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K3 SURFACES

Tesi di Laurea in Geometria Algebrica

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Introduction

Algebraic geometry is one of the most studied branch of mathematics. Before the 20th century, the central issue in this subject was the study and the classification of algebraic varieties (e.g. algebraic surfaces) as zero-locus of polynomials. Later, between the 1930s and 1960s, André Weil, Jean-Pierre Serre and Alexander Grothendieck contributed to a rewrite of the foundations of algebraic geometry using the sheaf theory and introducing the concept of scheme, a generalization of the classical notion of algebraic variety.

K3 surfaces were introduced by Weil in 1958, who named them in this way in honor of the three mathematicians Ernst Kummer, Erich Kähler, Kunihiko Kodaira and the K2 mountain, located in Himalaya. These objects represent one of the exceptional case in the classification of algebraic surfaces (Enriques-Kodaira classification).

In this thesis we describe K3 surfaces and their properties using the language of sheaves and schemes, giving many details of these huge theories.

In the first chapter of the elaborate we present presheaves, sheaves, morphisms of sheaves and the relative properties, with a great number of examples. Roughly speaking, a sheaf is a collection of objects (for example, abelian groups or commutative rings) for any open set of a fixed topological space, such that on the intersections of the sets, the objects are glueable in a well defined way (1.5 and 1.10).

In the second chapter we introduce schemes. The definition of a scheme requires several steps to be well understood.

Let $R = k[x_1, ..., x_n]$, the ring of polynomials in n variables with coefficents over an algebraically closed field k. The spectrum of R, denoted with $\operatorname{Spec} R$, is the set of prime ideals of R with the Zariski topology, that is the closed sets are zeros of a (finite) family of polynomials. By Hilbert's Nullstellensatz (see [1]), the points of $\operatorname{Spec} R$ are in 1-1 corrispondence with irreducible subvarieties of the affine space $\mathbb{A}^n$ (2.23). We can regard $\operatorname{Spec} R$ with a sheaf of rings, which represents the regular functions over the space (Chapter 2.2).

We can extend this construction to the spectrum of a generic commutative ring. This is necessary, since in this way any subvariety of $\mathbb{A}^n$ can be associated to the spectrum of a ring. For instance, if we have a prime ideal $\mathfrak{p}$ in $k[x_1, ..., x_n]$ and the variety $Y = \{P \in \mathbb{A}^n : f(P) = 0, \forall f \in \mathfrak{p}\}$, then Y corresponds to $\operatorname{Spec}(k[x_1, ..., x_n]/\mathfrak{p})$.

A scheme is a topological space which is locally isomorphic to the spectrum of a commutative ring (2.39). The most important examples of scheme are the projective ones, which are deepened in Chapter 2.4. Moreover we define abstract algebraic varieties to be a very particular case of scheme (2.87).

In the third chapter we complete the background on sheaves and schemes with an introduction on sheaves of modules on a scheme. These objects, together to some notion of category theory and homological algebra, allow to speak about sheaf cohomology, a very important invariant of schemes.

The fourth chapter is focused on three fundamentals points: the introduction of geometric and abstract divisors, the definition of canonical bundle and the adjunction formula. Divisors (4.7) are essentially hypersurfaces on a particular scheme and they're linked by a corrispondence to a class of sheaves of modules, Cartier divisors (4.25). The canonical bundle is a sheaf of modules which represents the n-forms on a scheme (4.49). The adjunction formula allows, among other things, to easily compute the canonical bundle of many divisors in the projective space (4.50 and 4.51).

In the fifth and last chapter we describe K3 surfaces. They're nonsingular and complete surfaces with no global section in the cotangent bundle and trivial canonical bundle (5.1). In the chapter we present many basic example and give a complete calculation of the Hodge diamond of a K3 surface, the set of dimensions (as vector space) of its cohomology group, through the Serre's duality (5.11).

Introduzione

La geometria algebrica è una delle materie più studiate della matematica. Prima del ventesimo secolo, l'obiettivo principale di questa materia era lo studio e la classificazione delle varietà algebriche (e delle superfici, in particolare) viste come il luogo degli zeri di polinomi. Successivamente, tra gli anni '30 e '60 del Novecento, André Weil, Jean-Pierre Serre e Alexander Grothendieck contribuirono ad una riscrittura dei fondamenti della geometria algebrica attraverso la teoria dei fasci e introducendo il concetto di schema, una generalizzazione della nozione classica di varietà algebrica.

Le superfici K3 sono state introdotte da Weil nel 1958, così battezzate in onore dei tre matematici Ernst Kummer, Erich Kähler e Kunihiko Kodaira e della montagna K2, nella catena dell'Himalaya. Questi oggetti rappresentano uno dei casi eccezionali nella classificazione delle superfici algebriche (la classificazione di Enriques-Kodaira).

In questa tesi descriviamo le superfici K3 e le loro proprietà utilizzando il linguaggio dei fasci e degli schemi, soffermandoci su numerosi dettagli di queste enormi teorie.

Nel primo capitolo dell'elaborato presentiamo prefasci, fasci, morfismi di fasci e le relative proprietà, fornendo un gran numero di esempi. Un fascio, sostanzialmente, è una collezione di oggetti (ad esempio gruppi abeliani o anelli commutativi) per ogni aperto di uno spazio topologico fissato, in modo che sulle intersezioni degli aperti gli oggetti si possano incollare bene (1.5 e 1.10).

Nel secondo capitolo introduciamo gli schemi, la cui definizione è piuttosto articolata e che richiede diversi passi per essere compresa appieno.

Sia $R = k[x_1, ..., x_n]$, l'anello dei polinomi in n variabili a coefficienti in un campo k che sia algebricamente chiuso. Lo spettro di R, denotato con $\operatorname{Spec} R$, è l'insieme degli ideali primi di R con la topologia di Zariski, ovvero i chiusi sono gli zeri di una famiglia (finita) di polinomi. Per il Nullstellsatz di Hilbert (vedi [1]), i punti di $\operatorname{Spec} R$ sono in corrispondenza 1-1 con le sottovarietà irriducibili dello spazio affine $\mathbb{A}^n$ (2.23). Possiamo assegnare a $\operatorname{Spec} R$ un fascio di anelli, che rappresenta le funzioni regolari sullo spazio (Capitolo 2.2).

Possiamo estendere questa costruzione sostituendo a $k[x_1, ..., x_n]$ un anello commutativo qualsiasi. Questo è necessario, poiché in tal modo ogni sottovarietà di $\mathbb{A}^n$ può essere associata allo spettro di un anello. Ad esempio, se $\mathfrak{p}$ è un ideale primo di $k[x_1, ..., x_n]$ e consideriamo la varietà $Y = \{P \in \mathbb{A}^n \colon f(P) = 0, \forall f \in \mathfrak{p}\}$, allora Y corrisponde a $\operatorname{Spec}(k[x_1, ..., x_n]/\mathfrak{p})$.

Uno schema è uno spazio topologico insieme ad un fascio di anelli, che sia localmente isomorfo allo spettro di un anello (2.39). L'esempio più importante di schema sono gli schemi proiettivi, che sono approfonditi nel Capitolo 2.4. Inoltre definiamo le varietà algebriche astratte come un caso molto particolare di schema (2.87).

Nel terzo capitolo completiamo la panoramica sui fasci e sugli schemi con un'introduzione ai fasci di moduli su uno schema. Questi oggetti, insieme a qualche nozione di teoria delle categorie e algebra omologica, consente di parlare di coomologia di fasci, un invariante degli schemi molto importante.

Il quarto capitolo si concentra su tre punti fondamentali: l'introduzione ai divisori geometrici e astratti, la definizione di fibrato canonico e la formula di aggiunzione. I divisori (4.7) sono essenzialmente ipersuperfici su schemi particolari e sono connessi attraverso una corrispondenza ad una classe di fasci di moduli, i divisori di Cartier (4.25). Il fibrato canonico è un fascio di moduli che rappresenta le n-forme su uno schema (4.49). La formula di aggiunzione consente, tra le altre cose, di calcolare facilmente il fibrato canonico dei divisori nello spazio proiettivo (4.50 e 4.51).

Nel quinto ed ultimo capitolo descriviamo le superfici K3. Queste sono superfici complete e nonsingolari, senza sezioni globali nel fibrato cotangente e con fibrato canonico banale (5.1). Nel capitolo presentiamo diversi esempi e svolgiamo il calcolo completo del diamante di Hodge di una superfice K3, ovvero l'insieme delle dimensioni (come spazi vettoriali) dei suoi gruppi di coomologia. Per farlo applichiamo la dualità di Serre (5.11).

Notation

In the whole elaborate, with *ring* we will mean a commutative ring with an identity element.

The notation $\subset$ is used for inclusion of sets. The strict inclusion will be never used.

We will assume the basic notion of General Topology, Commutative Algebra and Categories theory, with standard notations.

All the ideals of rings will be denoted with gothic letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{p}, \mathfrak{q}, ...$

Contents

Chapter 1

Sheaves

In this chapter we will present the category of sheaves on a generic topological space, the basic tool used in the whole elaborate.

In the first section we start defining presheaves and sheaves which have values in a particular class of categories, but we will just focus on sheaves of groups, rings and modules. Then, we illustrate stalks of sheaves, which characterize the sheaves up to isomorphisms (Proposition 1.22). A kind number of examples helps to make more readable this section.

In the second section we complete the definition of category of sheaves with morphisms of sheaves. We'll prove many basic results, which will be very useful in the next chapters.

1.1 Presheaves and Sheaves

Let us begin with some definitions from category theory.

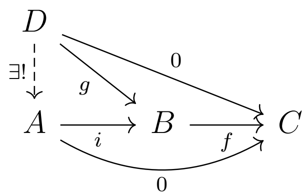
Definition 1.1. Let $\mathfrak{C}$ be a category. With $A \in \mathfrak{C}$ we will mean $A \in \text{ob}(\mathfrak{C})$, the class of objects of $\mathfrak{C}$. A *final object* of $\mathfrak{C}$ is an object $F \in \mathfrak{C}$ such that $\forall A \in \mathfrak{C}$ there exists a unique morphism $A \to F$.

An *initial object* of $\mathfrak{C}$ is an object $I \in \mathfrak{C}$ such that $\forall A \in \mathfrak{C}$ there exist a unique morphism $I \to A$.

A *0-object* is an object $0 \in \mathfrak{C}$ which is both a final and an initial object.

For any $A, B \in \mathfrak{C}$ there is a unique morphism in $\mathrm{Hom}(A, B)$ such that $A \to 0 \to B$. It is called *0-morphism* and it is denoted again with 0.

Definition 1.2. Let $\mathfrak{C}$ be a category with a 0-object; thus there is a 0-morphism 0. A *kernel* of a morphism $f\colon B \to C$ is an object A together a morphism $i\colon A \to B$ such that $f \circ i = 0$ and for every morphism $D \xrightarrow{g} B$ such that $f \circ g = 0$, there exists a unique morphism $D \to A$ such that the diagram



is commutative. A *cokernel* of $f\colon B \to C$ is defined dually by arrow-reversing.

We denote with $\ker f$ and $\operatorname{coker} f$ the kernel and the cokernel of f, respectively.

Definition 1.3. Let $\mathfrak{C}$ be a category. We say that $\mathfrak{C}$ is an abelian category if the following properties are satisfied.

1. For any $A, B \in \mathfrak{C}$, the set $\mathrm{Hom}(A, B)$ is an abelian group such that $\forall A, B, C \in \mathfrak{C}$ we have $f \circ (g + h) = f \circ g + f \circ h$ for every $g, h\colon A \to B$ and $f\colon B \to C$.

2. For any $A, B \in \mathfrak{C}$ it is defined the direct product $A \times B$.

3. Each morphism has a kernel and a cokernel.

4. Every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

Example 1.4. The following are all abelian categories.

1. Abelian groups $\mathfrak{Ab}$.

2. R-modules on a commutative ring R, $\mathfrak{Mod}_R$.

In the following, $\mathfrak{C}$ will always be one of the categories above.

Definition 1.5. Let X be a topological space and let $\mathfrak{C}$ be an abelian category. A *presheaf* $\mathcal{F}$ of objects of $\mathfrak{C}$ on X consists of

1. for every open subset $U \subset X$, an object $\mathcal{F}(U) \in \operatorname{ob}(\mathfrak{C})$;
2. for every $V \subset U$ open subsets of X, a morphism, called *restriction*, $\rho_V^U \in \operatorname{Hom}_{\mathfrak{C}}(\mathcal{F}(U), \mathcal{F}(V))$ such that
   a) $\mathcal{F}(\emptyset) = 0$, the 0-object;
   b) for every open $U \subset X$, $\rho_U^U = \operatorname{id}_{\mathcal{F}(U)}$;
   c) for every open subset $W \subset V \subset U$ of X, $\rho_W^U = \rho_W^V \circ \rho_V^U$.

We will denote $s|_V := \rho_V^U(s)$, for all $V \subset U$ and $s \in \mathcal{F}(U)$.

Equivalently, we can define a presheaf on X to be a controvariant functor $\mathcal{F}$ between $\mathfrak{Top}(X)$, the category of open subsets of X with morphisms of inclusion, and an abelian category $\mathfrak{C}$, such that $\mathcal{F}(\emptyset) = 0$.

In the following, with a presheaf on X we will mean a presheaf of abelian groups or commutative rings or R-modules on X.

Example 1.6. The *zero presheaf* is the presheaf $U \mapsto 0$ for any $U \subset X$.

Let A be a fixed object of a category. The *constant presheaf* $\mathcal{A}$ is the presheaf $U \mapsto A$, with restriction maps $\rho_V^U = \operatorname{id}_A$.

Example 1.7. Let $P \in X$ be a fixed point and let A be a fixed object of a category. The *skyscraper presheaf* at P is the presheaf $U \mapsto \underline{A}_P(U)$, where

$$\underline{A}_P(U) := \begin{cases} A, & \text{if } P \in U, \\ 0, & \text{if } P \notin U. \end{cases}$$

Here the morphisms of restiction are the zero or identity maps.

Example 1.8. Let X be a real topological manifold. We define the *presheaf of continuous maps* to be the presheaf of rings $U \mapsto C(U) = \{f : U \to \mathbb{R} : f \text{ continuous}\}$, with the natural restictions. If X is a differentiable manifold we can define the *presheaf of C^∞ maps* in the same way.

Example 1.9. Let k be a field and $\mathbb{A}^n \coloneqq \{(a_1, ..., a_n)\colon a_i \in k\}$ with the Zariski topology; thus, the closed subset of $\mathbb{A}^n$ are $\{(t_1, ..., t_n)\colon f_1(t_1, ..., t_n) = 0, ..., f_s(t_1, ...t_n) = 0\}$, where $f_1, ..., f_s \in k[x_1, ..., x_n]$. We note that with this topology, every open subset of $\mathbb{A}^n$ is dense. It is called *affine space n*-dimensional. An *affine variety* in $\mathbb{A}^n$ is a closed subset in the Zariski topology, endowed with the induced topology. A *quasi-affine variety* is an open subset of an affine variety.

Let U be a quasi-affine variety in $\mathbb{A}^n$. If $x \in U$, a function $f\colon U \to k$ is called *regular* at x if there is an open subset $V \subset U$ and polynomials $g, h \in k[x_1, ..., x_n]$ such that h is non-vanishing on V and $f|_V = g/h$. It is *regular* if it is regular at x, for any $x \in X$. The set of regular functions on U is denoted with $\mathcal{O}(U)$. It has a natural structure of ring.

Let X be an affine variety in $\mathbb{A}^n$. We define the presheaf of rings $U \mapsto \mathcal{O}(U)$ with natural restrictions.

Definition 1.10. A *sheaf* $\mathcal{F}$ on a topological space X is a presheaf such that, for every open $U \subset X$ and for every open covering $\bigcup_i V_i = U$ we have:

1. (*uniqueness*) if $s \in \mathcal{F}(U)$ and $s|_{V_i} = 0$ for each i, then $s = 0$;

2. (*glueing*) if we have an element $s_i \in \mathcal{F}(V_i)$ for each i, such that $\forall i \neq j$ $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$, for each i.

The elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ on U, while the ones of $\mathcal{F}(X)$ *global section* of $\mathcal{F}$.

We will often denote with $\Gamma(U, \mathcal{F})$ the set of the sections of $\mathcal{F}$ on U.

Example 1.11. The constant presheaf is not always a sheaf, because the glueing property falls. Indeed, let X be a Hausdorff space (with at least two points), and the presheaf $U \mapsto A$ on X with $A \neq 0$ a fixed group (for example). Since there exist open disjoint subsets U, V of X, we can take $x \neq y$ in A which match on $U \cap V = \emptyset$. Hence there cannot be $z \in A$ such that $z = x = y$.

Example 1.12. The skyscraper presheaf $\underline{A}_P$ is a sheaf. Let $\bigcup_i V_i = U$ be an open covering of an open subset $U \subset X$. If $P \notin U$ everything is 0, so nothing has to be proved. Otherwise, there exists k such that $P \in V_k$. If $s \in \underline{A}_P(U)$ such that $s|_{V_i} = 0$ for each i, then $0 = s|_{V_k} = s$, hence the uniqueness is checked. If we have $\{s_i \in \underline{A}_P(V_i)\}$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then $s_i = s_j$ for every i, j. Thus, $s_i = 0$ for any i, or $P \in V_i \cap V_j$ for any i, j. In both cases the section $s \in \underline{A}_P(U)$ such that $s = s_i \in A = \underline{A}_P(U)$ satisfies the glueing property.

Example 1.13. The presheaves of continuous maps on a topological manifold and C^∞ maps on a differentiable manifold are clearly sheaves.

Example 1.14. Let X be an affine variety in $\mathbb{A}^n$. The presheaf $\mathcal{O}$ on X is a sheaf of rings called *sheaf of regular functions* on X.

Let $U \subset X$ be an open subset and let $f \in \mathcal{O}(U)$. Then $f^{-1}(a)$ is a closed subset of U for any $a \in k$, indeed there exists an open covering $\bigcup_i V_i = U$ such that $f|_{V_i} = g_i/h_i$ as above. Thus, $V_i \cap f^{-1}(a) = \{x \in V_i \colon g_i - ah_i = 0\}$ is closed in V_i, since $g_i - ah_i$ is a continuous function. Then, $V_i \setminus f^{-1}(a)$ is an open set in V_i for any i and

$$U \setminus f^{-1}(a) = \bigcup_i (V_i \setminus f^{-1}(a))$$

is an open subset, so $f^{-1}(a)$ is closed in U.

Now, let U be an open subset of X. For any $s, t \in \mathcal{O}(X)$ such that $s|_U = g|_U$, we have that $s = t$, since $(s-t)^{-1}(0)$ is closed and dense in X. It easily follows that $\mathcal{O}$ verifies the uniqueness and glueing properties.

In the following X will be a fixed topological space and avery (pre)sheaf will be a (pre)sheaf on X.

We come to define stalks of (pre)sheaves. First, we have to introduce the notion of direct limit.

Definition 1.15. Let $(I, \leq)$ be a partially ordered set. A *direct system* on I is a family $\{A_i\}_{i \in I}$ of objects of an abelian category $\mathfrak{C}$ together to a morphism $r^i_j \colon A_i \to A_j$ for every $j \leq i$, such that

1. $r_i^i = \mathrm{id}_{A_i}$, $\forall i \in I$;

2. $r_k^i = r_k^j \circ r_j^i$, $\forall k \le j \le i$.

Given a direct system $\{A_i\}$ we define the *direct limit* $\varinjlim A_i$ of the direct system as $\bigsqcup_i A_i$ modulo an equivalence relation $\sim$, with $s_i \sim s_j$ if and only if $s_i \in A_i$, $s_j \in A_j$ and there exists $k \le i, j$ such that $r_k^i(s_i) = r_k^j(s_j)$.

Example 1.16. If $\mathcal{F}$ is a presheaf, then $\{\mathcal{F}(U) \colon U \subset X \text{ open}\}$ is a direct system with the restriction map. If $P \in X$, then $\{\mathcal{F}(U) \colon P \in U\}$ is again a direct system.

Definition 1.17. If $\mathcal{F}$ is a presheaf on X and $P \in X$, then we call *stalk* of $\mathcal{F}$ at P, denoted with $\mathcal{F}_P$, the direct limit

$$\varinjlim \{\mathcal{F}(U) \colon P \in U\}.$$

Thus, the elements of $\mathcal{F}_P$ are represented by pairs (U, s), with $P \in U$, $s \in \mathcal{F}(U)$ such that $(U, s) = (V, t)$ if there exists $W \subset U \cap V$ such that $s|_W = t|_W$. We will often use the notation s_P for the elements of $\mathcal{F}_P$.

Example 1.18. Let $\mathcal{F}$ be the sheaf of C^∞ maps on a differentiable manifold X and let $P \in X$. Then the direct limit of the direct system $\{\mathcal{F}(U) \colon P \in U\}$ is exactly the set of germs of function at P.

Example 1.19. Let $\mathcal{A}$ be the constant presheaf on X and let $P \in X$. Then the stalk $\mathcal{A}_P = A$.

