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REPRESENTATIONS OF SYMMETRIC GROUPS ON THE HOMOLOGY OF DUAL MATROIDS OF COMPLETE GRAPHS

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Introduction

This thesis investigates the representations of the symmetric group on the homology of the dual matroid of a complete graph. These representations arise as follows: with each graph we can associate a matroid, by taking the set of edges of the graph as ground set and the edge sets of simple cycles as the circuits of the matroid. We focus on the dual of the matroid of the complete graph K_n, which coincides with the dual matroid of the independent sets of the root system of type A_{n-1}. We calculate the homology of the simplicial complex L associated with this matroid.

Permuting the vertices of the complete graph induces a permutation on the edge set which is a vertex map of the simplicial complex. This vertex map sends independents to independents, thus inducing a simplicial map from the polytope of L to itself, hence on the homology spaces of L. This defines a representation of the symmetric group $\mathfrak{S}_n$ on the homology $H_i(L, \mathbb{C})$, which turns out in this case to be non trivial if $i = (n^2 - 3n)/2$. We show that the above representation is induced from a primitive representation of C_n, the cyclic subgroup of order n.

This problem has been suggested by the study of the Cattani-Kaplan-Schmid complex relative to a family of completely reducible spectral curves for the Hitchin fibration of type A_n, performed by de Cataldo, Heinloth and Migliorini ([7]). The dual graph of a spectral such curve is the complete graph, and the action of the symmetric group on the irreducible components of the curve yields an action on the vertices of the complete graph .

In chapter one we give some background regarding root systems and group representations. In the first section we focus on the root system of type A, whose Weyl group is the symmetric group. The action of the Weyl group $\mathfrak{S}_n$ on the root system A_{n-1}, without considering the sign, will correspond to the action of the symmetric group on the edges of the complete graph. In particular, in the second section, we define the induced representation and we compute an example that will be important for our purposes.

In chapter two we recall the necessary preliminaries regarding simplicial complexes, homology, and matroid theory. In the first section we prove Alexander duality, that will be crucial for the main result of this thesis. In the second section we see how the action on the ground set induces a simplicial map and furthermore a map on the homology. At the end of the chapter we describe the case of K_4 (or equivalently $\Phi^+(A_3)$).

In chapter three, following Stanley [20], we compute the representations on the homology of the partition lattice Π_n with the action induced from the permutations of $1, 2, \ldots, n$. We see that these representations are exactly the induced representations from C_n to $\mathfrak{S}_n$. At the end of the chapter we report the explicit calculation of the representations of Π_4. The reader can compare this example with the one at the end of chapter 2.

The results obtained in these two examples are generalized in chapter four. In the first section, given any simple matroid M, we apply Alexander duality to the following two abstract simplicial complexes: the independence set of the dual matroid M^* and the nonspanning simplicial complex of M, i.e. the set of all subsets of the groundset which do not contain any basis of M. In the second section we use a result due to Folkman to show the isomorphism between the homology of the non spanning simplicial complex and the homology of its lattice of flats. Combining the last two results we prove that the homology of the dual matroid M^* is isomorphic to the homology of the lattice of flats of M. On the other hand, the lattice of flats of the cycle matroid of K_n is exactly the partition lattice Π_n. Since these isomorphisms are natural, i.e. they do not depend on the choice of a basis, they are compatible with the action of $\mathfrak{S}_n$. Consequently the representations we have studied in chapter three coincide with the representations we want to study in this thesis.

It would be interesting to extend what has been done in this thesis to any type of root system. In chapter five we compute the representations of the Weyl group on the homology of the dual matroid associated to the root system of type B_2.

Contents

Chapter 1

Preliminaries in Algebra

1.1 Root systems

In this Section we recall the fundamental notions of root system and we explicitly construct the root systems of type A that will be fundamental for this thesis.

1.1.1 Basic definitions

Let E be a finite-dimensional real vector space endowed with a positive definite symmetric bilinear form $(\cdot,\cdot)$.
Given a non-zero vector $\alpha \in E$, let

$$P_\alpha = \{v \in E \mid (v, \alpha) = 0\}$$

be the hyperplane orthogonal to α, and let

$$\sigma_\alpha : \ E \ \longrightarrow \ E$$

be an invertible linear transformation such that

$$\sigma_\alpha(v) = v \ \ \forall v \in P_\alpha, \quad \sigma_\alpha(\alpha) = -\alpha.$$

It is sufficient to define σ_α on P_α and α, since $E = <\alpha> \oplus P_\alpha$.
It is easy to write down an explicit formula:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

We define $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$; notice that $\langle \beta, \alpha \rangle$ is linear only in the first variable.

Definition 1.1. A subset Φ of E is called a *root system* in E if the following axioms are satisfied:

(R1) Φ is finite, $0 \notin \Phi$, Φ spans E

(R2) If $\alpha \in \Phi$, then $c\alpha \in \Phi \Leftrightarrow c = \pm 1$

(R3) $\forall\ \alpha, \beta \in \Phi \quad \sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \ \in \Phi$

(R4) $\forall\ \alpha, \beta \in \Phi \quad \langle \beta, \alpha \rangle \in \mathbb{Z}$

The elements of Φ are called *roots*. The dimension of E, $\dim_{\mathbb{R}} E = l$, is called the *rank* of the root system.

The following statements give the first indication that the axioms for root systems are quite restrictive.
Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Then:

a) $(\alpha, \beta) = \|\alpha\| \|\beta\| \cos \widehat{\alpha\beta} \Longrightarrow \langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} = \frac{2\|\alpha\|}{\|\beta\|} \cos \widehat{\alpha\beta}$

b) $\langle \alpha, \beta \rangle = 0 \iff (\alpha, \beta) = 0$

c) $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ are always concordant.

d) $\|\alpha\| \leq \|\beta\| \Longrightarrow |\ \langle \alpha, \beta \rangle\ | \ \leq \ |\ \langle \beta, \alpha \rangle\ |$

e) $\langle \alpha, \beta \rangle\ \langle \beta, \alpha \rangle = 4 \cos^2 \widehat{\alpha\beta} \leq 3 \ \Longrightarrow \langle \alpha, \beta \rangle\ \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$

Definition 1.2. A subset Δ of Φ is called a *base* if:

(B1) Δ is a base of E.

(B2) Each root $\beta \in \Phi$ can be written as $\beta = \sum_{\gamma \in \Delta} n_\gamma \gamma$ with integral coefficients n_γ all nonnegative or all nonpositive.

The roots in Δ are called *simple*. If all $n_\gamma \geq 0$, we call β *positive* and write $\beta \succ 0$; if all $n_\gamma \leq 0$, we call β *negative* and write $\beta \prec 0$.
The collections of positive and negative roots (relative to Δ) will usually just be denoted by Φ^+ and Φ^-.

Definition 1.3. The root system Φ is called *irreducible* if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

Definition 1.4. Let Φ be a root system in E. Denote by $\mathcal{W}$ the subgroup of $GL(E)$ generated by the reflections σ_α with $\alpha \in \Phi$. $\mathcal{W}$ is called the *Weyl group* of Φ.

By (R3) $\mathcal{W}$ permutes the elements of Φ, so we can identify $\mathcal{W}$ with a subgroup of the symmetric group on Φ, in particular $\mathcal{W}$ is finite.

1.1.2 Root systems of type A

Let E be the n-dimensional subspace of $\mathbb{R}^{n+1}$ orthogonal to the vector $e_1 + \cdots + e_{n+1}$:

$$E = \left\{\mathbf{x} \in \mathbb{R}^{n+1} \mid (\mathbf{x}, e_1 + \cdots + e_{n+1}) = 0\right\}$$

Let also

$$I = \left\{\sum_{i=1}^{n+1} a_i e_i \mid a_i \in \mathbb{Z}\right\} \quad \text{and} \quad I' = I \cap E$$

Let Φ be the set of all vectors $\alpha \in I'$ such that $(\alpha, \alpha) = 2$:

$$\Phi = \left\{\alpha \in I' \mid (\alpha, \alpha) = 2\right\} = \left\{\alpha \in I' \mid \|\alpha\|^2 = 2\right\}$$

Let $\alpha \in \Phi \subseteq E$, we have for suitable $a_i \in \mathbb{Z}$:

$$\underbrace{0} = \Big(\alpha, \sum_{j=1}^{n+1} e_j\Big) = \Big(\sum_{i=1}^{n+1} a_i e_i, \sum_{j=1}^{n+1} e_j\Big) = \underbrace{a_1 + \ldots a_{n+1}}$$

Moreover $\|\alpha\|^2 = 2$,

$$\underbrace{2} = (\alpha, \alpha) = \Big(\sum_{i=1}^{n+1} a_i e_i, \sum_{i=1}^{n+1} a_i e_i\Big) = \underbrace{a_1^2 + \dots a_{n+1}^2}$$

For a fixed $\alpha \in \Phi$, there must be two different a_i, a_j such that:

$$a_i = 1 \quad a_j = -1 \quad a_k = 0, \ \ \forall k \neq i, j$$

or

$$a_i = -1 \quad a_j = 1 \quad a_k = 0, \ \ \forall k \neq i, j$$

Every element of Φ is of the form $e_i - e_j$ with $i \neq j$:

$$\Phi = \big\{e_i - e_j \mid i \neq j\big\}$$

Φ is obviously finite and $0 \notin \Phi$ by definition. It is evident that Φ spans E. Therefore (R1) is satisfied.
The choice of lengths we made make obvious that (R2) holds.
For (R3) it is enough to check that the reflection σ_α with $\alpha \in \Phi$ sends Φ to I', i.e $\sigma_\alpha(\Phi) \subseteq I'$, since then $\sigma_\alpha(\Phi)$ automatically consists of vectors of squared lengths equal to two (σ_α is an isometry, doesn't change lengths). But then (R3) follows directly from (R4):

$$\sigma_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\alpha \qquad \beta \in I', \alpha \in I'$$

If (R4) holds, we have that $\langle\beta, \alpha\rangle \in \mathbb{Z}$ and then $\sigma_\alpha(\beta) \in \Phi$.
Regarding (R4), let $\alpha, \beta \in \Phi \subseteq I'$ we have for suitable $a_i, b_i \in \mathbb{Z}$:

$$(\alpha, \beta) = \Big(\sum_{i=1}^{n} a_i e_i, \sum_{j=1}^{n} b_j e_j\Big) = a_1 b_1 + \dots + a_n b_n \quad \in \mathbb{Z}$$

It follows that

$$\langle\alpha, \beta\rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = \frac{2(\alpha, \beta)}{\|\alpha\|^2} = (\alpha, \beta)$$

Then $\langle\alpha, \beta\rangle \in \mathbb{Z}$. Therefore Φ is a root system and is called the root system of type A_n.

The vectors $\alpha_i = e_i - e_{i+1}$ $(1 \leqslant i \leqslant n)$ are linearly independent and for $i < j$:

$$e_i - e_j = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + (e_{j-1} + e_j)$$

The coefficients of the α_i are all positive, so the α_i form a base of Φ.

$$\Delta = \big\{\alpha_i \mid 1 \leqslant i \leqslant n\big\}$$

Finally, notice that the reflection σ_{α_i} with respect to the root α_i, permutes the subscripts $i, i+1$ and leaves all the others subscripts fixed.

$$\sigma_{\alpha_i}(\alpha_i) = -\alpha_i \qquad \sigma_{\alpha_i}(\alpha_{i+1}) = \alpha_{i+1} - \langle \alpha_{i+1}, \alpha_i \rangle \alpha_i = \alpha_{i+1} + \alpha_i$$

$$\sigma_{\alpha_i}(\alpha_j) = \alpha_j - \langle \alpha_j, \alpha_i \rangle \alpha_i = \alpha_j \qquad \forall j \neq i-1, i, i+1$$

Thus σ_{α_i} corresponds to the transposition $(i, i+1)$ in the symmetric group $\mathfrak{S}_{n+1}$. These transpositions generate $\mathfrak{S}_{n+1}$, so we obtain a natural isomorphism between $\mathcal{W}$ and $\mathfrak{S}_{n+1}$:

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathfrak{S}_{n+1} \\ \sigma_{\alpha_i} & \longmapsto & (i, i+1) \end{array}$$

If we think of A_n as embedded in $\mathbb{R}^{n+1}$, the action of the Weyl group corresponds to a permutation of the coordinates of $\mathbb{R}^{n+1}$. Each element σ of $\mathcal{W} = \mathfrak{S}^{n+1}$ induces a map on $\Phi^+_{A_n}$ in the following way: for each element $x_i \in \Phi^+_{A_n}$ consider the subspace $V_i = \text{Span}\{x_i\}$ of $\mathbb{R}^{n+1}$.

Since $\mathcal{W}$ permutes the roots of Φ_{A_n}, it also permutes the V_i's. By identifying the V_i with the x_i we get a map on $\Phi^+_{A_n}$; in other words we consider the action of the Weyl group on Φ_{A_n} without considering the sign.

Example 1. Let us consider the root system of type A_3:

$$\Phi_{A_3} = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm(\alpha_1 + \alpha_2), \pm(\alpha_2 + \alpha_3), \pm(\alpha_1 + \alpha_2 + \alpha_3)\}$$

$$\begin{array}{ccc} \mathcal{W}_{A_3} & \longrightarrow & \mathfrak{S}_4 \\ \sigma_{\alpha_1} & \longmapsto & (1, 2) \\ \sigma_{\alpha_2} & \longmapsto & (2, 3) \\ \sigma_{\alpha_3} & \longmapsto & (3, 4) \end{array}$$

If we consider $\sigma \in \mathcal{W}$, we call $\widetilde{\sigma}$ the induced map on $\Phi^+_{A_n}$:

$$\begin{array}{rccc} \sigma_{\alpha_1}: & \Phi_{A_3} & \longrightarrow & \Phi_{A_3} \\ & \alpha_1 & \longmapsto & -\alpha_1 \\ & \alpha_2 & \longmapsto & \alpha_1+\alpha_2 \\ & \alpha_3 & \longmapsto & \alpha_3 \end{array} \qquad \begin{array}{rccc} \widetilde{\sigma}_{\alpha_1}: & \Phi^+_{A_3} & \longrightarrow & \Phi^+_{A_3} \\ & \alpha_1 & \longmapsto & \alpha_1 \\ & \alpha_2 & \longmapsto & \alpha_1+\alpha_2 \\ & \alpha_3 & \longmapsto & \alpha_3 \end{array}$$

For example, if $f = \sigma_{\alpha_1} \circ \sigma_{\alpha_3}$ then:

$$\begin{array}{rccc} f: & \Phi_{A_3} & \longrightarrow & \Phi_{A_3} \\ & \alpha_1 & \longmapsto & -\alpha_1 \\ & \alpha_2 & \longmapsto & \alpha_1+\alpha_2+\alpha_3 \\ & \alpha_3 & \longmapsto & -\alpha_3 \end{array} \qquad \begin{array}{rccc} \widetilde{f}: & \Phi^+_{A_3} & \longrightarrow & \Phi^+_{A_3} \\ & \alpha_1 & \longmapsto & \alpha_1 \\ & \alpha_2 & \longmapsto & \alpha_1+\alpha_2+\alpha_3 \\ & \alpha_3 & \longmapsto & \alpha_3 \end{array}$$

1.2 Group representations

In this Section we recall the fundamental notions of the representations theory of finite groups. As an example, we focus on representations of the symmetric group, which is the most relevant for our purposes.

1.2.1 Basic definitions

Let V be a $\mathbb{C}$-vector space and let $\mathrm{GL}(V)$ be the group of isomorphisms of V onto itself. An element $a \in \mathrm{GL}(V)$ is an invertible linear transformation of V. We will denote its inverse by a^{-1}.
When V has a finite basis $\{e_1, \ldots, e_n\}$, each linear map:

$$a: \quad V \longrightarrow V$$

can be defined by a square matrix (a_{ij}) of order n. The coefficients a_{ij} are complex numbers; they are obtained by expressing the images $a(e_j)$ as linear combinations of the elements of the basis $\{e_1, \ldots, e_n\}$:

$$a(e_j) = \sum_{i=1}^{n} a_{ij} e_i$$

a is an isomorphism if and only if $\det(a) = \det(a_{ij}) \neq 0$.
In this way the group $GL(V)$ can be identified with the group of invertible square matrices of order n.
Suppose now G is a finite group, with identity element 1 and with composition:

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (s,t) & \longmapsto & st \end{array}$$

Definition 1.5. A *linear representation* of G in V is a group homomorphism ρ from the group G into the group $\mathrm{GL}(V)$:

$$\begin{array}{cccc} \rho: & G & \longrightarrow & \mathrm{GL}(V) \\ & s & \longmapsto & \rho_s \end{array}$$

If V has finite dimension n, we say also that n is the *degree* of the representation.

In other words, we associate each element $s \in G$ with an element ρ_s of $\mathrm{GL}(V)$ in such way that we have the following equality:

$$\rho_{st} = \rho(st) = \rho(s)\rho(t) = \rho_s\rho_t$$

Definition 1.6. Let ρ and ρ' be two representations of the same group G in two vector spaces V and V'. These representations are said to be isomorphic if there exists a linear isomorphism

$$\tau : \ V \ \longrightarrow \ V'$$

such that

$$\tau \circ \rho(s) = \rho'(s) \circ \tau \qquad \text{for all } s \in G$$

This is equivalent to say that the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\tau} & V' \\ {\scriptstyle \rho_s}\downarrow & & \downarrow{\scriptstyle \rho'_s} \\ V & \xrightarrow{\tau} & V' \end{array}$$

commutes for all $s \in G$.
When ρ and ρ' are given in matrix form by R_s and R'_s respectively, this means that there exists an invertible matrix T such that:

$$T\ R_s = R'_s\ T \qquad \text{for all } s \in G$$

Example 2. a) Let V be a vector space of dimension 1, then

$$\mathrm{GL}(V) \simeq \mathbb{C}^*$$

where $\mathbb{C}^*$ denotes the multiplicative group of nonzero complex numbers. A representation of degree 1 of a group G is a homomorphism

$$\rho : \ G \ \longrightarrow \ \mathbb{C}^*$$

Since each element of G has finite order, the values $\rho(s)$ of ρ are roots of unity; in particular

$$|\rho(s)| = |\rho_s| = 1$$

If we take $\rho(s) = 1$ for all $s \in G$, we obtain a representation of G which is called the *unit* (or *trivial*) representation.

b) Let g be the order of G, and let V be a vector space of dimension g, with a basis $(e_t)_{t \in G}$ indexed by the elements t of G. For $s \in G$, let ρ_s be the linear map of V into V such that:

$$\begin{aligned} \rho_s : \ V &\longrightarrow V \\ e_t &\longmapsto e_{st} \end{aligned}$$

This defines a linear representation of G, which is called the *regular representation* of G. Its degree is equal to the order of G. Note that:

$$e_s = \rho_s(e_1)$$

hence $\rho_s(e_1), s \in G$ form a basis of V.
Conversely, let W be a representation of G containing a vector w such that the elements $\rho_s(w)$ with $s \in G$, form a basis of W; then W is isomorphic to the regular representation under the following map:

$$\begin{aligned} \phi : \ V_{reg} &\longrightarrow W \\ e_s &\longmapsto \rho_s(w) \end{aligned}$$

We recall the definition of $\mathbb{K}$-algebra:

Definition 1.7. Let $\mathbb{K}$ be a field. A $\mathbb{K}$-algebra is a $\mathbb{K}$-vector space A equipped with an additional bilinear binary operation:

$$\begin{aligned} * : \ A \times A &\longrightarrow A \\ (x, y) &\longmapsto x * y \end{aligned}$$

A $\mathbb{K}$-algebra is said to be *associative* if the product $*$ is associative. An algebra is said to be *unitary* if it has an identity element with respect to the multiplication $*$.

Definition 1.8. Let $(A, *)$ and $(A', \star)$ be two $\mathbb{K}$-algebras. A $\mathbb{K}$-*algebra homomorphism* is a linear map

$$\phi : \ A \longrightarrow A'$$

such that

$$\phi(a * b) = \phi(a) \star \phi(b) \qquad \forall a, b \in A$$

Definition 1.9. Let A be an associative and unitary $\mathbb{K}$-algebra and let V be a vector space. A *representation* of A on V is a $\mathbb{K}$-algebra homomorphism

$$\widetilde{\rho}: \quad A \longrightarrow \mathrm{End}(V)$$

where $\mathrm{End}(V)$ is the associative algebra of endomorphism of V.

Definition 1.10. Let Λ be a ring with unity 1_Λ, an abelian group M is said to be a left Λ-module if there exists a map:

$$f: \quad \Lambda \times M \longrightarrow M$$

such that:

- $f(1_\Lambda, m) = m \qquad \forall m \in M$
- $f(\lambda_1 \lambda_2, m) = f(\lambda_1, f(\lambda_2, m)) \qquad \forall m \in M \quad \forall \lambda_1, \lambda_2 \in \Lambda$
- $f(\lambda_1 + \lambda_2, m) = f(\lambda_1, m) + f(\lambda_2, m) \qquad \forall m \in M \quad \forall \lambda_1, \lambda_2 \in \Lambda$
- $f(\lambda, m_1 + m_2) = f(\lambda, m_1) + f(\lambda, m_2) \qquad \forall m_1, m_2 \in M \quad \forall \lambda \in \Lambda$

We will often write $\lambda \,.\, m$ instead of $f(\lambda, m)$.

Let ϕ be a representation of an associative and unitary algebra A

$$\begin{aligned} \phi: \quad A &\longrightarrow \mathrm{End}(V) \\ a &\longmapsto \phi_a \end{aligned}$$

the vector space V can be seen as a left A-module through the action

$$a \,.\, v = \phi(a)(v) = \phi_a(v)$$

Definition 1.11. The *group algebra* $\mathbb{K}[G]$, where $\mathbb{K}$ is a field and G a group with operation $*$, is the set of all linear combinations of finitely many elements of G with coefficients in $\mathbb{K}$, hence all elements of the form:

$$a_1 \, g_1 + a_2 \, g_2 + \cdots + a_n \, g_n \qquad a_i \in \mathbb{K}, \quad g_i \in G \quad \forall i = 1, \ldots, n$$

This element can be denoted in general by

$$\sum_{g \in G} a_g \, g$$

where it assumed that $a_g = 0$ for all but finitely many element of G.

The group algebra $\mathbb{K}[G]$ is a $\mathbb{K}$-algebra with respect to the addition defined by the following rule:

$$\sum_{g\in G} a_g\, g + \sum_{g\in G} b_g\, g = \sum_{g\in G} (a_g + b_g)\, g;$$

the product by scalar is given by

$$\lambda \sum_{g\in G} a_g\, g = \sum_{g\in G} (\lambda a_g)\, g$$

and the multiplication is the following

$$\left(\sum_{g\in G} a_g\, g\right)\left(\sum_{h\in G} b_h\, h\right) = \sum_{g,h\in G} (a_g b_h)\; g * h.$$

From this definition, it follows that the identity element of G is the unity of $\mathbb{K}[G]$.

Remark. Every linear representation of a group G on V

$$\begin{array}{rccc} \rho: & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \longmapsto & \rho_g \end{array}$$

defines an algebra representation of $\mathbb{K}[G]$ on V in the following way:

$$\begin{array}{rccc} \widetilde{\rho}: & \mathbb{K}[G] & \longrightarrow & \mathrm{End}(V) \\ & \sum_{g\in G} \lambda_g\, g & \longmapsto & \sum_{g\in G} \lambda_g\, \rho_g \end{array}$$

Conversely, every algebra representation of $\mathbb{K}[G]$ on V defines a representation of G on V by considering the restriction of $\widetilde{\rho}$ to the elements of G.

1.2.2 Irreducible representations

Let $\rho: G \longrightarrow \mathrm{GL}(V)$ be a linear representation and let W be a vector subspace of V. Suppose that W is *stable* under the action of G, in other words suppose that for all $x \in W$, $\rho_s(x) \in W$ for all $s \in G$.
The restriction $\rho_s^{|W}$ of ρ_s is then an isomorphism of W onto itself, and we have:

$$\rho_{st}^{|W} = \rho_s^{|W}\; \rho_t^{|W}$$

Thus

$$\begin{array}{rccc} \rho^W : & G & \longrightarrow & \mathrm{GL}(W) \\ & s & \longmapsto & \rho_s^{|W} \end{array}$$

is a linear representation of G in W; ρ^W is said to be a *subrepresentation* of V.

Theorem 1.12. *Let* $\rho : G \longrightarrow \mathrm{GL}(V)$ *be a linear representation of* G *in* V *and let* W *be a vector subspace of* V *stable under* G*. Then there exists a complement* W^0 *of* W *which is stable under* G*. In other words, there exists a subspace* W^0 *such that:*

i) $V = W \oplus W^0$

ii) W^0 *is stable under the action of* G

Proof. See [19], Theorem 1; pag 6. □

Definition 1.13. Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a linear representation of G in V. We say that it is *irreducible* if V is not 0 and if no vector subspace of V is stable under G, except 0 and V.

By induction, Theorem 1.12 yields immediately the following:

Theorem 1.14. *Every representation is a direct sum of irreducible representations.*

Let V_1 and V_2 be two vector spaces. Let

$$\begin{array}{rccc} \rho^1 : & G & \longrightarrow & \mathrm{GL}(V_1) \\ & s & \longmapsto & \rho_s^1 \end{array} \qquad \begin{array}{rccc} \rho^2 : & G & \longrightarrow & \mathrm{GL}(V_2) \\ & s & \longmapsto & \rho_s^2 \end{array}$$

be two linear representations of a group G. For $s \in G$, define an element $\rho_s \in \mathrm{GL}(V_1 \otimes V_2)$ by the condition:

$$\rho_s(x_1 \otimes x_2) = \rho_s^1(x_1) \otimes \rho_s^2(x_2)$$

The existence and uniqueness of ρ_s follows from the definition of tensor product. We write:

$$\rho_s = \rho_s^1 \otimes \rho_s^2$$

We have thus defined a linear representation of G in $V_1 \otimes V_2$:

$$\begin{array}{rccc} \rho: & G & \longrightarrow & \mathrm{GL}(V_1 \otimes V_2) \\ & s & \longmapsto & \rho_s = \rho_s^1 \otimes \rho_s^2 \end{array}$$

which is called the *tensor product* of ρ^1 and ρ^2.
The tensor product of two irreducible representations is not in general irreducible.

1.2.3 The character of a representation

Let V be a vector space having a basis $\{e_1, \ldots, e_n\}$, and let a be a linear map of V into itself, with associated matrix (a_{ij}). We recall that the trace of a is the scalar

$$\mathrm{Tr}(a) = \sum_{i=1}^{n} a_{ii}$$

It is the sum of the eigenvalues of a counted with their multiplicities, thus it does not depend on the choice of the basis.

Definition 1.15. Let $\rho: \; G \longrightarrow \mathrm{GL}(V)$ be a linear representation of a finite group G in the vector space V. For each $s \in G$, we set:

$$\chi_\rho(s) = \mathrm{Tr}(\rho_s)$$

The complex valued function

$$\begin{array}{rccc} \chi_\rho: & G & \longrightarrow & \mathbb{C} \\ & s & \longmapsto & \mathrm{Tr}(\rho_s) \end{array}$$

is called the *character* of the representation ρ.

The following properties are straightforward:

Proposition 1.16. *If χ is the character of a representation ρ of degree n, we have:*

i) $\chi(1) = n$

ii) $\chi(s^{-1}) = \overline{\chi(s)}$ *for* $s \in G$

iii) $\chi(tst^{-1}) = \chi(s)$ *for* $s, t \in G$

Remark. A function

$$f: \; G \longrightarrow \mathbb{C}$$

satisfying identity *iii*) is called a *class function.*

Proposition 1.17. *Let*

$$\rho^1: \; G \longrightarrow \mathrm{GL}(V_1) \qquad \rho^2: \; G \longrightarrow \mathrm{GL}(V_2)$$

be two linear representations of a group G*, and let* χ_1 *and* χ_2 *be their characters. Then:*

i) The character χ_ρ *of the direct sum representation*

$$\rho: \; G \longrightarrow \mathrm{GL}(V_1 \oplus V_2)$$

is equal to $\chi_1 + \chi_2$*.*

ii) The character ψ_ρ *of the tensor product representation*

$$\rho: \; G \longrightarrow \mathrm{GL}(V_1 \otimes V_2)$$

is equal to $\chi_1 \cdot \chi_2$*.*

Proof. See [19], Proposition 2; pag 11. □

Proposition 1.18 (Schur's Lemma)**.** *Let*

$$\rho^1: \; G \longrightarrow \mathrm{GL}(V_1) \qquad \rho^2: \; G \longrightarrow \mathrm{GL}(V_2)$$

be two irreducible representations of G*, and let* f *be a linear mapping of* V_1 *into* V_2 *such that*

$$\rho^2_s \circ f = f \circ \rho^1_s \qquad \text{for all } s \in G$$

Then:

(1) *If* ρ^1 *and* ρ^2 *are not isomorphic, we have* $f = 0$*.*

(2) *If* $V_1 = V_2$ *and* $\rho^1 = \rho^2$*,* f *is a homothety i.e., a scalar multiple of the identity.*

Proof. See [19], Proposition 4; pag 13. □

Let

$$\phi : \ G \longrightarrow \mathbb{C} \qquad \psi : \ G \longrightarrow \mathbb{C}$$

two functions on G, set

$$\langle \phi, \psi \rangle = \frac{1}{g} \sum_{t \in G} \phi(t^{-1})\psi(t) = \frac{1}{g} \sum_{t \in G} \phi(t)\psi(t^{-1})$$

where $g = |G|$. We have that:

- $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$
- $\langle \phi, \psi \rangle$ is linear in ϕ and in ψ.

