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CORPUSCOLAR DESCRIPTION OF BOOTSTRAPPED NEWTONIAN GRAVITY

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When you know the fourfoil in all its seasons root and leaf and flower, by sight and scent and seed, then you may learn its true name, knowing its being: which is more than its use.

Ursula Le Guin,
"The Wizard of Earthsea"

Sommario

Una delle caratteristiche più importanti della Relatività Generale è che, sotto condizioni molto generali e per oggetti sufficientemente densi, porta sempre alla formazione di singolarità dello spaziotempo che sono difficili da gestire sia intuitivamente che matematicamente. Inoltre, queste singolarità sono in contrasto con il principio di indeterminazione di Heisenberg che impedisce la formazione di regioni di spazio con densità di energia infinita. Negli ultimi anni, Dvali e Gomez hanno tentato un nuovo modo di studiare la fisica dei buchi neri. In questa teoria il campo gravitazionale è visto come una collezione di gravitoni e diversi fenomeni caratteristici della gravità sono interpretati come effetti quantomeccanici legati ai gravitoni. Lo scopo di questa tesi è provare a costruire una descrizione corpuscolare autoconsistente del campo gravitazionale usando come teoria classica di riferimento la cosiddetta "bootstrapped Newtonian gravity", dove non si formano singolarità dello spaziotempo, e comparare i risultati ottenuti con le assunzioni fatte da Dvali e Gomez riguardo le propietà dei gravitoni.

Abstract

One of the main features of General Relativity is that, under very general assumptions and for sufficiently dense objects, it always leads to the creation of space-time singularities that are quite difficult to handle both from an intuitive and a mathematical point of view. Moreover, these singularities are in contrast with the Heisenberg uncertainty principle since they would form a region of space with infinite density of energy. In the last years, Dvali and Gomez suggested a novel approach to black holes' physics, according to which the gravitational field can be described as a collection of soft gravitons. In this picture some phenomena that are very well known in the geometrical picture of gravity are explained in terms of Quantum Mechanical effects related to gravitons. The aim of this work is to provide a self consistent quantum corpuscolar description of gravitational interactions with a classical background given by the bootstrapped Newtonian gravity, where no space-time singularities arise, and compare the results with the assumptions made by Dvali and Gomez about the properties of gravitons.

Contents

Introduction

Most astrophysical objects can be described using Newtonian Gravity, because the energy involved in large scale gravitational interaction is so weak that the effects of curvature of space-time are negligible. However, many astrophysical measurements made mostly in the second half of the twentieth century have shown the existence of some phenomena that could only be described by means of General Relativity. For example the huge luminescence of the QSO's can only be explained in terms of relativistic effects on the matter around the galactic nuclei, as well as the energy produced by the rotation of Neutron stars. We can also mention some recent achievements, such as the discovery of Gravitational Waves in 2015 and the image of the black hole M87 taken in April 2019, that seem to prove the validity of General Relativity at much smaller length scales. However, we have to consider that all these astonishing phenomena have been measured on the Earth where the gravitational effects are so weak that space-time can be assumed to be flat. This means that all the astrophysical measurements of these extreme objects have been made through the detection of particles (mainly photons and more recently GWs) which are classified as infinite dimensional representation of the Poincarè group that makes sense only in a Minkowskian space-time. Indeed, even though we are able to describe the effects of gravity at large distances from the source of the gravitational field, we are still unable to understand the physics where the gravitational field is so strong that no conventional notion of particle persists.

This is something similar to what happens in the Scattering theory in QFT. During the interaction we do not really know what happens because the fields that describe matter interact in a non linear way and so we do not even know if it is appropriate to talk about particles. All we can predict, using perturbation theory, is which particles are going to be detected very far away from where the interaction took place, namely when they can be considered completely free. The difference is that in General Relativity the impossibility to derive a particle description of matter does not necessarily come from the interaction between two or more particle, but from the interaction of a single particle with a curved space-time. The description of gravitational interactions becomes even more problematic in proximity of singularities, namely points of space-time where, according to General Relativity, the density of energy is infinite. However, such objects are not allowed in a theory that expects to be valid at any length scale since it is in contradiction with the Heisenberg uncertainty principle.

The research carried on in the recent years by Dvali and Gomez (Ref.[11]-[16]) gives a completely different descpription of gravitational interactions in which these problems are avoided since any geometrical picture of gravity is abandoned. Indeed, their model gives a completely quantum description of gravity where the interaction is carried by soft gravitons. In particular the black hole is seen as a *Bose condensate of weekly interacting soft gravitons.*

The main motivation of this dissertation is to try to derive a self-consistent corpuscolar description of gravity, starting from a classical background given by the so called bootstrapped Newtonian gravity described in Ref.[22]. There is shown that, for a spherically symmetric object, this theory avoids the formation of space-time singularities. Thus, in this framework, we are allowed to describe stable configurations with a radius $R \leq R_H$ which is impossible in General Relativity. This dissertation is organized as follows. In chapter 1 we derive the conditions under which a mass configuration in General Relativity inevitably gives rise to space-time singularities, deriving also an exact result for a spherically symmetric object in the form of the so called *Buchdal limit*.
Then in chapter 2 we recall the main features of the work carried on by Dvali and Gomez focusing on the way this model describes the main phenomena related to black holes.
Finally, in 3, after recalling the main features of Newtonian and bootstrapped Newtonian gravity, we build up a corpuscolar description of both the previously mentioned theories. In particular we derive, where possible, the form of the quantum coherent state that describes gravitons in a given configuration, the number of gravitons present in it and the mean wavelength of gravitons that compose the system, comparing these results with the assumptions made in Ref.[11]-[16].

Chapter 1

Collapse and Singularities in General Relativity

The presence of singularities in General relativity is far more problematic than in other theories and may represent a limit in its validity at the Plank length scale $l_p \approx 10^{-33}cm$. This happens because, differently from other theories where space-time structure is assumed and we just have to understand how matters moves inside it, in General Relativity space-time itself has its own dynamic described by the Einstein field equations. This means that a singularity in their solutions inevitably brings to *an ill-defined structure of space-time*. Thus differently from other theories, where the divergence of a physical quantity in a given point of space-time immediately signals the presence of a singularity in that same point, in General Relativity we are not even capable of saying "where and when" (whatever this means in proximity of a singularity) the predictability of the theory fails. For example in the Coulomb interaction there is a singularity in $r = 0$ that means we are able to know the electric field produced by a point-like charge in any place of space expect for $r = 0$, but when we consider the Schwarzschild metric the singularity in $r = 0$ we cannot say "where" the singularity is because space-time itself is singular in that point.

This means that in General Relativity we have to abandon the idea of thinking singularities as places of space-time[1] and try to characterize them in some other way. We could try to see if the curvature represented by the Riemann tensor $R^{\lambda}_{\mu\nu\rho}$ diverges at some point. However, the components of the curvature depend on the system of coordinates chosen. Then, we can consider $R_{abcd}R^{abcd}$ or R^{ab}_{ab} and see if they diverge. However not even this classification seems to be satisfactory because there are cases in which space-time is singular even though there are not divergences in the curvature

[1]There are some cases in which is still possible to characterize singularities as "places" such as in the Robertson Walker and Swarzschild solutions which rely in the definition of a topological space made which points represent the singularity.

tensor[2].This opens the possibility of the presence of many types of singularities, whose classification is however outside the scope of the present work. Indeed, we want just to recall the condition under which gravitational collapse inevitably occurs in General Relativity.

In the first section we focus our attention on the study of the collapse of spherically symmetric objects because they have many astrophysical applications and permit to find some exact solutions. Then, in the second section, we derive the Hawking-Penrose singularity theorems which state the conditions under which gravitational collapse occurs in not symmetric space-times.

1.1 Equation of stellar equilibrium

Roughly speaking, we can say that the equilibrium of a star is the result of the compensation of its own gravitational force and internal pressure. Considering this, we can easily see that the collapse occurs whenever there is a singularity in the expression of the equilibrium pressure. This is why we are more interested in the study of the equation of structure of the star than in the dynamics of the collapse itself.

Thus, we introduce the most general form of the proper time for an isotropic and static metric, written in the so called "standard" form, that is given by

$$d\tau^2 = -b(r)dt^2 + r^2(d\theta^2 + sen^2\theta d\phi^2) + a(r)r^2. \tag{1.1}$$

The energy-momentum tensor can be assumed to be that for a perfect fluid

$$T_{\mu\nu} = pg_{\mu\nu} + u_\mu u_\nu(p+\rho), \tag{1.2}$$

where p is the pressure, ρ is the density of energy and u_μ is the four-velocity of the fluid. Given that the fluid is at rest, the spatial components of the four-velocity disappear $u_r = u_\phi = 0 = u_\theta = 0$ and considering the equation

$$g_{\mu\nu}u^\mu u^\nu = -1, \tag{1.3}$$

valid for matter it is immediate to obtain

$$u_0 = \pm g_{tt}^{-1/2} = \pm\sqrt{b(r)}. \tag{1.4}$$

By definition, the Christoffel symbols are given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}), \tag{1.5}$$

[2]Minkowski space-time deprived of the points with azimuthal coordinate ϕ such that $0 < \phi < \phi_0$ is singular in $r = 0$ (Ref.[4]) even though the curvature tensor is null everywhere else

while the definitions of Riemann tensor, Ricci tensor and Ricci scalar are respectively

$$R^{\lambda}_{\mu\nu k} = \partial_k \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu k} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{k\eta} - \Gamma^{\eta}_{\mu k}\Gamma^{\lambda}_{\nu\eta} \tag{1.6}$$

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{1.7}$$

$$R = R^{\lambda}_{\lambda}. \tag{1.8}$$

Substituting the metric Eq.(1.1) into Eq.(1.5), we obtain

$$\begin{array}{rlr}
\Gamma^{r}_{rr} = \dfrac{1}{2a(r)}\dfrac{da(r)}{dr} & \Gamma^{r}_{\theta\theta} = \dfrac{-r}{a(r)} & \\
\Gamma^{r}_{\phi\phi} = \dfrac{-r\sin^2\theta}{a(r)} & \Gamma^{r}_{tt} = \dfrac{1}{2a(r)}\dfrac{db(r)}{dr} & \\
\Gamma^{\theta}_{r\theta} = \dfrac{1}{r} & \Gamma^{\phi}_{r\phi} = \dfrac{1}{r} & \\
\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta & \Gamma^{\phi}_{\phi\theta} = \dfrac{1}{r}\cot\theta & \Gamma^{t}_{tr} = \dfrac{1}{2b(r)}\dfrac{db(r)}{dr}.
\end{array} \tag{1.9}$$

Substituting Eq.(1.9) into Eq.(1.6), Eq.(1.7) and Eq.(1.8), the Einstein equations written in the form

$$R_{\mu\nu} = -8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} T^{\rho}_{\rho} g_{\mu\nu} \right), \tag{1.10}$$

become

$$\begin{cases}
R_{rr} = \dfrac{b''}{2b} - \dfrac{b'}{4b}\left(\dfrac{a'}{a} + \dfrac{b'}{b}\right) - \dfrac{a'}{ra} = -4\pi G_N(\rho - p)a & \text{(1.11a)} \\
R_{\theta\theta} = -1 + \dfrac{r}{2a}\left(-\dfrac{a'}{a} + \dfrac{b'}{b}\right) + \dfrac{1}{a} = -4\pi G_N(\rho - p)r^2 & \text{(1.11b)} \\
R_{tt} = \dfrac{b''}{2A} - \dfrac{b'}{4A}\left(\dfrac{a'}{a} + \dfrac{b'}{b}\right) - \dfrac{b'}{ar} = -4\pi G_N(\rho + 3p)b, & \text{(1.11c)}
\end{cases}$$

where the symbol $'$ states for first derivative respect to the radial coordinate r.
All the other equations give trivial identities or equations equivalent to the previous ones.

In order to obtain the condition under which a star is at equilibrium, it is essential to recall the equation of the pressure of a perfect fluid in hydrostatic equilibrium (Ref.[1], Eq.(5.4.5))

$$-\frac{\partial p}{\partial x^{\nu}} = (p + \rho)\frac{\partial}{x^{\lambda}} \ln{(-g_{00})^{1/2}}. \tag{1.12}$$

Using the metric Eq.(1.1), Eq.(1.12) becomes

$$\frac{b'}{b} = -\frac{2p'}{p+\rho}. \tag{1.13}$$

To solve this equation it is useful to compute the quantity

$$\frac{R_{rr}}{2a} + \frac{R_{\theta\theta}}{r^2} + \frac{R_{tt}}{2b} = -\frac{a'}{ra^2} - \frac{1}{r^2} + \frac{1}{a^2} = -8\pi G_N \rho, \tag{1.14}$$

that can be rewritten as the following differential equation

$$\left(\frac{r}{a}\right)' = 1 - 8\pi G_N \rho r^2. \tag{1.15}$$

Integrating Eq.(1.15) and isolating $a(r)$, we find

$$a(r) = [1 - 2M(r)/r]^{-1}, \tag{1.16}$$

with

$$M(r) = \int_0^r 4\pi r'^2 \rho(r') dr'. \tag{1.17}$$

Using Eq.(1.16) and Eq.(1.12), Eq.(1.11b) becomes

$$-1 + \left[1 - \frac{2G_N M(r)}{r}\right]\left(1 - \frac{rp'}{p+\rho}\right) + \frac{G_N M(r)}{r} - 4\pi G_N \rho r^2 = -4\pi G_N (\rho - p) r^2, \tag{1.18}$$

which can be rewritten as

$$-r^2 p'(r) = G_N M(r) \rho(r) \left(1 + \frac{p(r)}{\rho(r)}\right)\left(1 + \frac{4\pi r^3}{M(r)}\right)\left(1 - \frac{2G_N M(r)}{r}\right)^{-1}. \tag{1.19}$$

This is the fundamental equation that gives the pressure of a spherical source of given density as a function of the radial coordinate r.

Taking a compact source it is immediate to see that if one requires as initial condition $p(R) = 0$ where R is the radius of the source, then from Eq.(1.19) follows that $p(r) = 0$ for $r > R$ as one would expect. Knowing $p(r)$ and $M(r)$, we can easily compute $a(r)$ through Eq.(1.16). To compute $b(r)$, we substitute Eq.(1.19) in Eq.(1.12) obtaining

$$\frac{b'}{b} = \frac{2G_N}{r^2}(M(r) - 4\pi r^3 p)\left(1 - \frac{2G_N M(r)}{r}\right)^{-1}, \tag{1.20}$$

so that, for $b(\infty) = 1$

$$b(r) = \exp\left\{\int_r^\infty \frac{2G_N}{r'^2}(M(r') - 4\pi r'^3 p)\left(1 - \frac{2G_N M(r')}{r'}\right)^{-1}\right\}. \tag{1.21}$$

We notice that for $r > R$, $p(r) = 0$ so that we recover the usual Schwarzchild solution

$$b(r) = \left(1 - \frac{2G_N M(R)}{R}\right)^{-1}. \tag{1.22}$$

1.2 Stars of Uniform Density

In this section we want to study the solution of Eq.(1.19) when

$$\rho = cost.$$

Altough there are not stars described by a constant density the solution of this problem is important because it is simple enough to solve it exactly and it also fixes an upper limit for the radius of a given star of mass M in order not to collapse. With a constant density Eq.(1.19) becomes

$$-\frac{p'(r)}{(p + \rho(r))(\rho/3 + p(r))} = 4\pi G r\left(1 - \frac{8\pi G_N \rho r^2}{3}\right)^{-1}. \tag{1.23}$$

Integrating this equation from $r = R$ where $p = 0$ to r inside the source we obtain (Ref.[2])

$$p(r) = \frac{3M}{4\pi R^3}\left[\frac{(1 - 2MG_N/R)^{1/2} - (1 - 2MG_N r^2/R^3)^{1/2}}{-3(1 - 2MG_N/R)^{1/2} + (1 - 2MG_N r^2/R^3)^{1/2}}\right]. \tag{1.24}$$

where

$$M = \frac{4\pi}{3}\rho r^3, \tag{1.25}$$

while a and b can be computed immediately through Eq.(1.16) and Eq.(1.21) that give

$$a(r) = \left(1 - \frac{2G_N M r^2}{R^3}\right)^{-1} \tag{1.26}$$

$$b(r) = \frac{1}{4}\left[3\left(1 - \frac{2G_N M}{R}\right)^{1/2} - \left(1 - \frac{2G_N M r^2}{R^3}\right)^{1/2}\right]^2. \tag{1.27}$$

The interesting thing about Eq.(1.27) is that it is not valid for every value of r and, in particular for

$$r_\infty^2 = 9R^2 - \frac{4R^3}{MG_N}, \tag{1.28}$$

the pressure becomes infinite. This means that in order to have a stable configuration it has to be satisfied the condition

$$\frac{MG_N}{R} < \frac{4}{9}. \tag{1.29}$$

Now we want to show that this is an universal bound valid for any star described by a density subject to the following requirements:

- The radius R is fixed and

$$\rho(r) = 0 \qquad r > R. \tag{1.30}$$

- The mass M is fixed and

$$M(r) = \int_0^r dr' \rho(r') 4\pi r'^2, \tag{1.31}$$

- $a(r)$ has to be not singular and so from Eq.(1.16)

$$M(r) < r/2G_N \tag{1.32}$$

-

$$\rho'(r) \leq 0. \tag{1.33}$$

These are very weak requirements because the density ρ of every real star decreases with r. Recalling Eq.(1.11a) and Eq.(1.11b) we can write

$$3R_{rr}b + R_{tt}a = b'' - \frac{b'}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right) - \frac{3ba'}{ra} - \frac{b'}{r} = -16\pi G_N \rho\, ab. \tag{1.34}$$

Replacing b with a new variable α such that $b = \alpha^2$, after some tedious algebra, Eq.(1.34) becomes

$$\left[\frac{1}{r}\left(1 - \frac{2G_N M(r)}{r}\right)^{1/2} \alpha'(r)\right] = G_N \left[1 - 2G_N M(r)/r\right]^{-1/2} \left[\frac{M(r)}{r^3}\right]' \alpha(r). \tag{1.35}$$

Using Eq.(1.21) we come up with

$$\alpha(R) = \left[1 - \frac{2G_N M}{R}\right]^{1/2} \tag{1.36}$$

$$\alpha'(R) = \frac{G_N M}{r^2}\left(1 - \frac{2G_N M}{R}\right)^{1/2}. \tag{1.37}$$

We notice that α can never vanish because, from Eq.(1.21) it would mean that p is singular. As we have chosen α to be positive for $r = R$, the right hand side of Eq.(1.35) must be negative because $3M(r)/4\pi r^3$ is the mean density within the radius r which by hypotesis decreases with r. Thus we have

$$\left[\frac{1}{r}\left(1 - \frac{2G_N M(r)}{r}\right)^{1/2} \alpha'(r)\right] \leq 0. \tag{1.38}$$

Integrating Eq.(1.38) from 0 to R and using Eq.(1.37) we obtain

$$\alpha'(r) \geq \frac{MG_N r}{R^3}\left(1 - \frac{2G_N M(r)}{r}\right)^{-1/2}. \tag{1.39}$$

Finally, integrating Eq.(1.39) from 0 to R and using Eq.(1.36) we come up with

$$\alpha(0) \leq \left(1 - \frac{2G_N M}{R}\right)^{1/2} - \frac{MG_N}{R^3}\int_0^R dr \frac{r}{(1 - 2G_N M(r)/r)^{1/2}}. \tag{1.40}$$

From Eq.(1.40) we see that the configuration that maximizes $\alpha(0)$ is the one with the smallest $M(r)$ possible. At the same time it can be easily checked that the configuration with a given M and R that minimizes $M(r)$ is

$$M(r) = \frac{Mr^3}{R^3}, \tag{1.41}$$

that is the configuration of a ball of constant density.

Now if we compute the right hand side of Eq.(1.40) using Eq.(1.41) we obtain an upper limit for $\alpha(0)$ for every static configuration of given mass and radius.

$$\alpha(0) \leq \frac{3}{2}\left(1 - \frac{2MG_N}{R}\right)^{1/2} - \frac{1}{2}. \tag{1.42}$$

From Eq.(1.39) we see that the derivative of $\alpha(r)$ is always positive and so $r = 0$ is the point of minimum. This means that if $\alpha(0) > 0$, $\alpha(r)$ will be different from 0 everywhere and there are not singular points for the pressure p. For this reason, the condition of a stability for a configuration of given mass and radius is

$$\frac{MG_N}{R} < \frac{4}{9}, \tag{1.43}$$

which is the same we obtained for a star of constant density, but now we know that this is a general statement valid for every configuration. This is the so called *Buchdahl limit* that fixes the lower limit of the radius R of a spherical source in order to have a static non singular distribution of matter.

1.3 Gravitational collapse

In the previous section we have derived the limit of compactness for a star, but there are other solutions of Einstein equations with the same symmetry that give rise to singularities such as the Swarzschild solution of vacuum and the Robertson-Walker solution of a homogeneous universe. Then, one could be tempted to ask what would happen if the symmetry of the system was lowered, introducing a not symmetric density function for the source. An analytical solution would be very hard to construct for an arbitrary density function and using numerical solutions would become tedious, since Einstein equations should be solved for each possible configuration and possible initial data. This is why we need a more general statement, able to tells us when an object is going to collapse. This is what the genius of Penrose and Hawking (Ref.[7] - Ref.[10]) achieved for us formulating the so called *singularity theorems*.

1.3.1 Casual structure of space-time

In General Relativity the concept of "past" and "future" events $(\mathcal{M},\, g_{\mu\nu})$ cannot be *globally* defined as simply as in Minkownski space-time because g does not always describe a trivial topology. Physical observations and intuition suggest that in ordinary space-times this definition is always possible, but when singularities occur we should rely on a more rigorous treatment of the problem. Thus, we shall give some basic definitions and results concerning the *causal structure* of space-time in General relativity for which we remand to Ref.[4] and Ref.[3] for a more detailed treatment.

First of all we define a space-time as a couple $(\mathcal{M},\, g_{\mu\nu})$ where $\mathcal{M}$ is a C^∞ Hausdroff manifold with a metric $g_{\mu\nu}$. Now we have that in each point $p \in \mathcal{M}$ the tangent space V_p is isomorphic to Minkowski space-time and, as a consequence of the *Equivalence Principle*, we can construct a Minkowski like light cone in this space and call an half of it "future" and the other half "past" and do the same for every point p of $\mathcal{M}$. If a continuous choice of this labelling can be made, we say that $(\mathcal{M},\, g_{\mu\nu})$ is *time orientable.* Then we shall introduce some important objects whose definition is possible only in a time orientable space-time.

