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OPTIMAL MODEL POINTS IN TERM LIFE INSURANCE

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To my family

"Eterno studente,
perché la materia di studio sarebbe infinita
e soprattutto perché so di non sapere niente"
(Addio - F.Guccini)

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Abstract

This thesis is focused on the problem of seeking an optimal set of the model points selection when dealing with a portfolio of term insurance policies and a LIBOR Market Model that determines the dynamics of the forward rates. Specifically, the study is associated to the problem of minimizing a specific risk functional which measures the average discrepancy between two portfolios: the given portfolio of policies and the model points, a small group of representative contracts which substitute the first one, without misrepresenting its inherent risk structure. This optimization process is aimed to reducing the computation difficulties of the valuation of the performance of any portfolios of policies, projections to be made daily by life insurance companies.

In particular, in the present thesis, after a brief reference to some basic concepts in the interest rate field, there are described the LIBOR Market Model and the risk functional in a Banach space. The portfolio representation problem is also examined, because it allows to define the dynamics of those portfolios within a certain class that best represents the inherent risk structure of a given financial exposure. Finally, it is analyzed the particular case of the term life insurance.

Sommario

Questo lavoro di tesi è incentrato sulla ricerca di un portafoglio ottimale di model points quando consideriamo un portafoglio di polizze di assicurazione sulla vita a termine e il LIBOR Market Model che disciplina le dinamiche dei tassi forward. In particolare, lo studio è correlato alla ricerca del minimo di uno specifico funzionale di rischio il quale misura la discrepanza media tra due portafogli: il portafoglio di polizze iniziale e il model points, un piccolo gruppo di contratti rappresentativi con il quale il primo viene sostituito senza alterare la sua struttura di rischio inerente. Questo processo di ottimizzazione ha come scopo di ridurre il costo computazione della valutazione della performance di una portafoglio di polizze, proiezioni che quotidianamente le compagnie assicurative devono effettuare.

Nello specifico, nel presente lavoro, dopo un breve accenno ad alcuni concetti base nell'ambito dei tassi di interesse, vengono descritti il modello LIBOR e il funzionale di rischio in uno spazio di Banach. Viene, inoltre, approfondito il problema della rappresentazione di un portafoglio fissato, in quanto quest'ultimo ci permette di definire le dinamiche di questi portafogli all'interno di una determinata classe che meglio rappresenta il rischio inerente di una data esposizione finanziaria. Infine, viene analizzato il caso particolare delle assicurazioni della vita a termine.

Abbreviations

s.t. = such that

w.r.t = with respect to

a.s = almost surely

i.e. = id est = that is

B.m. = Brownian motion

FRA = Forward Rate Agreement

IRS = Interest Rate Swap

LMM = LIBOR Market Model

LFM = Lognormal Forward LIBOR Model

PSF = Payer (Forward) Interest Rate Swap

RSF = Reicever(Forward) Interest Rate Swap

Contents

Introduction

The progressive economic growth of the last decades, driven by technological and financial developments, has led to significant changes in the life insurance sector. Nowadays, insurance companies treat a large number of different securities, which are traded in order to hedge their exposure.
In particular, life insurance companies are allowed by regulators to estimate the performance of any portfolio of policies on the basis of suitable model points in order to reduce the computational difficulties of the operation. This procedure is permitted under suitable conditions, i.e. when the inherent risk structure of the original portfolio is not misrepresented and there are not loss of any significant attribute of the portfolio itself.

The point of this work is analyzing in detail the model points risk functional when dealing with a term life insurance portfolio and the dynamics of the forward rate is determined by the LIBOR Market Model. The main advantage of the latter is that it can be made consistent with an arbitrage free term structure model; this is possible because each rate is lognormal under the forward to the settlement date arbitrage free measure rather than under spot arbitrage free measure.

This thesis is structured as follows:

Chapter 1 We present some basic financial instruments, in particular we define different types of interest rates and the main derivatives, as caps floors and swaps with the respective ones Black's formula.

Chapter 2 We introduce the LMM, describing the dynamics of LIBOR forward rates, also under different numeraire. Then we present the Black volatilities implied by the cap market and show that the risk neutral valuation formula of caps gives the same prices as the Black's cap formula.

Chapter 3 We describe the risk functional, after that we have fixed a specified set. We demonstrate the Theorem 3.0.8 that give us two different formulations of the risk functional. Then, we consider the portfolio representation problem, highlighted the sensitivity based hedging approach and the case when we work with correlate risk factors.

Chapter 4 We discuss the problem of determining an optimal model points for a term life insurance portfolio, when a LIBOR Market Model determines the dynamics of forward rates. So, we make some assumptions for the LIBOR forward rates and then we define a model points risk functional relative to term insurance portfolio for a class of individuals.

Chapter 1

Standard definitions and concepts in the interest-rate world

1.1 The Bank account and short rate

Definition 1.1.1. We define a **Bank account or Money-market account** $B(t)$ to be the value of a bank account at time $t \geq 0$. We assume $B(0) = 1$ and the money account process is described by

$$dB(t) = r_t B(t) dt \qquad B(0) = 1$$

where r_t is a positive function of time, called *instantaneous short rate* at which the bank account grows up.
By solving:

$$B(t) = exp(\int_0^t r_s ds) \tag{1.1}$$

Definition 1.1.2. The **(stochastic) discount factor** $D(t, T)$, for $t \leq s \leq T$ time istants, is the amount at time t that is equivalent to one unit of currency payable at time T:

$$D(t, T) = \frac{B(t)}{B(T)} = exp(-\int_t^T r_s ds) \tag{1.2}$$

The probabilistic nature of r_t is important. In many pricing application r is assumed to be a deterministic function, so (1.1) and (1.2) at any future time are deterministic functions. However, when dealing to interest rate products, the main variability that matters is clearly that of interest rates themselves. It is therefore necessary to consider the evolution of r in time through a stochastic process, so the bank account (1.1) and the discount factors (1.2) will be stochastic processes, too.

1.2 Zero-Coupon Bonds and Spot Interest Rates

Definition 1.2.1. A **zero coupon bond** (pure discount bond) with **maturity date** T, also known as **T-bond**, is a contract which guarantees its owner the payment of 1 unit of currency at the date T, with no intermediate payments. The price at time t of a bond with maturity date T is denoted by $p(t, T)$.

Assumption 1.2.2. *We make some assumptions to guarantee the existence of the regular bond market:*

- *fot every $T > 0$ exists a (frictionless) market for T-bonds;*
- *$p(t, t) = 1$ for all t, in order to avoid arbitrage;*
- *for each fixed t, the bond price $p(t, T)$ is differentiable rispect to time of maturity $T > t$.*

Remark 1.2.3. We analize now the relationship between the discount factor $D(t, T)$ and the zero coupon bond $p(t, T)$. The difference is inherent in the definition of the two objects being respectively an "equivalent amount of currency" and a "value of a contract". If the rates r are deterministic, then D is deterministic and necessarily $D(t, T) = p(t, T)$. However, if rates are stochastic, $D(t, T)$ is a random quantity at time t depending on the future evolution of rates r between t and T. Whereas, $p(t, T)$ is the t-value of a contract with maturity date T, so it has to be known, then deterministic, at time t.

Definition 1.2.4. The **time to maturity** $T - t$ is the quantity of time, in years, from the present time t to the maturity time $T > t$. We denote by $\tau(t, T)$ the measure between t and T, that is usually expressed to as *year fraction* between the dates.

So, every time we want to know the present value of a future-time payment we have to consider the price of zero coupon bonds for that future time.
Zero coupon bond prices are the basic quantities in interest-rate theory, and all interest rates can be defined in terms of zero coupon bond prices.
Therefore, zero coupon bond are theoretical instruments that are not directy observable in the market. To talk about zero coupon bonds in terms of interest rates, and vice versa, we need to define two features of the rates: the compounding type and the day-count convention. The latter is the convention according to which a particular choice is made to measure the time between two dates. To understand how it works we mention three generic examples of the day-count convention:

- *Actual/365*: with this convention a year is 365 days long and the *year fraction* between two dates, expressed by day/month/year, $D_1(d_1, m_1, y_1)$ and $D_2(d_2, m_2, y_2)$, is the actual number of days between them divided by 365.
$$\frac{D_1 - D_2}{365}$$
- *Actual/360*: in this case a year is 360 days long, so the *year fraction* is:
$$\frac{D_1 - D_2}{360}$$
- *30/360*: with this convention months are assumed 30 days long and years 360 days. The *year fraction* is given by:
$$\frac{max(30 - d_1, 0) + min(d_2, 30) + 360 * (y_2 - y_1) + 30 * (m_2 - m_1 - 1)}{360}$$

We define now the compounding type.

Definition 1.2.5. The **continuously-compounded spot interest rate** prevailing at time t for the maturity T is denoted by $R(t,T)$, it is a constant rate at which an investment of $p(t,T)$ unity of currency at time t accrues continuously to yield a unity quantity of currency at maturity time T, i.e.

$$R(t,T) := -\frac{\ln p(t,T)}{\tau(t,T)} \tag{1.3}$$

or equivalently

$$e^{R(t,T)\tau(t,T)}p(t,T) = 1.$$

From this formula we can express the bond price as:

$$p(t,T) = e^{-R(t,T)\tau(t,T)}.$$

Definition 1.2.6. The **simply-compounded spot interest rate** prevailing at time t for the maturity time T is denoted by $L(t,T)$, it is a constant rate at which an investment of $p(t,T)$ unity of currency at initial time t accrues proportionally to the investment time to produce a unity quantity of currency at maturity time T. In formulas:

$$L(t,T) := \frac{1 - p(t,T)}{\tau(t,T)p(t,T)}. \tag{1.4}$$

They are denoted with L because the most importart interbank rate *market LIBOR rates* are simple-compouned rate. LIBOR rates are connected to zero bound prices by *Actual/360* day count covention for computing $\tau(t,T)$.
Similarly to before we have:

$$p(t,T)(1 + L(t,T)\tau(t,T)) = 1$$

and so we can express a bond price in terms of L as:

$$p(t,T) = \frac{1}{1 + L(t,T)\tau(t,T)}.$$

Definition 1.2.7. The **annually-compounded spot interest rate** prevailing at time t for the maturity T is denoted by $Y(t,T)$, it is a constant rate at which an investment of $p(t,T)$ unity of currency at initial time t, has to be made to produce a quantity of one unity of currency at maturity T, when reinvesting the obtained amounts once a year; i.e.

$$Y(t,T) := \frac{1}{[p(t,T)]^{\frac{1}{\tau(t,T)}}} - 1 \tag{1.5}$$

equivalently:

$$p(t,T)(1 + Y(t,T))^{\tau(t,T)} = 1$$

and so the bond prices in terms of this rates are defined by:

$$p(t,T) = \frac{1}{(1 + Y(t,T))^{\tau(t,T)}}$$

Example 1.2.8. We can understand better how the annually compounding works considering this example. If we invest today a unit of currency at the simply-compoundend rate Y, in one year we will obtain the amount $A = 1(1+Y)$. If we repeat the process and we invest a quantity for another year at the same rate Y, we have $A(1+Y) = (1+Y)^2$ in two years. And so, by reiterating the investment for n years, the final amount we obtain is $(1+Y)^n$.

Definition 1.2.9. The **k-times-per-year compounded spot interest rate** prevailing at time t for the maturity T is denoted by $Y^k(t,T)$, it is a constant rate (referred to a one year period) at which an investment of $p(t,T)$ unity of currency at initial time t has to be made to produce a quantity of one unity of currency at maturity T, when reinvesting the obtained amounts k times a year. In formulas:

$$Y^k(t,T) := \frac{k}{[p(t,T)]^{\frac{1}{k\tau(t,T)}}} - k \tag{1.6}$$

equivalently:

$$p(t,T)\left(1 + \frac{Y^k(t,T)}{k}\right)^{k\tau(t,T)} = 1$$

and so the bond prices in terms of this rates are defined by:

$$p(t,T) = \frac{1}{\left(1 + \frac{Y^k(t,T)}{k}\right)^{k\tau(t,T)}}.$$

Remark 1.2.10. We can observe that if we consider the limit of *k-times-per-year* compounded rates for the number k of compounding times going to infinity we obtain the *continuously-compounded* rates:

$$\lim_{k\to+\infty} \frac{k}{[p(t,T)]^{\frac{1}{k\tau(t,T)}}} - k = -\frac{ln(p(t,T))}{\tau(t,T)} = R(t,T)$$

Remark 1.2.11. The previous definitions of spot interest rates are equivalent in infinitesimal time intervals:

$$
\begin{aligned}
r(t) &= \lim_{T \to t^+} R(t,T) \\
&= \lim_{T \to t^+} L(t,T) \\
&= \lim_{T \to t^+} Y(t,T) \\
&= \lim_{T \to t^+} Y^k(t,T) \quad \forall k
\end{aligned}
$$

1.3 Forward rate

We define now the **forward rates** that are characterized by three time instants t, T and S, so related $t \leq T \leq S$, which represent respectively: t is the time instant at which the rate is considered, T is the expiry and S is the maturity of the rate.
Forward rates are interest rates that can be locked in today for an investment in a future time period, and they are agreed consistently with the current term structure of discount factors.

Definition 1.3.1. Given t, T and S, defined as above, we describe a process that starting at time t allows us to make an investment of one dollar at time T, and to have a deterministic rate of return, determined at the contract time t, over the range $[T, S]$. The proceedings is as follows:

1. at time t we sell one T-bond $\Longrightarrow$ this give us an income of $p(t,T)$ dollars;

2. with this income we buy $p(t,T)/p(t,S)$ S-bonds $\Longrightarrow$ so now the investment at t is 0;

3. at time T the T-bond matures $\Longrightarrow$ we have to pay out one dollar;

4. at time S the S-bond mature at one dollar at piece $\Longrightarrow$ we will receive exactly $p(t,T)/p(t,S)$ dollars at time S;

The actual effect is that, thanks to a contract fixed at time t, an investment of one dollar at time T give us a yield of $p(t,T)/p(t,S)$ dollars at time S. So, this kind of contract stipulated at time t guarantees us a **riskless** rate of interest over the future range $[T, S]$. This interest rate is called **forward rate**.