Let $\underline{A}_P$ the skyscraper sheaf at $P \in X$ and let $Q \in X$. If X is a T_1 space, that is every point of X is closed, then the stalk

$$(\underline{A}_P)_Q = \begin{cases} 0, & \text{if } P \neq Q, \\ A, & \text{if } P = Q. \end{cases}$$

Indeed, if $Q \neq P$ then $Q \in U = X \setminus \{P\}$ which is open for assumptions and $\underline{A}_P(U) = 0$.

1.2 Morphisms of Sheaves

Definition 1.20. Let $\mathcal{F}, \mathcal{G}$ be presheaves on X. A *morphism* of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ consists of a morphism $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ for each open subset $U \subset X$ and $\forall V \subset U$ the following diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ {\scriptstyle \rho^U_V}\downarrow & & \downarrow{\scriptstyle \rho^U_V} \\ \mathcal{F}(V) & \xrightarrow[\varphi(V)]{} & \mathcal{G}(V) \end{array}$$

is commutative. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves we say that φ is a morphism of sheaves.

Composition of (pre)sheaves is again a morphism of (pre)sheaves, hence we can define an *isomorphism* of (pre)sheaves as a morphism which has a two-sided inverse.

Remark 1.21. We note that φ induces a morphism on the stalk $\varphi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ for every $P \in X$; namely $\varphi_P(s_P) = (\varphi(s))_P$ for any $s_P \in \mathcal{F}_P$, where s_P is represented by (U, s). It is well defined, since if $(U, s) = (V, t)$ in $\mathcal{F}_P$, then $s|_W = t|_W$ for some $W \subset U, V$. Thus, $\varphi(W)(s) = \varphi(W)(t)$ implies $(U, \varphi(U)(s)) = (V, \varphi(V)(t))$ in $\mathcal{G}_P$.

The importance of stalks of sheaves is evident in the following proposition, a sort of characterisation of sheaves, up to isomorphism.

Proposition 1.22. *A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $\varphi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism $\forall P \in X$.*

Proof. If φ is an isomorphism with inverse ψ, then ψ_P is the inverse of φ_P for any $P \in X$. Now let us suppose that φ_P is an isomorphism for every $P \in X$. If we show that $\varphi(U)$ is invertible for any U, then the inverse of φ will be the collection of maps $\varphi^{-1}(U)$.

Let $s \in \mathcal{F}(U)$ such that $t = \varphi(U)(s) = 0$. For any $P \in U$ we have $t_P = 0$, hence $s_P = 0$. Then we have $U = \bigcup_i V_i$ with V_i open neighborhood of some $P \in U$ and $s|_{V_i} = 0$. For the uniqueness property $s = 0$, so $\varphi(U)$ is injective.

Let $t \in \mathcal{G}(U)$. There exists $s_P \in \mathcal{F}_P$ such that $\varphi_P(s_P) = t_P$ for any $P \in U$, since φ_P is surjective. Glueing together the sections (V_i, s_i), where V_i is an open neighborhood of some $P \in U$ and s_P is represented by (V_i, s_i), we obtain $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$. But $\varphi_P(s_P) = t_P$ for any P, then $\varphi(U)(s) = t$ for the uniqueness property. Hence $\varphi(U)$ is an isomorphism. □

We would work with sheaves instead presheaves for the reasons above. Luckily, we can associate a sheaf to a given presheaf in a natural way.

Proposition 1.23. *Let $\mathcal{F}$ be a presheaf. There exist a unique sheaf $\mathcal{F}^+$ and a morphism of presheaves $\theta\colon \mathcal{F} \to \mathcal{F}^+$ such that, for every sheaf $\mathcal{G}$ and morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ there exists a unique morphism φ^+ such that*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^+ \\ {\scriptstyle\varphi}\downarrow & \swarrow {\scriptstyle\varphi^+} & \\ \mathcal{G} & & \end{array}$$

is commutative. $\mathcal{F}^+$ is called sheaf associated to the presheaf $\mathcal{F}$.

Proof. Let $U \subset X$ be an open subset of X. We define $\mathcal{F}^+(U)$ to be the set of functions $s\colon U \to \bigcup_{P \in U} \mathcal{F}_P$ such that $s(P) \in \mathcal{F}_P$, $\forall P \in U$ and $\forall P \in U$ there exists an open neighborhood $V \subset U$ of P and a section $t \in \mathcal{F}(V)$ such that $t_Q = s(Q)$ in $\mathcal{F}_Q$, for every $Q \in V$. With the natural restrictions $\mathcal{F}^+$ is a sheaf which satisfies the universal property. For a clearer proof see [9, Chapter 2.2]. □

Remark 1.24. If $\mathcal{F}$ is a sheaf, then $\mathcal{F}$ and $\mathcal{F}^+$ are canonically isomorphic. If we take $\mathcal{G} = \mathcal{F}$ and $\mathcal{F} \xrightarrow{\text{id}} \mathcal{F}$ then, by the universal property, there exists a unique morphism $\theta'\colon \mathcal{F}^+ \to \mathcal{F}$ such that $\theta' \circ \theta = \text{id}$; hence θ is an isomorphism.

Remark 1.25. It follows from the construction of $\mathcal{F}^+$ that if $\mathcal{F}$ is a presheaf on X, then $\forall P \in X$ the stalks $\mathcal{F}_P = \mathcal{F}^+_P$.

Now we come to describe injective and surjective morphisms of sheaves.

Definition 1.26. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. We define the *presheaf kernel* of φ, the *presheaf cokernel* of φ and the *presheaf image* of φ as the presheaf given by $U \mapsto \ker(\varphi(U))$, $U \mapsto \operatorname{coker}(\varphi(U))$ and $U \mapsto \operatorname{im}(\varphi(U))$ and we denote them as $\ker\varphi$, $\operatorname{coker}\varphi$ and $\operatorname{im}\varphi$ respectively.

Remark 1.27. If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the presheaf $\ker\varphi$ is a sheaf. Let $U \subset X$ an open subset and let $\bigcup_i V_i = U$ be an open covering of U.

Let $s \in \ker\varphi(U) \subset \mathcal{F}(U)$ such that $s|_{V_i} = 0$ for each i. Then $s = 0$ in $\mathcal{F}(U)$.

Let $s_i \in \ker\varphi(V_i)$ for each i, such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. Then, there exists $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for any i. Therefore $\varphi(s) = 0$, since $\varphi(V_i)(s_i) = 0 \in \mathcal{G}(V_i)$ for any i and there are commutative diagrams

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V_i) & \xrightarrow[\varphi(V_i)]{} & \mathcal{G}(V_i) \end{array}$$

for any i. Hence $s \in \ker\varphi(U)$.

On the contrary, $\operatorname{im}\varphi$ and $\operatorname{coker}\varphi$ are not sheaves.

Definition 1.28. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We define the *sheaf cokernel* of φ and the *sheaf image* of φ to be the sheaves associated to the respective presheaves. We denote them in the same way as above.

Definition 1.29. Let $\mathcal{F}$ and $\mathcal{F}'$ be sheaves on X. We say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{F}$ if for each open $U \subset X$, $\mathcal{F}'(U)$ is a substructure of $\mathcal{F}(U)$. This means that if $\mathcal{F}$ and $\mathcal{F}'$ are sheaves of abelian groups (for example) then $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$.

Example 1.30. If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, $\ker\varphi$ is a subsheaf of $\mathcal{F}$. We'll see later, Corollary 1.34, that the sheaf $\operatorname{im}\varphi$ is a subsheaf of $\mathcal{G}$.

Definition 1.31. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We say that φ is *injective* if $\ker\varphi = 0$, the zero sheaf. Equivalently, φ is injective if $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all $U \subset X$.

Lemma 1.32. *A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is injective if and only if the induced map $\varphi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ is injective $\forall P \in X$.*

Proof. Let us suppose that φ is injective, hence $\varphi(U)$ is injective for each $U \subset X$, and let $P \in X$. Let $s_P, s'_P \in \mathcal{F}_P$ represented by (U, s) and (V, s'), respectively, and suppose that $\varphi_P(s_P) = \varphi_P(s'_P) = t_P$ in $\mathcal{G}_P$. If (W, t) represents t_P and $T = U \cap V \cap W$, then $\varphi(T)(s|_T) = \varphi(T)(s'|_T) = t|_T$, but $\varphi(T)$ is injective by assumption, hence $s|_T = s'|_T$ and $s_P = s'_P$.

Conversely, let $U \subset X$ be an open subset and let $s \in \mathcal{F}(U)$ such that $\varphi(U)(s) = 0$. For any $P \in U$ we have $\varphi_P(s_P) = 0$, therefore $s_P = 0$ since φ_P is injective. Thus, there exists an open neighborhood V of P such that $s|_V = 0$. Repeating for every $P \in X$, we obtain an open covering $\bigcup_i V_i = X$ and $s|_{V_i} = 0$, for any i. Hence $s = 0$ and $\ker \varphi(U) = 0$. □

Lemma 1.33. *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. If $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for any open set $U \subset X$, then the associated morphism of sheaves $\varphi^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ is injective.*

Proof. For any $P \in X$, the induced map on the stalks $\varphi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ is trivially injective. But $\mathcal{F}^+_P = \mathcal{F}_P$ and $\mathcal{G}^+_P = \mathcal{G}_P$, hence $\varphi^+_P\colon \mathcal{F}^+_P \to \mathcal{G}^+_P$ is injective for any $P \in X$. Thus, we conclude for 1.32. □

Corollary 1.34. *The sheaf* $\operatorname{im}\varphi$ *is a subsheaf of $\mathcal{G}$.*

Proof. We have an injective morphism of presheaves $\operatorname{im}\varphi \hookrightarrow \mathcal{G}$. The associated morphism of sheaves is still injective for 1.33. □

Definition 1.35. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We say that φ is *surjective* if $\operatorname{im}\varphi = \mathcal{G}$.

Lemma 1.36. *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves and let $P \in X$. Then*

i) $\operatorname{im}(\varphi_P) = (\operatorname{im}\varphi)_P$.

ii) $\ker(\varphi_P) = (\ker\varphi)_P$.

Proof. *i)* The sheaf $\operatorname{im}\varphi$ is a subsheaf of $\mathcal{G}$, hence $(\operatorname{im}\varphi)_P \subset \mathcal{G}_P$. Moreover $\operatorname{im}\varphi_P \subset \mathcal{G}_P$, so we have to show that $\operatorname{im}(\varphi_P) = (\operatorname{im}\varphi)_P$ in $\mathcal{G}_P$.

Let $y \in (\operatorname{im}\varphi)_P$ and let us consider (U, t), where $t \in (\operatorname{im}\varphi)(U)$, which represents y. Thus, there exists $s \in \mathcal{F}(U)$ such that $\varphi(U)(s) = t$. If $x = s_P \in \mathcal{F}_P$, then $\varphi_P(x) = \varphi_P(s_P) = t_P = y$, that is $y \in \operatorname{im}\varphi_P$.

Conversely, if $y \in \operatorname{im}\varphi_P$ then there is $x \in \mathcal{F}_P$ such that $\varphi_P(x) = y$. If (U, s) represents x in $\mathcal{F}_P$ and (U, t) represents y in $\mathcal{G}_P$, then $(\varphi(U)(s))_P = t_P = y$, that is y is in the stalk $(\operatorname{im}\varphi)_P$ of the presheaf $\operatorname{im}\varphi$. But the stalks of presheaves and associated sheaves coincide, hence $y \in (\operatorname{im}\varphi)_P$.

ii) Analogue. □

Lemma 1.37. *A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is surjective if and only if the induced map $\varphi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ is surjective $\forall P \in X$.*

Proof. We have $\operatorname{im}(\varphi_P) = (\operatorname{im}\varphi)_P$ from 1.36. Thus φ is surjective $\iff$ $\operatorname{im}\varphi = \mathcal{G} \overset{1.22}{\iff} (\operatorname{im}\varphi)_P = \mathcal{G}_P$ for any $P \iff \operatorname{im}(\varphi_P) = (\operatorname{im}\varphi)_P = \mathcal{G}_P$ for any $P \iff \varphi_P$ is surjective for any P. □

Corollary 1.38. *A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if φ is injective and surjective.*

Proof. It follows from 1.22, 1.32 and 1.37. □

Remark 1.39. It's not true that a morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is surjective if and only if it is surjective on any open subset $U \subset X$.

If $\varphi(U)$ is surjective for each $U \subset X$, then φ is surjective on the stalks, hence it is surjective for 1.37. However, the converse is false. See 1.41 for an example.

Definition 1.40. Let $\mathcal{F}, \mathcal{G}$ be sheaves on X. Then the *direct sum* $\mathcal{F} \oplus \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ with restriction maps induced by the ones in $\mathcal{F}$ and $\mathcal{G}$. We have that the stalk $(\mathcal{F} \oplus \mathcal{G})_P$ in $P \in X$ is $\mathcal{F}_P \oplus \mathcal{G}_P$.

Example 1.41. We want to show an example of surjective morphism of sheaves on X which is not surjective on some open set $U \subset X$.

Let us consider $X = \mathbb{R}$ as topological space, $\mathcal{F}$ as the constant sheaf $\underline{\mathbb{Z}}$ and $\mathcal{G}$ as the sum of two skyscraper sheaves on two distinct point of $\mathbb{R}$, that is $\mathcal{G} = \underline{\mathbb{Z}}_P \oplus \underline{\mathbb{Z}}_Q$ with $P \neq Q$. The natural morphism of restriction $\underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_P \oplus \underline{\mathbb{Z}}_Q$ is not surjective for any open $U \subset X$, since if $P, Q \in U$ we have $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $z \mapsto (z, z)$. However $\forall S \in \mathbb{R}$ the map induced on the stalks $\underline{\mathbb{Z}}_S = \mathbb{Z} \to (\underline{\mathbb{Z}}_P \oplus \underline{\mathbb{Z}}_Q)_S = (\underline{\mathbb{Z}}_P)_S \oplus (\underline{\mathbb{Z}}_Q)_S$ is surjective, since the right side is 0 if $S \neq P, Q$ and $\mathbb{Z}$ otherwise. Then $\mathcal{F} \to \mathcal{G}$ is surjective for 1.37.

We end up the first chapter introducing exact sequences of sheaves, the inverse image sheaf and the restriction of a sheaf. The last two notions will be fundamental in the next chapter.

Definition 1.42. Let $\mathcal{F}$ be a sheaf of abelian groups or modules and let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$. We define the *quotient sheaf* $\mathcal{F}/\mathcal{F}'$ to be the sheaf associated to $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$. We have $(\mathcal{F}/\mathcal{F}')_P = \mathcal{F}_P/\mathcal{F}'_P$, for all $P \in X$.

In the following, when we talk about quotient sheaves we always suppose to have sheaves of abelian groups or modules.

Proposition 1.43. *Let* $\varphi\colon \mathcal{F} \to \mathcal{G}$ *be a morphism of sheaves. Then* $\operatorname{im}\varphi \cong \mathcal{F}/\ker\varphi$.

Proof. From 1.36 we have

$$(\operatorname{im}\varphi)_P = \operatorname{im}\varphi_P \cong \mathcal{F}_P/(\ker\varphi)_P = \mathcal{F}_P/\ker\varphi_P = (\mathcal{F}/\ker\varphi)_P.$$

Hence $\operatorname{im}\varphi \cong \mathcal{F}/\ker\varphi$ for 1.22. □

Definition 1.44. A sequence of sheaves and morphisms of sheaves

$$\ldots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \to \ldots$$

is *exact* if $\ker\varphi^i = \operatorname{im}\varphi^{i-1}$ for every i.

Proposition 1.45. *The sequence* $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ *is exact if and only if* φ *is injective and* $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to 0$ *is exact if and only if* φ *is surjective.*

Proof. The zero morphism has trivial image and kernel. □

Proposition 1.46. *Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Then the projection map $\mathcal{F} \to \mathcal{F}/\mathcal{F}'$ is surjective with kernel $\mathcal{F}'$.*

Proof. For any open subset $U \subset X$, the map $\mathcal{F}(U) \to \mathcal{F}(U)/\mathcal{F}'(U)$ is surjective, hence $\mathcal{F} \to \mathcal{F}/\mathcal{F}'$ is surjective.

The kernel of the map contains $\mathcal{F}'$, obviously. Conversely, let $s \in \mathcal{F}(U)$ such that $s \mapsto 0 \in \mathcal{F}(U)/\mathcal{F}'(U)$, for some $U \subset X$. Hence there exists an open covering $\bigcup_i V_i = U$ such that $s|_{V_i} \in \mathcal{F}'(V_i)$ for any i, so $s \in \mathcal{F}'(U)$. □

Corollary 1.47. *The sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$ is exact.*

Proof. It follows from the last two propositions. □

Definition 1.48. Let $f: X \to Y$ a continuous map of topological spaces and let $\mathcal{F}, \mathcal{G}$ be sheaves on X and Y, respectively. We define:

1. the *direct image* sheaf $f_*\mathcal{F}$ to be the sheaf on Y associated to the presheaf $V \mapsto \mathcal{F}(f^{-1}(V))$;

2. the *inverse image* sheaf $f^{-1}\mathcal{G}$ to be the sheaf on X associated to the presheaf $U \mapsto \varinjlim \mathcal{G}(V)$, where the direct limit is taken on the subsets $V \subset Y$ such that $f(U) \subset V$.

Definition 1.49. Let $i: Z \to X$ be an injection of topoogical spaces and $\mathcal{F}$ a sheaf on X. The *restriction* of $\mathcal{F}$ to Z is the sheaf $\mathcal{F}|_Z := i^{-1}\mathcal{F}$ on Z.

Remark 1.50. If $Z \subset X$ is an open subset of X and $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}|_Z(U) = \mathcal{F}(U \cap Z)$, for any open subset U of Z.

Chapter 2

Schemes

In this chapter we present the concept of schemes, the central notion to study abstract algebraic varieties. Roughly speaking, a scheme is a collection of affine schemes, a generalization of affine algebraic varieties (Example 1.9) deeply described in the first two sections.

There are two important classes of scheme: the projective ones and algebraic varieties. In the last two sections we explain many fundamental concepts to understand these kinds of schemes.

2.1 Spectrum of a Ring

We introduce the notion of spectrum of a ring, the first step to define schemes.

Definition 2.1. Let R be a ring. The *spectrum* of R is the set of the prime ideals of R, denoted with $\operatorname{Spec} R$. We define:

1. $V(E) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} R \colon \mathfrak{p} \supset E\}$ for every $E \subset R$. If $f \in R$, we write $V(f)$ instead of $V((f))$;

2. $D(f) \coloneqq V(f)^C = \{\mathfrak{p} \in \operatorname{Spec} R \colon f \notin \mathfrak{p}\}$ in $\operatorname{Spec} R$, for every $f \in R$.

Proposition 2.2. *We have the following:*

i) *if* $\mathfrak{a}$ *is the ideal generated by* E, *then* $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, *where* $\sqrt{\mathfrak{a}} = \{a \in R\colon a^n \in \mathfrak{a},$ *for some* $n \in \mathbb{N}\}$ *is the radical of* $\mathfrak{a}$*;*

ii) $V(0) = \operatorname{Spec} R$, $V(\operatorname{Spec} R) = \emptyset$*;*

iii) *if* $(E_i)_{i\in I}$ *is a family of subsets of* R, *then*

$$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i);$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ *for all ideals* $\mathfrak{a}, \mathfrak{b} \subset R$.

These results show that the sets $V(E)$ *satisfy the closed axioms in topological spaces. The relative topology on* $\operatorname{Spec} R$ *is called Zariski topology.*

Proof. The proof is an easy check. See [6, Chapter II.2] for details. □

Proposition 2.3. *A base for Zarisky's topology in* $\operatorname{Spec} R$ *is given by the sets* $D(f)$ *with* $f \in R$. *They're called principal open subsets of* $\operatorname{Spec} R$.

Proof. We have $D(f) \cap D(g) = D(fg)$ for any $f, g \in R$, because $D(f) \cap D(g) = V(f)^C \cap V(g)^C = (V(f) \cup V(g))^C = (V(fg))^C = D(fg)$. Moreover, $V(1) = \operatorname{Spec} R$, so the set of $D(f)$ is a base for Zariski topology. □

Example 2.4. The space $\operatorname{Spec} \mathbb{Z}$ consists of all the prime ideals (p) with $p \in \mathbb{Z}$ prime number and (0). The Zariski topology on $\operatorname{Spec} \mathbb{Z}$ is the cofinite topology, where the closed sets are the finite ones. If C is a closed subset of $\operatorname{Spec} \mathbb{Z}$ then $C = V(\mathfrak{a})$ with $\mathfrak{a}$ ideal in $\mathbb{Z}$. But $\mathbb{Z}$ is a principal ideals domain, then $\mathfrak{a} = (a)$ with $a \in \mathbb{Z}$. If $p_1, .., p_n$ are the prime numbers which divide a, then $C = \{(p_1), ..., (p_n)\}$.

We observe that $\operatorname{Spec} R$ is not always a T_1 space, that is it may contain non-closed points.

