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Geodesic Motion and Raychaudhuri Equations

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Abstract

The work presented in this thesis is devoted to the study of geodesic motion in the context of General Relativity. The motion of a single test particle is governed by the geodesic equations of the given space-time, nevertheless one can be interested in the collective behavior of a family (congruence) of test particles, whose dynamics is controlled by the Raychaudhuri equations. In this thesis, both the aspects have been considered, with great interest in the latter issue.
Geometric quantities appear in these evolution equations, therefore, it goes without saying that the features of a given space-time must necessarily arise. In this way, through the study of these quantities, one is able to analyze the given space-time.
In the first part of this dissertation, we study the relation between geodesic motion and gravity. In fact, the geodesic equations are a useful tool for detecting a gravitational field. While, in the second part, after the derivation of Raychaudhuri equations, we focus on their applications to cosmology. Using these equations, as we mentioned above, one can show how geometric quantities linked to the given space-time, like expansion, shear and twist parameters govern the focusing or de-focusing of geodesic congruences. Physical requirements on matter stress-energy (i.e., positivity of energy density in any frame of reference), lead to the various energy conditions, which must hold, at least in a classical context. Therefore, under these suitable conditions, the focusing of a geodesics “bundle”, in the FLRW metric, bring us to the idea of an initial (big bang) singularity in the model of a homogeneous isotropic universe. The geodesic focusing theorem derived from both, the Raychaudhuri equations and the energy conditions acts as an important tool in understanding the Hawking-Penrose singularity theorems.

Sommario

L'elaborato presentato in questa tesi è incentrato sullo studio del moto geodetico nell'ambito della Relatività Generale. Il moto di una singola particella di prova è governato dall'equazione geodetica dello spazio-tempo in cui è immersa. Tuttavia, si può essere anche interessati al comportamento collettivo di una famiglia (congruenza) di particelle di prova, la cui dinamica è cotrollata dalle equazioni di Raychaudhuri. In questa tesi, entrambi gli aspetti sono stati presi in considerazione, con maggiore interesse nei confronti del secondo.
Le equazioni di evoluzione coinvolgono termini di natura geometrica caratterizzanti il dato spazio-tempo, perciò, è scontato che le proprietà di tale spazio-tempo debbano necessariamente emergere. In questo modo, attraverso lo studio di questi termini, si è in grado di analizzare la struttura di quest'ultimo.
Nella prima parte della tesi, si studia il legame tra moto geodetico e gravità. Infatti, le equazioni geodetiche sono un utile strumento mediante il quale rilevare un campo gravitazionale. Nella seconda parte, invece, in seguito alla derivazione delle equazioni di Raychaudhuri, ci si concentra sulle loro applicazioni in cosmologia. Tramite queste equazioni, come si è accennato, si può mostrare in che modo quantità di natura geometrica legate allo spazio-tempo considerato, quali espansione, taglio e torsione governino la focalizzazione o de-focalizzazione delle congruenze geodetiche. Requisiti fisici sulla materia (la richiesta di positività della densità di energia in ogni sistema di riferimento), portano alle cosiddette condizioni sull'energia, le quali devono valere almeno in un contesto classico. Sotto queste condizioni, la focalizzazione di un insieme di geodetiche, considerando la metrica FLRW, conduce all'idea dell'esistenza di una singolarità iniziale (big bang), in un modello cosmologico omogeneo e isotropo. Il teorema di focalizzazione geodetica derivato da entrambe, le equazioni di Raychaudhuri e le condizioni sull'energia riveste un ruolo importante nella comprensione dei teoremi sulle singolarità di Hawking e Penrose.

Contents

Introduction

The aim of this thesis is to show the fundamental role of geodesic motion in the fields of General Relativity and Cosmology.
In order to do so, we start from scratch, and define a special topological space, the manifold $\mathcal{M}$. This structure is essentially a space which is locally similar to Euclidean space, in that it can be covered by coordinate patches. In such a space, we define a way of parallel transporting quantities along a given path, and from that, we come up with geodesics as curves whose parallel vectors are parallely transported along them. Further, given a generic point $P \in \mathcal{M}$ and its neighborhood U_P, the notions of distance and angles can be introduced through the metric tensor g_{ab}. If we consider a manifold endowed with a metric, namely $(\mathcal{M}, g_{ab})$, this manifold corresponds to our intuitive ideas of the continuity of space and time. So, this brief introduction on differential geometry is fundamental in building a suitable mathematical environment, where the General Theory of Relativity can be developed.
The crucial step from Special to General Relativity is embedded in the definition of observer and of reference frame. From a mathematical point of view, Special Relativity is based on the principle that "*The laws of physics are the same for all inertial observers and the speed of light in vacuum is invariant*". This is realized by assuming the existence of global (inertial) reference frames. In contrast, General Relativity is based on the principle that "*The laws of physics are the same in all reference frames (for all observers)*" and so these laws have to be expressed in a form which can be adapted to any measuring apparatus, regardless of its inertial nature. This is achieved by the use of tensorial quantities of the space-time manifold, in particular by using the general metric g_{ab} instead of the Minkowski metric $\eta_{ab} = diag(-1, 1, 1, 1)$ and the covariant derivative ∇_a instead of the partial derivative ∂_a. In this way, the problem of General Relativity consists in finding the metric field g_{ab} describing the manifold. By means of the relation between the Ricci and stress-energy tensor, the former describing the curvature of the metric, the latter the distribution of all forms of matter, we come up with the Einstein's field equations.
Since in General Relativity (GR) a freely falling observer is "inertial", the local metric in its own reference frame is the canonical Minkowski one. So, test particles follow geodesics of the given space-time metric. It is for this reason that the study of geodesic motion is

fundamental. As a matter of fact, through the evolution of geodesic curves, we are able to study the properties of the given space-time.

While the motion of a single test particle is governed by the geodesic equations, the collective behavior of a family (congruence) of test particles is governed by the Raychaudhuri equations. The Raychaudhuri equations are evolution equations along a given congruence for certain variables (expansion, shear and rotation), and were first obtained by A. K. Raychaudhuri. His main motivation behind obtaining these equations was, at that time, restricted to cosmology. It is well known that, in GR, the Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology, which is a homogeneous and isotropic cosmological model of our universe, has an initial big bang singularity. It was during the middle of the twentieth century, Raychaudhuri tried to address the question whether the occurrence of the initial singularity in the FLRW model, was an artifact of the space-time symmetry encoded in the assumptions of homogeneity and isotropy or a generic feature of the gravity theory (GR). One of his aims was to see whether non-zero rotation (spin), anisotropy (shear) and/or a cosmological constant can be successful in avoiding the initial singularity. This is how the quest towards constructing a singularity free model of our universe led Raychaudhuri to arrive at his well-known and celebrated Raychaudhuri equations.

As mentioned above, the Raychaudhuri equations are evolution equations along a congruence for a set of kinematic variables known as expansion θ, shear σ_{ab} and rotation (twist) ω_{ab} (ESR). These are respectively, the trace, symmetric trace-less and anti-symmetric parts of the covariant gradient of the normalized velocity field $B_{ab} = \nabla_b u_a$. Along the congruence, θ measures the rate of change of the cross sectional area enclosing a family of geodesics, σ_{ab} measures the shear, and ω_{ab} measures the rotation or twist. In other words, the Raychaudhuri equations and their solutions demonstrate how a congruence evolves as a whole in a given space-time background. A geodesic congruence converges when θ is negative and diverges when θ is positive. A complete convergence implies $\theta \to -\infty$ which is termed as geodesic focusing. Geometric quantities such as the metric g_{ab} , the Riemann tensor R^a_{bcd}, the Ricci tensor R_{ab} and the Weyl tensor C^a_{bcd}, appear in the evolution equations along with the velocity field. It is therefore obvious that generic geometric features of a given space-time must necessarily be reflected in the evolution of a congruence.
The above discussion explains the usefulness of the Raychaudhuri equations in examining the features of a given space-time geometry through studies on the behavior of geodesic congruences. In addition, we know that the field equations of a given theory of gravity relates geometry to matter stress energy. Thus, a condition for geodesic focusing, which in a purely geometric context involves geometric quantities such as the Ricci tensor, may be translated into conditions on matter stress energy. Such conditions on matter are generally known as the energy conditions and are considered as independent conditions

on matter, largely motivated by simple physical requirements such as the positivity of energy density in a purely classical context. Since their introduction, the Raychaudhuri equations have appeared and been analyzed in a variety of contexts. Among their applications within GR, the most important and prominent one is their utility in understanding the space-time singularity problem. The geodesic focusing theorem derived from these equations, with the assumption that the convergence condition must be satisfied (i.e., $R_{ab}u^a u^b \geq 0$), acts as an important tool in understanding the Hawking-Penrose singularity theorems. In GR, physically reasonable classical matter satisfying all the energy conditions, also satisfies the convergence condition. This, together with a minimal set of assumptions such as Lorentz signature metrics, causality and the existence of trapped surfaces, lead to the inevitable existence of a space-time singularity; a result encoded in the singularity theorems.

Raychaudhuri equations have also been studied and analyzed beyond the context of gravity theories. It has been conjectured that "wherever there are vector fields describing a physical or geometrical quantity, there must be corresponding Raychaudhuri equations". In other words, wherever one can think of a well defined congruence (i.e., a family of world-lines) generated by a vector field, there must be corresponding Raychaudhuri equations.

Chapter 1

Geodesic motion on manifolds

In this first chapter, we introduce some basic concepts of differential geometry. Starting off with the definition of parallel transport, we proceed with that of geodesic curve, which will be the key tool of our study. Immediately after, we describe the main properties inherent geodesic curves, like: normal frames, metric connection and geodesic deviation. What we are going to develop in this first chapter is a necessary introduction for our study. In fact, as we will see later, the Raychaudhuri equations, which will be employed in different occasions, describe the dynamical evolution of a family of geodesics, and this makes their motion in a given space(-time) of crucial importance.

1.1 Parallelism and covariant derivatives

On a manifold without the notion of angles (namely, without a metric), the only definition of parallelism can be given at a point P: two vectors of the tangent space at P, T_P are parallel if they are linearly dependent. But one then needs a way to confront vectors belonging to the tangent spaces at different points. In particular, one can define a rule for transporting a vector "parallely" along a given path. This rule is named affine connection.

Suppose we have a curve γ defined on a manifold $\mathcal{M}$ and a connection, a rule for parallel transport. Let the tangent to γ be $\vec{V} = \frac{\mathrm{d}}{\mathrm{d}\lambda}$. At the point P, pick an arbitrary vector $\vec{W}$ from T_P. Then the connection allows us to define a vector field $\vec{W}$ along the curve γ, which is obtained by parallel-transporting $\vec{W}$. Since we can say that $\vec{W}$ does not change along γ, we can define a derivative with respect to which $\vec{W}$ has zero rate of change. This is called the covariant derivative along $\vec{V}$, $\nabla_{\vec{V}}$, and we write:

$$\nabla_{\vec{V}}\vec{W} = 0 \Longleftrightarrow \vec{W} \text{ is parallel-transported along } \gamma. \tag{1.1}$$

If $\vec{W}$ is a vector field defined everywhere on γ, we can define its covariant derivative along γ. This operation will associate to $\vec{W}$ a second vector $\vec{W}''_{-\triangle\lambda} \in T_{P(\lambda_0)}$, where $\lambda = \lambda_0 + \triangle\lambda$ identifies a displaced point on the curve (see Figure 1.1).
We then define the covariant derivative of the vector field $\vec{W}$ with respect to $\vec{V}$ at the point $P(\lambda_0)$ as the vector given by the limiting process:

$$\nabla_{\vec{V}}\vec{W}\Big|_{\lambda_0} = \lim_{\triangle\lambda\to 0}\frac{\vec{W}''_{-\triangle\lambda} - \vec{W}(\lambda_0)}{\triangle\lambda}, \tag{1.2}$$

whose result is a vector, by definition, and vanishes if the parallely transported vector coincides with the original vector in P.