Let

$$\phi : \ G \longrightarrow \mathbb{C} \qquad \psi : \ G \longrightarrow \mathbb{C}$$

two complex-valued functions on G, and set

$$(\phi, \psi) = \frac{1}{g} \sum_{t \in G} \phi(t)\overline{\psi(t)}$$

where $g = |G|$. Then:

- (ϕ, ψ) is linear in ϕ, semilinear in ψ
- $(\phi, \phi) > 0$ for all $\phi \neq 0$

Thus, (ϕ, ψ) is a hermitian product.

If $\widetilde{\psi}$ is the function defined by the formula $\widetilde{\psi}(t) = \overline{\psi(t^{-1})}$, we have:

$$(\phi, \psi) = \frac{1}{g} \sum_{t \in G} \phi(t)\overline{\psi(t)} = \frac{1}{g} \sum_{t \in G} \phi(t)\widetilde{\psi}(t^{-1}) = \langle \phi, \widetilde{\psi} \rangle$$

In particular if χ is the character of a representation of G, we have, by Proposition 1.16, that $\widetilde{\chi} = \chi$ hence

$$(\phi, \chi) = \langle \phi, \chi \rangle \qquad \text{for all functions } \phi \text{ on } G$$

So we can use at will (ϕ, χ) or $\langle \phi, \chi \rangle$, so long as we are concerned with characters.

Theorem 1.19. *i) If χ is the character of an irreducible representation, we have:*

$$(\chi, \chi) = 1$$

ii) If χ and χ' are the characters of two non isomorphic irreducible representation, we have:

$$(\chi, \chi') = 0$$

Proof. See 19, Theorem 3; pag 15. □

This result has important consequences, for example:

Theorem 1.20. *Let V be a linear representation of G, with character ϕ, and suppose that V decomposes into a direct sum of irreducible representation:*

$$V = W_1 \oplus \cdots \oplus W_k$$

Then, if W is an irreducible representation with character χ, the number of W_i isomorphic to W is equal to the scalar product $(\phi, \chi) = \langle \phi, \chi \rangle$.

Corollary 1.21. *The number of W_i isomorphic to W does not depend on the chosen decomposition. (This number is called the number of times that W occurs in V.)*

Proof. Indeed, (ϕ, χ) does not depend on the decomposition. □

Corollary 1.22. *Two representations with the same character are isomorphic.*

Proof. Indeed, Corollary 1.21 and Theorem 1.20 show that they contain each given irreducible representation the same number of times. □

The above results reduce the study of representations to that of their characters. If $\chi_1, \dots, \chi_h$ are the distinct irreducible characters of G, and if $W_1, \dots, W_h$ denote the corresponding representations, each representation V of G is isomorphic to a direct sum

$$V = m_1 W_1 \oplus \cdots \oplus m_h W_h \qquad m_i \text{ integers } \geqslant 0.$$

The character ϕ of V is equal to $\phi = m_1\chi_1 + \cdots + m_h\chi_h$ and we have

$$m_i = (\phi, \chi_i) \qquad \forall i = 1, \dots, h.$$

The orthogonality relations among the χ_i imply in addition:

$$(\phi, \phi) = \sum_{i=1}^{h} m_i^2.$$

Theorem 1.23. *If ϕ is the character of a representation V, then (ϕ, ϕ) is a positive integer and we have $(\phi, \phi) = 1$ if and only if V is irreducible.*

Proof. Since $(\phi, \phi) = \sum_{i=1}^{h} m_i^2$ is a positive integer, (ϕ, ϕ) is equal to 1 if and only if one of the m_i's is equal to 1 and the others to 0, that is, if and only if V is isomorphic to one of the W_i. □

1.2.4 Decomposition of the regular representation

For the rest of Subsection 1.2.4, the irreducible characters of G are denoted by $\chi_1, \dots, \chi_h$; their degrees are written $n_1, \dots, n_h$, we have $n_i = \chi_i(1)$.

Let V_{reg} be the regular representation of G, i.e. $V_{reg} = \langle e_g, g \in G \rangle$,

$$\rho : \begin{array}{ccc} G & \longrightarrow & \mathrm{GL}(V_{reg}) \\ s & \longmapsto & \rho_s \end{array} \qquad \rho_s : \begin{array}{ccc} V_{reg} & \longrightarrow & V_{reg} \\ e_t & \longmapsto & e_{st} \end{array}$$

If $s = 1_G$ we have:

$$\chi_{reg}(1_G) = \dim(V_{reg}) = |G| = g.$$

On the other hand, if $s \neq 1_G$, we have $st \neq t$ for all t, which implies that the diagonal terms of the matrix associated with ρ_s are zero. In particular we have $\mathrm{Tr}(\rho_s) = 0$:

$$\chi_{reg}(s) = 0$$

We can summarize the above results in the following Proposition:

Proposition 1.24. *The character χ_{reg} of the regular representation is given by the formulas:*

$$\chi_{reg}(1_G) = g$$

$$\chi_{reg}(s) = 0 \qquad \forall s \neq 1_G$$

Corollary 1.25. *Every irreducible representation W_i of G is contained in the regular representation with multiplicity equal to its degree n_i:*

$$V_{reg} = n_1 W_1 \oplus \cdots \oplus n_h W_h$$

Proof. According to Theorem 1.20, the number of times each representation W_i of G is contained in the regular representation is equal to (χ_{reg}, χ_i) and we have:

$$(\chi_{reg}, \chi_i) = \langle \chi_{reg}, \chi_i \rangle = \frac{1}{g} \sum_{t \in G} \chi_{reg}(t^{-1}) \chi_i(t) = \frac{1}{g}\, g\, \chi_i(1_G) = \chi_i(1_G) = n_i$$

□

Corollary 1.26. *a) The degrees n_i satisfy the relation*

$$\sum_{i=1}^{h} n_i^2 = g$$

b) If $s \in G$ and $s \neq 1_G$, we have:

$$\sum_{i=1}^{h} n_i\, \chi_i(s) = 0$$

Proof. By Corollary 1.25 we have:

$$\chi_{reg}(s) = \sum_{i=1}^{h} n_i \, \chi_i(s) \qquad \text{for all } s \in G$$

Taking $s = 1_G$ we obtain *a)*, and taking $s \neq 1_G$, we obtain *b)*. □

Remark. The above result can be used in determining the irreducible representations of a group G: suppose we have constructed some mutually non isomorphic irreducible representations of degrees $n_1, \ldots, n_k$; in order to check whether they are *all* the irreducible representations of G (up to isomorphism), it is necessary and sufficient to check whether $n_1^2 + \cdots + n_k^2 = g$.

1.2.5 Number of irreducible representations

Recall that a function f on G is called a *class function* if $f(tst^{-1}) = f(s)$ for all $s, t \in G$.

We introduce now the space H of class functions on G; the irreducible characters $\chi_1, \ldots, \chi_h$ belong to H.

Theorem 1.27. *The characters $\chi_1, \ldots, \chi_h$ form an orthonormal basis of H.*

Proof. See [19], Theorem 6; pag 19. □

Remark. Recall that two elements t and t' of G are said to be conjugate if there exists $s \in G$ such that $t' = sts^{-1}$:

$$t \sim t' \quad \Longleftrightarrow \quad \exists s \in G \mid t' = sts^{-1}.$$

This is an equivalence relation which partitions G into classes, called *conjugacy classes*. Here are some properties:

- The identity element of G is always the only element of its class:

$$[1_G] = \{1_G\}$$

- If G is abelian, then $gag^{-1} = a$ for all a and g in G, hence:

$$[g] = \{g\} \qquad \forall g \in G$$

- If two elements a, b belong to the same conjugacy class, then they have the same order.

Theorem 1.28. *The number of irreducible representations of G (up to isomorphism) is equal to the number of conjugacy classes of G.*

Proof. Let $C_1, \ldots, C_k$ be the distinct classes of G. To say that a function f on G is a class function is equivalent to saying that it is constant on each of $C_1, \ldots, C_k$; it is thus determined by its values λ_i on the C_i, and these can be chosen arbitrarily. Consequently, the dimension of the space H of class function is equal to k. On the other hand, this dimension is, by Theorem 1.27, equal to the number of irreducible representations of G (up to isomorphism). The result follows. □

Example 3. Take $G = \mathfrak{S}_4$. We have $|\mathfrak{S}_4| = 24$, and there are 5 conjugacy classes:

$$[id] \qquad [(12)] \qquad [(123)] \qquad [(1234)] \qquad [(12)(34)]$$

Thus, there are up to isomorphism 5 irreducible representations of $\mathfrak{S}_4$:

$$V_{reg} = n_1 V_1 \oplus n_2 V_2 \oplus n_3 V_3 \oplus n_4 V_4 \oplus n_5 V_5 \tag{1.1}$$

Let us describe these representations.
The following two representations of degree 1

$$\begin{array}{rccc} \rho^1: & G & \longrightarrow & \mathrm{GL}(V_1) \simeq \mathbb{C}^* \\ & \sigma & \longmapsto & 1 \end{array} \qquad \begin{array}{rccc} \rho^2: & G & \longrightarrow & \mathrm{GL}(V_2) \simeq \mathbb{C}^* \\ & \sigma & \longmapsto & \mathrm{sgn}(\sigma) \end{array}$$

are irreducible.

Let us consider $\mathbb{C}^4$ with its canonical basis $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$. Then we can define the following representation of $\mathfrak{S}_4$:

$$\begin{array}{rccc} \rho': & G & \longrightarrow & \mathrm{GL}(\mathbb{C}^4) \\ & \sigma & \longmapsto & \rho'_\sigma \end{array} \qquad \begin{array}{rccc} \rho'_\sigma: & \mathbb{C}^4 & \longrightarrow & \mathbb{C}^4 \\ & e_i & \longmapsto & e_{\sigma(i)} \end{array}$$

The representation above is not an irreducible representation of $\mathfrak{S}_4$. Indeed, if we consider the vector subspace V_3 of $\mathbb{C}^4$:

$$V_3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\},$$

then it is stable under the action of $\mathfrak{S}_4$. In fact for $(x_1, x_2, x_3, x_4) \in V_3$

$$\rho'_\sigma((x_1, x_2, x_3, x_4)) = \rho'_\sigma(x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4) =$$

$$= x_1 \rho'_\sigma(e_1) + x_2 \rho'_\sigma(e_2) + x_3 \rho'_\sigma(e_3) + x_4 \rho'_\sigma(e_4) = 0$$

So we consider the subrepresentation $\rho^3 = \rho'_{|V_3}$ of ρ':

$$\begin{array}{rccc} \rho^3 : & G & \longrightarrow & \mathrm{GL}(V_3) \\ & \sigma & \longmapsto & \rho^3_\sigma \end{array} \qquad \begin{array}{rccc} \rho^3_\sigma : & V_3 & \longrightarrow & V_3 \\ & v & \longmapsto & \rho'_\sigma(v) \end{array}$$

Let $\mathcal{B}_{V_3} = \{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$ be a basis of V_3, we now find the character of the representation ρ^3 by computing it on a representative of each conjugacy class:

$$\begin{array}{rccc} \rho^3_{(12)} : & V_3 & \longrightarrow & V_3 \\ & e_1 - e_2 & \longmapsto & e_2 - e_1 \\ & e_2 - e_3 & \longmapsto & e_1 - e_3 \\ & e_3 - e_4 & \longmapsto & e_3 - e_4 \end{array} \qquad \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \chi_3((12)) = 1$$

$$\begin{array}{rccc} \rho^3_{(123)} : & V_3 & \longrightarrow & V_3 \\ & e_1 - e_2 & \longmapsto & e_2 - e_3 \\ & e_2 - e_3 & \longmapsto & e_3 - e_1 \\ & e_3 - e_4 & \longmapsto & e_1 - e_4 \end{array} \qquad \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \chi_3((123)) = 0$$

$$\begin{array}{rccc} \rho^3_{(1234)} : & V_3 & \longrightarrow & V_3 \\ & e_1 - e_2 & \longmapsto & e_2 - e_3 \\ & e_2 - e_3 & \longmapsto & e_3 - e_4 \\ & e_3 - e_4 & \longmapsto & e_4 - e_1 \end{array} \qquad \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \qquad \chi_3((1234)) = -1$$

$$\begin{array}{cccc} \rho^3_{(12)(34)}: & V_3 & \longrightarrow & V_3 \\ & e_1 - e_2 & \longmapsto & e_2 - e_1 \\ & e_2 - e_3 & \longmapsto & e_1 - e_4 \\ & e_3 - e_4 & \longmapsto & e_4 - e_3 \end{array} \qquad \begin{pmatrix} \text{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & \text{-1} \end{pmatrix} \qquad \chi_3((12)(34)) = -1$$

We observe that:

$$(\chi_3, \chi_3) = \frac{1}{24}(9 \cdot 1 + 1 \cdot 6 + 0 \cdot 8 + 1 \cdot 6 + 1 \cdot 3) = 1$$

This shows that ρ^3 is irreducible.
Thus the equation 1.1 becomes:

$$V_{reg} = V_1 \oplus V_2 \oplus 3V_3 \oplus n_4 V_4 \oplus n_5 V_5$$

Therefore the dimensions of the last two irreducible representations of $\mathfrak{S}_4$ satisfy the equation:

$$1^2 + 1^2 + 3^2 + n_4^2 + n_5^2 = 24$$

One can easily see that one of the two irreducible representations must have dimension 3 and the other must have dimension 2.
Consider the tensor product $V_4 = V_3 \otimes V_2$, we have:

$$\mathcal{B}_{V_4} = \{(e_1 - e_2) \otimes v_2, (e_2 - e_3) \otimes v_2, (e_3 - e_4) \otimes v_2\}$$

where we supposed $V_2 = \langle v_2 \rangle$. Consider the following representation:

$$\rho^4 : \quad G \quad \longrightarrow \quad \mathrm{GL}(V_3 \otimes V_2).$$

By Proposition 1.17 we have $\chi_4 = \chi_3 \cdot \chi_2$ and so:

$\chi_4(\mathrm{id}) = 3 \quad \chi_4((12)) = \text{-1} \quad \chi_4((123)) = 0 \quad \chi_4((1234)) = 1 \quad \chi_4((12)(34)) = \text{-1}$

We see that the character of ρ^4 is different from the character of any other irreducible representation we have already constructed. So ρ^4 is not isomorphic to any of the previous ones. We observe that:

$$(\chi_4, \chi_4) = \frac{1}{24}(9 \cdot 1 + 1 \cdot 6 + 0 \cdot 8 + 1 \cdot 6 + 1 \cdot 3) = 1$$

which shows that ρ^4 is irreducible.
The last irreducible character χ_5 can be derived from the following table taking into consideration the character of the regular representation:

Table 1.1: Character table of $\mathfrak{S}_4$

	id	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	1	-1
χ_5	2	0	-1	0	2
χ_{reg}	24	0	0	0	0

1.2.6 Abelian subgroups and the cyclic group C_n

Let G be a group. G is abelian if $st = ts$ for all $s, t \in G$. This amounts to saying that each conjugacy class of G consists of a single element and that each function on G is a class function. The linear representations of such a group are particularly simple:

Theorem 1.29. *The following properties are equivalent:*

i) G is abelian.

ii) All the irreducible representations of G have degree 1.

Proof. See [19], Theorem 9; pag 25. □

Corollary 1.30. *Let A be an abelian subgroup of G, let a be its order and let g be that of G. Each irreducible representation of G has degree $\leqslant g/a$.*

Proof. See [19], Corollary; pag 25. □

We now consider C_n the cyclic group of order n consisting of the powers

$$1, r, \dots, r^{n-1}$$

of an element r such that $r^n = 1$. It can be realized as the group of rotations through angles $2k\pi/n$ around an axis. It is an abelian group.
According to Theorem 1.29, the irreducible representations of C_n are of degree 1. Such a representation associates with r a complex number w:

$$\begin{array}{rccc} \rho: & C_n & \longrightarrow & \mathbb{C}^* \\ & r & \longmapsto & w \end{array}$$

This representation associates with r^k the number w^k; since $r^n = 1$, we have $w^n = 1$, that is:

$$w = e^{2\pi i h/n} \qquad \text{with } h = 0, 1, \ldots, n-1$$

We thus obtained n *irreducible representations of degree 1* whose characters $\chi_0, \chi_1, \ldots, \chi_{n-1}$ are given by:

$$\chi_h(r^k) = e^{2\pi i hk/n}$$

For n=3, for example, the character table is the following:

	id	r	r^2
χ_0	1	1	1
χ_1	1	w	w^2
χ_2	1	w^2	w

where

$$w = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

1.2.7 Induced representations

We begin this subsection by studying the tensor product of two modules over a ring:

Definition 1.31. Given a ring R, a right R-module M, a left R-module N and an abelian group G, a map

$$\begin{array}{rccc} \phi: & M \times N & \longrightarrow & G \\ & (m, n) & \longmapsto & \phi(m, n) \end{array}$$

is said to be *R-balanced* if for all $m, m' \in M$ and $n, n' \in N$ and $r \in R$ we have:

- $\phi(m, n + n') = \phi(m, n) + \phi(m, n')$
- $\phi(m + m', n) = \phi(m, n) + \phi(m', n)$
- $\phi(m \cdot r, n) = \phi(m, r \cdot n)$

If R is abelian then the left R-module coincide with the right R-module.

Definition 1.32. Given a ring R, a right R-module M and a left R-module N, the *tensor product of two R-modules* $M \otimes_R N$ is an abelian group together with a R-balanced product

$$\otimes : \quad M \times N \quad \longrightarrow \quad M \otimes_R N$$

which is *universal* in the following sense: for any abelian group G and for any R-balanced product $f : \; M \times N \; \longrightarrow \; G$ there is only one group homomorphism $\widetilde{f} : \; M \otimes_R N \; \longrightarrow \; G$ such that:

$$\widetilde{f} \circ \otimes = f$$

If R is abelian, then $M \otimes_R N$ can be equipped with this map

$$\begin{array}{ccc} R \times \; M \otimes_R N & \longrightarrow & M \otimes_R N \\ r \cdot (x \otimes y) & \longmapsto & (r \cdot x) \otimes y = x \otimes (r \cdot y) \end{array}$$

With this structure $M \otimes_R N$ becomes an R-module.

Let R, S be rings. Suppose that the ring R is a subring of the ring S. If N is a left S-module, then N can also be naturally considered as a left R-module since the elements of R (being elements of S) act on N by assumption. More generally, if $f : \; R \; \longrightarrow \; S$ is a ring homomorphism from R to S with $f(1_R) = f(1_S)$ (for example the injection map if R is a subring of S as above) then it is easy to see that N can be considered as an R-module with

$$r \cdot n = f(r) \cdot n \qquad \forall r \in R \quad \text{and} \quad \forall n \in N.$$

In this situation S can be considered as an extension of the ring R and the resulting R-module is said to be obtained from N by *restriction of scalars* from S to R.

Now we want to try to do the opposite: suppose that R is a subring of S, we start with an R-module N and attempt to define an S-module structure on N that extends the action of R on N to an action of S on N (hence "extending the scalars" from R to S). In general this is impossible: for example $\mathbb{Z}$ is a $\mathbb{Z}$-module but it cannot be made into a $\mathbb{Q}$-module, (if it could, then $\frac{1}{2} \cdot 1 = z \in \mathbb{Z}$ and z would be an element of $\mathbb{Z}$ with $z + z = 1$, which is impossible). Although $\mathbb{Z}$ itself cannot be made into a $\mathbb{Q}$-module it is *contained* in a $\mathbb{Q}$-module, namely $\mathbb{Q}$ itself.

We now construct for a general R-module N an S-module that is the best possible target in which one can try to embed N. Consider the tensor product of the two R-modules $S \otimes_R N$. The elements of $S \otimes_R N$ can be written (non-uniquely in general) as finite sums of elements of the form $s \otimes n$ with $s \in S$, $n \in N$.
The tensor product $S \otimes_R N$ is naturally a left S-module under the action defined by:

$$\begin{array}{ccc} S \times \ S \otimes_R N & \longrightarrow & S \otimes_R N \\ \left(s, \sum_{\text{finite}} \ s_i \otimes n_i\right) & \longmapsto & \sum_{\text{finite}} (s\, s_i) \otimes n_i \end{array}$$

The module $S \otimes_R N$ is called the S-module obtained by *extension of scalars* from the R-module N.

Example 4.
- $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n = \mathbb{Q}^n$
- Let A be an abelian finite group, then $\mathbb{Q} \otimes_{\mathbb{Z}} A$=0

We can now give the definition of induced representation.

Recall that, as seen in Subsection 1.2.1, if $\rho: G \longrightarrow \mathrm{GL}(V)$ is a representation of G, this is equivalent to saying that V is a $\mathbb{C}[G]$-module.

Definition 1.33. Let G be a finite group and let H be a subgroup of G. Let W be a (left) $\mathbb{C}[H]$-module. Furthermore $\mathbb{C}[G]$ is a (right) $\mathbb{C}[H]$-module. Let

$$W' = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

be the $\mathbb{C}[G]$-module obtained by scalar extension from $\mathbb{C}[H]$ to $\mathbb{C}[G]$. Then we call W' the *induced representation* of G induced from W. We denote the induced representation of G from W by:

$$\mathrm{Ind}_H^G(W)$$

The elements of W' will be of the form:

$$\sum_{\text{finite}} c \otimes w \qquad c \in \mathbb{C}[G], \quad w \in W$$

Let $R = \{s_1, \ldots, s_n\}$ be a system of left representatives for G/H, then for any $g \in G$ there exist $s_i \in R$ and $h \in H$ such that $g = s_i h$. Then we have:

$$g \otimes w = s_i h \otimes w = s_i \cdot h \otimes w = s_i \otimes h \cdot w = s_i \otimes w'.$$

This implies that each element of $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ can be written as:

$$\sum_{\substack{s_i \in R \\ w \in W}} s_i \otimes w$$

From this we also deduce that:

$$\dim(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W) = [G : H] \cdot \dim(W)$$

Indeed, W' is a $\mathbb{C}[G]$-module defined by

$$\begin{array}{ccc} \mathbb{C}[G] \times W' & \longrightarrow & W' \\ (g, c \otimes w) & \longmapsto & g\, c \otimes w \end{array}$$

Remark. i) We can note that from this definition of induced representation, the existence and the uniqueness of the induced representations are obvious.

ii) Induction is *transitive*: if G is a subgroup of a group K, we have:

$$\mathrm{Ind}_G^K(\mathrm{Ind}_H^G(W)) \simeq \mathrm{Ind}_H^K(W)$$

Proposition 1.34. *Let V be a $\mathbb{C}[G]$-module which is a direct sum of vector subspaces transitively permuted by G:*

$$V = \bigoplus_{i\in I} W_i \qquad I \subseteq \mathbb{N}$$

Let $i_0 \in I$, $W = W_{i_0}$ and let H be the stabilizer of W in G:

$$H = \{g \in G \mid g \cdot W = W\} = \{g \in G \mid \rho_g(W) = W\}.$$

Then:

i) H is a subgroup of G

ii) The $\mathbb{C}[G]$-module V is induced by the $\mathbb{C}[H]$-module W.

Theorem 1.35. *Let H be a subgroup of G. Let (W, θ) be a linear representation of H and let (V, ρ) be the induced representation on G from W. Let h be the order of H and let R be a system of representatives of G/H. For each $u \in G$, we have:*

$$\chi_\rho(u) = \sum_{\substack{r\in R \\ r^{-1}ur\in H}} \chi_\theta(r^{-1}ur) = \frac{1}{h} \sum_{\substack{s\in G \\ s^{-1}us\in H}} \chi_\theta(s^{-1}us)$$

Proof. See [19], Theorem 12; pag 30. □

Theorem 1.36. *Let χ be the character of the representation ρ of G induced by the representation θ of H whose character is χ_θ.*
Let x be an element of G and C_j its conjugacy class in G with h_j elements, and let $g = g_j h_j$ where g is the order of G. Let h be the order of H. Then:

$$\chi(x) = \frac{g_j}{h} \sum_{z\in C_j\cap H} \chi_\theta(z)$$

Proof. If G is a finite group, for every $a \in G$ the elements in the conjugacy class of a are in $1-1$ correspondence with the cosets of the centralizer:

$$C_G(a) = \{g \in G \mid ga = ag\}$$

This can be seen by observing that any two elements b, c belonging to the same coset of $C_G(a)$, i.e. there exists an element z in $C_G(a)$ such that $b = zc$, give rise to the same element when conjugating a:

$$b^{-1}ab = c^{-1}z^{-1}azc = c^{-1}ac$$

Thus the number of elements in the conjugacy class of a is the index $[G : C_G(a)]$. The cardinality of $|C_G(a)|$ and its cosets is $g/h_j = g_j$. We have seen that two elements that belong to the same coset of $C_G(a)$ give rise to the same element when conjugating a.
We define:

$$\chi_1(w) = \begin{cases} \chi_\theta(w) & w \in H \\ 0 & w \notin H. \end{cases}$$

From Theorem 1.35 we know that:

$$\chi(x) = \frac{1}{h} \sum_{\substack{y \in G \\ y^{-1}xy \in H}} \chi_\theta(y^{-1}xy) = \frac{1}{h} \sum_{y \in G} \chi_1(y^{-1}xy) \tag{1.2}$$

As y ranges over G, $y^{-1}xy$ ranges over C_j and give the same $z \in C_j$ exactly g_j times. From equation (1.2) we obtain:

$$\chi(x) = \frac{1}{h}\, g_j \sum_{z \in C_j} \chi_1(z) = \frac{1}{h}\, g_j \sum_{z \in C_j \cap H} \chi_\theta(z)$$

□

Definition 1.37. If f is a class function on H, we consider the function f' on G defined by the formula

$$f'(s) = \frac{1}{h} \sum_{\substack{t \in G \\ t^{-1}st \in H}} f(t^{-1}st), \qquad h = |H|.$$

We say that f' is *induced* by f and denote it by $\mathrm{Ind}_H^G(f)$.

Proposition 1.38. *i) The function* $\mathrm{Ind}_H^G(f)$ *is a class function on* G.

ii) If f *is the character of a representation* W *of* H, $\mathrm{Ind}_H^G(f)$ *is the character of the induced representation* $\mathrm{Ind}_H^G(W)$ *of* G.

Proof. Assertion *ii*) follows from Theorem 1.35.
Regarding assertion *i*) see [19], Proposition 20; pag 56. □

If ϕ is a function on G, we denote by $\mathrm{Res}(\phi)$ its *restriction* to the subgroup H.

Theorem 1.39 (Frobenius reciprocity)**.** *If* ψ *is a class function on* H *and* ϕ *a class function on* G, *we have*

$$\langle\ \psi, \mathrm{Res}(\phi)\ \rangle_H = \langle\ \mathrm{Ind}_H^G(\psi), \phi\ \rangle_G$$

Proof. See [19], Theorem 13; pag 56. □

Corollary 1.40. *Let* H *be a subgroup of* G. *Let also* (θ, W) *be an irreducible representation of* H *and let* (ρ, V) *be an irreducible representation of* G. *Then the number of times that* W *occurs in* $\mathrm{Res}(V)$ *is equal to the number of times that* V *occurs in* $\mathrm{Ind}_H^G(W)$.

