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Reissner-Nordström Black Holes and Mass Inflation

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Abstract

The existence and structure of black holes are derived from Einstein's general theory of relativity. Mass inflation (an increase in mass) is found when the internal structure of black holes is studied. The objective of the present study is two-fold: (i) to obtain an understanding of the nature of Reissner-Nordström black holes and (ii) examine the mass inflation phenomenon. To do so, spherical symmetric solutions to Einstein's field equations are analyzed.The Schwarzschild solution is analyzed to show the most basic result of general relativity. The analytical (Kruskal) continuation of the Schwarzschild solution and the mechanism of gravitational collapse are also discussed. The Reissner-Nordström metric is then examined in detail analyzing both the general characteristics and the most generic field equations for a body with spherical symmetry. Moreover two important applications are considered: the Vaidya solutions and the Dray-'t Hooft-Redmount (DTR) relation. The mass inflation phenomenon is then formulated by formally integrating Einstein's field equations considering continuous infalling and outgoing radial fluxes of gravitational radiation. To evaluate the growth rate of the gravitational mass, a formal perturbation expansion in terms of the product of the flux luminosities is developed. Finally, the possibility that the asymmetries occurring during a realistic collapse could change the conclusions obtained for spherical symmetry is considered. The most striking features of the physics behind black holes and the mass inflation phenomenon are shown.

Sommario

L'esistenza e la struttura dei buchi neri deriva dalla teoria della relatività generale di Einstein. Il fenomeno dell'inflazione di massa (una crescita della massa) risulta dallo studio della struttura interna dei buchi neri. L'obiettivo del presente elaborato è duplice: (i) chiarire la natura dei buchi neri di Reissner-Nordström e (ii) esaminare il fenomeno dell'inflazione di massa. A questo scopo vengono analizzate le soluzioni a simmetria sferica delle equazioni di campo di Einstein. In particolare, viene studiata la soluzione di Schwarzschild per mostrare un risultato base della relatività generale. Vengono inoltre discusse la continuazione analitica (Kruskal) della soluzione di Schwarzschild e il meccanismo del collasso gravitazionale. Viene poi esaminata in dettaglio la metrica di Reissner-Nordström analizzando sia le caratteristiche generali che le più generiche equazioni di campo di un corpo a simmetria sferica. Vengono inoltre considerate due importanti applicazioni: le soluzioni di Vaidya e la relazione DTR. Il fenomeno dell'inflazione di massa viene poi formulato integrando formalmente le equazioni di Einstein considerando flussi continui entranti ed uscenti di radiazione gravitazionale. Per valutare il tasso di crescita della massa gravitazionale viene effettuata una espansione perturbativa in termini dei prodotti delle luminosità. Infine, viene considerata la possibilità che le asimmetrie presenti durante un collasso gravitazionale reale possano cambiare le conclusioni ottenute nel caso di simmetria sferica. Vengono mostrate le caratteristiche principali della fisica alla base dei buchi neri e del fenomeno dell'inflazione di massa.

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Chapter 1

Introduction

A black hole is a cosmological body that is formed by the collapse of a massive star. When such a star has exhausted the internal thermonuclear fuels in its core, the core becomes unstable and gravitationally collapses inward upon itself. The weight of the constituent matter falling in from all sides compresses the star to a point of zero volume and infinite density called the singularity. The structure of black holes is derived from Einstein's general theory of relativity.

Mass Inflation (an increase in mass) is found when we study the internal structure of black holes. The objective of the present study is two-fold: (i) to obtain an understanding of the nature of Reissner-Nordström black holes and (ii) to examine the mass inflation phenomenon. An analysis of the mass inflation phenomenon requires an understanding of Einstein's general theory of relativity and in particular of Reissner-Nordström black holes.

General Relativity is looked upon as one of the greatest achievements of theoretical physics conceived by a single mind. This is expressed clearly in a remarkable speech given by M. Born [1]:

> *(The general theory of relativity) seemed and still seems to me at present to be the greatest accomplishment of human thought about nature; it is a most remarkable combination of philosophical depth, physical intuition and mathematical ingenuity. I admire it as a work of art.*

Purely theoretical considerations led to the formulation of General Relativity. The Newtonian law of gravitation involved the assumption of an action-at-distance law, which was not compatible with the special theory of relativity. So, Einstein worked on a relativistic theory of gravitation and, after ten years of hard work was finally able to formulate the general theory of relativity. Here is a quotation about how he struggled during his research, written in a letter to A. Sommerfeld:

> *At present I occupy myself exclusively on the problem of gravitation and now believe that I shall master all difficulties with the help of a friendly mathemati-*

cian here (Marcel Grossmann). But one thing is certain, in all my life I have never labored nearly as hard, and I have become imbued with great respect for mathematics, the subtler part of which I had in my simple-mindedness regarded as pure luxury until now. Compared with this problem, the original theory of relativity is child's play.

The key point for the passage from Special to General Relativity is embedded in the definition of observer and of reference frame. From a mathematical point of view Special Relativity is based on the principle that "*The laws of physics are the same for all inertial observers and the speed of light in vacuum is invariant*" [2] and so it is realized by assuming the existence of global (inertial) reference frames connected by Lorentz transformations. In contrast, General Relativity is based on the principle that "*The laws of physics are the same in all reference frames (for all observers)*" and so these laws have to be expressed in a form which can be adapted to any measuring apparatus, regardless of its inertial nature. This is achieved by the use of tensor quantities of the space-time manifold, in particular by using the general metric $g_{\mu\nu}$ instead of the Minkowski metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ and the covariant derivative ∇_a instead of the partial derivative ∂_a.

To this end, the problem of General Relativity consists in finding the metric field $g_{\mu\nu}$ describing the manifold, and this must be related somehow to the energy-momentum distribution of all forms of matter. This relationship is encoded in *Einstein's field equations*

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu},$$

where $T_{\mu\nu}$ represents the energy-momentum tensor. This set of ten highly nonlinear partial differential operations relates the curvature of space-time to its matter content. These equations reduce to the Newtonian theory of gravity, so that it is embedded in General Relativity as a case of quasi-stationary weak fields and slowly changing matter sources. In addition, General Relativity provides corrections to the Newtonian theory of gravity, the most famous one being the anomalous precession of the perihelion of Mercury.

Black hole theory is without any doubt one of the greatest triumphs of General Relativity. It has been demonstrated [3] that the external gravitational field of a black hole relaxes to the Kerr-Newman field described by three parameters: the source's (i) mass, (ii) charge and (iii) angular momentum. This behavior is due to the fact that perturbations developing at the surface of a spherically collapsing star produce the emission of gravitational waves which carry away all the characteristics of the star's gravitational field except for the three parameters listed above. This radiation then interacts with the space-time curvature: a part of it escapes to infinity while the rest is backscattered and absorbed by the black hole.

Attempts have been made to find the internal structure of black holes [4]. For a Schwarzschild black hole the asymptotic portion of space-time near the singularity is virtually free of aspherical perturbations, since the gravitational radiation becomes infinitely diluted as it reaches the singularity. Things change if the charge or angular momentum do not vanish. The Kerr and Reissner-Nordström metric differ drastically from the Schwarzschild one since now the singularity is timelike and both these spaces possess a Cauchy horizon, that is a null hypersurface beyond which predictability breaks down.

The presence of a region beyond the Cauchy horizon is an embarrassment since there is no way of predicting the course of events in this region and signals coming from the singularity could alter the physics in an unforeseeable manner. There is also another problem at this internal horizon since it appears to be a surface of infinite blue shift. A free-falling observer would see the entire future history of the universe in a flash before encountering a wall of infinite density at the Cauchy horizon. It has been shown [5, 6, 7] that perturbations diverge to linear order near this horizon.

For the present study it is useful to analyze the perturbations beyond linear order and to evaluate whether or not the perturbations, allowed to act as a source in Einstein's field equations, can trigger the formation of a singularity of sufficient strength to effectively stop the evolution of space-time at the Cauchy horizon. What actually happens is that the combination of the outflux emitted from the surface of the collapsing star and its backscattered, blue shifted radiative tail provoke the inflation of the black hole's internal mass parameter, which becomes classically unbounded. Mass inflation, then, is responsible for the growth of curvature which goes to infinity at the Cauchy horizon.

In order to examine the behavior of mass inflation it is necessary to make a few assumptions to simplify the mathematics. The first assumption is to accept as a starting point the Reissner-Nordström solution to Einstein's field equations. Although this metric may appear to be too idealized and not realistic, it captures the essential physics behind the phenomenon since it shares the global structure with the Kerr solution and possesses the fundamental two characteristics needed: 1) the presence of a highly blue-shifted influx and 2) separation between the Cauchy and inner apparent horizons. The spherical model should allow a qualitative understanding of the phenomenon without introducing too many difficulties in the mathematical description. The second assumption is to model the in-/out- falling radiation as two intersecting radial streams of lightlike particles following null geodesics. Furthermore it is assumed that these streams do not interact with each other, so that they are separately conserved.

The most important solutions to Einstein's field equations are analyzed here in chapter 2. The Schwarzschild solution is analyzed to show the most basic result of General Relativity, in particular its analytical continuation (Kruskal) and the mechanism of gravitational collapse. The Reissner-Nordström metric is examined in detail and, in addition to its general characteristics, the most generic field equations for a spherical symmetric source are derived. The generalized Vaidya solutions, describing a charged, spherical black hole irradiated by a pure in- or outflux are also described. Finally, the generalized

Dray-'t Hooft-Redmount (DTR) relation, which describes the gravitational field before and after the collision of two spherical thin shells propagating at the speed of light, one expanding the other contracting, is derived.

Chapter 3 deals with the mass inflation phenomenon which is formulated by formally integrating Einstein's field equations considering continuous infalling and outgoing radial fluxes. To evaluate the growth rate of the gravitational mass, a formal perturbation expansion in terms of the product of the flux luminosities is developed. Finally, the possibility that the asymmetries occurring during a realistic collapse could change the conclusions obtained for spherical symmetry is discussed.

Chapter 2

Spherical Symmetric Solutions to Einstein's Field Equations

The mathematical background of the theory of General Relativity is given by differential geometry. This is a consequence of the principle of general relativity cited in the introduction and of the principle of general covariance [2]:

> *The laws of physics in a general reference frame are obtained from the laws of Special Relativity by replacing tensor quantities of the Lorentz group with tensor quantities of the space-time manifold.*

General Relativity and the geometry of space-time are thus deeply connected and this represents the key point of the theory. Einstein's field equations imply that matter sources determine the space-time curvature, which in turn affects the matter's motion. Since we are dealing with a generic metric, which will be flat (i.e. Minkowskian) only locally, another immediate consequence is that the concept of a straight line is substituted by that of a geodesic line.

2.1 Schwarzschild Solution

The first exact solution to Einstein's field equations was found by Schwarzschild in 1916 and describes the gravitational field generated by a spherical symmetric body [1].

In Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.1.1}$$

the component which encapsulates the characteristics of the source is the energy-momentum tensor $T_{\mu\nu}$. The description of the internal gravitational field of a celestial body is anything but easy so we focus on the external field generated by the source. In the outside region $T_{\mu\nu} = 0$ and so, taking the trace of Eq. (2.1.1) yields

$$R^{\mu}_{\ \mu} - \frac{1}{2} R g^{\mu}_{\ \mu} = 0. \tag{2.1.2}$$

We therefore obtain that outside the source $R = 0$ and so Eq. (2.1.1) becomes

$$R_{\mu\nu} = 0. \tag{2.1.3}$$

This is the so-called Ricci tensor [2], which is obtained by the contraction of two indices of the the Riemann tensor

$$R_{\mu\nu} = R^{k}_{\ \mu k\nu}. \tag{2.1.4}$$

To continue in the solution of the vacuum version of Einstein's field equation (2.1.3) we need to use the expression for the components of the Riemann tensor

$$R_{ij} = R^{k}_{\ ikj} = \partial_j \Gamma^{\rho}_{\ i\rho} - \partial_\rho \Gamma^{\rho}_{\ ij} + \Gamma^{l}_{\ i\rho} \Gamma^{\rho}_{\ jl} - \Gamma^{l}_{\ ij} \Gamma^{\rho}_{\ \rho l}, \tag{2.1.5}$$

which is related to the metric through the Christoffel symbol

$$\Gamma^{i}_{jk} = \frac{1}{2} g^{il} (g_{jl,k} + g_{kl,j} - g_{jk,l}). \tag{2.1.6}$$

The solution is made easier by the use of isometries, i.e. by the introduction of Killing vectors. Firstly, since the source is assumed to be static we are implicitly considering the existence of a time-like Killing vector K_t, associated to a suitable coordinate t such that

$$K_t = \frac{\partial}{\partial t}. \tag{2.1.7}$$

Since the source has spherical symmetry, it is possible to introduce three Killing vectors $K_i = d/d\theta_i$ corresponding to the three rotations centered on the source. These vectors must be preserved in time, and this means that they have to commute with K_t. We can therefore assume that in this metric rotations are orthogonal to K_t, like they were for the flat Minkowsky space-time, and use the familiar spherical coordinates on surfaces with the same t.

These considerations lead to a metric with a diagonal form

$$ds^2 = -A(r) dt^2 + B(r) dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1.8}$$

It is possible to rescale the radial coordinate to get $C(r) = r^2$. With this choice the coordinate r is identified through the area of a surface of coordinate radius r

$$A(r) = \int d\Omega^2 = 4\pi r^2. \tag{2.1.9}$$

It is interesting to note that r is no longer the classical radius, in fact the radius length of a sphere $A(r)$ is

$$R(r) = \int_0^r \sqrt{g_{rr}} dr. \tag{2.1.10}$$

Using the metric of Eq. (2.1.8) yields the following components of the Christoffel symbol

$$\begin{aligned}
&\Gamma^0_{10} = \frac{A'}{2A} \\
&\Gamma^1_{00} = \frac{A'}{2B} \quad \Gamma^1_{11} = \frac{B'}{2B} \quad \Gamma^1_{22} = -\frac{r}{B} \quad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B} \\
&\Gamma^2_{12} = \frac{1}{r} \quad \Gamma^2_{33} = -\sin\theta\cos\theta \\
&\Gamma^3_{13} = \Gamma^2_{12} \quad \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{2.1.11}$$

Using these components in Eq. (2.1.5) gives a system of four equations

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}(\frac{A'}{A} + \frac{B'}{B}) - \frac{A'}{rB} \tag{2.1.12a}$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'}{rB} \tag{2.1.12b}$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}(\frac{A'}{A} - \frac{B'}{B}) \tag{2.1.12c}$$

$$R_{33} = R_{22}\sin^2\theta. \tag{2.1.12d}$$

Considering these equations, firstly we find that

$$(AB)' = AB' + A'B = 0 \tag{2.1.13}$$

and so

$$A = B^{-1} \tag{2.1.14}$$

(note that the dimensionless constant here was taken to be one, which can always be done by rescaling the time variable). Secondly, by Eq. (2.1.12c) we find that

$$1 = A + rA' \tag{2.1.15}$$

which results in

$$A = 1 - \frac{2K}{r}. \tag{2.1.16}$$

The constant K can be determined by looking at the week field limit and this in turn yields the final form of the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.1.17}$$

This solution is asymptotically flat, since for $r \to \infty$ it reduces to the Minkowski metric $\eta_{\mu\nu} = diag(-1,1,1,1)$.

A feature which jumps straight to the eye is the presence of a singularity behavior in the proximity of $r = r_s = 2m$, the Schwarzschild radius. Here the g_{rr} element of the metric becomes singular so that the (t,r) coordinates cease to be valid. In particular, the time coordinate t reaches infinity, so that an observer watching a probe falling in the gravitational field sees it slowing down until reaching the Schwarzschild radius. If we consider the point of view of an observer on the probe, nothing prevents him from crossing the apparent singularity and continuing his voyage. This behavior is explained by the fact that $r = r_s$ is just a coordinate singularity, while $r = 0$ is the real one, indeed the Riemann tensor is finite at the Schwarzschild radius. A typical component is

$$R^1_{212} = R^1_{313} \sim \frac{1}{r^3}. \tag{2.1.18}$$

Hence, at $r = 2m$ the tidal forces remain finite, while they diverge for $r \to 0$. So, the Schwarzschild solution is exact until a position outside of the horizon is considered, i.e. where $r > r_s$.

2.1.1 Kruskal Continuation

The singular behavior of the Schwarzschild solution can be eliminated by a suitable change of coordinates. Let us start by introducing the rescaled radial coordinate

$$r^* = \int \frac{dr}{1 - 2m/r} = r + 2m \ln(r - 2m), \tag{2.1.19}$$

in order to define the Eddington-Finkelstein coordinates

$$v \equiv t + r^*, \tag{2.1.20a}$$
$$u \equiv t - r^*. \tag{2.1.20b}$$

Hence, v is an advanced null coordinate and u a retarded null coordinate. If we use (v, r, θ, ϕ) as coordinates we get a metric of the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \tag{2.1.21}$$

The manifold is the region $2m < r < \infty$ but it is easy to see that the metric of Eq. (2.1.21) is analytic in the larger manifold $0 < r < \infty$. By introducing the new coordinate v, the Schwarzschild metric has been extended so that it is no longer singular at $r = 2m$. It can also be noted that the surface $r = 2m$ is null, briefly a surface upon which every vector normal to it is null.

The Eddington-Finkelstein representation of the Schwarzschild solution has the odd feature of not being time symmetric. One could expect this from the cross term $dvdr$ in Eq. (2.1.21). A consequence of this behavior is that the $r = 2m$ surface acts as a one way membrane, letting pass only future-directed non space-like curves from the outside ($r > 2m$) to the inside ($r < 2m$). Further, any such curve which crosses the membrane will reach the singularity ($r = 0$) in a finite proper time (or affine distance if it is a null curve).

The same procedure can be followed by taking the coordinates (u, r, θ, ϕ), which leads to the metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)du^2 - 2dudr + r^2 d\Omega^2. \tag{2.1.22}$$

The observations made previously hold even for this choice, except for a few details. In fact, now the isometry of the Schwarzschild metric reverses the direction of time. The membrane $r = 2m$ is a surface that only past-directed time-like or null curves can cross from the outside to the inside.

Now it is possible to use Eqs. (2.1.20a, 2.1.20b) to make both the extensions simultaneously. In this way we are able to embed both manifolds described above in a larger one, so that they coincide in the region $r > 2m$. Let us start by considering the metric with the coordinates (v, u, θ, ϕ)

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dvdu + r^2 d\Omega^2, \tag{2.1.23}$$

where r is a function of (v, u) determined by

$$\frac{1}{2}(v - u) = r + 2m \ln(r - 2m). \tag{2.1.24}$$

Now it is possible to operate a further change of coordinates $V(v)$ and $U(u)$ leaving the two-dimensional space $(\theta, \phi) = const$ expressed in double null coordinates. The resulting metric is

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\frac{dv}{dV}\frac{du}{dU}dVdU + r^2 d\Omega^2. \tag{2.1.25}$$

The choice of the functions V and U determines the precise form of the metric. Let us consider the Kruskal coordinates

$$V \equiv -e^{-v/4m}, \tag{2.1.26a}$$

$$U \equiv -e^{-u/4m}, \tag{2.1.26b}$$

which yield the following metric

$$ds^2 = -e^{-r/2m}\frac{16m^2}{r}dVdU + r^2 d\Omega^2 \tag{2.1.27}$$

that can be reduced to a form corresponding to the Minkowski space-time, by introducing the coordinates x' and t' where

$$x^{'} = \frac{1}{2}(V - U) \quad \text{and} \quad t^{'} = \frac{1}{2}(V + U), \tag{2.1.28}$$

to obtain the metric

$$ds^2 = e^{-r/2m}\frac{16m^2}{r}(-dt^{'2} + dx^{'2}) + r^2 d\Omega^2. \tag{2.1.29}$$

Fig. 2.1.1 is a Kruskal diagram showing regions inside and outside of the event horizon of the Schwarzschild solution.

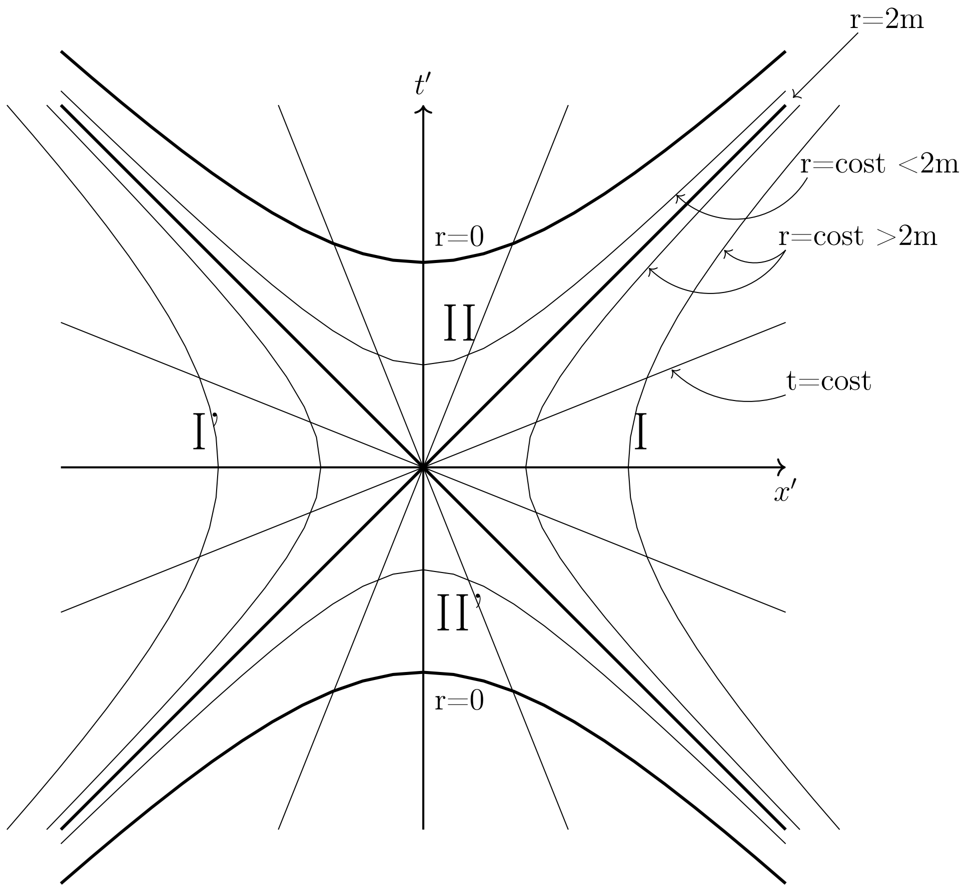

Figure 2.1.1: Kruskal diagram.