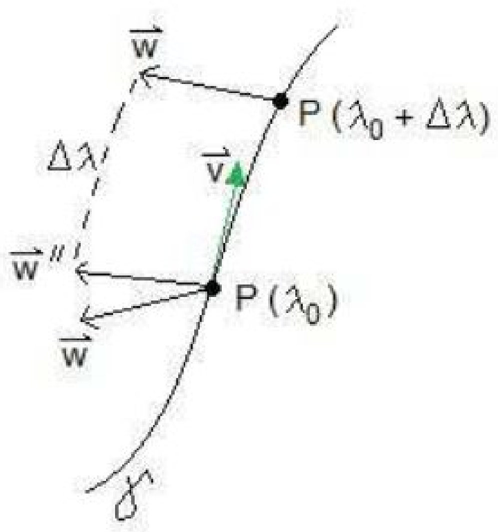

Figure 1.1: Parallel transport

We also assume that a change of parameterization of the curve $\gamma \to \gamma'$, that is $\lambda \to \mu(\lambda)$, does not affect the notion of parallelism. Let $\vec{V} = \frac{\mathrm{d}}{\mathrm{d}\lambda}$ and $\vec{V}' = \frac{\mathrm{d}}{\mathrm{d}\mu}$ be the tangent vectors to γ and γ' respectively,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\mu} \equiv h\frac{\mathrm{d}}{\mathrm{d}\mu} \ . \tag{1.3}$$

We then impose that

$$\nabla_{h\vec{V}}\vec{W} = h\nabla_{\vec{V}}\vec{W} \ , \tag{1.4}$$

for all smooth functions h, so that $\nabla_{\vec{V}}\vec{W} = 0$ implies $\nabla_{h\vec{V}}\vec{W} = 0$.

It is customary to name "covariant derivative of the vector $\vec{W}$" the formal operator associated with the above derivative acting on a given $\vec{W}$, but with no specific curve (and thus for all vectors $\vec{V}$). This object at a point P can be viewed as a type (1,1) tensor,

$$\nabla \vec{W} : \vec{V} \to \nabla_{\vec{V}} \vec{W}, \tag{1.5}$$

which associates to any $\vec{V}$ the corresponding derivative of $\vec{W}$.

1.2 Affine connection

Taken for granted the covariant derivative properties, one can obtain the components of the covariant derivative of a vector field in terms of the so-called Christoffel symbols. By expanding both $\vec{V}(\lambda_0)$ and the difference between $\vec{W}(\lambda_0)$ and $\vec{W}''(\lambda_0)$ on a basis of $T_{P(\lambda_0)}$ we get:

$$\begin{aligned} \nabla_{\vec{V}} \vec{W} &= \nabla_{V^i \vec{e}_i} (W^j \vec{e}_j) \\ &= V^i \nabla_{\vec{e}_i} (W^j \vec{e}_j) \\ &= V^i [(\nabla_{\vec{e}_i} W^j) \vec{e}_j + W^j (\nabla_{\vec{e}_i} \vec{e}_j)]. \end{aligned} \tag{1.6}$$

The second term in brackets is called the affine connection (or Christoffel Symbols). Since $\frac{\partial \vec{e}_j}{\partial x^i}$ is itself a vector and is just equivalent to $\nabla_{\vec{e}_i} \vec{e}_j$ we introduce the symbol $\Gamma^k_{ji} \vec{e}_k$ to denote the coefficients in this combination:

$$\nabla_{\vec{e}_i} \vec{e}_j = \Gamma^k_{ji} \vec{e}_k \tag{1.7}$$

The interpretation of $\Gamma^k_{ji} \vec{e}_k$ is that it is the k-th component of $\frac{\partial \vec{e}_j}{\partial x^i}$. It needs three indices: one (j) gives the basis vector being differentiated; the second (i) gives the coordinate with respect to which it is being differentiated; and the third (k) denotes the component of the resulting derivative vector.
Then, Eq. (1.6) can be written in the following manner:

$$\begin{aligned} \nabla_{\vec{V}} \vec{W} &= V^i [(\nabla_{\vec{e}_i} W^j) \vec{e}_j + W^j (\nabla_{\vec{e}_i} \vec{e}_j)] \\ &= V^i \Big[\frac{\partial W^k}{\partial x^i} + W^j \Gamma^k_{ji} \Big] \vec{e}_k \ . \end{aligned} \tag{1.8}$$

From which we can read out the components

$$\big(\nabla \vec{W}^k \big)_i = \frac{\partial W^k}{\partial x^i} + W^j \Gamma^k_{ji} \ . \tag{1.9}$$

Several different notations are in use for these components, for example

$$\nabla_i W^k = W^k_{;i} = W^k_{,i} + \Gamma^k_{ji} W^j \ . \tag{1.10}$$

Where the last equality underlines the difference between the covariant and partial derivatives, due to the presence of the Christoffel symbols.

1.3 Geodesic curves

A geodesic is a preferred curve along which the tangent vector to the curve itself is transported parallely. This notion allows us to extend to a general manifold the concept of "straight line" and, eventually, of extremal curve (on metric manifolds).
Let $\vec{V} = \frac{\mathrm{d}}{\mathrm{d}\lambda}$ be the tangent vector to a curve γ parameterized by $\lambda \in \mathbb{R}$. Then, γ is a geodesic if $\vec{V}$ satisfies:

$$\nabla_{\vec{V}} \vec{V} \Big|_{P} = 0, \ \forall P \in \gamma, \tag{1.11}$$

and λ is then called an affine parameter. From Eq. (1.4), it immediately follows that this definition is invariant under a change of parameterization of γ, which implies that the same geodesic can be described by different affine parameters.
Eq. (1.11) can be written in a local coordinate frame, in which $\gamma \in \mathcal{M}$ is mapped into $x^k = x^k(\lambda) \in \mathbb{R}^n$, as

$$\begin{aligned} \nabla_{\vec{V}} \vec{V} &= V^j \Big(\frac{\partial V^k}{\partial x^j} + \Gamma^k_{ji} V^i \Big) \\ &= \frac{\mathrm{d}V^k}{\mathrm{d}\lambda} + \Gamma^k_{ji} V^i V^j \\ &= \frac{\mathrm{d}^2 x^k}{\mathrm{d}\lambda^2} + \Gamma^k_{ji} \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda}, \end{aligned} \tag{1.12}$$

which is, given the definition $\vec{V} = V^k \vec{e_k} = dx^k \backslash d\lambda \cdot \partial \backslash \partial x^k = \frac{d}{d\lambda}$, a set of n second-order differential equations for the variables $x^k = x^k(\lambda)$.

1.4 Normal frames and metric connection

It is very helpful to use a coordinate system based on geodesics. To construct this system, we note that the geodesic curves through a point P give a 1-1 mapping of a neighborhood of P onto a neighborhood of the origin of T_P. This map arises because each element of T_P defines a unique geodesic curve through P, so we can associate the vector in T_P with the point an affine parameter $\triangle\lambda = 1$ along the curve from P. Using this map and choosing an arbitrary basis for T_P, one defines the normal coordinates of a point Q to be the components of the vector in T_P it is associated with. This map will generally be 1-1 only in some neighborhood of P, since geodesics may cross on a curved manifold. For our purposes the principal interest in the normal coordinates is that $\Gamma^k_{ij} = 0$ at P (but not elsewhere in the neighborhood of P). To see this, note that if a vector $\vec{V}$ with components $V^i(P)$ defines a geodesic curve, then the coordinates of the point with affine parameter λ along that curve are simply $x^i = \lambda V^i(P)$, with the convection that $\lambda = 0$ at P. Therefore $d^2x^i \backslash d\lambda^2$ vanishes, and (1.12) tells us that $\Gamma^k_{ij} V^i(P) V^j(P)$ must vanish along that whole curve. At P, however, V^i had an arbitrary direction, which means that $\Gamma^k_{ij}(P) = 0$. This bring us to the definition of a normal frame:

$$\Gamma^k_{ij}\Big|_P = 0 \Rightarrow \text{The system is gaussian (normal) around P.} \tag{1.13}$$

We are obviously really interested in the case in which parallel transport preserves lengths and angles, which requires the manifold $\mathcal{M}$ to be endowed with a metric tensor g_{ij}. Let us then consider two vectors $\vec{A}$ and $\vec{B}$, and assume they are transported parallely along a curve of tangent $\vec{V}$, that is $\nabla_{\vec{V}} \vec{A} = \nabla_{\vec{V}} \vec{B} = 0$. It is natural to demand that the scalar product between these two vectors does not change along the curve,

$$\nabla_{\vec{V}} \left[g(\vec{A}, \vec{B}) \right] = 0 \ , \quad \forall \quad \vec{A}, \vec{B}, \vec{V} \quad \text{such that} \quad \nabla_{\vec{V}} \vec{A} = \nabla_{\vec{V}} \vec{B} = 0 \ , \tag{1.14}$$

which, from the Leibniz rule, implies

$$\nabla_{\vec{V}} g = 0 \ , \quad \forall \vec{V} \quad \Rightarrow \nabla g = 0 \ . \tag{1.15}$$

Since g is symmetric by definition and through the following relation,

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lk,j} + g_{lj,k} - g_{jk,l}) \ , \tag{1.16}$$

one can immediately see that a metric connection is necessarily symmetric (namely $\Gamma^i_{jk} = \Gamma^i_{kj}$). All expressions can be simplified by assuming the metric is in canonical form at a point P, so that it can be expanded as

$$g_{ij} = \pm \delta_{ij} + \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}\Big|_P \delta x^k \delta x^l + \dots \ . \tag{1.17}$$

Once we have connected the parallel transport to the metric, we can also see that geodesics are indeed curves of local extremal length: suppose we take a specific geodesic γ of parameter λ and construct a Gaussian normal frame around it. In a (sufficiently small) neighborhood of γ, the metric will take the form (1.17), so that moving off the geodesic from the point $P = P(\lambda_P)$ along each Gaussian direction η_i, with $i = 1, \dots, n-1$, (from $ds^2 = g_{ij} dx^i dx^j$) one has

$$ds^2 \simeq \frac{1}{2} \frac{\partial^2 g_{ii}}{(\partial \eta^i)^2}\Bigg|_{\lambda = \lambda_P, \eta^1 = \dots = \eta^{n-1} = 0} (d\eta^i)^2 + \dots \ , \tag{1.18}$$

where there is no sum over the index i.

One can therefore conclude that each portion of a geodesic is a local extremum for the length of a curve. In fact, the above argument about geodesics can be used in order to derive Eq. (1.16). Let us consider the length of a curve between two fixed points a and b, namely:

$$s = \int_a^b \mathrm{d}s = \int_a^b \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\,\mathrm{d}\lambda \equiv \int_{\lambda_a}^{\lambda_b} \sqrt{2L(x^k, \dot{x}^l)}\,\mathrm{d}\lambda\ , \tag{1.19}$$

where a dot denotes the derivative with respect to the affine parameter λ; further we have $g_{ij} = g_{ij}(x^k)$. If we identify $\lambda = s$, we obviously have $2L = 1$ and, varying the above action is equivalent to varying the action without the square root, that is

$$\delta s = \delta \int_{s_a}^{s_b} \sqrt{2L}\mathrm{d}s = \int_{s_a}^{s_b} \frac{\delta L}{\sqrt{2L}}\,\mathrm{d}s = \delta \int_{s_a}^{s_b} L(x^k, \dot{x}^l)\,\mathrm{d}s\ , \tag{1.20}$$

By requiring $\delta s = 0$, we find the Euler-Lagrange equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}^m}\right) - \frac{\partial L}{\partial x^m} = 0\ . \tag{1.21}$$

In particular, one finds

$$\frac{\partial L}{\partial x^m} = \frac{1}{2}g_{jk,m}\dot{x}^j\dot{x}^k\ , \tag{1.22}$$

and

$$\frac{\partial L}{\partial \dot{x}^m} = g_{mj}\dot{x}^j\ , \tag{1.23}$$

from which

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^m}\right) &= g_{mj}\ddot{x}^j + g_{mj,k}\dot{x}^j\dot{x}^k \\ &= g_{mj}\ddot{x}^j + \frac{1}{2}(g_{mj,k} + g_{jm,k})\dot{x}^j\dot{x}^k\ .\end{aligned} \tag{1.24}$$

Putting the two parts together and multiplying by g^{im} we obtain

$$\ddot{x}^i + \frac{1}{2}g^{il}\big(g_{lk,j} + g_{lj,k} - g_{jk,l}\big)\dot{x}^j\dot{x}^k = 0\ , \tag{1.25}$$

which equals the geodesic equation

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0\ . \tag{1.26}$$

1.5 Geodesic Deviation

In a curved space, parallel lines when extended do not remain parallel. This can be formulated in terms of the Riemann tensor.