Proof. It follows directly from Theorem 1.39. □

Example 5. We compute the induced representations from C_4 to $\mathfrak{S}_4$.

$$C_4 = \{\mathrm{id}, (1234), (13)(24), (1432)\} \simeq \mathbb{Z}_4 \qquad [\mathfrak{S}_4 : C_4] = 6$$

The irreducible representation of C_4 are:

$$\begin{array}{rccc} \rho_1 : & C_4 & \longrightarrow & \mathrm{GL}(\widetilde{V_1}) \simeq \mathbb{C}^* \\ & (1234) & \longmapsto & 1 \end{array} \qquad \begin{array}{rccc} \rho_2 : & C_4 & \longrightarrow & \mathrm{GL}(\widetilde{V_2}) \simeq \mathbb{C}^* \\ & (1234) & \longmapsto & -1 \end{array}$$

$$\begin{array}{rccc} \rho_3 : & C_4 & \longrightarrow & \mathrm{GL}(\widetilde{V_3}) \simeq \mathbb{C}^* \\ & (1234) & \longmapsto & i \end{array} \qquad \begin{array}{rccc} \rho_4 : & C_4 & \longrightarrow & \mathrm{GL}(\widetilde{V_4}) \simeq \mathbb{C}^* \\ & (1234) & \longmapsto & -i \end{array}$$

And this is the character table of C_4:

Table 1.2: Character table of C_4

	id	(1234)	(13)(24)	(1432)
$\chi_{\widetilde{V_1}}$	1	1	1	1
$\chi_{\widetilde{V_2}}$	1	-1	1	-1
$\chi_{\widetilde{V_3}}$	1	i	-1	-i
$\chi_{\widetilde{V_4}}$	1	-i	-1	i

The representations induced from C_4 to $\mathfrak{S}_4$ are:

$$W_1 = \mathrm{Ind}_{C_4}^{\mathfrak{S}_4}(\widetilde{V_1}) \qquad W_2 = \mathrm{Ind}_{C_4}^{\mathfrak{S}_4}(\widetilde{V_2}) \qquad W_3 = \mathrm{Ind}_{C_4}^{\mathfrak{S}_4}(\widetilde{V_3}) \qquad W_4 = \mathrm{Ind}_{C_4}^{\mathfrak{S}_4}(\widetilde{V_4})$$

W_1, W_2, W_3, W_4 are representation of $\mathfrak{S}_4$ of degree 6. We want to apply Theorem 1.39 to get the irreducible decomposition of these modules.
In the example in Subsection 1.2.5 we computed the irreducible representations of $\mathfrak{S}_4$ and from Table 1.1 we get the table of their restriction to the subgroup C_4:

Table 1.3: Table of restrictions from $\mathfrak{S}_4$ to C_4

	id	(1234)	(13)(24)	(1432)
$\mathrm{Res}(\chi_{V_1})$	1	1	1	1
$\mathrm{Res}(\chi_{V_2})$	1	-1	1	-1
$\mathrm{Res}(\chi_{V_3})$	3	-1	-1	-1
$\mathrm{Res}(\chi_{V_4})$	3	1	-1	1
$\mathrm{Res}(\chi_{V_5})$	2	0	2	0

We can now apply Theorem 1.39:

$$(\chi_{W_1}, \chi_{V_1})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_1}}, \mathrm{Res}(\chi_{V_1}))_{C_4} = 1$$

$$(\chi_{W_1}, \chi_{V_2})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_1}}, \mathrm{Res}(\chi_{V_2}))_{C_4} = 0$$

$$(\chi_{W_1}, \chi_{V_3})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_1}}, \mathrm{Res}(\chi_{V_3}))_{C_4} = 0$$

$$(\chi_{W_1}, \chi_{V_4})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_1}}, \mathrm{Res}(\chi_{V_4}))_{C_4} = 1$$

$$(\chi_{W_1}, \chi_{V_5})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_1}}, \mathrm{Res}(\chi_{V_5}))_{C_4} = 1$$

We have calculated how many times the V_i occur in W_1, so we get:

$$W_1 \simeq V_1 \oplus V_4 \oplus V_5 \qquad \chi_{W_1} = \chi_{V_1} + \chi_{V_4} + \chi_{V_5}$$

We repeat the procedure for W_2, W_3, W_4:

$$(\chi_{W_2}, \chi_{V_1})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_2}}, \mathrm{Res}(\chi_{V_1}))_{C_4} = 0$$

$$(\chi_{W_2}, \chi_{V_2})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_2}}, \mathrm{Res}(\chi_{V_2}))_{C_4} = 1$$

$$(\chi_{W_2}, \chi_{V_3})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_2}}, \mathrm{Res}(\chi_{V_3}))_{C_4} = 1$$

$$(\chi_{W_2}, \chi_{V_4})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_2}}, \mathrm{Res}(\chi_{V_4}))_{C_4} = 0$$

$$(\chi_{W_2}, \chi_{V_5})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_2}}, \mathrm{Res}(\chi_{V_5}))_{C_4} = 1$$

And then:

$$W_2 \simeq V_2 \oplus V_3 \oplus V_5 \qquad \chi_{W_2} = \chi_{V_2} + \chi_{V_3} + \chi_{V_5}$$

$$(\chi_{W_3}, \chi_{V_1})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_3}}, \mathrm{Res}(\chi_{V_1}))_{C_4} = 0$$

$$(\chi_{W_3}, \chi_{V_2})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_3}}, \mathrm{Res}(\chi_{V_2}))_{C_4} = 0$$

$$(\chi_{W_3}, \chi_{V_3})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_3}}, \mathrm{Res}(\chi_{V_3}))_{C_4} = 1$$

$$(\chi_{W_3}, \chi_{V_4})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_3}}, \mathrm{Res}(\chi_{V_4}))_{C_4} = 1$$

$$(\chi_{W_3}, \chi_{V_5})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_3}}, \mathrm{Res}(\chi_{V_5}))_{C_4} = 0$$

Thus:

$$W_3 \simeq V_3 \oplus V_4 \qquad \chi_{W_3} = \chi_{V_3} + \chi_{V_4}$$

$$(\chi_{W_4}, \chi_{V_1})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_4}}, \mathrm{Res}(\chi_{V_1}))_{C_4} = 0$$

$$(\chi_{W_4}, \chi_{V_2})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_4}}, \mathrm{Res}(\chi_{V_2}))_{C_4} = 0$$

$$(\chi_{W_4}, \chi_{V_3})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_4}}, \mathrm{Res}(\chi_{V_3}))_{C_4} = 1$$

$$(\chi_{W_4}, \chi_{V_4})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_4}}, \mathrm{Res}(\chi_{V_4}))_{C_4} = 1$$

$$(\chi_{W_4}, \chi_{V_5})_{\mathfrak{S}_4} = (\chi_{\widetilde{V_4}}, \mathrm{Res}(\chi_{V_5}))_{C_4} = 0$$

Hence:

$$W_4 \simeq V_3 \oplus V_4 \qquad \chi_{W_4} = \chi_{V_3} + \chi_{V_4}$$

We note that $W_3 \simeq W_4$ and we report the character table of the induced representations from C_4 to $\mathfrak{S}_4$.

Table 1.4: Character table of the induced representations from C_4 to $\mathfrak{S}_4$

	id	(12)	(123)	(1234)	(12)(34)
χ_{W_1}	6	0	0	2	2
χ_{W_2}	6	0	0	-2	2
χ_{W_3}	6	0	0	0	-2
χ_{W_4}	6	0	0	0	-2

Chapter 2

Preliminaries in Topology and Combinatorics

2.1 Simplicial homology

In this Section we recall the fundamental definitions of simplicial homology. In the end of the Section we prove the Alexander duality.

2.1.1 Simplicial complexes

Definition 2.1. Given a set $\{a_0, \ldots, a_n\}$ of points of $\mathbb{R}^p$, this set is said to be *geometrically independent* if for any (real) scalars t_i, the equations

$$\sum_{i=0}^{n} t_i = 0 \qquad \text{and} \qquad \sum_{i=0}^{n} t_i a_i = \mathbf{0}$$

imply that $t_0 = t_1 = \cdots = t_n = 0$.

It is clear that a one point set is always geometrically independent. Elementary arguments show that in general $\{a_0, \ldots, a_n\}$ is geometrically independent if and only if the vectors

$$a_1 - a_0, \ldots, a_n - a_0$$

are linearly independent in the sense of ordinary linear algebra. Thus two distinct points in $\mathbb{R}^p$ form a geometrically independent set, as do three non-collinear points, four non-coplanar points, and so on.

Definition 2.2. Given a geometrically independent set of points $\{a_0, \ldots, a_n\}$, we define the *n-plane* P spanned by these points as the set of all points $x \in \mathbb{R}^p$ such that

$$x = \sum_{i=0}^{n} t_i a_i$$

for some scalars t_i with $\sum t_i = 1$. Since the a_i's are geometrically independent, the t_i's are uniquely determined by x. Note that each point a_i belongs to the plane P.

The plane P can also be described as the set of all points x such that

$$x = a_0 + \sum_{i=1}^{n} t_i(a_i - a_0)$$

for some scalars $t_1, \ldots, t_n$; in this form we speak of P as the plane through a_0 parallel to the vectors $a_i - a_0$.

Definition 2.3. Let $\{a_0, \ldots, a_n\}$ be a geometrically independent set in $\mathbb{R}^p$. We define the *n-simplex* σ spanned by $a_0, \ldots, a_n$ as the set of all points $x \in \mathbb{R}^p$ such that

$$x = \sum_{i=0}^{n} t_i a_i \qquad \text{where} \qquad \sum_{i=0}^{n} t_i = 1, \quad t_i \geqslant 0.$$

The numbers t_i's are uniquely determined by x; they are called the *barycentric coordinates* of the point x of σ with respect to $a_0, \ldots, a_n$.

Example 6. In low dimensions, one can easily draw a simplex. A 0-simplex is a point, of course. The 1-simplex spanned by a_0 and a_1 consists of all points of the form

$$x = ta_0 + (1-t)a_1 \qquad \text{where} \qquad 0 \leqslant t \leqslant 1.$$

This is just the line segment joining a_0 and a_1. Similarly, the 2-simplex σ spanned by a_0, a_1, a_2 equals the triangle having these three points as vertices.

Recall that a subset A of $\mathbb{R}^p$ is said to be *convex* if for each pair x, y of points of A, the line segment joining them lies in A.

Let σ be the n-simplex spanned by $\{a_0, \dots, a_n\}$, then the following properties hold:

1) σ is a compact, convex set in $\mathbb{R}^p$, which equals the intersection of all convex sets in $\mathbb{R}^p$ containing $a_0, \dots, a_n$.

2) Given a simplex σ, there is one and only one geometrically independent set of points spanning σ.

The points $a_0, \dots, a_n$ that span σ are called the *vertices* of σ; the number n is called the *dimension* of σ. Any simplex spanned by a subset of $\{a_0, \dots, a_n\}$ is called a *face* of σ. The faces of σ different from σ itself are called the *proper faces* of σ; their union is called the *boundary* of σ and denoted by $\mathrm{Bd}(\sigma)$. The *interior* of σ is defined as $\mathrm{Int}(\sigma) = \sigma \setminus \mathrm{Bd}(\sigma)$.

Since $\mathrm{Bd}(\sigma)$ consists of all points x of σ such that at least one of the barycentric coordinates $t_i(x)$ is zero, $\mathrm{Int}(\sigma)$ consists of those points of σ for which $t_i(x) \geqslant 0$ for all i.

Definition 2.4. A *simplicial complex* K in $\mathbb{R}^p$ is a collection of simplices in $\mathbb{R}^p$ such that:

i) Every face of a simplex of K is in K.

ii) The intersection of any two simplices of K is a face of each of them.

The following lemma is sometimes useful in verifying that a collection of simplices is a simplicial complex:

Lemma 2.5. *A collection K of simplices is a simplicial complex if and only if the following hold:*

(1) Every face of a simplex of K is in K

(2) Every pair of distinct simplices of K have disjoint interiors.

Proof. See [15], Lemma 2.1; pag 8. □

Definition 2.6. If L is a sub-collection of K that contains all faces of its elements, then L is a simplicial complex in its own right; it is called a *sub-complex* of K. One sub-complex of K is the collection of all simplices of K of dimension at most l; it is called the *l-skeleton* of K and is denoted by $K^{(l)}$. The points of the collection $K^{(0)}$ are called the *vertices* of K.

Definition 2.7. Let $|K|$ be the subset of $\mathbb{R}^p$ that is the union of the simplices of K. Giving each simplex its natural topology as a subspace of $\mathbb{R}^p$, we then topologize $|K|$ by declaring a subset A of $|K|$ to be closed in $|K|$ if and only if $A \cap \sigma$ is closed in σ, for each $\sigma \in K$. It is easy to see that this defines a topology on $|K|$. The space $|K|$ is called the *underlying space* of K, or the *polytope* of K.

Now we introduce the notion of a *simplicial map* of one complex into another.

Lemma 2.8. *Let K and L be complexes, and let*

$$f: \quad K^{(0)} \quad \longrightarrow \quad L^{(0)}$$

be a map such that whenever the vertices $v_0, \ldots, v_n$ of K span a simplex of K, the points $f(v_0), \ldots, f(v_n)$ are vertices of a simplex of L. Then f can be extended to a continuous map $g: \ |K| \ \to \ |L|$ as follows: for $x = \sum_{i=0}^{n} t_i v_i$

$$g(x) = \sum_{i=0}^{n} t_i f(v_i)$$

We call g the linear simplicial map induced by the vertex map f.

Proof. See [15], Lemma 2.7; pag 12. □

Lemma 2.9. *Suppose*

$$f: \quad K^{(0)} \quad \longrightarrow \quad L^{(0)}$$

is a bijective correspondence such that the vertices $v_0, \ldots, v_n$ *of* K *span a simplex of* K *if and only if* $f(v_0), \ldots, f(v_n)$ *span a simplex of* L. *Then the induced simplicial map*

$$g: \ |K| \ \rightarrow \ |L|$$

is a homeomorphism. Each simplex σ *of* K *is mapped by* g *onto a simplex* τ *of* L *of the same dimension as* σ. *The map* g *is called a simplicial homeomorphism of* K *with* L.

Proof. See [15], Lemma 2.8; pag 12. □

2.1.2 Abstract simplicial complex

Definition 2.10. An *abstract simplicial complex* is a collection $\mathcal{S}$ of finite nonempty sets, such that if A is an element of $\mathcal{S}$ and $B \subseteq A$ is a nonempty subset of A then $B \in \mathcal{S}$.

The element A of $\mathcal{S}$ is called a *simplex* of $\mathcal{S}$; its *dimension* is one less than the number of its elements. Each nonempty subset of A is called a *face* of A. The *dimension* of $\mathcal{S}$ is the largest dimension of one of its simplices, or it is infinite if there is no such largest dimension. The *vertex set* V of $\mathcal{S}$ is the union of the one-point elements of $\mathcal{S}$; we shall make no distinction between the vertex $v \in V$ and the 0-simplex $\{v\} \in \mathcal{S}$. A sub-collection of $\mathcal{S}$ that is itself a complex is called a *sub-complex* of $\mathcal{S}$.

Definition 2.11. Two abstract complexes $\mathcal{S}$ and $\mathcal{T}$ are said to be *isomorphic* if there is a bijective correspondence f mapping the vertex set of $\mathcal{S}$ to the vertex set of $\mathcal{T}$ such that:

$$\{a_0, \ldots, a_n\} \in \mathcal{S} \qquad \text{if and only if} \qquad \{f(a_0), \ldots, f(a_n)\} \in \mathcal{T}$$

Definition 2.12. If K is a simplicial complex, let $V = K^{(0)}$ be the vertex set of K. Let $\mathcal{R}$ be the collection of all subsets $\{a_0, \ldots, a_n\}$ of V such that the vertices $a_0, \ldots, a_n$ span a simplex of K. The collection $\mathcal{R}$ is called the *vertex scheme* of K.

The collection $\mathcal{R}$ is a particular example of an abstract simplicial complex.

Theorem 2.13. *a) Every abstract complex $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K.*

b) Two simplicial complexes are simplicial homeomorphic if and only if their vertex schemes are isomorphic as abstract simplicial complexes.

Proof. See [15], Theorem 3.1; pag 15. □

Definition 2.14. If the abstract simplicial complex $\mathcal{S}$ is isomorphic to the vertex scheme of the simplicial complex K, we call K a *geometric realization* of $\mathcal{S}$. It is uniquely determined up to simplicial homeomorphism.

Example 7. Let v_1, v_2 be two independent vectors in $\mathbb{R}^2$ and let

$$\Phi = \{v_1, v_2, v_1 + v_2, 2v_1 + v_2\}.$$

We now consider the collection of independent subsets of Φ:

$$\mathcal{S} = \Big\{\{v_1, v_2\}, \{v_1, v_1 + v_2\}, \{v_1, 2v_1 + v_2\}, \{v_2, v_1 + v_2\}, \{v_2, 2v_1 + v_2\},$$
$$\{v_1 + v_2, 2v_1 + v_2\}, \{v_1\}, \{v_2\}, \{v_1 + v_2\}, \{2v_1 + v_2\}\Big\}$$

The collection $\mathcal{S}$ is an abstract simplicial complex.

2.1.3 Homology groups

Definition 2.15. Let σ be a simplex (either geometrical or abstract). We define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If $\dim(\sigma) > 0$, the orderings of the vertices of σ then fall into two equivalence classes. Each of these classes is called an *orientation* of σ. (If σ is a 0-simplex, then there is only one class and hence only one orientation of σ.)
An *oriented simplex* is a simplex σ together with an orientation of σ.
If the points $v_0, \ldots, v_p$ are geometrically independent, we shall use the symbol:

$$v_0 \ldots v_p$$

to denote the simplex they span, and we shall use the symbol

$$[v_0 \dots v_p]$$

to denote the oriented simplex consisting of the simplex $v_0 \dots v_p$ and the equivalence class of the particular ordering $(v_0 \dots v_p)$.

Definition 2.16. Let K be a simplicial complex. A *p-chain* of K is a function c_p from the oriented p-simplices of K to the integers, such that:

i) $c_p(\sigma) = -c_p(\sigma')$ if σ and σ' are opposite orientations of the same simplex.

ii) $c_p(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values; the resulting group is denoted by $C_p(K)$ and is called the *group of (oriented) p-chains* of K. If $p < 0$ or $p > \dim(K)$ we let $C_p(K)$ denote the trivial group.

If σ is an oriented simplex, the *elementary chain* c corresponding to σ is a function defined as follows:

$c(\sigma) = 1$

$c(\sigma') = -1$ if σ' is the opposite orientation of σ,

$c(\tau) = 0$ for all other oriented simplices τ.

By abuse of notation, we often use the symbol σ to denote not only a simplex, or an oriented simplex, but also to denote the elementary p-chain c corresponding to the oriented simplex σ. With this convention, if σ and σ' are opposite orientation of the same simplex, then we can write $\sigma' = -\sigma$, because this equation holds when σ and σ' are interpreted as elementary chains.

Lemma 2.17. *$C_p(K)$ is free abelian: a basis for $C_p(K)$ can be obtained by orienting each p-simplex and using the corresponding elementary chain as a basis.*

Proof. See [15], Lemma 5.1; pag 28. □

The group $C_0(K)$ differs from the others, since it has a natural basis (since a 0-simplex has only one orientation). The group $C_p(K)$ has no natural basis if $p > 0$; one must orient the p-simplices of K in some arbitrary fashion in order to obtain a basis.

Definition 2.18. We now define a homomorphism

$$\partial_p : \ C_p(K) \ \longrightarrow \ C_{p-1}(K)$$

called the *boundary operator*. If $\sigma = [v_0, \ldots, v_p]$ is an oriented simplex with $p > 0$, we define

$$\partial_p(\sigma) = \partial_p([v_0, \ldots, v_p]) = \sum_{i=0}^{p} (-1)^p [v_0, \ldots, \hat{v}_i, \ldots, v_p] \tag{2.1}$$

where the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted from the array. Since $C_p(K)$ is the trivial group for $p < 0$, the operator ∂_p is the trivial homomorphism for $p \leqslant 0$

We must check that ∂_p is well-defined and that $\sigma_p(-\sigma) = -\partial_p(\sigma)$. For this purpose, it suffices to show that the right side of Equation (2.1) changes sign if we exchange two adjacent vertices in the array $[v_0, \ldots, v_p]$. So let us compare the expressions for

$$\sigma_p([v_0, \ldots, v_j, v_{j+1}, \ldots, v_p]) \quad \text{and} \quad \sigma_p([v_0, \ldots, v_{j+1}, v_j, \ldots, v_p])$$

For $i \neq j, j+1$, the ith terms in these two expressions differ precisely by a sign; the terms are identical except that v_j and v_{j+1} have been interchanged. What about the ith terms for $i = j$ and $i = j+1$? In the first expression, one has:

$$(-1)^j [\ldots, v_{j-1}, \hat{v}_j, v_{j+1}, v_{j+2}, \ldots] + (-1)^{j+1} [\ldots, v_{j-1}, v_j, \hat{v}_{j+1}, v_{j+2}, \ldots]$$

In the second expression, one has:

$$(-1)^j [\ldots, v_{j-1}, \hat{v}_{j+1}, v_j, v_{j+2}, \ldots] + (-1)^{j+1} [\ldots, v_{j-1}, v_{j+1}, \hat{v}_j, v_{j+2}, \ldots].$$

Hence these two expressions differ by a sign.

Example 8. For a 1-simplex, we have

$$\partial_1([v_0, v_1]) = v_1 - v_0$$

For a 2-simplex one has:

$$\partial_2([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

Lemma 2.19. $\partial_{p-1} \circ \partial_p = 0$

Proof. See [15], Lemma 5.3; pag 30. □

Definition 2.20. The kernel of

$$\partial_p : \quad C_p(K) \quad \longrightarrow \quad C_{p-1}(K)$$

is called the group of *p-cycles* and denoted by $Z_p(K)$. The image of

$$\partial_{p+1} : \quad C_{p+1}(K) \quad \longrightarrow \quad C_p(K)$$

is called the group of *p-boundaries* and is denoted by $B_p(K)$. By Lemma 2.19, each boundary of a $p+1$ chain is automatically a p-cycle. That is, $B_p(K) \subset Z_p(K)$. We define

$$H_p(K) = Z_p(K)/B_p(K)$$

and call it the pth *homology group* of K.

Theorem 2.21. *Let K be a complex. Then the group $H_0(K)$ is free abelian. If $\{v_\alpha\}$ is a collection consisting of one vertex from each connected component of $|K|$, then the homology classes of the chains v_α form a basis for $H_0(K)$.*

Proof. See [15], Theorem 7.1; pag 41. □

It is convenient to consider another version of the 0-dimensional homology.

Definition 2.22. Let

$$\epsilon : \quad C_0(K) \quad \longrightarrow \quad \mathbb{Z}$$

be the surjective homomorphism defined by $\epsilon(v) = 1$ for each vertex v of K. Then if c is a 0-chain, $\epsilon(c)$ equals the sum of the values of c on the vertices of K. The map ϵ is called the *augmentation map* for $C_0(K)$. We have just noted that $\epsilon(\partial(d)) = 0$ if d is a 1-chain. We define the *reduced homology group* of K in dimension 0, denoted by $\widetilde{H}_0(K)$, by the equation:

$$\widetilde{H}_0(K) = \ker(\epsilon)/\mathrm{Im}(\partial_1)$$

If $p > 0$, we let $\widetilde{H}_p(K)$ denote the usual group $H_p(K)$.

The relation between reduced and ordinary homology is as follows:

Theorem 2.23. *The group $\widetilde{H}_0(K)$ is free abelian, and*

$$\widetilde{H}_0(K) \oplus \mathbb{Z} \simeq H_0(K)$$

Thus $\widetilde{H}_0(K)$ vanishes if $|K|$ is connected. If $|K|$ is not connected, let $\{v_\alpha\}$ consist of one vertex from each component $|K|$, let α_0 be a fixed index. Then the homology classes of the chains $v_\alpha - v_{\alpha_0}$, for $\alpha \neq \alpha_0$, form a basis for $\widetilde{H}_0(K)$.

Proof. See [15], Theorem 7.2; pag 43. □

2.1.4 Homology groups with arbitrary coefficients

Definition 2.24. Let R be an abelian group. Let K be a simplicial complex. A *p-chain* of K with coefficients in R is a function c_p from the oriented p-simplices of K to R, such that:

i) $c_p(\sigma) = -c_p(\sigma')$ if σ and σ' are opposite orientations of the same simplex;

ii) $c_p(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values; the resulting group is denoted by $C_p(K; R)$ and is called the *group of (oriented) p-chains* of K with coefficients in R.

If σ is an oriented simplex and if $g \in R$, we denote by $g\sigma$ the *elementary chain* defined as follows:

- $g\sigma(\sigma) = g$
- $g\sigma(\sigma') = -g$ if σ' is the opposite orientation of σ
- $g\sigma(\tau) = 0$ for all other oriented simplices τ.

If one orients all the p-simplices of K, then each chain c_p can be written uniquely as a finite sum

$$c_p = \sum_i g_i \sigma_i$$

of elementary chains. Thus $C_p(K, R)$ is the direct sum of subgroups isomorphic to R, one for each p-simplex of K.

The boundary operator

$$\partial_p : \quad C_p(K, R) \quad \longrightarrow \quad C_{p-1}(K, R)$$

is defined easily by the formula:

$$\partial_p(g\sigma) = g\partial_p(\sigma)$$

where $\partial_p(\sigma)$ is the ordinary boundary, defined earlier. As before, $\sigma \circ \sigma = 0$ and we define $Z_p(K, R)$ to be the kernel of the homomorphism:

$$\partial_p : \quad C_p(K, R) \quad \longrightarrow \quad C_{p-1}(K, R).$$

Furthermore we define $B_p(K, R)$ to be the image of the following homomorphism:

$$\partial_{p-1} : \quad C_{p-1}(K, R) \quad \longrightarrow \quad C_p(K, R)$$

and

$$H_p(K, R) = Z_p(K, R)/B_p(K, R)$$

These groups are called the *group of the cycles*, the *group of the boundaries*, and the *homology group of K with coefficients in R* respectively.
Of course, one can also study *reduced* homology with coefficients in R. The details are clear.

2.1.5 Cohomology groups

Associated with any pair of abelian groups A, G is a third abelian group, the group $\mathrm{Hom}(A, G)$ of all homomorphism of A into G. In this subsection we study some of its properties.

Definition 2.25. If A and G are abelian groups, then the set $\mathrm{Hom}(A, G)$ of all homomorphism of A into G becomes an abelian group if we add two homomorphisms by adding their values in G.

Definition 2.26. A homomorphism $f: A \to B$ gives rise to a *dual homomorphism*

$$\widetilde{f}: \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G)$$

going in the reverse direction. The map $\widetilde{f}$ assigns to the homomorphism $\phi: B \to G$, the composite

$$\widetilde{f}(\phi): A \xrightarrow{f} B \xrightarrow{\phi} G.$$

That is, $\widetilde{f}(\phi) = \phi \circ f$.

Definition 2.27. In the context of group theory, a sequence

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \cdots \xrightarrow{f_n} G_n$$

of groups and group homomorphism is called *exact* if the image of each homomorphism is equal to the kernel of the next:

$$\mathrm{Im}(f_k) = \mathrm{Ker}(f_{k+1})$$

Note that the sequence of groups and homomorphisms may be either finite or infinite.

Example 9. If the following sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact then f is an injective homomorphism and g is a surjective homomorphism. Furthermore:

$$B/f(A) \simeq C$$

Theorem 2.28. *Let G be an abelian group. Let f be a homomorphism, let $\widetilde{f}$ be the dual homomorphism. Then:*

a) If f is an isomorphism, so is $\widetilde{f}$.

b) If f is the zero homomorphism, so is $\widetilde{f}$.

c) If f is surjective, then $\widetilde{f}$ is injective. That is the exactness of

$$B \xrightarrow{f} C \rightarrow 0$$

implies the exactness of

$$\mathrm{Hom}(B, G) \xleftarrow{\widetilde{f}} \mathrm{Hom}(C, G) \leftarrow 0$$

Proof. See [15], Theorem 41.1; pag 247. □

Now we can give the definition of cohomology groups.

Definition 2.29. Let K be a simplicial complex; let R be an abelian group. The group of *p-dimensional cochains* of K, with coefficients in R, is the group

$$C^p(K, R) = \mathrm{Hom}(C_p(K), R)$$

The *coboundary operator* δ_{p+1} is defined to be the dual of the boundary operator $\partial_{p+1} : \ C_{p+1}(K) \ \longrightarrow \ C_p(K)$. Thus

$$\delta_{p+1} : \ C^p(K, R) \ \longrightarrow \ C^{p+1}(K, R)$$

so that δ_{p+1} raises dimension by one. We define $Z^p(K; R)$ to the kernel of this homomorphism, $B^{p+1}(K, R)$ to be its image, and (noting that $\delta^2 = 0$ because $\partial^2 = 0$),

$$H^p(K; R) = Z^p(K; R)/B^p(K; R)$$

These groups are called the group of *cocycles*, the group of *coboundaries*, and the *cohomology group*, respectively, of K with coefficients in R.