As can be seen from Fig. 2.1.1 lines with $r = const$ are plotted as hyperbolas, while $t = const$ lines correspond to radial lines crossing the origin. Region I ($r > 2m$) and region II ($r < 2m$) are isometrically equivalent to the advanced Finkelstein extension, while regions I and II' are equivalent to the retarded extension. There is also a region I' which is again isometric to the outside Schwarzschild solution, but there are no time-like or null curves which go from I to I'. Every future-directed non space-like curve which crosses the surface $r = 2m$ reaches the singularity $r = 0$ at $t^{'} = \sqrt{2m + x^{'2}}$, while every past-directed non space-like curve which crosses $t^{'} = -|x^{'}|$ approaches another singularity at $t^{'} = \sqrt{2m + x^{'2}}$.

In the analysis of gravitational radiation, it is useful to construct a conformal compactification of Schwarzschild-Kruskal space-time, known as the Penrose diagram [8], which makes it possible to discuss the behavior at infinity of local differential geometric tools. In order to do so, let us define new advanced and retarded coordinates V' and U' where

$$V' = \arctan(V\sqrt{2m}), \quad U' = \arctan(U\sqrt{2m}) \tag{2.1.30}$$

for

$$-\pi < V' + U' < \pi \quad and \quad -\frac{1}{2}\pi < V' < \frac{1}{2}\pi, \; -\frac{1}{2}\pi < U' < \frac{1}{2}\pi \tag{2.1.31}$$

(see Fig. 2.1.2).

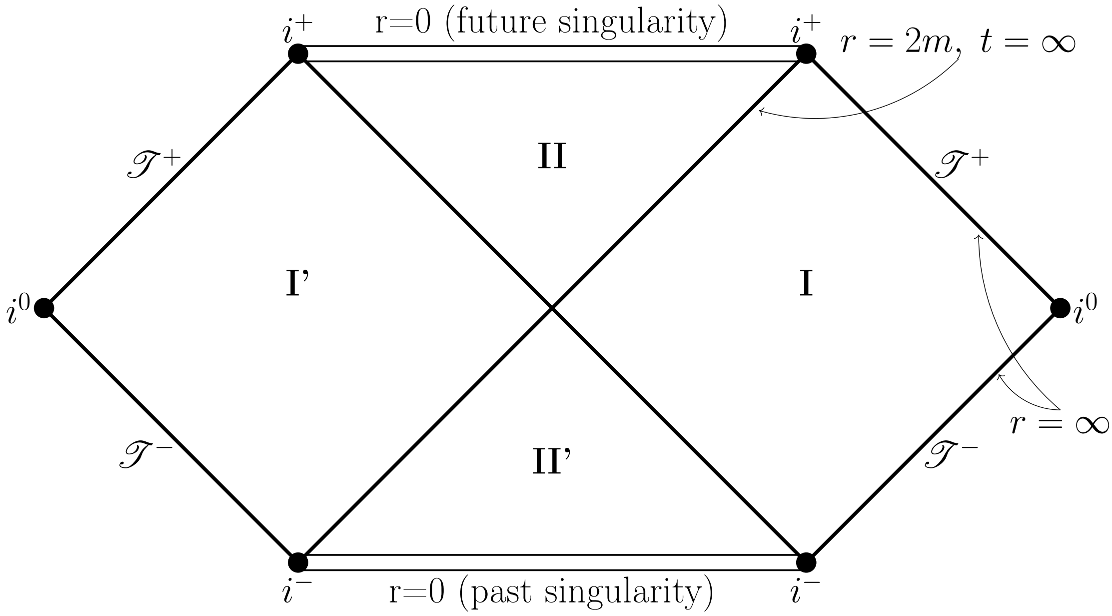

Figure 2.1.2: Penrose diagram for Schwarzschild-Kruskal space-time.

Now it is possible to see future, past and null infinities of the asymptotically flat regions I and I'. The fact that every particle which crosses the surface $r = 2m$ has to fall to the singularity is related to the property that each point inside region II represents a closed trapped surface. This means that if we consider any two-dimensional sphere

p (every point in these diagrams represents a two-dimensional sphere) and two two-dimensional spheres q and s formed by photons emitted respectively radially outwards and inwards from p, and if all of them lie in the region $r > 2m$, then q will be larger than p and s will be smaller than p. If they all lie in the region $r < 2m$ the areas of both q and s will be smaller than p, so that p is a closed trapped surface. The existence of singularities is closely related to the existence of the closed trapped surfaces.

Causal relations in the Schwarzschild-Kruskal space-time may be understood by considering that light rays are lines at 45°, as in the Minkowski manifold. Observers in I and I' can receive signals from II' and send them to II, but not the contrary.

2.1.2 Gravitational Collapse

Outside a spherically symmetric star the solution to Einstein's equations is necessarily that part of one of the asymptotically flat regions of the Schwarzschild solution where $r > r_0$ with r_0 corresponding to the surface of the star. There will be, for $r < r_0$, a solution depending on the radial distribution of density and pressure inside the star. In fact, even if the star is non-static the solution outside will be part of the Schwarzschild solution limited by the surface of the star, provided that it remains spherically symmetric.

If the star is static then the radius r_0 must be greater than $2m$ (with m the mass of the star), i.e. the Schwarzschild radius. In fact, the surface of the star must lie on the trajectory of a time-like Killing vector and in the Schwarzschild solution there exist such vectors only in the subspace $r > 2m$. If r_0 were less than $2m$ the star's surface would be expanding or contracting.

The life of a star generally consists of a long period of quasi-stationarity, where the star burns its nuclear fuel and supports itself against gravity by thermal and radiation pressure. Afterwards, when the nuclear fuel is exhausted, the pressure will decrease, the star will cool and it will start to contract (see Fig. 2.1.3). If this contraction can not be halted by any other force, then the star will collapse until reaching the Schwarzschild radius. Once this point has been crossed since the solution outside must be the Schwarzschild one, the star will be embedded in a closed trapped surface and thus a singularity will occur. Collapse to a singularity can not be avoided but at some point near it probably General Relativity will cease to be valid since it seems likely that quantum effects will arise.

But what happens to an observer on the surface of a collapsing star? This question can be answered if the collapse is exactly spherically symmetric so that the solution outside of the star will be given by the Schwarzschild solution. In this case, an observer O on the surface of the star, even if it may seem counterintuitive, will pass trough $r = 2m$ at a certain (proper) time, say τ' and will not notice anything special at that time. In fact, since tidal forces remain weak at the Schwarzschild radius, curvature is locally the same as it is elsewhere and the only difference will be that O will no longer be visible by an external observer, say O'. During the collapse, O' sees O's time increasingly slow

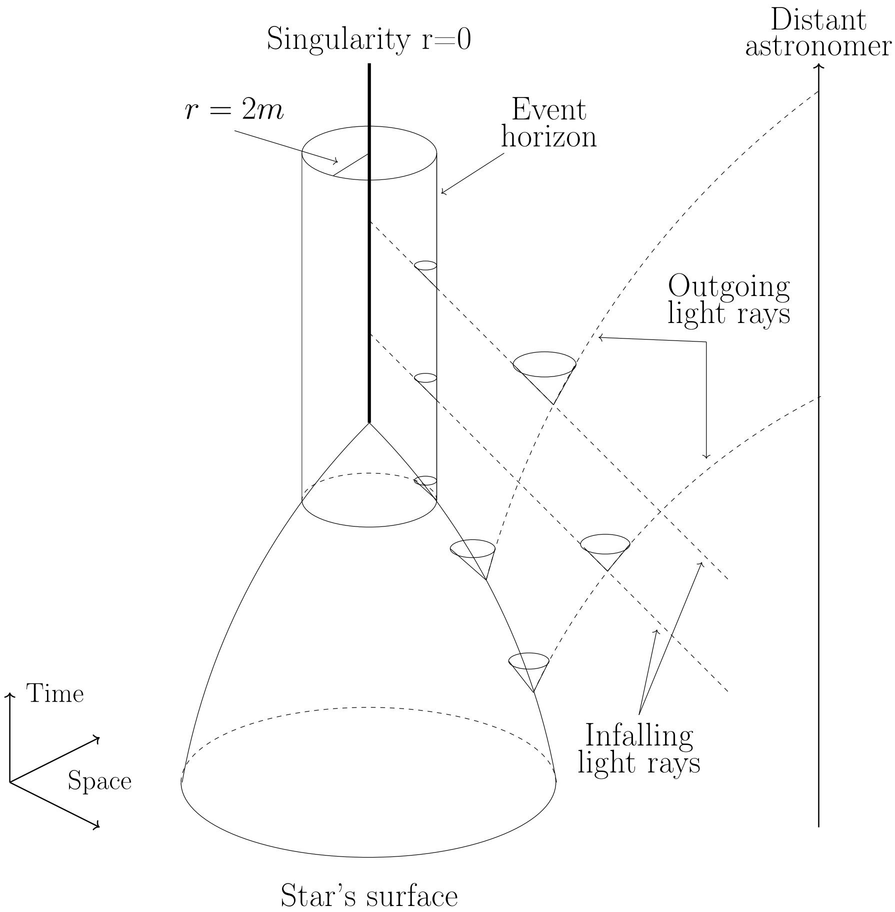

Figure 2.1.3: Diagram of the spherical symmetric collapse in Eddington-Finkelstein coordinates.

down and never reach the time τ', no matter how long he waits. This means that the light he receives from O will have a greater and greater shift of frequency to the red and, as a consequence, a greater and greater decrease in intensity. Thus, even if O' never stops to see the surface of the star, the star will start to become faint. First the center will become too faint to be seen, and then the rim. The characteristic time scale for this to happen is $\tau \sim R_s/c \sim 10^{-5}(M/M_s)$.

When the star has collapsed completely, we are dealing with a black hole and are thus left with an object which is, for all practical purposes, invisible. The "surface of a black hole" means its event horizon surface area $4\pi R_s^2$. In fact, the black hole still possesses the same Schwarzschild mass and produces the same space-time geometry as it did before the collapse. Thus, a possible way to "see" a black hole consists in the study of its gravitational effects, for instance the motion of nearby bodies, or the deflection of light rays passing near it. Because of its great gravitational attraction, a black hole acts like a cosmic vacuum cleaner. For example, if a black hole is part of a closed binary system, it can suck up matter from its partner and heat it to such a degree that a strong X-ray source results.

