Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

Backreaction from magnetogenesis in string inflation

Supervisor:
Prof. Michele Cicoli

Submitted by:
Sofia Panagiota Corbá

Academic Year 2017/2018

Abstract

This work is focused on the numerical study of the inflationary evolution in the presence of backreaction effects due to the production of primordial magnetic fields.
In the context of inflationary string models it is possible to generate a coupling of the inflaton with the electromagnetic field through the introduction of the Gauge Kinetic Function in the standard electromagnetic Lagrangian term. Since the inflaton changes monotonously during inflation it is natural to assume that the coupling function is a decreasing function of time, leading to the presence of a high electric energy density. This fact, known as the backreaction problem, does not allow to solve the background and Maxwell equations separately, and so it is often avoided.
Recent researches, however, showed that the backreaction can cause a slowdown in the evolution of the inflaton towards its minimum. This effect could play a crucial role in string inflationary models where the inflaton field range can be upper bounded by geometric conditions associated with the size of the extra dimensions. These Kähler cone conditions for the string moduli space can forbid the possibility to obtain enough e-foldings of inflation, and so a slowing-down of the inflaton due to backreaction effects can help to get a better phenomenology.
The analysis is carried out in two particular models of inflation, Kähler Inflation and Fibre Inflation, where the inflaton is a closed string mode belonging to the hidden sector. In both cases we obtain the desired slowdown, although in Kähler Inflation it is more difficult due to a stronger dependence of the potential on the microscopic parameters of the underlying compactification theory.

Sommario

Questo lavoro di tesi si focalizza sullo studio numerico dell'evoluzione inflazionaria in presenza di effetti di backreaction dovuti alla produzione di campi magnetici primordiali. Nell'ambito dei modelli inflazionari di stringa è possibile generare il coupling dell'inflatone con campi elettromagnetici tramite l'introduzione della Funzione Cinetica di Gauge nella Lagrangiana elettromagnetica ordinaria.
Dato che l'inflatone evolve in maniera monotona durante l'inflazione, è naturale assumere che la funzione di coupling sia decrescente nel tempo, il che implica la presenza di un'alta densità di energia elettrica. Come risultato, noto come problema del backreaction, le equazioni di background e quelle di Maxwell non possono più essere risolte separatamente e di conseguenza tale regime viene spesso evitato.
Recenti ricerche hanno tuttavia evidenziato come il backreaction possa causare un rallentamento dell'inflatone nella sua evoluzione verso il minimo del potenziale. Tale effetto potrebbe avere un ruolo molto importante nei modelli inflazionari di stringa dove l'intervallo dei valori permessi per l'inflatone può essere superiormente limitato da condizioni geometriche associate alle dimensioni extra. La presenza di tali condizioni del cono di Kähler per lo spazio dei moduli può essere in contrasto con la possibilità di avere abbastanza e-foldings di inflazione e quindi un rallentamento dell'inflatone, dovuto al backreaction, potrebbe essere utile per avere maggiore accordo con la fenomenologia.
A tal fine sono stati considerati due particolari modelli inflazionari di stringa, quello della Kähler Inflation e quello della Fibre Inflation, nei quali l'inflatone è un modo di stringa chiuso appartenente al settore nascosto. In entrambi i casi si ottiene alla fine il rallentamento desiderato, anche se nel caso della Kähler Inflation con maggiore difficoltà a causa della dipendenza più intensa del potenziale dai parametri microscopici provenienti dalla compattificazione delle dimensioni extra.

Contents

Introduction **1**

1 Inflation **3**
1.1 The Standard Cosmological Model . . . 4
1.2 Problems of the Standard Cosmological Model . . . 10
1.3 Inflation . . . 14
1.4 Quantum Fluctuations during Inflation . . . 19

2 Supersymmetry **27**
2.1 Basics of Supersymmetry . . . 30
2.2 Supersymmetry Breaking . . . 42
2.3 $\mathcal{N} = 1$ Supergravity . . . 44

3 String Compactifications **45**
3.1 Strings . . . 46
3.2 Extra dimensions and compactifications . . . 49
3.3 Calabi-Yau manifolds . . . 51
3.4 Type IIB compactifications on Calabi Yau three-folds . . . 54

4 Inflationary String Models **57**
4.1 Moduli Stabilisation . . . 58
4.2 Kähler Inflation . . . 60
4.3 Fibre Inflation . . . 66
4.4 General properties of inflationary string models . . . 73

5 Numerical analysis of backreaction from magnetogenesis **75**
5.1 The necessity of more e-foldings of Inflation . . . 76
5.2 System of equations . . . 78
5.3 Coupling Functions . . . 86
5.4 Numerical evolution . . . 88

Conclusions and outlook **101**

Introduction

One of the most fascinating questions in physics is related to the origin and evolution of the Universe in the first moments of its existence. Today, the most successful cosmological paradigm able to describe the physics at early times is that of primordial inflation. According to this theory, the Universe, before its standard evolution in the radiation and matter cosmological era, has undergone a rapid phase of accelerated expansion driven by a scalar field. In this way, one can explain successfully the temperature anisotropy observed in the Cosmic Microwave Background and can resolve many problems related with the initial conditions emerging in the Standard Cosmological Model.

The particular energy and density conditions that characterize this period of cosmic evolution make inflation an ultraviolet sensitive phenomenon. To address the critical questions about the inflationary dynamics it is required, therefore, to treat it in a complete theory of quantum gravity which unifies in a consistent way Quantum Field Theory and General Relativity.

The search for a unified theory of elementary particles and their interactions has culminated in the last years to the development of string theory. Since it is defined in a consistent way in ten dimensions, in order to extract any information about four-dimensional physics, we need to consider the six extra dimensions very small and compactified within a suitably six dimensional complex manifold. The low energy effective action of string compactifications on Calabi-Yau three-folds gives rise to a large number of uncharged massless scalar fields with a flat potential which lead to long range scalar forces unobserved in nature. It should, therefore, exist a mechanism that generates a potential for them. This makes string theory a particularly suitable context to study inflation since recent developments in moduli stabilisation techniques have revealed the emergence of many closed string modes which are excellent candidates to drive inflation.

