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THE CUT LOCUS OF A C^2 SURFACE IN THE HEISENBERG GROUP

Tesi di Laurea Magistrale in Analisi Matematica

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Introduzione

La mia tesi magistrale affronta alcune importanti proprietà relative alla distanza (naturale) di Carnot-Carathéodory nel gruppo di Heisenberg. In particolare vogliamo studiare quelle proprietà che riguardano il cut locus di una superficie liscia. Il cut locus di un sottoinsieme chiuso S in $\mathbb{R}^n$, denotato con $cut(S)$, è l'insieme dei punti finali dei segmenti massimali che realizzano la distanza da S (in altre parole, se si prende un segmento la cui lunghezza misura localmente la distanza da S, il punto dove tale segmento cessa di realizzare la distanza da S sta in $cut(S)$). In ambito Euclideo si conoscono già molte proprietà del cut locus quando S è il bordo di un aperto di $\mathbb{R}^n$ di classe C^2. Per esempio è noto che in questo caso $cut(S)$ è un insieme chiuso. Tali proprietà continuano a valere se si rimpiazza la metrica Euclidea con una qualsiasi metrica Riemanniana con coefficienti di classe C^2 (si vedano [1] e [2]).

Alcuni problemi riguardanti il cut locus che sono stati risolti in ambito Riemanniano, restano tuttavia aperti in geometria sub-Riemanniana. Il nostro obiettivo è quindi quello di studiare il cut locus nel gruppo di Heisenberg, che è la struttura sub-Riemanniana più elementare che esista. In particolare, stiamo cercando di provare che i cut loci di superfici che sono il bordo C^2 di sottoinsiemi aperti del gruppo di Heisenberg, sono insiemi chiusi. La scelta di una regolarità di classe C^2 viene fatta per analogia con il caso Riemanniano, dove è già stato trovato un controesempio per superfici di classe $C^{1,1}$ (si veda [11]).

Attualmente non siamo in grado di fornire una dimostrazione della chiusura del cut locus. Per questo motivo il nostro lavoro mira a trovare nuove proprietà della distanza di Carnot nel gruppo di Heisenberg, le quali pensiamo possano essere correlate al cut locus e che quindi possano risultare utili per arrivare alla dimostrazione finale. In particolare stiamo studiando i cosiddetti punti coniugati ad una superficie S di classe C^2, i quali presentano dei legami con il cut locus anche in campo Riemanniano. Alla fine di questo lavoro daremo alcuni piccoli ma nuovi risultati proprio riguardo i punti coniugati.

Questi sono gli argomenti che tratteremo nell'ultimo capitolo di questa tesi. D'altra parte avremo prima bisogno di presentare una teoria approfondita del gruppo di Heisenberg, partendo dalla definizione fino ad arrivare ai campi di Jacobi. Più precisamente nel primo capitolo, dopo aver definito il gruppo di Heisenberg, descriveremo la sua struttura di Lie e, soprattutto, parleremo della sua naturale metrica sub-Riemanniana. Tale metrica, denotata con g, induce esattamente la distanza di Carnot-Carathéodory (si vedano [6] e [12]).

Nel secondo capitolo continueremo descrivendo le curve di minima lunghezza relative alla metrica g, le cosiddette geodetiche (si vedano [12], [15] and [17]). Inoltre introdurremo la struttura pseudo-Hermitiana del gruppo di Heisenberg (si veda [18]), la quale ci consentirà di introdurre i campi relativi alle variazioni di geodetiche, ovvero i campi di Jacobi (si vedano [15] e [17]).

Infine il terzo capitolo, come già abbondantemente detto sopra, concluderà la tesi.

Introduction

My master thesis deals with some fine properties of the natural Carnot-Carathéodory distance in the Heisenberg group. In particular, we are interested in properties related to the cut locus of a smooth surface. The cut locus of a closed subset S of $\mathbb{R}^n$, denoted by $cut(S)$, is the set containing the endpoints of maximal segments that minimize the distance to S (i.e. if a segment starting at S locally minimize the distance from S, then the last distance-minimizing point of such segment is a point of $cut(S)$). In the Euclidean case many properties of $cut(S)$ are well known if S is the C^2 boundary of an open set of $\mathbb{R}^n$. For example the cut loci of such surfaces are closed (this is the fact we will be most interested in). Such results are still valid if the Euclidean metric is replaced with any Riemannian metric with coefficients of class C^2 (see [1], [2]).

Not all the known properties of the cut locus in Riemannian geometry are also known to hold in the sub-Riemannian case. So our goal is to generalize and prove some of them in the Heisenberg group, which has the simplest sub-Riemannian structure. In particular we are very interested in proving that the cut loci of surfaces which are the C^2 boundary of open sets in the Heisenberg group are closed sets. We choose a regularity of class C^2 because there is already a counterexample in the Riemannian $C^{1,1}$ case (see [11]).

At the moment we are not able to give a proof of the closure of the cut locus. So we are looking for new properties of the Carnot distance which may be related with the cut locus of a C^2 surface in the Heisenberg group and that may be useful to prove its closure. Precisely, we are investigating in points conjugate to the surface S, since they are strictly connected with the cut locus in the Riemannian case. Just about conjugate points, we will show some new small results at the end of this work.

These topics will be discussed in the last chapter. Before, we will present an in-depth theory of the Heisenberg group, starting from the definition up to talking about the Jacobi fields. Precisely, in the first chapter we will describe the Heisenberg group, its Lie structure and, above all, its natural sub-Riemannian metric. This metric, denoted with g, exactly induces the natural Carnot distance mentioned at the beginning of this paragraph (see [6] and [12]).

In the second chapter we will continue by describing the length-minimizing curves relating to the metric g, that is the so-called geodesics (see [12], [15] and [17]). Moreover, we will will introduce the pseudohermitian structure of the Heisenberg group (see [18]). This will allow us to describe the vector fields related to the variations of geodesics, that is the afromentioned Jacobi fields (see [15] and [17]).

Then the third chapter will conclude the thesis, as widely stated above.

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Chapter 1

An introduction to the Heisenberg group

In this chapter we will describe the main topics about the Heisenberg group. The principal references for such arguments are [6] and [12]. Let us start giving the definition of the Heisenberg group, that is

Definition 1.1. The Heisenberg group $\mathbb{H}$ is the unique analytic, nilpotent Lie group whose background manifold is $\mathbb{R}^3$ and whose Lie algebra $\mathfrak{h}$ has the following properties:

1. $\mathfrak{h} = V_1 \oplus V_2$, where V_1 has dimension 2 and V_2 has dimension 1

2. $[V_1, V_1] = V_2$, $[V_1, V_2] = 0$ and $[V_2, V_2] = 0$

Definition 1.1 is well posed thanks to the following

Proposition 1.2. *Let G be a simply connected, nilpotent Lie group and let $\mathfrak{g}$ be the Lie algebra of G. Then the exponential map* $\exp : \mathfrak{g} \to G$ *is a (global) diffeomorphism.*

So $\mathbb{H}$ is uniquely described by his Lie algebra. There are many ways of realizing this abstract structure. Now we describe one of them, that is the one which we refer to. First we need to recall the Baker-Campbell-Hausdorff formula for Lie algebras

Theorem 1.3 (Baker-Campbell-Hausdorff formula)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $X, Y \in \mathfrak{g}$. Then*

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]) + \dots) \tag{1.1}$$

For the complete formula and for a complete proof see [6].

Example 1.4. Using the notation of Definition 1.1, we fix an arbitrary basis X and Y of V_1 and we consider $T = [X, Y] \in V_2$. Then a generic point $P \in \mathfrak{h}$ has the form $P = xX + yY + tT$, $x, y, t \in \mathbb{R}$. Now let $P = x_1X + y_1Y + tT$ and $Q = x_2X + y_2Y + sT$ be two points in $\mathfrak{h}$. Using the Baker-Campbell-Hausdorff formula we are able to compute

$$\exp(P)\exp(Q) = \exp(P + Q + \frac{1}{2}[P, Q])$$

The formula is stopped at the second order because, by definition, $\mathfrak{h}$ is nilpotent of step two. Then the successive brackets are all equal to 0. Thanks to the fact that $T = [X, Y]$ we have

$$P + Q + \frac{1}{2}[P, Q] = (x_1 + x_2)X + (y_1 + y_2)Y + (t + s + \frac{1}{2}(x_1y_2 - x_2y_1))$$

From these facts we have a natural representation for $\mathbb{H}$: we identify $\mathbb{H} = \mathbb{C} \times \mathbb{R}$ through the exponential map, mapping a point $P \in \mathfrak{g}$ in the point $(z, t) = (x + iy, t) \in \mathbb{C} \times \mathbb{R}$. So it is immediate that the group law, denoted by $\cdot$, between two points $(z, t), (w, s) \in \mathbb{C} \times \mathbb{R}$ is given by

$$(z, t) \cdot (w, s) = (z + w, t + s - \frac{1}{2}Im(z\overline{w})) \tag{1.2}$$

We can also obviously identify $\mathbb{H} = (\mathbb{R}^3, \cdot)$. Finally the explicit representation of $\mathfrak{h}$ (and then of $\mathbb{H}$) is given by the left invariant vector fields

$$X = \partial_x - \frac{1}{2}y\partial_t, \quad Y = \partial_y + \frac{1}{2}x\partial_t \tag{1.3}$$

As a result we found $T = [X, Y] = \partial_t$ and so $V_1 = \text{span}\{X, Y\}$, $V_1 = \text{span}\{T\}$, $\mathfrak{h} = V_1 \oplus V_2$. To see the left invariance of X it is sufficient to compute:

$$L_Q(P) = QP = (x_2, y_2, s)(x_1, y_1, t) = (x_2 + x_1, y_2 + y_1, s + t + \frac{1}{2}(y_1 x_2 - x_1 y_2))$$

$$dL_Q = (\partial_a (QP)_j)_{a=x_1,y_1,t;\ j=1,2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}y_2 & \frac{1}{2}x_2 & 1 \end{pmatrix}$$

Now, by the standard identification between vector fields and points in $\mathbb{R}^3$, we have

$$X_P = (1, 0, -\frac{1}{2}y_1)$$
$$X_Q P = (1, 0, -\frac{1}{2}(y_2 + y_1)) = dL_Q X_P$$

The check is similar both for Y and T.

Remark 1.5. From now on we make a sligth abuse of notation by denoting with P both the point in $\mathbb{H}$ and the corresponding point $\exp^{-1}(P) \in \mathfrak{h}$. Sometimes it might also be useful to write a point $P = (x, y, t) = (x + iy, t) = (z, t)$ also as $x = (x_1, x_2, x_3) = (x_1 + ix_2, x_3) = (z, x_3)$. In this case the vector fields X, Y, T are replaced respectively by $X_1 = \partial_{x_1} - \frac{1}{2}x_2\partial_{x_3}$, $X_2 = \partial_{x_2} + \frac{1}{2}x_1\partial_{x_3}$ and $X_3 = \partial_{x_3}$. This change of notation will be adopted, for example, when the variable t will have to be used for denoting the time.

Example 1.6 (Polarizied representation)**.** Another way of seeing the Heisenberg group is to identify

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R}),\ x, y, t \in \mathbb{R} \right\} \tag{1.4}$$

as the subgroup of 3×3 reversal real matrix which have ones on the diagonal. Here the group law is the matrix product so, if we identify an element of $\mathbb{H}$ with the triplet (x, y, t), the group product becomes

$$(x_1, y_1, t)(x_2, y_2, s) = (x_1 + x_2, y_1 + y_2, t + s + x_1 y_2) \tag{1.5}$$

The tangent spaces of $\mathbb{H}$ can be described as follows: first consider left translation of a fixed point $P = (x, y, t)$ by one parameter families of matrices to form curves in $\mathbb{H}$, then take derivatives along those curves to determine the tangent vectors at the point P, that is

$$\begin{aligned} X_P &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\varepsilon, 0, 0)(x, y, t) = (1, 0, y) \\ Y_P &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (0, \varepsilon, 0)(x, y, t) = (0, 1, 0) \\ T_P &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (0, 0, \varepsilon)(x, y, t) = (0, 0, 1) \end{aligned} \tag{1.6}$$

It easy to see that $[X_P, Y_P] = T_P$, while all the other brackets are zero. This fact suggest us the explicit representation of $\mathfrak{h}$ in this case:

$$X = \partial_x + y\partial_t, \qquad Y = \partial_y, \qquad T = [X, Y] = \partial_t \tag{1.7}$$

A point $P = (x, y, t)$ can also be written as

$$\exp(tT_I \exp(yY_I)\exp(xX_I))$$

where exp in this case denoted the exponential of a matrix.

Remark 1.7. The representation of $\mathbb{H}$ given in Example 1.4 is called standard representation or representation of the first kind, while the one discussed in Example 1.6 is called polarizied representation or representation of the second kind. A canonical isomorphism between the two models is given by mapping a standard point $P_s = (x, y, t)$ onto the polarizied point $P_p = (x, y, t + \frac{1}{2}xy)$. From now on the model to which we refer is the standard one.

Remark 1.8 (!). Other infinite expicit representations of the Heisenberg group can be found as in Example 1.4. Indeed, it is sufficient to modify both the group product as

$$(z, t) \cdot (w, s) = (z + w, t + s - \alpha Im(z\overline{w})), \quad \alpha \in \mathbb{R} \tag{1.8}$$

and the left invariant vector fields X and Y in

$$X = \partial_x - \alpha y\partial_t, \quad Y = \partial_y + \alpha x\partial_t, \quad \alpha \in \mathbb{R} \tag{1.9}$$

depending on a real parameter α. As a result we found

$$T = [X, Y] = 2\alpha\partial_t \tag{1.10}$$

These representations are called standard representations of parameter α. The most used in literature are those with $\alpha = \pm 1/2, \pm 1, \pm 2$. For example, [6] use $\alpha = 1/2$, [15] have chosen $\alpha = -1$.

For geometric reasons we keep $\alpha = 1/2$ in the first two Chapters, while in the third one we will use $\alpha = -2$, according to [3].

We give now other elementary definitions and properties of the Heisenberg group:

i) The group identity is $0 = (0, 0, 0)$, while the group reverse is $P^{-1} = (-x, -y, -t)$.

ii) The Haar misure of $\mathbb{H}$ is simply the Lebesgue misure in $\mathbb{R}^3$ and this follows from the fact that the Euclidean volume form $dx \wedge dy \wedge dt$ is invariant under pull-back via left translation, that is $(L_P)^*(dx \wedge dy \wedge dt) = dx \wedge dy \wedge dt$.

iii) The group has a homogeneous structure given by the dilations $\delta_\lambda(P) = (\lambda x, \lambda y, \lambda^2 t)$, $\lambda > 0$, as a metter of fact $(\delta_\lambda)^*(dx \wedge dy \wedge dt) = \lambda^4 dx \wedge dy \wedge dt$. So the homogeneous dimension of $\mathbb{H}$ with respect to the group dilatations δ_λ is 4.

iv) The vector fields X and Y are left invariant, first-order differential operators, homogeneous of order 1 with respect to the dilations δ_λ.

Definition 1.9. The subbundle generated by the left invariant frame X, Y of the tangent bundle $T\mathbb{H}$ of $\mathbb{H}$ is called horizontal bundle. We denote it with $\mathcal{H}\mathbb{H}$ or simply with $\mathcal{H}$. Every frame given by a linear combination of X and Y is called horizontal and every frame having no component neither long X and long Y is called vertical. Similarly the subspace $\mathcal{H}_P := \text{span}\{X_P, Y_P\}$ of the tangent space $T_P\mathbb{H}$ is called horizontal section at the point P. Every vector in $\mathcal{H}_P$ is called horizontal and every vector having no components in $\mathcal{H}_P$ is called vertical.

Remark 1.10. The horizontal section at a point $P = (x, y, t)$ is at the same time also the kernel of a differential 1-form, that is of the 1-form $\omega = dt - \frac{1}{2}(xdy - ydx)$. The fact that both X and Y belong to the kernel is a simple check, whilw $\omega(T) = \omega(\partial_t) = 1$. So we have $\ker(\omega) = \text{span}\{X, Y\}$.

Definition 1.11. Let Ω be an open set of $\mathbb{H}$ and let $\phi : \Omega \to \mathbb{R}$ be a C^1 function. We call $\nabla_{\mathbb{H}}\phi \coloneqq (X\phi)X + (Y\phi)Y$ the horizontal gradient of ϕ.

