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SUPERSYMMETRIC QUANTUM MECHANICS AND APPLICATIONS

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Abstract

This thesis contains an introduction to Supersymmetric Quantum Mechanics and its possible applications to the solution of some common problems in Quantum Mechanics. After a brief discussion on the origins of Supersymmetric Quantum Mechanics, Lie superalgebras are introduced, since they constitute the mathematical apparatus required to develop this topic. Then, the model with $N = 2$ supersymmetry charges in $0 + 1$ dimensions is implemented and studied, and the concept of supersymmetry breaking is also tackled, along with the Witten index. Afterwards, some applications of this model are discussed, namely the chain of Hamiltonians, shape invariant potentials and the construction of a family of isospectral potentials. The thesis ends with a presentation of some explicit examples of these applications, where the methods developed in the framework of Supersymmetric Quantum Mechanics are used to solve some one dimensional problems.

Questa tesi contiene un'introduzione alla Meccanica Quantistica Supersimmetrica e alle sue possibili applicazioni nella risoluzione di problemi tipici della Meccanica Quantistica. Dopo una breve discussione sulle origini della Meccanica Quantistica Supersimmetrica, vengono introdotte le superalgebre di Lie, che costituiscono l'apparato matematico necessario per lo sviluppo di questo argomento. Viene poi implementato e studiato il modello con $N = 2$ cariche di supersimmetria in $0 + 1$ dimensioni, affrontando anche il concetto di rottura spontanea di supersimmetria e l'indice di Witten. In seguito, vengono discusse alcune applicazioni di questo modello, ovvero la catena di Hamiltoniane, i potenziali invarianti in forma e la costruzione di una famiglia di potenziali isospettrali. La tesi si conclude con esempi espliciti di tali applicazioni, in cui i metodi della Meccanica Quantistica Supersimmetrica vengono usati per risolvere alcuni problemi unidimensionali.

Contents

Introduction

1.1 The relevance of Supersymmetric Quantum Mechanics

Supersymmetry is defined as a symmetry between bosons and fermions. This idea was introduced in the attempt to achieve a unified description of these two kinds of particles or, in other words, of matter and interactions. Supersymmetry first emerged in 1971 in the work of physicists such as Gel'fand, Likhtman, Ramond, Neveu and Schwartz, and was soon developed using a new type of algebra, called Lie superalgebra. This algebra, which is characterized by anticommutator relations in addition to the usual commutator ones, allows for particles with different spins to be united in the same particle multiplet and to be transformed into each other using supersymmetry transformations. The presence of anticommutator relations is the key to overcome the powerful restrictions of the Coleman-Mandula theorem, which forbade the presence of particles with different spins in the same multiplet, if one used the regular Lie algebras [1]. This transition from Lie algebras to superalgebras to unite bosons and fermions under a multiplet is discussed in the next section.

Supersymmetry was soon appreciated because it can be introduced in Quantum Field Theory without adding any other assumption. Supersymmetric theories were then developed because of their potential to give a unified description of gravity and the other forces (the electroweak and the strong ones): this created the area of research called Supergravity. The interest of this field resides in the fact that supersymmetry can improve and hopefully solve divergence problems that arise when quantizing gravity. While nowadays Supergravity is mostly thought as the low energy limit of Superstring Theory (as the hope that some particular theory of supergravity might be finite is fading away), supersymmetric field theories offer the chance of studying exactly various quantum field theoretical properties. For example, the $N = 4$ super Yang-Mills theory is an interacting supersymmetric and conformal theory that is being study quite actively, with the hope of solving it exactly.

Supersymmetry is said to be unbroken if the lowest energy state of the theory is invariant under supersymmetry transformations. If this is the case, then supersymmetric theories predict the presence of particles with the same mass as the particles we know, but with a spin that differs for a half integer amount. Since no such particles have ever been detected, supersymmetry must be spontaneously broken. Because of the difficulties that arise when building models with broken supersymmetry in the four dimensional spacetime, supersymmetry was applied to the simpler one dimensional case: this is called Supersymmetric Quantum Mechanics. One of the main physicists that worked on this topic is Edward Witten, who also provided an indicator of supersymmetry breaking, called the Witten index [7]. In Chapter 2 a one dimensional supersymmetric model is presented, along with an introduction to the Witten index.

While studying these simpler supersymmetric models, it soon became clear that Supersymmetric Quantum Mechanics was an area of interest on its own, and not just a test field for theories in higher dimensions. First of all, the supersymmetric formalism shed

some light on the factorization method, an algebraic technique that is used to derive the spectrum and eigenfunctions for some one dimensional Hamiltonians. This method was first used by Schrödinger to solve the hydrogen atom, then it was formalized by Infeld and Hull in the 1950s to categorize solvable potentials. It turns out that this method is rather powerful, especially if it is applied to the particular (but not very restrictive) class of shape invariant potentials. All of this is more intuitive and appealing using the formalism of supersymmetry [9], as we are going to see in Chapter 3. The last section of Chapter 3 contains the study of isospectral deformation, an algebraic method that generates a family of potentials that are isospectral to a given one. Once again, this method was studied by mathematicians like Darboux, Abraham, Moses and Pursey before the advent of Supersymmetric Quantum Mechanics, but it can be better understood in terms of supersymmetry [9]. Finally, in Chapter 4 several examples of such applications of Supersymmetric Quantum Mechanics can be found.

1.2 Lie superalgebras

This section contains a brief overview of Lie superalgebras, the mathematical structure underlying Supersymmetric Quantum Mechanics. We will start by reviewing Lie algebras and their importance for Quantum Field Theories, as well as a crucial theorem due to Coleman and Mandula. Then we will extend our discussion to superalgebras and their role in the light of the Haag-Łopuszański-Sohnius theorem.
A Lie algebra $\mathcal{A}$ is defined as a vector space with a bilinear composition rule [,] that has the following additional properties:

$$[A, A] = 0, \tag{1.1}$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{1.2}$$

for all A, B and $C \in \mathcal{A}$ [2]. Property (1.1) is called skew-symmetry, and is equivalent to

$$[A, B] = -[B, A]; \tag{1.3}$$

property (1.2), on the other hand, is known as the Jacobi identity. For our purposes, the elements of $\mathcal{A}$ are going to be linear operators acting on Hilbert spaces, so the [,] bracket will be naturally identified with the commutator

$$[A, B] = AB - BA, \tag{1.4}$$

which is well known to satisfy properties (1.1) and (1.2).
As the structure underlying $\mathcal{A}$ is a vector space, we can take a basis $\{A_i\}$ $(i = 1, 2 \ldots \dim\mathcal{A})$ of this vector space, whose elements are the generators of the Lie algebra. Since the generators constitute a basis, any element of the algebra, including the commutator of two of them, can be written as their combination:

$$[A^a, A^b] = i f^{ab}{}_c A^c. \tag{1.5}$$

Here, we have adopted Einstein's convention for the summation over repeated indices, as we will do for the rest of the section. The $f^{ab}{}_c$ are called structure constants, because

they characterize every particular Lie algebra. Applying properties (1.1) and (1.2) to the generators gives two constraints on the structure constants:

$$\begin{aligned} f^{ab}{}_{c} &= -f^{ba}{}_{c}, \\ f^{ab}{}_{d}f^{dc}{}_{e} + f^{bc}{}_{d}f^{da}{}_{e} + f^{ca}{}_{d}f^{db}{}_{e} &= 0. \end{aligned} \tag{1.6}$$

Structure constants are also relevant because they define the adjoint representation: it is given by the matrices

$$(A^a)^b{}_{c} = -if^{ab}{}_{c}. \tag{1.7}$$

It is straightforward to check that the matrices defined this way indeed satisfy the algebra characterized by (1.5); moreover, their dimension is the same as the one of the algebra, since all the indices in this equation run from 1 to the dimension of the algebra. The interest for Lie algebras in modern Physics arises from the study of symmetries in elementary particle theories (e.g. isospin, the eightfold way). A particularly powerful result is the Coleman-Mandula theorem, which states that any group of (bosonic) symmetries for a relativistic field theory is the direct product of the Poincaré group $ISO(3,1)$ and an internal symmetry group G. The term "bosonic" indicates that we are only considering the usual Lie algebras with commutators relations we have just discussed, and the necessity to stress this shall become clear soon enough. So, the most general symmetry algebra, according to this theorem, goes as follows: if we denote with P_μ the Poincaré generators of translations, with $M_{\mu\nu}$ the Lorentz generators and with B^a the generators of the internal group G, then

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, M_{\lambda\rho}] &= i\eta_{\mu\lambda}M_{\nu\rho} - i\eta_{\mu\rho}M_{\nu\lambda} - i\eta_{\nu\lambda}M_{\mu\rho} + i\eta_{\nu\rho}M_{\mu\lambda} \\ [M_{\mu\nu}, P_\lambda] &= i\eta_{\mu\lambda}P_\nu - i\eta_{\nu\lambda}P_\mu \\ [B^a, B^b] &= if^{ab}{}_{c}B^c \\ [P_\mu, B^a] &= 0 \\ [M_{\mu\nu}, B^a] &= 0. \end{aligned} \tag{1.8}$$

Here, the first three relations are the usual Poincaré algebra, and the fourth one defines the algebra of G. The last two, by contrast, show that the B^a are invariant under spacetime transformations; the fact that there isn't a non zero relation involving both the Poincaré generators and the internal generators implies that $ISO(3,1)$ and G are "independent" groups, that is their transformations are not related: a spacetime transformation cannot affect an internal one and vice-versa.

The fact that the internal group G cannot influence spacetime features was quite restrictive for relativistic field theories trying to achieve a unified description of bosons and fermions. These theories aimeded to unite bosons and fermions in particle multiplets, and symmetry transformations would allow to switch between the two particle types. The problem is that bosons and fermions differ in spin, which is a spacetime property (since it is a form of angular momentum, connected to rotations): thus, symmetry transformations given by an internal group cannot connect particles with different spin. The same is true for mass, which is a spacetime property too. The conclusion is that one couldn't have particles with different mass and spin in the same multiplet according to the Coleman-Mandula theorem.

However, Coleman and Mandula only considered Lie groups with real parameters, thus an algebra involving just commutator relations, but it is possible to take into account

algebras with anticommutator relations too. Doing so (which roughly corresponds to lifting an hypothesis from the theorem) allows particles with different spin to be in the same multiplet. This can be understood because, if one has a "fermionic" transformation, carrying a half-integer spin, then when this transformation acts on a particle with integer spin the result will be a particle with half-integer spin. For example, if the original particle has spin s and the transformation has spin $\frac{1}{2}$, then the final particle can have spin $s + \frac{1}{2}$ or $s - \frac{1}{2}$ [1]. The presence of anticommutators in the algebra is then necessary, since fermionic systems are described by algebras involving anticommutator relations. The resulting algebra, including commutators and anticommutators, is called Lie superalgebra, and we will now give its definition.
Lie superalgebras can be thought of as extensions of Lie algebras that include two types of elements, obeying "even-like" and "odd-like" composition rules respectively. Mathematically, a Lie superalgebra is defined as an algebra $\mathcal{A}$ which is the direct sum of two algebras labelled with the elements 0 and 1 of the group $\mathbb{Z}_2$:

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1. \tag{1.9}$$

The elements of $\mathcal{A}_0$ will be called bosonic, while those of $\mathcal{A}_1$ will be called fermionic. This algebra has the bilinear composition rule denoted $[\ ,\ \}$, and its elements must satisfy the following properties:

$$[A_i, A_j\} \in \mathcal{A}_{i+j}, \tag{1.10}$$

$$[A_i, A_j\} = -(-1)^{i \cdot j}[A_j, A_i\}, \tag{1.11}$$

$$(-1)^{i \cdot k}[A_i, [A_j, A_k\}\} + (-1)^{j \cdot i}[A_j, [A_k, A_i\}\} + (-1)^{k \cdot j}[A_k, [A_i, A_j\}\} = 0. \tag{1.12}$$

for all $i, j, k \in \mathbb{Z}_2, A_i \in \mathcal{A}_i,\ A_j \in \mathcal{A}_j$ and $A_k \in \mathcal{A}_k$ [3] [4]. The dot $\cdot$ denotes standard multiplication.
The first property is called $\mathbb{Z}_2$ gradation, and tells us how the two algebras intertwine: the bracket between two bosonic or fermionic elements is bosonic, while the bracket beween a bosonic element and a fermionic one is fermionic.
The second property defines the "parity" of the bracket $[\ ,\ \}$: it reduces to the anticommutator

$$\{A_i, A_j\} = A_i A_j + A_j A_i \tag{1.13}$$

if A_i and A_j are both fermionic, while it reduces to the commutator $[\ ,\ ]$ if at least one of them is bosonic. This allows us to rewrite property (1.10) in terms of the bosonic and fermionic generators B and F:

$$[B^a, B^b] = i f^{ab}{}_c B^c \qquad [B^a, F^\alpha] = i g^{a\alpha}{}_\beta F^\beta \qquad \{F^\alpha, F^\beta\} = h^{\alpha\beta}{}_a B^a \tag{1.14}$$

where $f^{ab}{}_c$, $g^{a\alpha}{}_\beta$ and $h^{\alpha\beta}{}_a$ are constants that characterize the superalgebra. Here, we have used latin letters to index bosonic elements and greek letters to index fermionic ones: this difference is due to the fact that the algebras $\mathcal{A}_0$ and $\mathcal{A}_1$ do not need to have the same dimension, so the two types of indices possibly belong to different ranges. These realtions show that Lie superalgebras extend Lie algebras by introducing defining relations that involve anticommutators in addition to the usual commutators. Also, the first equality tells us that the bosonic elements form a Lie subalgebra. If we now insert the bosonic and fermionic generators into property (1.11) we obtain the following constraints on the constants f, h:

$$f^{ab}{}_c = -f^{ba}{}_c, \qquad h^{\alpha\beta}{}_a = h^{\beta\alpha}{}_a. \tag{1.15}$$

Nothing can be concluded about the $g^{a\alpha}{}_{\beta}$, because the two upper indices are different in nature (bosonic and fermionic), so they cannot be simply interchanged.
The third property in (1.12) is called the generalized Jacobi identity, since it clearly generalizes the classical Jacobi identity (1.2), taking into account the "parity" of the bracket [, }. Writing this identity for the bosonic and fermionic generators, one gets more constraints on the constants f, g, h:

$$\begin{aligned} f^{ab}{}_{d}f^{dc}{}_{e} + f^{bc}{}_{d}f^{da}{}_{e} + f^{ca}{}_{d}f^{db}{}_{e} = 0, \quad & f^{ab}{}_{c}g^{c\alpha}{}_{\gamma} - g^{b\alpha}{}_{\beta}g^{a\beta}{}_{\gamma} + g^{a\alpha}{}_{\beta}g^{b\beta}{}_{\gamma} = 0, \\ g^{a\alpha}{}_{\gamma}h^{\gamma\beta}{}_{c} + h^{\alpha\beta}{}_{b}f^{ba}{}_{c} + g^{a\beta}{}_{\gamma}h^{\gamma\alpha}{}_{c} = 0, \quad & h^{\alpha\beta}{}_{a}g^{a\gamma}{}_{\delta} + h^{\beta\gamma}{}_{a}g^{a\alpha}{}_{\delta} + h^{\gamma\alpha}{}_{a}g^{a\beta}{}_{\delta} = 0, \end{aligned} \tag{1.16}$$

where we can recognize the first equality as the subalgebra of bosonic generators.
We can see one last property of Lie superalgebras: the structure constants can be used to form the adjoint representation, and the matrices of the representation are

$$R(B^a) = \begin{pmatrix} F^a & 0 \\ 0 & G^a \end{pmatrix}, \qquad R(F^\alpha) = \begin{pmatrix} 0 & \Gamma^\alpha \\ H^\alpha & 0 \end{pmatrix}, \tag{1.17}$$

where the matrix elements are

$$\begin{aligned} (F^a)^b{}_c &= if^{ab}{}_c, \qquad & (G^a)^\alpha{}_\beta &= ig^{a\alpha}{}_\beta, \\ (\Gamma^\alpha)^a{}_\beta &= ig^{\alpha a}{}_\beta, \qquad & (H^\alpha)^\beta{}_a &= h^{\alpha\beta}{}_a. \end{aligned} \tag{1.18}$$

A notable example of Lie superalgebras are supersymmetry algebras (also called Poincaré superalgebras), which extend the Poincaré algebra by adding fermionic generators, as required by theories that provide a unified description of bosons and fermions. The relevance of supersymmetry algebras resides in the Haag-Łopuszański-Sohnius theorem, which states that those are the most general symmetry algebras for four-dimensional Quantum Field Theories (in addition to an internal symmetry group, possibly). To get an idea of what such an algebra is, let us see the commutator-anticommutator relations for the $d = 4$, $N = 1$ superalgebra: this means that the algebra is four-dimensional and that it has one fermionic generator, called supercharge (which has four real components).