In this thesis we focus, in particular, on two inflationary string models, Kähler Inflation (KI) and Fibre Inflation (FI), which arise in the context of Type IIB compactifications embedded in a promising moduli stabilisation scheme called the Large Volume Scenario (LVS).

The aim of our analysis is the numerical study of the inflationary evolution in presence of backreaction effects due to the coupling of the inflaton with the electromagnetic field. To this end we modify the standard electromagnetic Lagrangian term via the introduction of a coupling function which depends on time through the inflaton.

From the numerical analysis, it turns out that the presence of backreaction changes the ordinary inflationary dynamic and could cause a slowdown of the inflaton field. This fact is very useful because it could in principle make reconcile the geometrical limits that emerge in string theory due to the Kähler cone conditions of the moduli space with the phenomenological requirement, coming from the observations, of obtaining enough e-foldings of inflation.

In the first three chapters, we introduce the necessary tools for the understanding of inflation and the theory of compactification and moduli stabilisation. More precisely, in the first chapter we present the Standard Cosmological Model, its problems and how the introduction of an initial inflationary phase can resolve them, as well as Cosmological Perturbation Theory.
In the second chapter we describe the theory of Supersymmetry which is necessary for the internal consistency of string theory.
String theory compactifications are analyzed in the third chapter where we present all the necessary elements for its understanding while in the fourth chapter the techniques of compactification and stabilisation are applied to the two inflationary models which we are interested in.
In the fifth chapter there is the numerical analysis of the inflationary evolution and the comparison of the results coming from the two regimes with and without backreaction. After presenting our conclusions, we describe some proposals for future work that can be carried out based on the formalism developed in this thesis.

Chapter 1

Inflation

The Standard Cosmological Model, developed during the second half of the last century, provides a detailed description of the current state of the Universe and of its thermal history. It is based on General Relativity, the advent of which has opened the way for a systematic study of theoretical cosmology. Its main assumption, inspired by two fundamental observations i.e. the recession of galaxies, discovered by Hubble and the presence of the Cosmic Microwave Background, discovered by Penzias and Wilson, is that on sufficiently large scales the Universe appears homogeneous and isotropic. On smaller scales, instead, there is a strong inhomogeneity, due to the gravitational attraction that has concentrated matter into stars, galaxies and galaxy clusters.
According to the Standard Cosmological Model, after the Big Bang, the Universe began to cool down and expand until the present day. This evolution is divided into two phases, one dominated by radiation and the other dominated by matter whereas today we are experiencing a phase of accelerated expansion driven by the cosmological constant. In contrast to this well-defined picture we have no certain knowledge about what happened well before nucleosynthesis. Thanks to the new observations and the development of theoretical physics, we know that the theory must be completed by an initial period of accelerated expansion occurring before the radiation era, called inflation, in order to solve many phenomenological problems that emerge in the standard cosmology. In this way, we obtain a simple explanation for the homogeneity and flatness of the Universe as well as for the absence of cosmic relics. This rapid expansion at very early times provides also a mechanism for the origin of the observed large-scale structure based on the presence of quantum primordial fluctuations.
Inflation is an ultraviolet sensitive phenomenon, and so it cannot be described by ordinary effective theories but it has to be understood within the framework provided by an ultraviolet completion. String theory, is one of the most promising candidates to describe the initial evolution of our Universe as it is a theoretical model which unifies Quantum Field Theory and General Relativity in a consistent way. This chapter presents the basic notions of the standard and inflationary cosmology that we will use later in a string cosmology context.

1.1 The Standard Cosmological Model

The most important feature of modern cosmology is the Cosmological Principle according to which on sufficiently large scales the Universe is homogeneous and isotropic. This assumption together with the laws of General Relativity form the basis of the Standard Cosmological Model. In this context, the Universe can be viewed as a perfect fluid with two main components, matter and radiation.
The metric which is in agreement with the cosmological principle is found by foliating the spacetime with spatial three-dimensional homogeneous surfaces Σ_t of constant curvature K and by using comoving coordinates which identify the class of static geodetic observers as privileged. The result is the Friedmann-Robertson-Walker (FRW) metric which describes a spacetime manifold that admits spatial three-dimensional homogeneous and isotropic sections of constant curvature K.

$$dS^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta \;\; d\phi^2)\right) \tag{1.1}$$

All the dynamics, expressed in terms of the cosmic time, is contained in the scale factor $a(t)$, which depends on the distribution of matter in the Universe and it is determined by Einstein equations. On the basis of $k = K/|K|$ we can distinguish between three possibilities:

- $k = 0$

$$dS^2 = dt^2 - a^2(t)\left(dr^2 + r^2(d\theta^2 + sin^2\theta \;\; d\phi^2)\right) \tag{1.2}$$

With the change of coordinates

$$x = r\, cos\phi\, sin\theta \qquad y = r\, sin\phi\, sin\theta \qquad z = r\, cos\theta$$

$$\theta \in [0, \pi] \qquad \phi \in [0, 2\pi)$$

the spatial metric becomes

$$d\sigma^2 = dx^2 + dy^2 + dz^2 \tag{1.3}$$

which describes a spatially flat surface and therefore it gives rise to a flat universe. In this case the whole curvature is contained in time.

