8}\gamma_{db}F^{db}F_{ia}+\frac{1}{8}\gamma_{db}\gamma_{ac}\partial_i(A^c)F^{db}+\\
&\qquad-\frac{1}{8}\gamma_{db}\gamma_{ic}\partial_a(A^c)F^{db}-\frac{i}{8}\gamma_{db}\gamma_{ic}A_aA^cF^{db}+\frac{i}{8}\gamma_{db}\gamma_{ac}A_iA^cF^{db}+\frac{i}{8}\gamma_{db}\gamma_{ia}A^2F^{db}+\\
&\quad-\frac{i}{4}A^2F_{ia}-\frac{i}{4}\gamma_{ac}A^2\partial_i(A^c)+\frac{i}{4}\gamma_{ic}A^2\partial_a(A^c)-\frac{1}{4}\gamma_{ic}A_aA^cA^2+\frac{1}{4}\gamma_{ac}A_iA^cA^2+\frac{1}{4}\gamma_{ia}A^4\Big).
\end{aligned}
\tag{4.44}
$$

As we could notice, we decided to leave all the partial derivatives unexpanded: this makes its trace easier to calculate (see sec. F.3 of appendix F). As usual, we can trust the $c-$prescription to precisely produce the λ_c-analogous of (4.44):

$$
\begin{aligned}
b_{a,2}(x,R_{\lambda_c})=&-\frac{1}{60}\Box\partial^i\Big(\frac{i}{2}F_{ia}+\frac{i}{2}\gamma_{ac}\partial_i(A^c)-\frac{i}{2}\gamma_{ic}\partial_a(A^c)-\frac{1}{2}\gamma_{ic}A_aA^c+\frac{1}{2}\gamma_{ac}A_iA^c+\frac{1}{2}\gamma_{ia}A^2\Big)+\\
&+\frac{1}{6}\partial^i\Big(\frac{1}{4}\partial_b(A^b)F_{ia}+\frac{1}{4}\gamma_{ac}\partial_b(A^b)\partial_i(A^c)-\frac{1}{4}\gamma_{ic}\partial_b(A^b)\partial_a(A^c)+\frac{i}{4}\gamma_{ic}\partial_b(A^b)A_aA^c+\\
&\qquad-\frac{i}{4}\gamma_{ac}\partial_b(A^b)A_iA^c-\frac{i}{4}\gamma_{ia}A^2\partial_b(A^b)+\frac{1}{8}\gamma_{db}F^{db}F_{ia}+\frac{1}{8}\gamma_{db}\gamma_{ac}\partial_i(A^c)F^{db}+\\
&\qquad-\frac{1}{8}\gamma_{db}\gamma_{ic}\partial_a(A^c)F^{db}+\frac{i}{8}\gamma_{db}\gamma_{ic}A_aA^cF^{db}-\frac{i}{8}\gamma_{db}\gamma_{ac}A_iA^cF^{db}-\frac{i}{8}\gamma_{db}\gamma_{ia}A^2F^{db}+\\
&\quad+\frac{i}{4}A^2F_{ia}+\frac{i}{4}\gamma_{ac}A^2\partial_i(A^c)-\frac{i}{4}\gamma_{ic}A^2\partial_a(A^c)-\frac{1}{4}\gamma_{ic}A_aA^cA^2+\frac{1}{4}\gamma_{ac}A_iA^cA^2+\frac{1}{4}\gamma_{ia}A^4\Big).
\end{aligned}
\tag{4.45}
$$

In the next section we'll finally focus on the sheer determination of the quantum anomalies affecting the model, an aim we'll pursue through all the means acquired so far.

Chapter 5

Anomalies

Now possessing all the basic building blocks, we can start with the actual calculus of the anomalies connected to the transformations pondered in chapter 2, under which the lagrangian (2.1) revealed to be invariant. The gravitational anomaly will be addressed for last. The explicit evaluation of the anomalous vacuum expectation values can be found in appendix F.

5.1 Trace anomaly

As we illustrated in chapter 1, our starting point consists in considering Fujikawa's standard expression (2.49), i.e.

$$i\langle \delta S\rangle = -\lim_{M\to\infty} \mathrm{Tr}[Je^{-\frac{R}{M^2}}].$$

From here we proceed by simultaneously developing both the left and right hand side: in the former we replace the flat space restriction ($e \to 1$) of the action variation (2.10) with respect to the Weyl map

$$i\langle \delta_{\sigma(x)} S\rangle = i\int d^4x\, \langle T^a{}_a(x)\rangle \sigma(x)\ , \tag{5.1}$$

while in the latter we take advantage of the heat kernel formula (3.10), rewritten right here under

$$-\lim_{M\to\infty} \mathrm{Tr}[Je^{-\frac{R}{M^2}}] = -\int d^4x \frac{i}{(4\pi)^2}\mathrm{tr}[J(x)a_2(x,R)]\ .$$

Then we make use of the two non vanishing contributions, (2.39) and (2.30), participating in the Weyl-form of the infinitesimal jacobian (1.34),

$$J_W(x) = -\frac{3}{2}\sigma(x)\begin{pmatrix} P_L & 0 & 0 & 0\\ 0 & P_R & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} + 2\alpha\sigma(x)\begin{pmatrix} P_L & 0 & 0 & 0\\ 0 & P_R & 0 & 0\\ 0 & 0 & P_R & 0\\ 0 & 0 & 0 & P_L\end{pmatrix},$$

together with the matrix form (4.5) valid for $a_2(x,R)$. Hence, we get that

$$J_W(x)a_2(x,R) = \begin{pmatrix} \frac{-3+4\alpha}{2}\sigma(x)P_L & 0 & 0 & 0 \\ 0 & \frac{-3+4\alpha}{2}\sigma(x)P_R & 0 & 0 \\ 0 & 0 & 2\alpha\sigma(x)P_R & 0 \\ 0 & 0 & 0 & 2\alpha\sigma(x)P_L \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} a_2(x,R_\lambda)P_L & 0 & 0 & 0 \\ 0 & a_2(x,R_{\lambda_c})P_R & 0 & 0 \\ 0 & 0 & a_2(x,R_\rho)P_R & 0 \\ 0 & 0 & 0 & a_2(x,R_{\rho_c})P_L \end{pmatrix}$$
$$\Downarrow$$
$$\mathrm{tr}[J_W(x)a_2(x,R)] = \frac{-3+4\alpha}{2}\sigma(x)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] + \frac{-3+4\alpha}{2}\sigma(x)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] + \\ + 2\alpha\sigma(x)\,\mathrm{tr}[a_2(x,R_\rho)P_R] + 2\alpha\sigma(x)\,\mathrm{tr}[a_2(x,R_{\rho_c})P_L] \ ,$$

which, when substituted in (3.10), will cause the equality

$$- \lim_{M\to\infty} \mathrm{Tr}[J_W e^{-\frac{R}{M^2}}] \overset{(3.10)}{=} i \int d^4x \left\{ \frac{3-4\alpha}{2(4\pi)^2}\Big(\mathrm{tr}[a_2(x,R_\lambda)P_L] + \mathrm{tr}[a_2(x,R_{\lambda_c})P_R]\Big) + \right. \\ \left. - \frac{2\alpha}{(4\pi)^2}\Big(\mathrm{tr}[a_2(x,R_\rho)P_R] + \mathrm{tr}[a_2(x,R_{\rho_c})P_L]\Big)\right\}\sigma(x) \tag{5.2}$$

to hold true. Eventually, by inserting (5.2) and (5.1) in (2.49), we would forthwith attain

$$i\int d^4x\, \langle T^a{}_a(x)\rangle\sigma(x) = i \int d^4x \left\{ \frac{3-4\alpha}{2(4\pi)^2}\Big(\mathrm{tr}[a_2(x,R_\lambda)P_L] + \mathrm{tr}[a_2(x,R_{\lambda_c})P_R]\Big) + \right. \\ \left. - \frac{2\alpha}{(4\pi)^2}\Big(\mathrm{tr}[a_2(x,R_\rho)P_R] + \mathrm{tr}[a_2(x,R_{\rho_c})P_L]\Big)\right\}\sigma(x) \ ,$$

also realizing that a proper expression for the vacuum expectation value of the stress-energy tensor trace would be

$$\langle T^a{}_a\rangle = \left\{ \frac{3-4\alpha}{2(4\pi)^2}\Big(\mathrm{tr}[a_2(x,R_\lambda)P_L] + \mathrm{tr}[a_2(x,R_{\lambda_c})P_R]\Big) + \right. \\ \left. - \frac{2\alpha}{(4\pi)^2}\Big(\mathrm{tr}[a_2(x,R_\rho)P_R] + \mathrm{tr}[a_2(x,R_{\rho_c})P_L]\Big)\right\} = \\ \overset{(F.6)}{=} \frac{3-8\alpha}{6(4\pi)^2}\Big(-(\partial_a A^a)^2 + \frac{1}{2}F_{ab}F^{ab} - \Box(A)^2 + (\partial_a A_b)(\partial^a A^b)\Big). \tag{5.3}$$

This is the trace anomaly obtained with the Dirac mass PV regularization. Of course, we expect that most of the terms in (5.3) could be erased by varying appropriate local counterterms to be added to the effective action of the model.

5.2 Chiral anomaly

Repeating the same procedure we abided by right above, except for the fact that this time we'll be using the flat limit of the variation (2.9), we'll get that it's

$$i\langle \delta_{\zeta(x)} S \rangle = -i \int d^4x \, \langle \, \partial_\mu \left(i\overline{\lambda}(x)\gamma^\mu \lambda(x) \right) \, \rangle \zeta(x) \tag{5.4}$$

that has to be equaled to the trace expansion (3.10). In this respect, the combination of (2.28) and (2.37) now leads us to an infinitesimal jacobian (1.34)

$$J_C(x) = i\zeta(x) \begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & -P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

whose product with $a_2(x, R)$ reads as

$$J_C(x) a_2(x,R) = i\zeta(x) \begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & -P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} a_2(x,R_\lambda)P_L & 0 & 0 & 0 \\ 0 & a_2(x,R_{\lambda_c})P_R & 0 & 0 \\ 0 & 0 & a_2(x,R_\rho)P_R & 0 \\ 0 & 0 & 0 & a_2(x,R_{\rho_c})P_L \end{pmatrix}$$

$$\Downarrow$$

$$\mathrm{tr}[J_C(x) a_2(x,R)] = i\zeta(x) \, \mathrm{tr}[a_2(x,R_\lambda)P_L] - i\zeta(x) \, \mathrm{tr}[a_2(x,R_{\lambda_c})P_R] \, .$$

Then, the heat kernel formula (3.10) tells us that

$$- \lim_{M\to\infty} \mathrm{Tr}[J_C e^{-\frac{R}{M^2}}] \overset{(3.10)}{=} -i \int d^4x \, \frac{i\zeta(x)}{(4\pi)^2} \Big(\mathrm{tr}[a_2(x,R_\lambda)P_L] - \mathrm{tr}[a_2(x,R_{\lambda_c})P_R] \Big) \, , \tag{5.5}$$

making it clear that (2.49) should now look as

$$- i \int d^4x \, \langle \, \partial_\mu \left(i\overline{\lambda}(x)\gamma^\mu \lambda(x) \right) \, \rangle \zeta(x) = -i \int d^4x \, \frac{i\zeta(x)}{(4\pi)^2} \Big(\mathrm{tr}[a_2(x,R_\lambda)P_L] - \mathrm{tr}[a_2(x,R_{\lambda_c})P_R] \Big) \, , \tag{5.6}$$

from which the chiral anomaly itself

$$\begin{aligned} \langle \, \partial_\mu \left(i\overline{\lambda}\gamma^\mu \lambda \right) \, \rangle &= \frac{i}{(4\pi)^2} \Big(\mathrm{tr}[a_2(x,R_\lambda)P_L] - \mathrm{tr}[a_2(x,R_{\lambda_c})P_R] \Big) = \\ &\overset{(F.7)}{=} \frac{1}{3(4\pi)^2} \Big(\Box(\partial_s A^s) - \partial_a (A^2 A^a) + \frac{1}{2} \epsilon^{abcd} F_{ab} F_{cd} \Big) \end{aligned} \tag{5.7}$$

emerges. This is the correct chiral anomaly for the Weyl fermion, as it is well-known that counterterms can remove the even-parity terms.

5.3 Lorentz anomaly

In a similar way to what was done before, we can now move on to the Lorentz anomaly. So, given equation (2.49), we begin by replacing its LHS with the flat space limit ($e \to 1$) of the legitimate Lorentz variation we carried out on the action in (2.13):

$$i\langle \delta_{\omega(x)} S\rangle = i\int d^4x\, \langle \underbrace{T^{fe}(x)}_{\text{only its antisym. part will contrib.}}\rangle \omega_{ef}(x) = \frac{i}{2}\int d^4x\, \langle T^{fe}(x) - T^{ef}(x)\rangle \omega_{ef}(x). \tag{5.8}$$

Thereafter, we firstly rely on (3.10), so that we can manipulate the RHS of (2.49) in accordance to the heat kernel expansion we worked out in chapter 3: in this case the two pieces which comprise Fujikawa's jacobian are (2.32) and (2.41), thus getting

$$J_L(x) = \frac{1}{4}\omega_{ef}(x)\gamma^{ef}\begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.9}$$

Its matrix multiplication with (4.5)

$$J_L(x)a_2(x,R) = \begin{pmatrix} \frac{1}{4}\omega_{ef}(x)\gamma^{ef}P_L & 0 & 0 & 0 \\ 0 & \frac{1}{4}\omega_{ef}(x)\gamma^{ef}P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\cdot$$

$$\cdot\begin{pmatrix} a_2(x,R_\lambda)P_L & 0 & 0 & 0 \\ 0 & a_2(x,R_{\lambda_c})P_R & 0 & 0 \\ 0 & 0 & a_2(x,R_\rho)P_R & 0 \\ 0 & 0 & 0 & a_2(x,R_{\rho_c})P_L \end{pmatrix} =$$

$$\Downarrow$$

$$\mathrm{tr}[J_L(x)a_2(x,R)] = \frac{1}{4}\omega_{ef}(x)\,\mathrm{tr}[\gamma^{ef}a_2(x,R_\lambda)P_L] + \frac{1}{4}\omega_{ef}(x)\,\mathrm{tr}[\gamma^{ef}a_2(x,R_{\lambda_c})P_R]$$

then leads us to

$$\begin{aligned} -\lim_{M\to\infty}\mathrm{Tr}[J_L e^{-\frac{R}{M^2}}] &\overset{(3.10)}{=} -\int d^4x \frac{i}{(4\pi)^2}\mathrm{tr}[J_L(x)a_2(x,R)] = \\ &= -\frac{i}{4(4\pi)^2}\int d^4x\, \omega_{ef}(x)\Big(\mathrm{tr}[\gamma^{ef}a_2(x,R_\lambda)P_L] + \mathrm{tr}[\gamma^{ef}a_2(x,R_{\lambda_c})P_R]\Big). \end{aligned} \tag{5.10}$$

In the end, complying with (2.49), we do manage to learn what expression describes the anomalous expectation value of the stress-energy tensor's antisymmetric part:

$$\begin{aligned} &\frac{i}{2}\int d^4x\, \langle T^{fe}(x) - T^{ef}(x)\rangle\omega_{ef}(x) = \\ &\qquad = -\frac{i}{4(4\pi)^2}\int d^4x\, \omega_{ef}(x)\Big(\mathrm{tr}[\gamma^{ef}a_2(x,R_\lambda)P_L] + \mathrm{tr}[\gamma^{ef}a_2(x,R_{\lambda_c})P_R]\Big) \end{aligned}$$

$$\Downarrow$$

$$\frac{1}{2}\langle T^{ef}(x) - T^{fe}(x)\rangle = \frac{1}{4(4\pi)^2}\Big(\mathrm{tr}[\gamma^{ef} a_2(x, R_\lambda) P_L] + \mathrm{tr}[\gamma^{ef} a_2(x, R_{\lambda_c}) P_R]\Big) =$$
$$\stackrel{(F.10)}{=} \frac{1}{2}(4\pi)^2\Big(\frac{1}{2}\partial_c(A^c)F^{ef} + \frac{1}{3}\partial^b(A^e F^f{}_b) - \frac{1}{3}\partial^b(A^f F^e{}_b) - \frac{1}{6}\partial^e(A^d F^f{}_d) +$$
$$- \frac{1}{6}A^d\partial^e(F^f{}_d) + \frac{1}{6}\partial^f(A^d F^e{}_d) + \frac{1}{6}A^d\partial^f(F^e{}_d) +$$
$$+ \frac{1}{12}\epsilon^{cdef}\Box(F_{cd}) - \frac{1}{6}\epsilon^{cbef}A^a A_c F_{ab}\Big) \,. \tag{5.11}$$

Thus, a local Lorentz anomaly indeed emerges. Presumably it can be canceled by adding local counterterms to the effective action.

5.4 Gravitational anomaly

Finally, we can focus on the only anomaly still missing: the gravitational one. As usual, we proceed by entering the flat action variation (2.15) in the LHS of (2.49), producing

$$i\langle\delta_\xi S\rangle = -i\int d^4x\, \xi^e(x)\langle\partial^b T_{be}(x) + i\bar{\lambda}\gamma^b\lambda F_{be}(x)\rangle \,, \tag{5.12}$$

whereas, as regards its RHS, we must be a little more cautious than before. This time, in fact, we can't simply make use of one of the heat kernel expansions of chapter 3: after putting (2.34) and (2.43) together to compose the general flat Einstein jacobian

$$J_E(x) = \xi^e(x)\partial_e \begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & P_R & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix} \,, \tag{5.13}$$

we are obliged to single out its differential part,

$$J_1(x) = \xi^e(x)\partial_e \begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,, \tag{5.14}$$

from the standard one,

$$J_2(x) = \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & P_R & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix} \,. \tag{5.15}$$

Hence, when we replace (5.13) in the regular Fujikawa's limit of equation (2.49), we get

$$
\begin{aligned}
-\lim_{M\to\infty}\mathrm{Tr}[J_E e^{-\frac{R}{M^2}}] &= \underbrace{-\lim_{M\to\infty}\mathrm{Tr}[J_1 e^{-\frac{R}{M^2}}]}_{(3.28)} \underbrace{-\lim_{M\to\infty}\mathrm{Tr}[J_2 e^{-\frac{R}{M^2}}]}_{(3.10)} = \\
&= -\int d^4x \frac{i}{(4\pi)^2}\xi^e(x)\mathrm{tr}[b_{e,2}(x,R)]\Big|_{R_\rho=R_{\rho c}=0} + \\
&\quad -\int d^4x \frac{i}{(4\pi)^2}\mathrm{tr}[Q_0(x)a_2(x,R)]\Big|_{R_\rho=R_{\rho c}=0} + \\
&\quad -\int d^4x \frac{i}{(4\pi)^2}\mathrm{tr}[J_2(x)a_2(x,R)]\ ,
\end{aligned}
\tag{5.16}
$$

where, as (5.14) dictates, the use of (3.28) has been restricted to the left sectors only, since no differential operator is found to affect the right ones. Moreover, we have that

$$
Q_0(x) = -\frac{1}{2}\partial_e\big(\xi^e(x)\big)\begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & P_R & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix} - \xi^e(x)\begin{pmatrix} W_e(x)P_L & 0 & 0 & 0 \\ 0 & \widetilde{W}_e(x)P_R & 0 & 0 \\ 0 & 0 & Z_e(x)P_R & 0 \\ 0 & 0 & 0 & \widetilde{Z}_e(x)P_L \end{pmatrix}\ ,
\tag{5.17}
$$

as one could verify from(3.18). The reckoning of

$$
\begin{aligned}
Q_0(x)a_2(x,R) &= -\frac{1}{2}\partial_e\big(\xi^e(x)\big)\begin{pmatrix} P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 \\ 0 & 0 & P_R & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix}\cdot \\
&\quad \cdot\begin{pmatrix} a_2(x,R_\lambda)P_L & 0 & 0 & 0 \\ 0 & a_2(x,R_{\lambda_c})P_R & 0 & 0 \\ 0 & 0 & a_2(x,R_\rho)P_R & 0 \\ 0 & 0 & 0 & a_2(x,R_{\rho_c})P_L \end{pmatrix} + \\
&\quad -\xi^e(x)\begin{pmatrix} W_e(x)P_L & 0 & 0 & 0 \\ 0 & \widetilde{W}_e(x)P_R & 0 & 0 \\ 0 & 0 & Z_e(x)P_R & 0 \\ 0 & 0 & 0 & \widetilde{Z}_e(x)P_L \end{pmatrix}\cdot \\
&\quad \cdot\begin{pmatrix} a_2(x,R_\lambda)P_L & 0 & 0 & 0 \\ 0 & a_2(x,R_{\lambda_c})P_R & 0 & 0 \\ 0 & 0 & a_2(x,R_\rho)P_R & 0 \\ 0 & 0 & 0 & a_2(x,R_{\rho_c})P_L \end{pmatrix}
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
\mathrm{tr}[Q_0(x)a_2(x,R)]\Big|_{R_\rho=R_{\rho c}=0} &= -\frac{1}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] - \frac{1}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] + \\
&\quad -\xi^e(x)\,\mathrm{tr}[\underbrace{W_e(x)}_{(4.9)}\,a_2(x,R_\lambda)P_L] - \xi^e(x)\,\mathrm{tr}[\underbrace{\widetilde{W}_e(x)}_{(4.18)}\,a_2(x,R_{\lambda_c})P_R] =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] - \frac{1}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] +\\
&\quad + \frac{i}{2}\xi^e(x)A_e(x)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] + \frac{i}{2}\xi^e(x)A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_\lambda)P_L] +\\
&\quad - \frac{i}{2}\xi^e(x)A_e(x)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] - \frac{i}{2}\xi^e(x)A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_{\lambda_c})P_R]\ ,
\end{aligned}
$$

and then

$$
J_2(x)a_2(x,R) = \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\begin{pmatrix} P_L & 0 & 0 & 0\\ 0 & P_R & 0 & 0\\ 0 & 0 & P_R & 0\\ 0 & 0 & 0 & P_L\end{pmatrix}\cdot
$$

$$
\cdot\begin{pmatrix} a_2(x,R_\lambda)P_L & 0 & 0 & 0\\ 0 & a_2(x,R_{\lambda_c})P_R & 0 & 0\\ 0 & 0 & a_2(x,R_\rho)P_R & 0\\ 0 & 0 & 0 & a_2(x,R_{\rho_c})P_L\end{pmatrix}
$$

$$\Downarrow$$

$$
\begin{aligned}
\mathrm{tr}[J_2(x)a_2(x,R)] &= \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] + \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] +\\
&\quad + \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_\rho)P_R] + \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_{\rho_c})P_L]\ ,
\end{aligned}
$$

allows (5.16) to be expanded as

$$
\begin{aligned}
-\lim_{M\to\infty}\mathrm{Tr}[J_E e^{-\frac{R}{M^2}}] &= -\int d^4x\frac{i}{(4\pi)^2}\Big(\xi^e(x)\mathrm{tr}[b_{e,2}(x,R_\lambda)P_L] + \xi^e(x)\mathrm{tr}[b_{e,2}(x,R_{\lambda_c})P_R] +\\
&\quad - \frac{1}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] - \frac{1}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] +\\
&\quad + \frac{i}{2}\xi^e(x)A_e(x)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] + \frac{i}{2}\xi^e(x)A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_\lambda)P_L] +\\
&\quad - \frac{i}{2}\xi^e(x)A_e(x)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] - \frac{i}{2}\xi^e(x)A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_{\lambda_c})P_R] +\\
&\quad + \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] + \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] +\\
&\quad + \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_\rho)P_R] + \frac{\alpha}{2}\partial_e\big(\xi^e(x)\big)\,\mathrm{tr}[a_2(x,R_{\rho_c})P_L]\Big) =\\
&\doteq -\int d^4x\frac{i}{(4\pi)^2}\xi^e(x)\Big(\mathrm{tr}[b_{e,2}(x,R_\lambda)P_L] + \mathrm{tr}[b_{e,2}(x,R_{\lambda_c})P_R] +\\
&\quad + \frac{1}{2}\partial_e\big(\mathrm{tr}[a_2(x,R_\lambda)P_L]\big) + \frac{1}{2}\partial_e\big(\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]\big) +\\
&\quad + \frac{i}{2}A_e(x)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] + \frac{i}{2}A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_\lambda)P_L] +\\
&\quad - \frac{i}{2}A_e(x)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] - \frac{i}{2}A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_{\lambda_c})P_R] +\\
&\quad - \frac{\alpha}{2}\partial_e\big(\mathrm{tr}[a_2(x,R_\lambda)P_L]\big) - \frac{\alpha}{2}\partial_e\big(\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]\big) +
\end{aligned}
$$

$$- \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_\rho)P_R]\Big) - \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_{\rho_c})P_L]\Big)\Big) \,. \qquad (5.18)$$

Thus, by piecing together the two expanded sides of (2.49), or rather, (5.12) and (5.18), we realize that the Fujikawa's approach brings about

$$-i\int d^4x\,\xi^e(x)\langle\partial^b\,T_{be}(x) + i\bar{\lambda}\gamma^b\lambda F_{be}(x)\rangle = \qquad (5.19)$$

$$\begin{aligned}
= -\int d^4x \frac{i}{(4\pi)^2}\xi^e(x)\Big(&\mathrm{tr}[b_{e,2}(x,R_\lambda)P_L] + \mathrm{tr}[b_{e,2}(x,R_{\lambda_c})P_R] + \\
&+ \frac{1}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_\lambda)P_L]\Big) + \frac{1}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]\Big) + \\
&+ \frac{i}{2}A_e(x)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] + \frac{i}{2}A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_\lambda)P_L] + \\
&- \frac{i}{2}A_e(x)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] - \frac{i}{2}A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_{\lambda_c})P_R] + \\
&- \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_\lambda)P_L]\Big) - \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]\Big) + \\
&- \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_\rho)P_R]\Big) - \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_{\rho_c})P_L]\Big)\Big) \qquad (5.20)
\end{aligned}$$

to be valid, whence the gravitational anomaly finally follows:

$$\begin{aligned}
\langle\partial^b\,T_{be}(x) + i\bar{\lambda}\gamma^b\lambda F_{be}(x)\rangle = \frac{1}{(4\pi)^2}\Big(&\mathrm{tr}[b_{e,2}(x,R_\lambda)P_L] + \mathrm{tr}[b_{e,2}(x,R_{\lambda_c})P_R] + \\
&+ \frac{1}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_\lambda)P_L]\Big) + \frac{1}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]\Big) + \\
&+ \frac{i}{2}A_e(x)\,\mathrm{tr}[a_2(x,R_\lambda)P_L] + \frac{i}{2}A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_\lambda)P_L] + \\
&- \frac{i}{2}A_e(x)\,\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] - \frac{i}{2}A^f(x)\,\mathrm{tr}[\gamma_{ef}a_2(x,R_{\lambda_c})P_R] + \\
&- \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_\lambda)P_L]\Big) - \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]\Big) + \\
&- \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_\rho)P_R]\Big) - \frac{\alpha}{2}\partial_e\Big(\mathrm{tr}[a_2(x,R_{\rho_c})P_L]\Big)\Big) = \\
\stackrel{(F.13)}{=} \frac{1}{(4\pi)^2}\Big[\frac{1}{6}\Big(&\partial^i\big(\partial_b(A^b)F_{ie}\big) - \partial^i\big(\partial_i(A^c)F_{ec}\big) + \partial^i\big(\partial_e(A^c)F_{ic}\big) + \\
&+ \frac{1}{2}\epsilon_{dbic}\partial^i\big(A_eA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbec}\partial^i\big(A_iA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbie}\partial^i\big(A^2F^{db}\big)\Big) + \\
&+ \frac{1}{6}\partial_e\Big(-(\partial_aA^a)^2 + \frac{1}{2}F_{ab}F^{ab} - \Box(A^2) + \partial_b(A_c)\partial^b(A^c)\Big) + \\
&- \frac{1}{6}A_e\Big(-\Box(\partial_sA^s) + \partial_a(A^2A^a) - \frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big) + \\
&- A^f\Big(\frac{1}{6}\Box(F_{ef}) + \frac{1}{6}A^aA_fF_{ae} - \frac{1}{6}A^aA_eF_{af} - \frac{1}{4}\epsilon_{abef}\partial_c(A^c)F^{ab} + \\
&+ \frac{1}{3}\epsilon_{acef}\partial_b(A^cF^{ab}) + \frac{1}{6}\epsilon_{acef}\partial^a(A_dF^{cd}) + \frac{1}{6}\epsilon_{acef}A_d\partial^a(F^{cd})\Big) + \\
&- \frac{\alpha}{3}\partial_e\Big(-(\partial_aA^a)^2 + \frac{1}{2}F_{ab}F^{ab} - \Box(A^2) + \partial_b(A_c)\partial^b(A^c)\Big)\Big] \,. \qquad (5.21)
\end{aligned}$$

Hence, the expression of the gravitational anomaly has been deduced, too.

5.5 Discussion

We have calculated all the anomalies of a Weyl fermion in an abelian gauge background, using PV fields with a Dirac mass and casting the calculation in a typical Fujikawa's fashion. The various computations have then been verified through the software of ref. [15].

In the end, the chiral anomaly is seen to coincide with the expected results, see also [11]. As for the anomalies of the stress tensor, all of them appear as a consequence of the non invariance of the Dirac mass term adopted. However, one expects that local counterterms may reproduce the trace anomaly as computed in [11], without the topological odd-parity contribution, reinstating at the same time the local Lorentz and general coordinate symmetries. The structure of these counterterms will not be analyzed here, as the lack of the symmetries makes this task far from being obvious, and is left for future work. In any case, we can consider ourselves satisfied with the achievement of the gravitational anomaly's formula, whose computation in 4 dimensions through heat kernel methods had never been pursued until now.

Chapter 6

Conclusions

In summary, during the first chapter we introduced the anomaly's issue, thoroughly depicting the schemes we used to approach it: we illustrated the Fujikawa's method improved by the procedure described in [13, 18]. Then, we devoted chapter 2 to explore all the symmetry transformations marking out the Weyl spinor model, evaluating the variation of its action under those mappings. We also built an unusual Pauli-Villars lagrangian to be used in the regularization process, equipping it with a Dirac mass term proportional to the $\alpha^{\text{th}}-$power of the vierbein

$$M(\overline{\lambda}\rho + \overline{\rho}\lambda)e^{\alpha}.$$

The chapter ended with the computation of the pieces composing the infinitesimal Fujikawa's jacobians.

In 3 we discussed how to implement the heat kernel expansion to express the traces that are responsible for the anomalous quantum expectation values. Thus, we presented the Seeley-DeWitt coefficients and their differential generalization, i.e. the Branson-Gilkey-Vassilevich coefficients, leaving their actual calculation to chapter 4.