Definition 1.3.1. *In a time orientable space-time we define the chronological future of a point p as the set of points $I^+(p)$ that can be connected to p through a timelike, future oriented curve.*

Definition 1.3.2. *In a time orientable space-time we define the causal future of a point p as the set of points $J^+(p)$ that can be connected to p through a causal, future oriented curve.*

In an analogous way we can define the chronological past $I^-(p)$ and the causal past $J^-(p)$ of a point $p \in \mathcal{M}$. These definitions can be trivially extended to an hypersurface $\Sigma \subset \mathcal{M}$ defining $I^+(\Sigma)$ as $I^+(\Sigma) = \bigcup_{p\in\Sigma} I^+(p)$ and $J^+(\Sigma)$ as $J^+(\Sigma) = \bigcup_{p\in\Sigma} J^+(p)$. We shall also define the concept of achronal set that will be important to define a Cauchy surface.

Definition 1.3.3. *A set $S \in \mathcal{M}$ is said to be achronal if for any two points p and r in S, $r \notin I^+(p)$ and there are not points in common between $I^+(S)$ and S.*

With these definitions we can now state the following theorem whose demonstration can be found in section 8.1 of Ref.[4] and that will be essential in the demonstration of the Penrose singularity theorem.

Theorem 1.3.4. *Consider $S \subset \mathcal{M}$ where $(\mathcal{M},\, g_{\mu\nu})$ is a time orientable space-time. Then $I^+(S)$ is an achronal. three dimensional, embedded, C^0 submanifold of $\mathcal{M}$.*

At the beginning of this section we have assumed to be working with time orientable space-times, but this is not enough to guarantee a globally physical behavior of causal curves. For example there would be closed causal curves, whose existence would lead to the possibility for an observer on such a trajectory to change the past. Then, in order to exclude these curves from the theory we should introduce the concept of *stable causality.*

Definition 1.3.5. *A space-time $(\mathcal{M},\, g_{\mu\nu})$ is said to be stably causal if there exists a function f such that ∇f is a past directed timelike vector field.*

We point out that, until this moment we have considered $J^+(p)$ and $I^+(p)$ that are sets of events that can be influenced by S. Now we focus our attention on events that are *completely determined* by an appropiate set of events S. To do so, we first define a future *end point* of a causal curve λ as a point $p \in \mathcal{M}$ such that, for every neighborhood $\mathcal{O}$ of p, there exist a time t_0 such that, for every $t \geq t_0$, $\lambda(t) \in \mathcal{O}$. Then we say that a curve is *future inextendible* if it has not any future endpoint. We can define in the same way a past endpoint and a past *inextendible* causal curve. Thus, we can introduce the concept of domain of dependence, which is essential for the definition of a *globally hyperbolic space-time.*

Definition 1.3.6. *The future domain of dependence $D^+(S)$ of a closed achronal set S is the set of points $p \in \mathcal{M}$ such that every past directed causal curve passing through p intersects S.*

We can define similarly the *past domain of dependence* $D^-(S)$ of S as the set of points $p \in \mathcal{M}$ such that every future directed causal curve passing through p intersects S. Then, the *domain of dependence* of S is obviously

$$D(S) = D^+(S) \cup D^-(S). \tag{1.44}$$

Thus, we can define a Cauchy surface Σ as a closed achronal set such that $D(\Sigma) = \mathcal{M}$. It can be interpreted as an "initial data" surface, meaning that, knowing the velocity u^μ of a probe mass (or a light ray) in any point of the surface Σ it would be possible to construct its whole dynamics at any time. To have a more intuitive idea we recall that in Minkowski space-time a Cauchy surface is the familiar three dimensional space. Thus, we can define the very important concept of *global hyperbolicity.*

Definition 1.3.7. *A space-time is said to be global hyperbolic if it possesses a Cauchy hypersurface* Σ.

We end this section recalling theorem (8.3.10) of Ref.[4] which states that a global hyperbolic space-time is also stably causal. This will be important in the next sections because, even though global hyperbolicity is assumed to be valid for most of space-times, it can happen that this condition is not respected. However, for the validity of some results, it is sufficient only stable causality of space-time as we shall see.

1.3.2 Raychaudhuri equation

Consider a *congruence* of geodesics in a subspace $\mathcal{O}$ of $\mathcal{M}$, namely a set of timelike geodesics $\gamma(s)$ such that through each point $p \in \mathcal{O}$ passes only one curve of the family. We can assume that these geodesics are parameterized by their proper time τ and the four-velocity u^μ is normaized as $u_\mu u^\mu = -1$. We assume the Minkowski metric with sign $(-,+,+,+)$. Now we define the respectively the *expansion*, *shear* and *twist* as

$$\Theta = \nabla_\mu u_\nu h^{\mu\nu} \tag{1.45}$$

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} - \frac{1}{3} h_{\mu\nu} \tag{1.46}$$

$$\omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]}, \tag{1.47}$$

where $(\mu\nu)([\mu\nu])$ states from symmetrization (antisymmetrization) of the two pedicies and $h_{\mu\nu}$ is the spatial metric defined as

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu. \tag{1.48}$$

We notice that the shear is a completely symmetric tensor while the twist is antisymmetric. In terms of this new quantities we can write $\nabla_\mu u_\nu$ as

$$\nabla_\mu u_\nu = \frac{1}{3}\Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{1.49}$$

By the very definition of the Riemann tensor Eq.(1.6) it is immediate to verify that

$$[\nabla_\nu, \nabla_\rho]\, u_\mu = R^\alpha_{\mu\rho\nu} u_\alpha, \tag{1.50}$$

so that we can write

$$\begin{aligned} u^\nu \nabla_\nu \nabla_\mu u_\rho &= u^\nu \nabla_\mu \nabla_\nu u_\rho + R^\lambda_{\rho\nu\mu} u_\lambda u^\nu \\ &= \nabla_\mu \left(u^\nu \nabla_\nu u_\rho\right) - \left(\nabla_\nu u_\rho\right)\left(\nabla_\mu u^\nu\right) + R^\lambda_{\rho\nu\mu} u_\lambda u^\nu \\ &= -\left(\nabla_\nu u_\rho\right)\left(\nabla_\mu u^\nu\right) + R^\lambda_{\rho\nu\mu} u_\lambda u^\nu, \end{aligned} \tag{1.51}$$

where we used the geodesic equation

$$u^\mu (\nabla_\mu u_\nu) = 0. \tag{1.52}$$

Contracting ρ and μ and using Eq.(1.49), Eq.(1.51) becomes

$$u^\nu \nabla_\nu \Theta = \frac{d\Theta}{d\tau} = R^{\lambda\nu} u_\lambda u_\nu - \frac{1}{3}\Theta^2 - \sigma_{\mu\nu}^{\ \mu\nu} + \omega_{\mu\nu}^{\ \mu\nu}, \tag{1.53}$$

which is called *Raychaudhuri equation*. Multiplying the Einstein equations (1.10) for $u^\mu u^\nu$ we immediately obtain

$$R_{\mu\nu} u^\mu u^\nu = -8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) u^\mu u^\nu = -8\pi G_N \left(T_{\mu\nu} u^\mu u^\nu + \frac{1}{2} T g_{\mu\nu}\right). \tag{1.54}$$

Now we have to assume that the density of energy seen by an observer that moves on a geodesic with velocity u^μ is positive, namely

$$T_{\mu\nu} u^\mu u^\nu \geq 0. \tag{1.55}$$

This is a quite reasonable request, valid for almost any kind of classic matter that is known as *weak energy condition*. However the crucial condition that has to be verified is the *strong energy condition*

$$T_{\mu\nu} u^\mu u^\nu \geq -\frac{1}{2} T \tag{1.56}$$

which implies $R_{\mu\nu} \leq 0$. This seems a physically reasonable request, since for most distribution of matter the pressure cannot be strongly negative enough to make the right hand side of Eq.(1.54) negative. It can be shown (Ref.[4]) that Eq.(1.55) and Eq.(1.56) are respectively equivalent to

$$\rho \geq 0 \qquad \rho + p_i \geq 0 \;\; (i = 1, 2, 3) \tag{1.57}$$

$$\rho + \sum_{i=1}^{3} p_i \geq 0 \qquad \rho + p_i \geq 0 \;\; (i = 1, 2, 3) \tag{1.58}$$

where ρ can be thought as the energy density of the source at rest, while p_i $i = 1, 2, 3$ are called *principal pressures*. In this way we obtained the energy conditions in terms of familiar variables that are the pressure and and the energy density.

Supposing that the congruence is hypersurface orthogonal[3] and the energy conditions hold, we obtain from the Raychaudhuri equation (1.53) that

$$\frac{d\Theta}{d\tau} + \frac{1}{3}\Theta^2 \leq 0 \tag{1.59}$$

This inequality can be easily solved giving

$$\Theta^{-1}(\tau) \geq \Theta(0)^{-1} + \frac{1}{3}\tau \tag{1.60}$$

Eq.(1.60) states that if the value of the expansion $\Theta(0)$ is negative at any time, then its value will diverge negatively within a proper time of $3/|\Theta(0)|$. This means that the geodesics will tend to converge into a point. However, this does not mean we have obtained a singularity of space-time because at this stage we are still considering a background geometry that it is not affected by the motion of particles.

1.3.3 Conjugate points and surfaces

Definition 1.3.8. *Two points p and q of a space-time ($\mathcal{M}$, $g_{\mu\nu}$) are said to be conjugate if there exists a deviation vector η^μ, solution of*

$$v^\mu \nabla_\mu (v^\nu \nabla_\nu \eta^\lambda) = -R^\lambda_{\mu\nu\rho} \eta^\nu v^\mu v^\rho \tag{1.61}$$

such that $\eta^\mu = 0$ at both p and q.

We want to obtain a statement that tells us whenever two points are conjugate. Thus we consider a geodesic γ, a point p and the congruence of timelike geodesics passing through p[4].We notice that the deviation vector can always be chosen to be orthogonal to the 4-velocity u^μ. For this reason it can be written as a linear expansion of three spacelike vectors e^μ_a with $a = 1, 2, 3$ orthogonal to u^μ and parallelly transported along γ so that the orthogonality condition is satisfied for any point of the curve.

Now we can write the differential equation for the deviation vector

[3]This condition is equivalent to $\omega_{\mu\nu} = 0$.

[4]This means that p is a singular point for the congruence.

$$\frac{d^2\eta^a}{d\tau^2} = -\sum_{b\,c\,d} R^a_{bdc} u^b \eta^c u^d. \tag{1.62}$$

This is a second order linear differential equation, which means the value of the deviation at each time has to be a linear combination of the initial conditions $\eta^a(0)$ and $d\eta^a(0)/d\tau$. By construction $\eta^a(0) = 0$ and so we can write

$$\eta^a(\tau) = \sum_b C^a_b(\tau) \frac{d\eta^b}{d\tau}(0) \tag{1.63}$$

We see that the only way to have a non trivial solution of the above equation is that $|\det C| = 0$. Substituting this expression into Eq.(1.62) we find an equation for C^a_b that is

$$\frac{d^2 C^a_b}{d\tau^2} = -\sum_{c\,d\,e} R^a_{c\,e\,d} u^c C^d_b u^e \tag{1.64}$$

The necessary condition to have a a point q conjugate to p is that η vanishes for some value of τ. Thus we note that

$$\begin{aligned} \frac{d\eta^a}{d\tau} &= u^\mu \nabla_\mu \eta^a = u^\mu \nabla_\mu [(e_a)_\nu \eta^\nu] \\ &= (e_a)_\nu u^\mu \nabla_\mu \eta^\nu = (e_a)_\nu \eta^\mu \nabla_\mu u^\nu \\ &= \sum_{b=1}^{3} \eta^b \nabla_b u^a \end{aligned} \tag{1.65}$$

where we used the fact that e^μ_a is parallely transported and that[5].

$$u^\mu \nabla_\mu \eta^\nu = \eta^\mu \nabla_\mu u^\nu. \tag{1.66}$$

Now, using Eq.(1.63) we find

$$\frac{d\eta^a}{d\tau} = \sum_{b=1}^{3} \nabla_b u^a C^b_d \frac{d\eta^d}{d\tau}(0) = \sum_b \frac{dC^a_b(\tau)}{d\tau} \frac{d\eta^b}{d\tau}(0), \tag{1.67}$$

that can be rewritten in a matrix notation as

$$(\nabla u) = \left(\frac{dC}{d\tau} C^{-1}\right)^T, \tag{1.68}$$

[5] This equality comes from the fact the Lie derivative of the deviation with respect to the four-velocity is null.

and

$$\Theta = \text{tr}(\nabla u) = \text{tr}\left(\frac{dC}{d\tau}C^{-1}\right) = \frac{d}{d\tau}\,\text{tr}\,(\ln C(\tau)) = \frac{1}{|\det C|}\frac{d}{d\tau}|\det C|. \tag{1.69}$$

Since C^{μ}_{ν} follows the differential equation Eq.(1.64), $d|\det C|/d\tau$ cannot be infinite anywhere. This means that Θ can diverge only if $|\det C(\tau)| = 0$ for some value of the proper time τ. To get convinced that in this case Θ diverges negatively we can imagine to "follow the evolution" of geodesics until we arrive in proximity of the value τ_0 for which $|\det C| = 0$. Thus we can write

$$\Theta = \lim_{\tau\to\tau_0^-} \frac{1}{|\det C(\tau)|}\frac{|\det C(\tau)| - |\det C(\tau_0)|}{\tau - \tau_0} = \lim_{\tau\to\tau_0^-} \frac{1}{\tau-\tau_0}. \tag{1.70}$$

Since we are approaching τ_0 from the left (time goes forward), Θ diverges negatively. Conversely if $\Theta \to -\infty$ then $|\det C| \to 0$. Thus the negative divergence of Θ is a sufficient and necessary condition in order to have $|\det C| = 0$.

In Ref.[3] it is demonstrated that the congruence of timelike geodesic passing through a point p is hypersurface orthogonal which means that $\omega_{\mu\nu} = 0$. Thus we can use the statement we demonstrated in the previous section and say that in a space-time where $u^\mu u^\nu R_{\mu\nu} \leq 0$ for every timelike vector u^μ, if the expansion of a congruence of geodesic with velocity u^μ passing through p assumes a negative value Θ_0 in at least one point, the congruence will approach a point q conjugate to p within a time of $3/|\Theta_0|$. It can be further proven (Ref.[3]) that in a space-time where $u^\mu u^\nu R_{\mu\nu} \leq 0$ and $R_{\mu\nu\rho\sigma}u^\mu u^\nu \neq 0$ for every complete geodesic with velocity u^μ there has to be necessarily a neighborhood of a point where the value of the expansion is negative. This condition goes under the name of *timelike generic condition*. If it holds and $R_{\mu\nu}u^\mu u^\nu \leq 0$ for any timelike u^μ, every complete timelike geodesic will have a pair of conjugate points.

Now we want to establish a connection between conjugate points and curves of maximum proper time. In order to do this let us consider a family of smooth curves parameterized by a variable β, λ_β such that $\lambda_\beta(a) = p$, $\lambda_\beta(b) = q$ for each α where $p, q \in \mathcal{M}$. The proper time for all these curves is given by

$$\tau_\beta = \int_a^b dt(-u_\mu u^\mu)^{1/2} \tag{1.71}$$

and the derivative

$$
\begin{aligned}
\frac{d\tau}{d\beta} &= -\int_a^b dt\, \frac{1}{F}\eta^\mu \nabla_\mu (u_\nu) u^\nu \\
&= -\int_a^b dt\, \frac{1}{F} u^\mu \nabla_\mu (\eta_\nu) u^\nu \\
&= -\int_a^b dt\, u^\mu \nabla_\mu \left(\frac{u_\nu \eta^\nu}{F}\right) + \int_a^b dt\, u^\mu \nabla_\mu \left(\frac{u^\nu}{F}\right)\eta_\nu \\
&= \int_a^b dt\, u^\mu \nabla_\mu \left(\frac{u^\nu}{F}\right)\eta_\nu
\end{aligned}
\tag{1.72}
$$

where we used $u^\mu \nabla_\mu \eta^\nu = \eta^\mu \nabla_\mu u^\nu$ and the fact that the deviation vector is null in $t = a$, $t = b$. From Eq.(1.72) we obtain that a necessary condition for a curve to maximize τ is that it has to be a geodesic. Then, the second variation of the proper time reads

$$
\begin{aligned}
\left.\frac{d^2\tau}{d\beta^2}\right|_{\alpha=0} &= \int_a^b dt\, \eta^\gamma \nabla_\gamma \left[u^\mu \nabla_\mu \left(\frac{u^\nu}{F}\right)\eta_\nu\right] \\
&= \int_a^b dt\, \eta^\gamma \eta^\nu \nabla_\gamma \left[u^\mu \nabla_\mu \left(\frac{u^\nu}{F}\right)\right] \\
&= \int_a^b dt\, \eta^\gamma \eta_\nu u^\mu \nabla_\gamma \nabla_\mu \left(\frac{u^\nu}{F}\right) + \int_a^b dt\, \eta^\gamma \eta_\nu \nabla_\mu \left(\frac{u^\nu}{F}\right)\nabla_\gamma u^\mu \\
&= \int_a^b dt\, \eta^\gamma \eta_\nu u^\mu \nabla_\mu \nabla_\gamma \left(\frac{u^\nu}{F}\right) + \int_a^b dt\, u^\mu \eta^\gamma \eta^\nu R^\lambda_{\nu\mu\gamma}\left(\frac{u_\lambda}{F}\right) \\
&\quad + \int_a^b dt\, \eta_\nu u^\gamma \nabla_\gamma \eta^\mu \nabla_\mu \left(\frac{u^\nu}{F}\right) \\
&= \int_a^b dt\, \eta^\nu u^\mu \nabla_\mu \left[\eta^\gamma \nabla_\gamma \left(\frac{u^\nu}{F}\right)\right] + \int_a^b dt\, u^\mu \eta^\gamma \eta^\nu R^\lambda_{\nu\mu\gamma}\left(\frac{u_\lambda}{F}\right)
\end{aligned}
\tag{1.73}
$$

where we supposed the curve λ_0 is a geodesic. Finally, choosing the parameterization in such a way to have $F = 1$ along the geodesic, we can write the second derivative as

$$
\left.\frac{d^2\tau}{d\beta^2}\right|_{\alpha=0} = \int_a^b dt\, \eta^\nu u^\mu \nabla_\mu \left(\eta^\gamma \nabla_\gamma u^\nu\right) + \int_a^b dt\, u^\mu \eta^\gamma \eta^\nu R^\lambda_{\nu\mu\gamma} u_\lambda \tag{1.74}
$$

Thus we are able to establish the following theorem.

Theorem 1.3.9. *Consider a smooth timelike curve γ that connects two points of space-time $p, q \in \mathcal{M}$ where $(\mathcal{M}, g_{\mu\nu})$ is a space-time such that $R_{\mu\nu} \leq 0$ for every timelike vector u^μ. Then γ locally maximizes the proper time between p and q if, and only if, γ is a geodesic and there are not conjugate points to p between p and q.*

Proof. If γ is not a geodesic, Eq.(1.72) is not 0 and so it will not maximize the proper time Eq.(1.71). Then if γ is a geodesic but there is a point r between p and q, conjugate to p, it means that there exists a deviation vector field η^0_μ such that η^0_μ is 0 at both p and q. Thus we can define a new continuous deviation vector $\eta_\mu = \eta^0_\mu$ between p and r and $\eta_\mu = 0$ between r and q. In that region, the second derivative of proper time with respect β will be 0. We can think of making an infinitesimal change of the path in such a way that $d^2\tau/d\beta^2 > 0$ when $\alpha = 0$ meaning that γ minimizes the proper time. For further details see Ref.[3]. In this way we have proven that a necessary condition for γ to maximize the proper time is to be a geodesic and to have not conjugate points. Conversely, if γ is a geodesic with no conjugate points between p and q, the matrix C defined previously is not singular in any point of the curve. Thus substituting the expression $\eta^\mu = C_{\mu\nu}x^\nu$ in Eq.(1.74) it can be demonstrated Ref.[3] that $d\tau/d\beta$ is definite negative. □

Now we want to study the conjugacy between points and surfaces. Thus consider a spacelike hypersurface Σ and a congruence of timelike geodesics with velocity u^μ orthogonal to Σ. We define the extrinsic curvature $K_{\mu\nu}$ as

$$K_{\mu\nu} = \nabla_\mu u_\nu \tag{1.75}$$

By definition $K_{\mu\nu}$ is spacelike and the congruence is manifestly hypersurface orthogonal. Moreover $K_{\mu\nu}$ is symmetric and using

$$\mathcal{L}_u g_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu \tag{1.76}$$

we can write

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_u g_{\mu\nu} = \frac{1}{2}\mathcal{L}_u \left(h_{\mu\nu} - u_\mu u_\nu\right) = \frac{1}{2}\mathcal{L}_u h_{\mu\nu} \tag{1.77}$$

where in the last step we used the geodesic equation Eq.(1.52). Now, recalling the definition of the expansion Eq.(1.45) we immediately find

$$K_{\mu\nu}g^{\mu\nu} = K_{\mu\nu}h^{\mu\nu} = K = \Theta \tag{1.78}$$

Thus, we can define the conjugacy between a point and a surface.

Definition 1.3.10. *A point $p \in \mathcal{M}$ is said to be conjugate to an hypersurface Σ if there exist a deviation vector that vanishes in p but is not null on the surface.*

Thus following the same arguments discussed previously in the case of conjugate points we can state the following two theorems.

Proposition 1.3.11. *Consider a metric $g_{\mu\nu}$ defined on space-time such that $R_{\mu\nu}u^\mu u^\nu \leq 0$ for each timelike u^μ. Let Σ be an hypersurface such that there exist a point $q \in \Sigma$ where*

Θ assumes a negative value Θ_0. Then within a proper time $\tau < 3/|\Theta_0|$there will be a point p conjugate to Σ along the geodesic γ orthogonal to Σ in the point q.