- $k = 1$

$$dS^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-r^2} + r^2(d\theta^2 + sin^2\theta \;\; d\phi^2)\right) \tag{1.4}$$

In order to remove singularities on $r = \pm 1$, it is convenient to use $r = sin\chi$ in terms of which the spatial metric becomes

$$d\sigma^2 = d\chi^2 + sin^2\chi\,(d\theta^2 + sin^2\theta \;\; d\phi^2) \tag{1.5}$$

Through the change of coordinates

$$x = a_0\, sin\chi\, cos\phi\, sin\theta \qquad y = a_0\, sin\chi\, sin\phi\, sin\theta \qquad z = a_0\, sin\chi\, cos\theta \qquad w = a_0\, cos\chi$$

$$\chi \in [0, \pi] \qquad \theta \in [0, \pi] \qquad \phi \in [0, 2\pi)$$

we obtain

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2 \tag{1.6}$$

which is a three-dimensional sphere immersed in $\mathbb{R}^4$. In this case the Universe has positive curvature and it is compact with finite volume.

- $k = -1$

$$dS^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1+r^2} + r^2(d\theta^2 + sin^2\theta \;\; d\phi^2)\right) \tag{1.7}$$

We have again singularities which are removed by the redefinition $r = sinh\chi$

$$d\sigma^2 = d\chi^2 + sinh^2\chi\,(d\theta^2 + sin^2\theta \;\; d\phi^2) \tag{1.8}$$

Through the change of coordinates

$$x = a_0\, sinh\chi\, cos\phi\, sin\theta \quad y = a_0\, sinh\chi\, sin\phi\, sin\theta \quad z = a_0\, sinh\chi\, cos\theta \quad w = a_0\, cosh\chi$$

$$\chi \in [0, \infty) \qquad \theta \in [0, \pi] \qquad \phi \in [0, 2\pi)$$

we get

$$d\sigma^2 = dx^2 + dy^2 + dz^2 - dw^2 \tag{1.9}$$

This is a hyperboloid immersed in the Minkowski space which gives rise to non compact spaces with constant negative curvature

More generally, in the coordinates (t, χ, θ, ϕ) the metric becomes

$$dS^2 = dt^2 - a^2(t)\left(d\chi^2 + S_k^2(\chi)\,(d\theta^2 + sin^2\theta \;\; d\phi^2)\right) \tag{1.10}$$

with

$$S_k(\chi) = \begin{cases} sin\chi, & k = 1 \\ \chi, & k = 0 \\ \sinh\chi, & k = -1 \end{cases} \tag{1.11}$$

The worldlines of isotropic observers $(\chi, \theta, \phi) =$ constant are the geodetics and so are at rest with respect to the matter contained in the Universe. In the view of the Universe as a cosmological fluid, the role of isotropic observers is played by galaxies, which move away from each other with a relative velocity given by Hubble's law

$$v = Hr \tag{1.12}$$

where the Hubble parameter $H(t) = \dot{a}(t)/a(t)$ measures the expansion rate of the Universe and r is the proper distance of the galaxies. The *dot* always represents the derivative with respect to the cosmic time t.
The expansion is studied also through the redshift z, which determines the difference between the wavelength of a signal λ_e emitted at time t_e and that measured by an observer in motion with respect to the source λ_0, at the time $t_0 > t_e$.

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} \tag{1.13}$$

We find

$$1 + z = \frac{a(t_0)}{a(t_e)} \tag{1.14}$$

In order to understand the dynamics of the Universe, it is necessary to determine the evolution of the scale factor through the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1.15}$$

$R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature and $T_{\mu\nu}$ is the stress-energy tensor of a perfect fluid

$$T_{\mu\nu} = -p g_{\mu\nu} + (\rho + p) u_\mu u_\nu \tag{1.16}$$

where u_μ is the 4-velocity, ρ is the energy density and p the pressure. Homogeneity implies that ρ and p are functions only of time. There are three unknown functions $a(t)$, $\rho(t)$, $p(t)$ and therefore we need an equation of state that relates the energy density and pressure. The sources of the cosmic gravitational field can be described as perfect barotropic fluids with equation of state

$$p = w\rho \tag{1.17}$$

where w is a constant. From the conservation law of the stress-energy tensor ${T^{\mu\nu}}_{;\mu} = 0$ we find

$$\dot{\rho} = -3H(t)(\rho + p) \tag{1.18}$$

which can be rewritten like

$$\frac{d}{dt}(\rho a^3) = -p \frac{d}{dt}(a^3) \tag{1.19}$$

From the generic equation of state (1.17) we have

$$\frac{d}{dt}(\rho a^3) = -\rho w \frac{d}{dt}(a^3) \Rightarrow \frac{d\rho}{dt}a^3 + 3\rho a^2 \frac{da}{dt} = -3\rho w a^2 \frac{da}{dt} \Rightarrow \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = -\int_{a_0}^{a} \frac{da}{a} 3(1+w) \Rightarrow \frac{\rho(t)}{\rho_0} = \left(\frac{a(t)}{a_0}\right)^{-3(1+w)} \Rightarrow \rho(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^{3(1+w)} \tag{1.20}$$

where a_0 and ρ_0 are the current values of the respective quantities. Looking on (1.20) we can distinguish between three cases corresponding to the three different relevant fluids: Barionic matter is grouped into stars, galaxies and galaxy clusters. On a cosmic scale the single galaxy is described by a grain of dust and so the pressure can be neglected. Galaxies in addition to gas, intergalactic and interstellar powders and black matter are contribute to the matter energy density ρ_m. In this case

$$w_m = 0 \qquad p_m = 0 \qquad \rho_m(t) = \rho_{0,m}\left(\frac{a_0}{a(t)}\right)^3 \tag{1.21}$$

or

$$\rho_m(t) = \rho_{0,m}(1+z)^3 \tag{1.22}$$

so matter density decreases with the expansion of the Universe as $\rho_m(t) \propto a^{-3}$.
The radiation energy density ρ_r, on the other hand, is composed mostly by the CMB but also from the cosmic neutrinos background and it is characterized by

$$w_r = \frac{1}{3} \qquad p_r = \frac{\rho_r}{3} \qquad \rho_r(t) = \rho_{0,r}\left(\frac{a_0}{a(t)}\right)^4 \tag{1.23}$$

or

$$\rho_r(t) = \rho_{0,r}(1+z)^4 \tag{1.24}$$

so decreases with the expansion of the universe as $\rho_r(t) \propto a^{-4}$.
Finally, we have the energy density of dark energy the contribution of which is equivalent to that of a positive cosmological constant characterized by

$$w_\Lambda = -1 \qquad p_\Lambda = -\rho_\Lambda \tag{1.25}$$

The energy density $\rho_\Lambda = \frac{\Lambda}{8\pi G}$ remains constant regardless of the evolution of the scale factor. The total energy density is given by the sum of all these components

$$\rho_{tot} = \rho_m + \rho_r + \rho_\Lambda \tag{1.26}$$

but the evolution of the Universe is driven by the one that dominates in each cosmological period.

