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Study Of Gaseous Structures in Axisymmetric Rotation in presence of a Black Hole

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“To my parents ”

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Sommario

L'oggetto di tale tesi é lo studio di sistemi gassosi in rotazione assisimmetrica con distribuzioni barocline.

Nella prima parte dell'elaborato, vengono discussi i concetti fondamentali della dinamica dei fluidi; le equazioni del moto di fluidi in rotazione soggetti ad un potenziale gravitazionale vengono derivate e studiate.

Successivamente viene introdotta l'analisi sulla stabilità lineare e vengono derivati i criteri di Schwarzschild e di Rayleigh. Tali criteri vengono poi generalizzati introducendo il criterio di Solberg-Hoiland.

Soluzioni stazionarie per fluidi in rotazione assisimmetrica vengono introdotte e vengono discusse le differenze nei casi barotropico e baroclino.

Viene poi presentata una famiglia di sistemi baroclini in equilibrio in un potenziale gravitazionale assisimmetrico.

Successivamente ho considerato una famiglia di tori gassosi con una distribuzione di densità a legge di potenza. Sono stati studiati due casi: tori auto-gravitanti e tori in equilibrio in presenza della loro auto-gravità e di un buco nero centrale.

Infine, il criterio di Solberg-Hoiland é stato utilizzato per determinare in quali condizioni i sistemi toroidali auto-gravitanti sono linearmente stabili e per ottenere una formulazione analitica della condizione di stabilità per tori in equilibrio in presenza della loro auto-gravità e di un buco nero centrale.

Abstract

The subject of this thesis is the study of axisymmetric rotating gaseous systems with baroclinic distributions.

In the first part of the thesis, the basic concepts of fluid dynamics are discussed. The equations of motion of a rotating fluid subjected to a gravitational potential are derived.

Furthermore the linear stability analysis is presented and the Schwarzschild and Rayleigh criteria are derived. These criteria are then generalized introducing the Solberg-Hoiland criterion.

Stationary solutions for axisymmetric rotating fluids are introduced and the differences between the baroclinic and barotropic cases are discussed.

A family of baroclinic systems in equilibrium in an axisymmetric gravitational potential is presented.

Then I have considered a family of gaseous tori with power-law density distribution. Two cases have been studied: self-gravitating tori and tori in equilibrium in the presence of their self-gravity and a central black hole.

Finally the Solberg-Hoiland criterion is used to determine whether the considered self-gravitating toroidal systems are linearly stable and to obtain an analytical formulation of the stability condition for tori in equilibrium in the presence of their self-gravity and a central black hole.

Introduction

The goal of this thesis is to construct physical models of astrophysical fluids rotating in equilibrium in a gravitational potential.

In an intuitive way, we can define fluids as those objects that flow. A first question we can ask ourselves is therefore how these objects flow and under what conditions they reach equilibrium. Fluids are composed of particles at the microscopic level, but their description through the fluid dynamics equations treats them as a continuous medium with well-defined macroscopic properties. Such a description therefore presupposes that we are dealing with such large numbers of particles locally that it is meaningful to average their properties rather than following individual particle trajectories.

From an astrophysical point of view, many objects can be modeled as fluids: some examples are the stars, the gas components of the interstellar medium, the intracluster medium, the stellar winds, jets and accretion disks. These objects are distributed on scales that can vary from tens of kilometers to Mpc.

In details, in this work I will use a fluid stationary model to study toroidal gaseous systems in baroclinic configuration. These tori were originally proposed as collisionless stellar systems, but the isotropic case can be interpreted as a gaseous system. One of the possible astrophysical applications of these toroidal structures are the Active Galactic Nuclei (AGNs).

From observational constraints and statistical arguments, current belief is that every galaxy contains a super massive black hole (SMBH) at its gravitational center. Most of these SMBHs are quiescent, but approximately 1% - 10% are called "active". One of the most important characteristics of these AGNs is that their bolometric luminosities ranges from $\sim 10^{42}$ to $\sim 10^{48}$ erg/s. [1]

Although there are numerous AGN classes, a unified scheme has been emerging (e.g. [Antonucci 1993], [Heckman & Best 2014]). According to this unification scheme one can construct two different categories: IR-optical-UV-X-ray unification and radio unification. Here we consider the IR-optical-UV-X-ray unified models.

For a typical AGN the nuclear activity is powered by a supermassive black hole and its accretion disk, and this central engine is surrounded by a dusty toroidal gaseous structure. Much of the observed diversity is simply explained as the result of viewing

[1] The luminosity of an AGN can be even higher than that of the host galaxy.

this axisymmetric structure from different angles.

The current model to explain the origin of the AGN emitted energy admits the presence of hot accretion disk surrounding a SMBH. In details energy is generated by gravitational infall of rotating gas that converts its gravitational and kinetic energy into electromagnetic emission.

The geometric structure of a typical AGN is showed in Fig.(1).

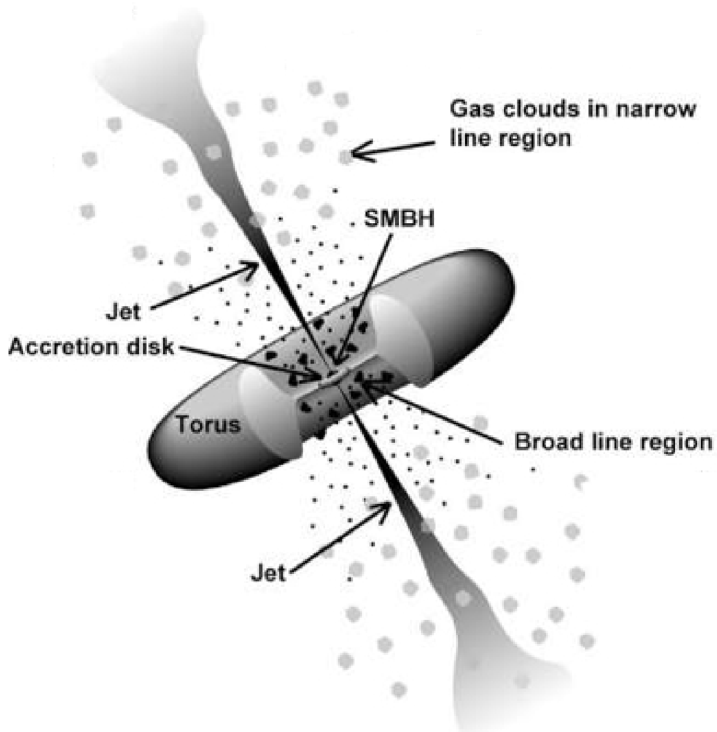

Figure 1: Typical structure of an AGN.
(Source: https:fermi.gsfc.nasa.gov/science/eteu/agn/)

In Fig. (1) we can identify the principal components of most AGNs:

- SMBH: a super massive black hole is an extremely massive object whose typical masses[2] are 10^6- 10^9 $M_\odot$ [Peterson 2004];
- Accretion disk: a sub-pc rotating structure around the black hole that transports gas into the center;
- Jets: collimated outflows of relativistic accelerated particles (associated to γ-ray emission) whose emission is mainly attributed to the inverse Compton mechanisms and to synchrotron radiation;
- Broad-Line Region (BLR): it is a zone above the accretion disk in which massive ionized clouds can be found, and it becomes noticeable with broad spectral lines. These gas clouds move at distance of $0.01 - 1\,pc$ from the SMBH;

[2] $M_\odot = 1.99 \times 10^{33}\ g$ is the mass of the sun.

- Narrow-Line Region (NLR): it is beyond the BLR. NLR extends to hundreds and even thousand of parsec along the general direction of the opening in the torus ("ionization cones"). It is formed by massive ionized clouds, with velocity dispersion lower than the BLR;

- Molecular torus: a circum-nuclear toroidal structure. Current belief is that these tori can be composed by a molecular-gas component of solar metallicity and by a dusty part [Fritz et al. 2005]. For instance, the typical size is $\sim 10\ pc$ for $M_{BH} \sim 10^7\ M_\odot$.

From observational data, we know that these toroidal structures can assume different orientations along the line of sight and from this it is possible to build various astronomical classes: type-I AGNs, type-II AGNs, LINERs, Lineless AGNs, radio galaxies and radio loud quasar [Netzer 2015].

- Type-I AGNs: sources showing broad $(1000 - 20000\, km\, s^{-1})$ semi-permitted or forbidden emission lines and a bright, non stellar, central point source visible at all wavelength.

- Type-II AGNs: sources containing strong narrow $(300 - 1000\, kms^{-1})$ NIR-optical-UV emission lines that are broader than the emission line galaxies of similar type observed.

- LINERs: AGNs that are characterized by their low ionization, narrow emission lines from gas which is ionized by non stellar sources.

- Lineless AGNs: in this group one can find AGNs with extremely weak emission lines.

In the simplest possible model the molecular tori can be considered in equilibrium in the gravitational potential of the central black hole. In some cases the mass of the torus can be of the order of the SMBH, so the self-gravity of the torus can be important. Therefore these models studied in this thesis could be used as simple models of AGN tori. Though idealized, these models are physical, in the sense that they are consistent stationary solutions of the fluid equations.

We will discuss how gaseous toroidal structures modify their kinematic properties as a function of the assigned density distribution in a self-gravitating configuration and in the presence of a black hole. Among these models we will identify restricted families of tori that are linearly stable.

Finally we note that some of these methods studied in this thesis allow to build general baroclinic rotating distributions, which can be applied to astrophysical systems different from AGN tori (e.g. galactic coronae).

Structure of the thesis

In Chapter 1, I discuss how one can apply the fluid theory to the astrophysical systems. I introduce the gravitational potential and the Poisson's equation that will permit to construct density-potential pairs. The fluid equations of mass and momentum are discussed in their most general formulation. Then, since most of astrophysical objects have non-negligible angular momentum, I focus the case of axial symmetry to study rotating flows. The perfect gas model is introduced and its behavior under isothermal and adiabatic transformations is described.

Chapter 2 introduces the fluid instability theory. Considering small (linear) perturbations, I study the dispersion relation in two cases: a uniform medium and a stratified atmosphere. For both of them I discuss the stability conditions of the system. Then, I present the Schwarzschild criterion. Afterwards, the focus of the chapter moves on the incompressible shear flow in axisymmetric conditions and the Rayleigh stability criterion is derived. Lastly, I show the behavior of a compressible rotating flow in axisymmetric conditions and I derive the Solberg-Hoiland stability criterion.

In Chapter 3, I present families of hydrostatic models for the rotation of gas in axisymmetric conditions. I start from the Poincaré-Wavre theorem that permits to link the gas distribution (barotropic or baroclinic) to the velocity field. With particular reference to baroclinic models, I derive the sufficient condition to have a physical acceptable solution by knowing the gas density distribution and the gravitational potential. This chapter concludes the introductory theories that will be used in chapter four to present the original results of the thesis.

In Chapter 4, I present toroidal gaseous system. I focus on power-law tori where the analytic density-potential pair is given by Ciotti and Bertin [Ciotti, Bertin 2005]. Two physical conditions are discussed: a self-gravitating case and tori in equilibrium in the presence of their self-gravity and a central black hole. For both cases, by solving the fluid-dynamics equations, I determine the rotation velocity field and a family of physical solutions is found. Then I use the Solberg-Hoiland criterion to determine whether these self-gravitating tori are linearly stable or unstable, and finally I obtain an analytic formulation in cylindrical coordinates of the Solberg-Hoiland criterion for the self-gravitating fluid in presence of the black hole.

In Appendix A, by means of the Fourier analysis, I discuss the axisymmetric instability and I obtain, with an alternative approach, the instability criteria, already presented in Chapter 2, from the dispersion relation.

In Appendix B, I show how to construct the projection of the vector $\boldsymbol{\nabla} \cdot \mathbf{F}$ (where $\mathbf{F}$ is a generic vector) on the $\mathbf{n}$ direction, that is useful when one needs to express this quantity on a generic mathematical basis, as, for instance, for the case of the transformation in cylindrical coordinates of the Euler equation.

Chapter 1

Astrophysical fluid Dynamics

In this chapter we will analyze the fundamental concepts, taken by theoretical physics, that permit to formulate a consistent theory to understand the behavior of many astrophysical objects. In fact, as we will show, the study of classical fluids makes it possible to create models in agreement with observational phenomena.

We will show the conservative equations, Poisson's equation and we will study how an axial symmetry reduces the degrees of freedom of the fluid rotating systems.

In this chapter we follow the treatment from "Principles of Astrophysical Fluid Dynamics" [Clarke, Carswell 2007].

1.1 A fluid approach to astrophysics

Since many astronomical objects can be discussed with the fluid theory, it is important to fix in which situations hydrodynamic approach is able to simplify the physical problems and under which conditions it is applicable. To this purpose we need to introduce what in the literature is called "fluid element".

The equations of dynamics governing fluids are based on the concept of fluid element, which is a region of space in which we can define macroscopic variables such as pressure, temperature and density. We assume this area is such that:

1. it is small enough that we can ignore systematic variations across it for any variable we are interested in, for example the region size l_{region} is much smaller than a scale length for change of any relevant variable q.

   So
   $$l_{region} \ll l_{scale} \sim \frac{q}{|\nabla q|}, \tag{1.1.1}$$

2. it is large enough that the element contains sufficient particles for us to ignore fluctuations due to the finite number of particles; thus if we define the number density of particles per unit volume n, we require that:
   $$n l_{region}^3 \gg 1, \tag{1.1.2}$$

The description of fluid-dynamics equations

To formulate the equations of fluid-dynamics for mass density, momentum and energy in the literature two different descriptions can be adopted:

i. Eulerian description

   We consider a small volume at a fixed spatial position. The fluid flows through the volume with physical variables specified as functions of time and the position, which is kept fixed. This means, for example, that we can write the density as $\rho = \rho(\mathbf{r}, t)$ or the temperature as $T = T(\mathbf{r}, t)$. So, the change of any measurable quantity as a function of time is $\partial/\partial t$ of the quantity, evaluated at the fixed position $\mathbf{r}$.

ii. Lagrangian description

   In this approach the spatial reference system is comoving with the fluid; in fact one chooses a particular fluid element and studies the change in variables in that. Thus, adopting this description, we examine the behavior of $\rho = \rho(\mathbf{a}, t)$, where $\mathbf{a}$ is a label for a particular fluid element. Following this approach the time derivative is a partial one at a fixed $\mathbf{a}$, and the rate of chance with respect to the time for a fixed element is denoted by the Lagrangian derivative: D/Dt. In this description, position is not an independent variable because $\mathbf{r} = \mathbf{r}(\mathbf{a}, t)$.

In conclusion the Eulerian description refers to the world as seen at a fixed spatial position, while the Lagrange description refers to the world as seen by an observer riding on a fluid element.

Relation between the Eulerian and Lagrangian descriptions. The two presented approaches are obviously connected to each other. We now see how one can move from one to another.

Consider a quantity Q in a fluid element which is at position $\mathbf{r}$ at a time t. At a time $t + \delta t$ the element is at $\mathbf{r} + \delta\mathbf{r}$, and from the definition

$$\frac{DQ}{Dt} = \lim_{\delta t \to 0} \left[\frac{Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t)}{\delta t} \right]. \tag{1.1.3}$$

The numerator in the square brackets can be expressed as

$$\begin{aligned} Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t) =& Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t) + Q(\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t + \delta t) = \\ =& Q(\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t) + Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t + \delta t) = \\ =& \frac{\partial Q(\mathbf{r}, t)}{\partial t}\delta t + \delta\mathbf{r} \cdot \boldsymbol{\nabla} Q(\mathbf{r}, t + \delta t) = \\ =& \frac{\partial Q(\mathbf{r}, t)}{\partial t}\delta t + \delta\mathbf{r} \cdot \left[\boldsymbol{\nabla} Q(\mathbf{r}, t) + \frac{\partial \boldsymbol{\nabla} Q}{\partial t}\delta t + \dots \right]. \end{aligned}$$

If we neglet second order terms we get in the limit:

$$\begin{aligned} \frac{DQ}{Dt} &= \lim_{\delta t \to 0} \left[\frac{Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - Q(\mathbf{r}, t)}{\delta t} \right] = \\ &= \lim_{\delta t \to 0} \left[\frac{1}{\delta t} \left(\frac{\partial Q(\mathbf{r}, t)}{\partial t}\delta t + \delta\mathbf{r} \cdot \boldsymbol{\nabla} Q(\mathbf{r}, t) \right) \right] = \\ &= \frac{\partial Q(\mathbf{r}, t)}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} Q(\mathbf{r}, t), \end{aligned}$$

where $\mathbf{u}$ is the fluid velocity.

So the relation between the two descriptions is given by the following equation

$$\frac{DQ(\mathbf{r}, t)}{Dt} = \frac{\partial Q(\mathbf{r}, t)}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} Q(\mathbf{r}, t). \tag{1.1.4}$$

We point out that the Lagrangian time derivative has two terms. The first one is due to the rate of change at a fixed location (which is the Eulerian time derivative), while the second one is due to the fact that the fluid element has moved to a new location where the variable assumes a different value. The Lagrangian derivative is called also 'convective derivative'.

1.2 The gravitational potential

In order to describe the gravitational potential, we must take into account conservative forces. These are defined as $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ and can be written, thanks to Stokes' theorem, as

$$\mathbf{F} = \boldsymbol{\nabla}\Psi, \tag{1.2.1}$$

where Ψ is a scalar potential.

In this particular case we define a scalar gravitational potential Φ such that the acceleration $\mathbf{g}$ is given by

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi. \tag{1.2.2}$$

Since the result is independent of the path taken, the work required to take a point object of unitary mass from the position r to infinity is given by

$$-\int_r^\infty \mathbf{g} \cdot d\mathbf{l} = \int_r^\infty \boldsymbol{\nabla}\Phi \cdot d\mathbf{l} = \Phi(\infty) - \Phi(r), \tag{1.2.3}$$

where $\Phi(\infty)$ is the value of the potential far from any gravitational source, and it is often taken to be zero in many relevant astrophysical applications: in fact it is only potential differences and gradients that have physical significance.

1.2.1 Poisson's equation

Poisson's equation is a partial differential equation of elliptic type with broad utility in theoretical physics. It arises, for instance, to describe the potential caused by a given charge or mass density distribution; with the potential known, one can then calculate gravitational field.

Consider a surface S and a point P located in the space. We define the solid angle subtended at P by $d\mathbf{S}$ as

$$d\Omega = \frac{d\mathbf{S} \cdot \hat{\mathbf{r}}}{r^2}. \tag{1.2.4}$$

Integrating over the whole surface

$$\int_S \frac{\hat{\mathbf{r}} \cdot d\mathbf{S}}{r^2} = \begin{cases} 4\pi & \text{if P is anywhere inside } S, \\ 0 & \text{if P is anywhere outside } S. \end{cases}$$

Proof. We may derive this equation by placing the origin of coordinates at P.

The left hand side may be written as

$$\int_S \frac{\hat{\mathbf{r}} \cdot d\mathbf{S}}{r^2} = \int_S \mathbf{f} \cdot d\mathbf{S},$$

where

$$\frac{\hat{\mathbf{r}}}{r^2} = \mathbf{f}.$$

Provided that $\mathbf{f}$ is finite over the volume enclosed by S (hence P is outside S), we can use the divergence theorem in Cartesian coordinates:

$$\begin{aligned}
\int_S \mathbf{f}\cdot d\mathbf{S} &= \int_V \boldsymbol{\nabla}\cdot\mathbf{f}\, dV = \\
&= \int_V \boldsymbol{\nabla}\cdot\left(\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}\right) dV = \\
&= \int_V \left[\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)+\right.\\
&\qquad\left.+\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)\right] dV = \\
&= \int_V \left[\frac{1}{(x^2+y^2+z^2)^3}\Big((x^2+y^2+z^2)^{3/2}-3x^2(x^2+y^2+z^2)^{1/2}+\right.\\
&\qquad +(x^2+y^2+z^2)^{3/2}-3y^2(x^2+y^2+z^2)^{1/2}+\\
&\qquad \left.+(x^2+y^2+z^2)^{3/2}-3z^2(x^2+y^2+z^2)^{1/2}\Big)\right] dV = \\
&\int_V \left[\frac{1}{(x^2+y^2+z^2)^3}\Big(3(x^2+y^2+z^2)^{3/2}-\right.\\
&\qquad \left.-3(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2)\Big)\right] dV = 0.
\end{aligned}$$

So we obtain, if P is outside S:

$$\int_S \frac{\hat{\mathbf{r}}\cdot d\mathbf{S}}{r^2} = 0. \tag{1.2.5}$$

In the case that P is inside S, then $\mathbf{f}$ is undefined at P, in order to apply the divergence theorem we need to place a small interior surface, named S', around P so that P is excluded from the volume of integration. We can write

$$\int_S \frac{\hat{\mathbf{r}}\cdot d\mathbf{S}}{r^2} + \int_{S'} \frac{\hat{\mathbf{r}}\cdot d\mathbf{S}}{r^2} = 0. \tag{1.2.6}$$

We are free to choose S' as we please so long as it excludes P; i.e. we could choose S' to be a sphere of radius a. In this case we have [1]

$$\frac{\hat{\mathbf{r}}\cdot d\mathbf{S}}{r^2} = \frac{-4\pi a^2}{a^2} = -4\pi.$$

Hence if P is inside S we get

$$\int_S \frac{\hat{\mathbf{r}}\cdot d\mathbf{S}}{r^2} = 4\pi. \tag{1.2.7}$$

□

[1]The negative sign results from the fact that the outward normal from S' is in the inward radial direction with respect P.

Suppose we put a mass M placed at the point P inside S. Since $\mathbf{g} = -(GM/r^2)\hat{\mathbf{r}}$, $\mathbf{g} \cdot d\mathbf{S} = -GM d\Omega$, and since the product GM is constant[2], we must have

$$\int_S \mathbf{g} \cdot d\mathbf{S} = -4\pi GM. \tag{1.2.8}$$

If we distribute masses throughout the volume enclosed by S, we obtain

$$\int_S \mathbf{g} \cdot d\mathbf{S} = -4\pi G \sum_i M_i = -4\pi G \int_V \rho \, dV. \tag{1.2.9}$$

Using the Gauss theorem we get

$$\int_S \mathbf{g} \cdot d\mathbf{S} = \int_V \boldsymbol{\nabla} \cdot \mathbf{g} \, dV, \tag{1.2.10}$$

that is

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{g} + 4\pi G\rho) \, dV = 0. \tag{1.2.11}$$

Since this must hold for any volume,

$$\boldsymbol{\nabla} \cdot \mathbf{g} + 4\pi G\rho = 0. \tag{1.2.12}$$

Now considering eq. (1.2.2), we get

$$\boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla}\Phi) + 4\pi G\rho = 0, \tag{1.2.13}$$

which can be written in the usual Poisson equation formulation

$$\boxed{\nabla^2 \Phi = 4\pi G\rho.} \tag{1.2.14}$$

1.3 The fluid equations

The equations that describe the motion of fluid elements are based upon conservation principles such as conservation of mass and momentum.

1.3.1 Conservation of mass

Consider a region of fixed volume V enclosed by a surface S.

If the mass density of the fluid inside the surface is given by ρ, the rate of change of mass of the fluid contained in the volume V is

$$\frac{\partial}{\partial t} \int_V \rho \, dV. \tag{1.3.1}$$

If there are no sources or sinks for matter, this quantity must be equal to the net inflow of mass integrated over the whole surface.

[2] G is the Newtonian constant of gravitation.

The outward mass flow across an infinitesimal element dS is $\rho\mathbf{u} \cdot d\mathbf{S}$.

Suppose the velocity vector $\mathbf{u}$ forms an angle θ with the surface element vector $d\mathbf{S}$. The distance traveled by the fluid particles per unit time in the direction of $d\mathbf{S}$ is then $u \cos\theta = \mathbf{u} \cdot d\mathbf{S}/|d\mathbf{S}|$. The mass of fluid crossing the surface in that time is given by $\rho\mathbf{u} \cdot d\mathbf{S}$.

The mass gained by the volume is obtained by integrating over the surface and applying the divergence theorem one gets

$$-\int_S \rho\, \mathbf{u} \cdot d\mathbf{S} = -\int_V \boldsymbol{\nabla} \cdot (\rho\mathbf{u})\, dV. \tag{1.3.2}$$

Hence combining eq. (1.3.1) and eq. (1.3.2) we get

$$\frac{\partial}{\partial t}\int_V \rho\, dV = -\int_V \boldsymbol{\nabla} \cdot (\rho\mathbf{u})\, dV. \tag{1.3.3}$$

Since this must be true for all volumes, we obtain the Eulerian form of the continuity equation:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) = 0, \tag{1.3.4}$$

To write this in Lagrangian form, we recall eq. (1.1.4), so setting $Q = \rho$, we obtain

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\rho = -\boldsymbol{\nabla} \cdot (\rho\mathbf{u}) + \mathbf{u} \cdot \boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla} \cdot \mathbf{u}. \tag{1.3.5}$$

That is the Lagrangian form of the conservation of mass:

$$\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{1.3.6}$$

For what we have said, we can define an incompressible flow as a fluid that has the property $D\rho/Dt = 0$. This means that individual fluid elements preserve their density along their paths, hence this does not imply that the density is constant everywhere, but we can affirm $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. Therefore incompressible flows have the special and useful property of being divergence free.

1.3.2 The Euler equation

We want to set up the momentum equations which involve forces within fluid: this means we have to take into account collisional fluid elements and their interactions through the laws of thermodynamics.

From a microscopical point of view, at a finite temperature, molecules or atoms in a gas are in a state of random motion and we can define the pressure as the one-side momentum flux associated with these motions. We notice that in a fluid with uniform properties, this momentum flux is balanced by an equal and opposite momentum flux through the other side of the hypothetical surface and there is no net acceleration of the fluid. However the pressure is different from zero.

If we define the stress tensor σ_{ij} and the unit normal vector $\hat{s}_j$, the forces across a surface are given by

$$F_i = \sigma_{ij}\hat{s}_j. \tag{1.3.7}$$

One can decompose any stress tensor into the sum of a diagonal one with equal elements plus a residual tensor. This mathematical transformation permits to define the pressure as the elements of the diagonal tensor, which can be written as $p\delta_{ij}$, where δ_{ij} is the Kronecker delta.

Taking advantage of this definition we can say that any effect on a microscopic scale that involves the appearance of a term of stress in this form can be inserted in the equations of fluids as a term of pressure.

We now discuss about the momentum equation. Consider a fluid subject to

1. gravity, with local acceleration due to $\mathbf{g}$;

2. pressure, from the surrounding fluid.

On the infinitesimal surface element $d\mathbf{S}$ of the fluid we are going to consider, the force caused by the pressure of the surrounding elements can be written as $-pd\mathbf{S}$ [3].

If we are interested on the force component in a generic direction $\hat{\mathbf{n}}$, we will have $-p\hat{\mathbf{n}} \cdot d\mathbf{S}$; therefore, using the divergence theorem, the net force acting over the whole surface in the direction $\hat{\mathbf{n}}$ is

$$F = -\int_S p\hat{\mathbf{n}} \cdot d\mathbf{S} \tag{1.3.8}$$

$$= -\int_V \boldsymbol{\nabla} \cdot (p\hat{\mathbf{n}})\, dV. \tag{1.3.9}$$

In order to find the equation of motion for the fluid element, in the direction $\hat{\mathbf{n}}$, we have to impose that the rate of change of momentum for the element must be equal to the force in that direction:

$$\left(\frac{D}{Dt}\int_V \rho\mathbf{u}\, dV\right) \cdot \hat{\mathbf{n}} = -\int_V \boldsymbol{\nabla} \cdot (p\hat{\mathbf{n}})\, dV + \int_V \rho\mathbf{g} \cdot \hat{\mathbf{n}}\, dV. \tag{1.3.10}$$

We can develop the integrand in the first integral on the right hand side of the equation (1.3.10) as

$$\boldsymbol{\nabla} \cdot (p\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}p + p\boldsymbol{\nabla} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}p.$$

In the limit when the fluid lump is small, we can replace $\int dV$ by δV, and we get

$$\frac{D}{Dt}(\rho\mathbf{u}\delta V) \cdot \hat{\mathbf{n}} = -\delta V\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}p + \delta V\rho\mathbf{g} \cdot \hat{\mathbf{n}}. \tag{1.3.11}$$

Using the product rule we obtain:

$$\hat{\mathbf{n}} \cdot \mathbf{u}\frac{D}{Dt}(\rho\delta V) + \rho\delta V\hat{\mathbf{n}} \cdot \frac{D\mathbf{u}}{Dt} = -\delta V\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}p + \delta V\rho\mathbf{g} \cdot \hat{\mathbf{n}}. \tag{1.3.12}$$

[3] The minus sign arises because the surface element vector is outwards, and the force acting on the element is inwards.

The first term is the rate of change of mass of the lump of fluid we are following and since mass is conserved, this is zero, so we are left with

$$\rho\delta V\hat{\mathbf{n}}\cdot\frac{D\mathbf{u}}{Dt} = -\delta V\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\rho\mathbf{g}\cdot\hat{\mathbf{n}}. \tag{1.3.13}$$

This is true for all δV and $\hat{\mathbf{n}}$, so we conclude that in a Lagrangian formulation the momentum equation reads

$$\rho\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla}p + \rho\mathbf{g}. \tag{1.3.14}$$

To transform to the Eulerian form we use the relation (1.1.4) and then

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\boldsymbol{\nabla}p + \rho\mathbf{g}, \tag{1.3.15}$$

which is called the Euler or momentum conservative equation.

1.3.3 Fluid equations in cylindrical coordinates

Most astrophysical bodies have an angular momentum associated with them, hence there are many problems in which one have to consider fluid flows in presence of rotation.

Therefore it is useful to express the fluid equations in cylindrical coordinates: (R, φ, z). Equation (1.3.4) gives

$$\frac{\partial\rho}{\partial t} + (\nabla\cdot\rho)\mathbf{u} + \rho\left(\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\varphi}{\partial\varphi} + \frac{\partial u_z}{\partial z}\right) = 0, \tag{1.3.16}$$

while the momentum equation we have (see Appendix B)

$$\begin{aligned}
&\frac{\partial u_R}{\partial t} + u_R\frac{\partial u_R}{\partial R} + u_z\frac{\partial u_R}{\partial z} + \frac{u_\varphi}{R}\frac{\partial u_R}{\partial\varphi} - \frac{u_\varphi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{\partial\Phi}{\partial R},\\
&\frac{\partial u_\varphi}{\partial t} + u_R\frac{\partial u_\varphi}{\partial R} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{u_\varphi}{R}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_R u_\varphi}{R} = -\frac{1}{\rho R}\frac{\partial p}{\partial\varphi} - \frac{\partial\Phi}{\partial\varphi},\\
&\frac{\partial u_z}{\partial t} + u_R\frac{\partial u_z}{\partial R} + u_z\frac{\partial u_z}{\partial z} + \frac{u_\varphi}{R}\frac{\partial u_z}{\partial\varphi} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\partial\Phi}{\partial z}.
\end{aligned} \tag{1.3.17}$$

1.4 The ideal gases

To solve eq. (1.3.4) and eq. (1.3.15), we need relationships that link Φ and p to the other variables ρ and $\mathbf{u}$. Thanks to the Poisson's equation we have seen, in principle, how to get the potential starting from a distribution ρ, but we still have to find out how to deal with the pressure term.

The relationship between p and other thermodynamic properties of the system is called the equation of state.

Ideal gases are theoretical fluids composed of many randomly moving point-like particles whose only interactions are perfectly elastic collisions. They are characterized by the fact that the other kinds of interaction among their constituents are negligible: most of the internal energy of these systems is contained into the kinetic term and therefore makes it a function of the only temperature.

From an astrophysical point of view we know that the gas in the Universe is very different in its properties, but it is almost all extremely dilute by terrestrial standards and the ideal gas condition is readily met [4].

1.4.1 The equation of state

The ideal-gas law of a classical system that contains n moles of the gas can be written as

$$pV = Nk_BT = n\mathcal{R}T, \qquad (\mathcal{R} = k_BN_A) \tag{1.4.1}$$

where p is the gas pressure, $N = nN_A$, N_A is the Avogadro number and $\mathcal{R}$ is the gas constant per mole.

Equation (1.4.1) can be written as

$$p = \frac{\mathcal{R}_*}{\mu_*}\rho T \tag{1.4.2}$$

where μ_* is the mean molecular weight of the constituents of the gas and $\mathcal{R}_* \equiv 1000 \times \mathcal{R}$ is the modified gas constant with physical measures of joules per molecular weight of the substance in kilograms.

Here we present a special class of gas transformations in which pressure is a function of density only, the so called "barotropic cases".

The isothermal case. An isothermal transformation requires that the temperature can be considered constant. This can represent the case in which the cooling and the heating process force the temperature to lie within a narrow range. For what concern an ideal gas we can simplify the equation of state as $p \propto \rho$.

[4] Some exceptions to this behavior are realized in the interior of giant planets, where the high pressure and density imply a significant deviation from the ideal conditions, or the interior of neutron stars and white dwarfs where the densities are such that the distribution of particle energies becomes restricted by quantum mechanical requirements on the number of particles that can populate a given energy level.

In the case of time-dependent problems, it is also necessary that the system can relax to this constant temperature thermal equilibrium on timescales that are short compared with the flow times.

The adiabatic case. We start with the first law of thermodynamics, which is an expression of energy conservation:

$$đQ = d\mathcal{E} + pdV. \tag{1.4.3}$$

In this equation $đQ$ represents the quantity of heat absorbed by unit mass of fluid from the reservoir, $d\mathcal{E}$ is the change in the internal energy content of unit mass of the fluid and $p\,dV$ is the work done by unit mass of fluid if its volume changes by dV.