Proposition 2.5. *If* R *is a ring and* $\mathfrak{p} \in \operatorname{Spec} R$, *then* $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. *In particular* $\{\mathfrak{p}\}$ *is closed in* $\operatorname{Spec} R$ *if and only if* $\mathfrak{p}$ *is maximal in* R.

Proof. The set $V(\mathfrak{p})$ is closed in $\operatorname{Spec} R$ and $\{\mathfrak{p}\} \subset V(\mathfrak{p})$. Let $V = V(\mathfrak{a})$ be a closed subset of $\operatorname{Spec} R$ such that $\mathfrak{p} \in V$. Then $\mathfrak{p} \supset \mathfrak{a}$, so for any $\mathfrak{q} \in V(\mathfrak{p})$ we have $\mathfrak{q} \supset \mathfrak{p} \supset \mathfrak{a}$, that is $\mathfrak{q} \in V(\mathfrak{a})$. Therefore $V(\mathfrak{p}) \subset V$. □

Example 2.6. If k is a field, then $\operatorname{Spec} k$ is one point.

Let us consider $R = k[x]$, with k algebraically closed. Then, the *affine line* over k is the space $\mathbb{A}^1_k := \operatorname{Spec} R = \{(x - a) \colon a \in k\} \cup \{(0)\}$. Its closed points (or maximal ideals) are in 1-1 corrispondence with the points of $k = \mathbb{A}^1$. Moreover, there is a dense point (0), since $\overline{\{(0)\}} = V((0)) = \mathbb{A}^1_k$.

The *affine n-dimensional space* over k is $\mathbb{A}^n_k := \operatorname{Spec} k[x_1, ..., x_n]$. If k is algebraically closed, the closed points are $(x_1 - a_1, ..., x_n - a_n)$, where $a_1, ..., a_n \in k$ (this is not obvious, but a consequence of Hilbert's Nullstellensatz, see [1]). Therefore, closed points of $\mathbb{A}^n_k$ are in 1-1 corrispondence with points of $\mathbb{A}^n$.

Before moving forward in the description of spectrum of rings, we need a few results of Commutative Algebra. See [1, Chapter 1] for proofs.

Proposition 2.7. *Let $R \neq 0$ be a ring. Then, there exists a maximal ideal in R. Hence, for every non-unit $f \in R$, there is a maximal ideal $\mathfrak{m}$ such that $f \in \mathfrak{m}$.*

Definition 2.8. Let R be a ring. The *nilradical* $\mathcal{N}(R)$ of R is the set of the nilpotent elements of R.

Proposition 2.9. *The nilradical of a ring R is the intersection of all prime ideals of R.*

Proposition 2.10. *The radical of an ideal $\mathfrak{a} \subset R$ is the intersection of all prime ideals in R which contain $\mathfrak{a}$.*

Proposition 2.11. *In every ring $R \neq 0$ we have:*

i) $D(f) = \emptyset \iff f$ *is nilpotent;*

ii) $D(f) = \operatorname{Spec} R \iff f$ *is a unit;*

iii) $D(f) = D(g) \iff \sqrt{(f)} = \sqrt{(g)}$.

Proof. $i)$ $D(f) = \emptyset \iff V(f) = \operatorname{Spec} R \iff f \in \mathfrak{p}$, for every prime ideal $\mathfrak{p} \iff f \in \mathcal{N}$.

$ii)$ $D(f) = \operatorname{Spec} R \iff V(f) = \emptyset \iff f$ is a unit.

$iii)$ $D(f) = D(g) \iff V(f) = V(g) \iff \sqrt{(f)} = \{\mathfrak{p} \supset (f)\} = \{\mathfrak{p} \supset (g)\} = \sqrt{(g)}$. □

Remark 2.12. Let $\varphi\colon R \to S$ be a ring homomorphism. If $\mathfrak{q} \in \operatorname{Spec} S$, then $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ is a prime ideal of R. Then φ induces a map

$$f\colon \operatorname{Spec} S \to \operatorname{Spec} R, \qquad \mathfrak{q} \mapsto \mathfrak{p} = \varphi^{-1}(\mathfrak{q}).$$

However images of ideals are not always ideals.

Definition 2.13. Let $\varphi\colon R \to S$ be a ring homomorphism. If $\mathfrak{a} \subset R$ is an ideal we define the *extended ideal* $\mathfrak{a}^e$ as the ideal generated by $\varphi(\mathfrak{a})$.

Proposition 2.14. *Let* $\varphi\colon R \to S$ *be a ring homomorphism. Then, the induced map* $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ *is continuous. More precisely:*

i) $\forall \mathfrak{a} \in \operatorname{Spec} R$, $f^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$;

ii) $\forall g \in R$, $f^{-1}(D(g)) = D(\varphi(g))$.

Proof. $i)$ Let $\mathfrak{a}$ be an ideal of R. Then

$$\begin{aligned} f^{-1}(V(\mathfrak{a})) &= \{\mathfrak{p} \in \operatorname{Spec} S\colon f(\mathfrak{p}) \in V(\mathfrak{a})\} = \{\mathfrak{p}\colon \varphi^{-1}(\mathfrak{p}) \supset \mathfrak{a}\} \\ &= \{\mathfrak{p}\colon \mathfrak{p} \supset \varphi(\mathfrak{a})\} = V(\varphi(\mathfrak{a})) = V(\mathfrak{a}^e). \end{aligned}$$

$ii)$ Let $g \in R$. Then

$$\begin{aligned} f^{-1}(D(g)) &= \{\mathfrak{p} \in \operatorname{Spec} S\colon f(\mathfrak{p}) \in D(g)\} = \{\mathfrak{p}\colon g \notin \varphi^{-1}(\mathfrak{p})\} \\ &= \{\mathfrak{p}\colon \varphi(g) \notin \mathfrak{p}\} = D(\varphi(g)). \end{aligned}$$

□

Proposition 2.15. *Let $\varphi\colon R \to S$ be a surjective ring homomorphism. Let $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ be the induced map. Then $\operatorname{Spec} S$ is homeomorphic to $V(\ker\varphi)$ via f.*

Proof. By the surjectivity of φ, we have $S \cong R/\ker\varphi$. We know there is a 1-1 corrispondence between prime ideals of S and prime ideals of R which contain $\ker\varphi$. Hence $f\colon \operatorname{Spec} S \to V(\ker\varphi)$ is bijective and continuous by 2.14. We show that f is a closed map.

Let $V = V(\mathfrak{a})$ be a closed subset of $\operatorname{Spec} S$ and let $\mathfrak{b} = \varphi^{-1}(\mathfrak{a})$. Then,

$$\begin{aligned} f(V) &= \{\mathfrak{p} \in \operatorname{Spec} R\colon \mathfrak{p} = f(\mathfrak{q}), \mathfrak{q} \supset \mathfrak{a}\} = \{\mathfrak{p}\colon \mathfrak{p} = \varphi^{-1}(\mathfrak{q}), \varphi^{-1}(\mathfrak{q}) \supset \varphi^{-1}(\mathfrak{a})\} \\ &= \{\mathfrak{p}\colon \mathfrak{p} \supset \mathfrak{b}\} = V(\mathfrak{b}), \end{aligned}$$

so f is a homeomorphism. □

Corollary 2.16. *Let R be a ring and let $\mathfrak{a}$ be an ideal of R. Then $\operatorname{Spec} R/\mathfrak{a}$ is homeomorphic to $V(\mathfrak{a})$.*

Now we introduce the concept of irreducible topological space.

Definition 2.17. A topological space X is *irreducible* if for every non-empty open subsets $U, V \subset X$ of X, $U \cap V = \emptyset$, or, equivalently, if every open subset of X is dense.

We note that in locally euclidean spaces this is an irrelevant definition, since any T_2 space is not irreducible.

Proposition 2.18. *Let R be a ring. Then $\operatorname{Spec} R$ is irreducible if and only if $\mathcal{N}(R)$, the nilradical of R, is a prime ideal.*

Proof. Let us suppose $\operatorname{Spec} R$ irreducible. If $f, g \notin \mathcal{N}(R)$, then $D(f), D(g) \neq \emptyset$. By proof of 2.3, we have $D(f) \cap D(g) = D(fg)$ for any $f, g \in R$, thus $D(fg) \neq \emptyset$ by assumption, so $fg \notin \mathcal{N}$.

Conversely, if the radical of R is a prime ideal, it lies in every open subset $U \subset \operatorname{Spec} R$, hence $\operatorname{Spec} R$ is irreducible. □

Example 2.19. The spaces $\operatorname{Spec}\mathbb{Z}$ and $\mathbb{A}^n_k$ are irreducible, since the nilradicals of $\mathbb{Z}$ and $k[x_1, ..., x_n]$ are zero ideals in integral domains, therefore prime.

Example 2.20. Let $f = xy \in R = k[x, y]$. Then, $\operatorname{Spec} R/(f)$ is not irreducible, since $\mathcal{N}(R/(f)) = (xy)$ which is not a prime ideal.

Lemma 2.21. *Let X be a topological space and $Y \subset X$. If Y is irreducible, then $\overline{Y}$ is irreducible.*

Proof. Let us suppose there exist proper open subsets $U, V \subset \overline{Y}$ such that $U \cap V = \emptyset$. Then $Y \subset U$ or $Y \subset V$ by assumption. If $Y \subset U$, $V^C \supset Y$ and it is a closed subset of $\overline{Y}$, which is closed in X. Hence Y^C is a closed subset of X which contains Y, so $Y^C = \overline{Y}$ and $U = \emptyset$. This is absurd. □

Proposition 2.22. *Let X be a topological space and let $Y \subset X$ be irreducible. Then, Y is contained in a maximal irreducible subspace of X. These maximal subspaces are closed and cover X. They're called irreducible components of X.*

Proof. Let $\Sigma = \{Z \subset X : Z \text{ irreducible}, Y \subset Z\} \neq \emptyset$, since $Y \in \Sigma$. We want to apply Zorn's Lemma (see [1, Chapter 1]) to Σ equipped with the inclusion relation. Let $\{Z_i\}_{i\in I}$ be a chain in Σ, that is $Z_i \subset Z_j$ or $Z_j \subset Z_i$ for any $i, j \in I$, and let $T = \bigcup_{i\in I} Z_i$. The space T is irreducible, since for any open subset $U \subset T$ and for every $x \in T \setminus \overline{U}$, there exists $i \in I$ such that $x \in Z_i \setminus \overline{U}$, but $U \cap Z_i$ is open in Z_i and $\overline{U \cap Z_i} = \overline{U} \cap Z_i$ in Z_i. Since Z_i is irreducible, x cannot exists, so every open subset of T is dense. Clearly $T \supset Y$, hence $T \in \Sigma$ and it is an upper bound of the chain. By Zorn's Lemma, Σ has maximal elements.

Every irreducible component is closed, since the closure of an irreducible subspace is again irreducible by 2.21. Finally, every $\{x\} \in X$ is an irreducible subspace, hence it is contained in a maximal irreducible subspace of X. □

Example 2.23. Let k be a field and let $\mathbb{A}^n$ be the affine space (Definition 1.9). Thus, every irreducible affine subvariety of $\mathbb{A}^n$ can be defined by polynomials $f_1, ..., f_n$ such that $(f_1, ..., f_n)$ is a prime ideal in $k[x_1, ..., x_n]$. Conversely, every prime ideal in $k[x_1, ..., x_n]$ defines an irreducible subvariety of $\mathbb{A}^n$ (see [6, Chapter I.1]). Thus, there is a 1-1 corrispondence between irreducible subvarieties of $\mathbb{A}^n$ and points of $\operatorname{Spec} k[x_1, ..., x_n]$.

2.2 Structure Sheaves

Let R be a ring. Our target is to define a sheaf of rings on $\operatorname{Spec} R$. To do this, let us briefly recall the notion of localization of rings.

Definition 2.24. Let R be a ring. We call R *local ring* if there exists a unique maximal ideal $\mathfrak{m}$ in R.

If R, S are local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively, we say that $\varphi \colon R \to S$ is a *local homomorphism of local rings* if it is a ring homomorphism such that $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

Proposition 2.25. *Let R be a ring and let $\mathfrak{m}$ be a maximal ideal in R. If each $a \in R \setminus \mathfrak{m}$ is a unit, then R is a local ring with maximal ideal $\mathfrak{m}$.*

Proof. If $\mathfrak{m}'$ is a maximal ideal in R, then $\mathfrak{m}' \not\subset \mathfrak{m}$, so $1 \in \mathfrak{m}'$. □

Definition 2.26. Let R be a ring. A subset $C \subset R$ is called *closed multiplicatively system* if $1 \in C$ and $\forall a, b \in C$, $ab \in C$. We define the *ring of fractions of R with respect to C* to be the set $C^{-1}R := (R \times C)/_\sim$ where $(a, s) \sim (b, t)$ if and only if there exists $u \in C$ such that $u(at - bs) = 0$, $\forall a, b \in R$, $\forall s, t \in C$. The equivalence class of (a, s) is denoted with a/s.

The set $C^{-1}R$ has a structure of ring given by the well defined operations

$$\frac{a}{s} + \frac{b}{t} := \frac{at + bs}{st}, \qquad \frac{a}{s} \cdot \frac{b}{t} := \frac{ab}{st}, \qquad \forall a, b \in R, \quad \forall s, t \in C.$$

If M is a R-module, we can define in the same way the R-module $C^{-1}M$.

Definition 2.27. Let R be a ring, $\mathfrak{p} \subset R$ a prime ideal and $f \in R$. Then $C_1 = R \setminus \mathfrak{p}$ and $C_2 = \{f^n, n \in \mathbb{N}\}$ are closed multiplicatively systems. We denote with $R_{\mathfrak{p}} := C_1^{-1}R$ and with $R_f := C_2^{-1}R$. In particular, $R_{\mathfrak{p}}$ is called *localization of* R at $\mathfrak{p}$.

Let M be a R-module. We denote with $M_{\mathfrak{p}} := C_1^{-1}M$ and with $M_f := C_2^{-1}M$.

Proposition 2.28. *Let R be a ring and C a closed multiplicatively system in R. Then, the prime ideals in $C^{-1}R$ are in 1-1 corrispondence with the prime ideals of R which don't meet C. This map is $\mathfrak{p} \mapsto C^{-1}\mathfrak{p}$, where $C^{-1}\mathfrak{p}$ is the ring of fractions of the R-module $\mathfrak{p}$.*

Proof. See [1, Chapter 3]. □

Remark 2.29. The set $R_{\mathfrak{p}}$ consists of all the fractions a/f with $f \notin \mathfrak{p}$. Every prime ideal $C^{-1}\mathfrak{q}$ in $R_{\mathfrak{p}}$ is in corrisponcence with a prime ideal $\mathfrak{q}$ of R such that $\mathfrak{q} \subset \mathfrak{p}$. Therefore, $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}} = C^{-1}\mathfrak{p}$.

Definition 2.30. Let R be a ring and $\operatorname{Spec} R$ its spectrum. For every open subset $U \subset \operatorname{Spec} R$ we define $\mathcal{O}(U)$ to be the set of functions $s\colon U \to \bigsqcup_{\mathfrak{p}\in U} R_{\mathfrak{p}}$ such that

1. for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in R_{\mathfrak{p}}$;
2. for each $\mathfrak{p} \in U$ there exists $V \subset U$ neighborhood of $\mathfrak{p}$ and $a, f \in R$ such that $\forall \mathfrak{q} \in V$ we have $s(\mathfrak{q}) = a/f$ and $f \notin \mathfrak{q}$.

With the natural sum and product of functions, $\mathcal{O}(U)$ is a ring $\forall U \subset X$. This presheaf (with the natural restriction maps) is a sheaf called *structure sheaf* of $\operatorname{Spec} R$.

Proposition 2.31. *Let R be a ring. Then, for every $\mathfrak{p} \in \operatorname{Spec} R$ we have $\mathcal{O}_{\mathfrak{p}} \cong R_{\mathfrak{p}}$. Therefore $\mathcal{O}_{\mathfrak{p}}$ is a local ring.*

Proof. Let $\varphi\colon \mathcal{O}_{\mathfrak{p}} \to R_{\mathfrak{p}}$ defined by $\varphi(s_{\mathfrak{p}}) = s(\mathfrak{p}) \in R_{\mathfrak{p}}$, where $s \in \mathcal{O}(U)$ and (U, s) is a representative of $s_{\mathfrak{p}}$. It is clearly well defined and it is a morphism of rings.

Let $a/f \in R_{\mathfrak{p}}$, with $a, f \in R$ and $f \notin \mathfrak{p}$. On $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R\colon f \notin \mathfrak{p}\}$, $s = a/f$ is a well defined section of $\mathcal{O}(D(f))$ and $\varphi(s_{\mathfrak{p}}) = s(\mathfrak{p}) = a/f$. Hence φ is surjective.

Let $s_{\mathfrak{p}}, t_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ such that $\varphi(s_{\mathfrak{p}}) = \varphi(t_{\mathfrak{p}})$. We may assume there exists an open subset U such that $s = a/f$ and $t = b/g$ are representatives of $s_{\mathfrak{p}}$ and $t_{\mathfrak{p}}$ on U. Thus $a/f = b/g$ in $R_{\mathfrak{p}}$, so there is $h \notin \mathfrak{p}$ such that $h(ag - bf) = 0$. For any $\mathfrak{q} \in V = D(f) \cap D(g) \cap D(h)$ we have $f, g, h \notin \mathfrak{q}$, so $s(\mathfrak{q}) = t(\mathfrak{q})$. Since V is an open neighborhood of $\mathfrak{p}$ we obtain $s_{\mathfrak{p}} = t_{\mathfrak{p}}$, so φ is an isomorphism. □

Proposition 2.32. *Let R be a ring. Then, for every $f \in \operatorname{Spec} R$ we have $\mathcal{O}(D(f)) \cong R_f$. In particular $\Gamma(\operatorname{Spec} R, \mathcal{O}) \cong R$.*

Proof. See [6, Chapter II.2]. □

Definition 2.33. A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that for each $P \in X$ the stalk $\mathcal{O}_{X,P}$ is a local ring.

Example 2.34. Let R be a ring. Then $(\operatorname{Spec} R, \mathcal{O})$ is a locally ringed space. It follows from 2.31.

Definition 2.35. A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$, where $f\colon X \to Y$ is a continuous map and $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves on Y, with $f^{\#}(V)\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ for any open $V \subset Y$ (Definition 1.48).

Remark 2.36. Let $(f, f^{\#})\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and let $P \in X$. We have that $f^{\#}$ induces a ring homomorphism $f^{\#}(V)\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ for each $V \subset Y$ such that $f(P) \in V$. Then we have a map

$$\mathcal{O}_{Y,f(P)} = \varinjlim \mathcal{O}_Y(V) \longrightarrow \varinjlim \mathcal{O}_X(f^{-1}(V)).$$

The direct system in the rightside direct limit is "smaller" then the direct system of the neighborhood of P, but there is a natural projection $\varinjlim \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_{X,P}$.

This shows that $f^\#$ induces a ring homomorphism on the stalks, denoted with $f^\#_P\colon \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ for any $P \in X$.

Definition 2.37. A *morphism of locally ringed spaces* is a morphism of ringed spaces $(f, f^\#)$ such that the induced map $f^\#_P\colon \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism of local rings for any $P \in X$.

A *isomorphism of locally ringed spaces* is a morphism with two-sided inverse, that is a morphism $(f, f^\#)$ with f a homeomorphism of topological spaces and $f^\#$ an isomorphism of sheaves.

Proposition 2.38. *Let R, S be two rings.*

i) Any homomorfism of rings $\varphi\colon R \to S$ induces a natural morphism of locally ringed space $(f, f^\#)\colon (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \to (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

ii) Any morhism of locally ringed spaces from $\operatorname{Spec} S$ *to* $\operatorname{Spec} R$ *is induced by a homomorphism of rings $\varphi\colon R \to S$.*

Proof. See [6, Chapter II.2]. □

2.3 Schemes

Finally, we can define the category of schemes.

Definition 2.39. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R.

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is locally an affine scheme, that is for each $P \in X$ there exists an open neighborhood U of P such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. Each one of these open sets are called *affine open* subset of X.

A *morphism* of schemes is a morphism of locally ringed spaces and an *isomorphism* of schemes is a morphism with a two-sided inverse.

Example 2.40. For any ring R, $(\operatorname{Spec} R, \mathcal{O})$ is a scheme, since it is a locally ringed space.

Let U be an open subset of a scheme X. Then $(U, \mathcal{O}_X|_U)$ is a scheme (see [5, Chapter 3.2]).

We outline the glueing of schemes, which allows us to build non-affine schemes.

Definition 2.41. Let X_1, X_2 be schemes and let $U_1 \subset X_1$ and $U_2 \subset X_2$ be affine open sets with an isomorphism $\varphi\colon (U_1, \mathcal{O}_{X_1}|_{U_1}) \to (U_2, \mathcal{O}_{X_2}|_{U_2})$ of locally ringed spaces. We define the *glueing* of X_1 and X_2 along U_1 and U_2 via φ to be a ringed space in the following way.

The topological space is $X = (X_1 \sqcup X_2)/_\sim$ where $x_1 \sim \varphi(x_1)$ for each $x_1 \in U_1$, with the quotient topology.