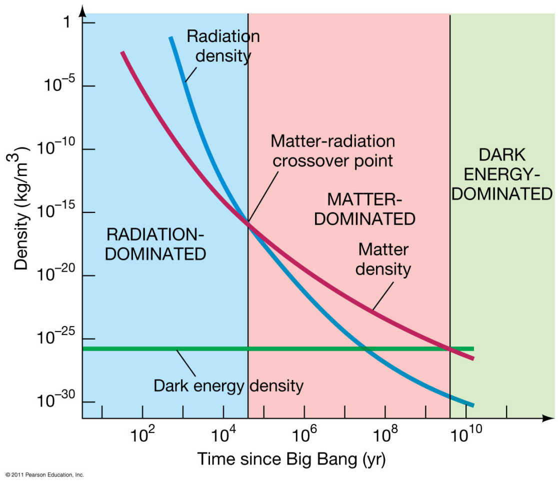


Figure 1.1: Different Cosmological Epochs

Since the radiation energy density, which is the dominant one immediately after the Big Bang, decreases faster than that of matter while that of the cosmological constant remains constant, we can distinguish three different eras that have followed one another as it is described in Figure (1.1). Right after the Big Bang, the Universe was dominated by radiation, next has entered in the matter era, and today we are in the dark energy era with the Universe that undergoes accelerated expansion. The transition from the radiation era to the matter era has occurred when $\rho_r = \rho_m$ at $z_{eq} = 3360 \pm 70$ about 61,000 years after the Big Bang.

By introducing the stress-energy tensor of the perfect fluid as the source of the Einstein equations we obtain the Friedmann equations

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 \tag{1.27}$$

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a \tag{1.28}$$

These equations in addition to the equation of state allow us to determine the three unknown functions $a(t)$, $\rho(t)$ and $p(t)$. In terms of the cosmological parameter defined as

$$\Omega = \frac{\rho}{\rho_c} \tag{1.29}$$

where $\rho_c = 3H^2/8\pi G$ is the critical density corresponding to the flat universe the equation (1.27) can be rewritten as

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.30}$$

It follows that

$$\Omega - 1 = \frac{k}{H^2 a^2} \tag{1.31}$$

which is a very useful equation for the comparison with observations. More precisely, studying the cosmological parameter we can get information on the geometry of our Universe.

$$\begin{cases} \Omega > 1 \Rightarrow \rho > \rho_c & k = 1 \\ \Omega < 1 \Rightarrow \rho < \rho_c & k = -1 \\ \Omega = 1 \Rightarrow \rho = \rho_c & k = 0 \end{cases} \tag{1.32}$$

The actual value of Ω, found through observations, is $\Omega_0 = 1,000(7)$, which means that our Universe is almost flat.

1.2 Problems of the Standard Cosmological Model

To study the propagation of a signal in the FRW metric instead of physical time, t, it is better to use the conformal time η,

$$\eta = \int \frac{dt}{a(t)} \tag{1.33}$$

in terms of which the (1.1) becomes

$$dS^2 = a^2(\eta)\Big(d\eta^2 - d\chi^2 - \Sigma_k^2(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)\Big) \tag{1.34}$$

A massless particle is characterized by $dS^2 = 0$, and so

$$d\eta^2 = d\chi^2 \Rightarrow \chi(\eta) = \pm\eta + \text{const} \tag{1.35}$$

where we have chosen, without loss of generality, $d\theta = d\phi = 0$. We can obtain, therefore, the maximum comoving distance traveled from the moment of Big Bang, t_i until the moment t under consideration that is called Particle Horizon:

$$\chi_p(t) = \chi(t) - \chi(t_i) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a(t)} \tag{1.36}$$

At time η the events which are at $\chi > \chi_p(\eta)$ are not accessible to the observer at $\chi = 0$. The physical dimension of the Particle Horizon is

$$d_p(t) = a(t)\chi_p = a(t)\int_{t_i}^{t} \frac{dt}{a(t)} \tag{1.37}$$

It is interesting to note that in reality photons have started to propagate at the time of decoupling t_d, thus after the Big Bang and consequently the real Horizon that limits the light signals is the Optical Horizon

$$d_{opt} = a(t)\int_{t_d}^{t} \frac{dt}{a(t)} \tag{1.38}$$

Although the two Horizons do not differ so much between them, the fact that the photons are released after the Big Bang hides very useful information about the primordial Universe. When the matter content of the Universe satisfies the strong energy condition $w > 1/3$ then the particle Horizon has a value very similar to the Hubble scale $1/H$ which is of order of the 4-curvature scale and it characterizes the size of the locally inertial reference system. However, these two quantities are conceptually different since the first is identified by kinematic reasoning while the second is a dynamical quantity related to the expansion rate. When the strong energy condition is violated the Particle Horizon grows faster than Hubble scale and thus they differ significantly. More precisely, if $w < 1/3$ then the expansion of the Universe is accelerated $\ddot{a} > 0$ and the Particle Horizon is

$$d_p(t) = a(t)\int^t \frac{dt}{a(t)} = a(t)\int^a \frac{da}{a\dot{a}} \tag{1.39}$$