1.1 The Carnot-Carathéodory (or CC) distance

In this section we want to discuss the natural metric structure with which we will equip the Heisenberg group. It is the so-called Carnot-Carathéodory metric and there are three equivalent ways to define such metric. Let us start by defining the concept of horizontal and $C(\delta)$ paths

Definition 1.12. An absolutely continuous path $\gamma : [0, 1] \to \mathbb{H}$ is called horizontal if there exist measurable functions $a, b : [0, 1] \to \mathbb{R}$ such that

$$\dot{\gamma}(t) = a(t)(X_1)_{\gamma(t)} + b(t)(X_2)_{\gamma(t)}, \quad \text{a.e in } [0, 1] \tag{1.11}$$

Now we want to write explicitly (1.11), so consider $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. Since $X_1 = (1, 0, -\frac{1}{2}x_2)$, $X_2 = (0, 1, \frac{1}{2}x_1)$, we have

$$X_1(\gamma) = (1, 0, -\frac{1}{2}\gamma_2), \quad X_2(\gamma) = (1, 0, \frac{1}{2}\gamma_1)$$

and then (1.11) becomes

$$\dot{\gamma}(t) = (a(t), b(t), -\frac{1}{2}(a(t)\gamma_2(t) - b(t)\gamma_1(t))) \tag{1.12}$$

that is

$$\dot{\gamma_3} = -\frac{1}{2}(\dot{\gamma}_1\gamma_2 - \dot{\gamma}_2\gamma_1) \tag{1.13}$$

So the following definition is equivalent to the 1.12 one.

Definition 1.13. An absolutely continuous path $\gamma : [0,1] \to \mathbb{H}$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ is called horizontal if

$$\omega_\gamma(\dot\gamma) = \dot\gamma_3 - \frac{1}{2}(\dot\gamma_2\gamma_1 - \dot\gamma_1\gamma_2) = 0, \quad \text{a.e. in } [0,1], \tag{1.14}$$

where $\omega = dx_3 - \frac{1}{2}(x_1dx_2 - x_2dx_1)$.

Definition 1.14. For $\delta > 0$ we say that an horizontal path γ as in definition 1.12 is of class $C(\delta)$ if $a(t)^2 + b(t)^2 \leq \delta^2$, a.e in $[0,1]$. The set of all paths of class $C(\delta)$ joining two fixed points $x, y \in \mathbb{H}$ is denoted by $C_{(x,y)}(\delta)$

Naturally, these definitions remain the same for every path γ defined in a generical interval $[a,b] \subset \mathbb{R}$. If it is necessary, to avoide mistakes, we will write $C_{(x,y)}(\delta, [a,b])$ to specify the interval of definition for the curves of class $C(\delta)$ joining two points x and y.

Theorem 1.15 (Chow's Theorem)**.** *Every pair of points in the Heisenberg group can be connected by an horizontal curve.*

Proof. Definition 1.13 gives a natural way to lift every planar curve in $\mathbb{C}$ to an horizontal path in $\mathbb{H}$. Let α be a planar curve $\alpha : [0,1] \to \mathbb{C}$ and let $\pi : \mathbb{H} \to \mathbb{C}$ the canonical projection. We want a (continuous) path $\gamma : [0,1] \to \mathbb{H}$ such that $\pi(\gamma) = \alpha$. Such a curve is given by placing $\gamma_i = \alpha_i$ for $i = 1, 2$ and

$$\gamma_3(t) = \gamma_3(0) - \frac{1}{2}\int_0^t (\dot\alpha_1\alpha_2 - \dot\alpha_2\alpha_1)(s)ds, \quad h \in \mathbb{R},\ t \in [0,1] \tag{1.15}$$

The geometric meaning of (1.15) is the key to complete the proof. Indeed, according to Stokes' Theorem, we have

$$-\frac{1}{2}\int_0^1 (\dot\alpha_1\alpha_2 - \dot\alpha_2\alpha_1)(s)ds = -\frac{1}{2}\int_{\tilde\alpha} x_2dx_1 - x_1dx_2 = \int_S dx_1 \wedge dx_2 = \text{Area}(S)$$

where $\tilde\alpha$ is the closed planar curve obtained by travelling α and the segment joining $\alpha(1)$ and $\alpha(0)$ and S is the region of the plane bounded by $\tilde\alpha$.

Now, we want to join two arbitrary points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ of $\mathbb{H}$ with an horizontal path. To do this, take a planar curve

$\alpha : [0,1] \to \mathbb{C}$ joining (x_1, x_2) and (y_1, y_2) in the plane such that the region S defined as above has signed Area equal to $y_3 - x_3$. Thus, the horizontal curve $\gamma : [0,1] \to \mathbb{H}$ starting at x and given by (1.15) prove the statement, since $\gamma_3(1) = x_3 + \text{Area}(S) = y_3$. $\square$

As a consequence we have that the set $C_{(x,y)}(\delta)$ is non empty if δ is large enough. This fact allow us to define the CC metric

Definition 1.16. Let $x, y \in \mathbb{H}$. We define the Carnot-Carathéodory (or CC) distance between x and y as

$$d_{cc}(x,y) = \inf\{\delta > 0 | C_{(x,y)}(\delta) \neq \emptyset\} \tag{1.16}$$

We denote it by $d_{cc}(x,y)$ to avoide ambiguity with the Euclidean distance, which will be denoted by $d_e(x,y)$ (but when it is clear what distance we refer to, we denote both of them simply with $d(x,y)$)

An equivalent definition of the CC distance can be obtained as follows: consider the $C_{(x,y)}(1)$ paths defined on $[0,T]$, $T > 0$, i.e. the absolutely continuous horizontal paths $\gamma : [0,T] \to \mathbb{H}$, such that $a^2 + b^2 \leq 1$ a.e. Then take the infimum on $T > 0$, that is

$$d_{cc}(x,y) = \inf\{T > 0 | \exists \gamma : [0,T] \to \mathbb{H},\ \gamma \in C_{(x,y)}(1, [0,T])\} \tag{1.17}$$

In other words $d_{cc}(x,y)$ is the shortest time that it takes to go from x to y, travelling at unit speed along horizontal paths.

Proof of the equality of the two distances. Let $\delta > 0$ be such that $C_{(x,y)}(\delta) \neq \emptyset$ and $\gamma \in C_{(x,y)}(\delta)$. Then just taking $\sigma = s/\delta$ in (1.15) we obtain

$$\gamma_3(t) = h - \frac{1}{2\delta}\int_0^t (\dot\gamma_1\gamma_2 - \dot\gamma_2\gamma_1)(\sigma)d\sigma = h - \frac{1}{2}\int_0^t (\frac{\dot\gamma_1}{\delta}\gamma_2 - \frac{\dot\gamma_2}{\delta}\gamma_1)(\sigma)d\sigma \tag{1.18}$$

where $t \in [0, 1/\delta]$. Then, if $T = 1/\delta$, the path $\eta : [0,T] \to \mathbb{H}$ defined by $\eta(\sigma) := \frac{1}{\delta}\gamma(\delta\sigma)$ is in $C_{(x,y)}(1, [0,T])$, since

$$\dot\eta_1^2 + \dot\eta_2^2 = \left(\frac{\dot\gamma_1(s)}{\delta}\right)^2 + \left(\frac{\dot\gamma_1(s)}{\delta}\right)^2 = \frac{\dot\gamma_1(s)^2 + \dot\gamma_2(s)^2}{\delta^2} \leq 1$$

$\square$

Proposition 1.17. *Let d be the CC distance and $x, y \in \mathbb{H}$. Then*

(i) $d(x, y) = d(y^{-1}x, 0)$, *that is the Carnot-Carathéodory metric is left-invariant.*

(ii) $d(\delta_s(x), \delta_s(y)) = sd(x, y)$, $\forall s > 0$, *i.e. d is homogeneous of degree 1 w.r.t. the group dilatations.*

(iii) The function $x \mapsto d(x, 0)$ is continuous in the following sense: let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that

$$\forall \varepsilon > 0\ \exists n_\varepsilon \in \mathbb{N} : x_n \in \delta_\varepsilon(B_1(0)),\ \text{if } n > n_\varepsilon,$$

with $B_1(0)$ the Euclidean unitary ball. Then $d(x_n, 0) \to 0$, $n \to \infty$.

Proof. (i) Since the vector fields X_1 and X_2 are left invariant, left translates of horizontal curves are still horizontal and then $C_{(x,y)}(\delta) = C_{(y^{-1}x,0)}(\delta)$.

(ii) First we prove that if a path γ is horizontal, then also $\delta_s(\gamma)$ is. Indeed let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be horizontal, then $\dot\gamma = (\dot\gamma_1, \dot\gamma_2, \dot\gamma_3)$ with $\dot\gamma_3 = -\frac{1}{2}(\dot\gamma_1\gamma_2 - \dot\gamma_2\gamma_1)$. So
$$\frac{d}{dt}(\delta_s(\gamma_3)) = -s^2\frac{1}{2}(\dot\gamma_1\gamma_2 - \dot\gamma_2\gamma_1) = -\frac{1}{2}(s\dot\gamma_1 s\gamma_2 - s\dot\gamma_2 s\gamma_1) = -\frac{1}{2}(\delta_s(\dot\gamma_1)\delta_s(\gamma_2) - \delta_s(\dot\gamma_2)\delta_s(\gamma_1))$$
shows that $\delta_s(\gamma)$ is horizontal. Moreover if $\gamma \in C_{(x,y)}(\delta)$, then $\delta_s(\gamma) \in C_{(x,y)}(s\delta)$ and the endpoints must be dilatated as well. This prove the second statement.

(iii) It is a direct consequence of (ii).

□

An equivalent metric of the CC one is the so-called Korányi metric, which is given in the following

Definition 1.18. Let $x, y \in \mathbb{H}$. We define the Korányi norm as

$$\|x\|_0^4 := (x_1^2 + x_2^2)^2 + 16x_3^2 \tag{1.19}$$

and the Korányi distance between x and y as

$$d_0(x, y) := \|y^{-1}x\|_0 \tag{1.20}$$

Proposition 1.19. *The distance d_0 is equivalent to the d_{cc} one defined in* (1.17).

Proof. To prove the triangular inequality for $\|\cdot\|_0$ write $x = (z, x_3)$, $y = (w, y_3)$ and compute

$$\begin{aligned}\|xy\|_0^4 &= |z+w|^4 + 16(x_3 + y_3 - \frac{1}{2}\mathrm{Im}(z\overline{w}))^2 \\ &= \left| |z+w|^2 + 4i(x_3 + y_3 - \frac{1}{2}\mathrm{Im}(z\overline{w})) \right|^2 \\ &= \left| |z|^2 + |w|^2 + 2\mathrm{Im}(z\overline{w}) + 4i(x_3 + y_3) \right|^2 \\ &\leq (\|x\|_0^2 + 2|z||w| + \|y\|_0^2)^2 \leq (\|x\|_0 + \|y\|_0)^4\end{aligned}$$

Consquently we have the triangular inequality for $d_{\mathbb{H}}$: to prove that $\|y^{-1}x\|_0 \leq \|z^{-1}x\|_0 + \|y^{-1}z\|_0$ just take $z^{-1}x = u$ and $y^{-1}z = v$. Finally it is clear from the definition that $\|\cdot\|_0$ is homogeneous of degree 1 w.r.t. the dilatations δ_s (i.e. $\|\delta_s(x)\|_0 = s\|x\|_0$). So from this fact, from the compactness of the unit Korányi sphere $B_{\mathbb{H}} := \{x \in \mathbb{H} | \|x\|_0 = 1\}$ and from the continuity of the distance proved in (ii) of Proposition 1.17 1.16 follows immediatly the existence of some constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_0 \leq d_{cc}(x, 0) \leq C_2\|x\|_0, \ \forall x \in \mathbb{H} \tag{1.21}$$

that is the equivalence of the two metrics. □

Corollary 1.20. *The topologies induced by the CC distance and the Euclidean one are the same.*

Proof. By (1.19) it follows that every Euclidean ball contains and is contained by a Korányi one and viceversa. We obmit the analytic details.

□

The final step of this section is to introduce the sub-Riemannian length of a curve in the Heisenberg group, to then show that the metric induced by the infimum of these lengths is the same of the CC one.

Remark 1.21. A sub-Riemannian metric on $\mathbb{H}$ is determined by choosing an inner product on the horizontal subbundle of the Lie algebra. If $\langle\cdot,\cdot\rangle$ is such an inner product, one can define the length of an horizontal curve $\gamma : [a,b] \to \mathbb{H}$ as

$$\ell_{\langle\cdot,\cdot\rangle}(\gamma) = \int_a^b \sqrt{\langle\dot\gamma(t), \dot\gamma(t)\rangle} dt \tag{1.22}$$

and then define also the distance between two points x and y to be

$$d_{\langle\cdot,\cdot\rangle}(x,y) = \inf\{\ell(\gamma) | \gamma \text{ is an horizontal path joining } x \text{ and } y\} \tag{1.23}$$

We want find, if it exists, the sub-Riemannian structure on $\mathbb{H}$ which is the same of the CC distance in the sense that

$$d_{\langle\cdot,\cdot\rangle}(x,y) = d_{cc}(x,y) \tag{1.24}$$

Since we have already fixed some arbitrary coordinates to present $\mathbb{H}$, and since we have used such coordinates to define the CC distance, one expects that $\langle\cdot,\cdot\rangle$ is the inner product that makes X_1 and X_2 an orthonormal frame for the horizontal bundle. Actually, this is true and this inner product will be denote by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$. Such sub-Riemannian metric is also denoted with g_H and it is the one with which we equip $\mathbb{H}$. Moreover, g_H can be extended to an inner product defined on the full tangent bundle by requiring that X_1, X_2 and X_3 form an orthonormal system. This new inner product will be again denote by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and defines the Riemannian metric with which we equip $\mathbb{H}$. Such Riemannian metrin is also denoted by g. If there is no possibility of misunderstanding with the Euclidean inner product, we could write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ to lighten the notation.

Definition 1.22. Let $\gamma : [a,b] \to \mathbb{H}$ be an horizontal path. We define the horizontal length of γ to be

$$\ell(\gamma) := \int_a^b \sqrt{\langle\dot\gamma(t), (X_1)_{\gamma(t)}\rangle_{\mathbb{H}}^2 + \langle\dot\gamma(t), (X_2)_{\gamma(t)}\rangle_{\mathbb{H}}^2} dt \tag{1.25}$$

Notice that writing $\dot\gamma = aX_1 + bX_2$, equation (1.25) becomes

$$\ell(\gamma) = \int_a^b \sqrt{a^2+b^2} dt$$

Proposition 1.23. *For all $x, y \in \mathbb{H}$ we have*

$$d_{cc}(x, y) = \inf_{\gamma} \ell(\gamma) \tag{1.26}$$

where the infimum is taken over all horizontal curves joining x and y.

Proof. See [6], Chapter 2, §2.2.2. □

Definition 1.24. i) Let P be a point of $\mathbb{H}$. The horizontal or CC norm of P, denoted with $\|P\|_{\mathbb{H}}$ is

$$\|P\|_{\mathbb{H}} = d_{cc}(0, P) \tag{1.27}$$

ii) Let v be an horizontal vector of the tangent space $T_P\mathbb{H}$ of $\mathbb{H}$ at a point $P \in \mathbb{H}$. The horizontal or CC norm of v, denoted again with $\|v\|_{\mathbb{H}}$ is

$$\|v\|_{\mathbb{H}} = \sqrt{\langle v, v\rangle_{\mathbb{H}}} \tag{1.28}$$

Also in Definition 1.24, if this does not create confusion with the Euclidean case, we may write $|\cdot|$ instead of $\|\cdot\|$, both in (i) and in (ii).

The next lemma shows that the Korányi and CC metrics generate the same infinitesimal structure.

Lemma 1.25. *Let $\gamma : [0, 1] \to \mathbb{H}$ be a C^1 curve and $t_i = i/n$, $i = 0, \dots n$, be a partition of* $[0, 1]$. *Then*

$$\limsup_{n\to\infty} \sum_{i=1}^{n} d_0(\gamma(t_i), \gamma(t_{i-1})) = \begin{cases} \ell(\gamma) & \text{if } \gamma \text{ is horizontal} \\ \infty & \text{otherwise} \end{cases} \tag{1.29}$$

Proof. See [6], Chapter 2, Lemma 2.4. □

1.2 Higher dimensional Heisenberg groups $\mathbb{H}^n$

In this short section we will give a quick description of the higher-dimensional analogs of the Heisenberg group. They represent the natural generalization of the Heisenberg group when the background manifold is $\mathbb{R}^{2n+1}$ or $\mathbb{C}^n \times \mathbb{R}^n$.

Precisely, the n-dimensional Heisenberg group $\mathbb{H}^n$ is the Lie group which have as background manifold $\mathbb{C}^n \times \mathbb{R}^n$, and whose Lie algebra has a step two stratification $\mathfrak{h}^n = V_1 \oplus V_2$, where V_1 has dimension $2n$, V_2 has dimension 1, and $[V_1, V_1] = V_2$, $[V_1, V_2] = [V_2, V_2] = 0$.

As in the case of the first Heisenberg group, we write a point $P \in \mathbb{H}^n$ as $P = (z_1, \ldots, z_n, t) = (x_1 + iy_1, \ldots, x_n + iy_n, t)$. Then the group law is given by

$$P \cdot Q = (z_1 + w_1, \ldots, z_n + w_n, t + s - \frac{1}{2} \sum_{i=1}^{n} \mathrm{Im}(z_i \overline{w_i})) \tag{1.30}$$

where $P = (z_1, \ldots, z_n, t)$ and $Q = (w_1, \ldots, w_n, s)$.