For an ideal gas, we can express the internal energy as a function of the only temperature $\mathcal{E} = \mathcal{E}(T)$, so eq. (1.4.3) can be written as

$$đQ = \frac{d\mathcal{E}}{dT}dT + pdV, \tag{1.4.4}$$

or, using (1.4.2)

$$đQ = C_V dT + \frac{\mathcal{R}_* T}{\mu_* V}dV, \tag{1.4.5}$$

where we have introduced the specific heat capacity at constant volume as

$$C_V = \frac{d\mathcal{E}}{dT}. \tag{1.4.6}$$

If the adiabatic transformation is reversible ($đQ = 0$) we have

$$C_V dT + \frac{\mathcal{R}_* T}{\mu_* V}dV = 0. \tag{1.4.7}$$

That is

$$C_V \int \frac{dT}{T} = -\frac{\mathcal{R}_*}{\mu_*}\int \frac{dV}{V}$$

$$C_V \ln T + A = -\frac{\mathcal{R}_*}{\mu_*}\ln V + A'$$

where A and A' are integration constants. This implies

$$V \propto T^{-\frac{C_V \mu_*}{\mathcal{R}_*}}. \tag{1.4.8}$$

Substituting the equation of state we obtain the scaling relations

$$p \propto T^{1+\frac{C_V}{\mathcal{R}_*/\mu}} \tag{1.4.9}$$

and

$$p \propto V^{-1+\frac{\mathcal{R}_*}{\mu_* C_V}}. \tag{1.4.10}$$

Usually these relations are expressed as the ratio between specific heat at constant pressure and specific heat at constant volume, denoted by γ. From the perfect gas equation of state we get

$$pV = \frac{\mathcal{R}_*}{\mu_*}T$$

$$d(pV) = d\left(\frac{\mathcal{R}_*}{\mu_*}T\right)$$

$$\implies V\,dp + pdV = \frac{\mathcal{R}_*}{\mu_*}dT.$$

Thus

$$\begin{aligned} đQ &= \frac{d\mathcal{E}}{dT} + pdV = \\ &= \frac{d\mathcal{E}}{dT}dT + \frac{\mathcal{R}_*}{\mu_*}dT - V\,dp = \\ &= \left(\frac{d\mathcal{E}}{dT} + \frac{\mathcal{R}_*}{\mu_*}\right)dT - V\,dp, \end{aligned} \tag{1.4.11}$$

from which we can define

$$C_P \equiv \left(\frac{d\mathcal{E}}{dT} + \frac{\mathcal{R}_*}{\mu_*}\right) \tag{1.4.12}$$

and we get the Mayer's relation for ideal gases:

$$C_P - C_V = \frac{\mathcal{R}_*}{\mu_*}. \tag{1.4.13}$$

We now define γ as

$$\gamma = \frac{C_P}{C_V} \tag{1.4.14}$$

and we may write the relations between p, V and T for reversible adiabatic changes as

$$V \propto T^{-1/(\gamma-1)}, \tag{1.4.15}$$

$$p \propto T^{\gamma/(\gamma-1)}, \tag{1.4.16}$$

$$p \propto V^{-\gamma}. \tag{1.4.17}$$

Since the volume V in the above equations refers to that occupied by unit mass of gas, the density is just the reciprocal of V and we can write the equation for a gas undergoing reversible adiabatic changes in the barotropic form:

$$p = K\rho^\gamma \tag{1.4.18}$$

where K is a constant.

Chapter 2

Fluid instability theory

In its most general sense, the stability theory refers to the stability of the solutions of the differential equations and the trajectories of the dynamical systems in the presence of small perturbations of initial conditions.

The importance of this mathematical branch in astrophysics is clear. As we said many astrophysical objects can be modeled as fluid, so let us suppose we have to study a fluid in a steady state. If we find that the small perturbations to this configuration grow with time, then our chance of finding the initial configuration are very small, and the configuration is said to be unstable with respect those perturbations. Instead the stable configurations are those in which the perturbations diminish or there is the possibility of oscillations about the equilibrium configuration.

In this chapter we follow the treatment of "Principles of Astrophysical Fluid Dynamics" [Clarke, Carswell 2007], " The Physics of Fluids and Plasma: an Introduction for Astrophysicists" [Choudhuri 1998], and "Astrophysical Flow" [Pringle, King 2007].

2.1 Hydrostatic equilibrium

In order to discuss fluid instabilities we need to introduce the concept of equilibrium.

Hydrostatic equilibrium is a particular configuration reached by the system in which $\mathbf{u} = 0$ everywhere and that $\partial/\partial t = 0$.

Since the continuity equation is trivially satisfied under these conditions, the only one to be solved is the momentum equation in which the only non-zero terms are gravity and pressure, which must therefore balanced. So we have

$$\frac{1}{\rho}\boldsymbol{\nabla} p = \mathbf{g}, \tag{2.1.1}$$

or, using the relation (1.2.2)

$$\frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\Phi. \tag{2.1.2}$$

This means that if we are dealing with a barotropic distribution ($p = p(\rho)$), we can use the Poisson's equation to solve for the density distribution $\rho(\mathbf{r})$ corresponding to hydrostatic equilibrium everywhere.

2.2 Sound waves

Sounds waves provide the principal mechanism by which disturbances propagate in fluids.

2.2.1 Sound waves in a uniform medium

Assuming that $\mathbf{g} = 0$, the Eulerian form of the continuity (1.3.4) and momentum equations (1.3.15) can be written as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0, \tag{2.2.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p. \tag{2.2.2}$$

Let us assume that the unperturbed state of the fluid represents an equilibrium state, characterized by uniform density ρ_0, pressure p_0 and zero velocity $\mathbf{u} = 0$. Let us consider perturbations to this equilibrium in the form

$$\begin{aligned} p &= p_0 + \Delta p, \\ \rho &= \rho_0 + \Delta\rho, \\ \mathbf{u} &= \Delta\mathbf{u}. \end{aligned} \tag{2.2.3}$$

We note that these perturbed equations are written in a Lagrangian form, but in this case Eulerian and Lagrangian perturbations are the same because

$$\nabla p_0 = \nabla \rho_0 = \nabla \mathbf{u}_0 = 0. \tag{2.2.4}$$

If we consider a general property of the flow X, then the perturbation to X according to (2.2.3) causes its value at a point P to change from its unperturbed value X_0 for two reasons:

(a) the perturbation may have changed the value of X of the local fluid element,

(b) the perturbation may have moved a fluid element with a different unperturbed value of X so as to be located at a point P.

If the displacement of a fluid element at P is denoted by the vector $\boldsymbol{\xi}$ then, to the first order, the change in X at a point P is given by [1]

$$\delta X = \Delta X - \boldsymbol{\xi} \cdot \boldsymbol{\nabla} X. \tag{2.2.5}$$

In this case, the unperturbed quantities are all uniform, hence the term $\boldsymbol{\xi} \cdot \boldsymbol{\nabla} X = 0$ and all δ quantities are equal to the Δ quantities.

Substituting in the fluid equations, and retaining only first order terms in the perturbed quantities, we have for the mass equation:

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\Delta \mathbf{u}) = 0. \tag{2.2.6}$$

Proof. From (2.2.1) we get

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho_0 + \Delta\rho) + \boldsymbol{\nabla} \cdot ((\rho_0 + \Delta\rho)\Delta \mathbf{u}) = 0$$

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\Delta \mathbf{u}) + \boldsymbol{\nabla} \cdot (\Delta\rho \Delta \mathbf{u}) = 0$$

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\Delta \mathbf{u}) = 0.$$

□

For the momentum equation we get

$$\frac{\partial \Delta \mathbf{u}}{\partial t} = -\left(\frac{dp}{d\rho}\right)_0 \frac{\boldsymbol{\nabla} \Delta\rho}{\rho_0}. \tag{2.2.7}$$

Proof. From (2.2.2), assuming a barotropic fluid for which a given change in density maps onto a unique change in pressure, and keeping just the first order in the

[1] we use Δ to represent Lagrangian perturbations and δ for Eulerian perturbations

expansion in serie we have

$$
\begin{aligned}
\frac{\partial \Delta \mathbf{u}}{\partial t} + \Delta \mathbf{u} \cdot \boldsymbol{\nabla}(\Delta \mathbf{u}) &= -\frac{1}{\rho_0 + \Delta\rho} \boldsymbol{\nabla}(p_0 + \Delta p), \\
\frac{\partial \Delta \mathbf{u}}{\partial t} &= -\frac{1}{\rho_0 + \Delta\rho} \boldsymbol{\nabla} \Delta p \\
&= -\frac{1}{\rho_0 \left(1 + \frac{\Delta\rho}{\rho_0}\right)} \boldsymbol{\nabla} \Delta p \\
&= -\frac{1}{\rho_0} \left(1 + \frac{\Delta\rho}{\rho_0}\right)^{-1} \boldsymbol{\nabla} \Delta p \\
&= -\frac{1}{\rho_0} \left(1 - \frac{\Delta\rho}{\rho_0}\right) \boldsymbol{\nabla} \Delta p \\
&= -\frac{1}{\rho_0} \boldsymbol{\nabla} \Delta p \\
&= -\frac{1}{\rho_0} \left(\frac{dp}{d\rho}\right)_0 \boldsymbol{\nabla} \Delta \rho.
\end{aligned}
$$

□

Combining equations (2.2.6) and (2.2.7), one obtain

$$
\boxed{\frac{\partial^2 \Delta\rho}{\partial t^2} = \left(\frac{dp}{d\rho}\right)_0 \nabla^2 \Delta\rho.} \tag{2.2.8}
$$

Equation (2.2.8) is a wave equation, and in one dimension has the solution

$$
\Delta\rho = \Delta\rho_0 \exp\{i(kx - \omega t)\}, \tag{2.2.9}
$$

where ω is the angular frequency and k is the wavenumber. The general form of (2.2.9) in higher dimensions can be written as

$$
\Delta\rho = \Delta\rho_0 \exp\{i(\mathbf{k} \cdot \mathbf{x} - \omega t)\}. \tag{2.2.10}
$$

Substituting this solution into the (2.2.8) we see that

$$
\frac{\omega^2}{k^2} = \left(\frac{dp}{d\rho}\right)_0. \tag{2.2.11}
$$

One can recognize the l.h.s. of eq. (2.2.11) the squared value of the speed of the propagation of points of constant phase. Hence the wave travels at speed

$$
c_s = \sqrt{\left(\frac{dp}{d\rho}\right)_0}.
$$

We can use this mathematical formulation for the other perturbed quantities. Hence the perturbed quantity ΔX can be put in form

$$
\Delta X = \Delta X_0 \exp\{i(\mathbf{kx} - \omega t)\}.
$$

2.2.2 Sound waves in a stratified atmosphere

Now we examine what happens if there are external forces by considering sound waves propagating in an isothermal atmosphere with a potential defined by

$$g = -\nabla\Phi = -\frac{d\,\Phi}{d\,z}, \tag{2.2.12}$$

acting in the z direction.

Considering the z-dependent terms, the fluid equations assume the form

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0, \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \end{aligned} \tag{2.2.13}$$

and the equilibrium conditions are

$$\begin{aligned} u_0 &= 0, \\ \rho_0(z) &= \tilde{\rho}\exp\Big\{-\frac{z}{H}\Big\}, \end{aligned} \tag{2.2.14}$$

where the scale height $H = \frac{\mathcal{R}_* T}{g\mu_*}$ has been introduced. Furthermore we have to define the pressure as

$$p_0(z) = \tilde{p}\exp\Big\{-\frac{z}{H}\Big\}. \tag{2.2.15}$$

We now apply the perturbative theory using the transformations

$$u \to \Delta u, \qquad \rho_0 \to \rho_0 + \Delta\rho \qquad \text{and} \qquad p_0 \to p_0 + \Delta p.$$

In order to evaluate the perturbed equations we need to find two useful relationships between $\Delta\mathbf{u}$, $\Delta\rho$ and the derivatives of the Lagrangian displacement $\boldsymbol{\xi}$.

The first,

$$\Delta\mathbf{u} = \frac{d\boldsymbol{\xi}}{dt} = \frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} \tag{2.2.16}$$

is the time derivative of the perturbed displacement of a particular fluid element that is just the perturbed velocity of the element. Since in the present case the unperturbed velocity is zero, this reduces to

$$\Delta\mathbf{u} = \frac{\partial\boldsymbol{\xi}}{\partial t}. \tag{2.2.17}$$

To gain a link between $\Delta\rho$ and the derivative of $\boldsymbol{\xi}$, we have to consider the Lagrangian continuity equation

$$\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{2.2.18}$$

Over a time Δt, we have

$$\Delta\rho + \rho_0\Big(\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}\Big)\Delta t = 0. \tag{2.2.19}$$

Therefore we obtain

$$\Delta\rho + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0. \tag{2.2.20}$$

We can substitute the expressions of the Eulerian perturbations

$$\delta\mathbf{u} = \Delta\mathbf{u}, \qquad \delta\rho = \Delta\rho - \xi_z \frac{\partial\rho_0}{\partial z}, \qquad \delta p = \Delta p - \xi_z \frac{\partial p_0}{\partial z} \tag{2.2.21}$$

into the conservation equations and we get the following results

$$\frac{\partial\Delta\rho}{\partial t} + \rho_0 \frac{\partial\Delta u_z}{\partial z} = 0 \tag{2.2.22}$$

that is the perturbated continuity equation.

If we assume that the perturbations obey an equation of state in which $p = p(\rho)$, and use the formula $c_s = \sqrt{\left(dp/d\rho\right)_0}$ we get the momentum equation

$$\frac{\partial\Delta u_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial\Delta p}{\partial z} = -\frac{c_s^2}{\rho_0}\frac{\partial\Delta\rho}{\partial z}. \tag{2.2.23}$$

If we now differentiate the continuity equation (2.2.22) with respect time, we get

$$\frac{\partial^2\Delta\rho}{\partial t^2} - c_s^2\frac{\partial^2\Delta\rho}{\partial z^2} - \frac{c_s^2}{H}\frac{\partial\Delta\rho}{\partial z} = 0. \tag{2.2.24}$$

Supposing that a solution can be written as

$$\Delta\rho \propto \exp\{i(kz - \omega t)\}, \tag{2.2.25}$$

we find

$$\omega^2 = c_s^2\left(k^2 - \frac{ik}{H}\right). \tag{2.2.26}$$

This relation is called *dispersion relation.*

In the case of a disturbance of real k, the dispersion relation indicates whether ω is real or imaginary and hence determines the stability of the system. In details we have

- if ω is a real quantity, we have an oscillating solution,
- if ω has a negative imaginary part, we have an exponentially decaying (damped) solution,
- if ω has a positive imaginary part, we have an exponentially growing (unstable) solution.

For real ω the dispersion relation indicates the spatial properties of the wave:

- if k is a real quantity, we have an oscillating solution,
- if k has a negative imaginary part, we have an exponentially growing solution,
- if k has a positive imaginary part, we have an exponentially decaying solution.

2.3 Instability criteria

In this section we are going to present some applications of stability theory to the fluids in a gravitational field in order to find out how they react to perturbations.

In Appendix A we show a different method to obtain the instability criteria here presented.

2.3.1 The Schwarzschild criterion

The first case we are going to analyze is the convective instability.

Consider an ideal gas in hydrostatic equilibrium in a uniform gravitational field. We are free to choose the z-axis so that gravity acts in the z direction; this choice has been made in order to have a decreasing pressure $p(z)$ and a decreasing density $\rho(z)$ as z increase. Let us take a fluid element at the same density and pressure as its surrounding, and displace it upward by a small amount δz, and we suppose that the surrounding density and pressure are respectively ρ' and p'. Pressure imbalances are removed very quickly by acoustic waves, but heat exchange takes considerably longer, so initially we can consider that the region of gas will change adiabatically to be in pressure equilibrium at the new position.

As a result, we can define a new density ρ^* at the new position. If $\rho^* < \rho'$, thanks to Archimedes' principle, we can affirm that the displaced region will be buoyant and given that it will continue to move away from the initial position, we can assert that the system is unstable. Instead, if $\rho^* > \rho'$ the region will try to come back to its original position, so the system will be stable.

As we said, the region is displaced adiabatically, so

$$\rho^* = \rho\left(\frac{p'}{p}\right)^{\frac{1}{\gamma}}, \tag{2.3.1}$$

and, recalling that the pressure gradient is $\frac{dp}{dz}$, to first order we can expand

$$p' = p + \frac{dp}{dz}\delta z. \tag{2.3.2}$$

Hence

$$\rho^* = \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\delta z. \tag{2.3.3}$$

For the medium outside the displaced element we can expand density in a similar way as

$$\rho' = \rho + \frac{d\rho}{dz}\delta z. \tag{2.3.4}$$

So the condition of instability becomes

$$\frac{\rho}{\gamma p}\frac{dp}{dz} < \frac{d\rho}{dz}. \tag{2.3.5}$$

Conversely, the system is stable if

$$\frac{\rho}{\gamma p}\frac{dp}{dz} > \frac{d\rho}{dz}.$$

This stability criterion is named "Schwarzschild criterion".

We would like to convert this criterion, in which we are dealing with pressure and density, to one linking the entropy and pressure gradients.

Now we show that the instability condition can be written as

$$\boxed{\frac{dp}{dz}\frac{dS}{dz} < 0.} \tag{2.3.6}$$

Proof. From (2.3.5) we get

$$\frac{\rho}{p\gamma}\frac{dp}{dz} < \frac{d\rho}{dz},$$

$$\frac{1}{p}\left(\frac{dp}{dz}\right)^2 < \gamma\frac{dp}{dz}\frac{1}{\rho}\frac{d\rho}{dz},$$

$$\frac{dp}{dz}\frac{1}{p}\frac{dp}{dz} - \gamma\frac{dp}{dz}\frac{1}{\rho}\frac{d\rho}{dz} < 0, \tag{2.3.7}$$

$$\frac{dp}{dz}\frac{d\ln p}{dz} - \gamma\frac{dp}{dz}\frac{d\ln\rho}{dz} < 0,$$

$$\frac{dp}{dz}\left(\frac{d\ln p}{dz} - \frac{d\ln\rho^\gamma}{dz}\right) < 0.$$

If we define

$$S \propto \ln\frac{p}{\rho^\gamma}, \tag{2.3.8}$$

we get eq. (2.3.6). □

2.4 Incompressible shear flow in axisymmetric conditions

In this section we take into account an incompressible fluid, and we analyze its rotational behavior within an hypothetical rigid cylinder.

Hence we adopt cylindrical coordinates and write down the unperturbed flow velocity as

$$\mathbf{u}_0 = (0, V(R), 0), \tag{2.4.1}$$

where the relation between the azimuthal velocity $V(R)$ and the angular velocity $\Omega(R)$ holds:

$$V(R) = R\Omega(R). \tag{2.4.2}$$

For such a flow, with fluid velocity $\mathbf{u} = (u_R, u_\varphi, u_z)$ the equations of motion are given by (1.3.17). In addition we require the mass conservative equation, which for an incompressible fluid is just given by (1.3.16).

We now perturb this solution, so that the velocity field becomes

$$\mathbf{u} = (u'_R, V(R) + u'_\varphi, u'_z). \tag{2.4.3}$$

Since ρ is constant, the equilibrium pressure distribution is given by

$$p(R) = \rho \int \frac{V^2}{R}\, dR, \tag{2.4.4}$$

and we define the useful quantity

$$W \equiv \frac{p'}{\rho}. \tag{2.4.5}$$

Therefore to first order, the perturbed Euler equations are given by

$$\begin{aligned}
&\frac{\partial u_R}{\partial t} + \frac{V}{R}\frac{\partial u_R}{\partial \varphi} - \frac{2Vu_\varphi}{R} = -\frac{\partial W}{\partial R},\\
&\frac{\partial u_\varphi}{\partial t} + \frac{V}{R}\frac{\partial u_\varphi}{\partial \varphi} + \left(\frac{V}{R} + \frac{dV}{dR}\right)u_R = -\frac{1}{R}\frac{\partial W}{\partial \varphi},\\
&\frac{\partial u_z}{\partial t} + \frac{V}{R}\frac{\partial u_z}{\partial \varphi} = -\frac{\partial W}{\partial z}.
\end{aligned} \tag{2.4.6}$$

Proof. We have for the R-component:

$$\begin{aligned}
&\frac{\partial u_R}{\partial t} + \mathbf{u}\cdot(\nabla u_R) - \frac{u_\varphi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R},\\
&\frac{\partial u'_R}{\partial t} + \mathbf{u}\cdot(\nabla u'_R) - \frac{(V(R)+u'_\varphi)^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R},\\
&\frac{\partial u'_R}{\partial t} + \frac{V(R)}{R}\frac{\partial u'_R}{\partial \varphi} - \frac{V^2(R) + u'^2_\varphi + 2V(R)u'_\varphi}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R},\\
&\frac{\partial u'_R}{\partial t} + \frac{V(R)}{R}\frac{\partial u'_R}{\partial \varphi} - \frac{2V(R)u'_\varphi}{R} = -\frac{\partial W}{\partial R}.
\end{aligned} \tag{2.4.7}$$

For the φ-component:

$$\begin{aligned}
&\frac{\partial u_\varphi}{\partial t} + \mathbf{u}\cdot(\nabla u_\varphi) + \frac{u_\varphi u_R}{R} = -\frac{1}{R\rho}\frac{\partial p}{\partial \varphi},\\
&\frac{\partial u'_\varphi}{\partial t} + \mathbf{u}\cdot\big(\nabla(V(R)+u'_\varphi)\big) + \frac{(V(R)+u'_\varphi)u'_R}{R} = -\frac{1}{R\rho}\frac{\partial p}{\partial \varphi},\\
&\frac{\partial u'_\varphi}{\partial t} + u'_R\frac{dV(R)}{dR} + \frac{V(R)}{R}\frac{\partial u'_\varphi}{\partial \varphi} + \frac{V(R)}{R}u'_R = -\frac{1}{R}\frac{\partial W}{\partial \varphi},\\
&\frac{\partial u'_\varphi}{\partial t} + \frac{V(R)}{R}\frac{\partial u'_\varphi}{\partial \varphi} + \left(\frac{V(R)}{R} + \frac{dV(R)}{dR}\right)u'_R = -\frac{1}{R}\frac{\partial W}{\partial \varphi}.
\end{aligned} \tag{2.4.8}$$

Finally for the z-component

$$\frac{\partial u_z}{\partial t} + \mathbf{u} \cdot (\nabla u_z) = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$
$$\frac{\partial u_z'}{\partial t} + \frac{V(R)}{R}\frac{\partial u_z'}{\partial \varphi} = -\frac{\partial W}{\partial z}. \tag{2.4.9}$$

We have dropped the primes in order to simplify the notation in the eq. (2.4.6). □

We write explicitly the mass conservation equation as

$$\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} = 0 \tag{2.4.10}$$

to point out that is already linearized.

Since the equilibrium configuration is independent of time, and of φ and z, we may now Fourier analyze in t, φ and z. Thus we assume that all the linear variables are of the form[2]

$$q'(R, \varphi, z, t) \to q'(R) \exp\{i(\omega t + m\varphi + kz)\}. \tag{2.4.11}$$

We define the local Doppler-shifted frequency as

$$\sigma(R) = \omega + m\Omega(R). \tag{2.4.12}$$

The linearized equations can now be written in the form

$$i\sigma u_R - 2\Omega u_\varphi = -\frac{dW}{dR}, \tag{2.4.13}$$

$$i\sigma u_\varphi + \left[\Omega + \frac{d}{dR}(R\Omega)\right] u_R = -\frac{imW}{R}, \tag{2.4.14}$$

$$i\sigma u_z = -ikW, \tag{2.4.15}$$

$$\frac{du_R}{dR} + \frac{u_R}{R} + \frac{imu_\varphi}{R} + iku_z = 0. \tag{2.4.16}$$

Proof. In fact we have to replace into the equations (2.4.6), for example

$$\frac{\partial u_R}{\partial t} = i\omega u_R = i\big(\sigma(R) - m\Omega(R)\big)u_R,$$
$$\frac{\partial u_R}{\partial \varphi} = imu_R, \tag{2.4.17}$$

for the radial component, and

$$\frac{\partial u_\varphi}{\partial t} = i\omega u_\varphi = i\big(\sigma(R) - m\Omega(R)\big)u_\varphi,$$
$$\frac{\partial u_\varphi}{\partial \varphi} = imu_\varphi, \tag{2.4.18}$$
$$-\frac{1}{R}\frac{\partial W}{\partial \varphi} = -\frac{imW}{R},$$

[2] We note that to keep q' a single-valued function of azimuth φ we require that m is an integer.

for the φ-component. Using this approach one can obtain the two others equations. □

We now look for an expression to describe the Eulerian velocity field perturbation in terms of the Lagrangian displacement $\boldsymbol{\xi}$.

The velocity field of the perturbed fluid is given by

$$\mathbf{u}(\mathbf{r}, t) = \frac{D\mathbf{r}}{Dt}. \tag{2.4.19}$$

In a similar way, we can write for a fluid particle in motion in the unperturbed fluid with a time function trajectory defined as $\mathbf{r}_0(t)$ the function

$$\mathbf{u_0}(\mathbf{r_0}, t) = \frac{D\mathbf{r_0}}{Dt}. \tag{2.4.20}$$

Hence, in considering the difference, we have by definition

$$\delta\mathbf{u} = \mathbf{u}(\mathbf{r}, t) - \mathbf{u_0}(\mathbf{r_0}, t). \tag{2.4.21}$$

We note that to the first order in $\boldsymbol{\xi}$ eq. (1.1.4) holds and we can write it as

$$\delta\mathbf{u} = \frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\xi}. \tag{2.4.22}$$

Now, for definition we have $\mathbf{u}' = \mathbf{u}(\mathbf{r}, t) - \mathbf{u_0}(\mathbf{r_0}, t)$ and given that to the first order $\delta\mathbf{u} = \mathbf{u}' + \boldsymbol{\xi}\cdot\nabla\mathbf{u}$, we obtain

$$\mathbf{u}' = \frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{u}_0\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{u}_0. \tag{2.4.23}$$

In the case we are considering, with the unperturbed velocity field in the azimuthal direction and dependent only on radius R, we have

$$u_R = i\sigma\xi_R, \tag{2.4.24}$$

$$u_z = i\sigma\xi_z \tag{2.4.25}$$

and

$$u_\varphi = i\sigma\xi_\varphi - R\frac{d\Omega}{dR}\xi_R. \tag{2.4.26}$$

Using these relations to write the linearized equations in terms of ξ we obtain

$$(\sigma^2 - 2R\Omega\Omega')\xi_R + 2i\sigma\Omega\xi_\varphi = \frac{dW}{dR}, \tag{2.4.27}$$

$$\sigma^2\xi_\varphi - 2i\sigma\Omega\xi_R = \frac{imW}{R}, \tag{2.4.28}$$

$$\sigma^2\xi_z = ikW, \tag{2.4.29}$$

$$\frac{d\xi_R}{dR} + \frac{\xi_R}{R} + \frac{im}{R}\xi_\varphi + ik\xi_z = 0, \tag{2.4.30}$$

where $\Omega' = d\Omega/dR$.

Combining these equations to eliminate ξ_φ and ξ_z we get

$$\sigma^2\left(\frac{d\xi_R}{dR} + \frac{\xi_R}{R}\right) - \frac{2m\Omega\sigma}{R}\xi_R = \left(\frac{m^2}{R^2} + k^2\right)W, \tag{2.4.31}$$

and

$$[\sigma^2 - \mathcal{R}(R)]\xi_R = \frac{dW}{dR} + \frac{2m\Omega}{\sigma R}W, \tag{2.4.32}$$

where $\mathcal{R}$ is the Rayleigh discriminant defined as

$$\mathcal{R}(R) = \frac{2\Omega}{R}\frac{dJ(R)}{dR}, \tag{2.4.33}$$

where we have introduced the specific angular momentum $J(R) = R^2\Omega$.

2.4.1 The Rayleigh's criterion

Here we consider axisymmetric modes in which $m = 0$ and therefore $\sigma = \omega$. Then we have from eq. (2.4.31)

$$\frac{1}{R}\frac{d}{dR}(R\xi_R) = \frac{k^2}{\omega^2}W, \tag{2.4.34}$$

and using the equation (2.4.32) we obtain

$$[\omega^2 - \mathcal{R}]\xi_R = \frac{dW}{dR}. \tag{2.4.35}$$

If we eliminate W between these two equations we get

$$\frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}(R\xi_R)\right) - k^2\xi_R = -\frac{k^2\mathcal{R}(R)}{\omega^2}\xi_R. \tag{2.4.36}$$

This equation is an example of a Sturm-Liuville problem with eigenvalue $\lambda = k^2/\omega^2$, if we assume the boundary conditions $\xi_R = 0$ at $R = R_1, R_2$.

Solving eq. (2.4.36) we have

$$\frac{\omega^2}{k^2} = \frac{I_1}{I_2}, \tag{2.4.37}$$

where we have defined

$$I_1 = \int_{R_1}^{R_2} \mathcal{R}(R)R\xi_R^2\, dR,$$

and

$$I_2 = \int_{R_1}^{R_2} \left[\frac{1}{R}\left(\frac{d(R\xi_R)}{dR}\right)^2 + k^2R\xi_R^2\right] dR.$$

Now we discuss about the stability of the models in which this dispersion relation holds true. Hence we are going to give the Rayleigh's criterion for the stability of a circular shear flow.

For any k, the second integral is positive definite, and therefore the sign of ω^2 depends on the sign of I_1. Since the sign of I_1 depends on $\mathcal{R}(R)$ we need to focus on the Rayleigh discriminant.

If $\mathcal{R} > 0$, then $\omega^2 > 0$ and the flow is stable to axisymmetric modes. Conversely, if $\mathcal{R} < 0$ at some point in the flow, we may choose a trial function $\xi(R)$ which makes $I_1 < 0$, which implies the existence of an unstable mode. To conclude, this criterion states that the flow is stable to axisymmetric disturbances if and only if the specific angular momentum $J(R)$ increase outwards.

2.5 Compressible rotating flow in axisymmetric conditions

Since realistic fluids are generally compressible, we must consider the case in which density variations are allowed.

In rotating astrophysical gases we must take into account two contributes: on one hand there is the contribute given by an effective radial gravitational force that requires we must consider the Schwarzschild stability criterion, and on the other hand we have seen that axisymmetric incompressible fluids stability is ruled by the Rayleigh criterion. Hence we can suppose that a generic rotating compressible fluid stability criterion will be a sort of combination of these two criteria. We will find out that this generalization is the called Solberg-Hoiland criterion.

We start by considering an axisymmetric fluid flow in a fixed axisymmetric gravitational potential $\Phi(R)$. The total energy of the unperturbed flow is given by the sum of the kinetic energy, thermal energy and gravitational potential energy and can be written as

$$\mathcal{E} = \frac{1}{2}\int_V \frac{J^2}{R^2}\, dm + \int_V e\, dm + \int_V \Phi\, dm. \tag{2.5.1}$$

The angular velocity and the corresponding angular momentum are $\Omega(R)$ and $J(R) = R^2\Omega$, respectively. In eq. (2.5.1) $dm = \rho\, dV$ is the mass of a fluid element and e is the internal energy.

We want to compute the change $\delta\mathcal{E}$ when the flow is subject to an axially symmetric perturbation $\boldsymbol{\xi}(R)$. The angular momentum J of each fluid element is conserved because the perturbation is taken axisymmetric, therefore the variation in the kinetic term occurs because the radius of each fluid element is changed; this is

$$\delta\frac{1}{2}\int_V \frac{J^2}{R^2}\, dm = -\int_V \frac{J^2}{R^3}\xi_R\, dm. \tag{2.5.2}$$

Moreover if we assume that the change is isentropic ($\frac{DS}{Dt} = 0$) we have

$$\frac{De}{Dt} + \frac{p}{\rho}\boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{2.5.3}$$

Proof. From the first thermodynamic law:

$$TdS = de + pdV, \tag{2.5.4}$$

we get

$$TdS = de - \frac{p}{\rho^2} d\rho. \tag{2.5.5}$$

Hence the rate of change of the heat content of a particular fluid element of unit mass is given by

$$\begin{aligned} T\frac{DS}{Dt} &= \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\big(- \rho \nabla \cdot \mathbf{u}\big) = \\ &= \frac{De}{Dt} + \frac{p}{\rho}\nabla \cdot \mathbf{u}. \end{aligned} \tag{2.5.6}$$

Therefore in adiabatic condition relation (2.5.3) holds. □

Hence we can write the change of internal energy as

$$\delta e = -\frac{p}{\rho} \boldsymbol{\nabla} \cdot \boldsymbol{\xi}. \tag{2.5.7}$$

Integrating this equation, we find that

$$\delta \int_V e \, dm = \int_V \frac{1}{\rho} \boldsymbol{\xi} \cdot \nabla p \, dm. \tag{2.5.8}$$

In addition, since the potential Φ is fixed, we obtain

$$\delta \Phi = \boldsymbol{\xi} \cdot \nabla \Phi, \tag{2.5.9}$$

this can be integrated as

$$\delta \int_V \Phi \, dm = \int_V \boldsymbol{\xi} \cdot \nabla \Phi \, dm. \tag{2.5.10}$$

In conclusion our change in the energy is given by

$$\delta \mathcal{E} = \int_V \boldsymbol{\xi} \cdot \left(\nabla \Phi + \frac{1}{\rho} \nabla p - \frac{J^2}{R^3} \hat{\mathbf{R}} \right) dm. \tag{2.5.11}$$

For a fluid in equilibrium we require that $\delta\mathcal{E} = 0$ for all vector fields $\boldsymbol{\xi}$. This means that the net force per unit mass acting on any fluid element must vanish everywhere

$$\mathbf{F} \equiv -\nabla \Phi - \frac{1}{\rho} \nabla p + \frac{J^2}{R^3} \hat{\mathbf{R}} = 0. \tag{2.5.12}$$

Hence equation (2.5.11) in equilibrium condition becomes

$$\delta \mathcal{E} = - \int_V \boldsymbol{\xi} \cdot \mathbf{F} \, dm = 0. \tag{2.5.13}$$

In order to study the stability of the configuration we need to consider the second-order perturbation to the energy. Using the equilibrium condition (2.5.12) this is given by

$$\delta(\delta \mathcal{E}) = - \int_V \boldsymbol{\xi} \cdot \delta \mathbf{F} \, dm. \tag{2.5.14}$$

The configuration is stable if $\delta^2\mathcal{E} > 0$.