Finally, in chap. 5, we tracked down the exact anomalies connected to the main symmetries of the action, though, to be fair, we didn't check for the existence of local counterterms which either prevent those to survive or simplify their expressions. Every result of this chapter has been then tested using a software developed in wolfram language [15, 21, 22], enhanced with several functions we wrote to better handle the data flow arising from trace computations.

Providing the reader with anomalies acquired in this new setup, we aimed the current thesis to pave the way for future projects: for example, thanks to the α parameter escorting our mass term, one could try to verify that, irrespective of the regulator's class adopted, only the chiral and trace anomalies survive and take the expected form.

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Appendix A

Spinor conventions

The following notation is the one we adopted throughout the paper.
As is well-known, a 4−dimensional Dirac spinor field ψ can be expressed by the direct sum of its indipendent 2−dimensional Weyl components, l and r, as

$$\psi = l \oplus r = \begin{pmatrix} l \\ r \end{pmatrix} . \tag{A.1}$$

Alternatively, one could insert the two previous chiral spinors into higher dimensional Dirac vectors,

$$\lambda = \begin{pmatrix} l \\ 0 \end{pmatrix} \tag{A.2}$$

and

$$\rho = \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{A.3}$$

respectively, thus restating (A.1) via

$$\psi = \lambda + \rho \, . \tag{A.4}$$

At this stage, the projectors (2.23), namely

$$P_L = \frac{\mathbb{1} + \gamma^5}{2} \qquad \text{and} \qquad P_R = \frac{\mathbb{1} - \gamma^5}{2}$$

enable (A.2) and (A.3) to be rewritten in terms of ψ

$$\lambda = P_L \psi \qquad \text{and} \qquad \rho = P_R \psi \ ,$$

where γ^5 and the other γ matrices are defined in appendix B.

Then, the conjugate Dirac spinor field $\overline{\psi}$ descends from the customary definition that sees $\beta = i\gamma^0$ involved

$$\overline{\psi} = \psi^\dagger \beta,$$

which in turn leads to the charge conjugated spinor ψ_c

$$\psi_c = C^{-1} \overline{\psi}^T . \tag{A.5}$$

In the last line, the charge conjugation operator C appeared: it acts on γ^j by binding it to its transposed

$$C\gamma^j C^{-1} = -\gamma^{j\,T}, \tag{A.6}$$

and satisfies the ensuing chain of equality

$$C = -C^T = -C^{-1} = -C^\dagger = C^* \ . \tag{A.7}$$

The only operator fulfilling both (A.6) and (A.7) is the one determined by a particular product of γ matrices, scilicet

$$C = \gamma^2\beta \ . \tag{A.8}$$

This completes appendix A.

Appendix B

Gamma matrices: conventions and formulae

We devote this appendix to the main results and properties holding true for gamma marices and their product. Before we start, let's recall the Clifford algebra's definition

$$\{\gamma^a, \gamma^b\} = 2g^{ab}(\mathbb{1}), \tag{B.1}$$

and, for a matter of completeness, we likewise introduce the antisymmetric tensor γ^{ab}

$$[\gamma^a, \gamma^b] = 2\gamma^{ab}, \tag{B.2}$$

as well as the fifth gamma matrix, obtained from the combination of all the others:

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d, \tag{B.3}$$

where ϵ_{abcd} is the complete antisymmetric Levi-Civita tensor, defined by the condition that

$$\epsilon_{0123} = -1. \tag{B.4}$$

γ^5 satisfies two essential relations that will be largely employed in the following. First of all, by its own definition (B.3), it anticommutes with any other gamma

$$\gamma^5\gamma^a = -\gamma^a\gamma^5\ , \tag{B.5}$$

and then its multiplication with itself gives

$$(\gamma^5)^2 = \mathbb{1}\ . \tag{B.6}$$

However, if we used the chiral representation to express the γs, they would read as follows

$$\gamma^0 = -i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^j = -i\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{B.7}$$

where the σ^j are the usual Pauli matrices.

With that in mind, we have

- **decomposition into symmetric and antisymmetric part:**

$$\gamma_a\gamma_b = \frac{\{\gamma_a, \gamma_b\}}{2} + \frac{[\gamma_a, \gamma_b]}{2} = g_{ab} + \gamma_{ab}\ ; \tag{B.8}$$

- **simple contraction:**

$$\gamma^a\gamma_a \overset{\text{(B.1)}}{=} g^a{}_a(\mathbb{1}) = 4(\mathbb{1}) \; ; \tag{B.9}$$

- **contraction (with one gamma matrix in between):**

$$\gamma^a\gamma^b\gamma_a \overset{\text{R:(B.1)}}{=} 2g^{ab}\gamma_a - \gamma^b \underbrace{\gamma^a\gamma_a}_{\text{(B.9)}} = 2\gamma^b - 4\gamma^b = -2\gamma^b \; , \tag{B.10}$$

where the symbol $\overset{\text{R:(B.}n\text{)}}{=}$ (with $n \in \mathbb{N}$) introduces a rearrangement of the gamma matrices which takes place according to the rules reported in equation (B.n). From now on its meaning will be so understood;

- **contraction (with two gamma matrices in between):**

$$\begin{aligned}\gamma^a\gamma^b\gamma^c\gamma_a &\overset{\text{R:(B.1)}}{=} 2g^{ab}\gamma^c\gamma_a - \gamma^b\gamma^a\gamma^c\gamma_a \overset{\text{R:(B.1)}}{=} 2g^{ab}\gamma^c\gamma_a - 2\gamma^b g^{ac}\gamma_a + \gamma^b\gamma^c \underbrace{\gamma^a\gamma_a}_{\text{(B.9)}} = \\ &= 2\gamma^c\gamma^b - 2\gamma^b\gamma^c + 4\gamma^b\gamma^c \overset{\text{(B.1)}}{=} 4g^{cb}(\mathbb{1}) - 2\gamma^b\gamma^c - 2\gamma^b\gamma^c + 4\gamma^b\gamma^c = \\ &= 4g^{cb}(\mathbb{1}) \; ; \end{aligned} \tag{B.11}$$

- **single antisymmetric contraction:**

$$\begin{aligned}\gamma^{ab}\gamma_a{}^c &= \left[\frac{\gamma^a\gamma^b - \gamma^b\gamma^a}{2}\right]\left[\frac{\gamma_a\gamma^c - \gamma^c\gamma_a}{2}\right] = \\ &= \frac{1}{4}\Big[\underbrace{\gamma^a\gamma^b\gamma_a}_{\text{(B.10)}}\gamma^c - \gamma^b \underbrace{\gamma^a\gamma_a}_{\text{(B.9)}}\gamma^c - \underbrace{\gamma^a\gamma^b\gamma^c\gamma_a}_{\text{(B.11)}} + \gamma^b \underbrace{\gamma^a\gamma^c\gamma_a}_{\text{(B.10)}}\Big] = \\ &= \frac{1}{4}\left[-2\gamma^b\gamma^c - 4\gamma^b\gamma^c - 4g^{cb}(\mathbb{1}) - 2\gamma^b\gamma^c\right] = \\ &= \frac{1}{4}\left[-8\gamma^b\gamma^c - 4g^{cb}(\mathbb{1})\right] = \\ &= -2\underbrace{\gamma^b\gamma^c}_{\text{(B.8)}} - g^{cb}(\mathbb{1}) = \\ &= -2\gamma^{bc} - 3g^{bc}(\mathbb{1}) \; ; \end{aligned} \tag{B.12}$$

- **double antisymmetric contraction:**

$$\begin{aligned}\gamma^{ab}\gamma_{ab} &= g_{bc}\underbrace{\gamma^{ab}\gamma_a{}^c}_{\text{(B.12)}} = g_{bc}\big(-2\gamma^b\gamma^c - g^{cb}(\mathbb{1})\big)\big) = \\ &= -2\underbrace{\gamma^b\gamma_b}_{\text{(B.9)}} - \underbrace{g^b{}_b}_{4}(\mathbb{1}) = \\ &= -12(\mathbb{1}) \; ; \end{aligned} \tag{B.13}$$

- **antisymmetric commutation**:

$$
\begin{aligned}
[\gamma_{ab},\gamma_{cd}] &= \left(\frac{\gamma_a\gamma_b-\gamma_b\gamma_a}{2}\right)\left(\frac{\gamma_c\gamma_d-\gamma_d\gamma_c}{2}\right)-\left(\frac{\gamma_c\gamma_d-\gamma_d\gamma_c}{2}\right)\left(\frac{\gamma_a\gamma_b-\gamma_b\gamma_a}{2}\right)= \\
&= \frac{1}{4}\big(\gamma_a\gamma_b\gamma_c\gamma_d-\gamma_a\gamma_b\gamma_d\gamma_c-\gamma_b\gamma_a\gamma_c\gamma_d+\gamma_b\gamma_a\gamma_d\gamma_c+ \\
&\qquad -\gamma_c\gamma_d\gamma_a\gamma_b+\gamma_c\gamma_d\gamma_b\gamma_a+\gamma_d\gamma_c\gamma_a\gamma_b-\gamma_d\gamma_c\gamma_b\gamma_a\big)= \\
\begin{pmatrix}\text{reordering the}\\ \text{last 4 terms}\end{pmatrix} &\overset{\text{R:(B.1)}}{=} \frac{1}{4}\big(\gamma_a\gamma_b\gamma_c\gamma_d-\gamma_a\gamma_b\gamma_d\gamma_c-\gamma_b\gamma_a\gamma_c\gamma_d+\gamma_b\gamma_a\gamma_d\gamma_c+ \\
&\quad -2g_{ad}\gamma_c\gamma_b+2g_{ac}\gamma_d\gamma_b-2g_{bd}\gamma_a\gamma_c+2g_{bc}\gamma_a\gamma_b-\gamma_a\gamma_b\gamma_c\gamma_d+ \\
&\quad +2g_{bd}\gamma_c\gamma_a-2g_{bc}\gamma_d\gamma_a+2g_{ad}\gamma_b\gamma_c-2g_{ac}\gamma_b\gamma_d+\gamma_b\gamma_a\gamma_c\gamma_d+ \\
&\quad +2g_{ac}\gamma_d\gamma_b-2g_{ad}\gamma_c\gamma_b+2g_{bc}\gamma_a\gamma_d-2g_{bd}\gamma_a\gamma_c+\gamma_a\gamma_b\gamma_d\gamma_c+ \\
&\quad -2g_{bc}\gamma_d\gamma_a+2g_{bd}\gamma_c\gamma_a-2g_{ac}\gamma_b\gamma_d+2g_{ad}\gamma_b\gamma_c-\gamma_b\gamma_a\gamma_d\gamma_c\big)= \\
&= \frac{1}{4}\big(4g_{ad}\gamma_b\gamma_c-4g_{ad}\gamma_c\gamma_b+4g_{ac}\gamma_d\gamma_b-4g_{ac}\gamma_b\gamma_d+ \\
&\qquad +4g_{bd}\gamma_c\gamma_a-4g_{bd}\gamma_a\gamma_b+4g_{bc}\gamma_a\gamma_d-4g_{bc}\gamma_d\gamma_a\big)\overset{\text{(B.2)}}{=} \\
&= 2g_{ad}\gamma_{bc}-2g_{ac}\gamma_{bd}-2g_{bd}\gamma_{ac}+2g_{bc}\gamma_{ad}\ ;
\end{aligned}
\tag{B.14}
$$

- **antisymmetric commutation with single contraction**:

$$
\begin{aligned}
[\gamma_{ab},\gamma^a{}_d] &= g^{ac}\underbrace{[\gamma_{ab},\gamma_{cd}]}_{\text{(B.14)}}= \\
&= 2g^c{}_d\gamma_{bc}-2\underbrace{g^c{}_c}_{4}\gamma_{bd}-2g_{bd}\underbrace{\gamma^c{}_c}_{0}+2g_{bc}\gamma^c{}_d= \\
&= 2\gamma_{bd}-8\gamma_{bd}+2\gamma_{bd}= \\
&= -4\gamma_{bd}\ ;
\end{aligned}
\tag{B.15}
$$

- **cyclic property of the trace:**

given any number of matrices A, B, C, $\cdots$, it is well known that the trace of their product is invariant under cyclic permutations:

$$
\mathrm{tr}[ABC\cdots]=\mathrm{tr}[BC\cdots A]=\mathrm{tr}[C\cdots AB]=\cdots \tag{B.16}
$$

In a nutshell, this is due to the basic rules of the matrix multiplication process;

- **trace of a single γ (γ^a):**

$$
\mathrm{tr}[\gamma^a]=\mathrm{tr}\Big[\gamma^a\underbrace{\frac{\gamma^b\gamma^b}{g^{bb}}}_{(\mathbb{1})}\Big]\overset{\text{R:(B.1)}}{=}-\mathrm{tr}\Big[\frac{\gamma^b\gamma^a\gamma^b}{g^{bb}}\Big]\overset{\text{(B.16)}}{=}-\mathrm{tr}\Big[\underbrace{\frac{\gamma^b\gamma^b}{g^{bb}}}_{(\mathbb{1})}\gamma^a\Big]=-\mathrm{tr}[\gamma^a]=0\ . \tag{B.17}
$$

An explanation is in order: inside the previous trace (first equality), we added the identity operator ($\mathbb{1}$), whose proper expression had been determined by inverting (B.1). Furthermore, we chose $\gamma^b \neq \gamma^a$ so that we could be free to use their anticommutation relation, always encoded in (B.1), and write $\gamma^a\gamma^b = -\gamma^b\gamma^a$ (second equality). Anything else should be crystal-clear. The same arguments hold for γ^5, too:

$$\mathrm{tr}[\gamma^5] = 0 \; ; \tag{B.18}$$

- **trace of the product of two γs (γ^a, γ^b):**

  by reorganizing the two outermost sides of the equality chain

$$\mathrm{tr}[\gamma^a\gamma^b] \overset{\mathrm{R:(B.1)}}{=} \mathrm{tr}[2g^{ab}(\mathbb{1}) - \gamma^b\gamma^a] = 2g^{ab}\underbrace{\mathrm{tr}[(\mathbb{1})]}_{4} - \mathrm{tr}[\gamma^b\gamma^a] \overset{\mathrm{(B.16)}}{=} 8g^{ab} - \mathrm{tr}[\gamma^a\gamma^b] \; ,$$

  it would be easy to see that the ensuing relation holds true

$$\mathrm{tr}[\gamma^a\gamma^b] = 4g^{ab} \; ; \tag{B.19}$$

- **trace of the product of an odd number of γs (γ^a, γ^b, …):**

$$\begin{aligned} \mathrm{tr}[\gamma^a\gamma^b\dots] &= \mathrm{tr}[\underbrace{\gamma^5\gamma^5}_{\mathrm{(B.6)}}\gamma^a\gamma^b\dots] \overset{\mathrm{R:(B.5)}}{=} -\mathrm{tr}[\gamma^5\gamma^a\gamma^b\dots\gamma^5] \overset{\mathrm{(B.16)}}{=} -\mathrm{tr}[\underbrace{\gamma^5\gamma^5}_{\mathrm{(B.6)}}\gamma^a\gamma^b\dots] = \\ &= -\mathrm{tr}[\gamma^a\gamma^b\dots] = 0 \; ; \end{aligned} \tag{B.20}$$

- **trace of the product of γ^5 and an odd number of γs (γ^a, γ^b, …):**

$$\mathrm{tr}[\gamma^5\gamma^a\gamma^b\dots] \overset{\mathrm{(B.5)}}{=} -\mathrm{tr}[\gamma^a\gamma^b\dots\gamma^5] \overset{\mathrm{(B.16)}}{=} -\mathrm{tr}[\gamma^5\gamma^a\gamma^b\dots] = 0 \; ; \tag{B.21}$$

- **trace of the product of γ^5 and two γs (γ^a, γ^b):**

$$\begin{aligned} \mathrm{tr}[\gamma^5\gamma^a\gamma^b] &= \mathrm{tr}\Big[\gamma^5\gamma^a\gamma^b\underbrace{\frac{\gamma^c\gamma^c}{g^{cc}}}_{(\mathbb{1})}\Big] \overset{\mathrm{R:(B.1)}}{=} \mathrm{tr}\Big[\gamma^5\gamma^c\gamma^a\gamma^b\frac{\gamma^c}{g^{cc}}\Big] \overset{\mathrm{R:(B.5)}}{=} -\mathrm{tr}\Big[\gamma^c\gamma^5\gamma^a\gamma^b\frac{\gamma^c}{g^{cc}}\Big] = \\ &\overset{\mathrm{(B.16)}}{=} -\mathrm{tr}\Big[\gamma^5\gamma^a\gamma^b\underbrace{\frac{\gamma^c\gamma^c}{g^{cc}}}_{(\mathbb{1})}\Big] = 0 \; , \end{aligned} \tag{B.22}$$

  where, similarly to what has been done in (B.17), γ^c must be different from γ^a and γ^b so that we can take advantage of its anticommutation property. Anyway, as long as we consider less than four gamma matrices, it's always possible to select γ^c such that $\gamma^c \neq \gamma^a$ and $\gamma^c \neq \gamma^b$;

- **trace of a single antisymmetric γ (γ^{ab}):**

$$\mathrm{tr}[\gamma^{ab}] = \frac{1}{2}\Big(\mathrm{tr}[\gamma^a\gamma^b - \gamma^b\gamma^a]\Big) = \frac{1}{2}\Big(\mathrm{tr}[\gamma^a\gamma^b] - \mathrm{tr}[\gamma^b\gamma^a]\Big) \overset{\mathrm{(B.16)}}{=} \frac{1}{2}\Big(\mathrm{tr}[\gamma^a\gamma^b] - \mathrm{tr}[\gamma^a\gamma^b]\Big) = 0 \; ; \tag{B.23}$$

- **trace of the product of γ^5 and a single antisymmetric γ (γ^{ab}):**

$$\mathrm{tr}[\gamma^5\gamma^{ab}] = \frac{1}{2}\Big(\mathrm{tr}[\gamma^5\gamma^a\gamma^b - \gamma^5\gamma^b\gamma^a]\Big) \overset{\mathrm{R:(B.5)}}{=} \frac{1}{2}\Big(\mathrm{tr}[\gamma^5\gamma^a\gamma^b + \gamma^b\gamma^5\gamma^a]\Big) \overset{\mathrm{(B.16)}}{=} \underbrace{\mathrm{tr}[\gamma^5\gamma^a\gamma^b]}_{\mathrm{(B.22)}} = 0 \; ; \tag{B.24}$$

- **identity - product of three γs (γ_a, γ_b, γ_c):**

  we intend to derive an intriguing identity holding for Dirac matrices that will come in handy during the evaluation of the traces of multiple $\gamma-$products. We begin by reporting every relation we'll make use of in the course of the present point. First of all, we recall that the flat space-time metric g_{ab} and its inverse g^{ab}, by which we lower ($\gamma_a = g_{ab}\gamma^b$) and raise ($\gamma^a = g^{ab}\gamma_b$) the tensor's indices respectively, are defined to be

$$g_{ab} \equiv g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1}_{3x3} \end{pmatrix} . \tag{B.25}$$

  Hence, from (B.1) and (B.25), we can gather

$$(\gamma^0)^2 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 = \mathbb{1}\,, \tag{B.26}$$

  and

$$\begin{aligned} &\gamma_0 = \gamma^0 \\ &\gamma_i = -\gamma^i, \quad (i = 1, 2, 3) \end{aligned} \tag{B.27}$$

  as well.

  Now, since we got everything settled, we can proceed by considering the unusual contraction that will permit us to achieve our purpose, that is $i\epsilon_{sabc}\gamma^s\gamma^5$. In fact, if this object were expanded, it would result in

$$\begin{aligned} i\epsilon_{sabc}\gamma^s\gamma^5 &\overset{(B.3)}{=} \underbrace{\epsilon_{sabc}\gamma^s}_{\text{expand}}\gamma^0\gamma^1\gamma^2\gamma^3 = \\ &= \epsilon_{0abc}\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + \epsilon_{1abc}\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 + \epsilon_{2abc}\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 + \epsilon_{3abc}\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = \\ &\overset{R:(B.1)}{=} \epsilon_{0abc}\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 - \epsilon_{1abc}\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 + \epsilon_{2abc}\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 - \epsilon_{3abc}\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = \\ &\overset{(B.26)}{=} \epsilon_{0abc}\gamma^1\gamma^2\gamma^3 + \epsilon_{1abc}\gamma^0\gamma^2\gamma^3 - \epsilon_{2abc}\gamma^0\gamma^1\gamma^3 + \epsilon_{3abc}\gamma^0\gamma^1\gamma^2, \end{aligned} \tag{B.28}$$

  which in turn, by testing every possible combination of the three indices a, b and c, would beget

$$i\epsilon_{sabc}\gamma^s\gamma^5 = \frac{1}{3!}\big(\gamma_a\gamma_b\gamma_c - \gamma_a\gamma_c\gamma_b + \gamma_b\gamma_c\gamma_a - \gamma_b\gamma_a\gamma_c + \gamma_c\gamma_a\gamma_b - \gamma_c\gamma_b\gamma_a\big) \tag{B.29}$$

  (for its thorough derivation see appendix G). Anyway, the preceding equality can be further processed by suitably reordering the three negative terms appearing in the RHS:

$$\begin{aligned} i\epsilon_{sabc}\gamma^s\gamma^5 &= \frac{1}{3!}\big(\gamma_a\gamma_b\gamma_c - \gamma_a\gamma_c\gamma_b + \gamma_b\gamma_c\gamma_a - \gamma_b\gamma_a\gamma_c + \gamma_c\gamma_a\gamma_b - \gamma_c\gamma_b\gamma_a\big) = \\ \begin{pmatrix}\text{reordering the}\\ \text{negative terms}\end{pmatrix} &\overset{R:(B.1)}{=} \frac{1}{6}\big(\gamma_a\gamma_b\gamma_c + \gamma_b\gamma_c\gamma_a + \gamma_c\gamma_a\gamma_b + \\ &\qquad - 2g_{ac}\gamma_b + 2g_{ab}\gamma_c - 2g_{bc}\gamma_a + \gamma_b\gamma_c\gamma_a + \\ &\qquad - 2g_{ba}\gamma_c + 2g_{bc}\gamma_a - 2g_{ca}\gamma_b + \gamma_c\gamma_a\gamma_b + \end{aligned}$$

$$- 2g_{cb}\gamma_a + 2g_{ca}\gamma_b - 2g_{ab}\gamma_c + \gamma_a\gamma_b\gamma_c\big) =$$

$$= \frac{1}{6}\big(2\gamma_a\gamma_b\gamma_c + 2\gamma_b\gamma_c\gamma_a + 2\gamma_c\gamma_a\gamma_b - 2g_{ac}\gamma_b - 2g_{ba}\gamma_c - 2g_{cb}\gamma_a\big) =$$

$$\begin{pmatrix}\text{reordering the} \\ 2^{\text{nd}} \text{ and } 3^{\text{rd}} \text{ term}\end{pmatrix} \overset{\text{R:(B.1)}}{=} \frac{1}{6}\big(2\gamma_a\gamma_b\gamma_c +$$
$$+ 4g_{ac}\gamma_b - 4g_{ab}\gamma_c + 2\gamma_a\gamma_b\gamma_c +$$
$$+ 4g_{ac}\gamma_b - 4g_{bc}\gamma_a + 2\gamma_a\gamma_b\gamma_c +$$
$$- 2g_{ac}\gamma_b - 2g_{ba}\gamma_c - 2g_{cb}\gamma_a\big) =$$
$$= \gamma_a\gamma_b\gamma_c + g_{ac}\gamma_b - g_{ab}\gamma_c - g_{bc}\gamma_a \ ,$$

and then rearranged in the much-discussed identity

$$\gamma_a\gamma_b\gamma_c = g_{ab}\gamma_c - g_{ac}\gamma_b + g_{bc}\gamma_a + i\epsilon_{sabc}\gamma^s\gamma^5 \ ; \tag{B.30}$$

- **trace of the product of four γs (γ^a, γ^b, γ^c, γ^d):**

$$\begin{aligned}\mathrm{tr}[\gamma^a\gamma^b\gamma^c\gamma^d] &\overset{\text{(B.30)}}{=} \mathrm{tr}[(g^{ab}\gamma^c - g^{ac}\gamma^b + g^{bc}\gamma^a + i\epsilon_s{}^{abc}\gamma^s\gamma^5)\gamma^d] = \\ &= g^{ab}\underbrace{\mathrm{tr}[\gamma^c\gamma^d]}_{\text{(B.19)}} - g^{ac}\underbrace{\mathrm{tr}[\gamma^b\gamma^d]}_{\text{(B.19)}} + g^{bc}\underbrace{\mathrm{tr}[\gamma^a\gamma^d]}_{\text{(B.19)}} + i\epsilon_s{}^{abc}\underbrace{\mathrm{tr}[\gamma^s\gamma^5\gamma^d]}_{\text{(B.22)}} = \\ &= 4g^{ab}g^{cd} - 4g^{ac}g^{bd} + 4g^{bc}g^{ad}\end{aligned} \tag{B.31}$$

- **trace of the product of γ^5 and four γs (γ^a, γ^b, γ^c, γ^d):**

$$\begin{aligned}\mathrm{tr}[\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d] &\overset{\text{(B.30)}}{=} \mathrm{tr}[\gamma^5(g^{ab}\gamma^c - g^{ac}\gamma^b + g^{bc}\gamma^a + i\epsilon_s{}^{abc}\gamma^s\gamma^5)\gamma^d] = \\ &= g^{ab}\underbrace{\mathrm{tr}[\gamma^5\gamma^c\gamma^d]}_{\text{(B.22)}} - g^{ac}\underbrace{\mathrm{tr}[\gamma^5\gamma^b\gamma^d]}_{\text{(B.22)}} + g^{bc}\underbrace{\mathrm{tr}[\gamma^5\gamma^a\gamma^d]}_{\text{(B.22)}} + i\epsilon_s{}^{abc}\underbrace{\mathrm{tr}[\gamma^5\gamma^s\gamma^5\gamma^d]}_{\text{(B.5)}} = \\ &= -i\epsilon_s{}^{abc}\mathrm{tr}[\underbrace{\gamma^5\gamma^5}_{\text{(B.6)}}\gamma^s\gamma^d] = -i\epsilon_s{}^{abc}\underbrace{\mathrm{tr}[\gamma^s\gamma^d]}_{\text{(B.6)}} = -4i\epsilon_s{}^{abc}g^{sd} = -4i\epsilon^{dabc} = \\ &= 4i\epsilon^{abcd}\end{aligned} \tag{B.32}$$

- **trace of the product of two antisymmetric γs (γ^{ab}, γ^{cd}):**

$$\begin{aligned}\mathrm{tr}[\gamma^{ab}\gamma^{cd}] &= \mathrm{tr}\Big[\Big(\frac{\gamma^a\gamma^b - \gamma^b\gamma^a}{2}\Big)\Big(\frac{\gamma^c\gamma^d - \gamma^d\gamma^c}{2}\Big)\Big] = \\ &= \frac{1}{4}\mathrm{tr}[\gamma^a\gamma^b\gamma^c\gamma^d - \gamma^a\gamma^b\gamma^d\gamma^c - \gamma^b\gamma^a\gamma^c\gamma^d + \gamma^b\gamma^a\gamma^d\gamma^c] = \\ &= \frac{1}{4}\mathrm{tr}[\gamma^a\gamma^b\gamma^c\gamma^d] - \frac{1}{4}\mathrm{tr}[\gamma^a\gamma^b\gamma^d\gamma^c] - \frac{1}{4}\mathrm{tr}[\gamma^b\gamma^a\gamma^c\gamma^d] + \frac{1}{4}\mathrm{tr}[\gamma^b\gamma^a\gamma^d\gamma^c] = \\ &\overset{\text{(B.31)}}{=} g^{ab}g^{cd} - g^{ac}g^{bd} + g^{bc}g^{ad} - g^{ab}g^{cd} + g^{ad}g^{bc} - g^{bd}g^{ac} + \\ &\quad - g^{ab}g^{cd} + g^{bc}g^{ad} - g^{ac}g^{bd} + g^{ab}g^{cd} - g^{bd}g^{ac} + g^{ad}g^{bc} = \\ &= 4g^{ad}g^{bc} - 4g^{ac}g^{bd}\end{aligned} \tag{B.33}$$

- **trace of the product of γ^5 and two antisymmetric γs (γ^{ab}, γ^{cd}):**

$$\begin{aligned}
\mathrm{tr}[\gamma^5\gamma^{ab}\gamma^{cd}] &= \mathrm{tr}\Big[\gamma^5\Big(\frac{\gamma^a\gamma^b-\gamma^b\gamma^a}{2}\Big)\Big(\frac{\gamma^c\gamma^d-\gamma^d\gamma^c}{2}\Big)\Big] = \\
&= \frac{1}{4}\mathrm{tr}[\gamma^5(\gamma^a\gamma^b\gamma^c\gamma^d-\gamma^a\gamma^b\gamma^d\gamma^c-\gamma^b\gamma^a\gamma^c\gamma^d+\gamma^b\gamma^a\gamma^d\gamma^c)] = \\
&= \frac{1}{4}\mathrm{tr}[\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d]-\frac{1}{4}\mathrm{tr}[\gamma^5\gamma^a\gamma^b\gamma^d\gamma^c]-\frac{1}{4}\mathrm{tr}[\gamma^5\gamma^b\gamma^a\gamma^c\gamma^d]+\frac{1}{4}\mathrm{tr}[\gamma^5\gamma^b\gamma^a\gamma^d\gamma^c] = \\
&\overset{(B.32)}{=} i\epsilon^{abcd}-i\epsilon^{abdc}-i\epsilon^{bacd}+i\epsilon^{badc} = \\
&= 4i\epsilon^{abcd}
\end{aligned} \tag{B.34}$$

- **single contraction of two ϵs (ϵ^{sdef}, ϵ^{sabc}):**

what we are now interested in is determining whether or not the contraction

$$\begin{aligned}
\epsilon_s{}^{def}\epsilon^{sabc} &= g^{dl}g^{em}g^{fn}\underbrace{\epsilon_{slmn}\epsilon^{sabc}}_{\text{expand}} = \\
&= g^{dl}g^{em}g^{fn}\big(\epsilon_{0lmn}\epsilon^{0abc}+\epsilon_{1lmn}\epsilon^{1abc}+\epsilon_{2lmn}\epsilon^{2abc}+\epsilon_{3lmn}\epsilon^{3abc}\big)\ ,
\end{aligned} \tag{B.35}$$

which will shortly be employed in (B.38), does possess some elegant formula in terms of the sole metric. In this regard, we immediately notice that, in each of the four terms of the second line of (B.35), the set of indices $\Big\{a,b,c\Big\}$ is certainly forced to be chosen from the same pool of values assumed by $\Big\{l,m,n\Big\}$. Therefore, starting from the situation where $l=a$, $m=b$ and $n=c$, which would correspond to

$$\epsilon_s{}^{def}\epsilon^{sabc} = g^{dl}g^{em}g^{fn}\epsilon_{slmn}\epsilon^{sabc} = g^{dl}g^{em}g^{fn}\big(-\delta_l{}^a\delta_m{}^b\delta_n{}^c\big) \qquad \text{if} \quad l=a\ \wedge\ m=b\ \wedge\ n=c, \tag{B.36}$$