One of the most striking features of the spherical symmetric collapse to a black hole is that the singularity appears in the region $r < 2m$ where light rays cannot escape to infinity. This means that the singularity, which is the place where physics and predictability cease to be valid, is excluded from the outer asymptotically flat region, so that here predictability still holds. It has been conjectured by Penrose that this is the behavior of all "realistic" singularities ("*Cosmic censorship hypothesis*"). The horizon is thus the boundary of the region which is causally connected to a distant observer. It acts as a one way membrane through which energy and information can pass to the inside but not to the outside. The existence of such membranes is another striking feature of General Relativity.

2.2 Reissner-Nordström Black Holes

The solution of interest for the present work is the one obtained for a charged body with spherical symmetry, i.e. a spherical charged black hole. This solution can be found using the same expressions for the Christoffel symbols in Eq. (2.1.11). In fact these are the same for every spherical symmetric space-time. The difference with the Schwarzschild solution in encoded in the energy-momentum tensor, which now has a Maxwellian contribution given by

$$E^{\mu}{}_{\nu} = \frac{e^2}{4\pi r^4}\, diag(-1,-1,1,1). \tag{2.2.1}$$

The considerations about isometries made in section 2.1 are still valid, so we are dealing with a metric of diagonal form

$$g_{\mu\nu} = diag(-A(r), B(r), r^2, r^2 \sin^2\theta). \tag{2.2.2}$$

From Einstein's field equations we therefore find two linearly independent equations

$$\frac{1}{B}\left(\frac{1}{r^2} - \frac{B'}{rB}\right) - \frac{1}{r^2} = -\frac{Ge^2}{r^4} \tag{2.2.3a}$$

$$\frac{1}{B}\left(\frac{1}{r^2} + \frac{A'}{rA}\right) - \frac{1}{r^2} = -\frac{Ge^2}{r^4}. \tag{2.2.3b}$$

From the two above equations we find, as in the Schwarzschild solution, $(AB)' = cost$ and so

$$A = B^{-1}. \tag{2.2.4}$$

Resolving Eqs. (2.2.3a, 2.2.3b) yields

$$A = 1 - \frac{2m}{r} + \frac{e^2}{r^2}. \tag{2.2.5}$$

The Reissner-Nordström solution is thus given by

$$ds^2 = -f_0 dt^2 + f_0^{-1} dr^2 + r^2 d\Omega^2, \tag{2.2.6}$$

$$f_0 = 1 - \frac{2m}{r} + \frac{e^2}{r^2}, \tag{2.2.7}$$

where e represents the body's total charge and m its mass, measured by an observer at great distance from the black hole.

The space-time generated by a spherical charged source is given by the solution to Einstein's field equations with an energy-momentum tensor given by

$$E^{\mu}{}_{\nu} = \frac{e^2}{4\pi r^4}\; diag(-1, -1, 1, 1), \tag{2.2.8}$$

which is the contribution given by the electromagnetic field.

The main difference with the Schwarzschild metric is the presence of two horizons, an external event horizon and an internal Cauchy horizon. This characteristic is evident in Fig. 2.2.1 where it is possible to observe the different behavior of the g_{tt} element of the metric in the two solutions. Astrophysically, charged black holes are of little interest since macroscopic bodies do not posses sizable net electric charge.

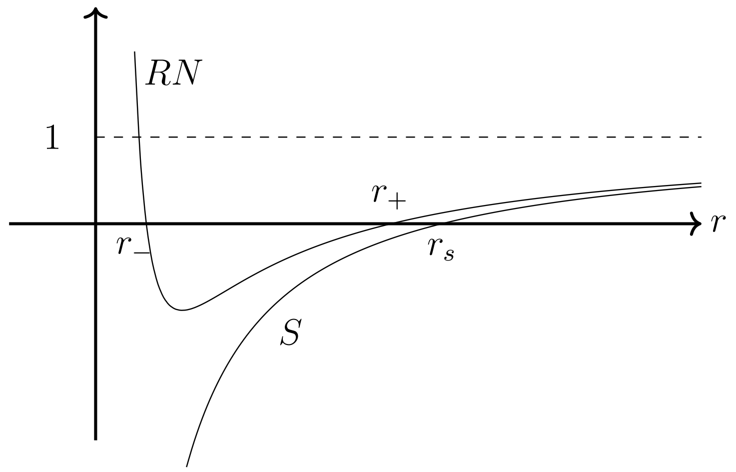

Figure 2.2.1: Behaviour of the g_{tt} metric element in Schwarzschild (S) and Reissner-Nordström (RN) solutions.

The two solutions to the equation $f_0 = 0$ represent the two horizons which arise in the metric and are given by

$$r_{\pm} = m \pm \sqrt{m^2 - e^2}. \tag{2.2.9}$$

We are only interested in the case where $m^2 > e^2$, so that the two solutions exist and are different. $r_\pm$ divide space-time into three regions: $\infty > r > r_+$, $\quad r_+ > r > r_-$ and $r_- > r > 0$.

It is interesting to look at the behavior of a time-like geodesic inside the black hole. In order to do so let us introduce a new coordinate $\tilde{t}$ defined by $t - r = \tilde{t} - r^*$, where now r^* is given by

$$r^* = \int \frac{dr}{1 - 2m/r + e^2/r^2} = \int \frac{dr}{f_0} = r + \frac{r_+^2}{r_+ - r_-} \ln(r - r_+) - \frac{r_-^2}{r_+ - r_-} \ln(r - r_-). \tag{2.2.10}$$

Thus, we obtain

$$dt = d\tilde{t} + (1 - f_0)dr, \tag{2.2.11}$$

which inserted into the metric of Eq. (2.2.6) gives

$$ds^2 = -f_0 d\tilde{t}^2 + 2(1 - f_0)d\tilde{t}dr + (2 - f_0)dr^2 + r^2 d\Omega^2. \tag{2.2.12}$$

If now we try to see how the radial null geodesics ($ds^2 = 0$, $d\Omega^2 = 0$) behave, we find that

$$(\tilde{t} + dr)[f_0 d\tilde{t} - (2 - f_0)dr] = 0. \tag{2.2.13}$$

Thus, we have found two families of radial null geodesics

$$\tilde{t} + r = cost, \tag{2.2.14a}$$

$$\frac{d\tilde{t}}{dr} = \frac{2 - f_0}{f_0}, \tag{2.2.14b}$$

which are plotted in Fig. 2.2.2.

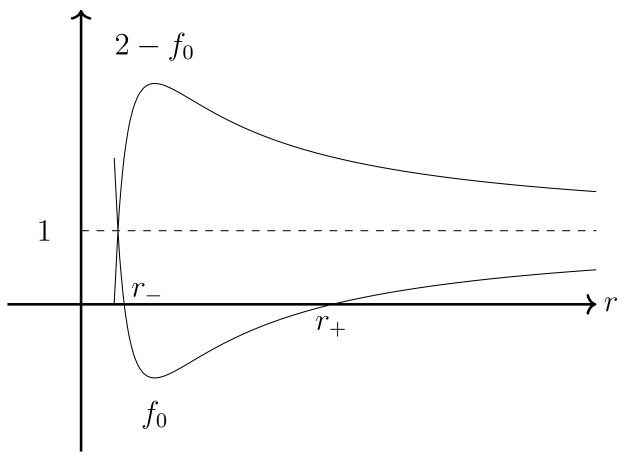

Figure 2.2.2: Graph of the slopes f_0 and $2 - f_0$.

From Fig. 2.2.3 we see that at $r_{\pm}$ the slope of Eq. (2.2.14b) becomes vertical and the surface $r = r_+$ is an event horizon, similar to the one found for the Schwarzschild solution. A surprising fact is that light cones in the region $r_- > r > 0$ are no longer oriented to the singularity $r = 0$ so that a time-like curve will never reach it but will return to the surface $r = r_-$ in a finite proper time.

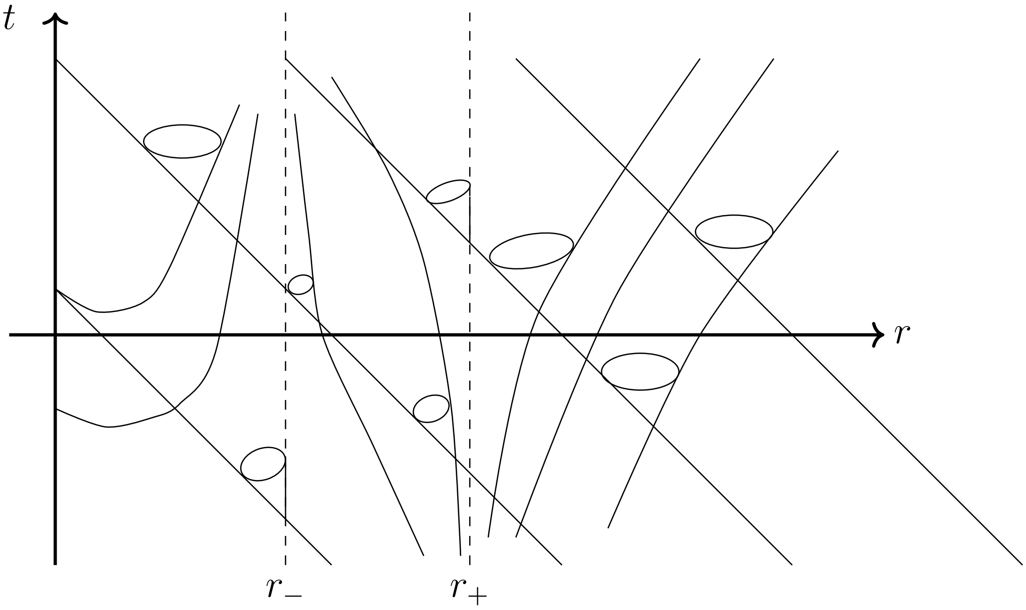

Figure 2.2.3: Geodesics and light cones.

To find an extension like the one found for the Schwarzschild solution in section 2.1, we start by introducing Eddington-Finkelstein coordinates as in Eq. (2.1.20), where now r^* is given by Eq. (2.2.10) and then we define the null coordinates (see Eq. 2.1.26) related to the inner horizon $r = r_-$

$$U = -e^{-k_0 u}, \tag{2.2.15a}$$

$$V = -e^{-k_0 v}, \tag{2.2.15b}$$

where k_0 is the surface gravity of the inner horizon: $k_0 \equiv -f_0'(r_-)/2 = (m_0^2 - e^2)^{1/2}/r_-^2$. This constant parameter is necessary so that the metric becomes regular at $r = r_-$. Now it is possible to operate a conformal compactification by defining the coordinates

$$V' = \arctan\left(exp\left(\frac{r_+ - r_-}{4r_+^2}v\right)\right), \quad U' = \arctan\left(-exp\left(\frac{-r_+ + r_-}{4r_+^2}u\right)\right). \tag{2.2.16}$$

By taking the metric with these new coordinates we obtain the maximal extension of the Reissner-Nordström space-time, which is shown in Fig. 2.2.4.