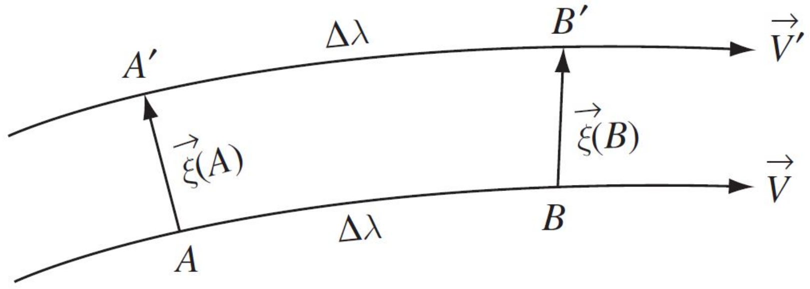

Figure 1.2: Geodesic Deviation

Consider two geodesics (with tangents $\vec{V}$ and $\vec{V}'$) that begin parallel and near each other, as in Fig. 1.2, at points A and A'. Let the affine parameter on the geodesics be called λ. We define a "connecting vector" $\vec{\xi}$ which "reaches" from one geodesic to another, connecting points at equal intervals in λ.
For simplicity, let us adopt a normal coordinate system at A (which, as we will see later in Chapter 2, is a locally inertial reference frame), in which the coordinate x^0 points along the geodesics and advances at the same rate as λ there.
Then because $V^\alpha = dx^\alpha/d\lambda$ we have at A, $V^\alpha = \delta^\alpha_0$. The equation of the geodesic at A is:

$$\left.\frac{d^2x^\alpha}{d\lambda^2}\right|_A = 0, \tag{1.27}$$

since all Christoffel symbols vanish at A. The Christoffel symbols do not vanish at A', so the equation of the geodesic $\vec{V}'$ at A' is

$$\left.\frac{d^2x^\alpha}{d\lambda^2}\right|_{A'} + \Gamma^\alpha_{00}(A') = 0, \tag{1.28}$$

where again at A' we have arranged the coordinates so that $V^\alpha = \delta^\alpha_0$. But, since A and A' are separated by $\vec{\xi}$, we have upon Taylor expanding:

$$\Gamma^\alpha_{00}(A') \cong \Gamma^\alpha_{00,\beta}\xi^\beta, \tag{1.29}$$

the right-hand side being evaluated at A. With Eq. (1.28) this gives

$$\left.\frac{d^2x^\alpha}{d\lambda^2}\right|_{A'} = -\Gamma^\alpha_{00,\beta}\xi^\beta. \tag{1.30}$$

Now, the difference $x^\alpha(\lambda, geodesic\ \vec{V}) - x^\alpha(\lambda, geodesic\ \vec{V'})$ is just the component ξ^α of the vector $\vec{\xi}$. Therefore, at A, we have

$$\frac{d^2\xi^\alpha}{d\lambda^2} = \frac{d^2x^\alpha}{d\lambda^2}\bigg|_{A'} - \frac{d^2x^\alpha}{d\lambda^2}\bigg|_{A} = -\Gamma^\alpha_{00,\beta}\xi^\beta. \tag{1.31}$$

This, then gives how the components of $\vec{\xi}$ change. But since the coordinates are to some extent arbitrary, we want to have, not merely the second derivative of ξ^α but the full second covariant derivative $\nabla_{\vec{V}}\nabla_{\vec{V}}\vec{\xi}$. We can use the general form

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\xi^\alpha \approx \nabla_\beta B^\mu_\nu = B^\mu_{\nu,\beta} + B^\alpha_\nu\Gamma^\mu_{\alpha\beta} - B^\mu_\alpha\Gamma^\alpha_{\nu\beta} \tag{1.32}$$

so the following equation yields,

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\xi^\alpha = \nabla_{\vec{V}}(\nabla_{\vec{V}}\xi^\alpha) = \nabla_{\vec{V}}(\xi^\alpha{}_{;\mu}) \tag{1.33}$$

with $\nu = 0$ and the affine parameter $\lambda \propto x^0$, Eq. (1.34). Considering that, at point A, the Christoffel symbols vanish, we simply obtain for the double covariant derivative of $\vec{\xi}$:

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\xi^\alpha = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\xi^\alpha + \Gamma^\alpha_{\beta 0}\xi^\beta\right) + 0 - 0 = \frac{d^2\xi^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta 0,0}\xi^\beta \tag{1.34}$$

We have also used $\xi^\beta{}_{,0} = 0$ at A, which is the condition that curves begin parallel. So we get

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\xi^\alpha = \left(\Gamma^\alpha_{\beta 0,0}\xi^\beta - \Gamma^\alpha_{00,\beta}\xi^\beta\right)\xi^\beta = R^\alpha_{00\beta}\xi^\beta = R^\alpha_{\mu\nu\beta}V^\mu V^\nu\xi^\beta \tag{1.35}$$

In the last passage, we remind the relation $V^\alpha = \delta^\alpha_0$.

The Riemann tensor $R^\alpha_{\beta\mu\nu}$, is the geometric object which describes in a covariant way the curvature properties of a manifold. The third equality follows from the identity (with again vanishing Christoffel symbols):

$$R^\alpha_{\beta\mu\nu} := \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}. \tag{1.36}$$

The final expression is frame invariant, and since A is an arbitrary point, we have, in any basis,

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\xi^\alpha = R^\alpha_{\mu\nu\beta}V^\mu V^\nu\xi^\beta. \tag{1.37}$$

We note that the geodesic deviation is zero if $R^\alpha_{\beta\mu\nu} = 0$ i.e $\iff$ the manifold is flat. Geodesics in flat space maintain their separation; those in curved spaces do not. Eq. (1.37), called equation of geodesic deviation, determines the relative covariant acceleration of two infinitesimally separated geodesics parameterized by $\vec{\xi}$.

Chapter 2

General Relativity

In this Chapter, we are going to sum up some basic ideas, concepts and a few equations of General Relativity. The importance of this theory stems directly from intuitive thoughts and observations. Its mathematical interpretation is based on the study of differential geometry, whose key concepts have been analyzed in the previous Chapter. In particular, we are going to analyze the geodesic motion of a particle embedded in a curved space-time and the relation between curvature and gravity. Further, we are going to introduce the stress-energy tensor and discuss its role inside the Einstein's equations. The same tensor is going to be essential even later, during the study of the Raychaudhuri's equation for the expansion parameter.

2.1 Basic Principles

Principle of General Relativity: "*The laws of physics are the same in all reference frames (for all observers).*"

Assuming that to each physical observer can be associated a reference frame (and, quite ideally, also the other way around), the principle of General Relativity can be translated into the mathematical requirement that all physical laws must involve only tensors and tensorial operations in the sense of differential geometry.

Speaking about gravity, it is a fact that the gravitational attraction between two bodies cannot be made to vanish; however, gravitational effects can be eliminated from the picture by considering a freely falling observer, which will not measure any gravitational acceleration in whatever experiment he or she carries on. The latter two observations are encoded in the following principle.

Equivalence Principle: "*For all physical objects, the gravitational charge (mass) m_g equals the inertial mass m_i.*"

From Newton's second law for a massive particle in a homogeneous and constant gravitational acceleration field $\vec{g}$, one has

$$m_g \vec{g} = m_i \vec{a} \quad \Rightarrow \quad \vec{a} = \vec{g} \ , \tag{2.1}$$

if $m_g = m_i$ for all bodies. In particular, both an observer (a physical apparatus) and a hypothetical test body will sustain the same acceleration, and one cannot devise any local observation that can tell whether one is not subject to any gravitational attraction at all (observer inside an elevator situated in free space), or if one is falling freely towards the center of a gravitational field (observer inside an elevator falling freely towards the ground).

From this follows that the gravitational effects are locally indistinguishable from the physical ones experienced in an accelerated frame. Such an equivalence enables us to eliminate gravity in a sufficiently small region of space-time. This is done by introducing a suitable chart which supports a locally inertial frame.
One can therefore assume that freely falling, inertial frames can be defined in a sufficiently small neighborhood U_P of each space-time point P, and the laws of Special Relativity, which may strictly hold only at each point P, will also be sufficiently good approximations of the true laws inside U_P for all inertial observers defined therein.
A typical example of such a possibility is provided by a free-falling elevator in the gravitational field of the Earth. In fact, a test body inside the falling elevator will fluctuate freely as if the elevator would be placed in empty space, in a region free from any gravitational field.

Principle of General Covariance: "*The laws of physics in a general reference frame are obtained from the laws of Special Relativity by replacing tensor quantities of the Lorentz group with tensor quantities of the space-time manifold.*"

A fundamental implication of this principle is the following: since a freely falling observer is "inertial", the local metric in its own reference frame is the canonical Minkowski metric all along (of course, only in a sufficiently small neighborhood of each point P of the observer's trajectory). Let $u^\mu = dx^\mu/d\tau$ denote the four-velocity of a test particle subject to no other force (but gravity). In the freely falling frame, it must then move along a straight line,

$$0 = \frac{d^2x^\alpha}{d\tau^2} = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = u^\mu \nabla_\mu u^\alpha \ , \tag{2.2}$$

where we remind the reader of the relation $\Gamma^{\alpha}_{\mu\nu} \sim g_{\mu\nu,\beta} = 0$. This comes straight from the definition of Gaussian (normal) system (1.13) and Eq. (1.16). A locally inertial frame of reference is in fact of the exact same type of the one mentioned above. The final result (2.2) is frame independent and we can simply say that test particles follow geodesics of the given space-time metric. These trajectories are usually referred to as world-lines. This argument further implies that the inertial observer itself moves along a geodesic of the space-time metric. So, if we consider a free particle instantaneously at rest, since the spatial components of the four-velocity vanish, the geodesic equation then reduces to

$$\ddot{x}^{\alpha} = -\Gamma^{\alpha}_{00} \; . \tag{2.3}$$

Hence, the Christoffel symbols represent the acceleration of gravity in the chosen reference frame and finally, we can tell, through (1.16), that the metric $g_{\mu\nu}$ can be viewed as a potential for the gravitational interaction.

2.2 Geodesic motion in General Relativity

In Special Relativity, light propagates along the null cone, which is a geodesic of the Minkowski metric, and one can easily show that this result generalizes to any space-time metric. In fact, the modulus $u^{\mu}u_{\mu} = 0$ of the (parallel transported) tangent vector u^{μ} to a geodesic is conserved along the geodesic itself, since

$$u^{\nu}\partial_{\nu}C = u^{\nu}(\nabla_{\nu}u^{\mu})u_{\mu} + u^{\nu}(\nabla_{\nu}u_{\mu})u^{\mu} = 2u^{\nu}u^{\mu}\nabla_{\nu}u_{\mu} = 2u^{\mu}(u^{\nu}\nabla_{\nu}u_{\mu}) = 0 \; . \tag{2.4}$$

Given a point P along a physical geodesic, its four-velocity must satisfy $u^{\mu}u_{\mu} = C$. For massive particles $C = -1$, which corresponds to time-like geodesic curves. $C = 0$ for light, which corresponds to light-like geodesic curves. And by convection we take $C = 1$ for space-like curves. Further, we also underline the implicit presence of the metric tensor $g_{\mu\nu}$ in Eq. (2.4), in this way it determines the causal structure of space-time by governing the propagation of light and of any other signal.

Previously, we mentioned the Einstein's conceptual experiment in the case of one test body inside the free-falling elevator.
However, if we put two test bodies (instead of one) inside the elevator, then there is an important physical difference between the two configurations previously mentioned i.e., free-fall in a given field and real absence of field, which soon clearly emerges.
Suppose, for instance, that the two bodies are initially at rest at the initial time t_0. Then, for $t > t_0$, they will keep both at rest in the absence of a real external field; on the contrary, they will start approaching each other with a relative accelerated motion if the elevator is free falling.
The relative motion is unavoidable, in the second case, due to the fact that the test bodies

are falling along geodesic trajectories which are not parallel, but converging toward the source of the physical field. So, even if the relative velocity of the two bodies is initially vanishing, $v(t_0) = 0$, their initial relative acceleration, $a(t_0)$, is always non-vanishing. Here we arrive at the point which is crucial for our discussion.