Therefore we must consider the integral expression

$$\delta^2\mathcal{E} = \int_V \boldsymbol{\xi} \cdot \left[\delta\left(\nabla\Phi + \frac{1}{\rho}\nabla p\right) + \frac{3J^2}{R^4}\xi_R \hat{\mathbf{R}}\right] dm. \tag{2.5.15}$$

Identifying with a prime the Eulerian perturbation, we can write the first term in the square brackets of (2.5.15) in the form

$$\delta\left[\nabla\Phi + \frac{1}{\rho}\nabla p\right] = \left[\nabla\Phi + \frac{1}{\rho}\nabla p\right]' + \boldsymbol{\xi} \cdot \left[\nabla\Phi + \frac{1}{\rho}\nabla p\right]. \tag{2.5.16}$$

For the first term in the r.h.s. of the (2.5.16), we have that

$$\left[\nabla\Phi + \frac{1}{\rho}\nabla p\right]' = \frac{1}{\rho}\nabla p' - \frac{\rho'}{\rho^2}\nabla p. \tag{2.5.17}$$

While in virtue of eq. (2.5.12), the second term of (2.5.15) can be written as

$$\boldsymbol{\xi} \cdot \nabla\left[\nabla\Phi + \frac{1}{\rho}\nabla p\right] = \boldsymbol{\xi} \cdot \nabla\left[\frac{J^2}{R^3}\hat{\mathbf{R}}\right]. \tag{2.5.18}$$

Hence by noting that

$$\frac{3J^2}{R^4}\xi_R\hat{\mathbf{R}} + \boldsymbol{\xi} \cdot \nabla\left[\frac{J^2}{R^3}\hat{\mathbf{R}}\right] = (\boldsymbol{\xi} \cdot \nabla J^2)\frac{\hat{\mathbf{R}}}{R^3}, \tag{2.5.19}$$

we discover that for axisymmetric perturbations the following equation holds

$$\delta^2\mathcal{E} = -\int_V \boldsymbol{\xi} \cdot (\mathcal{L}\boldsymbol{\xi})\, dm, \tag{2.5.20}$$

in which we have introduced the linear operator $\mathcal{L}$ defined as

$$\mathcal{L}\boldsymbol{\xi} = -\frac{\nabla p'}{\rho} + \frac{\rho'}{\rho^2}\nabla p - (\boldsymbol{\xi} \cdot \nabla J^2)\frac{\hat{\mathbf{R}}}{R^3}, \tag{2.5.21}$$

where (from the mass conservation)

$$\rho' = -\rho\boldsymbol{\nabla} \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla\rho, \tag{2.5.22}$$

and (from the energy conservation)

$$p' = -\gamma p\, \boldsymbol{\nabla} \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla\, p. \tag{2.5.23}$$

Integrating by parts equation (2.5.20) and using eq. (2.5.21), eq. (2.5.22) and eq. (2.5.23) we have

$$\delta^2\mathcal{E} = \int_V \xi_i \mathcal{M}_{ij} \xi_j\, dm + \int_V \left[\frac{p'^2}{\gamma p \rho}\right] dm \tag{2.5.24}$$

in which we defined the second order tensor $\mathcal{M}_{ij}$ as

$$\mathcal{M}_{ij} = \left[\frac{1}{\rho}\nabla p\right]_i \left[\frac{1}{\rho}\nabla\rho - \frac{1}{\gamma p}\nabla p\right]_j + \frac{1}{R^3}(\nabla R)_i(\nabla J^2)_j. \tag{2.5.25}$$

2.5.1 The Solberg-Hoiland criterion

In order to find the stable configurations we need to study the positivity of eq. (2.5.24). The second term of the r.h.s is positive definite, hence we will deal with those perturbations for which this term vanishes. We restrict our analysis to the first integral in the r.h.s of eq. (2.5.24).

We first note that $\mathcal{M}$ is a symmetric tensor.

Proof. Defining the following vectors:

$$\begin{aligned} \mathbf{A} &= \frac{1}{\gamma p}\nabla p - \frac{1}{\rho}\nabla \rho, \\ \mathbf{A}' &= -\frac{1}{\rho}\nabla p, \\ \mathbf{B} &= \frac{1}{R^3}\nabla J^2, \\ \mathbf{B}' &= \nabla R = \hat{\mathbf{R}}, \end{aligned} \tag{2.5.26}$$

we may break up the tensor as

$$\mathcal{M}_{ij} = A_i A'_j + B_i B'_j. \tag{2.5.27}$$

Now we take the curl of the eq. (2.5.12) to obtain

$$\nabla\left(\frac{1}{\rho}\right) \wedge \nabla p = \frac{1}{R^3}\nabla J^2 \wedge \hat{\mathbf{R}}, \tag{2.5.28}$$

which implies that

$$\mathbf{A} \wedge \mathbf{A}' + \mathbf{B} \wedge \mathbf{B}' = 0. \tag{2.5.29}$$

And in suffix notation this means that

$$\epsilon_{ijk}\mathcal{M}_{jk} = 0. \tag{2.5.30}$$

Therefore $\mathcal{M}$ is symmetric. □

The condition for a real symmetric second-rank tensor to give a positive definite expression when contracted twice with a vector is simply that $tr(\mathcal{M}) > 0$ and $\det(\mathcal{M}) > 0$. This give us the Solberg-Hoiland criterion.

Theorem 1 (Solberg-Hoiland criterion)**.** *A fluid configuration is stable to axisymmetric, adiabatic perturbations if and only if*

$$\frac{1}{R^3}\frac{\partial J^2}{\partial R} + \frac{1}{c_p}(-\mathbf{g} \cdot \nabla S) > 0 \tag{2.5.31}$$

and

$$-g_z\left(\frac{\partial J^2}{\partial R}\frac{\partial S}{\partial z} - \frac{\partial J^2}{\partial z}\frac{\partial S}{\partial R}\right) > 0 \tag{2.5.32}$$

Where S is the specific entropy, we have defined gravity $\mathbf{g}$ *as*

$$\mathbf{g} = \frac{1}{\rho}\nabla p = -\mathbf{A}' \tag{2.5.33}$$

and we have noted that

$$\mathbf{A} = \frac{\nabla S}{c_P}. \tag{2.5.34}$$

Thus we obtain a blend of the Rayleigh and Schwarzschild stability criteria as we can see by successively assuming that either S or J^2 are spatially constant.

Chapter 3

Stationary models of rotating gas with baroclinic distribution

In this chapter we present stationary models of gas rotating with baroclinic distributions in axisymmetric conditions.

For these models we derive the sufficient condition to have a physical acceptable solutions by knowing the gas density distribution and the gravitational potential.

Families of baroclinic models in the recent astrophysical literature have been studied, for instance, by [Barnabé et al. 2006] and [Sormani et al 2018]. In this chapter we follow the treatment of Barnabé.

3.1 The Poicaré-Wavre theorem

We consider a gaseous axisymmetric distribution $\rho(R, z)$ in permanent rotation, under the influence of an axisymmetric gravitational potential $\Phi_{tot}(R, z)$. The axial symmetry imposes that all the physical variables depend only on the cylindrical coordinates R and z.

We exclude the possibility of meridional motion, so we assume $u_R = u_z = 0$. This assumption implies that the continuity equation is always satisfied, whereas the stationary Euler equation can be written as

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{\partial \Phi}{\partial z} \tag{3.1.1}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial R} = -\frac{\partial \Phi}{\partial R} + \Omega^2 R \tag{3.1.2}$$

where ρ, p and Ω denote the gas density, pressure and angular velocity, respectively.

The gas rotational velocity is $u_\varphi = \Omega R$, where in general $\Omega = \Omega(R, z)$. The rotational velocity is said to be constant over cylinders when $\Omega = \Omega(R)$. In general Φ represents the total gravitational potential, including also the gas contribution; in this chapter we will assume that the gas is not self-gravitating, so that Φ is an external gravitational potential. For instance, in the case of a disk galaxy, Φ is the sum of the dark halo and the stellar potentials.

We recall that a fluid is said barotropic if its density is a function of pressure only:

$$p = p(\rho), \tag{3.1.3}$$

whereas it is said baroclinic if pressure is not expressible just as a function of ρ:

$$p \neq p(\rho). \tag{3.1.4}$$

In order to better understand the behavior of these fluid systems, we recall the *Poincaré-Wavre Theorem*[1].

Theorem 2 (Poincaré-Wavre theorem). *For an ideal axisymmetric fluid in a stationary azimuthal rotation the following statements are equivalent:*

1. *the fluid is in cylindrical rotation;*

2. *the pressure is stratified over the density;*

3. *an effective potential $\Phi_{eff}(R, z)$ exists.*

When these conditions occur

$$\Phi_{eff}(R, z) = \Phi(R, z) - \int_{R_0}^{R} \Omega^2(R')R'\, dR', \tag{3.1.5}$$

where R_0 is an arbitrary cylindrical radius. In addition the isobaric, isopycnic and effective isopotential surfaces coincide.

[1]For a detailed proof of this theorem see [Tassoul 1978].

Now we verify that when $\Omega = \Omega(R)$ eq. (3.1.1) and (3.1.2) can be written as

$$\nabla p = -\rho \nabla \Phi_{eff}. \tag{3.1.6}$$

Proof. Using the equations of motion (3.1.1) and (3.1.2)

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial z} &= -\frac{\partial}{\partial z}\left[\Phi_{eff}(R,z) + \int_{R_0}^{R} \Omega^2(R')R'\, dR'\right] = \\
&= -\frac{\partial \Phi_{eff}(R,z)}{\partial z} - \frac{\partial}{\partial z}\int_{R_0}^{R} \Omega^2(R')R'\, dR' = \\
&= -\frac{\partial \Phi_{eff}(R,z)}{\partial z},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial R} &= -\frac{\partial}{\partial R}\left[\Phi_{eff}(R,z) + \int_{R_0}^{R} \Omega^2(R')R'\, dR'\right] + \Omega^2 R = \\
&= -\frac{\partial \Phi_{eff}(R,z)}{\partial R} - \frac{\partial}{\partial R}\int_{R_0}^{R} \Omega^2(R')R'\, dR' + \Omega^2 R = \\
&= -\frac{\partial \Phi_{eff}(R,z)}{\partial R},
\end{aligned}$$

and we can write the equation of motion for an effective potential in the form

$$\frac{1}{\rho}\nabla p = -\nabla \Phi_{eff}. \tag{3.1.7}$$

□

The Poincaré-Wavre theorem states that the baroclinic stratification is a necessary and sufficient condition to obtain a rotational velocity vertical gradient. In fact, by definition, for any displacement on a level surface $\Phi_{eff} = constant$, one has $\nabla \Phi_{eff} = 0$. Since eq. (3.1.7) shows that $\nabla p = 0$ on the same surface, it follows at once that the isobaric surfaces coincide with the level surfaces. So we can write

$$p = p(\Phi_{eff}), \qquad \Phi_{eff} = \Phi_{eff}(p). \tag{3.1.8}$$

By virtue of eq. (3.1.7), one readily sees that

$$\frac{1}{\rho} = -\frac{d\Phi_{eff}(p)}{dp}, \qquad \rho = \rho(p). \tag{3.1.9}$$

When the distribution is barotropic, one fixes the gravitational potential Φ and a specific function $p(\rho)$, and integrates the equation (3.1.1) for example by assigning $\rho(R,0)$ or imposing $\rho(R,\infty) = 0$; the angular velocity Ω is then obtained from the equation (3.1.2).

3.2 The construction of a rotating barotropic equilibrium

The construction of a rotating barotropic equilibrium is quite straightforward: by assuming polytropic distribution $p = k\rho^{\gamma'}$, the solution of eq. (3.1.1) and (3.1.2) is:

$$\rho = \rho_0 \left[1 + \frac{\gamma' - 1}{\gamma'} \beta_0 (\Phi_{eff,0} - \Phi_{eff})\right]^{\frac{1}{\gamma'-1}} \tag{3.2.1}$$

where $\rho_0 \equiv \rho(x_0)$, $\Phi_{eff,0} \equiv \Phi_{eff}(x_0)$, $\beta_0 = \frac{\mu' m_p}{k_B T_0}$ and x_0 is a defined fixed point in the space and T_0 is a fixed temperature.

Proof. Starting from the first equation,

$$\frac{1}{\rho}\partial_z p = -\partial_z \Phi_{eff}$$

and using the identity $p = k\rho^{\gamma'}$ we have

$$\frac{1}{\rho}\partial_z (k\rho^{\gamma'}) = -\partial_z \Phi_{eff}.$$

The left-hand side of this equation can be written as

$$\frac{1}{\rho}\partial_z (k\rho^{\gamma'}) = \frac{1}{\rho} k\gamma' \rho^{\gamma'-1} \partial_z \rho = k\gamma' \rho^{\gamma'-2} \partial_z \rho,$$

so

$$\rho^{\gamma'-2}\partial_z \rho = -\frac{1}{k\gamma'}\partial_z \Phi_{eff}.$$

From the equality

$$\partial_z \rho^{\gamma'-1} = (\gamma' - 1)\rho^{\gamma'-2}\partial_z \rho,$$

we have

$$\frac{\partial_z \rho^{\gamma'-1}}{\gamma' - 1} = -\frac{1}{k\gamma'}\partial_z \Phi_{eff},$$

or

$$\partial_z \rho^{\gamma'-1} = -\frac{\gamma' - 1}{k\gamma'}\partial_z \Phi_{eff}.$$

Integrating this equation from z_0 to z, one finds

$$\int_{z_0}^{z} dz\, \partial_z \rho^{\gamma'-1} = -\frac{\gamma'-1}{k\gamma'} \int_{z_0}^{z} dz\, \partial_z \Phi_{eff},$$

$$\rho^{\gamma'-1}(z) - \rho^{\gamma'-1}(z_0) = \frac{\gamma'-1}{k\gamma'}(\Phi_{eff}(z_0) - \Phi_{eff}(z)),$$

$$\rho^{\gamma'-1}(z) = \rho^{\gamma'-1}(z_0) + \frac{\gamma'-1}{k\gamma'}\big(\Phi_{eff}(z_0) - \Phi_{eff}(z)\big),$$

$$\rho(z) = \left[\rho^{\gamma'-1}(z_0) + \frac{\gamma'-1}{k\gamma'}\big(\Phi_{eff}(z_0) - \Phi_{eff}(z)\big)\right]^{\frac{1}{\gamma'-1}}.$$

So we get

$$\rho(z) = \rho(z_0)\left[1 + \frac{\gamma'-1}{\gamma'}\frac{1}{k\rho^{\gamma'-1}(z_0)}\Big(\Phi_{eff}(z_0) - \Phi_{eff}(z)\Big)\right]^{\frac{1}{\gamma'-1}}.$$

Using equation in (3.1.2),

$$\frac{1}{\rho}\partial_R p = -\partial_R \Phi_{eff} + \Omega^2 R,$$

in analogy with the calculus showed in the first part of this proof we get

$$\int_{R_0}^{R} dR\, \partial_R \rho^{\gamma'-1} = \frac{\gamma'-1}{\gamma' k}\left[\int_{R_0}^{R} dR\,(-\partial_R \Phi_{eff}) + \int_{R_0}^{R} dR\, \Omega^2 R\right],$$

$$\rho^{\gamma'-1}(R) - \rho^{\gamma'-1}(R_0) = \frac{\gamma'-1}{\gamma' k}\Big(\Phi_{eff} - \Phi_{eff}(R)\Big) + \frac{\gamma'-1}{\gamma' k}\int_{R_0}^{R} dR\, \Omega^2 R,$$

$$\rho^{\gamma'-1}(R) - \rho^{\gamma'-1}(R_0) = \frac{\gamma'-1}{\gamma' k}\Big[\Phi_{eff}(R_0) - \Phi_{eff}(R) + \Phi_{eff}(R) - \Phi_{eff}(R)\Big],$$

$$\rho^{\gamma'-1}(R) = \rho^{\gamma'-1}(R_0) + \frac{\gamma'-1}{\gamma' k}\big(\Phi_{eff}(R_0) - \Phi_{eff}(R)\big),$$

$$\rho(R) = \rho(R_0)\left[1 + \frac{\gamma'-1}{\gamma'}\frac{1}{k\rho^{\gamma'-1}(R_0)}\big(\Phi_{eff}(R_0) - \Phi_{eff}(R)\big)\right]^{\frac{1}{\gamma'-1}}.$$

□

3.3 Baroclinic models

3.3.1 Baroclinic solution

We focus on the case of a baroclinic distribution of gas rotating with angular velocity $\Omega = \Omega(R, z)$.

The total potential $\Phi(R, z)$ and the gas density $\rho(R, z)$ are assigned functions with $\rho(R, z)$ vanishing at infinity. The equation (3.1.1) is integrated in full generality for the pressure as

$$p(R, z) - p(R, z_0) = \int_z^{z_0} \rho \frac{\partial \Phi}{\partial z'}\, dz', \tag{3.3.1}$$

where $p(R) \equiv p(R, z_0)$ is an arbitrary function and the height z_0 is fixed, for instance common choices are $z_0 = 0$ or $z_0 = \infty$.

For instance if we assume $p(R, \infty) = 0$ we obtain

$$p(R, z) = \int_z^{\infty} \rho \frac{\partial \Phi}{\partial z'}\, dz'. \tag{3.3.2}$$

In general the obtained pressure p cannot be expressed as a function of ρ only, so the system is *baroclinic*.

Accordingly, the rotational velocity field

$$u_\varphi^2(R, z) = \frac{R}{\rho}\frac{\partial p}{\partial R} + R\frac{\partial \Phi}{\partial R} \tag{3.3.3}$$

depends both on R and z.

Proof. Here we show how to get equation (3.3.3).

Multiplying by R eq. (3.1.2) we get

$$\frac{R}{\rho}\frac{\partial p}{\partial R} = -\, R\frac{\partial \Phi}{\partial R} + \Omega^2 R^2 = \tag{3.3.4}$$

$$= -R\frac{\partial \Phi}{\partial R} + u_\varphi^2. \tag{3.3.5}$$

□

The major drawbacks posed by construction of baroclinic stratifications following this approach is the fact that the existence of physically acceptable solutions ($u_\varphi^2 \geq 0$ everywhere) is not guaranteed. Due to the arbitrariness of the chosen density field, a negative radial pressure gradient in equation (3.3.3) can be dominant for some values of R and z. By combining equations (3.3.1) and (3.3.3), and integrating by parts, we obtain

$$\frac{\rho u_\varphi^2}{R} = \frac{\rho(R, z_0) u_\varphi^2(R, z_0)}{R} + \mathcal{C}[\rho, \Phi], \tag{3.3.6}$$

where

$$\mathcal{C}[\rho, \Phi] \equiv \int_z^{z_0} \left(\frac{\partial \rho}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial \rho}{\partial z'}\frac{\partial \Phi}{\partial R}\right) dz'. \tag{3.3.7}$$

Proof. For a generic z_0 we have

$$
\begin{aligned}
u_\varphi^2 =& \frac{R}{\rho}\frac{\partial p}{\partial R} + R\frac{\partial \Phi}{\partial R} = \frac{R}{\rho}\frac{\partial}{\partial R}\left[\int_z^{z_0} \rho(R,z')\frac{\partial \Phi(R,z')}{\partial z'}dz'\right] + R\frac{\partial \Phi(R,z)}{\partial R} = \\
=& \frac{R}{\rho}\left[\frac{\partial}{\partial R}\left[\rho(R,z_0)\Phi(R,z_0)\right] - \frac{\partial}{\partial R}\left[\rho(R,z)\Phi(R,z) - \right.\right. \\
& \left.\left. - \int_z^{z_0} dz' \frac{\partial \Phi(R,z')}{\partial R}\frac{\partial \rho(R,z')}{\partial z'} + \Phi(R,z')\frac{\partial^2 \rho(R,z')}{\partial R \partial z'}\right]\right] + R\frac{\partial \Phi(R,z)}{\partial R} = \\
=& \frac{R}{\rho}\left[\frac{\partial}{\partial R}\left[\rho(R,z_0)\Phi(R,z_0)\right] - \frac{\partial}{\partial R}\left[\rho(R,z)\Phi(R,z) - \right.\right. \\
& \left.\left. - \int_z^{z_0} dz' \frac{\partial \Phi(R,z')}{\partial R}\frac{\partial \rho(R,z')}{\partial z'} - \int_z^{z_0} dz' \Phi(R,z')\frac{\partial^2 \rho(R,z')}{\partial R \partial z'}\right]\right] + R\frac{\partial \Phi(R,z)}{\partial R} = \\
=& \frac{R}{\rho}\left[\frac{\partial}{\partial R}\left[\rho(R,z_0)\Phi(R,z_0)\right] - \frac{\partial}{\partial R}\left[\rho(R,z)\Phi(R,z)\right] - \int_z^{z_0} dz' \frac{\partial \Phi(R,z')}{\partial R}\frac{\partial \rho(R,z')}{\partial z'} - \right. \\
& \left. - \left[\Phi(R,z')\frac{\partial \rho(R,z')}{\partial R}\right]_z^{z_0} + \int_z^{z_0} dz' \frac{\partial \Phi(R,z')}{\partial z'}\frac{\partial \rho(R,z')}{\partial R}\right] + R\frac{\partial \Phi(R,z)}{\partial R} = \\
=& \frac{R}{\rho}\left\{\frac{\partial}{\partial R}\left[\rho(R,z')\Phi(R,z')\right]_z^{z_0} - \left[\Phi(R,z')\frac{\partial \rho(R,z')}{\partial R}\right]_z^{z_0} + \right. \\
& \left. + \int_z^{z_0} dz' \left(\frac{\partial \rho(R,z')}{\partial R}\frac{\partial \Phi(R,z')}{\partial z'} - \frac{\partial \rho(R,z')}{\partial z'}\frac{\partial \Phi(R,z')}{\partial R}\right)\right\} + R\frac{\partial \Phi(R,z)}{\partial R}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\rho u_\varphi^2}{R} =& \frac{\partial}{\partial R}\left[\rho(R,z')\Phi(R,z')\right]_z^{z_0} - \left[\Phi(R,z')\frac{\partial \rho(R,z')}{\partial R}\right]_z^{z_0} + \rho(R,z)\frac{\partial \Phi(R,z)}{\partial R} + \mathcal{C}[\rho,\Phi] = \\
=& \frac{\partial \rho(R,z_0)}{\partial R}\Phi(R,z_0) + \rho(R,z_0)\frac{\partial \Phi(R,z_0)}{\partial R} - \frac{\partial \rho(R,z)}{\partial R}\Phi(R,z) - \rho(R,z)\frac{\partial \Phi(R,z)}{\partial R} - \\
& - \frac{\partial \rho(R,z_0)}{\partial R}\Phi(R,z_0) + \Phi(R,z)\frac{\partial \rho(R,z)}{\partial R} + \rho(R,z)\frac{\partial \Phi(R,z)}{\partial R} + \mathcal{C}[\rho,\Phi] = \\
=& \rho(R,z_0)\frac{\partial \Phi(R,z_0)}{\partial R} + \mathcal{C}[\rho,\Phi] = \\
=& \frac{\rho(R,z_0)u_\varphi^2(R,z_0)}{R} + \mathcal{C}[\rho,\Phi].
\end{aligned}
$$

□

Here we note that, assuming $z_0 = \infty$, the positivity of the integrand in equation (3.3.7) is a sufficient condition to obtain $u_\varphi^2 \geq 0$ everywhere. Therefore, physically acceptable solutions are obtained if one assumes a potential Φ for which $\partial\Phi/\partial R \geq 0$ and $\partial\Phi/\partial z \geq 0$ and a density distribution so that $\partial\rho/\partial z \leq 0$ and $\partial\rho/\partial R \geq 0$.

An important property of the commutator-like relation $\mathcal{C}[\rho,\Phi]$ is its bilinearity that can be used to construct more complicated solutions starting from simple, physically acceptable "building block" configurations. For instance the rotational velocity associated with $\rho = \rho_1 + \rho_2$ is

$$u_\varphi^2 = \frac{\rho_1 u_{\varphi,1}^2 + \rho_2 u_{\varphi,2}^2}{\rho_1 + \rho_2}, \tag{3.3.8}$$

where $\rho_1 u_{\varphi,1}^2/R \equiv \mathcal{C}[\rho_1, \Phi]$ and $\rho_2 u_{\varphi,2}^2/R \equiv \mathcal{C}[\rho_2, \Phi]$.

Proof.

$$\begin{aligned}(\rho_1+\rho_2)\frac{u_\varphi^2}{R} &= \int_z^{z_0} \left(\frac{\partial(\rho_1+\rho_2)}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial(\rho_1+\rho_2)}{\partial z'}\frac{\partial \Phi}{\partial R}\right) dz' = \\ &= \int_z^{z_0} \left(\frac{\partial \rho_1}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial \rho_1}{\partial z'}\frac{\partial \Phi}{\partial R}\right) dz' + \int_z^{z_0} \left(\frac{\partial \rho_2}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial \rho_2}{\partial z'}\frac{\partial \Phi}{\partial R}\right) dz' = \\ &= \mathcal{C}[\rho_1, \Phi] + \mathcal{C}[\rho_2, \Phi].\end{aligned} \tag{3.3.9}$$

Thus we have

$$u_\varphi^2 = \frac{R}{\rho_1+\rho_2}\big(\mathcal{C}[\rho_1, \Phi] + \mathcal{C}[\rho_2, \Phi]\big) = \frac{\rho_1 u_{\varphi,1}^2 + \rho_2 u_{\varphi,2}^2}{\rho_1 + \rho_2}. \tag{3.3.10}$$

□

In addition, for a given ρ, if $\Phi = \Phi_1 + \Phi_2$ we have

$$u_\varphi^2 = u_{\varphi,1}^2 + u_{\varphi,2}^2, \tag{3.3.11}$$

where

$$u_{\varphi,1}^2 = \frac{R}{\rho}\mathcal{C}[\rho, \Phi_1], \tag{3.3.12}$$

and

$$u_{\varphi,2}^2 = \frac{R}{\rho}\mathcal{C}[\rho, \Phi_2]. \tag{3.3.13}$$

Proof.

$$\begin{aligned}\frac{\rho u_\varphi^2}{R} &= \int_z^{z_0} \left(\frac{\partial \rho}{\partial R}\frac{\partial(\Phi_1+\Phi_2)}{\partial z'} - \frac{\partial \rho}{\partial z'}\frac{\partial(\Phi_1+\Phi_2)}{\partial R}\right) dz' = \\ &= \int_z^{z_0} \left(\frac{\partial \rho}{\partial R}\frac{\partial \Phi_1}{\partial z'} - \frac{\partial \rho}{\partial z'}\frac{\partial \Phi_1}{\partial R}\right) dz' + \int_z^{z_0} \left(\frac{\partial \rho}{\partial R}\frac{\partial \Phi_2}{\partial z'} - \frac{\partial \rho}{\partial z'}\frac{\partial \Phi_2}{\partial R}\right) dz' = \\ &= \mathcal{C}[\rho, \Phi_1] + \mathcal{C}[\rho, \Phi_2].\end{aligned} \tag{3.3.14}$$

Using (3.3.12) and (3.3.13) we get

$$u_\varphi^2 = u_{\varphi,1}^2 + u_{\varphi,2}^2. \tag{3.3.15}$$

□

3.4 Families of baroclinic density distributions

We are going to show some simple families of physical baroclinic models of rotating gas discussed in [Barnabé et al. 2006].

3.4.1 Gas density distributions with a factor stratified on the effective potential

Let us consider the gas density distribution

$$\rho(R, z) = h(R, z)\, \rho_e(\Phi_{eff}), \tag{3.4.1}$$

where h is a non negative function and

$$\Phi_{eff} \equiv \Phi - \int_{R_0}^{R} \Omega^2(R')\, R'\, dR' \tag{3.4.2}$$

is the effective potential, Φ is the gravitational potential, $\Omega(R)$ is a given cylindrical rotation law and we define R_0 as an arbitrary fixed radius. Moreover, we assume that $\rho_e(\Phi_{eff})$ is a solution of the Euler equation

$$\nabla p = -\rho \nabla \Phi_{eff}, \tag{3.4.3}$$

with assigned pression $p = p(\rho)$. For $\Omega \neq 0$ one has a barotropic solution (cylindrical rotation), while for $\Omega = 0$ one obtains a hydrostatic solution in the potential Φ that we can indicate as ρ_h.

We have already showed how to construct the density field in the barotropic case (see eq. (3.2.1)). A different approach to the construction of $\rho_e(\Phi_{eff})$ is however possible, where the specific density field is prescribed. This can be done by using the fact that solutions of $\nabla p = -\rho \nabla \Phi_{eff}$ are stratified in Φ_{eff}; thus if one fixes Φ_{eff} and a prescribed function $\rho_e(\Phi_{eff})$, the pressure field is obtained by direct integration as

$$p(\Phi_{eff}) = p(\Phi_{eff,0}) - \int_{\Phi_{eff,0}}^{\Phi_{eff}} \rho_e(t)\, dt. \tag{3.4.4}$$

We point out that this approach can be used only for density stratifications such that $p > 0$ everywhere.

We can now demonstrate that for a gas density distribution with a factor stratified on the effective potential (eq. 3.4.1), assuming $z_0 = \infty$ we have

$$\begin{aligned} \frac{\rho u_\varphi^2}{R} = \int_z^\infty dz' \left[\frac{\partial h(R,z')}{\partial R} \frac{\partial \Phi}{\partial z'} - \frac{\partial h(R,z')}{\partial z'} \frac{\partial \Phi}{\partial R} \right] \rho_e\big(\Phi_{eff}\big) - \\ - R\Omega^2(R) \int_z^\infty dz' \left[h(R,z') \frac{\partial \rho_e(\Phi_{eff})}{\partial z'} \right]. \end{aligned} \tag{3.4.5}$$

Proof. Combining equation (3.4.1) and eq. (3.3.6) we obtain

$$
\begin{aligned}
\frac{\rho u_\varphi^2}{R} &= \int_z^\infty dz' \left(\frac{\partial \rho}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial \rho}{\partial z'}\frac{\partial \Phi}{\partial R} \right) = \\
&= \int_z^\infty dz' \Bigg\{ \left[\frac{\partial}{\partial R}\Big(h(R,z')\rho_e\big(\Phi_{eff}\big)\Big) \frac{\partial}{\partial z'}\Big(\Phi_{eff} + \int_{R_0}^R \Omega^2(R')R'\, dR \Big) \right] - \\
&\qquad - \left[\frac{\partial}{\partial z'}\Big(h(R,z')\rho_e\big(\Phi_{eff}\big)\Big) \frac{\partial}{\partial R}\Big(\Phi_{eff} + \int_{R_0}^R \Omega^2(R')R'\, dR \Big) \right] \Bigg\} = \\
&= \int_z^\infty dz' \Bigg\{ \left[\frac{\partial h(R,z')}{\partial R}\rho_e\big(\Phi_{eff}\big) + h(R,z')\frac{\partial \rho_e\big(\Phi_{eff}\big)}{\partial R} \right] \frac{\partial}{\partial z'}(\Phi) - \\
&\qquad - \left[\frac{\partial h(R,z')}{\partial z'}\rho_e\big(\Phi_{eff}\big) + h(R,z')\frac{\partial \rho_e\big(\Phi_{eff}\big)}{\partial z'} \right] \left[\frac{\partial}{\partial R}\big(\Phi_{eff}\big) + \Omega^2(R)R \right] \Bigg\} = \\
&= \int_z^\infty dz' \left[\frac{\partial h(R,z')}{\partial R}\frac{\partial \Phi_{eff}}{\partial z'} - \frac{\partial h(R,z')}{\partial z'}\frac{\partial \Phi_{eff}}{\partial R} \right] \rho_e\big(\Phi_{eff}\big) - \\
&\quad - R\Omega^2(R) \int_z^\infty dz' \left[h(R,z') \frac{\partial \rho_e(\Phi_{eff})}{\partial z'} \right] + \\
&\quad + \int_z^\infty dz' \left[h(R,z')\, [\rho_e(\Phi_{eff}), \Phi_{eff}] - \frac{\partial h(R,z')}{\partial z'}\rho_e(\Phi_{eff})\Omega^2(R)R \right] = \\
&= \int_z^\infty dz' \left[\frac{\partial h(R,z')}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial h(R,z')}{\partial z'}\frac{\partial \Phi}{\partial R} \right] \rho_e\big(\Phi_{eff}\big) - \\
&\quad - R\Omega^2(R) \int_z^\infty dz' \left[h(R,z') \frac{\partial \rho_e(\Phi_{eff})}{\partial z'} \right] + \\
&\quad + \int_z^\infty dz' \left[h(R,z')\, [\rho_e(\Phi_{eff}), \Phi_{eff}] \right] = \\
&= \int_z^\infty dz' \left[\frac{\partial h(R,z')}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial h(R,z')}{\partial z'}\frac{\partial \Phi}{\partial R} \right] \rho_e\big(\Phi_{eff}\big) - \\
&\quad - R\Omega^2(R) \int_z^\infty dz' \left[h(R,z') \frac{\partial \rho_e(\Phi_{eff})}{\partial z'} \right].
\end{aligned} \tag{3.4.6}
$$

□

For instance, taking into account eq. (3.3.8), one can calculate the rotational velocity for a density distribution $\rho = \rho_h(\Phi) + \rho_e(\Phi_{eff})$:

$$u_\varphi^2 = \frac{\rho_e R^2 \Omega^2(R)}{\rho_h + \rho_e}. \tag{3.4.7}$$

3.4.2 Baroclinic models built using a function $\mathbf{f(R)}$

We consider here a particular case of eq. (3.4.1) in which we assume

$$h(R,z) = f(R). \tag{3.4.8}$$

We assume to have a barotropic model in a gravitational potential Φ and cylindrical velocity $u_{\varphi,0}(R) \equiv u_\varphi(R, z_0)$. We can calculate the effective potential Φ_{eff}, the density ρ_e and the pressure p_e.