(the minus sign is due to $\epsilon_{0123}\epsilon^{0123}=-1$), we realize that we can seize the complete solution we were looking for by antisymmetrizing (B.36) with respect to one of the two sets of indices, $\Big\{a,b,c\Big\}$ or $\Big\{l,m,n\Big\}$. This would also allow for the possibility where $\Big(l=b,\ m=a,\ n=c\Big)$, $\Big(l=a,\ m=c,\ n=b\Big)$ and so forth, in addition to the one explored in (B.36). Thus, we end up with

$$\begin{aligned}
\epsilon_s{}^{def}\epsilon^{sabc} &= g^{dl}g^{em}g^{fn}\big(-\delta_l{}^a\delta_m{}^b\delta_n{}^c+\delta_l{}^a\delta_m{}^c\delta_n{}^b-\delta_l{}^b\delta_m{}^c\delta_n{}^a+ \\
&\qquad +\delta_l{}^b\delta_m{}^a\delta_n{}^c-\delta_l{}^c\delta_m{}^a\delta_n{}^b+\delta_l{}^c\delta_m{}^b\delta_n{}^a\big) = \\
&= -g^{da}g^{eb}g^{fc}+g^{da}g^{ec}g^{fb}-g^{db}g^{ec}g^{fa}+g^{db}g^{ea}g^{fc}-g^{dc}g^{ea}g^{fb}+g^{dc}g^{eb}g^{fa}\ ;
\end{aligned} \tag{B.37}$$

- **trace of the product of six γs (γ^a, γ^b, γ^c, γ^d, γ^e, γ^f):**

$$\mathrm{tr}[\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f] \overset{(B.30)}{=} \mathrm{tr}[(g^{ab}\gamma^c-g^{ac}\gamma^b+g^{bc}\gamma^a+i\epsilon_s{}^{abc}\underbrace{\gamma^s\gamma^5}_{(B.5)})\gamma^d\gamma^e\gamma^f] =$$

$$
\begin{aligned}
&= g^{ab}\underbrace{\mathrm{tr}[\gamma^c\gamma^d\gamma^e\gamma^f]}_{\text{(B.31)}} - g^{ac}\underbrace{\mathrm{tr}[\gamma^b\gamma^d\gamma^e\gamma^f]}_{\text{(B.31)}} + \\
&\quad + g^{bc}\underbrace{\mathrm{tr}[\gamma^a\gamma^d\gamma^e\gamma^f]}_{\text{(B.31)}} - i\epsilon_s{}^{abc}\underbrace{\mathrm{tr}[\gamma^5\gamma^s\gamma^d\gamma^e\gamma^f]}_{\text{(B.32)}} = \\
&= 4g^{ab}(g^{cd}g^{ef} - g^{ce}g^{df} + g^{cf}g^{de}) - 4g^{ac}(g^{bd}g^{ef} - g^{be}g^{df} + g^{bf}g^{de}) + \\
&\quad + 4g^{bc}(g^{ad}g^{ef} - g^{ae}g^{df} + g^{af}g^{de}) + 4\underbrace{\epsilon_s{}^{abc}\epsilon^{sdef}}_{\text{(B.37)}} = \\
&= 4\big(g^{ab}g^{cd}g^{ef} - g^{ab}g^{ce}g^{df} + g^{ab}g^{cf}g^{de} + \\
&\qquad - g^{ac}g^{bd}g^{ef} + g^{ac}g^{be}g^{df} - g^{ac}g^{bf}g^{de} + \\
&\qquad\quad + g^{bc}g^{ad}g^{ef} - g^{bc}g^{ae}g^{df} + g^{bc}g^{af}g^{de} + \\
&\qquad\qquad - g^{ad}g^{be}g^{cf} + g^{ad}g^{ce}g^{bf} - g^{bd}g^{ce}g^{af} + \\
&\qquad\qquad\quad + g^{bd}g^{ae}g^{cf} - g^{cd}g^{ae}g^{bf} + g^{cd}g^{be}g^{af}\big) \; ; \qquad \text{(B.38)}
\end{aligned}
$$

- **trace of the product of γ^5 and six γs (γ^a, γ^b, γ^c, γ^d, γ^e, γ^f):**

$$
\begin{aligned}
\mathrm{tr}[\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f] &\overset{\text{(B.30)}}{=} \mathrm{tr}[\gamma^5(g^{ab}\gamma^c - g^{ac}\gamma^b + g^{bc}\gamma^a + i\epsilon_s{}^{abc}\underbrace{\gamma^s\gamma^5}_{\text{(B.5)}})\gamma^d\gamma^e\gamma^f] = \\
&= g^{ab}\underbrace{\mathrm{tr}[\gamma^5\gamma^c\gamma^d\gamma^e\gamma^f]}_{\text{(B.32)}} - g^{ac}\underbrace{\mathrm{tr}[\gamma^5\gamma^b\gamma^d\gamma^e\gamma^f]}_{\text{(B.32)}} + \\
&\quad + g^{bc}\underbrace{\mathrm{tr}[\gamma^5\gamma^a\gamma^d\gamma^e\gamma^f]}_{\text{(B.32)}} - i\epsilon_s{}^{abc}\mathrm{tr}[\underbrace{\gamma^5\gamma^5}_{\text{(B.6)}}\gamma^s\gamma^d\gamma^e\gamma^f] = \\
&= 4ig^{ab}\epsilon^{cdef} - 4ig^{ac}\epsilon^{bdef} + 4ig^{bc}\epsilon^{adef} - i\epsilon_s{}^{abc}\underbrace{\mathrm{tr}[\gamma^s\gamma^d\gamma^e\gamma^f]}_{\text{(B.31)}} = \\
&= 4ig^{ab}\epsilon^{cdef} - 4ig^{ac}\epsilon^{bdef} + 4ig^{bc}\epsilon^{adef} + \\
&\quad - i\epsilon_s{}^{abc}(4g^{sd}g^{ef} - 4g^{se}g^{df} + 4g^{sf}g^{de}) = \\
&= 4ig^{ab}\epsilon^{cdef} - 4ig^{ac}\epsilon^{bdef} + 4ig^{bc}\epsilon^{adef} + \\
&\quad + 4ig^{ef}\epsilon^{abcd} - 4ig^{df}\epsilon^{abce} + 4ig^{de}\epsilon^{abcf} \; ; \qquad \text{(B.39)}
\end{aligned}
$$

- **trace of the product of two antisymmetric γs (γ^{ab}, γ^{cd}) and two γs (γ^e, γ^f)**

$$
\begin{aligned}
\mathrm{tr}[\gamma^{ab}\gamma^{cd}\gamma^e\gamma^f] &= \mathrm{tr}\Big[\Big(\frac{\gamma^a\gamma^b - \gamma^b\gamma^a}{2}\Big)\Big(\frac{\gamma^c\gamma^d - \gamma^d\gamma^c}{2}\Big)\gamma^e\gamma^f\Big] = \\
&= \frac{1}{4}\mathrm{tr}[\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f - \gamma^a\gamma^b\gamma^d\gamma^c\gamma^e\gamma^f - \gamma^b\gamma^a\gamma^c\gamma^d\gamma^e\gamma^f + \gamma^b\gamma^a\gamma^d\gamma^c\gamma^e\gamma^f] = \\
&= \frac{1}{4}\mathrm{tr}[\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f] - \frac{1}{4}\mathrm{tr}[\gamma^a\gamma^b\gamma^d\gamma^c\gamma^e\gamma^f] + \\
&\quad - \frac{1}{4}\mathrm{tr}[\gamma^b\gamma^a\gamma^c\gamma^d\gamma^e\gamma^f] + \frac{1}{4}\mathrm{tr}[\gamma^b\gamma^a\gamma^d\gamma^c\gamma^e\gamma^f] = \\
&\overset{\text{(B.38)}}{=} \cancel{g^{ab}g^{cd}g^{ef} - g^{ab}g^{ce}g^{df} + g^{ab}g^{cf}g^{de} +}
\end{aligned}
$$

$$
\begin{aligned}
&- g^{ac}g^{bd}g^{ef} + g^{ac}g^{be}g^{df} - g^{ac}g^{bf}g^{de} + \\
&\quad + g^{bc}g^{ad}g^{ef} - g^{bc}g^{ae}g^{df} + g^{bc}g^{af}g^{de} + \\
&\qquad - g^{ad}g^{be}g^{cf} + g^{ad}g^{ce}g^{bf} - g^{bd}g^{ce}g^{af} + \\
&\qquad\quad + g^{bd}g^{ae}g^{cf} \cancel{- g^{cd}g^{ae}g^{bf} + g^{cd}g^{be}g^{af}} + \\
- \big(&\cancel{g^{ab}g^{cd}g^{ef} - g^{ab}g^{de}g^{cf} + g^{ab}g^{df}g^{ce}} + \\
&- g^{ad}g^{bc}g^{ef} + g^{ad}g^{be}g^{cf} - g^{ad}g^{bf}g^{ce} + \\
&\quad + g^{bd}g^{ac}g^{ef} - g^{bd}g^{ae}g^{cf} + g^{bd}g^{af}g^{ce} + \\
&\qquad - g^{ac}g^{be}g^{df} + g^{ac}g^{de}g^{bf} - g^{bc}g^{de}g^{af} + \\
&\qquad\quad + g^{bc}g^{ae}g^{df} \cancel{- g^{cd}g^{ae}g^{bf} + g^{cd}g^{be}g^{af}}\big) + \\
- \big(&\cancel{g^{ab}g^{cd}g^{ef} - g^{ab}g^{ce}g^{df} + g^{ab}g^{cf}g^{de}} + \\
&- g^{bc}g^{ad}g^{ef} + g^{bc}g^{ae}g^{df} - g^{bc}g^{af}g^{de} + \\
&\quad + g^{ac}g^{bd}g^{ef} - g^{ac}g^{be}g^{df} + g^{ac}g^{bf}g^{de} + \\
&\qquad - g^{bd}g^{ae}g^{cf} + g^{bd}g^{ce}g^{af} - g^{ad}g^{ce}g^{bf} + \\
&\qquad\quad + g^{ad}g^{be}g^{cf} \cancel{- g^{cd}g^{be}g^{af} + g^{cd}g^{ae}g^{bf}}\big) + \\
+ &\cancel{g^{ab}g^{cd}g^{ef} - g^{ab}g^{de}g^{cf} + g^{ab}g^{df}g^{ce}} + \\
&- g^{bd}g^{ac}g^{ef} + g^{bd}g^{ae}g^{cf} - g^{bd}g^{af}g^{ce} + \\
&\quad + g^{ad}g^{bc}g^{ef} - g^{ad}g^{be}g^{cf} + g^{ad}g^{bf}g^{ce} + \\
&\qquad - g^{bc}g^{ae}g^{df} + g^{bc}g^{de}g^{af} - g^{ac}g^{de}g^{bf} + \\
&\qquad\quad + g^{ac}g^{be}g^{df} \cancel{- g^{cd}g^{be}g^{af} + g^{cd}g^{ae}g^{bf}} = \\
= 4\big(&-g^{ac}g^{bd}g^{ef} + g^{ac}g^{be}g^{df} - g^{ac}g^{bf}g^{de} + g^{bc}g^{ad}g^{ef} - g^{bc}g^{ae}g^{df} + \\
&+ g^{bc}g^{af}g^{de} - g^{ad}g^{be}g^{cf} + g^{ad}g^{ce}g^{bf} - g^{bd}g^{ce}g^{af} + g^{bd}g^{ae}g^{cf}\big) \; ; \qquad \text{(B.40)}
\end{aligned}
$$

- **identity - sum of products of metric (g) and Levi-Civita (ϵ) tensors:**

  the cyclic property of the trace (B.16) and the anticommutation relation (B.5) satisfied by γ^5 can be easily adopted to assert the validity of

$$
\begin{aligned}
&- \mathrm{tr}[\gamma^5\gamma^d\gamma^c\gamma^a\gamma^b\gamma^e\gamma^f] + \mathrm{tr}[\gamma^5\gamma^d\gamma^c\gamma^a\gamma^b\gamma^f\gamma^e] + \mathrm{tr}[\gamma^5\gamma^d\gamma^c\gamma^b\gamma^a\gamma^e\gamma^f] - \mathrm{tr}[\gamma^5\gamma^d\gamma^c\gamma^b\gamma^a\gamma^f\gamma^e] = \\
&= \mathrm{tr}[\gamma^5\gamma^f\gamma^d\gamma^c\gamma^a\gamma^b\gamma^e] - \mathrm{tr}[\gamma^5\gamma^e\gamma^d\gamma^c\gamma^a\gamma^b\gamma^f] - \mathrm{tr}[\gamma^5\gamma^f\gamma^d\gamma^c\gamma^b\gamma^a\gamma^e] + \mathrm{tr}[\gamma^5\gamma^e\gamma^d\gamma^c\gamma^b\gamma^a\gamma^f].
\end{aligned}
$$

  Since the two sides must coincide, we expand them both, term by term, through (B.39), attaining

$$
\begin{aligned}
&- 4ig^{dc}\epsilon^{abef} + 4ig^{da}\epsilon^{cbef} - 4ig^{ca}\epsilon^{dbef} - \cancel{4ig^{ef}\epsilon^{dcab}} + 4ig^{bf}\epsilon^{dcae} - 4ig^{be}\epsilon^{dcaf} + \\
&+ 4ig^{dc}\epsilon^{abfe} - 4ig^{da}\epsilon^{cbfe} + 4ig^{ca}\epsilon^{dbfe} + \cancel{4ig^{ef}\epsilon^{dcab}} - 4ig^{be}\epsilon^{dcaf} + 4ig^{bf}\epsilon^{dcae} + \\
&+ 4ig^{dc}\epsilon^{baef} - 4ig^{db}\epsilon^{caef} + 4ig^{cb}\epsilon^{daef} + \cancel{4ig^{ef}\epsilon^{dcba}} - 4ig^{af}\epsilon^{dcbe} + 4ig^{ae}\epsilon^{dcbf} + \\
&- 4ig^{dc}\epsilon^{bafe} + 4ig^{db}\epsilon^{cafe} - 4ig^{cb}\epsilon^{dafe} - \cancel{4ig^{ef}\epsilon^{dcba}} + 4ig^{ae}\epsilon^{dcbf} - 4ig^{af}\epsilon^{dcbe} = \\
&\qquad = 4ig^{fd}\epsilon^{cabe} - 4ig^{fc}\epsilon^{dabe} + 4ig^{dc}\epsilon^{fabe} + 4ig^{be}\epsilon^{fdca} - 4ig^{ae}\epsilon^{fdcb} + \cancel{4ig^{ab}\epsilon^{fdce}} + \\
&\qquad\; - 4ig^{ed}\epsilon^{cabf} + 4ig^{ec}\epsilon^{dabf} - 4ig^{dc}\epsilon^{eabf} - 4ig^{bf}\epsilon^{edca} + 4ig^{af}\epsilon^{edcb} - \cancel{4ig^{ab}\epsilon^{edcf}} +
\end{aligned}
$$

$$\begin{aligned}&-4ig^{fd}\epsilon^{cbae}+4ig^{fc}\epsilon^{dbae}-4ig^{dc}\epsilon^{fbae}-4ig^{ae}\epsilon^{fdcb}+4ig^{be}\epsilon^{fdca}-\cancel{4ig^{ab}\epsilon^{fdce}}+\\&+4ig^{ed}\epsilon^{cbaf}-4ig^{ec}\epsilon^{dbaf}+4ig^{dc}\epsilon^{ebaf}+4ig^{af}\epsilon^{edcb}-4ig^{bf}\epsilon^{edca}+\cancel{4ig^{ab}\epsilon^{edcf}}\;.\end{aligned}$$

Just a little more algebra (we add up all the identical terms, dividing the left and right side by $8i$) would help us simplify the prior equality:

$$\begin{aligned}&-\cancel{2g^{dc}\epsilon^{abef}}+g^{da}\epsilon^{cbef}-g^{ca}\epsilon^{dbef}+\cancel{g^{bf}\epsilon^{deac}}-\cancel{g^{be}\epsilon^{dcaf}}+\\&\quad-g^{db}\epsilon^{caef}+g^{cb}\epsilon^{daef}-\cancel{g^{af}\epsilon^{debc}}+\cancel{g^{ae}\epsilon^{debf}}=\\&\qquad=\cancel{2g^{dc}\epsilon^{fabe}}+g^{fd}\epsilon^{cabe}-g^{fc}\epsilon^{dabe}+\cancel{g^{be}\epsilon^{fdca}}-\cancel{g^{ae}\epsilon^{fdcb}}+\\&\qquad\quad-g^{ed}\epsilon^{cabf}+g^{ec}\epsilon^{dabf}-\cancel{g^{bf}\epsilon^{edca}}+\cancel{g^{af}\epsilon^{edcb}}\end{aligned}$$

$$\Downarrow$$

$$g^{da}\epsilon^{cbef}-g^{ca}\epsilon^{dbef}-g^{db}\epsilon^{caef}+g^{cb}\epsilon^{daef}=g^{fd}\epsilon^{cabe}-g^{fc}\epsilon^{dabe}-g^{ed}\epsilon^{cabf}+g^{ec}\epsilon^{dabf}\;; \qquad \text{(B.41)}$$

- **trace of the product of three antisymmetric γs (γ^{ab}, γ^{cd}, γ^{ef}):**

$$\begin{aligned}\mathrm{tr}[\gamma^{ab}\gamma^{cd}\gamma^{ef}]&=\mathrm{tr}\Big[\Big(\frac{\gamma^a\gamma^b-\gamma^b\gamma^a}{2}\Big)\Big(\frac{\gamma^c\gamma^d-\gamma^d\gamma^c}{2}\Big)\Big(\frac{\gamma^e\gamma^f-\gamma^f\gamma^e}{2}\Big)\Big]=\\&=\frac{1}{8}\mathrm{tr}[\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f-\gamma^a\gamma^b\gamma^d\gamma^c\gamma^e\gamma^f-\gamma^a\gamma^b\gamma^c\gamma^d\gamma^f\gamma^e+\gamma^a\gamma^b\gamma^d\gamma^c\gamma^f\gamma^e\\&\quad-\gamma^b\gamma^a\gamma^c\gamma^d\gamma^e\gamma^f+\gamma^b\gamma^a\gamma^d\gamma^c\gamma^e\gamma^f+\gamma^b\gamma^a\gamma^c\gamma^d\gamma^f\gamma^e-\gamma^b\gamma^a\gamma^d\gamma^c\gamma^f\gamma^e]=\\&=\frac{1}{8}\mathrm{tr}[\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f]-\frac{1}{8}\mathrm{tr}[\gamma^a\gamma^b\gamma^d\gamma^c\gamma^e\gamma^f]-\frac{1}{8}\mathrm{tr}[\gamma^a\gamma^b\gamma^c\gamma^d\gamma^f\gamma^e]+\\&\quad+\frac{1}{8}\mathrm{tr}[\gamma^a\gamma^b\gamma^d\gamma^c\gamma^f\gamma^e]-\frac{1}{8}\mathrm{tr}[\gamma^b\gamma^a\gamma^c\gamma^d\gamma^e\gamma^f]+\frac{1}{8}\mathrm{tr}[\gamma^b\gamma^a\gamma^d\gamma^c\gamma^e\gamma^f]+\\&\quad+\frac{1}{8}\mathrm{tr}[\gamma^b\gamma^a\gamma^c\gamma^d\gamma^f\gamma^e]-\frac{1}{8}\mathrm{tr}[\gamma^b\gamma^a\gamma^d\gamma^c\gamma^f\gamma^e]=\\&\overset{(B.38)}{=}\frac{1}{2}\big(\cancel{g^{ab}g^{cd}g^{ef}}-g^{ab}g^{ce}g^{df}+g^{ab}g^{cf}g^{de}-\cancel{g^{ac}g^{bd}g^{ef}}+g^{ac}g^{be}g^{df}-g^{ac}g^{bf}g^{de}+\\&\quad+\cancel{g^{bc}g^{ad}g^{ef}}-g^{bc}g^{ae}g^{df}+g^{bc}g^{af}g^{de}-g^{ad}g^{be}g^{cf}+g^{ad}g^{ce}g^{bf}-g^{bd}g^{ce}g^{af}+\\&\quad+g^{bd}g^{ae}g^{cf}-\cancel{g^{cd}g^{ae}g^{bf}}+\cancel{g^{cd}g^{be}g^{af}}+\\&\quad-\cancel{g^{ab}g^{cd}g^{ef}}+g^{ab}g^{de}g^{cf}-g^{ab}g^{df}g^{ce}+\cancel{g^{ad}g^{bc}g^{ef}}-g^{ad}g^{be}g^{cf}+g^{ad}g^{bf}g^{ce}+\\&\quad-\cancel{g^{bd}g^{ac}g^{ef}}+g^{bd}g^{ae}g^{cf}-g^{bd}g^{af}g^{ce}+g^{ac}g^{be}g^{df}-g^{ac}g^{de}g^{bf}+g^{bc}g^{de}g^{af}+\\&\quad-g^{bc}g^{ae}g^{df}+\cancel{g^{cd}g^{ae}g^{bf}}-\cancel{g^{cd}g^{be}g^{af}}\\&\quad-\cancel{g^{ab}g^{cd}g^{ef}}+g^{ab}g^{cf}g^{de}-g^{ab}g^{ce}g^{df}+\cancel{g^{ac}g^{bd}g^{ef}}-g^{ac}g^{bf}g^{de}+g^{ac}g^{be}g^{df}+\\&\quad-\cancel{g^{bc}g^{ad}g^{ef}}+g^{bc}g^{af}g^{de}-g^{bc}g^{ae}g^{df}+g^{ad}g^{bf}g^{ce}-g^{ad}g^{cf}g^{be}+g^{bd}g^{cf}g^{ae}+\\&\quad-g^{bd}g^{af}g^{ce}+\cancel{g^{cd}g^{af}g^{be}}-\cancel{g^{cd}g^{bf}g^{ae}}+\\&\quad+\cancel{g^{ab}g^{cd}g^{ef}}-g^{ab}g^{df}g^{ce}+g^{ab}g^{de}g^{cf}-\cancel{g^{ad}g^{bc}g^{ef}}+g^{ad}g^{bf}g^{ce}-g^{ad}g^{be}g^{cf}+\\&\quad+\cancel{g^{bd}g^{ac}g^{ef}}-g^{bd}g^{af}g^{ce}+g^{bd}g^{ae}g^{cf}-g^{ac}g^{bf}g^{de}+g^{ac}g^{df}g^{be}-g^{bc}g^{df}g^{ae}+\\&\quad+g^{bc}g^{af}g^{de}-\cancel{g^{cd}g^{af}g^{be}}+\cancel{g^{cd}g^{bf}g^{ae}}+\\&\quad-\cancel{g^{ab}g^{cd}g^{ef}}+g^{ab}g^{ce}g^{df}-g^{ab}g^{cf}g^{de}+\cancel{g^{bc}g^{ad}g^{ef}}-g^{bc}g^{ae}g^{df}+g^{bc}g^{af}g^{de}+\end{aligned}$$

$$
\begin{aligned}
&-\cancel{g^{ac}g^{bd}g^{ef}}+g^{ac}g^{be}g^{df}-g^{ac}g^{bf}g^{de}+g^{bd}g^{ae}g^{cf}-g^{bd}g^{ce}g^{af}+g^{ad}g^{ce}g^{bf}+\\
&-g^{ad}g^{be}g^{cf}+\cancel{g^{cd}g^{be}g^{af}}-\cancel{g^{cd}g^{ae}g^{bf}}+\\
&+\cancel{g^{ab}g^{cd}g^{ef}}-g^{ab}g^{de}g^{cf}+g^{ab}g^{df}g^{ce}-\cancel{g^{bd}g^{ac}g^{ef}}+g^{bd}g^{ae}g^{cf}-g^{bd}g^{af}g^{ce}+\\
&+\cancel{g^{ad}g^{bc}g^{ef}}-g^{ad}g^{be}g^{cf}+g^{ad}g^{bf}g^{ce}-g^{bc}g^{ae}g^{df}+g^{bc}g^{de}g^{af}-g^{ac}g^{de}g^{bf}+\\
&+g^{ac}g^{be}g^{df}-\cancel{g^{cd}g^{be}g^{af}}+\cancel{g^{cd}g^{ae}g^{bf}}+\\
&+\cancel{g^{ab}g^{cd}g^{ef}}-g^{ab}g^{cf}g^{de}+g^{ab}g^{ce}g^{df}-\cancel{g^{bc}g^{ad}g^{ef}}+g^{bc}g^{af}g^{de}-g^{bc}g^{ae}g^{df}+\\
&+\cancel{g^{ac}g^{bd}g^{ef}}-g^{ac}g^{bf}g^{de}+g^{ac}g^{be}g^{df}-g^{bd}g^{af}g^{ce}+g^{bd}g^{cf}g^{ae}-g^{ad}g^{cf}g^{be}+\\
&+g^{ad}g^{bf}g^{ce}-\cancel{g^{cd}g^{bf}g^{ae}}+\cancel{g^{cd}g^{af}g^{be}}+\\
&-\cancel{g^{ab}g^{cd}g^{ef}}+g^{ab}g^{df}g^{ce}-g^{ab}g^{de}g^{cf}+\cancel{g^{bd}g^{ac}g^{ef}}-g^{bd}g^{af}g^{ce}+g^{bd}g^{ae}g^{cf}+\\
&-\cancel{g^{ad}g^{bc}g^{ef}}+g^{ad}g^{bf}g^{ce}-g^{ad}g^{be}g^{cf}+g^{bc}g^{af}g^{de}-g^{bc}g^{df}g^{ae}+g^{ac}g^{df}g^{be}+\\
&-g^{ac}g^{bf}g^{de}+\cancel{g^{cd}g^{bf}g^{ae}}-\cancel{g^{cd}g^{af}g^{be}}\Big)=\\
&=-\cancel{2g^{ab}g^{ce}g^{df}}+\cancel{2g^{ab}g^{cf}g^{de}}+2g^{ac}g^{be}g^{df}-2g^{ac}g^{bf}g^{de}-2g^{bc}g^{ae}g^{df}+\\
&\quad+2g^{bc}g^{af}g^{de}-2g^{ad}g^{be}g^{cf}+2g^{ad}g^{ce}g^{bf}-2g^{bd}g^{ce}g^{af}+2g^{bd}g^{ae}g^{cf}+\\
&\quad+\cancel{2g^{ab}g^{ce}g^{df}}-\cancel{2g^{ab}g^{cf}g^{de}}-2g^{bc}g^{ae}g^{df}+2g^{bc}g^{af}g^{de}+2g^{ac}g^{be}g^{df}+\\
&\quad-2g^{ac}g^{bf}g^{de}+2g^{bd}g^{ae}g^{cf}-2g^{bd}g^{ce}g^{af}+2g^{ad}g^{ce}g^{bf}-2g^{ad}g^{be}g^{cf}=\\
&=+4g^{ac}g^{be}g^{df}-4g^{ac}g^{bf}g^{de}-4g^{bc}g^{ae}g^{df}+4g^{bc}g^{af}g^{de}+\\
&\quad-4g^{ad}g^{be}g^{cf}+4g^{ad}g^{ce}g^{bf}-4g^{bd}g^{ce}g^{af}+4g^{bd}g^{ae}g^{cf}\ ;
\end{aligned}
\tag{B.42}
$$