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, M_{\lambda\rho}] &= i\eta_{\mu\lambda}M_{\nu\rho} - i\eta_{\mu\rho}M_{\nu\lambda} - i\eta_{\nu\lambda}M_{\mu\rho} + i\eta_{\nu\rho}M_{\mu\lambda} \\ [M_{\mu\nu}, P_\lambda] &= i\eta_{\mu\lambda}P_\nu - i\eta_{\nu\lambda}P_\mu \\ [P_\mu, Q_\alpha] &= 0 \\ [M_{\mu\nu}, Q_\alpha] &= i(\Sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \\ \{Q_\alpha, Q_\beta\} &= (\gamma^\mu C)_{\alpha\beta} P_\mu. \end{aligned} \tag{1.19}$$

Here, the P_μ and the $M_{\mu\nu}$ are the bosonic Poincaré generators we already encountered in (1.8), while Q_α are the fermionic supercharges; further, $\eta_{\mu\nu}$ is the Minkowski metric, $\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ and C is the charge conjugation matrix. To this, we should add that the generators of the internal symmetry group commute both with the Poincaré generators and the supercharges. Thus, we can see that supersymmetry transformations indeed are spacetime transformations, not internal symmetries.
The first three relations in (1.19) are the usual Poincaré algebra, which forms a subalgebra of this superalgebra. The fourth relation shows that the supercharges are invariant under spacetime translations, while the fifth one tells us that they transform like spin $\frac{1}{2}$ spinors under the action of the Lorentz group. Note that, from our previous discussion,

we already knew that supersymmetry transformations had to have half-integer spin, but this equation adds the restriction that transformations with spin different from $\frac{1}{2}$ are not allowed. Finally, the sixth relation shows that a composition of two supersymmetry transformations generates a spacetime translation: this is a peculiar feature that characterizes supersymmetry in general.
Here we have shown the $N = 1$ supersymmetry algebra, but one could add multiple supercharges ($N = 2, 3...$) instead of just one, which gives the so called extended supersymmetry algebras.
We now know how and why supersymmetry algebras were introduced. However, as it has been explained in the introduction, we are only going to use them in their most simple form: we are going to study of one-dimensional (time only) models[1]. In these models, since we keep just one dimension, the only bosonic generator is the hamiltonian H (which generates time translations); we can then add N fermionic generators Q_α, which will be related to H because of property (1.14). For example, for $N = 1$ (one fermionic generator Q), one obtains this rather simple algebra:

$$[H, H] = 0, \quad [H, Q] = 0, \quad \{Q, Q\} = 2H. \tag{1.20}$$

The $N = 2$ model, on the contrary, will be the object of the next chapter.

[1]These are called models and not theories because experimental evidence for supersymmetry is lacking at the moment.

The $d = 1,\ N = 2$ supersymmetric model

2.1 Supersymmetric charges and Hamiltonian

We are now going to introduce the main characteristic elements of the $d = 1,\ N = 2$ supersymmetric model [5] [6] [8]. First, let us set set $\hbar = 1$ for convenience. We will deal with a point-like particle of mass $m = 1$ moving on the real axis; the particle will have two possible "polarization states", so its state will be represented by a two-component wave function

$$\Phi = \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \end{pmatrix}. \tag{2.1}$$

We can identify wave functions of the form

$$\Phi_B = \begin{pmatrix} \phi_0(x) \\ 0 \end{pmatrix} \quad \text{and} \quad \Phi_F = \begin{pmatrix} 0 \\ \phi_1(x) \end{pmatrix} \tag{2.2}$$

as bosonic and fermionic states respectively: the reason for this shall become clearer as we proceed in this section. The Hilbert space $\boldsymbol{H}$ to which Φ belongs is the direct product of a "bosonic" part, namely the $\boldsymbol{L}^2$ space of square integrable functions, and a "fermionic" part, namely the two-dimensional Fock space $\boldsymbol{F}_2$:

$$\boldsymbol{H} = \boldsymbol{L}^2 \otimes \boldsymbol{F}_2 = \boldsymbol{L}^2 \oplus \boldsymbol{L}^2. \tag{2.3}$$

As for the operators, we will have the standard position and momentum ones, x and $p = -i\frac{\partial}{\partial x}$ respectively (acting on $\boldsymbol{L}^2$), which are bosonic and satisfy the canonical commutation relation

$$[x, p] = i. \tag{2.4}$$

We also add the fermionic annihilation and creation operators ψ and $\psi^\dagger$, acting on $\boldsymbol{F}_2$: from the theory of fermionic systems, we know that they follow the rules

$$\{\psi, \psi\} = 0, \quad \{\psi^\dagger, \psi^\dagger\} = 0, \quad \{\psi, \psi^\dagger\} = 1. \tag{2.5}$$

Every other possible bracket $[\ ,\ \}$ between the operators x, p, ψ and $\psi^\dagger$ vanishes: x and p commute with ψ and $\psi^\dagger$, since the first two act on $\boldsymbol{L}^2$ and the last two act on $\boldsymbol{F}_2$.
It is useful to recall another property of the fermionic creation and annihilation operators: if the basis vectors of the Fock space $|0\rangle$ and $|1\rangle$ are defined as the state with no fermionic excitations and the state with one fermionic excitation respectively, then the action of ψ and $\psi^\dagger$ on these states is

$$\psi\,|0\rangle = 0, \quad \psi^\dagger\,|0\rangle = |1\rangle\,, \quad \psi\,|1\rangle = |0\rangle\,, \quad \psi^\dagger\,|1\rangle = 0. \tag{2.6}$$

This explicitly shows that $\psi^\dagger$ adds a fermionic excitation, while ψ removes it. Also, there can be no other fermionic excitations, because the rules (2.5) imply that $\left(\psi^\dagger\right)^2 = 0$, so

we have a Pauli exclusion principle. For the Hilbert space we are dealing with, the $|0\rangle$ and $|1\rangle$ states can be realized as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.7}$$

which is in agreement with our previous definition of bosonic and fermionic states (2.2). We can now have an explicit expression for ψ and $\psi^\dagger$: in the basis $\{|0\rangle, |1\rangle\}$ we find

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \psi^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

It is now possible to define the fermionic supercharges Q and $Q^\dagger$ as

$$Q \equiv (W(x) - ip)\psi = \begin{pmatrix} 0 & W(x) - \frac{\partial}{\partial x} \\ 0 & 0 \end{pmatrix}, \tag{2.9}$$

$$Q^\dagger \equiv (W(x) + ip)\psi^\dagger = \begin{pmatrix} 0 & 0 \\ W(x) + \frac{\partial}{\partial x} & 0 \end{pmatrix}, \tag{2.10}$$

where W is a function of x called the superpotential. We also define the hamiltonian H as

$$\begin{aligned} H &\equiv \frac{1}{2}\left(p^2 + W^2(x)\right)\mathbb{1} + \frac{1}{2}W'(x)[\psi^\dagger, \psi] = \\ &= \frac{1}{2}\begin{pmatrix} -\frac{\partial^2}{\partial x^2} + W^2(x) - W'(x) & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} + W^2(x) + W'(x) \end{pmatrix} \equiv \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}, \end{aligned} \tag{2.11}$$

where $\mathbb{1}$ is the identity matrix and the prime denotes a derivative with respect to x. H_+ and H_- are called partner Hamiltonians and, according to our definition (2.2), they act on the bosonic and fermionic part of the wave function, respectively. We can see that in the supersymmetric model the potential energy is written in terms of the superpotential:

$$V = \frac{1}{2}\left(W^2(x)\mathbb{1} + W'(x)[\psi^\dagger, \psi]\right). \tag{2.12}$$

The two non vanishing components of this matrix are called partner potentials, since they are the potentials for the partner Hamiltonians H_- and H_+:

$$\begin{aligned} V_- &= \frac{1}{2}\left(W^2(x) - W'(x)\right), \\ V_+ &= \frac{1}{2}\left(W^2(x) + W'(x)\right). \end{aligned} \tag{2.13}$$

It is now straightforward to check that H, Q and $Q^\dagger$ satisfy the superalgebra defined in section (1.2), in particular:

$$\begin{aligned} [H, H] = 0, \quad [Q, H] = 0, \quad [Q^\dagger, H] = 0, \\ \{Q, Q^\dagger\} = 2H, \quad \{Q, Q\} = 0, \quad \{Q^\dagger, Q^\dagger\} = 0. \end{aligned} \tag{2.14}$$

The second and third identities tell us that the supercharges are conserved, as they commute with the Hamiltonian, while the fourth one exhibits one of the fundamental properties of supersymmetry: the Hamiltonian is generated by the composition of two supercharges, which means that time translations are obtained from the composition of

two supersymmetry transformations.
One can also define the fermion number operator

$$F \equiv \psi^\dagger \psi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.15}$$

This operator satisfies the algebra

$$[F, H] = 0, \quad [Q, F] = Q, \quad [Q^\dagger, F] = -Q^\dagger, \quad \{F, F\} = 2F, \tag{2.16}$$

where the first identity shows that the fermion number is also conserved. We can now note that the bosonic and fermionic wave functions Φ_B and Φ_F from (2.2) are eigenfunctions of F with eigenvalues 0 and 1 respectively: it is then consistent to call F the fermion number operator. F also exhibits the exclusion principle: it is idempotent, so its eigenvalues can only be 0 and 1.
Finally, let us just mention that an alternative formulation of the $N = 2$ supersymmetric model can be achieved using the hermitian supercharges Q_1 and Q_2 instead of Q and $Q^\dagger$. Q_1 and Q_2 are defined as follows:

$$Q_1 = \frac{Q + Q^\dagger}{\sqrt{2}},\ Q_2 = \frac{Q - Q^\dagger}{\sqrt{2}i} \quad \Longleftrightarrow \quad Q = \frac{Q_1 + iQ_2}{\sqrt{2}},\ Q^\dagger = \frac{Q_1 - iQ_2}{\sqrt{2}}. \tag{2.17}$$

Then, the algebra for Q_1 and Q_2 is

$$\{Q_j, Q_k\} = \delta_{jk} H, \quad [Q_j, F] = i\epsilon_{jk} Q_k, \tag{2.18}$$

where ϵ_{jk} is the totally antisymmetric 2×2 tensor.

Example: the supersymmetric harmonic oscillator

This simple example is meant to illustrate and clarify the structure of the $N = 2$ model presented above, which at first can appear somewhat abstract [6] [8] [9].
We start with the bosonic harmonic oscillator, which is described by the Hamiltonian[1]

$$H_B = \frac{1}{2}p^2 + \frac{1}{2}{\omega_B}^2 x^2. \tag{2.19}$$

We have the usual bosonic annihilation and creation operators

$$a = \frac{1}{\sqrt{2\omega_B}}(\omega_B x + ip), \quad a^\dagger = \frac{1}{\sqrt{2\omega_B}}(\omega_B x - ip), \tag{2.20}$$

whose commutator and anticommutator are

$$[a, a^\dagger] = 1, \quad \frac{\omega_B}{2}\{a^\dagger, a\} = H_B. \tag{2.21}$$

Those can be used to define the boson number operator

$$N_B \equiv a^\dagger a = \frac{p^2}{2\omega_B} + \frac{\omega_B x^2}{2} - \frac{1}{2}; \tag{2.22}$$

[1] We set $m = \hbar = 1$ as before.

its eigenvalues will be called n_B ($n_B = 0, 1, 2...$), and the corresponding (normalized) eigenkets will be denoted as $|n_B\rangle$. This allows us to rewrite the Hamiltonian as

$$H_B = \omega_B \left(N_B + \frac{1}{2} \right), \tag{2.23}$$

which means that its eigenkets are the $|n_B\rangle$ and their eigenvalues are $E_{n_B} = \omega_B \left(n_B + \frac{1}{2} \right)$. Consider then the fermionic harmonic oscillator, whose Hamiltonian is

$$H_F = \frac{\omega_F}{2} [\psi^\dagger, \psi] = \frac{\omega_F}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.24}$$

Here, ψ and $\psi^\dagger$ are the same fermionic annihilation and creation operators we have analyzed in the previous section, so they obey the algebra (2.5) and can be represented by the matrices in (2.8). We can define the fermion number operator as

$$N_F = \psi^\dagger \psi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.25}$$

and its eigenkets will be the $|0\rangle$ and $|1\rangle$ we encountered above, with eigenvalues 0 and 1 respectively. Using the rules in (2.5) the Hamiltonian becomes

$$H_F = \omega_F \left(N_F - \frac{1}{2} \right), \tag{2.26}$$

so its eigenkets are $|0\rangle$ and $|1\rangle$, with eigenvalues $-\frac{\omega_F}{2}$ and $\frac{\omega_F}{2}$ respectively.
Now, let us create a system composed by a bosonic and a fermionic harmonic oscillator with the same frequency $\omega_B = \omega_F = \omega$. The Hilbert space associated with this new system is the direct product of the spaces belonging to the two oscillators, so we have a basis $|n_B, n_F\rangle$ formed by the common eigenkets of N_B and N_F. We can study the action of a, $a^\dagger$, ψ and $\psi^\dagger$ on this basis:

$$\begin{array}{ll} a\,|n_B, n_F\rangle \propto |n_B - 1, n_F\rangle\,, & a^\dagger\,|n_B, n_F\rangle \propto |n_B + 1, n_F\rangle\,, \\ \psi\,|n_B, 0\rangle = 0, & \psi^\dagger\,|n_B, 0\rangle = |n_B, 1\rangle\,, \\ \psi\,|n_B, n_F\rangle = |n_B, n_F - 1\rangle\,, & \psi^\dagger\,|n_B, n_F\rangle = 0. \end{array} \tag{2.27}$$

The Hamiltonian of the system is

$$\begin{aligned} H = H_B \mathbb{1} + H_F = \frac{\omega}{2} (\{a^\dagger, a\} \mathbb{1} + [\psi^\dagger, \psi]) = \omega (N_B \mathbb{1} + N_F) = \\ = \frac{1}{2} \begin{pmatrix} \omega^2 x^2 + p^2 - \omega & 0 \\ 0 & \omega^2 x^2 + p^2 + \omega \end{pmatrix}, \end{aligned} \tag{2.28}$$

which has exactly the form (2.11) with $W(x) = \omega x$. We can use this knowledge to construct the supercharges

$$Q = \sqrt{2\omega} a^\dagger \psi = \begin{pmatrix} 0 & \omega x - ip \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \sqrt{2\omega} a \psi^\dagger = \begin{pmatrix} 0 & 0 \\ \omega x + ip & 0 \end{pmatrix} \tag{2.29}$$

and check that they have the same form as (2.9), (2.10) and follow the algebra in (2.14). It is interesting to remark that the supercharges are compositions of fermionic creation

and bosonic annihilation operators and vice-versa.
Further, according to (2.28) the eigenvalues of H have the form

$$E = \omega(n_B + n_F), \tag{2.30}$$

which stays the same as long as the sum of bosonic and fermionic quanta is constant. Thus, the action of the supercharges doesn't change the energy of the system, since it creates a bosonic quantum and destroys a fermionic one or vice-versa. One could finally check that the operator N_F has the same role as F defined in (2.15) and follows the same algebra (2.16).

2.2 Energy eigenvalues and eigenfunctions

In this section we are going to carry out a more detailed study of the supersymmetric Hamiltonian introduced above (formula (2.11)) [5] [9].
First, its spectrum is entirely non negative. That is because, if $|\Phi\rangle$ is a generic eigenket of H, then

$$\langle\Phi|H|\Phi\rangle = \frac{1}{2}\left(\langle\Phi|QQ^\dagger|\Phi\rangle + \langle\Phi|Q^\dagger Q|\Phi\rangle\right) = \frac{1}{2}\left(\left|Q^\dagger|\Phi\rangle\right|^2 + \left|Q|\Phi\rangle\right|^2\right) \geq 0, \tag{2.31}$$

which shows that H is positive semi-definite. This, however, does not mean that H always has an eigenstate with energy $E = 0$; if such a state exists, we say that supersymmetry is unbroken, otherwise we say that supersymmetry is spontaneously broken.
If the $E = 0$ state, often called supersymmetric vacuum, exists, then it is obviously the ground state of the system, as states with $E < 0$ are not allowed. Moreover, if $|\Phi_0\rangle$ is an $E = 0$ eigenfunction, then $\langle\Phi_0|H|\Phi_0\rangle = 0$, so according to (2.31)

$$\left(\left|Q^\dagger|\Phi_0\rangle\right|^2 + \left|Q|\Phi_0\rangle\right|^2\right) = 0. \tag{2.32}$$

For this equality to hold, both norms must vanish, which means that their arguments must vanish too, since the Hilbert space has a positive definite norm; thus, $E = 0$ eigenfunctions satisfy

$$Q|\Phi_0\rangle = Q^\dagger|\Phi_0\rangle = 0. \tag{2.33}$$

This shows that the supersymmetric vacuum is invariant under transformations generated by the supercharges (hence the adjective "supersymmetric"):

$$\mathrm{e}^{\alpha Q + \beta Q^\dagger}|\Phi_0\rangle = |\Phi_0\rangle. \tag{2.34}$$

Also, property (2.33) is very useful if one wants to find the vacuum wave function $|\Phi_0\rangle$, because one can solve the first order equations $Q|\Phi_0\rangle = 0$ and $Q^\dagger|\Phi_0\rangle = 0$ instead of the second order equation $H|\Phi_0\rangle = 0$. In fact, these two equations can be easily solved as follows:

$$\begin{aligned} Q|\Phi\rangle &= \begin{pmatrix} 0 & W(x) - \frac{\partial}{\partial x} \\ 0 & 0 \end{pmatrix}\begin{pmatrix}\phi_0 \\ \phi_1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \frac{\mathrm{d}\phi_1}{\mathrm{d}x} = W(x)\phi_1 \Rightarrow \phi_1 = c_1\mathrm{e}^{U(x)} \\ Q^\dagger|\Phi\rangle &= \begin{pmatrix} 0 & 0 \\ W(x) + \frac{\partial}{\partial x} & 0 \end{pmatrix}\begin{pmatrix}\phi_0 \\ \phi_1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \frac{\mathrm{d}\phi_0}{\mathrm{d}x} = -W(x)\phi_0 \Rightarrow \phi_0 = c_0\mathrm{e}^{-U(x)} \end{aligned} \tag{2.35}$$

where $U(x)$ is a primitive of $W(x)$. So the general ground state eigenfunction is

$$\Phi_0 = \begin{pmatrix} c_0 \mathrm{e}^{-U(x)} \\ c_1 \mathrm{e}^{U(x)} \end{pmatrix}. \tag{2.36}$$

However, if $E = 0$ belongs to the bound spectrum (which can be ckecked applying the spectral structure theorem to H_- and H_+), the solution is acceptable only if it is normalizable: three cases then arise, according to the behaviour of $U(x)$.