To construct the baroclinic model, we build the density field as a variation of the barotropic density. This means that the baroclinic model density field can be written as

$$\rho = f(R)\rho_e \qquad \text{where } f(R) > 0. \tag{3.4.9}$$

Since the choice of p is arbitrary, we request the boundary condition $p(R, z_0) = 0$.

From eq. (3.3.1) and eq. (3.3.6) we can obtain the pressure and the velocity field of the baroclinic model:

$$p = f p_e, \tag{3.4.10}$$

$$u_\varphi^2(R, z) = u_{\varphi,0}^2(R) + \frac{p_e}{\rho_e}\frac{d\ln f(R)}{d\ln R}, \tag{3.4.11}$$

where $u_{\varphi,0}$ is the cylindrical velocity at $z = z_0$.

Proof. Equation (3.4.10) can be derived as follows:

$$p(R, z) = \int_z^{z_0} \rho \frac{\partial \Phi}{\partial z'} dz' = f(R) \int_z^{z_0} \rho_e \frac{\partial \Phi}{\partial z'} dz' = f(R) p_e.$$

While equation (3.4.11) can be derived as follows:

$$\begin{aligned}
\frac{\rho u_\varphi^2}{R} =& \frac{\rho(R, z_0) u_\varphi^2(R, z_0)}{R} + \int_z^{z_0} \left(\frac{\partial \rho}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial \rho}{\partial z'}\frac{\partial \Phi}{\partial R} \right) dz' = \\
=& \frac{\rho(R, z_0) u_\varphi^2(R, z_0)}{R} + \int_z^{z_0} \left(\frac{\partial (f(R)\rho_e)}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial (f(R)\rho_e)}{\partial z'}\frac{\partial \Phi}{\partial R} \right) dz' = \\
=& \frac{\rho(R, z_0) u_\varphi^2(R, z_0)}{R} + \int_z^{z_0} \frac{\partial f(R)}{\partial R} \rho_e \frac{\partial \Phi}{\partial z'} dz' = \\
=& \frac{\rho(R, z_0) u_\varphi^2(R, z_0)}{R} + \frac{\partial f(R)}{\partial R} \int_z^{z_0} \rho_e \frac{\partial \Phi}{\partial z'} dz' = \\
=& \frac{\rho(R, z_0) u_\varphi^2(R, z_0)}{R} + \frac{\partial f(R)}{\partial R} p_e \\
\Longrightarrow u_\varphi^2 =& u_{\varphi,0}^2 + \frac{\partial f(R)}{\partial R} p_e \frac{R}{\rho_e f(R)} = u_{\varphi,0}^2 + \frac{p_e}{\rho_e}\frac{\partial \ln f(R)}{\partial \ln R}.
\end{aligned}$$

□

3.4.3 Homeoidal potential

We present here a specific example of baroclinic models built as described in (3.4.1). Let $\Phi(l)$ be a homeoidally stratified potential with

$$l^2 = R^2 + \frac{z^2}{q_\Phi^2}, \qquad \text{where } 0 < q_\Phi \leq 1. \tag{3.4.12}$$

Assuming

$$h(R, z) = A(R)B(m), \tag{3.4.13}$$

we can write eq. (3.4.1) assuming a hydrostatic density factor $\rho_h(\Phi)$ as

$$\rho(R, z) = A(R)B(m)\rho_h(\Phi), \tag{3.4.14}$$

where $m^2 \equiv R^2 + \frac{z^2}{q_g^2}$ and $0 < q_g \leq 1$; $A(R)$ and $B(m)$ are positive functions.

From eq. (3.4.5), with $z_0 = \infty$, we get

$$\begin{aligned} \frac{\rho u_\varphi^2}{R} = \left(\frac{1}{q_\Phi^2} - \frac{1}{q_g^2}\right) A(R) R \int_z^\infty \frac{B'(m)}{m} \rho_h(\Phi)\Phi'(l) \frac{z'}{l}\, dz' + \\ + \frac{A'(R)}{q_\Phi^2} \int_z^\infty B(m)\rho_h(\Phi)\Phi' \frac{z'}{l}\, dz' \end{aligned} \tag{3.4.15}$$

and so $u_\Phi^2 \geq 0$ if

$$\Phi'(l) \geq 0, \qquad A'(R) \geq 0, \qquad B'(m) \leq 0, \qquad q_g \leq q_\Phi. \tag{3.4.16}$$

The flattening condition ($q_g \leq q_\Phi$) requires that the gas density distribution must be stratified on homeoids which are flatter than the isopotential surfaces.

3.4.4 Razor-thin uniform disk

As another example of application of the models introduced in section 3.4.2, we consider here the case of a razor-thin uniform disk.

We consider a density distribution in the form

$$\rho = A(R)\rho_h(\Phi) \tag{3.4.17}$$

and

$$\Phi = 2\pi\Sigma_0 z. \tag{3.4.18}$$

So from eq. (3.4.5) we can calculate:

$$\begin{aligned} \frac{\rho u_\varphi^2}{R} &= \int_z^{z_0} \left(\frac{\partial A(R)}{\partial R}\frac{\partial \Phi}{\partial z'} - \frac{\partial A(R)}{\partial z'}\frac{\partial \Phi}{\partial R}\right)\rho_h(\Phi)\, dz' = \\ &= \int_z^{z_0} \Big(A'(R) 2\pi G\Sigma_0\Big)\rho_h(\Phi)\, dz' = \\ &= 2\pi G\Sigma_0 A'(R) \int_z^{z_0} \rho_h(\Phi)\, dz'. \end{aligned} \tag{3.4.19}$$

Having $A'(R) \geq 0$ is the necessary and sufficient condition to have physical acceptable solutions in this case.

The reason for this condition is that a gas distribution $\rho(R, z)$ that not stratified on Φ must be rotating and its pressure must be radially increasing. We can note (from equation (3.3.2)) that in the present case the pressure is proportional to the gas column density and this means that in a vertical gravitational field $u_\varphi^2 \geq 0$ every time the column density is radially increasing.

Chapter 4

Power-law tori

In this chapter we focus on rotating toroidal gaseous systems. We take as starting point models that were originally introduced as stellar systems in [Ciotti, Bertin 2005]. In the isotropic case the governing equations are formally identical to the hydrodynamics equations, so the solutions can be interpreted as fluid systems.

These systems are rotating and have baroclinic distributions.

I will use the Euler equations and the results presented in the previous chapter to discriminate what models are physically acceptable and how these tori react to linear perturbations.

In the first part of the chapter we will investigate self-gravitating tori and then we will examine the case in which a black hole gravitational potential is added.

These latter systems can be considered idealized models of toroidal gaseous structures in AGN.

4.1 Construction of the models

We consider the family of power-law tori presented by L. Ciotti and G. Bertin [Ciotti, Bertin 2005], in which the dimensionless [1] density distribution is defined as

$$\tilde{\rho} = \frac{\tilde{R}^2}{\tilde{r}^\alpha}, \qquad (\alpha > 0) \tag{4.1.1}$$

where $\tilde{R} = R/a$, $\tilde{r} = r/a$, R and z are cylindrical coordinates, $r = \sqrt{R^2 + z^2}$ is the spherical radius and a is a scale parameter. Figure 4.1 shows the isodensity contours in the meridional plane for various values of α.

In this chapter we focus on models with

$$2 < \alpha < 5. \tag{4.1.2}$$

We will not consider here the interval $0 < \alpha < 2$ because we exclude cases in which the density diverges for $r \to \infty$. Moreover we will not study case with $\alpha > 5$ because they are characterized by an infinite central mass [Ciotti, Bertin 2005].

The dimensionless gravitational potential generated by the density distribution (4.1.1) for $2 < \alpha < 5$ is (see [Ciotti, Bertin 2005])

$$\tilde{\Phi} = \begin{cases} -\dfrac{\tilde{r}^{2-\alpha}}{(\alpha-2)(7-\alpha)}\left[\dfrac{4\tilde{r}^2}{(\alpha-4)(5-\alpha)} + \tilde{R}^2\right] & (\alpha \neq 4) \\ \dfrac{1}{3}\left(2\ln\tilde{r} - \dfrac{1}{2}\dfrac{\tilde{R}^2}{\tilde{r}^2}\right) & (\alpha = 4), \end{cases} \tag{4.1.3}$$

such that

$$\tilde{\nabla}^2\tilde{\Phi} = \tilde{\rho} \tag{4.1.4}$$

where $\tilde{\nabla}^2 = a^2\nabla^2$ is the dimensionless Laplace operator.

Physical normalization constant. Let us begin our study by calculating the physical normalization constant that permits to express the density-potential pair (eq. (4.1.1), eq. (4.1.3)) in physical units.

One can write the density distribution as

$$\rho = \rho_0\tilde{\rho} = \rho_0 a^{\alpha-2}\frac{R^2}{r^\alpha}, \tag{4.1.5}$$

where ρ_0 is a constant with dimension of a mass density.

If we introduce a constant Φ_0, with dimension of a gravitational potential (or of a velocity squared), and we define

$$\Phi = \Phi_0\tilde{\Phi}, \tag{4.1.6}$$

we can write, by virtue of equation (1.2.14),

$$\begin{aligned} \nabla^2\Phi =& \Phi_0\nabla^2\tilde{\Phi} = a^{-2}\Phi_0\tilde{\nabla}^2\tilde{\Phi} = \\ =& 4\pi G\rho = 4\pi G\rho_0\tilde{\rho}. \end{aligned} \tag{4.1.7}$$

[1] Throughout this chapter the symbol " ˜ " indicates dimensionless quantities.

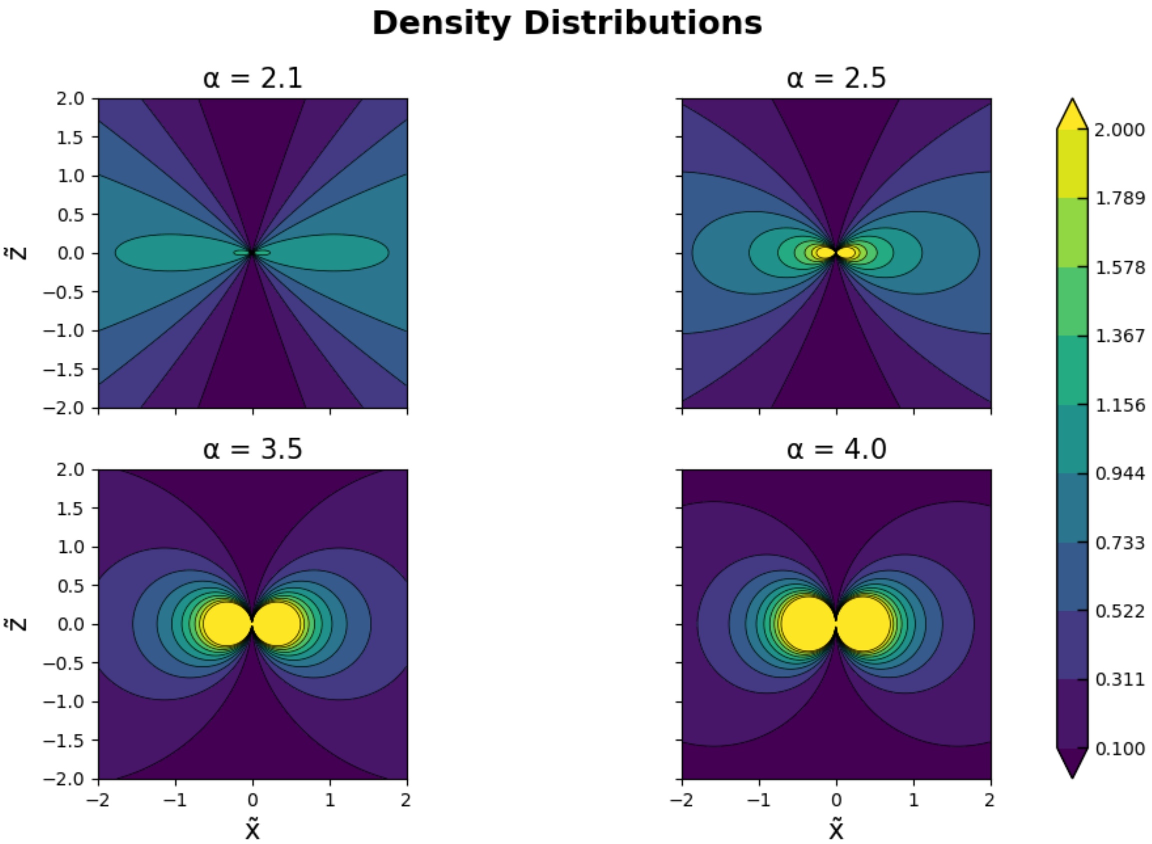

Figure 4.1: Isodensity contours in the meridional plane $y = 0$ of tori with density distribution (4.1.1) for different values of α. The color bar indicates the values of the normalized density $\tilde{\rho}$. We use as horizontal label $\tilde{x} = x/a$ and as vertical label $\tilde{z} = z/a$. Here x, y, z are Cartesian dimensionless coordinates ($R^2 = x^2 + y^2$).

Hence we find from equation (4.1.7) that the physical normalization constant for the potential is

$$\Phi_0 = 4\pi G \rho_0 a^2. \tag{4.1.8}$$

4.2 Self-gravitating case

In this section we are going to discuss about self-gravitating rotating toroidal gaseous systems, for which the gravitational potential is generated only by its density distribution (equation (4.1.1)).

4.2.1 Solution of the stationary Euler equation

If we suppose that the gas in the torus configuration can be modeled as an ideal fluid, it follows from the z component of the stationary Euler equation that

$$p(R,z) = \int_z^\infty \rho \frac{\partial \Phi}{\partial z} dz. \tag{4.2.1}$$

For $\alpha \neq 4$ and assuming that $p(R,\infty) = 0$ we have in cylindrical coordinates

$$p(R,z) = \frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}} \frac{R^2(R^2+z^2)^{1-\alpha}}{7-\alpha} \left[\frac{2(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} + \frac{R^2}{2(\alpha-1)} \right], \tag{4.2.2}$$

whereas if $\alpha = 4$ one obtains

$$p(R,z) = \frac{2}{3}\pi\rho_0^2 G a^4 \left[\frac{R^2}{(R^2+z^2)^2} + \frac{R^4}{3(R^2+z^2)^3} \right]. \tag{4.2.3}$$

Moreover, the R-component of the stationary Euler equation gives

$$\Omega^2 R = \begin{cases} \dfrac{8\pi G\rho_0}{(7-\alpha)a^{2-\alpha}} (R^2+z^2)^{(1-\frac{\alpha}{2})} \left[\dfrac{2(R^2+z^2)}{R(\alpha-2)^2(5-\alpha)} - \dfrac{R}{(\alpha-2)(\alpha-1)} \right], & \text{for } \alpha \neq 4 \\ \dfrac{4}{3}\pi\rho_0 G a^2 \left[\dfrac{1}{R} - \dfrac{1}{3}\dfrac{R}{R^2+z^2} \right], & \text{for } \alpha = 4, \end{cases} \tag{4.2.4}$$

where Ω is the angular frequency.

4.2.2 Selection of the physical models

By using the spherical coordinate r, from equation (4.2.4), with $u_\varphi = \Omega R$, we have

$$u_\varphi^2 = \frac{4}{3}\pi G\rho_0 a^2 \left[1 - \frac{R^2}{3r^2} \right], \qquad \text{for } \alpha = 4 \tag{4.2.5}$$

and

$$u_\varphi^2 = \frac{8\pi G\rho_0 r^{2-\alpha}}{(7-\alpha)a^{2-\alpha}} \left[\frac{2r^2}{(\alpha-2)^2(5-\alpha)} - \frac{R^2}{(\alpha-1)(\alpha-2)} \right], \qquad \text{for } \alpha \neq 4. \tag{4.2.6}$$

We look for those values of α that give physically acceptable solutions, i.e. solutions for which the rotational velocity field squared is positive defined or null for all (r, R) .

Using $R = r\sin\theta$, where θ is the colatitude in spherical coordinates ($0 \le \theta \le \pi$; $0 \le \sin\theta \le 1$), and factorizing r^2 in front of the square brackets, we can write

$$u_\varphi^2 = \frac{8\pi G\rho_0 r^{4-\alpha}}{(7-\alpha)(\alpha-2)a^{2-\alpha}}\left[\frac{2}{(\alpha-2)(5-\alpha)} - \frac{\sin^2\theta}{\alpha-1}\right]. \tag{4.2.7}$$

$u_\varphi^2 \ge 0$ for all r and θ when

$$\boxed{2 < \alpha < 5.} \tag{4.2.8}$$

Proof. We need to find those values of α such that

$$\frac{1}{(7-\alpha)(\alpha-2)}\left[\frac{2}{(\alpha-2)(5-\alpha)} - \frac{\sin^2\theta}{\alpha-1}\right] \ge 0, \qquad \text{for } 0 \le \theta \le \pi. \tag{4.2.9}$$

For

$$2 < \alpha < 5, \tag{4.2.10}$$

the first factor of (4.2.9) is positive, while the second is positive when

$$\frac{2}{(\alpha-2)(5-\alpha)} \ge \frac{\sin^2\theta}{\alpha-1}. \tag{4.2.11}$$

In the case (4.2.8) we have

$$\frac{2(\alpha-1)}{(\alpha-2)(5-\alpha)} \ge \sin^2\theta, \tag{4.2.12}$$

where $\sin^2\theta \in [0,1]$.

This must be true for every value of $\sin^2\theta$, so we have to solve the inequality

$$\frac{2(\alpha-1)}{(\alpha-2)(5-\alpha)} > 1. \tag{4.2.13}$$

Solutions of (4.2.13) are all in the interval

$$2 < \alpha < 5. \tag{4.2.14}$$

□

The dimensionless rotational velocity. One can define the dimensionless rotation velocity field squared as

$$\tilde{u}_\varphi^2 \equiv \frac{u_\varphi^2}{u_0^2} = \frac{u_\varphi^2}{8\pi G\rho_0 a^2} = \frac{1}{6}\left[1 - \frac{\tilde{R}^2}{3\tilde{r}^2}\right], \qquad \text{for } \alpha = 4 \tag{4.2.15}$$

and

$$\tilde{u}_\varphi^2 \equiv \frac{u_\varphi^2}{u_0^2} = \frac{u_\varphi^2}{8\pi G\rho_0 a^2} = \frac{\tilde{r}^{2-\alpha}}{(7-\alpha)(\alpha-2)}\left[\frac{2\tilde{r}^2}{(\alpha-2)(5-\alpha)} - \frac{\tilde{R}^2}{\alpha-1}\right], \qquad \text{for } \alpha \neq 4, \tag{4.2.16}$$

where

$$u_0^2 = 8\pi G\rho_0 a^2. \tag{4.2.17}$$

We have from (4.2.15) and (4.2.16) that the dimensionless rotation velocity field in the equatorial plane (at $z = 0$) reads

$$\tilde{u}_\varphi^2(R,0) = \frac{1}{9}, \qquad \text{for } \alpha = 4 \tag{4.2.18}$$

and

$$\tilde{u}_\varphi^2(R,0) = \frac{\tilde{R}^{4-\alpha}}{(7-\alpha)(\alpha-2)}\left[\frac{2}{(\alpha-2)(5-\alpha)} - \frac{1}{\alpha-1}\right], \qquad \text{for } \alpha \neq 4. \tag{4.2.19}$$

In figure (4.2) we show rotation velocity fields in the equatorial plane for models with different values of α.

4.2.3 General linear stability analysis

Here we focus on $\alpha \neq 4$. The case $\alpha = 4$ is studied in §4.2.4. In order to study the linear stability of the self-gravitating system the conditions of Solberg-Hoiland criterion have been used. For an ideal gas we can write the inequalities (2.5.31) and (2.5.32) as

$$\frac{1}{R^3}\frac{\partial J_z^2}{\partial R} - \frac{1}{\gamma\rho}\nabla p \cdot \nabla \ln\left(\frac{p}{\rho^\gamma}\right) \geq 0 \tag{4.2.20}$$

and

$$-\frac{\partial p}{\partial z}\left[\frac{\partial J_z^2}{\partial R}\frac{\partial \ln\left(p/\rho^\gamma\right)}{\partial z} - \frac{\partial J_z^2}{\partial z}\frac{\partial \ln\left(p/\rho^\gamma\right)}{\partial R}\right] \geq 0 \tag{4.2.21}$$

where J_z is the specific angular momentum and the entropy has be written as

$$\begin{aligned} S =& C_V \ln\left(p/\rho^\gamma\right) \\ =& C_V\Bigg\{\ln\left[\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}\frac{R^2(R^2+z^2)^{1-\alpha}}{7-\alpha}\left[\frac{2(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} + \frac{R^2}{2(\alpha-1)}\right]\right] + \\ & -\gamma\ln\left[\rho_0 a^{\alpha-2}\frac{R^2}{(R^2+z^2)^{\alpha/2}}\right]\Bigg\}, \end{aligned} \tag{4.2.22}$$

where we have used $C_P/C_V = \gamma$.

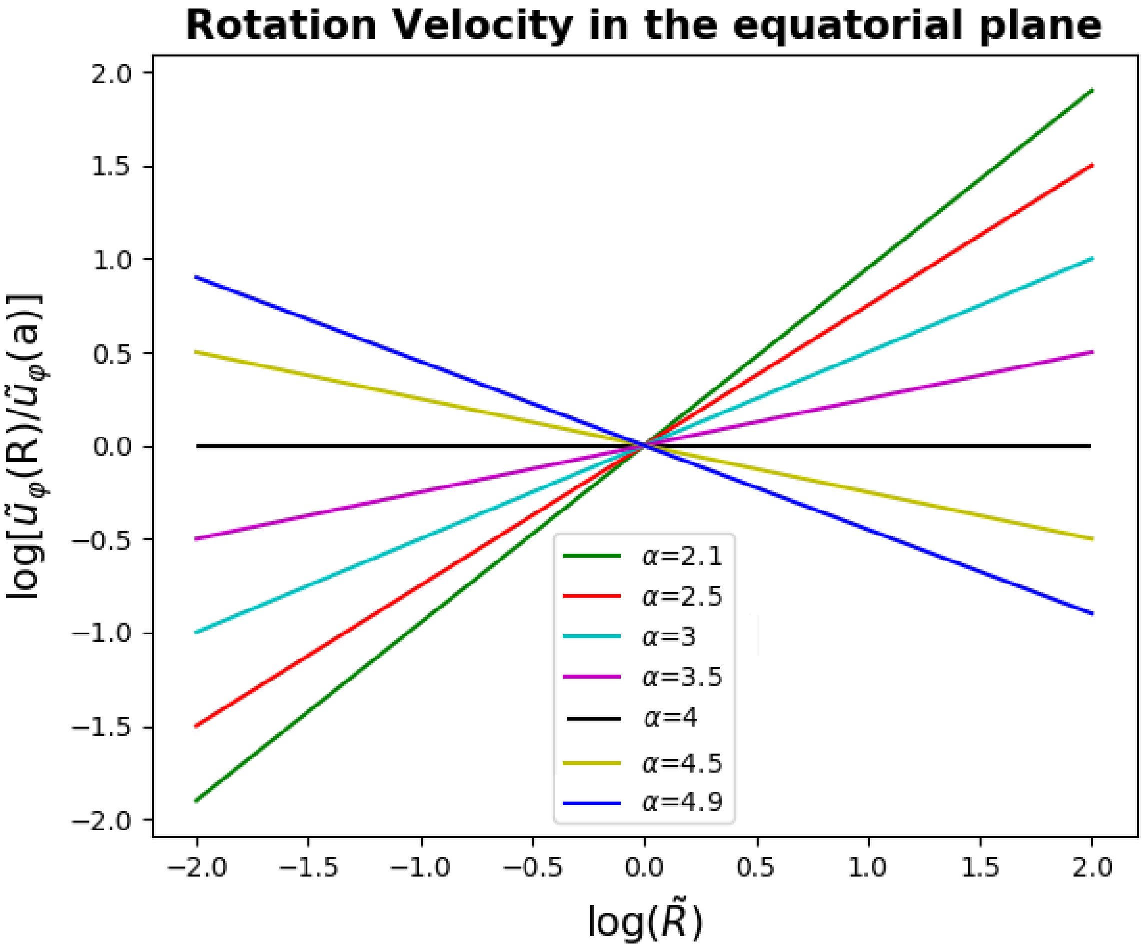

Figure 4.2: Rotation curves of velocity in the equatorial plane for self-gravitating power-law tori with $\alpha = 2.1$, $\alpha = 2.5$, $\alpha = 3$, $\alpha = 3.5$, $\alpha = 4$, $\alpha = 4.5$ and $\alpha = 4.9$. On the x-axis we have the logarithm of the cylindrical radius normalized to a; on the vertical axis we have the logarithm of the dimensionless rotation velocity field normalized to $\tilde{u}_\varphi(a)$.

Study of inequality (4.2.20). Let us take into account formula (4.2.20) by using cylindrical coordinates (R, z). We have for the first addend

$$\frac{1}{R^3}\frac{\partial J_z^2}{\partial R} = \frac{1}{R^3}\frac{\partial}{\partial R}\left\{\frac{4\pi G\rho_0}{(7-\alpha)a^{2-\alpha}}\left[\frac{4R^2(R^2+z^2)^{2-\alpha/2}}{(\alpha-2)^2(5-\alpha)} - \frac{2R^4(R^2+z^2)^{1-\alpha/2}}{(\alpha-2)(\alpha-1)}\right]\right\}. \tag{4.2.23}$$

Hence

$$\begin{aligned}\frac{1}{R^3}\frac{\partial J_z^2}{\partial R} = \frac{4\pi G\rho_0}{(7-\alpha)a^{2-\alpha}}\Bigg[&\frac{8(R^2+z^2)^{2-\alpha/2}}{R^2(\alpha-2)^2(5-\alpha)} + \frac{4(4-\alpha)(R^2+z^2)^{1-\alpha/2}}{(\alpha-2)^2(5-\alpha)} - \\ &- \frac{8(R^2+z^2)^{1-\alpha/2}}{(\alpha-2)(\alpha-1)} + \frac{2R^2(R^2+z^2)^{-\alpha/2}}{\alpha-1}\Bigg].\end{aligned} \tag{4.2.24}$$

Therefore we can develop the second term of the inequality as

$$\nabla p \cdot \nabla \ln(p/\rho^\gamma) = \frac{\partial p}{\partial R}\frac{\partial \ln(p/\rho^\gamma)}{\partial R} + \frac{\partial p}{\partial z}\frac{\partial \ln(p/\rho^\gamma)}{\partial z}, \tag{4.2.25}$$

where

$$\begin{aligned}\frac{\partial p}{\partial R} = \frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}(7-\alpha)}(R^2+z^2)^{-\alpha}\Bigg[&\frac{4R(R^2+z^2)^2}{(\alpha-2)^2(5-\alpha)} - \frac{4R^3(R^2+z^2)}{(\alpha-2)(5-\alpha)} + \\ &+ \frac{2R^3(R^2+z^2)}{\alpha-1} - R^5\Bigg],\end{aligned} \tag{4.2.26}$$

$$\frac{\partial p}{\partial z} = -\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}(7-\alpha)}z\left[\frac{4R^2(R^2+z^2)^{1-\alpha}}{(\alpha-2)(5-\alpha)} + R^4(R^2+z^2)^{-\alpha}\right], \tag{4.2.27}$$

$$\begin{aligned}\frac{\partial \ln(p/\rho^\gamma)}{\partial R} =& \frac{2}{R} + \frac{2R(1-\alpha)}{R^2+z^2} + \frac{8R(\alpha-1)+2R(\alpha-2)^2(5-\alpha)}{4(\alpha-1)(R^2+z^2)+R^2(\alpha-2)^2(5-\alpha)} \\ &- \gamma\left(\frac{2}{R} - \frac{\alpha R}{R^2+z^2}\right),\end{aligned} \tag{4.2.28}$$

and

$$\begin{aligned}\frac{\partial \ln(p/\rho^\gamma)}{\partial z} =& \frac{2z(1-\alpha)}{R^2+z^2} + \frac{8z(\alpha-1)}{4(\alpha-1)(R^2+z^2)+R^2(\alpha-2)^2(5-\alpha)} \\ &+ \gamma\frac{\alpha z}{R^2+z^2}.\end{aligned} \tag{4.2.29}$$

Expanding the r.h.s. of eq. (4.2.25), we get

$$\frac{4\pi G\rho_0^2}{(7-\alpha)a^{2(2-\alpha)}}\left[\frac{4R(R^2+z^2)^{2-\alpha}}{(\alpha-2)^2(5-\alpha)}-\frac{4R^3(R^2+z^2)^{1-\alpha}}{(\alpha-2)(5-\alpha)}+\right.$$
$$\left.+\frac{2R^3(R^2+z^2)^{1-\alpha}}{\alpha-1}-R^5(R^2+z^2)^{-\alpha}\right]\times$$
$$\times\left[\frac{2}{R}+\frac{2R(1-\alpha)}{R^2+z^2}+\frac{8(\alpha-1)R+2R(\alpha-2)^2(5-\alpha)}{4(\alpha-1)(R^2+z^2)+(\alpha-2)^2(5-\alpha)R^2}\right.$$
$$\left.-\gamma\left(\frac{2}{R}-\frac{\alpha R}{R^2+z^2}\right)\right]+$$
$$+\frac{4\pi G\rho_0^2}{(7-\alpha)a^{2(2-\alpha)}}z\left[-\frac{4R^2(R^2+z^2)^{1-\alpha}}{(\alpha-2)(5-\alpha)}-R^4(R^2+z^2)^{-\alpha}\right]\times$$
$$\times\left[\frac{2z(1-\alpha)}{R^2+z^2}+\frac{8z(\alpha-1)}{4(\alpha-1)(R^2+z^2)+(\alpha-2)^2(5-\alpha)R^2}+\right.$$
$$\left.+\gamma\left(\frac{\alpha z}{R^2+z^2}\right)\right]. \tag{4.2.30}$$

By solving the Solberg-Hoiland inequality (4.2.20) in spherical coordinates and using $t\equiv\sin\theta$ $(0\le t\le 1)$ we get

$$\frac{8\pi G\rho_0}{7-\alpha}\tilde{r}^{2-\alpha}\left\{\frac{8}{t^2(\alpha-2)^2(5-\alpha)}+\frac{2(\alpha^3-12\alpha^2+17\alpha-2)}{(\alpha-2)^2(\alpha-5)(\alpha-1)}+\left(\frac{\alpha}{2}-2\right)t^2+\alpha t^4+\right.$$
$$+\frac{1}{\gamma}\left[\frac{4}{t^2(\alpha-2)^2(\alpha-5)}+\frac{2(\alpha^3-3\alpha^2-4\alpha+10)}{(\alpha-5)(\alpha-2)^2(\alpha-1)}-\right.$$
$$-\frac{1}{4(\alpha-1)+t^2(\alpha-2)^2(5-\alpha)}\Bigg((\alpha^3-9\alpha^2+16\alpha-12)t^4-$$
$$\left.\left.-\frac{2(\alpha^3-9\alpha^2+26\alpha-22)}{\alpha-1}t^2+\frac{4(\alpha^3-5\alpha^2+8\alpha-8)}{(\alpha-2)^2(\alpha-5)}\Bigg)\right]\right\}\ge 0. \tag{4.2.31}$$

We can study some particular cases of equation (4.2.31). For example

- for $t\to 0$ ($\theta\to 0$ and $\theta\to\pi$) we have

$$\frac{8\pi G\rho_0}{7-\alpha}\tilde{r}^{2-\alpha}\left\{\frac{4}{t^2(\alpha-2)^2(5-\alpha)}\left[2-\frac{1}{\gamma}\right]+\frac{2(\alpha^3-12\alpha^2+17\alpha-2)}{(\alpha-2)^2(\alpha-5)(\alpha-1)}+\right.$$
$$\left.+\frac{2(\alpha^3-3\alpha^2-4\alpha+10)}{\gamma(\alpha-5)(\alpha-2)^2(\alpha-1)}-\frac{\alpha^3-5\alpha^2+8\alpha-8}{\gamma(\alpha-1)(\alpha-2)^2(\alpha-5)}\right\}\ge 0 \tag{4.2.32}$$

that can be written as

$$\frac{8\pi G\rho_0}{7-\alpha}\tilde{r}^{2-\alpha}\left\{\frac{4\left[2-\frac{1}{\gamma}\right]}{t^2(\alpha-2)^2(5-\alpha)}+\frac{\mathcal{P}(\alpha^3)}{\mathcal{P}(\alpha^4)}\right\}\ge 0 \tag{4.2.33}$$

where

$$\mathcal{P}(\alpha^3)=\alpha^3\left(2+\frac{1}{\gamma}\right)-\alpha^2\left(24+\frac{1}{\gamma}\right)+\alpha\left(34-\frac{16}{\gamma}\right)-4+\frac{28}{\gamma} \tag{4.2.34}$$

and

$$\mathcal{P}(\alpha^4) = (\alpha-2)^2(\alpha-5)(\alpha-1). \tag{4.2.35}$$

The ratio

$$\frac{\mathcal{P}(\alpha^3)}{\mathcal{P}(\alpha^4)} \tag{4.2.36}$$

gives no contributes in this limit.