Theorem 1.3.12. *Let γ be a smooth timelike curve connecting two points $p \in \mathcal{M}$ and $q \in \Sigma$ where Σ is a spacelike hypersurface. Then γ is the curve that locally maximizes the proper time between p and q if and only if γ is a geodesic orthogonal to Σ with no conjugate points between p and Σ.*

1.3.4 Null geodesics

In this section we want to write down some results concerning conjugate points and surfaces for null geodesics, pointing out the main differences with the timelike case.
First of all, we notice that in the case of null curves Eq.(1.54) becomes

$$R_{\mu\nu}k^\mu k^\nu = -8\pi G_N \left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) k^\mu k^\nu = -8\pi G_N T_{\mu\nu} k^\mu k^\nu, \tag{1.79}$$

where we used $k_\mu k^\mu = 0$.
Then, in order to have $R_{\mu\nu}k^\mu k^\nu \leq 0$, we have to impose on the matter source only the weak energy condition that is

$$T_{\mu\nu}k^\mu k^\nu \geq 0. \tag{1.80}$$

Eq.(1.80) can be also written, in terms of the principal pressures and the energy density, as

$$\rho + p_i \geq 0 \qquad i = 1, 2, 3. \tag{1.81}$$

Thus in the case of null curves the validity of the strong energy condition (1.56) is not necessary. However, the main difference is that in the case of null curves the space of physically interesting deviation vectors is two-dimensional. To see why this happens, we introduce the space of vectors orthogonal to k^μ in a point p, $\tilde{V}_p$, and the restriction of the metric $g_{\mu\nu}$ on that space $\tilde{h}_{\mu\nu}$. Then, since $\tilde{h}_{\mu\nu}k^\mu = 0$ and k_μ is orthogonal to himself and thus belongs to $\tilde{V}_p$, $\tilde{h}_{\mu\nu}$ is not a metric on $\tilde{V}_p$. For this reason we have to introduce a class of equivalence in which two vectors of $\tilde{V}_p$ belong to the same class if their difference is proportional to k^μ. The vector space obtained considering only one vector from the equivalence class just defined will be denoted as $\hat{V}_p$. In Ref.[4] it is shown that $g_{\mu\nu}$ gives rise to a *positive definite metric $\hat{h}_{\mu\nu}$ when its action is restricted on $\hat{V}_p$* that is a two-dimensional space.
Finally, we recall that the *null generic condition* is satisfied if either $R_{\mu\nu}k^\mu k^\nu \neq 0$ or $k_{[\eta}C_{\alpha]\beta\gamma[\delta}k_{\omega]}k^\beta k^\gamma \neq 0$ and that we cannot use the proper time τ to parameterize null geodesics as we did in the timelike case, but we shall use the *affine length* λ.

Now we define a *Jacobi field* η^μ on a null geodesic with tangent k^μ as

$$k^\gamma \nabla_\gamma \left[k^\beta \nabla_\beta (k^\alpha \eta_\alpha)\right] = 0 \tag{1.82}$$

We see that if η^μ is a deviation vector field, then also $\eta'^\mu = \eta^\mu + (a + b\lambda)k^\mu$ will be a solution of Eq.(1.82). Thus we can write the following result.

Proposition 1.3.13. *Let ($\mathcal{M}$, $g_{\mu\nu}$) be a space-time satisfying $R_{\mu\nu} \leq 0$ for all null geodesics k^μ. Consider a point $p \in \mu$ where μ is a null geodesic. If the convergence Θ of null geodesics emanating from p attains a negative value at some point $r \in \mu$, then within affine length $\lambda \leq 2/|\Theta_0|$ along μ there exists a point q conjugate to p assuming that μ can be extended that far.*

Following an argument similar to the one used to prove theorem 1.3.9 for timelike geodesics (the complete proof is given in Ref.[3])we can prove the following statement.

Theorem 1.3.14. *Consider a curve γ and two points p,$q \in \gamma$. Then γ cannot be smoothly deformed to a timelike curve if and only if γ is a null geodesic with no conjugate points between p and q.*

Now we shall define a notion of conjugacy for a null geodesics and a two dimensional spacelike surface S.For each point of S we can choose two linearly independent future directed null vectors orthogonal to S. If a continuous choice can be made throuhout S, then we can call one of the two vector fields "ingoing" and the other "outgoing". We will say that p is conjugate to S if there is a deviation vector $\hat{\eta}^\mu$ that is different from zero on S, but vanishes at p. Then we have all the elements necessary to state the following theorem, analogous to 1.3.10 and 1.3.11 of the previous section.

Proposition 1.3.15. *Consider a space-time ($\mathcal{M}$, $g_{\mu\nu}$) satisfying $R_{\mu\nu} \leq 0$ for all null k^μ. Consider a point $q \in S$, where S is a two dimensional spacelike submanifold of $\mathcal{M}$, such that the value of the expansion Θ_0 of the "outgoing" ("ingoing") set of null curves orthogonal to S in q is negative. Then within affine length $2/|\Theta_0|$ there exist a point p conjugate to S along the "outgoing" ("ingoing") null geodesic γ passing through q.*

Theorem 1.3.16. *Consider a smooth causal curve γ from a smooth two dimensional spacelike surface to a point p. Then γ cannot be deformed to a timelike curve connecting S and p if, and only if, γ is a null geodesic, orthogonal to S and with no points conjugate between p and S.*

As a consequence of this theorem we get the following statement, whose demonstration can be found in Ref.[4].

Theorem 1.3.17. *Consider a global hyperbolic space-time ($\mathcal{M}$, $g_{\mu\nu}$) and a two-dimensional, compact, orientable and spacelike surface S in $\mathcal{M}$. Then every point $p \in \partial I^+(S)$ belongs to a future directed null geodesic orthogonal to S and with no points conjugate between p and S.*

1.3.5 Singularity theorems

Now we have developed all the theory necessary to understand the problem of the formation on singularities by gravitational collapse. We shall focus in particular on two theorems, that are the most important in the context of gravitational collapse. These are respectively the Penrose singularity theorem (1965) that was the first to be develped and the Hawking-Penrose singularity theorem (1970) that is the most complete and strengthens the result of other theorems.

First of all we introduce the concept of a *trapped surface*, that is a compact, two dimensional, smooth spacelike submanifold S for which both the outgoing and ingoing sets of null curves have a value of the expansion Θ negative throughout S. For example, in the case of Shwarzschild metric, all the surfaces with Schwarzschild radius $r < R_H$ are trapped surface.

Now we are ready to state the *Penrose singularity theorem* (Ref.[10],1965), which states that under very general hypotheses a singularity must occur when a trapped surface is formed.

Theorem 1.3.18. *Consider a compact globally hyperbolic space-time ($\mathcal{M}$, $g_{\mu\nu}$) with a non-compact Cauchy surface Σ. Suppose that the weak or strong energy condition are verified and that ($\mathcal{M}$,$g_{\mu\nu}$) is a solution of Einstein equations. We further suppose that $\mathcal{M}$ contains a trapped surface T. Then there is at least one future directed, inextendible, orthogonal null geodesic starting from T, with affine length $\lambda \leq 2/|\Theta_0|$ where Θ_0 is the greatest value of Θ for both sets of "ingoing" and "outgoing" geodesics.*

Proof. Suppose that all the future directed null geodesics from T have affine length longer than $2/|\Theta_0|$. Then we can define two functions f^+ and f^-, such that $f^+ : [T; 0, 2/|\Theta_0|] \rightarrow \mathcal{M}$ where $f^+(q, \lambda)$ is the point corresponding to the parameter λ of the orthogonal outgoing null geodesic γ starting at $q \in T$. We define in the same way f^- for ingoing geoedsics. Since $[T; 0, 2/|\Theta_0|]$ is a compact set and f^+ and f^- are both continuous, the union of their images, that we will denote as A must also be compact.
Now, using proposition 1.3.15 and theorem 1.3.17 we come up to the conclusion that $\partial I^+(T)$ must be a closed subset of A. Then, as a consequence of theorem 1.3.4, $\partial I^+(T)$ will be a closed C^0 manifold and, since it is contained in a limited subset of $\mathcal{M}$ it will be limited and so compact.
Now, we shall show how the compactness of $\partial I^+(T)$ implies the compactness of Σ.
Since ($\mathcal{M}$, $g_{\mu\nu}$) is globally hyperbolic it will be also time orientable. Thus, there exist a timelike vector field u^μ defined throughout $\mathcal{M}$. Since $\partial I^+(T)$ is achronal, the curves of this vector field can intersect it at most once, while they intersect Σ exactly once. Thus we can define a function $F : I^+(T) \rightarrow \Sigma$ that associates to each point of $I^+(T)$ the corresponding point of Σ lying on the same timelike curve u^μ. If we restrict the function to the image of F on Σ that we will call S, then we obtain an homeomorphism between $I^+(T)$ and S $F : I^+(T) \rightarrow S$. Since F is an homeomorphism and $I^+(T)$ is

compact, so it will be S, that will be also closed. On the other hand, since by theorem 1.3.4 $I^+(T)$ is a three dimensional C^0 manifold, it can be covered by opens that are in one-to-one correspondence with open balls in $\mathbb{R}^3$. Thus, also S must be open. But since $\mathcal{M}$ is connected, so has to be Σ. This means that we have $S = \Sigma$, but this is impossible because S is compact and Σ is not. □

We end this section stating the *Hawking-Penrose singularity theorem* (Ref.[8],1970), which generalizes the results of other singularity theorems requiring less restricting conditions. We refer to [3] or [8] for the proof.

Theorem 1.3.19. *Consider a space-time ($\mathcal{M}$,$g_{\mu\nu}$) satisfying the following four conditions.*

- *($\mathcal{M}$,$g_{\mu\nu}$) is a solution of the Einstein equations and both weak and strong energy conditions hold.*
- *The timelike and null generic conditions are satisfied.*
- *No closed timelike curve exists.*
- *At least one of these three conditions holds:*
  - *($\mathcal{M}$,$g_{\mu\nu}$) possesses a closed achronal set without edge, namely ($\mathcal{M}$,$g_{\mu\nu}$) is a closed universe.*
  - *($\mathcal{M}$,$g_{\mu\nu}$) possesses a trapped surface.*
  - *There exist a point $p \in \mathcal{M}$ such that the expansion Θ of all future or past directed null geodesics emanating from p assume a negative value.*

Then ($\mathcal{M}$,$g_{\mu\nu}$) possesses at least one timelike or null incomplete geodesic.

We see from 1.3.19 that, adding the strong energy condition and the generic conditions to the hypothesis, we can eliminate the condition of global hyperbolicity of space-time that is necessary in 1.3.18. Thus 1.3.19 can be used in a wider range of situations than theorem 1.3.18, but it has weaker consequences, because it only states that there exists an incomplete geodesic without specifying its nature.

1.4 Post-Newtonian approximation

In this section we want to recall the main steps that lead the expression for the *post-Newtonian corrections* that describe particles moving in a weak gravitational field at very low velocities. When we consider a system bounded by the gravitational force in Newtonian theory we usually have a relation between the characteristic size of the system r, the velocity v and the mass M given by

$$\frac{G_N M}{r} \approx v^2. \tag{1.83}$$

This means that if we suppose to be sufficiently far away from the source of the field we can can expand all the quantities that we need in order to find the corrections to the Newtonian potential using as a parameter only the characteristic velocity of the system v and the relation Eq.(1.83). Since we are interested in finding the first order correction to the dynamics of particles in a gravitational field we want to find the correction to the Newtonian correction up to $(G_N M/r)^2$ that means we have to expand the Einstein equations Eq.(1.10) up to fourth order in the variable v.

Now, we shall notice that the derivatives with respect space and time behaves as

$$\frac{\partial}{\partial x^i} \approx \frac{1}{r} \qquad \frac{\partial}{\partial t} = \frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i} \approx \frac{v}{r}. \tag{1.84}$$

Then, recalling the physical interpretation of T^{00}, T^{0i} and T^{ij} that are respectively the energy density, momentum density and momentum flux we find that their series expansions in the variable Mv^n/r^3(we are taking c=1) read

$$\begin{aligned} T^{00} &= T^{00}_{(0)} + T^{00}_{(2)} + ... \\ T^{0i} &= T^{0i}_{(1)} + T^{0i}_{(3)} + ... \\ T^{ij} &= T^{ij}_{(2)} + T^{ij}_{(4)} + ... \ . \end{aligned} \tag{1.85}$$

On the other hand the expansions of the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ in terms of powers of v read

$$\begin{aligned} g_{00} &= -1 + g^{(2)}_{00} + g^{(4)}_{00} + ... \\ g_{0i} &= g^{(3)}_{0i} + g^{(5)}_{0i} + ... \\ g_{ij} &= \delta_{ij} + g^{(2)}_{ij} + g^{(4)}_{ij} + ... \, , \end{aligned} \tag{1.86}$$

and

$$\begin{aligned} g^{00} &= -1 + g^{00}_{(2)} + g^{00}_{(4)} + ... \\ g^{0i} &= g^{0i}_{(3)} + g^{0i}_{(5)} + ... \\ g^{ij} &= \delta^{ij} + g^{(2)}_{ij} + g^{ij}_{(4)} + ... \, , \end{aligned} \tag{1.87}$$

where the expansion in odd powers of g_{0i} comes from the change of sign that the component g_{0i} has to get in a time reversal trasformation $t \to -t$. We also notice that by the definition of $g^{\mu\nu}$, $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$ we get

$$g_{00}^{(0)} = g^{00}_{(0)} \qquad g_j^{(0)} = g^{ij}_{(0)} \qquad g_{00}^{(2)} = -g^{00}_{(2)} \qquad g_{0i}^{(3)} = g^{0i}_{(3)} \;\; \text{and so on}\,. \tag{1.88}$$

Thus the expansion of $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}$ reads

$$\begin{aligned} S_{00}^{(0)} &= \frac{1}{2} T^{00}_{(0)} \\ S_{00}^{(2)} &= \frac{1}{2}\left(T^{00}_{(2)} - 2 g_{00}^{(2)} T^{00}_{(0)} + T^{ii}_{(2)}\right) \\ S_{0i}^{(1)} &= -T^{i0}_{(1)} \\ S_{ij}^{(0)} &= \frac{1}{2}\delta_{ij} T^{00}_{(0)}. \end{aligned} \tag{1.89}$$

In order to get the expansion of the Ricci tensor we have to study at first the behavior of the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$. By their very definition Eq.(1.5) and using Eq.(1.86), Eq.(1.87), and Eq.(1.84) we easily find that the expansion of $\Gamma^{\lambda}_{\mu\nu}$ in terms of v/r reads

$$\begin{aligned} \Gamma^{i\;(2)}_{00} &= -\frac{1}{2}\frac{\partial g_{00}^{(2)}}{\partial x^i} \\ \Gamma^{i\;(4)}_{00} &= -\frac{1}{2}\frac{\partial g_{00}^{(4)}}{\partial x^i} + \frac{\partial g_{0i}^{(3)}}{\partial t} + \frac{1}{2}\frac{\partial^2 g_{00}^{(2)}}{\partial x^i \partial x^j} g_{ij}^{(2)} \\ \Gamma^{i\;(3)}_{0j} &= \frac{1}{2}\left[\frac{\partial g_{i0}^{(3)}}{\partial x^j} + \frac{\partial g_{ij}^{(2)}}{\partial t} - \frac{\partial g_{j0}^{(3)}}{\partial x^i}\right] \\ \Gamma^{i\;(3)}_{jk} &= \frac{1}{2}\left[\frac{\partial g_{ij}^{(2)}}{\partial x^k} + \frac{\partial g_{ik}^{(2)}}{\partial x^j} - \frac{\partial g_{jk}^{(2)}}{\partial x^i}\right] \\ \Gamma^{0\;(3)}_{00} &= -\frac{1}{2}\frac{\partial g_{00}^{(2)}}{\partial t} \\ \Gamma^{0\;(2)}_{0i} &= -\frac{1}{2}\frac{\partial g_{00}^{(2)}}{\partial x^i} \\ \Gamma^{0\;(1)}_{ij} &= 0. \end{aligned} \tag{1.90}$$

Using Eq.(1.7) and Eq.(1.90) we see that the expansion of the Ricci tensor $R_{\mu\nu}$ up to fourth order in the variable v/r^2 reads

$$
\begin{aligned}
R^{(2)}_{00} &= \frac{1}{2}\Delta g^{(2)}_{00} \\
R^{(3)}_{i0} &= \frac{1}{2}\frac{\partial^2 g_{jj}}{\partial x^i \partial t} - \frac{1}{2}\frac{\partial^2 g_{j0}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^i \partial t} + \frac{1}{2}\Delta g^{(3)}_{i0} \\
R^{(2)}_{ij} &= -\frac{1}{2}\frac{\partial^2 g^{(2)}_{00}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial^2 g^{(2)}_{kk}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial^2 g^{(2)}_{ik}}{\partial x^j \partial x^k} - \frac{1}{2}\frac{\partial^2 g^{(2)}_{kj}}{\partial x^k \partial x^i} + \frac{1}{2}\Delta g^{(2)}_{ij} \\
R^{(4)}_{00} &= \frac{1}{2}\Delta g^{(4)}_{00} - \frac{1}{2}\frac{\partial g^{(2)}_{00}}{\partial t^2} - \frac{1}{2} g^{(2)}_{ij}\frac{\partial^2 g^{(2}_{00}}{\partial x^i \partial x^j} \\
&- \frac{1}{2}\left(\frac{\partial g^{(2)}_{ij}}{\partial x^j}\right)\left(\frac{\partial g^{(2)}_{00}}{\partial x^i}\right) + \frac{1}{4}\left(\frac{\partial g^{(2)}_{00}}{\partial x^i}\right)\left(\frac{\partial g^{(2)}_{00}}{\partial x^i}\right) \\
&+ \frac{1}{4}\left(\frac{\partial g^{(2)}_{00}}{\partial x^i}\right)\left(\frac{\partial g^{(2)}_{jj}}{\partial x^i}\right).
\end{aligned}
\tag{1.91}
$$

Using the De Donder gauge

$$
g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0, \tag{1.92}
$$

we find for $g^{\mu\nu}\Gamma^{0}_{\mu\nu}$

$$
\frac{1}{2}\frac{\partial g^{(2)}_{00}}{\partial t} - \frac{\partial g^{(3)}_{0i}}{\partial x^i} + \frac{1}{2}\frac{\partial g^{(2)}_{ii}}{\partial t} = 0, \tag{1.93}
$$

while for $g^{\mu\nu}\Gamma^{i}_{\mu\nu}$ we get

$$
\frac{1}{2}\frac{\partial g^{(2)}_{00}}{\partial x^i} + \frac{\partial g^{(2)}_{ij}}{\partial x^j} - \frac{1}{2}\frac{\partial g^{(2)}_{jj}}{\partial x^i} = 0. \tag{1.94}
$$

Deriving Eq.(1.93) with respect time we get

$$
\frac{1}{2}\frac{\partial^2 g^{(2)}_{00}}{\partial t^2} - \frac{\partial^2 g^{(3)}_{0i}}{\partial x^i \partial t} + \frac{1}{2}\frac{\partial^2 g^{(2)}_{ii}}{\partial t^2} = 0. \tag{1.95}
$$

Then deriving Eq.(1.4) with respect x^j and symmetrysing with respect the indices i and j we get

$$
\frac{\partial^2 g^{(2)}_{00}}{\partial x^i x^j} + \frac{\partial^2 g^{(2)}_{ik}}{\partial x^k \partial x^j} + \frac{\partial^2 g^{(2)}_{jk}}{\partial x^k \partial x^i} - \frac{\partial^2 g^{(2)}_{kk}}{\partial x^i \partial x^j} = 0, \tag{1.96}
$$

while deriving Eq.(1.93) with respect x^i and Eq.(1.4) with respect t and subtracting we get

$$\frac{\partial^2 g^{(2)}_{jj}}{\partial x^i \partial t} + \frac{\partial^2 g^{(2)}_{j0}}{\partial x^i \partial x^j} - \frac{\partial^2 g^{(2)}_{ij}}{\partial x^j \partial t} = 0. \tag{1.97}$$

Substituting equation from Eq.(1.95), Eq.(1.96) and Eq.(1.97) into Eq.(1.91) we come up with

$$\begin{aligned}
R^{(2)}_{00} &= \frac{1}{2}\Delta g^{(2)}_{00} \\
R^{(3)}_{i0} &= \frac{1}{2}\Delta g^{(3)}_{i0} \\
R^{(2)}_{ij} &= \frac{1}{2}\Delta g^{(2)}_{ij} \\
R^{(4)}_{00} &= \frac{1}{2}\Delta g^{(4)}_{00} - \frac{1}{2}\frac{\partial g^{(2)}_{00}}{\partial t^2} - \frac{1}{2} g^{(2)}_{ij}\frac{\partial^2 g^{(2}_{00}}{\partial x^i \partial x^j} + \frac{1}{2}(\nabla g^{(2)}_{00})^2,
\end{aligned} \tag{1.98}$$

so that we can finally write Eq.(1.10) as

$$\begin{aligned}
\Delta g^{(2)}_{00} &= -8\pi G_N T^{00}_{(0)} \\
\Delta g^{(4)}_{00} &= \frac{\partial g^{(2)}_{00}}{\partial t^2} + g^{(2)}_{ij}\frac{\partial^2 g^{(2}_{00}}{\partial x^i \partial x^j} - (\nabla g^{(2)}_{00})^2 8\pi G_N \left(T^{00}_{(2)} - 2 g^{(2)}_{00} T^{00}_{(0)} + T^{ii}_{(2)}\right) \\
\Delta g^{(3)}_{i0} &= 16\pi G_N T^{i0}_{(1)} \\
\Delta g^{(3)}_{ij} &= -8\pi G_N \delta_{ij} T^{00}_{(0)}.
\end{aligned} \tag{1.99}$$

Thus we obtain as expected

$$g^{(2)}_{00} = -2V_N, \tag{1.100}$$

where V_N is the Newtonian potential whose general form is

$$V_N(\mathbf{x}, t) = -G_N \int d^3x' \frac{T^{00}_{(0)}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}. \tag{1.101}$$

Substituting Eq.(1.101) in the last equation of Eq.(1.99) we get

$$g^{(2)}_{ij} = -2\delta_{ij} V_N, \tag{1.102}$$

while $g^{(3)}_{i0}$ is given by

$$g^{(3)}_{i0} = -G_N \int d^3x' \frac{T^{0i}_{(1)}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}. \tag{1.103}$$

Finally using

$$||\nabla V_N||^2 = \frac{1}{2}\Delta V_N^2 - V_N \Delta V_N, \tag{1.104}$$

we obtain

$$-2V = g_{00}^{(2)} + g_{00}^{(4)} = -2V_N - 2V_N^2 - 2\delta V, \tag{1.105}$$

where δV is given by

$$\delta V(\mathbf{x}, t) = -\int \frac{d^3x'}{|\mathbf{x} - \mathbf{x'}|} \left[G_N T_{(2)}^{00}(\mathbf{x}', t) + G_N T_{(2)}^{ii}(\mathbf{x}, t) + \frac{1}{4}\frac{\partial^2 V_N(\mathbf{x}, t)}{\partial t^2} \right]. \tag{1.106}$$

1.4.1 Multipole expansion

If we want to compute the potential in a point $\mathbf{x}$ very far away from any point $\mathbf{x}'$ where the source of the gravitational field is not null we can expand $|\mathbf{x} - \mathbf{x}'|$ that appears in both Eq.(1.105) and Eq.(1.101) as

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^3} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^3}\right), \tag{1.107}$$

so that

$$V_N(\mathbf{x}, t) = -\frac{G_N M_{(0)}}{r} - \frac{G_N\, \mathbf{x} \cdot \mathbf{D}_{(0)}}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{1.108}$$

$$\delta V(\mathbf{x}, t) = -\frac{G_N M_{(2)}}{r} + \frac{G_N\, \mathbf{x} \cdot \mathbf{D}_{(2)}}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{1.109}$$

where $r = |\mathbf{x}|$ and

$$\begin{aligned} M_{(0)} &= \int d^3x T_{(0)}^{00}(\mathbf{x}) \\ \mathbf{D}_{(0)} &= \int d^3x' \left[\mathbf{x}' T_{(0)}^{00}(\mathbf{x}', t)\right] \\ M_{(2)} &= \int d^3x' \left[T_{(2)}^{00}(\mathbf{x}', t) + T_{(2)}^{ii}(\mathbf{x}', t)\right] \\ \mathbf{D}_{(2)} &= \int d^3x' \mathbf{x}' \left[T_{(2)}^{00}(\mathbf{x}', t) + T_{(2)}^{ii}(\mathbf{x}', t)\right]. \end{aligned} \tag{1.110}$$

Now, if we choose the reference frame of the center of energy where $\mathbf{D} = \mathbf{D}_{(0)} + \mathbf{D}_{(2)} = 0$, then we finally obtain

$$
\begin{aligned}
V(\mathbf{x}) =& V_N(\mathbf{x}) + V_N^2(\mathbf{x}) + \delta V(\mathbf{x}) \\
=& -\frac{G_N M}{r} + \left(\frac{G_N M}{r}\right)^2 + \frac{G_N\, \mathbf{x}\cdot\mathbf{D}}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right) \\
=& -\frac{G_N M}{r} + \left(\frac{G_N M}{r}\right)^2 + \mathcal{O}\left(\frac{1}{r^3}\right).
\end{aligned}
\tag{1.111}
$$

where $M = M_{(0)} + M_{(2)}$.