If c^p is a p-dimensional cochain, and c_p is a p-dimensional chain, we commonly use the notation

$$\langle c^p, c_p\rangle = c^p(c_p).$$

Proposition 2.30. *Using the same notation as above, the definition of the coboundary operator becomes:*

$$\langle \delta_{p+1}(c^p), d_{p+1}\rangle = \langle c^p, \partial_{p+1}(d_{p+1})\rangle$$

Proof. We have:

$$\begin{array}{rccc} \partial_{p+1}: & C_{p+1}(K) & \longrightarrow & C_p(K) \\ & d_{p+1} & \longmapsto & \partial_{p+1}(d_{p+1}) \end{array}$$

and δ_{p+1} is its dual map:

$$\begin{array}{rccc} \delta_{p+1}: & C^p(K) & \longrightarrow & C^{p+1}(K) \\ & c^p & \longmapsto & \delta_{p+1}(c^p) \end{array}$$

By definition we have:

$$\delta_{p+1}(c^p) = c^p \circ \partial_{p+1}$$

and so:

$$\delta_{p+1}(c^p)(d_{p+1}) = (c^p \circ \partial_{p+1})(d_{p+1}) = c^p(\partial_{p+1}(d_{p+1}))$$

□

Definition 2.31. Given a complex K, we dualize the standard augmentation map

$$C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\epsilon} \mathbb{Z}$$

and obtain a homomorphism $\widetilde{\epsilon}$:

$$C^1(K) \xleftarrow{\delta_1} C^0(K) \xleftarrow{\widetilde{\epsilon}} R$$

called a *coaugmentation*. It is injective, and $\delta_1 \circ \widetilde{\epsilon} = 0$. We define the *reduced cohomology* of K by setting:

$$\widetilde{H}^q(K; R) = H^q(K; R) \qquad \text{if } q > 0$$

and

$$\widetilde{H}^0(K; R) = \ker(\delta_1)/\mathrm{Im}(\widetilde{\epsilon})$$

Theorem 2.32. *If $|K|$ is connected, then $\widetilde{H}^0(K;R) = 0$. More generally, for any complex K, we have:*

$$H^0(K;R) \simeq \widetilde{H}^0(K;R) \oplus R$$

Proof. See [15], Theorem 42.2; pag 256. □

Proposition 2.33. *Let K be a simplicial complex. Let $\mathbb{F}$ be a field. Then*

$$\widetilde{H}^p(K,\mathbb{F}) \quad \textit{and} \quad \widetilde{H}_p(K,\mathbb{F})$$

have the structure of vector spaces over $\mathbb{F}$. They are dual of each other as vector spaces. Besides, both δ and ∂ are vector space homomorphisms (linear transformations).

Proof. See [15]; pag 324. □

2.1.6 Homomorphism induced by a simplicial map

If f is a simplicial map of $|K|$ into $|L|$, then f maps each p-simplex σ_i of K onto a simplex τ_i of L of the same or lower dimension. We shall define a homomorphism of p-chains that carries a formal sum $\sum_i m_i\sigma_i$ of oriented p-simplices of K onto the formal sum $\sum_i m_i\tau_i$ of their images. (We delete form the latter sum those simplices τ_i whose dimension is less than p.) This map in turn induces a homomorphism of homology groups. As a general notation, we shall use the phrase

$$'' \; f: \; K \longrightarrow L \quad \text{is a simplicial map} \; ''$$

to mean that f is a continuous map of $|K|$ into $|L|$ that maps each simplex of K linearly onto a simplex of L. Thus f maps each vertex of K to a vertex of L, and it equals the simplicial map induced by this vertex map.

Definition 2.34. Let $f: \; K \longrightarrow L$ be a simplicial map. If $v_0 \dots v_p$ is a simplex of K, then the points $f(v_0), f(v_1), \dots, f(v_p)$ span a simplex of L. We define a homomorphism

$$f_{\#}: \; C_p(K) \longrightarrow C_p(L)$$

by defining it on oriented simplices as follows:

$$f_{\#}([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_0), \dots, f(v_p) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases}$$

This map is clearly well-defined; exchanging two vertices in the expression $[v_0, \dots, v_p]$ changes the sign of the right side of the equation. The family of homomorphisms $\{f_{\#}\}$, one in each dimension, is called the *chain map induced by the simplicial map* f.

Lemma 2.35. *The homomorphism* $f_{\#}$ *commutes with* ∂*; therefore* $f_{\#}$ *induces a homomorphism*

$$f_* : \quad H_p(K) \quad \longrightarrow \quad H_p(L)$$

Proof. See [15], Lemma 12.1; pag 62. □

Lemma 2.36. *The chain map* $f_{\#}$ *preserves the augmentation map* ϵ*; therefore it induces a homomorphism*

$$f_* : \quad \widetilde{H}_p(K) \quad \longrightarrow \quad \widetilde{H}_p(L)$$

of reduced homology groups.

Proof. See [15], Lemma 12.3; pag 63. □

2.1.7 Alexander Duality

Let K be an abstract simplicial complex with ground set V.
For $\sigma \in K$, let

$$\overline{\sigma} = V \smallsetminus \sigma$$

Definition 2.37. The *Alexander dual* of K is the simplicial complex on the same ground set defined by

$$K^* = \{\sigma \subseteq V \mid \overline{\sigma} \notin K\}$$

It is easy to see that $K^{**} = K$.

Let K be a simplicial complex with ground set $V = \{1, 2, \dots, n\}$. For $j \in \sigma \in K$, we define the *sign*

$$\operatorname{sgn}(j, \sigma) = (-1)^{i-1}$$

where j is the ith smallest element of the set σ. The following simple property of the sign function will be needed.

Lemma 2.38. *Let $k \in \sigma \subseteq \{1, 2, \dots, n\}$ and $p(\sigma) = \prod_{i \in \sigma} (-1)^{i-1}$. Then*

$$\operatorname{sgn}(k, \sigma)\ p(\sigma \smallsetminus k) = \operatorname{sgn}(k, \overline{\sigma} \cup k)\ p(\sigma)$$

Proof.

$$\operatorname{sgn}(k, \sigma) \operatorname{sgn}(k, \overline{\sigma} \cup k) = \prod_{\substack{i \in \sigma \\ i < k}} (-1) \prod_{\substack{i \in \overline{\sigma} \\ i < k}} (-1) = (-1)^{k-1}$$

and

$$p(\sigma) p(\sigma \smallsetminus k) = \prod_{i \in \sigma} (-1)^{i-1} \prod_{i \in \sigma \smallsetminus k} (-1)^{i-1} = (-1)^{k-1}$$

We have also:

$$\frac{\operatorname{sgn}(k, \sigma) \operatorname{sgn}(k, \overline{\sigma} \cup k)}{p(\sigma) p(\sigma \smallsetminus k)} = \frac{(-1)^{k-1}}{(-1)^{k-1}} = 1$$

$$\operatorname{sgn}(k, \sigma) \operatorname{sgn}(k, \overline{\sigma} \cup k)\ p(\sigma) p(\sigma \smallsetminus k) = (-1)^{2k-2} = 1$$

We obtain then:

$$p^2(\sigma) p^2(\sigma \smallsetminus k) = 1 \quad \Longrightarrow \quad p^2(\sigma) = 1 \ \text{ and } \ p^2(\sigma \smallsetminus k) = 1$$

$$\operatorname{sgn}^2(k, \sigma) \operatorname{sgn}^2(k, \overline{\sigma} \cup k) = 1 \quad \Longrightarrow \quad \operatorname{sgn}^2(k, \sigma) = 1 \ \text{ and } \ \operatorname{sgn}^2(k, \overline{\sigma} \cup k) = 1$$

$$\operatorname{sgn}(k, \sigma)\ p(\sigma \smallsetminus k) = \operatorname{sgn}(k, \sigma)\ p(\sigma \smallsetminus k)\ \operatorname{sgn}^2(k, \overline{\sigma} \cup k)\ p^2(\sigma) =$$

$$= (-1)^{k-1}\ (-1)^{k-1}\ \operatorname{sgn}(k, \overline{\sigma} \cup k)\ p(\sigma) = \operatorname{sgn}(k, \overline{\sigma} \cup k)\ p(\sigma)$$

□

We now review the definitions and notation used for (co)homology. Throughout this subsection we suppose that R is a commutative ring containing a unit element. In particular R could also be a field $\mathbb{F}$.

Definition 2.39. Let $C_i = C_i(K)$ be a free R-module with the free basis

$$\{e_\sigma \mid \sigma \in K,\ \dim(\sigma) = i\}.$$

The *reduced chain complex* of K over R is the complex

$$\widetilde{C}_*(K) = \widetilde{C}_*(K,R) = \cdots C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \to \cdots, \qquad i \in \mathbb{Z}$$

whose mappings ∂_i are defined as

$$\partial_i(e_\sigma) = \sum_{j\in\sigma} \operatorname{sgn}(j,\sigma) e_{\sigma\smallsetminus j}.$$

The complex $\widetilde{C}_*(K)$ is formally infinite; however $C_i = 0$ for $i < -1$ or $i > \dim(K)$. The nth reduced homology group of K over R is

$$\widetilde{H}_n(K) = \widetilde{H}_n(K;R) = \ker(\partial_n)/\operatorname{Im}(\partial_{n+1}).$$

Definition 2.40. Let $C^i = C^i(K)$ be a free R-module with free basis

$$\{e^*_\sigma \mid \sigma \in K,\ \dim(\sigma) = i\}.$$

The *reduced cochain complex* of K over R is the complex

$$\widetilde{C}^*(K) = \widetilde{C}^*(K,R) = \cdots C^{i-1} \xrightarrow{\delta_i} C^i \xrightarrow{\delta_{i+1}} C^{i+1} \to \cdots, \qquad i \in \mathbb{Z}$$

where $\delta_i = \partial_i^*$ are maps dual to ∂_i, explicitly stated:

$$\delta_i(e^*_\sigma) = \sum_{\substack{j\notin\sigma \\ \sigma\cup j\in K}} \operatorname{sgn}(j,\sigma\cup j) e^*_{\sigma\cup j}$$

The nth reduced cohomology group over R is

$$\widetilde{H}^n(K) = \widetilde{H}^n(K;R) = \ker(\delta_{n+1})/\operatorname{Im}(\delta_n)$$

We now give the definition of relative homology.

Definition 2.41. Suppose that K is a simplicial complex and A is a subcomplex of K. Let

$$\mathcal{R}_i = \mathcal{R}_i(K, A) = C_i(K)/C_i(A)$$

where C_i was defined in Definition 2.39. The *relative reduced chain complex* of (K, A) over R is the complex:

$$\widetilde{C}_*(K, A) = \widetilde{C}_*(K, A; R) = \cdots \mathcal{R}_{i+1} \xrightarrow{d_{i+1}} \mathcal{R}_i \xrightarrow{d_i} \mathcal{R}_{i-1} \to \cdots, \qquad i \in \mathbb{Z}$$

where d_i is defined as follows:

$$d_i(e_\sigma + C_i(A)) = \sum_{j \in \sigma} \operatorname{sgn}(j, \sigma)(e_{\sigma \smallsetminus j} + C_{i-1}(A)).$$

The *nth relative homology group* of (K, A) over R is defined as:

$$\widetilde{H}_n(K, A) = \widetilde{H}_n(K, A; R) = \ker(d_n)/\operatorname{Im}(d_{n+1})$$

Remark. When we intend to compute relative homology groups, we can identify $\mathcal{R}_i = C_i(K)/C_i(A)$ with a free R-module with the free basis

$$\{e_\sigma \mid \sigma \in K,\ \sigma \notin A,\ \dim(\sigma) = i\}.$$

Then d_i can be rewritten as:

$$d_i(e_\sigma) = \sum_{\substack{j \in \sigma \\ \sigma \smallsetminus j \notin A}} \operatorname{sgn}(j, \sigma)\ e_{\sigma \smallsetminus j}$$

One of the important properties of relative homology groups is that they fit into a long exact sequence.

Lemma 2.42 (Long exact sequence of a pair)**.** *Let K be a complex; let A be a subcomplex. Then there is a long exact sequence:*

$$\cdots \to \widetilde{H}_n(A) \to \widetilde{H}_n(K) \to \widetilde{H}_n(K, A) \to \widetilde{H}_{n-1}(A) \to \cdots$$

Proof. See [15], Theorem 23.3; pag 133. □

Suppose that K is a simplicial complex with ground set V. Let 2^V be the full simplex with vertex set V.

Lemma 2.43. *Let K be a simplicial complex with ground set V. Then:*

$$\widetilde{H}_i(K) \simeq \widetilde{H}_{i+1}(2^V, K)$$

Proof. This follows from Lemma 2.42. We have the long exact sequence of the pair $(2^V, K)$:

$$\cdots \to \widetilde{H}_{i+1}(2^V) \to \widetilde{H}_{i+1}(2^V, K) \to \widetilde{H}_i(K) \to \widetilde{H}_i(2^V) \to \cdots$$

Since 2^V is the full simplex the groups $\widetilde{H}_{i+1}(2^V)$ and $\widetilde{H}_i(2^V)$ are zero. Hence, the sequence becomes:

$$\cdots \to 0 \to \widetilde{H}_{i+1}(2^V, K) \to \widetilde{H}_i(K) \to 0 \to \cdots$$

It follows that the groups $\widetilde{H}_{i+1}(2^V, K)$ and $\widetilde{H}_i(K)$ are isomorphic. □

Lemma 2.44. *Let K be a simplicial complex with ground set V of size n. Then*

$$\widetilde{H}_{i+1}(2^V, K) \simeq \widetilde{H}^{n-i-3}(K^*)$$

Proof. Suppose that $V = \{1, 2, \ldots, n\}$. The chain complex for reduced homology of the pair $(2^V, K)$ is the complex:

$$\cdots \mathcal{R}_{j+1} \xrightarrow{d_{j+1}} \mathcal{R}_j \xrightarrow{d_j} \mathcal{R}_{j-1} \xrightarrow{d_{j-1}} \cdots, \qquad j \in \mathbb{Z}$$

where $\mathcal{R}_j = \langle e_\sigma \mid \sigma \subseteq V,\ \sigma \notin K,\ \dim(\sigma) = j\rangle$, and the d_j's are the unique homomorphisms satisfying:

$$d_j(e_\sigma) = \sum_{\substack{k\in\sigma \\ \sigma\smallsetminus k\notin K}} \operatorname{sgn}(k, \sigma)\ e_{\sigma\smallsetminus k}.$$

The cochain complex for reduced cohomology of K^* is the complex:

$$\cdots \xrightarrow{\delta_{j-1}} C^{j-1} \xrightarrow{\delta_j} C^j \xrightarrow{\delta_{j+1}} \cdots, \qquad j \in \mathbb{Z}$$

where

$C^j = \langle e^*_\sigma \mid \sigma \subseteq V,\ \dim(\sigma) = j,\ \sigma \in K^* \rangle = \langle e^*_\sigma \mid \sigma \subseteq V,\ \dim(\overline{\sigma}) = n-j-2,\ \overline{\sigma} \notin K \rangle$

and the δ_j's are the unique homomorphisms satisfying:

$$\delta_j(e^*_\sigma) = \sum_{\substack{k \notin \sigma \\ \sigma \cup k \in K^*}} \operatorname{sgn}(k, \sigma \cup k) e^*_{\sigma \cup k} = \sum_{\substack{k \in \overline{\sigma} \\ \overline{\sigma} \smallsetminus k \notin K}} \operatorname{sgn}(k, \sigma \cup k) e^*_{\overline{\overline{\sigma} \smallsetminus k}}.$$

Define $p(\sigma)$ as in Lemma 2.38 and let ϕ_j be the following isomorphism:

$$\begin{aligned} \phi_j :\ \mathcal{R}_j &\longrightarrow C^{n-j-2} \\ e_\sigma &\longmapsto p(\sigma)\ e^*_{\overline{\sigma}} \end{aligned}$$

for $\sigma \notin K$ with $\dim(\sigma) = j$. (Note that these two conditions are equivalent to $\dim(\overline{\sigma}) = n - j - 2,\ \overline{\sigma} \in K^*$). We then have the following diagram:

$$\begin{array}{ccccccc} \xrightarrow{d_{j+1}} & \mathcal{R}_j & \xrightarrow{d_j} & \mathcal{R}_{j-1} & \xrightarrow{d_{j-1}} \\ & \downarrow{\scriptstyle \phi_j} & & \downarrow{\scriptstyle \phi_{j-1}} & \\ \xrightarrow{\delta_{n-j-2}} & C^{n-j-2} & \xrightarrow{\delta_{n-j-1}} & C^{n-j-1} & \xrightarrow{\delta_{n-j}} \end{array}$$

We want to check that:

$$\phi_{j-1} \circ d_j = \delta_{n-j-1} \circ \phi_j.$$

Let $\sigma \subseteq V,\ \ \sigma \notin K,\ \ \dim(\sigma) = j$. Then:

$$(\phi_{j-1} \circ d_j)(e_\sigma) = \phi_{j-1}\Bigg(\sum_{\substack{k \in \sigma \\ \sigma \smallsetminus k \notin K}} \operatorname{sgn}(k, \sigma)\ e_{\sigma \smallsetminus k} \Bigg) = \sum_{\substack{k \in \sigma \\ \sigma \smallsetminus k \notin K}} \operatorname{sgn}(k, \sigma)\ p(\sigma \smallsetminus k)\ e^*_{\overline{\sigma \smallsetminus k}}$$

$$(\delta_{n-j-1} \circ \phi_j)(e_\sigma) = \delta_{n-j-1}\big(p(\sigma)\ e^*_{\overline{\sigma}}\big) = \sum_{\substack{k \in \sigma \\ \sigma \smallsetminus k \notin K}} \operatorname{sgn}(k, \sigma \cup k)\ p(\sigma)\ e^*_{\overline{\sigma \smallsetminus k}}$$

These two sums are equal term by term, due to Lemma 2.38. Thus ϕ is an isomorphism of the complexes, implying:

$$\widetilde{H}_{i+1}(2^V, K) \simeq \widetilde{H}^{n-i-3}(K^*)$$

□

Theorem 2.45 (Combinatorial Alexander duality). *Let K be a simplicial complex with ground set of size n. Then*

$$\widetilde{H}_i(K) \simeq \widetilde{H}^{n-i-3}(K^*).$$

Here $\widetilde{H}$ stands for reduced homology (resp. cohomology) over a given ring R.

Proof. Combining the results of Lemma 2.43 and Lemma 2.44 we obtain the proof of the theorem. □

Example 10. Let $V = \{v_1, v_2, v_3, v_4\}$ be the ground set of the following abstract simplicial complex

$$\Delta = \Big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\Big\}$$

We now find the dual simplicial complex of Δ:

$$\Delta^* = \{A \subseteq V \mid V \smallsetminus A \notin \Delta\} = \Big\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_3\}\Big\}$$

We have the following geometric realizations:

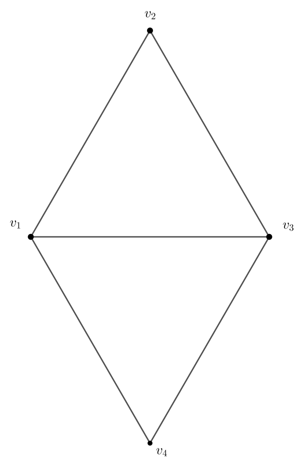


Figure 2.1: Δ

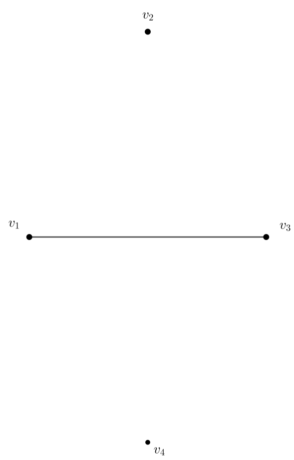


Figure 2.2: Δ^*

We deduce from the Alexander duality with $n = 4$ working with coefficients over any field $\mathbb{F}$ that

$$\widetilde{H}_i(\Delta, \mathbb{F}) \simeq \widetilde{H}^{1-i}(\Delta^*, \mathbb{F})$$

In particular we have:

$$\widetilde{H}_1(\Delta, \mathbb{F}) \simeq \widetilde{H}^0(\Delta^*, \mathbb{F})$$

Geometrically we see that the dimension of $\widetilde{H}_1(\Delta, \mathbb{F})$ is two. The simplicial complex Δ^* has 3 connected components then the dimension of $\widetilde{H}_0(\Delta^*, \mathbb{F})$ is two, therefore the dimension of $\widetilde{H}^0(\Delta^*, \mathbb{F})$ is also two.

2.2 Matroid theory

In this Section we recall the fundamental definitions of matroid theory. In the end of the section we explicitly calculate the representations on the top homology space of the simplicial complex associated with the dual matroid of K_4.

2.2.1 Basic definitions

Definition 2.46. A *matroid* M is an ordered pair (E, I) consisting of a finite set E and a collection I of subsets of E satisfying the three following conditions:

(I1) $\emptyset \in I$

(I2) *If* $A \in I$ *and* $A' \subseteq A$*, then* $A' \in I$

(I3) *If* A *and* B *are in* I *and* $|A| < |B|$*, then there is an element* $e \in B \setminus A$ *such that* $A \cup \{e\} \in I$*.*

The first two properties define an abstract simplicial complex (See Definition 2.10).

The members of I are the *independent sets* of M, and E is the *ground set* of M. We shall often write $IN(M)$ for I and $E(M)$ for E. A subset of E that is not in I is called *dependent*.

Proposition 2.47. *Let* E *be the set of column labels of an* $m \times n$ *matrix* A *over a field* $\mathbb{K}$*, and let* I *be the set of subsets* X *of* E *for which the multiset of columns labeled by* X *is linearly independent in the vector space* $V(m, \mathbb{K})$*. Then* $M = (E, I)$ *is a matroid.*

The matroid obtained as above from the matrix A will be denoted by $M[A]$. This matroid is called the *vector matroid* of A.

Example 11. Let A be the matrix:

$$\begin{matrix} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{matrix}$$

over the field $\mathbb{R}$. Then $E = \{1, 2, 3, 4, 5\}$ and

$$I = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$$

Thus the set of dependent sets of this matroid is

$$\{\{3\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\} \ \cup \ \{X \subseteq E \ : \ |X| \geqslant 3\}$$

The set of *minimal dependent sets*, that is, dependent sets all of whose proper subsets are independent is:

$$\{\{3\}, \{1, 4\}, \{1, 2, 5\}, \{2, 4, 5\}\}.$$

Definition 2.48. A minimal dependent set in an arbitrary matroid M will be called a *circuit* of M and we shall denote the set of circuits of M by $\mathcal{C}$ or $\mathcal{C}(M)$. A circuit of M having n elements will also be called an *n-circuit*.

Definition 2.49. A matroid is called *simple* if it has no circuits consisting of one or two elements.

Evidently, as in the last example, once I has been specified, $\mathcal{C}(M)$ can be determined. Similarly, I can be determined from $\mathcal{C}(M)$: the members of I are those subsets of E that contain no member of $\mathcal{C}(M)$. Thus a matroid is uniquely determined by its set $\mathcal{C}$ of circuits.
We now examine some properties of $\mathcal{C}$ with a view to characterizing those subsets of 2^E that can occur as the set of circuits of a matroid on E. It is easy to see that:

(C1) $\emptyset \in \mathcal{C}$;

(C2) *If* C_1 *and* C_2 *are members of* $\mathcal{C}$ *and* $C_1 \subseteq C_2$*, then* $C_1 = C_2$*.*

Lemma 2.50. *The set* $\mathcal{C}$ *of circuits of a matroid has the following property:*

(C3) *If* C_1 *and* C_2 *are distinct members of* $\mathcal{C}$ *and* $e \in C_1 \cap C_2$*, then there is a member* $C_3 \in \mathcal{C}$ *such that*

$$C_3 \subseteq (C_1 \cup C_2) \smallsetminus \{e\}$$

Proof. See [16], Lemma 1.1.3; pag 9. □

Theorem 2.51. *Let* E *be a set and* $\mathcal{C}$ *be a collection of subsets of* E *satisfying* (C1)-(C3)*. Let* I *be the collection of subsets of* E *that contain no member of* $\mathcal{C}$*. Then* (E, I) *is a matroid having* $\mathcal{C}$ *as its collection of circuits.*

Proof. See [16], Theorem 1.1.4; pag 10. □

Corollary 2.52. *Let* $\mathcal{C}$ *be a set of subsets of a set* E*. Then* $\mathcal{C}$ *is the collection of circuits of a matroid on* E *if and only if* $\mathcal{C}$ *satisfies* (C1)-(C2)-(C3).

We can associate a matroid to a graph:

Proposition 2.53. *Let* E *be the set of edges of a graph* Γ *and* $\mathcal{C}$ *be the set of edge sets of simple cycles of* Γ*. Then* $\mathcal{C}$ *is the set of circuits of a matroid on* E*.*

Proof. See [16], Proposition 1.1.7; pag 11. □

Definition 2.54. The matroid derived above from the graph Γ is called the *cycle matroid* of Γ. It is denoted by $M(\Gamma)$. Clearly a set X of edges is independent in $M(\Gamma)$ if and only if X does not contain the edge set of a cycle or, equivalently, $\Gamma[X]$, the subgraph induced by X, is a forest.

Definition 2.55. Two matroids M_1 and M_2 are *isomorphic*, written $M_1 \cong M_2$, if there is a bijection:

$$\psi : \quad E(M_1) \quad \longrightarrow \quad E(M_2)$$

such that, for all $X \subseteq E(M_1)$, $\psi(X)$ is independent in M_2 if and only if X is independent in M_1.

Example 12. Let Γ be the graph shown in figure 2.3 and let $M = M(\Gamma)$.

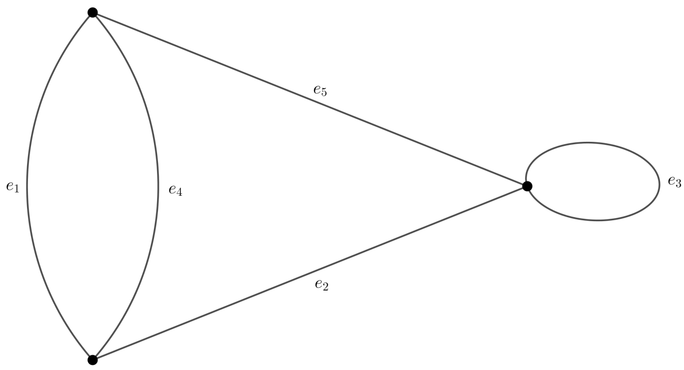


Figure 2.3

Then:

$$E(M) = \{e_1, e_2, e_3, e_4, e_5\} \quad \mathcal{C}(M) = \{\{e_3\}, \{e_1, e_4\}, \{e_1, e_2, e_5\}, \{e_2, e_4, e_5\}\}.$$

Comparing M with the matroid $M[A]$ in the previous example, we see that, under the bijection:

$$\begin{array}{rccc} \psi: & \{1,2,3,4,5\} & \longrightarrow & \{e_1, e_2, e_3, e_4, e_5\} \\ & i & \longmapsto & e_i \end{array}$$

a set X is a circuit in $M[A]$ if and only if $\psi(X)$ is a circuit in M. Equivalently, a set Y is independent in $M[A]$ if and only if $\psi(Y)$ is independent in M. Thus the matroids M and $M[A]$ are isomorphic.

A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*. So for instance the matroid $M[A]$ is graphic.

Definition 2.56. If Γ is a graph, we can form a directed graph $D(\Gamma)$ by arbitrarily assigning a direction to each edge. Let $A_{D(\Gamma)}$ denote the *incidence matrix* of $D(\Gamma)$, that is, $A_{D(\Gamma)}$ is the matrix $[a_{ij}]$ whose rows and columns are indexed by the vertices and edges, respectively, of $D(\Gamma)$, where:

$$a_{ij} = \begin{cases} 1, & \text{if vertex } i \text{ is the tail of non-loop edge } j \\ -1, & \text{if vertex } i \text{ is the head of non-loop edge } j \\ 0, & \text{otherwise} \end{cases}$$

Proposition 2.57. *If Γ is a graph, then $M(\Gamma) \cong M[A_{D(\Gamma)}]$ over any field $\mathbb{K}$ for any $D(\Gamma)$ formed by Γ.*

Proof. See [16], Proposition 5.1.2; pag 138. □

2.2.2 Basis, Rank and Closure Operator

Definition 2.58. A maximal independent set in a matroid M is called *basis* of M.

Lemma 2.59. *If B_1 and B_2 are bases of a matroid M, then $|B_1| = |B_2|$.*

Proof. See [16], Lemma 1.2.1; pag 16. □

If M is a matroid and $\mathcal{B}$ is its collection of bases, then, by (I1):

(B1) *$\mathcal{B}$ is non-empty.*

Lemma 2.60. *$\mathcal{B}$ satisfies the following condition:*

(B2) *If B_1 and B_2 are members of $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_2 \setminus B_1$ such that:*

$$(B_1 \setminus \{x\}) \cup \{y\} \ \in \mathcal{B}$$

Proof. See [16], Lemma 1.2.2; pag 17. □

Theorem 2.61. *Let E be a set and $\mathcal{B}$ be a collection of subsets of E satisfying* (B1) *and* (B2). *Let I be the collection of subsets of E that are contained in some member of $\mathcal{B}$. Then (E, I) is a matroid having $\mathcal{B}$ as its collection of bases.*

Proof. See [16], Theorem 1.2.3; pag 17. $\square$

Corollary 2.62. *Let $\mathcal{B}$ a set of subsets of a set E. Then $\mathcal{B}$ is the collection of bases of a matroid on E if and only if it satisfies* (B1)-(B2).