The main forward rates are continuously compounded rates or simple rates, defined as follows.

Definition 1.3.2. The **continuously compounded forward rate R** for $[T, S]$, where $T > t$ is the expiry and $S > T$ the maturity time, contracted at time t, is defined by:

$$R(t, T, S) := -\frac{ln(p(t, S)) - ln(p(t, T))}{S - T} \tag{1.7}$$

solution to the equation:

$$e^{R(S-T)} = \frac{p(t, T)}{p(t, S)}.$$

Whereas, the **simple compounded forward rate**, or **LIBOR** rate **F**, defined on the same three time instants $t \leq T \leq S$, is given by:

$$F(t, T, S) := -\frac{p(t, S) - p(t, T)}{(S - T)p(t, S)} \tag{1.8}$$

solution to the equation:

$$1 + (S - T)F = \frac{p(t, T)}{p(t, S)}$$

We remember that $S - T = \tau(T, S)$.

1.3.1 Forward rate agreement FRA

We give, now, a definition of forward rate through the **forward rate agreement (FRA)**, which is an agreement made to assure that a determined interest rate will apply to either borrowing or lending a certain capital during a specified future period of time. A FRA is a contract featured by three time instants: t the current time, T the expiry time and S the maturity time, with $t < T < S$. The contract provides that at the maturity time S, a fixed payment based on a fixed rate K is exchanged against a floating payment based on the spot rate $L(T, S)$ resetting in T. Hence, this contract allows to lock the interest rate for the range $[T, S]$ at a desired value K, with the rates that are simply compounded. Formally, the value of the contract in S is given by the difference between the fixed rate multiplied for the duration of the contract and the nominal value N, that represents units of currency that one receives, and same product for the floating rate, that is the amount to pay. In formulas:

$$N\tau(T, S)(K - L(T, S)).$$

For the definition of LIBOR rate (1.4), we have:

$$N\left[\tau(T,S)K - \frac{1}{p(T,S)} + 1\right] \tag{1.9}$$

Considering $1/p(T,S)$ the quantity of currency at time S, its value at time T is given by the product $p(T,S)(1/p(T,S)) = 1$, where $p(T,S)$ is the zero coupon price. Furthermore, one unit of currency at time T is given by $p(t,T)$ units of currency at time t. And so, the quantity $1/p(T,S)$ in S is equivalent to the amount $p(t,T)$ in t.
Replacing these observations in (1.9), we obtain that the value of FRA at time t is:

$$FRA(t,T,S,\tau(T,S),N,K) = N[p(t,S)\tau(T,S)K - p(t,T) + p(t,S)]. \tag{1.10}$$

So, remembering (1.8), we can express the value of FRA in terms of simply compounded forward interest rate:

$$\begin{aligned} FRA(t,T,S,\tau(T,S),N,K) &= N\left[p(t,S)\tau(T,S)K + p(t,S)\left(1 - \frac{p(t,T)}{p(t,S)}\right)\right] = \\ &= N[p(t,S)\tau(T,S)K + p(t,S)(-\tau(T,S)F(t,T,S))] \end{aligned}$$

$$\implies \quad FRA(t,T,S,\tau(T,S),N,K) = Np(t,S)\tau(T,S)(K - F(t,T,S)). \tag{1.11}$$

Remark 1.3.3. As we have done for the spot rate, we can define the *instantaneous forward rate* as the limit of the previous forward rates:

$$\begin{aligned} \lim_{S\to T^+} L(t,T,S) &= -\lim_{S\to T^+} \frac{1}{p(t,S)} \frac{p(t,S)-p(t,T)}{S-T} \\ &= -\frac{1}{p(t,T)} \frac{\partial p(t,T)}{\partial T} \\ &= \frac{\partial ln(p(t,T))}{\partial T} \end{aligned}$$

1.3.2 Interest Rate Swaps

A generalization of the FRA is the **Interest Rate Swap (IRS)**. A swap is a commitment to exchange the payments originating from a fixed and a floating leg.

Definition 1.3.4. Given a space of dates $\mathcal{T} = \{T_\alpha, \ldots, T_\beta\}$ and a set of year fractions $\tau = \{\tau_\alpha, \ldots, \tau_\beta\}$, at a specified instant T_i the amount of the fixed leg is made up of the product between the fixed interest rate K, the nominal value N and a year fraction τ_i between T_{i+1} and T_i, so it is given by

$$N\tau_i K,$$

on the other hand, the floating part payment is the quantity

$$N\tau_i L(T_i - 1, T_i)$$

where $L(T_i - 1, T_i)$ is the interest rate resetting at the instant $T_{i-1} \in T_\alpha, T_{\alpha+1}, \ldots, T_{\beta-1}$ for the maturity given by the current payment instant T_i into the dates $T_{\alpha+1}, \ldots, T_\beta$.

Remark 1.3.5. The payer of the fixed leg is termed **PFS** (i.e. Payer(Forward-Start) Interest Rate Swap) whereas the floating leg is received by the Receiver IRS (**RFS**). The *discounted payoff* of a PFS at a time $t < T_\alpha$ is given by

$$\sum_{t=\alpha+1}^{\beta} D(t, T_i) N\tau_i (L(T_{i-1}, T_i) - K)$$

and accordingly, the *discounted payoff* of a RFS at a time $t < T_\alpha$ can be expressed as

$$\sum_{t=\alpha+1}^{\beta} D(t, T_i) N\tau_i (K - L(T_{i-1}, T_i)).$$

We can see this contract as a portfolio of FRAs, we can evaluate the latter sum as a sum of (1.10) or (1.11):

$$\begin{aligned}
\mathbf{RFS}(t, \mathcal{T}, \tau, N, K) &= \sum_{t=\alpha+1}^{\beta} \mathbf{FRA}(t, T_{i-1}, T_i, \tau_i, N, K) = \\
&= N \sum_{t=\alpha+1}^{\beta} \tau_i p(t, T_i)(K - F(t, T_{i-1}, T_i)) = \\
&= -Np(t, T_\alpha) + Np(t, T_\beta) + N \sum_{t=\alpha+1}^{\beta} \tau_i K p(t, T_i).
\end{aligned} \tag{1.12}$$

Definition 1.3.6. The **forward swap rate** $S_{\alpha,\beta}(t)$ at time t, for the sets of times $\mathcal{T}$ and year fractions τ, is the fixed rate K calculated when $\mathbf{RFS}(t, \mathcal{T}, \tau, N, K) = 0$, i.e. it is chosen such that the value of the swap equals zero at the present time t. So from the latter equation we have:

$$S_{\alpha,\beta}(t) = \frac{p(t,T_\alpha) - p(t,T_\beta)}{\sum_{t=\alpha+1}^{\beta} \tau_i p(t,T_i)} \tag{1.13}$$

We can write this definition in terms of forward rate(LIBOR) dividing both the numerator and the denominator by $p(t, T_\alpha)$

$$\frac{p(t,T_k)}{p(t,T_\alpha)} = \prod_{j=\alpha+1}^{k} \frac{p(t,T_j)}{p(t,T_{j-1})} = \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)} \quad \forall k > \alpha$$

where $F_j(t) = F(t, T_{j-1}, T_j)$.
Therefore, we obtain in (1.13)

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1,\ldots,\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{t=\alpha+1,\ldots,\beta} \tau_i \prod_{j=\alpha+1,\ldots,i} \frac{1}{1+\tau_j F_j(t)}}$$

1.4 Derivative of Interest-Rate

1.4.1 Interest rate Caps/Floors

Interest rates **cap** or **floor** are financial insurance contracts that exchange payments at the end of each period only if it has a positive value. The cap can be viewed as a payer IRS whereas a floor is equivalent to a receiver IRS.
Given a set of dates $\mathcal{T} = \{T_\alpha, \ldots, T_\beta\}$ with associated the set of year fractions $\tau = \{\tau_{\alpha+1}, \ldots, \tau_\beta\}$, the cap discouted payoff is given by

$$\sum_{t=\alpha+1}^{\beta} D(t,T_i) N \tau_i (L(T_{i-1}, T_i) - K)^+.$$

On the other hand, the floor discounted payoff is given by

$$\sum_{t=\alpha+1}^{\beta} D(t,T_i) N \tau_i (K - L(T_{i-1}, T_i))^+.$$

Where "+" denotes the fact that we take the max between the difference of fixed and floating rate or otherwise 0, i.e. $max\big((L(T_{i-1}, T_i) - k), 0\big)$ for the cap and $max\big((K - L(T_{i-1}, T_i)), 0\big)$.
Technically, a cap is the sum of a number of basic contracts, called *caplet*, it is determined at T_{i-1} but not payed our until at time T_i, it is given by,

$$X_i := D(t, T_i) N \tau_i (L(T_{i-1}, T_i) - K)^+$$

and analogously a floor can be viewed as a portfolio of *floorlet* contracts.

Remark 1.4.1. The market practice is to price a cap by using the Black's formulas at time t

$$\mathbf{Cap}^{Black}(t, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) = N \sum_{t=\alpha+1}^{\beta} p(t, T_i) \tau_i Bl(K, F(t, T_{i-1}, T_i), v_i, 1) \tag{1.14}$$

where, denoting by Φ the standard Gaussian distribution function,

$$\begin{aligned} Bl(K, F, v, w) &= Fw\Phi(wd1(K, F, v)) - Kw\Phi(wd2(K, F, v)) \\ d1(K, F, v) &= \frac{ln(F/K) + v^2/2}{v} \\ d2(K, F, v) &= \frac{ln(F/K) - v^2/2}{v} \\ v_t &= \sigma_{\alpha,\beta}\sqrt{T_{t-1}} \end{aligned} \tag{1.15}$$

the costant $\sigma_{\alpha,\beta}$ is known as the **Black volatility** for caplets.
Analogously,the floor is priced by using the formula:

$$\mathbf{Flr}^{Black}(t, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) = N \sum_{t=\alpha+1}^{\beta} p(t, T_i) \tau_i Bl(K, F(t, T_{i-1}, T_i), v_i, -1) \tag{1.16}$$

In the market, therefore, cap prices are not quoted in monetary terms but instead in terms of implied Black volatilities.
It is implicit in the Black formula that the forward rate are lognormal, under some probability measures.

1.4.2 Swaptions

A **European swaption** is defined as the option, purchased at time t, to enter at time T_α a swap spanning the period (called tenor) $T_\beta - T_\alpha$, with the fixed leg paying at the swaption maturity with a fixed swap rate K.
The option will be exercised only if this value is positive, so the payer swaption payoff, discounted from the maturity time T_α to the current time is equal to

$$ND(t,T_\alpha)\Big(\sum_{t=\alpha+1}^{\beta} p(T_\alpha,T_i)\tau_i(F(T_\alpha,T_{i-1},T_i)-K)\Big)^+.$$

Contrary to the cap, this payoff cannot be decomposed in elementary product, each depending on a single forward rate.

Remark 1.4.2. The market practice is to price swaptions by using the Black's formula, the price of the payer swaption at time t is:

$$\mathbf{PS}^{Black}(t,\mathcal{T},\tau,N,K,\sigma_{\alpha,\beta}) = N\sum_{t=\alpha+1}^{\beta} p(t,T_i)\tau_i Bl(K,S_{\alpha,\beta}(t),\sigma_{\alpha,\beta}\sqrt{T_\alpha},-1) \qquad (1.17)$$

where $\sigma_{\alpha,\beta}$ is a volatility parameter quoted in the market, it is different from the caps/floors Black volatility.

Chapter 2

The LIBOR market models

In this chapter, we present one of the most popular interest-rate models: the LIBOR market models. LIBOR is short for *London Interbank Offered Rate* and it serves as the first step to calculating interest rates on various loans throughout the world. The British Bankers' Association publishes daily LIBOR values for a range of different currencies. The actual determination of these values is obtained by averaging quotes for loans from a number of contributing banks.

The market practice is to price the option (call(through the cap) or put(through the floor) option on a forward rate) assuming that the underlying forward rate process is lognormally distributed with zero drift. Before market models were introduced, there was no interest-rate dynamics compatible with either Black's formula for caps or Black's formula for swaptions. The introduction of market models provided a new derivation of Black's formulas based on rigorous interest-rate dynamics.

The main advantage of this market practice is that it can be made consistent with an arbitrage-free term structure model. Consecutive quarterly or semiannual forward rates can all be lognormal while the model will remain arbitrage free. This is possible because each rate is lognormal under the forward to the settlement date arbitrage-free measure rather than under spot arbitrage-free measure. Lognormality under the appropriate measure is needed to justify the use of the Black futures formula with discount for caplet pricing.

The market models will thus produce pricing formulas for caps and floors (the LIBOR models), and swaptions (the Swap market models) which are of the Black-76 model type and so conforming to the market practice. The LIBOR model market(LMM) is also known as "Log-normal Forward LIBOR Model" or "Brace-Gatarek-Musiela 1997 Model" (BGM model), from the names of the authors that described it.