Chapter 2

Quantum Corpuscolar Black Holes

In this chapter we recall the main feaures of the corpuscolar model of black holes stated by Dvali and Gomez (Ref.[11]-[16]) in which they are described as a *Bose Einstein Condendsate* of soft gravitions and the geometric picture of gravity is seen just as a classical manifestation of the quantum theory of gravitational interactions.
Making this assumption about the true nature of black holes, it is possible to achieve a very simple explanation of phenomena related to their physics such as Hawking effect and Bekenstein area law in terms of a single parameter that is the number of gravitons N_G in the condensate. More specifically we want to show that all these phenomena can be interpreted as consequences of effects really well known in the context of ordinary Quantum Mechanics.

In the first section we explain the meaning of *classicality*, *self-completeness* and *universality* of the number of gravitons and how this properties permit to study black holes at any energy scale only in terms of the number of gravitons N_G. Then we study how the Hawking effect emerges from this model as a depletion of gravitons from the condensate at such a rate to give a temperature $T = \hbar/(l_p\sqrt{N_G})$ and how the entropy of a black hole is related to N_G.

2.1 Self-Completeness of General Relativity

The graviton coupling constant to the energy source is given by

$$h_{\mu\nu}\frac{T^{\mu\nu}}{m_p^2}. \tag{2.1}$$

This sets m_p as the natural energy scale for which the coupling becomes big. Then one would be tempted to say that Einstein gravity is valid as long as the energy of the system considered is little enough. However, when the energy becomes of the order of m_p we should introduce in our theory new degrees of freedom to describe gravitons' dynamics at different energy scales, as it happens in QCD where we have to introduce quarks in the description of the strong interaction once we go beyond the GV scale. However, as stated in Ref.[11] this is not the correct way to study physics beyond the plank scale l_p. In fact there is no need to introduce new degrees of freedom since when we try to describe physics at a length scale $l \ll l_p$ in reality we are describing it at a length scale l_p^2/l. In other words there is a correspondence between

$$l \leftrightarrow \frac{l_p^2}{l}, \tag{2.2}$$

that is nothing but a correspondence between deep infrared gravity and deep ultraviolet gravity

$$\text{deep IR gravity} \leftrightarrow \text{deep UV gravity}\,. \tag{2.3}$$

This statement goes often under the name of *generalized uncertainty principle* (Ref.[17]-[18]). This means that the more we try to describe physics at short distances, the more our system classicalizes and can be understood in terms of classical gravity. The significance of this phenomenon is particularly clear when we consider black holes physics. We know that, given a black hole of a given size R_H, it is impossible to describe physics inside that radius for an external static observer. Now, since from the Heisenberg uncertainty principle the amount of energy that can be localized in a space of dimension l has to respect the condition $E \gtrsim (l_p m_p)/l$ we immediately obtain

$$R_H = \frac{l_p}{m_p} M = \frac{l_p}{m_p} E \gtrsim \frac{l_p^2}{l}. \tag{2.4}$$

Thus we see that when we try to probe distances beyond the plank scale the amount of energy contained in such a region is big enough to form a black hole that does not allow to study physics at those scales.

In other terms we can say that the minimum information stored in a region of space of dimension l is equivalent to the information stored in a black hole of dimension l_p^2/l. We can also think to introduce new degrees of freedom at length scales $l \ll l_p$. From a Quantum Field theoretic point of view this means that we have to add new poles to the theory with $p^2 = \hbar^2 l^{-2} \gg m_p^2$. The contribution of such a pole becomes important only when we consider an experiment with momentum transfered l^{-1} such as a scattering process. Thus we should localize an energy of order $\hbar l^{-1}$ within a space of dimension l, but from Eq.(2.4) it is clear that a classical black hole forms much earlier than we can probe such a distance so that the introduction of the new pole does not show any new physics.

2.2 Classicality

One of the main features that makes General Relativity different from any other fundamental force is that it is characterized by the gravitational constant G_N that has dimension *length/mass* in units of c since

$$G_N = \frac{l_p}{m_p}, \tag{2.5}$$

where l_p and m_p are the Plank length and Plank mass, respectively defined as

$$l_p = \sqrt{\hbar G_N} \qquad m_p = \sqrt{\frac{\hbar}{G_N}}. \tag{2.6}$$

This is one of the difficulties that are encountered when one tries to give a QFT like description of gravity.
In order to get a dimensionless parameter that gives the coupling parameter of gravitons we have to use other physical quantities, namely

$$\alpha_{gr} = \frac{\hbar G_N}{\lambda^2} = \frac{l_P^2}{\lambda^2}, \tag{2.7}$$

where λ is the wavelength of gravitons and we have defined the Plank length as

$$l_p = \sqrt{\hbar G_N}, \tag{2.8}$$

that is the length scale for which quantum fluctuations become important.
The length scale for which gravitational effects become important is set by the Shwarzschild radius

$$R_H = 2 G_N M, \tag{2.9}$$

which is a pure classical parameter of black holes since it does not depend on $\hbar$ differently from the Plank length Eq.(2.8). In order to define the concept of classicality it is necessary to introduce the gravitons occupation number N_G of gravitons.
For this purpose we consider a spherical object of constant density of mass M whose radius R is much greater than its gravitational radius R_H.
For such a source the gravitational field outside the body can be approximately given by the well known Newtonian potential. The formula that gives the number of gravitons in a given gravitational configuration is, without considering numerical factors,

$$N_G = \frac{M R_H}{\hbar}. \tag{2.10}$$

We recall that the gravitational energy of a compact object in Newtonian theory is just

$$E_{grav} \approx \frac{MR_H}{R}. \tag{2.11}$$

In our quantum picture the energy of the system is given by the sum of the energy of single gravitons. Since we are considering them as weakly interacting and massless bosons we can simply write

$$E_{grav} \approx \sum_{\lambda} N_\lambda \hbar \lambda^{-1}. \tag{2.12}$$

Assuming that the peak of the distribution is given by

$$\lambda = R, \tag{2.13}$$

and that the contribution of other wavelengths is negligible we immediately get Eq.(2.10) comparing Eq.(2.11) and Eq.(2.12). A more detailed justification of Eq.(2.10) will be clarified later when we will study the relation between the entropy of a black hole and the number of gravitons that compose it. Since the gravitational self-coupling constant α decreases with the wavelength, in our approximation in which λ can be replaced by R which is very large, we can safely say that the interactions among gravitons are suppressed and so the condensate is not self sustained.

Then we can define the criterion of *classicality* as

$$N_G \gg 1, \tag{2.14}$$

that simply states that a system is classical when there are many gravitons in it. Thus, substituting $R = R_H$ in Eq.(2.7) we obtain

$$\alpha_{bh} = \frac{l_p^2}{R_H^2} = \frac{\hbar}{R_H M} = N_G^{-1}, \tag{2.15}$$

that means the coupling constant depends only on the number of gravitons for a black hole as well as the wave-length

$$\lambda = l_p \sqrt{N_G} = G_N M, \tag{2.16}$$

and the mass

$$M = \sqrt{N_G} m_p. \tag{2.17}$$

From this point of view the black hole is the simplest object possible since all the phenomena related to it can be explained in terms of the number of gravitons that form it. Finally, we point out that black holes are the most classical among objects of a given size R as it can be easily verified using Eq.(2.10) and Eq.(2.14).

2.3 Universality

One of the main features of the quantum portrait of black holes is that the number of gravitons N_G depends only on the mass (or energy of the center of mass of the source $\sqrt{s}$) and not on its composition. This property does not affect the energy description of objects with a very large radius R since in that case the energy of gravitons is very weak compared to the energy of the source and can be neglected, but when the radius R approaches a radius comparable to R_H, then the energy contribution of gravitons becomes enormous and we can treat our quantum state as an N particles state. This means that in gravitational interactions, the scattering of a state composed by a certain number of particles N_s with energy $\sqrt{s} \gg m_p$ *can be described as a large N particle interaction* where N is the sum of the number of gravitons and the number of particles that compose the source.

Thus the universality of N_G plays a key role in the description of classicalization of gravitational interaction in deep UV since the large N description of gravity that is valid when $R \approx R_H$ is insensible to the nature of the source. Indeed, in both cases of a semiclassical source composed by a large amount of particles or a quantum source composed by few highly energetic particles, the result will be always a large-N state when we consider gravitons. Then, one would be tempted to describe the electromagnetic interaction in the same way since gravitons seem to have the same relation with mass sources that photons have with electromagnetic sources. However we should never forget that the main difference between gravity and all other foundamental forces relies on the fact that the the gravitational force is *energy sourced* while other fundamental forces are not.

For example in an electron-electron scattering process the mean number of photons in the system is fixed by the fine structure constant $N_\gamma \approx 1/137$ in such a way that this system cannot be considered as a large-N system. To be more specific, we try to evaluate the maximal number of particles that can construct a gravitational source of mass M and dimension R. In the case of a static configuration the energy of the system is simply the mass M of the source. At the same time, since the system is confined in a space with dimension R, the Heisenberg uncertainty principle states that the minimal energy of each particle has to be $E \simeq \hbar/R$ so that the maximal number of particles that can compose the source is given by

$$N_s^{max} = \frac{MR}{\hbar}. \tag{2.18}$$

If we compare Eq.(2.18) with Eq.(2.10) we see that as long as $R \gg R_H$ the number of sources can be greater than the number of gravitons, but when $R < R_H$, then $N_G > N_s$. Thus, any source system classicalizes as one considers gravitational effects. This property is just what allows us to describe every phenomenon reguarding black holes in terms of a single parameter that is the number of gravitons N_G.

2.4 Hawking effect

Each graviton of the black hole is linked to the condensate with an energy

$$E_{link} = \frac{\hbar}{\sqrt{N_G} l_p}. \tag{2.19}$$

The depletion originates from scattering processes with characteristic energy given by the inverse of gravitons' wavelength in such a way that the depletion rate can be estimated as

$$\Gamma = \frac{1}{N_G^2} N_G^2 \frac{\hbar}{\sqrt{N_G} l_p}. \tag{2.20}$$

In the previous formula the first factor is given by the coupling strength α between two gravitons, the second is a combinatoric one while the third represents the characteristic energy of the process. This means that, on average, a graviton escapes from the condensate every $\Delta t = \hbar \Gamma^{-1} = \sqrt{N_G} l_p$, giving a reduction of the mass of the system of $\Delta M = -\hbar/\lambda_{esc}$ and so

$$\frac{dM}{dt} = -\frac{\Gamma}{\lambda_{esc}} = -\frac{\hbar}{N_G l_p^2}. \tag{2.21}$$

We notice that substituting Eq.(2.10) into Eq.(2.21) we obtain

$$\frac{dM}{dt} = -\frac{\Gamma}{\lambda_{esc}} = -\frac{m_p^3 \hbar}{M^2 l_p}, \tag{2.22}$$

that, except for a numerical factor, is equal to the expression (3.250) of Ref.[19] obtained following a semiclassical treatment. In terms of the number of gravitons N_G, Eq.(2.21) becomes

$$\frac{dN_G}{dt} = \frac{d(M^2)}{dt} \frac{l_p^2}{\hbar^2} = \frac{\sqrt{N_G}\hbar}{l_p} \frac{dM}{dt} \frac{l_p^2}{\hbar^2} = -\frac{1}{\sqrt{N_G} l_p}. \tag{2.23}$$

Integrating the previous differential equation it is immediate to see that the half time of the condensate is

$$\int_{N_G}^{0} -l_p \sqrt{N_G} dN_G = \int_0^{\tau} dt, \tag{2.24}$$

which gives

$$\tau_{bh} = N_G^{3/2} l_p, \tag{2.25}$$

that again reproduces the expected semiclassical result (3.251) of Ref.[19].

Finally, defining the temperature T of the black hole as

$$T = \frac{\hbar}{l_p\sqrt{N_G}}, \tag{2.26}$$

we can write (2.21) and (2.25) as

$$\frac{dM}{dt} = -\frac{T^2}{\hbar} \qquad \tau = \frac{\hbar^2}{T^3 G_N}. \tag{2.27}$$

Thus we have seen that the Hawking effect emerges entirely as a quantum mechanical effect analogous to a quantum depletion of a leaky condensate of bosons. This reproduces the semiclassical Hawking effect for large N_G, although we did not rely on any classical element such as the existence of an event horizon.

2.5 Entropy

In this section we should clarify the relation between the number of gravitons N_G and the entropy S of a black hole proving that

$$S \propto N_G. \tag{2.28}$$

We drop all numerical factors and focus only on the dependence between physical quantities. Recalling that the entropy a therodinamic system is computed is proportional to the number of states in which the system can be

$$S \propto \ln(n), \tag{2.29}$$

we should count all the possible ways in which the N_G gravitons that form the Bose condensate can arrange. If gravitons were non-interacting, then there would N_G^α states, where α is the number of state in which a single graviton can be. But since gravitons interact with each other this number grows really fast with N_G. To estimate it we make use of the concept of flavor clarified in Ref.[11]. Then the number of states will be given by

$$n = \prod_{j=1}^{N_{flavors}} \epsilon_j, \tag{2.30}$$

where ϵ_j is the characteristic number of states of a single flavor.

Now we should remember that a black hole could form for each value of N_G so that we can define an union as a set of N_j constituents that form a black hole with mass $M_\alpha = \hbar\sqrt{N_\alpha}/l_p$ and whose gravitons have a characteristic wavelength $\lambda = \sqrt{N_\alpha} l_p$. The *flavor* is defined as a set of $\alpha = 1, 2, ...n_j$ unions such that their sum is equal to the total number of gravitons. The number of this unions is cutoff by the case when $N_\alpha \approx N_G$, implying that the number of flavors grows as N_G. Thus the wave function representing the black hole, at first order in the variable $1/N_G$, is given by a product of not interacting one flavor states

$$\Psi_{BH} = \prod_{j}^{N_{flavors}} \psi_j. \tag{2.31}$$

Finally, taking a characteristic ϵ valid for each j, we can write

$$n = \prod_{j}^{N_{flavors}} \epsilon_j \simeq \epsilon_j^N, \tag{2.32}$$

so that we have

$$S \propto \ln(n) \propto N_G, \tag{2.33}$$

as we wanted to show. Looking at Eq.(2.10), we see the relation (2.28) between the entropy and the number of gravitons is in accordance with the semiclassical result (3.245) of Ref.[19] in the large N_G limit.

Chapter 3

Corpuscolar bootstrapped gravity

So far we have discussed the conditions under which a star collapses in General Relativity and the way singularities inevitably arise from this theory. As we already mentioned, this is not acceptable in the framework of Quantum Mechanics due to the Heisenberg uncertainty principle which prohibits the formation of singular objects with finite energy. Thus we have to give a different description of gravity that does not bring to formation of singularities. This theory is obtained from the variation of the Einstein-Hilbert action in the next leading order after Newtonian limit. In General relativity this would lead to the post-Newtonian approximation, but we claim to take the resulting equation valid at an arbitrary length scale obtaining bootstrapped Newtonian gravity of Ref.[22].
The physical interpretation of this procedure is that we are adding to the Poisson equation of Newtonian gravity a graviton-graviton interaction term to the usual source contribution.

More specifically, in the first section we briefly recall the classical solution of Newtonian gravity, then we derive the defining equation of bootrstrapped Newtonian gravity and study its solution for some configurations, verifying that for this theory the pressure is finite for each value of r. In the third section we develop a quantum description of both Newtonian and bootstrapped Newtonian theories, obtaining also some explicit results concerning the quantum coherent states that describe gravity in the Newtonian case. Finally in the fourth section we develop a general procedure that allows to compute the number of gravitons and mean wavelength of gravitons for a wide range of models relying only on some general features of the potential and compare our results with the assumptions made by Dvali and Gomez about the collective properties of gravitons.

3.1 Classical solutions of Newtonian Gravity

The main equation that describes Newtonian gravity is the Laplace equation

$$\Delta V_N = 4\pi G_N \rho, \tag{3.1}$$

where V_N is the Newtonian potential and ρ is the density of mass of the system. We point out that this is a linear equation and that every potential obtained adding a constant to V_N is still a solution of Eq.(3.1). If we know restrict our discussion to spherically symmetric sources

$$\rho = \rho(r),$$

it is convenient to introduce the Bessel functions of first kind

$$j_0(kr) = \frac{\sin(kr)}{kr}, \tag{3.2}$$

that satisfy

$$\Delta j_0(kr) = -k^2 j_0(kr), \tag{3.3}$$

and

$$\int_0^\infty dr\, 4\pi r^2 j_0(kr) j_0(pr) = \frac{2\pi^2}{k^2}\delta(k-p). \tag{3.4}$$

Thus, since the Bessel functions (3.2) are a complete orthogonal basis, any function f depending only on the variable r (that in the Newtonian theory coincides with the distance from the centre) can be written as

$$f(r) = \int_0^\infty dk\, \frac{k^2}{2\pi^2} \tilde{f}(k)\, j_0(kr), \tag{3.5}$$

while the Fourier transform is

$$\tilde{f}(k) = 4\pi \int_0^\infty dr\, r^2 f(r)\, j_0(kr). \tag{3.6}$$

Expanding $V(r)$ and $\rho(r)$ with Eq.(3.5) and using Eq.(3.3) and the fact that the Bessel functions of first kind are linearly independent it is easy to obtain

$$\tilde{V}_N(k) = -4\pi \frac{G_N \tilde{\rho}(k)}{k^2}, \tag{3.7}$$

substituting Eq.(3.6) in Eq.(3.7) we find

$$\tilde{V}_N(k) = -16\pi^2 \frac{G_N}{k^2} \int_0^\infty dr\, r^2 \rho(r) j_0(kr), \tag{3.8}$$

that is a particular case of Eq.(1.101). We notice that it has been possible to obtain such a simple expression for the Fourier components of the potential in terms of the density ρ only because of the linearity of the Laplace equation Eq.(3.1).

We shall now give some examples for which the Fourier transform can be exactly computed. For a point-like particle we have

$$\rho(r) = M_0\,\delta(\mathbf{x}) = \frac{M_0\delta(r)}{4\pi r^2}, \tag{3.9}$$

which leads to

$$\tilde{V}_N(k) = -4\pi\frac{G_N}{k^2}\int_0^\infty dr M_0\delta(r)j_0(kr) = -4\pi\frac{G_N M_0}{k^2}. \tag{3.10}$$

Then, for a Gaussian distribution of matter the density $\rho(r)$ reads

$$\rho(r) = \frac{M_0\ \mathrm{e}^{-r^2/\sigma^2}}{\pi^{3/2}\sigma^3}, \tag{3.11}$$

that leads to the Fourier transform

$$\tilde{V}(k) = -\frac{16\pi^2 G_N}{k^2}\int_0^\infty dr\, r^2 j_0(kr)\frac{M_0\ \mathrm{e}^{-r^2/\sigma^2}}{\pi^{3/2}\sigma^3} = -\frac{4\pi G_N}{k^2}M_0\,\mathrm{e}^{\frac{-\sigma^2k^2}{4}}, \tag{3.12}$$

so that the potential becomes

$$V_N(r) = -2\frac{G_N}{\pi}\int_0^\infty dk M_0\,\mathrm{e}^{\frac{-\sigma^2k^2}{4}}\ j_0(kr). \tag{3.13}$$

Finally for an homogeneous ball $\rho(r)$ is given by

$$\rho(r) = \frac{3M_0}{4\pi R^3}\Theta(R-r), \tag{3.14}$$

which, using (3.1), brings to the gravitational potential

$$\begin{cases} \tilde{V}_N = \dfrac{G_N M_0}{R}(r^2 - 3R^2) \qquad r \leq R & \text{(3.15a)} \\ \tilde{V}_N = -\dfrac{G_N M}{R} \qquad r > R, & \text{(3.15b)} \end{cases}$$

The Fourier transform of $\rho(r)$ reads

$$\tilde{\rho}(k) = -\frac{3M_0}{R^2k^2}\left[\cos(kR) - \frac{\sin(kR)}{kR}\right], \tag{3.16}$$

so that $\tilde{V}_N(k)$ becomes

$$\tilde{V}_N(k) = \frac{12\pi M_0}{R^2 k^4}\left[\cos(kR) - \frac{\sin(kR)}{kR}\right]. \tag{3.17}$$

3.2 Bootstrapped Newtonian gravity

3.2.1 Static field equation

We start by defining the *Einstein-Hilbert action*[1]

$$S = S_{EH} + S_M = \int d^4x\sqrt{-g}\left(-\frac{m_p}{16\pi l_p}R + \mathcal{L}_M\right), \tag{3.18}$$

where the Ricci scalar is given by Eq.(1.8) and

$$g = \det(g_{ab}), \tag{3.19}$$

while $\mathcal{L}_M$ is the matter Lagrangian. We notice that the action is invariant under a generic transformation of coordinates since d^4x is a scalar density of weight 1, $\sqrt{-g}$ a scalar density of weight -1 and the term in parentheses is a scalar.