1. If $\lim_{x\to\pm\infty} U(x) = +\infty$, only ϕ_0 is normalizable and we must set $c_1 = 0$, so we have a bosonic ground state

$$\Phi_{0B} = \begin{pmatrix} c_0 \mathrm{e}^{-U(x)} \\ 0 \end{pmatrix}. \tag{2.37}$$

2. If $\lim_{x\to\pm\infty} U(x) = -\infty$, only ϕ_1 is normalizable and we must set $c_0 = 0$, so we have a fermionic ground state

$$\Phi_{0F} = \begin{pmatrix} 0 \\ c_1 \mathrm{e}^{U(x)} \end{pmatrix}. \tag{2.38}$$

3. If $\lim_{x\to\pm\infty} U(x) = \pm\infty$ or $\lim_{x\to\pm\infty} U(x) = \mp\infty$, neither ϕ_0 nor ϕ_1 are normalizable, so the ground state has $E > 0$.

On the other hand, if there is no bound spectrum and $E = 0$ belongs to the continuous spectrum, then there are no normalization issues, and the general solution (2.36) is acceptable. In this case, the ground state is doubly degenerate, and the space of eigenfunctions is spanned by a bosonic and a fermionic state, namely

$$\Phi_{0B} = \begin{pmatrix} c_0 \mathrm{e}^{-U(x)} \\ 0 \end{pmatrix} \text{ and } \Phi_{0F} = \begin{pmatrix} 0 \\ c_1 \mathrm{e}^{U(x)} \end{pmatrix}. \tag{2.39}$$

It is easy to check that the components $\phi_0 = c_0 \mathrm{e}^{-U(x)}$ and $\phi_1 = c_1 \mathrm{e}^{U(x)}$ are eigenfunctions of H_- and H_+ respectively, with eigenvalue $E_\pm = 0$:

$$\begin{aligned} \frac{1}{2} Q Q^\dagger \left|\Phi_{0B}\right\rangle &= \begin{pmatrix} H_- & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow H_- \left|\phi_0\right\rangle = 0, \\ \frac{1}{2} Q^\dagger Q \left|\Phi_{0F}\right\rangle &= \begin{pmatrix} 0 & 0 \\ 0 & H_+ \end{pmatrix} \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow H_+ \left|\phi_1\right\rangle = 0. \end{aligned} \tag{2.40}$$

Moving on to eigenstates with $E > 0$, we can see that they are doubly degenerate, as for every bosonic state there is a corresponding fermionic one with the same energy, and vice-versa. To prove this, consider a bosonic eigenstate $|b\rangle$ with eigenvalue E, then define

$$\left|f\right\rangle = Q^\dagger \left|b\right\rangle. \tag{2.41}$$

Recalling that bosonic and fermionic states are eigenfunctions of F with eigenvalues 0 and 1 respectively, we can calculate (using the commutation relation in (2.16))

$$F\left|f\right\rangle = F Q^\dagger \left|b\right\rangle = Q^\dagger F \left|b\right\rangle + Q^\dagger \left|b\right\rangle = Q^\dagger \left|b\right\rangle = \left|f\right\rangle, \tag{2.42}$$

which shows that $|f\rangle$ is fermionic. We can also check that $|f\rangle$ is an eigenstate with the same energy E:

$$H|f\rangle = HQ^\dagger|b\rangle = Q^\dagger H|b\rangle = EQ^\dagger|b\rangle = E|f\rangle\,. \tag{2.43}$$

Moreover, if $|b\rangle$ belongs to the bound spectrum and is normalized with $\langle b|b\rangle = 1$, then the normalized fermionic eigenstate is $\frac{1}{\sqrt{2E}}|f\rangle$:

$$\frac{1}{\sqrt{2E}}\langle f|\frac{1}{\sqrt{2E}}|f\rangle = \frac{1}{2E}\langle b|QQ^\dagger|b\rangle = \frac{1}{2E}\langle b|2E|b\rangle = 1. \tag{2.44}$$

Here, we have used the fact that

$$QQ^\dagger = \begin{pmatrix} 2H_- & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.45}$$

and the action of this operator on $|b\rangle$ is the same as the action of H, since the part H_+ doesn't affect a bosonic state.
Analoguous calculations show that, if we have a normalized fermionic eigenstate $|f\rangle$ with eigenvalue E, then

$$|b\rangle = \frac{1}{\sqrt{2E}}Q|f\rangle \tag{2.46}$$

is a normalized bosonic eigenstate with the same eigenvalue E:

$$F|b\rangle = \frac{1}{\sqrt{2E}}FQ|f\rangle = \frac{1}{\sqrt{2E}}(QF|f\rangle - Q^\dagger|f\rangle) = \frac{1}{\sqrt{2E}}(Q|f\rangle - Q|f\rangle) = 0, \tag{2.47}$$

$$H|b\rangle = \frac{1}{\sqrt{2E}}HQ|f\rangle = \frac{1}{\sqrt{2E}}QH|f\rangle = \frac{1}{\sqrt{2E}}EQ|f\rangle = E|b\rangle\,, \tag{2.48}$$

$$\langle b|b\rangle = \frac{1}{2E}\langle f|Q^\dagger Q|f\rangle = \frac{1}{2E}\langle f|2Ef\rangle = 1. \tag{2.49}$$

For the last one, we have used

$$Q^\dagger Q = \begin{pmatrix} 0 & 0 \\ 0 & 2H_+ \end{pmatrix}, \tag{2.50}$$

along with the fact that H_+ is the only part of H that acts on fermionic states.
For the continuous spectrum, the same calculations show that there still is this one-one correspondance between bosonic and fermionic eigenstates; the only difference is the "normalization" convention, which will be different since the eigenkets are no longer normalizable.
It is also clear from the structure of the Hamiltonian H in (2.11) that the components ϕ_0 and ϕ_1 of bosonic and fermionic states (see (2.2)) are eigenfunctions of H_- and H_+ respectively, with eigenvalue E.
Note that to "transform" a bosonic state into a fermionic one we have used the operator $Q^\dagger = (W(x) + ip)\psi^\dagger$, where $\psi^\dagger$ is the fermionic creation operator, and the $(W(x) + ip)$ part can be seen as a bosonic annihilation operator[2]. So, we can say that the supercharge $Q^\dagger$ destroys a boson and creates a fermion with the same energy, and similarly the supercharge Q destroys a fermion and creates a boson with the same energy.

[2]This is clear if we compare it with the corresponding operator for the bosonic harmonic oscillator: $a \propto \omega_B x + ip$.

We can conclude this section by seeing how these general properties apply to the supersymmetric harmonic oscillator discussed above. Recalling (2.30), it is clear that the spectrum is non negative, since n_B and n_F are non negative. The ground state is achieved for $n_B = n_F = 0 \Rightarrow E = 0$, which means that supersymmetry is unbroken. We can compute the ground state eigenfunction using (2.35): for the oscillator $W(x) = \omega x$, so $U(x) = \int \omega x \, dx = \frac{\omega}{2} x^2 \xrightarrow[x \to \pm\infty]{} +\infty$. This means that we have a bosonic ground state

$$\Phi_0 = \sqrt[4]{\frac{\omega}{\pi}} \begin{pmatrix} \mathrm{e}^{-\omega x^2/2} \\ 0 \end{pmatrix}, \tag{2.51}$$

which is the composition of the ground states of the bosonic and fermionic oscillators, that is $\sqrt[4]{\omega/\pi}\mathrm{e}^{-\omega x^2/2}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ respectively. Finally, from (2.30) we can see why the eigenvalues $E > 0$ are doubly degenerate: n_F can only take the values 0 and 1 because it is the eigenvalue of the idempotent operator N_F, so there are only two combinations of n_B and n_F that add up to the same value.

2.3 Supersymmetry breaking and Witten index

We are now going to analyze in further detail the phenomenon of supersymmetry breaking that was introduced in the previous section [7]. To do so, it is useful to introduce the parity operator

$$(-1)^F \tag{2.52}$$

where F is the fermion number operator. For the $N = 2$ supersymmetric model, $(-1)^F$ can be easily computed using the explicit representation of F in (2.15):

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.53}$$

The proof of this is straightforward once one recalls that F is idemponent and writes -1 using Euler's formula:

$$\begin{aligned} (-1)^F = \mathrm{e}^{i\pi F} &= \mathbb{1} + i\pi F + \frac{i^2\pi^2}{2} F + \ldots = \\ &= \mathbb{1} - F + \left(1 + i\pi + \frac{i^2\pi^2}{2} + \ldots\right) F = \\ &= \mathbb{1} - F + \mathrm{e}^{i\pi} F = \mathbb{1} - 2F, \end{aligned} \tag{2.54}$$

which gives the matrix in (2.53) if one inserts the representation of F found in (2.15). Moreover, one can check that the parity operator is fermionic and satisfies the relations

$$\{(-1)^F, Q\} = 0, \quad \{(-1)^F, Q^\dagger\} = 0, \quad [(-1)^F, H] = 0. \tag{2.55}$$

An important property of $(-1)^F$, which holds for supersymmetric models in general, is that its eigenfunctions are bosonic and fermionic states, with eigenvalues 1 and -1 respectively:

$$(-1)^F |b\rangle = |b\rangle, \quad (-1)^F |f\rangle = -|f\rangle. \tag{2.56}$$

This can be easily checked using the expression for $(-1)^F$ we found at the end of (2.54):

$$(-1)^F |b\rangle = (\mathbb{1} - 2F) |b\rangle = |b\rangle , \tag{2.57}$$

$$(-1)^F |f\rangle = (\mathbb{1} - 2F) |f\rangle = |f\rangle - 2 |f\rangle = - |f\rangle . \tag{2.58}$$

This property allows us to introduce the general concept of Witten index, which is defined as follows:

$$\Delta \equiv n_B^{E=0} - n_F^{E=0} = \mathrm{Tr}(-1)^F. \tag{2.59}$$

To show why the second equality holds, consider the case where we have n_B bosonic states and n_F fermionic ones: we can write the operator $(-1)^F$ in the basis they form, and then it is clear from (2.56) that its trace gives the difference between the number of bosonic and fermionic states. However, we have seen in the previous section that eigenstates with $E > 0$ always come in bosonic-fermionic pairs, so these states cancel out in $\mathrm{Tr}(-1)^F$, leaving only the ones with $E = 0$.
It turns out that $\mathrm{Tr}(-1)^F$ is not a good definition of the Witten index, because the sum does not converge absolutely. To fix this, one can instead define it as

$$\Delta = \mathrm{Tr}\left((-1)^F \mathrm{e}^{-\beta H}\right), \tag{2.60}$$

which is a good regulation, since the sum does not depend on the parameter β and it clearly reduces to our previous definition as $\beta \to 0$.
The Witten index is an indicator of supersymmetry breaking: if $\Delta \neq 0$, then there is at least one bosonic or fermionic state with $E = 0$, so supersymmetry is unbroken. However, if $\Delta = 0$, one cannot distinguish between the absence of $E = 0$ states (broken supersymmetry) and the presende of the same number of bosonic and fermionic $E = 0$ states (unbroken supersymmetry).
We can now compute the Witten index for the $N = 2$ supersymmetric model. Recalling the three cases that arise from (2.36) (bound spectrum), we have:

1. if $\lim_{x\to\pm\infty} U(x) = +\infty$, there is one bosonic $E = 0$ state, so $(-1)^F$=1 and $\Delta = 1$,

2. if $\lim_{x\to\pm\infty} U(x) = -\infty$, there is one fermionic $E = 0$ state, so $(-1)^F = -1$ and $\Delta = -1$,

3. if $\lim_{x\to\pm\infty} U(x) = \pm\infty$ or $\lim_{x\to\pm\infty} U(x) = \mp\infty$, there are no $E = 0$ states, so $(-1)^F = 0$ and $\Delta = 0$.

These cases show that one could change the form of $U(x)$ while keeping its asymptotic behaviour, and the Witten index would remain unchanged. This is an example of a general topological property of the Witten index: as one changes the parameters of the theory, some boson-fermion pairs could leave or enter the $E = 0$ state, but the Witten index stays the same. This is because, as we have seen above, $E > 0$ states must be doubly degenerate, and to preserve this degeneration transitions from and to the $E = 0$ state must happen by pairs of bosonic and fermionic excitations.

Applications

3.1 Factorization and chain of Hamiltonians

The method of factorization has been used for a long time to solve the Schrödinger equation; it turns out that it easily arises in the framework of Supersymmetric Quantum Mechanics, as we are going to see in this section [9] [10] [11]. We will set $\hbar = m = 1$ once again.
Suppose that we have a one dimensional Hamiltonian H_1 and that we know its ground state eigenvalue $E_1^{(0)}$ and eigenfunction $\psi_1^{(0)}$ (belonging to the bound spectrum). First of all, the Schrödinger equation makes it possible to express the potential V_1 in terms of $\psi_1^{(0)}$:

$$-\frac{1}{2}\frac{\mathrm{d}^2\psi_1^{(0)}}{\mathrm{d}x^2} + (V_1(x) - E_1^{(0)})\psi_1^{(0)} = 0 \quad \Rightarrow \quad V_1(x) = E_1^{(0)} + \frac{1}{2}\frac{\psi_1^{(0)\prime\prime}}{\psi_1^{(0)}}, \tag{3.1}$$

where the prime denotes a derivative with respect to x. Note that the potential is well defined because the lowest energy eigenfunction $\psi_1^{(0)}$ is nodeless.
It is also possible to factor H_1, that is to write it in the form

$$H_1 = E_1^{(0)} + \frac{1}{2}{A_1}^\dagger A_1, \tag{3.2}$$

where A_1 and $A_1^\dagger$ are defined as

$$A_1 = W_1(x) - \frac{\mathrm{d}}{\mathrm{d}x}, \quad {A_1}^\dagger = W_1(x) + \frac{\mathrm{d}}{\mathrm{d}x}. \tag{3.3}$$

Here, W_1 is a function of x that must be found in order for (3.2) to hold. One immediately sees that A_1 and ${A_1}^\dagger$ correspond to the only non-vanishing matrix elements of the supercharges Q and $Q^\dagger$, respectively (see (2.9) and (2.10)).
It is useful to compute the products ${A_1}^\dagger A_1$ and $A_1{A_1}^\dagger$, since we will encounter them in future calculations:

$${A_1}^\dagger A_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + {W_1}^2(x) + {W_1}'(x) = 2H_+, \tag{3.4}$$

$$A_1{A_1}^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + {W_1}^2(x) - {W_1}'(x) = 2H_-. \tag{3.5}$$

Here we have highlighted the connection between these products and the partner Hamiltonians introduced in (2.11). A more explicit form of H_1 then is

$$H_1 = E_1^{(0)} + \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + {W_1}^2(x) + {W_1}'(x)\right), \tag{3.6}$$

and its potential is

$$V_1 = E_1^{(0)} + \frac{1}{2}\left({W_1}^2(x) + {W_1}'(x)\right). \tag{3.7}$$

This shows that H_+ is just H_1 with a shifted potential $V_+ = V_1 - E_1^{(0)}$, which means that H_1 and H_+ have the same eigenfunctions, but the eigenvalues of H_+ are shifted so that the ground state has $E = 0$. In particular, $\psi_1^{(0)}$ is an eigenfunction of H_+ with eigenvalue $E_+ = 0$:

$$H_+\psi_1^{(0)} = 0. \tag{3.8}$$

We can now compute the expression of $W_1(x)$ for which (3.6) (or (3.2)) holds: we just need to compare (3.6) with the standard expression of the Hamiltonian H_1:

$$E_1^{(0)} + \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + {W_1}^2(x) + {W_1}'(x)\right) = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x). \tag{3.9}$$

Inserting the expression for V_1 found in (3.1) we get

$${W_1}^2(x) + {W_1}'(x) = \frac{\psi_1^{(0)\prime\prime}}{\psi_1^{(0)}}, \tag{3.10}$$

which is a differential equation for $W_1(x)$, known as the Riccati equation: one solution is

$$W_1(x) = \frac{\psi_1^{(0)\prime}}{\psi_1^{(0)}} = \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_1^{(0)}. \tag{3.11}$$

However, a way to calculate $W(x)$ without solving a differential equation is provided by supersymmetry. We know from the previous chapter (equations (2.33) and (2.40)) that eigenfunctions of H_+ with $E = 0$ can be found by solving $A_1\psi_1^{(0)} = 0$ (first order) instead of $H_+\psi_1^{(0)} = 0$ (second order). Expanding the first order order equation leads to

$$\left(W_1(x) - \frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_1^{(0)} = 0, \tag{3.12}$$

which immediately reduces to (3.11).