Therefore in range (4.2.8), we have that (4.2.33) is verified, thus the inequality (4.2.31) holds for $\theta \to 0$ and $\theta \to \pi$.

- for $t \to 1$ $(\theta \to \pi/2)$ we get

$$\begin{aligned}\frac{8\pi G\rho_0}{7-\alpha}\tilde{r}^{2-\alpha}\Bigg\{&\frac{-72+288\alpha-312\alpha^2+143\alpha^3-34\alpha^4+3\alpha^5}{2\alpha^4-20\alpha^3+66\alpha^2-88\alpha+40}+\\&+\frac{1}{\gamma}\left[\frac{-1344+3264a-3360a^2+1924a^3-681a^4+151a^5-19a^6+a^7}{a^7-19a^6+143a^5-557a^4+1236a^3-1588a^2+1104a-320}\right]\Bigg\}\geq 0.\end{aligned} \tag{4.2.37}$$

Let us define

$$\begin{aligned}W(\alpha,\gamma) =&\frac{-72+288\alpha-312\alpha^2+143\alpha^3-34\alpha^4+3\alpha^5}{2\alpha^4-20\alpha^3+66\alpha^2-88\alpha+40}+\\&+\frac{1}{\gamma}\left[\frac{-1344+3264a-3360a^2+1924a^3-681a^4+151a^5-19a^6+a^7}{a^7-19a^6+143a^5-557a^4+1236a^3-1588a^2+1104a-320}\right].\end{aligned} \tag{4.2.38}$$

We fix $\gamma = 5/3$ and look for the roots of equation (4.2.38):

$$\begin{aligned}501\alpha^8-9987\alpha^7+81203\alpha^6-&358409\alpha^5+949300\alpha^4-1484264\alpha^3+\\&+1231800\alpha^2-357216\alpha-76416=0.\end{aligned} \tag{4.2.39}$$

Considering the fundamental theorem of algebra, we have 8 roots and one can prove that 4 of these are in $\mathbb{R}$.

We have real solutions

$$\alpha \simeq -0.14; \qquad \alpha \simeq 1.46; \qquad \alpha \simeq 5.47; \qquad \alpha \simeq 6.24. \tag{4.2.40}$$

A change of sign of (4.2.38) occurs at these values. In particular we have a negative solution for

$$-0.14 < \alpha < 1.46 \qquad \bigcup \qquad 5.47 < \alpha < 6.24 \tag{4.2.41}$$

and positive solutions for

$$1.46 < \alpha < 5.47 \qquad \bigcup \qquad \alpha > 6.24 \tag{4.2.42}$$

By virtue of the the physical conditions (4.2.8) we have that for $\theta \to \pi/2$, the configuration is stable.

Following the same mathematical steps, if we assume $\gamma = 4/3$, eq. (4.2.38) is positive if

$$1.49 < \alpha < 5.44 \qquad \bigcup \qquad \alpha > 6.26 \tag{4.2.43}$$

Using the range conditions (4.2.8), we get positive solutions for every physical α.

Therefore (4.2.37) for $\gamma = 4/3$ is true in

$$2 < \alpha < 5. \tag{4.2.44}$$

Finally, if we consider a biatomic gas ($\gamma = 1.4$) in eq. (4.2.38) we have that (4.2.37) is always satisfied if

$$2 < \alpha < 5. \tag{4.2.45}$$

We can conclude that in

$$2 < \alpha < 5, \tag{4.2.46}$$

the linear stability criterion (4.2.20) is always satisfied for $\theta \to k\pi$ ($k = 0, 1$) and for $\theta \to \pi/2$. However, based on these calculations, we cannot exclude that the instability occurs at intermediate θ, for a generic α.

Study of inequality (4.2.21). For the formula (4.2.21) we have:

$$-\frac{\partial p}{\partial z} = \frac{4\pi G\rho_0^2}{(7-\alpha)a^{2(2-\alpha)}} zR^2(R^2+z^2)^{-\alpha}\left[\frac{4(R^2+z^2)}{(\alpha-2)(5-\alpha)} + R^2\right], \tag{4.2.47}$$

$$\begin{aligned} &\frac{\partial J_z^2}{\partial R}\frac{\partial \ln\left(p/\rho^\gamma\right)}{\partial z} = \\ &= \left[\frac{8\pi G\rho_0}{(7-\alpha)a^{2-\alpha}} R(R^2+z^2)^{-\frac{\alpha}{2}}\left(\frac{4(R^2+z^2)^2}{(\alpha-2)^2(5-\alpha)} + \frac{2R^2(4-\alpha)(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} - \right.\right. \\ &\qquad \left.\left. - \frac{4R^2(R^2+z^2)}{(\alpha-2)(\alpha-1)} - \frac{R^4}{\alpha-1}\right)\right] \times \\ &\quad \times z\left[\frac{2(1-\alpha)}{R^2+z^2} - \frac{8(1-\alpha)}{4(\alpha-1)(R^2+z^2)+R^2(\alpha-2)^2(5-\alpha)} + \gamma\frac{\alpha}{R^2+z^2}\right] \end{aligned} \tag{4.2.48}$$

and

$$\begin{aligned} &\frac{\partial J_z^2}{\partial z}\frac{\partial \ln\left(p/\rho^\gamma\right)}{\partial R} = \\ &= \left[\frac{8\pi G\rho_0 R^2}{(7-\alpha)a^{2-\alpha}} z(R^2+z^2)^{-\frac{\alpha}{2}}\left(\frac{2(4-\alpha)(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} + \frac{R^2}{\alpha-1}\right)\right] \times \\ &\quad \times \left[\frac{2}{R} + \frac{2R(1-\alpha)}{R^2+z^2} + 2R\frac{4(\alpha-1)+(\alpha-2)^2(5-\alpha)}{4(\alpha-1)(R^2+z^2)+R^2(\alpha-2)^2(5-\alpha)} - \right. \\ &\qquad \left. - \gamma\left(\frac{2}{R} - \frac{\alpha R}{R^2+z^2}\right)\right]. \end{aligned} \tag{4.2.49}$$

Therefore we have

$$\frac{\partial J_z^2}{\partial R}\frac{\partial \ln\left(p/\rho^\gamma\right)}{\partial z} - \frac{\partial J_z^2}{\partial z}\frac{\partial \ln\left(p/\rho^\gamma\right)}{\partial R} =$$

$$= \frac{8\pi G\rho_0}{(7-\alpha)a^{2-\alpha}} Rz(R^2+z^2)^{-\frac{\alpha}{2}}\Bigg\{ -\frac{4(\alpha+2)(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} -$$

$$- \frac{4(R^2+z^2)}{(\alpha-2)^2(5-\alpha)[4(\alpha-1)(R^2+z^2)+R^2(\alpha-2)^2(5-\alpha)]}\big(8(1-\alpha)(R^2+z^2)+$$

$$+ R^2(\alpha^4-13\alpha^3+60\alpha^2-116\alpha+80)\big) + \frac{8R^2}{\alpha-2} -$$

$$- \frac{2R^2}{4(\alpha-1)(R^2+z^2)+R^2(\alpha-2)^2(5-\alpha)}\times$$

$$\times\left(\frac{16(\alpha-1)(R^2+z^2)-R^2(\alpha^4-11\alpha^3+34\alpha^2-44\alpha+24)}{(\alpha-2)(\alpha-1)}\right) + \frac{4R^2}{R^2+z^2} - \frac{2R^2}{\alpha-1} +$$

$$+ \gamma\left[-\frac{4R^2\alpha}{(\alpha-2)(\alpha-1)} - \frac{2\alpha R^4}{(\alpha-1)(R^2+z^2)} + \frac{16(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} + \frac{2R^2}{\alpha-1}\right]\Bigg\}. \tag{4.2.50}$$

We can write the (4.2.21) in spherical coordinates, using $t \equiv \sin\theta$, as

$$\frac{32\pi^2 G^2\rho_0^3}{(7-\alpha)^2 a^{3(2-\alpha)}} r^{3(3-\alpha)} t^3(1-t^2)\left[\frac{4}{(\alpha-2)(5-\alpha)} + t^2\right]\times$$

$$\times\Bigg\{\frac{4(2+\alpha)}{(\alpha-2)^2(\alpha-5)} - \frac{4}{(\alpha-2)^2(5-\alpha)}\frac{8(1-\alpha)+t^2(\alpha-2)^2(\alpha-5)(\alpha-4)}{4(\alpha-1)+t^2(\alpha-2)^2(5-\alpha)} -$$

$$- \frac{2t^2}{(\alpha-2)(\alpha-1)}\frac{16(\alpha-1)-t^2(\alpha-2)(\alpha^3-9\alpha^2+16\alpha-12)}{4(\alpha-1)+t^2(\alpha-2)^2(5-\alpha)} +$$

$$+ \frac{2(3\alpha-2)}{(\alpha-2)(\alpha-1)}t^2 + 4t^4 +$$

$$+ 2\gamma\left[\frac{8}{(\alpha-2)^2(5-\alpha)} - \frac{\alpha+2}{(\alpha-2)(\alpha-1)}t^2 - \frac{\alpha}{\alpha-1}t^4\right]\Bigg\} \geq 0. \tag{4.2.51}$$

Ignoring the positive quantity, if we consider the limit $t \to 0$ in (4.2.51) we get

$$\frac{16(4\gamma-\alpha)}{(\alpha-2)^3(\alpha-5)^2} \geq 0. \tag{4.2.52}$$

Hence in the range (4.2.8) we have that this condition is always true. As we said before, we cannot exclude that the instability occurs at intermediate θ for generic α just from these calculations.

4.3 Linear stability analysis for models of given α

Here we determine whether some models of given α are stable or unstable.

In considering these models we note that, according to [Ciotti, Bertin 2005], the circular velocity vanishes for $\alpha = \alpha_n \simeq 2.44$. Moreover for $\alpha < \alpha_n$ we would have a negative squared circular velocity, in the sense that in this range the radial force field in the equatorial plane is directed outward.

We are going to consider monoatomic ($\gamma = 5/3$) physical gases for given values of α in order to determine whether they are stable or not by using inequality (4.2.20).

- For $\alpha = 2.1$, the inequality (2.5.31) is verified.

  From (4.2.31), we get

$$\frac{8.8 \times 10^{39} + 2.9 \times 10^{40}t^2 + 2.0 \times 10^{38}t^4 + 1.5 \times 10^{39}t^6 + 6.3 \times 10^{38}t^8}{3.0 \times 10^{35}t^4 + 4.6 \times 10^{37}t^2} \geq 0, \tag{4.3.1}$$

  that is true for every t.

- For $\alpha = 2.5$, the inequality (2.5.31) is verified.

  From (4.2.31), we get

$$\frac{129024 + 617472t^2 + 60160t^4 + 53055t^6 + 3750t^8}{1500t^4 + 14400t^2} \geq 0, \tag{4.3.2}$$

  that is true for every t.

- For $\alpha = 3$, the inequality (2.5.31) is verified.

  From (4.2.31), we get

$$\frac{30t^8 + 169t^6 + 152t^4 + 680t^2 + 112}{10t^4 + 40} \geq 0, \tag{4.3.3}$$

  that is true for every t.

- For $\alpha = 3.5$, the inequality (2.5.31) is verified.

  From (4.2.31), we get

$$\frac{358400 + 2467840t^2 + 746496t^4 + 1040715t^6 + 255150t^8}{72900t^4 + 216000t^2} \geq 0, \tag{4.3.4}$$

  that is true for every t.

- For $\alpha = 4.1$ the inequality (2.5.31) is verified.

  From (4.2.31), we get a positive function for every t.

- $\alpha = 4$, the inequality (2.5.31) is verified.

  For this model inequality (4.2.20) can be written as

$$\frac{8\pi\rho_0 G}{3\tilde{r}^2}\left[\frac{2}{\sin^2\theta} + 4\sin^2\theta + \frac{1}{\gamma(3+\sin^2\theta)}\left(-\frac{3}{\sin^2\theta} - 4 + \frac{2}{3}\sin^3\theta + \sin^4\theta\right)\right] \geq 0. \tag{4.3.5}$$

  Introducing $t = \sin\theta$, and neglecting the trivial positive terms, we get

$$\frac{2\,(1053 - 99t^2 + 3006t^4 + 100t^5 + 1152t^6)}{501\,(t^2+3)\,t^2} \geq 0, \tag{4.3.6}$$

  that is always verified.

Here we investigate the stability of some models according to (4.2.21) for monoatomic gases ($\gamma = 5/3$).

- For $\alpha = 2.1$, the inequality (2.5.32) is verified.

  From (4.2.51) we get

$$\frac{(t-1)\,t^3\,(1+t)\,(400+29t^2)\,(-44334400000+8310588000t^2+654774470t^4+2486837t^6)}{462550\,(29t^2+4400)}, \tag{4.3.7}$$

  that is a positive quantity for every $t \in [0,1]$.

- For $\alpha = 2.5$, the inequality (2.5.32) is verified.

  From (4.2.51) we get

$$\frac{(t-1)\,t^3\,(1+t)\,(16+5t^2)\,(-963072+561120t^2+285550t^4+17125t^6)}{3750\,(5t^2+48)}, \tag{4.3.8}$$

  that is a positive quantity for every $t \in [0,1]$.

- For $\alpha = 3$.

  From (4.2.51) we get

$$\frac{(t-1)\,t^3\,(1+t)\,(2+t^2)\,(-2944+2004t^2+2639t^4+401t^6)}{100\,(t^2+4)}. \tag{4.3.9}$$

  Here we note that for

  - $0.847 < t < 1$ the system is unstable,
  - $0 < t < 0.847$ the system is linearly stable.

- For $\alpha = 3.5$.

  From (4.2.51) we obtain

$$\frac{(t-1)\,t^3\,(1+t)\,(16+9t^2)\,(-678400+400800t^2+1020474t^4+223317t^6)}{20250\,(27t^2+80)}. \tag{4.3.10}$$

  Hence we have

  - $0.782 < t < 1$ the distribution is unstable,
  - $0 < t < 0.782$ the distribution is linearly stable.

- For $\alpha = 4.1$.

  In this case we have

  - $0.776 < t < 1$ the distribution is unstable,
  - $0 < t < 0.776$ the distribution is linearly stable.

- $\alpha = 4$

  In this case (4.2.21) assumes the form:

$$\frac{32\pi^2\rho_0^3G^2a^6}{9r^3}\left[\frac{2\sin^2\theta\cos\theta-\sin^4\theta\cos\theta}{3-\sin^2\theta}(12\sin\theta\cos\theta+\frac{8}{3}\sin^5\theta\cos\theta)-\right.$$
$$-\gamma(2\sin^2\theta\cos\theta-\sin^4\theta\cos\theta)(4\sin\theta\cos\theta-\frac{10}{3}\sin^3\theta\cos\theta+$$
$$\left.+\frac{8}{3}\sin^5\theta\cos\theta)\right]\geq 0. \tag{4.3.11}$$

  By using the parametrization $\sin\theta = t$ we have

$$\frac{\left(-2412+10632t^2-16605t^4+12160t^6-4443t^8+668t^{10}\right)t^3}{150\left(t^2-3\right)}. \tag{4.3.12}$$

  This means that we have

  - a linearly stable distribution in the area designed by $0 \leq t \leq 0.7$,
  - an instable distribution when $0.7 < t \leq 1$.

We summarize in table 4.1 the discussed linear stability analysis. We note that for small α the gas distribution is stable for every θ, while if α grows there are some regions in which the same model assume an instable behavior when perturbed.

α	**Behavior if perturbed**	**Range of coordinates**
2.1	Stable	$0 < t < 1$
2.5	Stable	$0 < t < 1$
3	Stable	$0 < t < 0.847$
	Unstable	$0.847 < t < 1$
3.5	Stable	$0 < t < 0.782$
	Unstable	$0.782 < t < 1$
4	Stable	$0 < t < 0.7$
	Unstable	$0.7 < t < 1$
4.1	Stable	$0 < t < 0.776$
	Unstable	$0.776 < t < 1$

Table 4.1: Results of the linear stability analysis for a selection of models with given α and $\gamma = 5/3$. Here we use $t \equiv \sin\theta$.

4.4 Black Hole potential

We consider here gaseous tori with density (4.1.1) in equilibrium in a gravitational potential given by the sum of their own gravitational potential and that of a central black hole. The gravitational potential of the black hole written in cylindrical coordinates is

$$\Phi_\bullet = -\frac{GM_\bullet}{(R^2+z^2)^{1/2}}, \tag{4.4.1}$$

where $M_\bullet$ is the mass of the central black hole.

4.4.1 Solution of the stationary Euler equation

Considering that the potential is additive and the derivative operator is linear, we can write the pressure as

$$p = p_{self} + p_\bullet, \tag{4.4.2}$$

where p_{self} is given by (4.2.2) and

$$\begin{aligned} p_\bullet &= \int_z^\infty \rho \frac{\partial \Phi_\bullet}{\partial z}\, dz = \\ &= \frac{GM_\bullet \rho}{(\alpha+1)(R^2+z^2)^{1/2}}. \end{aligned} \tag{4.4.3}$$

Proof.

$$\begin{aligned} p_\bullet &= \int_z^\infty \rho \frac{\partial \Phi_\bullet}{\partial z}\, dz = \\ &= \int_z^\infty \rho_0 a^{\alpha-2} R^2 (R^2+z^2)^{-\alpha/2} \Big(GM_\bullet z (R^2+z^2)^{-3/2} \Big)\, dz \\ &= \rho_0 a^{\alpha-2} R^2 GM_\bullet \int_z^\infty z \big(R^2+z^2\big)^{-\frac{3+\alpha}{2}}\, dz = \\ &= \frac{GM_\bullet \rho_0 a^{\alpha-2} R^2 (R^2+z^2)^{-\frac{\alpha+1}{2}}}{\alpha+1} = \\ &= \frac{GM_\bullet}{\alpha+1} (R^2+z^2)^{-1/2} \frac{\rho_0 \tilde{R}^2}{\tilde{r}^\alpha} = \\ &= \frac{GM_\bullet}{\alpha+1} \frac{\rho}{(R^2+z^2)^{1/2}}, \end{aligned} \tag{4.4.4}$$

where we have used eq. (4.1.5). □

The total gravitational potential is

$$\begin{aligned} \Phi_{tot} &= \Phi_{self} + \Phi_\bullet = \\ &= -\frac{4\pi G \rho_0}{a^{2-\alpha}} \frac{(R^2+z^2)^{1-\alpha/2}}{(\alpha-2)(7-\alpha)} \left(\frac{4(R^2+z^2)}{(\alpha-4)(5-\alpha)} + R^2 \right) - \\ &\quad - \frac{GM_\bullet}{(R^2+z^2)^{1/2}}, \qquad \text{for } \alpha \neq 4, \end{aligned} \tag{4.4.5}$$

and

$$\Phi_{tot} = \Phi_{self} + \Phi_{\bullet} = \\ = \frac{4\pi G\rho_0 a^2}{3}\left[2\ln\left(\frac{\sqrt{R^2+z^2}}{a}\right) - \frac{1}{2}\frac{R^2}{R^2+z^2}\right] - \frac{GM_{\bullet}}{(R^2+z^2)^{1/2}}, \qquad \text{for } \alpha = 4. \tag{4.4.6}$$

The pressure is

$$p = p_{self} + p_{\bullet} = \\ = \frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}\frac{R^2(R^2+z^2)^{1-\alpha}}{7-\alpha}\left[\frac{2(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} + \frac{R^2}{2(\alpha-1)}\right] + \\ + \frac{GM_{\bullet}\rho_0}{(\alpha+1)a^{2-\alpha}}\frac{R^2}{(R^2+z^2)^{\frac{1}{2}(\alpha+1)}}, \qquad \text{for } \alpha \neq 4 \tag{4.4.7}$$

and

$$p = p_{self} + p_{\bullet} = \frac{2\pi G\rho_0^2 a^4}{3}\left[\frac{R^2}{(R^2+z^2)^2} + \frac{R^4}{3(R^2+z^2)^3}\right] + \\ + \frac{GM_{\bullet}\rho_0}{5a^{-2}}\frac{R^2}{(R^2+z^2)^{\frac{5}{2}}}, \qquad \text{for } \alpha = 4. \tag{4.4.8}$$

If we take into account the radial component of the Euler equation (eq. 3.1.2) we have

$$\frac{1}{\rho}\frac{\partial p}{\partial R} = \frac{4\pi G\rho_0}{(7-\alpha)a^{2-\alpha}}(R^2+z^2)^{-\alpha/2}\left[\frac{4(R^2+z^2)^2}{R(\alpha-2)^2(5-\alpha)} - \frac{4R(R^2+z^2)}{(\alpha-2)(5-\alpha)} \right. \\ \left. + \frac{2R(R^2+z^2)}{\alpha-1} - R^3\right] + \\ + \frac{2GM_{\bullet}}{R(R^2+z^2)^{1/2}(\alpha+1)} - \frac{GM_{\bullet}R}{(R^2+z^2)^{3/2}}, \qquad \text{for } \alpha \neq 4, \tag{4.4.9}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial R} = \frac{4\pi\rho_0 G a^2}{3}\left[\frac{1}{R} - \frac{4R}{R^2+z^2} - \frac{R^3}{(R^2+z^2)^2}\right] \\ + \frac{2GM_{\bullet}}{5R(R^2+z^2)^{1/2}} - \frac{GM_{\bullet}R}{(R^2+z^2)^{3/2}}, \qquad \text{for } \alpha = 4, \tag{4.4.10}$$

and

$$\frac{\partial\Phi_{tot}}{\partial R} = -\frac{4\pi G\rho_0}{a^{2-\alpha}(\alpha-2)(7-\alpha)}(R^2+z^2)^{-\alpha/2}\left[-\frac{4R(R^2+z^2)}{5-\alpha} + 2R(R^2+z^2) + \right. \\ \left. + R^3(2-\alpha)\right] + \\ + \frac{GM_{\bullet}R}{(R^2+z^2)^{3/2}}, \qquad \text{for } \alpha \neq 4. \tag{4.4.11}$$

$$\begin{aligned}\frac{\partial \Phi_{tot}}{\partial R} =& \frac{4\pi\rho_0 G a^2}{3}\left[\frac{R}{R^2+z^2}+\frac{R^3}{(R^2+z^2)^2}\right]+ \\ &+\frac{GM_\bullet R}{(R^2+z^2)^{3/2}}, \qquad \text{for } \alpha = 4.\end{aligned} \tag{4.4.12}$$

Therefore we can conclude that the radial component of the acceleration can be expressed as function of (R, z) as

$$\begin{aligned}\Omega^2 R =& \frac{8\pi G\rho_0}{(7-\alpha)(\alpha-2)a^{2-\alpha}}(R^2+z^2)^{(1-\frac{\alpha}{2})}\left[\frac{2(R^2+z^2)}{R(\alpha-2)(5-\alpha)}-\frac{R}{(\alpha-1)}\right]+ \\ &+\frac{2GM_\bullet}{R(R^2+z^2)^{1/2}(\alpha+1)}, \qquad \text{for } \alpha \neq 4\end{aligned} \tag{4.4.13}$$

and

$$\Omega^2 R = \frac{4\pi G\rho_0 a^2}{3}\left[\frac{1}{R}-\frac{R}{3(R^2+z^2)}\right]+\frac{2GM_\bullet}{5R(R^2+z^2)^{1/2}}, \qquad \text{for } \alpha = 4. \tag{4.4.14}$$

4.4.2 Selection of the physical models

Using the same considerations as in the self-gravitating case, we can write rotation velocity field in presence of a black hole as

$$u_\varphi^2 = \frac{4}{3}\pi G\rho_0 a^2\left[1-\frac{R^2}{3r^2}\right]+\frac{2GM_\bullet}{5r}, \qquad \text{for } \alpha = 4, \tag{4.4.15}$$

that is a positive quantity, and

$$u_\varphi^2 = \frac{8\pi G\rho_0 r^{2-\alpha}}{(7-\alpha)a^{2-\alpha}}\left[\frac{2r^2}{(\alpha-2)^2(5-\alpha)}-\frac{R^2}{(\alpha-1)(\alpha-2)}\right]+\frac{2GM_\bullet}{(\alpha+1)r}, \qquad \text{for } \alpha \neq 4. \tag{4.4.16}$$

In order to study the positivity of (4.4.16) we need to analyze the following inequality:

$$\frac{4\pi\rho_0 r^{5-\alpha}}{(7-\alpha)(\alpha-2)a^{2-\alpha}}\left[\frac{2}{(\alpha-2)(5-\alpha)}-\frac{\sin^2\theta}{\alpha-1}\right] \geq -\frac{M_\bullet}{(\alpha+1)}, \tag{4.4.17}$$

where we have used $R = r\sin\theta$.

From the self-gravitating case, we know that the l.h.s. of (4.4.17) is a positive quantity if conditions (4.2.8) are imposed over α. The r.h.s. of (4.4.17) is always negative because $\alpha > 0$. Hence values of α in the range (4.2.8) are acceptable also in the presence of a black hole.

4.4.3 Dimensionless physical quantities

Here we present some dimensionless physical quantities that can be used to describe the properties of the gas distribution (4.1.1).

We write these quantities in the presence of a black hole, but the case without BH is obtained simply for $M_\bullet = 0$.

If we use $\tilde{R} = R/a$ and $\tilde{r} = r/a$ and we introduce the dimensionless acceleration defined by $\widetilde{\Omega^2 R} = (\Omega^2 R)/(8\pi G\rho_0 a)$ and the dimensionless black hole mass $\mu = M_\bullet/(4\pi\rho_0 a^3)$, we can write eq. (4.4.13) as

$$\begin{aligned}\widetilde{\Omega^2 R} = \frac{\Omega^2 R}{8\pi G\rho_0 a} =& \frac{(\tilde{R}^2+\tilde{z}^2)^{(1-\frac{\alpha}{2})}}{(7-\alpha)(\alpha-2)}\left[\frac{2(\tilde{R}^2+\tilde{z}^2)}{\tilde{R}(\alpha-2)(5-\alpha)} - \frac{\tilde{R}}{(\alpha-1)}\right]+\\ &+\frac{\mu}{\tilde{R}(\tilde{R}^2+\tilde{z}^2)^{1/2}(\alpha+1)}, \qquad \text{for } \alpha \neq 4\end{aligned} \tag{4.4.18}$$

and eq. (4.4.14) as

$$\widetilde{\Omega^2 R} = \frac{\Omega^2 R}{8\pi G\rho_0 a} = \frac{1}{6}\left[\frac{1}{\tilde{R}} - \frac{\tilde{R}}{3(\tilde{R}^2+\tilde{z}^2)}\right] + \frac{\mu}{5\tilde{R}(\tilde{R}^2+\tilde{z}^2)^{1/2}}, \qquad \text{for } \alpha = 4. \tag{4.4.19}$$

From equations (4.4.18) and (4.4.19) we can obtain the dimensionless angular frequencies,

$$\begin{aligned}\tilde{\Omega} = \Bigg\{&\frac{(\tilde{R}^2+\tilde{z}^2)^{(1-\frac{\alpha}{2})}}{(7-\alpha)(\alpha-2)}\left[\frac{2(\tilde{R}^2+\tilde{z}^2)}{\tilde{R}^2(\alpha-2)(5-\alpha)} - \frac{1}{(\alpha-1)}\right]+\\ &+\frac{\mu}{\tilde{R}^2(\tilde{R}^2+\tilde{z}^2)^{1/2}(\alpha+1)}\Bigg\}^{1/2}, \qquad \text{for } \alpha \neq 4\end{aligned} \tag{4.4.20}$$

and

$$\tilde{\Omega} = \left\{\frac{1}{6}\left[\frac{1}{\tilde{R}^2} - \frac{1}{3(\tilde{R}^2+\tilde{z}^2)}\right] + \frac{\mu}{5\tilde{R}^2(\tilde{R}^2+\tilde{z}^2)^{1/2}}\right\}^{1/2}, \qquad \text{for } \alpha = 4. \tag{4.4.21}$$

Using the equation of state for an ideal gas we can write

$$p = \frac{\rho k_B T}{\mu' m_p} \tag{4.4.22}$$

where k_B is the Boltzmann constant, μ' is the mean gas particle mass in units of the proton mass m_p and m_p is the proton mass at rest.

From (4.4.22) we get

$$\begin{aligned}T &= \frac{p\mu' m_p}{\rho k_B} =\\ &= \frac{p}{\rho}\frac{1}{u_0^2}\frac{\mu' m_p}{k_B}u_0^2,\end{aligned} \tag{4.4.23}$$

where

$$u_0^2 \equiv 8\pi G\rho_0 a^2. \tag{4.4.24}$$

Hence if we define

$$T_0 \equiv \frac{\mu' m_p}{k_B}u_0^2, \tag{4.4.25}$$

and we get

$$\tilde{T} = \frac{T}{T_0} = \frac{p}{\rho}\frac{1}{u_\varphi^2}. \tag{4.4.26}$$

If now we introduce

$$\tilde{p} = \frac{p}{p_0}, \tag{4.4.27}$$

where

$$p_0 \equiv 4\pi G\rho_0^2 a^2, \tag{4.4.28}$$

we can write (4.4.26), for $\alpha \neq 4$ as

$$\tilde{T} = \frac{1}{2}\frac{\tilde{p}}{\tilde{\rho}} = \\ = \frac{1}{2}\left[\frac{(\tilde{R}^2+\tilde{z}^2)^{1-\alpha/2}}{7-\alpha}\left(\frac{2(\tilde{R}^2+\tilde{z}^2)}{(\alpha-2)^2(5-\alpha)} + \frac{\tilde{R}^2}{2(\alpha-1)}\right) + \frac{\mu}{(\alpha+1)(\tilde{R}^2+\tilde{z}^2)^{1/2}}\right]. \tag{4.4.29}$$

and for $\alpha = 4$ as

$$\tilde{T} = \frac{1}{2}\left[\frac{1}{6}\left(1 + \frac{\tilde{R}^2}{3(\tilde{R}^2+\tilde{z}^2)}\right) + \frac{\mu}{5(\tilde{R}^2+\tilde{z}^2)^{1/2}}\right]. \tag{4.4.30}$$

Finally we use the dimensionless spherical coordinates ($\tilde{r} = r/a$), we can write eq. (4.4.16) as

$$\tilde{u}_\varphi^2 = \frac{\tilde{r}^{4-\alpha}}{(7-\alpha)(\alpha-2)}\left[\frac{2}{(\alpha-2)(5-\alpha)} - \frac{\sin^2\theta}{\alpha-1}\right] + \frac{\mu}{\tilde{r}(\alpha+1)}, \tag{4.4.31}$$

where

$$\tilde{u}_\varphi^2 = u_\varphi^2/u_0^2. \tag{4.4.32}$$

We can write the dimensionless rotation velocity field squared as a function of $(\tilde{r}, \tilde{R})$ as

$$\tilde{u}_\varphi^2 = \frac{1}{6}\left[1 - \frac{\tilde{R}^2}{3\tilde{r}^2}\right] + \frac{\mu}{5\tilde{r}}, \qquad \text{for } \alpha = 4 \tag{4.4.33}$$

and

$$\tilde{u}_\varphi^2 = \frac{\tilde{r}^{2-\alpha}}{(7-\alpha)(\alpha-2)}\left[\frac{2\tilde{r}^2}{(\alpha-2)(5-\alpha)} - \frac{\tilde{R}^2}{\alpha-1}\right] + \frac{\mu}{\tilde{r}(\alpha+1)}, \qquad \text{for } \alpha \neq 4 \tag{4.4.34}$$

From eq. (4.4.33) and eq. (4.4.34) we can get the dimensionless rotation velocity field at $z = 0$ that reads

$$\tilde{u}_\varphi^2(R,0) = \frac{1}{9} + \frac{\mu}{5\tilde{R}}, \qquad \text{for } \alpha = 4 \tag{4.4.35}$$

and

$$\tilde{u}_\varphi^2(R,0) = \frac{\tilde{R}^{4-\alpha}}{(7-\alpha)(\alpha-2)}\left[\frac{2}{(\alpha-2)(5-\alpha)} - \frac{1}{\alpha-1}\right] + \frac{\mu}{\tilde{R}(\alpha+1)}, \qquad \text{for } \alpha \neq 4. \tag{4.4.36}$$

In figures (4.3) and (4.5) we show the distributions of the temperature $\tilde{T}$ in the meridional plane for models $\alpha = 2.5$ and $\alpha = 4$ respectively. For the same models we show the trend at different z and in presence of different black hole masses in figures (4.4) and (4.6). In these figures one can observe a significant difference between the two models in the self-gravitating case: for $\alpha = 4$ the temperature distribution in spherical coordinates reads

$$\tilde{T} = \frac{1}{12}\left(1 + \frac{\sin^2\theta}{3}\right), \tag{4.4.37}$$

therefore it dependents just on the colatitude θ. Moreover we note that from the contour plot, the introduction of a black hole potential term modifies the temperature distribution that assume a spherical form in the meridional plane that diverges in the origin.

In figures (4.7) and (4.9) we show the distributions of the angular frequency $\tilde{\Omega}$ in the meridional plane for models $\alpha = 2.5$ and $\alpha = 4$ respectively. For the same models we show the trend at different z and in presence of different black hole masses in figures (4.8) and (4.10). In both models we note that as $\tilde{R} \to \infty$ the angular frequency tends to zero, whereas if $\tilde{R} \to 0$, no matter the value assumed by z or if we are in presence of a black hole potential term, the angular frequency profiles diverge to infinity. For both models ($\alpha = 2.5$ and $\alpha = 4$), comparing the self-gravitating toroidal distribution to the case in which the gas is influenced also by an external black hole potential, the figures show a different trend in the angular frequency profiles. In particular here we note that for $\mu = 0$, Ω increase with z and for $\mu = 10$, Ω decrease with z.