2.1 Description of the Lognormal Forward-LIBOR Model (LFM)

We make the usual mathematical assumptions. All processes are defined on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where the filtration $\{\mathcal{F}_t; t \leq 0\}$ is the natural filtration generated by M-dimensional Brownian motion $Z = \{Z(t); t \leq 0\}$.
Let $t = 0$ be the current time. Let a set of increasing expiries-maturities $\{T_0, \dots, T_M\}$ from which are taken pair of dates (T_{k-1}, T_k) for the forward rates. We consider also the corresponding set of year fractions $\{\tau_0, \dots, \tau_M\}$, where τ_k is the year fraction associated with the expiry-maturity pair (T_{k-1}, T_k) for $k > 0$, in a most popular caps it is equal to a quarter of a year; besides τ_0 is the year fraction from expiry T_0. We set $T_{-1} := 0$.
We also consider:

- $F_k(t) := F(t, T_{k-1}, T_k)$, $k = 1, \dots, M$ the generic forward rate between times T_k e T_{k-1} where in the latter time it coincides with the simply-compounded spot rate prevailing at time T_{k-1} for the maturity T_k, i.e. $F_k(T_{k-1}) = L(T_{k-1}, T_k)$;

- Q^k the probability measure associated with the numeraire $p(\cdot, T_k)$, it is often called the *forward (adjusted) measure for the maturity* T_k. And we can prove Q^k is a martingale measure for F_k, indeed we have the following result.

  Lemma 2.1.1. *For every* $k = 1, \dots, M$ *the LIBOR process* F_k *is a martingale under the corresponding forward measure* Q^k *on the interval* $[0, T_{k-1}]$.

  Proof. By the definition of simple compounding, we have

$$F_k(t)p(t, T_k) = \frac{p(t, T_{k-1}) - p(t, T_k)}{\tau_k}$$

being $p(t, T_{k-1}) - p(t, T_k)$ a difference between two price of discount bonds and $1/\tau_k$ a notional, results that $F_k(t)p(t, T_k)$ is the price of a tradable asset. So, when it is normalized respect to the numeraire $p(\cdot, T_k)$, and then divided by $p(t, T_k)$, F_k has to be a martingale under the measure Q^k. □

- $Z^k(t)$ a M-dimensional Brownian motion, under the measure Q^k, with instantaneous covariance $\rho = (\rho_{i,j})_{i,j=1,\dots,M}$

$$E[dZ^k(t)dZ^k(t)'] = \rho dt$$

- $\sigma_k(t)$ the instantaneous *volatility* coefficient for the forward rate $F_k(t)$

Under these hypotheses, it follows that F_k is modeled according to a driftless dynamics under Q^k.

Definition 2.1.2. If the LIBOR forward rates have the dynamics

$$dF_k(t) = \sigma_k(t)F_k(t)dZ_k^k(t) \qquad t \leq T_{k-1} \tag{2.1}$$

where Z_k^k is k-th component of Q^k-Wiener as described above, lower indices indicate the components whereas upper indice show under which measure we are working. Then we say that (2.1) describe a discrete **LIBOR market model** with volatilities σ_k for all $k = 1, \dots, M$

Remark 2.1.3. We consider now a scalar notation of (2.1)

$$dF_k(t) = \sigma_k(t)F_k(t)dZ_k(t) \qquad t \leq T_{k-1}$$

and if σ_k is bounded, it has a unique strong solution that describes a geometric Brownian motion. Indeed through Itô's formuala we obtain

$$dlnF_k(t) = -\frac{\sigma_k(t)^2}{2}dt + \sigma_k(t)dZ_k(t) \qquad t \leq T_{k-1}$$

$$\Longrightarrow lnF_k(T) = lnF_k(0) - \int_0^T \frac{\sigma_k(t)^2}{2}dt + \int_0^T \sigma_k(t)dZ_k(t)$$

$$\Longrightarrow F_k(T) = F_k(0) \cdot e^{-\frac{1}{2}\int_0^T \sigma_k(s)^2 ds + \int_0^T \sigma_k(s)dZ_k(s)} \qquad t \leq T_{k-1}$$

It is now easy to see that if σ_k is assumed to be deterministic, under this measure F_k has lognormal distribution indeed we can write:

$$F_k(T) = F_k(0) \cdot e^{Y_k(0,T)}$$

where $Y_k(0,T)$ is normally distribute with expected value

$$m_k(0,T) = -\frac{1}{2}\int_0^T \sigma_k(s)^2 ds$$

and variance

$$\Sigma_k^2(0,T) = \int_0^T \sigma_k(s)^2 ds.$$

2.1.1 Forward Rate Dynamics under different Numeraires

We calculate the dynamics of $F_k(t)$ under a measure Q^i different from Q^k for $t \leq min(T_i, T_{k-1})$, that have to be alive in t.

Proposition 2.1.4. *Under the lognormal assumption, we obtain that the dynamics of F_k under the forward measure Q^i in three cases are:*

$$\begin{aligned}
i < k \quad t \leq T_i: &\quad dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=i+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}dt + \sigma_k(t)F_k(t)dZ_k(t)\\
i = k \quad t \leq T_{k-1}: &\quad dF_k(t) = \sigma_k(t)F_k(t)dZ_k(t)\\
i > k \quad t \leq T_{k-1}: &\quad dF_k(t) = -\sigma_k(t)F_k(t)\sum_{j=k+1}^{i}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}dt + \sigma_k(t)F_k(t)dZ_k(t)
\end{aligned}$$

where we remember that $\rho_{k,j}$ is a correlation matrix of differents Brownian motions, and $Z = Z^i$ is a Brownian motion under Q^i.

Proof. Consider the forward rate $F_k(t) = F(t, T_{k-1}, T_k)$ under the forward measure Q^i with $i < k$. By remembering that the dynamics under the T_k-forward measure Q^k has null drift, starting from the latter we compute the dynamics under Q^i.

In order to prove this proposition, we consider the result of Girsanov Theorem, which shows that it is possible to arbitrarily replace the drift of a Itô process by suitably modifying the probability measurement and the Brownian motion. So we find the appropriate

Girsanov transformation that will take us from Q^k to Q^i. Denoting by $\eta_k^i(t)$ the Radon Nikodym derivative:

$$\eta_k^i(t) := \frac{dQ^i}{dQ^k}\bigg|_{\mathcal{F}_t} = \frac{p(0,T_k)p(t,T_i)}{p(0,T_i)p(t,T_k)}$$

and then in particular, from (1.8) we have

$$\eta_i^{i-1} = \frac{p(0,T_i)}{p(0,T_{i-1})} \cdot (1 + \tau_i F_i(t))$$

We compute the dynamics of η_i^{i-1} under Q^i, assuming that Q^i satisfies (2.1), is:

$$\begin{aligned} d\eta_i^{i-1}(t) &= \frac{p(0,T_i)}{p(0,T_{i-1})}\tau_i dF_i(t) = \frac{p(0,T_i)}{p(0,T_{i-1})}\tau_i F_i(t)\sigma_i(t)dZ^i(t) \\ &= \frac{p(0,T_i)}{p(0,T_{i-1})}\tau_i\frac{1}{\tau_i}\Big(\frac{p(t,T_{i-1})}{p(t,T_i)} - 1\Big)\sigma_i(t)dZ^i(t) \\ &= \eta_i^{i-1}\frac{p(0,T_i)}{p(0,T_{i-1})}\frac{1}{\eta_i^{i-1}}\Big(\frac{p(t,T_{i-1})}{p(t,T_i)} - 1\Big)\sigma_i(t)dZ^i(t) \end{aligned}$$

$$\Longrightarrow d\eta_i^{i-1}(t) = \eta_i^{i-1}(t)\frac{\tau_i F_i(t)}{1+\tau_i F_i(t)}\sigma_i(t)dZ^i(t) \tag{2.2}$$

Thus, the Girsanov kernel $\eta_i^{i-1}(t)$ for the transition from Q^i to Q^{i-1}, that represents the drift correction for forward rate $F_i(t)$, is given by:

$$\lambda := \frac{\tau_i F_i(t)}{1+\tau_i F_i(t)}\sigma_i^{'}(t)$$

so from the Girsanov theorem we have:

$$dZ^i(t) = \frac{\tau_i F_i(t)}{1+\tau_i F_i(t)}\rho_{i,i-1}\sigma_i^{'}(t)dt + dZ^{i-1}(t)$$

And applying this inductively we obtain:

$$i<k: \quad dZ^i(t) = -\sum_{j=i+1}^{k}\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\rho_{k,j}\sigma_j^{'}(t)dt + dZ^k(t) \tag{2.3}$$

$$i>k: \quad dZ^i(t) = \sum_{j=k+1}^{i}\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\rho_{i,j}\sigma_j^{'}(t)dt + dZ^k(t) \tag{2.4}$$

□

Corollario 2.1.5. *Consider a given volatility structure* $\sigma_1, \dots, \sigma_N$*, where each* σ_i *is bounded, a probability measure* Q^N *and a* Q^N*-Brownian motion* Z^N*. We define the processes* $F_1, \dots, F_N$ *by*

$$dF_i(t) = -F_i(t)\sigma_i(t)\left(\sum_{j=i+1}^{N} \frac{\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}\rho_{k,j}\right)dt + F_i(t)\sigma_i(t)dZ^N(t) \tag{2.5}$$

for $i = 1, \dots, N$*.*
Then the Q^i*-dynamics of* F_i *are given by (2.1) and thus exists a LIBOR model with the given volatility structure.*

Proof. First of all, we prove that (2.5) admits solution, for $i = N$ it is simple to see that

$$dF_N(t) = \sigma_N(t)F_N(t)dZ^N(t)$$

which is an exponential martingale, because we know that the measure-dependent drift correction turns an arbitrary forward rate into a forward rate driftless (i.e. an exponential martingale); since σ_N is bounded the solution does exist.
We suppose now that (2.5) admits a solution for $k = i+1, \dots, N$, then we can write the i-th component $(i < k)$ of (2.5) as

$$dF_i(t) = -F_i(t)\mu_i[t, F_{i+1}(t), \dots, F_N(t)]dt + F_i(t)\sigma_i(t)dZ^N(t)$$

where $\mu_i[t, F_i(t)] = \sigma_i(t)\left(\sum_{j=i+1}^{N} \frac{\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}\rho_{k,j}\right)$ the percentage drift of dF_k under Q^i and it only depends on F_k with $k = i+1, \dots, N$. Denoting the transpose vector $(F_{i+1}(t), \dots, F_N(t))$ by F_{i+1}^N, using the Itô formula, we thus have the explicit solution:

$$\begin{aligned} dlnF_i(t) &= \frac{dF_i(t)}{F_i(t)} - \frac{1}{2F_i(t)^2}\sigma_i(t)^2F_i(t)^2dt = \\ &= \mu_i[s, F_{i+1}^N(t)]dt + \sigma_i(t)dZ_i^N(t) - \frac{1}{2}\sigma_i(t)^2dt \end{aligned}$$

$$\Longrightarrow \quad lnF_i(t) = lnF_i(0) + \int_0^t \left(\mu_i[s, F_{i+1}^N(s)] - \frac{1}{2}\sigma_i^2(s)\right)ds + \int_0^t \sigma_i(s)dZ_i^N(s)$$

$$\Longrightarrow \quad F_i(t) = F_i(0)exp\left\{\int_0^t \left(\mu_i[s, F_{i+1}^N(s)] - \frac{1}{2}\sigma_i^2(s)\right)ds\right\} \cdot exp\left\{\int_0^t \sigma_i(s)dZ_i^N(s)\right\}$$

for $0 \le t \le T_{i-1}$. So this proves the existence by induction. Now analogously to the previous proposition, knowing that λ, the Girsanov kernel, is also bounded this involves that it satisfies the Novikov condition. $\square$

2.2 Pricing Caps in the LIBOR model

2.2.1 Equivalence between LFM and Black's caplet prices

We are now able to show that Black's model can be used to price European options in terms of the forward price of the underlying asset when interest rates are stochastic. Consider a European call option (cap) on an asset with strike price K and payment dates $T_{\alpha+1}, \ldots, T_\beta$. The option's price at time 0 of a cap can be obtained by considering the risk neutral expectation E of its discounted payoff, so it is given by

$$\begin{aligned} & E\left\{ \sum_{t=\alpha+1}^{\beta} D(0,T_i)\tau_i(F_i(T_{i-1}) - K)^+ \right\} \\ &= \sum_{t=\alpha+1}^{\beta} P(0,T_i)\tau_i E^i[(F_i(T_{i-1}) - K)^+] \end{aligned}$$

with the nominal $N = 1$, and where E^i denotes expectation under the forward measure Q^i. It is a sum of contract called T_{i-1}-caplet. Recall that a T_{i-1}-caplet is a contract paying at time T_i the difference between the T_i maturity spot rate at time T_{i-1} and a strike rate K, so the price at initial time 0 is:

$$p(0,T_i)E^i[max(F_i(T_{i-1}) - K, 0)]$$

Then given the lognormal distribution for F_i, we can compute a Black and Scholes price for a stock call option whose underlying stock is F_i, struck at K, with maturity T_{i-1}, with zero constant "risk-free rate" and instantaneous percentage volatility $\sigma_i(t)$. The price of the T_{i-1}-caplet implied by the LFM coincides with that given by the corresponding Black caplet formula, i.e.

$$\begin{aligned} \mathbf{Cpl}^{LFM}(0,T_{i-1},T_i,K) &= \mathbf{Cpl}^{Black}(0,T_{i-1},T_i,K,v_i) && (2.6) \\ &= p(0,T_i)\tau_i Bl(K, F_i(0), v_i) && (2.7) \end{aligned}$$

$$
\begin{aligned}
Bl(K, F_i(0), v_i) &= E^i(F_i(T_{i-1}) - k)^+ \\
&= F_i(0)\Phi(d_1(K, F_i(0), v_i)) - K\Phi(d_2(K, F_i(0), v_i))
\end{aligned}
$$

$$
\begin{aligned}
d_1(K, F, v) &= \frac{ln(F/K) + v^2/2}{v}, \\
d_2(K, F, v) &= \frac{ln(F/K) - v^2/2}{v},
\end{aligned}
$$

where

$$
\begin{aligned}
v_i^2 &= T_{i-1} v^2_{T_{i-1}-caplet} \\
v^2_{T_{i-1}-caplet} &:= \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i^2(t)dt.
\end{aligned}
$$

The quantity $v^2_{T_{i-1}-caplet}$ is called T_{i-1}- volatility and it has been defined as the square root of the average percentage variance of the forward rate $F_i(t)$ for $t \in [0, T_{i-1}]$.