where the integral is convergent for $t \to \infty$ and $a \to \infty$. At large times, therefore, the particle Horizon is proportional to $a(t)$, while the Hubble scale is $H^{-1} = a(t)/\dot{a}(t)$. Since during accelerated expansion $\dot{a}(t)$ is an increasing function of time the two quantities are quite different.
The high degree of homogeneity of the cosmic background radiation that is observed experimentally it is difficult to reconcile with the presence of a Particle Horizon. In fact, isotropy implies that at decoupling the plasma was at thermal equilibrium with temperature fluctuations of order $\sim 10^{-5}$. However, to achieve thermal equilibrium, the various regions of a system must have been in contact for a sufficiently long time. Considering that the signals propagate at the speed of light and that the Universe has not existed for an infinite time but began with the Big Bang, the standard model of cosmology cannot explain such high degree of homogeneity in the CMB. More precisely it is reasonable to wonder if the radiation coming from opposite directions of the sky comes from regions that have had enough time to make causal contact and thus thermalize or not. The size of the regions that have had enough time to thermalize at decoupling can be approximated with the distance traveled by light emitted at the time of the Big Bang up to that moment. This size must then be compared with the size of the Horizon which corresponds approximately to the distance traveled by the photons of the CMB before reaching us. The comoving distance made from a light signal from the moment when the scale factor was a_1 to the moment when the scale factor became a_2, in the case of a fluid with parameter w, is

$$\Delta\chi = \int_{a_1}^{a_2} \frac{da}{a\dot{a}} = \int_{a_1}^{a_2} \frac{a_0^{-3(1+w)/2}}{H_0} a^{-\frac{1}{2}+\frac{3}{2}w} = \frac{2}{3w+1}\frac{1}{a_0 H_0}\left[\left(\frac{a_2}{a_0}\right)^{\frac{1+3w}{2}} - \left(\frac{a_1}{a_0}\right)^{\frac{1+3w}{2}}\right]$$

where we have used the (1.27), and so

$$\Delta\chi = \frac{2\chi_H}{1+3w}\left[\left(\frac{a_2}{a_0}\right)^{\frac{1+3w}{2}} - \left(\frac{a_1}{a_0}\right)^{\frac{1+3w}{2}}\right] \quad \text{where} \quad \chi_H = \frac{1}{a_0 H_0} \tag{1.40}$$

which becomes

$$\Delta\chi_d = 2\chi_H\big(\sqrt{a_2} - \sqrt{a_1}\big) \qquad\qquad \Delta\chi_r = \chi_H\big(a_2 - a_1\big) \tag{1.41}$$

for dust and radiation respectively. As matter becomes dominant at $z_{mat} = 8800$ and decoupling takes place at $z_{dec} = 1100$ we can approximate the whole scale factor evolution with the one dominated by non relativistic matter. The Horizon at decoupling, with $w = 0$, is

$$\Delta\chi_{dec} = \int_0^{t_{dec}} \frac{dt}{a(t)} = \int_0^{a_{dec}} \frac{da}{a\dot{a}} = \int_0^{a_{dec}} \frac{1}{a_0^{3/2} a^{1/2} H_0} da = 2\chi_H \left(\frac{a_{dec}}{a_0}\right)^{1/2} \tag{1.42}$$

and since $1 + z_{dec} = a_0/a_{dec}$

$$\Delta\chi_{dec} = \frac{2\chi_H}{\sqrt{1+z_{dec}}} \tag{1.43}$$

The Hubble radius at decoupling is

$$\Delta\chi_H = \int_{a_{dec}}^{a_0} \frac{da}{a\dot{a}} = \frac{2\chi_H}{a_0^{1/2}}\left(a_0^{1/2} - a_{dec}^{1/2}\right) = 2\chi_H\left(1 - \frac{1}{\sqrt{1+z_{dec}}}\right) \tag{1.44}$$

By confronting the two quantities

$$\frac{\Delta\chi_{dec}}{\Delta\chi_H} \sim 1.7^\circ \tag{1.45}$$

it turns out that at decoupling only regions which today form an angle 1.7° have had sufficient time to make thermal contact. Outside the Hubble radius, bodies expand at speeds higher than light's and therefore the signals that they send will never reach us. In other words, the Hubble scale corresponds to the maximum distance of the most distant observers who could send us signals. The Particle Horizon, on the other hand, gives us the size of the regions in causal connection at a given moment, which in this case is the moment of decoupling. It is clear that at the time of decoupling the region in causal contact, i.e. the one enclosed in $\Delta\chi_{dec}$, is very small compared to the size of the region we see today. The uniformity of the CMB, however, implies that at decoupling the casual connected region had to be greater than the Hubble scale at the same time. This problem known as Horizon Problem can be solved by adding an initial phase of accelerated expansion that allows to put in causal contact the entire Universe by the time of decoupling.

Another problem that cannot be solved in the Standard Cosmological Model is the Flatness Problem. The observations suggest that the actual density of the Universe is very close to the critical one which means that our Universe is almost flat. In the FRW model, however, the cosmological evolution moves the density parameter away from unity and therefore to reproduce the observational evidences it is necessary a unnatural fine-tuning of the initial conditions. More precisely the evolution of the density parameter is given by (1.31) and so

$$\frac{d\Omega}{da} = -\frac{k}{(a^2H^2)^2}\frac{d}{da}\left(a^2H^2\right) = -\frac{2k}{a^2H^2}\left(\frac{1}{a} + \frac{1}{H}\frac{dH}{da}\right) \tag{1.46}$$

$$\frac{dH}{da} = \frac{dH}{dt}\frac{dt}{da} = \frac{\ddot{a}a - \dot{a}^2}{\dot{a}a^2} = \frac{1}{\dot{a}a}\left(-\frac{4\pi G}{3}(\rho+3p)\right) - \frac{1}{a}H = -\frac{1}{aH}\left(\frac{4\pi G}{3}(\rho+3p) + H^2\right)$$

where was used equation (1.28).