For any arbitrary gravitational field we can always (and entirely) eliminate the gravitational acceleration at any given point in space and at any given time, but there is no way to eliminate the acceleration between two different points, no matter how separated at a given instant of time.

If we take two points on two different geodesics they will be always characterized by a relative acceleration which cannot be eliminated (not even locally), and which is due to gravity, whose action tends to distort and focalize the trajectories. In the absence of gravity, on the contrary, the geodesics of all free bodies, quite independently of the chosen chart, are the straight lines of the Minkowski space-time, and their relative acceleration is vanishing.

Given a metric defined on the space-time manifold, and given a bundle of geodesic curves associated to that metric, the relative acceleration between points belonging to different geodesics only depends on the bending of the world-lines produced by the gravitational interaction, and can be used to denote, unambiguously, the presence (or the absence) of a physical gravitational field.

On the basis of the above, the geodesic deviation equation is of crucial importance in revealing the presence of a gravitational field. And by the equation

$$u^\mu u^\nu \nabla_\mu \nabla_\nu \xi^\alpha = R^\alpha_{\mu\nu\beta} u^\mu u^\nu \xi^\beta. \tag{2.5}$$

we can see the fundamental role of the Riemann tensor $R^\alpha_{\mu\nu\beta}$. In fact, in the case $R^\alpha_{\mu\nu\beta} = 0$, the geodesic deviation vanishes too, and we conclude there is no gravitational field. In this way, we have just demonstrated mathematically that the tidal forces of a gravitational field can be represented by the curvature of a space-time in which particles follow geodesics.

2.3 Brief look at Einstein's equations

Before discussing the general properties of the Einstein's equations we define some geometrical quantities derived from the Riemann tensor and the stress-energy tensor.

- Ricci (Curbastro) tensor:
$$R_{\mu\nu} := R^{\alpha}_{\mu\alpha\nu} = R_{\nu\mu} \ , \tag{2.6}$$
- Curvature Scalar:
$$R := R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \ , \tag{2.7}$$
- Einstein tensor:
$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\nu\mu} \ , \tag{2.8}$$

Further we mention the Bianchi identity,

$$\nabla_\nu G^{\mu\nu} = 0 \ . \tag{2.9}$$

In General Relativity, continuous matter distributions and fields are described by a stress-energy tensor $T_{\mu\nu}$.

Properties of the stress-energy tensor

This tensor for a perfect fluid has the following expression:

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \tag{2.10}$$

and satisfies the equation:

$$\nabla_\nu T^{\mu\nu} = 0 \ . \tag{2.11}$$

Where $u^\mu = (1, 0, 0, 0)$ in a co-moving coordinate system. In this context, u^μ is known as the velocity field of the fluid, and the co-moving coordinates are those with respect to which the fluid is at rest. ρ is the energy-density of the perfect fluid and p is the pressure.

In general, the matter and field distributions has to be supplemented by an equation of state. This is usually assumed to be that of a barotropic fluid, i.e. one whose pressure depends only on its density, $p = p(\rho)$. The most useful of cosmological fluids arise from considering a linear relationship between p and ρ:

$$p = \omega \rho \ , \tag{2.12}$$

where ω is known as the equation of state parameter.

- Dust:
For non-interacting particles, there is no pressure, $p = 0$, i.e. $\omega = 0$, the energy-momentum tensor has the simple form

$$T_{\mu\nu} = \rho u_\mu u_\nu \tag{2.13}$$

and such matter is usually referred to as dust,

$$\text{dust:} \quad p = 0 \quad \Rightarrow \quad \omega = 0 \ . \tag{2.14}$$

- Radiation:
This corresponds to $\omega = 1/3$ (in 1+3 dimensions). One way to see this is to note that the trace of a perfect fluid energy-momentum tensor is

$$T^\mu_\mu = -\rho + 3p \ . \tag{2.15}$$

Considering electro-magnetic radiation, for example, the energy-momentum tensor is that of Maxwell theory and hence trace-less. Therefore we obtain

$$\text{radiation:} \quad p = \rho/3 \quad \Rightarrow \quad \omega = 1/3 \ . \tag{2.16}$$

- Cosmological Constant:
A cosmological constant Λ, on the other hand, corresponds to a matter contribution with $p = -\rho$, i.e. $\omega = -1$,

$$\text{cosmological constant:} \quad p = -\rho \quad \Rightarrow \quad \omega = -1 \ . \tag{2.17}$$

Hence, either ρ is negative or p is negative. Then, from Eq. (2.18), one can deduce that a cosmological constant Λ is tantamount to adding matter with $p = -\rho$.

If the tensor in Eq. (2.8) is to be the left hand side of the equation which determines the metric, the source on the right hand side must have the same mathematical properties: it must be a symmetric and covariantly conserved (0,2) tensor built out of the matter content of the system. This tensor corresponds to the just mentioned stress-energy tensor.

The entire content of general relativity may be summarized as follows: "*Space-time is a manifold $\mathcal{M}$ on which there is defined a Lorentz metric $g_{\mu\nu}$. The curvature of $g_{\mu\nu}$ is related to the matter distribution in space-time by Einstein's equation.*"

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu} \tag{2.18}$$

- $G_{\mu\nu}$: The Einstein's tensor represents the geometrical curvature of the manifold.

- $\Lambda g_{\mu\nu}$: Term containing the Cosmological constant.

- $8\pi G_N/c^4$: Constant required by demanding that Newton's gravitational field equation comes out right.

- $T_{\mu\nu}$: Stress-momentum tensor, it encapsulates the characteristics of the source.

Since both sides are symmetric, these form a set of ten coupled non-linear partial differential equations in the metric and its first and second derivatives. However the covariant divergence of each side vanishes

$$(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu})_{;\mu} = 0 \tag{2.19}$$

and

$$T^{\mu\nu}_{;\mu} = 0 \tag{2.20}$$

independently of the field equations. Thus, the field equations really provide only six independent differential equations for the metric. This is, in fact, the correct number of equations to determine the space-time, since four of the ten components of the metric can be given arbitrary values by use of the four degrees of freedom to make coordinate transformations.

Chapter 3

Raychaudhuri Equations

In classical mechanics, usually, we formulate and solve the equations of motion of a system for certain initial conditions. In every framework, Newtonian, Lagrangian or Hamiltonian, the ultimate goal is to develop our understanding of the system through such solutions, which may be analytical or numerical, or a combination of both. Thus, the focus is on the behavior of a single trajectory beginning from a given initial condition. No attempt is made to study the collective behavior of a family of trajectories. We believe that such a study provides a new perspective towards qualitatively understanding the configuration space of a dynamical system.
Based on the above observation, we wish to frame a somewhat different question as follows: how do we understand the behavior of a properly defined family of trajectories? One way of defining a family of trajectories (i.e. a trajectory congruence) is to vary the initial conditions on position and velocity around specific values. The family of trajectories thus obtained may be treated as flow-lines generated by the velocity field u^a in the configuration space of the system. Flows are generated by a vector field; they are the integral curves of the given vector field. These curves may be geodesic or non-geodesic, though the former is more useful in our context.

In other words, the aim of this chapter is to study the behavior of a geodesic congruence defined in a generic space-time ($\mathcal{M}$, g_{ab}). Once we have defined the velocity gradient tensor B_{ab}, we decompose it into its trace θ (expansion), symmetric trace-less σ_{ij} (shear) and anti-symmetric part ω_{ij} (rotation). After that, we analyze the dynamical evolution of these three parameters; and, from θ, in particular, we obtain the principal Raychaudhuri equation. A more precise description of that follows.

3.1 Flow as a congruence of geodesics

Let $\mathcal{M}$ be a manifold and let $\mathbf{O} \subset \mathcal{M}$ be open. A congruence in $\mathbf{O}$ is a family of curves such that through each point $P \in \mathbf{O}$ there passes precisely one curve in this family. No two trajectories within the family can intersect each other in the course of time evolution. If they do, then the definition of congruence breaks down.

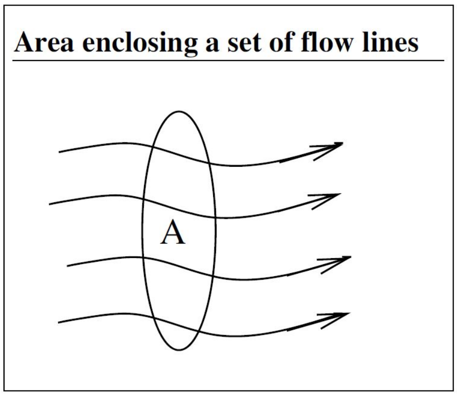

Figure 3.1: The cross-sectional area enclosing a congruence of geodesics

What quantities characterize a flow? If λ denotes the parameter labeling points on the curves in the flow, then, in order to characterize the flow, we must have different functions of λ. Once we have defined the velocity field u^a with the parameter λ, we shall decompose the velocity second rank gradient tensor B_{ab} into its trace θ (expansion), symmetric trace-less σ_{ab} (shear) and anti-symmetric part ω_{ab} (rotation). Mathematically, we have

$$B_{ab} = \frac{1}{n}\theta\delta_{ab} + \sigma_{ab} + \omega_{ab} \tag{3.1}$$

The corresponding evolution equations in time (Raychaudhuri equations), can thereby be obtained, for these kinematic variables. Here, n is the dimension of the configuration space. The initial conditions on these kinematic quantities are then translated into the perturbed initial conditions on the positions and velocities of all the trajectories within the family.

The geometric meaning of these quantities is shown through Fig. 3.1 and Fig. 3.2. The expansion, shear and twist (rotation) are related to the geometry of the cross-sectional area (enclosing a fixed number of geodesics) orthogonal to the flow lines (Fig. 3.1). As one moves from one point to another, along the flow, the shape of this area changes. It still includes the same set of geodesics in the bundle but may be isotropically smaller (or larger), sheared or twisted. The analogy with elastic deformations or fluid flow is, usually, a good visual aid for understanding the change in the geometry of this area. The kinematic evolution of the family may thus be studied through the solutions of the evolution equations for the kinematic variables. Through such studies, one can analyze how the family evolves as a whole.

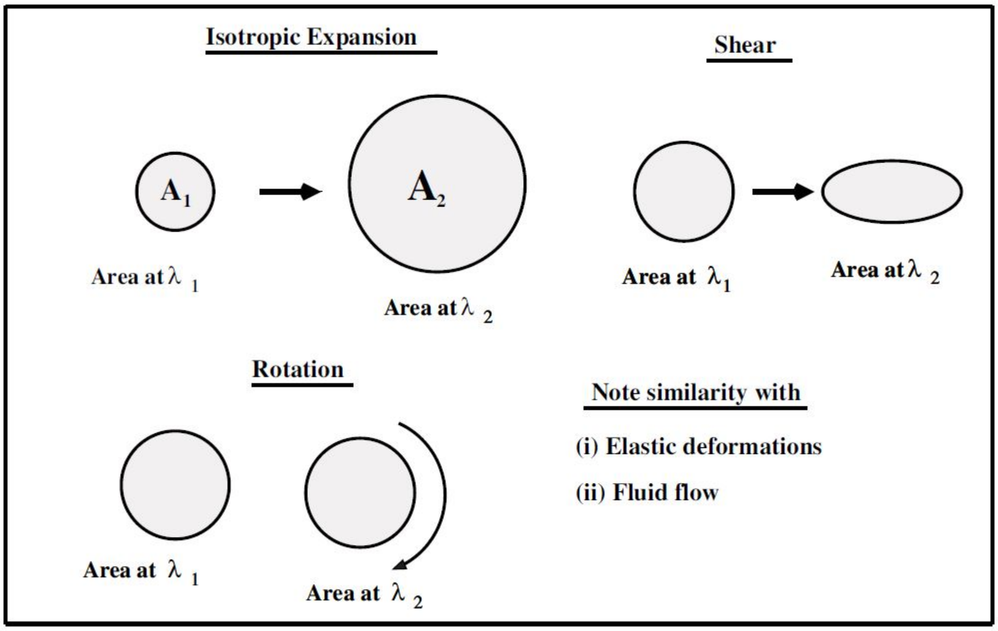

Figure 3.2: Illustration of expansion, rotation and shear

For example, one may ask; do the trajectories in the family diverge, or converge and intersect after some finite time? If not, is there a relative shearing or a twist of the area enclosing the family, or do they remain parallel? Through answers to such questions, we hope to gain newer insights on the behavior of a given system.