2.2.2 Spot vs Flat Volatilities

It is therefore clear that in the market the cap prices are quoted in terms of Black volatilities, and these volatilities can furthermore be quoted as **flat** volatility or as **forward or spot** volatility as a function of maturity. In the first, the maturity is the maturity of a cap or floor, instead in the case of spot volatility, it is the maturity of a caplet or floorlet.
Given the fixed set of dates $\mathcal{T}_j = \{T_0, \ldots, T_N\}$, we assume that for each $i = 1, \ldots, N$ there is a cap with resettlement dates $T_0, \ldots, T_N$ and we denote the market price by $\mathbf{Cap}^m$, the *implied Black volatilities* are defined as follows

- the *flat volatilities* $v_{T_1-cap}, \ldots, v_{T_N-cap}$ are the solutions of the equation:

$$
\mathbf{Cap}^m(t, \mathcal{T}_j, K) = \sum_{i=1}^{j} \mathbf{Cap}^{Black}(t, T_{i-1}, T_i, K, v_{T_j-cap}) \qquad j = 1, \ldots, N
$$

- the *spot or forward volatilities* $v_{T_1-caplet}, \ldots, v_{T_N-caplet}$ are the solutions of the equation:

$$
\mathbf{Capl}^m(t, T_{i-1}, T_i, K) = \mathbf{Capl}^{Black}(t, T_{i-1}, T_i, K, v_{T_i-caplet}) \qquad i = 1, \ldots, N
$$

where $\mathbf{Capl}^m(t, T_{i-1}, T_i, K) = \mathbf{Cap}^m(t, \mathcal{T}_i, K) - \mathbf{Cap}^m(t, \mathcal{T}_{i-1}, K)$

The following graph shows the typical pattern for spot volatilities and flat volatilities as a function of maturity, moreover we remind that the caps whose implied volatilities are quoted by the market typically have either T_0 equal to three months for $\alpha = 0$ and the other T_i's are equally three months spaced , or T_0 equal to six months for $\alpha = 0$ and all other T_i's equally six months spaced.

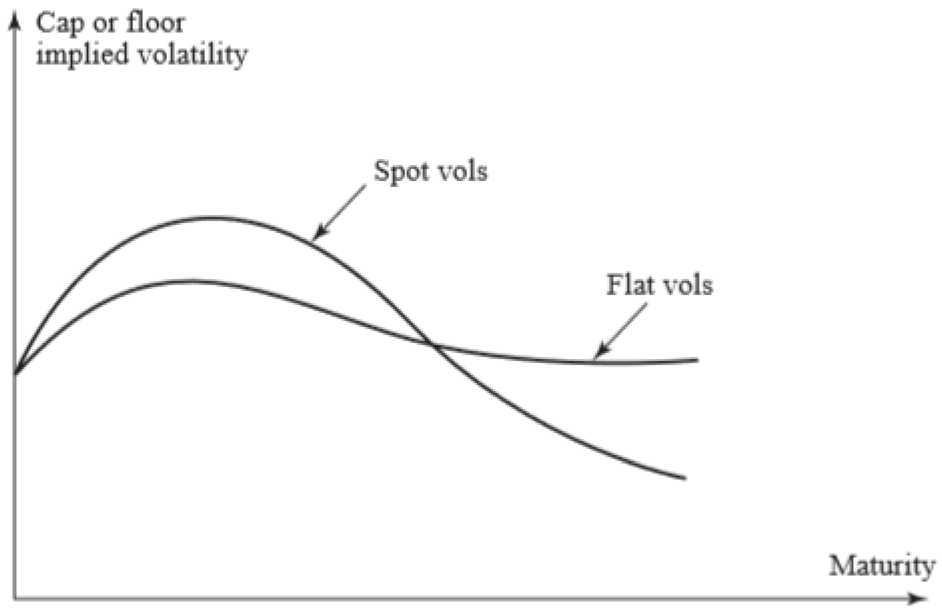

Figure 2.1: The volatily hump (J.Hull)

Chapter 3

Risk Functional

In this chapter, we introduce the problem of seeking an optimal set of model points associated to a fixed portfolio. First of all, the problem of efficient substitution of a portfolio of market securities is closely related to the portfolio representation problem. It consists in defining the dynamics of those portfolios within a certain class that best represents the inherent risk structure of a given financial exposure.
To generalize this problem, a specific risk functional is minimized, which measures the average discrepancy between the two portfolios in terms of stochastic variation of the underlying risk factors within a given time frame.

We first study the general case on the Banach space and in the next chapter we examine the problem in term life insurance.
This chapter is based on [8].

We fix a separable Hilbert space H and we write $\langle \cdot, \cdot \rangle_H$ to denote its inner product. We also fix a Banach space E with its dual E^*, the duality pairing of $x^* \in E^*$ and $x \in E$ is denoted by $\langle x^*, x \rangle_E$. Furthermore, we denote the space of bounded linear operators from H to E with $\mathcal{L}(H, E)$ and $\mathcal{L}(E^*, H)$ is its adjoint. We write I to denote the unit interval on the real line. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a reference complete probability space and for any $X \in L^1(\Omega, E)$, a Banach space, $E[X]$ is the expected value of X with respect to $\mathbb{P}$. We fix an H-cylindrical Wiener process $W = \{W(t) : t \in I\}$, toghether with the augmented filtration $\mathcal{G}^W$ that is generates, and we fix $\mathcal{H} = L^2(I, H)$. Where a H-

cylindrical Wiener process W is a one-parameter family of bounded and linear operator from H to $L^2(\Omega)$, that satisfy:

- for any $h \in H$ the process $Wh = \{W_t h : t \in I\}$ is a standard Brownian motion;
- for any $t, s \in I$ and $h_1, h_2 \in H$ results $\mathbb{E}\{W_t h_1 W_s h_2\} = (s \wedge t)\langle h_1, h_2\rangle_H$.

In the sequel, a process X is adapted if it is adapted to the filtration $\mathcal{G}^W$. Moreover, we say X to be stochastically integrable if it is so with respect to W.

Let $\xi_0 \in L^2(\Omega, E)$ be a strongly $\mathcal{G}_0^W$-measurable random variable, consider an adapted and strongly measurable E-value stochastic process $b = \{b_t : t \in I\}$ that belongs to $L^2(\Omega, L^2(I, E))$. Where an E-valued process is a one parameter family of E-valued random variables indexed by I. Further, we fix an adapted and H-strongly measurable $\mathcal{L}(H, E)$-value process $\sigma = \{\sigma_t : t \in I\}$ that belongs to $L^2(\Omega, L^2(I, \gamma(H, E)))$, where $\gamma(H, E)$ is the operator ideal in $\mathcal{L}(H, E)$.
Furthermore, the processes satisfy:

$$\xi_0 \in \mathbb{H}^{1,2}(E) \qquad b \in \mathbb{H}^{1,2}(L^2(I, E)) \qquad \sigma \in \mathbb{H}^{2,2}(L^2(I, \gamma(H, E))).$$

Then we define the E-valued process $\xi = \{\xi_t : t \in I\}$ by setting

$$\xi_t = \xi_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s \qquad for\ \ any\ \ t \in I. \tag{3.1}$$

Definition 3.0.1. Let D be some UMD Banach space. Let $\psi : I \times E \to D$ be a function of class $\mathcal{C}^{1,2}$, i.e. it is differentiable in the first variable and twice continuously Fréchet differentiable in the second variable. We say that ψ is a $BS-function$ relative to ξ, if the following condition is satisfied a.s.

$$\nabla_1 \psi(t, \xi_t) + \frac{1}{2} tr(\nabla_2^2 \psi(t, \xi_t); \sigma_t) = 0, \qquad for\ \ any\ \ t \in I. \tag{3.2}$$

where $\nabla_k \psi$ denotes the derivative of ψ with respect to the k-th component, for any $k = 1, 2$.

Lemma 3.0.2. *Let $\psi : I \times E \longrightarrow D$ be a function of class $\mathcal{C}^{1,2}$. If ψ is assumed to be a $BS-function$ relative to ξ, then the dynamics of the process $\psi(t,\xi_t)$ is so characterized:*

$$\psi(t,\xi_t) = \psi(0,\xi_0) + \int_0^t \nabla_2\psi(s,\xi_s)b_s ds + \int_0^t \nabla_2\psi(s,\xi_s)\sigma_s dW_s, \quad a.s.\ for\ any\ t \in I. \tag{3.3}$$

Definition 3.0.3. By a P-set relative to ξ we understand any set Φ of E^*-valued and adapted process $\phi = \{\phi_t : t \in I\}$ such that the following conditions are satisfied:

(i) for any $\phi \in \Phi$ results $\parallel \phi \parallel_\infty < \infty$, where

$$\parallel \phi \parallel_\infty = inf\{C \leq 0 : \parallel \phi_t \parallel_{E^*} \leq C\ \ a.s.\ for\ any\ t \in I\};$$

(ii) for any $\phi \in \Phi$, there exists a function $\varphi : I \times E \to E^*$ of class $\mathcal{C}_b^{1,2}$, i.e. φ satisfies $\parallel \nabla_2\varphi \parallel_\infty = \sup_{(t,x)\in I\times E} \parallel \nabla_2\varphi(t,x) \parallel_{\mathcal{L}(E,D)} < \infty$, such that

$$\phi_t = \varphi(t,\xi_t), \qquad a.s.\ for\ any\ t \in I;$$

(iii) for any $\phi \in \Phi$ the following identity holds true

$$\langle \phi_t, \xi_t \rangle_E = \langle \phi_0, \xi_0 \rangle_E + \int_0^t \langle \phi_s, b_s \rangle_E ds + \int_0^t \langle \phi_s, \sigma_s \rangle_E dW_s \qquad a.s. \tag{3.4}$$

Definition 3.0.4. Fix a P-set Φ relative to ξ and consider a function $f : I \times E \to \mathbb{R}$. We define the **discrepancy process** between f and ϕ relative to ξ, for any $\phi \in \Phi$ as $F(\phi) = \{F_t(\phi) : t \in I\}$ where

$$F_t(\phi) = f(t,\xi_t) - \langle \phi_t, \xi_t \rangle_E \qquad for\ any\ t \in I. \tag{3.5}$$

If not otherwise specified, when f and Φ are fixed, we write $F(\phi)$ to denote the discrepancy process.

Lemma 3.0.5. *Let Φ be a P-set relative to ξ , if $f : I \times E \to \mathbb{R}$ is of classe $\mathcal{C}_b^{1,2}$ then $F_t(\phi) \in L^1(\Omega)$, for any $t \in I$ and $\phi \in \Phi$ with*

$$\sup_{t\in I} \parallel F_t(\phi) \parallel^2_{L^2(\Omega)} < \infty$$

Lemma 3.0.6. *Let Φ be a P-set relative to ξ and $f : I \times E \to \mathbb{R}$ a function of class $\mathcal{C}_b^{1,2}$ then $F_t(\phi) \in \mathbb{H}^{1,2}$ for any $t \in I$ and $\phi \in \Phi$ with*

$$DF_t(\phi) = (\nabla_2 f(t, \xi_t) - \phi_t) D\xi_t \qquad a.s. \tag{3.6}$$

where $D : L^2(\Omega, E) \to L^2(\Omega, \gamma(\mathcal{H}, E))$ is a closed operator that it represents the Malliavin derivative; moreover $\mathbb{H}^{1,2} = \mathbb{H}^{1,2}(\mathbb{R})$.
We also point up that for any $t \in I$, (3.6) is equivalent to:

$$DF_t(\phi) = \nabla_2 f(t, \xi_t) D\xi_t - \langle \phi_t, D\xi_t \rangle_E \qquad a.s. \tag{3.7}$$

Definition 3.0.7. Let Φ be a P-set relative to ξ and let $f : I \times E \to \mathbb{R}$ be a function of class $\mathcal{C}_b^{1,2}$, we define the **risk functional**, relative to ξ induced by f over Φ, $\mathcal{F} : \Phi \to \mathbb{R}$ given by

$$\mathcal{F}(\phi) = \int_I \mathbb{E}\{[|F_t(\phi) - \mathbb{E}F_t(\phi)|^2]\} dt \qquad for\ any\ \phi \in \Phi \tag{3.8}$$

If not otherwise specified, when f and Φ are fixed, we use $\mathcal{F}$ to denote the risk functional relative to ξ induced by f over Φ.
The following theorem gives us two different equivalent representations of the risk functional.

Theorem 3.0.8. *Given Φ a P-set relative to ξ and $f : I \times E \to \mathbb{R}$ a function of class $\mathcal{C}_b^{1,2}$. Then:*

(i) the functional $\mathcal{F}$ admits the following representation

$$\mathcal{F}(\phi) = \int_I \mathbb{E}\Big\{ \int_0^t \| \mathbb{E}\{(\nabla_2 f(t, \xi_t) - \phi_t) D_s \xi_t | \mathcal{G}_s^W\} \|_H^2 \, ds \Big\} dt \qquad for\ any\ \phi \in \Phi; \tag{3.9}$$

(ii) if f is assumed to be a $BS - function$ relative to ξ and $b_t = 0$ a.s. for any $t \in I$ the functional $\mathcal{F}$ boils down to

$$\mathcal{F}(\phi) = \mathbb{E}\Big\{ \int_I \| (\nabla_2 f(t, \xi_t) - \phi_t) \sigma_t \|_H^2 \, (1 - t) dt \Big\} \qquad for\ any\ \phi \in \Phi; \tag{3.10}$$

Since σ takes values in $\gamma(H, E)$ we obtain in the last equation

$$(\nabla_2 f(t, \xi_t) - \phi_t)\sigma_t = \nabla_2 f(t, \xi_t)\sigma_t - \langle \phi_t \sigma_t \rangle_E$$

The proof of the theorem is based on the following lemma.