Definition 2.63. Let M be the matroid (E, I) and suppose that $X \subseteq E$. Define:

$$I|X = \{A \subseteq X \ : \ A \in I\}.$$

Then it is easy to see that the pair $(E, I|X)$ is a matroid. We call this matroid the *restriction of* M to X. It is denoted by $M|X$.
As $M|X$ is a matroid, Lemma 2.59 implies that all its bases are equicardinal. We define the *rank* $\mathrm{rk}(X)$ of X to be the size of a basis B of $M|X$.

It is clear that rk has the following properties:

(R1) *If $X \subseteq E$, then* $\mathrm{rk}(X) \leqslant |X|$.

(R2) *If $X \subseteq Y \subseteq E$, then* $\mathrm{rk}(X) \leqslant \mathrm{rk}(Y)$.

Lemma 2.64. *The rank function* rk *of a matroid M on a set E satisfies the following condition:*

(R3) *If X and Y are subsets of E, then:*

$$\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leqslant \mathrm{rk}(X) + \mathrm{rk}(Y).$$

Proof. See [16], Lemma 1.3.1; pag 23. $\square$

Theorem 2.65. *Let E be a set and* rk *be a function that maps 2^E into the set of non-negative integers and satisfies* (R1)-(R3). *Let I be the collection of subsets X of E for which* $\mathrm{rk}(X) = |X|$. *Then (E, I) is a matroid having rank function* rk.

Proof. See [16], Theorem 1.3.2; pag 23. □

Corollary 2.66. *Let E be a set. A function*

$$\mathrm{rk}: \; 2^E \longrightarrow \mathbb{Z}^+$$

is the rank function of a matroid on E if and only if rk *satisfies* (R1)-(R3).

Independent sets, bases and circuits are easily characterized in terms of the rank function:

Proposition 2.67. *Let M be a matroid with rank function* rk *and suppose that $X \subseteq E(M)$. Then:*

i) X is independent if and only if $|X| = \mathrm{rk}(X)$

ii) X is a basis if and only if $|X| = \mathrm{rk}(X) = \mathrm{rk}(M)$

iii) X is a circuit if and only if X is non-empty and, for all $x \in X$,

$$\mathrm{rk}(X \smallsetminus \{x\}) = |X| - 1 = \mathrm{rk}(X)$$

Definition 2.68. Let M be an arbitrary matroid having ground set E and rank function rk. Let cl be the function from 2^E into 2^E defined for all $X \subseteq E$, by

$$\mathrm{cl}(X) = \big\{x \in E \;:\; \mathrm{rk}(X \cup \{x\}) = \mathrm{rk}(X)\big\}.$$

This function is called the *closure operator* of M.

Lemma 2.69. *The closure operator of a matroid on the set E has the following properties:*

(CL1) *If $X \subseteq E$, then $X \subseteq \mathrm{cl}(X)$*

(CL2) *If $X \subseteq Y \subseteq E$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$*

(CL3) *If $X \subseteq E$, then $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$*

(CL4) *If $X \subseteq E$, $x \in E$, and $y \in (\mathrm{cl}(X \cup \{x\}) \smallsetminus \mathrm{cl}(X))$, then $x \in \mathrm{cl}(X \cup \{y\})$*

Proof. See [16], Lemma 1.4.2; pag 28. □

Theorem 2.70. *Let* E *be a set and* cl *be a function from* 2^E *into* 2^E *satisfying* (CL1)-(CL4). *Let*

$$I = \{X \subseteq E \ : \ x \notin \mathrm{cl}(X \smallsetminus \{x\}) \text{ for all } x \in X\}$$

Then (E, I) *is a matroid having closure operator* cl.

Proof. See [16], Theorem 1.4.4; pag 29. □

Corollary 2.71. *Let* E *be a set. A function*

$$\mathrm{cl}: \ 2^E \ \longrightarrow \ 2^E$$

is the closure operator of a matroid on E *if and only if it satisfies* (CL1)-(CL4).

Definition 2.72. If M is a matroid and $X \subseteq E(M)$, we call $\mathrm{cl}(X)$ the *closure* of X in M. If $X = \mathrm{cl}(X)$, then X is called a *flat* of M. A *hyperplane* of M is a flat of rank $(\mathrm{rk}(M) - 1)$.

2.2.3 Duality

In this subsection we define the dual of a matroid.

Theorem 2.73. *Let* M *be a matroid and define*

$$\mathcal{B}^* = \{E(M) \smallsetminus B \ : \ B \in \mathcal{B}(M)\}.$$

Then $\mathcal{B}^*$ *is the set of bases of a matroid on* $E(M)$. *This matroid is called dual matroid and is denoted by* M^*.

Proof. See [16], Theorem 2.1.1; pag 68. □

The bases of M^* are called *cobases* of M. A similar convention applies to other distinguished subsets of $E(M^*)$. Hence, for example, the circuits, hyperplanes, independent set of M^* are called *cocircuits, cohyperplanes, coindipendent sets* of M.

Remark. If Γ is a planar graph, and Γ^* is its dual, then:

$$M(\Gamma^*) = M^*(\Gamma).$$

If Γ is not planar, then the dual graph is not defined, but we still have a dual matroid $M^*(\Gamma)$.

In general, we attach an asterisk to a symbol to denote association with the dual. Thus, for example, rk^* will denote the rank function of M^* while $\mathcal{C}^*$ denotes its set of circuits. Evidently:

$$\mathrm{rk}(M) + \mathrm{rk}^*(M) = |E(M)| \tag{2.2}$$

The next result generalizes Equation (2.2) to give a formula for rk^*, the *corank function* of M.

Lemma 2.74. *Let $M = (E, I)$ be a matroid and $M^* = (E, I^*)$ its dual. Let A be a subset of the ground set E, then:*

$$\mathrm{rk}^*(A) = \mathrm{rk}(A^c) + |A| - \mathrm{rk}(E)$$

Proof. See [16], Proposition 2.1.9; pag 72. □

2.2.4 Lattice of flats

We now examine more closely the structure of the set of flats of a matroid. We shall need some more terminology.

Definition 2.75. A partially ordered set (POSET) is a set X taken together with a partial order on it. Formally, a partially ordered set is defined as an ordered pair $P = (X, \leqslant)$ where X is called the ground set of P and $\leqslant$ the partial order of P.

Definition 2.76. Given two posets $(S, \leqslant_S)$ and $(T, \leqslant_T)$, an *order isomorphism* from $(S, \leqslant_S)$ to $(T, \leqslant_T)$ is a bijective function f from S to T with the property that, for every $x, y \in S$:

$$x \leqslant_S y \iff f(x) \leqslant_T f(y).$$

An order isomorphism from a poset to itself is called order automorphism.

An *upper-bound* of a subset X of a poset P is an element $a \in P$ such that $a \geqslant x$, $\forall x \in X$. An upper-bound b of a subset X is called *least upper-bound* (*join*) if for all upper bounds z of X in P, $z \geqslant b$.
The notions of *lower bound* of X and *greatest lower bound* (*meet*) of X are defined dually.

Definition 2.77. In a poset P an element p *covers* an element q when $\nexists z \in P$ such that:

$$q < z < p.$$

An *atom* in P is an element that covers a minimal element $\widehat{0}$.

A *coatom* in P is an element that is covered by a maximal element $\widehat{1}$.

Definition 2.78. A *lattice* is a poset for which any two elements x and y have a least upper-bound (join) $x \vee y$ and greatest lower bound (meet) $x \wedge y$.

A finite lattice is *semi-modular* if whenever x and y cover $x \wedge y$ (i.e, $\nexists z$ such that $x \wedge y \ < z < x$ or $x \wedge y \ < z < y$), then $x \vee y$ covers both x and y.

A finite lattice is *geometric* if it is semimodular and every element is a join of *atoms* (elements covering $\widehat{0}$).

Definition 2.79. The *Möbius function* μ of a finite lattice L is a function of two lattice-variables which for all $x, y \in L$ satisfies the following properties:

$$\mu(x,y) = \begin{cases} 1, & \text{if } x = y \\ -\sum_{x \leqslant z < y} \mu(x,z), & \text{if } x < y \\ 0, & \text{if } x \nleqslant y \end{cases}$$

We now introduce a poset that will be fundamental for the work of this thesis.

Definition 2.80. Let Π_n denote the poset of all partitions of $[n]$, ordered by refinement. Thus the elements of Π_n are sets:

$$\beta = \{B_1, \ldots, B_k\}$$

where the B_i's are pairwise-disjoint non empty subsets of $[n]$ with union $[n]$. Moreover:

$$\{B'_1, \ldots, B'_j\} \leqslant \{B_1, \ldots, B_k\}$$

if and only if every B'_r is contained in some B_s.

Example 13. Let Π_3 be the partition lattice of $\{1,2,3\}$, then:

$$\Pi_3 = \{1|2|3, 12|3, 13|2, 23|1, 123\}$$

The maximal chains are:

$$a_1 = 1|2|3 \leqslant 12|3 \leqslant 123 \qquad a_2 = 1|2|3 \leqslant 13|2 \leqslant 123 \qquad a_3 = 1|2|3 \leqslant 23|1 \leqslant 123$$

Theorem 2.81. Π_n *is a geometric lattice of rank* $n-1$*.*

Proof. See [3], Theorem 12; pag 95. □

If M is a matroid, then $\mathcal{L}(M)$ will denote the poset of flats of M ordered by inclusion $(\mathcal{L}(M), \subseteq)$.

Lemma 2.82. $(\mathcal{L}(M), \subseteq)$ *is a geometric lattice and, for all flats* X *and* Y *of* M*, we have:*

$$X \wedge Y = X \cap Y \qquad X \vee Y = \mathrm{cl}(X \cup Y)$$

Proof. See [16], Lemma 1.7.3 and Theorem 1.7.5 ; pag 54/55. □

2.2.5 Lattice of flats of the complete graph K_n

Let K_n be the complete graph on n vertices.
A particularly important example of geometric lattice is the lattice of flats of the matroid $M(K_n)$. Let V be the vertex set of K_n. If F is a flat of $M(K_n)$, we denote by π_F the partition of V in which i and j are in the same partition if and only if the edge ij is in F. Conversely, if $\beta \in \Pi_n$ we denote by F_β the flat of $M(K_n)$ in which the edge ij is in F_β if and only if i and j are in the same partition of β.

This determines a map from the set $\mathcal{L}(M(K_n))$ of flats of $M(K_n)$ and the partition lattice Π_n of the n-set V:

$$\begin{array}{rccc} \phi: & \mathcal{L}(M(K_n)) & \longrightarrow & \Pi_n \\ & F & \longmapsto & \pi_F \end{array}$$

Moreover, ϕ is easily shown to be an order isomorphism. For $F_1, F_2 \in \mathcal{L}(M(K_n))$ we have:

$$F_1 \subseteq F_2 \quad \Longleftrightarrow \quad \pi_{F_1} \leqslant \pi_{F_2}$$

where $\leqslant$ indicates the order relationship introduced in Definition 2.80. Let's see an example:

Example 14. Consider K_3:

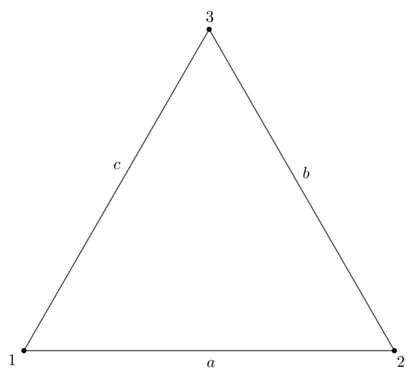


Figure 2.4

Let $M(K_3)$ be the matroid associated to the graph K_3 with:

$$E = \{a, b, c\} \quad I = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$$

$$\mathcal{L}(M(K_3)) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$$

In this case the order isomorphism between $\mathcal{L}(M(K_3))$ and Π_3 is the following:

$$\begin{array}{rccc} \phi: & \mathcal{L}(M(K_3)) & \longrightarrow & \Pi_3 \\ & \emptyset & \longmapsto & 1|2|3 \\ & \{a\} & \longmapsto & 12|3 \\ & \{b\} & \longmapsto & 23|1 \\ & \{c\} & \longmapsto & 13|1 \\ & \{a,b,c\} & \longmapsto & 123 \end{array}$$

2.2.6 Isomorphism between $M(\Phi^+(A_{n-1}), I)$ and $M(K_n)$

Due to the construction described in Section 1.1.2 we have:

$$\Phi^+(A_{n-1}) = \{e_i - e_j \mid i < j\}$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. We also have that:

$$|\Phi^+(A_{n-1})| = \binom{n-1+1}{2} = \frac{n(n-1)}{2}$$

We form a matrix A by placing each element of $\Phi^+(A_{n-1})$ as a column of A:

$$M(\Phi^+(A_{n-1}), I) = M[A]$$

Let K_n be the complete graph on n vertices. We call $E(K_n)$ the set of edges of K_n. Recall that the number of edges in K_n is:

$$|E(K_n)| = \frac{n(n-1)}{2}.$$

We label the vertices of K_n with $\{1, \ldots, n\}$ and we give a direction to each edge of K_n in the following way: if $e \in E(K_n)$ then there exist unique $i, j \in \{1, \ldots, n\}$ such that $e = \overline{ij}$. If $i < j$ let's make i the tail of the edge e and j its head. In this way we have formed a directed graph $D(K_n)$. By construction, up to a permutation of columns, we have that:

$$A = A_{D(K_n)}.$$

By Proposition 2.57 we have that:

$$M(K_n) = M[A_{D(K_n)}].$$

Combining the last two results we have that:

$$M(\Phi^+(A_{n-1}), I) = M(K_n).$$

Example 15. If we consider A_3 embedded in $\mathbb{R}^4$ we claim that:

$$\Phi^+(A_3) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} =$$

$$= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

We now consider K_4, label the vertices by $\{1, 2, 3, 4\}$ and form the directed graph $D(K_4)$ with the orientation discussed before:

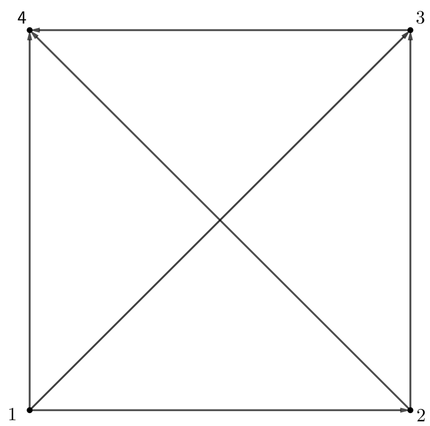


Figure 2.5: $D(K_4)$

$$A_{D(K_4)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix}$$

We note that if we permute the fourth with the sixth column we obtain exactly the matrix A, then we have:

$$M[A] = M[A_{D(K_4)}]$$

2.2.7 The $\mathfrak{S}_n$-action on the vertices of K_n

The Weyl group of A_{n-1} acts on $\Phi_{A_{n-1}}$ by permuting the coordinates of its elements.

We have already seen in Subsection 1.1.2 that each $\sigma \in \mathfrak{S}_n$ induces a permutation τ_σ on $\Phi^+_{A_{n-1}}$ given by considering the action of the Weyl group without the sign. The map τ_σ is a permutation of $E(M(\Phi^+_{A_{n-1}})) = \Phi^+(A_{n-1})$, the ground set of the matroid $M(\Phi^+_{A_{n-1}}, I)$.

Equivalently, in the case of the complete graph K_n, a vertex permutation σ induces a permutation on the edges τ_σ. The map τ_σ is a permutation of $E(M(K_n)) = E(K_n)$, the ground set of $M(K_n)$ where $E(K_n)$ is the set of edges of K_n.

If we want to think of the vector matroid $M[A_{D(K_n)}]$ associated with the complete graph K_n, we see that it is exactly the same as what we saw above with A_{n-1}: a permutation of vertices σ, which coincides with a permutation of coordinates (rows of the matrix), induces a permutation τ_σ of the columns of the matrix without considering the sign of the column vectors. Two column vectors that differ from the sign correspond to the same edge considered with different direction.

We now consider for simplicity $M(\Phi^+_{A_{n-1}}, I)$; some considerations can be expressed in terms of the isomorphic matroid $M(K_n)$.
Considering $IN(M(\Phi^+_{A_{n-1}}))$ as an abstract simplicial complex we have that τ_σ is a vertex map:

$$\tau_\sigma : \quad E(M(\Phi^+_{A_{n-1}})) \quad \longrightarrow \quad E(M(\Phi^+_{A_{n-1}}))$$

The map τ_σ also induces a simplicial map φ_{τ_σ}: since τ_σ is an element of the Weyl group acting on $\Phi^+_{A_{n-1}}$ without considering the sign, we have that a set of linearly independent vectors is mapped into a set of linearly independent vectors of the same cardinality. This implies that τ_σ satisfies the hypotheses of Lemma 2.8 and therefore induces a simplicial map φ_{τ_σ} from $|IN(M(\Phi^+_{A_{n-1}}))|$ to itself. Since τ_σ is a bijective map, Lemma 2.9 implies that the simplicial map φ_{τ_σ} is a simplicial homeomorphism.

We now consider $M^*(\Phi^+_{A_{n-1}}, I)$, the dual matroid of $M(\Phi^+_{A_{n-1}}, I)$, it has the same ground set of $M(\Phi^+_{A_{n-1}}, I)$, and therefore τ_σ is also a vertex map of $IN(M^*(\Phi^+_{A_{n-1}}))$.
Since the τ_σ-action permutes the bases of $M(\Phi^+_{A_{n-1}}, I)$, it will also permutes the bases of $M^*(\Phi^+_{A_{n-1}}, I)$ and therefore τ_σ satisfies the hypothesis of Lemma 2.9 and induces a simplicial map φ'_{τ_σ} from $|IN(M^*(\Phi^+_{A_{n-1}})|$ to itself.

From now on we denote $IN(M^*(\Phi^+_{A_{n-1}}) = IN(M^*(A_{n-1}))$.

As seen in Subsection 2.1.6 the map φ'_{τ_σ} induces a linear map $\varphi'_{\tau_\sigma *}$ on $\widetilde{H}_p(IN(M^*(A_{n-1}))), \mathbb{C})$:

$$\varphi'_{\tau_\sigma *} : \quad \widetilde{H}_p(IN(M^*(A_{n-1})); \mathbb{C}) \quad \longrightarrow \quad \widetilde{H}_p(IN(M^*(A_{n-1})); \mathbb{C}).$$

Working with complex coefficients the $\widetilde{H}_p(IN(M^*(A_{n-1})))$ are $\mathbb{C}$-vector spaces.

We can therefore study the following representations:

$$\begin{array}{rccc} \rho_p: & \mathcal{W} \simeq \mathfrak{S}_n & \longrightarrow & \mathrm{GL}(\widetilde{H}_p(IN(M^*(A_{n-1}))) \\ & \sigma & \longmapsto & \varphi'_{\tau_\sigma *} \end{array}$$

In particular we are interested in the study of representations on the top homology spaces. Let

$$|E| = |E(M^*(A_{n-1}))| = |\Phi^+(A_{n-1})| = \frac{n(n-1)}{2}$$

be the cardinality of the vertex set of $IN(M^*(A_{n-1}))$; in any matroid the top homology space is the $(k-1)$-th where k is the cardinality of a basis. Therefore in $IN(M^*(A_{n-1}))$ the cardinality of a basis is $|E| - (n-1)$ and then

$$|E| - (n-1) - 1 = |E| - n.$$

Therefore the top homology group is the $(|E| - n)$-th. Hence:

$$\begin{array}{rccc} \rho_{|E|-n}: & \mathfrak{S}_n & \longrightarrow & \mathrm{GL}(\widetilde{H}_{|E|-n}(IN(M^*(A_{n-1}))) \\ & \sigma & \longmapsto & \varphi'_{\tau_\sigma *} \end{array}$$

Due to the equivalence seen in Subsection 2.2.6, the above representations coincide with the representations on the top homology of the dual matroid of the complete graph K_n, i.e.

$$\begin{array}{rccc} \rho_{|E|-n}: & \mathfrak{S}_n & \longrightarrow & \mathrm{GL}(\widetilde{H}_{|E|-n}(IN(M^*(K_n))) \\ & \sigma & \longmapsto & \varphi'_{\tau_\sigma *} \end{array}$$

The purpose of this thesis is to study the above representations.

Example 16. Let us consider $M^*(\Phi^+_{A_3}, I) = M(K_4)$ and let us explicitly calculate the representations we want to study. Our purpose is to find the following representation:

$$\rho_2: \ \mathfrak{S}_4 \longrightarrow \mathrm{GL}(\widetilde{H}_2(IN(M^*(A_3)))$$

We have:

$$E(M^*) = \Phi^+_{A_3} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

First we calculate the bases of the matroid $M(\Phi^+_{A_3}, I)$:

$$B_M = \Big\{\{\alpha_1, \alpha_2, \alpha_3\}, \{\alpha_1, \alpha_2, \alpha_2+\alpha_3\}, \{\alpha_1, \alpha_2, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_1, \alpha_1+\alpha_2, \alpha_3\},$$

$$\{\alpha_1, \alpha_1+\alpha_2, \alpha_2+\alpha_3\}, \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_2, \alpha_1+\alpha_2, \alpha_3\},$$

$$\{\alpha_2, \alpha_1+\alpha_2, \alpha_2+\alpha_3\}, \{\alpha_2, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_3, \alpha_2, \alpha_1+\alpha_2+\alpha_3\},$$

$$\{\alpha_3, \alpha_2+\alpha_3, \alpha_1\}, \{\alpha_3, \alpha_2+\alpha_3, \alpha_1+\alpha_2\}, \{\alpha_3, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\},$$

$$\{\alpha_1, \alpha_3, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}\Big\}$$

Now we calculate the bases of the dual matroid $M^*(\Phi^+_{A_3}, I)$:

$$B_{M^*} = \Big\{\{\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_3, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3\},$$

$$\{\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_2, \alpha_3, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_2, \alpha_3, \alpha_2+\alpha_3\},$$

$$\{\alpha_1, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_1, \alpha_3, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_1, \alpha_3, \alpha_2+\alpha_3\}, \{\alpha_1, \alpha_1+\alpha_2, \alpha_2+\alpha_3\}$$

$$\{\alpha_2, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_1, \alpha_2, \alpha_1+\alpha_2+\alpha_3\}, \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\},$$

$$\{\alpha_2, \alpha_1+\alpha_2, \alpha_2+\alpha_3\}, \{\alpha_1, \alpha_2, \alpha_3\}, \{\alpha_1, \alpha_3, \alpha_1+\alpha_2\}\Big\}$$

The independence set I of M^* is composed of all the sets above together with all their subsets. Let's calculate the bases of the two free-modules C_2 and C_1. As for C_2 we just need to order the simplices:

$$\mathcal{B}_{C_2} = \Big\{[v_4, \overset{a_1}{v_5}, v_6], [v_3, \overset{a_2}{v_4}, v_6], [v_3, \overset{a_3}{v_4}, v_5], [v_2, \overset{a_4}{v_5}, v_6], [v_2, \overset{a_5}{v_3}, v_6], [v_2, \overset{a_6}{v_3}, v_5]$$

$$[v_1, \overset{a_7}{v_5}, v_6], [v_1, \overset{a_8}{v_3}, v_6], [v_1, \overset{a_9}{v_3}, v_5], [v_1, \overset{a_{10}}{v_4}, v_5], [v_2, \overset{a_{11}}{v_4}, v_6], [v_1, \overset{a_{12}}{v_2}, v_6], [v_1, \overset{a_{13}}{v_2}, v_4],$$

$$[v_2, \overset{a_{14}}{v_4}, v_5], [v_1, \overset{a_{15}}{v_2}, v_3], [v_1, \overset{a_{16}}{v_3}, v_4]\Big\} = \{a_1, a_2, \ldots, a_{16}\}$$

$$\mathcal{B}_{C_1} = \Big\{[v_1, v_2], [v_1, v_3], [v_1, v_4], [v_1, v_5], [v_1, v_6, [v_2, v_3], [v_2, v_4], [v_2, v_5],$$

$$[v_2, v_6], [v_3, v_4], [v_3, v_5], [v_3, v_6], [v_4, v_5], [v_4, v_6], [v_5, v_6]\Big\}$$

$$C_2 \simeq \mathbb{C}^{16} \qquad C_1 \simeq \mathbb{C}^{15}$$

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \qquad \widetilde{H}_2(IN(M^*(A_3))) = \ker(\partial_2)/\mathrm{Im}(\partial_3) = \ker(\partial_2)$$

$$\partial_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\widetilde{H}_2(IN(M^*(A_3))) = \mathrm{Span}\Big\{-a_1 - a_2 + a_4 - a_5 \overset{b_1}{+} a_6 - a_9 + a_{10} + a_{16}; -a_5 + a_8 \overset{b_2}{-} a_{12} + a_{15};$$
$$-a_1 + a_4 \overset{b_3}{-} a_{11} + a_{14}; -a_1 + a_4 - a_5 + a_6 + a_8 \overset{b_4}{-} a_9 + a_{10} - a_{11} - a_{12} + a_{13};$$
$$-a_4 + a_5 - a_6 \overset{b_5}{+} a_7 - a_8 + a_9; -a_1 - a_2 + a_3 \overset{b_6}{+} a_4 - a_5 + a_6\Big\} =$$
$$= \{b_1, b_2, b_3, b_4, b_5, b_6\}$$

Every σ induces a permutation τ_σ on the vertex set. For example:

$$\begin{array}{rccc} \tau_{(12)}: & C_0 & \longrightarrow & C_0 \\ & v_1 & \longmapsto & v_1 \\ & v_2 & \longmapsto & v_6 \\ & v_3 & \longmapsto & v_3 \\ & v_4 & \longmapsto & v_5 \\ & v_5 & \longmapsto & v_4 \\ & v_6 & \longmapsto & v_2 \end{array} \qquad \begin{array}{rccc} \tau_{(12)(34)}: & C_0 & \longrightarrow & C_0 \\ & v_1 & \longmapsto & v_1 \\ & v_2 & \longmapsto & v_4 \\ & v_3 & \longmapsto & v_3 \\ & v_4 & \longmapsto & v_2 \\ & v_5 & \longmapsto & v_6 \\ & v_6 & \longmapsto & v_5 \end{array}$$