Let $\psi_1\colon X_1 \to X$ and $\psi_2\colon X_2 \to X$ be the natural immersions. The structure sheaf $\mathcal{O}_X$ is $V \mapsto \mathcal{O}_X(V)$, where $\mathcal{O}_X(V)$ is the set of pairs (s_1, s_2), where $s_1 \in \mathcal{O}_{X_1}(i_1^{-1}(V))$ and $s_2 \in \mathcal{O}_{X_2}(i_2^{-1}(V))$, such that $\varphi(s_1|_{U_1 \cap i_1^{-1}(V)}) = s_2|_{U_2 \cap i_2^{-1}(V)}$.

We can generalize this construction glueing n schemes $X_1, ..., X_n$ along open subsets $U_1, ..., U_n$ if, for any i, j, we have open subsets $U_{ij} \subset U_i$ and isomorphisms of locally ringed spaces $\varphi_{ji}\colon (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$, such that

$$\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki} \text{ on } U_{ij} \cap U_{ik}. \tag{2.1}$$

For more details see [12, Chapter 3.2] or [5, Chapter 3.5].

Example 2.42. Let k be a field, $X_1 = \operatorname{Spec} k[s]$, $X_2 = \operatorname{Spec} k[t]$. Let us consider the open subsets $U_1 = X_1 \setminus (s)$ and $U_2 = X_2 \setminus (t)$. We note that $U_1 = D(s) = \operatorname{Spec} k[s, 1/s]$, indeed, from 2.28, there is a homeomorphism $D(s) \cong \operatorname{Spec} k[s]_s = \operatorname{Spec} k[s, 1/s]$. In the same way $U_2 \cong \operatorname{Spec} k[t, 1/t]$.

Let $\varphi_1, \varphi_2\colon U_1 \to U_2$ be isomorphisms of locally ringed spaces defined by $\varphi_1(f(s)) = (f(t))$ and $\varphi_2(f(s)) = (f(1/t))$.

We define:

1. the *line with two origins* to be the glueing of X_1 and X_2 along U_1 and U_2 via φ_1;

2. the *projective line* $\mathbb{P}^1_k$ over k to be the glueing of X_1 and X_2 along U_1 and U_2 via φ_2.

   We note that both of these schemes have $\mathbb{A}^1_k \cup \{P\}$ as set, but later we will see they have very different properties.

3. the *projective n-space* $\mathbb{P}^n_k$ over k is obtained by glueing $n+1$ copies of $\operatorname{Spec} k[x_1, ..., x_n]$ in the following way. We take

$$U_i = \operatorname{Spec} k \left[\frac{s_0}{s_i}, ..., \frac{\widehat{s_i}}{s_i}, ..., \frac{s_n}{s_i} \right],$$

   to distinguish the spaces,

$$U_{ij} = \operatorname{Spec} k \left[\frac{s_0}{s_i}, ..., \frac{\widehat{s_i}}{s_i}, ..., \frac{s_n}{s_i}, \frac{s_i}{s_j} \right]$$

   and $\varphi_{ji} \colon U_{ij} \to U_{ji}$ is defined by $s_k/s_i \mapsto (s_i/s_j)(s_k/s_i)$, for any k and $s_i/s_j \mapsto s_j/s_i$.

4. the *projective n-space* $\mathbb{P}^n_R$ over R, where R is a ring, to be the scheme obtained by the construction of $\mathbb{P}^n_k$, with an arbitrary ring R instead of k.

Proposition 2.43. *Let R be a ring. Then, the global sections of $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n_R}$ are $\Gamma(X, \mathcal{O}_X) = R$.*

Proof. Let $t \in \Gamma(X, \mathcal{O}_X)$ be a global section of X. We set $y_1 = s_0/s_i, ..., y_n = s_n/s_i$ and $z_1 = s_0/s_j, ..., z_n = s_n/s_j$, where i and j are fixed. We have $t = (t_0, ..., t_n)$ with $t_i \in \mathcal{O}_{U_i}(U_i) = R[s_0/s_i, ..., \widehat{s_i/s_i}, ..., s_n/s_i] = R[y_1, ..., y_n]$ and $\varphi_{ji}(t_i|_{U_{ij}}) = t_j|_{U_{ji}}$, for any i, j. Therefore

1. $t_i|_{U_{ij}} \in R[y_1, ..., y_n, 1/y_j]$,

2. $t_j|_{U_{ji}} \in R[z_1, ..., z_n, 1/z_j]$,

3. $t_i|_{U_{ij}}(y_1, ..., y_n, 1/y_j) = t_j|_{U_{ji}}(z_1, ..., z_n, 1/z_i)$.

If we rename the variables z_k, we have

$$t_i(y_1, ..., y_j, ..., y_n) = t_j(y_1, ..., 1/y_j, ..., y_n),$$

so the variable y_j doesn't occur in t_i and t_j. Repeating for any i, j and applying the condition (2.1), we obtain $t_0 = ... = t_n = c \in R$. □

Corollary 2.44. *The projective space $\mathbb{P}^n_R$ is not an affine scheme.*

Proof. If we suppose $\mathbb{P}^n_R = \operatorname{Spec} S$, for some ring S, then $\Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}) = S$, so $\mathbb{P}^n_R = \operatorname{Spec} R$ by the Proposition above. By construction of $\mathbb{P}^n_R$, this is absurd. □

Definition 2.45. Let X be a scheme. It's *irreducible* if it's irreducible as a topological space (Definition 2.17). It is *reduced* if for every open set U, the ring $\mathcal{O}_X(U)$ is reduced, that is it has no nilpotent elements. It's *integral* if for every open set U, the ring $\mathcal{O}_X(U)$ is an integral domain.

Remark 2.46. Let $U = \operatorname{Spec} R$ be an affine open subset of an integral scheme X. The ring $\mathcal{O}_X(U)$ is an integral domain, but $\mathcal{O}_X(U) \cong R$ by 2.32, so R is an integral domain.

Proposition 2.47. *A scheme X is integral if and only if it is reduced and irreducible.*

Proof. See [6, Chapter II.3]. □

Corollary 2.48. *Let R be an integral domain. Then* $\operatorname{Spec} R$ *is integral.*

Proof. From 2.18 $\operatorname{Spec} R$ is irreducible. For any closed multiplicatively system C in R, $C^{-1}R$ has no nilpotent element since R does (See [1, Chapter 3]). Hence $\mathcal{O}_{\operatorname{Spec} R}(D(f))$ has no nilpotent element for any $f \in R$, and the open sets $D(f)$ cover $\operatorname{Spec} R$. □

Now we want to explain in a clearer way morphisms of schemes. First, we need the following definition.

Definition 2.49. Let S be a scheme. A *scheme over* S is a scheme X together with a morphism $f\colon X \to S$. A *scheme over a field* k is a scheme over $\operatorname{Spec} k$.

Proposition 2.50. *Let X, Y be schemes. Let $\bigcup_i U_i = X$ be an open covering of X and let $f_i\colon U_i \to Y$ be morphisms of schemes for any i. Then, there is a unique morphism of schemes $f\colon X \to Y$ such that $f|_{U_i} = f_i$.*

Proof. See [5, Chapter 3.3]. □

Remark 2.51. Let R be a ring and let $f\colon X \to \operatorname{Spec} R$ a morphism of schemes. Thus, we have a morphism of sheaves $f^\#\colon \mathcal{O}_{\operatorname{Spec} R} \to f_*\mathcal{O}_X$. In particular we have a morphism of rings $R \to \Gamma(X, \mathcal{O}_X)$. Hence, the global section of a scheme over a field k form a k-algebra.

Conversely, let $R \to \Gamma(X, \mathcal{O}_X)$ be a morphism of rings. If $\bigcup_i \operatorname{Spec} R_i = X$ is an affine open covering of X, then we have morphisms $R \to \mathcal{O}_X(\operatorname{Spec} R_i) = R_i$ for any i, by using restriction morphisms of $\mathcal{O}_X$. By Propostion 2.38, we have morphisms of schemes $\operatorname{Spec} R_i \to \operatorname{Spec} R$ for any i. Such morphisms match on the intersections, hence it is induced a unique morphism $X \to \operatorname{Spec} R$ by 2.50.

Example 2.52. The projective space $\mathbb{P}^n_k$ is a scheme over k, since $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = k$. The morphism $\mathbb{P}^n_k \to \operatorname{Spec} k$ is induced by id_k, accordingly with the remark above.

Corollary 2.53. *The affine scheme* $\operatorname{Spec} \mathbb{Z}$ *is the final object in the category of the schemes.*

Proof. It follows from Remark 2.51, since $\mathbb{Z}$ is the initial object in the category of commutative rings. □

We conclude this section with the definition of closed immersion and closed subscheme of a scheme.

Definition 2.54. Let $(f, f^\#)\colon X \to Y$ be a morphism of schemes. We call $(f, f^\#)$ a *closed immersion* if $f(X)$ is closed in Y, $f\colon X \to f(X)$ is a homeomorphism and $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

A *closed subscheme* of X is a closed subset $Z \subset X$ together with a structure of scheme $(Z, \mathcal{O}_Z)$ and a closed immersion $Z \to X$.

Proposition 2.55. *Let $\varphi\colon R \to S$ be a surjective morphism of rings. Then, the induced morphism $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is a closed immersion.*

Proof. We know f is a homeomorphism on its image from 2.15. We have to show that $f^{\#}\colon \mathcal{O}_{\operatorname{Spec} R} \to f_*\mathcal{O}_{\operatorname{Spec} S}$ is surjective. The induced map on the stalks gives rise to a map $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Spec} R$, which is surjective since surjectivity is a local property of rings (see [1, Chapter 3]). Thus, $f^{\#}$ is surjective for the properties of morphism of sheaves (Lemma 1.37). □

Remark 2.56. Let R be a ring and $\mathfrak{a} \subset R$ an ideal. The projection $R \to R/\mathfrak{a}$ is surjective, so $\operatorname{Spec} R/\mathfrak{a} \to \operatorname{Spec} R$ is a closed immersion. We note that there are many scheme structures on $\operatorname{Spec} R/\mathfrak{a}$. Indeed we know that $\operatorname{Spec} R/\mathfrak{a} \cong V(\mathfrak{a})$ as topological space (Corollary 2.16), so we can take $\mathfrak{b}$ such that $V(\mathfrak{a}) = V(\mathfrak{b})$. As example, in $k[x]$ we have $V(x) = V(x^2)$, but in the second case the scheme associated is not reduced ($x \in \mathcal{O}_{\operatorname{Spec} R/(x^2)}$ is nilpotent). We will give more details in Chapter 3 (Remark 3.11 and Proposition 3.12).

Definition 2.57. A morphism of schemes $f\colon X \to Y$ is *locally of finite type* if there exists a covering of Y given by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, the set $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, with A_{ij} a finitely generated B_i-algebra.

We say f is *of finite type* if it's locally of finite type with finite subsets U_{ij} which cover $f^{-1}(V_i)$ for each i.

Remark 2.58. Let R be a ring and let $X = \operatorname{Spec} R$. We consider $f\colon X \to \operatorname{Spec} k$ a morphism of affine schemes of finite type. By definition, R is a finitely generated k-algebra, that is $R \cong k[x_1, ..., x_n]/\mathfrak{a}$, with $\mathfrak{a}$ an ideal of $k[x_1, ..., x_n]$. Hence, X is a closed subscheme of $\mathbb{A}^n_k$, for some n.

Proposition 2.59. *Let X be a scheme of finite type over a field k and let Y be a closed subscheme of X. Then Y is of finite type over k.*

Proof. See [5, Chapter 3.16]. □

2.4 Projective Spaces

In the following, we will focus on the projective space $\mathbb{P}^n$, describing it without glueing. We need to some notions of Commutative Algebra on graded rings.

Definition 2.60. A *graded ring* is a ring S together with a family of subgroups $\{S_d\}_{d\in\mathbb{N}}$ of S such that

$$S = \bigoplus_{d\geq 0} S_d,$$

and $S_d \cdot S_{d'} \subset S_{dd'}$, $\forall d, d' \geq 0$. Thus, S_0 is a subring of S and each S_d is a S_0-submodule of S.

The elements of S_d are called *homogeneous of degree* d and we write $\deg a = d$ for each $a \in S_d$.

An ideal $\mathfrak{a} \subset S$ is an *homogeneous ideal* of S if $\mathfrak{a} = \bigoplus_{d\geq 0}(S_d \cap \mathfrak{a})$.

We denote with $S_+ \coloneqq \bigoplus_{d>0} S_d$ the homogeneous ideal called *irrilevant ideal* of S.

We have the following properties on homogeneous ideals. See [4, Chapter 8.3] or [5, Chapter 13.1] for the proof.

Proposition 2.61. *Let S be a graded ring.*

i) Let $\mathfrak{a}$ be an ideal of S. Then, $\mathfrak{a}$ is homogeneous if and only if it can be generated by homogeneous elements of S.

ii) Sum, product, intersection and radical of homogeneous ideals are homogeneous ideals.

iii) Let $\mathfrak{a}$ be a homogeneous ideal of S. Then, $\mathfrak{a}$ is prime if and only if for every $f, g \in S$ homogeneous, $fg \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.

Definition 2.62. Let S be a graded ring. We define $\operatorname{Proj} S \coloneqq \{\mathfrak{p} \subset S \text{ prime ideal} \colon \mathfrak{p} \not\supseteq S_+\}$.

We define:

1. $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj} S \colon \mathfrak{p} \supset \mathfrak{a}\}$ for every $\mathfrak{a} \subset S$ homogeneous ideal;

2. $D(f) := V(f)^C$ in $\operatorname{Proj} S$ for every $f \in S_+$ homogeneous.

Proposition 2.63. *We have that:*

i) $V(0) = \operatorname{Proj} S$, $V(\operatorname{Proj} S) = \emptyset$;

ii) if $(\mathfrak{a}_i)_{i\in I}$ *is a family of homogeneous ideals of* S*, then*

$$V\left(\bigcup_{i\in I}\mathfrak{a}_i\right) = \bigcap_{i\in I} V(\mathfrak{a}_i);$$

iii) $V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a})\cup V(\mathfrak{b})$ *for every homogeneous ideals* $\mathfrak{a}, \mathfrak{b} \subset S$.

These results show that the sets $V(\mathfrak{a})$ *satisfy the closed axioms in topological spaces. The relative topology on* Proj S *is called Zariski topology. Moreover, a base for this topology is given by the sets* $D(f)$*, where* $f \in S_+$ *is homogeneous.*

Proof. The proof is the same as in 2.2 and 2.3. □

Definition 2.64. Let R be a ring and let $S = R[x_0, ..., x_n]$ be the graded ring with $S_d = \{f \in S \colon \deg(f) = d\}$ for each $d \geq 0$. We call $\mathbb{P}^n_R := \operatorname{Proj} S$ the n-dimensional *projective space* on R.

Remark 2.65. Let k be a closed algebraically field.

1. $\mathbb{P}^0_k = \{(0)\}$.

2. $\mathbb{P}^1_k = \{(ax_0 + bx_1) \colon a, b \in k\} \cup \{(0)\}$, that is the closed points of $\mathbb{P}^1_k$ are in 1-1 corrispondence with the set of lines in k^2, the standard 1-dimensional projective space.

Definition 2.66. Let S be a graded ring, C a closed multiplicatively system in S. We call the *degree* of $a/f \in C^{-1}S$ the integer $\deg(a/f) := \deg a - \deg f$.

Let $\mathfrak{p} \in \operatorname{Proj} S$ and $f \in S_d$ homogeneous of degree d.

1. If $C = \{g \in S \colon g \text{ homogeneous}, f \notin \mathfrak{p}\}$ in S, we denote with $S_{(\mathfrak{p})}$ the set of fractions in $C^{-1}S$ of degree 0.

2. If $C = \{g^n : n \in \mathbb{N}\}$ in S, then we denote with $S_{(f)}$ the set of fractions in $T^{-1}S$ of degree 0.

Example 2.67. If R is a ring, $S = R[x_0, ..., x_n]$ and $\mathfrak{p}$ is a homogeneous ideal in S, then

$$S_{(\mathfrak{p})} = \left\{ \frac{f}{g} : f, g \in R[x_0, ..., x_n] \text{ homogeneous, } g \notin \mathfrak{p} \text{ and } \deg f = \deg g \right\}.$$

If $g \in S_d$, then

$$S_{(g)} = \left\{ \frac{f}{g^n} : f \in R[x_0, ..., x_n] \text{ homogeneous, } \deg f = nd, n \in \mathbb{N} \right\}.$$

Let us define a structure sheaf on $\operatorname{Proj} S$.

Definition 2.68. Let S be a graded ring. For any open subset $U \subset \operatorname{Proj} S$ we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \bigsqcup_{\mathfrak{p} \in U} S_{(\mathfrak{p})}$ such that

1. for any $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$;
2. for any $\mathfrak{p} \in U$ there exists an open neighborhood $V \subset U$ of $\mathfrak{p}$ and homogeneous elements $a, f \in S$, such that $\deg a = \deg f$ and $\forall \mathfrak{q} \in V$ $s(\mathfrak{q}) = a/f$, with $f \notin \mathfrak{q}$.

With the natural sum and product of functions, and the natural restrictions, $\mathcal{O}$ is a presheaf of rings. It is a sheaf called *structure sheaf* of $\operatorname{Proj} S$.

Proposition 2.69. *Let S be a graded ring. Then, for every $\mathfrak{p} \in \operatorname{Proj} S$ we have $\mathcal{O}_{\mathfrak{p}} \cong S_{(\mathfrak{p})}$.*

Proof. The proof is the same as in 2.31. □

Proposition 2.70. *Let S be a graded ring and let $f \in S_+$ be a homogeneous element. Then $(D(f), \mathcal{O}|_{D(f)})$ is a ringed space for the Proposition above. We have $(D(f), \mathcal{O}|_{D(f)}) \cong \operatorname{Spec} S_{(f)}$.*

Proof. See [6, Chapter II.2]. □

Corollary 2.71. *The ringed space* $(\operatorname{Proj} S, \mathcal{O})$ *is a scheme, where* S *is a graded ring.*

Proof. It follows from 2.70, since the open subsets $D(f)$ cover $\operatorname{Proj} S$ from 2.63. □

We have the following properties.

Proposition 2.72. *Let* $\varphi\colon S \to T$ *a morphism of graded rings (a preserving degree homomorphism of rings). Let* $f\colon \operatorname{Proj} T \to \operatorname{Proj} S$ *be a function defined by* $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$*, for any* $\mathfrak{p} \in \operatorname{Proj} T$*. Then,* f *is a well defined morphism of schemes and it is a closed immersion which induces an isomorphism* $\operatorname{Proj} T \cong \operatorname{Proj}(S/\ker\varphi)$*.*

Proof. See [5, Chapter 13.2]. □

Corollary 2.73. *Let* S *be a graded ring and let* $\mathfrak{a}$ *be an homogeneous ideal of* S*. Then,* $\operatorname{Proj} S/\mathfrak{a}$ *is a closed subscheme of* $\operatorname{Proj} S$*.*

Proposition 2.74. *Let* R *be a ring and let us consider the projective space* $X = \mathbb{P}^n_R$ *over* R*. Let* Y *be a closed subscheme of* X*. Then, there exists a homogeneous ideal* $\mathfrak{a} \subset S = R[x_0, ..., x_n]$*, such that* $Y = \operatorname{Proj} S/\mathfrak{a}$*.*

Proof. See [6, Chapter II.5]. □

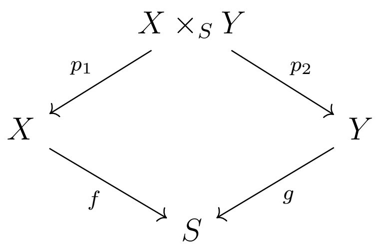
Corollary 2.75. *For every closed subscheme* Y *of* $\mathbb{P}^n_R$*, there exists a unique homogeneous ideal* $\mathfrak{a}$ *of* $R[x_0, ..., x_n]$ *such that* $Y \cong V(\mathfrak{a})$*.*

2.5 Algebraic Varieties

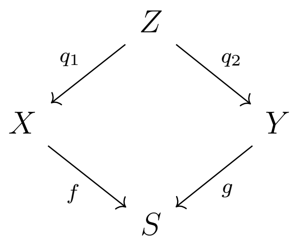
In this section we can define (abstract) algebraic varieties. To do this, we will explain the concept of separatedness and properness.

Finally, we give a short introduction of the dimensional theory of rings and topological spaces.

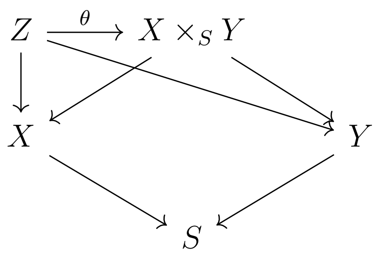
Definition 2.76. Let S be a scheme and let $f\colon X \to S$, $g\colon Y \to S$ be schemes over S. The *fibred product* of X and Y over S is a scheme, denoted with $X \times_S Y$, together with morphisms $p_1\colon X \times_S Y \to X$ and $p_2\colon X \times_S Y \to Y$ such that the diagram



is commutative and such that for each scheme Z and morphisms $q_1\colon Z \to X$ and $q_2\colon Z \to Y$ such that



is commutative, there exists a unique morphism $\theta\colon Z \to X \times_S Y$ such that



is commutative.