Lemma 3.0.9. *Let Φ be a P-set to ξ and $f : I \times E \to \mathbb{R}$ a function of class $\mathcal{C}_b^{1,2}$; in the particular case when f is assumed to be a $BS-function$ relative to ξ and $b_t = 0$ a.s. for any $t \in I$ we have a.s.*

$$F_t(\phi) = F_0(\phi) + \int_0^t (\nabla_2 f(s, \xi_s) - \phi_s)\sigma_s dW_s \qquad for\ any\ \phi \in \Phi\ t \in I. \tag{3.11}$$

Proof of Lemma 3.0.9. Fix $t \in I$. Since the function f is assumed to be a $BS-function$, for the lemma 3.0.2 we have

$$f(t, \xi_t) = f(0, \xi_0) + \int_0^t \nabla_2 f(s, \xi_s) b_s ds + \int_0^t \nabla_2 f(s, \xi_s)\sigma_s dW_s \qquad a.s.$$

And then for any $\phi \in \Phi$ from (3.4) we know that $\mathcal{F}_t(\phi)$ for $t \in I$ has the following representation :

$$F_t(\phi) = F_0(\phi) + \int_0^t (\nabla_2 f(s, \xi_s) - \phi_s) b_s ds + \int_0^t (\nabla_2 f(s, \xi_s) - \phi_s)\sigma_s dW_s \qquad a.s.$$

thus from the condition $b_t = 0$ a.s. we obtain (3.11). $\square$

Proof of Theorem 3.0.8. (i) Since f is assumed to be of class $\mathcal{C}_b^{1,2}$, from the lemma 3.0.6 we obtain that $F_t(\phi) \in \mathbb{H}^{1,2}$ and that for any $\phi \in \Phi$ and $t \in I$ we have

$$DF_t(\phi) = (\nabla_2 f(t, \xi_t) - \phi_t) D\xi_t \qquad a.s \tag{3.12}$$

Then, fix $t \in I$ and since the variable $F_t(\phi)$ is $\mathcal{G}_t^W$-measurable, for the Clarke-Ocone representation formula we obtain:

$$F_t(\phi) - \mathbb{E}F_t(\phi) = \int_0^t \mathbb{E}\{D_s F_t(\phi) | \mathcal{G}_s^W\} dW_s \qquad a.s. \tag{3.13}$$

Moreovere, as we have seen above, being that $F_t(\phi)$ is adapted and strongly measurable, we can write:

$$\mathbb{E}\left\{ \left| \int_0^t \mathbb{E}\{D_s F_t(\phi) | \mathcal{G}_s^W\} dW_s \right|^2 \right\} = \mathbb{E}\left\{ \int_0^t \| \mathbb{E}\{D_s F_t(\phi) | \mathcal{G}_s^W\} \|_H^2 \, ds \right\}. \tag{3.14}$$

Then, if we rewrite (3.14) in terms of (3.12) and (3.13), we have

$$\mathbb{E}\{|F_t(\phi) - \mathbb{E}F_t(\phi)|^2\} = \mathbb{E}\left\{ \int_0^t \| \mathbb{E}\{(\nabla_2 f(t, \xi_t) - \phi_t) D_s \xi_t | \mathcal{G}_s^W\} \|_H^2 \, ds \right\} \tag{3.15}$$

At last, if we integrate both sides of (3.15) with respect to $t \in I$, we obtain the thesis (3.9).

(ii) We notice that if we fix $\phi \in \Phi$ results $\mathbb{E}F_t(\phi) = \mathbb{E}F_0(\phi)$ a.s., for any $t \in I$. Since f is assumed to be a $BS - function$ relative to ξ and $b_t = 0$ a.s., for any $t \in I$, the previous lemma 3.0.9 gives us the following representation

$$F_t(\phi) - \mathbb{E}F_t(\phi) = F_0(\phi) - \mathbb{E}F_0(\phi) + \int_0^t (\nabla_2 f(s,\xi) - \phi_s)\sigma_s dW_s \qquad a.s. for \;\; any \;\; t \in I. \tag{3.16}$$

Then, let $p(x) = |x|^2$, for any $x \in \mathbb{R}$ and for any $t \in I$, as we have seen, from the strong measurability, according to (3.16), we obtain

$$|F_t(\phi) - \mathbb{E}F_t(\phi)|^2 = |F_0(\phi) - \mathbb{E}F_0(\phi)|^2 +$$
$$+\frac{1}{2}\int_0^t tr(\nabla_2^2 p(F_s(\phi)) - \mathbb{E}F_s(\phi); (\nabla_2 f(s,\xi_s) - \phi_s)\sigma_s)ds + \int_0^t \kappa_s(\phi)dW_s \qquad a.s. \tag{3.17}$$

where for any $s \leq t$ we set

$$\kappa_s(\phi) = 2(F_s(\phi) - \mathbb{E}F_s(\phi))(\nabla_2 f(s,\xi_s) - \phi_s)\sigma_s.$$

Let $h_1, h_2, \ldots$ be an orthonormal basis of H and considering that $\nabla^2(p(x)) = 2$ for any $x \in \mathbb{R}$. So, from the definition of the trace operator $tr(\cdot,\cdot)$ we have

$$tr(\nabla_2^2 p(F_s(\phi) - \mathbb{E}F_s(\phi)); (\nabla_2 f(s,\xi_s) - \phi_s)\sigma_s) = 2\sum_{n\geq 1}((\nabla_2 f(s,\xi_s) - \phi_s)\sigma_s h_n)^2 =$$
$$= 2 \parallel (\nabla_2 f(s,\xi_s) - \phi_s)\sigma_s \parallel_H^2 .$$

Considering that $\mathbb{E}|F_0(\phi) - \mathbb{E}F_0(\phi)|^2 = 0$ and that $F_0(\phi)$ is $\mathcal{G}_s^W$- measurable, and so $F_0(\phi) = \mathbb{E}F_0(\phi)$ a.s. In this way, from (3.17) we obtain

$$\mathbb{E}\{|F_t(\phi) - \mathbb{E}F_t(\phi)|^2\} = \mathbb{E}\left\{\int_0^t \parallel (\nabla_2 f(s,\xi_s) - \phi_s)\sigma_s \parallel_H^2 ds\right\}. \tag{3.18}$$

Thus, integrating both sides of (3.18) with respect to $t \in I$, we obtain the thesis (3.10).

□

Definition 3.0.10. Let Υ be some set and a functional $G : \Upsilon \to \mathbb{R}$. We say that any element $v^* \in \Upsilon$ is G-optimal is the following inequality holds true

$$G(v^*) \leq G(v) \qquad for \;\; any \;\; v \in \Upsilon. \tag{3.19}$$

3.1 Portfolio Representation Problem

Now we take in consideration the problem of substituting a financial exposure without misleading its performance in terms of the related inherent risk structure. The risk factors of a given portfolio are represented by tradable observables, such as the market price of a stock, a commodity itself or some major market benchmark assessing the value of an entire class of securities. The only market risk that we consider is the one that affects financial exposure.

Any element of E represents the overall discounted value of the risk factors at a determined time. Furthermore, I is the reference time range, that is taken as a period of one year and its fractions are determined with respect to the day-count convention.
We assume that the process (3.1) gives us dynamics for the overall discounted value of the risk factors. We also consider $f : I \times E \to \mathbb{R}$ of class $\mathcal{C}_b^{1,2}$ as a fixed financial exposure and thus $f(t, \xi_t)$ is its discounted value at any time $t \in I$. Moreover, for a fixed P-set Φ relative to ξ, we will suppose that any $\phi \in \Phi$ is the dynamics of a certain portfolio of risk factors handled by the trader, then $\langle \phi_t, \xi_t \rangle_E$ is its discounted value at time $t \in I$, which depends on the overall discounted value ξ_t and the portfolio composition ϕ_t.
We consider the risk functional $\mathcal{F}$ relative to ξ induced by f over Φ as the error that happens when substituting the exposure represented by f with some portfolio in Φ. Hence, the functional measures the sensitivity of the financial exposure to little variations of the underlying risk factors.

Remark 3.1.1. We remember that a integrable process $X = \{X_t : t \in I\}$ is a market price if

$$\mathbb{E}X_t = \mathbb{E}X_0 \qquad for\ \ any\ \ t \in I.$$

The process ξ in (3.1) is a *market price* when we take $b_t = 0$ a.s., for any $t \in I$. In the case when f is assumed to be a $BS-function$ relative to ξ and $b_t = 0$ a.s. for any $t \in I$, for the Lemma 3.0.2 we have

$$\mathbb{E}f(t, \xi_t) = \mathbb{E}f(0, \xi_0) \qquad for\ \ any\ \ t \in I.$$

Hence, in the case of ξ is a market price and f is a a $BS - function$ relative to ξ, we obtain that, for any $t \in I$, also $f(t, \xi_t)$ is a market price. From this it results that the assumption (3.2) is a risk-free condition. Moreover, from the Lemma 3.0.9 we can write

$$\mathbb{E}F_t(\phi) = \mathbb{E}F_0(\phi) \qquad for\ any\ t \in I\ and\ \phi \in \Phi.$$

Then, when $f(0, \xi_0) = \langle \phi_0, \xi_0 \rangle_E$ a.s., for any $\phi \in \Phi$, we have $F_t(\phi) = 0$ a.s. for any $t \in I$ and $\phi \in \Phi$ and so, the risk-functional $\mathcal{F}$ induced by f over Φ is given by

$$\mathcal{F}(\phi) = \parallel F(\phi) \parallel^2_{L^2(\Omega \times I)} \qquad for\ any\ \phi \in \Phi.$$

Therefore, the $\mathcal{F}$-optimal portfolio $\phi^* \in \Phi$ minimizes the average squared discrepancy when the financial exposure is market valued and it is perfectly hedged by any portfolio $\phi \in \Phi$ at time $t = 0$.

3.1.1 Constrained Hedging

Let $f : I \times E \to \mathbb{R}$ be a function of class $\mathcal{C}_b^{1,2}$ and fix a P-set Φ relative to ξ. In the theorem 3.0.8 we have two different representation of the functional $\mathcal{F}$ in terms of the operator $\nabla_2 f$, if there is a process $\phi^* \in \Phi$ such that a.s. results:

$$\phi_t^* = \nabla_2 f(t, \xi_t) \qquad for\ any\ t \in I \tag{3.20}$$

then ϕ^* is the $\mathcal{F}$-optimal, indeed in this case we have $\mathcal{F}(\phi^*) = 0$. In this respect, the optimization of the functional is correlated with the notion of portfolio immunization via sensitivity-based hedging approach.

We take E coincident with the Euclidean space $\mathbb{R}^2$, where $\langle \cdot, \cdot \rangle_E$ is the standard product and $e = (e_1, e_2)$ is its canonical basis. Furthermore, we consider H coincident with $\mathbb{R}$ and then the H-cylindrical process W is a standard one dimension Brownian motion. Fixed this, the space $\gamma(H, E)$ boils down to E itself. Then, we represent the process $\sigma \in L^2(\Omega, L^2(I, E))$ as $\sigma = (\sigma_1, \sigma_2)$, where each $\sigma_i = \{\sigma_{i,t} : t \in I\}$, for any $i = 1, 2$, is equal to

$$\sigma_{i,t} = \langle e_i, \sigma_t \rangle_E \qquad for\ any\ t \in I.$$

Analogously, $\xi_i = \{\xi_{i,t} : t \in I\}$, for any $i = 1, 2$, are the components of the process ξ such that

$$\xi_{i,t} = \langle e_i, \xi_t \rangle_E \qquad for\ any\ t \in I.$$

Moreover, here and in the sequel, we assume that a.s. $b_t = 0$ and $\sigma_{2,t} = 0$ for any $t \in I$, further we fix $\xi_{2,0} = 1$ a.s. Then for $i = 1, 2$ we obtain

$$\langle e_i, \int_0^t \sigma_s dW_s \rangle_E = \int_0^t \langle e_i, \sigma_s \rangle_E dW_s \qquad for\ \ any\ \ t \in I$$

thus the representation of the dynamics is as follow

$$\xi_{1,t} = \xi_{1,0} + \int_0^t \sigma_{1,s} dW_s \quad a.s. \qquad \xi_{2,t} = 1 \quad a.s. \tag{3.21}$$

Consider $f : I \times E \to \mathbb{R}$ of class $\mathcal{C}^{1,2}$ and a P-set Φ relative to ξ. We can write, for any $\phi \in \Phi$, the variable ϕ_t in terms of its components, and then we have, for $i = 1, 2$,

$$\phi_{i,t} = \langle e_i, \phi_t \rangle_E \qquad for\ \ any\ \ t \in I.$$

Proposition 3.1.2. *Let Φ be a P-set relative to ξ and f a $BS-function$ relative to ξ. If the components of the process ξ are given by (3.21), then*

$$\mathcal{F}(\phi) = \mathbb{E}\left\{ \int_I |\partial_{x_1} f(t, \xi_{1,t}, \xi_{2,t}) - \phi_{1,t}\sigma_{1,t}\sigma_{1,t}|^2 (1-t) dt \right\} \qquad for\ \ any\ \ \phi \in \Phi. \tag{3.22}$$

Proof. Being $E = \mathbb{R}^2$, for any $t \in I$ and $x = (x_1, x_2)$, we obtain that

$$\nabla_2 f(t, x) = (\partial_{x_1} f(t, x_1, x_2), \partial_{x_2} f(t, x_1, x_2))$$

and so we can set

$$\nabla_2 f(t, x) y = \langle \nabla_2 f(t, x), y \rangle_E \qquad for\ \ any\ \ y \in E$$

then the thesis follow from (3.21) and the statement (ii) of the Theorem 3.0.8, with $\sigma_{2,t} = 0$ a.s., for any $t \in I$. □

We may interpret the first component of the process ξ as a risk-neutral dynamics for the discounted price of some risky asset and, on the other hand, the second component as a risk-neutral model for the discounted value of the bank account. Moreover, f represents a European contingent claim written on the risky asset and the P-set Φ relative to ξ stands for the entire class of the hedging portfolios.