3.2 Introduction to Raychaudhuri equations

In this section we shall define the expansion, shear, and twist of a time-like and null geodesic congruence for a generic space-time ($\mathcal{M}$, g_{ab}), and derive equations for their rate of change as one moves along the curves in the congruence.
In order to do that, we are going to start off from the "geodesic formalism" we have introduced in Chapter 1.

The starting point is a geodesic curve with tangent vector field u^a,

$$u^b \nabla_b u^a = 0 \ , \tag{3.2}$$

and a deviation vector field ξ^a characterized by the (symmetric connection) condition

$$[u, \xi]^a = u^b \nabla_b \xi^a - \xi^b \nabla_b u^a = 0 \quad \Leftrightarrow \quad D_\tau \xi^a = \xi^b \nabla_b u^a \ \ . \tag{3.3}$$

The rationale for this condition is that, if $x^a(\tau, s)$ is a family of geodesics labeled by s, one has the identifications

$$u^a = \frac{\partial}{\partial \tau} x^a(\tau, s) \quad , \quad \xi^a = \frac{\partial}{\partial s} x^a(\tau, s) \ \ . \tag{3.4}$$

Since second partial derivatives commute, this implies the relation

$$\frac{\partial}{\partial \tau} \xi^a(\tau, s) = \frac{\partial}{\partial s} u^a(\tau, s) \ \ . \tag{3.5}$$

Condition (3.3) is nothing other than the covariant way of writing (3.5). Now, introducing the tensor

$$B_{ab} = \nabla_b u_a \tag{3.6}$$

we can write (3.3) as

$$D_\tau \xi^a = B^a_b \xi^b \ \ , \tag{3.7}$$

in this way, the matrix B_{ab} describes the evolution and deformation of the deviation vector ξ^a along the geodesic. Further, it satisfies

$$B_{ab} u^b = u^b \nabla_b u_a = 0 \tag{3.8}$$

and

$$u^a B_{ab} = \frac{1}{2} \nabla_b (u^a u_a) = 0 \ \ , \tag{3.9}$$

so B_{ab} is transverse to u^a. As a consequence one has

$$u_a D_\tau \xi^a = 0 \tag{3.10}$$

(i.e. $D_\tau \xi^a$ is transverse to u^a) and therefore

$$\frac{d}{d\tau}(u_a \xi^a) = D_\tau (u_a \xi^a) = u_a D_\tau \xi^a = 0 \quad . \tag{3.11}$$

So, this means that $u_a \xi^a$ is simply constant and contains no interesting information about the geodesic itself.

In the time-like case this means that a vector of the form $\xi^a = \xi u^a$ is a deviation vector only if ξ is constant, and then ξ^a is simply a translation along the geodesic and therefore not a deviation vector of interest (and certainly anyhow not a vector of the kind one has in mind when thinking about a deviation vector, which should point away from the geodesic). In the null case, the interpretation is slightly different (and we will return to this later), but the fact that $u_a \xi^a$ is simply constant for a deviation vector remains, and we can, without loss of information, choose the deviation vector to satisfy the condition $\xi^a u_a = 0$.

Given this set-up, we now want to calculate

$$\begin{aligned} (D_\tau)^2 \xi^a &= (D_\tau B^a_b)\xi^b + B^a_b D_\tau \xi^b \\ &= (D_\tau B^a_\gamma + B^a_b B^b_\gamma)\xi^\gamma \quad . \end{aligned} \tag{3.12}$$

Note that, along with $D_\tau \xi^a$, also $D^2_\tau \xi^a$ is automatically transverse to u^a, $u_a D^2_\tau \xi^a = 0$, regardless of whether or not one imposes the condition $\xi^a u_a = 0$. For the term in brackets we find, using the geodesic equation for u^a,

$$\begin{aligned} D_\tau B^a_\gamma + B^a_b B^b_\gamma &= u^b \nabla_b \nabla_\gamma u^a + (\nabla_\gamma u^b)\nabla_b u^a \\ &= u^b \nabla_b \nabla_\gamma u^a + \nabla_\gamma (u^b \nabla_b u^a) - u^b \nabla_\gamma \nabla_b u^a \\ &= u^b (\nabla_b \nabla_\gamma - \nabla_\gamma \nabla_b) u^a = R^a_{\delta b \gamma} u^b u^\delta \quad , \end{aligned} \tag{3.13}$$

and plugging this back into (3.12), we obtain straightaway the covariant version of the geodesic deviation equation in the form

$$(D_\tau)^2 \xi^a = R^a_{\delta b \gamma} u^b u^\delta \xi^\gamma \quad . \tag{3.14}$$

In this way, we have been able to obtain the geodesic deviation equation through the tensor B_{ab}, which plays a central role in deriving the Raychaudhuri equations.

3.3 Derivation of the Raychaudhuri equations

Manipulations similar to those of the previous section allow one to derive an equation for the rate of change of the divergence $\nabla_a u^a$ of a family of geodesics along themselves. This simple result, known as the Raychaudhuri equation, has important implications and ramifications in General Relativity, especially in the context of the so-called singularity theorems of Hawking and Penrose.

Let us consider u^a, it now denotes a tangent vector field to an affinely parameterized geodesic congruence, $u^a \nabla_a u^b = 0$ (and $u^a u_a = -1$ or $u^a u_a = 0$ everywhere for a time-like or null congruence). As in Section 3.2, we introduce the tensor field (3.6)

$$B_{ab} = \nabla_b u_a \quad . \tag{3.15}$$

We recall from Eqs. (3.8) and (3.9) that B_{ab} has components only in the directions transverse to u^a. Its trace

$$\theta = B^a_a = g^{ab} B_{ab} = \nabla_a u^a \tag{3.16}$$

is the divergence of u^a and is known as the expansion of the (affinely parameterized) geodesic congruence.

The key equation governing B_{ab} evolution along the integral curves of the geodesic vector field is (3.13), which comes straight from the geodesic deviation Eq. (3.12):

$$D_\tau B^a_\gamma + B^a_b B^b_\gamma = R^a_{b\delta\gamma} u^b u^\delta \quad . \tag{3.17}$$

By taking the trace of this equation, multiplying both sides by $g^{\gamma\delta} g_{\delta a}$ and adjusting the indexes, we evidently obtain an evolution equation for the expansion θ, namely

$$\frac{d}{d\tau}\theta = -(\nabla_a u_b)(\nabla^b u^a) - R_{ab} u^a u^b \quad . \tag{3.18}$$

To gain some more insight into the geometric significance of this equation, we now consider the case that the geodesic congruence u^a is time-like and normalized in the standard way as $u^a u_a = -1$ (so that τ is proper time).

Given this time-like geodesic congruence, we can introduce the tensor

$$h_{ab} = g_{ab} + u_a u_b \quad . \tag{3.19}$$

Point-wise, h_{ab} can be interpreted as a metric on the space of vectors transverse to the geodesic bundle. Its purely algebraic properties are identical to those of the induced metric.

In particular,

- h_{ab} has the characteristic property that it is orthogonal to u^a,

$$u^a h_{ab} = h_{ab} u^b = 0 \quad . \tag{3.20}$$

- It can therefore be interpreted as the spatial projection of the metric in the directions orthogonal to the time-like vector field u^a. This can be seen more explicitly in terms of the projectors

$$\begin{aligned} h^a_b &= \delta^a_b + u^a u_b \\ h^a_b h^b_\gamma &= h^a_\gamma \quad . \end{aligned} \tag{3.21}$$

On directions tangential to u^a they act as

$$h^a_b u^b = 0 \quad , \tag{3.22}$$

whereas on vectors ξ^a orthogonal to u^a, $u_a \xi^a = 0$ (space-like vectors), one has

$$h^a_b \xi^b = \xi^a \quad . \tag{3.23}$$

- Thus, acting on an arbitrary vector field $\vec{V}$, $V^a = h^a_b V^b$ is the projection of this vector into the plane orthogonal to u^a. In the same way one can project an arbitrary tensor to a spatial or transverse tensor. E.g. one has

$$t_{a...b} = T_{\gamma...\delta} h^\gamma_a ... h^\delta_b \tag{3.24}$$

which satisfies

$$u^a t_{a...b} = ... = u^b t_{a...b} = 0 \quad . \tag{3.25}$$

- In particular, the projection of the metric is

$$g_{ab} \to g_{\gamma\delta} h^\gamma_a h^\delta_b = g_{ab} + u_a u_b = h_{ab} \quad , \tag{3.26}$$

as anticipated above. Whereas for the space-time metric one obviously has $g^{ab} g_{ab} = \delta^a_a = 4$, the trace of h_{ab} is (in the 4-dimensional case)

$$\begin{aligned} g^{ab} h_{ab} &= g^{ab} g_{ab} + g^{ab} u_a u_b = 4 - 1 = 3 \\ &= (g^{ab} + u^a u^b)(g_{ab} + u_a u_b) \\ &= g^{ab} g_{ab} + g^{ab} u_a u_b + g_{ab} u^a u^b + u^a u^b u_a u_b \\ &= 4 - 1 - 1 + 1 = 3 = h^{ab} h_{ab} \quad . \end{aligned} \tag{3.27}$$

Thus for an affinely parameterized congruence the properties (3.8) and (3.9) show that B_{ab} is automatically a spatial or transverse tensor in the sense above,

$$b_{ab} \equiv h_a^\gamma h_b^\delta B_{\gamma\delta} = B_{ab} \quad . \tag{3.28}$$

Note that the affine parameterization of the time-like geodesic congruence, expressed by the normalization condition $u^a u_a = -1$, is crucial for this entire set-up, since the projection operator requires a unit vector field. This is in contrast with the case of the null geodesic congruence k^a, where the property $k^a k_a = 0$ is independent of the parameterization.

We now decompose b_{ab} into its trace, symmetric trace-less and anti-symmetric part,

$$b_{ab} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} \quad , \tag{3.29}$$

with

$$\theta = h^{ab} b_{ab} = g^{ab} B_{ab} = \nabla_a u^a \quad , \tag{3.30}$$

$$\sigma_{ab} = \frac{1}{2}(b_{ab} + b_{ba}) - \frac{1}{3}\theta h_{ab} \quad , \tag{3.31}$$

$$\omega_{ab} = \frac{1}{2}(b_{ab} - b_{ba}) \quad . \tag{3.32}$$

The quantities θ, σ_{ab} and ω_{ab} are known, like we have already said in the previous sections, as the expansion, shear and rotation tensor of the congruence (family) of geodesics defined by u^a.

Now, returning to the evolution equation for the expansion θ (3.18) we get:

$$\begin{aligned}
\frac{d}{d\tau}\theta &= -(\nabla^b u^a)(\nabla_a u_b) - R_{ab} u^a u^b \\
&= -B^{ab} B_{ba} - R_{ab} u^a u^b \\
&= -(\frac{1}{3}\theta h^{ab} + \sigma^{ab} + \omega^{ab})(\frac{1}{3}\theta h_{ba} + \sigma_{ba} + \omega_{ba}) - R_{ab} u^a u^b \\
&= -\frac{1}{9}\theta^2 h^{ab} h_{ba} - \sigma^{ab}\sigma_{ba} - \omega^{ab}\omega_{ba} - R_{ab} u^a u^b \\
&= -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} - R_{ab} u^a u^b \\
&= -\frac{1}{3}\theta^2 - \sigma^2 + \omega^2 - R_{ab} u^a u^b \quad .
\end{aligned} \tag{3.33}$$

What we have obtained here, is the Raychaudhuri expansion equation for a time-like geodesic congruence.