Now we want to rewrite the action Eq.(3.18) for small potentials, namely for a metric tensor written in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{3.20}$$

where $\epsilon \ll 1$ and $\eta_{\mu\nu}$ is the Minkowski metric with signature $(-,+,+,+)$. Using Eq.(3.19) and Eq.(3.20) we find

$$\sqrt{-g} = 1 + \frac{\epsilon}{2}h + \frac{\epsilon^2}{8}(h^2 - 2h^\nu_\mu h^\mu_\nu) + \mathcal{O}(\epsilon^3). \tag{3.21}$$

The Christoffel's symbols Eq.(1.5) in terms of the metric Eq.(3.20) up to third order are given by

$$\Gamma^\lambda_{\mu\nu} \simeq \frac{\epsilon}{2}(\eta^{\lambda\rho} - \epsilon h^{\lambda\rho} + \epsilon^2 h^{\lambda\sigma}h^\rho_\sigma)(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}). \tag{3.22}$$

In the De Donder gauge

$$2\partial_\mu h^{\mu\nu} = \partial^\nu h, \tag{3.23}$$

the Lagrangian that describes Newtonian gravity can be seen as the sum of two terms

$$L[V_N] = \epsilon^2 L_{FP} + \epsilon L_M, \tag{3.24}$$

[1] To be precise the Einstein-Hilbert action is only the part labeled as S_{EH} which brings to Einstein equations in vacuum.

where

$$
\begin{aligned}
L_{FP} &= \frac{m_p}{16\pi l_p}\int_0^{\infty} d^3x\left(-\sqrt{-g}R\right)_{(2)} \\
&= \frac{m_p}{16\pi l_p}\int d^3x\left(\frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\mu h_{\nu\sigma}\partial^\mu h^{\nu\sigma} + \partial_\mu h_{\nu\sigma}\partial^\nu h^{\mu\sigma} - \partial_\mu h\partial_\sigma h^{\mu\sigma}\right) \\
&= \frac{m_p}{16\pi l_p}\int d^3x\left(-\frac{1}{2}\partial_\mu h_{\nu\sigma}\partial^\mu + \partial_\mu h_{\nu\sigma}\partial^\nu h^{\mu\sigma}\right) \\
&\simeq \frac{m_p}{32\pi l_p}\int d^3x\,\partial_\mu h_{00}\partial^\mu h^{00} = \frac{m_p}{32\pi l_p}\int d^3x\,\partial_r h_{00}\partial^r h^{00} \\
&= \frac{m_p}{32\pi l_p}\int d^3x\, h_{00}\Delta h^{00} \\
&= \frac{m_p}{32\pi l_p}4\pi\int dr\, r^2 h_{00}\Delta h^{00}.
\end{aligned}
\tag{3.25}
$$

Here, we are assuming that the source of the gravitational field is spherically symmetric and moving at non relativistic speed in such a way that the only relevant component of the curvature metric is $h_{00}(r)$.
The suffix (n) means that the function between parentheses has been spanned up to n^{th} order using Eq.(3.22), Eq.(3.20) and Eq.(1.7).

We recall that the Lagrangian matter $\mathcal{L}_M$ is defined in such a way to describe the source term in the Einstein equations

$$T_{\mu\nu} = \delta_{\mu 0}\,\delta_{\nu 0}\,\rho(\mathbf{x}) = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} - \mathcal{L}_M g_{\mu\nu}, \tag{3.26}$$

where we used the non relativistic definition of the energy-momentum tensor and

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}. \tag{3.27}$$

Substituting the expression for the variation of the density ρ

$$\delta\rho = \frac{1}{2}\rho(g_{\mu\nu} + u_{\mu\nu})\delta g^{\mu\nu}, \tag{3.28}$$

in Eq.(3.26) we immediately find

$$\mathcal{L}_M \simeq -\rho(r). \tag{3.29}$$

Thus we can write

$$L_M = \int d^3x\left(\sqrt{-g}\mathcal{L}_M\right)_{(1)} = 4\pi\int_0^{\infty} r^2 dr\frac{h_{00}}{2}\rho. \tag{3.30}$$

Varying the effective Lagrangian(3.24) we find

$$2\pi \int_0^\infty dr\, r^2 \left(\frac{m_p}{8\pi l_p} \Delta h_{00} + \rho \right) \delta h_{00}, \tag{3.31}$$

so that the field equation for h_{00} reads

$$\Delta h_{00} = -\frac{8\pi l_p}{m_p} \rho. \tag{3.32}$$

Comparing Eq.(3.32) with Eq.(3.1) we find

$$h_{00} = -2V_N, \tag{3.33}$$

that is the same result Eq.(1.100) that we obtained in the general case starting directly from Einstein equations.

We notice that, in order to recover the Laplace equation, we have compered the second order expansion of the Einstein-Hilbert Lagrangian with the first order expansion of the matter Lagrangian. This means that the gravitational constant l_p/m_p has to be proportional to ϵ in order to recover an equality between two quantities of the same order of magnitude. Writing explicitly this rescaling factor of G_N and defining $h_{00} = -2V$ also in the not Newtonian case we come up with

$$\begin{aligned} S[V] &= \int \epsilon dt \left[(L_{FP} + L_M) + \epsilon \int d\mathbf{x} (-\sqrt{-g}\mathcal{R})_{(3)} + (\sqrt{-g}\mathcal{L}_M)_{(2)} \right] \\ &= 4\pi \int \epsilon dt \int_0^\infty r^2 dr \left\{ \frac{m_p}{8\pi l_p} V \Delta V - \rho V + \frac{\epsilon}{2} \left[\frac{m_p}{4\pi l_p} (V')^2 + V\rho \right] V \right\}, \end{aligned} \tag{3.34}$$

where we used

$$-\left(\sqrt{-g}R\right)_{(3)} = V(V')^2 \tag{3.35}$$

$$\left(\sqrt{-g}\mathcal{L}_M\right)_{(2)} = \frac{1}{8} h_{00}^2 T_{00} = \frac{1}{2} V^2 \rho. \tag{3.36}$$

From Eq.(3.34) we see that the choice of ϵ fixes the time parameter used to describe the evolution of the system. It does not affect the dynamics in Newtonian gravity but changes the corrections to the Newtonian potential. In particular, to recover the correct form of the post-Newtonian potential (A.11) seen from a static observer we have to put $\epsilon = 4$ in the action (3.34). Thus the Lagrangian will be given by

$$
\begin{aligned}
L[V] =& 4\pi \int_0^\infty r^2 dr \left\{ \frac{m_p}{8\pi l_p} V \Delta V - \rho V + 2 \left[\frac{m_p}{4\pi l_p} (V')^2 + V\rho \right] V \right\} \\
=& -4\pi \int_0^\infty r^2 dr \left[\frac{m_p}{8\pi l_p} V \Delta V (-1 + 4V) + \rho V (1 - 2V) \right].
\end{aligned} \tag{3.37}
$$

Using the Euler-Lagrange equations

$$
\frac{d}{dr} \frac{\delta \mathcal{L}}{\delta V'} = \frac{\delta \mathcal{L}}{\delta V}, \tag{3.38}
$$

we come up with the *bootstrapped Newtonian field equation* (Ref.[22])

$$
(1 - 4V) \left(\Delta V - 4\pi \frac{l_p}{m_p} \rho \right) = 2 (V')^2 . \tag{3.39}
$$

Using this equation for every distance r means we have completely abandoned any geometric view of gravity typical of General Relativity. However, we notice that Eq.(3.39) is not linear as its Newtonian counterpart(3.1). This means that there is not a correspondence between V and ρ as simple as Eq.(3.7). So we have to try to solve Eq.(3.39) without relying in the method of Fourier transform.

The bootstrapped vacuum equation

$$
\Delta V = \frac{2(V')^2}{1 - 4V}, \tag{3.40}
$$

is exactly solved by

$$
V_c = \frac{1}{4} \left[1 - c_1 \left(1 + \frac{6c_2}{r} \right)^{2/3} \right]. \tag{3.41}
$$

The two arbitrary constants c_1 and c_2 can be fixed by requiring that Eq.(3.41) reproduces the same result of found in appendix A in the large r limit. The large r expansion of Eq.(3.41) reads

$$
V_c \underset{r\to\infty}{\simeq} \frac{1}{4}(1 - c_1) - \frac{c_1 c_2}{r} + \frac{c_1 c_2^2}{r^2} + \mathcal{O}\left(\frac{G_N M}{r} \right)^3 . \tag{3.42}
$$

Thus we have that

$$
\begin{aligned}
c_1 =& 1 \\
c_2 =& G_N M.
\end{aligned} \tag{3.43}
$$

Using these values the potential assumes the form

$$V_c = \frac{1}{4}\left[1 - \left(1 + \frac{6G_N M}{r}\right)^{2/3}\right]. \tag{3.44}$$

We notice that the potential V_c diverges slower then V_N for $r \simeq 0$

$$\left(\frac{V_c}{V_N}\right) \underset{r\to 0}{\simeq} \left(\frac{r}{G_N M}\right)^{1/3}. \tag{3.45}$$

3.2.2 Uniform spherical source

In this section we will study in detail the solution of Eq.(3.39) for a ball of constant density and see if a limit for the size of the source like (1.43) arises from this theory. Thus let us consider a ball of radius R and constant density ρ

$$\rho = \frac{3M_0}{4\pi R^3}\Theta(R - r), \tag{3.46}$$

where Θ is the Heaviside distribution and M_0 is the source contribution to the energy given by

$$M_0 = 4\pi \int_0^\infty r^2 dr \rho(r). \tag{3.47}$$

It is very difficult to find an exact solution of Eq.(3.39) even for this simple model. For this reason we have to find an analytical approximations of V. For this purpose it is convenient to introduce some dimensionless parameters, namely

$$r = R\tilde{r} \qquad G_N M_0 = R\tilde{M}_0 \qquad G_N M = R\tilde{M}. \tag{3.48}$$

Substituting these scaled variables in Eq.(3.39) we obtain

$$(1 - 4\tilde{V})(\tilde{\Delta}\tilde{V} - 4\pi G_N \tilde{\rho}) = 2(\tilde{V}')^2, \tag{3.49}$$

where $\tilde{\Delta} = \tilde{r}^{-2}\partial_{\tilde{r}}\left(\tilde{r}^2\partial_{\tilde{r}}\right)$, $\tilde{V}(\tilde{r}) = V(r)$ and

$$\tilde{\rho} = 3\frac{\tilde{M}_0}{4\pi}\Theta(1 - \tilde{r}). \tag{3.50}$$

Outside the ball we simply have the vacuum solution rewritten in terms of the new variables

$$\tilde{V}_c(\tilde{r}) = \frac{1}{4}\left[1 - \left(1 + \frac{6\tilde{M}}{\tilde{r}}\right)^{2/3}\right] \qquad \tilde{r} > 1, \tag{3.51}$$

while inside the ball we have to use an analytical approximations, namely we write

$$\tilde{V}_c = \tilde{V}_N + \tilde{W}, \tag{3.52}$$

where

$$\tilde{V}_N = \frac{\tilde{M}_0}{2}\,(\tilde{r}^2 - 3C), \tag{3.53}$$

is the Newtonian solution and

$$|\tilde{W}| \ll |\tilde{V}_N| \qquad \tilde{r} \simeq 0. \tag{3.54}$$

Replacing $\tilde{V}_c$ with Eq.(3.52) into Eq.(3.51) we obtain

$$\tilde{\Delta}\tilde{W} = 2\frac{(\tilde{V}_N' + \tilde{W}')}{1 - 4\tilde{V}_N}, \tag{3.55}$$

where we discarded $-4\tilde{W}$ in the denominator because of the inequality Eq.(3.54). The solution around $\tilde{r} \simeq 0$ can be written as

$$\tilde{W} \simeq \frac{{\tilde{M}_0}^2 \tilde{r}^4}{10(1 + 6C\tilde{M}_0)}\left[1 + \frac{20\tilde{M}_0\tilde{r}^2}{21(1 + 6C\tilde{M}_0)}\right] = \tilde{W}_4 + \tilde{W}_6, \tag{3.56}$$

so that

$$\left|\frac{\tilde{W}_4}{\tilde{V}_N}\right| \leq \frac{\tilde{M}_0}{15C(1 + 6C\tilde{M}_0)}. \tag{3.57}$$

This means that also Eq.(3.54) is verified since numerical solutions suggest that $C \geq 1$ as it is shown in Ref.[22].

The solution of Eq.(3.49) is obtained by requiring the continuity of the potential and its derivative on the surface of the ball obtaining $\tilde{M}$ and C in terms of $\tilde{M}_0$, namely

$$\tilde{V}_N^- + \tilde{W}^- = \tilde{V}_c^+ \tag{3.58}$$

$$\tilde{V}_N^{-\prime} + \tilde{W}^{-\prime} = \tilde{V}_c^{+\prime}. \tag{3.59}$$

Inserting the expression Eq.(3.57)(with $\tilde{W} \simeq \tilde{W}_4$) and Eq.(3.51) in Eq.(3.58) and Eq.(3.59) we come up with

$$2\tilde{M}_0(1 - 3C) + \frac{2\tilde{M}_0^2}{5(1 + 6C\tilde{M}_0)} \simeq 1 - (1 + 6\tilde{M})^{2/3} \tag{3.60}$$

$$\tilde{M}_0 + \frac{2\tilde{M}_0^2}{5(1 + 6C\tilde{M}_0)} \simeq \frac{\tilde{M}}{(1 + 6\tilde{M})^{1/3}}. \tag{3.61}$$

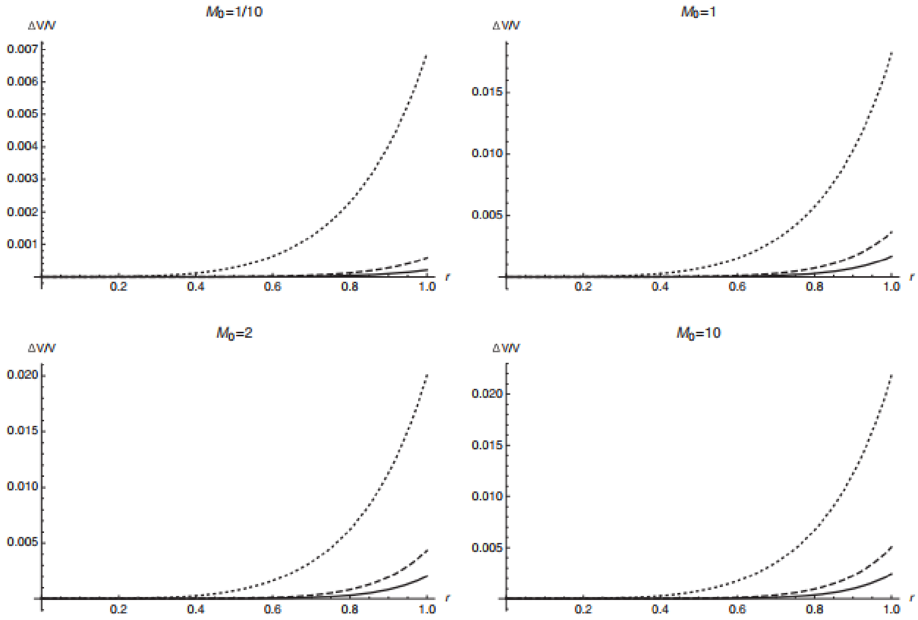

Figure 3.1: Relative error $(\tilde{V}_c - \tilde{V}_n)/\tilde{V}_n$ with respect to the numerical solution $\tilde{V}_n$ for $\tilde{V}_c = \tilde{V}_N + \tilde{W}_4 + \tilde{W}_6$ (solid line) vs $\tilde{V}_c = \tilde{V}_N + \tilde{W}_4$ (dashed line) vs Newtonian case $\tilde{V}_c = \tilde{V}_N$ (dotted line) for $C = 1$ and $\tilde{M}_0 = 1/10$ (top left), $\tilde{M}_0 = 1$ (top right), $\tilde{M}_0 = 2$ (bottom left) and $\tilde{M}_0 = 10$ (bottom right).

When $\tilde{M}_0 \ll 1$ Eq.(3.61) yields

$$\tilde{M} \simeq \tilde{M}_0 \left(1 + \frac{12}{5}\tilde{M}_0\right), \tag{3.62}$$

and from Eq.(3.60)

$$C \simeq 1 + \tilde{M}_0. \tag{3.63}$$

On the other hand for $\tilde{M} \gg 1$, Eq.(3.61) yields

$$\tilde{M} \simeq \sqrt{6}\tilde{M}_0^{3/2}, \tag{3.64}$$

while Eq.(3.60) gives (Ref.[22])

$$C = \frac{\tilde{V}_c(0)}{\tilde{V}_N(0)} \simeq 1.34. \tag{3.65}$$

We notice that the previous analysis becomes more inaccurate as long as we increase the value of $\tilde{M}_0$ while, in the opposite regime, we can make a more detailed analysis using an approximated expression for Eq.(3.49), namely

$$\tilde{\Delta}\tilde{V}_{WF} \simeq 4\pi\rho + 2(\tilde{V}'_{WF})^2, \tag{3.66}$$

where we used the fact that if $\tilde{M}_0$ is small we expect that also $\tilde{V} \ll 1$. In vacuum(3.66) becomes

$$\tilde{\Delta}\tilde{V}_{WF} \simeq 2(\tilde{V}_{WF}{}'), \tag{3.67}$$

that is exactly solved by

$$\tilde{V}_{WF} = -\frac{1}{2}\ln\left(1 + \frac{2\tilde{M}}{\tilde{r}}\right), \tag{3.68}$$

where the two integration constants have been computed requiring the expected Newtonian behavior for large $\tilde{r}$.
The interior of the homogeneous source is now described by the following equation

$$\tilde{V}_{WF} \simeq 3\tilde{M}_0 + 2(\tilde{V}'_{WF}), \tag{3.69}$$

which is solved exactly by

$$\tilde{V}_{WF} = A - \frac{1}{2}\ln\left[\frac{\sin(b + \sqrt{6\tilde{M}_0}\tilde{r})}{\tilde{r}}\right]. \tag{3.70}$$

Requiring regularity in $\tilde{r} = 0$ and on the surface $\tilde{r} = 1$, we come up with the following form of the potential for the bootstrapped potential generated by a ball of small density

$$\begin{cases} \tilde{V}_{WF} = A - \dfrac{1}{2}\ln\left[\dfrac{\sin(\sqrt{6\tilde{M}_0}\tilde{r})}{\tilde{r}}\right] & \tilde{r} \leq 1 \quad (3.71a) \\ \tilde{V}_{WF} = -\dfrac{1}{2}\ln\left(1 + \dfrac{2\tilde{M}}{\tilde{r}}\right) & \tilde{r} > 1, \quad (3.71b) \end{cases}$$

with $\tilde{M}$ and C given by Eq.(3.62) and Eq.(3.63).

3.2.3 The pressure

The main reason for which bootstrapped Newtonian gravity was introduced is to have a theory that describes gravity also in very extreme situations without generating singularities. We want to show this is indeed the case by studying the behavior of pressure and showing that we never meet singularities. Thus we briefly recall the main steps that lead to the expression for the Newtonian pressure. The Newtonian equation that describes hydrostatic equilibrium is

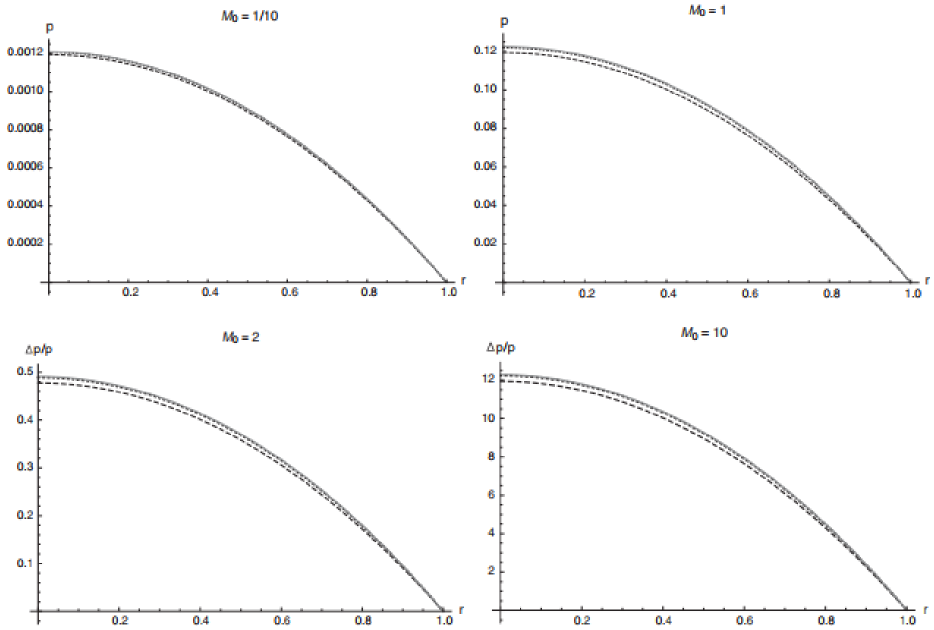

Figure 3.2: Numerical solution (solid gray line) vs analytical approximation (3.76) with $C = 1.34$ (dotted line) vs Newtonian pressure (3.74) (dashed line) for $\tilde{M}_0 = 1 = 10$ (top left), $\tilde{M}_0 = 1$ (top right), $\tilde{M}_0 = 2$ (bottom left) and $\tilde{M}_0 = 10$ (bottom right).

$$p_N'(r) = -V_N'(r)\rho(r) = -\frac{G_N m(r)}{r^2}\rho(r), \tag{3.72}$$

where

$$m(r) = 4\pi \int_0^\infty dr\, r^2 \rho(r). \tag{3.73}$$

This equation is very easy to solve for a homogeneous ball of radius R. Requiring that the pressure vanishes on the surface, namely $p(R) = 0$, the solution reads

$$p_N(r) = \frac{3G_N M_0^2(r^2 - R^2)}{8\pi R^6}. \tag{3.74}$$

For bootstrapped Newtonian theory we follow the same steps replacing $\tilde{V}_N$ with the bootstrapped potential Eq.(3.52)

$$p'(r) = -V_c'(r)\rho(r). \tag{3.75}$$

Substituting Eq.(3.52) into Eq.(3.75) and requiring $p(R) = 0$ we come up with (Ref.[22])

$$p(r) = \frac{3G_N M_0^2 (R^2 - r^2)\{5R^3 + G_N M_0[r^2 + (1+30C)R^2]\}}{40\pi R^8 (R + 6CG_N M_0)}. \tag{3.76}$$

Unlike Eq.(1.24), the equation we obtained for bootstrapped Newtonian pressure holds for any value R of the star showing that it is possible to have configurations with dimensions below the Buchdal limit (1.43).