In figures (4.11) and (4.13) we show the distributions of the rotation velocity field $\tilde{u}_\varphi$ in the meridional plane for models $\alpha = 2.5$ and $\alpha = 4$ respectively. For the same models we show the trend at different z and in presence of different black hole masses in figures (4.12) and (4.14). Fig. (4.13) shows the dependence only on θ of the rotation velocity field in the self-gravitating model $\alpha = 4$ with no black hole. Both models show that the black hole potential term assumes an important role when it is present: the maps in the right panels of (4.11) and (4.13) tends to circular symmetry with a rotation velocity that diverges in the center.

Finally in figure (4.15) we report the rotation velocity curves in the equatorial plane for different tori and different black hole potentials.

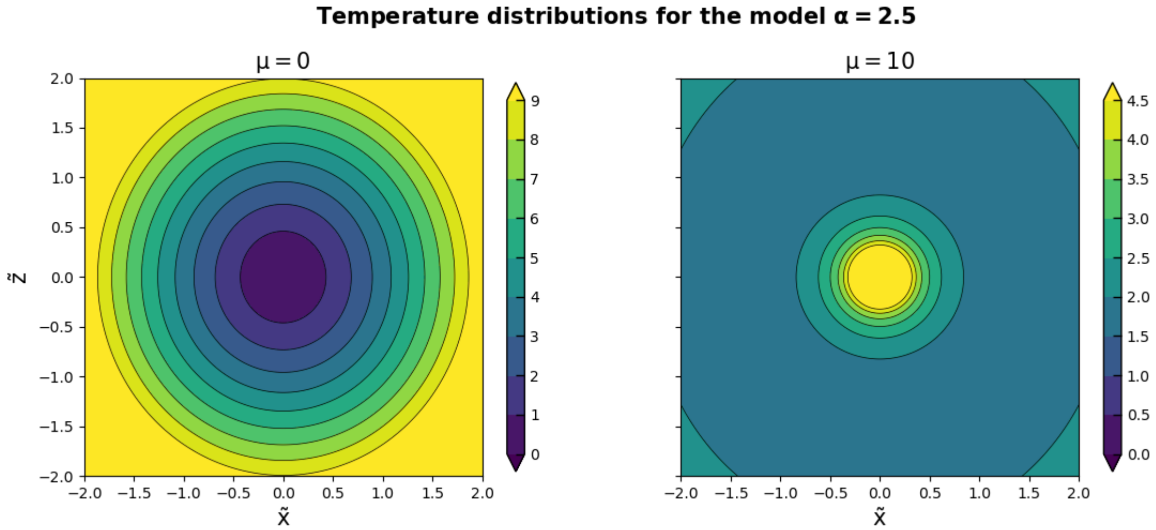

Figure 4.3: Temperature $\tilde{T} = T/T_0$ contours in the meridional plane $\tilde{y} = 0$ for tori with density distribution (4.1.1) for $\alpha = 2.5$ ($\tilde{x} = x/a$ and $\tilde{z} = z/a$). In the left we show the self-gravitating model, while in the right panel we present a toroidal distribution in presence of a black hole of mass $M_\bullet = 4\pi\rho_0 a^3 \mu$.

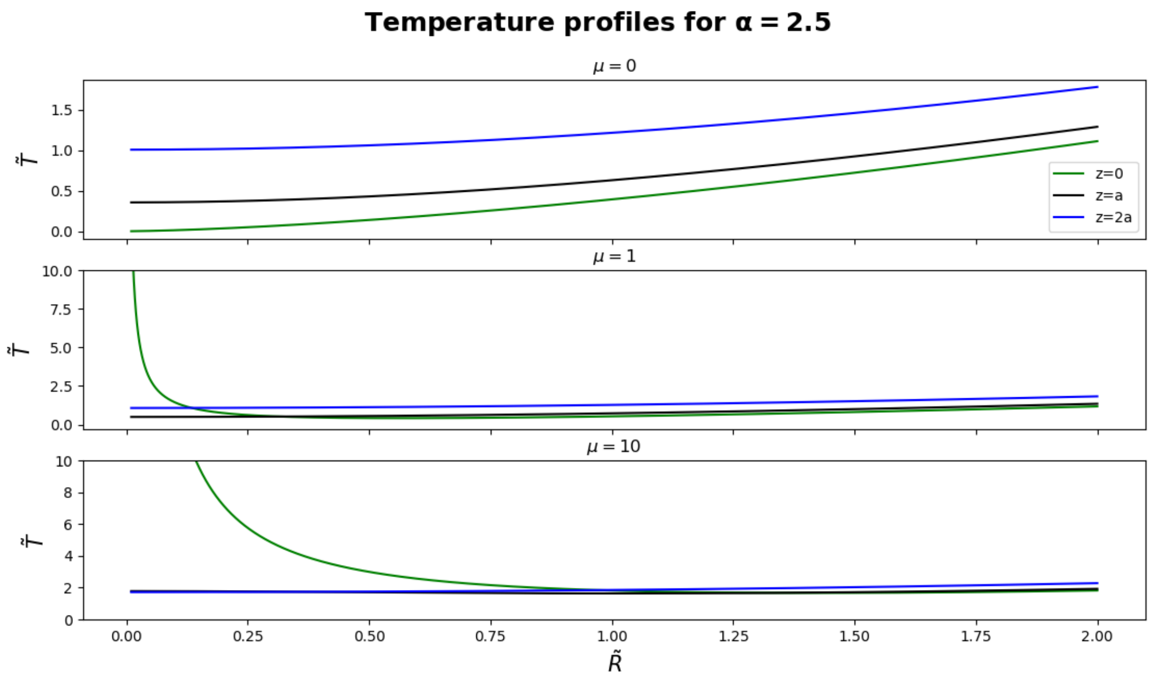

Figure 4.4: Radial temperature profiles at different heights z for power-law tori with $\alpha = 2.5$ for three values of the black hole mass ($\mu = 0$ is the case without black hole). Here we use $\tilde{R} = R/a$ and $\tilde{T} = T/T_0$

.

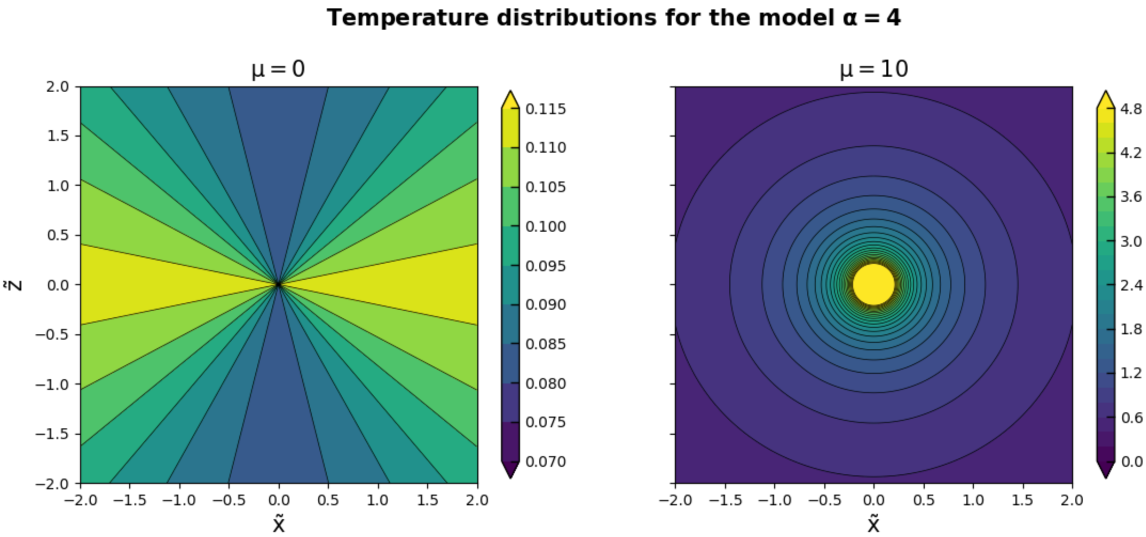

Figure 4.5: Same as fig. (4.3), but for $\alpha = 4$.

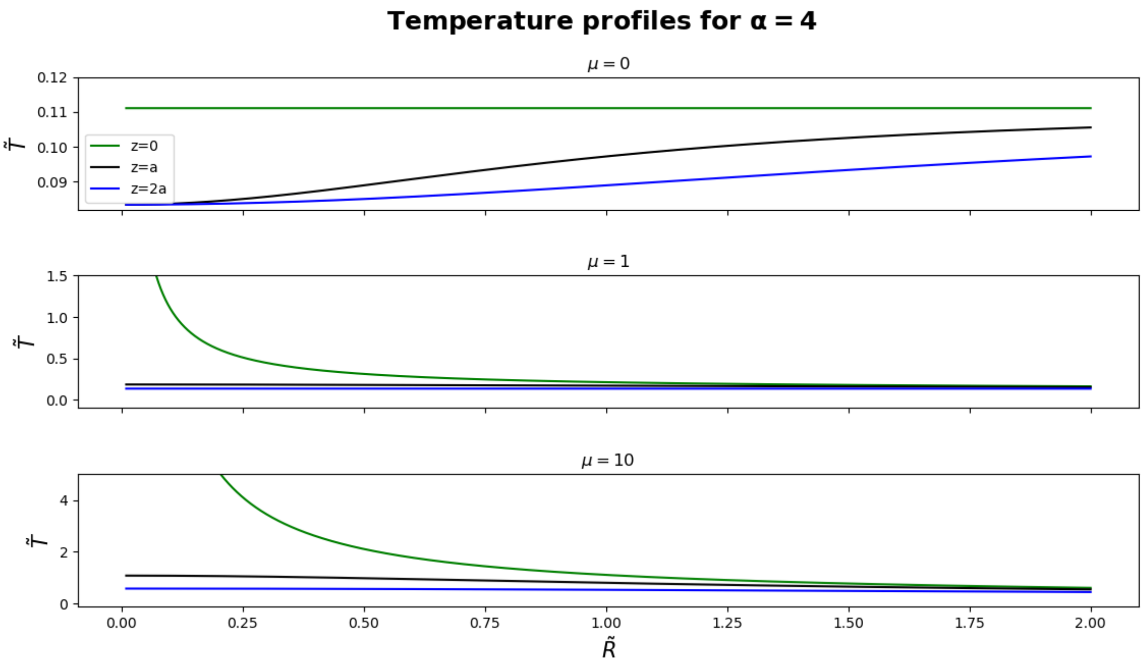

Figure 4.6: Same as Fig. (4.4), but for $\alpha = 4$.

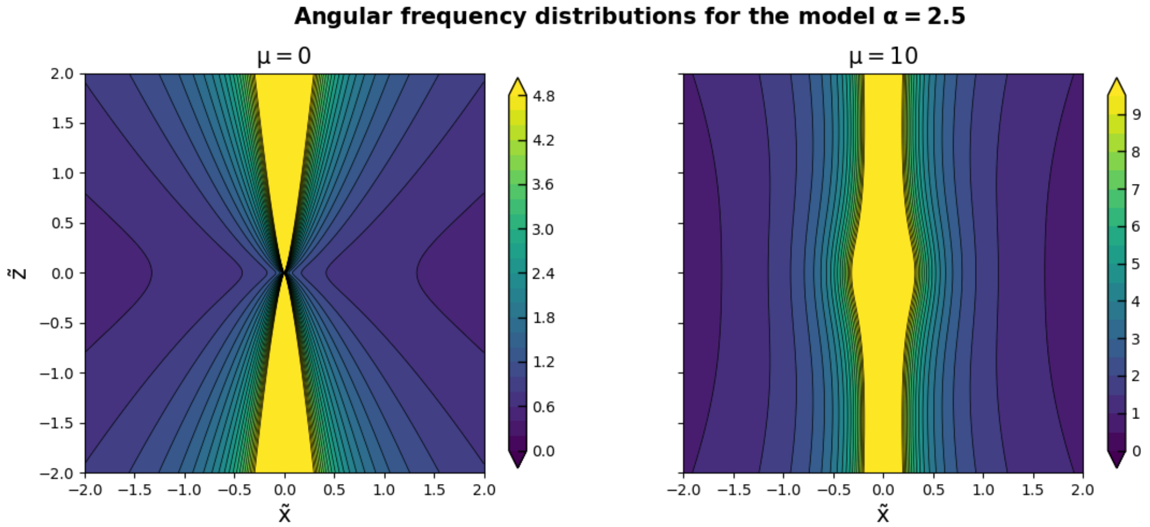

Figure 4.7: Angular frequency ($\tilde{\Omega} = \Omega/\Omega_0$) contours in the meridional plane $y = 0$ for tori with density distribution (4.1.1) for $\alpha = 2.5$ ($\tilde{x} = x/a$ and $\tilde{z} = z/a$). In the left we show the self-gravitating model, while in the right panel we present a toroidal distribution in presence of a black hole of mass $M_\bullet = 4\pi\rho_0 a^3 \mu$.

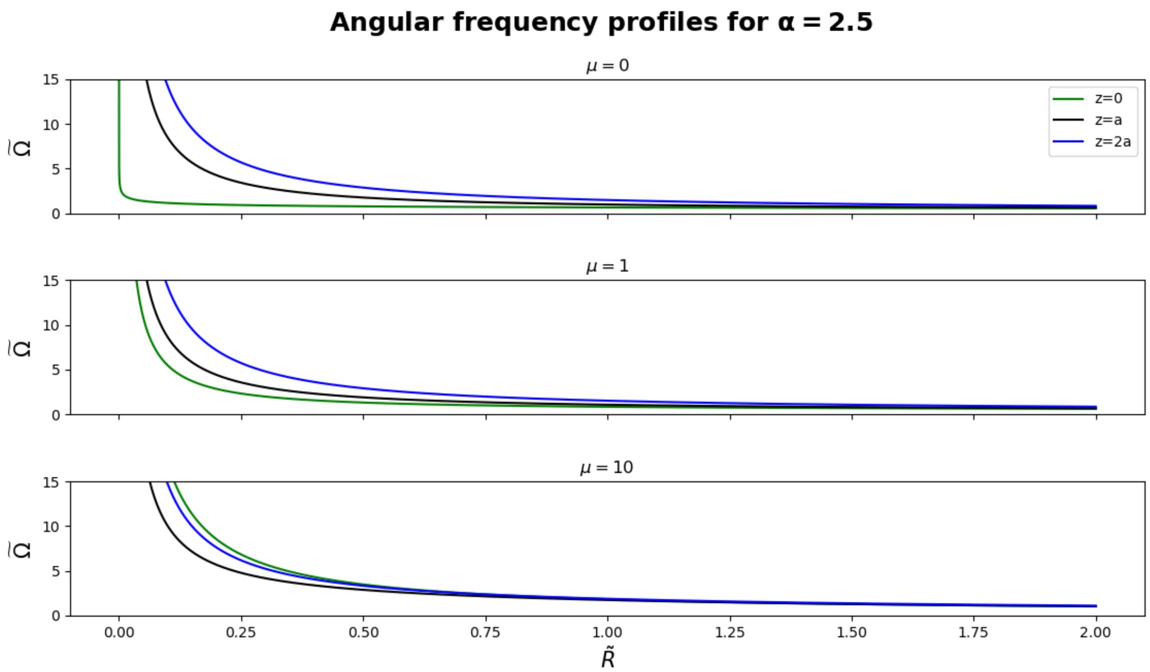

Figure 4.8: Angular frequency profiles at different heights z for power-law tori with $\alpha = 2.5$ for three values of the black hole mass ($\mu = 0$ is the case without black hole). Here we use $\tilde{\Omega} = \Omega/\Omega_0$ and $\tilde{R} = R/a$.

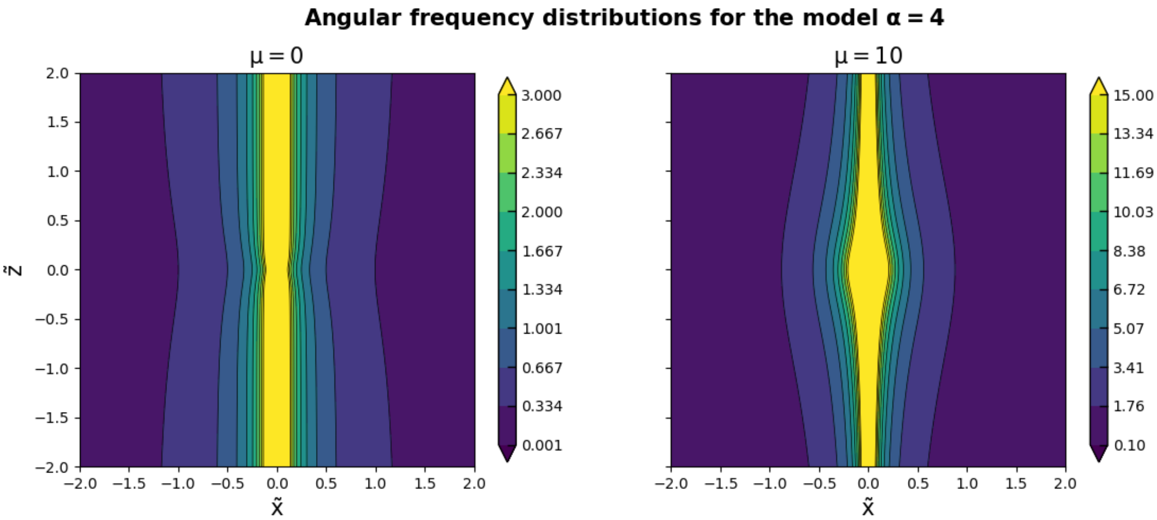

Figure 4.9: Same as figure 4.7, but for $\alpha = 4$.

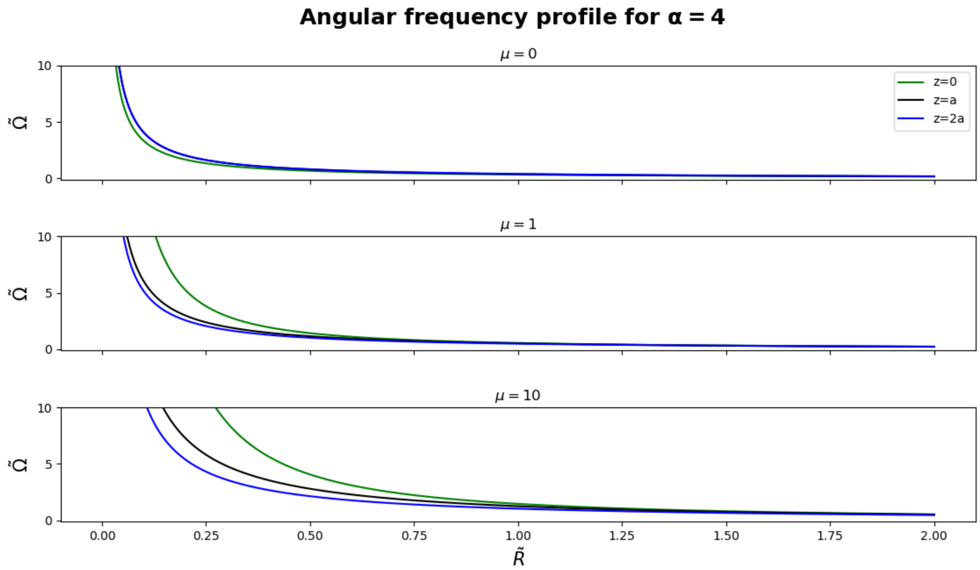

Figure 4.10: Same as figure (4.8), but for $\alpha = 4$.

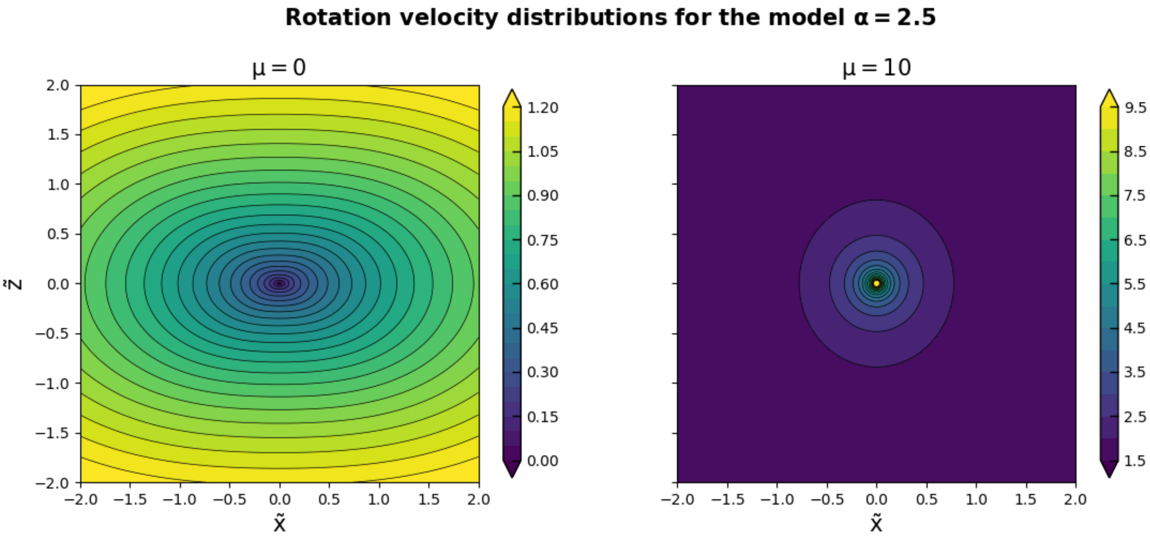

Figure 4.11: Rotation velocity ($\tilde{u}_\varphi = u_\varphi/u_0$) contours in the meridional plane $y = 0$ for tori with density distribution (4.1.1) for $\alpha = 2.5$ ($\tilde{x} = x/a$ and $\tilde{z} = z/a$). On the left we show the self-gravitating model, while in the right panel we present a toroidal distribution in presence of a black hole of mass $M_\bullet = 4\pi\rho_0 a^3 \mu$.

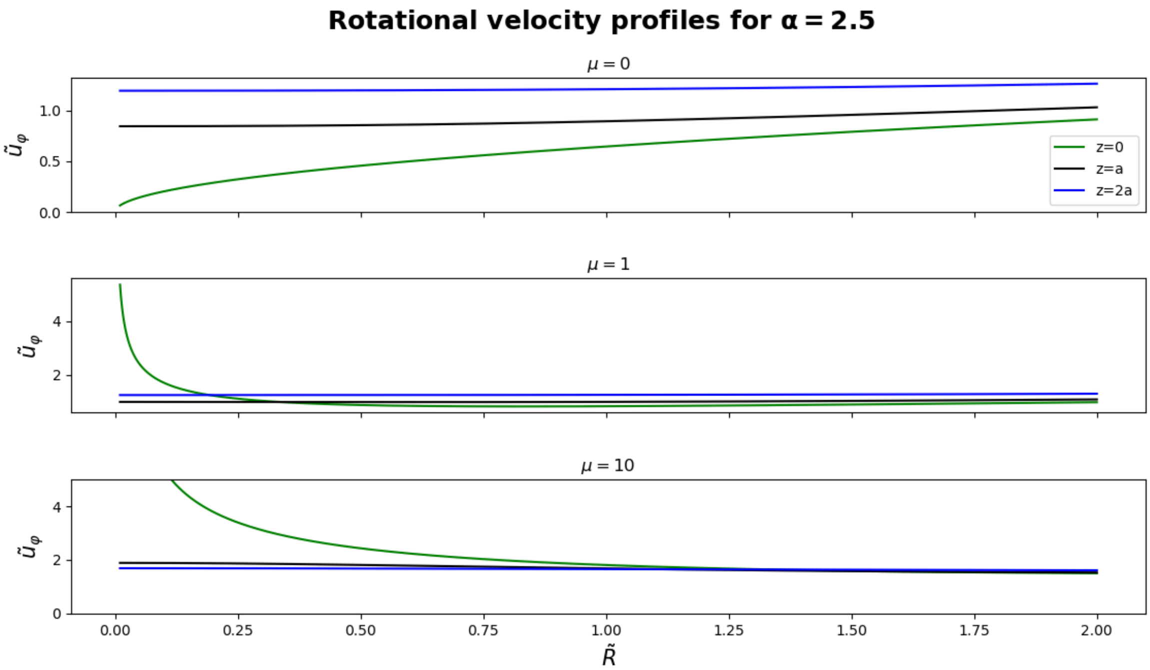

Figure 4.12: Radial rotational velocity profiles for the model $\alpha = 2.5$. On the top panel we show the self-gravitating case for different values of z, while lower panel we show the radial rotation velocity profiles of the gaseous distribution in presence of a black hole with $\mu = 1$ and $\mu = 10$.

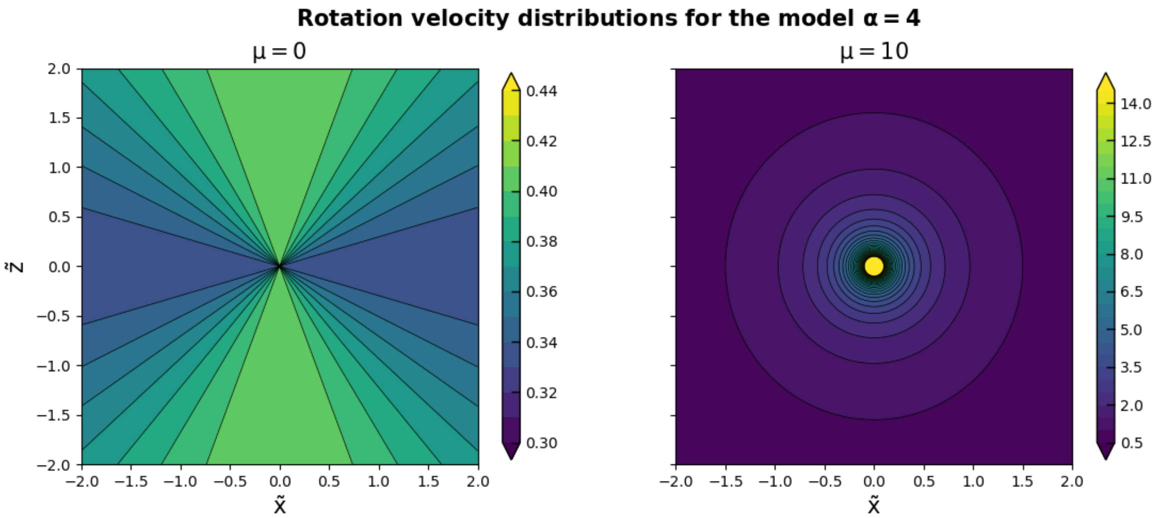

Figure 4.13: Same as figure (4.11), but for $\alpha = 4$.

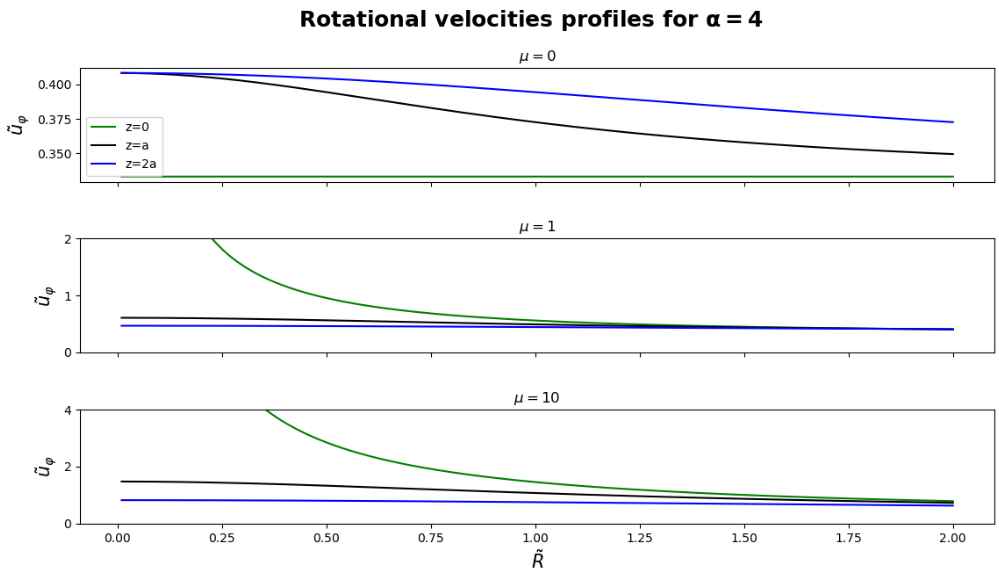

Figure 4.14: Same as figure (4.12), but for $\alpha = 4$.

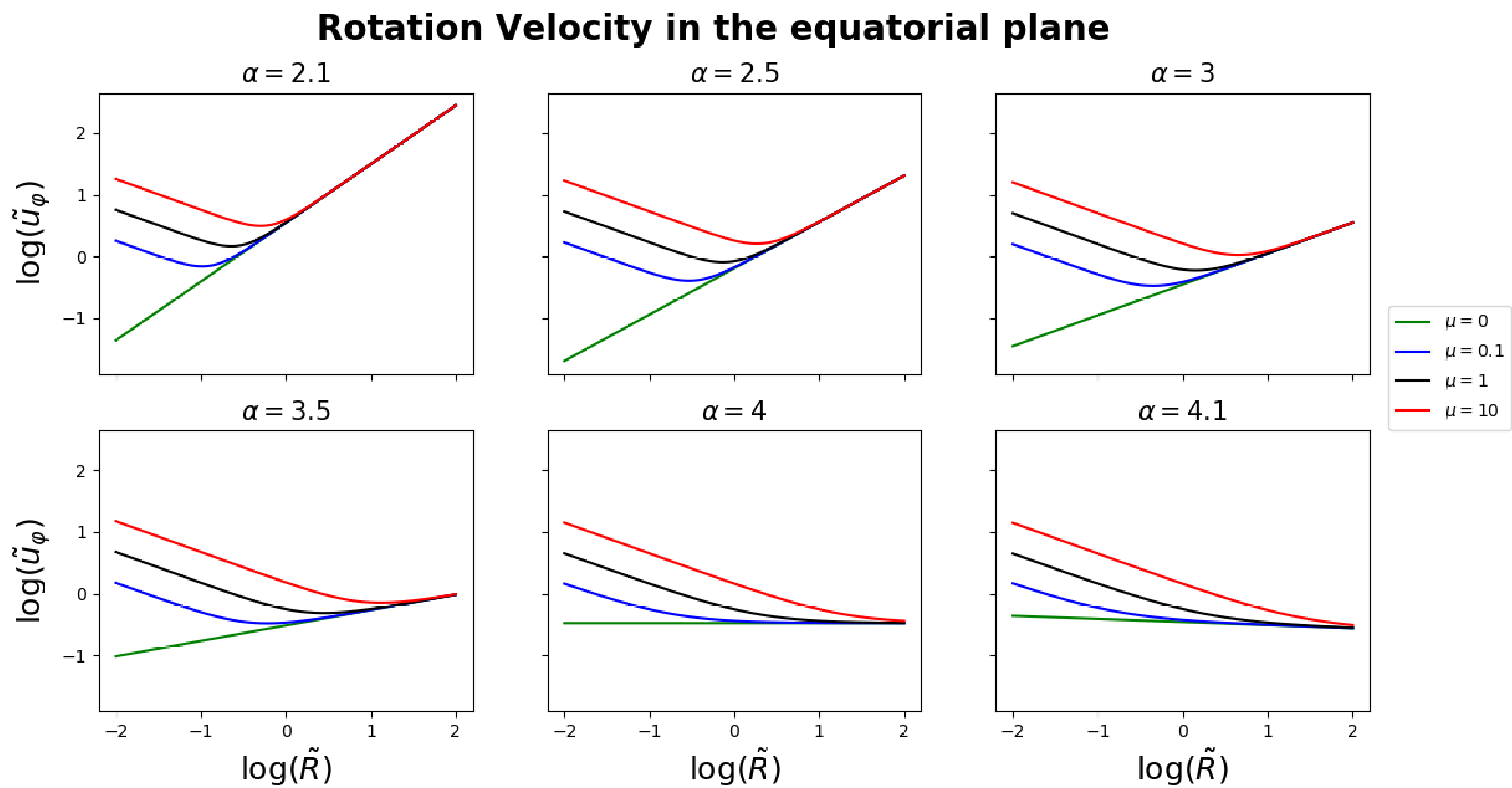

Figure 4.15: Rotation velocity profiles in the equatorial plane for different values of α and in presence of different masses of black hole, $\mu = 0$ is the case with no black hole ($\tilde{R} = R/a$ and $\tilde{u}_\varphi = u_\varphi/u_0$).

4.4.4 The linear stability

Here we make some comment about the Solberg-Hoiland conditions for tori in equilibrium in the gravitational potential given by their self gravity plus a black hole.