Where we have used, by virtue of construction, the relations:

- $$h^{ab}\sigma_{ab} = 0 \tag{3.34}$$

- $$h^{ab}\omega_{ab} = 0 \tag{3.35}$$

following the two above we derive:

- $$u^a\sigma_{ab} = u^a\omega_{ab} = 0 \tag{3.36}$$

then by summing over the two indexes:

- $$\sigma^{ab}\omega_{ba} + \omega^{ab}\sigma_{ba} = -\sigma^{ab}\omega_{ab} + \omega^{ab}\sigma_{ab} = 0 \tag{3.37}$$

We here mention also the symmetric trace-less and anti-symmetric parts of Eq. (3.17) (for the sake of completeness, we re-write the well-known trace part, too). However, the two equations are not relevant for our aims:

$$\frac{d}{d\tau}\theta = -\frac{1}{3}\theta^2 - \sigma^2 + \omega^2 - R_{ab}u^a u^b \ , \tag{3.38}$$

$$u^\gamma \nabla_\gamma \sigma_{ab} + \frac{2}{3}\theta\sigma_{ab} + \sigma_{a\gamma}\sigma^\gamma_b + \omega_{a\gamma}\omega^\gamma_b - \frac{1}{3}h_{ab}(\sigma^2 - \omega^2) - C_{\gamma ba\delta}u^\gamma u^\delta - \frac{1}{2}\widetilde{R}_{ab} = 0 \ , \tag{3.39}$$

$$u^\gamma \nabla_\gamma \omega_{ab} + \frac{2}{3}\theta\omega_{ab} + \sigma^\gamma_b\omega_{a\gamma} - \sigma^\gamma_a\omega_{b\gamma} = 0 \ . \tag{3.40}$$

Where $C_{\gamma ba\delta}$ is the Weil tensor and $\widetilde{R}_{ab} = (h_{a\gamma}h_{b\delta} - \frac{1}{3}h_{ab}h_{\gamma\delta})R^{\gamma\delta}$ is the transverse trace-free part of R_{ab}.

Similarly, one can derive a set of Raychaudhuri equations for the congruence of null geodesics. Though, we do not go through a very detailed derivation of the equations for the null geodesic congruence.

The central issue in the case of null geodesic congruences is the construction of the transverse part of the deviation vector and the space-time metric.
Assuming an affine parameterization in the sense $dx^a = k^a d\lambda$ with $k^a k_a = 0$ and $k^a\xi_a = 0$ (with ξ^a being the deviation vector) we realize that we are in trouble because of the

above two orthogonality relations. Naively writing $h_{ab} = g_{ab} + k_a k_b$ will not work here ($k^a h_{ab} \neq 0$). The transverse metric is thus defined by introducing an auxiliary null vector N^a with $k_a N^a = -1$ (the choice of -1 is by convention, the essence is that the quantity must be non-zero). Hence we can choose h_{ab} to have the form:

$$h_{ab} = g_{ab} + k_a N_b + k_b N_a \quad . \tag{3.41}$$

This satisfies $k^a h_{ab} = 0$ and $N^a h_{ab} = 0$. Note that h_{ab} now is entirely spatial two-dimensional, so we get

$$\begin{aligned} h_{ab}h^{ab} &= h_{ab}(g^{ab} + k^a N^b + k^b N^a) \\ &= h_{ab}g^{ab} + 0 \\ &= g_{ab}g^{ab} + k^a N^b g^{ab} + k^b N^a g^{ab} \\ &= 4 - 1 - 1 = 2 \quad . \end{aligned} \tag{3.42}$$

Keeping this transverse metric in mind, we can proceed in the same way as for the time-like case by constructing $\widehat{B}_{ab} = \nabla_b k_a$. We quote below the equation for the expansion:

$$\frac{d}{d\tau}\widehat{\theta} = -\frac{1}{2}\widehat{\theta}^2 - \widehat{\sigma}^2 + \widehat{\omega}^2 - R_{ab}k^a k^b \quad . \tag{3.43}$$

Note that the structure and geometric features of a given space-time encoded in the metric g_{ab}, the Riemann tensor $R^a_{b\gamma\delta}$, the Ricci tensor R_{ab} and the Weyl tensor $C^a_{b\gamma\delta}$, appears in the evolution equations. Therefore, the structure and geometric features of a given space-time must necessarily be reflected in the evolution of a congruence.
Using these equations, kinematic evolution of geodesic congruences have been studied in different space-time backgrounds. These studies explore the role of initial conditions as well as space-time curvature on the evolution of geodesic congruences using Raychaudhuri equations.

Chapter 4

Application to Cosmology

In this chapter we discuss the applications of what we have achieved in Chapter 3. In particular, after introducing some basic concepts of modern cosmology, we talk about the solutions of Einstein's field equations in the context of a FLRW universe. In order to do that, we substitute the space-time metric into Einstein's equations. In such a way, we obtain predictions for the dynamical evolution of the homogeneous and isotropic universe. Then, in the same context, we consider a family of time-like geodesics (i.e., a congruence) and use the Raychaudhuri equations in order to get the same evolution equations derived through the previous method. Hence, we show the dynamical equivalence between the two different approaches.
Then, through these equations and under simple considerations, we come up with the existence of a space-time singularity, namely the big-bang singularity.
Immediately after, we present the energy conditions on the stress-energy tensor T_{ab} and explain the importance of these conditions on matter. Finally, given the previous conditions and some others, we point out, one more time, the existence of a singularity in the congruence of time-like geodesics in a FLRW space-time. In other words, we demonstrate the divergence to ∞ of the expansion parameter θ. This is the same thing as saying that the geodesic's affine parameter τ is finite and its path is inexetendible (i.e., the geodesic is incomplete). As we will see, it is the most effective way of detecting a singularity. In conclusion, we briefly comment the concept of a space-time singularity, and the fundamental contribution of the Raychaudhuri equations to the Penrose-Hawking singularity theorems.

4.1 Basic concepts of Cosmology

Modern cosmology is mainly based on the following two principles:

Copernican Principle: "*We are not a preferred observer in the Universe.*"

In other words, it is reasonable to assume that the Universe would look to any other observer like it looks to us. From the practical point of view, this principle is of limited use. However, and although it goes a long way ahead to infer from the above, one eventually relies on the

Cosmological Principle: "*The Universe is homogeneous and isotropic.*"

Isotropy is here taken as an observational statement, whereas homogeneity follows from assuming that isotropy is independent of the observation point according to the Copernican principle.

Loosely speaking, homogeneity means that at any given "instant of time" each point of "space" should "look like" any other point. A precise formulation can be given as follows: a space-time is said to be (spatially) homogeneous if there exists a one-parameter family of space-like hyper-surfaces Σ_t foliating the space-time (see Fig. 4.1) such that for each t and for any points p, $q \in \Sigma_t$ there exists an isometry (a bijective map between two metric spaces that preserves distances) of the space-time metric, g_{ab}, which takes p into q.

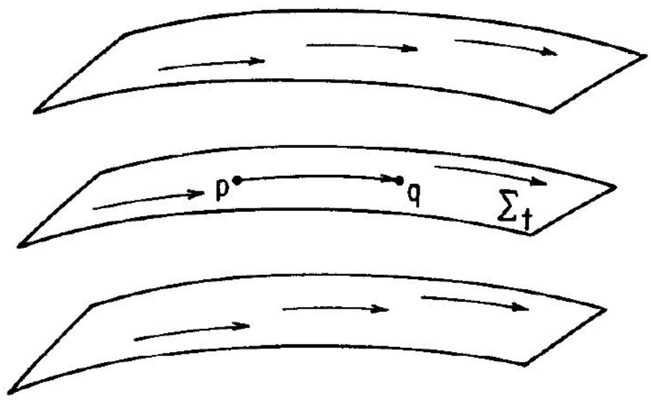

Figure 4.1: The hyper-surfaces of spatial homogeneity in space-time. By definition of homogeneity, for each t and each p, $q \in \Sigma_t$ there exists an isometry of the space-time which takes p into q

A space-time is said to be (spatially) isotropic at each point if there exists a congruence of time-like curves (i.e., observers), with tangents denoted u^a, filling the space-time (see Fig. 4.2) and satisfying the following property. Given any point p and any two unit "spatial" tangent vectors s_1^a, $s_2^a \in V_p$ (i.e., vectors at p orthogonal to u^a), there exists an isometry of g_{ab} which leaves p and u^a at p fixed but rotates s_1^a into s_2^a. Thus, in an isotropic universe it is impossible to construct a geometrically preferred tangent vector orthogonal to u^a. In the present case of a homogeneous and isotropic space-time, the surfaces Σ_t of homogeneity must be orthogonal to the tangents, u^a, to the world lines of the isotropic observers.

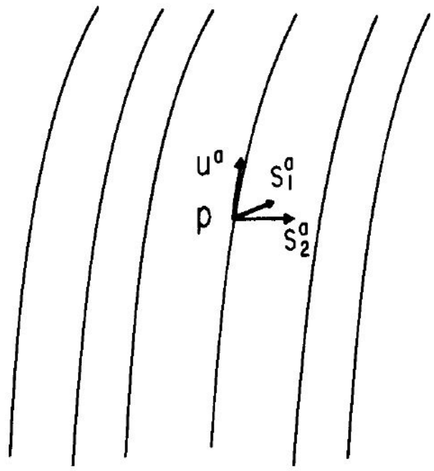

Figure 4.2: The world lines of isotropic observers in space-time. By definition of isotropy, for any two vectors s_1^a, s_2^a at p which are orthogonal to u^a, there exists an isometry of the space-time which leaves p fixed and rotates s_1^a into s_2^a

Because of that, we may express the four-dimensional space-time metric g_{ab} as

$$g_{ab} = -u_a u_b + h_{ab}(t) \ , \tag{4.1}$$

where for each t, $h_{ab}(t)$ is the metric of either a sphere, a flat Euclidean space or a hyperboloid, on Σ_t. We can choose convenient coordinates on the four-dimensional space-time as follows.
We choose, respectively, either (a) spherical coordinates, (b) Cartesian coordinates, or (c) hyperbolic coordinates on one of the homogeneous hyper-surfaces. We then "carry" these coordinates to each of the other homogeneous hyper-surfaces by means of our isotropic observers; i.e., we assign a fixed spatial coordinate label to each observer. Finally, we label each hyper-surface by the proper time, τ, of a clock carried by any of the isotropic observers. (By homogeneity, all the isotropic observers must agree on the time difference

between any two hyper-surfaces). Thus, τ and our spatial coordinates label each event in the universe. Expressed in these coordinates, the space-time metric takes the form

$$ds^2 = -d\tau^2 + a^2(\tau) \begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2) \\ dx^2 + dy^2 + dz^2 \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2) \end{cases} \tag{4.2}$$

where the three possibilities in the bracket correspond to the three possible spatial geometries. [The metric for the spatially flat case could be made to look more similar to the other cases by writing it in spherical coordinates as $d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta d\phi^2)$]. The general form of the metric, equation (4.2) is called a Robertson-Walker cosmological model. Thus, our assumptions of homogeneity and isotropy alone have determined the space-time metric up to three discrete possibilities of spatial geometry and the arbitrary positive function $a(\tau)$ named (cosmic) scale factor.
Equivalently, we can express the above metric (4.2) in the following Friedmann–Lemaître –Robertson–Walker form:

$$ds^2 = -d\tau^2 + a(\tau)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \right] \ . \tag{4.3}$$

where $k = +1$ for the 3-sphere, $k = 0$ for flat space and $k = -1$ for the hyperboloid.

Our aim is now to substitute the space-time metric, Eq. (4.3), into Einstein's equations (2.18) in order to obtain predictions for the dynamical evolution of the universe. The first step is to describe the matter content of the universe in terms of its stress-energy tensor, T_{ab}, which enters the right-hand side of Einstein's equation. Most of the mass-energy in the present universe is believed to be found in ordinary matter, concentrated in galaxies. On the cosmic scales with which we are dealing, each galaxy can be idealized as a "grain of dust". The random velocities of the galaxies are small, so the "pressure" of this dust of galaxies is negligible. Thus, to a good approximation, the stress-energy tensor of matter of the present universe takes the form

$$T_{ab} = \rho u_a u_b \ , \tag{4.4}$$

where ρ is the (average) mass density of matter. However, other forms of mass-energy are also present in the universe. In fact, a thermal distribution of radiation at a temperature of about $3K$ fills the universe. This radiation can also be described by a perfect fluid stress-energy tensor, but its pressure is nonzero; indeed, for mass-less thermal radiation, we have $p = \rho/3$. The contribution of this radiation to the stress-energy of the present universe is negligible, but, this radiation is predicted to make the dominant contribution to T_{ab} in the early universe.