3.3 Corpuscolar desciption of compact sources

In order to give a quantum description of gravity we define a classical scalar field $\Phi(t, r)$ proportional to the potential V in such a way to reproduce the correct dimensions $\sqrt{mass/length}$. Working with $c = 1$ the potential V is dimensionless and so

$$\begin{cases} \Phi(r) = \sqrt{\dfrac{m_p}{l_p}} V(r) & \text{(3.77a)} \\ J_B(r) = \sqrt{\dfrac{l_p}{m_p}} \rho(r), & \text{(3.77b)} \end{cases}$$

where J_B is a field proportional to ρ with the same dimensions of Φ.
Substituting the potential V with the scalar field Φ into Eq.(3.37) we obtain

$$L[\Phi] = 4\pi \int_0^\infty dr\, r^2 \left[\frac{1}{2}\Phi\Box\Phi - J_B\Phi\left(1 - 2\Phi\sqrt{\frac{l_p}{m_p}}\Phi\right) + 2\sqrt{\frac{l_p}{m_p}}\left(\partial_\mu\Phi\right)^2\Phi\right]. \tag{3.78}$$

At this level we have simply rewritten the Lagrangian that describes the static potential analyzed in the previous sections in terms of a scalar field adding a time dependence implicitly assumed by replacing Δ with $\Box$.

3.3.1 Compact spherical sources

Now we want to study the free equation of the scalar potential Φ

$$\Box\Phi = 0. \tag{3.79}$$

In order to find the general solution of this equation and quantize it we introduce the functions $u_k(r, t)$ defined as

$$u_k(r, t) = j_0(kr)\, \mathrm{e}^{-ikt}\,. \tag{3.80}$$

These are a complete orthogonal set of solutions of Eq.(3.79) in the space of spherically symmetric functions that satisfy

$$4\pi \int_0^\infty dr\, r^2\, u_k(r,t) u_h^*(r,t) = \frac{2\pi^2}{k^2}\, \delta(k-h). \tag{3.81}$$

Now, if we introduce the annihilation and creation operators $\hat{a}_k$, $\hat{a}_k^\dagger$ which satisfy

$$\left[\hat{a}_k,\, \hat{a}_h^\dagger\right] = \frac{2\pi^2}{k^2}\, \delta(k-h), \tag{3.82}$$

we can write the operator valued scalar field $\hat{\Phi}(t,r)$ and its conjugate momentum $\hat{\Pi}(t,r)$ as

$$\hat{\Phi}(t,r) = \int_0^\infty dk\, \frac{k^2}{2\pi^2}\, A(k) \left[\hat{a}_k u_k(r,t) + \hat{a}_k^\dagger u_k^*(r,t)\right] \tag{3.83}$$

$$\hat{\Pi}(t,r) = i \int_0^\infty dk\, \frac{k^2}{2\pi^2}\, A(k) k \left[\hat{a}_k u_k(r,t) - \hat{a}_k^\dagger u_k^*(r,t)\right]. \tag{3.84}$$

The factor $A(k)$ can be easily computed requiring the validity of canonical equal time commutation relation between $\hat{\Phi}(t,r)$ and $\hat{\Pi}(t,r)$, namely

$$\left[\hat{\Phi}(t,r_1),\, \hat{\Pi}(t,r_2)\right] = i\, \frac{m_p l_p}{4\pi r^2} \delta(r_1 - r_2). \tag{3.85}$$

Substituting Eq.(3.83) and Eq.(3.84) into Eq.(3.85) we easily get

$$A(k) = \sqrt{\frac{l_p m_p}{2k}}. \tag{3.86}$$

Then we define the coherent state $|g\rangle$ which has the main property to be an eigenvalue of the annihilation operator a_k

$$a_k \left|g\right\rangle = g_k\, \mathrm{e}^{i\gamma_k(t)} \left|g\right\rangle. \tag{3.87}$$

In order to compute g_k and $\gamma_k(t)$ we impose that

$$\left\langle g\right| \hat{\Phi}(t,r) \left|g\right\rangle = \Phi(r). \tag{3.88}$$

Using relations Eq.(3.82), Eq.(3.87), the expansion of the quantum scalar field Eq.(3.83) and the fact that the coherent state $|g\rangle$ is a normalized state we find

$$\left\langle g\right| \hat{\Phi}(t,r) \left|g\right\rangle = \sqrt{l_p m_p} \left\langle g\right| \int_0^\infty dk\, \frac{k^{3/2}}{2\sqrt{2}\pi^2} \left[\hat{a}_k u_k(r,t) + \hat{a}_k^\dagger u_k^*(r,t)\right] \left|g\right\rangle \tag{3.89}$$

$$= \sqrt{l_p m_p} \left\langle g | g\right\rangle \int_0^\infty dk\, \frac{k^{3/2}}{2\sqrt{2}\pi^2}\, g_k \left[\mathrm{e}^{i\gamma_k(t)}\, u_k(r,t) + \mathrm{e}^{-i\gamma_k(t)}\, u_k^*(r,t)\right] \tag{3.90}$$

$$= \int_0^\infty dk\, \frac{k^2}{2\pi^2} \sqrt{\frac{2 l_p m_p}{k}}\, g_k\, \cos[\gamma_k(t) - \omega t] j_0(kr), \tag{3.91}$$

so that using the expansion (3.5) of the Newtonian potential and the fact that the Bessel function $j_0(kR)$ are linearly independent we come up with

$$g_k = \sqrt{\frac{k}{2l_p m_p}}\tilde{\Phi}(k) \qquad \gamma_k(t) = kt. \tag{3.92}$$

The physical meaning of the equality Eq.(3.88) is that in the Quantum interpretation of gravity the potential that we measure is just the mean value of an operator computed on the state $|g\rangle$. At this level we can think that the uncertainty of this measure that inevitably has to come out from a quantum theory is covered by the fact that $|g\rangle$ is a coherent state which is the is the "most classical" among quantum states.

So far we have just defined some quantum properties of gravity by simply imposing the connection Eq.(3.88) between the classical description in which we have a well defined potential and the quantum description. Now we want to use these variables we have introduced *ad hoc* to understand some characteristics of gravitons that give rise to the potentials we are studying and see if they reproduce the results of Dvali and Gomez. For this purpose it is useful to write the coherent state explicitly as

$$|g\rangle = \mathrm{e}^{-N_G/2} \exp\left\{\int_0^\infty dk\,\frac{k^2}{2\pi^2} g_k\; \mathrm{e}^{i\gamma_k(t)}\, \hat{a}_k^\dagger\right\} |0\rangle\,, \tag{3.93}$$

where $|0\rangle$ is the vacuum of the theory which as the main property

$$\hat{a}_k\,|0\rangle = \langle 0|\,\hat{a}_k^\dagger = 0. \tag{3.94}$$

Imposing the normalization condition $\langle g|g\rangle = 1$ and using the well-known Baker-Campbell-Hausdorff formulas we have

$$\begin{aligned} 1 = \langle g|g\rangle &= \mathrm{e}^{-N_G}\,\langle 0|\exp\left\{\int_0^\infty dk\,\frac{k^2}{2\pi^2} g_k \hat{a}_k\right\} \exp\left\{\int_0^\infty dp\,\frac{p^2}{2\pi^2} g_p \hat{a}_p^\dagger\right\}|0\rangle \\ &= \mathrm{e}^{-N_G}\,\langle 0|\exp\left\{\int_0^\infty dp\,\frac{p^2}{2\pi^2}\int_0^\infty dk\,\frac{k^2}{2\pi^2} g_k g_p [\hat{a}_k, \hat{a}_p^\dagger]\right\}|0\rangle\,, \end{aligned}$$

where we dropped away the other factors coming from Baker-Campbell-Hausdorff formulas because they act with annihilation operator on $|0\rangle$. Using Eq.(3.82) and $\langle 0|0\rangle = 1$ we come up with

$$\mathrm{e}^{-N_G}\;\exp\left\{\int_0^\infty dk\,\frac{k^2}{2\pi^2}\, g_k^2\right\} = 1, \tag{3.95}$$

and so

$$N_G = \int_0^\infty dk\,\frac{k^2}{2\pi^2}\, g_k^2. \tag{3.96}$$

Using Eq.(3.96) and (3.92) we can study the quantum corpuscolar description of some sources. First, we compute(3.96) for a Newtonian point-like particle. Substituting Eq.(3.10) into Eq.(3.92) we obtain

$$g_k = -4\pi \frac{M_0}{m_p}\sqrt{\frac{1}{2k^3}}, \tag{3.97}$$

so that Eq.(3.96) becomes

$$N_G = 4\,\frac{M_0^2}{m_p^2}\int_0^\infty dk\,\frac{1}{k} = 4\,\frac{M_0^2}{m_p^2}\int_{k_0}^\Lambda dk\,\frac{1}{k}, \tag{3.98}$$

where in the last passage we introduced an infrared cut off k_0 and an ultraviolet one, Λ. The cutoff $k_0 = \pi/R_\infty$ is related to the fact that that gravitons cannot have an arbitrary large wavelength because if we assume that the source has lived for a finite amount of time R_∞ its field cannot have propagated out of a ball of radius R_∞. On the other hand the vector number Λ is associated to the fact tha,t in realistic models, we have to consider sources with a non vanishing dimension R.
This means that the number of gravitons in a coherent state for a point-like particle is given by

$$N_G = 4\,\frac{M_0^2}{m_p^2}\,\ln\left(\frac{R_\infty}{R}\right), \tag{3.99}$$

where $R \approx \pi/\Lambda$ can be thought as the radius of a very tiny ball that replaces the initial point-like particle.
Now, the mean wave vector of gravitons in a coherent state will be

$$\frac{\langle k\rangle}{N_G} = \frac{1}{N_G}\int_0^\infty dk\,\frac{k^3}{2\pi^2}\,g_k^2 = \frac{1}{N_G}\int_0^\infty dk\,\frac{k^4}{4\pi^2}\,\tilde{\Phi}^2(k), \tag{3.100}$$

where $\langle k\rangle$ has been computed using

$$\langle k\rangle = \langle g|\int_0^\infty dkk\,\hat{n}_k\,|g\rangle = \langle g|\int_0^\infty dk\,\frac{k^3}{2\pi^2}\hat{a}_k^\dagger\hat{a}_k\,|g\rangle\,. \tag{3.101}$$

and $\hat{n}_k$ is the density of states with wave vector k. Substituting Eq.(3.12) in Eq.(3.92) we obtain the expression of g_k for a Gaussian distribution of matter

$$g_k = -4\pi \frac{M_0}{m_p}\sqrt{\frac{1}{2k^3}}\,\mathrm{e}^{-\frac{\sigma^2 k^2}{4}}, \tag{3.102}$$

so that N_G becomes

$$N_G = 4\,\frac{M_0^2}{m_p^2}\int_0^\infty dk\,\frac{\mathrm{e}^{-\sigma^2k^2/2}}{k} = 4\,\frac{M_0^2}{m_p^2}\int_{\sigma k_0}^\infty dz\,\frac{\mathrm{e}^{-z^2/2}}{z}, \tag{3.103}$$

where we have substituted k with $z = \sigma k$. In this case we only need an infrared cutoff to regularize the integral and if we choose it in such a way that $\sigma k_0 \ll 1$ the contribution to the integral given by wave vectors above the order of $1/\sigma$ is very small . Then we do not commit a great error if we approximate the previous equation as

$$N_G \simeq 4\frac{M_0^2}{m_p^2} \ln\left(\frac{R_\infty}{\sigma}\right), \tag{3.104}$$

$\langle k \rangle$ can be exactly computed using Eq.(3.12) as

$$\langle k \rangle = 4\,\frac{M_0^2}{m_p^2} \int_0^\infty dk\ \mathrm{e}^{-\sigma^2 k^2/2} = \frac{4}{\sigma}\frac{M_0^2}{m_p^2}\sqrt{\frac{\pi}{2}}, \tag{3.105}$$

while the mean wave vector is

$$\frac{\langle k \rangle}{N_G} = \frac{\sqrt{\pi/2}}{\sigma}\frac{1}{\ln\left(\frac{R_\infty}{\sigma}\right)} + \mathcal{O}\left[\frac{1}{\ln^2\left(\frac{R_\infty}{\sigma}\right)}\right]. \tag{3.106}$$

For a ball of constant density we obtain g_k substituting Eq.(3.17) into Eq.(3.92) that gives

$$g_k = 12\pi\frac{M_0}{m_p}\frac{1}{\sqrt{2}R^3 k^{9/2}}\left[kRcos(kR) - \sin(kR)\right], \tag{3.107}$$

and so Eq.(3.96) becomes

$$N_G = 36\,\frac{M_0^2}{m_p^2}\int_0^\infty dk\,\frac{1}{R^4 k^5}\left[\cos(kR) - \frac{\sin(kR)}{kR}\right]^2. \tag{3.108}$$

It is immediate to see that for large k the previous integral is convergent. For little wave vectors we have instead

$$\cos(kR) - \frac{\sin(kR)}{kR} \simeq -(kR)^2/3. \tag{3.109}$$

This means that for small values of kR we can approximate the previous integral as

$$N_G \simeq 36\frac{M_0^2}{m_p^2}\left\{\int_0^K dk\frac{1}{9k} + \int_K^\infty dk\,\frac{1}{R^4k^5}\left[\cos(kR) - \frac{\sin(kR)}{kR}\right]^2\right\}, \tag{3.110}$$

where K is some wave vector such that $K \ll 1/R$.
The second integral in the right hand side of Eq.(3.110) is convergent while to compute the first one we need again an infrared regulator $k_0 = \pi/R_\infty$ so that the number of gravitons can be estimated as

$$N_G \simeq 4\frac{M_0^2}{m_p^2}\ln\left(\frac{R_\infty}{R}\right). \tag{3.111}$$

$\langle k\rangle$ can be exactly computed as

$$\langle k\rangle = 36\,\frac{1}{R}\frac{M_0^2}{m_p^2}\int_0^\infty dz\,\frac{1}{z^4}\left[\cos(z)-\frac{\sin(z)}{z}\right]^2. \tag{3.112}$$

The computation of this integral is trivial but very tedious and is done in B. The final result is

$$\langle k\rangle = \frac{12\pi}{5}\frac{1}{R}\frac{M_0^2}{m_p^2}, \tag{3.113}$$

and the mean wave vector of gravitons becomes

$$\frac{\langle k\rangle}{N_G} = \frac{3\pi}{5}\frac{1}{R}\frac{1}{\ln\left(\frac{R_\infty}{R}\right)} + \mathcal{O}\left[\frac{1}{\ln^2(\frac{R^\infty}{R^*})}\right]. \tag{3.114}$$

3.3.2 Number of gravitons

We shall now generalize the results of the previous section deriving an expression for the number of gravitons in a coherent state for a generic matter source without relying on any assumption about its symmetry.
Thus consider a static compact source (or even gaussian) with density $\rho = \rho(\mathbf{x})$ that will generate a potential $V(\mathbf{x})$, seen by a static observer, according to the generic law

$$\Delta V(\mathbf{x}) = F[\rho(\mathbf{x}), V(\mathbf{x}), \nabla V(\mathbf{x})]. \tag{3.115}$$

For now, we only assume that F is positive and that it behaves in such a way that

$$r^2\frac{\partial V(\mathbf{x})}{\partial r} \to 0 \qquad r \to 0 \tag{3.116}$$

$$V(\mathbf{x}) = -\frac{G_N M}{r} + \mathcal{O}\left(\frac{R^{*2}}{r^2}\right) \qquad r \gg R^*, \tag{3.117}$$

where $r = |\mathbf{x}|$, R is the radius of a ball within which is contained all the matter source ($R = \sigma$ in the case of a gaussian distribution), M is the energy of the source measured by a static external observer and R^* is defined as

$$R^* = \max\{G_N M, R\} \tag{3.118}$$

This is the length scale within which the value of the Laplacian of the potential is not negligible. The second condition is quite reliable since it only states that in the large r limit we recover the Newtonian limit.

Now, introducing the functions

$$u_{\mathbf{k}}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3.119}$$

we can write the Fourier transform of the potential as

$$\tilde{V}(\mathbf{k}) = \int d^3x\big[V(\mathbf{x})u^*_{\mathbf{k}}(\mathbf{x})\big]. \tag{3.120}$$

From the orthonormality of the functions $u_{\mathbf{k}}(\mathbf{x})$

$$\int d^3x[u_{\mathbf{k}}(\mathbf{x})u^*_{\mathbf{h}}(\mathbf{x})] = \delta(\mathbf{k}-\mathbf{h}), \tag{3.121}$$

while the anti Fourier transform becomes

$$V(\mathbf{x}) = \int d^3k\big[\tilde{V}(\mathbf{k})u_{\mathbf{k}}(\mathbf{x})\big] = \frac{1}{2}\int d^3k\big[\tilde{V}(\mathbf{k})u_{\mathbf{k}}(\mathbf{x}) + c.c\big], \tag{3.122}$$

where the second equality comes from the reality conditions on the potential $V(\mathbf{x})$.

Following the same steps of section 3.3.1 we can define a scalar field $\Phi(\mathbf{x})$ with dimension $\sqrt{mass/length}$ as

$$\Phi(\mathbf{x}) = \sqrt{\frac{m_p}{l_p}}V(\mathbf{x}). \tag{3.123}$$

The free equation of the massless scalar potential $\Phi(\mathbf{x})$ reads

$$\Box\Phi(\mathbf{x},t) = 0. \tag{3.124}$$

A complete set of orthonormal and linearly independent solutions of Eq.(3.124) are given by

$$u_{\mathbf{k}}(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\,\mathrm{e}^{-ikt+i\mathbf{k}\cdot\mathbf{x}}, \tag{3.125}$$

that respect the condition

$$\int d^3x[u_{\mathbf{k}}(t,\mathbf{x})u^*_{\mathbf{h}}(t,\mathbf{x})] = \delta(\mathbf{k}-\mathbf{h}). \tag{3.126}$$

Thus the most general solution of Eq.(3.124) is given by

$$\hat{\Phi}(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}A(\mathbf{k})\left(a_{\mathbf{k}}\,\mathrm{e}^{-ikt+i\mathbf{k}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{k}}\,\mathrm{e}^{ikt-i\mathbf{k}\cdot\mathbf{x}}\right), \tag{3.127}$$

where we have implicitly quantized the scalar field $\hat{\Phi}$ introducing the annihilation and creation operators $a_{\mathbf{k}}$ and $a^{\dagger}_{\mathbf{k}}$. $A(\mathbf{k})$ in Eq.(3.127) can be computed imposing the canonical equal time commutation relations

$$\left[\hat{\Phi}(t,\mathbf{x}),\hat{\Pi}(t,\mathbf{y})\right] = i\hbar\delta(\mathbf{x}-\mathbf{y}), \tag{3.128}$$

where $\hbar = l_p m_p$ and $\hat{\Pi}(t,\mathbf{x})$ is given by the time derivative of $\hat{\Phi}(t,\mathbf{x})$, namely

$$\hat{\Pi}(t,\mathbf{x}) = i\int \frac{d^3k}{(2\pi)^{3/2}} A(\mathbf{k})k\left(-a_{\mathbf{k}}\,\mathrm{e}^{-ikt+i\mathbf{k}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{k}}\,\mathrm{e}^{ikt-i\mathbf{k}\cdot\mathbf{x}}\right). \tag{3.129}$$

Thus, substituting Eq.(3.129) and Eq.(3.127) into Eq.(3.128) and using

$$\left[a_{\mathbf{k}},a^{\dagger}_{\mathbf{p}}\right] = \delta(\mathbf{k}-\mathbf{p}), \tag{3.130}$$

we obtain

$$\begin{aligned}\left[\hat{\Phi}(t,\mathbf{x}),\hat{\Pi}(t,\mathbf{y})\right] &= \frac{i}{(2\pi)^3}\int d^3k\int d^3p A(\mathbf{k})A(\mathbf{p})(p+k)[a_{\mathbf{k}},a^{\dagger}_{\mathbf{p}}]\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{p}\cdot\mathbf{y}-it(k-p)}\\ &= 2i\int d^3k\,A^2(\mathbf{k})k\frac{1}{(2\pi)^3}\,\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,.\end{aligned} \tag{3.131}$$

From the well known form of the Fourier transform of the δ function

$$\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^3}\int d^3k\,\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{3.132}$$

we immediately see that

$$A(\mathbf{k}) = \sqrt{\frac{l_p m_p}{2k}}. \tag{3.133}$$

Thus $\hat{\Phi}$ becomes

$$\hat{\Phi}(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{l_p m_p}{2k}}\left(a_{\mathbf{k}}\,\mathrm{e}^{-ikt+i\mathbf{k}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{k}}\,\mathrm{e}^{ikt-i\mathbf{k}\cdot\mathbf{x}}\right). \tag{3.134}$$