A basis for the Lie algebra of $\mathbb{H}^n$ is given by the left-invariant vector fields $X_i = \partial_{x_i} - \frac{1}{2} y_i \partial_t$, $Y_i = \partial_{y_i} + \frac{1}{2} x_i \partial_t$, $i = 1, \ldots, n$, and $T = \partial_t$. It is easy to see that $[X_i, Y_i] = T$, $i = 1, \ldots, n$, while all the other brackets are null. Then we say as in Definition 1.9 that the first $2n$ vector fields $X_1, Y_1, \ldots, X_n, Y_n$ span the horizontal bundle of $\mathbb{H}^n$.

As in Remark 1.8, we can obtain other infinite parametrization of $\mathbb{H}^n$ depending on a real parameter α. It is sufficient to replace the coefficient $\frac{1}{2}$ just with α both in the group law (1.30) and in the equations of the horizontal vector fields. As a result we will have $[X_i, Y_i] = 2\alpha \partial_t$, $i = 1, \ldots, n,$.

Finally, also the results regarding the Carnot-Carathéodory distance remain valid. In particular, the sub-Riemannian metric compatible with the CC distance is the metric which makes orthonormal the horizontal vector fields $X_1, Y_1, \ldots, X_n, Y_n$, $i = 1, \ldots, n$.

Chapter 2

Geodesics in the Heisenberg group

In this Chapter we will talk about the length-minimizing curves joining pairs of points in $\mathbb{H}$, that is the so-called geodesics. Thanks to the left-invariance of the CC metric, we may assume, without loss of generality, that one of the two points is the origin (denoted with 0=(0,0,0)). We will also deal with variations of such curves to obtain the equations of Jacobi fields as in [15] and [17]. We point out that we will make an extensive use of Riemannian and differential geometry, assuming these subjects to be known (see, for example, [8]).

Remark 2.1. Let us recall the geometric meaning of the proof of Chow's Theorem: let $\gamma : [0,1] \to \mathbb{H}$ be an horizontal path joining 0 and a point $x = (x_1, x_2, x_3)$ and $\pi : \mathbb{H} \to \mathbb{C}$ be the standard projection. So we consider the closed curve $\tilde{\gamma}$ in the complex plane obtained by closing $\pi(\gamma)$ and with the segment joining $\pi(x)$ and 0. Let S denote the $\mathbb{C}$-region bounded by $\tilde{\gamma}$. Then by Stokes' theorem and by (1.15) we have

$$\begin{aligned} x_3 &= \int_0^1 \dot{\gamma}_3(t)dt = -\frac{1}{2}\int_0^1 (\dot{\gamma}_1\gamma_2 - \dot{\gamma}_2\gamma_1)(t)dt \\ &= -\frac{1}{2}\int_{\tilde{\gamma}} x_2dx_1 - x_1dx_2 = \int_S dx_1 \wedge dx_2 = \text{Area}(S). \end{aligned} \tag{2.1}$$

In view of Remark 2.1 we can rephrase the problem of finding the horizontal curve from 0 to $x = (x_1, x_2, x_3)$ with minimal length with the following problem:
Find the plane curve from the origin to (x_1, x_2) with minimum length, subject to the constraint that the region S delimited by the curve and the segment joining 0 to (x_1, x_2) has fixed area.
This is one formulation of Dido's problem, thanks to which, we can move a sub-Riemannian problem to a bonded Euclidean one.

The solution of this problem is well known and is given by arcs of a circle joining 0 and x. The proof of this fact can be found at [9]. In particular, in the limit case of $x_3 = \text{Area}(S) = 0$ we have the straight line from 0 to x as a solution. As a conclusion, we have the following

Proposition 2.2. *A length minimizing curve, or a geodesic, between 0 and x is given by the lift of a circular arc joining 0 and (x_1, x_2) in $\mathbb{C}$, whose convex hull has area x_3.*

Corollary 2.3. *The geodesics in $\mathbb{H}$ are curves of class C^∞.*

Proof. Take an horizontal curve $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ joining 0 and a point x. Suppose its projection $\pi(\gamma) = (\gamma_1, \gamma_2)$ is an arc of a circle. This means that γ_1 and γ_2 are C^∞. Thus, Remark (1.15) immediately implies that also γ_3 is C^∞, since both γ_1 and γ_2 are. □

This is a geometric way to prove the smoothness of the geodesics in $\mathbb{H}$. Actually, the same result is true in the higher order Heisenberg groups $\mathbb{H}^n$. The proof of this fact can be found at [13]. Here the author uses some elements of Control Theory to minimize the length of horizontal curves. Make also attention to the different parametrization of $\mathbb{H}^n$ used in [13], which is the one with $\alpha = -2$ (see Remark 1.8)

Remark 2.4 (!). Based on the Heisenberg group model, one can think that the geodesics in sub-Riemannian spaces are always curves of class C^∞. But this is false! The C^∞-regularity of the geodesics in sub-Riemannian geometry is not a foregone fact, differently from what we know about Riemannian manifolds. Actually, it is an open problem for several years!

2.1 The equations of geodesics

The following theorem provides the explicit analytic form of Heisenbrg's geodesics with unit-speed and which initial point is the origin

Theorem 2.5. *Let v be an horizontal vector with unitary CC norm. The family of unit-speed geodesics having starting point 0 and initial velocity v is given explicitly by*

$$\gamma_{0,c,v} = \left(e^{i\phi} \frac{1 - e^{-ics}}{ic}, -\frac{cs - \sin(cs)}{2c^2} \right) \tag{2.2}$$

where $c \in \mathbb{R}$ is said to be the curvature of the geodesic and $\phi \in [0, 2\pi)$ is unique such that $v = \cos\phi X_v + \sin\phi Y_v$. Moreover γ is length-minimizing over any interval of length $2\pi/|c|$.

To prove Theorem 2.5, keeping in mind Corollary 2.3, we compute the equations of the geodesics by thinking of a geodesic as a smooth horizontal curve that is a critical point of length under any smooth variation by horizontal curves with fixed endpoints. To do this we first need some notions of differential geometry about $\mathbb{H}$.

Definition 2.6. We denote by D the Levi-Civita connection of $(\mathbb{H}, g)$, which we remember to be the unique connection such that

i) D is metric-preserving, i.e. $D_U(g(V, W)) = g(D_U V, W) + g(V, D_U W)$

ii) D is torsion-free, i.e. $D_U V - D_V U = [U, V]$

for any vector fields $U, V, W \in T\mathbb{H}$. Here g is the Riemannian metric with which we have equipped $\mathbb{H}$ in Remark 1.21.

Definition 2.7. We define J to be the field endomorphism $J : T\mathbb{H} \to T\mathbb{H}$, $J(U) := -2D_U(T)$.

From the very definition it may not be clear what J and D are. So we need some calculations which allow us to compute both the values of J and D applied to the orthonormal (w.r.t. the metric g) frame X, Y, T. This will give us explicit expressions for the two operators when applied to left-invariant vector fields.

Let us recall the Koszul formula, that is

$$\begin{aligned}2\langle \nabla_V W, Z\rangle = V\langle W, Z\rangle + W\langle V, Z\rangle - Z\langle V, W\rangle \\ + \langle [V, W], Z\rangle - \langle [W, Z], V\rangle - \langle [V, Z], W\rangle \\ + \langle \mathrm{Tor}_\nabla(V, W), Z\rangle - \langle \mathrm{Tor}_\nabla(W, Z), V\rangle - \langle \mathrm{Tor}_\nabla(V, Z), W\rangle\end{aligned} \tag{2.3}$$

for any vector fields V, W, Z (not necessarily left-invariant) and for any metric-preserving connection ∇ in (H, g). Here we write $\langle \cdot, \cdot\rangle$ instead of $\langle \cdot, \cdot\rangle_{\mathbb{H}}$. Thus, the Levi-Civita connection D trivially satisfies (2.3) without the terms involving the torsion. Since the scalar product of left-invariant vector field is a constant function, by (2.3) and we obtain

$$2\langle D_V W, Z\rangle = \langle [V, W], Z\rangle - \langle [W, Z], V\rangle - \langle [V, Z], W\rangle \tag{2.4}$$

for any left-invarint vector fields V, W, Z. Now, by (2.4), the following derivatives can be easily computed

$$\begin{aligned} &D_X X = 0 \quad D_X Y = \frac{1}{2}T \quad D_X T = -\frac{1}{2}Y \\ &D_Y X = -\frac{1}{2}T \quad D_Y Y = 0 \quad D_Y T = \frac{1}{2}X \\ &D_T X = -\frac{1}{2}Y \quad D_T Y = \frac{1}{2}X \quad D_T T = 0 \end{aligned} \tag{2.5}$$

For example we show $D_X X = 0$. We have, for Z left invariant

$$\langle D_X X, Z\rangle = -\langle [X, Z], X\rangle$$

and then

$$\langle D_X X, X\rangle = 0, \quad \langle D_X X, Y\rangle = -\langle T, X\rangle = 0, \quad \langle D_X X, T\rangle = 0$$

from which it follows $D_X X = 0$. The check is similar for the other derivatives.

Remark 2.8. We collect some important properties about the endomorphism J

i) By (2.5) it follows $J(X) = Y$ and $J(Y) = -X$. That is, the endomorphism J is an involution of the horizontal distributions or rather $(J|_{\mathcal{H}})^2 = -\mathrm{Id}_{\mathcal{H}}$.

ii) Observe that $1 = \langle Y, Y\rangle = \langle J(X), Y\rangle = -\langle X, J(Y)\rangle = \langle X, X\rangle$, while $\langle X, J(X)\rangle = \langle Y, J(Y)\rangle = 0$. Thus,

$$\langle J(U), V\rangle + \langle U, J(V)\rangle = 0 \tag{2.6}$$

for any vector fields U, V.

iii) We also have

$$[V, W] = -\langle V, J(W)\rangle T \tag{2.7}$$

for any V,W left-invariant. Indeed, $[X, Y] = T = -(-1)T = -4(-|X|^2)T = -\langle X, -X\rangle T = -\langle X, J(Y)\rangle T$ and the check is similar for the other brackets.

Remark 2.9. Equations (2.4) and (2.7) provide also an explicit formula to calculate $D_V W$ for all left-invariant vector fields V and W, that is

$$D_V W = \omega(V)J(W) + \omega(W)J(V) - \frac{1}{2}\langle V, J(W)\rangle T \tag{2.8}$$

Here ω is the contact form in $\mathbb{H}$ we have already discussed in the first Chapter.

Now we are ready to collect all the pieces of the proof of Theorem 2.5. We point out that not all the previous remarks are preparatory for the proof, but they will be useful when we will introduce the Jacobi fields.

Definition 2.10. Let $\gamma : I \to \mathbb{H}$ be a C^2 horizontal curve defined on a compact interval $I \subset \mathbb{R}$. A variation of γ is a C^2 map $F : I \times I' \to \mathbb{H}$, where I' is an open interval of $\mathbb{R}$ around the origin, such that $F(s, 0) = \gamma(s)$. We denote $\gamma_\varepsilon(s) = F(s, \varepsilon)$. We say that the variation is admissible if the curves γ_ε are horizontal and have fixed boundary points.

Lemma 2.11. *Let γ_ε be an admissible C^2 variation of an horizontal curve γ as in Definition 2.10. Let V_ε be a vector field along γ_ε. Then $V_\varepsilon(s) = (\partial F/\partial\varepsilon)(s,\varepsilon)$ if and only if V vanishes at the endpoints of γ and*

$$\dot\gamma(\langle V, T\rangle) = -\langle V_H, J(\dot\gamma)\rangle. \tag{2.9}$$

Here $\gamma = \gamma_0$, $V = V_0$ and $V_H = V - \langle V, T\rangle$ denotes the horizontal projection of V_0.

Proof. The proof is given in [17], §3. Just make attention that in [17], the authors have chosen the parametrization of the Heisenberg group given by $\alpha = -1$ (see Remark 1.8). □

Proposition 2.12. *Let $\gamma : I \to \mathbb{H}$ a C^2 horizontal curve parametrized by arc-length. Then γ is a critical point of length for any admissible variation if and only if there is $c \in \mathbb{R}$ such that γ satisfies the second order ordinary differential equation*

$$D_{\dot\gamma}\dot\gamma - cJ(\dot\gamma) = 0. \tag{2.10}$$

Notice that in Riemannian geometry the equation of the geodesics was given by $D_{\dot\gamma}\dot\gamma = 0$. Here the sub-Riemannian structure of the Heisenberg group causes the presence of the endomorphism J in such equation.

Proof. Let V be the vector field of an admissible variation γ_ε of γ as in Lemma 2.11. Since γ is parametrized by arc-length, by the first-variation formula (see [8], Theorem 3.31) we have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\ell(\gamma_\varepsilon)) = -\int_I \langle D_{\dot\gamma}\dot\gamma, V\rangle$$

because V vanishes at the endpoints. Suppose that γ is a critical point of length for any admissible variation. We then want to prove that γ satifies (2.10). Since $|\dot\gamma| = 1$ and $\langle D_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = \dot\gamma(|\dot\gamma|) - \langle D_{\dot\gamma}\dot\gamma, \dot\gamma\rangle$, we can start by observing $\langle D_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = 0$. As γ is horizontal, we also have by (2.4) $\langle D_{\dot\gamma}\dot\gamma, T\rangle = 0$. Moreover $\langle J(\dot\gamma), \dot\gamma\rangle = 0$, then $D_{\dot\gamma}\dot\gamma$ is obliged to be proportional to $J(\dot\gamma)$ at any point of γ. In light of this, we need to show that $\langle D_{\dot\gamma}\dot\gamma, J(\dot\gamma)\rangle$ is costant to complete the proof. Assume, without loss of generality, that $I = [0, a]$. Consider a C^1 function $f : I \to \mathbb{R}$ vanishing at the endpoints and such that $\int_I f = 0$. Finally, we take the vector field V on γ so that $V_H = -fJ(\dot\gamma)$ and $\langle V, T\rangle(s) = \int_0^s f$. Indeed such a V satifies (2.9), because $\dot\gamma(\langle V, T\rangle(s)) = \frac{d}{ds}(\int_0^s f) = f = f|\dot\gamma|^2 = f\langle J(\dot\gamma), J(\dot\gamma)\rangle = -\langle V_H, J(\dot\gamma)\rangle$. Inserting in the first variation formula, since $D_{\dot\gamma}\dot\gamma$ is horizontal, we have

$$\int_I f\langle D_{\dot\gamma}\dot\gamma, J(\dot\gamma) = 0$$

But $\int_I f = 0$, so $\langle D_{\dot\gamma}\dot\gamma, J(\dot\gamma)$ is necessarly costant. Hence we have proved that $D_{\dot\gamma}\dot\gamma$ is a scalar multiple of$J(\dot\gamma)$.

The proof of the converse follows taking into account (2.9) and the first variation formula. □

Definition 2.13. An horizontal path $\gamma : I \to \mathbb{H}$ is said to be a geodesic of curvature c if it satisfies equation (2.10). Given a point $P \in \mathbb{H}$, an horizontal vector $v \in T_P\mathbb{H}$ and $c \in \mathbb{R}$, we denote by $\gamma_{P,c,v}$ the unique solution to (3.2) with initial conditions $\gamma(0) = P$, $\dot\gamma(0) = v$. We call the real parameter c the curvature of the geodesic.

Proof of Theorem 2.5. Let v be a unit horizontal vector and $c \in \mathbb{R}$ fixed. We have to compute the equation of an horizontal geodesic γ assuming that $\gamma(0) = 0$ and $\dot\gamma(0) = v$. Then suppose such a geodesic γ has equations $\gamma(s) = (x(s), y(s), t(s))$. Since γ is horizontal we have

$$\dot\gamma(s) = \dot x(s)X_{\gamma(s)} + \dot y(s)Y_{\gamma(s)}$$
$$\dot t(s) = \frac{1}{2}(x\dot y - \dot x y)(s)$$

Let us explicit (2.10) to find the differential equations that the coordinates of $\dot\gamma$ satisfy. We have (see [8], Theorem 2.68)

$$\begin{aligned} D_{\dot\gamma}\dot\gamma &= D_{\dot\gamma}(\dot xX_\gamma + \dot yY_\gamma) = \frac{D}{dt}(\dot xX_\gamma + \dot yY_\gamma) \\ &= \ddot xX_\gamma + \ddot yY_\gamma + \dot x\frac{D}{dt}X_\gamma + \dot y\frac{D}{dt}Y_\gamma = \ddot xX_\gamma + \ddot yY_\gamma + \dot xD_{\dot\gamma}X + \dot yD_{\dot\gamma}Y \\ &= \ddot xX_\gamma + \ddot yY_\gamma \end{aligned}$$

where in the last equation it follows from (2.5) that $\dot{x}D_{\dot{\gamma}}X + \dot{y}D_{\dot{\gamma}}Y$ is 0. On the other hand from the very definition $cJ(\dot{\gamma}) = cJ(\dot{x}X_\gamma + \dot{y}Y_\gamma) = c\dot{x}Y_\gamma - c\dot{y}X_\gamma$ and then we obtain

$$\ddot{x} = c\dot{y}$$
$$\ddot{y} = -c\dot{x}$$

with initial conditions $x(0) = y(0) = 0$ and $\dot{x}(0) = a$, $\dot{y}(0) = b$, where a and b are the two horizontal components of v.