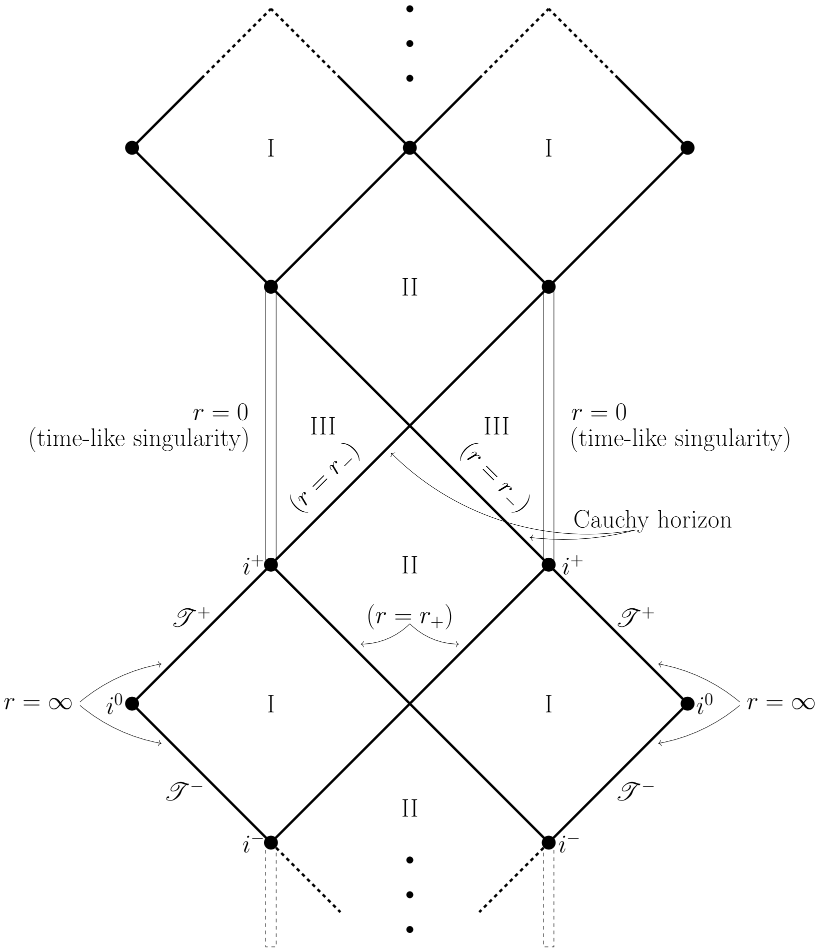

Figure 2.2.4: Penrose diagram of the maximal extension of Reissner-Nordström space-time.

As can be seen, there are an infinite number of asymptotically flat regions where $r > r_+$ (denoted by I) which are connected by intermediate regions where $r_+ > r > r_-$ and $r_- > r > 0$ (respectively denoted by II and III). The fundamental aspect of this representation is that now the real singularity $r = 0$ is time-like, while for the Schwarzschild-Kruskal manifold it was space-like. Thus, the singularity can be avoided by a future-directed time-like or null curve; such a curve can pass through regions II, III and again II to re-emerge into another asymptotically flat region I. It is important to remember that once a particle from region I has crossed the event horizon it is forever lost for the observers in that asymptotically flat universe. For instance, consider a particle P crossing $r = r_+$: it would appear as infinitely red-shifted to an observer whose world line stays outside the horizon, and will approach the future infinity i^+. Now P is in region II, where curves of constant r are space-like so that every point of the figure is a closed trapped surface. When P crosses $r = r_-$ it sees the entire history of one of the regions I, so that objects here appear to be infinitely blue-shifted as they reach i^+. This is a suggestion that the surface $r = r_-$ will be unstable under small perturbations, which could probably lead to a singularity, as we will see in chapter 3.

A few remarks about the two horizons are now in order. First, by looking at Fig. 2.2.2 it is possible to notice that the outer horizon $r = r_+$ is an event horizon. Thus, everything which crosses this surface is forever lost for the asymptotically flat universe. The situation is different for the inner horizon, called Cauchy horizon. In fact, this one can be described in terms of causal properties of the space-time, for which it is worth introducing some terminology. Infinite spacial surfaces in the space-time are called Cauchy surfaces. Points through which every time-like curve would intersect a Cauchy surface constitutes its domain of dependence. Solutions to the wave equation can be constructed within the domain of dependence of their Cauchy data. Finally, a Cauchy horizon is the boundary of the domain of dependence of a Cauchy surface. This means that predictability breaks down at the inner horizon of a Reissner-Nordström black hole for every Cauchy surface in our asymptotically flat universe. Thus, it seems that the *Cosmic censorship hypothesis* holds even for charged black holes since the singularity remains covered by the horizon.

2.3 Field Equations for Spherical Space-Times

We are now going to derive the field equations for spherical space-times. It is more convenient to use a coordinate system of the form (x^a, θ, ϕ), with $a = 1, 2$ and the coordinates x^a left unspecified. The metric then takes the generic form

$$ds^2 = g_{ab}dx^a dx^b + r^2 d\Omega^2, \tag{2.3.1}$$

where g_{ab} is the metric in the two-space $(\theta, \phi) = cost.$

Let us start by considering Einstein's field equations

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi(T_{\alpha\beta} + E_{\alpha\beta}), \tag{2.3.2}$$

where $E_{\alpha\beta}$ is the Maxwellian contribution to the stress-energy tensor and $T_{\alpha\beta}$ the non-Maxwellian contribution which will describe the cross flow of light-like radiation and $\alpha, \beta = 0, 1, 2, 3$. From the metric of Eq. (2.3.1) it is possible to derive the components of the Einstein tensor [1]

$$G_{ab} = -[2rr_{;ab} + g_{ab}(1 - r^{,a}r_{,a} - 2r\Box r)]/r^2 \tag{2.3.3a}$$

$$G_{\theta\theta} = \sin^{-2}\theta G_{\phi\phi} = r\Box r - \frac{1}{2}r^2 R. \tag{2.3.3b}$$

[1] With $\Box$ of a scalar quantity f intended as $\Box f = g^{ab} f_{;ab}$.

The Maxwellian contribution to the stress-energy tensor $E_{\alpha\beta}$ has to be thought of as representing a static electric field generated by a point charge e located in the origin $r = 0$ and is expressed as

$$E^{\alpha}_{\beta} = \frac{e^2}{8\pi r^4} diag(-1, -1, 1, 1). \tag{2.3.4}$$

About the non-Maxwellian contribution $T_{\alpha\beta}$ it is most convenient to leave it generic and just decompose it according to

$$T^a{}_a = T, \quad T^\theta{}_\theta = T^\phi{}_\phi = P. \tag{2.3.5}$$

Substitution into Eq. (2.3.2) then gives

$$2rr_{;ab} + g_{ab}(1 - r^{,a}r_{,a} - 2r\Box r - e^2/r^2) = -8\pi r^2 T_{ab}, \tag{2.3.6a}$$

$$r\Box r - \frac{1}{2}r^2 R - e^2/r^2 = 8\pi r^2 P. \tag{2.3.6b}$$

It is now useful to introduce the scalar fields $f(x^a)$, $m(x^a)$ and $k(x^a)$

$$g^{ab}r_{,a}r_{,b} \equiv f \equiv 1 - 2m/r + e^2/r^2, \tag{2.3.7a}$$

$$k \equiv -\frac{1}{2}\partial_r f = -m/r^2 + e^2/r^3 = -(m - e^2/r)/r^2, \tag{2.3.7b}$$

which inserted into Eqs. (2.3.6) give

$$r_{;ab} - g_{ab}(k + \Box r) = -4\pi r T_{ab}. \tag{2.3.8}$$

Now it is possible, taking the trace of Eq. (2.3.8), to obtain

$$\Box r = -2k + 4\pi r T, \tag{2.3.9}$$

which inserted into Eq. (2.3.8) and Eq. (2.3.6b) furnish

$$r_{;ab} + k g_{ab} = -4\pi r(T_{ab} - T g_{ab}), \tag{2.3.10a}$$

$$R - 2\partial_r k = 8\pi(T - 2P). \tag{2.3.10b}$$

Equations (2.3.10a and 2.3.10b) are the basic field equations for any spherical system.

Now it is time to consider a special case of Einstein's field equations by taking the non-Maxwellian component of the stress-energy tensor so that it describes in- and outgoing fluxes of radiation, i.e. of the form

$$T_{ab} = \rho_{in} l_a l_b + \rho_{out} n_a n_b \tag{2.3.11}$$

where l_a is a radial null vector pointing inwards and n_a a radial null vector pointing outwards. It is important to notice that the scalars ρ_{in} and ρ_{out} do not have a direct operational meaning since the null vectors can be arbitrarily normalized.

Upon noting that in this case $P = T = 0$, from Eq. (2.3.10) we obtain

$$r_{;ab} + kg_{ab} = -4\pi r T_{ab}, \tag{2.3.12a}$$
$$R = 2\partial_r K = 2(2m - 3e^2/r)/r^3, \tag{2.3.12b}$$
$$m_{,a} = 4\pi r^2 T_a^{\ b} r_{,b}, \tag{2.3.12c}$$
$$(r^2 T^{ab})_{;b} = 0. \tag{2.3.12d}$$

The last equation was obtained by using the conservation equation $(r^2T^{ab})_{;b} = (r^2)^{;a}P$ and the fact that, by Eq. (2.3.7a), $f_{,a} = -(2/r)m_{,a}$ and therefore

$$m_{,a} = 4\pi r^2 (T_a^{\ b} - \delta_a^{\ b} T) r_{,b}. \tag{2.3.13}$$

Equations (2.3.12a, 2.3.12b and 2.3.12c) may now be used to derive three one-dimensional scalar wave equations of the form $\Box\psi = \rho$.

First, taking the trace of Eq. (2.3.12a) and noting that the trace of T_{ab} is zero yields

$$\Box r = -2k. \tag{2.3.14}$$

Taking the derivative of Eq. (2.3.12c) and using the conservation relation gives

$$\Box m = -4\pi r^2 T^{ab} r_{;ab}, \tag{2.3.15}$$

and substituting the value of $r_{;ab}$ from Eq. (2.3.12a) yields the second one-dimensional scalar wave equation

$$\Box m = -(4\pi)^2 r^3 T^{ab} T_{ab}. \tag{2.3.16}$$

This is a very useful equation because it does not depend on the singular contributions of ρ_{in} and ρ_{out} but just on the bilinear one $\rho_{in}\rho_{out}$.