Finally, if there is a process $\phi^* \in \Phi$ such that

$$\phi^*_{1,t} = \partial_{x_1} f(t, \xi_t) \qquad for\ \ any\ \ t \in I \tag{3.23}$$

condition called *delta-hedging* for the contingent claim f, then ϕ^* is $\mathcal{F}$-optimal, since from (3.22) results $\mathcal{F}(\phi^*) = 0$.
Supposing that the first component of $\phi^* \in \Phi$ satisfy the delta hedging condition (3.23), when the condition

$$f(0, \xi_0) = \langle \phi_0^*, \xi_0 \rangle_E \tag{3.24}$$

then $F_0(\phi^*) = 0$ a.s. Thus the latter equation (3.24) implies that the financial exposure is perfectly hedged by the portfolio $\phi^* \in \Phi$ at time $t = 0$. Besides, from the lemma 3.0.9 we have that $F_t(\phi^*) = 0$ a.s. for any $t \in I$ and we obtain the second component of ϕ^* from the identity

$$\phi_{2,t}^* = f(t, \xi_t) - \phi_{1,t}^* \xi_{1,t} \qquad a.s.\ for\ any\ t \in I$$

Given this framework, the identity (3.3) boils down to

$$\nabla_1 f(t, \xi_{1,t}, \xi_{2,t}) + \frac{1}{2} \partial_{x_1, x_1} f(t, \xi_{1,t}, \xi_{2,t}) \sigma_{1,t}^2 = 0 \quad a.s.\ for\ any\ t \in I$$

and it corresponds to a Black-Scholes type equation, whereby the risk-free rate is set to be null at any time.

3.1.2 Correlation and residual risk

We now consider the particular case when the condition $\phi_t^* = \nabla_2 f(t, \xi_t)$ is not satisfied. This is interesting when the financial exposure and the replication portfolio actually depends upon two different but correlated risk factors.
We assume E and H coincident with the Euclidean space $\mathbb{R}^2$ and, as we have seen previously, $\langle \cdot, \cdot \rangle_E$ and $e = (e_1, e_2)$ are respectively the standard product and the canonical basis. We fix $\rho \in (-1, 1)$ and we define the inner product on H as $\langle e_i, e_j \rangle_H = 1$ if $i = j$ and $\langle e_i, e_j \rangle_H = \rho$ when $i \neq j$. Let $f : I \times E \to \mathbb{R}$ be a function given by

$$f(t, x) = \langle e_1, x \rangle_E \qquad for\ any\ t \in I\ \ x \in E \tag{3.25}$$

f is of class $\mathcal{C}_b^{1,2}$ and for any $t \in I$ and $x \in E$ results $\nabla_2 f(t, x) = e_1$. Let Φ a P-set relative to ξ, we assume that for any $\phi \in \Phi$ exists a real-valued process $\phi_2 = \{\phi_{2,t} : t \in I\}$, with whom any ϕ is identified, which satisfies the following identity,

$$\phi_t = \phi_{2,t} e_2 \qquad a.s.\ for\ any\ t \in I \tag{3.26}$$

Proposition 3.1.3. *Let f be the function given by (3.25) and Φ be a P-set relative to ξ that verifies the condition (3.26). If $b_t = 0$ a.s., for any $t \in I$ then*

$$\mathcal{F}(\phi) = \mathbb{E}\left\{ \int_I \| \langle e_1, \sigma_t \rangle_E - \phi_{2,t} \langle e_2, \sigma_t \rangle_E \|_H^2 \, (1-t)dt \right\} \qquad for \;\; any \;\; \phi \in \Phi \tag{3.27}$$

Proof. Notice that $\nabla_1 f(t,x) = 0$ and $\nabla_2^2 f(t,x) = 0$ for any $t \in I$ and $x \in I$, so f is a $BS - function$ relative to ξ. Furthermore, we have

$$\nabla_2 f(t, \xi_t)\sigma_t = \langle e_1, \sigma_t \rangle_E \qquad for \;\; any \;\; t \in I$$

and for any $\phi \in \Phi$

$$\langle \phi_t, \sigma_t \rangle_E = \phi_{2,t} \langle e_2, \sigma_t \rangle_E \qquad for \;\; any \;\; t \in I$$

then we obtain the thesis directly from the Theorem 3.0.8. □

Proposition 3.1.4. *Let f be the function given by (3.25) and Φ a P-set relative to ξ that verifies the condition (3.26). If $b_t = 0$ a.s., for any $t \in I$ then the process $\phi^* \in \Phi$ is defined by*

$$\phi^*_{2,t} = \rho(\sigma_{1,t}/\sigma_{2,t}) \qquad for \;\; any \;\; t \in I \tag{3.28}$$

and it is $\mathcal{F}$-optimal; where for any $i = 1, 2$ the real process $\sigma_i = \{\sigma_{i,t} : t \in I\}$ is such that

$$\langle e_i \sigma_i \rangle_E = \sigma_{i,t} e_i \qquad for \;\; any \;\; t \in I.$$

Remark 3.1.5. The H-Wiener process W can be understood as a 2-dimensional Wiener process where each component W_i is given by $W_{i,t} = W_t e_i$ for any $t \in I$, and with ρ that is their instantaneous correlation,

$$\mathbb{E}\{W_t e_1 \cdot W_t e_2\} = \langle e_1, e_2 \rangle_H t = \rho t \qquad for \;\; any \;\; t \in I.$$

Hence, we can interpret the components of the process ξ as the dynamics if the discounted prices of two correlated risky assets, moreover, a perfect hedged may not be recovered. In this case, we have that given $\phi^* \in \Phi$ as in (3.28), $\mathcal{F}(\phi^*)$ is strictly positive for $\rho < 1$ and it vanishes for $\rho = 1$, where in the latter case the two assets are completely correlated. Then we may understand $\mathcal{F}(\phi^*)$ as the *residual hedging risk*.

Chapter 4

Optimal Model Points in Term Life Insurance

In this chapter, we discuss the problem of determining an optimal model points portfolio related to some fixed policies portfolio. Specifically, we describe the model points associated to a given term life insurance portfolio as a group of policies that minimizes a certain risk functional, as we have seen in general in the previous chapter.
The model points are suitable representative contracts with which the insurance companies replace any homogeneous group of policies in order to compute the cash flow projections. These projections must be carried out by insurance companies in order to evaluate the value of their portfolios and to demonstrate the compliance of their portfolios, by considering the sensitivity analysis. Indeed, life insurance companies are allowed by regulators to estimate the performance of any portfolio of policies on the basis of suitable model points in order to reduce the computational difficulties of the operation. This procedure is permitted under suitable conditions, i.e. when the inherent risk structure of the original portfolio is not misrepresented and there is not a loss of any significant attribute of the portfolio itself.

In particular, we assess the problem of the model points selection when dealing with a portfolio of term insurance policies and a LIBOR Market Model that determines the dynamics of the forward rates. Term life insurance, also known as pure life insurance, is life insurance that guarantees payment of a lump sum benefit on the death of the

policy owner, provided that it occurs until a specific term that is defined in the contract. Once the term expires, the policyholder can either renew for another term, convert to permanent coverage, or allow the policy to terminate.
For the sake of simplicity, we assume that the benefit related to each policy is always represented by a unit amount of a certain currency. We consider only policies that are unaffected by credit risk, in other words, the insurance company always guarantees the entire benefit that is provided for in the contract. Moreover, we do not analyze the revenues received by the insurance company and thus we do not take into account the premiums stream of the contract nor any further expense that is the responsibility of the client.

Assumption 4.0.1. ***for the LIBOR Market Model***
As we have seen in the chapter 2, we recall some assumptions for the LIBOR Market Model, which governes the time evolution of the forward rates:

- $(\Omega, \mathcal{F}, \mathbb{P})$ *a complete probability space;*
- $\mathcal{T} = \{T_0, T_1, \ldots, T_N\}$ *a finite set of maturity time with* $T_0 = 1$ *and* $T_0 < \ldots < T_N$*;*
- $\tau_n = T_n - T_{n-1}$ *a year fraction for any* $n = 0, \ldots, N$*;*
- $I = (0, 1)$ *unit interval on the real line, i.e. a period of one year;*
- $p_n(t) := p(t, T_n)$ *the risk-neutral discounted price at time* $t \in I$ *for a market valued bond maturing at future time* T_n*, we highlighted also that, denoting* $F_n(t)$ *the LIBOR forward rate associated to the period* $(T_{n-1}, T_n]$*, we can write*

$$p_n(t) = p_0(t) \prod_{k=1}^{n} \frac{1}{1 + F_k(t)\tau_k} \qquad for\ \ any\ \ t \in I, t < T_n \tag{4.1}$$

- $H = \mathbb{R}^N$ *with* $h_1, \ldots h_n$ *components of any* $h \in H$ *with respect to orthonormal basis*

$$\| h \|_H = \left\{ \sum_{n,k} \rho_{nk} h_n h_k \right\}^{1/2} \qquad for\ \ any\ \ h \in H$$

  where $\rho_{nk} = e^{-\delta(n-k)}$ *for some constant* $\delta \in \mathbb{R}^+$*;*

- *H-cylindrical process $W = \{W(t) : t \in I\}$ which is assumed as a correlated N-correlated Weiner process $W(t) = (W_1(t), \dots, W_N(t))$ for $t \in I$ s.t.*

$$\mathbb{E}\{W_n(t)W_k(t)\} = \rho_{nk};$$

- *$\mathcal{G}^W = \{\mathcal{G}_t^W : t \in I\}$ the augmented filtration generated by W, we remember that a process is adapted if it is adapted w.r.t. $\mathcal{G}^W$;*

- *$dF(t) = \mu(t)dt + \Sigma(t)dW(t)$ dynamics of the N-dimensional process $F(t) = (F_1(t), \dots, F_N(t))$, with $F(0) = (F_1(0), \dots, F_N(0))$ where for any $n = 1, \dots, N$, the dynamics of the forward rate $F_n(t)$ is given by*

$$dF_n(t) = \mu_n(t)dt + \Sigma_{nk}dW_n(t) \tag{4.2}$$

where $\mu_n(t)$ is the N-components of $\mu(t)$ and Σ_{nk} is a matrix

$$\begin{aligned} \mu_n(t) &= \sigma_n(t)F_n(t)\sum_{k=1}^{n}\frac{\rho_{nk}\tau_k\sigma_kF_k(t)}{1+F_k(t)\tau_k} \\ \Sigma_{nk}(t) &= \sigma_n(t)F_n(t)\delta_{nk} \end{aligned}$$

- *$\tilde{p}_n(t) = \frac{p_n(t)}{p_0(t)}$ the discounted bond price, for any $t \in I$, associated to the bond expiring at the date $T_n \in \mathcal{T}$, $\tilde{p}_n(t)$ admits the following dynamics*

$$d\tilde{p}_n(t) = -\epsilon_n(t)\tilde{p}_n(t)dW(t) \tag{4.3}$$

where

$$\epsilon_n(t) = \sum_{k=1}^{n}\frac{\tau_k}{1+F_k(t)\tau_k}\Sigma_k(t). \tag{4.4}$$

4.1 Term Insurance Portfolio

We consider a generic term insurance policy in a given portfolio that have to be labelled by both the age of the policy owner at time $t = 0$ and the maturity date of the contract. Hence, taken two finite index sets $\mathcal{I}$ and $\mathcal{J}$, let $\mathcal{X} = \{x_i : i \in \mathcal{I}\}$ and $\mathcal{Y} = \{y_j : j \in \mathcal{J}\}$ be finite sequences of real values such that $x_i \leq 0$ for any $i \in \mathcal{I}$ and $y_j \leq 1$ for any $j \in \mathcal{J}$. So, each class of individuals with age labelled by $x_i \in \mathcal{X}$ and term

date $y_i \in \mathcal{Y}$ are uniquely identified by the couple (x_i, y_j).
We denote the force of mortality at time $s \geq 0$ related to the class of individuals labelled by $x_i \in \mathcal{X}$, i.e. the instantaneous rate of mortality at time s and relative to an individual that has x_i age at time $t = 0$, with

$$\mu(s, x_i + s) = a(s)exp(x_i + s)b(s) \qquad for\ any\ s \geq 0\ and\ x_i \in \mathcal{X} \tag{4.5}$$

where $a(s)$ and $b(s)$ are functions that can be observed deterministically.

Assumption 4.1.1. *We take $a(s) = 0$ for any $s \in I$*

$$\Longrightarrow \quad \mu(s, x_i + s) = 0 \qquad for\ any\ s \in I\ and\ any\ i \in \mathcal{I} \tag{4.6}$$

this assumption ensures that the eventual death of policyholders does not bring changes to any portfolio of term insurance policies within the time range I. The latter is assumed as acceptable hypothesis because all the events that happen in the first years affect hardly on the performance of the overall portfolio.