- **trace of the product of γ^5 and three antisymmetric γs (γ^{ab}, γ^{cd}, γ^{ef}):**

$$
\begin{aligned}
\mathrm{tr}[\gamma^5\gamma^{ab}\gamma^{cd}\gamma^{ef}] &= \mathrm{tr}\Big[\gamma^5\Big(\frac{\gamma^a\gamma^b-\gamma^b\gamma^a}{2}\Big)\Big(\frac{\gamma^c\gamma^d-\gamma^d\gamma^c}{2}\Big)\Big(\frac{\gamma^e\gamma^f-\gamma^f\gamma^e}{2}\Big)\Big]=\\
&=\frac{1}{8}\mathrm{tr}[\gamma^5(\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f-\gamma^a\gamma^b\gamma^d\gamma^c\gamma^e\gamma^f-\gamma^a\gamma^b\gamma^c\gamma^d\gamma^f\gamma^e+\\
&\qquad+\gamma^a\gamma^b\gamma^d\gamma^c\gamma^f\gamma^e-\gamma^b\gamma^a\gamma^c\gamma^d\gamma^e\gamma^f+\gamma^b\gamma^a\gamma^d\gamma^c\gamma^e\gamma^f+\\
&\qquad+\gamma^b\gamma^a\gamma^c\gamma^d\gamma^f\gamma^e-\gamma^b\gamma^a\gamma^d\gamma^c\gamma^f\gamma^e)]=\\
&=\frac{1}{8}\mathrm{tr}[\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f]-\frac{1}{8}\mathrm{tr}[\gamma^5\gamma^a\gamma^b\gamma^d\gamma^c\gamma^e\gamma^f]-\frac{1}{8}\mathrm{tr}[\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d\gamma^f\gamma^e]+\\
&\quad+\frac{1}{8}\mathrm{tr}[\gamma^5\gamma^a\gamma^b\gamma^d\gamma^c\gamma^f\gamma^e]-\frac{1}{8}\mathrm{tr}[\gamma^5\gamma^b\gamma^a\gamma^c\gamma^d\gamma^e\gamma^f]+\frac{1}{8}\mathrm{tr}[\gamma^5\gamma^b\gamma^a\gamma^d\gamma^c\gamma^e\gamma^f]+\\
&\quad+\frac{1}{8}\mathrm{tr}[\gamma^5\gamma^b\gamma^a\gamma^c\gamma^d\gamma^f\gamma^e]-\frac{1}{8}\mathrm{tr}[\gamma^5\gamma^b\gamma^a\gamma^d\gamma^c\gamma^f\gamma^e]=\\
&\overset{(B.39)}{=}\frac{i}{2}\Big(\cancel{g^{ab}\epsilon^{cdef}}-g^{ac}\epsilon^{bdef}+g^{bc}\epsilon^{adef}+\cancel{g^{ef}\epsilon^{abcd}}-g^{df}\epsilon^{abce}+g^{de}\epsilon^{abcf}+\\
&\quad-\cancel{g^{ab}\epsilon^{dcef}}+g^{ad}\epsilon^{bcef}-g^{bd}\epsilon^{acef}-\cancel{g^{ef}\epsilon^{abdc}}+g^{cf}\epsilon^{abde}-g^{ce}\epsilon^{abdf}+\\
&\quad-\cancel{g^{ab}\epsilon^{cdfe}}+g^{ac}\epsilon^{bdfe}-g^{bc}\epsilon^{adfe}-\cancel{g^{ef}\epsilon^{abcd}}+g^{de}\epsilon^{abcf}-g^{df}\epsilon^{abce}+\\
&\quad+\cancel{g^{ab}\epsilon^{dcfe}}-g^{ad}\epsilon^{bcfe}+g^{bd}\epsilon^{acfe}+\cancel{g^{ef}\epsilon^{abdc}}-g^{ce}\epsilon^{abdf}+g^{cf}\epsilon^{abde}+\\
&\quad-\cancel{g^{ab}\epsilon^{cdef}}+g^{bc}\epsilon^{adef}-g^{ac}\epsilon^{bdef}-\cancel{g^{ef}\epsilon^{bacd}}+g^{df}\epsilon^{bace}-g^{de}\epsilon^{bacf}+\\
&\quad+\cancel{g^{ab}\epsilon^{dcef}}-g^{bd}\epsilon^{acef}+g^{ad}\epsilon^{bcef}+\cancel{g^{ef}\epsilon^{badc}}-g^{cf}\epsilon^{bade}+g^{ce}\epsilon^{badf}+
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \cancel{g^{ab}\epsilon^{cdfe}} - g^{bc}\epsilon^{adfe} + g^{ac}\epsilon^{bdfe} + \cancel{g^{ef}\epsilon^{bacd}} - g^{de}\epsilon^{bacf} + g^{df}\epsilon^{bace} + \\
&\quad - \cancel{g^{ab}\epsilon^{dcfe}} + g^{bd}\epsilon^{acfe} - g^{ad}\epsilon^{bcfe} - \cancel{g^{ef}\epsilon^{badc}} + g^{ce}\epsilon^{badf} - g^{cf}\epsilon^{bade}) = \\
&= 2i(-g^{ac}\epsilon^{bdef} + g^{bc}\epsilon^{adef} + g^{ad}\epsilon^{bcef} - g^{bd}\epsilon^{acef}) + \\
&\quad - 2i \underbrace{(g^{df}\epsilon^{abce} - g^{de}\epsilon^{abcf} - g^{cf}\epsilon^{abde} + g^{ce}\epsilon^{abdf})}_{\text{(B.41)}} = \\
&= 2i(-g^{ac}\epsilon^{bdef} + g^{bc}\epsilon^{adef} + g^{ad}\epsilon^{bcef} - g^{bd}\epsilon^{acef}) + \\
&\quad - 2i(g^{ad}\epsilon^{cbef} - g^{ac}\epsilon^{dbef} - g^{bd}\epsilon^{caef} + g^{bc}\epsilon^{daef}) = \\
&= -4i(g^{ac}\epsilon^{bdef} - g^{bc}\epsilon^{adef} - g^{ad}\epsilon^{bcef} + g^{bd}\epsilon^{acef}) \; ;
\end{aligned}
\tag{B.43}
$$

- **projectors' (P_L, P_R) commutation with an antisymmetric γ (γ^{ab}):**

$$
\begin{aligned}
[P_L, \gamma^{ab}] &= \underbrace{P_L\gamma^{ab}}_{\text{expand}} - \gamma^{ab}P_L = \Big(\frac{\mathbb{1} + \gamma^5}{2}\Big)\Big(\frac{\gamma^a\gamma^b - \gamma^b\gamma^a}{2}\Big) - \gamma^{ab}P_L = \\
&= \frac{1}{4}\big(\gamma^a\gamma^b - \gamma^b\gamma^a\big) + \frac{1}{4}\big(\underbrace{\gamma^5\gamma^a\gamma^b}_{\text{(B.5)}} - \underbrace{\gamma^5\gamma^b\gamma^a}_{\text{(B.5)}}\big) - \gamma^{ab}P_L = \\
&= \frac{1}{4}\big(\gamma^a\gamma^b - \gamma^b\gamma^a\big) + \frac{1}{4}\big(\gamma^a\gamma^b\gamma^5 - \gamma^b\gamma^a\gamma^5\big) - \gamma^{ab}P_L = \\
&= \Big(\frac{\gamma^a\gamma^b - \gamma^b\gamma^a}{2}\Big)\frac{\mathbb{1}}{2} + \Big(\frac{\gamma^a\gamma^b - \gamma^b\gamma^a}{2}\Big)\frac{\gamma^5}{2} - \gamma^{ab}P_L = \\
&= \gamma^{ab}\frac{\mathbb{1}}{2} + \gamma^{ab}\frac{\gamma^5}{2} - \gamma^{ab}P_L \\
&= \gamma^{ab}P_L - \gamma^{ab}P_L \\
&= 0 \; .
\end{aligned}
\tag{B.44}
$$

A similar reasoning also applies to P_R

$$
[P_R, \gamma^{ab}] = 0 \; ; \tag{B.45}
$$

- **product C^{-1} – (antisymmetric γ transposed) – C:**

$$
\begin{aligned}
C^{-1}(\gamma^{ef})^T C &= C^{-1}\Big(\frac{\gamma^e\gamma^f - \gamma^f\gamma^e}{2}\Big)^T C = C^{-1}\Big(\frac{(\gamma^f)^T(\gamma^e)^T - (\gamma^e)^T(\gamma^f)^T}{2}\Big)C = \\
&\overset{\text{(A.6)}}{=} C^{-1}\Big(\frac{(-C\gamma^f C^{-1})(-C\gamma^e C^{-1}) - (-C\gamma^e C^{-1})(-C\gamma^f C^{-1})}{2}\Big)C = \\
&= -\gamma^{ef} \; .
\end{aligned}
\tag{B.46}
$$

Appendix C

Complete and explicit calculations

We created this appendix in order to explicitly show the carrying out of the main calculations that compose the body of the paper. In this way, the interested reader may check the validity of the results in person.

C.1 Integration by parts of eq. (3.26)

Here below are reported the series of integrations by parts undergone by the Branson-Gilkey-Vassilevich coefficients of section 3.2.1:

$$
\begin{aligned}
b_0(x,H) &= 0 = \xi^a(x) b_{a,0}(x,H) \ ,\\
b_1(x,H) &= -\frac{1}{6}\underbrace{G^{ij}(x)}_{(3.22)}\mathcal{F}_{ij}(x) = \frac{1}{12}\big(\partial^i\xi^j(x) - \partial^j\xi^i(x)\big)(\mathbb{1})\underbrace{\mathcal{F}_{ij}(x)}_{\text{antisym.}} = \\
&= \frac{1}{6}\partial^i\xi^j(x)\mathcal{F}_{ij}(x) \doteq \underbrace{-\frac{1}{6}\xi^j(x)\nabla^i\mathcal{F}_{ij}(x)}_{j\to a} = \\
&= -\frac{1}{6}\xi^a(x)\nabla^i\mathcal{F}_{ia}(x) = \xi^a(x) b_{a,1}(x,H) \ ,\\
b_2(x,H) &= \frac{1}{45}\nabla_k\big(\mathcal{F}_{ij}(x)\big)\nabla^k\big(G^{ij}(x)\big) - \frac{1}{90}\nabla^j\big(\mathcal{F}_{ij}(x)\big)\nabla_k\big(G^{ik}(x)\big) + \frac{1}{6}V(x)\mathcal{F}_{ij}(x)G^{ij}(x) = \\
&\doteq -\frac{1}{45}\nabla_k\nabla^k\big(\mathcal{F}_{ij}(x)\big)\underbrace{G^{ij}(x)}_{(3.22)} + \frac{1}{90}\nabla_k\nabla^j\big(\mathcal{F}_{ij}(x)\big)\underbrace{G^{ik}(x)}_{(3.22)} + \frac{1}{6}V(x)\mathcal{F}_{ij}(x)\underbrace{G^{ij}(x)}_{(3.22)} = \\
&= \frac{1}{90}\nabla_k\nabla^k\underbrace{\big(\mathcal{F}_{ij}(x)\big)}_{\text{antisym.}}\big(\partial^i\xi^j(x) - \partial^j\xi^i(x)\big) - \frac{1}{180}\nabla_k\nabla^j\big(\mathcal{F}_{ij}(x)\big)\big(\partial^i\xi^k(x) - \partial^k\xi^i(x)\big) + \\
&\qquad - \frac{1}{12}V(x)\underbrace{\mathcal{F}_{ij}(x)}_{\text{antisym.}}\big(\partial^i\xi^j(x) - \partial^j\xi^i(x)\big) = \\
&= \frac{1}{45}\nabla_k\nabla^k\big(\mathcal{F}_{ij}(x)\big)\partial^i\xi^j(x) - \frac{1}{180}\nabla_k\nabla^j\big(\mathcal{F}_{ij}(x)\big)\partial^i\xi^k(x) +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{180}\nabla_k\nabla^j\big(\mathcal{F}_{ij}(x)\big)\partial^k\xi^i(x)-\frac{1}{6}V(x)\mathcal{F}_{ij}(x)\partial^i\xi^j(x)=\\
&\doteq-\frac{1}{45}\underbrace{\nabla^i\nabla_k\nabla^k\big(\mathcal{F}_{ij}(x)\big)\xi^j(x)}_{j\to a}+\frac{1}{180}\underbrace{\nabla^i\nabla_k\nabla^j\big(\mathcal{F}_{ij}(x)\big)\xi^k(x)}_{k\to a}+\\
&\quad-\frac{1}{180}\underbrace{\nabla^k\nabla_k\nabla^j\big(\mathcal{F}_{ij}(x)\big)\xi^i(x)}_{i\to a}+\frac{1}{6}\underbrace{\nabla^i\big(V(x)\mathcal{F}_{ij}(x)\big)\xi^j(x)}_{j\to a}=\\
&=\xi^a(x)\Big(-\frac{1}{45}\nabla^i\nabla_k\nabla^k\big(\mathcal{F}_{ia}(x)\big)+\frac{1}{180}\nabla^i\nabla_a\nabla^j\big(\mathcal{F}_{ij}(x)\big)+\\
&\quad-\frac{1}{180}\nabla^k\nabla_k\nabla^j\big(\mathcal{F}_{aj}(x)\big)+\frac{1}{6}\nabla^i\big(V(x)\mathcal{F}_{ia}(x)\big)\Big)=\\
&=\xi^a(x)b_{a,2}(x,H)\;,
\end{aligned}
\tag{C.1}
$$

where "$\doteq$" denotes an equality that holds up to vanishing boundary terms.

C.2 Computations of chapter 4

C.2.1 W_aW^a and Z_aZ^a

Here is the contraction of the gauge connection W_a (4.9) with itself

$$
\begin{aligned}
W^aW_a&=\left(-\frac{i}{2}A^a-\frac{i}{2}\gamma^{ab}A_b\right)\left(-\frac{i}{2}A_a-\frac{i}{2}\gamma_{ac}A^c\right)=\\
&=-\frac{1}{4}A^aA_a-\frac{1}{4}\underbrace{\gamma_{ac}A^aA^c}_{0}-\frac{1}{4}\underbrace{\gamma^{ab}A_aA_b}_{0}-\frac{1}{4}\underbrace{\gamma^{ab}\gamma_a{}^c}_{(B.12)}A_bA_c=\\
&=-\frac{1}{4}A^aA_a+\frac{1}{4}\left[2\cancel{\gamma^{bc}}+3g^{bc}\right]\underbrace{A_bA_c}_{\text{sym. in } bc}=\\
&=-\frac{1}{4}A^aA_a+\frac{3}{4}\underbrace{A^bA_b}_{(b\to a)}=\\
&=\frac{1}{2}A^aA_a\;.
\end{aligned}
\tag{C.2}
$$

The square contraction of Z_a (4.27) is provided by a similar calculation.

C.2.2 ∇_aV and ∇_aU

Down below are reported the gauge covariant derivatives of V (4.10) and U (4.28)

$$
\begin{aligned}
\nabla_aV&=\partial_aV+[W_a,V]=\\
&=\partial_a\left(\frac{i}{2}\partial_sA^s+\frac{i}{4}\gamma^{cd}F_{cd}+\frac{1}{2}A^2\right)+\left[\underbrace{W_a}_{(4.9)},\left(\frac{i}{2}\partial_sA^s+\frac{i}{4}\gamma^{cd}F_{cd}+\frac{1}{2}A^2\right)\right]=
\end{aligned}
$$

$$= \frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) + \underbrace{\left[-\frac{i}{2}\gamma_{ab}A^b, \frac{i}{4}\gamma_{cd}F^{cd}\right]}_{(B.14)} =$$

$$= \frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) + \frac{1}{8}A^b F^{cd}\left(2g_{ad}\gamma_{bc} - 2g_{ac}\gamma_{bd} - 2g_{bd}\gamma_{ac} + 2g_{bc}\gamma_{ad}\right) =$$

$$= \frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) + \frac{1}{4}\big(\gamma_{bc}A^b F^c{}_a - \underbrace{\gamma_{bd}A^b F_a{}^d}_{d\to c} - \gamma_{ac}A_d F^{cd} + \underbrace{\gamma_{ad}A_c F^{cd}}_{d\to c;\ c\to d}\big) =$$

$$= \frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) + \frac{1}{2}\gamma_{bc}A^b F^c{}_a + \frac{1}{2}\gamma_{ca}A_d F^{cd}\,, \tag{C.3}$$

where, in the third equality, only the non vanishing contribution to the commutator $[W_a, V]$ has been written. Likewise, we have

$$\nabla_a U = \partial_a U + [Z_a, U] =$$

$$= \partial_a\left(-\frac{i}{2}\partial_s A^s + \frac{i}{4}\gamma^{cd}F_{cd} + \frac{1}{2}A^2\right) + \left[\underbrace{Z_a}_{(4.27)}, \left(-\frac{i}{2}\partial_s A^s + \frac{i}{4}\gamma^{cd}F_{cd} + \frac{1}{2}A^2\right)\right] =$$

$$= -\frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) + \underbrace{\left[\frac{i}{2}\gamma_{ab}A^b, \frac{i}{4}\gamma_{cd}F^{cd}\right]}_{(B.14)} =$$

$$= -\frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) - \frac{1}{8}A^b F^{cd}\left(2g_{ad}\gamma_{bc} - 2g_{ac}\gamma_{bd} - 2g_{bd}\gamma_{ac} + 2g_{bc}\gamma_{ad}\right) =$$

$$= -\frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) - \frac{1}{4}\big(\gamma_{bc}A^b F^c{}_a - \underbrace{\gamma_{bd}A^b F_a{}^d}_{d\to c} - \gamma_{ac}A_d F^{cd} + \underbrace{\gamma_{ad}A_c F^{cd}}_{d\to c;\ c\to d}\big) =$$

$$= -\frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) - \frac{1}{2}\gamma_{bc}A^b F^c{}_a - \frac{1}{2}\gamma_{ca}A_d F^{cd}\,. \tag{C.4}$$

C.2.3 $\nabla^a\nabla_a V$ and $\nabla^a\nabla_a U$

We dedicate this section to the explicit calculus of the gauge covariant d'Alembertians of V (4.10) and U (4.28)

$$\nabla^a\nabla_a V = \partial^a\big(\underbrace{\nabla_a V}_{(4.12)}\big) + \Big[\underbrace{W^a}_{(4.9)}, \underbrace{\nabla_a V}_{(4.12)}\Big] =$$

$$= \frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) + \frac{1}{2}\gamma_{bc}\partial^a(A^b F^c{}_a) + \frac{1}{2}\gamma_{ca}\partial^a(A_d F^{cd}) +$$

$$+ \underbrace{\left[-\frac{i}{2}\gamma^{al}A_l, \frac{i}{4}\gamma^{cd}\partial_a(F_{cd})\right]}_{(B.14)} + \underbrace{\left[-\frac{i}{2}\gamma^{al}A_l, \frac{1}{2}\gamma^{bc}A_b F_{ca}\right]}_{(B.14)} + \underbrace{\left[-\frac{i}{2}\gamma^{al}A_l, -\frac{1}{2}\gamma_a{}^c A_d F_c{}^d\right]}_{(B.15)} =$$

$$= \frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) + \frac{1}{2}\gamma_{bc}\partial^a(A^b F^c{}_a) + \frac{1}{2}\gamma_{ca}\partial^a(A_d F^{cd}) +$$

$$+ \frac{1}{8}A_l\partial_a(F_{cd})\big(2g^{ad}\gamma^{lc} - 2g^{ac}\gamma^{ld} - 2g^{ld}\gamma^{ac} + 2g^{lc}\gamma^{ad}\big) +$$

$$
\begin{aligned}
&\quad - \frac{i}{4} A_l A_b F_{ca}\big(2g^{ac}\gamma^{lb} - 2g^{ab}\gamma^{lc} - 2g^{lc}\gamma^{ab} + 2g^{lb}\gamma^{ac}\big) + \\
&\quad + \frac{i}{4} A_l A_d F_c{}^d\big(-4\gamma^{lc}\big) = \\
&= \frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) + \frac{1}{2}\gamma_{bc}\underbrace{\partial^a(A^b F^c{}_a)}_{\text{expand}} + \frac{1}{2}\gamma_{ca}\underbrace{\partial^a(A_d F^{cd})}_{\text{expand}} + \\
&\quad + \frac{1}{4}\Big(\gamma^{lc} A_l \partial^d(F_{cd}) - \underbrace{\gamma^{ld} A_l \partial^a(F_{ad})}_{d\to c;\ a\to d} - \gamma^{ac} A^d \partial_a(F_{cd}) + \underbrace{\gamma^{ad} A^c \partial_a(F_{cd})}_{c\to d;\ d\to c}\Big) + \\
&\quad - \frac{i}{2}\Big(\underbrace{\gamma^{lb} A_l A_b}_{0} F^a{}_a - \gamma^{lc} A_l A^a F_{ca} - \underbrace{\gamma^{ab} A^c A_b F_{ca}}_{b\to l;\ c\to a;\ a\to c} + \gamma^{ac} A^b A_b F_{ca}\Big) + \\
&\quad - i\gamma^{lc} A_l A_d F_c{}^d = \\
&= \frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) + \frac{1}{2}\underbrace{\gamma^{bc}\partial^a(A_b) F_{ca}}_{a\to b;\ b\to c;\ c\to a} + \frac{1}{2}\gamma^{bc} A_b \partial^a(F_{ca}) + \\
&\quad + \frac{1}{2}\gamma^{ca}\partial_a(A^d) F_{cd} + \frac{1}{2}\gamma^{ca} A^d \partial_a(F_{cd}) + \frac{1}{2}\underbrace{\gamma^{lc} A_l \partial^d(F_{cd})}_{l\to b;\ d\to a} - \frac{1}{2}\gamma^{ac} A^d \partial_a(F_{cd}) + \\
&\quad + \underbrace{\cancel{i\gamma^{lc} A^a A_l F_{ca}}}_{a\to d} + \frac{i}{2}\gamma^{ac} A^2 F_{ac} - \cancel{i\gamma^{lc} A_l A_d F_c{}^d} = \\
&= \frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) + \frac{1}{2}\gamma^{ca}\partial^b(A_c) F_{ab} + \gamma^{bc} A_b \partial^a(F_{ca}) + \\
&\quad + \frac{1}{2}\gamma^{ca}\partial_a(A^d) F_{cd} - \gamma^{ac} A^d \partial_a(F_{cd}) + \frac{i}{2}\gamma^{ac} A^2 F_{ac}\ ,
\end{aligned} \tag{C.5}
$$

where, in writing the second line, we reported all and only the non-zero terms arising from $[W^a, \nabla_a V]$. Analogously, it is also true that

$$
\begin{aligned}
\nabla^a \nabla_a U &= \partial^a\big(\underbrace{\nabla_a U}_{(4.30)}\big) + \Big[\underbrace{Z^a}_{(4.27)}, \underbrace{\nabla_a U}_{(4.30)}\Big] = \\
&= -\frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) - \frac{1}{2}\gamma_{bc}\partial^a(A^b F^c{}_a) - \frac{1}{2}\gamma_{ca}\partial^a(A_d F^{cd}) + \\
&\quad + \underbrace{\Big[\frac{i}{2}\gamma^{al} A_l, \frac{i}{4}\gamma^{cd}\partial_a(F_{cd})\Big]}_{(B.14)} + \underbrace{\Big[\frac{i}{2}\gamma^{al} A_l, -\frac{1}{2}\gamma^{bc} A_b F_{ca}\Big]}_{(B.14)} + \underbrace{\Big[\frac{i}{2}\gamma^{al} A_l, \frac{1}{2}\gamma_a{}^c A_d F_c{}^d\Big]}_{(B.15)} = \\
&= -\frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) - \frac{1}{2}\gamma_{bc}\partial^a(A^b F^c{}_a) - \frac{1}{2}\gamma_{ca}\partial^a(A_d F^{cd}) + \\
&\quad - \frac{1}{8} A_l \partial_a(F_{cd})\big(2g^{ad}\gamma^{lc} - 2g^{ac}\gamma^{ld} - 2g^{ld}\gamma^{ac} + 2g^{lc}\gamma^{ad}\big) + \\
&\quad - \frac{i}{4} A_l A_b F_{ca}\big(2g^{ac}\gamma^{lb} - 2g^{ab}\gamma^{lc} - 2g^{lc}\gamma^{ab} + 2g^{lb}\gamma^{ac}\big) + \\
&\quad + \frac{i}{4} A_l A_d F_c{}^d\big(-4\gamma^{lc}\big) =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) - \frac{1}{2}\gamma_{bc}\underbrace{\partial^a(A^b F^c{}_a)}_{\text{expand}} - \frac{1}{2}\gamma_{ca}\underbrace{\partial^a(A_d F^{cd})}_{\text{expand}} + \\
&\quad - \frac{1}{4}\Big(\gamma^{lc}A_l\partial^d(F_{cd}) - \underbrace{\gamma^{ld}A_l\partial^a(F_{ad})}_{d\to c;\ a\to d} - \gamma^{ac}A^d\partial_a(F_{cd}) + \underbrace{\gamma^{ad}A^c\partial_a(F_{cd})}_{c\to d;\ d\to c}\Big) + \\
&\quad - \frac{i}{2}\Big(\underbrace{\gamma^{lb}A_lA_b}_{0}F^a{}_a - \gamma^{lc}A_lA^aF_{ca} - \underbrace{\gamma^{ab}A^cA_bF_{ca}}_{b\to l;\ c\to a;\ a\to c} + \gamma^{ac}A^bA_bF_{ca}\Big) + \\
&\quad - i\gamma^{lc}A_lA_dF_c{}^d = \\
&= -\frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) - \frac{1}{2}\underbrace{\gamma^{bc}\partial^a(A_b)F_{ca}}_{a\to b;\ b\to c;\ c\to a} - \frac{1}{2}\gamma^{bc}A_b\partial^a(F_{ca}) + \\
&\quad - \frac{1}{2}\gamma^{ca}\partial_a(A^d)F_{cd} - \frac{1}{2}\gamma^{ca}A^d\partial_a(F_{cd}) - \frac{1}{2}\underbrace{\gamma^{lc}A_l\partial^d(F_{cd})}_{l\to b;\ d\to a} + \frac{1}{2}\gamma^{ac}A^d\partial_a(F_{cd}) + \\
&\quad + \underbrace{\cancel{i\gamma^{lc}A^aA_lF_{ca}}}_{a\to d} + \frac{i}{2}\gamma^{ac}A^2F_{ac} - \cancel{i\gamma^{lc}A_lA_dF_c{}^d} = \\
&= -\frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) - \frac{1}{2}\gamma^{ca}\partial^b(A_c)F_{ab} - \gamma^{bc}A_b\partial^a(F_{ca}) + \\
&\qquad - \frac{1}{2}\gamma^{ca}\partial_a(A^d)F_{cd} + \gamma^{ac}A^d\partial_a(F_{cd}) + \frac{i}{2}\gamma^{ac}A^2F_{ac}\ . \qquad \text{(C.6)}
\end{aligned}
$$

C.2.4 $\mathcal{F}_{ab}$ and $\mathcal{E}_{ab}$

$\mathcal{F}_{ab}$ and $\mathcal{E}_{ab}$ embody the gauge covariant derivative commutator, $[\nabla_a, \nabla_b]$, in the $\lambda-$ and $\rho-$sector respectively. Their full expressions are

$$
\begin{aligned}
\mathcal{F}_{ab} &= \partial_a\underbrace{(W_b)}_{(4.9)} - \partial_b\underbrace{(W_a)}_{(4.9)} + \Big[\underbrace{W_a}_{(4.9)}, \underbrace{W_b}_{(4.9)}\Big] = \\
&= -\frac{i}{2}\partial_a(A_b) - \frac{i}{2}\gamma_{bc}\partial_a(A^c) + \frac{i}{2}\partial_b(A_a) + \frac{i}{2}\gamma_{ac}\partial_b(A^c) + \underbrace{\Big[-\frac{i}{2}\gamma_{ac}A^c, -\frac{i}{2}\gamma_{bd}A^d\Big]}_{(B.14)} = \\
&= -\frac{i}{2}\Big(\underbrace{\partial_a(A_b) - \partial_b(A_a)}_{(2.6)}\Big) - \frac{i}{2}\gamma_{bc}\partial_a(A^c) + \frac{i}{2}\gamma_{ac}\partial_b(A^c) + \\
&\qquad\qquad - \frac{1}{4}\underbrace{A^cA^d}_{\text{sym. in } cd}\big(2g_{ad}\gamma_{cb} - \cancel{2g_{ab}\gamma_{cd}} - 2g_{cd}\gamma_{ab} + 2g_{cb}\gamma_{ad}\big) = \\
&= -\frac{i}{2}F_{ab} - \frac{i}{2}\gamma_{bc}\partial_a(A^c) + \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_bA^c + \frac{1}{2}\gamma_{bc}A_aA^c + \frac{1}{2}\gamma_{ab}A^cA_c\ , \qquad \text{(C.7)}
\end{aligned}
$$

and

$$
\mathcal{E}_{ab} = \partial_a\underbrace{(Z_b)}_{(4.27)} - \partial_b\underbrace{(Z_a)}_{(4.27)} + \Big[\underbrace{Z_a}_{(4.27)}, \underbrace{Z_b}_{(4.27)}\Big] =
$$

$$= -\frac{i}{2}\partial_a(A_b) + \frac{i}{2}\gamma_{bc}\partial_a(A^c) + \frac{i}{2}\partial_b(A_a) - \frac{i}{2}\gamma_{ac}\partial_b(A^c) + \underbrace{\left[\frac{i}{2}\gamma_{ac}A^c, \frac{i}{2}\gamma_{bd}A^d\right]}_{\text{(B.14)}} =$$

$$= -\frac{i}{2}\Big(\underbrace{\partial_a(A_b) - \partial_b(A_a)}_{\text{(2.6)}}\Big) + \frac{i}{2}\gamma_{bc}\partial_a(A^c) - \frac{i}{2}\gamma_{ac}\partial_b(A^c) +$$

$$- \frac{1}{4}\underbrace{A^c A^d}_{\text{sym. in } cd}\big(2g_{ad}\gamma_{cb} - \cancel{2g_{ab}\gamma_{cd}} - 2g_{cd}\gamma_{ab} + 2g_{cb}\gamma_{ad}\big) =$$

$$= -\frac{i}{2}F_{ab} + \frac{i}{2}\gamma_{bc}\partial_a(A^c) - \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_bA^c + \frac{1}{2}\gamma_{bc}A_aA^c + \frac{1}{2}\gamma_{ab}A^cA_c\ ; \qquad \text{(C.8)}$$

C.2.5 $\mathcal{F}_{ab}\mathcal{F}^{ab}$

Let's expand the product defining the contraction

$$\mathcal{F}_{ab}\mathcal{F}^{ab} = \Big(-\frac{i}{2}(\mathbb{1})F_{ab} - \frac{i}{2}\gamma_{bc}\partial_a(A^c) + \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_bA^c + \frac{1}{2}\gamma_{bc}A_aA^c + \frac{1}{2}\gamma_{ab}A^2\Big)\cdot$$
$$\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab} - \frac{i}{2}\gamma^{bd}\partial^a(A_d) + \frac{i}{2}\gamma^{ad}\partial^b(A_d) - \frac{1}{2}\gamma^{ad}A^bA_d + \frac{1}{2}\gamma^{bd}A^aA_d + \frac{1}{2}\gamma^{ab}A^2\Big)$$

by writing all the thirty-six terms that arise from it, that is:

① $\Big(-\frac{i}{2}(\mathbb{1})F_{ab}\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = -\frac{1}{4}(\mathbb{1})F_{ab}F^{ab}$

② $\Big(-\frac{i}{2}\gamma_{bc}\partial_a(A^c)\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = -\frac{1}{4}\underbrace{\gamma_{bc}\partial_a(A^c)F^{ab}}_{a\to b;\ b\to a} = \frac{1}{4}\gamma_{ac}\partial_b(A^c)F^{ab}$

③ $\Big(+\frac{i}{2}\gamma_{ac}\partial_b(A^c)\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = \frac{1}{4}\gamma_{ac}\partial_b(A^c)F^{ab}$

④ $\Big(-\frac{1}{2}\gamma_{ac}A_bA^c\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = \frac{i}{4}\underbrace{\gamma_{ac}A_bA^cF^{ab}}_{a\to b;\ b\to a} = \frac{i}{4}\gamma_{cb}A_aA^cF^{ab}$

⑤ $\Big(+\frac{1}{2}\gamma_{bc}A_aA^c\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = \frac{i}{4}\gamma_{cb}A_aA^cF^{ab}$

⑥ $\Big(+\frac{1}{2}\gamma_{ab}A^2\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = -\frac{i}{4}\gamma_{ab}A^2F^{ab}$