For the third wave equation more work is necessary. The purpose is to find an expression for $\Box \ln f$ which will be used in section 2.5 to derive the generalized DTR relation. Let us start by noting that, for every scalar function ψ,

$$\Box \ln \psi = (\psi \Box \psi - \psi^{,a}\psi_{,a})/\psi^2. \tag{2.3.17}$$

By taking the second derivative of Eq. (2.3.7a) and $k_{,a} = -m_{,a}/r^2 + Rr_{,a}/2$ it is possible to calculate $\Box f$ as

$$\begin{aligned} \Box f &= \nabla_a(-2m_{,a}/r + 2mr_{,a}/r^2 - 2e^2 r_{,a}/r^3) \\ &= -2(\Box m/r - r^{,a}m_{,a}/r^2 + k\Box r + k_{,a}r^{,a}) \\ &= -2\Box m/r + 4m^{,a}r_{,a}/r^2 - 2k\Box r - Rr_{,a}r^{,a} \\ &= -2\Box m/r + 4m^{,a}r_{,a}/r^2 + 4k^2 - fR. \end{aligned} \tag{2.3.18}$$

The product $f^{,a}f_{,a}$ yields

$$\begin{aligned} f^{,a}f_{,a} &= 4m^{,a}m_{,a}/r^2 - 4(2m/r^2 - 2e^2/r^3)m_{,a}r^{,a}/r + (2m/r^2 - 2e^2/r^3)r_{,a}r^{,a} \\ &= 4m^{,a}m_{,a}/r^2 + 8km_{,a}r^{,a}/r + 4fk^2. \end{aligned} \tag{2.3.19}$$

Therefore, it is necessary to determine the product $m_{,a}m^{,a}$ which can be calculated as follows

$$\begin{aligned} m_{,a}m^{,a} &= (4\pi)^2 r^4 T_a{}^b r_{,b} T^a{}_c r^{,c} \\ &= (4\pi)^2 r^4 T^{db} T_{ec} r_{,b} r^{,c} g_{da} g^{ea} \\ &= \frac{1}{2}(4\pi)^2 r^4 T^{eb} T_{ec} r_{,b} r^{,c} \\ &= \frac{1}{2}(4\pi)^2 r^4 T^{ec} T_{ec} r_{,b} r^{,c} g_c{}^b \\ &= -\frac{1}{2} r \Box m r_{,b} r^{,b} = -\frac{1}{2} r \Box m\, f. \end{aligned} \tag{2.3.20}$$

Inserting Eq. (2.3.20) into Eq. (2.3.19) yields

$$f^{,a}f_{,a} = -2f\Box m/r + 8km_{,a}r^{,a} + 4fk^2. \tag{2.3.21}$$

Finally, substituting Eqs. (2.3.21 and 2.3.18) into Eq. (2.3.22) yields, after a few calculations, the third wave equation

$$\Box \ln f = 16\pi f^{-2}(f - 2kr)T^{ab}r_{,a}r_{,b} - R. \tag{2.3.22}$$

Note that $\Box \ln f$ is linear in T^{ab}.

All these wave functions will be very useful in chapter 3 since they can be formally integrated.

2.4 Charged Vaidya Solutions

Until now we have considered only solutions with a vanishing non-Maxwellian component of the stress-energy tensor T_{ab}. However, by taking

$$T_{ab} = \rho_{in} l_a l_b, \tag{2.4.1}$$

we are led to the ingoing charged Vaidya solution

$$ds^2 = dv(2dr - f_{in}dv) + r^2 d\Omega^2, \tag{2.4.2a}$$

$$f_{in} = 1 - 2m_{in}/r + e^2/r^2, \tag{2.4.2b}$$

where v is taken as the Eddington-Finkelstein advanced time coordinate defined in Eq. (2.1.20a), with the normalization condition $l_a = -\partial_a v$. This solution represents the situation of a spherical, charged black hole which is irradiated by a light-like radiation falling inside from the infinite past of the universe.

The same can be said for the outgoing charged Vaidya solution, with $T_{ab} = \rho_{out} n_a n_b$, which is given by

$$ds^2 = du(2dr - f_{out}du) + r^2 d\Omega^2, \tag{2.4.3a}$$

$$f_{out} = 1 - 2m_{out}/r + e^2/r^2, \tag{2.4.3b}$$

where now u is the retarded time coordinate defined in Eq. (2.1.20b), with the normalization condition $n_a = -\partial_a u$. Now, conversely, we are dealing with a spherical, charged black hole irradiated by an outgoing radiation coming from the other universe (in a physical sense this radiation may be thought of as coming from the surface of the collapsing star). Using the generalized field equation $m_{,a} = 4\pi r^2 T_a{}^b r_{,b}$ (Eq. 2.3.12c) and the above Vaidya solutions the relationships between m_{in}/m_{out} and ρ_{in}/ρ_{out} can be determined:

$$dm_{in}(v)/dv = 4\pi r^2 \rho_{in}, \tag{2.4.4a}$$

$$dm_{out}(u)/du = 4\pi r^2 \rho_{out}. \tag{2.4.4b}$$

Note that according to Eq. (2.4.4b) the mass parameter actually increases even though the star loses mass.

Eddington-Finkelstein coordinates represent a good choice in describing the system since they reduce to classical advanced and retarded time far from the source. Furthermore, since mass parameters are objects which can be measured at infinity, so that they have a direct operational meaning, their derivatives in Eqs. (2.4.4a and 2.4.4b) also have a direct operational meaning as do the energy densities ρ_{in} and ρ_{out}.

2.5 Dray-'t Hooft-Redmount (DTR) Relation

Until now only the cases of single in- or outgoing fluxes of gravitational radiation have been considered. Let us now consider the case in which a Reissner-Nordström space-time is perturbed by a crossflow of infalling and outgoing light-like gravitational radiation which can be modeled as two spherical thin shells, one contracting and the other expanding, as shown in Fig. 2.5.1. The shells divide space-time into four different sectors, each one possessing a mass m and a function f. The generalized Dray-'t Hooft-Redmount (DTR) relation describes the relationship between the masses of these four sectors (the original DTR relation is valid in the case of vanishing charge).

The first step in determining the DTR relation requires expressing the field equations in double null coordinates (V, U), which for now are left unspecified, giving the general form

$$ds^2 = -2e^{2\sigma} dV dU, \tag{2.5.1}$$

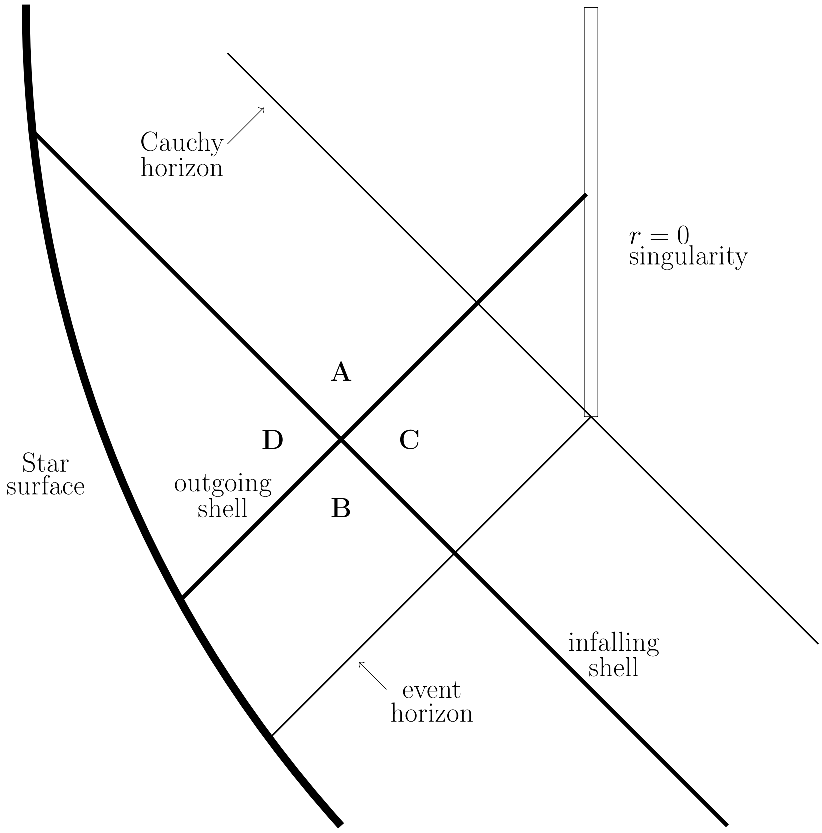

Figure 2.5.1: Two concentric null shells colliding at event q which divide space-time into four different regions **A**, **B**, **C**, **D**. The energy content of the infalling shell is given by $m_C - m_B$ while that of the outgoing shell by $m_D - m_B$.

where $\sigma = \sigma(V, U)$. Using this metric it is possible to find that for a scalar function ϕ

$$\Box\phi = -2e^{-2\sigma}\phi_{,UV}. \tag{2.5.2}$$

The stress-energy tensor takes the usual form $T_{ab} = \rho_{in} l_a l_b + \rho_{out} n_a n_b$, but now the normalization condition is chosen to be

$$l_a = -\partial_a V, \quad n_a = -\partial_a U. \tag{2.5.3}$$

Recalling Eq. (2.3.12c) it is clear that $r^2\rho_{in}(\rho_{out})$ do not depend on $U(V)$, therefore

$$\rho_{in} = \frac{L_{in}(V)}{4\pi r^2} \quad \text{and} \quad \rho_{out} = \frac{L_{out}(U)}{4\pi r^2}, \tag{2.5.4}$$

where $L_{in}(L_{out})$ represent the luminosities (note that they do not have direct operational meaning since they depend on the choice of the coordinates U and V). Using these results in Eq. (2.3.16) yields

$$\Box m = -2(re^{4\sigma})^{-1}L_{in}(V)L_{out}(U). \tag{2.5.5}$$

Considering Eq. (2.3.22), it is possible to appreciate that $\Box \ln f$ does not depend on the bilinear contribution $L_{in}L_{out}$ but just on the single contributions. This property is fundamental for the integration of the field equations in order to obtain the DTR relation. Before doing so, it is important to consider a mathematical result which is a generalization of Green's identity, as discussed in the 1990 article of E. Poisson and W. Israel [4].

Any equation of the form $\Box\phi = \rho$ can be formally integrated (note that we are using the metric of Eq. (2.5.1)). Suppose we are interested in the solution for ϕ with the boundary conditions located in a characteristic sector Ω described by the equations $U = U_1$ and $V = V_1$. From an application of Green's identity it follows that the solution at event (U, V), which has to be in the future of sector Ω, will be given by

$$\phi(V,U) = -\frac{1}{2}\int_{U_1}^{U}\int_{V_1}^{V} e^{2\sigma'}\rho' dV' dU' + \phi(V,U_1) + \phi(V_1,U) - \phi(V_1,U_1). \tag{2.5.6}$$

The simplicity of this result follows from the fact that the Green function of the operator $\Box$, if expressed in double null coordinates, is given by the superposition of two Heavyside step functions. Note that the solution at event (U, V) is given by the sum of the boundary conditions specified on the sector Ω and by a surface integral over the past radial light cone of the event.

The DTR relation is derived simply by the application of Eq. (2.5.6) to Eq. (2.3.22) [9], since ρ is linear in L_{in} and L_{out}, which are now represented by δ functions. In fact, if we integrate over an arbitrarily small light-like rhombus around the collision point (q in Fig. 2.5.1) the absence of any bilinear term guarantees that the integral contribution will be arbitrarily small. Therefore, we get $\ln f_A = \ln f_C + \ln f_D - \ln f_B$ and so

$$f_A f_B = f_C f_D. \tag{2.5.7}$$

In absence of gravity, the expression takes a trivial linear form, expressing the conservation of material energy in the collision. Its actual non linear form encases mathematically a number of surprising nonlinear effects normally hidden in Einstein's field equations, the most surprising of which is mass inflation.