Given this framework, the survival index, i.e. the proportion of the individuals labelled by $x_i \in \mathcal{X}$ which survive to age $x_i + T_n$, is defined by

$$S(x_i, T_n) = exp\left\{-\int_1^{T_n} \mu(s, x_i + s)ds\right\} \qquad for\ any\ x_i \in \mathcal{X}\ and\ any\ n = 0, \ldots, N \tag{4.7}$$

and deriving S w.r.t. its second component we obtain

$$S_T(x_i, T_n) = -S(x_i, T_n)\mu(T_n, x_i + T_n).$$

Definition 4.1.2. For any $i \in \mathcal{I}$ and $j \in \mathcal{J}$, we define the discounted risk-free value at time $t \in I$ of a term insurance policy owned by an individual labelled by $x_i \in \mathcal{X}$ and with maturity $y_j \in \mathcal{Y}$ as

$$z_{ij}(t) = -\sum_{n=1}^{N} S_T(x_i, T_n)\tilde{p}_n(t)\mathbb{1}_{\{T_n \leq y_j\}} \tag{4.8}$$

Definition 4.1.3. We define a **term insurance portfolio** relative to $\mathcal{X}$ and $\mathcal{Y}$ any matrix $v = \{v_{ij} : x_i \in \mathcal{X}\ and\ y_j \in \mathcal{Y}\}$. Besides, the discounted risk-free value $v(t)$ of v at time $t \in I$ is given by any linear combination of (4.8), i.e. it is defined as

$$v(t) = \sum_{ij} z_{ij}(t)v_{ij}. \tag{4.9}$$

where any component v_{ij} is considered as the quantity of policies in v that are owner by the class of inidividuals labelled by $x_i \in \mathcal{X}$ and $y_j \in \mathcal{Y}$.

Definition 4.1.4. We call *dimension* of a term insurance portfolio v relative to $\mathcal{X}$ and $\mathcal{Y}$ the amount

$$dim(v) = \sum_{ij} v_{ij}$$

Moreover, for any couple of term insurance policy portfolio we write

$$(v_1 - v_2)(t) = v_1(t) - v_2(t) \qquad for\ any\ t \in I.$$

4.1.1 Model points risk functional

We consider a policy portfolio v relative to $\mathcal{X}$ and $\mathcal{Y}$ and we fix a set $\mathcal{W}$ of term insurance policy portfolios relative to $\mathcal{X}$ and $\mathcal{Y}$.
Given this setup, we can define.

Definition 4.1.5. The **model points risk functional** induced by a model points portfolio v over $\mathcal{W}$ is defined by the functional $\mathcal{V} : \mathcal{W} \to \mathbb{R}$ such that

$$\mathcal{V}(w) = \int_{\mathcal{I}} \mathbb{E}|(v-w)(t) - \mathbb{E}(v-w)(t)|^2 dt \qquad for\ any\ w \in \mathcal{W} \tag{4.10}$$

As we have seen in section 3.1 of the previous chapter, we can regard the risk functional $\mathcal{V}(w)$ as the error that happens if we substitute the policy portfolio v with the model points portfolio $w \in \mathcal{W}$. We observe that when $v \in \mathcal{W}$ we have $\mathcal{V}(v) = 0$.

Definition 4.1.6. A model points portfolio $w^* \in \mathcal{W}$ is said $\mathcal{V}$-optimal relative to v if

$$\mathcal{V}(w^*) \leq \mathcal{V}(w) \qquad for\ any\ w \in \mathcal{W} \tag{4.11}$$

We can represent the model points risk functional in another form, according to the following proposition.

Proposition 4.1.7. *The model points risk functional induced by a portfolio v over $\mathcal{W}$ admits the following form*

$$\mathcal{V}(w) = \mathbb{E}\left\{\int_I \| \sum_n \left\{ \sum_{ij} (v_{ij} - w_{ij}) S_T(x_i, T_n) \mathbb{1}_{\{T_n \leq y_j\}} \right\} \epsilon_n(t) \tilde{p}_n(t) \|_H^2 (1-t) dt \right\} \qquad for\ any\ w \in \mathcal{W}. \tag{4.12}$$

Proof. Let $E = \mathbb{R}^N$, we set, for any $t \in I$, $\tilde{p}(t) = (\tilde{p}_1(t), \ldots, \tilde{p}_N(t))$. We define $U = (\mathcal{X} \times \mathcal{Y})$ as the class of real matrices $r = \{r(x_i, Y_j) : i \in \mathcal{I} \ and \ j \in \mathcal{J}\}$. We take the functional $Z \in \mathcal{L}(E, U)$ such that for any $x_i \in \mathcal{X}$ and $y_j \in \mathcal{Y}$ resuls

$$Z(q)(x_i, y_j) = -\sum_n q_n S_T(x_i, T_n)\mathbb{1}_{\{T_n \leq y_j\}} \qquad \forall q \in E$$

and, we consider the process given by (4.9) $z(t) = \{z(t) : t \in I\}$ which can also obtain from

$$z(t) = Z(\tilde{p}(t)) \qquad for \ any \ t \in I.$$

Z is linear and then for any $q \in E$ its Frechét derivative $\nabla Z(q) \in \mathcal{L}(E, U)$ satisfies, for any $x_i \in \mathcal{X}$ and $y_j \in \mathcal{Y}$

$$(\nabla Z(q)q')(x_i, y_j) = -\sum_n q'_n S_T(x_i, T_n)\mathbb{1}_{\{T_n \leq y_j\}} \qquad for \ any \ q' \in E.$$

Besides,for any $x_i \in \mathcal{X}$ and $y_j \in \mathcal{Y}$, it is true the following identity

$$(\nabla Z(\cdot)\epsilon)(x_i, y_j) = -\sum_n \tilde{p}_n(t)\epsilon_n(t)S_T(x_i, T_n)\mathbb{1}_{\{T_n \leq y_j\}} \tag{4.13}$$

where ϵ is a matrix whose n-th row is given by $\tilde{p}_n(t)\epsilon_n$, as defined in (4.4). Finally, we consider the function $\zeta : I \times E \to U$ defined as

$$\zeta(t, q) = Z(q) \qquad for \ any \ t \in I \ and \ q \in E. \tag{4.14}$$

This definition is possible because we have assumed, imposing (4.6) ,that the survival index (4.7) does not depend on $t \in I$. We observe also that ζ is $\mathcal{C}_b^{1,2}$ and then it is a $BS - function$ relative to $\tilde{p}$. Then we can rewrite (4.3) as follow

$$d\tilde{p}(t) = -\epsilon(t)dW(t) \tag{4.15}$$

then

$$\mathcal{V}(w) = \mathbb{E}\left\{\int_I \| \sum_{ij}(v_{ij} - w_{ij})(\nabla Z(\tilde{p}(t))\epsilon(t))(x_i, y_j)\|_H^2 (1-t)dt\right\} \qquad for \ any \ w \in \mathcal{W} \tag{4.16}$$

the latter, jointly with (4.13) gives us the thesis.

□

4.1.2 Numerical considerations in Matlab

In this section, we represent the following functional from a numerical point of view:

$$\mathcal{V}(w) = \mathbb{E}\left\{ \int_I \| \sum_n \left\{ \sum_{ij} (v_{ij} - w_{ij}) S_T(x_i, T_n) \mathbb{1}_{\{T_n \le y_j\}} \right\} \epsilon_n(t) \tilde{p}_n(t) \|_H^2 (1-t) dt \right\} \qquad for\ any\ w \in \mathcal{W}. \tag{4.17}$$

The latter, for any $w \in \mathcal{W}$, is estimated as a combination of Monte Carlo simulation for computing the expectation, jointly with the discretization of the integral w.r.t. the time variable $t \in I$.

Here and in the sequel, we fix, for $n = 1 \ldots, N$, the set of maturity date $\mathcal{T} = \{10, \ldots, 90\}$ and so, for any $n = 1, \ldots, 9$, $F_n(0) = \{0.02, 0.03, 0.04, 0.04, 0.05, 0.05, 0.06, 0.06, 0.06\}$; furthermore, for every n and for any $t \in I = [0,1]$, we set $\sigma_n(t) = 0.1$.

First of all, we show one path of 1000 simulations that represents the dynamics of the Forward rate F_n relative to the LIBOR Market Model, described in the chapter 2 and given by

$$dF_n(t) = \mu_n(t)dt + \Sigma_n(t)dW_n(t)$$

then, for numerical simulation we have used the following scheme

$$ln(\hat{F}_n(t) + \Delta t) = ln(\hat{F}_n(t)) + \mu_n(t)\Delta t + \sigma_n(t)\Delta \hat{W}_n(t)$$

where

$$\mu_n(t) = \sigma_n(t) \sum_{m=1}^{n} \frac{\rho_{nm}\sigma_m(t)\hat{F}_m(t)}{+\tau_m \hat{F}_m(t)}$$

and for any $t \in I$ we identify with $\hat{F}_n(t)$ and $\Delta\hat{W}_n(t)$ the approximations of $F_n(t)$ and $dW_n(t)$ respectively.

Given this assumptions, we set the correlation matrix for the N-correlated Brownian motion, remembering that is computed with Cholesky decomposition, i.e. this means that the initial matrix must be symmetric and definite positive, with elements between -1 and 1; the dynamics of the forward rate is showed in the Figure 4.1.

We assume that for any $s \ge 1$, $a(s) = a = 0.03e - 13$ and $b(s) = b = 0.06e - 11$, and we also fix $X = \{25, 28, 31, \ldots, 67, 70, 73\}$ a set of ages with term date set $Y = \{55, 52, 49, \ldots, 13, 10, 7\}$ for the term insurance portfolios v.

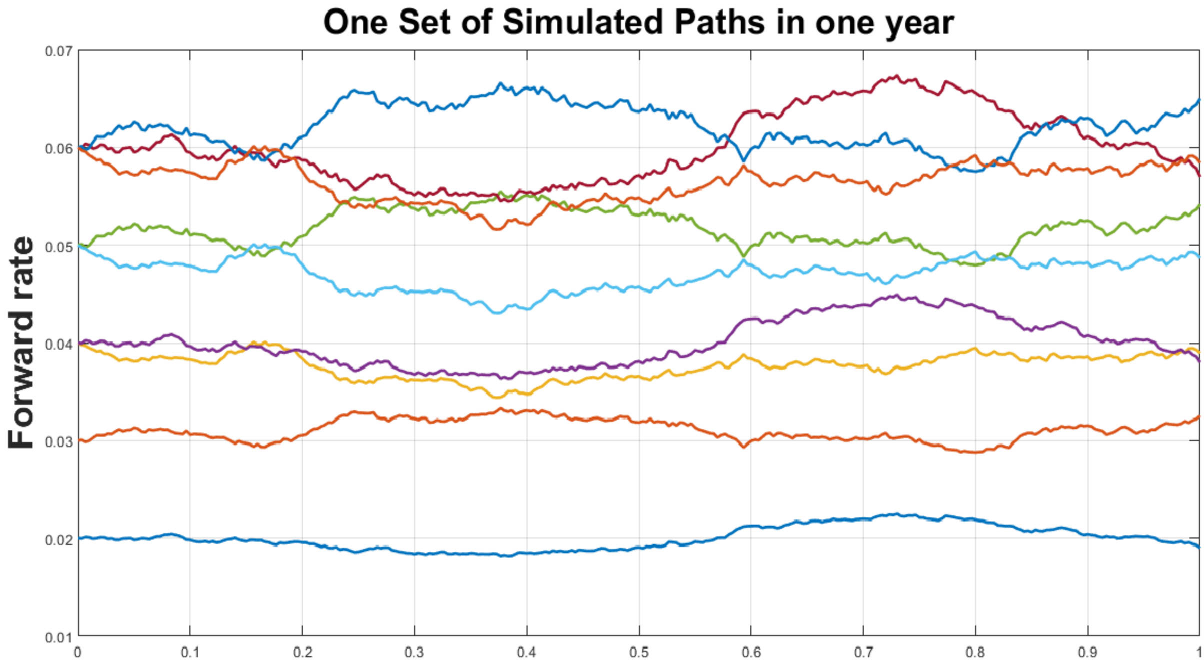

Figure 4.1: Forward rate LIBOR

On the other hand, we have to fix also a set $\mathcal{W}$ for the optimal model point w. It is assumed as $X_{\mathcal{W}} = \{25, 31, 37, 43, 49, 55, 61, 67, 73\}$ and $Y_{\mathcal{W}} = \{55, 49, 43, 37, 31, 25, 19, 13, 7\}$. Given this framework, we consider two examples:

Case 1: we fix $v_{ij} = (X(i), Y(j))$ and analogously $w_{ij} = (X_W(i), Y_W(j))$, so we print 10 simulations of the calculation of the functional and we get, in 0.397139 seconds, the following results collected in the table 4.1

$\mathcal{V} =$

0.0102
-0.0378
-0.0887
-0.0888
0.0655
-0.1285
0.0114
-0.0053
0.0033
-0.1267

Table 4.1: 10 simulations of the computation of the risk functional

Case 2 : we fix, analogously to before, $v_{ij} = (X(i), Y(j))$, but we compute all the values of $X_{\mathcal{W}}$ for every single $Y_{\mathcal{W}}$, then we obtain a table of $|Y_{\mathcal{W}}| = 9$ columns of 10 simulations in 2.369400 seconds:

$Y_{\mathcal{W}} = 55$	$Y_{\mathcal{W}} = 49$	$Y_{\mathcal{W}} = 43$	$Y_{\mathcal{W}} = 37$	$Y_{\mathcal{W}} = 31$	$Y_{\mathcal{W}} = 25$	$Y_{\mathcal{W}} = 19$	$Y_{\mathcal{W}} = 13$	$Y_{\mathcal{W}} = 7$
-0.0797	0.0169	0.1173	-0.1709	-0.0306	-0.3175	-0.0203	0.2764	-0.1440
-0.0172	-0.1396	-0.0936	-0.1820	-0.1461	0.0295	0.2102	0.1369	-0.2258
-0.1411	0.1353	-0.0546	0.1840	-0.1005	0.0519	-0.3487	0.2657	-0.2593
0.0091	0.0144	-0.1035	-0.0258	-0.3116	-0.0212	-0.0432	0.4872	-0.0117
-0.0897	-0.0429	0.1987	-0.1324	0.0769	0.0867	-0.1026	-0.0898	0.3428
0.0885	0.1231	0.0341	-0.0870	-0.0264	-0.0263	-0.0986	-0.2008	-0.4094
-0.0299	0.0174	-0.3365	-0.3465	-0.2964	-0.1634	0.1050	0.0801	0.1543
0.0181	0.1422	-0.2696	0.1294	-0.0095	-0.2139	0.0629	-0.1749	0.0671
0.0203	-0.0313	-0.1281	0.0560	0.2736	0.0743	-0.1398	-0.3718	-0.1516
-0.0267	-0.1299	0.1177	0.0765	-0.0988	0.1717	0.1598	0.0969	0.0666

Table 4.2: 10 simulations of the computation of the risk functional

Appendix A

Math Toolbox

In this chapter we recall some elements of the theory of stochastic processes, useful for the description of financial models.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $I = [0, T]$ or $I = \mathbb{R}_{\geq 0}$ a real interval.