Finding the superpotential W_1 is the first step of the factorization method. So far, we have shown how to derive it starting from the knowledge of the ground state eigenvalue $E_1^{(0)}$ and eigenfunction $\psi_1^{(0)}$. However, looking at the form of the potential V_1, one might be able to guess a function W_1 that satisfies (3.7), without needing to know anything about the ground state of the system. Managing to do so has two benefits: first, requiring that (3.7) holds makes the ground state energy level emerge, usually in the form of an additive constant needed to adjust the equality; second, one can immediately compute $\psi_1^{(0)}$ by inverting (3.11):

$$\psi_1^{(0)}(x) = N\mathrm{e}^{\int^x W_1(y)\,dy}, \tag{3.13}$$

where N is a normalization constant. We are going to see this process in action with some later examples.

Now that we know the superpotential W_1 (either having derived it from (3.7) or having "guessed" it), it is possible to construct the partner Hamiltonian of H_1, which we shall denote as H_2, just by reversing the order of ${A_1}^\dagger$ and A_1:

$$H_2 = E_1^{(0)} + \frac{1}{2}A_1{A_1}^\dagger = H_- + E_1^{(0)}. \tag{3.14}$$

H_1 and H_2 are called partner Hamiltonians because their relationship mirrors the one between H_+ and H_- with just a shift of $E_1^{(0)}$ in the spectrum. We can now get an explicit form of H_2 using (3.5) and (3.11):

$$\begin{aligned} H_2 &= E_1^{(0)} + \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + {W_1}^2(x) - {W_1}'(x)\right) = \\ &= H_1 - {W_1}'(x) = H_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\psi_1^{(0)}. \end{aligned} \tag{3.15}$$

This means that the relation between the potentials V_1 and V_2 (often called partner potentials) is

$$V_2 = V_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\psi_1^{(0)}, \tag{3.16}$$

or, in terms of the superpotential,

$$V_2 = V_1 - {W_1}'. \tag{3.17}$$

We can now use the knowledge of H_+ and H_- developed in the previous chapter to deduce some properties of H_1 and H_2. For the bound spectrum, we know that if H_+ has an eigenstate with $E = 0$, then H_- cannot have one too; further, for eigenstates with $E > 0$, H_+ and H_- share the same spectrum, and one can use the action of the supercharges to obtain an eigenfunction of H_- from one of H_+ and vice-versa. Since $H_1 = H_+ + E_1^{(0)}$ and $H_2 = H_- + E_1^{(0)}$, it follows that H_2 has the same spectrum as H_1, except for the ground state $E = E_1^{(0)}$. Moreover, if

$$\left|\psi_1^{(n)}\right\rangle \tag{3.18}$$

is a normalized eigenket of H_1 with eigenvalue $E_1^{(n)}$, then

$$\left|\psi_2^{(n-1)}\right\rangle = \frac{1}{\sqrt{2\left(E_1^{(n)} - E_1^{(0)}\right)}} A_1 \left|\psi_1^{(n)}\right\rangle \tag{3.19}$$

is a normalized eigenket of H_2 with the same eigenvalue

$$E_2^{(n-1)} = E_1^{(n)}. \tag{3.20}$$

Conversely, if

$$\left|\psi_2^{(n)}\right\rangle \tag{3.21}$$

is a normalized eigenket of H_2 with eigenvalue $E_2^{(n)}$, then

$$\left|\psi_1^{(n+1)}\right\rangle = \frac{1}{\sqrt{2\left(E_2^{(n)} - E_1^{(0)}\right)}} {A_1}^\dagger \left|\psi_2^{(n)}\right\rangle \tag{3.22}$$

is a normalized eigenket of H_1 with the same eigenvalue

$$E_1^{(n+1)} = E_2^{(n)}. \tag{3.23}$$

For the continuous spectrum, we can deduce that there is a complete correspondence between the eigenkets of H_1 and H_2 with the same energy, since the ground state isn't affected by normalization issues; once again, the operators A_1 and ${A_1}^\dagger$ realize this correspondence. We can thus write for the continuous spectrum some relations that are analoguous to the ones for the bound spectrum:

$$E_1^{(\lambda)} = E_2^{(\lambda)}, \tag{3.24}$$

$$\left|\psi_2^{(\lambda)}\right\rangle \propto A_1 \left|\psi_1^{(\lambda)}\right\rangle, \tag{3.25}$$

$$\left|\psi_1^{(\lambda)}\right\rangle \propto {A_1}^\dagger \left|\psi_2^{(\lambda)}\right\rangle, \tag{3.26}$$

where λ is now a continuous index and the proportionality symbol indicates that one may choose the normalization he wishes for the kets he obtains.
We can summarize this by saying that A_1 generates an eigenket of H_2 form one of H_1 with the same eigenvalue, while ${A_1}^\dagger$ does the opposite (with the exception of the ground state of H_1, if it belongs to the bound spectrum). For the bound spectrum, thanks to a well-known property of nodes for its eigenfunctions, we can add that A_1 destroys a node in the wave function, while ${A_1}^\dagger$ creates one.
It can be interesting to derive the relation between the reflection and transmission amplitudes of the partner Hamiltonians. For scattering, we are dealing with the continuous spectrum, so we are going to use the relations (3.24) to (3.26) to switch between eigenfunctions of the two Hamiltonians with the same energy. First, let us define

$$W_1^\pm = \lim_{x\to\pm\infty} W_1(x); \tag{3.27}$$

thus, we can say that

$$\lim_{x\to\pm\infty} V_{1,2}(x) = \frac{1}{2}\big(W^\pm\big)^2. \tag{3.28}$$

Assuming that the incident wave from $x \to -\infty$ is plane, the asymptotic forms for the eigenfunctions ψ_1 and ψ_2 of H_1 and H_2 are

$$\begin{aligned} \lim_{x\to-\infty} \psi_{1,2}(x) &= \mathrm{e}^{ikx} + r_{1,2}\,\mathrm{e}^{-ikx}, \\ \lim_{x\to+\infty} \psi_{1,2}(x) &= t_{1,2}\,\mathrm{e}^{ik'x}, \end{aligned} \tag{3.29}$$

where

$$k = \sqrt{2\left(E - E_1^{(0)}\right) - \left(W_1^-\right)^2} \quad \text{and} \quad k' = \sqrt{2\left(E - E_1^{(0)}\right) - \left(W_1^+\right)^2}. \tag{3.30}$$

We can now write, according to (3.26), $\psi_1(x) = N{A_1}^\dagger\psi_2(x)$, where N is a normalization constant; then the asymptotic forms of ψ_1 are also given by

$$\begin{aligned} \lim_{x\to-\infty} \psi_1(x) &= N\left(W_1^- + ik\right)\mathrm{e}^{ikx} + Nr_2\left(W_1^- - ik\right)\mathrm{e}^{-ikx}, \\ \lim_{x\to+\infty} \psi_1(x) &= Nt_2\left(W_1^- + ik'\right)\mathrm{e}^{ik'x}. \end{aligned} \tag{3.31}$$

Comparing the two asymptotic forms for ψ_1 from (3.29) and (3.31) allows us to eliminate N and find the relations between $r_{1,2}$ and $t_{1,2}$ we were looking for:

$$r_2 = \frac{W_1^- + ik}{W_1^- - ik}\,r_1, \qquad t_2 = \frac{W_1^- + ik}{W_1^+ + ik'}\,t_1. \tag{3.32}$$

We can also compute

$$\begin{aligned}|r_2|^2 &= \frac{\left|W_1^- + ik\right|^2}{\left|W_1^- - ik\right|^2}|r_1|^2 = \frac{\left(W_1^-\right)^2 + 2E - \left(W_1^-\right)^2}{\left(W_1^-\right)^2 + 2E - \left(W_1^-\right)^2}|r_1|^2 = |r_1|^2,\\ |t_2|^2 &= \frac{\left|W_1^- + ik\right|^2}{\left|W_1^+ + ik'\right|^2}|t_1|^2 = \frac{\left(W_1^-\right)^2 + 2E - \left(W_1^-\right)^2}{\left(W_1^+\right)^2 + 2E - \left(W_1^+\right)^2}|t_1|^2 = |t_1|^2.\end{aligned} \tag{3.33}$$

This shows that partner potentials have the same reflection and transmission probabilities.
After creating a partner Hamiltonian H_2 for H_1, we can go on and create a partner Hamiltonian H_3 for H_2. To do so, we must write H_2 in the form

$$H_2 = E_2^{(0)} + \frac{1}{2}A_2{}^\dagger A_2 = E_2^{(0)} + \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + {W_2}^2(x) + {W_2}'(x)\right), \tag{3.34}$$

with

$$A_2 = W_2(x) - \frac{\mathrm{d}}{\mathrm{d}x}, \quad A_2{}^\dagger = W_2(x) + \frac{\mathrm{d}}{\mathrm{d}x}, \quad W_2(x) = \frac{\psi_2^{(0)\prime}}{\psi_2^{(0)}} = \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_2^{(0)}. \tag{3.35}$$

Here, $E_2^{(0)}$ and $\psi_2^{(0)}$ denote the ground state eigenvalue and eigenfunction of H_2. We can also see that the potential V_2 has the form

$$V_2 = E_2^{(0)} + \frac{1}{2}\left({W_2}^2(x) + {W_2}'(x)\right). \tag{3.36}$$

The partner H_3 can now be defined as

$$H_3 = E_2^{(0)} + \frac{1}{2}A_2 A_2{}^\dagger = E_2^{(0)} + \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + {W_2}^2(x) - {W_2}'(x)\right). \tag{3.37}$$

Its potential V_3 then is

$$V_3 = V_2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\psi_2^{(0)} = V_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\left(\psi_1^{(0)}\psi_2^{(0)}\right). \tag{3.38}$$

We also have the following relations between the eigenvalues and eigenfunctions of H_3, H_2 and H_1:

$$E_3^{(n)} = E_2^{(n+1)} = E_1^{(n+2)}, \tag{3.39}$$

$$\begin{aligned}\left|\psi_3^{(n)}\right\rangle &= \frac{1}{\sqrt{2\left(E_2^{(n+1)} - E_2^{(0)}\right)}}A_2\left|\psi_2^{(n+1)}\right\rangle =\\ &= \frac{1}{\sqrt{2^2\left(E_1^{(n+2)} - E_1^{(1)}\right)\left(E_1^{(n+2)} - E_1^{(0)}\right)}}A_2 A_1\left|\psi_1^{(n+2)}\right\rangle,\end{aligned} \tag{3.40}$$

$$\left|\psi_2^{(n)}\right\rangle = \frac{1}{\sqrt{2\left(E_2^{(n+1)} - E_2^{(0)}\right)}}A_2{}^\dagger\left|\psi_2^{(n-1)}\right\rangle, \tag{3.41}$$

$$\left|\psi_1^{(n)}\right\rangle = \frac{1}{\sqrt{2^2\left(E_1^{(n+2)}-E_1^{(1)}\right)\left(E_1^{(n+2)}-E_1^{(0)}\right)}} A_1{}^\dagger A_2{}^\dagger \left|\psi_3^{(n-2)}\right\rangle . \tag{3.42}$$

This means that H_3 has the same spectrum as H_2 except the ground state as the latter, and also the same spectrum as H_1 except for its first two energy levels. Further, the operators A_2 and $A_2{}^\dagger$ can be used to obtain eigenfunctions of H_3 from ones of H_2 and vice-versa.

One can extend the construction of partner Hamiltonians as long as he wishes, building the so-called chain or hierarchy of Hamiltonians. Agreeing that symbols with a subscript k indicate quantities belonging to the k-th Hamiltonian in the chain and that a superscript (m) indicates the m-th eigenvalue or eigenfunction, the following general relations can be obtained:

$$H_k = E_k^{(0)} + \frac{1}{2} A_k{}^\dagger A_k = E_{k-1}^{(0)} + \frac{1}{2} A_{k-1} A_{k-1}{}^\dagger, \tag{3.43}$$

where

$$A_k = W_k(x) - \frac{\mathrm{d}}{\mathrm{d}x}, \quad A_k{}^\dagger = W_k(x) + \frac{\mathrm{d}}{\mathrm{d}x}, \quad W_k(x) = \frac{\psi_k^{(0)\prime}}{\psi_k^{(0)}} = \frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_k^{(0)}; \tag{3.44}$$

$$V_k = V_{k-1} - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \psi_{k-1}^{(0)} = ... = V_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left(\psi_1^{(0)} \psi_2^{(0)} ... \psi_{k-1}^{(0)}\right). \tag{3.45}$$

The eigenvalues and eigenfunctions are related by:

$$E_k^{(n)} = E_{k-1}^{(n+1)} = ... = E_1^{(n+k-1)}, \tag{3.46}$$

$$\left|\psi_k^{(n)}\right\rangle = \frac{A_{k-1} A_{k-2} ... A_1}{\sqrt{2^{k-1}\left(E_1^{(n+k-1)}-E_1^{(k-1)}\right) ... \left(E_1^{(n+k-1)}-E_1^{(0)}\right)}} \left|\psi_1^{(n+k-1)}\right\rangle . \tag{3.47}$$

$$\left|\psi_1^{(n)}\right\rangle = \frac{A_1{}^\dagger ... A_{k-2}{}^\dagger A_{k-1}{}^\dagger}{\sqrt{2^{k-1}\left(E_1^{(n+k-1)}-E_1^{(k-1)}\right) ... \left(E_1^{(n+k-1)}-E_1^{(0)}\right)}} \left|\psi_k^{(n-k+1)}\right\rangle . \tag{3.48}$$

Equation (3.46) shows that, as one "goes up" in the chain of Hamiltonians by n steps (that is, k increases to $k+n$), the Hamiltonians have the same spectrum except the first n levels, which are removed. On the other hand, if one wants to switch between eigenfunctions of H_k and H_l (suppose $k<l$) with the same energy, one needs to act on eigenfunctions of H_k with the operators $A_{l-1} ... A_k$, or on eigenfunctions of H_l with the operators $A_k{}^\dagger ... A_{l-1}{}^\dagger$. A pictorial representation of this mechanism is given in Figure 3.1. Note that, while the hierarchy of Hamiltonians could virtually go on ad infinitum, the relations between eigenvalues and eigenfunctions suggest that this construction is only useful until we reach the Hamiltonian whose only eigenvalue is the last level in the bound spectrum of H_1.

Finally, one can obtain the reflection and transmission amplitudes for the n-th Hamiltonian in the chain by applying formula (3.32) for every potential in the chain:

$$\begin{aligned} r_n &= \left(\frac{W_1^- + ik_1}{W_1^- - ik_1}\right) ... \left(\frac{W_{n-1}^- + ik_{n-1}}{W_{n-1}^- - ik_{n-1}}\right) r_1, \\ t_n &= \left(\frac{W_1^- + ik_1}{W_1^+ + ik_1'}\right) ... \left(\frac{W_{n-1}^- + ik_{n-1}}{W_{n-1}^+ + ik_{n-1}'}\right) t_1, \end{aligned} \tag{3.49}$$

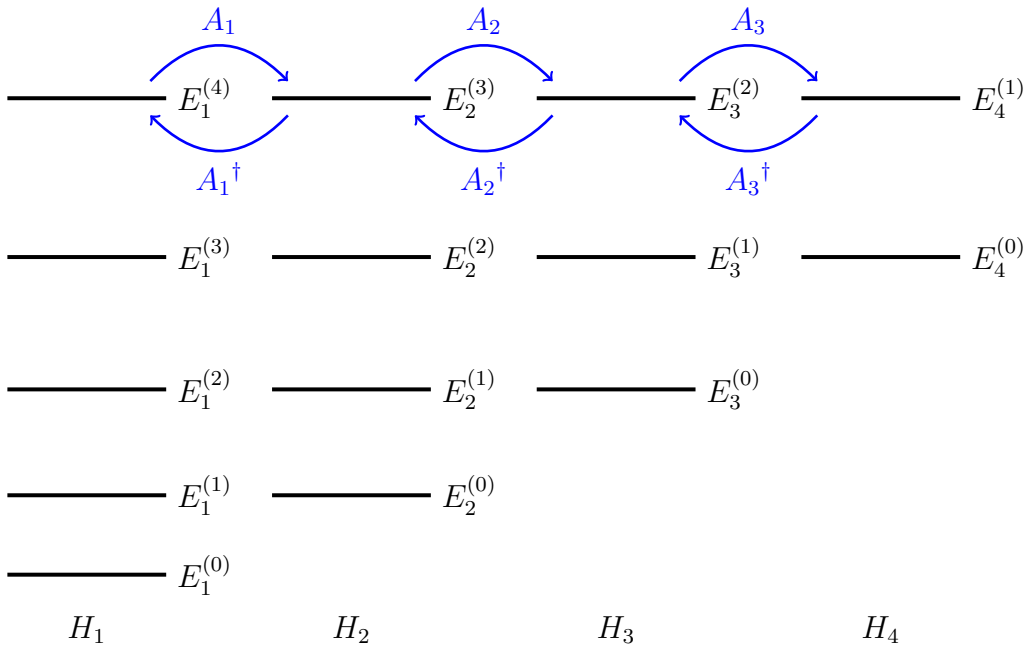

Figure 3.1: Illustration of the energy levels E of four partner Hamiltonians H, including the operators A and $A^\dagger$ that allow to switch between eigenfunctions of different Hamiltonians.

where we have set

$$W_n^\pm = \lim_{x\to\pm\infty} W_n(x), \tag{3.50}$$

$$k_n = \sqrt{2\left(E - E_n^{(0)}\right) - (W_n^-)^2}\,, \quad k_n' = \sqrt{2\left(E - E_n^{(0)}\right) - (W_n^+)^2}. \tag{3.51}$$

It is now possible to check that all the transmission and reflection probabilities are the same (the calculation is basically the same as the one we saw for the first two partner potentials):

$$|r_1|^2 = |r_2|^2 = ... = |r_n|^2 = ... \qquad |t_1|^2 = |t_2|^2 = ... = |t_n|^2 = ... \tag{3.52}$$

The creation of a chain of Hamiltonians will be particularly useful to find the bound spectrum (and its eigenfunctions) for a class of potentials which is the object of the next section. Furthermore, building a chain starting from a Hamiltonian with known spectrum and eigenfunctions generates a series of new exactly solvable potentials, whose spectrum is almost the same as the starting one and whose eigenfunctions can be easily computed using formula (3.47). This is demonstrated in the next chapter, with the potential box example. One could also pick a superpotential and obtain two partner Hamiltonians with closely related spectra and eigenfunctions. This can be particularly interesting if the partner potentials happen to have rather different forms, like the ones shown in Figure 3.2, because it can look quite surprising that they almost share the same spectrum and eigenfunctions.