Chapter 3

Mass Inflation

The necessary characteristics for mass inflation to occur are (i) a surface of infinite blue shift and (ii) a separation between the Cauchy and apparent horizon. The need for the second characteristic is not immediately evident. In fact, one would expect that the infinitely blue shifted influx could be enough to generate the inflation of the mass parameter and that the outflux could hypothetically be switched off without altering the situation. However, in this case no mass inflation occurs. This apparent mystery is simply explained: while the Cauchy horizon is a surface of infinite blue shift for our asymptotically flat universe, the inner apparent horizon is a surface of infinite red shift for other asymptotically flat universes and so the two effects cancel each other out because the two surfaces coincide. Nevertheless, if the outflux gets switched on it crosses the Cauchy horizon and focuses its generators so that the apparent horizon starts deflating very fast and the surfaces become distinct. In conclusion, the infinitely blue shifted influx is a key element for explaining the phenomenon, but it needs to be triggered by an arbitrarily small amount of outgoing radiation which creates the necessary separation between the Cauchy and apparent horizons.

If mass inflation occurs then the curvature also inflates. This is so because, even if the internal mass parameter has no global meaning, it always has a local one. In fact, it determines the "Coulomb" component of the local curvature. An increase in the mass parameter would be manifested locally, for example, by an increase in the tidal forces felt by an (extended) observer falling radially inward near the Cauchy horizon, even though this observer would register the energy density due to the infalling radiative tail as almost not blue shifted and vanishingly small.

3.1 Derivation of Mass Inflation

For the derivation of a mathematical expression for the mass inflation phenomenon let us start by considering a Reissner-Nordström black hole perturbed by a cross flow of infalling and outgoing radiation. Consider, in particular, switching on the infalling flux at advanced time V_1 and the outgoing flux at retarded time U_1 (see Fig. 3.1.1). In this way space-time is divided into four different regions, depending on the values of V and U, with four different mass parameters. Numbering these regions as in Fig. 3.1.1 it is possible to see that for $U < U_1$ and $V < V_1$ the space-time geometry will be described by the Reissner-Nordström solution with mass m_1. For $U < U_1$ and $V > V_1$ or $U > U_1$ and $V < V_1$ the geometry will be given respectively by the ingoing or outgoing Vaidya solution. Furthermore, for the cross-flow region where $U > U_1$ and $V > V_1$ the space-time geometry will have a solution given by Eqs. (2.3.12) with suitable boundary conditions.

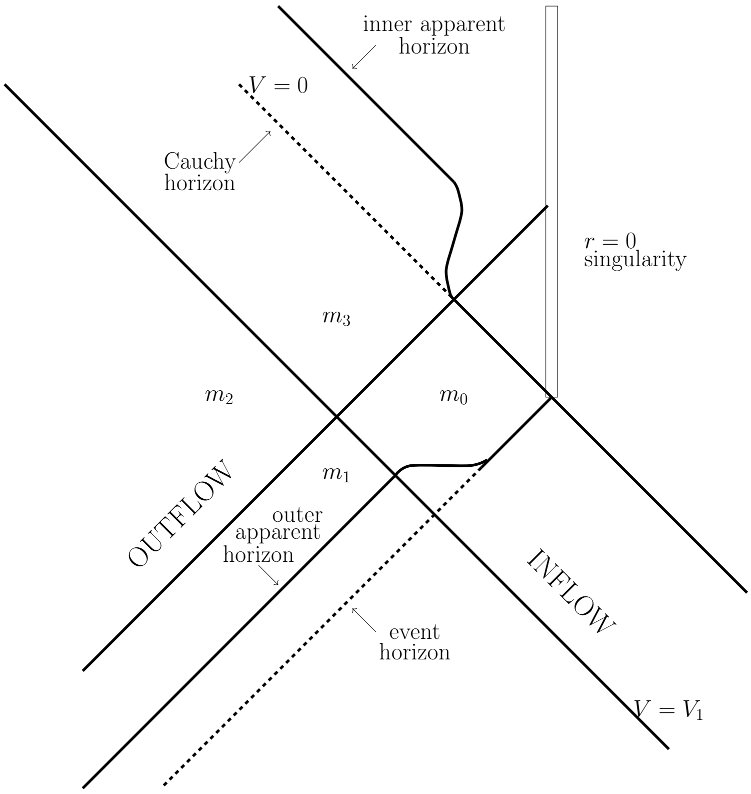

Figure 3.1.1: Background Reissner-Nordström space-time perturbed by cross-flowing streams of radial radiation.

Let us start again by considering double null coordinates, without specifying which ones, where

$$ds^2 = -2e^{2\sigma} dV dU + r^2 d\Omega^2, \tag{3.1.1}$$

with a non-Maxwellian contribution to the stress-energy tensor given by

$$T_{ab} = [L_{in}/4\pi r^2] l_a l_b + [L_{out}/4\pi r^2] n_a n_b. \tag{3.1.2}$$

$l_a = -\partial_a V$ and $n_a = -\partial_a U$ are the normalization conditions.

From Eq. (2.3.7a)

$$1 - 2m(U,V)/r + e^2/r^2 = -2e^{-2\sigma}(\partial_U r)(\partial_V r) \tag{3.1.3}$$

which, using Eq. (2.3.12c), yields the field equations for m:

$$\partial_U m = -L_{out}(U)e^{-2\sigma}(\partial_V r), \tag{3.1.4a}$$
$$\partial_V m = -L_{in}(V)e^{-2\sigma}(\partial_U r). \tag{3.1.4b}$$

Now let us notice that it is possible to derive the relation between L_{in} and $dm_{in}(v)/dv$ in the pure ingoing region where $m = m_{in}$ and $f = f_{in}$ so that there the solution is completely given by the ingoing Vaidya solution. For continuity, L_{in} will be the same as that of the cross flow region. In this way, using the second of Eqs. (3.1.4b) in terms of v and Eq. (2.3.7a), the result is

$$\partial_V m = \frac{\partial m}{\partial v}\frac{\partial v}{\partial V} = -L_{in}(V)e^{-2\sigma}\partial_U r = -L_{in}(V)e^{-2\sigma}\frac{2\partial_U r \partial_V r}{2\partial_V r} = L_{in}(V)\frac{f}{2\partial_V r}. \tag{3.1.5}$$

Considering a light-like ($ds^2 = 0$) flow within the ingoing Vaidya solution, it gives

$$2dr = f_{in}dv \quad \rightarrow f_{in} = 2\frac{\partial r}{\partial v}, \tag{3.1.6}$$

which substituted into Eq. (3.1.5) yields

$$\frac{\partial m}{\partial v}\frac{\partial v}{\partial V} = L_{in}\frac{\partial r}{\partial v}\frac{\partial V}{\partial r}. \tag{3.1.7}$$

Finally, the expression for L_{in} is given by

$$L_{in} = \left(\frac{\partial v}{\partial V}\right)^2 \frac{\partial m_{in}(v)}{\partial v}, \tag{3.1.8}$$

and similarly for L_{out}

$$L_{out} = \left(\frac{\partial u}{\partial U}\right)^2 \frac{\partial m_{out}(u)}{\partial u}. \tag{3.1.9}$$

It is now important to determine the expression for $m_{in}(v)$. In order to do so, it is useful to recall the analysis by Richard H. Price made nearly fifty years ago [10], which showed that the amplitude of the backscattered radiation varies as v^{-n} at late advanced times, where n is given by $2l+2$ with l representing the multi-pole order of the testing field. Now the energy density of the ingoing radiation, which is proportional to dm_{in}/dv, will vary as v^{-2n}, so that the ingoing mass function will reach the asymptotic limit $m_0 - m_{in}(v) \sim v^{-(4l+3)}$. The dominant contribution to the influx will come from the quadrupole moment $l = 2$ so that typically $m_0 - m_{in}(v) \sim v^{-11}$. Finally, we can take

$$dm_{in}(v)/dv \sim v^{-p}, \tag{3.1.10}$$

with $p = 4(l+1) \geqslant 12$. It will be seen that for L_{out} it is not important to know how it behaves but just to consider it as a positive quantity, which is reasonable since it determines the energy density of the radiation escaping from the surface of a collapsing star.

Eq. (2.5.5) and Eq. (2.5.6) can be used to obtain a formal expression for the mass function

$$m(V,U) = \int_{U_1}^{U} \int_{V_1}^{V} (r' e^{2\sigma'})^{-1} L_{in}(V') L_{out}(U') dV' dU' + m_{in}(V) + m_{out}(U) - m_1. \tag{3.1.11}$$

Until now the coordinates V and U have been left unspecified. In order to define the coordinates V and U, we start by assuming that the influx is switched on at $V = V_1$ and the outflux at $U = U_1$. This means that we also assume that the flux is turned off at V_2. In particular the case of interest here is where V_2 tends to the Cauchy horizon. In this model, space-time is divided into 4 static regions with mass m_0, m_1, m_2, m_3 and in inflow, outflow and cross-flow regions (see Fig. 3.1.1). Each static region is described by the Reissner-Nordström solution and has its couple of null Eddington-Finkelstein coordinates, as defined in Eqs. (2.1.20a and 2.1.20a). Let us define the coordinate v as the advanced time referred to the static region m_0, so that the Cauchy horizon will lie in the $v = \infty$ surface. In the same way the coordinate u is defined as the retarded time referred to the region m_1, so that the outer horizon will lie in the $u = -\infty$ surface. The coordinates V and U can now be defined as the Kruskalized advanced/retarded time associated with the inner horizons $r = r_0 = m_0 - (m_0^2 - e^2)^{1/2}$ and $r = r_1 = m_1 - (m_1^2 - e^2)^{1/2}$ respectively:

$$V = -e^{-k_0 v}, \tag{3.1.12a}$$
$$U = -e^{-k_1 u}, \tag{3.1.12b}$$

with $k_0 = (m_0^2 - e^2)^{1/2}/r_0^2$ and $k_1 = (m_1^2 - e^2)^{1/2}/r_1^2$. In this way it is possible to see that $V = 0$ on the Cauchy horizon and $U = -\infty$ on the outer horizon.

Using this system of coordinates in Eq. (3.1.8) gives

$$L_{in}(V) \sim \frac{[-\ln(-V)]^{-p}}{V^2} \sim v^{-p} e^{2k_0 v}, \tag{3.1.13}$$

and, for $V \to 0$,

$$\int_{V_1}^{V} L_{in}(V) dV \sim \frac{[-\ln(-V)]^{-p}}{-V} \sim v^{-p} e^{k_0 v}. \tag{3.1.14}$$

Considering Eq. (3.1.11) it can be seen that $m(V,U)$ will diverge as $V \to 0$, unless the factor $re^{2\sigma}$ goes to infinity quickly enough. Intuitively, this factor will instead tend to go to zero. This is the case since the product of the derivatives $\partial_U r \partial_V r$ remains well behaved probably because the coordinates U and V are well behaved in the vicinity of the Cauchy horizon and because this horizon should not deflate too quickly. In fact, the behavior of the Cauchy horizon is only ruled by the amount of outgoing radiation focusing its generators, since it is described by $r_{CH}(U) \equiv r(V=0,U)$. This is not the case of the inner *apparent* horizon, described by the scalar r_{AH} derived from $f(r_{AH}) = 0$, which collapses catastrophically as the mass inflates to infinity.