These τ_σ induce linear maps on C_2. In order to calculate ρ_2 we need to see how the representation on C_2 works (See Subsection 2.1.6):

$$\begin{array}{rccc} \alpha_\tau : & \mathfrak{S}_4 & \longrightarrow & \mathrm{GL}(C_2) \\ & (12) & \longmapsto & \alpha_{(12)} \\ & (123) & \longmapsto & \alpha_{(123)} \\ & (1234) & \longmapsto & \alpha_{(1234)} \\ & (12)(34) & \longmapsto & \alpha_{(12)(34)} \end{array}$$

$$\alpha_{(12)}(a_1) = -a_4 \quad \alpha_{(12)}(a_2) = -a_6 \quad \alpha_{(12)}(a_3) = -a_5 \quad \alpha_{(12)}(a_4) = -a_1 \quad \alpha_{(12)}(a_5) = -a_3$$

$$\alpha_{(12)}(a_6) = -a_2 \quad \alpha_{(12)}(a_7) = -a_7 \quad \alpha_{(12)}(a_8) = a_9 \quad \alpha_{(12)}(a_9) = a_8 \quad \alpha_{(12)}(a_{10}) = a_{12}$$

$$\alpha_{(12)}(a_{11}) = -a_{14} \quad \alpha_{(12)}(a_{12}) = a_{10} \quad \alpha_{(12)}(a_{13}) = -a_{13} \quad \alpha_{(12)}(a_{14}) = -a_{11}$$

$$\alpha_{(12)}(a_{15}) = -a_{16} \quad \alpha_{(12)}(a_{16}) = -a_{15}$$

$$\alpha_{(123)}(a_1) = a_9 \quad \alpha_{(123)}(a_2) = a_7 \quad \alpha_{(123)}(a_3) = a_8 \quad \alpha_{(123)}(a_4) = -a_3 \quad \alpha_{(123)}(a_5) = -a_1$$

$$\alpha_{(123)}(a_6) = a_2 \quad \alpha_{(123)}(a_7) = a_6 \quad \alpha_{(123)}(a_8) = -a_4 \quad \alpha_{(123)}(a_9) = -a_5$$

$$\alpha_{(123)}(a_{10}) = -a_{15} \quad \alpha_{(123)}(a_{11}) = -a_{10} \quad \alpha_{(123)}(a_{12}) = a_{14} \quad \alpha_{(123)}(a_{13}) = a_{13}$$

$$\alpha_{(123)}(a_{14}) = a_{16} \quad \alpha_{(123)}(a_{15}) = a_{11} \quad \alpha_{(123)}(a_{16}) = a_{12}$$

$$\alpha_{(1234)}(a_1) = -a_{10} \quad \alpha_{(1234)}(a_2) = -a_7 \quad \alpha_{(1234)}(a_3) = -a_1 \quad \alpha_{(1234)}(a_4) = a_{16}$$

$$\alpha_{(1234)}(a_5) = a_8 \quad \alpha_{(1234)}(a_6) = -a_2 \quad \alpha_{(1234)}(a_7) = a_{13} \quad \alpha_{(1234)}(a_8) = a_{12}$$

$$\alpha_{(1234)}(a_9) = -a_{11} \quad \alpha_{(1234)}(a_{10}) = -a_{14} \quad \alpha_{(1234)}(a_{11}) = a_9 \quad \alpha_{(1234)}(a_{12}) = a_{15}$$

$$\alpha_{(1234)}(a_{13}) = a_6 \quad \alpha_{(1234)}(a_{14}) = -a_3 \quad \alpha_{(1234)}(a_{15}) = a_5 \quad \alpha_{(1234)}(a_{16}) = -a_4$$

$$\alpha_{(12)(34)}(a_1) = -a_{14} \quad \alpha_{(12)(34)}(a_2) = a_6 \quad \alpha_{(12)(34)}(a_3) = -a_3 \quad \alpha_{(12)(34)}(a_4) = -a_{11}$$

$$\alpha_{(12)(34)}(a_5) = -a_5 \quad \alpha_{(12)(34)}(a_6) = a_2 \quad \alpha_{(12)(34)}(a_7) = -a_{13} \quad \alpha_{(12)(34)}(a_8) = -a_{15}$$

$$\alpha_{(12)(34)}(a_9) = a_{16} \quad \alpha_{(12)(34)}(a_{10}) = -a_{10} \quad \alpha_{(12)(34)}(a_{11}) = -a_4 \quad \alpha_{(12)(34)}(a_{12}) = -a_{12}$$

$$\alpha_{(12)(34)}(a_{13}) = -a_7 \quad \alpha_{(12)(34)}(a_{14}) = -a_1 \quad \alpha_{(12)(34)}(a_{15}) = -a_8 \quad \alpha_{(12)(34)}(a_{16}) = a_9$$

$$\begin{array}{rccc}
\rho_2 : & \mathfrak{S}_4 & \longrightarrow & \mathrm{GL}(\widetilde{H}_2(IN(M^*(A_3)))) \\
& \mathrm{Id} & \longmapsto & I_6 \\
& (12) & \longmapsto & \rho_{(12)} \\
& (123) & \longmapsto & \rho_{(123)} \\
& (1234) & \longmapsto & \rho_{(1234)} \\
& (12)(34) & \longmapsto & \rho_{(12)(34)}
\end{array}$$

$$\begin{array}{rccc}
\rho_{(12)} : & \widetilde{H}_2(IN(M^*(A_3))) & \longrightarrow & \widetilde{H}_2(IN(M^*(A_3))) \\
& b_1 & \longmapsto & \text{-}b_2 + b_6 \\
& b_2 & \longmapsto & \text{-}b_1 + b_6 \\
& b_3 & \longmapsto & b_3 \\
& b_4 & \longmapsto & b_3\text{-}b_4 + b_6 \\
& b_5 & \longmapsto & \text{-}b_5\text{-}b_6 \\
& b_6 & \longmapsto & b_6
\end{array}$$

$$\rho_{(12)} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\begin{array}{rccc} \rho_{(123)}: & \widetilde{H}_2(IN(M^*(A_3))) & \longrightarrow & \widetilde{H}_2(IN(M^*(A_3))) \\ & b_1 & \longmapsto & \text{-}b_2\text{-}b_5\text{-}b_6 \\ & b_2 & \longmapsto & \text{-}b_3 \\ & b_3 & \longmapsto & b_1\text{-}b_6 \\ & b_4 & \longmapsto & \text{-}b_2\text{-}b_3 + b_4\text{-}b_6 \\ & b_5 & \longmapsto & b_6 \\ & b_6 & \longmapsto & \text{-}b_5\text{-}b_6 \end{array}$$

$$\rho_{(123)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & -1 & 1 & -1 \end{pmatrix}$$

$$\begin{array}{rccc} \rho_{(1234)}: & \widetilde{H}_2(IN(M^*(A_3))) & \longrightarrow & \widetilde{H}_2(IN(M^*(A_3))) \\ & b_1 & \longmapsto & b_1\text{-}b_3 + b_5 \\ & b_2 & \longmapsto & \text{-}b_2 \\ & b_3 & \longmapsto & b_1\text{-}b_6 \\ & b_4 & \longmapsto & b_1\text{-}b_2\text{-}b_3 \\ & b_5 & \longmapsto & \text{-}b_1 + b_4 \\ & b_6 & \longmapsto & b_1 + b_5 \end{array}$$

$$\rho_{(1234)} = \begin{pmatrix} 1 & 0 & 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{rccc} \rho_{(12)(34)}: & \widetilde{H}_2(IN(M^*(A_3))) & \longrightarrow & \widetilde{H}_2(IN(M^*(A_3))) \\ & b_1 & \longmapsto & \text{-}b_1 + b_3 \\ & b_2 & \longmapsto & \text{-}b_2 \\ & b_3 & \longmapsto & b_3 \\ & b_4 & \longmapsto & \text{-}b_1\text{-}b_2 + b_3\text{-}b_5 \\ & b_5 & \longmapsto & b_1 + b_2\text{-}b_4 \\ & b_6 & \longmapsto & \text{-}b_6 \end{array}$$

$$\rho_{(12)(34)} = \begin{pmatrix} -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{array}{rccc} \chi_{\rho_2}: & \mathfrak{S}_4 & \longrightarrow & \mathbb{C} \\ & \mathrm{Id} & \longmapsto & 6 \\ & (12) & \longmapsto & 0 \\ & (123) & \longmapsto & 0 \\ & (1234) & \longmapsto & 0 \\ & (12)(34) & \longmapsto & -2 \end{array}$$

Comparing this result with Example 5 we note that the representation ρ_2 we found is isomorphic to $\mathrm{Ind}_{C_4}^{\mathfrak{S}_4}(i)$.

In the next two chapters we will generalize this result.

Chapter 3

Representations on the homology of the partition lattice

3.1 On the homology of a poset

Let P be a finite poset. A *chain* is a totally ordered subset of a poset P. The *length* of a finite chain C is

$$l(C) = |C| - 1$$

We assume that P has a unique minimal element $\widehat{0}$, a unique maximal element $\widehat{1}$ and that every maximal chain has the same length n.
Define the rank function:

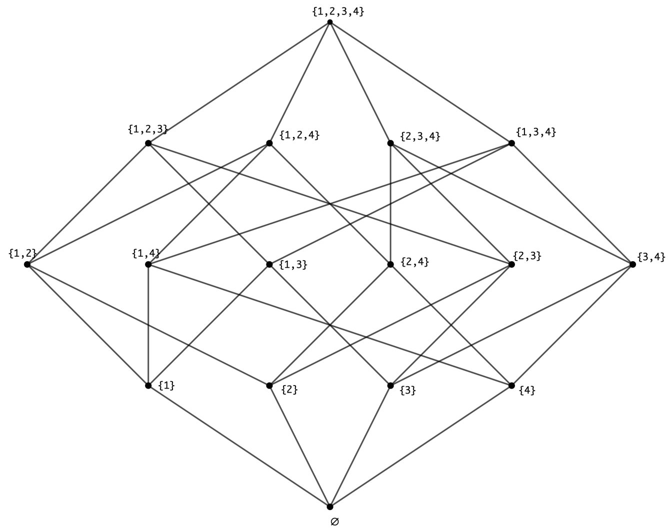
$$r: \quad P \quad \longrightarrow \quad [n] = \{1, 2, \ldots, n\}$$

by setting $r(x)$ equal to the length of any saturated chain in the interval $\left[\widehat{0}, x\right] = \{y \mid \widehat{0} \leqslant y \leqslant x\}$.

Definition 3.1. If $S \subseteq [n-1] = \{1, 2, \ldots, n-1\}$ then define the *rank-selected subposet* P_S of P by:

$$P_S = \{x \in P \mid r(x) \in S\} \cup \{\widehat{0}, \widehat{1}\}$$

Example 17. $\mathcal{P}_4 = (2^{\{1,2,3,4\}}, \subseteq)$, if we take $S = \{1, 2\}$ we obtain:



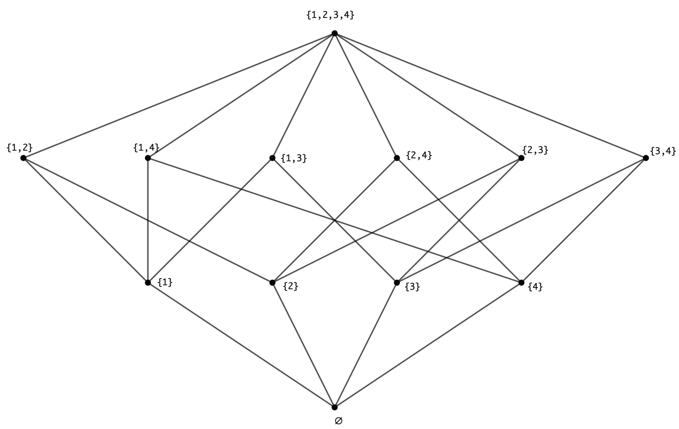


Figure 3.1: $\mathcal{P}_S$ subposet of P

Definition 3.2. Let Q be any poset with $\widehat{0}$ and $\widehat{1}$, then define the *order complex* $\Delta(Q)$ to be the abstract simplicial complex whose vertices are the elements of $\overline{Q} = Q \setminus \{\widehat{0}, \widehat{1}\}$ and whose faces (or simplices) are the chains

$$x_0 < x_1 < \cdots < x_k \qquad \text{in } Q \setminus \{\widehat{0}, \widehat{1}\}$$

Denote by $\widetilde{H}_i(Q)$ the reduced simplicial homology group $\widetilde{H}_i(\Delta(Q), \mathbb{C})$. Recall that for any simplicial complex Δ, $\widetilde{H}_{-1}(\Delta, \mathbb{C}) = 0$ unless $\Delta = \emptyset$, while by definition $\widetilde{H}_{-1}(\emptyset, \mathbb{C}) \simeq \mathbb{C}$ and $\widetilde{H}_{-1}(\emptyset, \mathbb{C}) = 0$ for $i \geqslant 0$.

Now suppose G is a group of order automorphism of P (see Definition 2.76). For any $S \subseteq [n-1]$ G permutes the maximal chain of P_S. Let C_S be the free module over $\mathbb{C}$ on the set of maximal chains of P_S:

$$C_S = \langle a_1, \dots, a_r \rangle \qquad a_i \text{ maximal chains of } P_S$$

Let α_S^P denote the permutation representation of G on C_S:

$$\begin{array}{rccc} \alpha_S^P: & G & \longrightarrow & \mathrm{GL}(C_S) \\ & g & \longmapsto & \alpha_g : C_S \longrightarrow C_S \end{array}$$

Example 18. Consider as before $\mathcal{P}_4 = (\{1,2,3,4\}, \subseteq)$, this time however with $S = \{1\} \subseteq [3]$:

$$P_S = \mathcal{P}_{4_S} = \{\emptyset, \{1,2,3,4\}, \{1\}, \{2\}, \{3\}, \{4\}\}$$

The maximal chains of P_S are:

$$a_1 = \emptyset \subseteq \{1\} \subseteq \{1,2,3,4\} \qquad a_2 = \emptyset \subseteq \{2\} \subseteq \{1,2,3,4\}$$

$$a_3 = \emptyset \subseteq \{3\} \subseteq \{1,2,3,4\} \qquad a_4 = \emptyset \subseteq \{4\} \subseteq \{1,2,3,4\}$$

Let C_S be the free-module over $\mathbb{C}$ having $\{a_1, a_2, a_3, a_4\}$ as basis:

$$C_S = \{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4; \quad \lambda_i \in \mathbb{C}\}$$

G is a group of order automorphisms of $\mathcal{P}_4$, so G is a subgroup of $\mathfrak{S}_4$. Suppose $G = \mathfrak{S}_4$.

$$\begin{array}{rccc} \alpha_S^P: & \mathfrak{S}_4 & \longrightarrow & \mathrm{GL}(C_S) \\ & (123) & \longmapsto & \alpha_{(123)} \\ & (34) & \longmapsto & \alpha_{(34)} \end{array}$$

$$\begin{array}{rccc} \alpha_{(123)}: & C_S & \longrightarrow & C_S \\ & a_1 & \longmapsto & a_2 \\ & a_2 & \longmapsto & a_3 \\ & a_3 & \longmapsto & a_1 \\ & a_4 & \longmapsto & a_4 \end{array} \qquad \alpha_{(123)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rccc} \alpha_{(34)}: & C_S & \longrightarrow & C_S \\ & a_1 & \longmapsto & a_1 \\ & a_2 & \longmapsto & a_2 \\ & a_3 & \longmapsto & a_4 \\ & a_4 & \longmapsto & a_3 \end{array} \qquad \alpha_{(34)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\chi_{\alpha_S^P}((123)) = 1 \qquad \text{Number of maximal chains fixed by } (123)$$

$$\chi_{\alpha_S^P}((34)) = 2 \qquad \text{Number of maximal chains fixed by } (34)$$

As we have seen in the previous example, $\chi_{\alpha_S^P}((g))$ is the number of maximal chains of P_S fixed by g. In particular, $\chi_{\alpha_S^P}((\mathrm{Id}))$ is just the number of maximal chains of P_S.
G also acts on each reduced homology group $\widetilde{H}_i(P_S)$ with $-1 \leqslant i \leqslant |S|-1$ (see Lemma 2.9 and Subsection 2.1.6).
Let $\gamma_{S,i}$ denote this representation of G:

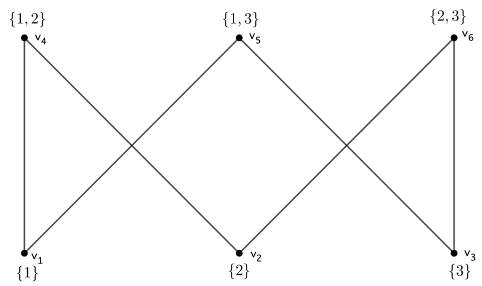
$$\gamma_{S,i}: \ G \ \longrightarrow \ \mathrm{GL}(\widetilde{H}_i(P_S))$$

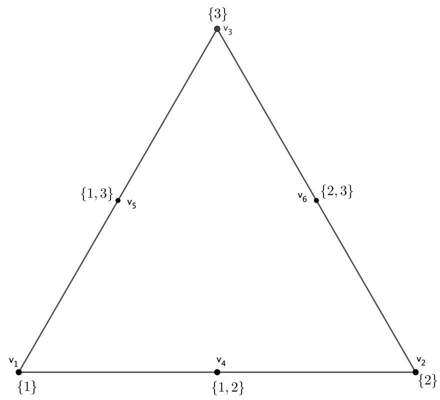
Now define a virtual representation $\beta_S = \beta_S^P$ of G by:

$$\beta_S = \sum_{i=-1}^{|S|-1} (-1)^{|S|-1-i} \, \gamma_{S,i} \tag{3.1}$$

In particular, when $S = \emptyset$ then β_S is the trivial representation, i.e
$\beta_S(g) = \mathrm{Id} \ \ \forall g \in G$.

Example 19. Consider $\mathcal{P}_3 = (2^{\{1,2,3\}}, \subseteq)$ and $S = [2]$. In this case we have that $P_S = \mathcal{P}_3$.
To calculate the homology of $\mathcal{P}_3$ we have to consider $\overline{\mathcal{P}_3}$ and his order complex:





Observing the order complex of $\mathcal{P}_3$ we immediately notice that:

$$\widetilde{H}_i(\mathcal{P}_3) = 0; -1 \leqslant i \leqslant 0 \qquad \text{and} \qquad \widetilde{H}_1(\mathcal{P}_3) \simeq \mathbb{C}$$

We explicitly calculate $\widetilde{H}_1(\mathcal{P}_3)$ to see the action of the group on it.
The vertex set of $\Delta(\mathcal{P}_3)$ is $\{v_1, \dots, v_6\}$. The 1-chains of $\overline{\mathcal{P}_3}$ are:

$$\overline{a_1} = [v_1, v_4] \qquad \overline{a_2} = [v_1, v_5] \qquad \overline{a_3} = [v_2, v_4]$$

$$\overline{a_4} = [v_2, v_6] \qquad \overline{a_5} = [v_3, v_5] \qquad \overline{a_6} = [v_3, v_6]$$

Hence:

$$C_0 = \Big\{ \sum_{i=1}^{6} \lambda_i v_i, \ \lambda_i \in \mathbb{C} \Big\} \qquad C_1 = \Big\{ \sum_{i=1}^{6} \lambda_i \overline{a_i}, \ \lambda_i \in \mathbb{C} \Big\} \qquad C_2 = 0$$

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

$$\begin{array}{rccc} \partial_1 : & C_1 & \longrightarrow & C_0 \\ & \overline{a_1} & \longmapsto & v_4 - v_1 \\ & \overline{a_2} & \longmapsto & v_5 - v_1 \\ & \overline{a_3} & \longmapsto & v_4 - v_2 \\ & \overline{a_4} & \longmapsto & v_6 - v_2 \\ & \overline{a_5} & \longmapsto & v_5 - v_3 \\ & \overline{a_6} & \longmapsto & v_6 - v_3 \end{array} \qquad \partial_1 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\widetilde{H}_1(\mathcal{P}_3) = \ker(\partial_1)/\mathrm{Im}(\partial_2) = \ker(\partial_1) \qquad \ker(\partial_1) = \begin{cases} x_1 = -t \\ x_2 = t \\ x_3 = t \\ x_4 = -t \\ x_5 = -t \\ x_6 = t \end{cases}$$

$$\widetilde{H}_1(\mathcal{P}_3) = \mathrm{Span}\{\underbrace{-[v_1, v_4] + [v_1, v_5] + [v_2, v_4] - [v_2, v_6] - [v_3, v_5] + [v_3, v_6]}_{l}\}$$

Let G be a group of order automorphism of $\mathcal{P}_3$, so G is a subgroup of $\mathfrak{S}_3$. Suppose $G = \mathfrak{S}_3$ and let's calculate $\gamma_{[2],1}$:

$$\begin{array}{rccc} \gamma_{[2],1} : & \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(\widetilde{H}_1(\mathcal{P}_3)) \\ & \mathrm{Id} & \longmapsto & \gamma_{\mathrm{Id}} \\ & (12) & \longmapsto & \gamma_{(12)} \\ & (123) & \longmapsto & \gamma_{(123)} \end{array}$$

$$\begin{array}{rccc} \gamma_{\mathrm{Id}} : & \widetilde{H}_1(\mathcal{P}_3) & \longrightarrow & \widetilde{H}_1(\mathcal{P}_3) \\ & l & \longmapsto & l \end{array} \qquad \begin{array}{rccc} \gamma_{(12)} : & \widetilde{H}_1(\mathcal{P}_3) & \longrightarrow & \widetilde{H}_1(\mathcal{P}_3) \\ & l & \longmapsto & -l \end{array}$$

$$\begin{array}{rccc} \gamma_{(123)} : & \widetilde{H}_1(\mathcal{P}_3) & \longrightarrow & \widetilde{H}_1(\mathcal{P}_3) \\ & l & \longmapsto & l \end{array}$$

$\gamma_{[2],1}$ is isomorphic to the sign representation:

$$\begin{array}{rccc} \rho: & \mathfrak{S}_3 & \longrightarrow & \mathbb{C}^* \\ & g & \longmapsto & \operatorname{sgn}(g) \end{array}$$

And in this case we have $\beta_{[2]} = \gamma_{[2],1}$.

To be able to enunciate the next theorem we need two results due to Baclawsky and Björner; we begin by setting some notation.

Definition 3.3. Given a poset P and a order automorphism f, we write P^f for the *fixed point set*:

$$P^f = \{x \in P \mid x = f(x)\}$$

Definition 3.4. Let P be a poset and let $\epsilon_i(P)$ be the number of i-chains of $\overline{P} = P \setminus \{\widehat{0}, \widehat{1}\}$. The Euler-characteristic $\mathcal{E}(P)$ is defined by:

$$\mathcal{E}(P) = \sum_{i=0}^{+\infty} (-1)^i \, \epsilon_i(P)$$

In particular $\mathcal{E}(\emptyset) = 0$.

The well known Euler-Poincaré formula states that

$$\mathcal{E}(P) = \sum_{n=0}^{+\infty} (-1)^n \dim_{\mathbb{C}} H_n(P, \mathbb{C})$$

We can also introduce the definition of reduced Euler characteristic:

$$\widetilde{\mathcal{E}}(P) = \sum_{n=-1}^{+\infty} (-1)^n \dim_{\mathbb{C}} \widetilde{H}_n(P, \mathbb{C})$$

It is easy to see that:

$$\mathcal{E}(P) = \widetilde{\mathcal{E}}(P) + 1$$

It is a theorem of P. Hall ([17], Prop 6, pag. 346) that:

$$\mathcal{E}(P) = \mu(P) + 1, \quad \text{i.e.} \quad \widetilde{\mathcal{E}}(P) = \mu(P) \tag{3.2}$$

with $\mu(P) = \mu\big(\widehat{0}, \widehat{1}\big)$ the Möbius function of P.

Definition 3.5 (Lefschetz Number)**.** Let P be a finite poset. For an order automorphism f of P let:

$$f_n : \ H_n(P,\mathbb{C}) \ \longrightarrow \ H_n(P,\mathbb{C}) \qquad \widetilde{f}_n : \ \widetilde{H}_n(P,\mathbb{C}) \ \longrightarrow \ \widetilde{H}_n(P,\mathbb{C})$$

be the linear maps which are induced on homology and reduced homology respectively. The *Lefschetz number* of f is:

$$\Lambda(f) = \sum_{n=0}^{+\infty} \ (-1)^n \ \mathrm{Tr}(f_n)$$

and the *reduced Lefschetz number* of f is:

$$\widetilde{\Lambda}(f) = \sum_{n=-1}^{+\infty} \ (-1)^n \ \mathrm{Tr}(\widetilde{f}_n)$$

Theorem 3.6 (Hopf-Lefschetz fixed point theorem)**.** *Let P be a finite poset and let f be an order automorphism of P. Then:*

$$\Lambda(f) = \mathcal{E}(P^f) \qquad \widetilde{\Lambda}(f) = \widetilde{\mathcal{E}}(P^f)$$

In particular, if $\Lambda(f) \neq 0$, then $P^f \neq \emptyset$.

Proof. See [2], Theorem 1.1; pag. 265.
The result is stated for ordinary simplicial homology, but the proof works just as well for reduced simplicial homology. □

Now we can state the following theorem:

Theorem 3.7. *The representation α_S and the virtual representation β_S are related by the formulas:*

$$\alpha_S = \sum_{T \subseteq S} \ \beta_T \tag{3.3}$$

$$\beta_S = \sum_{T \subseteq S} \ (-1)^{|S \setminus T|} \ \alpha_T \tag{3.4}$$

Proof. Let P be a finite poset and let G be a group of order automorphism of P. Let $\widetilde{\Lambda}_S(g)$ be the Lefschetz number of the map $g \in G$ working in P_S subposet of P:

$$\widetilde{\Lambda}_S(g) = \sum_{n=-1}^{+\infty} (-1)^n \operatorname{Tr}(\widetilde{g}_n) \qquad \widetilde{g}_n : \widetilde{H}_n(P_S, \mathbb{C}) \longrightarrow \widetilde{H}_n(P_S, \mathbb{C})$$

Recall that:

$$\beta_S = \sum_{n=-1}^{|S|-1} (-1)^{|S|-1-n} \gamma_{S,n} \qquad \begin{array}{rccc} \gamma_{S,n} : & G & \longrightarrow & \mathrm{GL}(\widetilde{H}_n(P_S)) \\ & g & \longmapsto & \widetilde{g}_n \end{array}$$

The character of the virtual representation β_S is:

$$\chi_{\beta_S}(g) = \sum_{n=-1}^{|S|-1} (-1)^{|S|-1-n} \chi_{\gamma_{S,n}}(g) = \sum_{n=-1}^{|S|-1} (-1)^{|S|-1-n} \operatorname{Tr}(\widetilde{g}_n) =$$

$$= \sum_{n=-1}^{|S|-1} (-1)^{|S|-1} (-1)^{-n} \operatorname{Tr}(\widetilde{g}_n) = (-1)^{1-|S|} \sum_{n=-1}^{|S|-1} (-1)^n \operatorname{Tr}(\widetilde{g}_n)$$

Since $\widetilde{H}_n(P_S, \mathbb{C}) = 0$ for all $n > |S| - 1$, we have that:

$$\widetilde{\Lambda}_S(g) = \sum_{n=-1}^{|S|-1} (-1)^n \operatorname{Tr}(\widetilde{g}_n)$$

But as far as we see before we get:

$$\chi_{\beta_S}(g) = (-1)^{1-|S|} \widetilde{\Lambda}_S(g) \qquad \widetilde{\Lambda}_S(g) = (-1)^{|S|-1} \chi_{\beta_S}(g) \tag{3.5}$$

Let $\widetilde{\mathcal{E}}(P_S^g)$ be the reduced Euler-characteristic of the subposet P_S^g of P_S. By applying Theorem 3.6 to the poset P_S we have that:

$$\widetilde{\Lambda}_S(g) = \widetilde{\mathcal{E}}(P_S^g) \tag{3.6}$$

By definition of the Euler characteristic, recalling that $\chi_{\alpha_S}(g)$ is the number of maximal chains of P_S fixed by g, we claim that:

$$\widetilde{\mathcal{E}}(P_S^g) = \sum_{T \subseteq S} (-1)^{|T|-1} \chi_{\alpha_T}(g)$$

Hence for (3.6):

$$(-1)^{|S|-1}\ \chi_{\beta_S}(g) = \sum_{T \subseteq S} (-1)^{|T|-1}\ \chi_{\alpha_T}(g)$$

$$\chi_{\beta_S}(g) = \sum_{T \subseteq S} (-1)^{|T|-|S|}\ \chi_{\alpha_T}(g) = \sum_{T \subseteq S} (-1)^{|S \smallsetminus T|}\ \chi_{\alpha_T}(g)$$

for all $g \in G$, so Equation (3.4) follows. For obtaining Equation (3.3), it suffices to apply the Inclusion-Exclusion principle. □

Example 20. Consider $\mathcal{P}_3 = (2^{\{1,2,3\}}, \subseteq)$. We want to calculate explicitly β_S with the new characterization provided by Theorem 3.7 and compare the result with that of the previous example. Since length$(P) = 3$, we take $S = \{1, 2\} \subseteq [2]$.

In this case we have that $P_S = \mathcal{P}_3$. We want to calculate:

$$\beta_{[2]} = \sum_{T \subseteq [2]} (-1)^{|[2] \smallsetminus T|}\ \ \alpha_T$$

i) For the first element of the sum let us consider $T = S = [2]$.

The maximal chains of P_S are:

$$a_1 = \emptyset \subseteq \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \qquad a_2 = \emptyset \subseteq \{1\} \subseteq \{1,3\} \subseteq \{1,2,3\}$$

$$a_3 = \emptyset \subseteq \{2\} \subseteq \{1,2\} \subseteq \{1,2,3\} \qquad a_4 = \emptyset \subseteq \{2\} \subseteq \{2,3\} \subseteq \{1,2,3\}$$

$$a_5 = \emptyset \subseteq \{3\} \subseteq \{1,3\} \subseteq \{1,2,3\} \qquad a_6 = \emptyset \subseteq \{3\} \subseteq \{2,3\} \subseteq \{1,2,3\}$$

Let C_S be the free-module over $\mathbb{C}$ with $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ as basis:

$$C_S = \{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 + \lambda_5 a_5 + \lambda_6 a_6;\quad \lambda_i \in \mathbb{C}\}$$

Suppose $G = \mathfrak{S}_3$.

$$\begin{array}{rccc} \alpha_S : & \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(C_S) \\ & \mathrm{Id} & \longmapsto & I_6 \\ & (12) & \longmapsto & \alpha^S_{(12)} \\ & (123) & \longmapsto & \alpha^S_{(123)} \end{array}$$

$$\begin{array}{rccc} \alpha^S_{(12)}: & C_S & \longrightarrow & C_S \\ & a_1 & \longmapsto & a_3 \\ & a_2 & \longmapsto & a_4 \\ & a_3 & \longmapsto & a_1 \\ & a_4 & \longmapsto & a_2 \\ & a_5 & \longmapsto & a_6 \\ & a_6 & \longmapsto & a_5 \end{array} \qquad \alpha^S_{(12)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{rccc} \alpha^S_{(123)}: & C_S & \longrightarrow & C_S \\ & a_1 & \longmapsto & a_4 \\ & a_2 & \longmapsto & a_3 \\ & a_3 & \longmapsto & a_6 \\ & a_4 & \longmapsto & a_5 \\ & a_5 & \longmapsto & a_1 \\ & a_6 & \longmapsto & a_2 \end{array} \qquad \alpha^S_{(123)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

It turns out that α_S is the regular representation of $\mathfrak{S}_3$.

ii) We now consider $T = T_1 = \emptyset \subseteq S$.
The only maximal chain of P_{T_1} is:

$$b_1 = \emptyset \subseteq \{1, 2, 3\}$$

Let C_{T_1} be the free-module over $\mathbb{C}$ with $\{b_1\}$ as basis:

$$C_{T_1} = \{\lambda_1 b_1;\quad \lambda_i \in \mathbb{C}\} \qquad \begin{array}{rccc} \alpha_{T_1}: & \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(C_{T_1}) \\ & g & \longmapsto & 1 \end{array}$$

iii) Let $T = T_2 = \{1\} \subseteq S$.