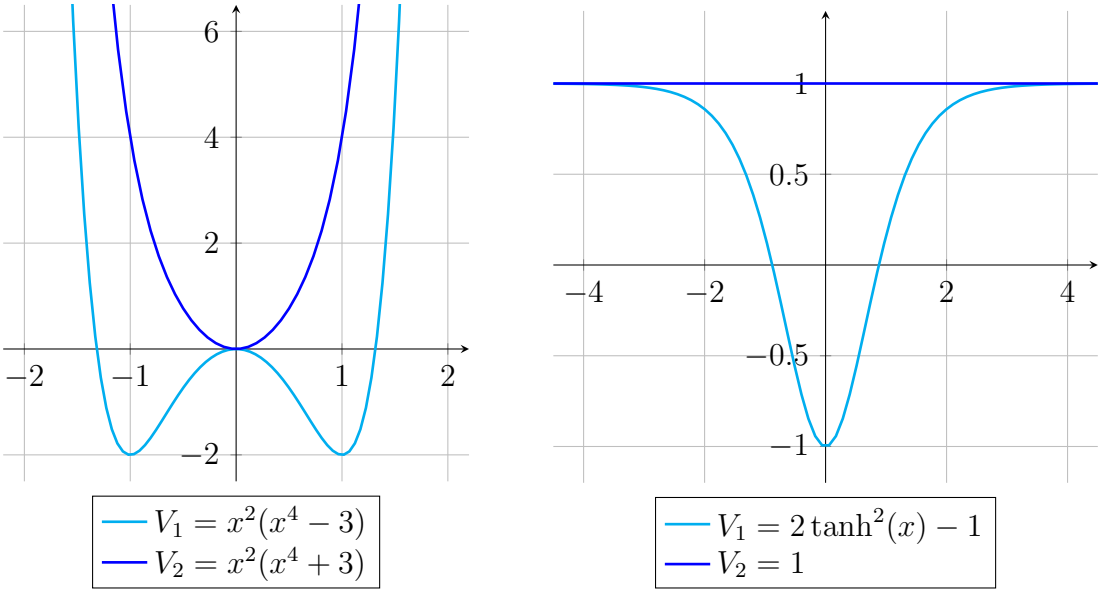

Figure 3.2: Two pairs of partner potentials that are quite different in shape. The left one is generated by the superpotential $W = x^3$, while the right one is generated by $W = \tanh(x)$. The latter is particularly interesting, because just from the properties of partner potentials and the spectral structure theorem one can deduce that the potential $V = 2\tanh^2(x) - 1$ has only one bound state, with $E = 0$.

3.2 Shape invariant potentials

Two potentials V_1 and V_2 are said to be shape invariant if they satisfy

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1), \tag{3.53}$$

where a_1 and a_2 are sets of parameters related by

$$a_2 = f(a_1) \tag{3.54}$$

for some function f, and the remainder R does not depend on x. This means that V_1 and V_2 have the same functional form, but can differ for a set of parameters and for an additive constant [9] [11].
It turns out that this class includes a wide variety of common potentials encountered in Quantum Mechanics. This is very fortunate, because applying the method of factorization and the knowledge of supersymmetry to these potentials provides an easy way to derive their spectrum.
Let us assume that we want to find the spectrum of the Hamiltonian $H_1(x; a_1)$. If we know its lower energy eigenvalue $E_1^{(0)}$ and eigenfunction $\psi_1^{(0)}$, we can use formula (3.11) to compute the superpotential

$$W_1(x; a_1) = \frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_1^{(0)}(x; a_1), \tag{3.55}$$

or alternatively we can "guess" $W_1(x; a_1)$ as discussed above; then we can build a chain of Hamiltonians H_n using the method developed in the previous section. If the partner

potentials V_n appearing in the chain are shape invariant, it means that they satisfy

$$V_n(x; a_{n-1}) = V_{n-1}(x; a_n) + R(a_{n-1}), \tag{3.56}$$

where

$$a_n = f(a_{n-1}) \tag{3.57}$$

and the function f is the same for every n. Actually, if one wants to check wether the potentials of the chain are shape invariant, he just needs to check it for the first pair of Hamiltonians, so it is sufficient that (3.53) holds. This is clear because, if the second potential differs from the first one by just a set of parameters, then the same will be true for the corresponding superpotentials W_1 and W_2, so W_2 will in turn generate a similar partner potential V_3, with the same functional change of parameters as the one between V_1 and V_2. In other words, when we generate V_3 from V_2 we are just repeating the same calculations used to build V_2 from V_1, the only difference is the set of parameters. This implies that the functional form of the remainders will always be the same, too.
Once we have checked that the first two potentials are shape invariant, we can apply condition (3.56) to every potential in the chain, and this allows us to write V_n in terms of V_1 and the remainders:

$$V_n(x; a_1) = V_1(x; a_n) + \sum_{k=1}^{n-1} R(a_k). \tag{3.58}$$

Consequently, the n-th Hamiltonian reads

$$H_n(x; a_1) = H_1(x; a_n) + \sum_{k=1}^{n-1} R(a_k). \tag{3.59}$$

This shows that the spectrum of H_n is very similar to one of H_1, but it depends on the set of parameters a_n instead of a_1 and it is shifted by the sum of the remainders. In particular, the lowest energy level of H_n is given by

$$E_n^{(0)} = E_1^{(0)}(a_n) + \sum_{k=1}^{n-1} R(a_k). \tag{3.60}$$

Further, we know from the previous section that $E_n^{(0)} = E_1^{(n-1)}$, so we can use (3.60) to get the whole spectrum of the original Hamiltonian H_1:

$$E_1^{(n)} = E_1^{(0)}(a_{n+1}) + \sum_{k=1}^{n} R(a_k). \tag{3.61}$$

(3.59) also shows that $H_n(x; a_1)$ has the same eigenfunctions as $H_1(x; a_n)$, which is just our starting Hamiltonian with the parameters changed form a_1 to $a_n = f^n(a_1)$. In particular, its ground state eigenfunction is

$$\psi_n^{(0)}(x; a_1) = \psi_1^{(0)}(x; a_n) = N \mathrm{e}^{\int^x W_1(y; a_n)\, dy}, \tag{3.62}$$

where N is a normalization constant. We also know from the previous section that to "transform" an eigenket of H_n into one of H_1 with the same energy we just need to apply the operators $A^\dagger$, so the eigenkets of H_1 can be written as

$$\left|\psi_1^{(n)}(a_1)\right\rangle = N' A^\dagger(a_1) A^\dagger(a_2) \dots A^\dagger(a_n) \left|\psi_1^{(0)}(a_{n+1})\right\rangle, \tag{3.63}$$

where N' is another normalization constant. Note that, because of the shape invariance condition, the operators $A^\dagger$ all have the same form and only differ in the parameters. The same is true for the A, of course.
The shape invariance condition also simplifies relation (3.49) for the reflection and transmission amplitudes, because the superpotentials W_n are just the same as the first one, W_1, but with shifted parameters, so one doesn't need to compute all the superpotentials (and the same is true for the k and the k'):

$$\begin{aligned} r_n &= \left(\frac{W_1^-(a_1)+ik(a_1)}{W_1^-(a_1)-ik(a_1)}\right)...\left(\frac{W_1^-(a_{n-1})+ik(a_{n-1})}{W_1^-(a_{n-1})-ik(a_{n-1})}\right) r_1, \\ t_n &= \left(\frac{W_1^-(a_1)+ik(a_1)}{W_1^+(a_1)+ik'(a_1)}\right)...\left(\frac{W_1^-(a_{n-1})+ik(a_{n-1})}{W_1^+(a_{n-1})+ik'(a_{n-1})}\right) t_1. \end{aligned} \tag{3.64}$$

To summarize, we have seen that, if the first two partner potentials V_1 and V_2 in the chain of Hamiltonians of H_1 are shape invariant, then all the other potentials of the chain will be shape invariant as well. In this case, we just need to study their relations in terms of the parameters a_n and the remainders $R(a_n)$, and the spectrum and eigenfunctions of H_1 follow directly from (3.61) and (3.63). We can see this method as a generalization (for shape invariant potentials) of the solution of the one dimensional harmonic oscillator involving raising and lowering operators: here, the counterparts of those operators are A and $A^\dagger$, because the shape invariance condition allows us to use them to switch between higher and lower energy eigenkets.

3.3 Isospectral deformation

In this section we shall see, starting from a given potential V_1, how to use supersymmetric partner potentials to build a family of strictly isospectral potentials, that is potentials with the same spectrum but also the same transmission and reflection probabilities as V_1. Indeed, the search for strictly isospectral potentials is motivated by practical applications, for example in the $\alpha - \alpha$ scattering some ambiguities in the potential were observed, even if the transmission and reflection coefficients were fixed [9] [10] [11].
We start by asking this question: given a potential V_1 (written in terms of a superpotential W_1)

$$V_1 = E_1^{(0)} + \frac{1}{2}\left({W_1}^2 + {W_1}'\right), \tag{3.65}$$

and given its partner potential

$$V_2 = E_1^{(0)} + \frac{1}{2}\left({W_1}^2 - {W_1}'\right), \tag{3.66}$$

is the superpotential W_1 unique? That is, can we find other superpotentials $\widetilde{W_1}$ (and corresponding potentials $\widetilde{V_1}$) that generate the same partner V_2? If such superpotentials $\widetilde{W_1}$ exist, we assume that they have the form

$$\widetilde{W_1}(x) = W_1(x) + \phi(x). \tag{3.67}$$

Requiring that V_2 is generated by W_1 as well as $\widetilde{W_1}$ means that the following equation must hold:

$$V_2 = E_1^{(0)} + \frac{1}{2}\left({W_1}^2 - {W_1}'\right) = E_1^{(0)} + \frac{1}{2}\left(\widetilde{W_1}^2 - \widetilde{W_1}'\right). \tag{3.68}$$

Inserting (3.67) into (3.68), one gets

$$\phi' = \phi^2 + 2W_1\phi, \tag{3.69}$$

which is a Bernoulli differential equation. It is known from the theory of differential equations that a useful substitution to solve this is

$$y(x) = \frac{1}{\phi(x)}; \tag{3.70}$$

inserting it into (3.69) gives a linear differential equation for y:

$$y' + 2W_1 y + 1 = 0. \tag{3.71}$$

The solution to this can be easily found using the integrating factor method: the result is

$$y(x) = -\mathrm{e}^{-2\int^x W_1(t)\,dt}\left(\int^x \mathrm{e}^{2\int^u W_1(t)\,dt}\,du + \lambda_1\right), \tag{3.72}$$

where λ_1 is an integration constant. It is interesting to note that this solution can be recast in terms of the ground state wave function ${\psi_1}^{(0)}$ of V_1 (using formula (3.13)):

$$y(x) = -\left({\psi_1}^{(0)}(x)\right)^{-2}\left[\int^x \left({\psi_1}^{(0)}(u)\right)^2 du + \lambda_1\right]. \tag{3.73}$$

Here, we have omitted the normalization constant for the wave function, since its presence would just rescale the real parameter λ_1. We now have the function $\phi(x)$ we were looking for:

$$\phi(x) = -\left({\psi_1}^{(0)}(x)\right)^2\left[\int^x \left({\psi_1}^{(0)}(u)\right)^2 du + \lambda_1\right]^{-1} = -\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[I_1(x) + \lambda_1\right], \tag{3.74}$$

where we have set

$$I_1(x) = \int^x \left({\psi_1}^{(0)}(u)\right)^2 du. \tag{3.75}$$

This means we have found an entire family of superpotentials $\widetilde{W_1}$, depending on the real parameter λ_1, which generate V_2 as a partner potential:

$$\widetilde{W_1}(x) = W_1(x) - \frac{\mathrm{d}}{\mathrm{d}x}\ln\left[I_1(x) + \lambda_1\right]. \tag{3.76}$$

Note that, since in the bound spectrum $\lim_{x\to\pm\infty}{\psi_1}^{(0)}(x) = 0$, the superpotentials W_1 and $\widetilde{W_1}$ have the same asymptotic limits. Now, the corresponding potentials $\widetilde{V_1}$ can be found using the properties of partner Hamiltonians:

$$\widetilde{V_1} = V_2 + \widetilde{W_1}' = V_1 - W_1' + \widetilde{W_1}' = V_1 + \phi', \tag{3.77}$$

which gives the final result

$$\widetilde{V_1} = V_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\left[I_1(x) + \lambda_1\right]. \tag{3.78}$$

The $\widetilde{V_1}$ all have V_2 as partner potential, so they have the same spectrum as the starting potential V_1; moreover, we know from (3.32) that partner potentials have the same

reflection and transmission probabilities, so the family of $\widetilde{V}_1$ has the same reflection and transmission probabilities as V_1. We can also see that the original potential V_1 is obtained in the limit $\lambda_1 \to \infty$.
If we are interested in the eigenfunctions $\widetilde{\psi}_n$ of the family $\widetilde{V}_1$, we can easily find them using the relations between partner potentials (3.19) and (3.22):

$$\widetilde{\psi_1}^{(n)} \propto \widetilde{A}_1^\dagger \psi_2^{(n-1)} \propto \widetilde{A}_1^\dagger A_1 {\psi_1}^{(n)}, \tag{3.79}$$

where, of course,

$$\widetilde{A}_1^\dagger = \widetilde{W}_1 + \frac{\mathrm{d}}{\mathrm{d}x}. \tag{3.80}$$

This leaves out the ground state wave function $\widetilde{\psi}_1^{(0)}$, since it cannot be obtained from eigenfunctions of V_2, but we have a simpler formula for it (see (3.13)):

$$\widetilde{\psi_1}^{(0)}(x) = N_1 \mathrm{e}^{\int^x \widetilde{W}_1(y)\,dy} = N_1 \mathrm{e}^{\int^x [W_1(y) - \frac{\mathrm{d}}{\mathrm{d}y}\ln(I_1(y)+\lambda_1)]\,dy} = N_1 \frac{{\psi_1}^{(0)}(x)}{I_1(x)+\lambda_1}. \tag{3.81}$$

There is a particular choice of the lower boundary of the integration range for I_1, namely $u = -\infty$, that allows us to easily compute the normalization constant N_1:

$$\begin{gathered} \int_{-\infty}^{+\infty} {N_1}^2 \left({\psi_1}^{(0)}(u)\right)^2 (I_1(u)+\lambda_1)^{-2}\,du = {N_1}^2 \int_{\lambda_1}^{\lambda_1+1} \frac{dt}{t^2} = \frac{{N_1}^2}{\lambda_1(\lambda_1+1)} = 1 \\ \Rightarrow N_1 = \sqrt{\lambda_1(\lambda_1+1)}. \end{gathered} \tag{3.82}$$

In the first line, we have made the substitution $I_1(u) + \lambda_1 = t$, so if $x = -\infty$ then $I_1 = 0$ and $t = \lambda_1$, and similarly if $x = +\infty$ then $I_1 = 1$ and $t = \lambda_1 + 1$. Now, because of the reality property of one dimensional wave functions, the expression for N_1 forces us to limit λ_1 to the ranges $\lambda_1 < -1$ and $\lambda_1 > 0$.
So, using this procedure of isospectral deformation, we have built a family of strictly isospectral potentials from a starting one, and we have obtained an expression for their eigenfunctions. Thus, if this is applied to a potential that can be analytically solved, one is able to generate a family of potentials which are also solvable, but without having to tackle the Schrödinger equation directly.
It is possible to extend this procedure starting from a "higher" potential in the chain of partner potentials, instead of V_2: we will see V_3 as an example, then we will generalize to V_n.
From our previous discussion we can realize that, starting from the knowledge of V_3, it is possible to generate a family of potentials $\widetilde{V}_2$ that are strictly isospectral to V_2. Recalling formula (3.78), this family is given by

$$\widetilde{V}_2 = V_2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left[I_2(x) + \lambda_2\right], \tag{3.83}$$

where

$$I_2(x) = \int^x \left[\psi_2^{(0)}(u)\right]^2 du \propto \int^x \left[A_1 \psi_1^{(1)}(u)\right]^2 du \tag{3.84}$$

and λ_2 is an integration constant, but also the parameter on which the family depends. Now, we can generate a family of potentials that are isospectral to V_1 starting from each one of the $\widetilde{V}_2$: this creates a two parameter family of potentials

$$\widetilde{V}_1 = V_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left[I_2(x) + \lambda_2\right]\left[I_1(x) + \lambda_1\right]. \tag{3.85}$$