Definition A.0.1. A measurable stochastic process on $\mathbb{R}^N$ is a collection $(X_t)_{t \in I}$ of random variables with values in $\mathbb{R}^N$ such that

$$X : I \times \Omega \to \mathbb{R}^N, \qquad X(t, \omega) = X_t(\omega)$$

is measurable w.r.t. the product σ-algebra $\mathcal{B}(I) \otimes \mathcal{F}$. We say that X is integrable if $X_t \in L^1(\Omega, \mathbb{P})$ for every $t \in I$.

Definition A.0.2. A stochastic process X is continuous (a.s.) if the paths $t \rightarrowtail X_t(\omega)$ are continuous functions for every $\omega \in \Omega$(for almost all $\omega \in \Omega$).

Definition A.0.3. A filtration $(\mathcal{F}_t)_{t \geq 0}$ in $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family of sub σ-algebras of $\mathcal{F}$.

Definition A.0.4. Given a stochasic process $X = (X_t)_{t \in I}$, the natural filtration for X is defined by

$$\tilde{\mathcal{F}}_t^X = \sigma(X_s | 0 \leq s \leq t) := \sigma(\{X_s^{-1}(H) | 0 \leq s \leq t, H \in \mathcal{B}\}) \qquad t \in I.$$

A stochastic process X is adapted to a filtration $(\mathcal{F}_t)$ is $\tilde{\mathcal{F}}_t^X \subseteq \mathcal{F}_t$ for every t, i.e. in other word if X_t if $\mathcal{F}_t$-measurable for every t.

Here and in the sequel, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$.

Definition A.0.5. Let M be an integrable adapted stochastic process on filtered probability space, defined as above. We say that M is

- a martingale w.r.t. $(\mathcal{F}_t)$ and to the measure $\mathbb{P}$ if

$$M_s = \mathbb{E}[M_t|\mathcal{F}_s] \qquad for\ \ every \quad 0 \leq s \leq t$$

  this mean that a martingale M is constant in time;

- a super-martingale if

$$M_s \geq \mathbb{E}[M_t|\mathcal{F}_s] \qquad for\ \ every \quad 0 \leq s \leq t;$$

- a sub-martingale if

$$M_s \leq \mathbb{E}[M_t|\mathcal{F}_s] \qquad for\ \ every \quad 0 \leq s \leq t;$$

Definition A.0.6. We denote by $\mathbb{L}^2_{loc}$ the family of processes $(u_t)_{t\in[0,T]}$ such that

- they are progressively measurable w.r.t. the filtration $(\mathcal{F}_t)_{t\in[0,T]}$;
- $\int_0^T u_t^2 dt < \infty$ a.s.

Remark A.0.7. We notice that the space $\mathbb{L}^2_{loc}$ is invariant w.r.t. changes of equivalent probability measures.

Definition A.0.8. A d-dimensional Brownian motion is a stochastic process $W = (W_t)_{t\geq 0}$ in $\mathbb{R}^d$ such that

(i) $W_0 = 0$ $\mathbb{P}$-a.s.;

(ii) W is $\mathcal{F}_t$-adapted and continuous;

(iii) for $t > s \geq 0$ the random variable $W_t - W_s$ has multi-normal distribution $\mathcal{N}_{0,(t-s)I_d}$, where I_d is the $(d \times d)$-identity matrix, and it is indipendent of $\mathcal{F}_s$

Remark A.0.9. We can observe that from the properties (i) and (ii) the paths of a B.m. start at time $t = 0$ from the origin a.s. and they are continuous. As a consequence for every $i = 1, \dots, d$ we have

$$W_t^i \sim \mathcal{N}_{0,t}$$

since $W_t^i = W_t^i - W_0$a.s.

Remark A.0.10 (**Correlated Brownian motion**). Given an $(N \times d)$-dimensional matric α with constant real entries, we fix

$$\rho = \alpha\alpha^* \tag{A.1}$$

where $\rho = (\rho^{ij})$ is a $(N \times N)$-dimensional matrix, symmetric, positive semi-definite and $\rho^{ij} = \langle \alpha^i, \alpha^j \rangle$ where α^i is the i-th row of the matrix α.
Given $\mu \in \mathbb{R}^N$ and a d-dimensional Brownian motion W, we set

$$B_t = \mu + \alpha W_t \rightarrow \quad dB_t = \alpha dW_t \tag{A.2}$$

We observe that $B_t \sim \mathcal{N}_{\mu,t\rho}$, with $Cov(B_t) = t\rho$ where

$$\mathbb{E}\left[(B_t^i - \mu_i)(B_t^j - \mu_j)\right] = t\rho^{ij}$$

In this case we say that B is a Brownian motion starting from μ with deterministic correlation matrix ρ.

Definition A.0.11. An N-dimensional Itô process is a stochastic process X of the form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \qquad t \in [0, T] \tag{A.3}$$

where X_0 is a $\mathcal{F}_0$ measurable random variable, W is a d-dimensional Brownian motion, $\mu \in \mathbb{L}^1_{loc}$is a $(N \times 1)$-vector and $\sigma \in \mathbb{L}^2_{loc}$is a $(N \times d)$-matrix. We can rewrite this formula in differential form

$$dX_t = \mu_t dt + \sigma_t dW_t \quad \Leftrightarrow \quad dX_t^i = \mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j \qquad i = 1, \dots, N$$

The process μ is called DRIFT coefficient, while σ DIFFUSION coefficient.

Lemma A.0.12. *Consider an Itô process X, defined as above, and fix $C = \sigma\sigma^*$. Then we have that the covariation process is the stochastic process $(\langle X^i, X^j \rangle_t)_{t\in[0,T]}$ given by*

$$\langle X^i, X^j \rangle_t = \int_0^t C_s^{ij} ds \quad t \geq 0 \qquad \Leftrightarrow \quad d\langle X \rangle_t = C_t dt.$$

Theorem A.0.13 (Itô formula). *Given an Itô process of the form (A.3) and let $f = f(t, x) \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$. Then*

$$df = \partial_t f dt + \nabla f \cdot dX_t + \frac{1}{2} \sum_{i,j=1}^{N} \partial_{x_i, x_j} f d\langle X^i, X^j \rangle_t \tag{A.4}$$

with $f = f(t, X_t)$ and $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_N} f)$.

Appendix B

Absence of Arbitrage

We present briefly the theory of the change of the probabily measure introducing a so-called **martingale measure** or a **risk neutral measure**. It has a central role in a interest rate theory because at every martingale measure corresponds to a market price of the risk and a price for the derivatives which avoids introducing arbitrage opportunities. Arbitrage is an opportunity to perform financial operations at no cost that produce a risk-free profit. In real markets, arbitrages exist but in the theoretical framework, it is clear that in a sensible financial model it must exclude these forms of profit. In fact, the principle of absence of arbitrage has become the dominant criterion for the valuation of financial derivatives.

B.1 Change of measure

Definition B.1.1 (Exponential martingales). We consider a d-dimensional Brownian motion $(W_t)_{t\in[0,T]}$ on the space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. Let $\lambda \in \mathbb{L}^2_{loc}$ be a d-dimendional process: we define an *exponential martingale associated to* λ as

$$Z_t^\lambda = exp\Big(- \int_0^t \lambda_s \cdot dW_s - \frac{1}{2} \int_0^t |\lambda_s|^2 ds \Big), \qquad t \in [0,T] \tag{B.1}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^d$. By the Itô formula we have

$$dZ_t^\lambda = -Z_t^\lambda \lambda_t \cdot dW_t \tag{B.2}$$

Z^λ is a local martingale. Since Z^λ is positive and Z^λ is a continuous adapted process, it is also a super-martingale:

$$E[Z_t^\lambda] \leq E[Z_0^\lambda] = 1, \qquad t \in [0,T]$$

and thus $(Z_t^\lambda)_{t\in[0,T]}$ is martingale if and only if $E[Z_t^\lambda] = 1$.

Lemma B.1.2. *If there exists a constant C such that*

$$\int_0^T |\lambda_t|^2 dt \leq C \qquad a.s.$$

then Z^λ in (B.1) is a martingale such that

$$E\left[\underset{0\leq t\leq T}{sup}(Z_t^\lambda)^p\right] < \infty, \qquad p \leq 1$$

In particular $Z^\lambda \in \mathbb{L}^p(\Omega, P)$ for each $p\ leq1$.

Theorem B.1.3 (Bayes' formula). *Let P, Q be probability measures in $(\Omega, \mathcal{F})$ with $Q \ll_{\mathcal{F}} P$. If $X \in L^1(\Omega, Q)$, $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$ and we set $L = \frac{dQ}{dP}\big|_{\mathcal{F}}$, then we have*

$$E^Q[X|\mathcal{G}] = \frac{E^P[XL|\mathcal{G}]}{E^P[L|\mathcal{G}]}$$

Then, supposing that Z^λ in (B.1) is a martingale and we define the measure Q on $(\Omega, \mathcal{F})$ by

$$\frac{dQ}{dP} = Z_T^\lambda \quad \Longleftrightarrow \quad Q(F) = \int_F Z_T^\lambda dP \qquad F \in \mathcal{F} \tag{B.3}$$

then from the Bayes' theorem we have for every $X \in L^1(\Omega, Q)$

$$E^Q[X|\mathcal{F}_t] = \frac{E^P[XZ_T^\lambda|\mathcal{F}_t]}{E^P[Z_T^\lambda|\mathcal{F}_t]} \qquad t \in [0,T]$$

Consequently we have the following lemma:

Lemma B.1.4. *Assume that Z^λ in (B.1) is a P-martingale and Q is the probability measure defined in (B.3). Then a process $(M_t)_{t\in[0,T]}$ is a Q-martingale id and only if $(M_t Z_t^\lambda)_{t\in[0,T]}$ is a P-martingale.*

Theorem B.1.5 (Girsanov's theorem). *Let Z^λ in (B.1) be an exponential martingale associated to the process $\lambda \in \mathbb{L}^2_{loc}$. We assume that Z^λ is a P-martingale and we consider the measure Q defined by*

$$\frac{dQ}{dP} = Z^\lambda_T$$

Then the process

$$W^\lambda_t = W_t + \int_0^t \lambda_s ds \qquad t \in [0,T] \tag{B.4}$$

is a Brownian motion on $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t))$

In general , in financial application, we assume that λ is a bounded process, but it is diffucult to prove, so we introduce this condition to ensure that Z^λ is a martingale.

Theorem B.1.6 (Novikov condition). *If $\lambda \in \mathbb{L}^2_{loc}$ is such that*

$$E\left[exp\left(\frac{1}{2}\int_0^T |\lambda_s|^2 ds\right)\right] < \infty$$

then the exponential martingale Z^λ in (B.1) is a strict martingale.

Proposition B.1.7 (Representation if Brownian martingales). *Let $(W_t)_{t\in[0,T]}$ a d-dimensionale Brownian motion on the space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. For every d-dimensional process $u \in \mathbb{L}^2_{loc}$ and $M_0 \in \mathbb{R}$ result that the real integral process*

$$M_t = M_0 + \int_0^t u_s \cdot dW_s \qquad t \in [0,T] \tag{B.5}$$

is a $\mathcal{F}^W$-martingale. Moreover, every real $\mathcal{F}^W$-martingale can be represent in this form.

Theorem B.1.8. *Under the assumtion of Girsanov's theorem if M is a local martingale in $(\Omega, \mathcal{F}, P, (\mathcal{F}^W_t))$ then there exists a unique $u \in \mathbb{L}^2_{loc}(\mathcal{F}^W)$ such that*

$$M_t = M_0 + \int_0^t u_s \cdot dW^\lambda_s \qquad t \in [0,T]$$

where W^λ is the Q-Brownian motion defined in (B.4).

Theorem B.1.9 (Change of drift). *Let Q be a probability measure equivalent to P. The Radon-Nikodym derivative of Q with respect to P is an exponential martingale*

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}^W_t} = Z^\lambda_t \qquad dZ^\lambda_t = -Z^\lambda_t \lambda_t \cdot dW_t$$

with $\lambda \in \mathbb{L}^2_{loc}$ *and the process* W^λ *is defined by*

$$dW_t = dW_t^\lambda - \lambda_t dt \tag{B.6}$$

is a Brownian motion on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t^W))$.

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