Thus, in treating Einstein's equations, we shall take T_{ab} to be of the general perfect fluid form,

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} \quad . \tag{4.5}$$

Computing G_{ab} from the metric and equating it with $8\pi T_{ab}$, we will get 10 equations corresponding to the 10 independent components of a symmetric two-index tensor. However, it is not difficult to see that on account of the space-time symmetries, there will be only two independent equations in this case. Namely, the vector $G^{ab}u_b$ cannot have off-diagonal components, or isotropy and homogeneity would be violated. Further, considering that the "space-space" components yield the same equations, the independent components of Einstein's equations are simply

$$G_{\tau\tau} - \Lambda = 8\pi T_{\tau\tau} = 8\pi\rho \quad , \tag{4.6}$$

$$G_{**} + \Lambda = 8\pi T_{**} = 8\pi p \quad , \tag{4.7}$$

where $G_{\tau\tau} = G_{ab}u^a u^b$ and $G_{**} = G_{ab}s^a s^b$, with s^a a unit vector tangent to the homogeneous hyper-surfaces.

We now want to compute $G_{\tau\tau}$ and G_{**} in terms of $a(\tau)$ for the case of flat spatial geometry. Through the equations of the Christoffel symbols and Riemann tensor as functions of the metric, which we here report,

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(g_{lk,j} + g_{lj,k} - g_{jk,l}\right) \tag{4.8}$$

$$R_{ij} = R^k_{ikj} = \partial_j \Gamma^\rho_{i\rho} - \partial_\rho \Gamma^\rho_{ij} + \Gamma^l_{i\rho}\Gamma^\rho_{jl} - \Gamma^l_{ij}\Gamma^\rho_{\rho l}. \tag{4.9}$$

we obtain the non-vanishing components of the Christoffel symbols, which are merely

$$\Gamma^\tau_{xx} = \Gamma^\tau_{yy} = \Gamma^\tau_{zz} = a\dot{a} \quad , \tag{4.10}$$

$$\Gamma^x_{x\tau} = \Gamma^x_{\tau x} = \Gamma^y_{y\tau} = \Gamma^y_{\tau y} = \Gamma^z_{z\tau} = \Gamma^z_{\tau z} = \frac{\dot{a}}{a} \quad , \tag{4.11}$$

hence, the independent Ricci tensor components:

$$R_{\tau\tau} = -3\ddot{a}/a \quad , \tag{4.12}$$

$$\begin{aligned} R_{**} &= R_{ab}s^a s^b \\ &= R_{ab}s^a g^{ab} s_b \\ &= a^{-2}R_{xx} = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \quad . \end{aligned} \tag{4.13}$$

Since we have

$$R = -R_{\tau\tau} + 3R_{**} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \quad , \tag{4.14}$$

we thus obtain

$$G_{\tau\tau} - \Lambda = R_{\tau\tau} + \frac{1}{2}R - \Lambda = 3\dot{a}^2/a^2 - \Lambda = 8\pi T_{\tau\tau} = 8\pi\rho \ , \tag{4.15}$$

$$G_{**} + \Lambda = R_{**} - \frac{1}{2}R + \Lambda = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \Lambda = 8\pi T_{**} = 8\pi p \ . \tag{4.16}$$

Using the first equation, we may write the second equation as

$$3\frac{\ddot{a}}{a} = -4\pi(\rho + 3p) + \Lambda \ . \tag{4.17}$$

Obtaining, in this manner, the equation that, as we will see later, characterize the evolution of $a(\tau)$. Then, considering the conservation of the stress-energy tensor, from the condition $\nabla_a T^{ab} = 0$, we note that the spatial components of this conservation law

$$\nabla_a T^{ai} = 0 \tag{4.18}$$

turn out to be identically satisfied. In fact, inserting expression (4.5), this is demonstrated by virtue of the fact that u^a are geodesics and that the functions ρ and p are only functions of time.

The only interesting conservation law is thus the zero-component

$$\nabla_a T^{a0} = \partial_a T^{a0} + \Gamma^a_{ab} T^{b0} + \Gamma^0_{ab} T^{ab} = 0 \ , \tag{4.19}$$

which for a perfect fluid with $T_{00} = \rho(t)$ and $T_{ij} = p(t) g_{ij}$ becomes

$$\partial_t \rho(t) + \Gamma^a_{a0}\rho(t) + \Gamma^0_{00}\rho(t) + \Gamma^0_{ij} T^{ij} = 0 \ . \tag{4.20}$$

Inserting the explicit expressions for the Christoffel symbols, and for the given metric, one finds the continuity equations.

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} \ . \tag{4.21}$$

4.2 On the initial singular state of the universe

Let's now consider the continuity equation of the perfect fluid energy-momentum tensor, just as in the previous paragraph. Further, let's consider a general velocity field u^a with $u^a u_a = -1$ and

$$u^a \nabla_a \rho = \dot{\rho} = \frac{d\rho}{d\tau} \quad . \tag{4.22}$$

Let us now see what the condition $\nabla_a T^{ab} = 0$ implies. We first consider the case $p = 0$, so this corresponds to a pressure free perfect fluid. Then one has

$$T_{ab} = \rho u_a u_b \quad \Rightarrow \quad \nabla_a T^{ab} = (\dot{\rho} + \rho \nabla_a u^a) u^b + \rho u^a \nabla_a u^b \quad . \tag{4.23}$$

Here $\nabla_a u^a = B^a_a = \theta$ is (and measures) the expansion of the velocity field u^a (introduced previously, in the context of the Raychaudhuri equations, Chapter 3), and the last term $u^a \nabla_a u^b = a^b$ is its acceleration, so that we can also write this equation as

$$(\dot{\rho} + \theta\rho) u^b + \rho a^b = 0 \quad . \tag{4.24}$$

Since u^a and a^a are orthogonal to each other,

$$u_a u^a = -1 \quad \Rightarrow \quad u_a a^a = 0 \quad , \tag{4.25}$$

this equation breaks up into two independent pieces,

$$\nabla_a T^{ab} = 0 \quad \Leftrightarrow \quad \dot{\rho} + \theta\rho = 0 \quad \text{and} \quad a^b = u^a \nabla_a u^b = 0 \quad . \tag{4.26}$$

Its time (energy flow) component is a continuity equation, while its space (momentum flow) part tells us that the particles have to move on geodesics.

Now what happens if we include pressure p? This corresponds to adding $p(g^{ab} + u^a u^b) \equiv p h^{ab}$, but this tensor is orthogonal to u^a (just like in Chapter 3),

$$u^a h_{ab} \equiv u^a (g_{ab} + u_a u_b) = u_b - u_b = 0 \quad . \tag{4.27}$$

Therefore through the equation $\nabla_a T^{ab}$, we arrive at

$$\nabla_a T^{ab} = (\dot{\rho} + \theta(\rho + p)) u^b + (\rho + p) a^b + (\nabla_a p) h^{ab} \tag{4.28}$$

The part tangent to u^b tells us that

$$\dot{\rho} + \theta(\rho + p) = 0 \quad , \tag{4.29}$$

so this is a conservation law, and the part orthogonal to u^a gives

$$(\rho + p) a^b + (\nabla_a p) h^{ab} = 0 \quad , \tag{4.30}$$

which is a curved-space relativistic generalization of the Euler equations for a perfect fluid.

In particular, now the velocity field is not composed of geodesics unless the derivative of p in the directions orthogonal to u^a (i.e. the spatial derivative) is zero, $h^{ab}\nabla_a p = 0$. This is precisely the situation we are considering in cosmology, where $\rho = \rho(t)$ and $p = p(t)$ depend only on t, which is the proper time of co-moving observers described by the velocity field u^a.

Returning thus to the cosmological setting, where we have chosen the matter to move along geodesics, we are left with the continuity equation, which is now the same as (4.21) because for $u^a = (1,0,0,0)$ in co-moving coordinates one has

$$\theta = \nabla_a u^a = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}u^a) = a(t)^{-3}\partial_t(a(t)^3) = 3\dot{a}(t)/a(t) \ . \tag{4.31}$$

This equality comes from the general expression for the divergence of a vector field, as follows:

$$\begin{aligned} \nabla \cdot \mathbf{V} &= V^i_{;i} \\ &= \partial_i V^i + \Gamma^l_{il} V^i \\ &= \partial_i V^i + \frac{1}{2}\Big(\Gamma^l_{ik}\delta^k_l + \Gamma^l_{ij}\delta^j_l\Big) V^i \\ &= \partial_i V^i + \frac{1}{2} g^{jk}\Big(\Gamma^l_{ik} g_{lj} + \Gamma^l_{ij} g_{kl}\Big) V^i \ . \end{aligned} \tag{4.32}$$

Then from the following relation

$$g_{kj;i} = 0 = g_{kj,i} - \Gamma^l_{ik} g_{lj} - \Gamma^l_{ij} g_{kl} \tag{4.33}$$

we have

$$g_{kj,i} = \Gamma^l_{ik} g_{lj} + \Gamma^l_{ij} g_{kl} \tag{4.34}$$

and going back to the divergence relation, we get

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \partial_i V^i + \frac{1}{2} g^{jk}\partial_i g_{kj} V^i \\ &= \partial_i V^i + \frac{1}{2} tr(g^{jk}\partial_i g_{kl}) V^i \\ &= \partial_i V^i + \frac{1}{2} tr(g^{-1}\partial_i g) V^i \\ &= \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}\, V^i\right) \ . \end{aligned} \tag{4.35}$$

Where we used the formula for any non-singular matrix A:

$$tr\Big[\frac{dA}{d\tau}A^{-1}\Big] = \frac{1}{detA}\frac{d}{d\tau}(detA) \tag{4.36}$$

For details concerning the derivation of this expression, see (Wald, 1984).

So, the derived expression implies:

$$\frac{d}{dt}\theta = 3(\ddot{a}/a - \dot{a}^2/a^2) \ . \tag{4.37}$$

This equation is a special case of the Rachaudhury Eq. (3.38) for time-like geodesic congruences. Indeed, specializing to the family of co-moving observers in a FLRW geometry and noting that

- the proper time τ is the cosmological time t;
- the rotation is zero (on symmetry grounds);
- the shear is zero (on symmetry grounds);
- the relevant component of the Ricci tensor is $R_{\tau\tau} = -3\ddot{a}/a$,

one sees that the Raychaudhuri equation (3.38) reduces to

$$\frac{d}{d\tau}\theta = -\frac{1}{3}\theta^2 - R_{\tau\tau} = -3\frac{\dot{a}^2}{a^2} + 3\frac{\ddot{a}}{a} \ . \tag{4.38}$$

Then, considering the 00-component of the Einstein's equations

$$R_{\tau\tau} = 8\pi(T_{\tau\tau} + \frac{1}{2}T^a_a) - \Lambda = 4\pi(\rho + 3p) - \Lambda \tag{4.39}$$

one finds that

$$\frac{d}{d\tau}\theta = -\frac{1}{3}\theta^2 - R_{\tau\tau} \quad \Leftrightarrow \quad -3\frac{\ddot{a}}{a} = 4\pi(\rho + 3p) - \Lambda \tag{4.40}$$

which is precisely Eq. (4.17). The aim of the preceding pages was to show that, through the Raychaudhuri's equation (3.38), we arrived at the same results achieved by means of the resolution of the Einstein's equations in presence of a FLRW metric. This illustrates the importance of the Raychaudhuri's equation; in fact, as one can see, we come up with the equation, which, is going to bring us to the initial singularity.