⑦ $\Big(-\frac{i}{2}(\mathbb{1})F_{ab}\Big)\cdot\Big(-\frac{i}{2}\gamma^{bd}\partial^a(A_d)\Big) = -\frac{1}{4}\underbrace{\gamma^{bd}\partial^a(A_d)F_{ab}}_{a\to b;\ b\to a;\ d\to c} = \frac{1}{4}\gamma^{ac}\partial^b(A_c)F_{ab}$

$$\text{⑧}\quad \left(-\frac{i}{2}\gamma_{bc}\partial_a(A^c)\right)\cdot\left(-\frac{i}{2}\gamma^{bd}\partial^a(A_d)\right) = -\frac{1}{4}\underbrace{\gamma^{bc}\gamma_b{}^d}_{\text{(B.12)}}\partial_a(A_c)\partial^a(A_d) =$$

$$= -\frac{1}{4}\left(-2\cancel{\gamma^{cd}} - 3g^{cd}(\mathbb{1})\right)\underbrace{\partial_a(A_c)\partial^a(A_d)}_{\text{sym. in } cd} =$$

$$= \frac{3}{4}(\mathbb{1})\underbrace{\partial_a(A_c)\partial^a(A^c)}_{a\to b} =$$

$$= \frac{3}{4}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)$$

$$\text{⑨}\quad \left(+\frac{i}{2}\gamma_{ac}\partial_b(A^c)\right)\cdot\left(-\frac{i}{2}\gamma^{bd}\partial^a(A_d)\right) = \frac{1}{4}\underbrace{\gamma^{bd}\gamma^{ac}\partial_b(A_c)\partial_a(A_d)}_{d\to c;\ c\to d} = \frac{1}{4}\gamma^{bc}\gamma^{ad}\partial_b(A_d)\partial_a(A_c)$$

$$\text{⑩}\quad \left(-\frac{1}{2}\gamma_{ac}A_bA^c\right)\cdot\left(-\frac{i}{2}\gamma^{bd}\partial^a(A_d)\right) = \frac{i}{4}\underbrace{\gamma^{ac}\gamma^{bd}A_bA_c\partial_a(A_d)}_{a\to b;\ b\to a} = \frac{i}{4}\gamma^{bc}\gamma^{ad}A_aA_c\partial_b(A_d)$$

$$\text{⑪}\quad \left(+\frac{1}{2}\gamma_{bc}A_aA^c\right)\cdot\left(-\frac{i}{2}\gamma^{bd}\partial^a(A_d)\right) = -\frac{i}{4}\gamma_b{}^c\gamma^{bd}A_aA_c\partial^a(A_d)$$

$$\text{⑫}\quad \left(+\frac{1}{2}\gamma_{ab}A^2\right)\cdot\left(-\frac{i}{2}\gamma^{bd}\partial^a(A_d)\right) = \frac{i}{4}\underbrace{\gamma_b{}^a\gamma^{bd}A^2\partial_a(A_d)}_{a\to c} = \frac{i}{4}\gamma_b{}^c\gamma^{bd}A^2\partial_c(A_d)$$

$$\text{⑬}\quad \left(-\frac{i}{2}(\mathbb{1})F_{ab}\right)\cdot\left(+\frac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = \frac{1}{4}\underbrace{\gamma^{ad}\partial^b(A_d)F_{ab}}_{d\to c} = \frac{1}{4}\gamma^{ac}\partial^b(A_c)F_{ab}$$

$$\text{⑭}\quad \left(-\frac{i}{2}\gamma_{bc}\partial_a(A^c)\right)\cdot\left(+\frac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = \frac{1}{4}\gamma^{bc}\gamma^{ad}\partial_a(A_c)\partial_b(A_d)$$

$$\text{⑮}\quad \left(+\frac{i}{2}\gamma_{ac}\partial_b(A^c)\right)\cdot\left(+\frac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = -\frac{1}{4}\underbrace{\gamma^{ac}\gamma_a{}^d}_{\text{(B.12)}}\partial_b(A_c)\partial^b(A_d) =$$

$$= -\frac{1}{4}\left(-2\cancel{\gamma^{cd}} - 3g^{cd}(\mathbb{1})\right)\underbrace{\partial_b(A_c)\partial^b(A_d)}_{\text{sym. in } cd} =$$

$$= \frac{3}{4}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)$$

$$\text{⑯}\quad \left(-\frac{1}{2}\gamma_{ac}A_bA^c\right)\cdot\left(+\frac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = -\frac{i}{4}\underbrace{\gamma_a{}^c\gamma^{ad}A_bA_c\partial^b(A_d)}_{a\to b;\ b\to a} = -\frac{i}{4}\gamma_b{}^c\gamma^{bd}A_aA_c\partial^a(A_d)$$

$$\text{⑰}\quad \left(+\frac{1}{2}\gamma_{bc}A_aA^c\right)\cdot\left(+\frac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = \frac{i}{4}\gamma^{bc}\gamma^{ad}A_aA_c\partial_b(A_d)$$

$$\text{⑱}\quad \left(+\frac{1}{2}\gamma_{ab}A^2\right)\cdot\left(+\frac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = \frac{i}{4}\underbrace{\gamma_a{}^b\gamma^{ad}A^2\partial_b(A_d)}_{a\to b;\ b\to c} = \frac{i}{4}\gamma_b{}^c\gamma^{bd}A^2\partial_c(A_d)$$

$$\text{⑲}\quad \left(-\frac{i}{2}(\mathbb{1})F_{ab}\right)\cdot\left(-\frac{1}{2}\gamma^{ad}A^bA_d\right) = \frac{i}{4}\underbrace{\gamma^{ad}A^bA_dF_{ab}}_{a\to b;\ b\to a;\ d\to c} = \frac{i}{4}\gamma^{cb}A^aA_cF_{ab}$$

$$⑳\quad \left(-\tfrac{i}{2}\gamma_{bc}\partial_a(A^c)\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) = \tfrac{i}{4}\gamma^{bc}\gamma^{ad}A_bA_d\partial_a(A_c)$$

$$㉑\quad \left(+\tfrac{i}{2}\gamma_{ac}\partial_b(A^c)\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) = -\tfrac{i}{4}\underbrace{\gamma_a{}^c\gamma^{ad}A^bA_d\partial_b(A_c)}_{a\to b;\ b\to a} = -\tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^aA_d\partial_a(A_c)$$

$$\begin{aligned}㉒\quad \left(-\tfrac{1}{2}\gamma_{ac}A_bA^c\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) &= \tfrac{1}{4}\underbrace{\gamma^{ac}\gamma_a{}^d}_{\text{(B.12)}}A_bA_cA^bA_d = \\ &= \tfrac{1}{4}\left(-2\cancel{\gamma^{cd}} - 3g^{cd}(\mathbb{1})\right)\underbrace{A_bA_cA^bA_d}_{\text{sym. in } cd} = \\ &= -\tfrac{3}{4}(\mathbb{1})A^4\end{aligned}$$

$$㉓\quad \left(+\tfrac{1}{2}\gamma_{bc}A_aA^c\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) = -\tfrac{1}{4}\gamma^{bc}\gamma^{ad}\underbrace{A_aA_cA_bA_d}_{\text{sym. in } bc} = 0$$

$$\begin{aligned}㉔\quad \left(+\tfrac{1}{2}\gamma_{ab}A^2\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) &= -\tfrac{1}{4}\underbrace{\gamma^{ab}\gamma_a{}^d}_{\text{(B.12)}}A_bA_dA^2 = \\ &= -\tfrac{1}{4}\left(-2\cancel{\gamma^{bd}} - 3g^{bd}(\mathbb{1})\right)\underbrace{A_bA_dA^2}_{\text{sym. in } bd} = \\ &= \tfrac{3}{4}(\mathbb{1})A^4\end{aligned}$$

$$㉕\quad \left(-\tfrac{i}{2}(\mathbb{1})F_{ab}\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = -\tfrac{i}{4}\underbrace{\gamma^{bd}A^aA_dF_{ab}}_{d\to c} = \tfrac{i}{4}\gamma^{cb}A^aA_cF_{ab}$$

$$㉖\quad \left(-\tfrac{i}{2}\gamma_{bc}\partial_a(A^c)\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = -\tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^aA_d\partial_a(A_c)$$

$$㉗\quad \left(+\tfrac{i}{2}\gamma_{ac}\partial_b(A^c)\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = \tfrac{i}{4}\underbrace{\gamma^{ac}\gamma^{bd}A_aA_d\partial_b(A_c)}_{a\to b;\ b\to a} = \tfrac{i}{4}\gamma^{bc}\gamma^{ad}A_bA_d\partial_a(A_c)$$

$$㉘\quad \left(-\tfrac{1}{2}\gamma_{ac}A_bA^c\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = -\tfrac{1}{4}\gamma^{ac}\gamma^{bd}\underbrace{A_bA_cA_aA_d}_{\text{sym. in } ac} = 0$$

$$\begin{aligned}㉙\quad \left(+\tfrac{1}{2}\gamma_{bc}A_aA^c\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) &= \tfrac{1}{4}\underbrace{\gamma^{bc}\gamma_b{}^d}_{\text{(B.12)}}A_aA_cA^aA_d = \\ &= \tfrac{1}{4}\left(-2\cancel{\gamma^{cd}} - 3g^{cd}(\mathbb{1})\right)\underbrace{A_cA_dA^2}_{\text{sym. in } cd} = \\ &= -\tfrac{3}{4}(\mathbb{1})A^4\end{aligned}$$

$$\textcircled{30} \qquad \left(+\tfrac{1}{2}\gamma_{ab}A^2\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = -\tfrac{1}{4}\underbrace{\gamma^{ba}\gamma_b{}^d}_{\text{(B.12)}}A_aA_dA^2 =$$

$$= -\tfrac{1}{4}\left(-2\cancel{\gamma^{ad}} - 3g^{ad}(\mathbb{1})\right)\underbrace{A_aA_dA^2}_{\text{sym. in } ad} =$$

$$= \tfrac{3}{4}(\mathbb{1})A^4$$

$$\textcircled{31} \qquad \left(-\tfrac{i}{2}(\mathbb{1})F_{ab}\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = -\tfrac{i}{4}\gamma^{ab}A^2F_{ab}$$

$$\textcircled{32} \quad \left(-\tfrac{i}{2}\gamma_{bc}\partial_a(A^c)\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = \tfrac{i}{4}\underbrace{\gamma_b{}^c\gamma^{ba}A^2\partial_a(A_c)}_{a\to d} = \tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^2\partial_d(A_c)$$

$$\textcircled{33} \quad \left(+\tfrac{i}{2}\gamma_{ac}\partial_b(A^c)\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = \tfrac{i}{4}\underbrace{\gamma_a{}^c\gamma^{ab}A^2\partial_b(A_c)}_{a\to b;\ b\to d} = \tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^2\partial_d(A_c)$$

$$\textcircled{34} \qquad \left(-\tfrac{1}{2}\gamma_{ac}A_bA^c\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = -\tfrac{1}{4}\underbrace{\gamma^{ac}\gamma_a{}^b}_{\text{(B.12)}}A_bA_cA^2 =$$

$$= -\tfrac{1}{4}\left(-2\cancel{\gamma^{cb}} - 3g^{cb}(\mathbb{1})\right)\underbrace{A_bA_cA^2}_{\text{sym. in } cb} =$$

$$= \tfrac{3}{4}(\mathbb{1})A^4$$

$$\textcircled{35} \qquad \left(+\tfrac{1}{2}\gamma_{bc}A_aA^c\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = -\tfrac{1}{4}\underbrace{\gamma^{bc}\gamma_b{}^a}_{\text{(B.12)}}A_aA_cA^2 =$$

$$= -\tfrac{1}{4}\left(-2\cancel{\gamma^{ca}} - 3g^{ca}(\mathbb{1})\right)\underbrace{A_aA_cA^2}_{\text{sym. in } ca} =$$

$$= \tfrac{3}{4}(\mathbb{1})A^4$$

$$\textcircled{36} \qquad \left(+\tfrac{1}{2}\gamma_{ab}A^2\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = \tfrac{1}{4}\underbrace{\gamma_{ab}\gamma^{ab}}_{\text{(B.13)}}A^4 = -3(\mathbb{1})A^4\ .$$

We decided to keep the spinor identity matrix, where needed, in order to preserve the aesthetic rigour of the calculus. By all means, we can now recognize the terms that sum up together by accurately reviewing the ones above, i.e.:

- $\textcircled{1} = -\frac{1}{4}(\mathbb{1})F_{ab}F^{ab}$
- $\textcircled{2} + \textcircled{3} + \textcircled{7} + \textcircled{13} = \gamma^{ac}\partial^b(A_c)F_{ab}$
- $\textcircled{4} + \textcircled{5} + \textcircled{19} + \textcircled{25} = i\gamma^{cb}A^aA_cF_{ab}$
- $\textcircled{6} + \textcircled{31} = -\frac{i}{2}\gamma^{ab}A^2F_{ab}$

- $\textcircled{8} + \textcircled{15} = \frac{3}{2}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)$
- $\textcircled{9} + \textcircled{14} = \frac{1}{2}\gamma^{bc}\gamma^{ad}\partial_a(A_c)\partial_b(A_d)$
- $\textcircled{10} + \textcircled{17} = \frac{i}{2}\gamma^{bc}\gamma^{ad}A_aA_c\partial_b(A_d)$
- $\textcircled{11} + \textcircled{16} = -\frac{i}{2}\gamma_b{}^c\gamma^{bd}A_aA_c\partial^a(A_d)$
- $\textcircled{12} + \textcircled{18} = \frac{i}{2}\gamma_b{}^c\gamma^{bd}A^2\partial_c(A_d)$
- $\textcircled{20} + \textcircled{27} = \frac{i}{2}\gamma^{bc}\gamma^{ad}A_bA_d\partial_a(A_c)$
- $\textcircled{21} + \textcircled{26} = -\frac{i}{2}\gamma_b{}^c\gamma^{bd}A^aA_d\partial_a(A_c)$
- $\textcircled{22} + \textcircled{24} + \textcircled{29} + \textcircled{30} + \textcircled{34} + \textcircled{35} + \textcircled{36} = -\frac{3}{2}(\mathbb{1})A^4$
- $\textcircled{23} + \textcircled{28} = 0$
- $\textcircled{32} + \textcircled{33} = \frac{i}{2}\gamma_b{}^c\gamma^{bd}A^2\partial_d(A_c)$.

Finally, altogether, it is possible to claim that

$$\begin{aligned}
\mathcal{F}_{ab}\mathcal{F}^{ab} = & -\tfrac{1}{4}(\mathbb{1})F_{ab}F^{ab} + \gamma^{ac}\partial^b(A_c)F_{ab} + i\gamma^{cb}A^aA_cF_{ab} - \tfrac{i}{2}\gamma^{ab}A^2F_{ab} + \tfrac{3}{2}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)+\\
& + \tfrac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big] - \tfrac{3}{2}(\mathbb{1})A^4+\\
& + \tfrac{i}{2}\underbrace{\gamma_b{}^c\gamma^{bd}}_{\text{(B.12)}}\Big[A^2\partial_c(A_d) + A^2\partial_d(A_c) - A_aA_d\partial^a(A_c) - A_aA_c\partial^a(A_d)\Big] =\\
= & -\tfrac{1}{4}(\mathbb{1})F_{ab}F^{ab} + \gamma^{ac}\partial^b(A_c)F_{ab} + i\gamma^{cb}A^aA_cF_{ab} - \tfrac{i}{2}\gamma^{ab}A^2F_{ab} + \tfrac{3}{2}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)+\\
& + \tfrac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big] - \tfrac{3}{2}(\mathbb{1})A^4+\\
& + \tfrac{i}{2}\Big(-2\cancel{\gamma^{cd}} - 3g^{cd}(\mathbb{1})\Big)\underbrace{\Big[A^2\partial_c(A_d) + A^2\partial_d(A_c) - A_aA_d\partial^a(A_c) - A_aA_c\partial^a(A_d)\Big]}_{\text{sym. in } cd} =\\
= & -\tfrac{1}{4}(\mathbb{1})F_{ab}F^{ab} + \gamma^{ac}\partial^b(A_c)F_{ab} + i\gamma^{cb}A^aA_cF_{ab} - \tfrac{i}{2}\gamma^{ab}A^2F_{ab} + \tfrac{3}{2}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)+\\
& + \tfrac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big] - \tfrac{3}{2}(\mathbb{1})A^4+\\
& - 3i\Big[A^2\partial^d(A_d) - A_aA_d\partial^a(A^d)\Big]\,. \qquad \text{(C.9)}
\end{aligned}$$

C.2.6 $\mathcal{E}_{ab}\mathcal{E}^{ab}$

What we previously did for $\mathcal{F}_{ab}\mathcal{F}^{ab}$ should also be repeated with $\mathcal{E}_{ab}\mathcal{E}^{ab}$. In particular, the thirty-six terms emerging from

$$\mathcal{E}_{ab}\mathcal{E}^{ab} = \Big(-\frac{i}{2}(\mathbb{1})F_{ab} + \frac{i}{2}\gamma_{bc}\partial_a(A^c) - \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_bA^c + \frac{1}{2}\gamma_{bc}A_aA^c + \frac{1}{2}\gamma_{ab}A^2\Big)\cdot$$
$$\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab} + \frac{i}{2}\gamma^{bd}\partial^a(A_d) - \frac{i}{2}\gamma^{ad}\partial^b(A_d) - \frac{1}{2}\gamma^{ad}A^bA_d + \frac{1}{2}\gamma^{bd}A^aA_d + \frac{1}{2}\gamma^{ab}A^2\Big)$$

are now:

① $\Big(-\frac{i}{2}(\mathbb{1})F_{ab}\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = -\frac{1}{4}(\mathbb{1})F_{ab}F^{ab}$

② $\Big(+\frac{i}{2}\gamma_{bc}\partial_a(A^c)\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = +\frac{1}{4}\underbrace{\gamma_{bc}\partial_a(A^c)F^{ab}}_{a\to b;\ b\to a} = -\frac{1}{4}\gamma_{ac}\partial_b(A^c)F^{ab}$

③ $\Big(-\frac{i}{2}\gamma_{ac}\partial_b(A^c)\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = -\frac{1}{4}\gamma_{ac}\partial_b(A^c)F^{ab}$

④ $\Big(-\frac{1}{2}\gamma_{ac}A_bA^c\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = \frac{i}{4}\underbrace{\gamma_{ac}A_bA^cF^{ab}}_{a\to b;\ b\to a} = \frac{i}{4}\gamma_{cb}A_aA^cF^{ab}$

⑤ $\Big(+\frac{1}{2}\gamma_{bc}A_aA^c\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = \frac{i}{4}\gamma_{cb}A_aA^cF^{ab}$

⑥ $\Big(+\frac{1}{2}\gamma_{ab}A^2\Big)\cdot\Big(-\frac{i}{2}(\mathbb{1})F^{ab}\Big) = -\frac{i}{4}\gamma_{ab}A^2F^{ab}$

⑦ $\Big(-\frac{i}{2}(\mathbb{1})F_{ab}\Big)\cdot\Big(+\frac{i}{2}\gamma^{bd}\partial^a(A_d)\Big) = +\frac{1}{4}\underbrace{\gamma^{bd}\partial^a(A_d)F_{ab}}_{a\to b;\ b\to a;\ d\to c} = -\frac{1}{4}\gamma^{ac}\partial^b(A_c)F_{ab}$

⑧ $\Big(+\frac{i}{2}\gamma_{bc}\partial_a(A^c)\Big)\cdot\Big(+\frac{i}{2}\gamma^{bd}\partial^a(A_d)\Big) = -\frac{1}{4}\underbrace{\gamma^{bc}\gamma_b{}^d}_{\text{(B.12)}}\partial_a(A_c)\partial^a(A_d) =$

$$= -\frac{1}{4}\Big(-2\cancel{\gamma^{cd}} - 3g^{cd}(\mathbb{1})\Big)\underbrace{\partial_a(A_c)\partial^a(A_d)}_{\text{sym. in } cd} =$$
$$= \frac{3}{4}(\mathbb{1})\underbrace{\partial_a(A_c)\partial^a(A^c)}_{a\to b} =$$
$$= \frac{3}{4}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)$$

⑨ $\Big(-\frac{i}{2}\gamma_{ac}\partial_b(A^c)\Big)\cdot\Big(+\frac{i}{2}\gamma^{bd}\partial^a(A_d)\Big) = \frac{1}{4}\underbrace{\gamma^{bd}\gamma^{ac}\partial_b(A_c)\partial_a(A_d)}_{d\to c;\ c\to d} = \frac{1}{4}\gamma^{bc}\gamma^{ad}\partial_b(A_d)\partial_a(A_c)$

⑩ $\Big(-\frac{1}{2}\gamma_{ac}A_bA^c\Big)\cdot\Big(+\frac{i}{2}\gamma^{bd}\partial^a(A_d)\Big) = -\frac{i}{4}\underbrace{\gamma^{ac}\gamma^{bd}A_bA_c\partial_a(A_d)}_{a\to b;\ b\to a} = -\frac{i}{4}\gamma^{bc}\gamma^{ad}A_aA_c\partial_b(A_d)$

⑪ $\Big(+\frac{1}{2}\gamma_{bc}A_aA^c\Big)\cdot\Big(+\frac{i}{2}\gamma^{bd}\partial^a(A_d)\Big) = \frac{i}{4}\gamma_b{}^c\gamma^{bd}A_aA_c\partial^a(A_d)$

$$⑫ \quad \left(+\tfrac{1}{2}\gamma_{ab}A^2\right)\cdot\left(+\tfrac{i}{2}\gamma^{bd}\partial^a(A_d)\right) = -\tfrac{i}{4}\underbrace{\gamma_b{}^a\gamma^{bd}A^2\partial_a(A_d)}_{a\to c} = -\tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^2\partial_c(A_d)$$

$$⑬ \quad \left(-\tfrac{i}{2}(\mathbb{1})F_{ab}\right)\cdot\left(-\tfrac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = -\tfrac{1}{4}\underbrace{\gamma^{ad}\partial^b(A_d)F_{ab}}_{d\to c} = -\tfrac{1}{4}\gamma^{ac}\partial^b(A_c)F_{ab}$$

$$⑭ \quad \left(+\tfrac{i}{2}\gamma_{bc}\partial_a(A^c)\right)\cdot\left(-\tfrac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = \tfrac{1}{4}\gamma^{bc}\gamma^{ad}\partial_a(A_c)\partial_b(A_d)$$

$$⑮ \quad \left(-\tfrac{i}{2}\gamma_{ac}\partial_b(A^c)\right)\cdot\left(-\tfrac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = -\tfrac{1}{4}\underbrace{\gamma^{ac}\gamma_a{}^d}_{\text{(B.12)}}\partial_b(A_c)\partial^b(A_d) =$$

$$= -\tfrac{1}{4}\left(-2\cancel{\gamma^{cd}} - 3g^{cd}(\mathbb{1})\right)\underbrace{\partial_b(A_c)\partial^b(A_d)}_{\text{sym. in } cd} =$$

$$= \tfrac{3}{4}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)$$

$$⑯ \quad \left(-\tfrac{1}{2}\gamma_{ac}A_bA^c\right)\cdot\left(-\tfrac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = \tfrac{i}{4}\underbrace{\gamma_a{}^c\gamma^{ad}A_bA_c\partial^b(A_d)}_{a\to b;\ b\to a} = \tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A_aA_c\partial^a(A_d)$$

$$⑰ \quad \left(+\tfrac{1}{2}\gamma_{bc}A_aA^c\right)\cdot\left(-\tfrac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = -\tfrac{i}{4}\gamma^{bc}\gamma^{ad}A_aA_c\partial_b(A_d)$$

$$⑱ \quad \left(+\tfrac{1}{2}\gamma_{ab}A^2\right)\cdot\left(-\tfrac{i}{2}\gamma^{ad}\partial^b(A_d)\right) = -\tfrac{i}{4}\underbrace{\gamma_a{}^b\gamma^{ad}A^2\partial_b(A_d)}_{a\to b;\ b\to c} = -\tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^2\partial_c(A_d)$$

$$⑲ \quad \left(-\tfrac{i}{2}(\mathbb{1})F_{ab}\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) = \tfrac{i}{4}\underbrace{\gamma^{ad}A^bA_dF_{ab}}_{a\to b;\ b\to a;\ d\to c} = \tfrac{i}{4}\gamma^{cb}A^aA_cF_{ab}$$

$$⑳ \quad \left(+\tfrac{i}{2}\gamma_{bc}\partial_a(A^c)\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) = -\tfrac{i}{4}\gamma^{bc}\gamma^{ad}A_bA_d\partial_a(A_c)$$

$$㉑ \quad \left(-\tfrac{i}{2}\gamma_{ac}\partial_b(A^c)\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) = \tfrac{i}{4}\underbrace{\gamma_a{}^c\gamma^{ad}A^bA_d\partial_b(A_c)}_{a\to b;\ b\to a} = \tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^aA_d\partial_a(A_c)$$

$$㉒ \quad \left(-\tfrac{1}{2}\gamma_{ac}A_bA^c\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) = \tfrac{1}{4}\underbrace{\gamma^{ac}\gamma_a{}^d}_{\text{(B.12)}}A_bA_cA^bA_d =$$

$$= \tfrac{1}{4}\left(-2\cancel{\gamma^{cd}} - 3g^{cd}(\mathbb{1})\right)\underbrace{A_bA_cA^bA_d}_{\text{sym. in } cd} =$$

$$= -\tfrac{3}{4}(\mathbb{1})A^4$$

$$㉓ \quad \left(+\tfrac{1}{2}\gamma_{bc}A_aA^c\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) = -\tfrac{1}{4}\gamma^{bc}\gamma^{ad}\underbrace{A_aA_cA_bA_d}_{\text{sym. in } bc} = 0$$

$$⑳\!4\quad \left(+\tfrac{1}{2}\gamma_{ab}A^2\right)\cdot\left(-\tfrac{1}{2}\gamma^{ad}A^bA_d\right) = -\tfrac{1}{4}\underbrace{\gamma^{ab}\gamma_a{}^d}_{\text{(B.12)}}A_bA_dA^2 =$$

$$= -\tfrac{1}{4}\left(-2\cancel{\gamma^{bd}} - 3g^{bd}(\mathbb{1})\right)\underbrace{A_bA_dA^2}_{\text{sym. in } bd} =$$

$$= \tfrac{3}{4}(\mathbb{1})A^4$$

$$㉕\quad \left(-\tfrac{i}{2}(\mathbb{1})F_{ab}\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = -\tfrac{i}{4}\underbrace{\gamma^{bd}A^aA_dF_{ab}}_{d\to c} = \tfrac{i}{4}\gamma^{cb}A^aA_cF_{ab}$$

$$㉖\quad \left(+\tfrac{i}{2}\gamma_{bc}\partial_a(A^c)\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = \tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^aA_d\partial_a(A_c)$$

$$㉗\quad \left(-\tfrac{i}{2}\gamma_{ac}\partial_b(A^c)\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = -\tfrac{i}{4}\underbrace{\gamma^{ac}\gamma^{bd}A_aA_d\partial_b(A_c)}_{a\to b;\ b\to a} = -\tfrac{i}{4}\gamma^{bc}\gamma^{ad}A_bA_d\partial_a(A_c)$$

$$㉘\quad \left(-\tfrac{1}{2}\gamma_{ac}A_bA^c\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = -\tfrac{1}{4}\gamma^{ac}\gamma^{bd}\underbrace{A_bA_cA_aA_d}_{\text{sym. in } ac} = 0$$

$$㉙\quad \left(+\tfrac{1}{2}\gamma_{bc}A_aA^c\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = \tfrac{1}{4}\underbrace{\gamma^{bc}\gamma_b{}^d}_{\text{(B.12)}}A_aA_cA^aA_d =$$

$$= \tfrac{1}{4}\left(-2\cancel{\gamma^{cd}} - 3g^{cd}(\mathbb{1})\right)\underbrace{A_cA_dA^2}_{\text{sym. in } cd} =$$

$$= -\tfrac{3}{4}(\mathbb{1})A^4$$

$$㉚\quad \left(+\tfrac{1}{2}\gamma_{ab}A^2\right)\cdot\left(+\tfrac{1}{2}\gamma^{bd}A^aA_d\right) = -\tfrac{1}{4}\underbrace{\gamma^{ba}\gamma_b{}^d}_{\text{(B.12)}}A_aA_dA^2 =$$

$$= -\tfrac{1}{4}\left(-2\cancel{\gamma^{ad}} - 3g^{ad}(\mathbb{1})\right)\underbrace{A_aA_dA^2}_{\text{sym. in } ad} =$$

$$= \tfrac{3}{4}(\mathbb{1})A^4$$

$$㉛\quad \left(-\tfrac{i}{2}(\mathbb{1})F_{ab}\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = -\tfrac{i}{4}\gamma^{ab}A^2F_{ab}$$

$$㉜\quad \left(+\tfrac{i}{2}\gamma_{bc}\partial_a(A^c)\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = -\tfrac{i}{4}\underbrace{\gamma_b{}^c\gamma^{ba}A^2\partial_a(A_c)}_{a\to d} = -\tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^2\partial_d(A_c)$$

$$㉝\quad \left(-\tfrac{i}{2}\gamma_{ac}\partial_b(A^c)\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = -\tfrac{i}{4}\underbrace{\gamma_a{}^c\gamma^{ab}A^2\partial_b(A_c)}_{a\to b;\ b\to d} = -\tfrac{i}{4}\gamma_b{}^c\gamma^{bd}A^2\partial_d(A_c)$$

$$\textcircled{34} \qquad \left(-\tfrac{1}{2}\gamma_{ac}A_bA^c\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = -\tfrac{1}{4}\underbrace{\gamma^{ac}\gamma_a{}^b}_{\text{(B.12)}}A_bA_cA^2 =$$

$$= -\tfrac{1}{4}\left(-2\cancel{\gamma^{cb}} - 3g^{cb}(\mathbb{1})\right)\underbrace{A_bA_cA^2}_{\text{sym. in } cb} =$$

$$= \tfrac{3}{4}(\mathbb{1})A^4$$

$$\textcircled{35} \qquad \left(+\tfrac{1}{2}\gamma_{bc}A_aA^c\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = -\tfrac{1}{4}\underbrace{\gamma^{bc}\gamma_b{}^a}_{\text{(B.12)}}A_aA_cA^2 =$$

$$= -\tfrac{1}{4}\left(-2\cancel{\gamma^{ca}} - 3g^{ca}(\mathbb{1})\right)\underbrace{A_aA_cA^2}_{\text{sym. in } ca} =$$

$$= \tfrac{3}{4}(\mathbb{1})A^4$$

$$\textcircled{36} \qquad \left(+\tfrac{1}{2}\gamma_{ab}A^2\right)\cdot\left(+\tfrac{1}{2}\gamma^{ab}A^2\right) = \tfrac{1}{4}\underbrace{\gamma_{ab}\gamma^{ab}}_{\text{(B.13)}}A^4 = -3(\mathbb{1})A^4 \; .$$

In the end, all these different operators, whose partial sums are listed right down here