The maximal chains of P_{T_2} are:

$$c_1 = \emptyset \subseteq \{1\} \subseteq \{1, 2, 3\} \quad c_2 = \emptyset \subseteq \{2\} \subseteq \{1, 2, 3\} \quad c_3 = \emptyset \subseteq \{3\} \subseteq \{1, 2, 3\}$$

Let C_{T_2} be the free-module over $\mathbb{C}$ with $\{c_1, c_2, c_3\}$ as basis:

$$C_{T_2} = \{\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3; \quad \lambda_i \in \mathbb{C}\}$$

$$\begin{array}{rccc} \alpha_{T_2} : & \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(C_{T_2}) \\ & \mathrm{Id} & \longmapsto & I_3 \\ & (12) & \longmapsto & \alpha^{T_2}_{(12)} \\ & (123) & \longmapsto & \alpha^{T_2}_{(123)} \end{array}$$

$$\begin{array}{rccc} \alpha^{T_2}_{(12)} : & C_{T_2} & \longrightarrow & C_{T_2} \\ & c_1 & \longmapsto & c_2 \\ & c_2 & \longmapsto & c_1 \\ & c_3 & \longmapsto & c_3 \end{array} \qquad \alpha^{T_2}_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rccc} \alpha^{T_2}_{(123)} : & C_{T_2} & \longrightarrow & C_{T_2} \\ & c_1 & \longmapsto & c_2 \\ & c_2 & \longmapsto & c_3 \\ & c_3 & \longmapsto & c_1 \end{array} \qquad \alpha^{T_2}_{(123)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

iv) Let $T = T_3 = \{2\} \subseteq S$.

The maximal chains of P_{T_3} are:

$$d_1 = \emptyset \subseteq \{1,2\} \subseteq \{1,2,3\} \qquad d_2 = \emptyset \subseteq \{1,3\} \subseteq \{1,2,3\}$$

$$d_3 = \emptyset \subseteq \{2,3\} \subseteq \{1,2,3\}$$

Let C_{T_3} be the free-module over $\mathbb{C}$ with $\{d_1, d_2, d_3\}$ as basis:

$$C_{T_3} = \{\lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3; \quad \lambda_i \in \mathbb{C}\}$$

$$\begin{array}{rccc} \alpha_{T_3} : & \mathfrak{S}_3 & \longrightarrow & \mathrm{GL}(C_{T_3}) \\ & \mathrm{Id} & \longmapsto & I_3 \\ & (12) & \longmapsto & \alpha^{T_3}_{(12)} \\ & (123) & \longmapsto & \alpha^{T_3}_{(123)} \end{array}$$

$$\begin{array}{rcccl} \alpha^{T_3}_{(12)}: & C_{T_3} & \longrightarrow & C_{T_3} \\ & d_1 & \longmapsto & d_1 \\ & d_2 & \longmapsto & d_3 \\ & d_3 & \longmapsto & d_2 \end{array} \qquad \alpha^{T_3}_{(12)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{rcccl} \alpha^{T_3}_{(123)}: & C_{T_3} & \longrightarrow & C_{T_3} \\ & d_1 & \longmapsto & d_3 \\ & d_2 & \longmapsto & d_1 \\ & d_3 & \longmapsto & d_2 \end{array} \qquad \alpha^{T_3}_{(123)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Now we can calculate the character of the representation $\beta_{[2]}$:

$$\beta_{[2]} = +\alpha_{T_1} - \alpha_{T_2} - \alpha_{T_3} + \alpha_S$$

$$\chi_{\beta_{[2]}}(\mathrm{Id}) = \chi_{\alpha_{T_1}}(\mathrm{Id}) - \chi_{\alpha_{T_2}}(\mathrm{Id}) - \chi_{\alpha_{T_3}}(\mathrm{Id}) + \chi_{\alpha_S}(\mathrm{Id}) = 1 - 3 - 3 + 6 = 1$$

$$\chi_{\beta_{[2]}}((12)) = 1 - 1 - 1 + 0 = -1$$

$$\chi_{\beta_{[2]}}((123)) = 1 - 0 - 0 + 0 = 1$$

$\beta_{[2]}$ is isomorphic to the sign representation:

$$\begin{array}{rccc} \rho: & \mathfrak{S}_3 & \longrightarrow & \mathbb{C}^* \\ & g & \longmapsto & \mathrm{sgn}(g) \end{array}$$

The result is consistent with that of the previous example. In this case, compute $\beta_{[2]}$ coincides with the calculation of $\gamma_{[2],1}$.

We try to generalize this result for some types of poset:

Definition 3.8. Define an arbitrary finite poset P with $\widehat{0}$ and $\widehat{1}$ to be *Cohen-Macaulay* (over $\mathbb{C}$) if for every interval $I = [x, y] = \{z \ : \ x \leqslant z \leqslant y\}$ of P we have:

$$\widetilde{H}_i(I) = 0 \qquad \text{whenever } i \neq \dim\,\Delta(I)$$

Theorem 3.9. *If P is a Cohen-Macaulay poset of rank n and if $S \subseteq [n-1]$, then P_S is also Cohen-Macaulay.*

Proof. See [1], Theorem 6.4; pag 247. □

Let P be a Cohen-Macaulay poset with $\widehat{0}$ and $\widehat{1}$, it follows from (3.1), that:

$$\beta_S = \gamma_{S,s-1} \qquad s = |S|$$

In other words:

Theorem 3.10. *If P is a Cohen-Macaulay poset then β_S is isomorphic to the natural representation $\gamma_{S,s-1}$ of G on the top reduced homology group $\widetilde{H}_{s-1}(P_S)$.*

3.2 On the homology of the partition lattice Π_n

Let Π_n denote the poset of all partitions of $[n]$, ordered by refinement.

Proposition 3.11. *Let μ be the Möbius function of the lattice of partitions Π_n, then:*

$$\mu(\widehat{0},\widehat{1}) = (-1)^{n-1}\ (n-1)!$$

Proof. See [17], Proposition 3; pag 359. □

Now we report a result due to Folkman which applies to any geometric lattice:

Theorem 3.12. *Let L be a geometric lattice of rank r and let μ denote the Möbius function of L. Let $F(k)$ denote the free $\mathbb{C}$-module of dimension k. Then:*

$$\widetilde{H}_i(L) \simeq \begin{cases} F(|\mu(\widehat{0},\widehat{1})|), & \text{if } i = r-2 \\ 0, & \text{if } i \neq r-2 \end{cases}$$

Proof. See [9], Theorem 4.1; pag 634. □

From the previous theorem it follows that every geometric lattice L is a Cohen-Macaulay poset and then the only non-zero reduced homology group of L is the $(r-2)$-th. Hence, Π_n is a Cohen-Macaulay poset.
The symmetric group $\mathfrak{S}_n$ acts as an order automorphism group on Π_n by permuting the letters of the partitions. For example, the transposition (12) acting on the partition $13|2|4$ yields the partition $23|1|4$. Let's see an example of how (12) acts on a chain:

$$1|2|3|4 < 23|1|4 < 234|1 < 1234 \quad \longmapsto \quad 1|2|3|4 < 13|2|4 < 134|2 < 1234$$

Our aim is to study the representation γ_{n-3} of $\mathfrak{S}_n$ on the top homology group $\widetilde{H}_{n-3}(\Pi_n)$ of Π_n .

If we take $S = [n-2]$, we obtain from Theorem 3.10:

$$(\Pi_n)_S = \{x \in \Pi_n \mid x = \widehat{0} \ \vee \ x = \widehat{1} \ \vee \ r(x) \in S\} = \Pi_n$$

$$\beta_S = \beta_{[n-2]} = \sum_{i=-1}^{n-3} (-1)^{n-3-i} \ \gamma_{S,i} = \gamma_{S,n-3} = \gamma_{n-3}$$

So in the case of the partition lattice the representation we are looking for is $\beta_{[n-2]} = \gamma_{n-3}$:

$$\beta_{[n-2]} = \gamma_{n-3} : \ \ \mathfrak{S}_n \ \longrightarrow \ \mathrm{GL}(\widetilde{H}_{n-3}(\Pi_n))$$

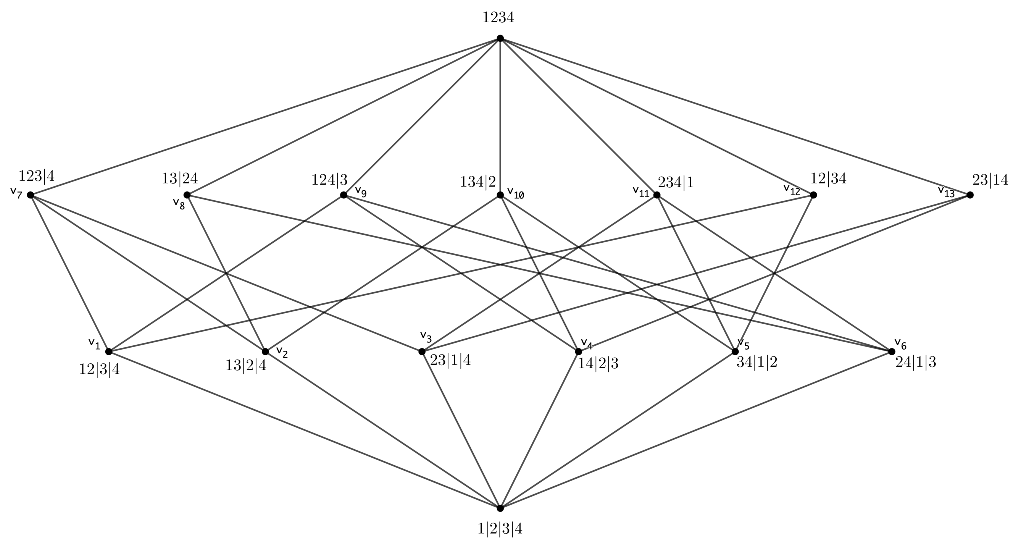
By Theorem 3.12 and Proposition 3.11 we have that:

$$\dim(\widetilde{H}_{n-3}(\Pi_n)) = |\mu(\widehat{0}, \widehat{1})| = |(-1)^{n-1} \ (n-1)!| = (n-1)!$$

Thus, $\beta_{[n-2]}$ is a representation of $\mathfrak{S}_n$ of degree $(n-1)!$.
Before describing $\beta_{[n-2]}$ more explicitly, let's make an example:

Example 21. Consider the partition lattice Π_4 of [4] with every maximal chain of length 3 and with $S = [2]$, we always consider S maximal since we want that:

$$P_S = P = \Pi_4$$



$$\mu(\Pi_4) = \mu(\widehat{0}, \widehat{1}) = -3! = -6 \qquad \widetilde{H}_1(\Pi_4) \simeq \mathbb{C}^6$$

$$C_1 \simeq \mathbb{C}^{18} \qquad C_0 \simeq \mathbb{C}^{13}$$

$$\mathcal{B}_{C_1} = \Big\{ \overset{a_1}{[v_1, v_7]}, \overset{a_2}{[v_1, v_9]}, \overset{a_3}{[v_1, v_{12}]}, \overset{a_4}{[v_2, v_7]}, \overset{a_5}{[v_2, v_8]}, \overset{a_6}{[v_2, v_{10}]}, \overset{a_7}{[v_3, v_7]}, \overset{a_8}{[v_3, v_{11}]}, \overset{a_9}{[v_3, v_{13}]},$$

$$, \overset{a_{10}}{[v_4, v_9]}, \overset{a_{11}}{[v_4, v_{10}]}, \overset{a_{12}}{[v_4, v_{13}]}, \overset{a_{13}}{[v_5, v_{10}]}, \overset{a_{14}}{[v_5, v_{11}]}, \overset{a_{15}}{[v_5, v_{12}]}, \overset{a_{16}}{[v_6, v_8]}, \overset{a_{17}}{[v_6, v_9]}, \overset{a_{18}}{[v_6, v_{11}]} \Big\}$$

$$\mathcal{B}_{C_0} = \Big\{ v_1, v_2, \ldots, v_{13} \Big\}$$

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \qquad \widetilde{H}_1(\Pi_4) = \ker(\partial_1)/\mathrm{Im}(\partial_2) = \ker(\partial_1)$$

$$\partial_1 = \begin{pmatrix}
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\begin{aligned}
\widetilde{H}_1(\Pi_4) = \mathrm{Span}\{ & -a_4 + a_5 + a_7 \overset{b_1}{-} a_8 - a_{16} + a_{18}; -a_4 + a_5 + a_7 - a_9 \overset{b_2}{-} a_{10} + a_{12} - a_{16} + a_{17}; \\
& ; -a_4 + a_6 + a_7 \overset{b_3}{-} a_9 - a_{11} + a_{12}; a_2 - a_3 - a_{10} \overset{b_4}{+} a_{11} - a_{13} + a_{15}; \\
& ; -a_8 + a_9 + a_{11} \overset{b_5}{-} a_{12} - a_{13} + a_{14}; a_1 - a_2 - a_7 \overset{b_6}{+} a_9 + a_{10} - a_{12}\} = \\
& = \{b_1, b_2, b_3, b_4, b_5, b_6\}
\end{aligned}$$

We indicate with $\overline{C_S}$ the free-module over $\mathbb{C}$ on the set of maximal chains of $P_S = \Pi_4$ without the minimal and maximal element. Thus,

$$\overline{C_S} = \overline{C_{[2]}} = C_1$$

Before we can calculate γ_1 we need to see how $\alpha = \alpha_{[2]}$ works on C_1:

$$\begin{aligned}
\alpha : \quad \mathfrak{S}_4 & \longrightarrow \mathrm{GL}(C_1) \\
(12) & \longmapsto \alpha_{(12)} \\
(123) & \longmapsto \alpha_{(123)} \\
(1234) & \longmapsto \alpha_{(1234)} \\
(12)(34) & \longmapsto \alpha_{(12)(34)}
\end{aligned}$$

$$\alpha_{(12)}(a_1) = a_1 \quad \alpha_{(12)}(a_2) = a_2 \quad \alpha_{(12)}(a_3) = a_3 \quad \alpha_{(12)}(a_4) = a_7 \quad \alpha_{(12)}(a_5) = a_9$$

$$\alpha_{(12)}(a_6) = a_8 \quad \alpha_{(12)}(a_7) = a_4 \quad \alpha_{(12)}(a_8) = a_6 \quad \alpha_{(12)}(a_9) = a_5 \quad \alpha_{(12)}(a_{10}) = a_{17}$$

$$\alpha_{(12)}(a_{11}) = a_{18} \quad \alpha_{(12)}(a_{12}) = a_{16} \quad \alpha_{(12)}(a_{13}) = a_{14} \quad \alpha_{(12)}(a_{14}) = a_{13}$$

$$\alpha_{(12)}(a_{15}) = a_{15} \quad \alpha_{(12)}(a_{16}) = a_{12} \quad \alpha_{(12)}(a_{17}) = a_{10} \quad \alpha_{(12)}(a_{18}) = a_{11}$$

$$\alpha_{(123)}(a_1) = a_7 \quad \alpha_{(123)}(a_2) = a_8 \quad \alpha_{(123)}(a_3) = a_9 \quad \alpha_{(123)}(a_4) = a_1 \quad \alpha_{(123)}(a_5) = a_3$$

$$\alpha_{(123)}(a_6) = a_2 \quad \alpha_{(123)}(a_7) = a_4 \quad \alpha_{(123)}(a_8) = a_6 \quad \alpha_{(123)}(a_9) = a_5 \quad \alpha_{(123)}(a_{10}) = a_{18}$$

$$\alpha_{(123)}(a_{11}) = a_{17} \quad \alpha_{(123)}(a_{12}) = a_{16} \quad \alpha_{(123)}(a_{13}) = a_{10} \quad \alpha_{(123)}(a_{14}) = a_{11}$$

$$\alpha_{(123)}(a_{15}) = a_{12} \quad \alpha_{(123)}(a_{16}) = a_{15} \quad \alpha_{(123)}(a_{17}) = a_{14} \quad \alpha_{(123)}(a_{18}) = a_{13}$$

$$\alpha_{(1234)}(a_1) = a_8 \quad \alpha_{(1234)}(a_2) = a_7 \quad \alpha_{(1234)}(a_3) = a_9 \quad \alpha_{(1234)}(a_4) = a_{18} \quad \alpha_{(1234)}(a_5) = a_{16}$$

$$\alpha_{(1234)}(a_6) = a_{17} \quad \alpha_{(1234)}(a_7) = a_{14} \quad \alpha_{(1234)}(a_8) = a_{13} \quad \alpha_{(1234)}(a_9) = a_{15} \quad \alpha_{(1234)}(a_{10}) = a_1$$

$$\alpha_{(1234)}(a_{11}) = a_2 \quad \alpha_{(1234)}(a_{12}) = a_3 \quad \alpha_{(1234)}(a_{13}) = a_{10} \quad \alpha_{(1234)}(a_{14}) = a_{11}$$

$$\alpha_{(1234)}(a_{15}) = a_{12} \quad \alpha_{(1234)}(a_{16}) = a_5 \quad \alpha_{(1234)}(a_{17}) = a_4 \quad \alpha_{(1234)}(a_{18}) = a_6$$

$$\alpha_{(12)(34)}(a_1) = a_8 \quad \alpha_{(12)(34)}(a_2) = a_7 \quad \alpha_{(12)(34)}(a_3) = a_9 \quad \alpha_{(12)(34)}(a_4) = a_{18}$$

$$\alpha_{(12)(34)}(a_5) = a_{16} \quad \alpha_{(12)(34)}(a_6) = a_{17} \quad \alpha_{(12)(34)}(a_7) = a_{14} \quad \alpha_{(12)(34)}(a_8) = a_{13}$$

$$\alpha_{(12)(34)}(a_9) = a_{15} \quad \alpha_{(12)(34)}(a_{10}) = a_1 \quad \alpha_{(12)(34)}(a_{11}) = a_2 \quad \alpha_{(12)(34)}(a_{12}) = a_3$$

$$\alpha_{(12)(34)}(a_{13}) = a_{10} \quad \alpha_{(12)(34)}(a_{14}) = a_{11} \quad \alpha_{(12)(34)}(a_{15}) = a_{12}$$

$$\alpha_{(12)(34)}(a_{16}) = a_5 \quad \alpha_{(12)(34)}(a_{17}) = a_4 \quad \alpha_{(12)(34)}(a_{18}) = a_6$$

$$\begin{array}{rccc} \gamma_1 : & \mathfrak{S}_4 & \longrightarrow & \mathrm{GL}(\widetilde{H}_1(\Pi_4)) \\ & \mathrm{Id} & \longmapsto & I_6 \\ & (12) & \longmapsto & \gamma^1_{(12)} \\ & (123) & \longmapsto & \gamma^1_{(123)} \\ & (1234) & \longmapsto & \gamma^1_{(1234)} \\ & (12)(34) & \longmapsto & \gamma^1_{(12)(34)} \end{array}$$

$$\begin{array}{rccc} \gamma^1_{(12)} : & \widetilde{H}_1(\Pi_4) & \longrightarrow & \widetilde{H}_1(\Pi_4) \\ & b_1 & \longmapsto & \text{-}b_3 \\ & b_2 & \longmapsto & \text{-}b_2 \\ & b_3 & \longmapsto & \text{-}b_1 \\ & b_4 & \longmapsto & b_1\text{-}b_2 + b_4\text{-}b_5 \\ & b_5 & \longmapsto & b_1\text{-}b_3\text{-}b_5 \\ & b_6 & \longmapsto & b_2 + b_6 \end{array} \qquad \gamma^1_{(12)} = \begin{pmatrix} 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rccc} \gamma^1_{(123)} : & \widetilde{H}_1(\Pi_4) & \longrightarrow & \widetilde{H}_1(\Pi_4) \\ & b_1 & \longmapsto & \text{-}b_3\text{-}b_4\text{-}b_6 \\ & b_2 & \longmapsto & \text{-}b_1\text{-}b_4 + b_5\text{-}b_6 \\ & b_3 & \longmapsto & \text{-}b_2\text{-}b_6 \\ & b_4 & \longmapsto & \text{-}b_1 + b_2 \\ & b_5 & \longmapsto & b_2\text{-}b_3 \\ & b_6 & \longmapsto & b_1 \end{array} \qquad \gamma^1_{(123)} = \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{rccc} \gamma^1_{(1234)} : & \widetilde{H}_1(\Pi_4) & \longrightarrow & \widetilde{H}_1(\Pi_4) \\ & b_1 & \longmapsto & \text{-}b_1 + b_3 + b_5 \\ & b_2 & \longmapsto & \text{-}b_1\text{-}b_4 + b_5\text{-}b_6 \\ & b_3 & \longmapsto & \text{-}b_1 + b_2\text{-}b_4 + b_5 \\ & b_4 & \longmapsto & \text{-}b_6 \\ & b_5 & \longmapsto & b_4 \\ & b_6 & \longmapsto & b_4\text{-}b_5 + b_6 \end{array} \qquad \gamma^1_{(1234)} = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rccc} \gamma^1_{(12)(34)}: & \widetilde{H}_1(\Pi_4) & \longrightarrow & \widetilde{H}_1(\Pi_4) \\ & b_1 & \longmapsto & \text{-}b_2 + b_3 \\ & b_2 & \longmapsto & \text{-}b_2 \\ & b_3 & \longmapsto & b_1\text{-}b_2 \\ & b_4 & \longmapsto & b_4\text{-}b_5 + b_6 \\ & b_5 & \longmapsto & \text{-}b_5 \\ & b_6 & \longmapsto & \text{-}b_6 \end{array} \qquad \gamma^1_{(12)(34)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

$$\begin{array}{rccc} \chi_{\gamma_1}: & \mathfrak{S}_4 & \longrightarrow & \mathbb{C} \\ & \text{Id} & \longmapsto & 6 \\ & (12) & \longmapsto & 0 \\ & (123) & \longmapsto & 0 \\ & (1234) & \longmapsto & 0 \\ & (12)(34) & \longmapsto & -2 \end{array}$$

To do less calculations you could directly find β_S using Theorem 3.7. Comparing this result with the Example 5, we note that the representation γ_1 we found is isomorphic to the induced representation $\text{ind}_{C_4}^{\mathfrak{S}_4}(i)$.

We want to show that the result we obtained is not a case but extends to all the partitions lattices $\Pi_n,\ n \in \mathbb{N}$.
The facts we need are the following results of P. Hall and P. Hanlon:

Corollary 3.13. *Let L be a finite lattice with atoms $\{a_1, \ldots, a_n\}$ and coatoms $\{b_1, \ldots, b_m\}$.*

a) If $\widehat{0}$ is not the meet of coatoms, i.e.

$$b_1 \wedge b_2 \wedge \cdots \wedge b_m \neq \widehat{0}$$

then:

$$\mu(\widehat{0}, \widehat{1}) = 0$$

b) If $\widehat{1}$ is not the join of atoms, i.e.

$$a_1 \vee a_2 \vee \cdots \vee a_m \neq \widehat{1}$$

then:

$$\mu(\widehat{0}, \widehat{1}) = 0$$

Proof. See [17], Corollary (Ph. Hall); pag 349. □

Lemma 3.14. *Let $\pi \in \mathfrak{S}_n$, and let Π_n^π denote the sublattice of Π_n fixed pointwise by π. Let μ_π denote the Möbius function of Π_n^π. Then:*

$$\mu_\pi(\widehat{0}, \widehat{1}) = \begin{cases} (-1)^{d-1}\mu(n/d)(d-1)!(n/d)^{d-1}, & \textit{if } \pi \textit{ is a product of } d \textit{ cycles of length } n/d \\ 0, & \textit{otherwise} \end{cases}$$

Here $\mu(n/d)$ denotes the usual number-theoretic Möbius function.

Proof. See [11], Theorem 4; pag 338.
Hanlon actually computes $\mu_\pi(x_\pi, \widehat{1})$, where x_π is the meet of the coatoms of Π_n^π. It follows from [11] Lemma 2 that:

$$x_\pi = \widehat{0} \qquad \Longleftrightarrow \qquad \text{all cycles of } \pi \text{ have the same length}$$

Combining the previous results with Corollary 3.13 we obtain the proof of the lemma. □

Theorem 3.15. *Let μ denotes the usual number-theoretic Möbius function. Then:*

$$\mu(n) = \sum_{\substack{1 \le h \le n \\ (h,n)=1}} e^{\frac{2\pi i h}{n}}$$

i.e. the sum of the primitive n-th roots of unity.

Proof. See [12], Theorem 271 and Equation (16.6.4), pag 239. □

Let C_n be a cyclic subgroup of $\mathfrak{S}_n$ of order n generated by an n-cycle σ. Let $\zeta = e^{2\pi i/n}$ and let ρ_n be the associated representation of C_n:

$$\begin{array}{rccc} \rho_n: & C_n & \longrightarrow & \mathrm{GL}(V) \simeq \mathbb{C}^* \\ & \sigma & \longmapsto & \zeta \end{array}$$

Define the induced representation:

$$\psi_n = \mathrm{ind}_{C_n}^{\mathfrak{S}_n}(\rho_n)$$

Lemma 3.16. *Let $\pi \in \mathfrak{S}_n$. Using the notations described above we claim:*

$$\chi_{\psi_n}(\pi) = \begin{cases} \mu(n/d)(d-1)!(n/d)^{d-1}, & \text{if } \pi \text{ is a product of } d \text{ cycles of length } n/d \\ 0, & \text{otherwise} \end{cases}$$

Proof. Theorem 1.36 on the character of induced representation yields

$$\chi_{\psi_n}(\pi) = \frac{|\mathfrak{S}_n|}{|C_n||C_\pi|} \sum_{\tau \in C_\pi \bigcap C_n} \chi_{\rho_n}(\tau) = \frac{(n-1)!}{|C_\pi|} \sum_{\tau \in C_\pi \bigcap C_n} \chi_{\rho_n}(\tau) \tag{3.7}$$

where C_π is the conjugacy class of $\mathfrak{S}_n$ containing π.
Suppose that π has d cycles, hence:

$$\chi_{\psi_n}(\pi) = 0 \qquad \text{unless } d|n \text{ and } \pi \text{ has } d \text{ cycles of length } n/d.$$

Indeed, if $d \nmid n$ and π has not d cycles of length n/d then $C_\pi \bigcap C_n = \emptyset$.

Let π have d cycles of length n/d, if $\tau \in C_\pi \bigcap C_n$ then:

$$\exists k \text{ with } \gcd(n,k) = d \quad \text{such that} \quad \sigma^k = \tau$$

$$\rho_n(\tau) = \chi_{\rho_n}(\tau) = \zeta^k$$

Let z be a primitive n-th root of unity. A power $w = z^k$ of z is a primitive a-th root of unity for

$$a = \frac{n}{\gcd(n,k)}$$

In our case ζ is a primitive n-th root of unity, then $\chi_{\rho_n}(\tau)$ is a primitive n/d-th root of unity, so $\chi_{\rho_n}(\tau)$ runs through all primitive n/d-th root of unity. From Theorem 3.15 we obtain that:

$$\sum_{\tau \in C_\pi \bigcap C_n} \chi_{\rho_n}(\tau) = \mu(n/d)$$

We can compute the size of the conjugacy class C_π using the Proposition 1.1.1 of [18] and we obtain:

$$|C_\pi| = \frac{n!}{(n/d)^d\ d!} = \frac{(n-1)!}{(n/d)^{d-1}\ (d-1)!}$$

Substituting what obtained in the equation (3.7) we have:

$$\chi_{\psi_n}(\pi) = \frac{(n-1)!\ (n/d)^{d-1}\ (d-1)!}{(n-1)!}\ \mu(n/d) = (n/d)^{d-1}\ (d-1)!\ \mu(n/d)$$

□

Theorem 3.17. *Let $G = \mathfrak{S}_n$ acts on $P = \Pi_n$ in the obvious way. Then:*

$$\beta_{[n-2]} = \gamma_{n-3} = (sgn)\ \psi_n$$

Proof. From Equation (3.5) and Equation (3.6) we get:

$$\widetilde{\Lambda}_{[n-2]}(\pi) = (-1)^{n-3}\ \chi_{\beta_{[n-2]}}(\pi) = (-1)^{n-1}\ \chi_{\beta_{[n-2]}}(\pi)$$

$$\widetilde{\Lambda}_{[n-2]}(\pi) = \widetilde{\mathcal{E}}(P^\pi_{[n-2]}) = \widetilde{\mathcal{E}}(P^\pi) = \widetilde{\mathcal{E}}(\Pi^\pi_n)$$

From Equation (3.2) we have also:

$$\widetilde{\mathcal{E}}(\Pi^\pi_n) = \mu_\pi(\Pi^\pi_n) = \mu_\pi(\widehat{0},\widehat{1})$$

By combining these last two results with Lemma 3.14 we get that:

$$\chi_{\beta_{[n-2]}}(\pi) = (-1)^{n-1}\ \widetilde{\Lambda}_{[n-2]}(\pi) = (-1)^{n-1}\ \mu_\pi(\widehat{0},\widehat{1})$$

$$\chi_{\beta_{[n-2]}}(\pi) = \begin{cases} (-1)^{n+d}\mu(n/d)(d-1)!(n/d)^{d-1}, & \text{if } \pi \text{ is a product of } d \text{ cycles of length } n/d \\ 0, & \text{otherwise} \end{cases}$$

Using Lemma 3.16 we have:

$$\chi_{\beta_{[n-2]}}(\pi) = (-1)^{n+d}\ \chi_{\psi_n}(\pi) = (-1)^{n-d}\ \chi_{\psi_n}(\pi)$$

□

Remark. Note that if $n \not\equiv 2 (\text{mod } 4)$, then either $(-1)^{n-d} = 1$ or $\mu(n/d) = 0$ for all $d \mid n$.