Integrating these equation, for $c \neq 0$, we obtain

$$x(s) = a\frac{\sin(cs)}{c} + b\frac{1-\cos(cs)}{c}$$
$$y(s) = -a\frac{1-\cos(cs)}{c} + b\frac{\sin(cs)}{c}$$
$$t(s) = -\frac{cs - \sin(cs)}{2c^2}$$

Now to obtain (2.2), just set $\phi = \arccos a = \arcsin b$. This is certainly possible, since $a^2 + b^2 = 1$ for hypothesis. Then $\phi \in [0, 2\pi)$, $a = \cos\phi$, $b = \sin\phi$ and so, by using the complex notation, we get

$$\begin{aligned} x(s) + iy(s) &= \frac{1-\cos(cs)}{c}(\sin\phi - i\cos\phi) + \frac{\sin(cs)}{c}(\cos\phi + i\sin\phi) \\ &= -i\frac{1-\cos(cs)}{c}e^{i\phi} + \frac{\sin(cs)}{c}e^{i\phi} \\ &= -ie^{i\phi}\frac{1-(\cos(cs) - i\sin(cs))}{c} \\ &= e^{i\phi}\frac{1-e^{-ics}}{ic} \end{aligned}$$

Integrating, for $c = 0$, we obtain $x(s) = as$, $y(s) = bs$ and $t = 0$ which are Euclidean straight lines in the xy-plane of $\mathbb{H}$. This result is exactly the limit for $c \to 0$ of the equations we have obtained by the previous integration for $c \neq 0$. We obmit the check for $\dot{t}$ and this complete the proof. □

Corollary 2.14. *Let $\gamma = \gamma_{0,c,v}$ b a nonunit-speed geodesic passing through the origin. Then it has equations*

$$\gamma_{0,c,v}(s) = \left(|v|e^{i\phi}\frac{1-e^{-ics}}{c}, -|v|^2\frac{cs - \sin(cs)}{2c^2}\right) \tag{2.11}$$

Here $|v| = \|v\|_{\mathbb{H}}$ denotes the horizontal length of the vector v.

Proof. In the proof of Theorem 2.5 we set $\phi = \arccos a = \arcsin b$. There we had $v = (a, b)$ with $|v|^2 = a^2 + b^2 = 1$. Here $|v|$ is other than 1, so just set $\phi = \arccos a/|v| = \arcsin b/|v|$. □

Note that in Corollary 2.14 we wrote $\gamma_{0,c,v}$ with a slight abuse of notation. Indeed, here, $\gamma_{0,c,v}$ has not the same meaning as in Theorem 2.5. That is, (2.11) should be rewritten as

$$\gamma_{0,c,v}(s) = \left(e^{i\phi} \frac{1 - e^{-i\frac{c}{|v|}s|v|}}{c/|v|}, -\frac{\frac{c}{|v|}s|v| - \sin\left(\frac{c}{|v|}s|v|\right)}{2(c/|v|)^2} \right) = \gamma_{0,\frac{c}{|v|},\frac{v}{|v|}}(|v|s) \quad (2.12)$$

where $\gamma_{0,c/|v|,v/|v|}(|v|s)$ has the same signification as in Theorem 2.5. Thus, from now on, $\gamma_{P,c,v}$ denotes a geodesic with curvature $c/|v|$ and costant speed $|v|$ and the notation is not in contrast with the previous one, since the geodesic has exactly curvature c if $|v| = 1$. In particular, $v = \dot{\gamma}(0)$ and the geodesic $\gamma_{P,c,v}$ is length-minimizing above any interval of length $2\pi|v|/c$.

Corollary 2.15. *Let $\gamma = \gamma_{0,c,v}$ be a geodesic passing through the origin. Then the equation of geodesic passing through a generic point $P = (x_0, y_0, t_0) = (z_0, t_0) \in \mathbb{H}$ with initial velocity $|v|$ and curvature $c/|v|$ is given by the left translation by P of γ. That is*

$$\gamma_{P,c,v} = L_P(\gamma_{0,c,v}) = \left(z_0 + |v|e^{i\phi}\frac{1 - e^{-ics}}{c}, t_0 - |v|^2\frac{cs - \sin(cs)}{2c^2} - z_0 e^{-i\phi}\frac{1 - e^{ics}}{2c} \right) \quad (2.13)$$

Proof. The proof is the left invariance of the vector fields X and Y. Moreover (2.13) is given by $\mathbb{H}$'s group law defined in (1.2). □

This provides a complete description of the geodesics in the Heisenberg group and shows that all the geodesics in $\mathbb{H}$ can be obtained by left translations and/or by reparametrizations of the unit speed geodesics passing through the origin. Since the initial velocity v and the parameter ϕ are in one-to-one correspondence, from now on we write indistinctly $\gamma_{P,c,v}$ or $\gamma_{P,c,\phi}$.

Remark 2.16. In this remark we point out some basic properties of the geodesics. That is

i) Any isometry in $(\mathbb{H}, g)$ preserving the horizontal distribution transforms geodesics in geodesics since it respects the Levi-Civita connection and commutes with J.

ii) It follows from a simple calculation that any group dilatation δ_λ, $\lambda > 0$, takes a geodesic $(\gamma_{P,c,v})(s)$ to the geodesic $\gamma_{P,c/\lambda,v}(\lambda s)$. I.e. the group dilatations modify the curvature and the wlaking time of the geodesics.

iii) Take a unit speed geodesic passing through the origin $\gamma = \gamma_{0,c,v}(s)$. Then, for $c \neq 0$, we have by (2.2) that the t-coordinate of γ is monotone increasing w.r.t. the time s. It follows that γ leaves every compact set in finite time. The same is true for any other horizontal geodesic, since it can be obtained by a left translation or a rotation of γ.

iv) Any geodesic $\gamma_{P,c,v}$ is uniquely determined by the three parameters P, $v = \dot{\gamma}(0)$ and c. This is a central difference with Riemannian metrics! Indeed, in any Riemannian manifold, a geodesic is well defined only by the starting point and the initial velocity. Here the additional parameter we need is the curvature c and we emphasize there are infinite geodesics having both the same starting point and the same initial velocity.

v) The geodesic $\gamma_{P,-c,-v}(-s)$ has the same support of $\gamma_{P,c,v}(s)$, but it is walked in the opposite direction.

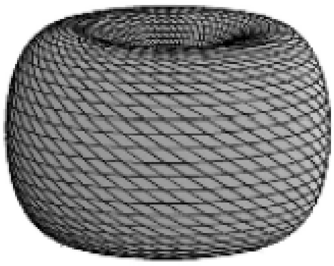
vi) If $\gamma_{P,c,v}$ and $\gamma_{P',c',v'}$ have an arc, of their supports, in common, then $c = \pm c'$.

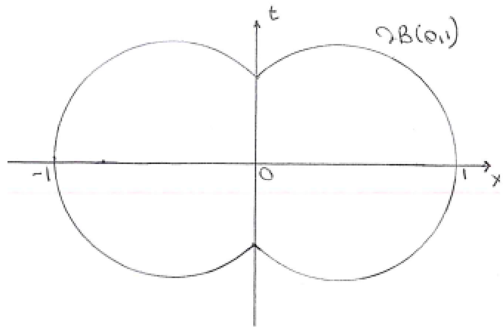
Example 2.17 (Carnot-Carathéodory balls in $\mathbb{H}$)**.** By Theorem 2.5 we can deduce the parametrization of the boundary of the CC ball having center in 0 and radius 1. Indeed, it is sufficient to set $s = 1$ in (2.2) since the time s

measure exactly the distance between 0 and $\gamma(s)$. As a result we obtain

$$\partial B_{cc}(0,1) = \begin{cases} z(c,\phi) = e^{i\phi}\frac{1-e^{-ic}}{c} \\ t(c,\phi) = -\frac{c-\sin c}{2c^2} \end{cases} \tag{2.14}$$

which is a surface in $\mathbb{H}$ depending on the two parameters c and ϕ. The following figures show the graphic of (2.14) and of its projection in the plane (x,t)





In particular, one can see that the boundary of the unit CC ball is C^∞ for $z \neq 0$. On the contrary, $\partial B_{cc}(0,1)$ is not even of class C^1 for $z = 0$, where it intersects the t-axis in two points called poles.

2.2 The pseudohermitian structure of $\mathbb{H}$

The topics covered in this short section are preparatory for studying Jacobi fields in $\mathbb{H}$. We will not enter into detailed arguments: just describe the basic facts we need. The reader can learn more in [18].

In the whole section M is a C^∞ manifold. If it is not specified the dimension of M is n. Before describing the pseudohermitian structure of the Heisenberg group, we need to recall some facts relating to the Frobenius theorem.

Definition 2.18. Let M be a C^∞ manifold of dimension n. An r-dimensional distribution $\mathcal{D}$ on M is a C^∞ r-dimensional subbundle of the tangent bundle of M (i.e. for any $p \in M$ $\mathcal{D}_p$ is a r-dimensional subspace of T_pM which admits a C^∞ basis $X_1, \ldots, X_r$ in a neighborhood of p).

A submanifold N of M is called an integral manifold of $\mathcal{D}$ if $T_pN = \mathcal{D}_p$ for an arbitrary $p \in N$.

The distribution $\mathcal{D}$ is said to be completely integrable if there exists an integral manifold of $\mathcal{D}$ for each point of M.

Definition 2.19. A distribution D on a smooth manifold M is said to be involutive if $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$.

Definitions 2.18 and 2.19 are equivalent thanks to the Frobenius theorem, that is

Theorem 2.20 (Frobenius theorem)**.** *A distribution $\mathcal{D}$ on a C^∞ manifold M is completely integrable if and only if it is involutive.*

Frobenius theorem can be rewritten in terms of differential forms on M. Just define $I(\mathcal{D})$ as the set of the differential forms on M which vanishes on $\mathcal{D}$. Actually, $I(\mathcal{D})$ is an ideal of the exterior algebra of M. Moreover, it is closed w.r.t. the exterior derivate, that is $dI(\mathcal{D}) \subset I\mathcal{D}$ iff $\mathcal{D}$ is involutive.

Now the r-dimensional distribution $\mathcal{D}$ can be represented locally by equations $\omega_1 = \cdots = \omega_{n-r} = 0$. That is, there is an open neighborhood U of any point p of M such that $\omega_1, \ldots, \omega_{n-r}$ are linearly indipendent on U and $\mathcal{D}_q = \{X \in T_qM | \omega_1(X) = \cdots = \omega_{n-r}(X) = 0\}$ for all $q \in U$. $\mathcal{D}$ is involutive (on U) iff there exists 1-forms ω_{ij} such that

$$d\omega_i = \sum_{i=1}^{n-r} \omega_i \wedge \omega_{ij} \tag{2.15}$$

Equation (2.15) is called integrability conditions for the 1-forms ω_i, $i = 1 \ldots n-r$.

Theorem 2.21 (Frobenius theorem)**.** *Let represent a distribution $\mathcal{D}$ locally as $\omega_1 = \cdots = \omega_{n-r} = 0$. Then $\mathcal{D}$ is completely integrable if and only if* (2.15) *holds.*

The proofs of Theorems 2.20 and 2.21 can be found in [14]. Here the reader can also learn more about the Frobenius theorem and the completely integrable distributions.

Now we are going to describe the intrinsic pseudohermitian structure of the Heisenberg group. As the very word "pseudo" suggests, we before need to know some basic topics of Hermitian geometry, which we resume below

i) A complex manifold of dimension n is a manifold with an atlas of charts to the open unit disk in $\mathbb{C}^n$, such that the transition maps are holomorphic.

ii) A manifold M is said to be an almost complex manifold if there exists a linear map $J : TM \to TM$ satisfying $J^2 = -Id$ and J is said to be an almost complex structure of M. Then $J^2 = -I_n$ for a suitable base of TM and $(-1)^n = det(J^2) = (detJ)^2 = 1$. Thus an almost complex manifold is even dimensional.

iii) Let $T^C M = TM \otimes \mathbb{C}$ be the complexification of TM. Let us denote $T^{(1,0)M} = T^1 M = \{V + iJV | V \in TM\}$ and $T^{(0,1)M} = T^{\bar{1}} M = \{V - iJV | V \in TM\}$. Then $TM = T^1 M \oplus T^{\bar{1}} M$. Furthermore, $JZ = -iZ$ iff $Z \in T^1 M$ and $JZ = iZ$ iff $Z \in T^{\bar{1}} M$. All these facts are also valid punctually.

iv) A Riemannian metric g on an almost complex manifold (M, J) is called almost Hermitian if

$$g(U, V) = g(U, V), \ \forall U, V \in \Gamma(TM) \tag{2.16}$$

If J define a complex structure on M, then g is a Hermitian metric and M is said to be a Hermitian manifold.

Remark 2.22. We will see soon that $\mathbb{H}$ is a manifold satisfying (ii), (iii) and (iv) just on the horizontal bundle $\mathcal{H}$ and not on all $T\mathbb{H}$. That is why the name of pseudohermitian: $\mathbb{H}$ has an Hermitian structure, but which is not defined on the whole tangent bundle.

From now until the end of the section we also assume that M is paracompact and that M is odd dimensional with, $\dim M = N = 2n+1$ (actually, the most general theory is with $N = 2n + d$, but for our purposes it is sufficient $d = 1$).

Definition 2.23. A contact structure (or bundle) ξ on M is a $2n$-dimensional completely nonintegrable distribution. A contact form θ is a 1-form annilathing ξ, i.e. $\theta(\xi) = 0$. If ξ is oriented, we say that θ is oriented if $\theta(U, V) > 0$ for any oriented basis of ξ.

We point out that, if M admits a contact (oriented) bundle ξ, there always exists a global (oriented) contact form defined on M, which can be obtained by patching together local ones with a partition of unity. Thus we assume that M has a contact (oriented) structure: we denote it with ξ and its relative contact (oriented) global form is denoted by θ.

Remark 2.24. There exist a unique vector field W such that $\theta(W) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(W, \cdot) = 0$. Here $\mathcal{L}$ denotes the Lie derivative. The vector field W is said to be the Reeb vector field of θ.

Definition 2.25. A CR-structure compatible with ξ is a smooth endomorphism $J : \xi \to \xi$ such that $J^2 = -Id$. We say that J is oriented if (V, JV) is an oriented basis of ξ for all $V \in \xi$.

Definition 2.26. A pseudohermitian structure on M compatible with ξ is a CR-structure J compatible with ξ together with a global contact form θ.

Since $J^2 = -Id$, then there exists a basis of $\xi \otimes \mathbb{C}$ (the complexification of ξ) w.r.t. J^2 is represented by the matrix $-I_{2n}$. So J^2 has only the two eigenvalues i and $-i$, both of multiplicity n. We denote by $Z_1, \dots, Z_n$ the eigenvector basis of the autospace relative to the eigenvalue i and by $Z_{\bar{1}}, \dots, Z_{\bar{n}}$ the eigenvector basis of the autospace relative to the eigenvalue $-i$.

Remark 2.27. With the notations just set $Z_{\bar{j}} = \overline{Z_j}$ for any $j = 1, \dots, n$. In particular, we have the decomposition

$$\xi \otimes \mathbb{C} = S \oplus \overline{S} \tag{2.17}$$

where $S = < Z_1, \dots, Z_n >$ and $\overline{S} = < Z_{\bar{1}}, \dots, Z_{\bar{n}} >$.

Remark 2.28. A simple calculation show that there exist an unique basis $\{U_1, \ldots, U_n, V_1, \ldots, V_n\}$ such that $JU_j = V_j$ and $Z_j = U_j - iV_j$. Indeed, it is given by

$$U_j = \frac{Z_j - Z_{\overline{j}}}{2}, \quad V_j = \frac{Z_{\overline{j}} - Z_j}{2i}, \qquad j = 1, \ldots, n \tag{2.18}$$

Remark 2.29. It is easy to calculate that, for $j, k = 1, \ldots, n$, $[Z_j, Z_k] = 0$, $[Z_j, Z_{\overline{k}}] = 0$ for $k \neq j$ and $[Z_j, Z\overline{j}] = 2i[U_j, V_j]$. Since ξ is noninvolutive by hypothesis (Definition 2.23), $[U_j, V_j]$ must be other than 0 for all j. Indeed, every restriction of TM to $< \xi_j, W >$ (here $\xi_j =< U_j, V_j >$) has a natural pseudohermitian structure provided by $J|_{\xi_j}$ and $\theta|_{<\xi_j,W>}$. Thus ξ_j is involutive and this implies $[U_j, V_j] \neq 0$. The only possible value for $[U_j, V_j]$ is a real scalar multiple of W, depending on the normalization of the basis. We choose, naturally, the one so that $[U_j, V_j] = W/2i$.

Remark 2.30. Although it may not be obvious, a sub-Riemannian metric g_ξ is natural defined on M by choosing g_ξ as the metric on ξ which makes $\{U_1, \ldots, U_n, V_1, \ldots, V_n\}$ an orthonormal basis for ξ. It can be soon extended to a metric g on TM making W orthonormal to ξ. Such metric g is Hermitian on ξ in the sense that $g_\xi(U, V) = g_\xi(JU, JV)$, for any $U, V \in \xi$.