- $\textcircled{1} = -\frac{1}{4}(\mathbb{1})F_{ab}F^{ab}$
- $\textcircled{2} + \textcircled{3} + \textcircled{7} + \textcircled{13} = -\gamma^{ac}\partial^b(A_c)F_{ab}$
- $\textcircled{4} + \textcircled{5} + \textcircled{19} + \textcircled{25} = i\gamma^{cb}A^aA_cF_{ab}$
- $\textcircled{6} + \textcircled{31} = -\frac{i}{2}\gamma^{ab}A^2F_{ab}$
- $\textcircled{8} + \textcircled{15} = \frac{3}{2}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)$
- $\textcircled{9} + \textcircled{14} = \frac{1}{2}\gamma^{bc}\gamma^{ad}\partial_a(A_c)\partial_b(A_d)$
- $\textcircled{10} + \textcircled{17} = -\frac{i}{2}\gamma^{bc}\gamma^{ad}A_aA_c\partial_b(A_d)$
- $\textcircled{11} + \textcircled{16} = \frac{i}{2}\gamma_b{}^c\gamma^{bd}A_aA_c\partial^a(A_d)$
- $\textcircled{12} + \textcircled{18} = -\frac{i}{2}\gamma_b{}^c\gamma^{bd}A^2\partial_c(A_d)$
- $\textcircled{20} + \textcircled{27} = -\frac{i}{2}\gamma^{bc}\gamma^{ad}A_bA_d\partial_a(A_c)$
- $\textcircled{21} + \textcircled{26} = \frac{i}{2}\gamma_b{}^c\gamma^{bd}A_aA_d\partial^a(A_c)$
- $\textcircled{22} + \textcircled{24} + \textcircled{29} + \textcircled{30} + \textcircled{34} + \textcircled{35} + \textcircled{36} = -\frac{3}{2}(\mathbb{1})A^4$

- $\textcircled{23}+\textcircled{28}=0$
- $\textcircled{32}+\textcircled{33}=-\frac{i}{2}\gamma_b{}^c\gamma^{bd}A^2\partial_d(A_c)$,

add up to

$$\begin{aligned}
\mathcal{E}_{ab}\mathcal{E}^{ab} &= -\tfrac{1}{4}(\mathbb{1})F_{ab}F^{ab}-\gamma^{ac}\partial^b(A_c)F_{ab}+i\gamma^{cb}A^aA_cF_{ab}-\tfrac{i}{2}\gamma^{ab}A^2F_{ab}+\tfrac{3}{2}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)+\\
&\quad+\tfrac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d)-iA_aA_c\partial_b(A_d)-iA_bA_d\partial_a(A_c)\Big]-\tfrac{3}{2}(\mathbb{1})A^4+\\
&\quad-\tfrac{i}{2}\underbrace{\gamma_b{}^c\gamma^{bd}}_{\text{(B.12)}}\Big[A^2\partial_c(A_d)+A^2\partial_d(A_c)-A_aA_d\partial^a(A_c)-A_aA_c\partial^a(A_d)\Big]=\\
&=-\tfrac{1}{4}(\mathbb{1})F_{ab}F^{ab}-\gamma^{ac}\partial^b(A_c)F_{ab}+i\gamma^{cb}A^aA_cF_{ab}-\tfrac{i}{2}\gamma^{ab}A^2F_{ab}+\tfrac{3}{2}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)+\\
&\quad+\tfrac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d)-iA_aA_c\partial_b(A_d)-iA_bA_d\partial_a(A_c)\Big]-\tfrac{3}{2}(\mathbb{1})A^4+\\
&\quad-\tfrac{i}{2}\Big(-2\cancel{\gamma^{cd}}-3g^{cd}(\mathbb{1})\Big)\underbrace{\Big[A^2\partial_c(A_d)+A^2\partial_d(A_c)-A_aA_d\partial^a(A_c)-A_aA_c\partial^a(A_d)\Big]}_{\text{sym. in } cd}=\\
&=-\tfrac{1}{4}(\mathbb{1})F_{ab}F^{ab}-\gamma^{ac}\partial^b(A_c)F_{ab}+i\gamma^{cb}A^aA_cF_{ab}-\tfrac{i}{2}\gamma^{ab}A^2F_{ab}+\tfrac{3}{2}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)+\\
&\quad+\tfrac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d)-iA_aA_c\partial_b(A_d)-iA_bA_d\partial_a(A_c)\Big]-\tfrac{3}{2}(\mathbb{1})A^4+\\
&\quad+3i\Big[A^2\partial^d(A_d)-A_aA_d\partial^a(A^d)\Big]\ .
\end{aligned}\tag{C.10}$$

C.2.7 $a_2(x,R_\lambda)$ and $a_2(x,R_\rho)$

In this last part we'll carry out the derivation of the two Seeley-DeWitt coefficients, $a_2(x,R_\lambda)$ and $a_2(x,R_\rho)$. The first is

$$\begin{aligned}
a_2(x,R_\lambda) &= \frac{1}{2}V^2-\frac{1}{6}\nabla^2V+\frac{1}{12}\mathcal{F}_{ab}\mathcal{F}^{ab}=\\
&=-\frac{1}{8}(\partial_aA^a)^2-\frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab}+\cancel{\frac{i}{4}A^2\partial_c(A^c)}+\cancel{\frac{i}{8}\gamma^{ab}A^2F_{ab}}-\frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd}+\cancel{\frac{1}{8}A^4}+\\
&\quad-\frac{i}{12}\Box(\partial_sA^s)-\frac{i}{24}\gamma^{cd}\Box(F_{cd})-\frac{1}{12}\Box(A^2)+\frac{1}{12}\gamma^{ac}\partial^b(A_c)F_{ab}-\frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca})+\\
&\qquad-\frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd}+\frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd})-\underbrace{\cancel{\frac{i}{12}\gamma^{ac}A^2F_{ac}}}_{c\to b}+\\
&\quad-\frac{1}{48}F_{ab}F^{ab}+\frac{1}{12}\gamma^{ac}\partial^b(A_c)F_{ab}+\frac{i}{12}\gamma^{cb}A^aA_cF_{ab}-\cancel{\frac{i}{24}\gamma^{ab}A^2F_{ab}}+\frac{1}{8}\partial_b(A_c)\partial^b(A^c)+\\
&\quad+\frac{1}{24}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d)+iA_aA_c\partial_b(A_d)+iA_bA_d\partial_a(A_c)\Big]-\cancel{\frac{1}{8}A^4}+\\
&\quad-\underbrace{\cancel{\frac{i}{4}A^2\partial^d(A_d)}}_{d\to c}+\frac{i}{4}A_aA_d\partial^a(A^d)=\\
&=-\frac{1}{8}(\partial_aA^a)^2-\frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab}-\frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd}-\frac{i}{12}\Box(\partial_sA^s)-\frac{i}{24}\gamma^{cd}\Box(F_{cd})+
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{12}\Box(A^2)+\frac{1}{6}\gamma^{ac}\partial^b(A_c)F_{ab}-\frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca})-\frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd}+\\
&+\frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd})-\frac{1}{48}F_{ab}F^{ab}+\frac{i}{12}\gamma^{cb}A^aA_cF_{ab}+\frac{1}{8}\partial_b(A_c)\partial^b(A^c)+\\
&+\frac{1}{24}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d)+iA_aA_c\partial_b(A_d)+iA_bA_d\partial_a(A_c)\Big]+\frac{i}{4}A_aA_d\partial^a(A^d)\,, \qquad \text{(C.11)}
\end{aligned}
$$

while the other one is found to be

$$
\begin{aligned}
a_2(x,R_\rho)&=\frac{1}{2}U^2-\frac{1}{6}\nabla^2U+\frac{1}{12}\mathcal{E}_{ab}\mathcal{E}^{ab}=\\
&=-\frac{1}{8}(\partial_aA^a)^2+\frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab}-\xcancel{\frac{i}{4}A^2\partial_c(A^c)}+\cancel{\frac{i}{8}\gamma^{ab}A^2F_{ab}}-\frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd}+\cancel{\frac{1}{8}A^4}+\\
&+\frac{i}{12}\Box(\partial_sA^s)-\frac{i}{24}\gamma^{cd}\Box(F_{cd})-\frac{1}{12}\Box(A^2)-\frac{1}{12}\gamma^{ac}\partial^b(A_c)F_{ab}+\frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca})+\\
&\qquad+\frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd}-\frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd})-\underbrace{\cancel{\frac{i}{12}\gamma^{ac}A^2F_{ac}}}_{c\to b}+\\
&-\frac{1}{48}F_{ab}F^{ab}-\frac{1}{12}\gamma^{ac}\partial^b(A_c)F_{ab}+\frac{i}{12}\gamma^{cb}A^aA_cF_{ab}-\cancel{\frac{i}{24}\gamma^{ab}A^2F_{ab}}+\frac{1}{8}\partial_b(A_c)\partial^b(A^c)+\\
&\quad+\frac{1}{24}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d)-iA_aA_c\partial_b(A_d)-iA_bA_d\partial_a(A_c)\Big]-\cancel{\frac{1}{8}A^4}+\\
&\qquad+\underbrace{\xcancel{\frac{i}{4}A^2\partial^d(A_d)}}_{d\to c}-\frac{i}{4}A_aA_d\partial^a(A^d)=\\
&=-\frac{1}{8}(\partial_aA^a)^2+\frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab}-\frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd}+\frac{i}{12}\Box(\partial_sA^s)-\frac{i}{24}\gamma^{cd}\Box(F_{cd})+\\
&-\frac{1}{12}\Box(A^2)-\frac{1}{6}\gamma^{ac}\partial^b(A_c)F_{ab}+\frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca})+\frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd}+\\
&-\frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd})-\frac{1}{48}F_{ab}F^{ab}+\frac{i}{12}\gamma^{cb}A^aA_cF_{ab}+\frac{1}{8}\partial_b(A_c)\partial^b(A^c)+\\
&+\frac{1}{24}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d)-iA_aA_c\partial_b(A_d)-iA_bA_d\partial_a(A_c)\Big]-\frac{i}{4}A_aA_d\partial^a(A^d). \qquad \text{(C.12)}
\end{aligned}
$$

This concludes appendix C.

Appendix D

Right−sector pseudo prescription

In section 4.3 we pointed out that we do lack a proper rule allowing us to move from the left to the right part of the model and vice versa. Even so, by studying the relative signs of operators having the same role within the left and right part of our model (such as W_a and Z_a, or V and U), that task could still be accomplished. Thus, let's start working right away.

It's clear that V (4.10) and U (4.28) only differ in the sign of the term containing the divergence $(\partial_a A^a)$, therefore if we evaluated U^2 the result would be the same as V^2 (4.11), except that every $(\partial_a A^a)$−term entails a change of sign:

$$V^2 = -\frac{1}{4}\underbrace{(\partial_a A^a)^2}_{(-1)^2} - \frac{1}{4}\gamma^{ab}\underbrace{\partial_c(A^c)}_{(-1)}F_{ab} + \frac{i}{2}A^2\underbrace{\partial_c(A^c)}_{(-1)} + \frac{i}{4}\gamma^{ab}A^2F_{ab} - \frac{1}{16}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} + \frac{1}{4}A^4$$
$$\Downarrow$$
$$U^2 = -\frac{1}{4}(\partial_a A^a)^2 + \frac{1}{4}\gamma^{ab}\partial_c(A^c)F_{ab} - \frac{i}{2}A^2\partial_c(A^c) + \frac{i}{4}\gamma^{ab}A^2F_{ab} - \frac{1}{16}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} + \frac{1}{4}A^4\ . \qquad \text{(D.1)}$$

Similar remarks may be made about the computation of $\nabla_a U$, $\nabla^2 U$, $\mathcal{E}_{ab}$ and $\mathcal{E}_{ab}\mathcal{E}^{ab}$, too. In these cases, we also need to keep in mind the sign modification of the contracted term $\gamma^{ab}A_b$, that occurs while going from W_a (4.9) to Z_a (4.27). This means that an expression for

$$\nabla_a U = \partial_a U + [Z_a, U]$$

is straight out deduced from that of $\nabla_a V$ (4.12), by jointly considering the changes of sign affecting both, the terms including the usual divergence $(\partial_a A^a)$ as well as those involving a gauge vector A^a contracted to the matrix γ_{ab}; this contraction might be direct, as in $\gamma^{ab}A_b$, or indirect through some other tensors[1], as it happens in $\gamma_{ca}A_dF^{cd}$. All of this brings about

$$\nabla_a V = \frac{i}{2}\partial_a(\underbrace{\partial_s A^s}_{(-1)}) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) + \frac{1}{2}\underbrace{\gamma_{bc}A^bF^c{}_a}_{(-1)} + \frac{1}{2}\underbrace{\gamma_{ca}A_dF^{cd}}_{(-1)}$$
$$\Downarrow$$
$$\nabla_a U = -\frac{i}{2}\partial_a(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\partial_a(F_{cd}) + \frac{1}{2}\partial_a(A^2) - \frac{1}{2}\gamma_{bc}A^bF^c{}_a - \frac{1}{2}\gamma_{ca}A_dF^{cd}. \qquad \text{(D.2)}$$

[1]This is due to the commutator operation, that acts mixing the indices.

Moreover, the exactly same rule also authorizes the extraction of $\nabla^2 U$ from $\nabla^2 V$ (4.13), so that

$$\begin{aligned}\nabla^2 U = -\frac{i}{2}\Box(\partial_s A^s) + \frac{i}{4}\gamma^{cd}\Box(F_{cd}) + \frac{1}{2}\Box(A^2) - \frac{1}{2}\gamma^{ca}\partial^b(A_c)F_{ab} - \gamma^{bc}A_b\partial^a(F_{ca}) + \\ - \frac{1}{2}\gamma^{ca}\partial_a(A^d)F_{cd} + \gamma^{ac}A^d\partial_a(F_{cd}) + \frac{i}{2}\gamma^{ac}A^2F_{ac}\ .\end{aligned} \tag{D.3}$$

De facto, we could object that the kind of contraction mentioned just above shows up every time something like

$$\gamma_{ab}F^{ab}O^{cd\dots}_{\ \ ef\dots} \quad \text{or} \quad \gamma_{ab}F^{sb}O^{cd\dots}_{\ \ ef\dots} \quad \text{(etc.)} \tag{D.4}$$

appears, since the definition (2.6) of F^{ab} engages the gauge four-potential A^b itself. But we must not be mistaken! We recall that the "rule" we are adopting right here is neither a proper prescription nor some sort of correspondence principle. We built it ad hoc, based on the sign inequalities existing between the graphical expressions of $\lambda-$ and $\rho-$quantities playing the same function. Therefore, for our purposes, F_{ab} and A_a are to be considered as two distinct basic tensors, completely indipendent of each other. It's in this context that the contractions in (D.4) are exempt from any sign transmutations.

An entirely different case is the one regarding the determination of

$$\mathcal{E}_{ab} = \partial_a Z_b - \partial_b Z_a + [Z_a, Z_b] \tag{D.5}$$

and $\mathcal{E}_{ab}\mathcal{E}^{ab}$. In fact, despite the several appearances of Z_a (4.27) throughout its definition, $\mathcal{E}_{ab}$ doesn't trivially descend from an overall sign change of every $\gamma^{ab}A_b-$term inside $\mathcal{F}_{ab}$ (4.14), as we could expect. Or rather, this does happen, but the commutator of Z_a with itself, satisfying

$$[Z_a, Z_b] = [W_a, W_b],$$

suppresses any sign mutations except those involving $\gamma^{ab}A_b-$terms appearing inside the derivative part of (D.5)[2]. In other words, this time the sign switches affect solely $\partial_c(\gamma^{ab}A_b)-$kind terms, producing

$$\mathcal{F}_{ab} = -\frac{i}{2}F_{ab} - \frac{i}{2}\underbrace{\gamma_{bc}\partial_a(A^c)}_{(-1)} + \frac{i}{2}\underbrace{\gamma_{ac}\partial_b(A^c)}_{(-1)} - \frac{1}{2}\gamma_{ac}A_bA^c + \frac{1}{2}\gamma_{bc}A_aA^c + \frac{1}{2}\gamma_{ab}A^cA_c$$

$$\Downarrow$$

$$\mathcal{E}_{ab} = -\frac{i}{2}F_{ab} + \frac{i}{2}\gamma_{bc}\partial_a(A^c) - \frac{i}{2}\gamma_{ac}\partial_b(A^c) - \frac{1}{2}\gamma_{ac}A_bA^c + \frac{1}{2}\gamma_{bc}A_aA^c + \frac{1}{2}\gamma_{ab}A^cA_c\ . \tag{D.6}$$

Of course, the same applies to $\mathcal{E}_{ab}\mathcal{E}^{ab}$ as well, the only difference is that, in addressing its derivation from $\mathcal{F}_{ab}\mathcal{F}^{ab}$ (4.15), we must extend the sign inversion to all derivatives of A^a, as a result of the

[2]Those arising from $\partial_a Z_b - \partial_b Z_a$, just to be clear.

complete contraction process:

$$\mathcal{F}_{ab}\mathcal{F}^{ab} = -\frac{1}{4}F_{ab}F^{ab} + \underbrace{\gamma^{ac}\partial^b(A_c)}_{(-1)}F_{ab} + i\gamma^{cb}A^aA_cF_{ab} - \frac{i}{2}\gamma^{ab}A^2F_{ab} + \frac{3}{2}\underbrace{\partial_b(A_c)\partial^b(A^c)}_{(-1)^2} +$$
$$+ \frac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\underbrace{\partial_a(A_c)\partial_b(A_d)}_{(-1)^2} + iA_aA_c\underbrace{\partial_b(A_d)}_{(-1)} + iA_bA_d\underbrace{\partial_a(A_c)}_{(-1)}\Big] - \frac{3}{2}A^4 +$$
$$- 3i\Big[A^2\underbrace{\partial^d(A_d)}_{(-1)} - A_aA_d\underbrace{\partial^a(A^d)}_{(-1)}\Big]$$
$$\Downarrow$$
$$\mathcal{E}_{ab}\mathcal{E}^{ab} = -\frac{1}{4}F_{ab}F^{ab} - \gamma^{ac}\partial^b(A_c)F_{ab} + i\gamma^{cb}A^aA_cF_{ab} - \frac{i}{2}\gamma^{ab}A^2F_{ab} + \frac{3}{2}\partial_b(A_c)\partial^b(A^c) +$$
$$+ \frac{1}{2}\gamma^{bc}\gamma^{ad}\Big[\partial_a(A_c)\partial_b(A_d) - iA_aA_c\partial_b(A_d) - iA_bA_d\partial_a(A_c)\Big] - \frac{3}{2}A^4 +$$
$$+ 3i\Big[A^2\partial^d(A_d) - A_aA_d\partial^a(A^d)\Big]. \tag{D.7}$$

Our intended purposes may be deemed fulfilled.

Appendix E

Null traces of section 4.5

The main objective of appendix E is to prove the validity of eq. (4.43). Of course, when tracing $b_{a,2}(x,R)$, we are forced to rely upon its matrix structure (4.6), where the exact projector always accompanies the coefficients defined in each sector. Therefore, we have

$$\mathrm{tr}[b_{a,2}(x,R)] = \mathrm{tr}[b_{a,2}(x,R_\lambda)P_L] + \mathrm{tr}[b_{a,2}(x,R_{\lambda_c})P_R] + \mathrm{tr}[b_{a,2}(x,R_\rho)P_R] + \mathrm{tr}[b_{a,2}(x,R_{\rho_c})P_L] \ , \quad \text{(E.1)}$$

With that in mind, it's easy for us to show that, given two operators A and B, the trace of the product between their commutator ($[A,B]$) and some projector (P) does vanish as long as

$$[A,P] = 0 \quad \text{(E.2)}$$

is respected, namely

$$\begin{aligned} \mathrm{tr}\big[[A,B]P\big] &= \mathrm{tr}\big[ABP - BAP\big] = \mathrm{tr}\big[ABP\big] - \mathrm{tr}\big[B\underbrace{AP}_{\text{(E.2)}}\big] = \\ &= \mathrm{tr}\big[ABP\big] - \underbrace{\mathrm{tr}\big[BPA\big]}_{\text{(B.16)}} = \\ &= 0 \ . \end{aligned} \quad \text{(E.3)}$$

Thus, if we explicitly wrote down all the four terms appearing in the last of (3.27)

$$\begin{aligned} b_{a,2}(x,R_\lambda) = -\frac{1}{45}\nabla^i\nabla_k\nabla^k\big(\mathcal{F}_{ia}(x)\big) + \frac{1}{180}\nabla^i\nabla_a\nabla^j\big(\mathcal{F}_{ij}(x)\big) - \frac{1}{180}\nabla^k\nabla_k\nabla^j\big(\mathcal{F}_{aj}(x)\big) + \\ + \frac{1}{6}\nabla^i\big(V(x)\mathcal{F}_{ia}(x)\big) \ , \end{aligned}$$

we would have:

$$\begin{aligned} -\frac{1}{45}\nabla^i\nabla_k\nabla^k\big(\mathcal{F}_{ia}\big) &= -\frac{1}{45}\nabla^i\nabla_k\big(\partial^k\mathcal{F}_{ia} + [W^k,\mathcal{F}_{ia}]\big) = \\ &= -\frac{1}{45}\nabla^i\big(\Box\mathcal{F}_{ia} + \partial_k[W^k,\mathcal{F}_{ia}] + [W_k,\partial^k\mathcal{F}_{ia}] + \big[W_k,[W^k,\mathcal{F}_{ia}]\big]\big) = \\ &= -\frac{1}{45}\Big(\partial^i\Box\mathcal{F}_{ia} + \underbrace{\partial^i\partial_k[W^k,\mathcal{F}_{ia}]}_{\text{null trace term}} + \underbrace{\partial^i[W_k,\partial^k\mathcal{F}_{ia}]}_{\text{null trace term}} + \underbrace{\partial^i\big[W_k,[W^k,\mathcal{F}_{ia}]\big]}_{\text{null trace term}} + \end{aligned}$$

$$+\underbrace{[W^i,\Box\mathcal{F}_{ia}]}_{\text{null trace term}}+\underbrace{[W^i,\partial_k[W^k,\mathcal{F}_{ia}]]}_{\text{null trace term}}+\underbrace{[W^i,[W_k,\partial^k\mathcal{F}_{ia}]]}_{\text{null trace term}}+$$
$$+\underbrace{\Big[W^i,\big[W_k,[W^k,\mathcal{F}_{ia}]\big]\Big]}_{\text{null trace term}}\Big)=$$
$$=-\frac{1}{45}\partial^i\Box\mathcal{F}_{ia}\;; \tag{E.4}$$

$$\frac{1}{180}\nabla^i\nabla_a\nabla^j\big(\mathcal{F}_{ij}\big)=\frac{1}{180}\nabla^i\nabla_a\big(\partial^j\mathcal{F}_{ij}+[W^j,\mathcal{F}_{ij}]\big)=$$
$$=\frac{1}{180}\nabla^i\big(\partial_a\partial^j\mathcal{F}_{ij}+\partial_a[W^j,\mathcal{F}_{ij}]+[W_a,\partial^j\mathcal{F}_{ij}]+\big[W_a,[W^j,\mathcal{F}_{ij}]\big]\big)=$$
$$=\frac{1}{180}\Big(\underbrace{\partial^i\partial_a\partial^j}_{\text{sym in }ij}\mathcal{F}_{ij}+\underbrace{\partial^i\partial_a[W^j,\mathcal{F}_{ij}]}_{\text{null trace term}}+\underbrace{\partial^i[W_a,\partial^j\mathcal{F}_{ij}]}_{\text{null trace term}}+\underbrace{\partial^i\big[W_a,[W^j,\mathcal{F}_{ij}]\big]}_{\text{null trace term}}+$$
$$+\underbrace{[W^i,\partial_a\partial^j\mathcal{F}_{ij}]}_{\text{null trace term}}+\underbrace{[W^i,\partial_a[W^j,\mathcal{F}_{ij}]]}_{\text{null trace term}}+\underbrace{[W^i,[W_a,\partial^j\mathcal{F}_{ij}]]}_{\text{null trace term}}+$$
$$+\underbrace{\Big[W^i,\big[W_a,[W^j,\mathcal{F}_{ij}]\big]\Big]}_{\text{null trace term}}\Big)=$$
$$=0\;; \tag{E.5}$$

$$-\frac{1}{180}\nabla^k\nabla_k\nabla^j\big(\mathcal{F}_{aj}\big)=-\frac{1}{180}\nabla^k\nabla_k\big(\partial^j\mathcal{F}_{aj}+[W^j,\mathcal{F}_{aj}]\big)=$$
$$=-\frac{1}{180}\nabla^k\big(\partial_k\partial^j\mathcal{F}_{aj}+\partial_k[W^j,\mathcal{F}_{aj}]+[W_k,\partial^j\mathcal{F}_{aj}]+\big[W_k,[W^j,\mathcal{F}_{aj}]\big]\big)=$$
$$=-\frac{1}{180}\Big(\underbrace{\Box\partial^j\mathcal{F}_{aj}}_{j\to i}+\underbrace{\Box[W^j,\mathcal{F}_{aj}]}_{\text{null trace term}}+\underbrace{\partial^k[W_k,\partial^j\mathcal{F}_{aj}]}_{\text{null trace term}}+\underbrace{\partial^k\big[W_k,[W^j,\mathcal{F}_{aj}]\big]}_{\text{null trace term}}+$$
$$+\underbrace{[W^k,\partial_k\partial^j\mathcal{F}_{aj}]}_{\text{null trace term}}+\underbrace{[W^k,\partial_k[W^j,\mathcal{F}_{aj}]]}_{\text{null trace term}}+\underbrace{[W^k,[W_k,\partial^j\mathcal{F}_{aj}]]}_{\text{null trace term}}+$$
$$+\underbrace{\Big[W^k,\big[W_k,[W^j,\mathcal{F}_{aj}]\big]\Big]}_{\text{null trace term}}\Big)=$$
$$=\frac{1}{180}\Box\partial^i\mathcal{F}_{ia}\;; \tag{E.6}$$

$$\frac{1}{6}\nabla^i\big(V\mathcal{F}_{ia}\big)=\frac{1}{6}\big(\partial^i(V\mathcal{F}_{ia})+\underbrace{[W^i,V\mathcal{F}_{ia}]}_{\text{null trace term}}\big)=$$
$$=\frac{1}{6}\partial^i(V\mathcal{F}_{ia})\;; \tag{E.7}$$

since all over (E.4)-(E.7) the underbraced operators do adhere to (E.2), they don't survive the trace operation (E.1). In any case, the surviving ones sum up to

$$b_{a,2}(x,R_\lambda)=-\frac{1}{60}\Box\partial^i\mathcal{F}_{ia}+\frac{1}{6}\partial^i(V\mathcal{F}_{ia})\;. \tag{E.8}$$

Nothing more need to be added.

Appendix F

Traces involving heat kernel coefficients

Considering the importance of the SDW coefficients, especially their massive use throughout the anomalous expectation values of chapter 5, we chose to gather in the present appendix the step by step computation of those traces defining the anomalies themselves. They will be presented by following the order of their appearance in chap. 5. In the following, as will be noted, we opted to restore the spinor identity operator wherever needed: this should enhance the clarity and readability of our work.