The eigenfunctions for this family will be given by

$$\begin{aligned}\widetilde{\psi}_1^{(0)}(x) &\propto \frac{{\psi_1}^{(0)}(x)}{I_1(x)+\lambda_1},\\ \widetilde{\psi}_1^{(1)}(x) &\propto \widetilde{A}_1^{\dagger} A_1 \frac{{\psi_1}^{(1)}(x)}{I_2(x)+\lambda_2},\\ \widetilde{\psi}_1^{(n)}(x) &\propto \widetilde{A}_1^{\dagger}\widetilde{A}_2^{\dagger} A_2 A_1 {\psi_1}^{(n)}(x) \quad n \geq 2,\end{aligned} \tag{3.86}$$

with

$$\widetilde{A}_2^{\dagger} = \widetilde{W}_2 + \frac{\mathrm{d}}{\mathrm{d}x}, \quad \widetilde{W}_2(x) = W_2(x) - \frac{\mathrm{d}}{\mathrm{d}x}\ln\left[I_2(x)+\lambda_2\right]. \tag{3.87}$$

Because of observations on normalization constants for these wave functions, similarly to what we have seen for λ_1, we must limit the parameters λ_i, with $i = 1, 2$, to the ranges $\lambda_i < -1$ and $\lambda_i > 0$.
So, starting from V_3 instead of V_2, we have built a two parameter family of isospectral potentials instead of a one parameter family. If we have a potential with n bound states, we can start this construction from the partner potential V_{n+1}, which is the first potential with no bound states, so that its partner V_n has just one bound state. In this case, we can define, for $i = 1, 2, \ldots n$,

$$I_i(x) = \int^x \left[\psi_i^{(0)}(u)\right]^2 du \propto \int^x \left[A_{i-1}\ldots A_1 \psi_1^{(i-1)}(u)\right]^2 du, \tag{3.88}$$

$$\widetilde{W}_i(x) = W_i(x) - \frac{\mathrm{d}}{\mathrm{d}x}\ln\left[I_i(x)+\lambda_i\right], \tag{3.89}$$

and

$$\widetilde{A}_i^{\dagger} = \widetilde{W}_i + \frac{\mathrm{d}}{\mathrm{d}x}. \tag{3.90}$$

Here, we have introduced n real parameters λ_i. Then the n parameter family of potentials which are strictly isospectral to V_1 is:

$$\widetilde{V}_1 = V_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\left[I_n(x)+\lambda_n\right]\ldots\left[I_1(x)+\lambda_1\right]. \tag{3.91}$$

The eigenfunctions for this family are

$$\begin{aligned}\widetilde{\psi}_1^{(0)}(x) &\propto \frac{{\psi_1}^{(0)}(x)}{I_1(x)+\lambda_1},\\ \widetilde{\psi}_1^{(1)}(x) &\propto \widetilde{A}_1^{\dagger} A_1 \frac{{\psi_1}^{(1)}(x)}{I_2(x)+\lambda_2},\\ &\vdots\\ \widetilde{\psi}_1^{(n-1)}(x) &\propto \widetilde{A}_1^{\dagger}\ldots\widetilde{A}_{n-1}^{\dagger} A_{n-1}\ldots A_1 \frac{{\psi_1}^{(n-1)}(x)}{I_n(x)+\lambda_n}.\end{aligned} \tag{3.92}$$

Finally, we address the uniqueness of isospectral deformation. We ask the following question: if we apply the technique we have just described to the newly obtained family $\widetilde{V}_1$, do we get another family of isospectral potentials or is $\widetilde{V}_1$ unique? The correct option is the second one, as we shall see with a brief calculation. We present this proof just for the one parameter family for the sake of simplicity, but the reasoning is exactly the

same for families with more parameters.
If we apply the isospectral deformation to the one parameter family $\widetilde{V}_1$, we get a new family parameterized by μ_1:

$$\widetilde{\widetilde{V}}_1 = V_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left[I_1(x) + \lambda_1\right][\widetilde{I}_1(x) + \mu_1], \tag{3.93}$$

where

$$\begin{aligned} \widetilde{I}_1(x) = \int_{-\infty}^{x} \left(\widetilde{\psi}_1^{(0)}(u)\right)^2 du = \lambda_1(\lambda_1+1) \int_{-\infty}^{x} \left(\frac{{\psi_1}^{(0)}(u)}{I_1(u)+\lambda_1}\right)^2 du = \\ = \lambda_1(\lambda_1+1)\left(\frac{1}{\lambda_1} - \frac{1}{I_1(x)+\lambda_1}\right). \end{aligned} \tag{3.94}$$

Inserting this expression into (3.93) we get

$$\widetilde{\widetilde{V}}_1 = V_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left[I_1(x) + \lambda_1\right]\left[\frac{(\lambda+1)I_1(x)}{I_1(x)+\lambda_1} + \mu_1\right] = V_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left[I_1(x) + \nu_1\right], \tag{3.95}$$

with

$$\nu_1 = \frac{\lambda_1\mu_1}{\lambda_1+\mu_1+1}. \tag{3.96}$$

It can be shown that, if λ_1 and μ_1 satisfy $\lambda_1 < -1 \vee \lambda_1 > 0$ and $\mu_1 < -1 \vee \mu_1 > 0$, then the same is true for ν_1: $\nu_1 < -1 \vee \nu_1 > 0$. This means that ν_1 just reparameterizes the original family $\widetilde{V}_1$ without adding any new potentials, thus the proof of the uniqueness of isospectral deformation is complete.
So, we have seen that the knowledge of supersymmetric partner potentials has provided a quite simple and straightforward procedure that, starting from a potential V_1 with n bound states, generates a unique n parameter family of potentials that are strictly isospectral to V_1. Indeed, the development of Supersymmetric Quantum Mechanics has been able to revive the interest in the search for isospectral potentials.

Examples of applications

4.1 Partner potentials of the one dimensional potential box

In this section we compute some partner Hamiltonians for the one-dimensional potential box, along with their eigenvalues and eigenfunctions, in order to get an intuition of the chain of Hamiltonians mechanism we have discussed in the previous chapter [9].
Let us take the potential box in the form

$$V_1(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & x \leq 0 \vee x \geq L \end{cases}. \tag{4.1}$$

The eigenvalue problem for this potential is well known in quantum mechanical theory: the energy levels are

$$E_1^{(n)} = \frac{(n+1)^2\pi^2}{2L^2} \qquad n = 0, 1, 2... \tag{4.2}$$

where we have adjusted the quantum number n so that the lowest eigenvalue has $n = 0$. The corresponding eigenfunctions are

$$\psi_1^{(n)} = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{(n+1)\pi x}{L} & 0 \leq x \leq L \\ 0 & x \leq 0 \vee x \geq L \end{cases}. \tag{4.3}$$

From now on, we are going to focus on the range $[0, L]$, since it is quite obvious that, outside of it, all the partner potentials will be infinite and all their eigenfunctions will vanish.
The first step in the construction of the chain of Hamiltonians is the computation of the superpotential W_1: using (3.11) we find

$$W_1 = \frac{\pi}{L} \frac{\cos \frac{\pi x}{L}}{\sin \frac{\pi x}{L}} = \frac{\pi}{L} \cot \frac{\pi x}{L}. \tag{4.4}$$

Inserting this in (3.16) allows us to obtain the first partner potential V_2:

$$V_2 = -\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\pi}{L} \cot \frac{\pi x}{L} \right) = \frac{\pi^2}{L^2} \csc^2 \frac{\pi x}{L} \tag{4.5}$$

We know that this potential will have the same energy levels as V_1 except the one with $n = 0$: we can write these levels as as

$$E_2^{(n)} = \frac{(n+2)^2\pi^2}{2L^2} \qquad n = 0, 1, 2... \tag{4.6}$$

The corresponding eigenfunctions $\psi_1^{(n)}$ can be found by acting on the $\psi_1^{(n)}$ with the operator

$$A_1 = \frac{\pi}{L} \cot \frac{\pi x}{L} - \frac{\mathrm{d}}{\mathrm{d}x}, \tag{4.7}$$

which, after adjusting the normalization according to (3.19), results in

$$\psi_2^{(n)} = \sqrt{\frac{2}{(n+1)(n+3)L}}\left[\cot\frac{\pi x}{L}\,\sin\frac{(n+2)\pi x}{L} - (n+2)\cos\frac{(n+2)\pi x}{L}\right]. \tag{4.8}$$

The general expression might look complicated, but we can compute the first three functions to see what they look like:

$$\begin{gathered}\psi_2^{(0)} = \sqrt{\frac{8}{3L}}\sin^2\frac{\pi x}{L}, \quad \psi_2^{(1)} = \sqrt{\frac{4}{L}}\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}, \\ \psi_2^{(2)} = \sqrt{\frac{32}{15L}}\sin^2\frac{\pi x}{L}\left(5 - 6\sin^2\frac{\pi x}{L}\right).\end{gathered} \tag{4.9}$$

These eigenfunctions are plotted in Fig. 4.1, where they are compared with the ones belonging to the original potential.
We can go one step further and compute the third partner potential V_3. To do so, let us find the superpotential W_2:

$$W_2 = \frac{2\pi}{L}\cot\frac{\pi x}{L}, \tag{4.10}$$

from which we have

$$V_2 = \frac{\pi^2}{L^2}\csc^2\frac{\pi x}{L} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2\pi}{L}\cot\frac{\pi x}{L}\right) = \frac{3\pi^2}{L^2}\csc^2\frac{\pi x}{L}. \tag{4.11}$$

As for the eigenvalues, one more energy level is removed from the bottom, so

$$E_3^{(n)} = \frac{(n+3)^2\pi^2}{2L^2} \qquad n = 0, 1, 2... \tag{4.12}$$

We can find the eigenfunctions by using

$$A_2 = \frac{2\pi}{L}\cot\frac{\pi x}{L} - \frac{\mathrm{d}}{\mathrm{d}x} \tag{4.13}$$

on the ones from the previous potential: the general expression is quite long, so we just show the first two,which are once again plotted in Fig. 4.1:

$$\begin{gathered}\psi_3^{(0)} = \sqrt{\frac{4}{5L}}\left(\cos\frac{\pi x}{L}\sin\frac{2\pi x}{L} - 2\sin\frac{\pi x}{L}\cos\frac{2\pi x}{L}\right), \\ \psi_3^{(1)} = \sqrt{\frac{128}{5L}}\sin^3\frac{\pi x}{L}\cos\frac{\pi x}{L}.\end{gathered} \tag{4.14}$$

We could go on building partner Hamiltonians as long as we like, since the spectrum of H_1 is infinite, but we stop here to avoid boring calculations, having already seen what partner potentials and their eigenfunctions look like. Note that, in this process, we have found the eigenvalues and eigenfunctions for potentials in the form $V = A\csc(ax)$, which can be rather complicated to compute if one tries to solve the Schrödinger equation.

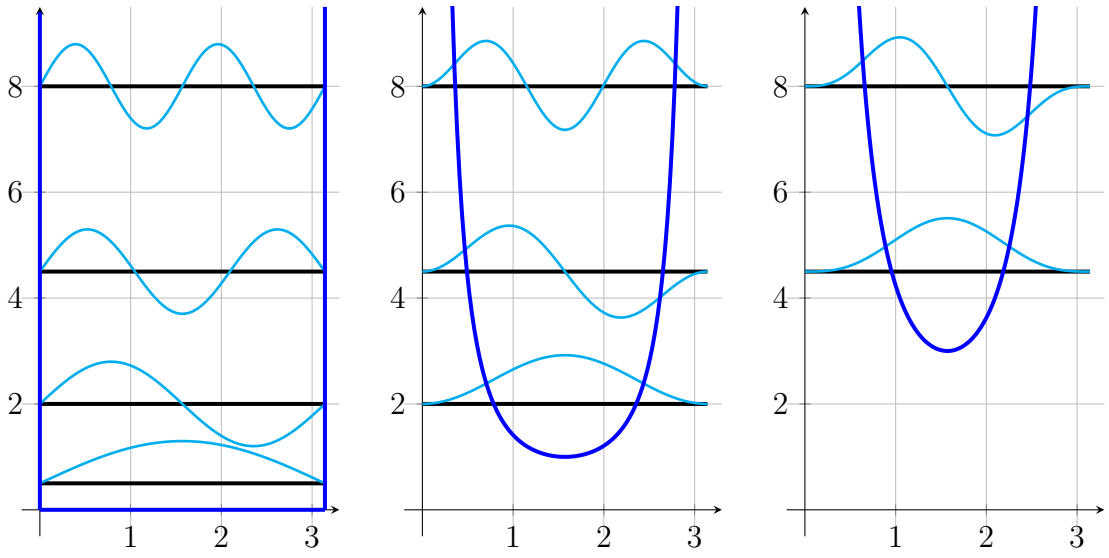

Figure 4.1: From left to right, the potential box with $L = \pi$ and its first two partner potentials. The low energy spectrum is also plotted, and every level is accompanied by its eigenfunction.

4.2 The one dimensional harmonic oscillator

We said in section 3.2 that the chain of Hamiltonians is a procedure that extends, for shape invariant potentials, the raising and lowering operators method developed for the harmonic oscillator. In this section we shall familiarize with shape invariant potentials by seeing how this factorization method indeed reduces to the annihilation and creation operators if it is applied to the harmonic oscillator [11].
Setting $\hbar = m = 1$, the original potential is

$$V_1 = \frac{1}{2}\omega^2 x^2. \tag{4.15}$$

We want to write it in the form (3.7): we can immediately notice that two superpotentials that can do the trick are

$$W_1 = \pm\omega x. \tag{4.16}$$

If we take the + option and we insert it into Eq. (3.7), we get $E_1^{(0)} = -\frac{\omega}{2}$: this is in contrast with a well known quantum mechanical theorem, which states that the energy levels of a system are always greater than the minimum of the potential, and in our case the minimum is zero. This means that we must rule out this superpotential[1] and take instead $W_1 = -\omega x$: now Eq. (3.7) gives

$$E_1^{(0)} = \frac{\omega}{2}, \tag{4.17}$$

which is the expected result for the harmonic oscillator. We can also compute the ground state eigenfunction

$$\psi_1^{(0)} = \mathrm{e}^{-\int^x \omega y\, dy} = N\mathrm{e}^{-\omega x^2/2}, \tag{4.18}$$

[1]Another indicator that $W_1 = \omega x$ is not a "good" superpotential is the fact that, if we insert it in formula (3.13), it generates a non normalizable ground state eigenfunction.

where N is a normalization constant (that can be found to be $N = \sqrt[4]{\omega/\pi}$). Now we can compute the partner potential

$$V_2 = V_1 - W_1' = \frac{1}{2}\omega^2 x^2 + \omega, \tag{4.19}$$

which is just the same as V_1, but shifted by ω: this means that this potential is shape invariant, with parameters that are all the same (the only possible parameter is ω) and identical remainders of ω:

$$a_1 = a_2 = ... = a_n = \omega, \qquad R(a_1) = R(a_2) = ... = R(a_n) = \omega. \tag{4.20}$$

Eq. (3.61) then allows us to get the entire energy spectrum of V_1:

$$E_1^{(n)} = \frac{\omega}{2} + \sum_{k=1}^{n} \omega = \omega\left(n + \frac{1}{2}\right). \tag{4.21}$$

Thanks to the shape invariance condition we know the ground state eigenfunction for every partner potential: in fact, they are identical to the first one since the parameters are all the same, so

$$\psi_n^{(0)} = \sqrt[4]{\frac{\omega}{\pi}} \mathrm{e}^{-\omega x^2/2}. \tag{4.22}$$

In order to compute the eigenfunctions for V_1, we need the operators

$$A_n{}^\dagger = W_1(a_n) + \frac{\mathrm{d}}{\mathrm{d}x} = -\omega x + \frac{\mathrm{d}}{\mathrm{d}x}; \tag{4.23}$$

if we apply them to the ground state eigenfunctions as in formula (3.63) we get:

$$\psi_1^{(n)} \propto \left(-\omega x + \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \mathrm{e}^{-\omega x^2/2}. \tag{4.24}$$

Note that the operators $A_n{}^\dagger$ are just the usual creation operators (and computing the A_n would give the annihilation ones), so the formula for the eigenfunctions is the same as the one resulting from the use of these operators. We can see that the shape invariance method reduces to the "classical" one because the A_n, the $A_n{}^\dagger$ and the ground state eigenfunctions are the same for all the Hamiltonians in the chain, so it looks like we are always working on the original Hamiltonian H_1.