Remark 2.31 (!). The natural pseudohermitian structure of the Heisenberg group $\mathbb{H}$ is, at this point, obvious. Indeed

i) The contact structure is the horizontal bundle $\mathcal{H}$ and the contact form is $\omega = dt - \frac{1}{2}(xdy - ydx)$.

ii) The ω's Reeb vector field is clearly $T = \partial/\partial_t$.

iii) The CR-structure is provided by the endomorphism J defined in Definition 2.7 (actually, by ist restriction to $\mathcal{H}$)

iv) Finally, the natural basis for $\mathcal{H}$ and $\mathcal{H} \otimes \mathbb{C}$ described in Remarks 2.27, 2.28 and 2.29 are, respectively, $\{X, Y\}$ and $\{Z, \overline{Z}\}$, with $Z = X - iY$.

The generalization to the higher dimensional Heisenberg groups $\mathbb{H}^n$ is equally obvious.

Before going on to the next section and proceeding with the description of the Jacobi fields, we want define the pseudohermitian connection of a pseudohermitian manifold M and see its relationship with the Levi-Civita's one. It is the natural covariant derivative that one can consider on a contact manifold M. Precisely

Proposition 2.32. *There exists an unique affine connection $\nabla : TM \to TM \otimes T^*M$ on M satisfying the following conditions*

i) The contact structure ξ is parallel, that is $\nabla_U \xi \subset \xi$, for all $U \in TM$.

ii) The Reeb vector field W, the endomorphism J and the 2-form $d\theta$ are parallel, that is $\nabla W = \nabla J = \nabla d\theta = 0$.

iii) The torsion Tor_∇ of ∇ satisfies

$$\mathrm{Tor}_\nabla(U, V) = d\theta(U, V)W, \ \forall U, V \in \xi \tag{2.19}$$

$$\mathrm{Tor}_\nabla(W, JU) = -J\mathrm{Tor}_\nabla(W, U) \ \forall U \in \xi \tag{2.20}$$

The proof of Proposition 2.32 can be found at [18] and thanks to it we can give the following

Definition 2.33. The connection ∇ of Proposition 2.32 is called pseudohermitian connection of M. Sometimes we may work with other different connections. In this case, the pseudohermitian one will be denoted with ∇^{ph} and its torsion will be denoted Tor_{ph}.

Remark 2.34. The proof of Proposition 2.32 is given providing the (unique) extension of ∇ as an operator, denoted again with ∇, from $T^C M$ to $T^C M \otimes T^{C*} M$. In other words, ∇ can be complexificated as $\nabla : T^C M \to T^C M \otimes T^{C*} M$ in a unique way. This extension is obtained by imposing

i) For any $A, B \in S$ we have $\nabla_{\overline{A}} B$, $\nabla_A B$ and $\nabla_W B$ are determined respectively by equations (2.21), (2.22) and (2.23) below

$$\nabla_{\overline{A}}B = \pi_S([\overline{A}, B]) \tag{2.21}$$

$$d\theta(\nabla_A B, \overline{C}) = Ad\theta(B, \overline{C}) - d\theta(B, \pi_{\overline{S}}([A, \overline{C}]), \quad \forall C \in S \tag{2.22}$$

$$\nabla_W B = \mathcal{L}_W B - \frac{1}{2}J(\mathcal{L}_W(JB)) \tag{2.23}$$

Here S, $\overline{S}$ are the same as in Remark 2.27 and π_S, $\pi_{\overline{S}}$ denote the canonical projection from $T^C M$ to, respectively, S, $\overline{S}$.

ii) For any $C \in T^C M$, $\nabla_C W = 0$.

iii) For any $A, B \in S$

$$\nabla_A \overline{B} = \overline{\nabla_{\overline{A}} B}, \quad \nabla_{\overline{A}} \overline{B} = \overline{\nabla_A B}, \quad \nabla_W \overline{B} = \overline{\nabla_W B} \tag{2.24}$$

Proposition 2.32 and Remark 2.34 allow us to calculate $\nabla_A B$ for every A, B both belonging to TM and to $T^C M$ and, then, to obtain other interessant equations for the p.h. connection ∇. Such properties can be found, for example, in [18], [7]. Regarding us, we are interested in calculating some of them in the Heisenberg group.

Remark 2.35 (The pseudohermitian connection in $\mathbb{H}$!). Let us recall we have fixed the orthonormal basis $X = \partial_x - \frac{1}{2}y\partial_t$ and $Y\partial_y + \frac{1}{2}x\partial_t$ of $\mathcal{H}$ and the contact form is $\omega = dt - \frac{1}{2}(xdy - ydx)$. So $d\omega = -dx \wedge dy$. We want to collect some remarkable properties of the p.h. connection ∇ of $\mathbb{H}$, that is

i) Writing any pair of left-invariant vector fields U, V as a linear combination with constant coefficients of the elements of the basis $\{X, Y, T\}$ we have

$$[U, V] = -\langle U, JV\rangle T \tag{2.25}$$

$$d\omega(U, V) = U(\omega(V)) - V(\omega(U)) - \omega([U, V]) \tag{2.26}$$

ii) Taking U and V as in (i) also horizontal, we get

$$d\omega(U, V) = -\omega([U, V]) = \langle U, JV\rangle \tag{2.27}$$

which implies that the quadratic form

$$U \in \mathcal{H} \mapsto -d\omega(U, JU) = -\langle U, J^2 U\rangle = |U|^2 \tag{2.28}$$

is positive definite (such quadratic form is exactly the sub-Riemannian metric, that is $-d\omega(\cdot, J\cdot) = \langle \cdot, \cdot \rangle$).

iii) Proposition 2.32-(ii) tells us the Reeb's vector field T is parallel with respect to ∇. This means that T is torsion-free with respect to ∇, that is $\mathrm{Tor}(U, T) = \mathrm{Tor}(T, U) = 0$ for any vactor field U. Hence, by (2.19) and (2.27) we deduce

$$\mathrm{Tor}(U, V) = \langle U, JV\rangle T \tag{2.29}$$

iv) Equations (2.19), (2.25) and (2.27) also implies

$$\mathrm{Tor}(U, V) = -[U, V] \tag{2.30}$$

v) Let R_∇ be the curvature operator associated to the p.h. connection ∇. equation (2.30) implies

$$R_\nabla(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W = 0 \tag{2.31}$$

for U, V, W left-invariant vector fields. This implies that the p.h. connection ∇ is flat.

vi) If U and V are left-invariant, then $J(V)$ is also left-invariant and

$$\nabla_U(JV) = J(\nabla_U V) \tag{2.32}$$

vii) As a consequence of the Koszul formula (2.3), the p.h. connection ∇ is related with the Levi-Civita one, D, by

$$2\langle \nabla_U V, W\rangle = 2\langle D_U V, W\rangle + \langle \mathrm{Tor}(U, V), W\rangle - \langle \mathrm{Tor}(U, W), V\rangle - \langle \mathrm{Tor}(V, W), U\rangle \tag{2.33}$$

We point out that (2.33) holds for any triple of vector fields U, V, W belonging to the tangent bundle $T\mathbb{H}$, not necessarily left-invariant or with costant coefficients.

2.3 Jacobi fields in $\mathbb{H}$

In this section we want to describe the Jacobi fields in the Heisenberg group. The same arguments are presented, in the Riemannian context, in [8]. We start by proving an analytic property for the vector field associated to the variation of curve which is a geodesic.

Lemma 2.36. *Let $\gamma : I \to \mathbb{H}$ be a geodesic of curvature c. Let γ_ε be a variation of γ given by horizontal curves parametrized by arc-length. Finally, let U be the C^1 vector field canonically associated by $U = \partial\gamma_\varepsilon/\partial\varepsilon$ as in Lemma 2.11. Then the function*

$$c\langle U, T\rangle + \langle U, \dot\gamma\rangle \tag{2.34}$$

is constant along γ.

Proof. First note that

$$\begin{aligned}
\dot\gamma(\langle U, T\rangle) &= \langle D_{\dot\gamma}U, T\rangle + \langle U, D_{\dot\gamma}T\rangle \overset{(*)}{=} \langle D_U\dot\gamma, T\rangle - \frac{1}{2}\langle U, J(\dot\gamma)\rangle \\
&= U(\langle\dot\gamma, T\rangle) - \langle\dot\gamma, D_U T\rangle + \frac{1}{2}\langle\dot\gamma, J(U)\rangle \\
&= U.0 + \frac{1}{2}\langle\dot\gamma, J(U)\rangle + \frac{1}{2}\langle\dot\gamma, J(U)\rangle \\
&= \langle\dot\gamma, J(U)\rangle
\end{aligned}$$

On the other hand, we have

$$\dot\gamma(\langle U, \dot\gamma\rangle) = \langle D_{\dot\gamma}U, \dot\gamma\rangle + \langle U, D_{\dot\gamma}\dot\gamma\rangle \overset{(*)}{=} \langle D_U\dot\gamma, \dot\gamma\rangle + \langle U, cJ(\dot\gamma)\rangle = -c\langle\dot\gamma, J(U)\rangle$$

where in the last equality we used $0 = U.1 = U(|\dot\gamma|^2) = 2\langle D_U\dot\gamma, \dot\gamma\rangle$, while the starred equalities follows from $[U, \dot\gamma] = 0$. Thus, $\dot\gamma(c\langle U, T\rangle + \langle U, \dot\gamma\rangle) = c\langle\dot\gamma, J(U)\rangle - c\langle\dot\gamma, J(U)\rangle = 0$ □

As in Riemannian geometry we may expect that the vector field associated to a variation of a given geodesic by geodesics of the same curvature satisfies a certain second order differential equation. In fact, we have

Lemma 2.37. *Let γ_ε be a variation of γ by geodesics of the same curvature c. Assume that the canonically associated vector field U is C^2. Then U satisfies*

$$\ddot{U} + R(U,\dot{\gamma})\dot{\gamma} - c(J(\dot{U}) + \frac{1}{2}\langle U,\dot{\gamma}\rangle T) = 0 \tag{2.35}$$

where R denotes the Riemannian curvature tensor in $(\mathbb{H}, g)$ and $\dot{U} = D_{\dot{\gamma}}U$, $\ddot{U} = D_{\dot{\gamma}}D_{\dot{\gamma}}U$.

Proof. As any γ_ε is a geodesic of curvature c, we have $D_{\dot{\gamma}_\varepsilon}\dot{\gamma}_\varepsilon - cJ(\dot{\gamma}_\varepsilon) = 0$. Deriving with respect to U and taking into account that $D_U D_{\dot{\gamma}}\dot{\gamma} = D_{\dot{\gamma}}D_U\dot{\gamma} + R(U,\dot{\gamma})\dot{\gamma} + D_{[U,\dot{\gamma}]}\dot{\gamma}$ and that $[U,\dot{\gamma}] = 0$, we deduce

$$\begin{aligned} 0 &= D_U D_{\dot{\gamma}}\dot{\gamma} - cD_U J(\dot{\gamma}) = D_{\dot{\gamma}}D_U\dot{\gamma} + R(U,\dot{\gamma})\dot{\gamma} - cD_U J(\dot{\gamma}) \\ &= D_{\dot{\gamma}}D_{\dot{\gamma}}U + R(U,\dot{\gamma})\dot{\gamma} - cD_U J(\dot{\gamma}) \end{aligned}$$

Now we need to compute $D_U J(\dot{\gamma})$. To do this, we first observe that from (2.5) we have

$$\begin{aligned} D^2_{X,Y}T &= D_X D_Y T - D_{D_X Y}T = 0 \\ D^2_{X,X}T &= D_X D_X T - D_{D_X X}T = -\frac{1}{2}D_X Y = -\frac{1}{4}T \\ D^2_{Y,Y}T &= D_Y D_Y T - D_{D_X X}T = \frac{1}{2}D_Y X = -\frac{1}{4}T \end{aligned}$$

where $D^2_{W,Z} := D_W D_Z - D_{D_W Z}$ is the second covariant derivative (see [8], Chapter III, §A). This implies for W and Z arbitrary vector fields, of which at least one horizontal, that

$$D^2_{W,Z} = -\frac{1}{4}\langle W,Z\rangle T$$

As a consequence the calculation of $D_U J(\dot{\gamma})$ is quite easy. Indeed

$$D_U J(\dot{\gamma}) = -2D_U D_{\dot{\gamma}}T = -2D_{D_U\dot{\gamma}}T - 2D^2_{U,\dot{\gamma}}T = J(D_U\dot{\gamma}) + \frac{1}{2}\langle U,\dot{\gamma}\rangle T$$

Setting $D_{\dot{\gamma}}U = \dot{U}$, $D_{\dot{\gamma}}D_{\dot{\gamma}}U = \ddot{U}$ and combining the results obtained above we effectively have (2.35) □

Definition 2.38. We call (2.35) the Jacobi equation for geodesics in $\mathbb{H}$ of curvature c. Any solution of (2.35) is a Jacobi Field along γ.

Remark 2.39. Among all the Jacobi fields, we want to quickly describe the tanget ones. That is, $U = f\dot\gamma$ is a Jacobi field if and only if

$$\ddot f \dot\gamma + c\dot f J(\dot\gamma) = 0, \quad (*)$$

where f is a real valued function. Indeed, remembering that $R(W, W')Z = D_W D_{W'} Z - D_{W'} D_W Z - D_{[W,W']}Z$, it is not difficult to see (it is just quite long) that if (*) holds, then $U = f\dot\gamma$ satisfy (2.36). Conversely (2.36) with $U = f\dot\gamma$ gives (*). Thus, any tangent Jacobi field to γ is of the form $(as+b)\dot\gamma$, with $a = 0$ when $c = 0$.

We want now to rewrite the Jacobi equation for $\mathbb{H}$'s geodesics in terms of the p.h. connection ∇. Since ∇ is flat (Remark 2.35), the cuvature tensor will disappear in the new Jacobi equation. Furthermore, we want to remove the hypothesis of costant curvature for the variation of geodesics. Let us start with the following

Lemma 2.40. *Let γ be an horizontal path in $\mathbb{H}$. Then we have*

$$D_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma \tag{2.36}$$

Proof. It is sufficient to prove that $\langle D_{\dot\gamma}\dot\gamma - \nabla_{\dot\gamma}\dot\gamma, W\rangle = 0$, for $W = X, Y, T$. We prove the statement with $W = X$ and the check is similar with Y and T.

Since γ is horizontal, we know $\dot\gamma = aX_\gamma + bY_\gamma$ (see Definition 1.12). Moreover, for U and V left-invariant vector fields and f, g C^∞ functions we have Now, replace both U and V with $\dot\gamma$ and W with X in (2.33). It follows

$$\langle D_{\dot\gamma}\dot\gamma - \nabla_{\dot\gamma}\dot\gamma, X\rangle = \langle \mathrm{Tor}(\dot\gamma, X), \dot\gamma\rangle \tag{2.37}$$

Actually (2.37) make sense only if X is calculated along γ, being $\dot\gamma$ a vector field along γ. Hence by (2.30) we have

$$\mathrm{Tor}(\dot\gamma, X_\gamma) = -[aX_\gamma + bY_\gamma, X_\gamma] = -bT_\gamma = -bT$$

which implies the right hand side of (2.37) is equal to $b\langle T, X\rangle = 0$. □

Remark 2.41. As a consequence of Lemma 2.40 and Proposition 2.12 we have that an horizontal curve γ is a geodesic of curvature c if and only if it satifies

$$\nabla_{\dot{\gamma}}\dot{\gamma} - cJ(\dot{\gamma}) = 0 \tag{2.38}$$

That is, it is indifferent to describe the equation of a geodesic by using the Levi-Civita connection D or the p.h. one ∇.

Thus, to obtain the Jacobi equation (of constant curvature) for the p.h. connection it is sufficient to repeat exactly the same steps made to get Lemma 2.37, but with ∇ instead of D. As a result we will not have the term involving the curvature tensore, but one involving the torsion will appear.