F.1 Trace and chiral anomaly

Therefore, let's start with the first trace showing up in sections 5.1 and 5.2:

$$
\begin{aligned}
\mathrm{tr}[a_2(x,R_\lambda)P_L] &\overset{\text{(B.16)}}{=} \mathrm{tr}[P_L a_2(x,R_\lambda)] = \\
&= \mathrm{tr}\Big\{\Big[\frac{\mathbb{1}+\gamma^5}{2}\Big]\Big[-\frac{1}{8}(\mathbb{1})(\partial_a A^a)^2 - \frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab} - \frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd} + \\
&\quad - \frac{i}{12}(\mathbb{1})\Box(\partial_s A^s) - \frac{i}{24}\gamma^{cd}\Box(F_{cd}) - \frac{1}{12}(\mathbb{1})\Box(A^2) + \frac{1}{6}\gamma^{ac}\partial^b(A_c)F_{ab} + \\
&\quad - \frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca}) - \frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd} + \frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd}) - \frac{1}{48}(\mathbb{1})F_{ab}F^{ab} + \\
&\quad + \frac{i}{12}\gamma^{cb}A^aA_cF_{ab} + \frac{1}{8}(\mathbb{1})\partial_b(A_c)\partial^b(A^c) + \frac{i}{4}(\mathbb{1})A_aA_d\partial^a(A^d) + \\
&\quad + \frac{1}{24}\gamma^{bc}\gamma^{ad}\Big(\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big)\Big]\Big\} = \\
\begin{pmatrix}\text{we can use (B.18),}\\ \text{(B.23) and (B.24)}\\ \text{to neglect all}\\ \text{the null terms}\end{pmatrix} &= -\frac{1}{16}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}(\partial_a A^a)^2 - \frac{1}{64}\underbrace{\mathrm{tr}[\gamma^{ab}\gamma^{cd}]}_{\text{(B.33)}}F_{ab}F_{cd} - \frac{i}{24}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}\Box(\partial_s A^s) - \frac{1}{24}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}\Box(A^2) + \\
&\quad - \frac{1}{96}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}F_{ab}F^{ab} + \frac{1}{16}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}\partial_b(A_c)\partial^b(A^c) + \frac{i}{8}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}A_aA_d\partial^a(A^d) + \\
&\quad + \frac{1}{48}\underbrace{\mathrm{tr}[\gamma^{bc}\gamma^{ad}]}_{\text{(B.33)}}\Big(\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big) +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{48}\underbrace{\mathrm{tr}[\gamma^5\gamma^{bc}\gamma^{ad}]}_{\text{(B.34)}}\Big(\partial_a(A_c)\partial_b(A_d)+iA_aA_c\partial_b(A_d)+iA_bA_d\partial_a(A_c)\Big)+\\
&-\frac{1}{64}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ab}\gamma^{cd}]}_{\text{(B.34)}}F_{ab}F_{cd}=\\
=&-\frac{1}{4}(\partial_aA^a)^2-\frac{1}{16}(g^{ad}g^{bc}-g^{ac}g^{bd})F_{ab}F_{cd}-\frac{i}{6}\Box(\partial_sA^s)-\frac{1}{6}\Box(A^2)+\\
&-\frac{1}{24}F_{ab}F^{ab}+\frac{1}{4}\partial_b(A_c)\partial^b(A^c)+\frac{i}{2}A_aA_d\partial^a(A^d)+\\
&+\frac{1}{12}(g^{bd}g^{ac}-g^{ba}g^{cd})\Big(\partial_a(A_c)\partial_b(A_d)+iA_aA_c\partial_b(A_d)+iA_bA_d\partial_a(A_c)\Big)+\\
&+\frac{i}{12}\underbrace{\epsilon^{bcad}}_{\text{fully antisym.}}\Big(\partial_a(A_c)\partial_b(A_d)+\cancel{iA_aA_c\partial_b(A_d)}+\cancel{iA_bA_d\partial_a(A_c)}\Big)+\\
&-\frac{i}{16}\epsilon^{abcd}F_{ab}F_{cd}=\\
=&-\frac{1}{4}(\partial_aA^a)^2-\frac{1}{16}F_{ab}F^{ba}+\frac{1}{16}F_{ab}F^{ab}-\frac{i}{6}\Box(\partial_sA^s)-\frac{1}{6}\Box(A^2)+\\
&-\frac{1}{24}F_{ab}F^{ab}+\frac{1}{4}\partial_b(A_c)\partial^b(A^c)+\frac{i}{2}A_aA_d\partial^a(A^d)+\\
&+\frac{1}{12}\big(\partial_aA^a\big)^2-\frac{1}{12}\partial_b(A_c)\partial^b(A^c)+\frac{i}{12}A^2\partial_a(A^a)+\\
&\qquad-\frac{i}{12}A_aA_d\partial^a(A^d)+\frac{i}{12}A^2\partial_a(A^a)-\frac{i}{12}A_aA_d\partial^a(A^d)+\\
&+\frac{i}{12}\epsilon^{bcad}\underbrace{\underbrace{\partial_a(A_c)}\,\underbrace{\partial_b(A_d)}}_{\text{only their antisym. part will contrib}}-\frac{i}{16}\epsilon^{abcd}F_{ab}F_{cd}=\\
=&-\frac{1}{6}(\partial_aA^a)^2+\frac{1}{12}F_{ab}F^{ab}-\frac{i}{6}\Box(\partial_sA^s)-\frac{1}{6}\Box(A^2)+\frac{1}{6}\partial_b(A_c)\partial^b(A^c)+\\
&\underbrace{+\frac{i}{3}A_aA_d\partial^a(A^d)+\frac{i}{6}A^2\partial_a(A^a)}_{\text{Leibniz's rule}}+\frac{i}{48}\underbrace{\epsilon^{bcad}F_{ac}F_{bd}}_{c\to b;\ b\to c}-\frac{i}{16}\epsilon^{abcd}F_{ab}F_{cd}=\\
=&\ \frac{1}{6}\Big(-(\partial_aA^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+\\
&+\frac{i}{6}\Big(-\Box(\partial_sA^s)+\partial_a(A^2A^a)-\frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big).
\end{aligned}
\tag{F.1}
$$

Then there is $\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]$, easily acquired from (F.1) by virtue of the combined action of the customary $c-$prescription (4.17) and the projector transmutation

$$
P_L=\frac{\mathbb{1}+\gamma^5}{2}\longrightarrow P_R=\frac{\mathbb{1}-\gamma^5}{2}\ ,
$$

that basically acts by modifying the sign of each term coming from a trace involving the γ^5 matrix, namely by mapping the Levi-Civita tensor in its opposite

$$
\epsilon^{abcd}\longrightarrow-\epsilon^{abcd}\ . \tag{F.2}
$$

This enables us to write

$$
\begin{aligned}
\mathrm{tr}[a_2(x,R_\lambda)P_L] &= \frac{1}{6}\Big(-\underbrace{(\partial_a A^a)^2}_{(-1)^2}+\frac{1}{2}\underbrace{F_{ab}F^{ab}}_{(-1)^2}-\Box\underbrace{(A^2)}_{(-1)^2}+\underbrace{\partial_b(A_c)\partial^b(A^c)}_{(-1)^2}\Big)+\\
&\quad+\frac{i}{6}\Big(-\Box\underbrace{(\partial_s A^s)}_{(-1)}+\underbrace{\partial_a(A^2A^a)}_{(-1)^3}-\frac{1}{2}\underbrace{\epsilon^{abcd}F_{ab}F_{cd}}_{(-1)^3}\Big)\\
&\Downarrow\\
\mathrm{tr}[a_2(x,R_{\lambda_c})P_R] &= \frac{1}{6}\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+\\
&\quad-\frac{i}{6}\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big).
\end{aligned}
\tag{F.3}
$$

Now it's the turn of $\mathrm{tr}[a_2(x,R_\rho)P_R]$, which must be computed from scratch. Fortunately, its calculation doesn't differ much from that of (F.1):

$$
\begin{aligned}
\mathrm{tr}[a_2(x,R_\rho)P_R] &\overset{(B.16)}{=} \mathrm{tr}[P_R a_2(x,R_\rho)] =\\
&= \mathrm{tr}\Big\{\Big[\frac{\mathbb{1}-\gamma^5}{2}\Big]\Big[-\frac{1}{8}(\mathbb{1})(\partial_a A^a)^2+\frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab}-\frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd}+\\
&\quad+\frac{i}{12}(\mathbb{1})\Box(\partial_s A^s)-\frac{i}{24}\gamma^{cd}\Box(F_{cd})-\frac{1}{12}(\mathbb{1})\Box(A^2)-\frac{1}{6}\gamma^{ac}\partial^b(A_c)F_{ab}+\\
&\quad+\frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca})+\frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd}-\frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd})-\frac{1}{48}(\mathbb{1})F_{ab}F^{ab}+\\
&\quad+\frac{i}{12}\gamma^{cb}A^aA_cF_{ab}+\frac{1}{8}(\mathbb{1})\partial_b(A_c)\partial^b(A^c)-\frac{i}{4}(\mathbb{1})A_aA_d\partial^a(A^d)+\\
&\quad+\frac{1}{24}\gamma^{bc}\gamma^{ad}\Big(\partial_a(A_c)\partial_b(A_d)-iA_aA_c\partial_b(A_d)-iA_bA_d\partial_a(A_c)\Big)\Big]\Big\} =\\
\begin{pmatrix}\text{we can use (B.18),}\\ \text{(B.23) and (B.24)}\\ \text{to neglect all}\\ \text{the null terms}\end{pmatrix}\quad &= -\frac{1}{16}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}(\partial_a A^a)^2-\frac{1}{64}\underbrace{\mathrm{tr}[\gamma^{ab}\gamma^{cd}]}_{(B.33)}F_{ab}F_{cd}+\frac{i}{24}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}\Box(\partial_s A^s)-\frac{1}{24}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}\Box(A^2)+\\
&\quad-\frac{1}{96}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}F_{ab}F^{ab}+\frac{1}{16}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}\partial_b(A_c)\partial^b(A^c)-\frac{i}{8}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}A_aA_d\partial^a(A^d)+\\
&\quad+\frac{1}{48}\underbrace{\mathrm{tr}[\gamma^{bc}\gamma^{ad}]}_{(B.33)}\Big(\partial_a(A_c)\partial_b(A_d)-iA_aA_c\partial_b(A_d)-iA_bA_d\partial_a(A_c)\Big)+\\
&\quad-\frac{1}{48}\underbrace{\mathrm{tr}[\gamma^5\gamma^{bc}\gamma^{ad}]}_{(B.34)}\Big(\partial_a(A_c)\partial_b(A_d)-iA_aA_c\partial_b(A_d)-iA_bA_d\partial_a(A_c)\Big)+\\
&\quad+\frac{1}{64}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ab}\gamma^{cd}]}_{(B.34)}F_{ab}F_{cd} =\\
&= -\frac{1}{4}(\partial_a A^a)^2-\frac{1}{16}(g^{ad}g^{bc}-g^{ac}g^{bd})F_{ab}F_{cd}+\frac{i}{6}\Box(\partial_s A^s)-\frac{1}{6}\Box(A^2)+\\
&\quad-\frac{1}{24}F_{ab}F^{ab}+\frac{1}{4}\partial_b(A_c)\partial^b(A^c)-\frac{i}{2}A_aA_d\partial^a(A^d)+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{12}(g^{bd}g^{ac}-g^{ba}g^{cd})\Big(\partial_a(A_c)\partial_b(A_d)-iA_aA_c\partial_b(A_d)-iA_bA_d\partial_a(A_c)\Big)+\\
&-\tfrac{i}{12}\underbrace{\epsilon^{bcad}}_{\text{fully antisym.}}\Big(\partial_a(A_c)\partial_b(A_d)-\cancel{iA_aA_c\partial_b(A_d)}-\cancel{iA_bA_d\partial_a(A_c)}\Big)+\\
&+\frac{i}{16}\epsilon^{abcd}F_{ab}F_{cd}=\\
=&-\frac{1}{4}(\partial_aA^a)^2-\frac{1}{16}F_{ab}F^{ba}+\frac{1}{16}F_{ab}F^{ab}+\frac{i}{6}\Box(\partial_sA^s)-\frac{1}{6}\Box(A^2)+\\
&-\frac{1}{24}F_{ab}F^{ab}+\frac{1}{4}\partial_b(A_c)\partial^b(A^c)-\frac{i}{2}A_aA_d\partial^a(A^d)+\\
&+\frac{1}{12}\big(\partial_aA^a\big)^2-\frac{1}{12}\partial_b(A_c)\partial^b(A^c)-\frac{i}{12}A^2\partial_a(A^a)+\\
&\qquad+\frac{i}{12}A_aA_d\partial^a(A^d)-\frac{i}{12}A^2\partial_a(A^a)+\frac{i}{12}A_aA_d\partial^a(A^d)+\\
&-\frac{i}{12}\epsilon^{bcad}\underbrace{\underbrace{\partial_a(A_c)}\,\underbrace{\partial_b(A_d)}}_{\text{only their antisym. part will contrib}}+\frac{i}{16}\epsilon^{abcd}F_{ab}F_{cd}=\\
=&-\frac{1}{6}(\partial_aA^a)^2+\frac{1}{12}F_{ab}F^{ab}+\frac{i}{6}\Box(\partial_sA^s)-\frac{1}{6}\Box(A^2)+\frac{1}{6}\partial_b(A_c)\partial^b(A^c)+\\
&\underbrace{-\frac{i}{3}A_aA_d\partial^a(A^d)-\frac{i}{6}A^2\partial_a(A^a)}_{\text{Leibniz's rule}}-\frac{i}{48}\underbrace{\epsilon^{bcad}F_{ac}F_{bd}}_{c\to b;\ b\to c}+\frac{i}{16}\epsilon^{abcd}F_{ab}F_{cd}=\\
=&\ \frac{1}{6}\Big(-(\partial_aA^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+\\
&+\frac{i}{6}\Big(\Box(\partial_sA^s)-\partial_a(A^2A^a)+\frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big).
\end{aligned}
\tag{F.4}
$$

Of course, for the last missing contribution to the trace anomaly, that is $\mathrm{tr}[a_2(x,R_{\rho_c})P_L]$, the reasoning is the same as before: we jointly apply the two maps (F.2) and (4.17) to (F.4), getting

$$
\begin{aligned}
\mathrm{tr}[a_2(x,R_\rho)P_R]=&\ \frac{1}{6}\Big(-\underbrace{(\partial_aA^a)^2}_{(-1)^2}+\frac{1}{2}\underbrace{F_{ab}F^{ab}}_{(-1)^2}-\Box\underbrace{(A^2)}_{(-1)^2}+\underbrace{\partial_b(A_c)\partial^b(A^c)}_{(-1)^2}\Big)+\\
&+\frac{i}{6}\Big(\Box\underbrace{(\partial_sA^s)}_{(-1)}-\underbrace{\partial_a(A^2A^a)}_{(-1)^3}+\frac{1}{2}\underbrace{\epsilon^{abcd}F_{ab}F_{cd}}_{(-1)^3}\Big)\\
&\Downarrow\\
\mathrm{tr}[a_2(x,R_{\rho_c})P_L]=&\ \frac{1}{6}\Big(-(\partial_aA^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+\\
&-\frac{i}{6}\Big(\Box(\partial_sA^s)-\partial_a(A^2A^a)+\frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big).
\end{aligned}
\tag{F.5}
$$

The combination of the previous results is first used to infer the trace anomaly,

$$
\langle T^a{}_a\rangle=\Big\{\frac{3-4\alpha}{2(4\pi)^2}\Big(\underbrace{\mathrm{tr}[a_2(x,R_\lambda)P_L]}_{\text{(F.1)}}+\underbrace{\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]}_{\text{(F.3)}}\Big)+
$$

$$
-\frac{2\alpha}{(4\pi)^2}\Big(\underbrace{\mathrm{tr}[a_2(x,R_\rho)P_R]}_{(F.4)}+\underbrace{\mathrm{tr}[a_2(x,R_{\rho_c})P_L]}_{(F.5)}\Big)\Big\} =
$$

$$
\begin{aligned}
&= \frac{3-4\alpha}{2(4\pi)^2}\Big[\frac{1}{6}\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+\\
&\quad+\frac{i}{6}\Big(\cancel{-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}}\Big)+\\
&\quad+\frac{1}{6}\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+\\
&\quad-\frac{i}{6}\Big(\cancel{-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}}\Big)\Big]+\\
&\quad-\frac{2\alpha}{(4\pi)^2}\Big[\frac{1}{6}\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+\\
&\quad+\frac{i}{6}\Big(\cancel{\Box(\partial_s A^s)-\partial_a(A^2A^a)+\tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}}\Big)+\\
&\quad+\frac{1}{6}\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+\\
&\quad-\frac{i}{6}\Big(\cancel{\Box(\partial_s A^s)-\partial_a(A^2A^a)+\tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}}\Big)\Big] =\\
&= \frac{3-4\alpha}{2(4\pi)^2}\Big[\frac{1}{3}\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)\Big]+\\
&\quad-\frac{2\alpha}{(4\pi)^2}\Big[\frac{1}{3}\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)\Big] =\\
&= \frac{3-8\alpha}{6(4\pi)^2}\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A)^2+(\partial_a A_b)(\partial^a A^b)\Big)\ ,
\end{aligned} \tag{F.6}
$$

and then the chiral one

$$
\begin{aligned}
\langle\, \partial_\mu\left(i\overline{\lambda}\gamma^\mu\lambda\right)\,\rangle &= \frac{i}{(4\pi)^2}\Big(\underbrace{\mathrm{tr}[a_2(x,R_\lambda)P_L]}_{(F.1)}-\underbrace{\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]}_{(F.3)}\Big) =\\
&= \frac{i}{(4\pi)^2}\Big[\frac{1}{6}\Big(\cancel{-(\partial_a A^a)^2+\tfrac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)}\Big)+\\
&\quad+\frac{i}{6}\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)+\\
&\quad-\frac{1}{6}\Big(\cancel{-(\partial_a A^a)^2+\tfrac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)}\Big)+\\
&\quad+\frac{i}{6}\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)\Big] =\\
&= \frac{1}{3(4\pi)^2}\Big(\Box(\partial_s A^s)-\partial_a(A^2A^a)+\frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)\ ,
\end{aligned} \tag{F.7}
$$

F.2 Lorentz anomaly

In this case there are only two trace terms that make a contribution to the Lorentz anomaly, as we can see from its defining expression in section 5.3, that is $\mathrm{tr}[\gamma^{ef}a_2(x,R_\lambda)P_L]$ and $\mathrm{tr}[\gamma^{ef}a_2(x,R_{\lambda_c})P_R]$.

In particular, we have

$$
\begin{aligned}
\mathrm{tr}[\gamma^{ef}a_2(x,R_\lambda)P_L] &\overset{(B.16)}{=} \mathrm{tr}[P_L\gamma^{ef}a_2(x,R_\lambda)] = \\
&= \mathrm{tr}\Big\{\Big[\frac{\gamma^{ef}+\gamma^5\gamma^{ef}}{2}\Big]\Big[-\frac{1}{8}(\mathbb{1})(\partial_a A^a)^2 - \frac{1}{8}\gamma^{ab}\partial_c(A^c)F_{ab} - \frac{1}{32}\gamma^{ab}\gamma^{cd}F_{ab}F_{cd}+ \\
&\quad -\frac{i}{12}(\mathbb{1})\Box(\partial_s A^s) - \frac{i}{24}\gamma^{cd}\Box(F_{cd}) - \frac{1}{12}(\mathbb{1})\Box(A^2) + \frac{1}{6}\gamma^{ac}\partial^b(A_c)F_{ab}+ \\
&\quad -\frac{1}{6}\gamma^{bc}A_b\partial^a(F_{ca}) - \frac{1}{12}\gamma^{ca}\partial_a(A^d)F_{cd} + \frac{1}{6}\gamma^{ac}A^d\partial_a(F_{cd}) - \frac{1}{48}(\mathbb{1})F_{ab}F^{ab}+ \\
&\quad +\frac{i}{12}\gamma^{cb}A^aA_cF_{ab} + \frac{1}{8}(\mathbb{1})\partial_b(A_c)\partial^b(A^c) + \frac{i}{4}(\mathbb{1})A_aA_d\partial^a(A^d)+ \\
&\quad +\frac{1}{24}\gamma^{bc}\gamma^{ad}\Big(\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big)\Big]\Big\} = \\
\begin{pmatrix}\text{we can use}\\ \text{(B.23) and (B.24)}\\ \text{to neglect all}\\ \text{the null terms}\end{pmatrix}
&= -\frac{1}{16}\underbrace{\mathrm{tr}[\gamma^{ef}\gamma^{ab}]}_{(B.33)}\partial_c(A^c)F_{ab} - \frac{1}{64}\underbrace{\mathrm{tr}[\gamma^{ef}\gamma^{ab}\gamma^{cd}]}_{(B.42)}F_{ab}F_{cd}+ \\
&\quad -\frac{i}{48}\underbrace{\mathrm{tr}[\gamma^{ef}\gamma^{cd}]}_{(B.33)}\Box(F_{cd}) + \frac{1}{12}\underbrace{\mathrm{tr}[\gamma^{ef}\gamma^{ac}]}_{(B.33)}\partial^b(A_c)F_{ab}+ \\
&\quad -\frac{1}{12}\underbrace{\mathrm{tr}[\gamma^{ef}\gamma^{bc}]}_{(B.33)}A_b\partial^a(F_{ca}) - \frac{1}{24}\underbrace{\mathrm{tr}[\gamma^{ef}\gamma^{ca}]}_{(B.33)}\partial_a(A^d)F_{cd}+ \\
&\quad +\frac{1}{12}\underbrace{\mathrm{tr}[\gamma^{ef}\gamma^{ac}]}_{(B.33)}A^d\partial_a(F_{cd}) + \frac{i}{24}\underbrace{\mathrm{tr}[\gamma^{ef}\gamma^{cb}]}_{(B.33)}A^aA_cF_{ab}+ \\
&\quad +\frac{1}{48}\underbrace{\mathrm{tr}[\gamma^{ef}\gamma^{bc}\gamma^{ad}]}_{(B.42)}\Big(\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big)+ \\
&\quad -\frac{1}{16}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ef}\gamma^{ab}]}_{(B.34)}\partial_c(A^c)F_{ab} - \frac{1}{64}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ef}\gamma^{ab}\gamma^{cd}]}_{(B.43)}F_{ab}F_{cd}+ \\
&\quad -\frac{i}{48}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ef}\gamma^{cd}]}_{(B.34)}\Box(F_{cd}) + \frac{1}{12}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ef}\gamma^{ac}]}_{(B.34)}\partial^b(A_c)F_{ab}+ \\
&\quad -\frac{1}{12}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ef}\gamma^{bc}]}_{(B.34)}A_b\partial^a(F_{ca}) - \frac{1}{24}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ef}\gamma^{ca}]}_{(B.34)}\partial_a(A^d)F_{cd}+ \\
&\quad +\frac{1}{12}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ef}\gamma^{ac}]}_{(B.34)}A^d\partial_a(F_{cd}) + \frac{i}{24}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ef}\gamma^{cb}]}_{(B.34)}A^aA_cF_{ab}+ \\
&\quad +\frac{1}{48}\underbrace{\mathrm{tr}[\gamma^5\gamma^{ef}\gamma^{bc}\gamma^{ad}]}_{(B.43)}\Big(\partial_a(A_c)\partial_b(A_d) + iA_aA_c\partial_b(A_d) + iA_bA_d\partial_a(A_c)\Big) = \\
&= -\frac{1}{4}(g^{eb}g^{fa} - g^{ea}g^{fb})\partial_c(A^c)F_{ab} - \frac{1}{16}(g^{ac}g^{be}g^{df} - g^{ac}g^{bf}g^{de} - g^{bc}g^{ae}g^{df}+ \\
&\quad + g^{bc}g^{af}g^{de} - g^{ad}g^{be}g^{cf} + g^{ad}g^{ce}g^{bf} - g^{bd}g^{ce}g^{af} + g^{bd}g^{ae}g^{cf})F_{ab}F_{cd}+
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{i}{12}(g^{ed}g^{fc}-g^{ec}g^{fd})\Box(F_{cd})+\frac{1}{3}(g^{ec}g^{fa}-g^{ea}g^{fc})\partial^b(A_c)F_{ab}+\\
&-\frac{1}{3}(g^{ec}g^{fb}-g^{eb}g^{fc})A_b\partial^a(F_{ca})-\frac{1}{6}(g^{ea}g^{fc}-g^{ec}g^{fa})\partial_a(A^d)F_{cd}+\\
&+\frac{1}{3}(g^{ec}g^{fa}-g^{ea}g^{fc})A^d\partial_a(F_{cd})+\frac{i}{6}(g^{eb}g^{fc}-g^{ec}g^{fb})A^aA_cF_{ab}+\\
&+\frac{1}{12}(+g^{ba}g^{ce}g^{df}-g^{ba}g^{cf}g^{de}-g^{ca}g^{be}g^{df}+g^{ca}g^{bf}g^{de}+\\
&\qquad -g^{bd}g^{ce}g^{af}+g^{bd}g^{ae}g^{cf}-g^{cd}g^{ae}g^{bf}+g^{cd}g^{be}g^{af})\cdot\\
&\qquad\quad\cdot\Big(\partial_a(A_c)\partial_b(A_d)+iA_aA_c\partial_b(A_d)+iA_bA_d\partial_a(A_c)\Big)+\\
&-\frac{i}{4}\epsilon^{efab}\partial_c(A^c)F_{ab}+\frac{i}{16}(g^{ac}\epsilon^{bdef}-g^{bc}\epsilon^{adef}-g^{ad}\epsilon^{bcef}+g^{bd}\epsilon^{acef})F_{ab}F_{cd}+\\
&+\frac{1}{12}\epsilon^{efcd}\Box(F_{cd})+\frac{i}{3}\epsilon^{efac}\partial^b(A_c)F_{ab}-\frac{i}{3}\epsilon^{efbc}A_b\partial^a(F_{ca})+\\
&-\frac{i}{6}\epsilon^{efca}\partial_a(A^d)F_{cd}+\frac{i}{3}\epsilon^{efac}A^d\partial_a(F_{cd})-\frac{1}{6}\epsilon^{efcb}A^aA_cF_{ab}+\\
&-\frac{i}{12}(g^{ba}\epsilon^{cdef}-g^{ca}\epsilon^{bdef}-g^{bd}\epsilon^{caef}+g^{cd}\epsilon^{baef})\cdot\\
&\qquad\cdot\Big(\partial_a(A_c)\partial_b(A_d)+iA_aA_c\partial_b(A_d)+iA_bA_d\partial_a(A_c)\Big)=\\
=&-\frac{1}{4}\partial_c(A^c)(F^{fe}-F^{ef})-\frac{1}{16}\Big(\cancel{F^{ce}F_c{}^f}-\cancel{F_c{}^fF^{ce}}-\cancel{F^{ec}F_c{}^f}+\\
&+\cancel{F^f{}_cF^{ce}}-\cancel{F^{de}F^f{}_d}+\cancel{F_d{}^fF^{ed}}-\cancel{F^{fd}F^e{}_d}+\cancel{F^e{}_dF^{fd}}\Big)+\\
&-\frac{i}{12}\Box(F^{fe}-F^{ef})+\frac{1}{3}\big(\partial^b(A^e)F^f{}_b-\partial^b(A^f)F^e{}_b\big)+\\
&-\frac{1}{3}\big(\underbrace{A^f\partial^a(F^e{}_a)-A^e\partial^a(F^f{}_a)}_{a\to b}\big)-\frac{1}{6}\big(\partial^e(A^d)F^f{}_d-\partial^f(A^d)F^e{}_d\big)+\\
&+\frac{1}{3}\big(A^d\partial^f(F^e{}_d)-A^d\partial^e(F^f{}_d)\big)+\frac{i}{6}\big(A^aA^fF_a{}^e-A^aA^eF_a{}^f\big)+\\
&+\frac{1}{12}\Big(\underbrace{\partial^b(A^e)\partial_b(A^f)-\partial^b(A^f)\partial_b(A^e)}_{0}-\underbrace{\cancel{\partial_a(A^a)\partial^e(A^f)}+\cancel{\partial_a(A^a)\partial^f(A^e)}}_{a\to b}+\\
&\qquad-\cancel{\partial^f(A^e)\partial_b(A^b)}+\cancel{\partial^e(A^f)\partial_b(A^b)}\underbrace{-\partial^e(A^d)\partial^f(A_d)+\partial^f(A^d)\partial^e(A_d)}_{0}+\\
&\qquad+\underbrace{\cancel{iA^bA^e\partial_b(A^f)}-\cancel{iA^bA^f\partial_b(A^e)}}_{b\to a}\underbrace{-iA^2\partial^e(A^f)+iA^2\partial^f(A^e)}_{\cancel{-iA^2F^{ef}}}+\\
&\qquad\underbrace{-iA^fA^e\partial_b(A^b)+iA^eA^f\partial_b(A^b)}_{0}\underbrace{\cancel{-iA^eA^d\partial^f(A_d)}+\cancel{iA^fA^d\partial^e(A_d)}}_{d\to c}+\\
&\qquad+\cancel{iA^aA^f\partial_a(A^e)}-\cancel{iA^aA^e\partial_a(A^f)}\underbrace{-iA^eA^f\partial_a(A^a)+iA^fA^e\partial_a(A^a)}_{0}+
\end{aligned}
$$

$$\underbrace{-iA^2\partial^f(A^e)+iA^2\partial^e(A^f)}_{\cancel{iA^2F^{ef}}}-\cancel{iA^fA^c\partial^e(A_c)}+\cancel{iA^eA^c\partial^f(A_c)}\Big)+$$

$$-\frac{i}{4}\epsilon^{abef}\partial_c(A^c)F_{ab}+\frac{i}{16}\big(\epsilon^{bdef}\underbrace{F^c{}_bF_{cd}}_{\text{sym. in } bd}-\epsilon^{adef}\underbrace{F_a{}^cF_{cd}}_{\text{sym. in } ad}-\epsilon^{bcef}\underbrace{F^d{}_bF_{cd}}_{\text{sym. in } bc}+$$

$$+\epsilon^{acef}\underbrace{F_a{}^dF_{cd}}_{\text{sym. in } ac}\big)+\frac{1}{12}\epsilon^{cdef}\Box(F_{cd})+\frac{i}{3}\epsilon^{acef}\partial^b(A_c)F_{ab}-\frac{i}{3}\underbrace{\epsilon^{bcef}A_b\partial^a(F_{ca})}_{b\to c;\ c\to a;\ a\to b}+$$

$$+\frac{i}{6}\epsilon^{acef}\partial_a(A^d)F_{cd}+\frac{i}{3}\epsilon^{acef}A^d\partial_a(F_{cd})-\frac{1}{6}\epsilon^{cbef}A^aA_cF_{ab}+$$

$$-\frac{i}{12}\Big(\epsilon^{cdef}\underbrace{\partial^b(A_c)\partial_b(A_d)}_{\text{sym. in } cd}-\cancel{\epsilon^{bdef}\partial_a(A^a)\partial_b(A_d)}-\underbrace{\cancel{\epsilon^{caef}\partial_a(A_c)\partial_b(A^b)}}_{a\to b;\ b\to a;\ c\to d}+$$

$$+\epsilon^{baef}\underbrace{\partial_a(A_c)\partial_b(A^c)}_{\text{sym. in } ab}+\cancel{i\epsilon^{cdef}A^bA_c\partial_b(A_d)}-\cancel{i\epsilon^{bdef}A^2\partial_b(A_d)}+$$

$$-i\epsilon^{caef}\underbrace{A_aA_c\partial_b(A^b)}_{\text{sym. in } ac}+\cancel{i\epsilon^{baef}A_aA^d\partial_b(A_d)}+\underbrace{\cancel{i\epsilon^{cdef}A^aA_d\partial_a(A_c)}}_{d\to c;\ c\to d;\ a\to b}+$$

$$-i\epsilon^{bdef}\underbrace{A_bA_d\partial_a(A^a)}_{\text{sym. in } bd}-\underbrace{\cancel{i\epsilon^{caef}A^2\partial_a(A_c)}}_{a\to b;\ c\to d}+\underbrace{\cancel{i\epsilon^{baef}A_bA^c\partial_a(A_c)}}_{a\to b;\ b\to a;\ c\to d}\Big)=$$

$$=\frac{1}{2}\partial_c(A^c)F^{ef}+\frac{i}{6}\Box(F^{ef})+\underbrace{\frac{1}{3}\partial^b(A^e)F^f{}_b+\frac{1}{3}A^e\partial^b(F^f{}_b)}_{\text{Leibniz's rule}}+$$

$$\underbrace{-\frac{1}{3}\partial^b(A^f)F^e{}_b-\frac{1}{3}A^f\partial^b(F^e{}_b)}_{\text{Leibniz's rule}}\underbrace{-\frac{1}{6}\partial^e(A^d)F^f{}_d-\frac{1}{3}A^d\partial^e(F^f{}_d)}_{\text{Leibniz's rule}}+$$

$$\underbrace{+\frac{1}{6}\partial^f(A^d)F^e{}_d+\frac{1}{3}A^d\partial^f(F^e{}_d)}_{\text{Leibniz's rule}}+\frac{i}{6}A^aA^fF_a{}^e-\frac{i}{6}A^aA^eF_a{}^f+$$

$$-\frac{i}{4}\epsilon^{abef}\partial_c(A^c)F_{ab}+\frac{1}{12}\epsilon^{cdef}\Box(F_{cd})+$$

$$\underbrace{+\frac{i}{3}\epsilon^{acef}\partial^b(A_c)F_{ab}+\frac{i}{3}\epsilon^{acef}A_c\partial^b(F_{ab})}_{\text{Leibniz's rule}}+$$

$$\underbrace{+\frac{i}{6}\epsilon^{acef}\partial_a(A^d)F_{cd}+\frac{i}{3}\epsilon^{acef}A^d\partial_a(F_{cd})}_{\text{Leibniz's rule}}-\frac{1}{6}\epsilon^{cbef}A^aA_cF_{ab}=$$

$$=\frac{1}{2}\partial_c(A^c)F^{ef}+\frac{1}{3}\partial^b(A^eF^f{}_b)-\frac{1}{3}\partial^b(A^fF^e{}_b)-\frac{1}{6}\partial^e(A^dF^f{}_d)+$$

$$-\frac{1}{6}A^d\partial^e(F^f{}_d)+\frac{1}{6}\partial^f(A^dF^e{}_d)+\frac{1}{6}A^d\partial^f(F^e{}_d)+$$

$$+\frac{1}{12}\epsilon^{cdef}\Box(F_{cd})-\frac{1}{6}\epsilon^{cbef}A^aA_cF_{ab}+$$

$$+ i\Big(\frac{1}{6}\Box(F^{ef}) + \frac{1}{6}A^a A^f F_a{}^e - \frac{1}{6}A^a A^e F_a{}^f - \frac{1}{4}\epsilon^{abef}\partial_c(A^c)F_{ab} + $$
$$+ \frac{1}{3}\epsilon^{acef}\partial^b(A_c F_{ab}) + \frac{1}{6}\epsilon^{acef}\partial_a(A^d F_{cd}) + \frac{1}{6}\epsilon^{acef}A^d\partial_a(F_{cd})\Big) \,. \qquad \text{(F.8)}$$