Again, we can define a coherent state $|g\rangle$ as

$$|g\rangle = \mathrm{e}^{-N_G/2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\int d^3k\,g_{\mathbf{k}}\,\mathrm{e}^{i\gamma_{\mathbf{k}}(t)}\,a^{\dagger}_{\mathbf{k}}\right)^n|0\rangle\,, \tag{3.135}$$

which has the main property to be an eigenfunction of the annihilation operator

$$a_{\mathbf{k}}\,|g\rangle = g_{\mathbf{k}}\,\mathrm{e}^{i\gamma_{\mathbf{k}}(t)}\,|g\rangle\,. \tag{3.136}$$

Following the same argument of section 3.3.1 we can evaluate the expression of $g_{\mathbf{k}}$ in terms of the potential imposing

$$\langle g|\,\hat{\Phi}(t,\mathbf{x})\,|g\rangle = \Phi(\mathbf{x}), \tag{3.137}$$

that gives

$$\begin{aligned}
\langle g|\,\hat{\Phi}(t,\mathbf{x})\,|g\rangle &= \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{l_p m_p}{2k}}\,\langle g|\left(a_{\mathbf{k}}\,\mathrm{e}^{-ikt+i\mathbf{k}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{k}}\,\mathrm{e}^{ikt-i\mathbf{k}\cdot\mathbf{x}}\right)|g\rangle \\
&= \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{l_p m_p}{2k}}\left(g_{\mathbf{k}}\,\mathrm{e}^{i\gamma_{\mathbf{k}}(t)-ikt+i\mathbf{k}\cdot\mathbf{x}} + c.c\right) \\
&= \sqrt{\frac{m_p}{l_p}}\int \frac{d^3k}{2(2\pi)^{3/2}}\left(\tilde{V}(\mathbf{k})\,\mathrm{e}^{+i\mathbf{k}\cdot\mathbf{x}} + c.c\right) = \Phi(\mathbf{x}),
\end{aligned} \tag{3.138}$$

where we used $\langle g|g\rangle = 1$.
Thus Eq.(3.137) brings to

$$g_{\mathbf{k}} = \frac{1}{l_p}\sqrt{\frac{k}{2}}\tilde{V}(\mathbf{k}) \qquad \gamma_{\mathbf{k}}(t) = kt. \tag{3.139}$$

It is a well known that in QFT that the number operator for a massless uncharged scalar field is given by

$$\hat{N} = \int d^3k\, a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}, \tag{3.140}$$

so that the number of gravitons in a coherent state is

$$N_G = \langle g|\,\hat{N}\,|g\rangle = \int d^3k\,|g_{\mathbf{k}}|^2 = \frac{1}{l_p^2}\int d^3k\,\frac{k}{2}|\tilde{V}(\mathbf{k})|^2, \tag{3.141}$$

while the mean wave vector can be computed as

$$\frac{\langle k\rangle}{N_G} = \frac{1}{N_G}\int d^3k\,k|g_{\mathbf{k}}|^2 = \frac{1}{N_G l_p^2}\int d^3k\,\frac{k^2}{2}|\tilde{V}(\mathbf{k})|^2, \tag{3.142}$$

where we used

$$\langle k\rangle = \langle g|\int d^3k\,k\hat{n}_{\mathbf{k}}\,|g\rangle = \langle g|\int d^3k\,k\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,|g\rangle\,. \tag{3.143}$$

Applying the Laplacian operator to both sides of Eq.(3.122) and recalling the property of the plane waves

$$\Delta u_{\mathbf{k}}(\mathbf{x}) = -k^2 u_{\mathbf{k}}(\mathbf{x}), \tag{3.144}$$

we can write

$$\begin{aligned}\int d^3p\left[-p^2\tilde{V}(\mathbf{p})\right]u_{\mathbf{p}}(\mathbf{x})&=\Delta V(\mathbf{x})\\ \int d^3p\left[-p^2\tilde{V}(\mathbf{p})\right]\int d^3x\,u_{\mathbf{k}}(\mathbf{x})u_{\mathbf{p}}(\mathbf{x})&=\int d^3x\big[\Delta V(\mathbf{x})u_{\mathbf{k}}(\mathbf{x})\big],\end{aligned}\tag{3.145}$$

in such a way that, using the orthogonality relation of the plane waves (3.126), we come up with the following expression for the Fourier transform of the potential $\tilde{V}(\mathbf{k})$

$$\tilde{V}(\mathbf{k})=-\frac{1}{k^2}\int d^3x\big[\Delta V(\mathbf{x})u_{\mathbf{k}}(\mathbf{x})\big].\tag{3.146}$$

Thus Eq.(3.141) becomes

$$\begin{aligned}N_G&=\frac{1}{2(2\pi)^3l_p^2}\int d^3x\int d^3y\,\Delta V(\mathbf{x})\Delta V(\mathbf{y})\int d^3k\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^3}\\ &=\frac{1}{(2\pi)^2l_p^2}\int d^3x\int d^3y\,\Delta V(\mathbf{x})\Delta V(\mathbf{y})\int_{k_0}^{\infty}dk\,\frac{\sin(kr')}{k^2r'},\end{aligned}\tag{3.147}$$

where $r'=|\mathbf{x}-\mathbf{y}|$ and we introduced the infrared cutoff $k_0=\pi/R_\infty$ to regularize the integral in the variable k. As we mentioned in section 3.3.1 the introduction of this regulator implies that the potential is defined only within a radius R_∞ while it identically vanishes outside. Thus from now on we will consider the domain of integration in the spatial coordinates as a ball of radius R_∞ that we will denote as U.

Focusing on the integral in the variable k we obtain

$$\begin{aligned}\int_{k_0}^{\infty}dk\,\frac{\sin(kr')}{k^2r'}&=\int_{r'k_0}^{\infty}dz\frac{\sin(z)}{z^2}=\frac{\sin(r'k_0)}{r'k_0}-C_i(r'k_0)\\ &=\frac{\sin(r'k_0)}{r'k_0}+\int_0^{r'k_0}dt\,\frac{1-\cos(t)}{t}-\gamma-\ln(r'k_0)\\ &=\ln\left(\frac{R_\infty}{R^*}\right)+\ln\left(\frac{R^*}{\pi r'}\right)+\mathcal{O}\left(\frac{r'}{R_\infty}\right)^0.\end{aligned}\tag{3.148}$$

where we have divided and multiplied the argument of the logarithm for the scale length R^* defined in (3.118).What we are actually doing is using the intrinsic length scale of the problem to split the divergent contribution in the limit $R_\infty\to\infty$ from all subleading terms. Then, substituting the first term of the last equation in Eq.(3.147) and using the *Gauss theorem* we come up with

$$\ln\left(\frac{R_\infty}{R^*}\right)\frac{1}{(2\pi)^2 l_p^2}\left[\int_U d^3x\Delta V(\mathbf{x})\right]^2 = \ln\left(\frac{R_\infty}{R^*}\right)\frac{1}{(2\pi)^2 l_p^2}\left[\int_{\partial U} d^2\mathbf{x}\cdot\nabla V(\mathbf{x})\right]^2. \quad (3.149)$$

The integral in the last expression is computed on the surface of a ball with radius R_∞. In this case, if $R_\infty \gg R^*$, $V(\mathbf{x})$ will always behave as Eq.(3.117) on ∂U whichever its analytical expression will be.
Since in spherical coordinates we have

$$\nabla = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \quad (3.150)$$

we obtain

$$\int_{\partial U} d^2\mathbf{x}\cdot\nabla V(\mathbf{x}) = R_\infty^2\int d\Omega\left[\frac{G_N M}{R_\infty^2} + \mathcal{O}\left(\frac{R^*}{R_\infty}\right)^3\right] = 4\pi G_N M + \mathcal{O}\left(\frac{R^*}{R_\infty}\right), \quad (3.151)$$

meaning that if R_∞ is larger than the characteristic length of the region where ΔV can be finite, we can treat the potential as it is Newtonian. Thus we can write

$$\ln\left(\frac{R_\infty}{R^*}\right)\frac{1}{4\pi^2 l_p^2}\left[\int_U d^3x\Delta V(\mathbf{x})\right]^2 = 4\ln\left(\frac{R_\infty}{R^*}\right)\left[\frac{M^2}{m_p^2} + \mathcal{O}\left(\frac{R^*}{R_\infty}\right)\right]. \quad (3.152)$$

Now we want to verify that the second and third term in Eq.(3.148) are subleading when we consider the whole integral Eq.(3.147), namely

$$\int_U d^3x\, d^3y\, \Delta V(\mathbf{x})\Delta V(\mathbf{y})\left[\ln\left(\frac{R^*}{\pi r'}\right) + \mathcal{O}\left(\frac{r'}{R_\infty}\right)^0\right] \ll \ln\left(\frac{R_\infty}{R^*}\right)\left[\int_U d^3x\, \Delta V(\mathbf{x})\right]^2. \quad (3.153)$$

To do this it is useful to rewrite Eq.(3.147) as

$$N_G = \frac{1}{4\pi^2 l_p^2}\int_U d^3x\int_U d^3y\, \Delta V(\mathbf{x})\Delta V(\mathbf{y}) f(r'), \quad (3.154)$$

and

$$f(r') = \int_{r'k_0}^{\infty} dz\, \frac{\sin(z)}{z^2}. \quad (3.155)$$

The function $f(r')$ has logarithmic divergence in $r' = 0$. However it does not generate a divergence of the whole integral since the points where the function is singular form a

three-dimensional hypersurface in a six-dimensional space of integration.
That means we can neglect the contribution of this *null measure region* and consider only the cases in which r' is finite. In particular we are interested in the region where $r' \approx R_\infty$, since when $r' \ll R_\infty$, $f(r')$ can be approximated by $\ln(R_\infty/R^*)$. In that case we see that

$$|f(r')| < \int_{r'k_0}^{\infty} dz\, \frac{1}{z^2} = \frac{1}{r'k_0} = \frac{R_\infty}{\pi r'} \leq D, \tag{3.156}$$

where $D \ll \ln\left(R_\infty/R^*\right)$ is some constant. Thus we can write

$$\begin{aligned}
&\frac{\pi}{(2\pi)^3 l_p^2} \int_S d^3x\, d^3y\, \Delta V(\mathbf{x}) \Delta V(\mathbf{y}) f(r') \\
\leq &\frac{\pi}{(2\pi)^3 l_p^2} \int_S d^3x\, d^3y\, \Delta V(\mathbf{x}) \Delta V(\mathbf{y}) |f(r')| \\
\leq &\frac{D\pi}{(2\pi)^3 l_p^2} \int_S d^3x\, d^3y\, \Delta V(\mathbf{x}) \Delta V(\mathbf{y}) \\
\leq &\frac{D\pi}{(2\pi)^3 l_p^2} \int_U d^3x\, d^3y\, \Delta V(\mathbf{x}) \Delta V(\mathbf{y}) \\
\leq &4 \frac{M^2 D}{m_p^2} \ll 4 \frac{M^2}{m_p^2} \ln\left(\frac{R_\infty}{R^*}\right),
\end{aligned} \tag{3.157}$$

where S can be thought as the region where $|\mathbf{x}-\mathbf{y}|$ assumes a value comparable to R_∞,

$$S = \left\{ (\mathbf{x}, \mathbf{y}) \in U \times U \middle| |\mathbf{x}-\mathbf{y}| \lesssim R_\infty \right\}. \tag{3.158}$$

Thus the expression for the number of gravitons in a coherent state produced by a compact source with arbitrary density is

$$N_G = 4 \frac{M^2}{m_p^2} \ln\left(\frac{R_\infty}{R^*}\right) + \mathcal{O}\left(\frac{R^*}{R_\infty}\right)^0, \tag{3.159}$$

where the term $\mathcal{O}(R^*/R_\infty)^0$ means that the corrections to the expression for the number of gravitons is a function of R_∞ limited by some constant.

We notice that in our computation of the number of gravitons *we are not making any assumption about the functional dependence of M from M_0*, but we care only about the value of M that is the parameter that characterizes the gravitational effects outside of the source and that can be measured by an external observer. Moreover the dependence from R^* can be neglected as long as we take $R_\infty \gg R^*$ so that M becomes the only parameter that affects the total number of gravitons in the system. This is a consequence of the fact that the computation of N_G required only the knowledge of the potential on a ball of radius R_∞ where we have assumed the validity of the asymptotic condition(3.117). Thus N_G is affected only by the region where the Newtonian approximation is allowed and any further information about the internal structure or the form of the function F in (3.115), that characterizes the theory underlying gravitational interactions, is lost.

3.3.3 Mean graviton wavelength

Following the same procedure we adopted for the computation of the number of gravitons it is immediate to see that Eq.(3.142) can be written as

$$\begin{aligned}\langle k\rangle &= \frac{1}{2(2\pi)^3 l_p^2}\int d^3x \int d^3y\, \Delta V(\mathbf{x})\Delta V(\mathbf{y}) \int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2} \\ &= \frac{1}{(2\pi)^2 l_p^2}\int d^3x \int d^3y\, \Delta V(\mathbf{x})\Delta V(\mathbf{y}) \int_{k_0}^{\infty} dk\, \frac{\sin(kr')}{kr'} \\ &= \frac{1}{(2\pi)^2 l_p^2}\int_U d^3x \int_U d^3y\, \frac{\Delta V(\mathbf{x})\Delta V(\mathbf{y})}{r'} \left[\frac{\pi}{2} - \mathrm{S_i}\left(\frac{\pi r'}{R_\infty}\right)\right],\end{aligned} \tag{3.160}$$

where $r' = |\mathbf{x}-\mathbf{y}|$.
We have introduced the cutoff k_0 even though it is not necessary to compute the integral in the variable k because of the physical interpretation that we gave to R_∞ as maximal length extension of the potential. Thus we have to limit the domain of integration to a ball of radius R_∞ and study which is the leading contribution to the value of $\langle k\rangle$ in the variable R^*/R_∞ following the same steps of the calculation of N_G. In particular, since

$$S_i\left(\frac{\pi r'}{R_\infty}\right) = \int_0^{\pi r'/R_\infty} \frac{\sin(t)}{t} dt \le \int_0^{\pi r'/R_\infty} dt = \frac{\pi r'}{R_\infty} \tag{3.161}$$

we can write

$$\begin{aligned}&\int_U d^3x \int_U d^3y\, \frac{\Delta V(\mathbf{x})\Delta V(\mathbf{y})}{r'}\, \mathrm{S_i}\left(\frac{\pi r'}{R_\infty}\right) \\ \le&\frac{\pi}{R_\infty}\int_U d^3x \int_U d^3y\, \Delta V(\mathbf{x})\Delta V(\mathbf{y}) \\ \simeq&\pi\frac{(4\pi G_N M)^2}{R_\infty} \le (16\pi^3 G_N M)\frac{R^*}{R_\infty} = \mathcal{O}\left(\frac{R^*}{R_\infty}\right)\end{aligned} \tag{3.162}$$

where we used Eq.(3.151) and the definition of R^* (3.118). Thus the second term in square brackets of (3.160), once integrated, gives a term of the form $\mathcal{O}(R^*/R_\infty)$ as we expected.

Now, differently from what happens in the case of N_G where the main contribution depends only on the boundary of the domain of integration, $\langle k\rangle$ depends on the value of the potential in any point of U. Thus we shall restrict our considerations to spherically symmetric potentials $V = V(r)$ for which a more detailed calculation can be developed. In this case $\langle k\rangle$ becomes

$$\begin{aligned}\langle k\rangle =&\frac{\pi}{2(2\pi)^2 l_p^2}\int_U d^3x\int_U d^3y\,\frac{\Delta V(\mathbf{x})\Delta V(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}+\mathcal{O}\left(\frac{R^*}{R_\infty}\right)\\ =&\frac{\pi}{2(2\pi)^2 l_p^2}\int_0^{R_\infty} dr_1\int_0^{R_\infty} dr_2\, r_1^2 r_2^2\Delta V(r_1)\Delta V(r_2)\int d\Omega_1\int d\Omega_2\,\frac{1}{|\mathbf{x}-\mathbf{y}|}\\ &+\mathcal{O}\left(\frac{R^*}{R_\infty}\right)\end{aligned} \tag{3.163}$$

The integral in the variable Ω_2 can be computed considering θ_2 as the angle between $\mathbf{x}$ and $\mathbf{y}$. In this way the integral in Ω_1 gives simply a factor 4π and that in $d\phi_2$ a factor 2π. Thus we come up with

$$\begin{aligned}\langle k\rangle =&\frac{\pi}{l_p^2}\int_0^{R_\infty} dr_1\int_0^{R_\infty} dr_2\, r_1^2 r_2^2\Delta V(r_1)\Delta V(r_2)\int_{-1}^{1} dx\frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2x}}+\mathcal{O}\left(\frac{R^*}{R_\infty}\right)\\ =&\frac{\pi}{l_p^2}\int_0^{R_\infty} dr_1\int_0^{R_\infty} dr_2\, r_1 r_2\Delta V(r_1)\Delta V(r_2)[(r_1+r_2)-|r_1-r_2|]\\ &+\mathcal{O}\left(\frac{R^*}{R_\infty}\right)\end{aligned} \tag{3.164}$$

Now, taking advantage of the symmetry of the previous expression in the exchange of the variables r_1 and r_2, we can write

$$\begin{aligned}\langle k\rangle =&\frac{2\pi}{l_p^2}\int_0^{R_\infty} dr_1\left\{\int_0^{r_1} dr_2\,\left[r_2^2\Delta V(r_2)\right] r_1\Delta V(r_1)\right\}\\ &+\frac{2\pi}{l_p^2}\int_0^{R_\infty} dr_1\left\{\int_{r_1}^{R_\infty} dr_2\,\left[r_2\Delta V(r_2)\right] r_1^2\Delta V(r_1)\right\}+\mathcal{O}\left(\frac{R^*}{R_\infty}\right)\\ =&\frac{2\pi}{l_p^2}\int_0^{R_\infty} dr_1\left[\left.\frac{dV(u)}{du}\right|_{1/r_1}^{\infty} r_1\Delta V(r_1)\right]\\ &+\frac{2\pi}{l_p^2}\int_0^{R_\infty} dr_1\left[\left(\left.u\frac{dV(u)}{du}\right|_0^{1/r_1}-\left.V(u)\right|_0^{1/r_1}\right) r_1^2\Delta V(r_1)\right]+\mathcal{O}\left(\frac{R^*}{R_\infty}\right).\end{aligned} \tag{3.165}$$

where we introduced the variable $u = 1/r_2$.
Finally, assuming as usual the asymptotic conditions (3.116) and (3.117), we obtain the following expression of $\langle k\rangle$ for a gravitational field generated by a density of mass $\rho = \rho(r)$

$$\langle k\rangle = -\frac{2\pi}{l_p^2}\int_0^{R_\infty} dr\,\left[r^2 V(r)\Delta V(r)\right] + \mathcal{O}\left(\frac{R^*}{R_\infty}\right). \tag{3.166}$$

For example, substituting Eq.(3.15) in Eq.(3.166), $\langle k\rangle$ for a system of gravitons generated by a spherical homogeneous source in Newtonian theory we obtain

$$\langle k\rangle = -\frac{3\pi}{R^6}\frac{M_0^2}{m_p^2}\int_0^R dr\,r^2(r^2-3R^2) + \mathcal{O}\left(\frac{R}{R_\infty}\right) = \frac{12\pi}{5R}\frac{M_0^2}{m_p^2} + \mathcal{O}\left(\frac{R}{R_\infty}\right). \tag{3.167}$$

Substituting (3.13) and (3.11) into (3.166), we obtain for a gaussian distribution

$$\begin{aligned}
\langle k\rangle &= \frac{16\pi^2 G_N^2}{l_p^2}\int_0^\infty dr\,r^2\int_0^\infty \frac{dk}{\pi}\left(j_0(kr)M_0\,\mathrm{e}^{-\frac{\sigma^2k^2}{4}}\right)\frac{M_0 e^{-r^2/\sigma^2}}{\pi^{3/2}\sigma^3} + \mathcal{O}\left(\frac{R^*}{R_\infty}\right)\\
&= \frac{16\pi G_N^2 M_0^2}{l_p^2}\int_0^\infty \frac{dk}{k}\int_0^\infty dr\,r\,\sin(kr)\frac{\mathrm{e}^{-r^2/\sigma^2}}{\pi^{3/2}\sigma^3}\,\mathrm{e}^{-\frac{\sigma^2k^2}{4}} + \mathcal{O}\left(\frac{R^*}{R_\infty}\right)\\
&= \frac{16\pi G_N^2 M_0^2}{l_p^2}\int_0^\infty \frac{dk}{2ik}\,\mathrm{e}^{-\frac{k^2\sigma^2}{2}}\int_{-\infty}^\infty dr\frac{\mathrm{e}^{-(\frac{r}{\sigma}-i\frac{k\sigma}{2})^2}}{\pi^{3/2}\sigma^3}r + \mathcal{O}\left(\frac{R^*}{R_\infty}\right)\\
&= 4\frac{M_0^2}{m_p^2}\int_0^\infty dk\,\mathrm{e}^{-\frac{k^2\sigma^2}{2}} + \mathcal{O}\left(\frac{R^*}{R_\infty}\right) = 4\frac{M_0^2}{\sigma m_p^2}\sqrt{\frac{\pi}{2}} + \mathcal{O}\left(\frac{R^*}{R_\infty}\right).
\end{aligned} \tag{3.168}$$

where we considered $R = \sigma$. Thus we see that Eq.(3.167) and Eq.(3.168) reproduces exactly the results Eq.(3.113) and Eq.(3.105) in the limit $R_\infty \to \infty$, computed using the exact expressions of the Fourier transform of the potential for those models.

More in general, in the case of Newtonian theory, since ΔV is given by (3.1), we can write

$$\begin{aligned}
\langle k\rangle &= -\frac{2\pi}{m_p l_p}\int_0^{R_\infty} dr\,\left[r^2 4\pi\rho(r)V(r)\right] + \mathcal{O}\left(\frac{R}{R_\infty}\right)\\
&= -\frac{2\pi}{m_p l_p}E_g(R) + \mathcal{O}\left(\frac{R}{R_\infty}\right),
\end{aligned} \tag{3.169}$$

where we recognized the integral in the variable r to be simply the gravitational energy of the system. Thus we found that *in the Newtonian theory the norm of the wave* *vector of gravitons is proportional to the gravitational energy of the system.* This can be interpreted as a consequence of the fact that, as long as the source is sufficiently rare, the interaction among gravitons can be considered negligible so that their energy will be proportional to their frequency $\omega = k$.