If $S = \operatorname{Spec} \mathbb{Z}$ we write $X \times Y$.

Proposition 2.77. *Let X, Y be schemes over a scheme S. There exists the fibred product $X \times_S Y$ and it is unique up to unique isomorphism.*

Proof. Let $X = \operatorname{Spec} U$, $Y = \operatorname{Spec} V$, $S = \operatorname{Spec} W$. Then $X \times_S Y = \operatorname{Spec}(U \otimes_W V)$. See [6, Chapter II.3] for the complete proof. □

Example 2.78. Let $X \to \operatorname{Spec} k$ be a scheme over a field k and let $k \hookrightarrow k'$ be a field extension. We have a morphism of affine schemes $\operatorname{Spec} k' \to \operatorname{Spec} k$, so we can consider the fibred product $X \times_k k'$.

$$\begin{array}{ccc} X \times_k k' & \longrightarrow & \operatorname{Spec} k' \\ \downarrow & & \downarrow \\ X & \longrightarrow & \operatorname{Spec} k \end{array}$$

In particular, there is a morphism $X \times_k k' \to \operatorname{Spec} k'$, that is $X \times_k k'$ is a scheme over k'. This construction is called *extension of scalars*.

Now we define the fiber of a morphism of scheme to be a particular case of fibred product.

Definition 2.79. Let Y be a scheme and let $Q \in Y$. We know $\mathcal{O}_{Q,Y}$ is a local ring with maximal ideal $\mathfrak{m}_Q$. We set $k(y) := \mathcal{O}_{Q,Y}/\mathfrak{m}_Q$. This field is called *residue field* of Q.

Let $p\colon X \to Y$ be a morphism of schemes. We define the *fiber* of p over Q to be the fibred product $X_Q := X \times_Y \operatorname{Spec} k(Q)$.

Proposition 2.80. *In the notations above, we have that X_Q is homeomorphic to $p^{-1}(Q)$ as topological spaces.*

Proof. See [5, Chapter 4.5]. □

Example 2.81. Let $X = \operatorname{Spec}(k[x,y]/(xy-1)) =: \operatorname{Spec} R$ and $Y = \mathbb{A}^1_k = \operatorname{Spec} k[x]$, with k field. We note that X corresponds to the hyperbole $\{xy = 1\}$ in the affine plane. Let $k[x] \hookrightarrow k[x,y] \to R$, where the latter morphism is the natural projection to the quotient, and we consider the associated morphism of schemes $f\colon X \to Y$. For any $\mathfrak{p} = (x-a) \in Y$, we have $\mathcal{O}_{\mathfrak{p},Y} \cong k[x]_{\mathfrak{p}}$, so the residue field of $\mathfrak{p}$ is $k[x]_{\mathfrak{p}}/\mathfrak{p}k[x]_{\mathfrak{p}} \cong (k[x]/\mathfrak{p})_{\mathfrak{p}}$ (see [1, Chapter 3]).

1. If $a \neq 0$, then

$$\begin{aligned} X_{\mathfrak{p}} &= \operatorname{Spec} R \times_{\operatorname{Spec} k[x]} \operatorname{Spec} k \\ &= \operatorname{Spec}(k[x,y]/(xy-1) \otimes_{k[x]} k[x]/(x-a)) \\ &= \operatorname{Spec}(k[x,y]/(x-a, xy-1)) \cong \operatorname{Spec} k, \end{aligned}$$

which is one point. For the details of the computation, see [3, Chapter 4].

2. If $a = 0$, in a similar way we have $X_{\mathfrak{p}} \cong \operatorname{Spec} k[x,y]/(1) = \emptyset$.

We observe that the projetion of the hyperbole on the x-axis has empty fiber if and only if the point is the origin.

We come to the definition of separated scheme.

Definition 2.82. Let $f\colon X \to Y$ be a morphism of schemes. The *diagonal morphism* $\Delta\colon X \to X \times_Y X$ is the unique morphism such that $p_1 \circ \Delta = p_2 \circ \Delta = id_X$. We say that f is *separated* if Δ is a closed immersion. In this case we say that X is *separated over* Y. A scheme X is *separated* if it's separated over $\operatorname{Spec} \mathbb{Z}$.

Proposition 2.83. *Let X be a scheme over a field k. X is separated over k if and only if for any affine open subsets U, V of X, $U \cap V$ is affine and the canonical homomorphism $\mathcal{O}_X(U) \otimes_k \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.*

Proof. See [11, Chapter 3.3]. □

Proposition 2.84. *We have the following:*

i) Each affine scheme $\operatorname{Spec} R$ is separated for any ring R.

ii) The line with two origins is not separated.

iii) The projective space $\mathbb{P}^n_R$ over any ring R is separated.

Proof. *i*) The diagonal morphism $\Delta\colon \operatorname{Spec} R \to \operatorname{Spec}(R \otimes R)$ is induced by $R \otimes R \to R$ such that $a \otimes b \mapsto ab$, which is surjective. Hence Δ is a closed immersion by 2.55.

ii) Let X be the line with two origins, obtained by glueing two copies of $Y = \mathbb{A}^1_k$ along $U = Y \setminus (0)$. The condition of the Proposition above fails, because if $U = \operatorname{Spec} k[x]$, $V = \operatorname{Spec} k[y]$ then $\mathcal{O}_X(U) \otimes_k \mathcal{O}_X(V) = k[x,y]$, but $k[x,y] \to \mathcal{O}_X(U \cap V) = k[x]_{(0)} = k[x, 1/x]$ is not surjective.

iii) See [5, Chapter 13.2]. □

Corollary 2.85. *The line with two origins is not an affine scheme.*

Proposition 2.86. *Let $f\colon X \to Y$ be a closed immersion. Then f is a separated morphism.*

Proof. See [6, Chapter II.4]. □

Definition 2.87. A *variety* is a reduced, separated scheme of finite type over an algebraically closed field k.

Corollary 2.88. *We have the following:*

i) Let A be a k-algebra, where k is an algebraically closed field. Then, $\operatorname{Spec} A$ is a variety.

ii) The projective space $\mathbb{P}^n_k$ over any algebraically closed field k is a variety.

Proof. It is a consequence of 2.48, 2.51 and 2.84. □

We come to the notion of complete variety.

Definition 2.89. A morphism of schemes is *closed* if the image of any closed set of X is closed.

Let $f\colon X \to Y$ be a morphism of schemes. It is *universally closed* if for morphism of schemes $Z \to Y$, the projection $X \times_Y Z \to Z$ is a closed map.

A morphism of schemes $f\colon X \to Y$ is *proper* if it is separated, of finite type and universally closed.

Definition 2.90. A variety X over a field k is *complete* if $X \to \operatorname{Spec} k$ is a proper morphism.

Proposition 2.91. *We have the following:*

i) The affine space $\mathbb{A}^n$ is not a complete variety.

ii) The projective space $\mathbb{P}^n_R$ over any ring R is is a complete variety.

Proof. *i*) Let us prove the case $n = 1$. The general case is analogous.

Let $X = \mathbb{A}^1_k$ and let uss consider $X \to \operatorname{Spec} k$. The projection $X \times_k X \to X$ is not a closed map. Indeed $X \times_k X = \mathbb{A}^2_k$, and if we take the closed subset $V(xy - 1) \in \mathbb{A}^2_k$, the image is $X \setminus (0)$, which is not closed in X.

ii) See [6, Chapter II.4]. □

Proposition 2.92. *A closed immersion of schemes $X \to Y$ is proper.*

Proof. See [6, Chapter II.4]. □

Corollary 2.93. *Let k be an algebraically closed field and let X be a reduced closed subscheme of $\mathbb{P}^n_k$. Then X is a complete variety over k.*

Proof. Let $f\colon X \to \mathbb{P}^n_k$ be a closed immersion. By 2.86, 2.92 and since composition of separated (proper) morphisms is a separated (proper) morphism, then X is a proper reduced scheme over k. Finally, X is of finite type over k by 2.59.

□

Proposition 2.94. *Let k be an algebraically closed field and let X be an irreducible complete variety over k. Then $\Gamma(X, \mathcal{O}_X) = k$.*

Proof. See [6, Chapter II.4]. □

We conclude this chapter with a short overview of dimensional theory.

Definition 2.95. Let R be a ring and let $\mathfrak{p} \subset R$ be a prime ideal. The *height* of $\mathfrak{p}$ is the supremum on all the integer n such that there exists a chain of distinct prime ideals $\mathfrak{p}_\mathfrak{o} \subset \ldots \subset \mathfrak{p}_\mathfrak{n} = \mathfrak{p}$.

The *dimension* of the ring R is the supremum of the heights of all its prime ideals. We denote it with $\dim R$.

Remark 2.96. If R is a local ring, then $\dim R$ is the height of its maximal ideal.

Definition 2.97. Let X be a topological space. The *dimension* of X, $\dim X$, is defined to be the supremum on all the integer n such that there exist an ascending chain $Z_0 \subset ... \subset Z_n$ of distinct irreducible closed subsets of X.

The *dimension* of a scheme X is its dimension as a topological space.

The *codimension* of a closed subset $Y \subset X$ of a scheme X is

$$\operatorname{codim}(Y, X) = \inf_{P \in Y} \dim \mathcal{O}_{P,X}$$

(Definition by [5, Chapter 5.8]).

Definition 2.98. Let X be a variety over k. If $\dim X = n$ we say that X is a n-*dimensional* variety.

Proposition 2.99. *Let k be a field. Then,* $\dim \mathbb{A}^n_k = \dim \mathbb{P}^n_k = n$.

Proof. See [6, Chapters I.1, I.2]. □

Chapter 3

Sheaves of Modules

In the first two chapters we have given the basic definitions of sheaves and schemes and their first properties. Now we complete the view on scheaves with sheaves of $\mathcal{O}_X$-modules on a scheme X. We will be able to explain two important invariants of schemes: the Picard Group and the cohomology of sheaves.

3.1 Quasi-Coherent Sheaves

In this section we define sheaves of modules on a scheme and we give many examples of them. In particular, quasi-coherent sheaves are a class of sheaves of modules which have many properties, as we will see in section 3.

Definition 3.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf of abelian groups $\mathcal{F}$ on X together with a morphism of sheaves $\mathcal{O}_X \to \mathcal{F}$. This means that for any open subset $U \subset X$, the group $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module and for any $V \subset U$ the diagram

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

is commutative.

A *morphism* of sheaves of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that for any $U \subset X$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules.

Example 3.2. Each structure sheaf $\mathcal{O}_X$ on a scheme X is trivially a sheaf of modules.

Let $U \subset X$ be an open subset of a scheme X. Then $\mathcal{O}_X|_U$ is a sheaf of modules on X, since for any open subset $V \subset X$ we have $\mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) = \mathcal{O}_X|_U(V)$.

Kernel, cokernel and image of a morphism of $\mathcal{O}_X$-modules are again $\mathcal{O}_X$-modules.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules and $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, then $\mathcal{F}/\mathcal{F}'$ is again a sheaf of $\mathcal{O}_X$-modules.

Example 3.3. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules.

1. The sheaf *direct sum* $\mathcal{F} \oplus \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$. The stalks are $(\mathcal{F} \oplus \mathcal{G})_P = \mathcal{F}_P \oplus \mathcal{G}_P$, for any $P \in X$.

2. The sheaf *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. The stalks are $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_P = \mathcal{F}_P \otimes_{\mathcal{O}_{P,X}} \mathcal{G}_P$, for any $P \in X$.

3. If we denote with $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ the group of morphisms of sheaves $\mathcal{F} \to \mathcal{G}$, then we define the sheaf $\mathscr{H}om$ to be the sheaf associated to the presheaf $U \mapsto \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$, denoted by $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. The stalks are $(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_P = \operatorname{Hom}_{\mathcal{O}_{P,X}}(\mathcal{F}_P, \mathcal{G}_P)$, for any $P \in X$.

4. Let R be a ring and let M be a R-module. We recall that the *exterior algebra* $\bigwedge M = \oplus_{n=0}^{\infty} \bigwedge^n M$ is the quotient of the tensor algebra $\mathrm{T}\, M$ by the ideal generated by the elements $x \otimes x$, for every $x \in M$ (here we assume the notion of tensor algebra). Moreover, the *symmetric algebra* $\mathrm{S}\, M = \oplus_{n=0}^{\infty} \mathrm{S}^{\mathrm{n}}\, M$ is the quotient of $\mathrm{T}\, M$ by the ideal generated by the expressions $x \otimes y - y \otimes x$, for any $x, y \in M$.

The sheaf *exterior algebra* $\bigwedge \mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto \bigwedge(\mathcal{F}(U)) = \oplus_{n=0}^{\infty} \bigwedge^n(\mathcal{F}(U))$. The stalks are $(\bigwedge \mathcal{F})_P = \bigwedge(\mathcal{F}_P)$, for any $P \in X$.

The sheaf *symmetric algebra* $\mathrm{S}\,\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto \mathrm{S}(\mathcal{F}(U)) = \oplus_{n=0}^{\infty} \mathrm{S}^{\mathrm{n}}(\mathcal{F}(U))$. The stalks are $(\mathrm{S}\,\mathcal{F})_P = \mathrm{S}(\mathcal{F}_P)$, for any $P \in X$.

All of them are sheaves of $\mathcal{O}_X$-modules.

Example 3.4. Let R be a ring and let M be an R-module. The *sheaf associated* to M on $X = \operatorname{Spec} R$ is the sheaf $\widetilde{M}$ defined as follows.

For any open set $U \subset \operatorname{Spec} R$ we define $\widetilde{M}(U)$ to be the set of functions $s \colon U \to \bigsqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}}$ such that

1. for any $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{\mathfrak{p}}$;
2. for any $\mathfrak{p} \in U$ there exists an open neighborhood $V \subset U$ of $\mathfrak{p}$ and $m \in M, f \in R$ such that $\forall \mathfrak{q} \in V$ we have $s(\mathfrak{q}) = m/f$, with $f \notin \mathfrak{q}$.

The restriction maps are the natural ones. This is a sheaf of $\mathcal{O}_X$-modules.

Proposition 3.5. *Let R be a ring and M a R-module. For any $\mathfrak{p} \in \operatorname{Spec} R$ we have $\widetilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}}$.*

Moreover, $\widetilde{M}(D(f)) \cong M_f$ for any $f \in R$. In particular $\Gamma(\operatorname{Spec} R, \widetilde{M}) = M$.

Proof. See [6, Chapter II.5]. □

Proposition 3.6. *Let M, N be R-modules, with R a ring. Then, the sheaf associated to $M \otimes_R N$ is isomorphic to the sheaf $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.*

Proof. Let us show that the sheaves are isomorphic on the stalks.

If C is a closed multiplicative system in R, then $C^{-1}(M \otimes_R N) \cong C^{-1}M \otimes_{C^{-1}R} C^{-1}N$ (see [1, Chapter 3]). By the proposition above, for any $\mathfrak{p} \in \operatorname{Spec} R$

$$(\widetilde{M \otimes_R N})_{\mathfrak{p}} \cong (M \otimes_R N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \cong \widetilde{M}_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p},X}} \widetilde{N}_{\mathfrak{p}} = (\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})_{\mathfrak{p}}.$$

□

Here we present quasi-coherent sheaves.

Definition 3.7. Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X is *quasi-coherent* if X can be covered by affine subsets $U_i = \operatorname{Spec} A_i$, such that $\forall i$ there exists an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$.

We say that $\mathcal{F}$ is *coherent* if it is quasi-coherent and M_i is a finitely generated A_i-module, for any i.

Example 3.8. The sheaf associated to a R-module on $\operatorname{Spec} R$ is trivially quasi-coherent. The structure sheaf of a scheme X is coherent.

Sheaves of ideals (see below) are sheaves which allow to connect sheaves of modules on a scheme X with closed subscheme of X.

Definition 3.9. A sheaf of $\mathcal{O}_X$-modules $\mathcal{I}$ which is a subsheaf of $\mathcal{O}_X$ is called *sheaf of ideals* on X. Actually, for any open subset $U \subset X$, $\mathcal{I}(U)$ is an ideal of $\mathcal{O}_X(U)$.

Let Y be a closed subscheme of X and let $i\colon Y \hookrightarrow X$ be the relative closed immersion. We define the *sheaf of ideals* of Y to be the sheaf kernel of the morphism $i^\#\colon \mathcal{O}_X \to i_*\mathcal{O}_Y$. We denote it with $\mathcal{I}_Y$.

Remark 3.10. Since $i^\#$ is surjective by definition, by 1.43 we have $i_*\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{I}_Y$. Thus, by 1.47 we have a short exact sequence

$$0 \to \mathcal{I}_{\mathcal{Y}} \xhookrightarrow{i} \mathcal{O}_X \xrightarrow{i^\#} i_*\mathcal{O}_Y \to 0. \tag{3.1}$$

Remark 3.11. Let R be a ring and let $\mathfrak{a} \subset R$ be an ideal. We know $Y = \operatorname{Spec} R/\mathfrak{a}$ is a closed subscheme of $X = \operatorname{Spec} R$ (Remark 2.56). The sheaf of ideals of Y is $\widetilde{\mathfrak{a}}$, since for any $\mathfrak{p} \in \operatorname{Spec} R/\mathfrak{a}$

$$(\mathcal{O}_X/\widetilde{\mathfrak{a}}_Y)_\mathfrak{p} \cong \mathcal{O}_{\mathfrak{p},X}/\widetilde{\mathfrak{a}}_{\mathfrak{p},Y} \cong R_\mathfrak{p}/\mathfrak{a}_\mathfrak{p} = (R/\mathfrak{a})_\mathfrak{p} = \mathcal{O}_{\mathfrak{p},Y} = (i_*\mathcal{O}_Y)_\mathfrak{p}.$$

Here we have used a property of rings of fractions: if C is a closed multiplicatively system in a ring R, and $\mathfrak{a}$ is an ideal of R, then C is a multiplicatively system in $R/\mathfrak{a}$ and $C^{-1}(R/\mathfrak{a}) = C^{-1}R/C^{-1}\mathfrak{a}$ (see [1, Chapter 3]).

Proposition 3.12. *Let X be a scheme. If Y is a closed subscheme of X, then the ideal sheaf $\mathcal{I}_Y$ is a quasi-coherent sheaf of ideals on X.*

Conversely, if $\mathcal{I}$ is a quasi-coherent sheaf of ideals on X there exists a unique closed subscheme Y of X, such that $\mathcal{I}_Y = \mathcal{I}$.

In particular, any closed subscheme of X is uniquely determined by its sheaf of ideals.

Proof. See [6, Chapter II.5]. □

3.2 Invertible Sheaves

Definition 3.13. Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is *free* if $\mathcal{F} \cong \bigoplus_{i \in I} \mathcal{O}_X$. If I is infinite we say that $\mathcal{F}$ has *infinite rank*. Otherwise the *rank* of $\mathcal{F}$ is $|I|$.

We say that $\mathcal{F}$ is a *locally free sheaf* if there exists an open covering $\bigcup_i U_i = X$ of X such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module for each i.

Remark 3.14. If X is connected and $\mathcal{F}$ is a locally free sheaf on X, then for each U of the covering above, the rank of $\mathcal{F}|_U$ is constant. Hence it is well defined the rank of a locally free sheaf on a connected scheme X.

Definition 3.15. A locally free sheaf of rank one is called *invertible sheaf* or *line bundle*. We denote with $\operatorname{Pic} X$ the set of invertible sheaf on a scheme X.

Let $\mathcal{F}$ be a locally free sheaf of finite rank on X. We define the *dual sheaf* of $\mathcal{F}$ to be $\mathcal{F}^\vee := \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Proposition 3.16. *Let X be a scheme. Then,* $\operatorname{Pic} X$ *is a commutative group with tensor product as operation. More precisely:*

i) tensor product of invertible sheaves is an invertible sheaf;

ii) $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{F}$, for any invertible sheaf $\mathcal{F}$;

iii) $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \cong \mathcal{O}_X$, for any invertible sheaf $\mathcal{F}$;

iv) $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$, *for any invertible sheaves* $\mathcal{F}, \mathcal{G}$.

It is called Picard group of X.

Proof. $i)$ is obvious, we need to take the intersections of the open subsets on which the sheaves are isomorphic to $\mathcal{O}_X$.

Associativity of tensor product, $ii)$ and $iv)$ follow by basic properties of tensor product of modules (see [1, Chapter 2]).

Let us prove $iii)$. First, $\mathcal{F}^\vee$ is an invertible sheaf. Indeed, if $U \subset X$ is an open subset such that $\mathcal{F}|_U \cong \mathcal{O}_X|_U$, we have $\mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{O}_X|_U) \cong \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U, \mathcal{O}_X|_U) \cong \mathcal{O}_X|_U$.