So we move in a similar way as in Lemma 2.36, but this time we consider a variation of geodesics γ_ε of a geodesic γ having no constant curvature. Hence each curve of the variation $\{\gamma_\varepsilon\}$ has curvature $c(\varepsilon)/|\dot{\gamma}_\varepsilon|$ and costant speed. In other words, we have

$$\nabla_{\dot{\gamma}_\varepsilon}\dot{\gamma}_\varepsilon - c(\varepsilon)J(\dot{\gamma}_\varepsilon) = 0 \tag{2.39}$$

Let $U_\varepsilon = \partial\gamma_\varepsilon/\partial\varepsilon$ the deformation vector field as in Lemma 2.11. Differentiating (2.39) along U_ε we obtain

$$\nabla_{U_\varepsilon}\nabla_{\dot{\gamma}_\varepsilon}\dot{\gamma}_\varepsilon - c'(\varepsilon)J(\dot{\gamma}_\varepsilon) - c(\varepsilon)J(\nabla_{U_\varepsilon}\dot{\gamma}_\varepsilon) = 0 \tag{2.40}$$

since $\nabla_{U_\varepsilon}c(\varepsilon) = \partial c(\varepsilon)/\partial\varepsilon = c'(\varepsilon)$ and also thanks to (2.32). Using the sub-Riemannian curvature tensor associated with ∇, which is null, we get

$$\nabla_{U_\varepsilon}\nabla_{\dot{\gamma}_\varepsilon}\dot{\gamma}_\varepsilon = R_\nabla(U_\varepsilon, \dot{\gamma}_\varepsilon)\dot{\gamma}_\varepsilon + \nabla_{\dot{\gamma}_\varepsilon}\nabla_{U_\varepsilon}\dot{\gamma}_\varepsilon + \nabla_{[U_\varepsilon,\dot{\gamma}_\varepsilon]}\dot{\gamma}_\varepsilon = \nabla_{\dot{\gamma}_\varepsilon}\nabla_{U_\varepsilon}\dot{\gamma}_\varepsilon$$

being also $[U_\varepsilon, \dot{\gamma}_\varepsilon] = \partial_\varepsilon\partial_s\gamma - \partial_s\partial_\varepsilon\gamma = 0$. Involving now the torsion, by (2.29) we have

$$\nabla_{\dot{\gamma}_\varepsilon}\nabla_{U_\varepsilon}\dot{\gamma}_\varepsilon = \nabla_{\dot{\gamma}_\varepsilon}\nabla_{\dot{\gamma}_\varepsilon}U_\varepsilon + \nabla_{\dot{\gamma}_\varepsilon}(\mathrm{Tor}(U_\varepsilon, \dot{\gamma}_\varepsilon)) = \nabla_{\dot{\gamma}_\varepsilon}\nabla_{\dot{\gamma}_\varepsilon}U_\varepsilon + \nabla_{\dot{\gamma}_\varepsilon}(\langle U_\varepsilon, J(\dot{\gamma}_\varepsilon)\rangle T)$$

As a last step, the Leibnitz rule implies

$$\nabla_{\dot{\gamma}_\varepsilon}(\langle U_\varepsilon, J(\dot{\gamma}_\varepsilon)\rangle T) = \dot{\gamma}_\varepsilon\langle U_\varepsilon, J(\dot{\gamma}_\varepsilon)\rangle T + \langle U_\varepsilon, J(\dot{\gamma}_\varepsilon)\rangle\nabla_{\dot{\gamma}_\varepsilon}T = \dot{\gamma}_\varepsilon\langle U_\varepsilon, J(\dot{\gamma}_\varepsilon)\rangle T$$

where in the last equality we have exploit $\nabla_{\dot\gamma_\varepsilon} T = 0$. Thus, collecting all the pieces, equation (2.40) becomes

$$\nabla_{\dot\gamma_\varepsilon}\nabla_{\dot\gamma_\varepsilon} U_\varepsilon - c(\varepsilon)J(\nabla_{\dot\gamma_\varepsilon} U_\varepsilon) - c'(\varepsilon)J(\dot\gamma_\varepsilon) + \dot\gamma_\varepsilon\langle U_\varepsilon, J(\dot\gamma_\varepsilon)\rangle T = 0$$

Evaluating at $\varepsilon = 0$ we obtain the Jacobi equation along the geodesic γ, that is

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} U - cJ(\nabla_{\dot\gamma} U) - c'J(\dot\gamma) + \dot\gamma\langle U, J(\dot\gamma)\rangle T = 0 \tag{2.41}$$

where $c = c(0)$ is the curvature of γ and $c' = c'(0)$. Equation (2.41) can also be written, to lighten the notation, as

$$\ddot U - cJ(\dot U) - c'J(\dot\gamma) + \dot\gamma\langle U, J(\dot\gamma)\rangle T = 0 \tag{2.42}$$

where we have set, naturally, $\ddot U = \nabla_{\dot\gamma}\nabla_{\dot\gamma} U$ and $\dot U = \nabla_{\dot\gamma} U$.

Remark 2.42. By observing the equation just obtained, we notice that the noncostance of the curvature of the geodesic variation $\{\gamma_\varepsilon\}$ had caused only the presence of the term $c'J(\dot\gamma)$ in (2.42). Hence, the equivalent of (2.35) in terms of the p.h. connection ∇ is

$$\ddot U - cJ(\dot U) + \dot\gamma\langle U, J(\dot\gamma)\rangle T = 0 \tag{2.43}$$

Equation (2.43) is the Jacobi equation for geodesics in $\mathbb{H}$ of curvature c with respect to the p.h. connection ∇.

Remark 2.43. Similarly to Remark 2.42, the Jacobi equation (of noncostant curvature) with respect to the Levi-Civita connection D will be

$$\ddot U + R(U, \dot\gamma)\dot\gamma - c'J(\dot\gamma) - c(J(\dot U) + \frac{1}{2}\langle U, \dot\gamma\rangle T) = 0 \tag{2.44}$$

where R is the curvature tensor of D. In other words, passing from the connection ∇ to the connection D for writing the Jacobi equations, the term $R(U, \dot\gamma)\dot\gamma - \frac{c}{2}\langle U, \dot\gamma\rangle T$ is trasformed into $\dot\gamma\langle U, J(\dot\gamma)\rangle T$.

The solutions of equation (2.42) can be explicitly computed in $\mathbb{H}$. We first need a small preliminary result

Lemma 2.44. *Let $\gamma : \mathbb{R} \to \mathbb{H}$ be a geodesic in $\mathbb{H}$ with curvature c and let V be a left-invariant vector field. Then*

$$\begin{aligned} \frac{d}{ds}\langle \dot\gamma(s), V_{\gamma(s)}\rangle &= c\langle \dot\gamma(s), JV_{\gamma(s)}\rangle \\ \frac{d}{ds}\langle \dot\gamma(s), JV_{\gamma(s)}\rangle &= -c\langle \dot\gamma(s), V_{\gamma(s)}\rangle \end{aligned} \tag{2.45}$$

where s is the arc-length parameter of γ. In particular

$$\begin{aligned} \langle \dot\gamma(s), V_{\gamma(s)}\rangle &= \langle \dot\gamma(0), V_{\gamma(0)}\rangle \cos(cs) + \langle \dot\gamma(0), JV_{\gamma(0)}\rangle \sin(cs) \\ \langle \dot\gamma(s), JV_{\gamma(s)}\rangle &= -\langle \dot\gamma(0), V_{\gamma(0)}\rangle \sin(cs) + \langle \dot\gamma(0), JV_{\gamma(0)}\rangle \cos(cs) \end{aligned} \tag{2.46}$$

Moreover, if V is a left-invariant vector field so that $V_{\gamma(0)} = \dot\gamma(0)$ then

$$\dot\gamma(s) = \cos(cs)V_{\gamma(s)} - \sin(cs)JV_{\gamma(s)} \tag{2.47}$$

Proof. The system (2.45) is obtained from the geodesic equation (2.38) taking into account that left-invariant vector fields in $\mathbb{H}$ are parallel for the pseudohermitian connection. Then it follows (2.46) and, in turn, also (2.47). □

Now we compute explicitly the Jacobi fields along a given sub-Riemannian geodesic. Let us introduce the following notation: if $v \in T_P\mathbb{H}$, then v^l is the only left-invariant vector field such that $(v^l)_P = v$.

Lemma 2.45. *Let $\gamma : \mathbb{R} \to \mathbb{H}$ be a sub-Riemannian geodesic of curvature c parameterized by arc-length, and let U be a Jacobi field along γ satisfying equation (2.42). Then U is given by $U(s) = U_h(s) + \lambda(s)T_{\gamma(s)}$, where U_h and λ satisfy the equations:*

$$\ddot U_h - cJ(\dot U_h) - c'JU = 0 \tag{2.48}$$

$$\dot\lambda = -b, \quad \text{where } b = \langle U, J(\dot\gamma)\rangle \tag{2.49}$$

Moreover, U_h is given by

$$\begin{aligned} U_h(s) = [U_h(0)^l]_{\gamma(s)} &+ \frac{\sin(cs)}{c}[\dot U_h(0)^l]_{\gamma(s)} - \frac{1-\cos(cs)}{c}[J(\dot U(0))^l]_{\gamma(s)} \\ &+ c'\left(\frac{cs - \sin(cs)}{c^2}\dot\gamma(s) + \frac{1-\cos(cs)}{c}J(\dot\gamma(s))\right) \end{aligned} \tag{2.50}$$

where we have used the following notation: if $v \in T_P\mathbb{H}$, then v^l is the only left-invariant vector field such that $(v^l)_P = v$.

Proof. See [15], Lemma 2.3. As already mentioned in the proof of Lemma 2.11, one need some attention to the different parametrizations of the Heisenberg group. □

Remark 2.46. Suppose U is a Jacobi field along a geodesic γ associated to a variation by arc-length parameterized curves. Then γ itself is arc-length parameterized, so

$$\begin{aligned}0 = \frac{1}{2}U|\dot\gamma|^2 = \frac{1}{2}U\langle\dot\gamma,\dot\gamma\rangle = \frac{1}{2}(\langle\nabla_U\dot\gamma,\dot\gamma\rangle + \langle\dot\gamma,\nabla_U\dot\gamma\rangle) = \langle\nabla_U\dot\gamma,\dot\gamma\rangle \\ = \langle\nabla_{\dot\gamma}U + \mathrm{Tor}(U,\dot\gamma),\dot\gamma\rangle = \dot\gamma\langle U,\dot\gamma\rangle - \langle U,\nabla_{\dot\gamma}\dot\gamma\rangle = \dot\gamma\langle U,\dot\gamma\rangle - c\langle U, J(\dot\gamma)\rangle,\end{aligned}$$

where c is the curvature of the geodesic γ and $\langle \mathrm{Tor}(U,\dot\gamma),\dot\gamma\rangle = 0$ thanks to (2.19). Hence,

$$\dot\gamma\langle U,\dot\gamma\rangle - c\langle U, J(\dot\gamma)\rangle = 0 \tag{2.51}$$

Moreover, using equations (2.49) and (2.51), we obtain

$$\dot\gamma\langle U,\dot\gamma\rangle + c\dot\gamma\langle U, T\rangle = 0 \tag{2.52}$$

This means $\langle U, \dot\gamma + cT\rangle$ is constant along γ.

Lemma 2.47. *Let $\gamma : \mathbb{R} \to \mathbb{H}$ be a geodesic of curvature c. Consider a Jacobi field U along γ given by*

$$U(s) = a(s)\dot\gamma(s) + b(s)J(\dot\gamma(s)) + d(s)T_{\gamma(s)} \tag{2.53}$$

Assume that the variation associated to U consists of arc-length parameterized geodesics, then the functions a, b, d, satisfy the differential equations

$$\begin{aligned}\dot a - cb = 0 \\ \ddot b + c\dot a - c' = 0 \\ \dot d + b = 0\end{aligned} \tag{2.54}$$

Proof. Take a Jacobi field U as in (2.53). It is always possible to write a Jacobi field in such a way because $\dot\gamma$ and $J(\dot\gamma)$ are horizontal and orthogonal and so they form, together with T, a basis for the vector fields along γ.

We want to rewrite (2.42) for U in terms of a, b and d, so we derive (2.53) along γ. By (2.32), (2.38) and $\nabla_{\dot\gamma}T = 0$ we get

$$\begin{aligned}\dot U = \nabla_{\dot\gamma}U &= \dot a\dot\gamma + a\nabla_{\dot\gamma}\dot\gamma + \dot b\dot\gamma + bJ(\nabla_{\dot\gamma}\dot\gamma) + \dot dT\\ &= \dot a\dot\gamma + acJ(\dot\gamma) + \dot b\dot\gamma + bcJ^2(\dot\gamma) + \dot dT\\ &= \dot a\dot\gamma + acJ(\dot\gamma) + \dot b\dot\gamma - bc\dot\gamma + \dot dT \qquad (*)\end{aligned}$$

and (here we obmit all the passages)

$$\ddot U = \nabla_{\dot\gamma}\nabla_{\dot\gamma}U = \nabla_{\dot\gamma}(*) = \ddot a\dot\gamma + 2c\dot aJ(\dot\gamma) - c^2a\dot\gamma + \ddot bJ(\dot\gamma) - 2c\dot b\dot\gamma - c^2bJ(\dot\gamma) + \ddot dT$$

Now we compute

$$\begin{aligned}\ddot U - cJ(\dot U) &= (\ddot a - c^2a - 2c\dot b + c^2a + c\dot b)\dot\gamma + (\ddot b + 2c\dot a - c^2b - c\dot a + c^2b)J(\dot\gamma) + \ddot dT\\ &= (\ddot a - c\dot b)\dot\gamma + (\ddot b + c\dot a)J(\dot\gamma) + \ddot dT\end{aligned}$$

and

$$\dot\gamma\langle U, J(\dot\gamma)\rangle = \langle\dot U, J(\dot\gamma)\rangle - c\langle U,\dot\gamma\rangle = ac + \dot b - ac = \dot b$$

from which (2.42) becomes

$$(\ddot a - c\dot b)\dot\gamma + (\ddot b + c\dot a - c')J(\dot\gamma) + (\ddot d + \dot b)T = 0$$

This prove the second equation of (2.54), while the first and the third one are weaker than we wanted.

The stronger equation $\dot a - c\dot b$ follows by (2.51), while to get the third one we differetiate

$$\dot d = \dot\gamma\langle U, T\rangle = \langle\nabla_{\dot\gamma}U, T\rangle = \langle\nabla_U\dot\gamma + \mathrm{Tor}(\dot\gamma, U), T\rangle = -\langle J(\dot\gamma), U\rangle = -b$$

□

Corollary 2.48. *Let γ and U as in Lemma 2.47. Then $d(s) = \langle U(s), T_{\gamma(s)} \rangle$ satisfies*

$$\dddot{d} + c^2 \dot{d} - c' = 0 \tag{2.55}$$

In particular,

$$d(s) = d(0) + \dot{d}(0)\frac{\sin(cs)}{c} + \ddot{d}(0)\frac{1-\cos(cs)}{c^2} + c'\frac{cs - \sin(cs)}{c^3} \tag{2.56}$$

Proof. It is a direct consequence of the previous Lemma and (2.52). For a little more detailed arguments see [15], Corollary 2.6. □

Remark 2.49 (!). We want to give an improvement of Remark 1.8. That is, we want to collect all the basic results obtained until now in function of different parametrizations of $\mathbb{H}$, depending on a real parameter α.

i) We have seen right in Remark 1.8 that the group product is given by

$$(z,t)\cdot(w,s) = (z+w, t+s-\alpha Im(z\overline{w})), \quad \alpha \in \mathbb{R} \tag{2.57}$$

and the left invariant vector fields X, Y and T are

$$X = \partial_x - \alpha y \partial_t, \quad Y = \partial_y + \alpha x \partial_t, \quad T = [X,Y] = 2\alpha\partial_t \tag{2.58}$$

ii) The 1-form of which the horizontal vector fields are the kernel is given by

$$\omega = dt - \alpha(xdy - ydx) \tag{2.59}$$

In this way the Reeb vector field of ω is ∂_t, that is $\omega(\partial_t) = 1$.

iii) The endomorphism J is defined as

$$J(U) = -\frac{1}{|\alpha|}D_U T = -\frac{1}{\alpha}D_U \partial_t \tag{2.60}$$

and then (2.5) becomes

$$\begin{aligned} &D_X X = 0 \quad D_X Y = \alpha T \quad D_X T = -\alpha Y \\ &D_Y X = -\alpha T \quad D_Y Y = 0 \quad D_Y T = \alpha X \\ &D_T X = -\alpha Y \quad D_T Y = \alpha X \quad D_T T = 0 \end{aligned} \tag{2.61}$$

iv) The geodesics' equation is at the same time given by

$$D_{\dot\gamma}\dot\gamma - 2\alpha c J(\dot\gamma) = 0 \tag{2.62}$$

$$\nabla_{\dot\gamma}\dot\gamma - 2\alpha c J(\dot\gamma) = 0 \tag{2.63}$$

As a consequence the analytic expression of a geodesic passing through the origin is modified in

$$\gamma_{0,c,v} = \left(e^{i\phi}\frac{1-e^{-ics}}{ic}, -\alpha\frac{cs - \sin(cs)}{c^2} \right) \tag{2.64}$$

v) The Jacobi equations (2.44) and (2.42), derived respectively by (2.62) and (2.63), remain the same, since the parameter α is contained in the vector field T.

vi) Finally, we rewrite the Lemma 2.47. That is, if we take a Jacobi field of the form $U = a\dot\gamma + bJ(\dot\gamma) + d$, then the functions a, b and d satisfies

$$\begin{aligned} &\dot a - 2\alpha c b = 0 \\ &\ddot b + 2\alpha(c\dot a - c') = 0 \\ &\dot d + 2\alpha b = 0 \end{aligned} \tag{2.65}$$

Chapter 3

The distance funtion and the cut locus

We are finally ready to study some fine properties of the distance function from a C^2 suface in the Heisenberg group. Moreover, we are going to investigate on the cut locus of such surfaces. Informally, it the set containing maximal segments' endpoints that minimize the distance to the surface. The reason of the C^2 regularity comes from the Riemannian context, as already explained in the introductory part.