Let us consider the physical quantities that compose inequality (4.2.20). We have

$$\begin{aligned} J_z^2 = \Omega^2 R^4 = R^3\left(\frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{\partial \Phi_{tot}}{\partial R}\right) = \left(\frac{R^3}{\rho}\frac{\partial (p_{self} + p_\bullet)}{\partial R} + R^3\frac{\partial(\Phi_{self} + \Phi_\bullet)}{\partial R}\right) = \\ = R^3\left(\frac{1}{\rho}\frac{\partial p_{self}}{\partial R} + \frac{\partial \Phi_{self}}{\partial R}\right) + R^3\left(\frac{1}{\rho}\frac{\partial p_\bullet}{\partial R} + \frac{\partial \Phi_\bullet}{\partial R}\right) = (J_z^2)_{self} + (J_z^2)_\bullet, \end{aligned} \tag{4.4.38}$$

where we have defined

$$(J_z^2)_{self} = R^3\left(\frac{1}{\rho}\frac{\partial p_{self}}{\partial R} + \frac{\partial \Phi_{self}}{\partial R}\right) \tag{4.4.39}$$

and

$$(J_z^2)_\bullet = R^3\left(\frac{1}{\rho}\frac{\partial p_\bullet}{\partial R} + \frac{\partial \Phi_\bullet}{\partial R}\right). \tag{4.4.40}$$

We also have

$$\begin{aligned} \nabla p \cdot \nabla \ln\left(\frac{p}{\rho^\gamma}\right) &= \left(\frac{\partial p_{self}}{\partial R} + \frac{\partial p_\bullet}{\partial R}\right)\left(\frac{\partial \ln(p_{self} + p_\bullet)}{\partial R} - \gamma\frac{\partial \ln\rho}{\partial R}\right) + \\ &+ \left(\frac{\partial p_{self}}{\partial z} + \frac{\partial p_\bullet}{\partial z}\right)\left(\frac{\partial \ln(p_{self} + p_\bullet)}{\partial z} - \gamma\frac{\partial \ln\rho}{\partial z}\right) = \\ &= \frac{\partial p_{self}}{\partial R}\frac{\partial \ln(p_{self} + p_\bullet)}{\partial R} - \gamma\frac{\partial p_{self}}{\partial R}\frac{\partial \ln\rho}{\partial R} + \\ &\quad + \frac{\partial p_\bullet}{\partial R}\frac{\partial \ln(p_{self} + p_\bullet)}{\partial R} - \gamma\frac{\partial p_\bullet}{\partial R}\frac{\partial \ln\rho}{\partial R} + \\ &\quad + \frac{\partial p_{self}}{\partial z}\frac{\partial \ln(p_{self} + p_\bullet)}{\partial z} - \gamma\frac{\partial p_{self}}{\partial z}\frac{\partial \ln\rho}{\partial z} + \\ &\quad + \frac{\partial p_\bullet}{\partial z}\frac{\partial \ln(p_{self} + p_\bullet)}{\partial z} - \gamma\frac{\partial p_\bullet}{\partial z}\frac{\partial \ln\rho}{\partial z}. \end{aligned} \tag{4.4.41}$$

Therefore the extended form of (4.2.20) is

$$\begin{aligned} \frac{1}{R^3}\frac{\partial (J_z^2)_{self}}{\partial R} + \frac{1}{R^3}\frac{\partial (J_z^2)_\bullet}{\partial R} - \frac{1}{\gamma\rho}\Bigg(&\frac{\partial p_{self}}{\partial R}\frac{\partial \ln(p_{self} + p_\bullet)}{\partial R} - \gamma\frac{\partial p_{self}}{\partial R}\frac{\partial \ln\rho}{\partial R} + \\ &+ \frac{\partial p_\bullet}{\partial R}\frac{\partial \ln(p_{self} + p_\bullet)}{\partial R} - \gamma\frac{\partial p_\bullet}{\partial R}\frac{\partial \ln\rho}{\partial R} + \\ &+ \frac{\partial p_{self}}{\partial z}\frac{\partial \ln(p_{self} + p_\bullet)}{\partial z} - \gamma\frac{\partial p_{self}}{\partial z}\frac{\partial \ln\rho}{\partial z} + \\ &+ \frac{\partial p_\bullet}{\partial z}\frac{\partial \ln(p_{self} + p_\bullet)}{\partial z} - \gamma\frac{\partial p_\bullet}{\partial z}\frac{\partial \ln\rho}{\partial z}\Bigg) \geq 0. \end{aligned} \tag{4.4.42}$$

Instead if we consider (4.2.21), we get

$$
\begin{aligned}
-\left[\frac{\partial p_{self}}{\partial z}+\frac{\partial p_\bullet}{\partial z}\right]\times\Bigg[&\frac{\partial (J_z^2)_{self}}{\partial R}\frac{\partial \ln(p_{self}+p_\bullet)}{\partial z}+\frac{\partial (J_z^2)_\bullet}{\partial R}\frac{\partial \ln(p_{self}+p_\bullet)}{\partial z}-\\
&-\frac{\partial (J_z^2)_{self}}{\partial z}\frac{\partial \ln(p_{self}+p_\bullet)}{\partial R}-\frac{\partial (J_z^2)_\bullet}{\partial z}\frac{\partial \ln(p_{self}+p_\bullet)}{\partial R}-\\
&-\gamma\Bigg(\frac{\partial (J_z^2)_{self}}{\partial R}\frac{\partial \ln\rho}{\partial z}+\frac{\partial (J_z^2)_\bullet}{\partial R}\frac{\partial \ln\rho}{\partial z}-\\
&\qquad-\frac{\partial (J_z^2)_{self}}{\partial z}\frac{\partial \ln\rho}{\partial R}-\frac{\partial (J_z^2)_\bullet}{\partial z}\frac{\partial \ln\rho}{\partial R}\Bigg)\Bigg]\geq 0.
\end{aligned}
\tag{4.4.43}
$$

Here we note that from (4.4.42) and (4.4.43), in general, we do not expect the same solutions that we obtained in the self-gravitating case.

In the case[2] $\alpha \neq 4$, for the assigned potential (see 4.4.5) and pressure (see 4.4.7), we have

$$
\begin{aligned}
J_z^2 = &\frac{8\pi G\rho_0}{a^{2-\alpha}(7-\alpha)}(R^2+z^2)^{-\alpha/2}\left[\frac{2R^2(R^2+z^2)^2}{(\alpha-2)^2(5-\alpha)}-\frac{R^4(R^2+z^2)}{(\alpha-2)(\alpha-1)}\right]+\\
&+\frac{2GM_\bullet R^2}{(\alpha+1)(R^2+z^2)^{1/2}}.
\end{aligned}
\tag{4.4.44}
$$

Therefore (4.4.38) can be written as

$$
\begin{aligned}
\frac{1}{R^3}\frac{\partial J_z^2}{\partial R} =&\frac{8\pi G\rho_0}{a^{2-\alpha}(7-\alpha)}(R^2+z^2)^{-\alpha/2}\left[\frac{4(R^2+z^2)^2}{R^2(\alpha-2)(5-\alpha)}+\frac{2(4-\alpha)(R^2+z^2)}{(\alpha-2)(5-\alpha)}-\right.\\
&\left.-\frac{4(R^2+z^2)}{(\alpha-1)(\alpha-2)}+\frac{2R^2}{\alpha-1}\right]+\\
&+\frac{2GM_\bullet}{R^2(R^2+z^2)(\alpha-1)}\left(2-\frac{R^2}{R^2+z^2}\right).
\end{aligned}
\tag{4.4.45}
$$

Then we have

$$
-\frac{1}{\gamma\rho} = -\frac{(R^2+z^2)^{\alpha/2}}{\gamma(\rho_0 a^{\alpha-2}R^2)}
\tag{4.4.46}
$$

[2]Similar equations can be built for the model $\alpha = 4$.

and equation (4.4.41) can be written as

$$\nabla p\cdot\nabla\ln(p/\rho^\gamma)=\frac{\partial p}{\partial R}\frac{\partial\ln(p/\rho^\gamma)}{\partial R}+\frac{\partial p}{\partial z}\frac{\partial\ln(p/\rho^\gamma)}{\partial z}$$
$$=\Bigg\{\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}(R^2+z^2)^{-\alpha}\Bigg[\frac{4R(R^2+z^2)+4R^3(2-\alpha)(R^2+z^2)}{(\alpha-2)^2(5-\alpha)}+$$
$$+\frac{2R^3(R^2+z^2)+R^5(1-\alpha)}{\alpha-1}\Bigg]+$$
$$+\frac{GM_\bullet\rho_0 R(R^2+z^2)^{-\alpha/2}}{(\alpha+1)a^{2-\alpha}}\Bigg[\frac{2}{(R^2+z^2)^{1/2}}+\frac{R^2(1-\alpha)}{(R^2+z^2)^{3/2}}\Bigg]\Bigg\}\times$$
$$\times\Bigg\{\frac{\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}(R^2+z^2)^{-\alpha}\Big[\frac{4R(R^2+z^2)+4R^3(2-\alpha)(R^2+z^2)}{(\alpha-2)^2(5-\alpha)}+\frac{2R^3(R^2+z^2)+R^5(1-\alpha)}{\alpha-1}\Big]}{\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}\frac{R^2(R^2+z^2)^{1-\alpha}}{7-\alpha}\Big[\frac{2(R^2+z^2)}{(\alpha-2)^2(5-\alpha)}+\frac{R^2}{2(\alpha-1)}\Big]+\frac{GM_\bullet\rho_0}{(\alpha+1)a^{2-\alpha}}\frac{R^2}{(R^2+z^2)^{\frac{\alpha+1}{2}}}}+$$
$$+\frac{\frac{GM_\bullet\rho_0R(R^2+z^2)^{-\alpha/2}}{(\alpha+1)a^{2-\alpha}}\Big[\frac{2}{(R^2+z^2)^{1/2}}+\frac{R^2(1-\alpha)}{(R^2+z^2)^{3/2}}\Big]}{\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}\frac{R^2(R^2+z^2)^{1-\alpha}}{7-\alpha}\Big[\frac{2(R^2+z^2)}{(\alpha-2)^2(5-\alpha)}+\frac{R^2}{2(\alpha-1)}\Big]+\frac{GM_\bullet\rho_0}{(\alpha+1)a^{2-\alpha}}\frac{R^2}{(R^2+z^2)^{\frac{\alpha+1}{2}}}}-\gamma\Bigg[\frac{2}{R}-\frac{\alpha R}{R^2+z^2}\Bigg]\Bigg\}+$$
$$+\Bigg\{z\Bigg[\frac{4\pi G\rho_0^2R^2}{a^{2(2-\alpha)}}(R^2+z^2)^{-\alpha}\Bigg[-\frac{4(R^2+z^2)}{(\alpha-2)(5-\alpha)}-R^2\Bigg]-\frac{GM_\bullet\rho_0R^2}{a^{2-\alpha}(R^2+z^2)^{\frac{\alpha+3}{2}}}\Bigg]\Bigg\}\times$$
$$\times\Bigg\{z\Bigg[\frac{\frac{4\pi G\rho_0^2R^2}{a^{2(2-\alpha)}}(R^2+z^2)^{-\alpha}\Big[-\frac{4(R^2+z^2)}{(\alpha-2)(5-\alpha)}-R^2\Big]-\frac{GM_\bullet\rho_0R^2}{a^{2-\alpha}(R^2+z^2)^{\frac{\alpha+3}{2}}}}{\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}\frac{R^2(R^2+z^2)^{1-\alpha}}{7-\alpha}\Big[\frac{2(R^2+z^2)}{(\alpha-2)^2(5-\alpha)}+\frac{R^2}{2(\alpha-1)}\Big]+\frac{GM_\bullet\rho_0}{(\alpha+1)a^{2-\alpha}}\frac{R^2}{(R^2+z^2)^{\frac{\alpha+1}{2}}}}-\gamma\frac{\alpha}{R^2+z^2}\Bigg]\Bigg\}. \tag{4.4.47}$$

In order to simplify the notation we can address with $A(R,z)$ the r.h.s. of equation (4.4.45), with $B(R,z)$ the r.h.s. of equation (4.4.46) and finally with $C(R,z)$ the r.h.s. of equation (4.4.47).

Hence we can formulate Solberg-Hoiland condition as

$$A(R,z)+B(R,z)C(R,z)\geq 0. \tag{4.4.48}$$

Taking into account (4.2.21) we have

$$-\frac{\partial p}{\partial z}=z\Bigg[\frac{4\pi G\rho_0^2R^2}{a^{2(2-\alpha)}}(R^2+z^2)^{-\alpha}\Bigg[\frac{4(R^2+z^2)}{(\alpha-2)(5-\alpha)}+R^2\Bigg]+\frac{GM_\bullet\rho_0R^2}{a^{2-\alpha}(R^2+z^2)^{\frac{\alpha+3}{2}}}\Bigg], \tag{4.4.49}$$

and the partial derivatives of eq. (4.4.44) reads

$$\frac{\partial J_z^2}{\partial R}=\frac{8\pi G\rho_0}{a^{2-\alpha}(7-\alpha)}R(R^2+z^2)^{-\alpha/2}\Bigg[2(R^2+z^2)\frac{2(R^2+z^2)+R^3(4-\alpha)}{(\alpha-2)^2(5-\alpha)}-$$
$$-R^2\frac{4(R^2+z^2)-R^3(2-\alpha)}{(\alpha-2)(\alpha-1)}\Bigg]+ \tag{4.4.50}$$
$$+\frac{2GM_\bullet R}{(\alpha+1)(R^2+z^2)^{1/2}}\Bigg(2-\frac{R^2}{R^2+z^2}\Bigg)$$

and

$$\frac{\partial J_z^2}{\partial z} = \frac{8\pi G\rho_0}{a^{2-\alpha}(7-\alpha)} R^2 z (R^2+z^2)^{-\alpha/2}\left[\frac{2(4-\alpha)(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} - \frac{R^4(2-\alpha)}{(\alpha-2)(\alpha-1)}\right] - \\ - \frac{2GM_\bullet R^2 z}{(\alpha+1)(R^2+z^2)^{3/2}}. \tag{4.4.51}$$

Moreover we have

$$\frac{\partial \ln(p/\rho^\gamma)}{\partial R} = \frac{\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}(R^2+z^2)^{-\alpha}\left[\frac{4R(R^2+z^2)+4R^3(2-\alpha)(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} + \frac{2R^3(R^2+z^2)+R^5(1-\alpha)}{\alpha-1}\right]}{\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}\frac{R^2(R^2+z^2)^{1-\alpha}}{7-\alpha}\left[\frac{2(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} + \frac{R^2}{2(\alpha-1)}\right] + \frac{GM_\bullet\rho_0}{(\alpha+1)a^{2-\alpha}}\frac{R^2}{(R^2+z^2)^{\frac{\alpha+1}{2}}}} + \\ + \frac{\frac{GM_\bullet\rho_0 R(R^2+z^2)^{-\alpha/2}}{(\alpha+1)a^{2-\alpha}}\left[\frac{2}{(R^2+z^2)^{1/2}} + \frac{R^2(1-\alpha)}{(R^2+z^2)^{3/2}}\right]}{\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}\frac{R^2(R^2+z^2)^{1-\alpha}}{7-\alpha}\left[\frac{2(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} + \frac{R^2}{2(\alpha-1)}\right] + \frac{GM_\bullet\rho_0}{(\alpha+1)a^{2-\alpha}}\frac{R^2}{(R^2+z^2)^{\frac{\alpha+1}{2}}}} - \\ - \gamma\left[\frac{2}{R} - \frac{\alpha R}{R^2+z^2}\right], \tag{4.4.52}$$

and

$$\frac{\partial \ln(p/\rho^\gamma)}{\partial z} = z\left[\frac{\frac{4\pi G\rho_0^2 R^2}{a^{2(2-\alpha)}}(R^2+z^2)^{-\alpha}\left[-\frac{4(R^2+z^2)}{(\alpha-2)(5-\alpha)} - R^2\right] - \frac{GM_\bullet\rho_0 R^2}{a^{2-\alpha}(R^2+z^2)^{\frac{\alpha+3}{2}}}}{\frac{4\pi G\rho_0^2}{a^{2(2-\alpha)}}\frac{R^2(R^2+z^2)^{1-\alpha}}{7-\alpha}\left[\frac{2(R^2+z^2)}{(\alpha-2)^2(5-\alpha)} + \frac{R^2}{2(\alpha-1)}\right] + \frac{GM_\bullet\rho_0}{(\alpha+1)a^{2-\alpha}}\frac{R^2}{(R^2+z^2)^{\frac{\alpha+1}{2}}}} - \right. \\ \left. - \gamma\frac{\alpha}{R^2+z^2}\right]. \tag{4.4.53}$$

The inequalities obtained in this section are cumbersome even when we substitute given values of α, so the stability analysis with black hole is beyond the purpose of the thesis.

4.5 Baroclinic models in an external keplerian potential

We show here how the method described in §3.4.2 can be used to build baroclinic models like those presented in this chapter. For simplicity we assume that the potential is external (we neglect self-gravity of the gas). We start from a barotropic configuration with cylindrical velocity $u_{\varphi,0}(R) \equiv u_\varphi(R, z_0) = 0$ and a black hole potential

$$\Phi(R, z) = -\frac{GM_\bullet}{(R^2+z^2)^{1/2}}. \tag{4.5.1}$$

In practice we build baroclinic models with

$$\rho(R,z) = f(R)\rho_h(\Phi), \tag{4.5.2}$$

where $\rho_h(\Phi)$ is a hydrostatic solution.

Under these assumptions the effective pressure is given by

$$p_h = \int_z^{z_0} GM_\bullet \rho_h \frac{z'}{(R^2+z'^2)^{\frac{3}{2}}}\, dz', \tag{4.5.3}$$

and we have

$$p(R,z) = f(R)\int_z^{z_0} GM_\bullet \rho_h \frac{z'}{(R^2+z'^2)^{\frac{3}{2}}}\, dz'. \tag{4.5.4}$$

We assume

$$f(R) = R^2 \tag{4.5.5}$$

and

$$\rho_h = \frac{GM_\bullet}{(R^2+z^2)^\kappa}, \tag{4.5.6}$$

in order to get

$$\rho(R,z) = f(R)\rho_h = \frac{GM_\bullet R^2}{(R^2+z^2)^\kappa}, \qquad \kappa \neq -1. \tag{4.5.7}$$

Assuming $z_0 = \infty$,

$$p(R,z) = \frac{(GM_\bullet)^2 R^2}{(2\kappa+1)(R^2+z^2)^{\frac{1}{2}+\kappa}}. \tag{4.5.8}$$

Hence if we suppose that $u^2_{\varphi,0}(R) = 0$ from (3.4.11) we get

$$u^2_\varphi(R,z) = \frac{2GM_\bullet}{(2\kappa+1)(R^2+z^2)^{\frac{1}{2}}} \tag{4.5.9}$$

As an example we demonstrate that if we choose

$$\kappa = \alpha/2, \tag{4.5.10}$$

we get

$$p(R,z) = p_\bullet. \tag{4.5.11}$$

Proof. We start from

$$p_\bullet = \frac{GM_\bullet \rho_0}{(\alpha+1)a^{2-\alpha}} \frac{R^2}{(R^2+z^2)^{\frac{1}{2}(\alpha+1)}}, \tag{4.5.12}$$

where by definition

$$\begin{aligned} \rho_0 &= \rho a^{2-\alpha}\frac{(R^2+z^2)^{\alpha/2}}{R^2} = \\ &= \frac{GM_\bullet R^2}{(R^2+z^2)^{\alpha/2}} a^{2-\alpha}\frac{(R^2+z^2)^{\alpha/2}}{R^2} = \\ &= GM_\bullet a^{2-\alpha}. \end{aligned} \tag{4.5.13}$$

Then we have

$$\begin{aligned} p_\bullet &= \frac{GM_\bullet \rho_0}{(\alpha+1)a^{2-\alpha}} \frac{R^2}{(R^2+z^2)^{\frac{1}{2}(\alpha+1)}} = \\ &= \frac{GM_\bullet GM_\bullet a^{2-\alpha}}{(\alpha+1)a^{2-\alpha}} \frac{R^2}{(R^2+z^2)^{\frac{1}{2}(\alpha+1)}} = \\ &= p(R,z). \end{aligned} \tag{4.5.14}$$

□

If we consider the rotation velocity field, equation (4.5.9) (when $\kappa = \alpha/2$) is formally identical to (4.4.16) in the case of a negligible self-gravitating tori term.

4.6 An example in physical units

In this section we present an example of the considered power-law tori in physical units. With the idea of representing toroidal structures of AGN, we choose values of the physical parameter inspired by the observational data presented in [Combes et al. 2019], who observed molecular tori AGN (Seyfert/LINER galaxies) with the Atacama Large Millimeter/submillimeter Array (ALMA).

In detail we assume:

$$M_\bullet = 10^{7.6} M_\odot \sim 7.9 \times 10^{40}\, g, \tag{4.6.1}$$

$$a = 19.7\, pc \sim 6.09 \times 10^{19}\, cm \tag{4.6.2}$$

The Schwarzschild radius of such black hole is

$$R_S = 3.79 \times 10^{-6}\, pc, \tag{4.6.3}$$

so we do not have to use relativistic corrections.

From the definition of μ, we can get the value of ρ_0:

$$\rho_0 \equiv \frac{M_\bullet}{4\pi\mu a^3}. \tag{4.6.4}$$

For

- $\mu = 1$ we get

$$\rho_0 = 2.78 \times 10^{-20}\, \frac{g}{cm^3}, \tag{4.6.5}$$

- $\mu = 10$ we get

$$\rho_0 = 2.78 \times 10^{-21}\, \frac{g}{cm^3} \tag{4.6.6}$$

Therefore from equation (4.1.5) we have that

- for $\mu = 1$

$$\begin{aligned}\rho = \rho_0 \tilde{\rho} &= 2.78 \times 10^{-20} \frac{\tilde{R}^2}{\tilde{r}^\alpha} \frac{g}{cm^3} \\ &= 412.16 \frac{\tilde{R}^2}{\tilde{r}^\alpha} \frac{M_\odot}{pc^3}\end{aligned} \tag{4.6.7}$$

- for $\mu = 10$

$$\begin{aligned}\rho = \rho_0 \tilde{\rho} &= 2.78 \times 10^{-21} \frac{\tilde{R}^2}{\tilde{r}^\alpha} \frac{g}{cm^3} = \\ &= 41.22 \frac{\tilde{R}^2}{\tilde{r}^\alpha} \frac{M_\odot}{pc^3}\end{aligned} \tag{4.6.8}$$

In figures (4.16) and (4.17) we show the density distribution for power-law index $\alpha = 3.5$ in cases $\mu = 1$ and for $\mu = 10$.

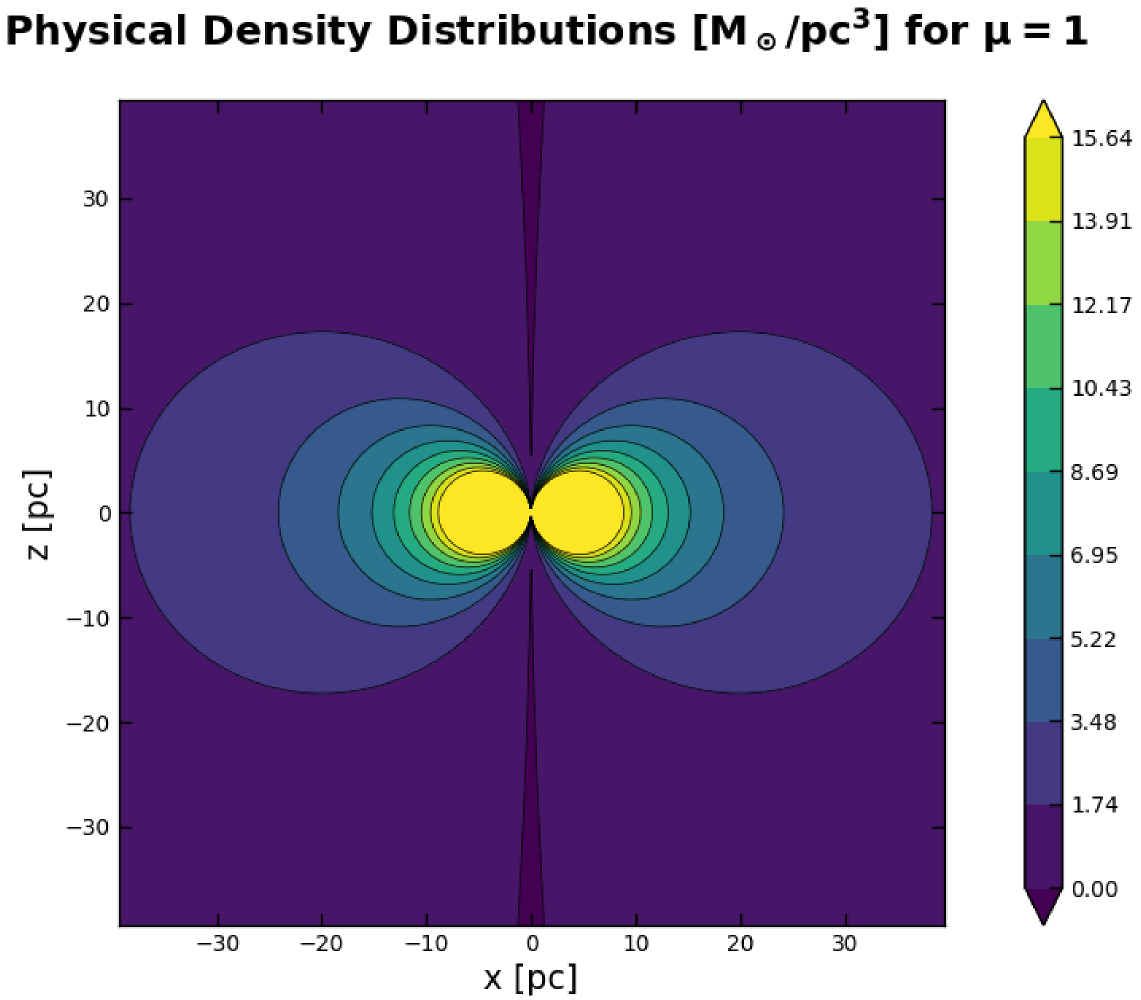

Figure 4.16: Density distribution contours in the meridional plane $y = 0$ for the torus model described in §4.6 with $\mu = 1$. The density distribution is in $M_\odot/pc^3$.

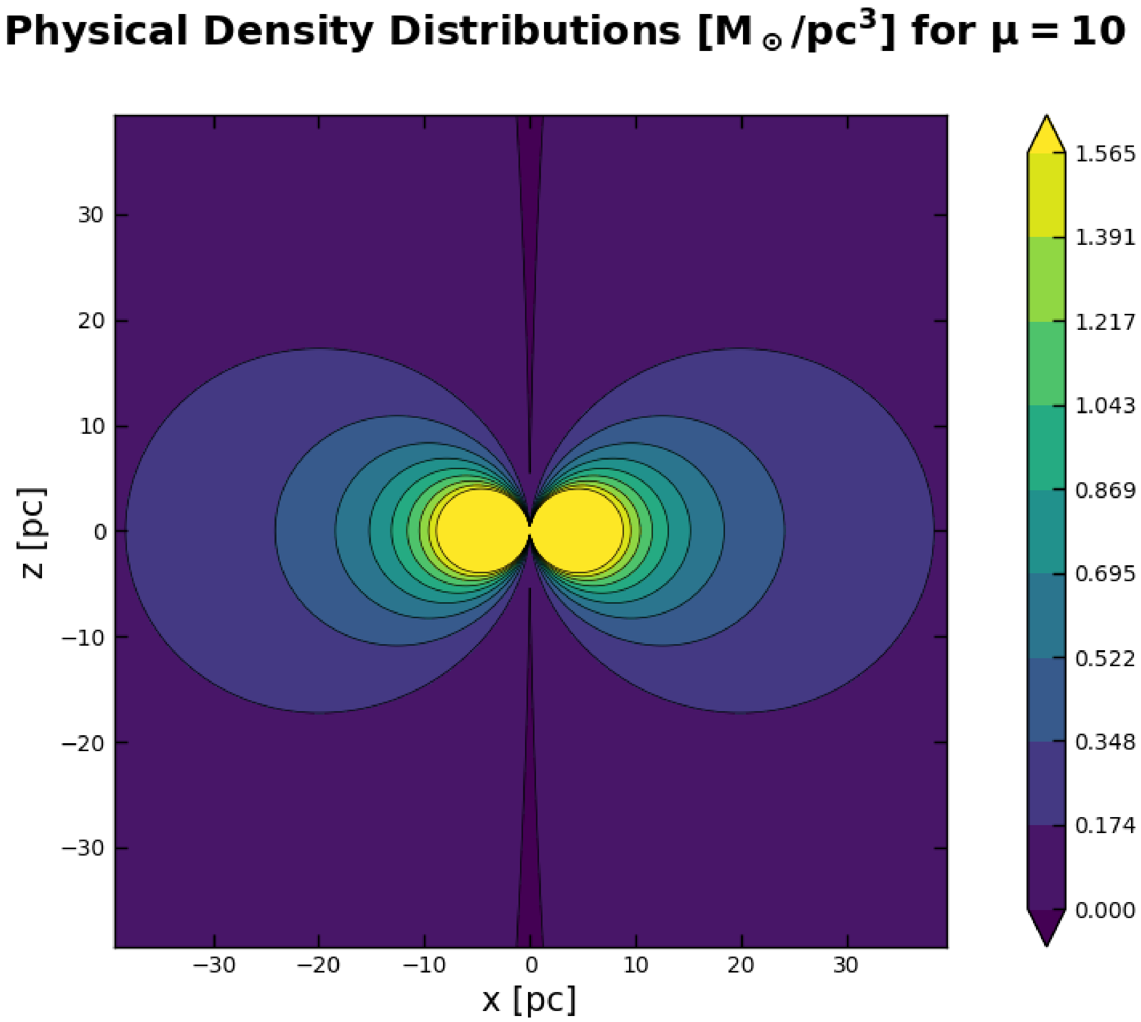

Figure 4.17: Same as fig. (4.16), but for $\mu = 10$

We can write the rotation velocity field of this model as

$$u_\varphi = u_0 \tilde{u}_\varphi = \sqrt{8\pi G \rho_0 a^2} \tilde{u}_\varphi \tag{4.6.9}$$

and we have

- for $\mu = 1$

$$u_\varphi = 131 \sqrt{\frac{4(\tilde{R}^2 + \tilde{z}^2)^{-3/4}}{21} \left[\frac{8(\tilde{R}^2 + \tilde{z}^2)}{9} - \frac{2\tilde{R}^2}{5} \right] + \frac{2}{9(\tilde{R}^2 + \tilde{z}^2)^{1/2}}} \frac{km}{s} \tag{4.6.10}$$

- for $\mu = 10$

$$u_\varphi = 41.6 \sqrt{\frac{4(\tilde{R}^2 + \tilde{z}^2)^{-3/4}}{21} \left[\frac{8(\tilde{R}^2 + \tilde{z}^2)}{9} - \frac{2\tilde{R}^2}{5} \right] + \frac{20}{9(\tilde{R}^2 + \tilde{z}^2)^{1/2}}} \frac{km}{s} \tag{4.6.11}$$

In figure (4.18) we show the distribution of the rotation velocity field (expressed in km/s) for $\mu = 1$ and $\mu = 10$ for the toroidal density distribution described in this section.

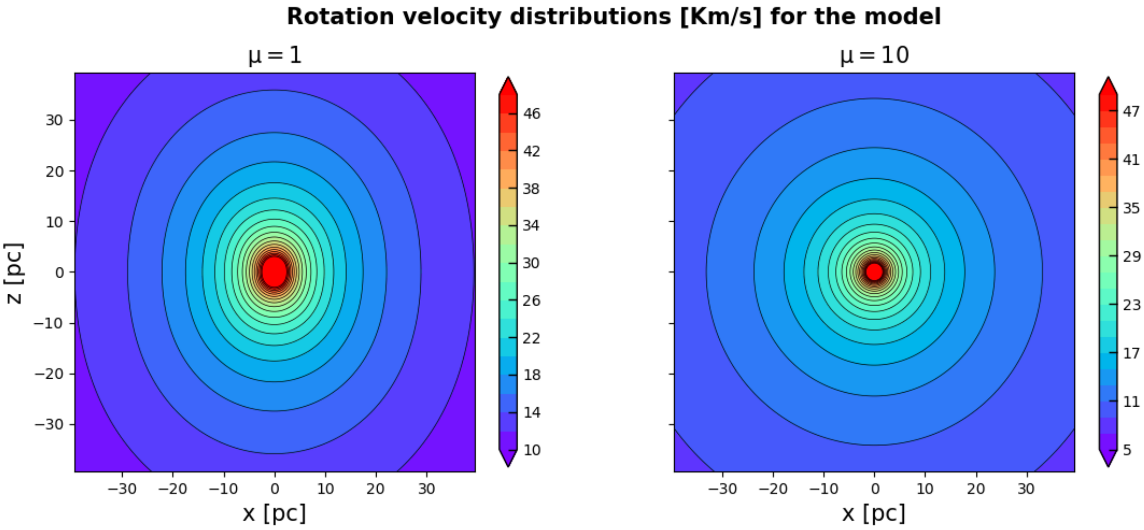

Figure 4.18: Rotation velocity contours in the meridional plane $y = 0$ for the torus model described in §4.6 with $\mu = 1$ and $\mu = 10$. The rotation velocity distribution is in km/s.