If $U \subset X$ is an open subset such that $\mathcal{F}|_U \cong \mathcal{O}_X|_U$, then $\mathcal{F}(U) \otimes \mathcal{F}^\vee(U) \cong \mathcal{O}_X(U) \otimes \mathcal{O}_X(U) \cong \mathcal{O}_X(U)$. □

Now we are going to explain an important example of invertible sheaf on a projective space, that is the sheaves $\mathcal{O}_X(n)$.

Definition 3.17. Let S be a graded ring, $S = \bigoplus_{d\in\mathbb{N}} S_d$. A S-module M is called *graded* S*-module* if there exist additive subgroups $M_n \subset M$ for every $n \in \mathbb{N}$, such that $M = \bigoplus_{n\in\mathbb{N}} M_n$ and $S_d \cdot M_n \subset M_{d\cdot n}$ for any $d, n \in \mathbb{N}$.

Definition 3.18. Let S be a graded ring and let M be a graded S-module. The *sheaf associated* to M on $\mathrm{Proj}\, S$ is the sheaf $\widetilde{M}$ defined as follows.

For any open set $U \subset \mathrm{Proj}\, S$ we define $\widetilde{M}(U)$ to be the set of functions $s\colon U \to \bigsqcup_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$ such that

1. for any $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{(\mathfrak{p})}$;

2. for any $\mathfrak{p} \in U$ there exists an open neighborhood $V \subset U$ of $\mathfrak{p}$ and $m \in M_d, f \in R_d$ for some d, such that $\forall \mathfrak{q} \in V$ we have $s(\mathfrak{q}) = m/f$, with $f \notin \mathfrak{q}$.

The restriction maps are again the natural ones.

Proposition 3.19. *Let* S *be a graded ring and* M *a graded* R*-module. For any* $\mathfrak{p} \in \mathrm{Proj}\, S$ *we have* $\widetilde{M}_{\mathfrak{p}} \cong M_{(\mathfrak{p})}$.

Moreover, for any homogeneous $f \in S_+$ *we have* $\widetilde{M}|_{D(f)} \cong \widetilde{M_{(f)}}$ *via the isomorphism* $D(f) \to \operatorname{Spec} S_{(f)}$ *(Proposition 2.70). In particular,* $\widetilde{M}$ *is a quasi-coherent sheaf.*

Proof. See [6, Chapter II.5]. □

Definition 3.20. Let S be a graded ring. We set $S(n) := \bigoplus_{d \geq n} S_d$, which is a graded S-module since S is a graded ring. If $X = \operatorname{Proj} S$, we define the sheaf of modules $\mathcal{O}_X(n) := \widetilde{S(n)}$, for any $n \in \mathbb{Z}$. Clearly, $\mathcal{O}_X(0) = \mathcal{O}_X$.

Proposition 3.21. *Let* S *be a graded ring generated by* S_1 *as* S_0*-algebra and let* $X = \operatorname{Proj} S$*. Then* $\mathcal{O}_X(n)$ *is an invertible sheaf for any* $n \in \mathbb{Z}$*. Moreover,* $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$ *for any* $n, m \in \mathbb{Z}$*. In particular,* $\mathcal{O}_X(n)^\vee = \mathcal{O}_X(-n)$*.*

Proof. Since S is generated by S_1, we have

$$\bigcup_{f \in S_1} D(f) = D\left(\sum_{f \in S_1} (f)\right) = D(S) = X. \tag{3.2}$$

Let $f \in S_1$. Then, $\mathcal{O}_X(n)|_{D(f)} \cong \widetilde{S(n)_{(f)}}$, by 3.19. If we show that $S(n)_{(f)}$ is a $S_{(f)}$-module of rank 1, the invertibility of $\mathcal{O}_X(n)$ will be proved. Indeed, we would have $\mathcal{O}_X|_{D(f)} \cong \widetilde{S}_{(f)}$ and so $\mathcal{O}_X(n)|_{D(f)} \cong \mathcal{O}_X|_{D(f)}$, with the subsets $D(f)$ which cover X by (3.2).

Let us consider the morphism of modules

$$S_{(f)} \to S(n)_{(f)}, \qquad \frac{a}{f^m} \mapsto \frac{f^n a}{f^m},$$

where $\deg a = m$. It is well defined and it is injective, because if $f^n a/f^m = f^n b/f^l$ in $S(n)_{(f)}$, where $\deg a = m$ and $\deg b = l$, then there exists $q \in \mathbb{N}$ such that $0 = f^q(f^{n+l}a - f^{n+m}b) = f^{q+n}(f^l a - f^m b)$. So $a/f^m = b/f^l$ in $S_{(f)}$. It is surjective since for any $a/f^m \in S(n)_{(f)}$, with $\deg a = m + n$, we have

$$\frac{a}{f^m} = f^n \frac{a}{f^{n+m}}.$$

Hence $\mathcal{O}_X(n)$ is an invertible sheaf for any $n \in \mathbb{Z}$.

For the second part of the Proposition, we note that the sheaf associated to the module $M \otimes_S N$ is isomorphic to $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, in the same way as 3.6. Hence, for any $f \in S_1$ we have $(\mathcal{O}_X(n) \otimes \mathcal{O}_X(m))_{(f)} = \mathcal{O}_X(n)_{(f)} \otimes \mathcal{O}_X(m)_{(f)}$, which is isomorphic to $\mathcal{O}_X(n+m)_{(f)}$ via the morphism

$$\frac{a}{f^k} \otimes \frac{b}{f^l} \mapsto \frac{ab}{f^{k+l}}, \qquad \text{with } \deg a = k+n \text{ and } \deg b = l+m.$$

□

Proposition 3.22. *Let R be a ring, $X = \mathbb{P}^n_R = \operatorname{Proj} S$, where $S = R[x_0, ..., x_n]$. Then, the global sections of $\mathcal{O}_X(k)$ are*

$$\Gamma(X, \mathcal{O}_X(k)) = \begin{cases} S_k, & \text{if } k \geq 0 \\ 0, & \text{if } k < 0. \end{cases}$$

Proof. The proof is exactly the same as in Proposition 2.43, taking $D(x_i)$ as open subsets of X. □

Proposition 3.23. *Let $X = \mathbb{P}^n_k$ and let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of modules. For any $l \in \mathbb{Z}$, we have a sequence $0 \to \mathcal{F}' \otimes \mathcal{O}_X(l) \to \mathcal{F} \otimes \mathcal{O}_X(l) \to \mathcal{F}'' \otimes \mathcal{O}_X(l) \to 0$ which is exact.*

Proof. See [6, Chapter II.5]. □

3.3 Cohomology of Sheaves

All the results can be found in [6, Chapter III].

First, we need to some notions from homological algebra.

Example 3.24. The following are all abelian categories (Definition 1.1).

1. Abelian groups.
2. R-modules on a commutative ring R.
3. Sheaves of abelian groups on a topological space X.

4. Sheaves of $\mathcal{O}_X$-modules on a scheme X.

5. Quasi-coherent sheaves on a scheme X.

In the following, $\mathfrak{C}$ will always be one of the categories above.

Definition 3.25. Let $f\colon A \to B$ be a morphism in $\mathfrak{C}$. Let $p\colon B \to C$ be the cokernel of f. We define the *image* of f to be $\operatorname{im} f = \ker p$.

Let

$$... \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to ...$$

be a collection of object and morphisms of $\mathfrak{C}$. We call it *exact sequence* if $f_i \circ f_{i-1} = 0$ and the natural morphism $\operatorname{im} f_{i-1} \to \ker f_i$ is an isomorphism, for any i.

Definition 3.26. We define a *complex* $\dot{A}$ in $\mathfrak{C}$ to be a collection of objects and morphisms

$$... \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to ... \qquad i \in \mathbb{Z},$$

such that $d^i \circ d^{i-1} = 0$ for any i. The morphisms are called *coboundary operators* and we will often omit their index.

We define the *ith cohomology object* of $\dot{A}$ to be $H^i(\dot{A}) := \ker d^i / \operatorname{im} d^{i-1}$. Indeed, in every abelian category above, the quotient of two objects is well defined.

Definition 3.27. A *morphism* of complexes $f\colon \dot{A} \to \dot{B}$ is a collection of morphisms $f^i\colon A^i \to B^i$ which commute with the coboundary operators.

Given morphisms of complexes $f, g\colon \dot{A} \to \dot{B}$, we say that f, g are *homotopic* if there exist morphisms $k^i\colon A^i \to B^{i-1}$ such that $f - g = dk + kd$. We say that k is an *homotopy* between f and g and we write $f \sim g$.

We say that two complexes $\dot{A}, \dot{B}$ are *homotopy equivalent* if there exist morphisms of complexes $f\colon \dot{A} \to \dot{B}$ and $g\colon \dot{B} \to \dot{A}$ such that $f \circ g$ and $g \circ f$ are homotopic with the identity on the relative complexe.

Definition 3.28. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a covariant functor between abelian categories. It is *additive* if for each $A, B \in \mathfrak{C}$, the induced map $\mathrm{Hom}_{\mathfrak{C}}(A, B) \to \mathrm{Hom}_{\mathfrak{D}}(FA, FB)$ is a homomorphism of groups. F is *left exact* if for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathfrak{C}$, the sequence $0 \to FA' \to FA \to FA''$ is exact in $\mathfrak{D}$.

If F is controvariant, we can give the same definitions as above in an analogous way.

Example 3.29. Let $(X, \mathcal{O}_X)$ be a ringed space. The functor $\Gamma(X, \cdot)$ of global sections on X from the sheaves of $\mathcal{O}_X$-modules to the abelian groups is a covariant left exact functor.

Definition 3.30. An object $I \in \mathfrak{C}$ is called *injective* if for any exact sequence $0 \to A \to B$ in $\mathfrak{C}$, for any morphism $A \to I$ there exists a morphism $B \to I$ such that

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & \downarrow & \swarrow & \\ & & I & & \end{array}$$

is commutative.

If $A \in \mathfrak{C}$, an *injective resolution* of A is a complex $I^0 \to I^1 \to \ldots$ together a morphism $\varepsilon\colon A \to I^0$, such that I^i is injective for each i and

$$0 \to A \xrightarrow{\varepsilon} I^0 \to I^1 \to \ldots$$

is an exact sequence.

We say that $\mathfrak{C}$ has *enough injectives* if every $A \in \mathfrak{C}$ has an injective resolution.

Example 3.31. All the categories in Example 3.24 have enough injectives.

Lemma 3.32. *Let $\mathfrak{C}$ be a category with enough injectives and let $A \in \mathfrak{C}$. If $\dot{I}$ and $\dot{J}$ are injective resolutions of A, then they are homotopy equivalent.*

Definition 3.33. Let $\mathfrak{C}$ be a category with enough injectives and let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a covariant left exact functor. We define the *right derived functors* of

F to be the functors R^iF, with $i \geq 0$, such that $R^iF(A) := h^i(F(\dot{I}))$ for any $A \in \mathfrak{C}$, where $\dot{I}$ is a injective resolution of A.

Theorem 3.34. *Let $\mathfrak{C}$ be a category with enough injectives and let $F: \mathfrak{C} \to \mathfrak{D}$ be a covariant left exact functor.*

i) The right derived functors R^iF are independent of the choice of the injective resolution.

ii) There is a natural isomorphism $F \cong R^0F$.

iii) For any short exact sequence $0 \to A' \to A \to A'' \to 0$, there exists a morphism $\delta^i: R^iF(A'') \to R^{i+1}(A')$ for each $i \geq 0$, such that the sequence

$$\ldots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to \ldots$$

is exact. This is called long exact sequence of cohomology.

iv) Given a morphism between two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

then the morphisms defined in iii) make the diagram

$$\begin{array}{ccc} R^iF(A'') & \xrightarrow{\delta^i} & R^iF(A') \\ \downarrow & & \downarrow \\ R^iF(B'') & \xrightarrow{\delta^i} & R^iF(B') \end{array}$$

commutative for each i.

v) For any injective object I of $\mathfrak{C}$, we have $R^iF(I) = 0$ for each $i > 0$.

Finally, we come to the definition of cohomology of sheaves on a scheme.

Definition 3.35. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. We denote with $H^i(X, \mathcal{F}) := R^i\Gamma(X, \cdot)(\mathcal{F}) = R^i\Gamma(X, \mathcal{F})$ the *i-th group of cohomology* of $\mathcal{F}$.

Remark 3.36. By *ii)* of the Theorem above, it is clear that $\Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$.

Definition 3.37. Let X be a topological space. We say that X is *noetherian* if for any descending sequence $Y_1 \supset ... \supset Y_n \supset ...$ of closed irreducible subsets of X, there exists $m \in \mathbb{N}$ such that $Y_i = Y_{i+1}$ for any $i \geq m$.

A scheme X is *noetherian* if it is a noetherian topological space.

Example 3.38. Any varietyis a noetherian scheme.

We have the following results.

Theorem 3.39. *Let X be a noetherian scheme of dimension n. Then, $H^i(X, \mathcal{F}) = 0$ for any sheaf $\mathcal{F}$ of abelian groups on X and for any $i > n$.*

Theorem 3.40. *Let R be a noetherian ring and let $X = \operatorname{Spec} R$. Then $H^i(X, \mathcal{F}) = 0$ for any $i > 0$ and for every quasi-coherent sheaf $\mathcal{F}$ on X.*

Theorem 3.41. *Let R be a noetherian ring and let $X = \mathbb{P}^r_R$, with $r \geq 1$.*

i) $H^i(X, \mathcal{O}_X(n)) = 0$ *for each* $0 < i < r$ *and* $n \in \mathbb{Z}$.

ii) $H^r(X, \mathcal{O}_X(-r-1)) \cong R$.

Chapter 4

Divisors and Differentials

The aim of this chapter is to present divisors and to establish an explicit connection between sheaves of modules and hypersurfaces of an algebraic variety. Moreover, we study differentials on a scheme and we will come to the adjunction formula (Theorem 4.50).

First of all, we need some notions from Commutative Algebra.

4.1 Discrete Valuation Rings

Definition 4.1. Let k be a field and let Γ be a totally ordered group, that is Γ is a group and a totally ordered set such that, if $a \leq b$, then $a + c \leq b + c$ for any $a, b, c \in \Gamma$. A *valuation* of k in Γ is a map $v\colon k \setminus \{0\} \to \Gamma$ such that

1. $v(xy) = v(x) + v(y)$ for every $x, y \in \Gamma$;
2. $v(x + y) \geq \min\{v(x), v(y)\}$ for every $x, y \in \Gamma$.

If $\Gamma = \mathbb{Z}$ the valuation is called *discrete*.

For convention, we set $v(0) \coloneqq +\infty$.

Proposition 4.2. *Let v be a valuation of a field k. Then the set $R = \{x \in k\colon v(x) \geq 0\} \cup \{0\}$ is a subring of k and the set $\mathfrak{m} = \{x \in k\colon v(x) > 0\} \cup \{0\}$ is a maximal ideal of R. Furthermore $(R, \mathfrak{m})$ is a local ring.*

Proof. It easily follows from the properties of valuations. □

Definition 4.3. The ring R is called *valuation ring* of v. If v is a discrete valuation, R is called *discrete valuation ring* (DVR).

Let R be an integral domain with field of fractions k. We call R *(discrete) valuation ring* if there exists a (discrete) valuation v of k such that R is the (discrete) valuation ring of v.

Example 4.4. Let $S = k[x_0, ..., x_n]$, where k is a field. Let f be a non-constant homogeneous irreducible polynomial of S and let $\mathfrak{p} = (f)$ be the homogeneous prime ideal generated by f. Then, the localized ring $S_{(\mathfrak{p})}$ is a DVR. Let us prove this claim.

The field of fractions of $S_{(\mathfrak{p})}$ is the field of homogeneous polynomial fractions $K \coloneqq k^h(x_0, ..., x_n) \coloneqq \{f/g \in k(x_0, ..., x_n) \colon f, g \text{ homogeneous}, \deg f = \deg g\}$. The ring S is a unique factorization domain, so for any homogeneous $g \in S_+$, there exists a unique $g' \in S_+$ such that $g = f^\alpha g'$ and $g' \notin (f)$, where $\alpha \in \mathbb{N}$. Let us consider the function

$$v_f \colon K \setminus \{0\} \to \mathbb{Z}, \qquad v_f\left(\frac{g}{h}\right) \coloneqq \alpha - \beta,$$

for any $g, h \in S_d$, $d > 0$. We have that v_f is a valuation of K, because

$$\begin{aligned} v_f\left(\frac{g}{h}\cdot\frac{l}{m}\right) &= v_f\left(\frac{f^\alpha g'}{f^\beta h'}\cdot\frac{f^\gamma l'}{f^\delta m'}\right) = v_f\left(f^{\alpha-\beta+\gamma-\delta}\frac{g'l'}{h'm'}\right) \\ &= \alpha - \beta + \gamma - \delta = v_f\left(\frac{g}{h}\right) + v_f\left(\frac{l}{m}\right), \end{aligned}$$

and

$$\begin{aligned} v_f\left(\frac{g}{h}+\frac{l}{m}\right) &= v_f\left(\frac{f^\alpha g'}{f^\beta h'}+\frac{f^\gamma l'}{f^\delta m'}\right) = v_f\left(\frac{f^{\alpha+\delta}g'm' + f^{\beta+\gamma}l'h'}{f^{\beta+\delta}h'm'}\right) \\ &= \begin{cases} \alpha-\beta, & \text{if } \alpha+\delta \le \beta+\gamma \\ \gamma-\delta, & \text{if } \beta+\gamma \le \alpha+\delta \end{cases} \\ &= \begin{cases} v_f(g/h), & \text{if } v_f(g/h) \le v_f(l/m) \\ v_f(l/m), & \text{if } v_f(l/m) \le v_f(g/h). \end{cases} \\ &= \min\{v_f(g/h), v_f(l/m)\}. \end{aligned}$$

Moreover, $\{x \in K : v_f(x) \geq 0\} \cup \{0\} = \{g/h \in K : h \notin \mathfrak{p}\} = S_{(\mathfrak{p})}$, hence $S_{(\mathfrak{p})}$ is a DVR with valuation v_f.

Definition 4.5. Let $(R, \mathfrak{m})$ be a local ring of dimension d. We call R a *regular local ring* if it is a noetherian local ring and $\mathfrak{m}$ can be generated by d elements.

Theorem 4.6. *Let R be a noetherian integral domain such that R is a local ring with $\mathfrak{m}$ as maximal ideal. If $\dim R = 1$, then R is a DVR if and only if R is a regular local ring.*

Proof. See [1, Chapter 9]. □

4.2 Weil Divisors

The divisors theory is not presented here in the most general case. We will study divisors on irreducible varieties nonsingular in codimension one, but Weil divisors can be defined on noetherian integral separated schemes nonsingular in codimension one. This because we didn't give any detail on noetherian schemes in the eleborate.

Definition 4.7. Let X be an irreducible variety. X is called *nonsingular in codimension 1* if any local ring $\mathcal{O}_{P,X}$ of dimension one is a regular local ring.

An integral subscheme Y of X of codimension one is called *prime divisor*. A collection of prime divisors $Y_1, ..., Y_r$ on X with assigned integer $k_1, ..., k_r$ is called a (*Weil*) *divisor* on X. Thus, a divisor can be written as a formal linear combination

$$D = \sum k_i Y_i.$$

If $k_i = 0$ for each i, we write $D = 0$. A divisor is said to be *effective* if $k_i \geq 0$ for each i and $D \neq 0$.

We define $\operatorname{Div} X$ to be the set of Weil divisors on X, that is, the free abelian group generated by prime divisors on X.

Example 4.8. The projective space $\mathbb{P}^n_k$ is nonsingular in codimension one.

Before going on, we need the notion of generic point and function field of a scheme.

Definition 4.9. Let X be a scheme, Y an irreducible closed subset of X and let $\eta \in Y$. If $\overline{\{\eta\}} = Y$, we call η *generic point* of Y.

Proposition 4.10. *Let X be a scheme and let Y be an irreducible closed subset of X, $Y \neq \emptyset$. Then Y has exactly one generic point.*

Proof. For the uniqueness, we suppose $Y = \overline{\{\eta_1\}} = \overline{\{\eta_2\}}$. If $U = \operatorname{Spec} R$ is an open neighborhood of η_1 and η_2, and we consider η_1 and η_2 as prime ideals of R, then $V(\eta_1) = V(\eta_2)$ by 2.5. Then $\sqrt{\eta_1} = \sqrt{\eta_2}$ by 2.11 *iii*), but they are prime ideals, so $\eta_1 = \eta_2$.

For the existence, we note that every open subset of Y is dense for the irreducibility of Y. Let $U = \operatorname{Spec} R$ be an open dense subset of Y. Clearly $\operatorname{Spec} R$ is irreducible, so the nilradical $\mathfrak{p} = \mathcal{N}(R)$ is a minimal prime ideal of R, by 2.18. Hence $V(\mathfrak{p}) = \operatorname{Spec} R$, but then $\overline{\{\mathfrak{p}\}}$ is a closed subset of Y which contains the open dense U. Therefore $\overline{\{\mathfrak{p}\}} = Y$. □

Example 4.11. Let $Y = V(f)$ in $\mathbb{P}^n_k$ be an irreducible closed subscheme, where f is a non-costant homogeneous irreducible polynomial of $k[x_0, ..., x_n]$. It is tautological that the unique generic point of Y is (f).