Although it is not possible to prove that the factor $re^{2\sigma}$ goes to zero, it can be shown that it cannot grow to infinity approaching the Cauchy horizon. In order to do so, let us call $\psi = re^{2\sigma}$ and derive an expression for $\Box \ln \psi$ which can be integrated using again Eq. (2.5.6). Any scalar function ψ verifies

$$\Box \ln \psi = (\psi \Box \psi - \psi^{,a}\psi_{,a})/\psi^2, \tag{3.1.15}$$

so that in this case

$$\begin{aligned}\Box \ln \psi &= [re^{2\sigma}(\Box r + 2r\Box\sigma + 2r\sigma_{,a})e^{2\sigma} - fe^{4\sigma} - r^2 e^{4\sigma} 4\sigma^{,a}\sigma_{,a}]/r^2 e^{4\sigma} = \\ &= \Box r/r + 2\Box\sigma - f/r^2.\end{aligned} \tag{3.1.16}$$

The Ricci scalar is calculated to be $R = -2\Box\sigma$, which yields

$$\Box\sigma = -\partial_r k = -(2m - 3e^2/r)/r^3. \tag{3.1.17}$$

Using Eqs. (3.1.17, 2.3.7a, 2.3.14 and 2.3.7b) in Eq. (3.1.16) gives

$$\begin{aligned}\Box \ln \psi &= \frac{2}{r^3}(m - e^2/r) - \frac{2}{r^3}(2m - 3e^2/r) - 1/r^2 + 2m/r^3 - e^2/r^4 = \\ &= 3e^2/r^4 - 1/r^2 = (3e^2 - r^2)/r^4.\end{aligned} \tag{3.1.18}$$

It is possible to integrate the above expression using Eq. (2.5.6),

$$\begin{aligned}\ln \psi &= -\frac{1}{2}\int_{U_1}^{U}\int_{V_1}^{V} e^{2\sigma'}(3e^2 - r'^2)/r'^4 \, dV'dU' + \ln\psi(U_1,V) + \ln\psi(U,V_1) - \ln\psi(U_1,V_1) = \\ &= \ln\frac{\psi_{in}(U_1,V)\psi_{out}(U,V_1)}{\psi(U_1,V_1)} - \frac{1}{2}\int_{U_1}^{U}\int_{V_1}^{V} \psi' r'^{-5}(3e^2 - r'^2)\, dV'dU',\end{aligned} \tag{3.1.19}$$

where ψ_{in} and ψ_{out} represent the function $re^{2\sigma}$, which can be obtained respectively from the known ingoing and outgoing charged Vaidya solutions.

Now the first term is finite because ψ_{in} is finite on the Cauchy horizon [4]. This is an effect of the fact that when only the influx is considered, the surfaces of infinite blue-shift and red-shift coincide, thus preventing the inflation of the mass parameter. The second term is more delicate since it might be unbounded and what is crucial is the sign of the contribution. If the integral is actually unbounded the dominant contribution will come from values near the Cauchy horizon. Since $|e| < m_0$ (indispensable for the creation of the black hole), $r_0 < |e|$ so that in the vicinity of the Cauchy horizon the contribution from the integral will be negative because $(3e^2 - r^2)$ remains positive. Considering this, $\ln \psi$ will be necessarily not bounded below, but bounded above. This in turn ensures that $re^{2\sigma}$ cannot go to infinity near the Cauchy horizon, hence it cannot stop the inflation phenomenon.

In summary, mass inflation is expressed through the divergence of Eq. (3.1.11). The physical interpretation is that the combined effect of the infalling flux, which is infinitely blue-shifted and piles up at the Cauchy horizon, and of the outgoing radiation, which produces the fundamental separation between Cauchy and apparent horizons, results in the inflation of the internal mass parameter.

The phenomenon can be understood further from an application of the DTR relation (Fig. 2.5.1): if the two shells cross through each other near the Cauchy horizon of sector B, then f_B will be very small but since $f_A f_B$ remains constant f_A will increase as the cross section goes near the Cauchy horizon. This can be interpreted as a violent increase of the mass parameter at the intersecting point. However, the DTR relation does not help in understanding the characteristic time over which mass inflation occurs.

No trace of the mass inflation phenomenon is perceptible externally. Outside the black hole the mass m does not inflate, but remains practically the same as the mass of the original star. News of the drastic change of the internal field must propagate with the speed of light as a gravitational wave, and it can never emerge from the event horizon. Classical physics provides no damping mechanism for this growth of curvature near the Cauchy horizon. However, as the curvature grows it is possible that new, unknown quantum effects come into play.

3.1.1 Growth Rate Estimation

It is possible to derive an estimation of the mass growth rate by expanding the integral of Eq. (3.1.11) in powers of bilinear $L_{in}L_{out}$. Keeping only the first-order term, it is possible to take for $re^{2\sigma}$ the background values in the Reissner-Nordström solution with mass m_0 (in first approximation $m_0 \simeq m_1$). Recalling the definition given for the coordinates U and V in the section of Reissner-Nordström black holes,

$$U = -e^{-k_0 u} \quad \text{and} \quad V = -e^{-k_0 v}, \tag{3.1.20}$$

with $k_0 = -f_0'(r_0) = (m_0^2 - e^2)^{1/2}/r_0^2$ the surface gravity of the inner horizon, it is possible to see that near the Cauchy horizon $f_0 \simeq -2UV$. This implies that the metric element

g_{UV} takes the form

$$g_{UV} = \frac{f_0}{2k_0^2 UV} \simeq -\frac{1}{k_0^2}. \tag{3.1.21}$$

Using Eq. (3.1.21) we can take

$$re^{2\sigma} \simeq \frac{r_0}{k_0^2}. \tag{3.1.22}$$

Substituting Eq. (3.1.13) and Eq. (3.1.14) into Eq. (3.1.11) yields an estimation of the internal mass parameter

$$\begin{aligned} m(U,V) &\simeq \int_{U_1}^{U}\int_{V_1}^{V} (r'e^{2\sigma'})^{-1} L_{in}(V')L_{out}(U')dV'dU' \\ &\simeq m_0 \frac{k_0}{m_0}\int_{U_1}^{U} L_{out}(U')dU' \frac{k_0}{r_0}\int_{V_1}^{V} L_{in}(V')dV' \\ &\simeq m_0 \Gamma(U)\epsilon^2 \frac{(k_0 v)^{-p} e^{k_0 v}}{k_0 r_0}, \end{aligned} \tag{3.1.23}$$

where $\Gamma(U) = k_0 m_0^{-1} \int_{U_1}^{U} L_{out}(U')dU'$ represents the fraction of the star's mass radiated away. Since, as was said above, the principal contribution comes from the quadrupole moment $p = 12$

$$m(U,V) \simeq m_0 \Gamma(U)\epsilon^2 \frac{(k_0 v)^{-12} e^{k_0 v}}{k_0 r_0}, \tag{3.1.24}$$

with ϵ representing a dimensionless quadrupole moment.

This crude estimate shows that mass inflates exponentially with Eddington-Finkelstein advanced time with a characteristic time scale of $1/k_0$. This is probably an underestimate since, as was shown above, the factor $re^{2\sigma}$ does not remain constant but will reasonably tend to go to zero thus precipitating further mass inflation.

3.1.2 Asymmetries

Some aspects about how asymmetries, which reasonably are present in any realistic collapse, influence the spherical symmetric phenomenon of mass inflation should be considered. The broad aspects of mass inflation should be generic since it is characterized mainly by the presence of a surface of infinite blue shift and a separation between the Cauchy and apparent horizons.

In the case of a rotating black hole, the angular momentum J is nearly conserved during the collapse and completely conserved in the case of axial symmetry. Thus, the Kerr parameter $a = J/m$ will be negligible compared to the mass in the vicinity of the Cauchy horizon. This shows that the geometry will be expected to be given by the Schwarzschild solution.

Let us now consider the effects of non rotational asymmetries. It has been shown by the work of E. Poisson and W. Israel in 1989 [11] that the tail of the spacelike singularity, corresponding to large values of advanced time v, developed at $r = 0$ during the collapse, relaxes asymptotically to a Schwarzschild-like form. It is straightforward to adapt it to the situation where a Cauchy horizon is present: a spacelike curve in the region near the horizon now corresponds to the curve in $r = 0$. Asymmetries can interfere with the tail just for a very brief period of the star's history and also become exponentially red shifted. Thus, the geometry of the Cauchy horizon's tail is determined only by the asymptotic values of the external field as $v \to \infty$. But it is known that it relaxes to a Kerr-Newman form which, among other things, has a constant inner-horizon surface gravity k_0. This means that the exponential mass inflation factor $e^{k_0 v}$ (Eq. (3.1.14)) is uniform and so the variations $\Delta m/m \sim \Delta\theta$ of the mass aspect $m(\theta, \phi, v)$ on the angular scale $\Delta\theta$ should not grow exponentially but remain nearly constant during inflation. In the vicinity of the Cauchy horizon these nonuniformities will be negligible on length scales comparable to the local curvature. Thus, the geometry of space-time should be indistinguishable from a Schwarzschild solution for a very large mass factor.

In conclusion, there are no reasons to expect that the effect of rotations or asymmetries should modify the results found about mass inflation.

Chapter 4

Conclusions

In the present study the most striking features of the physics behind black holes have been shown. In chapter 2, beginning with the Schwarzschild solution it is clear that the choice of the right coordinates to describe the metric is fundamental to understand the essential physics behind the various solutions to Einstein's field equations. This is a recurring concept in all general relativity and fundamental to the analysis of the mass inflation phenomenon.

The calculations presented in the section on Reissner-Nordström black holes show that in addition to the classic event horizon a new one with peculiar characteristics is created: a Cauchy horizon. The presence of this null surface gives importance to the study of the internal structure of black holes. In fact, this is a surface of infinite blue shift beyond which predictability breaks down. Indeed, initial data specified for example at the beginning of a gravitational collapse are not sufficient to predict what happens to the future of the Cauchy horizon.

Considering a Reissner-Nordström space-time perturbed by in- or outgoing fluxes of radiation different solutions to Einstein's field equations are possible: the Vaidya solutions and the DTR relation. The importance of this last relation is that it gave a first glance that something strange was happening at the Cauchy horizon. In fact two streams of gravitational radiation, modeled as two spherical thin shells one contracting and one expanding, divide space-time into four separated regions each one with a different mass m and function f (see Fig. 2.5.1). The relation of Eq. (2.5.7) implies that if the two shells are interacting near the inner horizon the smallness of the function of the outer region (B) will be balanced by the largeness of that of the inner region (A) and this can be interpreted as a violent increase of the mass parameter at the intersecting point.

In this study of the mass inflation phenomenon, the calculations show that the intersection of the outgoing gravitational radiation with its backscattered blue shifted radiative tail at the Cauchy horizon causes a classically unbounded inflation phenomenon of the internal mass parameter of the hole. Since this region is causally disconnected from the outside of the hole, the external mass parameter remains well behaved. Unfortunately the field equations do not permit the derivation of an explicit expression for $m(U, V)$. The estimation derived for the growth rate (Eq. (3.1.24)) is probably an underestimate since the factor $re^{2\sigma}$ will reasonably tend to vanish precipitating further mass inflation.

The mass inflation phenomenon is one of the most peculiar aspects of the non linearity of general relativity. The singularity created at the inner horizon suggests that the classical laws of general relativity will cease to be valid at the so called Planckian scale, where quantum effects will presumably occur.

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