The pressure of the model $\alpha = 3.5$ can be written as

$$\begin{aligned} p(R, z) =& \frac{8\pi G\rho_0 a^3}{7} R^2 (R^2 + z^2)^{-5/2} \left[\frac{16}{27}(R^2 + z^2) + \frac{R^2}{5} \right] + \\ &+ \frac{2GM_\bullet \rho_0 a^{3/2}}{9} \frac{R^2}{(R^2 + z^2)^{9/4}}. \end{aligned} \tag{4.6.12}$$

We note that

$$\lim_{R,\, z \to \infty} p(R, z) = 0. \tag{4.6.13}$$

Following [Nenkova et al. 2004] we assume that tori are clumpy structures, made of a distribution of clouds characterized by their velocity dispersion σ. For this reason we can define the velocity dispersion from the relation

$$p = \rho\sigma^2. \tag{4.6.14}$$

And we have

$$\sigma = 2a\sqrt{\pi G\rho_0}\sqrt{\frac{\tilde{p}}{\tilde{\rho}}}. \tag{4.6.15}$$

Using physical units we get

- for $\mu = 1$

$$\sigma = 93\sqrt{\frac{\tilde{p}}{\tilde{\rho}}} \frac{km}{s} \tag{4.6.16}$$

• for $\mu = 10$

$$\sigma = 29\sqrt{\frac{\tilde{p}}{\tilde{\rho}}}\,\frac{km}{s} \tag{4.6.17}$$

In figures (4.19) and (4.20), we show the distribution of the dispersion velocities (expressed in km/s) for $\mu = 1$ and for $\mu = 10$, for a toroidal distribution of density (4.6.7) and (4.6.8) with $\alpha = 3.5$ respectively.

In both case we observe that the dispersion velocity diverges as one takes the limit for $r \to 0$ and $r \to \infty$. This suggest that in the nearby of the black hole the system is fully governed by the black hole contribution.

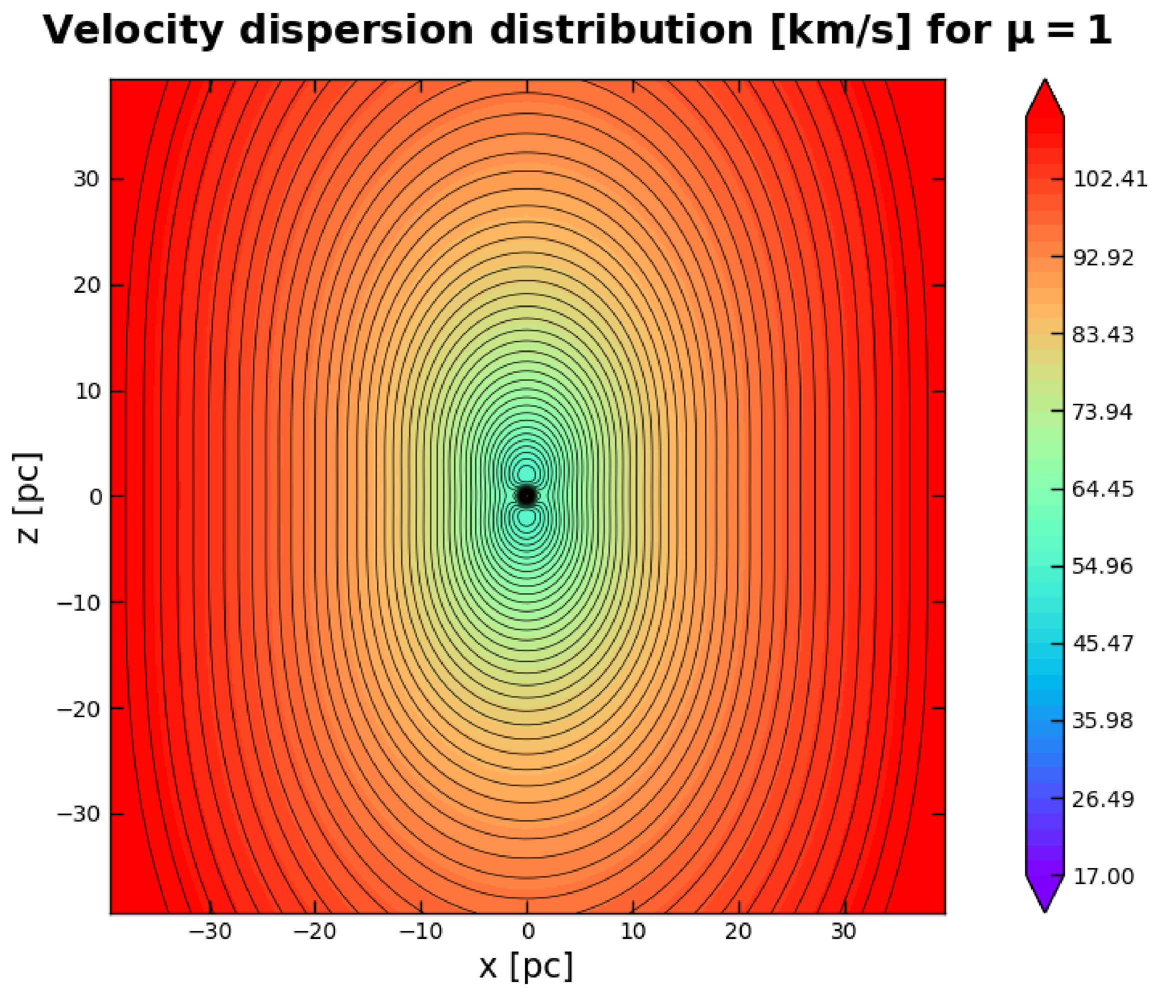

Figure 4.19: Velocity dispersion contours in the meridional plane $y = 0$ for the torus model described in §4.6 with $\mu = 1$. The velocity dispersion is in km/s.

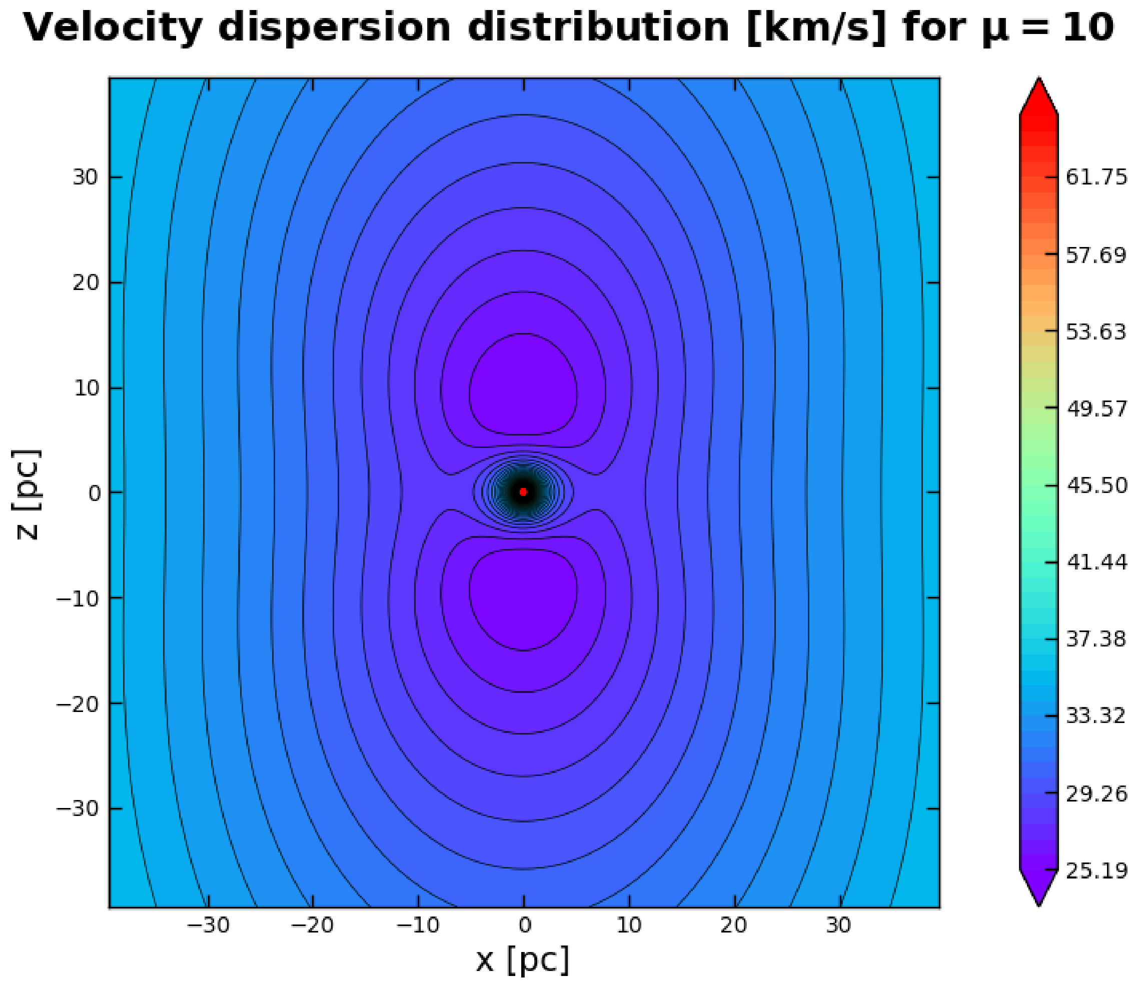

Figure 4.20: Same as fig. 4.19, but for $\mu = 10$.

Conclusions

In this thesis I have considered models of rotating astrophysical gaseous systems in equilibrium in a gravitational potential. In particular I focused on those baroclinic models that can be described by a perfect gas equation of state in axisymmetric conditions. I have derived the linear instability criterion that is named after Solberg and Hoiland, and I have verified a commutator-like relation that can be used in baroclinic distributions to state if the rotational velocity field is physically acceptable.

I have taken into account rotating toroidal gaseous systems. I focused on a family of tori that can be described by means of a power-law density distribution for which the corresponding gravitational potential is analytic. This approach leads to a density-potential pair that can be studied by means of the fluid dynamics equations.

Two cases have been analyzed in this thesis: self-gravitating power-law tori and power-law tori in equilibrium in the presence of their self-gravity and a central black hole. In particular for both the families of tori studied, the rotation velocity field has been discussed in order to discover the physically acceptable models.

I found out that both for self-gravitating tori and the case with the additional potential of a central black hole, models with power-law index α in the range $2 < \alpha < 5$ are physically acceptable.

Once the physical models have been found, the conditions for their linear stability have been studied by applying the Solberg-Hoiland criterion. Analytic results have been found for the case when no black hole is present and they allow to select power-law tori that can be in a stable equilibrium configuration. The distributions that are everywhere linearly stable are those one for which the power-law index α is small. I have carried out the full linear-stability analysis for a few specific values of α. In details, I have proved that for $\alpha = 2.1$ and $\alpha = 2.5$ the toroidal distribution is everywhere linearly stable, while for models $\alpha = 3$, $\alpha = 3.5$, $\alpha = 4$ and $\alpha = 4.1$ the region closest to the vertical axis is unstable.

Then I have found the analytical expression for the Solberg-Hoiland criterion for power-law tori in equilibrium in the presence of their self-gravity and a central black hole. The inequalities describing the stability criterion in this case are cumbersome even for fixed power-law index α. I defer a numerical study of these inequalities to future work.

The models presented in this work can be seen as idealized models of toroidal structures in AGNs. I have presented an example of these models in the specific case of a black hole with mass $\sim 10^7\, M_\odot$.

Appendix A

Axisymmetric instability in rotating galactic coronae

In this appendix we are going to show a different approach that can be used to obtain the instability criteria for an axisymmetric rotating galactic corona. We consider the general case of rotating fluids in the presence of cooling and thermal conduction.

We follow the treatment of [Nipoti 2010].

A.1 Remark: governing equations

Let us take into account a stratified, rotating, unmagnetized atmosphere in the presence of cooling and thermal conduction. This system is governed by the following equations for mass, momentum and energy conservations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{A.1.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{u} = -\frac{\nabla p}{\rho} - \nabla \Phi, \tag{A.1.2}$$

$$\frac{p}{\gamma - 1}\left[\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right] \ln(p\rho^{-\gamma}) = \nabla \cdot (\kappa T^{5/2} \nabla T) - \left(\frac{\rho}{\mu' m_p}\right)^2 \Lambda(T). \tag{A.1.3}$$

where $\rho, p, T, \mathbf{u}$ are the gas density, pressure, temperature and velocity respectively. Φ is the gravitational potential, γ is the ratio of the principal specific heats, Λ is the cooling function, μ' is the mean gas particle mass in units of the proton mass m_p and κ is the thermal conductivity. As we neglect magnetic fields, we treat the thermal conductivity as a scalar.

In cylindrical coordinates, if we assume an axisymmetric gravitational potential, the hydrodynamic equations can be written as:

$$\left.\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial R \rho u_R}{\partial R} + \frac{\partial \rho u_z}{\partial z} + \frac{1}{R}\frac{\partial \rho u_\Phi}{\partial \Phi} = 0\right\} \quad \text{Mass conservation,} \tag{A.1.4}$$

$$
\left.\begin{aligned}
&\frac{\partial u_R}{\partial t} + u_R\frac{\partial u_R}{\partial R} + u_z\frac{\partial u_R}{\partial z} + \frac{u_\varphi}{R}\frac{\partial u_R}{\partial \varphi} = -\frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{\partial \Phi}{\partial R} + \frac{u_\Phi^2}{R}\\
&\frac{\partial u_z}{\partial t} + u_R\frac{\partial u_z}{\partial R} + u_z\frac{\partial u_z}{\partial z} + \frac{u_\varphi}{R}\frac{\partial u_z}{\partial \varphi} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\partial \Phi}{\partial z}\\
&\frac{\partial u_\varphi}{\partial t} + u_R\frac{\partial u_\varphi}{\partial R} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{u_\varphi}{R}\frac{\partial u_\varphi}{\partial \varphi} = -\frac{1}{\rho R}\frac{\partial p}{\partial \varphi} - \frac{u_R u_\Phi}{R}
\end{aligned}\right\}\ \text{Momentum conservation,} \tag{A.1.5}
$$

$$
\left.\begin{aligned}
&\frac{p}{\gamma-1}\left[\frac{\partial}{\partial t} + u_R\frac{\partial}{\partial R} + u_R\frac{\partial}{\partial z} + \frac{u_\varphi}{R}\frac{\partial}{\partial \varphi}\right]\ln(p\rho^{-\gamma})\\
&\quad = \frac{1}{R}\frac{\partial}{\partial R}\left(R\kappa T^{5/2}\frac{\partial T}{\partial R}\right) + \frac{\partial}{\partial z}\left(\kappa T^{5/2}\frac{\partial T}{\partial z}\right) +\\
&\quad + \frac{1}{R}\frac{\partial}{\partial \varphi}\left(\frac{\kappa T^{5/2}}{R}\frac{\partial T}{\partial \varphi}\right) - \left(\frac{\rho}{\mu m_p}\right)^2\Lambda(T)
\end{aligned}\right\}\ \text{Energy conservation.} \tag{A.1.6}
$$

The unperturbed atmosphere is assumed to be axisymmetric and close to thermal and hydrostatic equilibrium, so that the system is approximately in a steady state over the time-scales of interest even in the presence of radiative cooling and thermal conduction. For this reason we can describe the unperturbed fluid by the time-independent axisymmetric parameters of density ρ_0, temperature T_0, pressure p_0 and velocity field $\mathbf{u_0}$ which satisfy equations (A.1.4),(A.1.5) and (A.1.6).

In general, the atmosphere is assumed to be differentially rotating with angular velocity $\Omega \equiv u_{0\varphi}/R$. But, in this case it is convenient to distinguish two cases:

- $\partial\Omega/\partial z = 0$,
- $\partial\Omega/\partial z \neq 0$.

In fact the Poincaré-Wavre theorem (Section 3.1) states that the surfaces of constant pressure and constant density coincide if and only if $\partial\Omega/\partial z = 0$: as a consequence, distributions with $\partial\Omega/\partial z = 0$ are said *barotropic*, and distributions with $\partial\Omega/\partial z \neq 0$ are said *barocline*. In the first case pressure is a function of only density while in the second case pressure is not a function of only density.

A.2 Axisimmetric perturbations

A.2.1 Fourier analysis and the dispersion relation

In this section we assume that the fluid rotates differentially with $\Omega = \Omega(R, z)$ and that the perturbations are axisymmetric. Linearizing the hydrodynamic equations (A.1.4),(A.1.5),(A.1.6) with perturbations of the form:

$$
F_0 + F\exp(-i\omega t + ik_R R + ik_z z), \qquad |F| \ll |F_0|
$$

in the limit of short-wavelength, low-frequency perturbations we get

$$
-i\hat{\omega}\rho + ik_R u_R \rho_0 + ik_z u_z \rho_0 = 0 \tag{A.2.1}
$$

for the mass,

$$\begin{aligned}
&-i\hat{\omega}u_R\rho_0 = -ik_R p + A_{pR}c_0^2\rho + 2\Omega u_\varphi \rho_0, \\
&-i\hat{\omega}u_z\rho_0 = -ik_z p + A_{pz}c_0^2\rho, \\
&-i\hat{\omega}u_\varphi\rho_0 + u_R\rho_0\Omega_R + u_z\rho_0\Omega_z = -\rho_0 u_R \Omega
\end{aligned} \tag{A.2.2}$$

for the momentum and

$$\frac{T_0}{T\gamma}\left[-i\hat{\omega}\frac{p}{p_0} + i\gamma\hat{\omega}\frac{\rho}{\rho_0} + u_R(A_{pR}-\gamma A_{\rho R}) + u_z(A_{pz}-\gamma A_{\rho z})\right] = -(\omega_c+\omega_{th}) \tag{A.2.3}$$

for energy. In these formulas we have introduced the Doppler-shifted frequency $\hat{\omega} = \omega - \mathbf{k}\cdot\mathbf{u_0} = \omega - (k_R u_{0R} + k_z u_{0z})$, the isothermal sound speed squared $c_0^2 \equiv p_0/\rho_0$, the inverse of the pressure scalelenght $A_{pR} \equiv (\partial p_0/\partial R)/p_0$ and the inverse of the pressure scalehight $A_{pz} \equiv (\partial p_0/\partial z)/p_0$. The following frequencies have been defined: $\Omega_R \equiv \partial(\Omega R)/\partial R$,$\Omega_z \equiv \partial(\Omega R)/\partial z$, the thermal-conduction frequency

$$\omega_c \equiv \left(\frac{\gamma-1}{\gamma}\right)\frac{k^2\kappa T_0^{7/2}}{p_0} \tag{A.2.4}$$

and the thermal-instability frequency

$$\omega_{th} \equiv -\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho_0^2\Lambda(T_0)}{p_0(\mu m_p)^2}\left[2-\frac{d\ln\Lambda(T_0)}{d\ln T_0}\right]. \tag{A.2.5}$$

In terms of the defined quantities, the assumption of short-wavelength perturbations gives $|k_R|, |k_z| \gg |A_{\rho R}|, |A_{\rho z}|, |A_{pR}|, |A_{pz}|$ and $\Omega^2, \Omega_R^2, \Omega_z^2 \ll c_0^2 k^2$, while the assumption of low-frequency perturbations gives $\omega^2 \ll c_0^2 k^2$.
The system of equations (A.2.1),(A.2.2) and (A.2.3) can be reduced to the following dispersion relation for $n \equiv -i\hat{\omega}$:

$$n^3 + n^2\omega_d + \left(\omega_{BV}^2 + \omega_{rot}^2\right)n + \omega_{rot}^2\omega_d = 0 \tag{A.2.6}$$

where $\omega_d \equiv \omega_{th} + \omega_c$ is the characteristic frequency of dissipative process,

$$\omega_{rot}^2 \equiv -\frac{k_z^2}{k^2}\frac{1}{R^3}\mathcal{D}(R^4\Omega^2) \tag{A.2.7}$$

is the differential rotation term,

$$\omega_{BV}^2 \equiv -\frac{k_z^2}{k^2}\frac{\mathcal{D}p_0}{\rho_0\gamma}\mathcal{D}S_0 \tag{A.2.8}$$

is the buoyancy term; and we introduced the unperturbed specific entropy $S_0 \equiv \ln(p_0\rho_0^{-\gamma})$ and the differential operator

$$\mathcal{D} \equiv \frac{k_R}{k_z}\frac{\partial}{\partial z} - \frac{\partial}{\partial R}$$

which can be seen as taking derivatives along surfaces of constant wave phase.

A.2.2 The stability criteria

Let us start from the simplest case, in which only one of the three characteristic frequencies ω_{BV}, ω_{rot}, ω_d is non-null.

Case with $\boldsymbol{\omega_d = \omega_{rot} = 0}$. In this case, there is no dissipation and the fluid is either non-rotating ($\Omega = 0$) or rotating differentially with vanishing gradient of the specific angular momentum [$d(\Omega R^2)/dR = 0$, when $\Omega = \Omega(R)$].

We have the dispersion relation

$$n^2 = -\omega_{BV}^2 \tag{A.2.9}$$

so we have stability if the square of the Brunt-Vaisala frequency $\omega_{BV}^2 > 0$. When $\Omega = \Omega(R)$, the Brunt-Vaisala frequency squared can be written as

$$\omega_{BV}^2 = \frac{k_z^2}{k^2}\frac{c_0^2 A_{pz}^2}{\gamma}\left(\frac{\gamma}{\gamma'} - 1\right)\left(\frac{k_R}{k_z} - \frac{A_{pR}}{A_{pz}}\right)^2 \tag{A.2.10}$$

where, using the fact that p_0 is a function of only ρ_0 (barotropic distribution), we defined

$$\gamma' \equiv \frac{d\ln p_0}{d\ln \rho_0}$$

which can be considered a local polytropic index. The condition for convective stability is $\gamma' < \gamma$ (i.e. the Schwarzschild's criterion; see Section 2.2.1).

Case with $\boldsymbol{\omega_{BV} = \omega_{rot} = 0}$. We assume that the characteristic frequencies associated with rotation and buoyancy are null. In the barotropic case, these conditions are met when $\gamma' = \gamma$ and $d(\Omega R^2)/dR = 0$ (i.e. the radial gradient of the specific entropy and the radial gradient of the specific angular momentum are zero). The dispersion relation is

$$n = -\omega_d \tag{A.2.11}$$

so we have thermal instability if $\omega_d < 0$, which is the Field's instability criterion. From the definition of $\omega_d = \omega_{th} + \omega_c$, it is clear that the condition for thermal instability is that the growth rate of the thermal perturbation must be faster than conductive damping. For fixed unperturbed gas temperature T_0 and pressure p_0, ω_c increases for increasing perturbation wavenumber k, while ω_{th} is independent of k, so there is a critical perturbation wavelength such that $\omega_d < 0$ for longer wavelength and $\omega_d > 0$ for shorter wavelengths.

Case with $\boldsymbol{\omega_{BV} = \omega_d = 0}$. Without buoyancy and dissipation we get the dispersion relation

$$n^2 = -\omega_{rot}^2 \tag{A.2.12}$$

so the stability criterion is $\omega_{rot}^2 > 0$. The value of ω_{rot}^2 depends on the ratio k_R/k_z, so it is positive if and only if $\partial\Omega/\partial z = 0$ and $d(\Omega R^2)/dR > 0$, i.e. the specific angular momentum must increase outwards. This is the Rayleigh's criterion (see §2.3.1).

In the following analysis of the dispersion relation, only one among ω_{BV}, ω_{rot}, ω_d is null.

Case with $\boldsymbol{\omega_{BV} = 0}$. In the absence of buoyancy, but for $\omega_{rot} \neq 0$ and $\omega_d \neq 0$, the dispersion relation is

$$(n^2 + \omega_{rot}^2)(\omega_d + n) = 0 \tag{A.2.13}$$

which is a combination of the Field's instability criterion and the Schwarzschild's criterion, so the presence of a gradient of the specific angular momentum does not modify the thermal-instability criterion in an interesting way. Specifically, when $\omega_d < 0$ the medium is thermally unstable, independent of the presence and properties of rotation, while rotation can destabilize an otherwise thermally stable medium($\omega_d > 0$) if $\omega_{rot}^2 < 0$. Thus, the condition for stability is $\omega_d > 0$ and $\omega_{rot}^2 > 0$, which holds for all values of k_R/k_z if $\partial\Omega/\partial z = 0$ and $d(\Omega R^2)/dR > 0$.

Case with $\boldsymbol{\omega_d = 0}$. In the absence of dissipation, one obtains the relation

$$n^2 = -(\omega_{BV}^2 + \omega_{rot}^2) \tag{A.2.14}$$

which leads to the convective stability criterion for a rotating, stratified fluid $\omega_{BV}^2 + \omega_{rot}^2 > 0$, showing the stabilizing effect of rotation against convection. The inequality is verified for all values of k_R/k_z if and only if

$$\begin{aligned} &-\frac{1}{\gamma\rho_0}\nabla p_0 \cdot \nabla S_0 + \frac{1}{R^3}\frac{\partial R^4\Omega^2}{\partial R} > 0 \\ &-\frac{\partial p_0}{\partial z}\left(\frac{\partial R^4\Omega^2}{\partial R}\frac{\partial S_0}{\partial z} - \frac{\partial R^4\Omega^2}{\partial z}\frac{\partial S_0}{\partial R}\right) > 0 \end{aligned} \tag{A.2.15}$$

which is the Solberg-Hoiland criterion (see Section 2.4.1).

Case with $\boldsymbol{\omega_{rot} = 0}$. In this case the the dispersion relation is

$$n^2 + \omega_d n + \omega_{BV}^2 = 0 \tag{A.2.16}$$

When $\omega_d > 0$ we have stability (damping by thermal conduction) if $\omega_{BV}^2 > 0$, while we have convective instability if $\omega_{BV}^2 < 0$. When $\omega_d < 0$, we have over-stability if $\tilde{\omega}_{BV}^2 > 1/4$, thermal instability if $0 < \tilde{\omega}_{BV}^2 < 1/4$ and convective instability if $\tilde{\omega}_{BV}^2 < 0$.

Appendix B

Calculation of the differential form $\mathbf{n} \cdot \nabla \mathbf{F}$ in general coordinates

Here we calculate the differential form $\mathbf{n} \cdot \nabla \mathbf{F}$, where $\mathbf{n}$ and $\mathbf{F}$ are vectors. In our specific case we will consider $\mathbf{n} = \mathbf{u}$ and $\mathbf{F} = \mathbf{u}$.

We follow the treatment in [Batchelor 1967].

Consider an orthogonal coordinate system ξ_1, ξ_2, ξ_3 and the triad $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of unit vectors parallel to the coordinate lines oriented in the directions of increase of ξ_1, ξ_2, ξ_3 respectively. We define a position vector $\mathbf{x}$ in this space.

Under this premises, the change of $\mathbf{x}$ can be written as

$$\delta \mathbf{x} = h_1 \delta \xi_1 \mathbf{a} + h_2 \delta \xi_2 \mathbf{b} + h_3 \delta \xi_3 \mathbf{c}, \tag{B.0.1}$$

where the positive scale factors h_1, h_2, h_3 have been introduced and, as for the unit vector, can be function of the coordinates.

We can obtain useful information about the derivatives of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, using the fact that the three family of coordinates lines introduces are orthogonal.

From

$$\frac{\partial \mathbf{x}}{\partial \xi_i} \cdot \frac{\partial \mathbf{x}}{\partial \xi_j} = 0 \qquad i, j = 1, 2, 3 \tag{B.0.2}$$

one has

$$\begin{aligned}
\frac{\partial}{\partial \xi_k}\left(\frac{\partial \mathbf{x}}{\partial \xi_i} \cdot \frac{\partial \mathbf{x}}{\partial \xi_j}\right) &= \frac{\partial^2 \mathbf{x}}{\partial \xi_k \partial \xi_i} \cdot \frac{\partial \mathbf{x}}{\partial \xi_j} + \frac{\partial \mathbf{x}}{\partial \xi_i} \cdot \frac{\partial^2 \mathbf{x}}{\partial \xi_k \partial \xi_j} = \\
&= \frac{\partial}{\partial \xi_i}\left(\frac{\partial \mathbf{x}}{\partial \xi_k}\right) \cdot \frac{\partial \mathbf{x}}{\partial \xi_j} + \frac{\partial \mathbf{x}}{\partial \xi_i} \cdot \frac{\partial}{\partial \xi_j}\left(\frac{\partial \mathbf{x}}{\partial \xi_k}\right) = \\
&= \frac{\partial}{\partial \xi_i}\left(\frac{\partial \mathbf{x}}{\partial \xi_k} \cdot \frac{\partial \mathbf{x}}{\partial \xi_j}\right) - \frac{\partial \mathbf{x}}{\partial \xi_k} \cdot \frac{\partial^2 \mathbf{x}}{\partial \xi_i \partial \xi_j} + \frac{\partial}{\partial \xi_j}\left(\frac{\partial \mathbf{x}}{\partial \xi_i} \cdot \frac{\partial \mathbf{x}}{\partial \xi_k}\right) - \frac{\partial \mathbf{x}}{\partial \xi_k} \cdot \frac{\partial^2 \mathbf{x}}{\partial \xi_i \partial \xi_j} = \\
&= -2 \frac{\partial \mathbf{x}}{\partial \xi_k} \frac{\partial^2 \mathbf{x}}{\partial \xi_i \partial \xi_j}.
\end{aligned} \tag{B.0.3}$$

Therefore we can affirm, for example, that

$$\frac{\partial^2 \mathbf{x}}{\partial\xi_1\partial\xi_2} = \frac{\partial}{\partial\xi_1}\frac{\partial\mathbf{x}}{\partial\xi_2} = \frac{\partial}{\partial\xi_2}\frac{\partial\mathbf{x}}{\partial\xi_1} = \frac{\partial(h_2\mathbf{b})}{\partial\xi_1} = \frac{\partial(h_1\mathbf{a})}{\partial\xi_2} \tag{B.0.4}$$

is a vector perpendicular to $\mathbf{c}$.

From relation (B.0.4) we note that

$$\begin{aligned} \frac{\partial\mathbf{a}}{\partial\xi_2} &= \frac{1}{h_1}\frac{\partial h_2}{\partial\xi_1}\mathbf{b}, \\ \frac{\partial\mathbf{b}}{\partial\xi_1} &= \frac{1}{h_2}\frac{\partial h_1}{\partial\xi_2}\mathbf{a}. \end{aligned} \tag{B.0.5}$$

One may compute the remaining four relations between the vectors.

Moreover we can write

$$\begin{aligned} \frac{\partial\mathbf{a}}{\partial\xi_1} &= \frac{\partial(\mathbf{b}\times\mathbf{c})}{\partial\xi_1} = \\ &= -\frac{1}{h_2}\frac{\partial h_1}{\partial\xi_2}\mathbf{b} - \frac{1}{h_3}\frac{\partial h_1}{\partial\xi_3}\mathbf{c} \end{aligned} \tag{B.0.6}$$

with analogous relations for the two other vectors.

Let be V a scalar function, we can express the gradient of V as

$$\nabla V = (\frac{\mathbf{a}}{h_1}\frac{\partial}{\partial\xi_1} + \frac{\mathbf{b}}{h_2}\frac{\partial}{\partial\xi_2} + \frac{\mathbf{c}}{h_3}\frac{\partial}{\partial\xi_3})V. \tag{B.0.7}$$

We can now calculate the components of $\mathbf{n}\cdot\nabla\mathbf{F}$, where $\mathbf{n}$ represents a fixed direction, while $\mathbf{F} = F_1\mathbf{a} + F_2\mathbf{b} + F_3\mathbf{c}$. From the relations we have follow that

$$\begin{aligned} \mathbf{n}\cdot\nabla\mathbf{F} = \mathbf{a}\Bigg[\mathbf{n}\cdot\nabla F_1 &+ \frac{F_2}{h_1h_2}\left(n_1\frac{\partial h_1}{\partial\xi_2} - n_2\frac{\partial h_2}{\partial\xi_1}\right) + \frac{F_3}{h_3h_1}\left(n_1\frac{\partial h_1}{\partial\xi_3} - n_3\frac{\partial h_3}{\partial\xi_1}\right)\Bigg] + \\ + \mathbf{b}\Bigg[\mathbf{n}\cdot\nabla F_2 &+ \frac{F_3}{h_2h_3}\left(n_2\frac{\partial h_2}{\partial\xi_3} - n_3\frac{\partial h_3}{\partial\xi_2}\right) + \frac{F_1}{h_1h_2}\left(n_2\frac{\partial h_2}{\partial\xi_1} - n_1\frac{\partial h_1}{\partial\xi_2}\right)\Bigg] + \\ + \mathbf{c}\Bigg[\mathbf{n}\cdot\nabla F_3 &+ \frac{F_1}{h_3h_1}\left(n_3\frac{\partial h_3}{\partial\xi_1} - n_1\frac{\partial h_1}{\partial\xi_3}\right) + \frac{F_2}{h_2h_3}\left(n_3\frac{\partial h_3}{\partial\xi_2} - n_2\frac{\partial h_2}{\partial\xi_3}\right)\Bigg]. \end{aligned} \tag{B.0.8}$$

B.1 Cylindrical coordinates.

We can apply the equation (B.0.8) to the cylindrical coordinates (z, R, φ). The corresponding scale factors are

$$h_1 = h_2 = 1, \qquad h_3 = R \tag{B.1.1}$$

hence

$$\begin{aligned} \mathbf{n}\cdot\nabla\mathbf{F} = \mathbf{a}\Big[\mathbf{n}\cdot\nabla F_1\Big] + \mathbf{b}\Bigg[\mathbf{n}\cdot\nabla F_2 + \frac{F_3}{R}\left(-n_3\frac{\partial R}{\partial R}\right)\Bigg] + \\ + \mathbf{c}\Bigg[\mathbf{n}\cdot\nabla F_3 + \frac{F_2}{R}\left(n_3\frac{\partial R}{\partial R}\right)\Bigg]. \end{aligned} \tag{B.1.2}$$

Using the common notation we can then write

$$\mathbf{n} \cdot \nabla \mathbf{F} = \mathbf{a}\left[\mathbf{n} \cdot \nabla F_z\right] + \mathbf{b}\left[\mathbf{n} \cdot \nabla F_R - \frac{n_R F_R}{R}\right] + \mathbf{c}\left[\mathbf{n} \cdot \nabla F_\varphi + \frac{n_\varphi F_\varphi}{R}\right]. \tag{B.1.3}$$

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