In particular, as we mentioned, the two different approaches bring us to the following fundamental result: given Eq. (4.17), we note that the model of our universe cannot be static, provided that $\rho > 0$ and $p \geq 0$ (i.e., under certain energy conditions), which tell us that $\ddot{a}(\tau) < 0$. Thus, the universe must always either be expanding ($\dot{a} > 0$) or contracting ($\dot{a} < 0$). Note the peculiar nature of this expansion/contraction: the distance scale between all isotropic observers (in particular, between galaxies) changes with time, but there is no preferred center of expansion/contraction. Indeed, if the distance between two isotropic observers at time τ is R, the rate of change of R is

$$v \equiv \frac{dR}{d\tau} = \frac{R}{a}\frac{da}{d\tau} = HR \ , \tag{4.41}$$

where $H(\tau) = \dot{a}/a$ is called Hubble's constant. Equation (4.41) is known as Hubble's law. The expansion of the universe in accordance to this equation has been confirmed by the observation of red-shifts of distant galaxies. Neglecting Λ and considering an expanding universe, $\dot{a} > 0$, and $\ddot{a} < 0$ because $\rho + 3p > 0$, it follows that there cannot have been a turning point in the past and $a(t)$ must be concave downwards. Therefore, if the universe had always expanded at its present rate, then at the time $T = \dot{a}/a = H^{-1}$ ago $a(t)$ must have reached $a = 0$. Thus, under the assumptions of homogeneity and isotropy, GR makes the striking prediction that at a time less than H^{-1} ago, the universe was in a singular state. The distance between all "points of space" was zero; the density of all forms of matter and the curvature of space-time was infinite. This singular state of the universe is referred to as the *big bang*.

4.3 Energy Conditions and The Geodesic Focusing Theorem

The conditions on the nature of matter present in the universe are essential for the study of the initial singularity in the FLRW metric.

A generic observer with a four-velocity u^a will measure the energy density $T^{ab}u_a u_b$.

The Weak Energy Condition (WEC): if the energy-momentum tensor obeys the inequality

$$T^{ab}u_a u_b \geq 0 \ , \tag{4.42}$$

for all time-like vectors u^a then we say that the energy-momentum tensor obeys the weak energy condition.
This condition for the energy-momentum tensor is satisfied by most fluids known. It is basically saying that all time-like observers will measure a positive energy density.

$$\text{WEC} \quad \Leftrightarrow \quad \rho \geq 0 \quad \text{and} \quad \rho + p_i \geq 0 \ \ (i = 1,2,3) \ . \tag{4.43}$$

The Null Energy Condition (NEC): if the energy-momentum tensor obeys the inequality

$$T^{ab}k_a k_b \geq 0 \ , \tag{4.44}$$

for all light-like vectors k^a then we say that the energy-momentum tensor obeys the null energy condition.

$$\text{NEC} \quad \Leftrightarrow \quad \rho + p_i \geq 0 \ \ (i = 1,2,3) \ . \tag{4.45}$$

The Strong Energy Condition (SEC): if the energy-momentum tensor obeys the inequality

$$\left(T^{ab} - \frac{1}{2}Tg^{ab}\right)u_a u_b \geq 0 \ , \tag{4.46}$$

for all time-like vectors u^a then we say that the energy-momentum tensor obeys the strong energy condition.
This condition for the energy-momentum tensor is satisfied by most fluids known. All time-like observers will measure a positive energy density.

$$\text{SEC} \quad \Leftrightarrow \quad \rho + \sum_i p_i \geq 0 \quad \text{and} \quad \rho + p_i \geq 0 \ \ (i = 1,2,3) \ . \tag{4.47}$$

If we have a barotropic perfect fluid

$$p = \omega\rho \tag{4.48}$$

and all the principal pressures are equal to p, the WEC is equivalent to $\omega \geq -1$. The SEC on the other hand, put the stronger constraint $\omega \geq -\frac{1}{3}$. Note from Eq. (4.17), that if SEC is satisfied then gravity is attractive for observers moving along time-like geodesics. If a space-time satisfies the Einstein's equations then we can replace the energy-momentum tensor with the Ricci tensor ($R_{ab} = T_{ab} - \frac{1}{2}g_{ab}T$). The SEC can therefore be written as

$$R^{ab}u_a u_b \geq 0 \tag{4.49}$$

for all time-like u^a. Hence, the space-time has a positive curvature for time-like vectors. If we have two neighboring parallel geodesics and if the SEC is satisfied, the geodesics will converge and at some point meet.

Assume that the matter obeys the SEC (i.e., $\rho + 3p \geq 0$), this implies that the last term of Eq. (3.38) will be negative. Furthermore, we will also assume that the congruence is non-rotating. Hence, from the principal Raychaudhuri equation we get the inequality

$$\dot{\theta} \leq -\frac{1}{3}\theta^2 \ . \tag{4.50}$$

The relation

$$\frac{d}{d\tau}\left[\theta(\tau)^{-1}\right] = -\frac{\dot{\theta}}{\theta^2} \ , \tag{4.51}$$

dividing by θ^2 yields

$$\frac{d}{d\theta}\left(\frac{1}{\theta}\right) \geq \frac{1}{3} \ , \tag{4.52}$$

considering $d\theta^{-1} = -\theta^{-2}d\theta$ and hence integrating in the interval $[\theta, \theta_0]$,

$$\frac{1}{\theta(\tau)} \leq \frac{1}{\theta_0} + \frac{1}{3}\tau \ . \tag{4.53}$$

Here, θ_0 is the value of θ at $\tau = 0$, and $\tau \leq 0$. Assume, further, that the geodesic congruence is expanding at $\tau = 0$, i.e. $\theta_0 > 0$ (which would be the case for an expanding universe). Then, according to Eq. (4.53), the function $\theta(\tau)^{-1}$ must have passed through zero at a finite time τ_s. In particular, τ_s is bounded by the inequality $|\tau_s| \leq 3\theta_0^{-1}$. This means that at the time τ_s, the expansion scalar was infinite $\theta(\tau_s) = \infty$, which indicates that there was a singularity at τ_s. Strictly speaking, this only tells that there is a singularity of the geodesic congruence, however this analysis is one of the key ingredients for proving the singularity theorem stated below. There are many global aspects that we have to consider, but we refer the reader to (Wald, 1984) or to (Hawking Ellis, 1973) for more details. Roughly speaking we can say that:

If the matter obeys the SEC and there exists a positive constant $C > 0$ such that $\theta > C$, everywhere in the past of some specific hyper-surface, then there exists a past singularity where all past directed geodesics end.

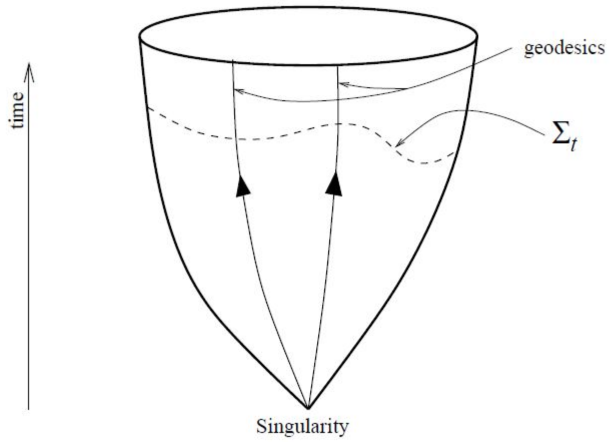

Figure 4.3: An expanding universe containing matter that obeys the SEC, means that the universe has a past singularity

We underline the fact that the singularity in θ represents merely a singularity in the congruence, not a singularity in the structure of space-time. It simply states that caustics will develop in a congruence if convergence occurs anywhere. This hold for congruences in Minkowski space-time and in many other singularity free space-times.

The initial singularity results from a homogeneous contraction of space down to "zero size". Since space-time structure itself is singular at the big bang, it does not make sense, either physically or mathematically, to ask about the state of the universe "before" the big bang; there is no natural way to extend the space-time manifold and the metric beyond the big bang singularity. Thus, GR leads to the viewpoint that the universe began at the big bang. For many years it was generally believed that the prediction of a singular origin of the universe was due merely to the assumptions of exact homogeneity and isotropy, that if these assumptions were relaxed one would get a non-singular "bounce" at small $a(\tau)$ rather than a singularity. However, the singularity theorems (see Hawking Ellis, 1973) of GR show that singularities are generic features of cosmological solutions. These theorems have ruled out the possibility of "bounce" models close to homogeneous and isotropic ones. Of course, at the extreme conditions very near the big bang singularity one expects that quantum effects will become important, and the predictions of classical GR are expected to break down.

As the reader may have noticed, we still have not defined what a singularity really is. Intuitively, a space-time singularity is a "place" where the curvature "blows up" or other "pathological behavior" of the metric takes place. The difficulty in making this notion into a satisfactory, precise definition of a singularity stems from the above terms

placed in quotes. In all physical theories except general relativity, the manifold and the metric structure of space-time is assumed in advance; we know the "where and when" of all space-time events, and our task is simply to determine the values of physical quantities at these events. If a physical quantity is infinite or otherwise undefined at a point in space-time, we have no difficulty in saying that there is a singularity at that point. Thus, for example, we easily may give precise meaning to the statement that the Coulomb solution of Maxwell's equations in special relativity has a singularity at the events labeled by $r = 0$.
The situation in GR is completely different. Here we are trying to solve for the manifold and metric structure of space-time itself. Since the notion of an event makes physical sense only when the manifold and metric structure are defined around it, the most natural approach in general relativity is to say that a space-time consists of a manifold $\mathcal{M}$ and a metric g_{ab} defined everywhere on $\mathcal{M}$. Thus, the big bang singularity of FLRW solution is not considered to be part of the space-time manifold; it is not a "place" or a "time". Similarly, only the region $r > 0$ is incorporated into the Schwarzschild space-time; unlike the Coulomb solution in special relativity, the singularity at $r = 0$ is not a "place".
Further, the general characterization of singularities by the "blowing up" of curvature is unsatisfactory, too. In fact, curvature is described by a tensor field R^a_{bcd}, and if one uses bad behavior of the components of this tensor or its derivatives as a criterion for singularities, one can get into trouble since this bad behavior of components could be due to bad behavior of the coordinate or tetrad basis rather than the curvature. The characterization of singularities by a detailed enumeration of the possible other types of "pathological behavior" of the space-time metric also appears to be a hopeless task because of the infinite variety of possible pathological behaviors.

How, then, can one characterize singular space-times? By far the most satisfactory idea proposed thus far is basically to use the "holes" left behind by the removal of singularities as the criterion for their presence. These "holes" should be detectable by the fact that there will be geodesics which have finite affine length; more precisely there should exist geodesics which are inextendible in at least one direction but have only a finite range of affine parameter. Such geodesics are said to be incomplete. Thus, we could define a space-time to be singular if it possesses at least one incomplete geodesic. In such a space-time, it is possible for at least one freely falling particle or photon to end its existence within a finite "time" (i.e. affine parameter) or to have begun its existence a finite time ago. Hence, even if one does not have a completely satisfactory general notion of singularities, one would be justified in calling such space-times physically singular. It is this property that is proven, by the singularity theorems, to hold in a wide class of space-times. These theorems, known as the Penrose-Hawking singularity theorems, have as one of the fundamental tools in their formulation and proof the Raychaudhuri equations. For a thorough description see (Hawking Ellis, 1973).

Conclusions

We began this thesis by introducing the idea of parallel transporting quantities along a given path, on a generic manifold. From this followed the definition of geodesic curve Eq. (1.11) and other useful properties involving normal frames and geodesic deviation (Chapter 1). This brief introduction about differential geometry aimed at giving the suitable basis for treating the following topics.

The geodesic motion and its properties, as we have noticed throughout our study, are crucial in analyzing the features of a generic space-time. This is done in Chapter 2, where we explained the relation between curvature of space-time and presence of a gravitational field. Besides, we also presented the main features and equations of the General Theory of Relativity, with particular interest on the stress-energy tensor.

In Chapter 3, we presented the Raychaudhuri equations (3.38) as dynamical equations describing the evolution of some parameters (expansion θ, shear σ_{ab} and twist ω_{ab}) characterizing the geodesic congruence (namely, a bundle of geodesic curves). Immediately after, we presented, in Chapter 4, the application for which these equations have been derived by Raychaudhuri. Specifically, the presence of an initial singularity in a homogeneous and isotropic model of the universe. In order to show that, we described the FLRW metric and pointed out the reasons leading to this singularity. Then, we discussed one of our main results: the geodesic focusing theorem. The focusing of a geodesic congruence, in the FLRW space-time, resulting in $\theta \to -\infty$, is together with few additional hypothesis, the proof of the initial (*big bang*) singularity; independently of any symmetries encoded in the assumptions of homogeneity and isotropy.

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