$$\frac{d\Omega}{da} = -\frac{2k}{a^2H^2}\left(\frac{1}{a} - \frac{1}{aH^2}(\frac{4\pi G}{3}(\rho+3p) + H^2)\right) \Rightarrow a\frac{d\Omega}{da} = \frac{2k}{a^2H^2}\left(\frac{4\pi G}{3H^2}(\rho+3p)\right)$$

$$\frac{d\Omega}{d\log a} = \frac{2k}{a^2H^4}\frac{4\pi G}{3}(\rho+3p) \qquad \Rightarrow \qquad \frac{d\Omega}{d\log a} = (\Omega-1)\Omega(1+\frac{3p}{\rho}) \tag{1.47}$$

From (1.47) it is clear that for ordinary matter $\rho + 3p > 0$, $\Omega(a)$ is increasing if $\Omega > 1$ and decreasing if $\Omega < 1$. This means that the solution $\Omega = 1$ is a repulsor and therefore in order to reproduce the observations we must require $|\Omega - 1| < 10^{-64}$ at Planck time, which corresponds to an extremely homogeneous initial condition. Even this problem can be solved by introducing an initial phase of accelerated expansion in which the strong energy condition is violated. In fact, if $w < -1/3$, i.e. $p < -\rho/3$ then from (1.47) the solution $\Omega = 1$ becomes an attractor and so the energy density assumes its critical value in a natural way. In order to confirm observations the accelerated expansion has to last long enough so that the subsequent decelerated expansion governed first by radiation and then by matter brings the density parameter to a value sufficiently close to one without any fine-tuning.
If the Hot Big Bang begins at very high temperatures, it emerges the problem of unwanted relics that may survive until the present day and that are forbidden by observations . An example is the gravitino, the particle occurring in supergravity as the spin-3/2 partner of the graviton. Depending on the theory, there may also be unwanted topological defects. If in the early Universe a GUT symmetry is restored then magnetic monopoles are produced when it is spontaneously broken. Their abundance typically is higher than allowed by observations unless they are connected by strings. Relic abundances can be reduced to a satisfactory level by the expansion during inflation, provided that they are produced before the inflationary epoch.
Apart from resolving all these problems the biggest success of inflation is to explain the origin of primordial inhomogeneities which are responsible for the small anisotropies in the CMB and that are required for structure formation. In fact, the homogeneous and isotropic geometry of the standard model does not provide the thermal fluctuations of cosmic radiation and consequently there are no density fluctuations that could have been condensed to form the actual observed structure. Today it is believed that the density and energy fluctuations currently existing on a cosmic scale are due to the quantum fluctuations of matter and geometry amplified during the inflationary phase and subsequently increased at a macroscopic level.

1.3 Inflation

Inflation is referred to the period of accelerated expansion governed by a scalar field that violates the strong energy condition. The inflationary cosmology is not a substitute for the Standard Cosmological Model but rather an add-on that occurs at very early times without disturbing any of its successes. To obtain inflation we need matter with negative pressure, i.e. a scalar field, called inflaton, with $T^{\mu}{}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p)$. In general there is no equation of state linking pressure and energy density. However, in the regime where we can neglect the kinetic energy of the field compared to the potential, it is approximately true that $p \approx -\rho$. A scalar field minimally coupled to gravity is described by the action

$$S[g,\phi] = -\frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}R + S_M[g,\phi] \tag{1.48}$$

where $M_{pl} = 2.4 \cdot 10^{18}$ GeV is the reduced Planck mass, R is the scalar curvature, $g = \det g_{\mu\nu}$ and $S_M[g,\phi]$ is the action of the scalar field modified accordingly to the Principle of General Covariance

$$S_M[g,\phi] = \int d^4x\mathcal{L}_M(\phi,\nabla\phi,g) = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right) \tag{1.49}$$

The stress-energy tensor of a scalar field in General Relativity becomes

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}_M = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{1.50}$$

From the stationary condition for the action we get the Einstein equation (1.15),

$$\begin{aligned}0 = \frac{\delta S}{\delta g^{\mu\nu}} &= -\frac{M_{pl}^2}{2}\int d^4x\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}}R + \frac{\delta R}{\delta g^{\mu\nu}}\sqrt{-g} + \frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}}\mathcal{L}_M + \sqrt{-g}\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}\\
&= -\frac{M_{pl}^2}{2}\int d^4x\big(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}R + R_{\mu\nu}\sqrt{-g}\big) - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\mathcal{L}_M + \frac{1}{2}\sqrt{-g}\partial_\mu\phi\partial_\nu\phi\\
&= -\frac{1}{2}\sqrt{-g}\big(-\frac{M_{pl}^2}{2}g_{\mu\nu}R + M_{pl}^2R_{\mu\nu} - T_{\mu\nu}\big)\end{aligned}$$

considering $M_{pl} = 1/8\pi G$. From the field equation for the scalar field, instead, we have

$$\partial_\alpha\frac{\delta\mathcal{L}_M}{\delta\partial_\alpha\phi} - \frac{\delta\mathcal{L}_M}{\delta\phi} = 0 \tag{1.51}$$

we find

$$\frac{\delta\mathcal{L}_M}{\delta\partial_\alpha\phi} = \sqrt{-g}g^{\alpha\mu}\partial_\mu\phi \qquad \frac{\delta\mathcal{L}_M}{\delta\phi} = -\sqrt{-g}V'(\phi)$$

and finally

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\mu}\partial_\mu\phi) + V'(\phi) = 0 \tag{1.52}$$