Proposition 4.12. *Let X be an irreducible variety and let $Y \subset X$ be an irreducible closed subset of X. Let η be the generic point of Y. Then*

$$\operatorname{codim}(Y, X) = \dim \mathcal{O}_{\eta,X}.$$

Proof. By Definition 2.97, we have

$$\operatorname{codim}(Y, X) = \inf_{P \in Y} \dim \mathcal{O}_{P,X}.$$

For any $P \in Y$, if $U = \operatorname{Spec} R$ is an open neighborhood of P, then $\dim \mathcal{O}_{P,X}$ is the height of P as prime ideal of R. By the proof of 4.10, the minimal height of such ideals is the height of η, since it is the minimal prime ideal of all the rings R, where $\operatorname{Spec} R$ is an affine open subset of Y. □

Proposition 4.13. *Let X be an integral scheme and let $\eta \in X$ be its generic point. Then $\mathcal{O}_{\eta,X}$ is a field.*

Proof. Let $U = \operatorname{Spec} R$ be an affine open neighborhood of η in X. Since X is integral, R is an integral domain, so the nilradical $\mathcal{N}(R) = (0)$. Hence $\mathcal{O}_{\eta,X} \cong R_{(0)}$, which is the field of fractions of R. □

Definition 4.14. Let X be an integral scheme with generic point η. We denote with $K(X)$ the field $\mathcal{O}_{\eta,X}$ and we call it *function field* of X. We call any $f \in K(X)$ *rational function* on X.

Example 4.15. If $X = \mathbb{P}^n_k$, then the generic point of X is (0) and $K(X) = k^h(x_0, ..., x_n)$.

Remark 4.16. Now, let X be an irreducible variety which is nonsingular in codimension 1 and let Y be a prime divisor on X with generic point η. Since 4.12 holds, $\mathcal{O}_{\eta,X}$ is a noetherian local integral domain of dimension one. In particular, η is a principal ideal.

Moreover, we note that the field of fractions of the integral domain $\mathcal{O}_{\eta,X}$ is the function field $K(X)$ on X. Indeed any affine open subset which meets η, contains the generic point of X too. Thus, by 4.6, $\mathcal{O}_{\eta,X}$ is a DVR with valuation $v_Y : K(X) \to \mathbb{Z}$, which depends only on Y.

Example 4.17. Let $X = \mathbb{P}^n_k$ and let $Y \subset X$ be a prime divisor. By Corollary 2.75, Y is uniquely determined by a prime homogeneous ideal $\mathfrak{p}$ of $k[x_0, ..., x_n]$, with $Y = V(\mathfrak{p})$. Thus, the generic point of Y is $\mathfrak{p}$ and by the remark above, there exists a homogeneous irreducible polynomial $f \in k[x_0, ..., x_n]$, such that $\mathfrak{p} = (f)$. Hence we have a valuation

$$v_Y : k^h(x_0, ..., x_n) \setminus \{0\} \to \mathbb{Z},$$

with $v_Y = v_f$, accordingly with the notation of Example 4.4.

In the following, X will always be an irreducible variety which is nonsingular in codimension 1.

Definition 4.18. Let $f \in K(X)$ and Y a prime divisor on X. Let $v_Y(f) = d \in \mathbb{Z}$. Then

1. if $d > 0$, we say f has a *zero along* Y of order d;
2. if $d < 0$, f has a *pole along* Y of order d;
3. if $d = 0$, f is invertible on Y.

Definition 4.19. Let $f \in K(X)$. We set $(f) := \sum v_Y(f) \cdot Y$, where the sum runs over all the prime divisors on X. We call (f) the *divisor of* f.

Proposition 4.20. *For any $f \in K(X)$, $(f) = \sum v_Y(f) \cdot Y$ is a well defined Weil divisor, that is the sum is finite.*

Proof. See [6, Chapter II.6]. □

Definition 4.21. Let $D \in \operatorname{Div} X$ a Weil divisor. It is *principal* if there exists $f \in K(X)$ such that $(f) = D$.

Two divisors D_1, D_2 are *linearly equivalent* if $D_1 - D_2$ is principal. The group of Weil divisors on X modulo the linear equivalence relation is denoted with $\operatorname{Cl} X$.

Example 4.22. Let $X = \mathbb{P}^n_k$ and let $Y = \sum_{i=1}^m k_i Y_i$ be a divisor on X. Thus, there exist homogeneous irreducible polynomial $f_1, ..., f_m \in k[x_0, ..., x_n]$, such that $Y_i = V(f_i)$ for any i. We define the *degree* of Y_i to be the degree of f_i and the degree of D to be $\deg D = \sum_{i=1}^m k_i \deg Y_i$.

If D is an effective divisor, then $D = (f)$, where $f = f_1^{k_1} \cdots f_m^{k_m}$ is homogeneous. If $\deg D = 1, 2, 3, 4, ...$ we call D, respectively, *hyperplane*, *conic*, *cubic*, *quartic*,...

Proposition 4.23. *Let $X = \mathbb{P}^n_k$. Then $\operatorname{Cl}(X) \cong \mathbb{Z}$ as groups.*

Proof. Let us consider the degree function $\deg\colon \operatorname{Div} X \to \mathbb{Z}$. Then, for any $f = g/h \in K(X)$ with $\deg g = \deg h$, we have $(f) = (g) - (h)$, by the properties of valuation, so $\deg(f) = 0$. Hence deg induces a map $\operatorname{Cl} X \to \mathbb{Z}$, which is clearly surjective. To show that it is injective we prove that each divisor of degree d is linearly equivalent to dH, where $H = (x_0)$.

Let $D = \sum k_i Y_i \in \operatorname{Div} X$ be a divisor with degree d. We set $D = D_1 - D_2$, where D_1 is the sum of prime divisor with positive coefficients and $D_2 = D - D_1$. Hence D_1 and D_2 are effective divisors. By the remark above, $D_1 = (g)$ and $D_2 = (h)$, where g and h are homogeneous polynomial such that $\deg g - \deg h = \deg D = d$. Then we have

$$D - dH = (g) - (h) - (x_0^d) = \left(\frac{g}{hx_0^d}\right) =: (f),$$

with $\deg f = 0$. Therefore $f \in K(X)$, so $D - dH$ is principal. □

4.3 Cartier Divisors and Invertible Sheaves

Now we want to link Weil divisors and line bundles. The first step is to establish a 1-1 corrispondence between Weil divisors and Cartier divisors, a generalization of Weil divisor which could be defined on a generic scheme. However, in our context we define them over an integral scheme, to simplify the notations.

Definition 4.24. Let X be an integral scheme and let $\mathcal{K}^*$ be the constant sheaf $K^* := K(X) \setminus \{0\}$ on X. Let $\mathcal{O}^*$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}^*(U)$, the (multiplicative) group of invertible elements of $\mathcal{O}(U)$. A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{K}^*/\mathcal{O}^*$ on X. Thus, a Cartier divisor on X is described by $\{(U_i, f_i)\}$, where $\bigcup_i U_i = X$ is an open covering of X and $f_i \in K(X)$ for any i, such that $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$, for any i, j.

A Cartier divisor is *principal* if it is represented by a single (X, f) with $f \in K^*$. It is *effective* if $f_i \in \mathcal{O}(U_i)$ for any i.

Two Cartier divisors $D_1 = \{(U_i, f_i)\}, D_2 = \{(V_j, g_j)\}$ are *linearly equivalent* if $D_1 - D_2 = \{(U_i \cap V_j, f_i/g_j)\}$ is principal. We use the additive notation instead the multiplicative one to preserve the analogy with Weil divisors.

We set the group of Cartier divisors on X modulo the linear equivalence relation with $\operatorname{CaCl} X$.

Proposition 4.25. *Let X be an irreducible variety which is nonsingular in codimension 1 and such that every local ring in X is a unique factorization domain. Then* $\operatorname{Cl} X \cong \operatorname{CaCl} X$.

Proof. We want to establish a 1-1 corrisponcence between Weil divisors and Cartier divisors in order to send principal Weil divisors to principal Cartier divisors.

Let $\{(U_i, f_i)\}$ be a Cartier divisor on X. For any prime divisor Y on X, we choose i such that $Y \cap U_i \neq \emptyset$ and we consider $v_Y(f_i)$. We have a well defined Weil divisor $D = \sum v_Y(f_i)Y$, since for any $i \neq j$, f_i/f_j is invertible on $U_i \cap U_j$, so $v_Y(f_i) - v_Y(f_j) = v_Y(f_i/f_j) = 0$. Moreover, if $\{(U_i, f_i)\}$ is principal, then $\{(U_i, f_i)\} = (X, f)$, with $f \in K(X)$. Hence, the associated Weil divisor is principal.

For the converse, we prove the theorem in the case $X = \mathbb{P}^n_k$. See [6, Chapter II.6] for the complete proof.

Let D be a Weil divisor of degree d on $X = \operatorname{Proj} k[x_0, ..., x_n]$. We split D in $D = D_1 - D_2$, where D_1 and D_2 are effective divisors, as in 4.23. Then $D_i = (g_i)$ where g_i is a homogeneous polynomial of $k[x_0, ..., x_n]$, for $i = 1, 2$. Now, we consider the open covering $\bigcup_{i=0}^n D(x_i) = X$ and we set $f_i := g_1/x_i^d g_2 \in K(X)$ for any $i = 0, ..., n$. For every $i \neq j$ we have $f_i/f_j = x_j^d/x_i^d$ on $U_i \cap U_j$, which is an invertible element of $\mathcal{O}_X(D(x_i) \cap D(x_j))$. Hence $\{(U_i, f_i)\}$ is a well defined Cartier divisor. We note that if D is principal, then $d = \deg D = 0$ and $f = g_1/g_2$ for any i, that is $\{(U_i, f_i)\} = (X, g_1/g_2)$ is principal.

Clearly the maps are inverse to each other. □

Now we associate to a Cartier divisor an invertible sheaf. In the following, X will be an integral scheme.

Definition 4.26. Let $D = \{(U_i, f_i)\}$ be a Cartier divisor on an integral scheme X. We define the *sheaf $\mathcal{L}(D)$ associated to D* in the following way.

For any i, we set $\mathcal{L}(D)(U_i)$ as the $\mathcal{O}_X(U_i)$-submodule of $\mathcal{K}$ generated by f_i^{-1}. For any open $U \subset X$, we define $\mathcal{L}(D)(U)$ to be the set of $\{s_i\}$, where $s_i \in \mathcal{L}(D)(U_i \cap U)$ such that $s_i = s_j$ on $U_i \cap U_j$, for any i, j. It is well defined, since $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module.

Remark 4.27. The sheaf $\mathcal{L}(D)$ is clearly an invertible sheaf, since we have isomorphisms $\mathcal{O}_X(U_i) \to \mathcal{L}(D)(U_i)$, generated by $1 \mapsto f_i^{-1}$. Conversely, if $\mathcal{L}$ is an invertible subsheaf of $\mathcal{K}$ we can build a Cartier divisor, taking on U_i the inverse of the generator of $\mathcal{L}(U_i)$.

Thus, we have a corrispondence between Cartier divisor on X and invertible subsheaves of $\mathcal{K}$.

Proposition 4.28. *Let X be an integral scheme. Then there exists an isomorphism of groups*

$$\operatorname{CaCl} X \longleftrightarrow \operatorname{Pic} X.$$

Proof. By the remark above, there is a 1-1 corrispondence between Cartier divisors and invertible subsheaves of $\mathcal{K}$. Let us show that this corrispondence is an isomorphism of groups which respect the linear equivalence of divisors.

Let D_1 and D_2 be Cartier divisors locally generated by f_i and g_j, respectively. Then $\mathcal{L}(D_1) \otimes \mathcal{L}(D_2) \cong \mathcal{L}(D_1 + D_2)$. Indeed $\mathcal{L}(D_1 + D_2)$ is locally generated by $f_i^{-1} g_j^{-1}$, while $\mathcal{L}(D_1) \otimes \mathcal{L}(D_2)$ is locally generated by $f_i^{-1} \otimes g_j^{-1}$, hence the sheaves are locally isomorphic. Moreover we have $\mathcal{L}(D_1) \otimes \mathcal{L}(-D_1) \cong \mathcal{O}_X$, clearly. Finally, we see that D is a principal Cartier divisor if and only if $\mathcal{L}(D) \cong \mathcal{O}_X$. Indeed, D is principal $\Leftrightarrow$ D is defined by a single $f \in K^*$ $\Leftrightarrow$ $\mathcal{O}_X \cong \mathcal{L}(D)$ via the morphism $1 \mapsto f^{-1}$ on global sections.

To conclude, we need to show that any invertible sheaf $\mathcal{L}$ of X is isomorphic to a subsheaf of $\mathcal{K}$. We note that for any open subset $U \subset X$ such that $\mathcal{L}$ is trivial on U, we have

$$(\mathcal{L} \otimes \mathcal{K})|_U = \mathcal{L}|_U \otimes \mathcal{K}|_U \cong \mathcal{O}_X \otimes \mathcal{K} \cong \mathcal{K},$$

which is a constant sheaf. Since X is irreducible, any two open subsets intersect, hence $\mathcal{L} \otimes \mathcal{K}$ is constant and it is isomorphic to $\mathcal{K}$. Thus, we have an exact sequence $0 \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$, that is $\mathcal{L}$ is a subsheaf of $\mathcal{K}$. $\square$

Proposition 4.29. *Every invertible sheaf on $\mathbb{P}^n_k$ is isomorphic to $\mathcal{O}(l)$, with $l \in \mathbb{Z}$.*

Proof. Let $X = \mathbb{P}^n_k$. By 4.23, we have $\operatorname{Cl} X \cong \mathbb{Z}$. Therefore

$$\operatorname{Pic} X \cong \operatorname{CaCl} X \cong \operatorname{Cl} X \cong \mathbb{Z}.$$

To conclude, we have to show that the composition of the isomorphisms in 4.25 and 4.28 sends $\mathcal{O}_X(1)$ to the class of hyperplanes. Indeed, $\operatorname{Cl} X$ is generated by this class, so $\operatorname{Pic} X$ is the free group generated by $\mathcal{O}(1)$.

If $X = \operatorname{Proj} S = \operatorname{Proj} k[x_0, ..., x_n]$ let us consider the covering of X given by the open subsets $D(x_i)$, $i = 0, ..., n$. By 3.19, the local sections of $\mathcal{O}_X(1)$ are $\mathcal{O}_X(1)|_{D(x_i)} = \{f/x_i^n \colon f \in S_{n+1}, n \in \mathbb{N}\}$, hence $\mathcal{O}_X|_{D(x_i)}$ is generated by x_i^{-1} as $\mathcal{O}_X|_{D(x_i)}$-module. The (class of) Cartier divisor associated to this sheaf via the isomorphism in 4.28 is $\{(D(x_i), x_i)\}_{i=0,...,n}$, so the (class of) Weil divisor associated to $\mathcal{O}_X(1)$ is the hyperplane's one.

□

Definition 4.30. Let $\{(U_i, f_i)\}$ be an effective Cartier divisor on an integral scheme X. Let us consider the sheaf of ideals $\mathcal{I}$ locally generated by f_i. We define the associated subscheme Y of X of codimension one to be the closed subscheme defined by $\mathcal{I}$.

Proposition 4.31. *Let $D = \{(U_i, f_i)\}$ be an effective Cartier divisor on an integral scheme X and let Y be the associated subscheme of X of codimension one. Then $\mathcal{I}_Y \cong \mathcal{L}(-D)$.*

Proof. By the proof of 4.28, we know that $\mathcal{L}(-D)$ is locally generated by f_i, thus the Proposition follows from the definition above. □

4.4 Differentials

In this section we introduce the differentials on a separated scheme X over a scheme Y. We could define differentials on generic schemes, but in this way we simplify the notations.

Our goal is to present the notions of nonsingular variety, of canonical bundle on a nonsingular variety and the adjunction formula.

All the proofs can be found in [10, Chapter 10.26] or in [6, Chapter II.8].

Definition 4.32. Let X be a scheme and let $P \in X$. We consider the local ring $\mathcal{O}_{P,X}$ with maximal ideal $\mathfrak{m}_P$. We define the *Zariski cotangent space* in P to be the $\mathcal{O}_{P,X}$-module $\mathfrak{m}_P/\mathfrak{m}_P^2$. We write $\mathfrak{m}/\mathfrak{m}^2$ if there is no misunderstanding on the base point.

We see that $\mathfrak{m}/\mathfrak{m}^2$ is a $k(P)$-vector space, where $k(P)$ is the residue field of P (Definition 2.79). Indeed, for any class $\bar{\lambda} = \lambda + \mathfrak{m} \in k(P)$ and $\bar{x} = x + \mathfrak{m}^2 \in \mathfrak{m}/\mathfrak{m}^2$ we have $\bar{\lambda}\bar{x} = \lambda x + \mathfrak{m}^2 \in \mathfrak{m}/\mathfrak{m}^2$.

Definition 4.33. Let R be a ring, A an R-algebra and M an A-module. An *R-derivation* of A in M is a map $d\colon A \to M$ such that for any $a, b \in A$ and for any $r \in R$:

$$d(a+b) = d(a) + d(b); \tag{4.1}$$

$$d(ab) = d(a)b + ad(b); \tag{4.2}$$

$$d(r) = 0. \tag{4.3}$$

Remark 4.34. Let X be a differentiable real manifold. We know that the tangent space $T_P X$ at $P \in X$ can be defined to be the vector space of derivations at P, that is the set of R-derivations $D\colon \mathcal{O}_P \to \mathbb{R}$, where $\mathcal{O}_P$ is the $\mathbb{R}$-vector space of germs of functions at P. In particular, $\mathcal{O}_P$ is a local ring with maximal ideal $\mathfrak{m} = \{f \in \mathcal{O}_P \colon f(P) = 0\}$.

We observe that for any constant function $c \in \mathcal{O}_P$ and for any derivation D, we have $D(c) = 0$, thus each derivations in uniquely determined by a morphism of modules $\mathfrak{m} \to \mathbb{R}$ and by the map $\mathcal{O}_P \to \mathfrak{m}$ defined by $f \mapsto f - f(P)$. Moreover, for any $f, g \in \mathfrak{m}$ we have $D(fg) = f(P)D(g) + g(P)D(f) = 0$, so it is induced a linear map $\mathfrak{m}/\mathfrak{m}^2 \to \mathbb{R}$ (see [14]).

Thus, to give an element of $T_P X$ is the same thing to give an element of $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. This justifies the definition of Zariski cotangent space.

Remark 4.35. Let R be a ring and A an R-algebra. Then we have a morphism of rings $R \to A$ which induces a morphism of schemes $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} R$. Our aim is to define a sheaf $\Omega_{A/R}$ of A-modules on $\operatorname{Spec} A$ such that for any $\mathfrak{p} \in \operatorname{Spec} R$, the stalks of $\Omega_{A/R}$ represents the Zariski cotangent space. Then, we want to generalize everything to separated schemes.

In the following, we always assume R a ring, A an R-algebra and M an A-module.

Definition 4.36. The *module of relative differential forms* of A over R is defined to be an A-module $\Omega_{A/R}$ together with an R-derivation $d\colon A \to \Omega_{A/R}$, such that for any A-module M and for any R-derivation $d'\colon A \to M$, there exists a unique A-module morphism $f\colon \Omega_{A/R} \to M$ such that $d' = f \circ d$.

Proposition 4.37. *The module of relative differential forms $\Omega_{A/R}$ of A over R exists and it is unique up to unique isomorphism. In particular, $\Omega_{A/R}$ is generated by $\{da\colon a \in A\}$.*

Proof. Let us suppose A is a R-module via $\varphi\colon R \to A$. We can construct $\Omega_{A/R}$ in the following way: we take the set of formal symbols $\{da\colon a \in A\}$ and we quotient it with the submodule generated by the expressions:

1. $d(a+b) - da - db$, for any $a, b \in A$;
2. $d(ab) - bda - adb$, for any $a, b \in A$;
3. dr, for any $r \in \varphi(R)$.

The map $d\colon A \to \Omega_{A/R}$ is defined by $a \mapsto da$.

The uniqueness follows by the universal property. □

Remark 4.38. Let us suppose that A is a finitely generated R-algebra. Then $A = R[x_1, ..., x_n]/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $R[x_1, .., x_n]$ generated by $\{f_i\}_{i \in I}$. Then, the module of relative differential forms of A over R is generated by $dx_1, ..., dx_n$ as A-module, modulo the relations (4.1), (4.2), (4.3) and $df_i = 0$ for any $i \in I$.

Example 4.39. Let $A = k[x_1, ..., x_n]$ and $R = k$, where k is a field. Then $\Omega_{A/R}$ is the R-module $Adx_1 \oplus ... \oplus Adx_n$.

Let $A = k[x_1, ..., x_n]/(f_1, ..., f_m)$. Then

$$\Omega_{A/R} = (Adx_1 \oplus ... \oplus Adx_n)/(Adf_1 \oplus ... \oplus Adf_m).$$

Hence, if $A = k[x, y]/(x^2 - 5y^3)$, then

$$\Omega_{A/R} = \{fdx + gdy\colon f, g \in k[x, y]\}/(2xdx - 15y^2dy).$$

If $A = R/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of R, then $\Omega_{A/R} = 0$, by condition 3. of the proof above.