Thus in this case:

$$\gamma_{n-3} = \beta_{[n-2]} = \psi_n$$

Chapter 4

Representations on the homology of dual matroid of complete graphs

4.1 Alexander duality for non spanning matroid and independence dual matroid

Let $M = (E, I)$ be a matroid and $M^* = (E, I^*)$ its dual (See Subsection 2.2.3).
$A \subseteq E$ is non-spanning in M if and only if it does not contain any basis of M, i.e. $\mathrm{rk}(A) < \mathrm{rk}(E)$. Let

$$NS(M) = \{A \subseteq E \mid A \text{ is non spanning in } M\}$$

It is easy to see that $NS(M)$ is an abstract simplicial complex.
The two rank functions are as follows:

$$\begin{array}{rccc} \mathrm{rk}: & \mathcal{P}(E) & \longrightarrow & \mathbb{Z}^+ \\ & U & \longmapsto & \max\limits_{\substack{A \subseteq U \\ A \in I}} |A| \end{array}$$

$$\begin{array}{rccc} \mathrm{rk}^* : & \mathcal{P}(E) & \longrightarrow & \mathbb{Z}^+ \\ & U & \longmapsto & \max\limits_{\substack{A \subseteq U \\ A \in I^*}} |A| \end{array}$$

Remark. $\mathrm{rk}(E)$ is equal to the number of elements of a basis of M.

Proposition 4.1. *$A \subseteq E$ is not-spanning in M^* if and only if A^c is dependent in M.*

Proof. If $A \subseteq E$ is not-spanning in M^* we have:

$$\mathrm{rk}^*(A) < \mathrm{rk}^*(E)$$

By Lemma 2.74, this is equivalent to:

$$\mathrm{rk}(A^c) + |A| - \mathrm{rk}(E) < \mathrm{rk}^*(E)$$

$$\Updownarrow$$

$$\mathrm{rk}(A^c) < -|A| + \underbrace{\mathrm{rk}(E) + \mathrm{rk}^*(E)}_{|E|} = |E| - |A| = |A^c| \Longleftrightarrow A^c \notin I$$

For every $A \subseteq E$, we have $\mathrm{rk}(A) \leqslant |A|$, thus A is independent if and only if $\mathrm{rk}(A) = |A|$. □

We have deduced from the Alexander's duality (Theorem 2.45) that for every simplicial complex Δ on vertex set V such that $V \notin \Delta$, with $n = |V|$:

$$\widetilde{H}_i(\Delta) \simeq \widetilde{H}^{n-3-i}(\Delta^*)$$

We are working with reduced (co)homology groups $\widetilde{H}(\Delta) = \widetilde{H}(\Delta, \mathbb{C})$ with coefficients in $\mathbb{C}$, and we claim that:

$$\widetilde{H}_j(\Delta) \simeq \widetilde{H}^j(\Delta)$$

In fact, working with complex coefficients the reduced cohomology group $\widetilde{H}^j(\Delta)$ is the dual vector space of the reduced homology group $\widetilde{H}_j(\Delta)$. Combining the two results we obtain:

$$\widetilde{H}_i(\Delta) \simeq \widetilde{H}_{n-3-i}(\Delta^*) \tag{4.1}$$

Proposition 4.2. *Let $\Delta = IN(M) = I$ be the abstract simplicial complex associated with the independents of the matroid $M = (E, I)$ and let Δ^* be the dual complex defined in the preliminaries chapter, then:*

$$\Delta^* = NS(M^*)$$

Proof. Using the result shown in Propostion 4.1 we claim that:

$$\Delta^* = \{A \subseteq V : (V \setminus A) \notin \Delta\} = \{A \subseteq E : A^c \notin I\} =$$

$$= \{A \subseteq E : A^c \text{ is dependent in } M = (E, I)\} =$$

$$= \{A \subseteq E : A \text{ is not spanning of } M^*\} = \text{NS}(M^*)$$

□

The previous result, together with Equation (4.1), implies the following:

$$\widetilde{H}_i(\text{IN}(M)) \simeq \widetilde{H}_{n-3-i}(\text{NS}(M^*))$$

Since $M^{**} = M$, by duality we obtain:

Theorem 4.3.

$$\widetilde{H}_i(NS(M)) \simeq \widetilde{H}_{n-3-i}(IN(M^*))$$

4.2 Isomorphism between homology of non spanning complex of a matroid and homology of its lattice of flats

Let L be a lattice with maximal and minimal element $\widehat{1}$ and $\widehat{0}$ respectively.

Definition 4.4. If L is a lattice with $\widehat{0}$ and $\widehat{1}$, a *cross-cut* of L is a set $C \subseteq L$ such that:

i) $\widehat{0}$, $\widehat{1} \notin C$.

ii) If $x, y \in C$ then $x \nless y$ and $y \nless x$. (*x and y are incomparable*)

iii) Any finite chain $x_1 < x_2 < \cdots < x_n$ in L can be extended to a chain which contains an element of C.

In particular, axiom iii) implies that every maximal chain contains an element of C.

Let L be a lattice with $\widehat{1}$ and $\widehat{0}$ and let C be a *cross-cut* of L.

Definition 4.5. A finite subset $\{x_1, \ldots, x_n\} \subseteq C$ *'spans'* if and only if

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n = \widehat{0} \qquad \text{and} \qquad x_1 \vee x_2 \vee \cdots \vee x_n = \widehat{1}$$

Here $x \wedge y$ denotes the largest element $\leq x$ and $\leq y$, and $x \vee y$ denotes the smallest element $\geqslant x$ and $\geqslant y$.
Let $K(C)$ be the abstract simplicial complex whose vertices are the elements of C and whose simplexes are all finite subsets of C which do not *'span'*.
We define $\widetilde{H}_i(C) = \widetilde{H}_i(K(C))$.
Let $K(L)$ be the abstract simplicial complex of the poset L as we have seen in the section 3.1 and define $\widetilde{H}_i(L) = \widetilde{H}_i(K(L)) = \widetilde{H}_i(\overline{L})$.

Theorem 4.6. $\widetilde{H}_i(C) \simeq \widetilde{H}_i(L)$

Proof. See [9], Theorem 3.1; pag 633. □

Let $M = (E, I)$ be a simple matroid with $E = \{a_1, \ldots a_m\}$. (See Definition 2.49) Let $\mathcal{L}(M)$ be the lattice of flats of M ordered by inclusion, we want to consider a *cross-cut* of $\mathcal{L}$.
Since M is simple we have that each singleton of E is a flat, so in $\mathcal{L}(M)$ we have $\emptyset, \{a_1\}, \{a_2\}, \ldots, \{a_m\} \in \mathcal{L}(M)$ and these corresponds to the **atoms** of the poset $(\mathcal{L}(M), \subseteq)$.
Consider

$$C = \{\{a_1\}, \{a_2\}, \ldots, \{a_m\}\} \subseteq \mathcal{L}(M)$$

C is a *cross-cut* of $\mathcal{L}$, indeed:

i) $\widehat{0}, \widehat{1} \notin C$

ii) All the atoms are incomparable with the inclusion.

iii) Every maximal chain have one element of C because $\{a_1\}, \{a_2\}, \ldots, \{a_m\}$ are the atoms of the lattice $\mathcal{L}(M)$.

We want to prove that:

$$K(C) = NS(M)$$

In the following proposition we perform a slight abuse of notation by identifying:

$$C = \{\{a_1\}, \{a_2\}, \ldots, \{a_m\}\} = \{a_1, a_2, \ldots, a_m\}$$

Proposition 4.7. *$A \subseteq C$ does not 'span' if and only if A is a non-spanning set in $M = (E, I)$.*

Proof. $\Longrightarrow$) In $\mathcal{L}(M)$ we have:

$$\widehat{0} = \emptyset \qquad \widehat{1} = E$$

Let $A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}$ be a subset of C. If $A \subseteq C$ does not *'span'*:

$$a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_n} = D \neq \widehat{1} \tag{4.2}$$

$D \in \mathcal{L}(M)$ and $D \neq \widehat{1}$ implies that D is a non-spanning subset of E because the only spanning subset in $\mathcal{L}(M)$ is $E = \widehat{1}$.
It follows from (4.2) that $A \subseteq D$; since D is a non-spanning subset of E then A is a non-spanning subset of E.

$\Longleftarrow$) In $NS(M)$ the bases are the maximal non-spanning subsets of E, (i.e, the subsets of E, such that if we add an element they become spanning set) so they are flats, in particular they correspond to the co-atoms of $(\mathcal{L}(M), \subseteq)$.
Let $A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}$ be a non-spanning subset of E, there exist a basis $\mathcal{B}$ of $NS(M)$ such that:

$$A \subseteq \mathcal{B} \qquad \mathcal{B} \text{ is a flat} \Rightarrow \mathcal{B} \in \mathcal{L}(M)$$

This implies:

$$a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_n} \subseteq \mathcal{B} \neq \widehat{1}$$

then A does not *'span'*.

□

Using the result of Proposition 4.7, we obtain:

$$K(C) = NS(M)$$

If we consider a *cross-cut* C of $\mathcal{L}(M)$ as follows:

$$C = \{\text{atoms of } \mathcal{L}(M)\}$$

we shall obtain by Theorem 4.6:

Theorem 4.8.

$$\widetilde{H}_i(\mathcal{L}(M)) \simeq \widetilde{H}_i(C) = \widetilde{H}_i(NS(M))$$

4.3 Representations on the homology of dual matroid of a complete graph

By combining the results of the two previous sections we get that, for every simple matroid, we have the isomorphism:

$$\widetilde{H}_{n-3-i}(IN(M^*)) \simeq \widetilde{H}_i(\mathcal{L}(M)) \tag{4.3}$$

We now consider the following equivalent simple matroids

$$M^*(\Phi^+_{A_{r-1}}, I) = M^*(K_r)$$

and we describe the representation we have seen in Subsection 2.2.7.

Theorem 4.9. $\widetilde{H}_{n-3-i}(IN(M^*(K_r)))$ *and* $\widetilde{H}_i(\Pi_r)$ *are isomorphic as* $\mathfrak{S}_r$*-modules.*

Proof. In particular from Subsection 2.2.5 we get that:

$$\mathcal{L}(M(\Phi^+_{A_{r-1}}, I)) = \mathcal{L}(M(K_r)) = \Pi_r$$

From Equation (4.3) with

$$n = |E(M^*(\Phi^+_{A_{r-1}}))| = |\Phi^+(A_{r-1})| = \binom{r}{2} = \frac{r(r-1)}{2}$$

we get these natural isomorphisms:

$$\widetilde{H}_{n-3-i}(IN(M^*(\Phi^+_{A_{r-1}}))) \simeq \widetilde{H}_i(\mathcal{L}(M(\Phi^+_{A_{r-1}}, I))) \simeq \widetilde{H}_i(\Pi_r)$$

or, rephrased in terms of the complete graph:

$$\widetilde{H}_{n-3-i}(IN(M^*(K_r))) \simeq \widetilde{H}_i(\mathcal{L}(M(K_r)) \simeq \widetilde{H}_i(\Pi_r)$$

Since the isomorphism $\widetilde{H}_{n-3-i}(IN(M^*(\Phi^+_{A_{r-1}}))) \simeq \widetilde{H}_i(\Pi_r)$ is natural, it also respects the action of the symmetric group $\mathfrak{S}_r$. □

Remark. We can make a dimensional calculation to better understand the dimensional shift.
The matroid $M(\Phi^+_{A_{r-1}}, I)$ has rank equal to $r-1$, i.e. each basis has $r-1$ elements. Therefore, the matroid $M^*(\Phi^+_{A_{r-1}}, I)$ has rank equal to:

$$\frac{r(r-1)}{2} - (r-1) = \frac{(r-1)(r-2)}{2}$$

Thus the dimension of the top homology of $IN(M^*(\Phi^+_{A_{r-1}})$ is one less than the number of the elements of a basis of $M^*(\Phi^+_{A_{r-1}}, I)$:

$$\frac{(r-1)(r-2)}{2} - 1$$

We have that:

$$\widetilde{H}_{n-3-i}(IN(M^*(\Phi^+_{A_{r-1}}))) \simeq \widetilde{H}_i(\Pi_r)$$

We impose

$$n - 3 - i = \frac{(r-1)(r-2)}{2} - 1$$

and we get the i:

$$\frac{r(r-1)}{2} - 3 - i = \frac{(r-1)(r-2)}{2} - 1$$

$$r^2 - r - 6 - 2i = r^2 - 2r - r + 2 - 2 \quad \Longrightarrow \quad i = r - 3$$

Indeed, $H_{r-3}(\Pi_r)$ is the only nonzero homology group of Π_r.

By Theorem 4.9 these two representations

$$\rho_{n-r} : \ \ \mathfrak{S}_r \ \ \longrightarrow \ \ \mathrm{GL}(\widetilde{H}_{n-r}(IN(M^*(\Phi^+_{A_{r-1}}))))$$

and

$$\gamma_{r-3} : \ \ \mathfrak{S}_r \ \ \longrightarrow \ \ \mathrm{GL}(\widetilde{H}_i(\Pi_r))$$

are isomorphic. As seen in Section 3.2 we get that

$$\rho_{n-r} \simeq \gamma_{r-3} \simeq (\mathrm{sgn}) \ \mathrm{ind}_{C_r}^{\mathfrak{S}_r}(e^{2\pi i/r})$$

Similarly,

$$\rho_{n-r} : \ \ \mathfrak{S}_r \ \ \longrightarrow \ \ \mathrm{GL}(\widetilde{H}_{n-r}(IN(M^*(K_r))))$$

and we get

$$\rho_{n-r} \simeq \gamma_{r-3} \simeq (\mathrm{sgn}) \ \mathrm{ind}_{C_r}^{\mathfrak{S}_r}(e^{2\pi i/r}).$$

Chapter 5

Representations on the homology for the root system B_2

Now we consider the root system of type B_2,

$$\Phi_{B_2} = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$$

which corresponds to the Lie algebra $SO(5,\mathbb{C})$. We find the representations on $\widetilde{H}(IN(M^*(\Phi^+_{B_2})))$ of the Weyl group on Φ_{B_2} without considering the sign, i.e what we did in the thesis for Φ_{A_n}.

First we study the homology group $\widetilde{H}(IN(M^*(\Phi^+_{B_2})))$:

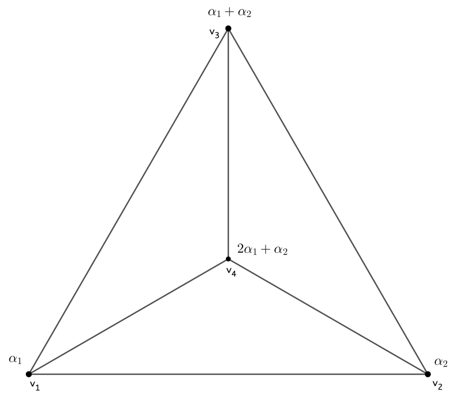
$$\Phi^+_{B_2} = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\} = \{v_1, v_2, v_3, v_4\}$$

$$IN(M(\Phi^+_{B_2})) = \Big\{\{\alpha_1,\alpha_2\},\{\alpha_1,\alpha_1+\alpha_2\},\{\alpha_1,2\alpha_1+\alpha_2\},\{\alpha_2,\alpha_1+\alpha_2\},\{\alpha_2,2\alpha_1+\alpha_2\},$$
$$,\{\alpha_1+\alpha_2,2\alpha_1+\alpha_2\},\{\alpha_1\},\{\alpha_2\},\{\alpha_1+\alpha_2\},\{2\alpha_1+\alpha_2\}\Big\}$$

$$IN(M^*(\Phi^+_{B_2})) = \Big\{\{\alpha_1+\alpha_2,2\alpha_1+\alpha_2\},\{\alpha_2,2\alpha_1+\alpha_2\},\{\alpha_2,\alpha_1+\alpha_2\},\{\alpha_1,2\alpha_1+\alpha_2\},$$

$$\{\alpha_1, \alpha_1+\alpha_2\}, \{\alpha_1, \alpha_2\}, \{\alpha_1\}, \{\alpha_2\}, \{\alpha_1+\alpha_2\}, \{2\alpha_1+\alpha_2\}\Big\}$$

The independence set $IN(M^*(\Phi^+_{B_2}))$ is a simplicial complex that we can embed in $\mathbb{R}^2$:



Geometrically we see that the dimension of $\widetilde{H}_1(IN(M^*(\Phi^+_{B_2})))$ is three, now we calculate a basis explicitly.

First, we write down the basis of the two free modules C_1 and C_0:

$$\mathcal{B}_{C_1} = \Big\{\overset{a_1}{[v_1, v_2]}, \overset{a_2}{[v_1, v_3]}, \overset{a_3}{[v_1, v_4]}, \overset{a_4}{[v_2, v_3]}, \overset{a_5}{[v_2, v_4]}, \overset{a_6}{[v_3, v_4]}\Big\} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$$

$$\mathcal{B}_{C_0} = \{v_1, v_2, v_3, v_4\}$$

$$C_1 \simeq \mathbb{C}^6 \qquad C_0 \simeq \mathbb{C}^4$$

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \qquad \widetilde{H}_1(IN(M^*(\Phi^+_{B_2}))) = \ker(\partial_1)/\mathrm{Im}(\partial_2) = \ker(\partial_1)$$

$$\begin{array}{rccc} \partial_1: & C_1 & \longrightarrow & C_0 \\ & a_1 & \longmapsto & v_2 - v_1 \\ & a_2 & \longmapsto & v_3 - v_1 \\ & a_3 & \longmapsto & v_4 - v_1 \\ & a_4 & \longmapsto & v_3 - v_2 \\ & a_5 & \longmapsto & v_4 - v_2 \\ & a_6 & \longmapsto & v_4 - v_3 \end{array} \qquad \partial_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$\widetilde{H}_1(IN(M^*(\Phi^+_{B_2}))) = \mathrm{Span}\Big\{a_2 - a_3 + a_6; a_1 - a_3 + a_5; a_1 - a_2 + a_4\Big\} = \{b_1, b_2, b_3\}$$

The Weyl group $\mathcal{W}_{B_2}$ is isomorphic to the dihedral group

$$D_4 = \langle r, s \mid r^4 = s^2 = (sr)^2 = 1\rangle$$

$$\begin{array}{rccc} \phi: & \mathcal{W}_{B_2} & \longrightarrow & D_4 \\ & \sigma_{\alpha_1} & \longmapsto & r^2 s \\ & \sigma_{\alpha_2} & \longmapsto & rs \end{array} \qquad \begin{array}{rccc} \psi: & D_4 & \longrightarrow & \mathcal{W}_{B_2} \\ & r & \longmapsto & \sigma_{\alpha_2} \circ \sigma_{\alpha_1+\alpha_2} \\ & s & \longmapsto & \sigma_{\alpha_1+\alpha_2} \end{array}$$

and we have $\phi \circ \psi = \psi \circ \phi = \mathrm{id}$.
The dihedral group D_4 has five conjugacy classes:

$$\{\mathrm{id}\} \quad \{r^2\} \quad \{r, r^3\} \quad \{s, r^2 s\} \quad \{rs, r^3 s\}$$

and therefore the following character table (See [19], Section 5.3; pag 36.):

Table 5.1: Character table of D_4

	id	r^2	$r \vert r^3$	$s \vert r^2 s$	$rs \vert r^3 s$
χ_{V_1}	1	1	1	1	1
χ_{V_2}	1	1	1	-1	-1
χ_{V_3}	1	1	-1	1	-1
χ_{V_4}	1	1	-1	-1	1
χ_{V_5}	2	-2	0	0	0
$\chi_{V_{reg}}$	8	0	0	0	0

The action of the Weyl group on Φ_{B_2} induces a permutation on the vertex set of the simplicial complex $IN(M^*(\Phi^+_{B_2}))$ as seen in Subsection 1.1.2 for $\Phi_{A_{n-1}}$. Since $\mathcal{W}$ is generated by r and s, the action is described as follows:

$$
\begin{array}{rccc}
s: & \Phi_{B_2} & \longrightarrow & \Phi_{B_2} \\
 & \alpha_1 & \longmapsto & \alpha_1 \\
 & \alpha_2 & \longmapsto & -(2\alpha_1+\alpha_2) \\
 & \alpha_1+\alpha_2 & \longmapsto & -(\alpha_1+\alpha_2) \\
 & 2\alpha_1+\alpha_2 & \longmapsto & -\alpha_2
\end{array}
\qquad
\begin{array}{rccc}
\tau_s: & \Phi^+_{B_2} & \longrightarrow & \Phi^+_{B_2} \\
 & v_1 & \longmapsto & v_1 \\
 & v_2 & \longmapsto & v_4 \\
 & v_3 & \longmapsto & v_3 \\
 & v_4 & \longmapsto & v_2
\end{array}
$$

$$
\begin{array}{rccc}
r: & \Phi_{B_2} & \longrightarrow & \Phi_{B_2} \\
 & \alpha_1 & \longmapsto & \alpha_1+\alpha_2 \\
 & \alpha_2 & \longmapsto & -(2\alpha_1+\alpha_2) \\
 & \alpha_1+\alpha_2 & \longmapsto & -\alpha_1 \\
 & 2\alpha_1+\alpha_2 & \longmapsto & \alpha_2
\end{array}
\qquad
\begin{array}{rccc}
\tau_r: & \Phi^+_{B_2} & \longrightarrow & \Phi^+_{B_2} \\
 & v_1 & \longmapsto & v_3 \\
 & v_2 & \longmapsto & v_4 \\
 & v_3 & \longmapsto & v_1 \\
 & v_4 & \longmapsto & v_2
\end{array}
$$

Our purpose is to find the following representation

$$\rho_1 : \; D_4 \;\longrightarrow\; \mathrm{GL}(\widetilde{H}_1(IN(M^*(\Phi^+_{B_2}))))$$

The maps $\{\tau_g,\ g \in \mathcal{W}\}$ induce linear maps on C_1. In order to calculate ρ_1 we need to see how the representation on C_1 works (See Subsection 2.1.6):

$$
\begin{array}{rccc}
\alpha_\tau: & D_4 & \longrightarrow & \mathrm{GL}(C_1) \\
 & r^2 & \longmapsto & \alpha_{r^2} \\
 & r & \longmapsto & \alpha_r \\
 & s & \longmapsto & \alpha_s \\
 & rs & \longmapsto & \alpha_{rs}
\end{array}
$$

The map α_{r^2} is the identity on C_1. Let's see the others:

$$\begin{array}{rccc}
\alpha_r: & C_1 & \longrightarrow & C_1 \\
& a_1 & \longmapsto & a_6 \\
& a_2 & \longmapsto & -a_2 \\
& a_3 & \longmapsto & -a_4 \\
& a_4 & \longmapsto & -a_3 \\
& a_5 & \longmapsto & -a_5 \\
& a_6 & \longmapsto & a_1
\end{array}
\qquad
\begin{array}{rccc}
\alpha_s: & C_1 & \longrightarrow & C_1 \\
& a_1 & \longmapsto & a_3 \\
& a_2 & \longmapsto & a_2 \\
& a_3 & \longmapsto & a_1 \\
& a_4 & \longmapsto & -a_6 \\
& a_5 & \longmapsto & -a_5 \\
& a_6 & \longmapsto & -a_4
\end{array}
\qquad
\begin{array}{rccc}
\alpha_{rs}: & C_1 & \longrightarrow & C_1 \\
& a_1 & \longmapsto & -a_4 \\
& a_2 & \longmapsto & -a_2 \\
& a_3 & \longmapsto & a_6 \\
& a_4 & \longmapsto & -a_1 \\
& a_5 & \longmapsto & a_5 \\
& a_6 & \longmapsto & a_3
\end{array}$$

We can now study the desired representation:

$$\begin{array}{rccc}
\rho_1: & D_4 & \longrightarrow & \mathrm{GL}(\widetilde{H}_1(IN(M^*(\Phi^+_{B_2})))) \\
& \mathrm{Id} & \longmapsto & I_3 \\
& r^2 & \longmapsto & \rho_{r^2} \\
& r & \longmapsto & \rho_r \\
& s & \longmapsto & \rho_s \\
& rs & \longmapsto & \rho_{rs}
\end{array}$$

$$\begin{array}{rccc}
\rho_{r^2}: & \widetilde{H}_1(IN(M^*(\Phi^+_{B_2})) & \longrightarrow & \widetilde{H}_1(IN(M^*(\Phi^+_{B_2})) \\
& b_1 & \longmapsto & b_1 \\
& b_2 & \longmapsto & b_2 \\
& b_3 & \longmapsto & b_3
\end{array}
\qquad
\rho_{r^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rccc}
\rho_r: & \widetilde{H}_1(IN(M^*(\Phi^+_{B_2})) & \longrightarrow & \widetilde{H}_1(IN(M^*(\Phi^+_{B_2})) \\
& b_1 & \longmapsto & b_3 \\
& b_2 & \longmapsto & b_1\text{-}b_2 + b_3 \\
& b_3 & \longmapsto & b_1
\end{array}
\qquad
\rho_r = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{rccc}
\rho_s: & \widetilde{H}_1(IN(M^*(\Phi^+_{B_2})) & \longrightarrow & \widetilde{H}_1(IN(M^*(\Phi^+_{B_2})) \\
& b_1 & \longmapsto & -b_3 \\
& b_2 & \longmapsto & -b_2 \\
& b_3 & \longmapsto & -b_1
\end{array}
\qquad
\rho_s = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{rccc} \rho_{rs}: & \widetilde{H}_1(IN(M^*(\Phi^+_{B_2})) & \longrightarrow & \widetilde{H}_1(IN(M^*(\Phi^+_{B_2})) \\ & b_1 & \longmapsto & \text{-}b_1 \\ & b_2 & \longmapsto & \text{-}b_1 + b_2\text{-}b_3 \\ & b_3 & \longmapsto & \text{-}b_3 \end{array} \qquad \rho_{rs} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\begin{array}{rccc} \chi_{\rho_1}: & D_4 & \longrightarrow & \mathbb{C} \\ & \mathrm{Id} & \longmapsto & 3 \\ & r^2 & \longmapsto & 3 \\ & r & \longmapsto & -1 \\ & s & \longmapsto & -1 \\ & rs & \longmapsto & -1 \end{array}$$

We now see how ρ_1 decomposes into irreducible representations:

$$(\chi_{\rho_1}, \chi_{V_1}) = \frac{1}{8}(3 + 3 - 2 - 2 - 2) = 0$$

$$(\chi_{\rho_1}, \chi_{V_2}) = \frac{1}{8}(3 + 3 - 2 + 2 + 2) = 1$$

$$(\chi_{\rho_1}, \chi_{V_3}) = \frac{1}{8}(3 + 3 + 2 - 2 + 2) = 1$$

$$(\chi_{\rho_1}, \chi_{V_4}) = \frac{1}{8}(3 + 3 + 2 + 2 - 2) = 1$$

$$(\chi_{\rho_1}, \chi_{V_5}) = \frac{1}{8}(3 - 3 + 0 + 0 + 0) = 0$$

Therefore:

$$\chi_{\rho_1} = \chi_{V_2} + \chi_{V_3} + \chi_{V_4} \qquad \widetilde{H}_1(IN(M^*(\Phi^+_{B_2})) \simeq V_2 \oplus V_3 \oplus V_4$$

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