From now on, S will denote a surface in $\mathbb{H}$. If it is not specified, we suppose S satisfies this two hypothesis

H1. S is the boundary of an open and connected set Ω and of $\mathbb{H} \setminus \overline{\Omega}$

H2. S is of class C^2 in the Euclidean sense

The hypothesis H2 reflects what we have just said above, while the reason of the assumption H1 will be soon clear, when we will need to give a sign to the distance function (see Definition 3.1).

Almost all the topics we are going to discuss have been proved by Arcozzi and Ferrari in [3]. It is the main work to which we refer the reader for this chapter. For this reason we change the parametrization of the Heisenberg group by choosing, according to Remark 2.49, $\alpha = -2$. In this way we have the same representation of $\mathbb{H}$ as in [3]. More, the authors in [3] proved some statements below with a little less restrictive hypothesis on the regularity of the surface, assuming a $C^{1,1}$ regularity.

These were the premises for this chapter. Now we can start giving the definition of the distance (and the signed one) from a C^2 surface

Definition 3.1. Let P be a point in $\mathbb{H}$. We define the distance from P to S as

$$d_S(P) = \inf_{Q \in S} d_{cc}(P, Q) \tag{3.1}$$

Thanks to H1, we can also define the so-called signed distance from P to S, that is

$$\delta_S(P) = \begin{cases} -d_S(P), & \text{if } P \in \Omega \\ d_S(P), & \text{if } P \notin \Omega \end{cases} \tag{3.2}$$

In Remark 2.16-(iv), we have underline as in the Heisenberg group there are infinite geodesics having both the same starting point and the same initial velocity. This leads to a serious problem. Indeed, in the Riemannian case the normal geodesics to surface are unique and they determine (locally) the set of minimal distance's points the surface. Here, the normal geodesic to a point $P \in S$ is not uniquely determined by the normal vector to S at P. So it is not clear which is the (locally) distance-minimizing geodesic which minimizes (locally) the distance from S. Such a geodesic will be called the metric normal.

We first need to introduce some new differential geometric notions about S, which will be used to define right the metric normal to S.

Definition 3.2. Let $P \in S$. Then $T_P S$ denote the Euclidean tangent space to S at P and $\Pi_S P$ denote the Euclidean plane in $\mathbb{H}$ tangent to S at P in the Euclidean sense. The direction tangent to S at P is $V_P S := T_P S \cap \mathcal{H}_P$, while the direction normal to S at P is $N_P S := \mathcal{H}_P \ominus V_P S$. Finally let ν be the Euclidean exterior normal to S at P and let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product. The Pansu exterior normal to S at P, denoted by $N_P^+ S$, is the unique vector $v \in N_P S$ such that $\langle v, \nu \rangle > 0$.

Remark 3.3. The directions tangent and the direction normal to S can be easly determined when the surface S is described implicitly by a function. Indeed, if S is implicitly defined by $g(x, y, t) = 0$ we have

$$V_P S = \text{span}\{(Yg)X - (Xg)Y\}, \quad N_P S = \text{span}\{\nabla_{\mathbb{H}} g\} \tag{3.3}$$

Definition 3.4. A point $C \in S$ is said to be characteristic if $T_C S = \mathcal{H}_C$. We denote by $\text{Char}(S)$ the set of all characteristic points of S.

Remark 3.5. Each Euclidean plane $\mathcal{P}$ of the form $t = ax + by + c$, $a, b, c \in \mathbb{R}$, has one and only one characteristic point, which is

$$C = (-b/2, a/2, c). \tag{3.4}$$

Indeed the tangent space of $\mathcal{P}$ at (x, y, t) is spanned by $(1, 0, a)$ and $(0, 1, b)$. They are both horizontal iff $a = 2y$ and $b = -2x$. Any other Euclidean plane which is not of this form, has equation $ax + by + c = 0$. Such a plane is called vertical and has no characteristic points. Sometimes, with a slight abuse of notation, we said that a vertical plane $\mathcal{P}$ has characteristic point C at infinity and that $d_{cc}(P, C) = \infty$ for all $P \in \mathcal{P}$.

We are now ready to define the metric normal to the surface S, that is

Definition 3.6. Let $P \in S$. The metric normal to S at P, denoted by $\mathcal{N}_P S$, is the set of the points $Q \in \mathbb{H}$ such that $d_S Q = d_{cc}(Q, P)$.

Note that from the very definition, a point $Q \in \mathcal{N}_P S$, $Q \neq P$, cannot belong to S, otherwise we should have $0 = d_S Q = d_{cc}(Q, P) > 0$ and it is impossible. So the metric normal describes which points of $\mathbb{H} \setminus S$ are “closest” to the points of S.

Similarly for $P, Q \in S$, one can ask what is the fastest path to go from P to Q staying within S. This is in a certain sense the sub-Riemannian metric “induced” in S by the sub-Riemannian one g_H (Remark 1.21) of $\mathbb{H}$. It is a very interesting problem, but it goes beyond what we intend to deal with in this thesis. For our purposes, in fact, we will only need to study a few situations concerning the induced metric and only if S is a plane.

Remark 3.7. If the distance between two pionts of S is realized by a geodesic γ of $\mathbb{H}$, as a consequence of Definition 3.2 it needs $\dot\gamma(0) \in V_P S$.

Example 3.8. Let's compute $V_P S$ and $N_P S$ if S is the plane $t - ax - by - c = 0$, $a, b, c \in \mathbb{R}$ and $P \in S$ is the point $P = (z_P, t_P) = (x_P, y_P, t_P) = (x_P, y_P, ax_P + by_P + c)$. We use the results given by Proposition 3.3.

$$g(x, y, t) = t - ax - by - c$$
$$Xg(P) = -a - 2y_P, \quad Yg(P) = -b + 2x_P$$
$$X_P = (1, 0, -2y_P), \quad Y_P = (0, 1, 2x_P)$$

By (3.4) the S's characteristic point is $C = (-b/2, a/2, c)$. Then we have

$$C - P = \frac{1}{2}(b - 2x_P, -a - 2y_P, -2ax_P - 2by_P)$$
$$N_P S = \text{span}\{Xg(P)X_P + Yg(P)Y_P\} = \text{span}\{(y_{C-P}, -x_{C-P}, t_{C-P} + 2|z_P|^2)\}$$
$$V_P S = \text{span}\{Xg(P)Y_P - Yg(P)X_P\} = \text{span}\{C - P\}$$

Lemma 3.9. *Let S be the plane $t-ax-by-c = 0$, $a, b, c \in \mathbb{R}$, with characteristic point $C = (-b/2, a/2, c)$. Let $P = (x_P, y_P, t_P) \in S$ be noncharacteristic. Then the CC distance between P and the characteristic point C of S is*

$$d_{cc}(P, C)^2 = |x_P + \frac{b}{2}|^2 + |y_P - \frac{a}{2}|^2 \tag{3.5}$$

that is the Euclidean distance between the projections of P and C in $\mathbb{C}$.

Proof. $V_P S = \text{span}\{C - P\}$, then the Euclidean segment joining C and P is an $\mathbb{H}$'s geodesic. Its length is the Euclidean length of the projection in $\mathbb{C}$. □

For example the $\mathbb{C}$-plane in $\mathbb{H}$ has the origin as a characteristic point. So, Lemma 3.9 tells us something we already know: the planar geodesics passing through the origin are straight lines and the distance between 0 and a point in $\mathbb{C}$ is the Euclidean one.

Note that Definitions 3.2 and 3.6 remain the same for any closed subset E of $\mathbb{H}$. The results below gives a first geometic description of $\mathcal{N}_P S$ and explain why we have introduced the notions of Definitions 3.2 and 3.4.

Lemma 3.10. *Let E be a closed subset of $\mathbb{H}$ and let $P \in E$. Let $Q \in \mathcal{N}_P E$ and $\gamma : I \to \mathbb{H}$ be a length minimizing geodesic from P to Q, with $I \subset \mathbb{R}$ an interval. Then $\gamma(I) \subset \mathcal{N}_P E$.*

Proof. Let A be any point in γ, then $d_S(A) \leq d(A, P)$. By absurd take a P' in S such that $d(A, P') < d(A, P)$. Then, by the triangle inequality, we have

$$d(Q, P') \leq d(Q, A) + d(P', A) < d(Q, A) + d(P, A) = d(Q, P)$$

contradicting $Q \in \mathcal{N}_P E$. □

Since any surface is a closed set, then Lemma 3.10 holds for S. Hence, as we expected, for each $P \in S$, there is effectively a normal geodesic to S at P, among the infinite normal ones, which locally minimize the distance of points of $\mathbb{H}$ from P. However, Lemma 3.10 is still a weak result concerning the metric normal of S. Indeed, we will soon have an improvement of this fact: we are going to prove that the metric normal to S at P coincides with the metric normal to $\Pi_P S$ at P, where $\Pi_P S$ is the Euclidean tangent plane to S at P as in Definition 3.2. This is a fundamental fact because we are able to give an explicit expression of the metric normal to a plane, thanks to the following statements.

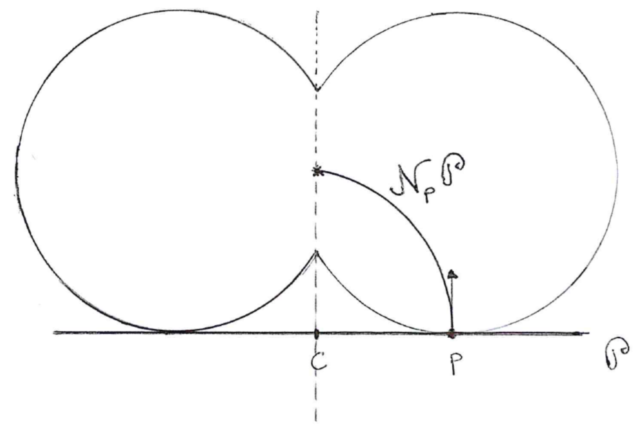
Theorem 3.11. *Let $\mathcal{P}$ be a plane in $\mathbb{H}$ and let $P \in \mathcal{P}$ be non-characteristic. If $\mathcal{P}$ has a characteristic point C and Ω is one of the half-spaces having $\mathcal{P}$ as boundary, then*

$$\mathcal{N}_P \mathcal{P} = \gamma_{P, \frac{2}{d_{cc}(P,C)}, N_P^+ \mathcal{P}} \left(\left[-\frac{\pi}{2} d_{cc}(P, C), \frac{\pi}{2} d_{cc}(P, C) \right] \right) \tag{3.6}$$

and $\mathcal{N}_C \mathcal{P} = \{C\}$ is degenerate. If $\mathcal{P}$ is a vertical plane, then $\mathcal{N}_P \mathcal{P}$ is the straight geodesic through P, in the direction $N_P \mathcal{P}$.

Proof. See [3], §3, Theorem 3.1. □

The following picture shows the geometic meaning of the metric normal to a point P of a plane $\mathcal{P}$ when P is noncharacteristic for $\mathcal{P}$ (when P is characteristic for $\mathcal{P}$ there is nothing to explain)



First, we consider the characteristic point C of $\mathcal{P}$ and the orthogonal axis to $\mathcal{P}$ passing through C. Then we take the unique CC ball which center belong to such axis and which is tangent to $\mathcal{P}$ at P. Thus, the metric normal is the geodesic joining P with the center of the ball.

By Theorem 3.11 and (2.13) we have an explicit expression for the metric normal to a plane.

Proposition 3.12. *The metric normal to $\mathcal{P} = \{t = 0\}$ at $P = (z, t)$, $z = x + iy$, is the support of the geodesic arc*

$$\gamma_P(\sigma) = (w(\sigma), s(\sigma)) = \left(\frac{1}{2}z(1 + e^{-ic\sigma}), \frac{|z|^2}{2}(c\sigma + \sin(c\sigma))\right) \tag{3.7}$$

with $c = 2/|z|$ and $|\sigma| \le \pi/c$.

Observe that by (3.7) we have

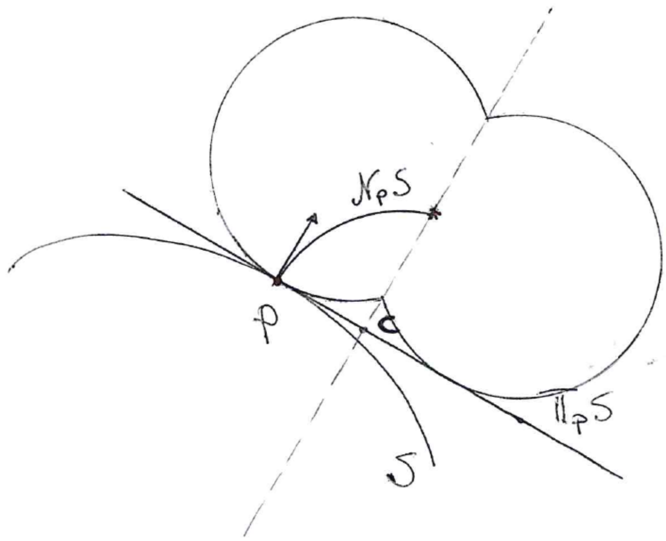
$$\begin{cases} |w(\sigma)| = |z|(1 + cos(c\sigma)) \\ s(\sigma) = \frac{|z|^2}{2}(c\sigma + \sin(c\sigma)) \end{cases} \tag{3.8}$$

which is, for σ fixed, $|\sigma| \le \pi/c$, a parametrization of the set of points having distance σ from $\mathcal{P} = t = 0$.

Theorem 3.13. *Let $P \in S$ be noncharacteristic and let $\Pi_P S$ be the Euclidean tangent plane to S at P. Then $\mathcal{N}_P S$ is a nontrivial geodesic arc* *having endpoints in* Ω *and* $\mathbb{H} \setminus \overline{\Omega}$. *Moreover,* $\mathcal{N}_P S \cap \mathcal{N}_P \Pi_P S$ *is an nontrivial geodesic arc containing* P.

Proof. See [3], §4, Theorem 4.1. □

The geometic construction of the metric normal to a surface is the same as in the case of a plane. Here, the plane we need to consider is just the Euclidean tangent one at P to the surface S



So we have an explicit expression for the metric normal to S at each of its points, given by Theorems 3.11, 3.12 and 3.13. This is the starting point for the definition of the exponential map of S: informally it is the map moving each point P of S along its metric normal.

Definition 3.14. Let $P \in S$. The oriented metric normal to S at P, denoted by $\mathcal{N}_P^+ S$, is the unique parametrization of $\mathcal{N}_P S$ such that $\delta_S(\mathcal{N}_P^+ S(\sigma), P) = \sigma$, $\forall \sigma \in D(\mathcal{N}_P S(\sigma))$.

This means that, if $\mathcal{N}_P^+ S$ is nontrivial, $\mathcal{N}_P^+ S(\sigma) \in \Omega$ for $\sigma < 0$, $\mathcal{N}_P^+ S(\sigma) \in \mathbb{H} \setminus \Omega$ for $\sigma > 0$ and $\mathcal{N}_P^+ S(0) = N_P^+ S$.

Now if C is characteristic for S, then by Theorems 3.11 and 3.13 we have $\mathcal{N}_C S = \mathcal{N}_C^+ S = \{C\}$.

Viceversa, suppose that $P \in S$ is noncharacteristic and that, locally near P, S has equation $g(x, y, t) = 0$, where $g : \mathbb{H} \to \mathbb{R}$ is C^2 in the Euclidean sense (according to H2) and $\nabla g \neq 0$ pointwise on S. The Euclidean tangent plane to S at P, $\Pi_P S$, has equation

$$\partial_x g(P)(x - x_P) + \partial_y g(P)(y - y_P) + \partial_t g(P)(t - t_P) = 0 \tag{3.9}$$

If $\partial_t g(P) \neq 0$, thanks to (3.4) we know the explicit expression of the characteristic point C of $\Pi_P S$. Then by Definition 3.2, Example 3.8 and Lemma 3.9, by a direct calculus we have

$$d_{cc}(P, C) = \frac{|\nabla_{\mathbb{H}} g(P)|}{2|\partial_t g(P)|} \tag{3.10}$$

and

$$N_P^+ S = \frac{\nabla_{\mathbb{H}} g(P)}{|\nabla_{\mathbb{H}} g(P)|} \tag{3.11}$$

Here $\nabla_{\mathbb{H}} g(P) \neq 0$ since P in noncharacteristic. So Theorems 3.11 and 3.13 allow us to write $\mathcal{N}_P^+ S$ in terms of g's partial derivatives: $\mathcal{N}_P^+ S(\sigma) = P \cdot \eta(\sigma)$, where

$$\eta(\sigma) = \left(e^{i\phi} \frac{1 - e^{-ic\sigma}}{ic}, 2\frac{c\sigma - \sin(c\sigma)}{c^2} \right) \tag{3.12}$$

and $c = 4\partial_t g(P)/|\nabla_{\mathbb{H}} g(P)|$, $\cos\phi = Xg(P)/|\nabla_{\mathbb{H}} g(P)|$, $\sin\phi = Yg(P)/|\nabla_{\mathbb{H}} g(P)|$.