Proposition 4.40. *Let $f\colon A\otimes_R A \to A$ be the morphism defined by $f(a\otimes b) = ab$ and let $I := \ker f$. We define the map $d\colon A \to I/I^2$, with $da = 1\otimes a - a\otimes 1$ (modulo I^2). Then $\Omega_{A/R} = I/I^2$ with d as R-derivation.*

Definition 4.41. Let $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} R$ be affine schemes and let $f\colon U \to V$. We define the *sheaf of relative differentials* of U over V to be the sheaf of modules $\Omega_{U/V} = \widetilde{\Omega_{A/R}} = \widetilde{I/I^2}$.

Now, we extend this definition to any separated scheme.

Definition 4.42. Let $X \to Y$ be a closed immersion of schemes and let $\mathcal{I}$ be the ideal sheaf associated to X. We call *conormal sheaf* of the closed immersion the sheaf of modules $\mathcal{I}/\mathcal{I}^2$ on X.

We define the *normal sheaf* to be $\mathcal{N}_{X/Y} = (\mathcal{I}/\mathcal{I}^2)^\vee$.

Definition 4.43. Let $X \to Y$ be a separated morphism of schemes and consider the diagonal morphism $\Delta\colon X \to X \times_Y X$. By definition of separated morphism, we have that $\Delta(X)$ is a closed subscheme of $X \times_Y X$. We define the *cotangent sheaf* $\Omega_{X/Y}$ of X over Y to be the conormal sheaf of Δ.

Remark 4.44. Let $f\colon X \to Y$ be a separated morphism of schemes. Let $U = \operatorname{Spec} A \subset X$ and $V = \operatorname{Spec} R \subset Y$ be affine schemes such that $f(U) \subset V$. Then we have that $U \times_V U \cong \operatorname{Spec}(A \otimes_R A)$ is an open affine subset of

$X \times_Y X$. The closed subscheme $Z = \Delta(X) \cap (U \times_V U)$ is defined to be the kernel of $\Delta|_U$, that is the kernel I of the morphism $A \otimes_R A \to A$. Hence, the cotangent sheaf of U over V is $\widetilde{\mathcal{I}/\mathcal{I}^2}$.

We can glue toghether each derivation $d\colon A \to \Omega_{A/R}$, obtaining a morphism of sheaves $\mathcal{O}_X \to \Omega_{X/Y}$. In particular, $\Omega_{X/Y}$ is a quasi-coherent sheaf of modules on X.

Proposition 4.45. *Let R be a ring, $Y = \operatorname{Spec} R$ and $X = \mathbb{P}^n_R$. Then we have an exact sequence of sheaves on X*

$$0 \to \Omega_{X/Y} \to \bigoplus_{i=1}^{n+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0.$$

Now we briefly introduce the concept of nonsingularity in the context of schemes.

Definition 4.46. An irreducible algebraic variety X over an algebraically closed field k is *nonsingular* if all its local rings are regular local rings (Definition 4.5).

Theorem 4.47. *An irreducible n-variety X over a field k is nonsingular if and only if the cotangent sheaf $\Omega_{X/k}$ is locally free of rank n.*

Proposition 4.48. *Let X be a nonsingular variety over k and let Y be an irreducible closed subscheme of X, with sheaf of ideals $\mathcal{I}$. Then Y is nonsingular if and only if $\Omega_{Y/k}$ is locally free and the sequence*

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$$

is exact.

Definition 4.49. Let X be a nonsingular n-variety over a field k. We define the *canonical* (or *determinant*) *bundle* of X to be the sheaf $\omega_X = \det \Omega_{X/k} := \bigwedge^n \Omega_{X/k}$ (Example 3.3).

Theorem 4.50 (Adjunction Formula)**.** *Let Y be a nonsingular subvariety of codimension r in a nonsingular variety X over k. Then $\omega_Y \cong \omega_X \otimes \bigwedge^m \mathcal{N}_{Y/X}$.*

Corollary 4.51. *Let X be a nonsingular variety over a field k and let Y be an effective Weil divisor on X, with $\mathcal{L}$ as associated invertible sheaf. Then $\omega_Y \cong \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_Y$.*

We conclude the chapter computing the canonical bundle of the projective space $\mathbb{P}^n_k$.

Lemma 4.52. *Let X be a scheme and let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be an exact sequence of locally free sheaves of rank n_1, n and n_2, respectively. Then*

$$\bigwedge^{n} \mathcal{F} \cong \bigwedge^{n_1} \mathcal{F}_1 \otimes \bigwedge^{n_2} \mathcal{F}_2.$$

Proof. See [11, Chapter 6.4.1]. □

Corollary 4.53. *Let k be a field and $X = \mathbb{P}^n_k$. Then $\omega_X \cong \mathcal{O}_X(-n-1)$.*

Proof. By 4.45 we have the exact sequence

$$0 \to \Omega_{X/Y} \to \bigoplus_{i=1}^{n+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0.$$

It follows from the Lemma that

$$\omega_X \cong \omega_X \otimes \mathcal{O}_X \cong \bigwedge^{n+1} \left(\bigoplus_{i=1}^{n+1} \mathcal{O}_X(-1) \right) \overset{*}{\cong} \bigotimes_{i=1}^{n+1} \mathcal{O}_X(-1) \cong \mathcal{O}_X(-n-1).$$

The isomorphism $*$ is a well known result of algebra. See [2], for instance. □

Chapter 5

K3 Surfaces

Finally, in the last chapter of the thesis we talk about K3 surfaces. in the first section give several examples of K3 surfaces, some of which are only mentioned.

In the second section we present a very important property of K3 surfaces, that is all of them have the same Hodge diamond. We take care to calculating it, using many results by cohomology theory (as Serre's Duality Theorem, 5.11).

5.1 Introduction

Definition 5.1. A *K3 surface* is a complete and nonsingular variety X of dimension 2 over an algebraically closed field k, such that $H^1(X, \mathcal{O}_X) = 0$ and the canonical bundle $\omega_X \cong \mathcal{O}_X$.

In the whole chapter, k will always be an algebraically closed field.

Nonsingular quartics in $\mathbb{P}^3_k$

Theorem 5.2. *Let k be an algebraically closed field and let X be a nonsingular quartic in $\mathbb{P}^3_k$ (Example 4.22). Then X is a K3 surface.*

For the proof we proceed in 4 steps. X will always be a nonsingular quartic in $\mathbb{P}^3_k$.

Lemma 1. X is a complete variety over k.

Proof. It follows directly by 2.93. □

Lemma 2. X is a nonsingular surface.

Proof. It is clear by definition. □

Lemma 3. $H^1(X, \mathcal{O}_X) = 0$.

Proof. We want to compute $H^1(X, \mathcal{O}_X)$ using the long exact sequence of cohomology (Theorem 3.34). Hence we need to start by a short exact sequence, which will be the sequence (3.1). Therefore, we need the sheaf of ideals of X (Definition 3.9).

By definition, X is a prime Weil divisor of degree 4. Since $\mathbb{P}^3_k$ is a nonsingular irreducible variety, we have an invertible sheaf associated to X, by 4.25 and 4.28 and such invertible sheaf is $\mathcal{O}(4)$ (here we omit we are in $\mathbb{P}^3_k$). By 4.31, the sheaf of ideals of X is $\mathcal{O}(-4)$. Thus we have the exact sequence

$$0 \to \mathcal{O}(-4) \to \mathcal{O} \to \mathcal{O}_X \to 0.$$

By the long exact sequence of cohomology we have, in particular,

$$\ldots \to H^1(\mathbb{P}^3_k, \mathcal{O}) \to H^1(X, \mathcal{O}_X) \to H^2(\mathbb{P}^3_k, \mathcal{O}(-4)) \to \ldots$$

But $H^1(\mathbb{P}^3_k, \mathcal{O}) = H^2(\mathbb{P}^3_k, \mathcal{O}(-4)) = 0$ by Theorem 3.41, so $H^1(X, \mathcal{O}_X) = 0$. □

Lemma 4. The canonical bundle is trivial, that is $\omega_X \cong \mathcal{O}_X$.

Proof. By the Adjunction Formula for Weil divisors (Corollary 4.51), we have

$$\omega_X \cong \omega_{\mathbb{P}^3_k} \otimes \mathcal{L} \otimes \mathcal{O}_X,$$

where $\mathcal{L}$ is the invertible sheaf associated to X. We have $\mathcal{L} \cong \mathcal{O}(4)$, by the proof above. Now, $\omega_{\mathbb{P}^3_k} \cong \mathcal{O}(-4)$ for Corollary 4.53, so

$$\omega_X \cong \mathcal{O}(-4) \otimes \mathcal{O}(4) \otimes \mathcal{O}_X \cong \mathcal{O}_X.$$

□

Nonsingular Complete Intersections

Definition 5.3. Let X be a closed subscheme in $\mathbb{P}^n_k$, with $\mathfrak{a} \subset k[x_0, ..., x_n]$ as homogeneous ideal associated (Corollary 2.75). We say that X is a *complete intersection* if $\mathfrak{a}$ can be generated by $r = \text{codim}(X, \mathbb{P}^n_k)$ elements.

In other words, X is a complete intersection if it is the intersection of r hypersurfaces of $\mathbb{P}^n_k$, that is if there exist r homogeneous polynomials $f_1, ..., f_r$ such that $X = V(f_1, ..., f_r) = \text{Proj}(k[x_0, ...x_n]/(f_1, ..., f_r))$. If $d_i = \deg f_i$ for $i = 1, ..., r$, we say that X is a *complete intersection of type* $(d_1, ..., d_r)$.

Example 5.4. We are interested in nonsingular complete intersections. As trivial example we can consider $V(f)$, where f is an irreducible homogeneous polynomial.

Theorem 5.5. *Let X be a nonsingular complete intersection of type $(d_1, ..., d_n)$ in $\mathbb{P} := \mathbb{P}^{n+2}_k$. Then, X is a K3 surface if and only if $\sum_i d_i = n + 3$.*

We proceed again by steps.

Lemma 1. X is a nonsingular complete variety over k of dimension 2.

Proof. Clear. □

Lemma 2. Let X be a complete intersection of type $(d_1, ..., d_n)$ in $\mathbb{P}^m_k$, with $m \geq 2$. Hence X has dimension $q = m - n$. Then, $H^i(X, \mathcal{O}_X(l)) = 0$ for any $l \in \mathbb{Z}$ and for any $i = 1, ..., q$. In particular, for $l = 0, q = 2$ and $i = 1$ we have $H^1(X, \mathcal{O}_X) = 0$.

Proof. In a similar way as in Lemma 2 we will need a short exact sequence of sheaves to get a long exact sequence of cohomology.

Let us prove the Lemma by induction on n. If $n = 0$, then $X = \mathbb{P}^m_k$, so the claim follows by 3.41.

Now, we assume the theorem for $n - 1$ and let Y be the scheme $Y = \text{Proj}(k[x_0, ..., x_m]/(f_1, ..., f_{n-1}))$, where $f_1, ..., f_n$ are homogeneous polynomials of degree $d_1, ..., d_n$, such that $X = \text{Proj}(k[x_0, ..., x_m]/(f_1, ..., f_n))$. Then X is a closed subscheme of Y. By 4.31 (applied on a generic scheme), the ideal sheaf of X in Y is $\mathcal{L}(-D)$, where D is the effective Cartier divisor associated to X (indeed $\operatorname{codim}(X,Y)=1$). Similarly to the proof of 4.29, we have $\mathcal{L}(-D)=\mathcal{O}_Y(-d_n)$. By 3.10 there is a short exact sequence of sheaves of modules

$$0 \to \mathcal{O}_Y(-d_n) \to \mathcal{O}_Y \to \mathcal{O}_X \to 0. \tag{5.1}$$

For any $l \in \mathbb{Z}$, tensoring (5.1) by $\mathcal{O}_Y(l)$ we have a sequence

$$0 \to \mathcal{O}_Y(-d_n+l) \to \mathcal{O}_Y(l) \to \mathcal{O}_X(l) \to 0,$$

which is again exact by Proposition 3.23. This exact sequence yields a long exact sequence of cohomology

$$\ldots \to H^i(Y,\mathcal{O}_Y(l)) \to H^i(X,\mathcal{O}_X(l)) \to H^{i+1}(Y,\mathcal{O}_Y(-d_n+l)) \to \ldots$$

for any $i=1,\ldots,m-n-1$. Since $\dim Y = \dim X+1$, by inductive assumption we have $H^i(Y,\mathcal{O}_Y(l)) = H^{i+1}(Y,\mathcal{O}_Y(-d_n+l)) = 0$ and so $H^i(X,\mathcal{O}_X(l))$. □

Lemma 3. The canonical sheaf of X in $\mathbb{P}=\mathbb{P}_k^{n+2}$ is

$$\omega_X \cong \mathcal{O}_X\left(-n-3+\sum_{i=1}^{n} d_i\right).$$

Proof. Let $X=\operatorname{Proj}(k[x_0,\ldots,x_{n+2}]/(f_1,\ldots,f_n))$, where f_i is a homogeneous polynomial of degree d_i for every i. Let $X_i \coloneqq \operatorname{Proj}(k[x_0,\ldots,x_{n+2}]/(f_1,\ldots,f_i))$ for any i, with $X_n=X$.

The hypersurface X_1 is a Weil divisor of degree d_1 in $\mathbb{P}_k^{n+2}$ and by the Adjunction Formula for divisors 4.51 we have $\omega_{X_1} \cong \omega_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(d_1) \otimes \mathcal{O}_{X_1} \cong \mathcal{O}_{X_1}(-n-3+d_1)$.

Now, X_2 is a hypersurface of X_1 of degree d_2, hence, again by 4.51, $\omega_{X_2} \cong \omega_{X_1} \otimes \mathcal{O}_{X_1}(d_2) \otimes \mathcal{O}_{X_2} \cong \mathcal{O}_{X_2}(-n-3+d_1+d_2)$. Repeating this procedure for any i we have the claim. □

Now we can give the proof of the theorem.

Proof of the theorem. We have seen that any nonsingular complete intersection X of type $(d_1, ..., d_n)$ in $\mathbb{P}_k^{n+2}$ is a nonsingular complete variety of dimension 2 with $H^1(X, \mathcal{O}_X) = 0$. By the previous Lemma, X is a K3 surface if and only if $\sum_i d_i = n + 3$. □

Remark 5.6. We observe that if there exists i such that $d_i = 1$, then our surface X lies on a hyperplane isomorphic to an affine space, hence it's not an interesting case. If we suppose $d_i > 1$ for any i, then we have a few of possible chance to obtain a K3 surface (we suppose $d_1 \geq ... \geq d_n$):

i) $n = 1$ and $d_1 = 4$, that is X is a quartic in $\mathbb{P}_k^3$ (the previous example);

ii) $n = 2$ and $d_1 = 3$ and $d_2 = 2$ (or $d_1 = 2$ and $d_2 = 3$). These are K3 surfaces of degree 6 in $\mathbb{P}_k^4$;

iii) $n = 3$ and $d_1 = d_2 = d_3 = 2$, K3 surfaces of degree 8 in $\mathbb{P}_k^5$.

Other examples

It is hard to give explicit descriptions of many others K3 surfaces with the notions exposed in this elaborate. We mention the following examples, which are presented in [8, Chapters 1.1 and 1.4].

1. Let A be an abelian surface on k, that is a surface with a structure of group. Let us consider the involution $\iota \colon A \to A$ given by $x \mapsto -x$. Then, the minimal resolution $X \to A/\iota$ of A/ι is a K3 surface. This kind of K3 surface is called *Kummer surface*.

2. Let C be a nonsingular curve of degree 6 on $\mathbb{P}^2$. Then, a double covering $\pi \colon X \to \mathbb{P}^2$ branched along C is a K3 surface.

3. Let X be a hypersuface of $\mathbb{P}^r \times \mathbb{P}^s$. We say X is of type (p, q) if the projection of X on $\mathbb{P}^r$ (resp. $\mathbb{P}^s$) is a hypersurface of degree p (resp. q). Any nonsingular hypersurfaces X of $\mathbb{P}^2 \times \mathbb{P}^1$ of type $(3, 2)$ is a K3 surface. Any nonsingular hypersurfaces X of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of type $(2, 2, 2)$ is a K3 surface.

5.2 Cohomology of K3 Surfaces

In this section we want to compute some cohomology group of K3 surfaces.

Definition 5.7. Let X be a K3 surface. Let us consider the cotangent sheaf $\Omega_X = \Omega_{X/k}$ of X over k. For any $p \in \mathbb{N}$ we set $\Omega_X^p := \bigwedge^p \Omega_X$, with $\Omega_X^0 = \mathcal{O}_X$ and $\Omega_X^2 = \omega_X$.

We call *Hodge number* of X each non-negative integer $h^{p,q}(X)$ which are by definition the dimension over k of the k-vector space $H^q(X, \Omega^p)$.

Since X has dimension 2, $h^{p,q}(X) = 0$ if $p > 2$ or $q > 2$. We call *Hodge diamond* of X the diagram

$$\begin{array}{ccccc} & & h^{0,0}(X) & & \\ & h^{1,0}(X) & & h^{0,1}(X) & \\ h^{2,0}(X) & & h^{1,1}(X) & & h^{0,2}(X) \\ & h^{2,1}(X) & & h^{1,2}(X) & \\ & & h^{2,2}(X) & & \end{array} \tag{5.2}$$

Our aim is to compute (5.2). Moreover, we will prove that any K3 surface has the same Hodge diamond.

i) By 2.94 we know that $\Gamma(X, \mathcal{O}_X) = k$. Since $H^0(X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$ (Theorem 3.34, *ii*)), it follows that $h^{0,0}(X) = 1$.

ii) By definition of K3 surface, $\omega_X \cong \mathcal{O}_X$, so $H^0(X, \omega_X) = H^0(X, \mathcal{O}_X) = k$. Hence $h^{2,0}(X) = 1$.

iii) By definition of K3 surface we have $H^1(X, \mathcal{O}_X) = 0$, so $h^{0,1}(X) = 0$.

iv) We have that $h^{1,0}(X) = 0$. See [8, Chapter 1.3] for details.

To compute the remaining Hodge numbers we need the Serre's Duality Theorem. First, we show that every K3 surface is a projective scheme.

Definition 5.8. Let X be a scheme over k. We say that X is a *projective scheme* if there exists a closed immersion $X \longrightarrow \mathbb{P}_k^n$, for some $n \in \mathbb{N}$.

Theorem 5.9. *Let X be a nonsingular complete surface over k. Then X is projective.*

Proof. See [6, Chapter II.4]. □

Corollary 5.10. *Any K3 surface is a projective scheme.*

Theorem 5.11 (Serre's Duality for a Nonsingular Projective Variety)**.** *Let X be a nonsingular projective variety of dimension n. For any $p, q = 0, ..., n$, we have*

$$H^q(X, \Omega_X^p) \cong H^{n-q}(X, \Omega_X^{n-p})^\vee. \tag{5.3}$$

Proof. See [6, Chapter III.7]. □

Remark 5.12. We know $h^{0,0}(X) = h^{0,2}(X) = 1$ and $h^{0,1}(X) = h^{1,0}(X) = 0$.

By Serre's Duality we have (since $k^\vee \cong k$):

i) $h^{2,2}(X) = h^{0,0}(X) = 1$;

ii) $h^{2,0}(X) = h^{0,2}(X) = 1$;

iii) $h^{2,1}(X) = h^{0,1}(X) = 0$;

iv) $h^{1,2}(X) = h^{1,0}(X) = 0$.

Now we compute $h^{1,1}(X)$.

Definition 5.13. Let X be a projective scheme over k and let $\mathcal{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathcal{F}$ to be

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} \dim_k H^i(X, \mathcal{F}).$$

Example 5.14. For the calculations above, we can say that for any K3 surface X, the Euler characteristic of $\mathcal{O}_X$ is $\chi(X, \mathcal{O}_X) = 1 - 0 + 1 = 2$.

Now we mention two direct Corollaries of the Hirzebruch-Riemann-Roch formula applied in this very particular situation. We use the formula only in the case of K3 surfaces, omitting the definion of Chern classes. The general version of the formula can be found in [6, Appendix A.4].

Theorem 5.15. *Let X be a K3 surface. Then*

$$\chi(X, \mathcal{O}_X) = \frac{c_2(X)}{12},$$

where $c_2(X)$ is the second Chern class of X. See [6, Appendix A.3] for the definition of Chern class.

Theorem 5.16. *Let X be a K3 surface and let us consider the cotangent bundle Ω_X. Then* $\dim_k H^1(X, \Omega_X) = c_2(X) - 4$.

Remark 5.17. Since $\chi(X, \mathcal{O}_X) = 2$, then $c_2(X) = 24$ by 5.15 and $h^{1,1}(X) = 20$ by 5.16.

Corollary 5.18. *Let X be a K3 surface. Then, the Hodge diamond of X is*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{5.4}$$

Corollary 5.19. *Every K3 surface has the same Hodge diamond (5.4).*

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