4.3 Hydrogenlike atoms

In this section we derive the spectrum and eigenfunctions for hydrogenlike atoms using the chain of Hamiltonians method and the shape invariance condition [12].
The potential energy of an electron in a hydrogenlike atom is given by the Coulomb potential:

$$V = -\frac{Ze^2}{r}. \tag{4.25}$$

Since this is a radial potential, its eigenfunctions can be expressed as

$$\psi(r, \theta, \phi) = \frac{\chi(r)}{r} Y_{l,m_l}(\theta, \phi), \tag{4.26}$$

where Y_{l,m_l} are the spherical harmonics. Inserting this into the Schrödinger equation gives the equation for χ (with $\hbar = m = 1$):

$$-\frac{1}{2}\frac{\mathrm{d}^2\chi}{\mathrm{d}r^2} + \left(\frac{l(l+1)}{2r^2} - \frac{Ze^2}{r}\right)\chi = E\chi. \tag{4.27}$$

This looks like a one-dimensional Schrödinger equation with the potential

$$V_1 = \frac{l(l+1)}{2r^2} - \frac{Ze^2}{r}, \tag{4.28}$$

which we are now going to study. First, we want to factor this potential: looking at its expression, we expect that its superpotential will have the form

$$W_1 = \alpha + \frac{\beta}{r}. \tag{4.29}$$

Inserting this ansatz into Eq. (3.7) gives

$$\frac{\alpha^2}{2} + \frac{\alpha\beta}{r} + \frac{\beta^2}{2r^2} - \frac{\beta}{2r^2} + E_1^{(0)} = \frac{l(l+1)}{2r^2} - \frac{Ze^2}{r}. \tag{4.30}$$

We can now equate the terms with the same power of r and solve for α, β and $E_1^{(0)}$:[2]

$$\begin{cases} l(l+1) = \beta^2 - \beta \\ \alpha\beta = -Ze^2 \\ E_1^{(0)} + \frac{\alpha^2}{2} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \beta = l+1 \\ \alpha = -\frac{Ze^2}{l+1} \\ E_1^{(0)} = -\frac{Z^2e^4}{2(l+1)^2} \end{cases} . \tag{4.31}$$

This already provides an expression for the ground state energy (which, for $l = 0$, is the same as the one known from the standard study of hydrogenlike atoms), as well as the superpotential

$$W_1 = -\frac{Ze^2}{l+1} + \frac{l+1}{r}. \tag{4.32}$$

The ground state eigenfunction then is

$$\chi_1^{(0)} = N_{0,l}\, r^{l+1} \mathrm{e}^{-Ze^2 r/(l+1)}. \tag{4.33}$$

We can now go on and compute the partner potential for V_1:

$$V_2 = \frac{l(l+1)}{2r^2} - \frac{Ze^2}{r} + \frac{l+1}{r^2} = \frac{(l+1)(l+2)}{2r^2} - \frac{Ze^2}{r}. \tag{4.34}$$

Its form is identical to the one of V_1, except that l is replaced with $l+1$, which means that the radial Coulomb potential is shape invariant. The parameters for the chain of partner potentials are

$$a_1 = l, \quad a_2 = l+1, \quad \ldots \quad a_\nu = l+\nu-1, \tag{4.35}$$

while the remainders R_ν all vanish. Formula (3.61) then gives the energy spectrum for hydrogenlike atoms:

$$E_1^{(\nu)} = E_1^{(0)}(l+\nu) = -\frac{Z^2e^4}{2(l+\nu+1)^2}. \tag{4.36}$$

[2]The first equation also admits the solution $\beta = -l$; however, this solution is not acceptable, because it would generate a non normalizable ground state eigenfunction $\psi_1^{(0)} \propto r^{-l}e^{Zer/l}$.

This expression might look somewhat confusing, since the indexing of energy levels that arises from this method is different from the typical one used for atoms. One can however restore the usual indexing by setting the principal quantum number

$$n = l + \nu + 1, \tag{4.37}$$

so the energy levels indeed are

$$E_1^{(n)} = -\frac{Z^2 e^4}{2n^2}. \tag{4.38}$$

The range of the azimuthal quantum number can simply be obtained by noting that, because of our construction,

$$\nu \geq 0 \;\Rightarrow\; n \geq l + 1 \;\Rightarrow\; l \leq n - 1. \tag{4.39}$$

We are now ready to derive the eigenfunctions. The first one was already found in (4.33): it corresponds to $\nu = 0$, so according to our definition of n (4.37) it holds for $l = n - 1$, that is for the highest possible angular momentum for a given n. We can then rewrite the eigenfunction in the form

$$\chi_1^{(l=n-1)} = N_{0,n}\, r^n \mathrm{e}^{-Ze^2 r/n}, \tag{4.40}$$

which is indeed the usual formula for eigenfunctions with $l = n-1$. Thanks to the shape invariance condition, we can immediately find the ground state eigenfunctions for any potential V_ν:

$$\chi_\nu^{(0)} = N'_{\nu,l}\, r^{l+\nu} \mathrm{e}^{-Ze^2 r/(l+\nu)}. \tag{4.41}$$

In order to transform them into higher states eigenfunctions for V_1, we need the "raising" operators: the first one is found from W_1:

$$A_1{}^\dagger = -\frac{Ze^2}{l+1} + \frac{l+1}{r} + \frac{\mathrm{d}}{\mathrm{d}r}, \tag{4.42}$$

so by the shape invariance condition

$$A_\nu{}^\dagger = -\frac{Ze^2}{l+\nu} + \frac{l+\nu}{r} + \frac{\mathrm{d}}{\mathrm{d}r}. \tag{4.43}$$

Now we just need to use formula (3.48) to get that, for $\nu \geq 1$,

$$\chi_1^{(\nu)} = N_{\nu,l} \left(-\frac{Ze^2}{l+1} + \frac{l+1}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\right) ... \left(-\frac{Ze^2}{l+\nu} + \frac{l+\nu}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\right) \left(r^{l+\nu+1} \mathrm{e}^{-Ze^2 r/(l+\nu+1)}\right). \tag{4.44}$$

Similarly to what we have seen for $\chi_1^{(0)}$, these are eigenfunctions for states with $l = n - \nu - 1$, so we can rewrite them in terms of the principal quantum number:

$$\chi_1^{(l=n-\nu-1)} = N_{\nu,n} \left(-\frac{Ze^2}{n-\nu} + \frac{n-\nu}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\right) ... \left(-\frac{Ze^2}{n-1} + \frac{n-1}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\right) \left(r^n \mathrm{e}^{-Ze^2 r/n}\right). \tag{4.45}$$

For example, for $\nu = 1$ this gives

$$\chi_1^{(l=n-2)} = N_{1,n}\, r^{n-1} \left[1 - \frac{Ze^2 r}{n(n-1)}\right] \mathrm{e}^{-Ze^2 r/n}. \tag{4.46}$$

We now have an expression for all the eigenfunctions of the bound spectrum, which is equivalent to the usual one containing Laguerre polynomials.

4.4 The Pauli equation

In this section we discuss the non relativistic description of an electron in an external magnetic field in terms of supersymmetry [9].
Since the electron has spin $\frac{1}{2}$, we must use the Pauli equation

$$H\left|\psi\right\rangle = \frac{1}{2}\left[\left(\vec{p}+\vec{A}\right)^2 + \left(\vec{\nabla}\times\vec{A}\right)\cdot\vec{\sigma}\right]\left|\psi\right\rangle = i\frac{\partial}{\partial t}\left|\psi\right\rangle, \tag{4.47}$$

where we have set $\hbar = m = e = 1$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ contains the three Pauli matrices. Let us make some simplifying assumptions: first, we are not going to deal with magnetic fields that change in time, so we will focus on the time independent equation only. Also, let us set the magnetic field along the z axis and restrict the motion of the electron on the xy plane. The Pauli equation now reads

$$H\left|\psi\right\rangle = \frac{1}{2}\left[(p_x+A_x)^2 + (p_y+A_y)^2 + (\vec{\nabla}\times\vec{A})_z\,\sigma_z\right]\left|\psi\right\rangle = E\left|\psi\right\rangle. \tag{4.48}$$

The form of the Hamiltonian closely resembles the supersymmetric one in (2.11)[3]; indeed one can define the supercharges

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}}\left[(p_x+A_x) - i(p_y+A_y)\right]\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \\ Q^\dagger &= \frac{1}{\sqrt{2}}\left[(p_x+A_x) + i(p_y+A_y)\right]\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \end{aligned} \tag{4.49}$$

and check that they satisfy the algebra in (2.14). Thus, the Pauli equation already has a supersymmetric form.
Let us now leave the general discussion and study a particular form of the magnetic field. We choose an asymmetric gauge for $\vec{A}$:

$$A_x = A_x(y), \quad A_y = 0, \tag{4.50}$$

so the Hamiltonian becomes

$$H = \frac{1}{2}\left[(p_x + A_x(y))^2 + {p_y}^2 - A'_x(y)\sigma_z\right]. \tag{4.51}$$

It no longer depends on x, so the wave function ψ can be factored as

$$\psi(x,y) = \mathrm{e}^{ikx}\phi(y), \tag{4.52}$$

where k is an eigenvalue of p_x, so it can take all the values in $-\infty < k < +\infty$. Inserting this form of ψ in the Pauli equation we obtain the equation for ϕ:

$$\frac{1}{2}\left[{p_y}^2 + (k+A_x)^2 - A'_x\sigma_z\right]\phi = E\phi, \tag{4.53}$$

where we can see that left term has exactly the form of a supersymmetric Hamiltonian, with superpotential

$$W = k + A_x. \tag{4.54}$$

[3]Note that, using (2.8), the commutator in (2.11) can be written as $[\psi^\dagger, \psi] = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = -\sigma_z$.

This already allows us to say that the spectrum will be non negative for any form of A_x and that, for the bound spectrum, we will have an $E = 0$ level only if

$$\lim_{y\to\pm\infty}\left(ky + \int^y A_x(u)\,du\right) = +\infty \quad \text{or} \quad \lim_{y\to\pm\infty}\left(ky + \int^y A_x(u)\,du\right) = -\infty. \tag{4.55}$$

If this level exists, it will be non degenerate, while $E > 0$ levels will be doubly degenerate. However, there is no way to fix k, since it comes from the "x" part of the problem, so this adds an infinite degeneracy to every level.
We shall now analyze a special case: let us set $A_x(y) = \omega y + c$, which corresponds to a uniform magnetic field $\vec{B} = -\omega\,\hat{z}$ and is known as the Landau problem. The Hamiltonian is

$$H = \frac{1}{2}\left[{p_y}^2 + (\omega y + c + k)^2 - \omega\sigma_z\right], \tag{4.56}$$

which can be split into the two partner Hamiltonians

$$\begin{aligned} H_- &= \frac{1}{2}\left[{p_y}^2 + (\omega y + c + k)^2 - \omega\right], \\ H_+ &= \frac{1}{2}\left[{p_y}^2 + (\omega y + c + k)^2 + \omega\right]. \end{aligned} \tag{4.57}$$

The superpotential is

$$W = \omega y + c + k, \tag{4.58}$$

so we can see that its primitive

$$U = \frac{\omega}{2}y^2 + (c + k)y \xrightarrow[x\to\pm\infty]{} +\infty, \tag{4.59}$$

which means that only H_- has the $E = 0$ level. The corresponding eigenfunction is

$$\phi_-^{(0)} = N_0 e^{-\frac{\omega}{2}y^2 - (c+k)y}. \tag{4.60}$$

We can get the rest of the spectrum for V_- using the chain of Hamiltonians, because this potential is shape invariant: we already have its partner potential V_+ in (4.57), from which we get

$$V_+ = V_- + \omega. \tag{4.61}$$

Thus, the parameters of the partner potentials are all the same, and all the remainders are $R_n = \omega$.[4] Consequently, the spectrum for V_- is

$$E_n = n\omega \qquad n = 0, 1, 2... \tag{4.62}$$

and the spectrum for V_+ is the same except for the missing $E = 0$ level. These are the well known Landau levels. Since the parameters are the same for all the partner potentials, all the ground state eigenfunctions will be the same as the one in (4.60); to get the n-th eigenfunction of H_- we need to apply n times the operator

$$A^\dagger = \omega y + c + k + \frac{\mathrm{d}}{\mathrm{d}y}, \tag{4.63}$$

[4]Note the analogy with the harmonic oscillator we have already discussed: this is due to the fact that this potential is quadratic as well.

while to get the the n-th eigenfunction of H_+ we need to apply it $n-1$ times. We conclude that the eigenfunctions for H are

$$\begin{aligned}\phi_0 &= N_0 \begin{pmatrix} \mathrm{e}^{-\frac{\omega}{2}y^2-(c+k)y} \\ 0 \end{pmatrix} \\ \phi_n &= N_n \begin{pmatrix} \left(\omega y + c + k + \frac{\mathrm{d}}{\mathrm{d}y}\right)^n \mathrm{e}^{-\frac{\omega}{2}y^2-(c+k)y} \\ \left(\omega y + c + k + \frac{\mathrm{d}}{\mathrm{d}y}\right)^{n-1} \mathrm{e}^{-\frac{\omega}{2}y^2-(c+k)y} \end{pmatrix} \quad n = 1, 2, 3...\end{aligned} \tag{4.64}$$

where the N_i are normalization constants. We can see that, actually, the parameter c is irrelevant in the solution, since it only appears in sums with k, which can already take any real value. This is consistent with the fact that c doesn't appear in the expression for the magnetic field, so it doesn't affect the electron's dynamics.

4.5 Isospectral deformation for the harmonic oscillator

In this section we compute the one parameter family of isospectral potentials that arises from deforming the harmonic oscillator [9], whose potential is

$$V_1 = \frac{1}{2}\omega^2 x^2. \tag{4.65}$$

First, let us compute the quantity I_1, which is going to appear in several calculations to come. To do this, we need the ground state wave function for V_1, which we found in section 4.2 to be

$$\psi_1^{(0)} = \sqrt[4]{\frac{\omega}{\pi}} \mathrm{e}^{-\omega x^2/2}. \tag{4.66}$$

We can now use formula (3.75) to find I_1:

$$I_1(x) = \sqrt{\frac{\omega}{\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\omega y^2}\, dy = \frac{1}{2}[1 + \mathrm{erf}\,(\sqrt{\omega}x)], \tag{4.67}$$

where of course

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\, dt. \tag{4.68}$$

Then, the deformed potential $\widetilde{V_1}$ is given by formula (3.78):

$$\begin{aligned}\widetilde{V_1} &= \frac{1}{2}\omega^2 x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left[\frac{1}{2}\,\mathrm{erf}\,(\sqrt{\omega}x) + \frac{1}{2} + \lambda_1\right] = \\ &= \frac{1}{2}\omega^2 x^2 - \sqrt{\frac{\omega}{\pi}} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\mathrm{e}^{-\omega x^2}}{\frac{1}{2}\,\mathrm{erf}\,(\sqrt{\omega}x) + \frac{1}{2} + \lambda_1} = \\ &= \frac{1}{2}\omega^2 x^2 + \frac{\omega}{\pi} \frac{\mathrm{e}^{-2\omega x^2}}{\left[\frac{1}{2}\,\mathrm{erf}\,(\sqrt{\omega}x) + \frac{1}{2} + \lambda_1\right]^2} + 2\omega\sqrt{\frac{\omega}{\pi}} \frac{x\mathrm{e}^{-\omega x^2}}{\frac{1}{2}\,\mathrm{erf}\,(\sqrt{\omega}x) + \frac{1}{2} + \lambda_1}.\end{aligned} \tag{4.69}$$

Some potentials for selected values of λ_1 are plotted in Fig. 4.2: note how the isospectral deformation of the harmonic potential acts mainly on the zone around $x = 0$, while the

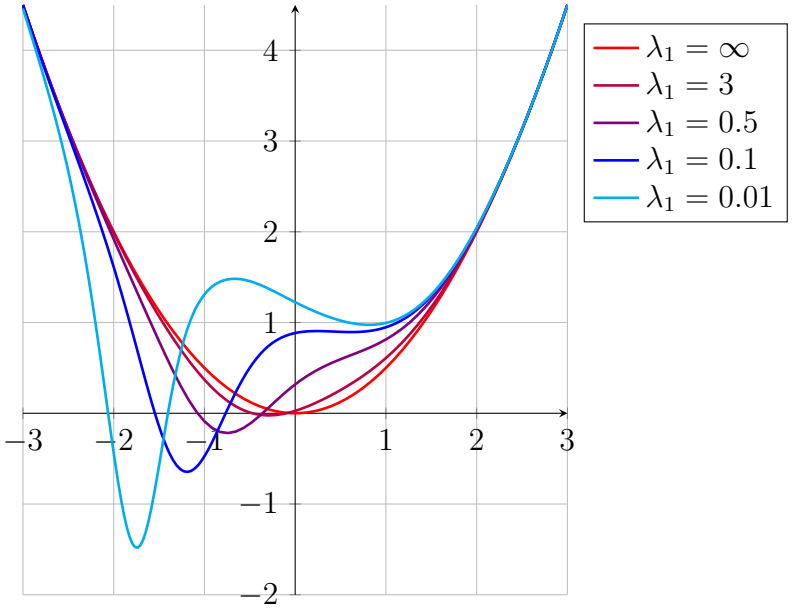

Figure 4.2: Isospectral potentials for the harmonic oscillator (with $\omega = 1$), plotted for some positive values of λ_1. For values of $\lambda_1 < -1$, the potentials look just the same as the ones above, but are reflected about the y axis.

asymptotic behaviour is unchanged.
We can now compute the eigenfunctions for the deformed potentials. The ground state one is given by formula (3.81):

$$\widetilde{\psi}_1^{(0)}(x) = \sqrt{\lambda_1(\lambda_1+1)}\sqrt[4]{\frac{\omega}{\pi}}\frac{\mathrm{e}^{-\omega x^2/2}}{\frac{1}{2}\operatorname{erf}(\sqrt{\omega}x)+\frac{1}{2}+\lambda_1}, \tag{4.70}$$

and some of these are plotted in Fig. 4.3.

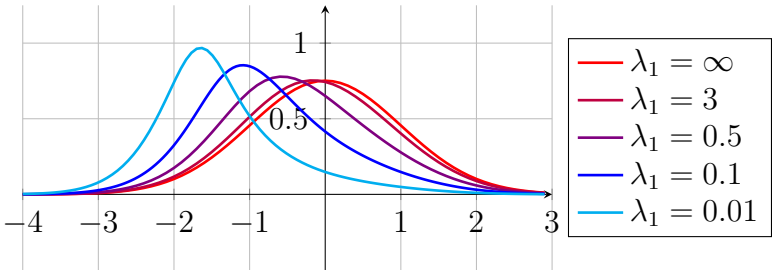

Figure 4.3: Ground state eigenfunctions belonging to isospectral potentials for the harmonic oscillator (with $\omega = 1$), plotted for some positive values of λ_1. Once again, for $\lambda_1 < -1$, the wave functions look the same as the ones above, but are reflected about the y axis.