3.3.4 Uniform spherical source

Now we focus on the case of bootstrapped Newtonian theory. Making again the change of variable $u = 1/r$ we can develop Eq.(3.166) obtaining

$$\begin{aligned}
\langle k \rangle &= -\frac{2\pi}{l_p^2}\int_0^\infty du\, V(u)\frac{d^2V(u)}{du^2} + \mathcal{O}\left(\frac{R^*}{R_\infty}\right) \\
&= -\frac{\pi}{l_p^2}\frac{d\,[V(u)]^2}{du}\bigg|_0^\infty + \frac{2\pi}{l_p^2}\int_0^\infty du\left[\frac{dV(u)}{du}\right]^2 + \mathcal{O}\left(\frac{R^*}{R_\infty}\right) \\
&= \frac{2\pi}{l_p^2}\int_0^\infty du\left[\frac{dV(u)}{du}\right]^2 + \mathcal{O}\left(\frac{R^*}{R_\infty}\right) \\
&= \frac{2\pi}{l_p^2}\int_0^\infty dr\, r^2\left[\frac{dV(r)}{dr}\right]^2 + \mathcal{O}\left(\frac{R^*}{R_\infty}\right),
\end{aligned} \tag{3.170}$$

where we integrated by parts and assumed that $r^2\partial_r V^2(r) \to 0$ when $r \to 0$ (we are excluding, for example, the point-like particle). Substituting Eq.(3.39) into Eq.(3.166) we come up with

$$\begin{aligned}
\langle k \rangle &= \frac{\pi}{l_p^2}\left\{\int_0^\infty dr\,\frac{d}{dr}\left[r^2\frac{dV(r)}{dr}\right](1-4V) - \int_0^\infty dr\,4\pi G_N\rho(r)r^2(1-4V)\right\} + \mathcal{O}\left(\frac{R^*}{R_\infty}\right) \\
&= \frac{\pi}{l_p^2}\left[r^2\frac{dV(r)}{dr}\bigg|_0^\infty - \int_0^\infty dr\,4\pi G_N\rho(r)r^2(1-4V)\right] + 2\langle k \rangle + \mathcal{O}\left(\frac{R^*}{R_\infty}\right).
\end{aligned} \tag{3.171}$$

Thus using Eq.(3.117), we get

$$\langle k \rangle = -\frac{\pi G_N}{l_p^2}\left[(M - M_0) + 4\int_0^\infty dr\,4\pi r^2\rho(r)V(r)\right] + \mathcal{O}\left(\frac{R^*}{R_\infty}\right), \tag{3.172}$$

where in the last passage we used

$$M_0 = 4\pi\int_0^\infty dr\, r^2\rho(r), \tag{3.173}$$

and we are still assuming the validity of the asymptotic conditions (3.116) and (3.117). The main difference between (3.159) and (3.172) is that $\langle k\rangle$ depends on both M and M_0 as well as from the internal structure of the source while N_G depends only on M. This happens because the first order contribution to the expression for N_G (3.159) comes from gravitons with a wavelength $\lambda \approx R_\infty$. This is evident if we consider that we had to introduce the infrared cutoff k_0 to compute N_G from the expression (3.147). However the wavelength of these gravitons is so large that their contribution to $\langle k\rangle$ is negligible. This means that those gravitons with $\lambda \ll R_\infty$ can be neglected in the computation of N_G, but represent the main contribution to $\langle k\rangle$.

Now we shall compute $\langle k\rangle$ for a coherent state of a ball of homogeneous density studied in section 3.2. Substituting the dimensionless variables (3.48) in Eq.(3.172) we obtain

$$\langle k\rangle = -\frac{\pi R}{l_p^2}\left[(\tilde{M}-\tilde{M}_0) + 4\int_0^\infty d\tilde{r} 4\pi\tilde{r}^2\tilde{\rho}(\tilde{r})\tilde{V}(\tilde{r})\right] + \mathcal{O}\left(\frac{R^*}{R_\infty}\right), \tag{3.174}$$

When $\tilde{M}_0 \ll 1$, $\tilde{M}$ and C are respectively given by Eq.(3.62) and Eq.(3.63). Thus Eq.(3.174) becomes

$$\begin{aligned}\langle k\rangle =& -\frac{\pi R}{l_p^2}\left[\frac{12\tilde{M}_0^2}{5} + 6\tilde{M}_0^2\int_0^\infty d\tilde{r}\left(\tilde{r}^4 - 3C\tilde{r}^2\right)\right] \\ =& \frac{12\pi}{5}\frac{\tilde{M}_0^2 R}{l_p^2} + \mathcal{O}(\tilde{M}_0^3) + \mathcal{O}\left(\frac{R^*}{R_\infty}\right) \\ =& \frac{12\pi}{5}\frac{M_0^2}{R m_p^2} + \mathcal{O}\left(\frac{G_N M_0}{R}\right)^3 + \mathcal{O}\left(\frac{R^*}{R_\infty}\right),\end{aligned} \tag{3.175}$$

which again reproduces the expected result of the Newtonian limit.
The mean wave vector will be

$$\frac{\langle k\rangle}{N_G} = \frac{3\pi}{5}\frac{1}{R} + \mathcal{O}\left(\frac{G_N M_0}{R}\right)^3 + \mathcal{O}\left[\frac{1}{\ln^2\left(\frac{R_\infty}{R^*}\right)}\right]. \tag{3.176}$$

On the other hand, when $\tilde{M}_0 \gg 1$, $\tilde{M}$ and C will be given by Eq.(3.64) and Eq.(3.65) so that

$$\begin{aligned}\langle k\rangle =& -\frac{\pi R}{l_p^2}\left[\left(\tilde{M} - \frac{\tilde{M}^{2/3}}{6^{1/3}}\right) + 6^{1/3}\tilde{M}^{4/3}\int_0^1 d\tilde{r}\;\tilde{r}^2\left(\tilde{r}^2 - 3C + \frac{\tilde{r}^4}{30C}\right)\right] + \mathcal{O}\left(\frac{R^*}{R_\infty}\right) \\ =& -\frac{\pi R}{l_p^2}\left[\left(\tilde{M} - \frac{\tilde{M}^{2/3}}{6^{1/3}}\right) + 6^{1/3}\tilde{M}^{4/3}\left(\frac{1}{5} - C + \frac{1}{210C}\right)\right] + \mathcal{O}\left(\frac{R^*}{R_\infty}\right),\end{aligned} \tag{3.177}$$

where we used the relation (3.64) between $\tilde{M}$ and $\tilde{M}_0$ in the case of large compactness. The leading term in the expression Eq.(3.174) becomes

$$\langle k \rangle \simeq \frac{6^{1/3}\pi R}{l_p^2}\tilde{M}^{4/3}\left(-\frac{1}{5}+C-\frac{1}{210C}\right) \tag{3.178}$$

while the mean wave vector will be

$$\frac{\langle k \rangle}{N_G} \simeq \frac{6^{1/3}\pi}{4}\left(-\frac{1}{5}+C-\frac{1}{210C}\right)\frac{1}{[(G_N M)^2 R]^{1/3}}\frac{1}{\ln\left(\frac{R_\infty}{R^*}\right)}+\mathcal{O}\left[\frac{1}{\ln\left(\frac{R_\infty}{R^*}\right)}\right]^2. \tag{3.179}$$

The comparison between (3.179) and (3.176) shows a very different behavior of the wavelength of gravitons for rare and dense sources. From (3.176) we see that, as long as the radius R remains bigger than the gravitational radius $G_N M$, the mean wavelength remains insensible to the change of mass of the system which affects only N_G, while in the opposite regime, it depends on both R and $G_N M$. This can be seen as a consequence the fact that the expression for the number of gravitons will remain proportional to M, while that for $\langle k \rangle$ will be proportional to $\tilde{M}_0^2$. Thus, the difference in the functional dependence of $\tilde{M}$ from $\tilde{M}_0$ for dense (Eq.(3.64)) and rare sources (Eq.(3.62)), which can be interpreted as a difference in the self interaction of gravitons in the two cases, becomes a difference in the behavior of the mean wavelength of gravitons.

Conclusions and outlooks

Following Ref.[22] we derived the defining equation of bootstrapped Newtonian gravity and studied its solution in the case of a ball of constant density showing that, differently from what happens in General Relativity, no singularity occurs in the expression for the equilibrium pressure. Then, we obtained the corpuscolar description of a generic static potential, defining a scalar massless quantum field, whose mean value, measured on a suitable coherent state of gravitons, gives the classical scalar potential. From this description, after having defined R^* as the length scale that signals when $\Delta V \approx 0$, we wrote down the expressions for the number of gravitons (3.141) and the mean wave vector (3.142) in a coherent state in terms of the Fourier transform of the potential. Developing the general expression for the number of gravitons, we obtained Eq.(3.159) that, as long as the potential is defined in a region very larger than that where $\Delta V \neq 0$, is in accordance with the expression (2.10) stated by Dvali and Gomez. The main property of the result (3.159) is that the number of gravitons, as long as we consider to take a very high value of R_∞, is independent from the nature of the source, but it is sensible only to the asymptotic behavior of the potential (3.117). This can be seen as a consequences of the fact that the value of N_G is dominated by the gravitons with wavelength $k \approx 1/R_\infty$. On the other hand, computing the mean wavelength for a bootstrapped Newtonian ball, we found that, when $R \gg G_N M$, both the expression obtained in this work (3.176) and that of the peak of the distribution of gravitons given by (2.13) in the model of Dvali and Gomez, are proportional to the radius of the source R, but are independent from M. However, the situation changes when $R \ll G_N M$. Indeed, comparing Eq.(3.179) and Eq.(2.16) we see that the mean wavelength of a spherical source of large compactness is proportional to $[(G_N M)^2 R]^{1/3}$ while according to the works of Dvali and Gomez each graviton in the black hole condensate should have a wavelength proportional to $G_N M$. Thus, in the model we described, the wavelength of gravitons still depends on the radius of the source even when $R \leq G_N M$ while, in Ref.[11] the mean wavelength of the gravitons is always proportional to the Schwarzschild radius once the black hole is formed. However we notice that the equivalence between the two models is recovered in the limit in which the radius of the source is proportional to its gravitational radius $G_N M$. This may signal the necessity to abandon the semiclassical treatment at some length scale and give a completely Quantum Mechanical description of the problem. Thus, in the future, it would be interesting to compute the variance of the gravitational potential to see when the Quantum fluctuations become so important that the semiclassical description we adopted is no longer valid.

In this case, we should describe gravitons as Quantum Mechanical waves with the usual property to vanish on both the origin and on the surface of a ball of radius R_∞. This would lead to a discretization of the wave vector $\mathbf{k}$. However these corrections are suppressed when $R_\infty \to \infty$, because in that case the continuum limit is recovered. Thus we can safely say that, limiting the domain of integration to a sphere of radius R_∞, we introduce just an error of order $\mathcal{O}(1/R_\infty)$.

Then, it could also be interesting to study a dynamical model in which $R \simeq R_H$ and the value of R_∞ becomes comparable to that of R^*. In that case the contribution of the other parameters of the black hole cannot be considered negligible and using Eq.(2.28) we could come up with some correction to the Hawking-Bekenstein entropy due to the factor R^*/R_∞.

We shall also spend some words about the expression $\langle k \rangle$ that we used in sections 3.3.1 and 3.3.3. By its very definition (3.143) one would be tempted to associate $\langle k \rangle$ with the energy of the coherent state, since each massless quantum mechanical wave carries an energy proportional to k. However, this is immediate to verify only in the case of Newtonian gravity as we can see from (3.169) ,while in the case of bootstrapped Newtonian gravity it becomes more problematic to find a clear connection between the energy of the system and the mean wavelength.

Finally, we point out that, in the study of bootstrapped Newtonian theory, we have not kept into account the energy contribution of the pressure to the system, but we have simply treated the source as a ball that produces a certain potential and verified the condition under which it can be stable. The introduction of the energy contribution due to pressure changes the dependence of M from M_0, as shown in Ref.[28], and it may also change the results we obtained about the collective properties of gravitons.

Appendix A

First order expansion of the Schwarzschild metric

In order to derive the fisrt order correction to the Newtonian potential of a point-like particle

$$V_N(r) = -\frac{G_N M}{r}, \tag{A.1}$$

we need to recall the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2G_N M}{\tilde{r}}\right) d\tilde{t}^2 + \left(1 - \frac{2G_N M}{\tilde{r}}\right)^{-1} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2. \tag{A.2}$$

Substituting

$$mu_0 = E = -\frac{d\tilde{t}}{d\tau}\left(1 - \frac{2G_N M}{\tilde{r}}\right)^{-1}, \tag{A.3}$$

in Eq.(1.3) we find

$$\left(\frac{d\tilde{r}}{d\tau}\right)^2 = \frac{2G_N M}{\tilde{r}^2} - 1 + \frac{E^2}{m^2}, \tag{A.4}$$

where m is the mass of a particle subject to the gravitational potential. This immediately leads to

$$\frac{d^2\tilde{r}}{d\tau^2} = -\frac{G_N M}{\tilde{r}^2}. \tag{A.5}$$

Now we write the equation of motion with respect to the proper time t measured by a static observer

$$d\tau = \left(1 - \frac{2G_N M}{r}\right)^{1/2} dt. \tag{A.6}$$

Using Eq.(A.3) we immediately find

$$\frac{d}{d\tau} = \frac{E}{m}\left(1 - \frac{2M}{r}\right)^{-1/2} \frac{d}{dt}. \tag{A.7}$$

Thus the gravitational strength with respect the proper time t is

$$\frac{d^2\tilde{r}}{dt^2} = -\frac{G_N M}{\tilde{r}^2}\left(1 - \frac{2G_N M}{\tilde{r}}\right)\left[\frac{1}{E^2} - \left(1 - \frac{2G_N M}{\tilde{r}}\right)^{-2}\left(\frac{d\tilde{r}}{dt}\right)^2\right]. \tag{A.8}$$

If we are dealing with an almost flat space-time and particles moving at non relativistic speed we have that both $M/\tilde{r}$ and $d\tilde{r}/dt$ are much less than 1 and we can safely approximate $\tilde{r}$ and with r that is the radial coordinate. Thus it is useful to introduce a variable $\epsilon \ll 1$ to keep track of the order of magnitude of the variables we are dealing with and write (A.8)

$$\epsilon\frac{d^2\tilde{r}}{dt^2} \simeq -\epsilon\frac{G_N M}{\tilde{r}^2}\left(1 - \frac{2\epsilon G_N M}{\tilde{r}}\right)\left[1 - \left(1 - \frac{2\epsilon G_N M}{\tilde{r}}\right)^{-2}\epsilon^2\left(\frac{d\tilde{r}}{dt}\right)^2\right]. \tag{A.9}$$

If we write only the first order correction to the gravitational strength we get

$$\epsilon\frac{d^2\tilde{r}}{dt^2} \simeq -\epsilon\frac{G_N M}{\tilde{r}^2} + \epsilon^2\frac{2G_N^2 M^2}{\tilde{r}^3}, \tag{A.10}$$

that comes from the correction to the potential

$$V(r) \simeq -\frac{G_N M}{\tilde{r}} + \frac{G_N^2 M^2}{\tilde{r}^2}. \tag{A.11}$$

Appendix B

Mean energy of uniform spherical source

In 3.3.1 we have seen that the mean energy of a system of gravitons generated by a spherically symmetric source is

$$\langle k \rangle = 36 \, \frac{1}{R} \frac{M_0^2}{m_p^2} \int_0^\infty dz \, \frac{1}{z^4} \left[\cos(z) - \frac{\sin(z)}{z} \right]^2 . \tag{B.1}$$

To comute the integral

$$I = \int_{z_0}^\infty dz \, \frac{1+\cos(2z)}{2z^4} + \frac{1-\cos(2z)}{2z^6} - \frac{\sin(2z)}{z^5}, \tag{B.2}$$

we compute separately

$$\begin{aligned} I_4 = \int_{z_0}^\infty dz \, \frac{1+\cos(2z)}{2z^4} &= \frac{1}{6z_0^3} + \frac{\cos(2z_0)}{6z_0^3} - \frac{\sin(2z_0)}{6z_0^2} \\ &\quad - \frac{\cos(2z_0)}{3z_0} + \frac{2}{3} \int_{z_0}^\infty dz \, \frac{\sin(z)}{z} \end{aligned} \tag{B.3}$$

$$\begin{aligned} I_6 = \int_{z_0}^\infty dz \, \frac{1-\cos(2z)}{2z^6} &= \frac{1}{10z_0^5} - \frac{\cos(2z_0)}{10z_0^5} + \frac{\sin(2z_0)}{20z_0^4} + \frac{\cos(2z_0)}{30z_0^3} \\ &\quad - \frac{\sin(2z_0)}{30z_0^2} - \frac{\cos(2z_0)}{15z_0} + \frac{2}{15} \int_{z_0}^\infty dz \, \frac{\sin(z)}{z} \end{aligned} \tag{B.4}$$

$$\begin{aligned} I_5 = -\int_{z_0}^\infty dz \, \frac{\sin(2z)}{z^5} &= -\frac{\sin(2z_0)}{4z_0^4} - \frac{\cos(2z_0)}{6z_0^3} \\ &\quad + \frac{\sin(2z_0)}{6z_0^2} - \frac{\cos(2z_0)}{3z_0} - \frac{2}{3} \int_{z_0}^\infty dz \, \frac{\sin(z)}{z} . \end{aligned} \tag{B.5}$$

Thus we have

$$
\begin{aligned}
I = I_4 + I_5 + I_6 =& \frac{\cos(2z_0)}{30z_0^3} - \frac{\sin(2z_0)}{5z_0^4} - \frac{\cos(2z_0)}{15z_0} - \frac{\cos(2z_0)}{10z_0^5} - \frac{\sin(2z_0)}{30z_0^2} \\
&+ \frac{1}{6z_0^3} - \frac{1}{10z_0^5} + \frac{2}{15}\int_{z_0}^{\infty} dz \frac{\sin(z)}{z} \\
=& \frac{2\pi}{15} + \mathcal{O}(z_0).
\end{aligned}
\tag{B.6}
$$

Substituting Eq.(B.6) in Eq.(3.112) we immediately find Eq.(3.113).

Bibliography

[1] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (Wiley and Sons, New York, 1972).

[2] H. Stephani, Relativity: An Introduction to Special and general relativity (Cambridge University Press, Cambridge, England, 2004).

[3] S. W. Hawking and G. F. R. Ellis, “The Large Scale Structure of Space-Time,” (Cambridge University Press, Cambridge, 1973)

[4] R. Wald, General Relativity, University of Chicago Press, 1984.

[5] Chandrasekhar, The highly collapsed configurations of a stellar mass, M. N. of Roy. Astr. Soc. 95, 207 (1935).

[6] J. R. Oppenheimer, H. Snyder, On Continued Gravitational Contraction, Phys. Rev. 56, 455 (1939).

[7] S. W Hawking, The Occurrence of Singularities in Cosmology, Proc. Roy. Soc. Lond. A 294, 511 (1966).

[8] S. W. Hawking, R. Penrose, The singularities of Gravitational Collapse and Cosmology, Proc. Roy. Soc. Lond. A 314, 529 (1970).

[9] S. W. Hawking and D. W. Sciama: ”Singularities in Collapsing Stars and Expanding Universes”(Comments on Astrophysics and Space Physics, Vol. 1, p.1)

[10] R. Penrose, Gravitational collapse and space-time singularities, Phys. Rev. Lett. 14, 57 (1965).

[11] G. Dvali, C. Gomez, Black hole’s quantum N-portrait, arXiv:1112.3359 [hep-th];

[12] G. Dvali and , C. Gomez, Self-Completeness of Einstein Gravity, arXiv:1005.3497 [hep-th];

[13] G. Dvali, C. Gomez, R. S. Isermann, D. Lust, S. Stieberger, Black Hole Formation and Classicalization in Ultra-Planckian 2 ! N Scattering, Nucl. Phys. B 893, 187 (2015).

[14] G. Dvali, Strong Coupling and Classicalization, arXiv:1607.07422 [hep-th].

[15] G. Dvali, C. Gomez, Black Holes as Critical Point of Quantum Phase Transitions, Eur. Phys. J. C 74, 2752 (2014).

[16] G. Dvali, C. Gomez, Landau-Ginzburg Limit of Black Holes's Quantum Portrait : Self-Similarity and Critical Exponent, Phys. Lett. B 716, 240 (2012).

[17] Y.J. Ng and H. van Dam, Mod. Phys. Lett. A 9, 335 (1994); Mod. Phys. Lett. A 10,2801 (1995); R. J. Adler and D. I. Santiago, Mod. Phys. Lett. A14, 1371 (1999); M. Maggiore, Phys. Lett. B304, 65 (1993); Phys. Rev. D49, 5182 (1994); F. Scardigli, Phys. Lett. B452, 39 (1999).

[18] G. Veneziano, Europhys. Lett. 2, 199 (1986); D. J. Gross and P. F. Mende, Nucl. Phys. B303, 407 (1988); D. Amati, M. Ciafolini and G. Veneziano. Phys. Lett. B216, 41 (1989); K. Konishi, G. Paffuti and P. Provero, Phys. Lett. B234, 276 (1990); E. Witten, Phys. Today, Apr. 24 (1996)

[19] A. Fabbri and J. Navarro-Salas, Modeling Black holes evaporation(Imperial College Press/World Scientific, London, 2005).

[20] J. D. Bekenstein, Black Holes andthe Second Law, Lett. Nuovo Cimento 4, 737 (1972).

[21] J. D. Bekenstein, Black Holes and Entropy, Phys. Rev. D 7, 2333 (1973).

[22] R. Casadio, M. Lenzi and O. Micu, Phys. Rev. D 98 (2018) 104016 [arXiv:1806.07639 [gr-qc]].

[23] V. Faraoni, Symmetry 7, 2038 (2015).

[24] R. Casadio, A. Giugno, O. Micu, A. Orlandi, Black Holes as Self-Sustained Quantum States and Hawking Radiation, Phys. Rev. D 90, 084040 (2014).

[25] R. Casadio, A. Giugno, A. Orlandi, Thermal Corpuscular Black Holes, Phys. Rev. D 91, 124069 (2015).

[26] R. Casadio, A. Giugno, A. Giusti, Matter and Gravitons in the Gravitational Collapse, Phys. Lett. B 763, 337 (2016).

[27] R. Casadio, A. Giugno, A. Giusti, and M. Lenzi, Phys. Rev. D 96, 044010 (2017).

[28] R. Casadio, M. Lenzi and O. Micu, "Bootstrapped Newtonian stars and black holes," arXiv:1904.06752 [gr-qc].