Starting from (F.8), as usual, we are allowed to gain knowledge about the other trace's value by exploiting the 2 transformations provided by (4.17) and (F.2):

$$\begin{aligned}
\mathrm{tr}[\gamma^{ef}a_2(x,R_\lambda)P_L] = {} & \frac{1}{2}\underbrace{\partial_c(A^c)F^{ef}}_{(-1)^2} + \frac{1}{3}\underbrace{\partial^b(A^e F^f{}_b)}_{(-1)^2} - \frac{1}{3}\underbrace{\partial^b(A^f F^e{}_b)}_{(-1)^2} - \frac{1}{6}\underbrace{\partial^e(A^d F^f{}_d)}_{(-1)^2} + \\
& - \frac{1}{6}\underbrace{A^d\partial^e(F^f{}_d)}_{(-1)^2} + \frac{1}{6}\underbrace{\partial^f(A^d F^e{}_d)}_{(-1)^2} + \frac{1}{6}\underbrace{A^d\partial^f(F^e{}_d)}_{(-1)^2} + \\
& + \frac{1}{12}\underbrace{\epsilon^{cdef}\Box(F_{cd})}_{(-1)^2} - \frac{1}{6}\underbrace{\epsilon^{cbef}A^a A_c F_{ab}}_{(-1)^4} + \\
& + i\Big(\frac{1}{6}\underbrace{\Box(F^{ef})}_{(-1)} + \frac{1}{6}\underbrace{A^a A^f F_a{}^e}_{(-1)^3} - \frac{1}{6}\underbrace{A^a A^e F_a{}^f}_{(-1)^3} - \frac{1}{4}\underbrace{\epsilon^{abef}\partial_c(A^c)F_{ab}}_{(-1)^3} + \\
& + \frac{1}{3}\underbrace{\epsilon^{acef}\partial^b(A_c F_{ab})}_{(-1)^3} + \frac{1}{6}\underbrace{\epsilon^{acef}\partial_a(A^d F_{cd})}_{(-1)^3} + \frac{1}{6}\underbrace{\epsilon^{acef}A^d\partial_a(F_{cd})}_{(-1)^3}\Big)
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\mathrm{tr}[\gamma^{ef}a_2(x,R_{\lambda_c})P_R] = {} & \frac{1}{2}\partial_c(A^c)F^{ef} + \frac{1}{3}\partial^b(A^e F^f{}_b) - \frac{1}{3}\partial^b(A^f F^e{}_b) - \frac{1}{6}\partial^e(A^d F^f{}_d) + \\
& - \frac{1}{6}A^d\partial^e(F^f{}_d) + \frac{1}{6}\partial^f(A^d F^e{}_d) + \frac{1}{6}A^d\partial^f(F^e{}_d) + \\
& + \frac{1}{12}\epsilon^{cdef}\Box(F_{cd}) - \frac{1}{6}\epsilon^{cbef}A^a A_c F_{ab} + \\
& - i\Big(\frac{1}{6}\Box(F^{ef}) + \frac{1}{6}A^a A^f F_a{}^e - \frac{1}{6}A^a A^e F_a{}^f - \frac{1}{4}\epsilon^{abef}\partial_c(A^c)F_{ab} + \\
& + \frac{1}{3}\epsilon^{acef}\partial^b(A_c F_{ab}) + \frac{1}{6}\epsilon^{acef}\partial_a(A^d F_{cd}) + \frac{1}{6}\epsilon^{acef}A^d\partial_a(F_{cd})\Big) \,. \qquad \text{(F.9)}
\end{aligned}$$

Finally, the Lorentz anomaly appears to be:

$$\begin{aligned}
\frac{1}{2}\langle T^{ef} - T^{fe}\rangle = {} & \frac{1}{4(4\pi)^2}\Big(\underbrace{\mathrm{tr}[\gamma^{ef}a_2(x,R_\lambda)P_L]}_{\text{(F.8)}} + \underbrace{\mathrm{tr}[\gamma^{ef}a_2(x,R_{\lambda_c})P_R]}_{\text{(F.9)}}\Big) = \\
= {} & \frac{1}{4(4\pi)^2}\Big[\frac{1}{2}\partial_c(A^c)F^{ef} + \frac{1}{3}\partial^b(A^e F^f{}_b) - \frac{1}{3}\partial^b(A^f F^e{}_b) - \frac{1}{6}\partial^e(A^d F^f{}_d) + \\
& - \frac{1}{6}A^d\partial^e(F^f{}_d) + \frac{1}{6}\partial^f(A^d F^e{}_d) + \frac{1}{6}A^d\partial^f(F^e{}_d) + \\
& + \frac{1}{12}\epsilon^{cdef}\Box(F_{cd}) - \frac{1}{6}\epsilon^{cbef}A^a A_c F_{ab} + \\
& + i\Big(\tfrac{1}{6}\Box(F^{ef}) + \tfrac{1}{6}A^a A^f F_a{}^e - \tfrac{1}{6}A^a A^e F_a{}^f - \tfrac{1}{4}\epsilon^{abef}\partial_c(A^c)F_{ab} +
\end{aligned}$$

$$+\frac{1}{3}\epsilon^{acef}\partial^b(A_c F_{ab}) + \frac{1}{6}\epsilon^{acef}\partial_a(A^d F_{cd}) + \frac{1}{6}\epsilon^{acef} A^d \partial_a(F_{cd})\Big) +$$

$$+ \frac{1}{2}\partial_c(A^c)F^{ef} + \frac{1}{3}\partial^b(A^e F^f{}_b) - \frac{1}{3}\partial^b(A^f F^e{}_b) - \frac{1}{6}\partial^e(A^d F^f{}_d) +$$

$$- \frac{1}{6}A^d\partial^e(F^f{}_d) + \frac{1}{6}\partial^f(A^d F^e{}_d) + \frac{1}{6}A^d\partial^f(F^e{}_d) +$$

$$+ \frac{1}{12}\epsilon^{cdef}\Box(F_{cd}) - \frac{1}{6}\epsilon^{cbef}A^a A_c F_{ab} +$$

$$- i\Big(\frac{1}{6}\Box(F^{ef}) + \frac{1}{6}A^a A^f F_a{}^e - \frac{1}{6}A^a A^e F_a{}^f - \frac{1}{4}\epsilon^{abef}\partial_c(A^c)F_{ab} +$$

$$+\frac{1}{3}\epsilon^{acef}\partial^b(A_c F_{ab}) + \frac{1}{6}\epsilon^{acef}\partial_a(A^d F_{cd}) + \frac{1}{6}\epsilon^{acef} A^d \partial_a(F_{cd})\Big)\Big] =$$

$$= \frac{1}{2(4\pi)^2}\Big(\frac{1}{2}\partial_c(A^c)F^{ef} + \frac{1}{3}\partial^b(A^e F^f{}_b) - \frac{1}{3}\partial^b(A^f F^e{}_b) - \frac{1}{6}\partial^e(A^d F^f{}_d) +$$

$$- \frac{1}{6}A^d\partial^e(F^f{}_d) + \frac{1}{6}\partial^f(A^d F^e{}_d) + \frac{1}{6}A^d\partial^f(F^e{}_d) +$$

$$+ \frac{1}{12}\epsilon^{cdef}\Box(F_{cd}) - \frac{1}{6}\epsilon^{cbef}A^a A_c F_{ab}\Big) \,. \tag{F.10}$$

That exhausts the Lorentz anomaly's part of appendix F.

F.3 Gravitational anomaly

The last quantities needed in chapter 5 are the traces of the two vectorial generalization of the heat kernel coefficients, $b_{a,2}(x, R_\lambda)$ and $b_{a,2}(x, R_{\lambda_c})$, both weighted by the relative projector. Of course, we'll make use of their simplified expressions, i.e. those introduced in 4.5 through (4.44) and (4.45) respectively. Thus, we have

$$\mathrm{tr}[b_{a,2}(x, R_\lambda)P_L] \overset{\text{(B.16)}}{=} \mathrm{tr}[P_L b_{a,2}(x, R_\lambda)] =$$

$$= \mathrm{tr}\Big\{\Big[\frac{\mathbb{1}+\gamma^5}{2}\Big]\cdot\Big[-\frac{1}{60}\Box\partial^i\Big(-\frac{i}{2}(\mathbb{1})F_{ia} - \frac{i}{2}\gamma_{ac}\partial_i(A^c) + \frac{i}{2}\gamma_{ic}\partial_a(A^c) +$$

$$- \frac{1}{2}\gamma_{ic}A_a A^c + \frac{1}{2}\gamma_{ac}A_i A^c + \frac{1}{2}\gamma_{ia}A^2\Big) +$$

$$+ \frac{1}{6}\partial^i\Big(\frac{1}{4}(\mathbb{1})\partial_b(A^b)F_{ia} + \frac{1}{4}\gamma_{ac}\partial_b(A^b)\partial_i(A^c) - \frac{1}{4}\gamma_{ic}\partial_b(A^b)\partial_a(A^c) +$$

$$- \frac{i}{4}\gamma_{ic}\partial_b(A^b)A_a A^c + \frac{i}{4}\gamma_{ac}\partial_b(A^b)A_i A^c + \frac{i}{4}\gamma_{ia}A^2\partial_b(A^b) +$$

$$+ \frac{1}{8}\gamma_{db}F^{db}F_{ia} + \frac{1}{8}\gamma_{db}\gamma_{ac}\partial_i(A^c)F^{db} - \frac{1}{8}\gamma_{db}\gamma_{ic}\partial_a(A^c)F^{db} +$$

$$- \frac{i}{8}\gamma_{db}\gamma_{ic}A_a A^c F^{db} + \frac{i}{8}\gamma_{db}\gamma_{ac}A_i A^c F^{db} + \frac{i}{8}\gamma_{db}\gamma_{ia}A^2 F^{db} +$$

$$- \frac{i}{4}(\mathbb{1})A^2 F_{ia} - \frac{i}{4}\gamma_{ac}A^2\partial_i(A^c) + \frac{i}{4}\gamma_{ic}A^2\partial_a(A^c) +$$

$$- \frac{1}{4}\gamma_{ic}A_a A^c A^2 + \frac{1}{4}\gamma_{ac}A_i A^c A^2 + \frac{1}{4}\gamma_{ia}A^4\Big)\Big]\Big\} =$$

$$\begin{pmatrix}\text{we can use (B.18),}\\ \text{(B.23) and (B.24)}\\ \text{to neglect all}\\ \text{the null terms}\end{pmatrix} = \frac{i}{240}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}\Box\partial^i(F_{ia}) + \frac{1}{48}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}\partial^i\big(\partial_b(A^b)F_{ia}\big) + \frac{1}{96}\underbrace{\mathrm{tr}[\gamma_{db}\gamma_{ac}]}_{\text{(B.33)}}\partial^i\big(\partial_i(A^c)F^{db}\big)+$$

$$- \frac{1}{96}\underbrace{\mathrm{tr}[\gamma_{db}\gamma_{ic}]}_{\text{(B.33)}}\partial^i\big(\partial_a(A^c)F^{db}\big) - \frac{i}{96}\underbrace{\mathrm{tr}[\gamma_{db}\gamma_{ic}]}_{\text{(B.33)}}\partial^i\big(A_aA^cF^{db}\big)+$$

$$+ \frac{i}{96}\underbrace{\mathrm{tr}[\gamma_{db}\gamma_{ac}]}_{\text{(B.33)}}\partial^i\big(A_iA^cF^{db}\big) + \frac{i}{96}\underbrace{\mathrm{tr}[\gamma_{db}\gamma_{ia}]}_{\text{(B.33)}}\partial^i\big(A^2F^{db}\big)+$$

$$- \frac{i}{48}\underbrace{\mathrm{tr}[\mathbb{1}]}_{4}\partial^i\big(A^2F_{ia}\big) + \frac{1}{96}\underbrace{\mathrm{tr}[\gamma^5\gamma_{db}\gamma_{ac}]}_{\text{(B.34)}}\partial^i\big(\partial_i(A^c)F^{db}\big)+$$

$$- \frac{1}{96}\underbrace{\mathrm{tr}[\gamma^5\gamma_{db}\gamma_{ic}]}_{\text{(B.34)}}\partial^i\big(\partial_a(A^c)F^{db}\big) - \frac{i}{96}\underbrace{\mathrm{tr}[\gamma^5\gamma_{db}\gamma_{ic}]}_{\text{(B.34)}}\partial^i\big(A_aA^cF^{db}\big)+$$

$$+ \frac{i}{96}\underbrace{\mathrm{tr}[\gamma^5\gamma_{db}\gamma_{ac}]}_{\text{(B.34)}}\partial^i\big(A_iA^cF^{db}\big) + \frac{i}{96}\underbrace{\mathrm{tr}[\gamma^5\gamma_{db}\gamma_{ia}]}_{\text{(B.34)}}\partial^i\big(A^2F^{db}\big) =$$

$$= \frac{i}{60}\Box\partial^i(F_{ia}) + \frac{1}{12}\partial^i\big(\partial_b(A^b)F_{ia}\big) + \frac{1}{24}(g_{dc}g_{ba} - g_{da}g_{bc})\partial^i\big(\partial_i(A^c)F^{db}\big)+$$

$$- \frac{1}{24}(g_{dc}g_{bi} - g_{di}g_{bc})\partial^i\big(\partial_a(A^c)F^{db}\big) - \frac{i}{24}(g_{dc}g_{bi} - g_{di}g_{bc})\partial^i\big(A_aA^cF^{db}\big)+$$

$$+ \frac{i}{24}(g_{dc}g_{ba} - g_{da}g_{bc})\partial^i\big(A_iA^cF^{db}\big) + \frac{i}{24}(g_{da}g_{bi} - g_{di}g_{ba})\partial^i\big(A^2F^{db}\big)+$$

$$- \frac{i}{12}\partial^i\big(A^2F_{ia}\big) + \frac{i}{24}\epsilon_{dbac}\partial^i\big(\partial_i(A^c)F^{db}\big) - \frac{i}{24}\epsilon_{dbic}\partial^i\big(\partial_a(A^c)F^{db}\big)+$$

$$+ \frac{1}{24}\epsilon_{dbic}\partial^i\big(A_aA^cF^{db}\big) - \frac{1}{24}\epsilon_{dbac}\partial^i\big(A_iA^cF^{db}\big) - \frac{1}{24}\epsilon_{dbia}\partial^i\big(A^2F^{db}\big) =$$

$$= \frac{i}{60}\Box\partial^i(F_{ia}) + \frac{1}{12}\partial^i\big(\partial_b(A^b)F_{ia}\big) + \frac{1}{24}\partial^i\big(\partial_i(A^c)F_{ca}\big) - \frac{1}{24}\partial^i\big(\partial_i(A^c)F_{ac}\big)+$$

$$- \frac{1}{24}\partial^i\big(\partial_a(A^c)F_{ci}\big) + \frac{1}{24}\partial^i\big(\partial_a(A^c)F_{ic}\big) - \frac{i}{24}\partial^i\big(A_aA^cF_{ci}\big) + \frac{i}{24}\partial^i\big(A_aA^cF_{ic}\big)+$$

$$+ \frac{i}{24}\partial^i\big(A_iA^cF_{ca}\big) - \frac{i}{24}\partial^i\big(A_iA^cF_{ac}\big) + \frac{i}{24}\partial^i\big(A^2F_{ai}\big) - \frac{i}{24}\partial^i\big(A^2F_{ia}\big)+$$

$$- \frac{i}{12}\partial^i\big(A^2F_{ia}\big) + \frac{i}{24}\epsilon_{dbac}\partial^i\big(\partial_i(A^c)F^{db}\big) - \frac{i}{24}\epsilon_{dbic}\partial^i\big(\partial_a(A^c)F^{db}\big)+$$

$$+ \frac{1}{24}\epsilon_{dbic}\partial^i\big(A_aA^cF^{db}\big) - \frac{1}{24}\epsilon_{dbac}\partial^i\big(A_iA^cF^{db}\big) - \frac{1}{24}\epsilon_{dbia}\partial^i\big(A^2F^{db}\big) =$$

$$= \frac{1}{12}\Big(\partial^i\big(\partial_b(A^b)F_{ia}\big) - \partial^i\big(\partial_i(A^c)F_{ac}\big) + \partial^i\big(\partial_a(A^c)F_{ic}\big)+$$

$$+ \frac{1}{2}\epsilon_{dbic}\partial^i\big(A_aA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbac}\partial^i\big(A_iA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbia}\partial^i\big(A^2F^{db}\big)\Big)+$$

$$+ \frac{i}{6}\Big(\frac{1}{10}\Box\partial^i(F_{ia}) + \frac{1}{2}\partial^i\big(A_aA^cF_{ic}\big) - \frac{1}{2}\partial^i\big(A_iA^cF_{ac}\big)+$$

$$- \partial^i\big(A^2F_{ia}\big) + \frac{1}{4}\epsilon_{dbac}\partial^i\big(\partial_i(A^c)F^{db}\big) - \frac{1}{4}\epsilon_{dbic}\partial^i\big(\partial_a(A^c)F^{db}\big)\Big)\ , \quad \text{(F.11)}$$

from which $\mathrm{tr}[b_{a,2}(x,R_{\lambda_c})P_R]$ promptly follows through the implementation of (4.17) and (F.2):

$$
\begin{aligned}
\mathrm{tr}[b_{a,2}(x,R_{\lambda_c})P_R] = \frac{1}{12}\Big(&\partial^i\big(\partial_b(A^b)F_{ia}\big) - \partial^i\big(\partial_i(A^c)F_{ac}\big) + \partial^i\big(\partial_a(A^c)F_{ic}\big) + \\
&+ \frac{1}{2}\epsilon_{dbic}\partial^i\big(A_aA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbac}\partial^i\big(A_iA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbia}\partial^i\big(A^2F^{db}\big)\Big) + \\
&- \frac{i}{6}\Big(\frac{1}{10}\Box\partial^i(F_{ia}) + \frac{1}{2}\partial^i\big(A_aA^cF_{ic}\big) - \frac{1}{2}\partial^i\big(A_iA^cF_{ac}\big) + \\
&- \partial^i\big(A^2F_{ia}\big) + \frac{1}{4}\epsilon_{dbac}\partial^i\big(\partial_i(A^c)F^{db}\big) - \frac{1}{4}\epsilon_{dbic}\partial^i\big(\partial_a(A^c)F^{db}\big)\Big)\ . \qquad \text{(F.12)}
\end{aligned}
$$

The gravitational anomaly then results in

$$
\begin{aligned}
\langle \partial^b T_{be}(x) + i\bar{\lambda}\gamma^b\lambda F_{be}(x)\rangle = \frac{1}{(4\pi)^2}\Big(&\underbrace{\mathrm{tr}[b_{e,2}(x,R_\lambda)P_L]}_{\text{(F.11)}} + \underbrace{\mathrm{tr}[b_{e,2}(x,R_{\lambda_c})P_R]}_{\text{(F.12)}} + \\
&+ \frac{1}{2}\partial_e\big(\underbrace{\mathrm{tr}[a_2(x,R_\lambda)P_L]}_{\text{(F.1)}}\big) + \frac{1}{2}\partial_e\big(\underbrace{\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]}_{\text{(F.3)}}\big) + \\
&+ \frac{i}{2}A_e(x)\ \underbrace{\mathrm{tr}[a_2(x,R_\lambda)P_L]}_{\text{(F.1)}} + \frac{i}{2}A^f(x)\ \underbrace{\mathrm{tr}[\gamma_{ef}a_2(x,R_\lambda)P_L]}_{\text{(F.8)}} + \\
&- \frac{i}{2}A_e(x)\ \underbrace{\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]}_{\text{(F.3)}} - \frac{i}{2}A^f(x)\ \underbrace{\mathrm{tr}[\gamma_{ef}a_2(x,R_{\lambda_c})P_R]}_{\text{(F.9)}} + \\
&- \frac{\alpha}{2}\partial_e\big(\underbrace{\mathrm{tr}[a_2(x,R_\lambda)P_L]}_{\text{(F.1)}}\big) - \frac{\alpha}{2}\partial_e\big(\underbrace{\mathrm{tr}[a_2(x,R_{\lambda_c})P_R]}_{\text{(F.3)}}\big) + \\
&- \frac{\alpha}{2}\partial_e\big(\underbrace{\mathrm{tr}[a_2(x,R_\rho)P_R]}_{\text{(F.4)}}\big) - \frac{\alpha}{2}\partial_e\big(\underbrace{\mathrm{tr}[a_2(x,R_{\rho_c})P_L]}_{\text{(F.5)}}\big)\Big) =
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{(4\pi)^2}\Big[&\frac{1}{12}\Big(\partial^i\big(\partial_b(A^b)F_{ie}\big) - \partial^i\big(\partial_i(A^c)F_{ec}\big) + \partial^i\big(\partial_e(A^c)F_{ic}\big) + \\
&+ \frac{1}{2}\epsilon_{dbic}\partial^i\big(A_eA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbec}\partial^i\big(A_iA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbie}\partial^i\big(A^2F^{db}\big)\Big) + \\
&+ \frac{i}{6}\Big(\cancel{\tfrac{1}{10}\Box\partial^i(F_{ie}) + \tfrac{1}{2}\partial^i\big(A_eA^cF_{ic}\big) - \tfrac{1}{2}\partial^i\big(A_iA^cF_{ec}\big) +} \\
&\cancel{- \partial^i\big(A^2F_{ie}\big) + \tfrac{1}{4}\epsilon_{dbec}\partial^i\big(\partial_i(A^c)F^{db}\big) - \tfrac{1}{4}\epsilon_{dbic}\partial^i\big(\partial_e(A^c)F^{db}\big)}\Big) + \\
&+ \frac{1}{12}\Big(\partial^i\big(\partial_b(A^b)F_{ie}\big) - \partial^i\big(\partial_i(A^c)F_{ec}\big) + \partial^i\big(\partial_e(A^c)F_{ic}\big) + \\
&+ \frac{1}{2}\epsilon_{dbic}\partial^i\big(A_eA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbec}\partial^i\big(A_iA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbie}\partial^i\big(A^2F^{db}\big)\Big) + \\
&- \frac{i}{6}\Big(\cancel{\tfrac{1}{10}\Box\partial^i(F_{ie}) + \tfrac{1}{2}\partial^i\big(A_eA^cF_{ic}\big) - \tfrac{1}{2}\partial^i\big(A_iA^cF_{ec}\big) +} \\
&\cancel{- \partial^i\big(A^2F_{ie}\big) + \tfrac{1}{4}\epsilon_{dbec}\partial^i\big(\partial_i(A^c)F^{db}\big) - \tfrac{1}{4}\epsilon_{dbic}\partial^i\big(\partial_e(A^c)F^{db}\big)}\Big) + \\
&+ \frac{1}{12}\partial_e\Big(-(\partial_aA^a)^2 + \frac{1}{2}F_{ab}F^{ab} - \Box(A^2) + \partial_b(A_c)\partial^b(A^c)\Big) +
\end{aligned}
$$

$$+\frac{i}{12}\partial_e\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)+$$

$$+\frac{1}{12}\partial_e\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+$$

$$-\frac{i}{12}\partial_e\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)+$$

$$+\frac{i}{12}A_e\Big(-(\partial_a A^a)^2+\tfrac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+$$

$$-\frac{1}{12}A_e\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)+$$

$$+\frac{i}{2}A^f\Big(\tfrac{1}{2}\partial_c(A^c)F_{ef}+\tfrac{1}{3}\partial^b(A_eF_{fb})-\tfrac{1}{3}\partial^b(A_fF_{eb})-\tfrac{1}{6}\partial_e(A^dF_{fd})+$$

$$-\tfrac{1}{6}A^d\partial_e(F_{fd})+\tfrac{1}{6}\partial_f(A^dF_{ed})+\tfrac{1}{6}A^d\partial_f(F_{ed})+$$

$$+\tfrac{1}{12}\epsilon_{cdef}\Box(F^{cd})-\tfrac{1}{6}\epsilon_{cbef}A_aA^cF^{ab}\Big)+$$

$$-\frac{1}{2}A^f\Big(\frac{1}{6}\Box(F_{ef})+\frac{1}{6}A^aA_fF_{ae}-\frac{1}{6}A^aA_eF_{af}-\frac{1}{4}\epsilon_{abef}\partial_c(A^c)F^{ab}+$$

$$+\frac{1}{3}\epsilon_{acef}\partial_b(A^cF^{ab})+\frac{1}{6}\epsilon_{acef}\partial^a(A_dF^{cd})+\frac{1}{6}\epsilon_{acef}A_d\partial^a(F^{cd})\Big)+$$

$$-\frac{i}{12}A_e\Big(-(\partial_a A^a)^2+\tfrac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+$$

$$-\frac{1}{12}A_e\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)+$$

$$-\frac{i}{2}A^f\Big(\tfrac{1}{2}\partial_c(A^c)F_{ef}+\tfrac{1}{3}\partial^b(A_eF_{fb})-\tfrac{1}{3}\partial^b(A_fF_{eb})-\tfrac{1}{6}\partial_e(A^dF_{fd})+$$

$$-\tfrac{1}{6}A^d\partial_e(F_{fd})+\tfrac{1}{6}\partial_f(A^dF_{ed})+\tfrac{1}{6}A^d\partial_f(F_{ed})+$$

$$+\tfrac{1}{12}\epsilon_{cdef}\Box(F^{cd})-\tfrac{1}{6}\epsilon_{cbef}A_aA^cF^{ab}\Big)+$$

$$-\frac{1}{2}A^f\Big(\frac{1}{6}\Box(F_{ef})+\frac{1}{6}A^aA_fF_{ae}-\frac{1}{6}A^aA_eF_{af}-\frac{1}{4}\epsilon_{abef}\partial_c(A^c)F^{ab}+$$

$$+\frac{1}{3}\epsilon_{acef}\partial_b(A^cF^{ab})+\frac{1}{6}\epsilon_{acef}\partial^a(A_dF^{cd})+\frac{1}{6}\epsilon_{acef}A_d\partial^a(F^{cd})\Big)+$$

$$-\frac{\alpha}{12}\partial_e\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+$$

$$-\frac{i\alpha}{12}\partial_e\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)+$$

$$-\frac{\alpha}{12}\partial_e\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+$$

$$+\frac{i\alpha}{12}\partial_e\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)+$$

$$-\frac{\alpha}{12}\partial_e\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+$$

$$+\frac{i\alpha}{12}\partial_e\Big(-\Box(\partial_s A^s)+\partial_a(A^2A^a)-\tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)+$$

$$-\frac{\alpha}{12}\partial_e\Big(-(\partial_a A^a)^2+\frac{1}{2}F_{ab}F^{ab}-\Box(A^2)+\partial_b(A_c)\partial^b(A^c)\Big)+$$

$$
\begin{aligned}
&-\frac{i\alpha}{12}\partial_e\Big(-\Box(\partial_s A^s) + \partial_a(A^2A^a) - \tfrac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)\Big] = \\
= \frac{1}{(4\pi)^2}\Big[&\frac{1}{6}\Big(\partial^i\big(\partial_b(A^b)F_{ie}\big) - \partial^i\big(\partial_i(A^c)F_{ec}\big) + \partial^i\big(\partial_e(A^c)F_{ic}\big)+ \\
&\quad + \frac{1}{2}\epsilon_{dbic}\partial^i\big(A_eA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbec}\partial^i\big(A_iA^cF^{db}\big) - \frac{1}{2}\epsilon_{dbie}\partial^i\big(A^2F^{db}\big)\Big)+ \\
&+\frac{1}{6}\partial_e\Big(-(\partial_aA^a)^2 + \frac{1}{2}F_{ab}F^{ab} - \Box(A^2) + \partial_b(A_c)\partial^b(A^c)\Big)+ \\
&-\frac{1}{6}A_e\Big(-\Box(\partial_sA^s) + \partial_a(A^2A^a) - \frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}\Big)+ \\
&-A^f\Big(\frac{1}{6}\Box(F_{ef}) + \frac{1}{6}A^aA_fF_{ae} - \frac{1}{6}A^aA_eF_{af} - \frac{1}{4}\epsilon_{abef}\partial_c(A^c)F^{ab}+ \\
&\quad + \frac{1}{3}\epsilon_{acef}\partial_b(A^cF^{ab}) + \frac{1}{6}\epsilon_{acef}\partial^a(A_dF^{cd}) + \frac{1}{6}\epsilon_{acef}A_d\partial^a(F^{cd})\Big)+ \\
&-\frac{\alpha}{3}\partial_e\Big(-(\partial_aA^a)^2 + \frac{1}{2}F_{ab}F^{ab} - \Box(A^2) + \partial_b(A_c)\partial^b(A^c)\Big)\Big]\ . \qquad \text{(F.13)}
\end{aligned}
$$

This puts an end to the current appendix.

Appendix G

Demonstration: from (B.28) to (B.29)

The main purpose of this section is to provide the detailed proof of equation (B.29), that should be attained by properly manipulating (B.28), i.e.

$$i\epsilon_{sabc}\gamma^s\gamma^5 = \epsilon_{0abc}\gamma^1\gamma^2\gamma^3 + \epsilon_{1abc}\gamma^0\gamma^2\gamma^3 - \epsilon_{2abc}\gamma^0\gamma^1\gamma^3 + \epsilon_{3abc}\gamma^0\gamma^1\gamma^2,$$

as anticipated in appendix B. Indeed, if we entered each of the sixty-four available combinations of the three indices a, b and c, we would observe that, whenever a, b and c differ from one another, $i\epsilon_{sabc}\gamma^s\gamma^5$ correctly reproduces $\gamma_a\gamma_b\gamma_c$, while it vanishes otherwise. All this is summarized in

$$i\epsilon_{sabc}\gamma^s\gamma^5 = \begin{cases} \gamma_a\gamma_b\gamma_c & \text{if } a \neq b \ \wedge \ a \neq c \ \wedge \ c \neq b \\ 0 & \text{otherwise} \end{cases}, \tag{G.1}$$

or, alternatively, through

$$i\epsilon_{sabc}\gamma^s\gamma^5 = \frac{1}{3!}\big(\gamma_a\gamma_b\gamma_c - \gamma_a\gamma_c\gamma_b + \gamma_b\gamma_c\gamma_a - \gamma_b\gamma_a\gamma_c + \gamma_c\gamma_a\gamma_b - \gamma_c\gamma_b\gamma_a\big).$$

Undoubtedly, the finest of the two expressions is the latter, which not only corresponds to (B.29), but also gathers in itself both solutions of (G.1) by simply antisymmetrizing the upper non-vanishing one. This ends the current discussion.