The other wave functions can be found through forumla (3.79); to use it, we first need the superpotential

$$\widetilde{W}_1 = -\omega x - \frac{\mathrm{d}}{\mathrm{d}x}\ln\left[\frac{1}{2}\operatorname{erf}(\sqrt{\omega}x)+\frac{1}{2}+\lambda_1\right] = -\omega x - \sqrt{\frac{\omega}{\pi}}\frac{\mathrm{e}^{-\omega x^2}}{\frac{1}{2}\operatorname{erf}(\sqrt{\omega}x)+\frac{1}{2}+\lambda_1}, \tag{4.71}$$

as well as the raising operator

$$\widetilde{A}_1^\dagger = -\omega x - \sqrt{\frac{\omega}{\pi}} \frac{\mathrm{e}^{-\omega x^2}}{\frac{1}{2}\operatorname{erf}(\sqrt{\omega}x) + \frac{1}{2} + \lambda_1} + \frac{\mathrm{d}}{\mathrm{d}x}. \tag{4.72}$$

Now we know everything we need to compute the wave functions:

$$\widetilde{\psi}_1^{(n)} \propto \left[\omega x + \sqrt{\frac{\omega}{\pi}} \frac{\mathrm{e}^{-\omega x^2}}{\frac{1}{2}\operatorname{erf}(\sqrt{\omega}x) + \frac{1}{2} + \lambda_1} - \frac{\mathrm{d}}{\mathrm{d}x}\right] \left[\omega x + \frac{\mathrm{d}}{\mathrm{d}x}\right] \left[-\omega x + \frac{\mathrm{d}}{\mathrm{d}x}\right]^n \mathrm{e}^{-\omega x^2/2}. \tag{4.73}$$

For example, it is fairly simple to compute the eigenfunction $\widetilde{\psi}_1^{(1)}$ for the the first excited state:

$$\widetilde{\psi}_1^{(1)} \propto 2\omega x \mathrm{e}^{-\omega x^2/2} + \sqrt{\frac{\omega}{\pi}} \frac{\mathrm{e}^{-3\omega x^2/2}}{\frac{1}{2}\operatorname{erf}(\sqrt{\omega}x) + \frac{1}{2} + \lambda_1}; \tag{4.74}$$

Fig. 4.4 shows some plots of this wave function for selected values of λ_1.
From now on, obtaining more explicit expressions of the eigenfunctions becomes quite tedious and not so interesting, so we will not proceed any further.

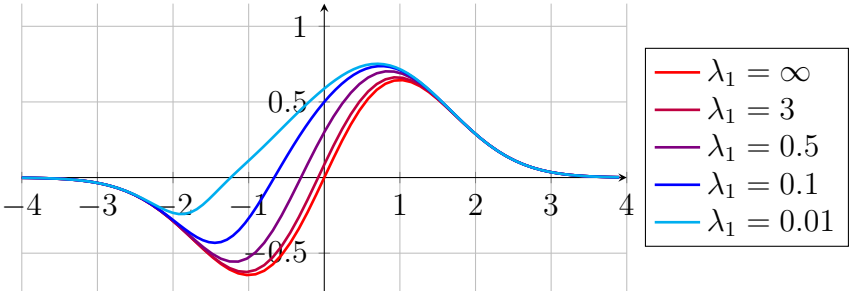

Figure 4.4: First excited state eigenfunctions belonging to isospectral potentials for the harmonic oscillator (with $\omega = 1$), plotted for some positive values of λ_1. Once again, for $\lambda_1 < -1$, the wave functions look the same as the ones above, but are reflected about the y axis.

4.6 Isospectral deformation for the one dimensional potential box

We now compute the isospectral deformation for the one dimensional potential box, whose potential is given by

$$V_1(x) = \begin{cases} 0 & 0 \le x \le L \\ \infty & x \le 0 \vee x \ge L \end{cases}. \tag{4.75}$$

As we did in section 4.1, we will focus on the interval $0 \le x \le L$, since it is clear that every potential will be infinite and every wave function will vanish outside of this range.

Let us recall the energy eigenvalues and eigenfunctions for this potential:

$$E_1^{(n)} = \frac{(n+1)^2\pi^2}{2L^2}, \qquad \psi_1^{(n)} = \sqrt{\frac{2}{L}} \sin\frac{(n+1)\pi x}{L}, \qquad n = 0, 1, 2... \tag{4.76}$$

We also found in Section 4.1 that the superpotential W_1 and the partner potential V_2 are

$$W_1 = \frac{\pi}{L}\cot\frac{\pi x}{L}, \qquad V_2 = \frac{\pi^2}{L^2}\csc^2\frac{\pi x}{L}. \tag{4.77}$$

As usual, we begin by computing I_1 using the ground state wave function:

$$I_1 = \int_0^x \frac{2}{L}\sin^2\frac{\pi y}{L}\,dy = \frac{x}{L} - \frac{1}{2\pi}\sin\frac{2\pi x}{L}. \tag{4.78}$$

Consequently, the family if isospectral potentials is given by

$$\begin{aligned}\widetilde{V}_1 &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\left(\frac{x}{L} - \frac{1}{2\pi}\sin\frac{2\pi x}{L} + \lambda_1\right) \\ &= -\frac{1}{L}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-\cos\frac{2\pi x}{L}}{\frac{x}{L} - \frac{1}{2\pi}\sin\frac{2\pi x}{L} + \lambda_1}\right) \\ &= \frac{1}{L^2}\frac{\left(1-\cos\frac{2\pi x}{L}\right)^2}{\left(\frac{x}{L} - \frac{1}{2\pi}\sin\frac{2\pi x}{L} + \lambda_1\right)^2} - \frac{2\pi}{L^2}\frac{\sin\frac{2\pi x}{L}}{\frac{x}{L} - \frac{1}{2\pi}\sin\frac{2\pi x}{L} + \lambda_1}.\end{aligned} \tag{4.79}$$

Some of these potentials are plotted in Fig. 4.5.

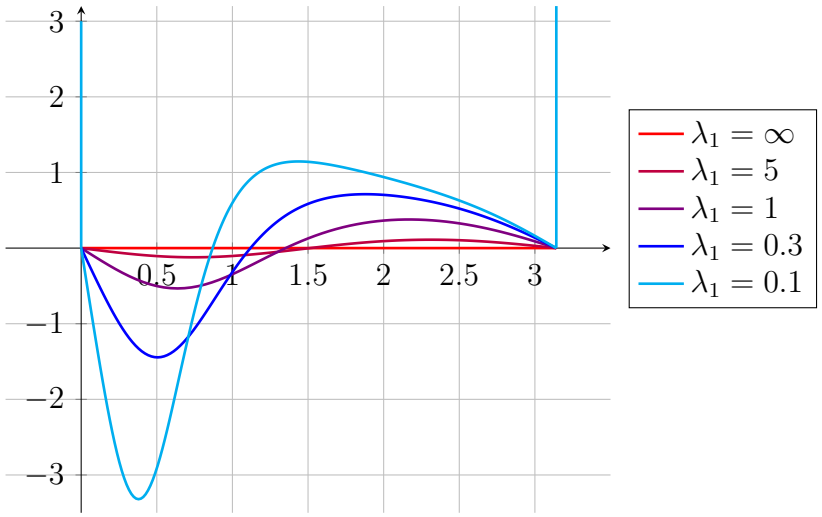

Figure 4.5: One parameter family of isospectral potentials for the potential box (with $L = \pi$), plotted for some positive values of λ_1. For $\lambda_1 < -1$, the potentials look the same, but are reflected about the line $y = \frac{\pi}{2}$.

We can now study the eigenfunctions for the deformed potentials. The grouns state one is

$$\widetilde{\psi}_1^{(0)} = \sqrt{\frac{2\lambda_1(\lambda_1+1)}{L}}\,\frac{\sin\frac{\pi x}{L}}{\frac{x}{L} - \frac{1}{2\pi}\sin\frac{2\pi x}{L} + \lambda_1}, \tag{4.80}$$

and it is shown in Fig. 4.6 for some values of λ_1. In order to compute the higher energy eigenfunctions we need the operators

$$\widetilde{A}_1^\dagger = W_1 - \frac{\mathrm{d}}{\mathrm{d}x}\ln(I_1+\lambda_1) + \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\pi}{L}\cot\frac{\pi x}{L} - \frac{1}{L}\frac{1-\cos\frac{2\pi x}{L}}{\frac{x}{L}-\frac{1}{2\pi}\sin\frac{2\pi x}{L}+\lambda_1} + \frac{\mathrm{d}}{\mathrm{d}x}; \tag{4.81}$$

then the eigenfunctions are given by

$$\widetilde{\psi}_1^{(n)} \propto \sqrt{\frac{2}{L}}\left(\frac{\pi}{L}\cot\frac{\pi x}{L} - \frac{1}{L}\frac{1-\cos\frac{2\pi x}{L}}{\frac{x}{L}-\frac{1}{2\pi}\sin\frac{2\pi x}{L}+\lambda_1} + \frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\frac{\pi}{L}\cot\frac{\pi x}{L} - \frac{\mathrm{d}}{\mathrm{d}x}\right)\sin\frac{(n+1)\pi x}{L}. \tag{4.82}$$

We are not going to carry out the calculation for a generic n explicitly, but we can find $\widetilde{\psi}_1^{(1)}$ as an example (also plotted in Fig. 4.6):

$$\widetilde{\psi}_1^{(1)} \propto \frac{3\pi}{2L}\sin\frac{2\pi x}{L} - \frac{2}{L}\frac{\sin^4\frac{\pi x}{L}}{\frac{x}{L}-\frac{1}{2\pi}\sin\frac{2\pi x}{L}+\lambda_1}. \tag{4.83}$$

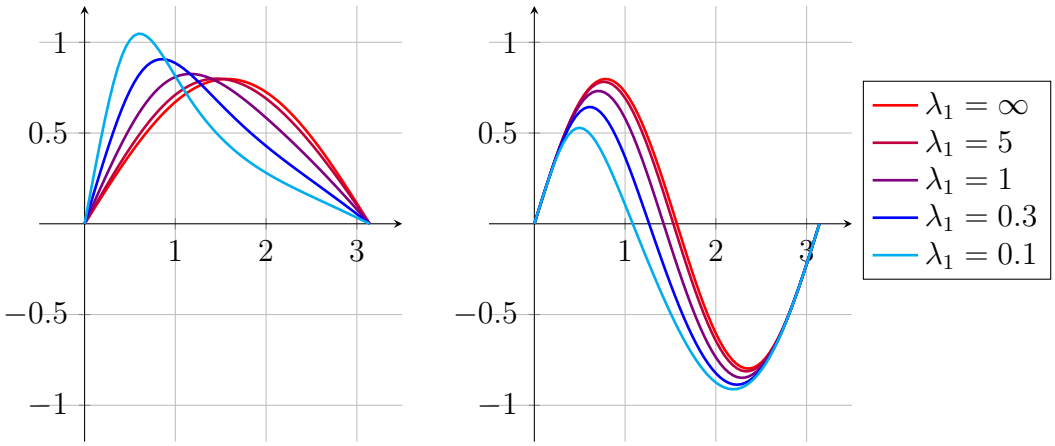

Figure 4.6: On the left, ground state eigenfunctions for some isospectral potentials of the potential box, with $L = \pi$. On the right, first excited state eigenfunctions for the same potentials.

Because of the simplicity of the eigenfunctions of V_1, it is easy to compute I_2:

$$I_2 = \int_0^x \frac{2}{L}\sin^2\frac{2\pi y}{L}\,dy = \frac{x}{L} - \frac{1}{4\pi}\sin\frac{4\pi x}{L}. \tag{4.84}$$

This allows us to build a two parameter family of isospectral potentials using formula (3.85):

$$\widetilde{V}_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\left(\frac{x}{L} - \frac{1}{2\pi}\sin\frac{2\pi x}{L} + \lambda_1\right)\left(\frac{x}{L} - \frac{1}{4\pi}\sin\frac{4\pi x}{L} + \lambda_2\right); \tag{4.85}$$

however, we will not show more calculations, but simply plot some isospectral potentials in Fig. 4.7.

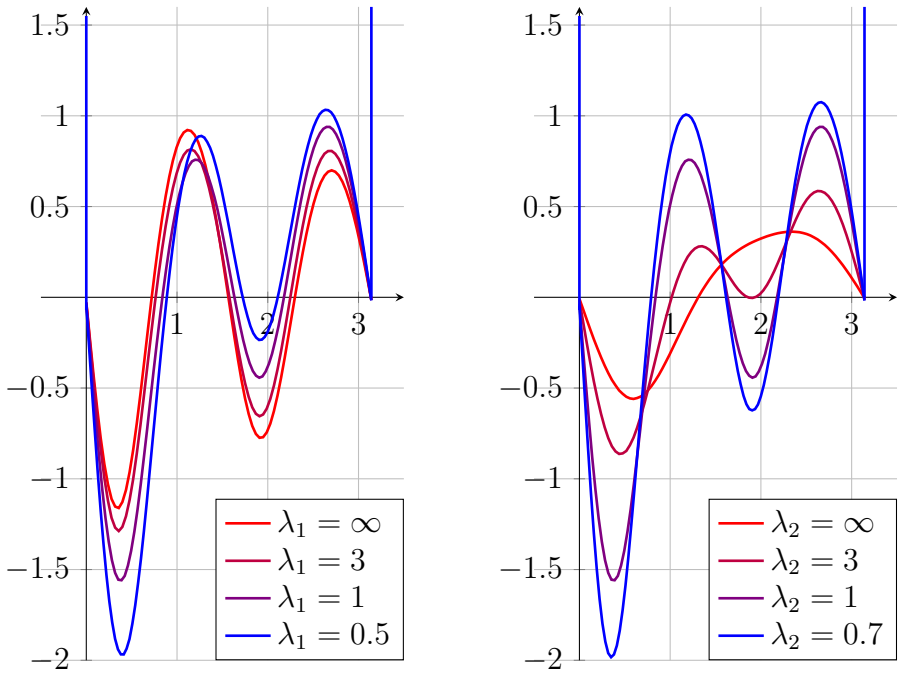

Figure 4.7: Some potentials belonging to the two parameter isospectral family for the potential box (with $L = \pi$). On the left, λ_2 is fixed to 1 and λ_1 varies, while on the right λ_1 is fixed to 1 and λ_2 varies.

Conclusions

In this thesis we have seen why and how Supersymmetric Quantum Mechanics has been developed, as well as some applications to the solution of quantum mechanical problems. Supersymmetric Quantum Mechanics is a reduction of supersymmetric quantum field theories to just one dimension (time only). These theories aim to achieve a unified description of bosons and fermions, but some mathematical difficulties arose in higher dimensions, so Supersymmetric Quantum Mechanics was born as a "test field" to conduct simpler studies. Just like its higher dimensional counterparts, it is built using supersymmetry algebras, which are extensions of the Poincaré algebras that include anticommutator relations in addition to the usual commutator ones. Every supersymmetric model has an algebra which is characterized by a number N of fermionic operators called supersymmetry charges. We have studied the $N = 2$ model by building its Hamiltonian and studying its eigenvalues and eigenfunctions. The spectrum is non negative and doubly degenerate, except possibly the ground state if it has energy $E = 0$; the degeneracy is produced by the presence of a bosonic and a fermionic state with the same energy, and that can be transformed into each other using the supersymmetric charges. We have also introduced the concept of supersymmetry breaking, which occurs when the ground state of the system has energy $E > 0$, and we have provided an indicator of supersymmetry breaking, the Witten index.

Then, we have shown that the concepts developed for this supersymmetric model can have interesting applications in Quantum Mechanics. Firstly, starting from a given one dimensional Hamiltonian, we have used the knowledge of the supersymmetric Hamiltonian to build a chain of Hamiltonians that have the same spectrum as the original one, except the first levels, and whose eigenfunctions are related by some well defined operators. Then, we have seen that, for shape invariant potentials, the chain of Hamiltonians is particularly useful because it provides simple formulae to compute their eigenvalues and eigenfunctions. In fact, this method is the counterpart of the operator method used to solve the harmonic oscillator, extended to the wider class of shape invariant potentials. Also, the chain of Hamiltonians proved useful to build a family of potentials which have the same spectrum and the same reflection/transmission probabilities as a given potential. Finally, we have given some explicit examples of these applications using common problems in Quantum Mechanics, like the potential box, the harmonic oscillator, hydrogenlike atoms and the Pauli equation.

Of course, the field of Supersymmetric Quantum Mechanics is much wider: for example, we haven't covered the superspace formalism that can be used to derive the supersymmetric Hamiltonian more formally [5]. Also, as we have mentioned above, there are supersymmetric models with a different number of supersymmetry charges, even though their main features are similiar to the ones we have discussed for the $N = 2$ model. The applications of Supersymmetric Quantum Mechanics are broader, too. First of all, we need to mention that the class of shape invariant potentials has received quite a lot of attention, and researchers have been able to develop a precise categorization for these potentials. A supersymmetric version of the WKB method has also been developed: it allows to find approximate solutions for the Schrödinger equation, and its results are usually better than those obtained from the standard method [9].

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