If $\partial_t g(P) = 0$, then $\Pi_P S$ is a vertical plane, with characteristic point at infinity. Then $d_{cc}(P, C) = \infty$ and (3.12) becomes

$$\eta(\sigma) = (e^{i\phi\sigma}, 0) \tag{3.13}$$

Definition 3.15. Let $\mathcal{C} := \{(P, s) : \ P \in S, \ s \in D(\mathcal{N}_P^+ S)\} \subset S \times \mathbb{R}$. The exponential map associated with S is the map

$$\exp_S : \mathcal{C} \to \mathbb{H}, \ \exp_S(P, s) := \mathcal{N}_P^+ S(s) \tag{3.14}$$

We call this map exponential because it extends the notion of exponential map of Riemannian manifolds. Clearly, using a slight abuse of notation,

$S = S \times \{0\} \subset \mathcal{C}$ and $\exp_S(P, 0) = P, \forall\, P \in S$. Then $\{\exp_S(P, 0)|\ P \in S\} = \exp_S(S, 0) = S$.

If we define

$$\mathrm{Unp}(S) \coloneqq \{P \in \mathbb{H} :\ \exists!\ Q \in S \text{ s.t. } d_S(P) = d_{cc}(P, Q)\} \tag{3.15}$$

then we have

Theorem 3.16. *The map* $\exp_S$ *is an homeomorphism of int($\mathcal{C}$) onto an open subset of* $int(\mathrm{Unp}(S))$*. Moreover,* $S \cap int(\mathrm{Unp}(S)) = S \cap \exp_S(int(\mathcal{C}))$*.*

Proof. See [3], §5, Theorem 5.2. □

Actually, we are able to improve Theorem 3.16. Indeed, it is not difficult to show that $\exp_S$ is a diffeomorphism in an open neighborhood of $S \setminus \mathrm{char}(S)$. To do this suppose that $P \in S$ is noncharacteristic and that $\mathcal{U} \subset S$ is an open neighborhood of P free of characteristic points, parametrized by

$$\mathcal{U} = \{(u, v, f(u, v)) :\ (u, v) \in A\}. \tag{3.16}$$

Here $A \in \mathbb{C}$ is open and $f : A \to \mathbb{H}$ is C^2. In particular P is described by $P = (u_P, v_P, f(u_P, v_P))$. Moreover the map

$$F : A \times \mathbb{R} \to \mathbb{H}, \quad F(u, v, \tau) \coloneqq \exp_S((u, v, f(u, v)), \tau) \tag{3.17}$$

is an expression of $\exp_S$ in local coordinates and we have

Lemma 3.17. *The matrix representing* $JF(u, v, 0)$ *with respect to the basis* $\{\partial_u, \partial_v, \partial_\tau\}$ *of* $A \times \mathbb{R}$ *and the basis* $\{X, Y, \partial_t\}$ *of* $\mathbb{H}$ *is*

$$\begin{pmatrix} 1 & 0 & \frac{Xf}{|\nabla_{\mathbb{H}} f|} \\ 0 & 1 & \frac{Yf}{|\nabla_{\mathbb{H}} f|} \\ -Xf & -Yf & 0 \end{pmatrix} \tag{3.18}$$

Here we have denoted $Xf = Xg$ *and* $Yf = Yg$ *with* $g(u, v, t) = t - f(u, v)$*.*

Sketch of proof. The main fact is that from (3.12) we can get an explicit expression of the map F. To do this, we fix the coordinates $(z,t) = (x,y,t)$ and $(z',t') = (x',y',t')$ for $\mathbb{H}$ and the coordinates $(u,v,\tau) = (u+iv,\tau) = (w,\tau)$ for $A \times \mathbb{R}$. By (3.12) we obtain

$$(z,t) = F(u,v,\tau) = \mathcal{N}_{(u,v,f(u,v))}S(\tau) = (u+iv, f(u,v)) \cdot \eta'(\tau), \qquad (3.19)$$

where

$$\eta'(\tau) = (z'(\tau), t'(\tau)) = \left(\frac{1}{4}(1-e^{-ic\tau}), \frac{|\nabla_{\mathbb{H}} f|^2}{8}(c\tau - \sin(c\tau))\right) \qquad (3.20)$$

and $c = 4/|\nabla_{\mathbb{H}} f|$. Actually $z'(\tau) = z'(u,v,\tau)$, since z' contain f in its expression. So

$$(z,t) = (u+iv+z', t'+f+2\mathrm{Im}((u+iv)\overline{z'})). \qquad (3.21)$$

Then to complete the proof of Lemma 3.17, just calculate the deivatives of F. We obmit such calculations, since they are very long and boring. However, the reader can be found them at [3], §6, Lemma 6.5. □

As a consequence of Lemma 3.17 we are able to calculate

$$|JF(u,v,0)| = \frac{(Xf)^2 + (Yf)^2}{|\nabla_{\mathbb{H}} f|} = |\nabla_{\mathbb{H}} f| = 2\frac{|\nabla_{\mathbb{H}} f|}{2|\partial_t g|} = 2d_{cc}(P,C)$$

where in the last two equalities we have used that $\partial_t g = 1$ and (3.10). This shows that $|JF(u,v,0)| = 2d_{cc}(P,C) \neq 0$, where C is the characteristic point of $\Pi_P S$. Then we have the desired improvement of Theorem 3.16.

Remark 3.18. The explicit expression of the exponential map in local coordinates shows that the map $F(u,v,t)$ is of class C^1, because it contains some partial derivatives of f, which is a C^2 function by hypothesis. Hence, the evolute of $S \setminus \mathrm{Char}(S)$ along the exponential map is, at least for small times, still a surface of class C^1. This Remark will be useful in the next section.

From Lemma 3.17 and some other number of Lemmata (see [3], §6) it follows

Theorem 3.19. *If S is C^k in the Euclidean sense, $k \geq 2$, then δ_S and $\nabla_{\mathbb{H}}\delta_S$ are of class C^{k-1} in the Euclidean sense, in an open neighborhood of $S \setminus \mathrm{Char}(S)$.*

Proof. See [3], §6, Theorem 6.1. □

In particular, according to our hypothesis H1 and H2, we will always have that δ_S and $\nabla_{\mathbb{H}}\delta_S$ are at least of class C^1.

3.1 The cut locus and our partial results

Now we are ready to define the cut locus of S. Moreover, we present some partial results we have obtained on the exponential map. We think that such results could be the starting point for a proof of the closure of the cut locus of S.

Definition 3.20. Let $P \in S$ and define $Q \in \overline{\Omega}$ to be the endpoint, other than P, of $\mathcal{N}_P S$. When $\mathcal{N}_P S$ reduces to the only point P, we set $Q = P$. The set of such points Q as P varies over S, denoted by K_S or $\mathrm{cut}(S)$, is the cut locus of S in $\overline{\Omega}$. The definition of the cut locus of S in $\mathbb{H} \setminus \overline{\Omega}$ is the same, with $Q \in (\mathbb{H} \setminus \overline{\Omega})$.

In other words $\mathrm{cut}(S)$ is the set of the metric normals' endpoint of S. Below K_S denote the cut locus of S in $\overline{\Omega}$ and $\mathcal{N}_P S$ refers to the portion of metric normal at P which lies in $\overline{\Omega}$.

Proposition 3.21. *The cut locus K_S of S has the following properties*

i) *K_S has empty interior*

ii) $\mathrm{Char}(S) \subset K_S$ *and each characteristic point of S is an accumulation point of K_S*

iii) *Let $R \in (\mathbb{H} \setminus K_S)$. Then there is a unique geodesic γ from R to S such that $L_{\mathbb{H}}(\gamma) = d_S(R)$, i.e.* $\mathrm{Unp}(S) \subset K_S$

Proof. See [3], §7, Propositions 7.1 and 7.2. □

As we have said at the beginning of this paper, we think that K_S is a closed subset of $\mathbb{H}$, but we are still unable to prove it. The first step we have taken is to find a link between the exponential map associated with S and the cut locus K_S: clearly, by (iii) of Proposition 3.21, the exponential map is differentiable in an open subset of $\mathbb{H} \setminus K_S$. So we define

Definition 3.22. Let $(P, s) \in \mathcal{C}$ such that $\exp_S$ is not a local diffeomorphism around (P, s) and, $\forall\ 0 < \varepsilon \leq s$, $\exp_S$ is a local diffeomorphism around $(P, s - \varepsilon)$. Then we call $\exp_S(P, s)$ conjugate point of S (or S-conjugate).

For example, characteristic points of S are conjugate points of S. We use the name "conjugate" for the similarity that is noted with the Riemannian case. The next Lemma gives a limit condition in order that a point of $\mathbb{H}$ is S-conjugate.

Lemma 3.23. *Let S_τ denote the surface $\exp_S(S, \tau)$. Let $P \in S$ be noncharacteristic and suppose that there is a $\sigma > 0$ such that $\exp_S(P, \sigma)$ is characteristic for S_σ. Then $\exp_S(P, \sigma + \varepsilon)$ can not be S-coniugate, $\forall\ \varepsilon > 0$.*

Before giving the proof we need some preparation. Suppose that A is an open subset of $\mathbb{C}$ and that S has implicit equation, around a noncharacteristic point P

$$S = \{(\alpha(u, v), \beta(u, v), \gamma(u, v)) : (u, v) \in A\} \tag{3.22}$$

We want compute the tangent plane, $\Pi_P S$, of S at P and the distance between P and the characteristic point C of $\Pi_P S$.

Since $P \in S$, we can write $P = (\alpha(u, v), \beta(u, v), \gamma(u, v))$. So the tangent plane is given by

$$\begin{vmatrix} x-\alpha & y-\beta & t-\gamma \\ \alpha_u & \beta_u & \gamma_u \\ \alpha_v & \beta_v & \gamma_v \end{vmatrix} = 0 \tag{3.23}$$

Equation (3.23) becomes

$$t = \frac{1}{\alpha_u\beta_v - \alpha_v\beta_u}((\gamma_u\beta_v - \gamma_v\beta_u)(x-\alpha) + (\alpha_u\gamma_v - \alpha_v\gamma_u)(y-\beta) + \gamma) \tag{3.24}$$

if $\alpha_u\beta_v - \alpha_v\beta_u \neq 0$, while we have the vertical plane

$$(\gamma_u\beta_v - \gamma_v\beta_u)(x-\alpha) + (\alpha_u\gamma_v - \alpha_v\gamma_u)(y-\beta) = 0 \tag{3.25}$$

if $\alpha_u\beta_v - \alpha_v\beta_u = 0$. In (28) the characteristic point is

$$C = \frac{1}{2(\alpha_u\beta_v - \alpha_v\beta_u)}(\alpha_v\gamma_u - \alpha_u\gamma_v, \gamma_u\beta_v - \gamma_v\beta_u, c) \tag{3.26}$$

where c contains everything that in (28) does not multiply x and y. The distance between P and C is (in the nonvertical case)

$$d_{cc}(P,C)^2 = \left(\alpha + \frac{\alpha_u\gamma_v - \alpha_v\gamma_u}{2(\alpha_u\beta_v - \alpha_v\beta_u)}\right)^2 + \left(\alpha + \frac{\beta_u\gamma_v - \beta_v\gamma_u}{2(\alpha_u\beta_v - \alpha_v\beta_u)}\right)^2 \tag{3.27}$$

Proof of Lemma 3.24. We denote $P_\tau := \exp_S(S,\tau)$ and we use the same notations of Lemma 3.17. Then we have a local parametrization of S around P of the type $(u,v,f(u,v))$, with $(u,v) \in A \subset \mathbb{C}$. Since $\exp_S$ is a local diffeomorphism, we can restrict A just enough to have S_τ of class C^1 for all τ. This allow us to derive F at each time τ.

Let $\tau > 0$ be fixed and suppose that $\Pi_{P_\tau} S_\tau$ is nonvertical. This is sure possible for a certain time $\tau < \sigma$, otherwise we have that the distance from P_τ and the characteristic point of $\Pi_{P_\tau} S_\tau$ should be ∞ for all $\tau \leq \sigma$. But this contradicts our hypothesis. With these assumptions the parametrization of S_τ is as in (3.22), with $\alpha = u + x'$, $\beta = v + y'$ and $\gamma = t'$. Then the Jacobian of the map F at the time τ is

$$JF(u,v,\tau) = \begin{pmatrix} 1 + x'_u & x'_v & x'_\tau \\ y'_u & 1 + y'_v & y'_\tau \\ t'_u & t'_v & t'_\tau \end{pmatrix} \tag{3.28}$$

and

$$\begin{aligned} \frac{|JF(u,v,\tau)|}{(1+x'_u)(1+y'_v) - x'_v y'_u} =& x'_\tau \frac{y'_u t'_v - t'_u(1+y'_v)}{(1+x'_u)(1+y'_v) - x'_v y'_u} + \\ &+ y'_\tau \frac{x'_v t'_u - t'_v(1+x'_u)}{(1+x'_u)(1+y'_v) - x'_v y'_u} + t'_\tau \end{aligned} \tag{3.29}$$

For $\tau = \sigma$ we have, by hypothesis, $d(P_\sigma, C) = 0$, where C is the characteristic point of $\Pi_{P_\tau} S_\tau$. So by (31) we have

$$\frac{y'_u t'_v - t'_u(1 + y'_v)}{(1 + x'_u)(1 + y'_v) - x'_v y'_u} = -2(u + x') \tag{3.30}$$

$$\frac{x'_v t'_u - t'_v(1 + x'_u)}{(1 + x'_u)(1 + y'_v) - x'_v y'_u} = 2(v + y') \tag{3.31}$$

Putting (3.30) and (3.31) into (3.29) and calculating all the derivatives (the explicit calculus is just made in [3], §6, proof of Lemma 6.5) we obtain $|JF(u, v, \sigma)| = 0$. □

Corollary 3.24. *Let P be a point of S and suppose that P_τ is characteristic for S_τ. Then P_τ is a conjugate point for the domain of F.*

Proof. As in Lemma 3.17 one have $|JF(u, v, \tau)| = 2d_{cc}(P_\tau, C_\tau)$ for all $\tau < \sigma$. Here C_τ is the characteristic point of the tanget plane at the time τ. Indeed we have seen that at the time τ, S_τ has a C^1 implicit expression as in (3.22). So we can restrict around the point P_τ to find an explicit expression of the kind $(u, v, f^{(\tau)}(u, v))$. Then just repeat the proof of Lemma 3.17 by replacing f with $f^{(\tau)}$. Since we have supposed that the point $\exp_S(P, \tau)$ is noncharacteristic, the Jacobian of F must be other than 0 for all $\tau < \sigma$. So the characteristic points which S develops at a certain time σ (if they exist!) are necessarily conjugate points of the domain of F. □

Remark 3.25. The opposite implication in Corollary 3.24 is not necessarly true. Indeed, if $\exp_S(P, \tau)$ is a conjugate points of the domain of F, by the inverse function Theorem we have $|JF(u, v, \tau)| = 0$. But we are not sure that S_τ is C^1 in a neighborhood of P_τ! So we cannot conclude that $|JF(u, v, \tau)| = 2d_{cc}(P_\tau, C_\tau) = 0$ (which would means the point $\exp_S(P, \tau)$ is characteristic for S_τ) because we are not sure that S_τ admits a tangent plane at P_τ.

So we need to explain what really happens to the conjugate points of S. We start by the following

Proposition 3.26. *The evolution S_τ of S cannot develops any characteristic points.*

Proof. Suppose that P_τ is characteristic for S_τ. Consider an Heisenberg ball $B = B(P, \tau)$ of radius τ and center in P having P_τ on its boundary. We want to first show that P_τ is necessarily a pole of B. Suppose P_τ is not a pole of B. Certainly, the surface S_τ cannot have any point in the interior of the ball B, otherwise a point of S_τ would be reached in a shorter time than τ. Since P_τ is characteristic for S_τ, the tangent plane $\Pi_{P_\tau} S_\tau$ is horizontal. But the geodesic $\mathcal{N}_P S$ can be extended for times longer than τ, because P_τ is not a pole. Thus, the normal vector $N_{P_\tau} S_\tau$ to S_τ at P_τ is horizontal too. This is impossible because we would have a 3-dimensional horizontal space in P_τ. Then the other chance for P_τ is to be a pole of B.

Since we have already seen that the surface S_τ cannot have any point in the interior of the ball B, we can conclude that S_τ develops a singularity at P_τ having the same behavior of the the pole of the CC ball B. Hence, P_τ cannot be characteristic for S_τ. □

Corollary 3.27. *Let P_τ be a conjugate point of S. Then S_τ loses its regularity of class C^1 in P_τ.*

Proof. If S_τ is C^1 in a neighborhood of P_τ, then by Remark 3.25 the converse of Corollary 3.24 holds. This means that P_τ needs to be characteristic for S_τ. But it is impossible by Proposition 3.26.

Moreover, by the proof of Proposition 3.26, we have that the singularity developed at P_τ has the same behavior as a CC ball's pole. □

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