Using (1.50) in the case of spatially homogeneous, isotropic and flat configurations

$$dS^2 = dt^2 - a^2(t)\delta_{ik}dx^i dx^k \qquad g_{\mu\nu} = diag(1, -a^2, -a^2, -a^2)$$

where the inflaton depends only on time $\phi = \phi(t)$, we have

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1.53}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{1.54}$$

while the (1.52) becomes

$$\frac{1}{a^3}\partial_0(a^3 g^{00}\partial_0\phi) + \frac{1}{a^3}\partial_i(a^3 g^{ii}\partial_i\phi) + V'(\phi) = 0 \Rightarrow \frac{1}{a^3}3a^2\dot{\phi}\dot{a} + \ddot{\phi} + V'(\phi) = 0$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{1.55}$$

Using (1.53), (1.54) in (1.27) with $k = 1$, we obtain

$$H^2 = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{1.56}$$

while by replacing them in the (1.28), we have

$$\frac{\ddot{a}}{\dot{a}} = -\frac{1}{3M_{pl}^2}\left(\dot{\phi}^2 - V(\phi)\right) \tag{1.57}$$

In the accelerated expansion $\ddot{a} > 0$ and so $V(\phi) > \dot{\phi}^2$. In particular, if the potential energy dominates over the kinetic one we can neglect the kinetic terms in the stress energy tensor: $\rho \approx V(\phi)$ and $p \approx -V(\phi)$ and consequently $\rho \approx -p$.

As long as the approximation

$$V(\phi) >> \frac{1}{2}\dot{\phi}^2 \tag{1.58}$$

is true, the Universe undergoes an accelerated expansion with exponential growth of the scale factor. To sustain this inflationary regime long enough in order to solve the problems seen above, while the scalar field moves towards the minimum of the potential $V(\phi)$, its kinetic energy must change slowly

$$|\ddot{\phi}| << 3H|\dot{\phi}| \tag{1.59}$$

As long as the slow roll conditions (1.58) and (1.59) are fulfilled the scalar field in its motion towards the minimum of the potential starting from a higher value generate a cosmological constant which drive the expansion. The equations (1.55), (1.56) become

$$3H\dot{\phi} = -V'(\phi) \tag{1.60}$$

$$H^2 = \frac{1}{M_{pl}^2} V(\phi) \tag{1.61}$$

The slow roll conditions can be reformulated as conditions for the potential. In fact, inflation begins if $\epsilon \ll 1$ and lasts long enough if $|\eta| \ll 1$, where ϵ and η are the slow roll parameters

$$\epsilon = \frac{M_{pl}^2}{2} \left(\frac{V'(\phi)}{V(\phi)} \right)^2 \tag{1.62}$$

$$\eta = M_{pl}^2 \frac{V''(\phi)}{V(\phi)} \tag{1.63}$$

The amount of inflation that occurs normally is quantified by the number of e-foldings, i.e. the ratio of the scalar factor at the final time to its value at some initial time.

$$N(t) \equiv \ln \frac{a(t_{end})}{a(t)} \tag{1.64}$$

where t_{end} is the moment of time when inflation ends. This measures the amount of inflation that still has to occur after time t, with N decreasing to 0 at the end of inflation. To solve the Horizon and flatness problems, around 50-60 e-foldings of inflation are required. In terms of the potential it becomes

$$N = -\frac{1}{M_{pl}^2} \int_{\phi}^{\phi_{end}} \frac{V(\phi)}{V'(\phi)} d\phi \tag{1.65}$$

because from (1.60) and (1.61) we have

$$V'(\phi) = \frac{3\dot{\phi}^2}{M_{pl}^2} \frac{V(\phi)}{V'(\phi)} \qquad H = -\frac{\dot{\phi}^2}{M_{pl}^2} \frac{V(\phi)}{V'(\phi)} \tag{1.66}$$

$$N = \ln \frac{a(t_{end})}{a(t)} = \int_t^{t_{end}} \frac{\dot{a}}{a} dt = \int_t^{t_{end}} H dt = -\frac{1}{M_{pl}} \int_{\phi}^{\phi_{end}} \frac{V(\phi)}{V'(\phi)} \dot{\phi} dt$$

The Universe, through the mechanism of inflation, is driven in a natural way in a state such that the subsequent evolution, dominated first by radiation and after by matter leads to that we observe today. The inflationary solutions are attractors and thus all the solutions approaching one another very quickly to the point of being indistinguishable. The subsequent evolution is, therefore, independent from the initial conditions before inflation and the flatness problem is resolved. The maximum distance at which a signal can be propagated from an initial moment t_i to a final one t, it is equal to the difference of the corresponding conformal time

$$\Delta\chi = \chi - \chi_i = \int_{t_i}^{t} \frac{dt'}{a(t')} = \eta(t) - \eta(t_i) \tag{1.67}$$

The conformal time depends on the Hubble radius

$$\eta = \int \frac{dt}{a(t)} = \int \frac{da}{a\dot{a}} = \int \frac{1}{a^2 H} da = \int \frac{1}{aH} d(\ln a) \tag{1.68}$$

From the (1.27) with $k = 0$ and (1.20) we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} = \frac{\rho_0}{\rho_{0c}} H_0^2 \left(\frac{a_0}{a}\right)^{3(1+w)} \Rightarrow \dot{a} = H_0 \frac{a_0^{\frac{3}{2}(1+w)}}{a^{\frac{1}{2}(1+3w)}}$$

and consequently

$$\frac{1}{aH} = \frac{1}{\dot{a}} = \frac{a^{\frac{1}{2}(1+3w)}}{H_0 a_0^{\frac{3}{2}(1+w)}} \tag{1.69}$$

which gives the Hubble radius in function of a. If $1 + 3w > 0$, the Hubble radius is an increasing function of a, while if $1 + 3w < 0$, it is decreasing.

$$\eta = \int \frac{1}{aH} d(\ln a) = \int \frac{a^{\frac{1}{2}(1+3w)}}{H_0 a_0^{\frac{3}{2}(1+w)}} d(\ln a) = \int \frac{1}{H_0 a_0^{\frac{3}{2}(1+w)}} e^{\ln a \frac{1}{2}(1+3w)} d(\ln a)$$

$$\eta = \frac{1}{H_0 a_0^{\frac{3}{2}(1+w)}} \frac{2}{1+3w} a^{\frac{1}{2}(1+3w)} \tag{1.70}$$

From this expression it is clear that for ordinary matter $w > -\frac{1}{3}$ the conformal time increases with a and at the beginning it is zero. Introducing inflation, on the other hand, the violation of the strong energy condition $1+3w < 0$, implies an initial conformal time $\eta_i \to -\infty$ and consequently we have a shrinking Hubble sphere, i.e. the decreasing of the comoving Hubble radius

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \tag{1.71}$$

This means that by violating the strong energy condition the elapsed conformal time between the initial singularity and the moment of decoupling is much longer. This explain how very distant points may have been in causal contact in the past resolving the Horizon Problem (Figure 1.2).

The end of inflation is identified with the end of the slow roll regime $\epsilon \sim 1$ and $\eta \sim 1$. By this time all classical inhomogeneities have been exponentially washed out and the Universe has became very flat. If the potential has a minimum the scalar field oscillates around the minimum and these oscillations are damped due to the Hubble expansion.

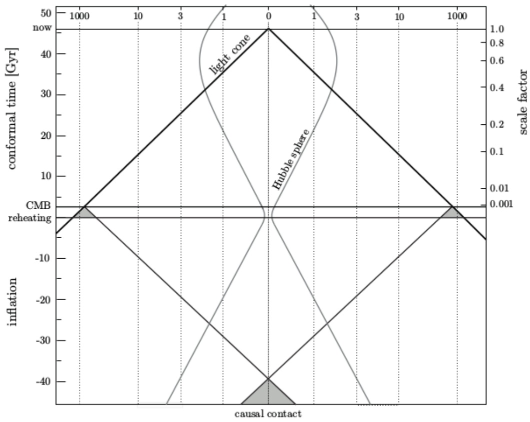


Figure 1.2: Resolution of the Horizon Problem

The Inflaton therefore decays and produces the known particles of the Standard Model (SM). This process, known as Reheating, generates the conditions for the particle densities and for the temperature which are at the beginning of the subsequent evolution of the Universe correctly described by the Standard Cosmological Model.
More precisely after the end of inflation the inflaton starts to oscillate around the global minimum of the potential and most of its inflationary energy is transformed first in kinetic energy and after in standard model degrees of freedom. If we assume a direct coupling of the inflaton ϕ with a bosonic field χ

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - g^2\phi^2\chi^2 \tag{1.72}$$

through the process of spontaneous symmetry breaking we obtain two types of interactions: $\phi\chi\chi$ and $\phi\phi\chi\chi$. The decays $\phi \to \chi\chi$ and $\phi\phi \to \chi\chi$ transfer the energy at χ particles which belong to the Standard Model. The equation is

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + m^2\phi^2 = 0 \tag{1.73}$$

where the additional friction term Γ represents the energy exchange and marks the end of inflation when it equals the decreasing Hubble parameter H. This description of reheating ignores quantum corrections, i.e. the backreaction of the classical inflaton oscillations on the quantum mechanical production of the SM fields. In fact the coupling between the two fields changes the equation of motion of the χ-particles and this can lead to resonance phenomena. In this context we can distinguish two different phases: the initial preheating which takes place in a regime of broad parametric resonance which eventually becomes narrow, and the subsequent slow reheating and thermalization where the classical theory of reheating can be successfully applied to the products of the preheating process.

1.4 Quantum Fluctuations during Inflation

In the context of the Standard Cosmological Model it is impossible to explain the presence of thermal variations in the CMB and the large density fluctuations of the cosmic matter that have gave rise to the galactic and extragalactic structures which we observe today.
Inflationary models provide an answer also to this problem. In fact, density and energy fluctuations currently existing on a cosmic scale are caused by quantum fluctuations of matter and geometry amplified during the inflationary phase and subsequently grown to the macroscopic level. More precisely, being the inflaton a quantum entity, it will have spatially varying fluctuations which imply that different regions of space inflate by different amounts. These differences in the local expansion histories lead to different local densities at the end of inflation and consequently to fluctuations in the CMB temperature.
In order to study the evolution of quantum fluctuations of the scalar field, it is in general convenient to do a Fourier decomposition of the modes that compose it and study the evolution of each comoving wavenumber. During inflation the comoving Hubble radius is decreasing whereas in the rest of the whole evolution of the Universe it is increasing. A fixed comoving scale $\lambda = 1/k$ therefore may begins its evolution smaller than the comoving Hubble radius $1/aH$ while by the end of inflation it becomes larger. When

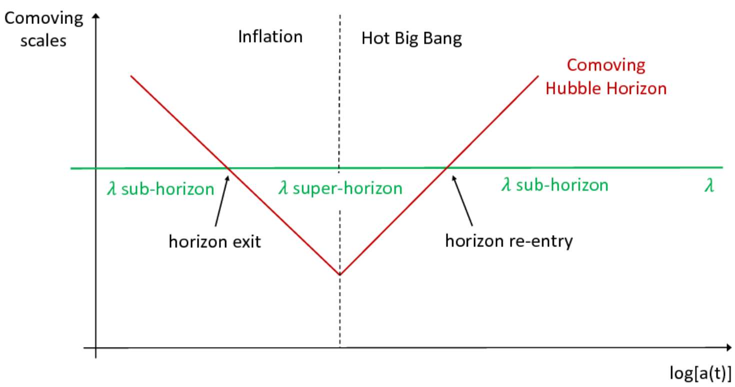


Figure 1.3: Evolution of cosmic scales

$k = aH$ the corresponding mode crosses the Horizon. It, therefore, stops to oscillate and begins its amplification.
In order to understand the amplification mechanism we must refer to the theory of cosmological perturbations. In a generic gravitational model with two type of sources, a scalar field and a perfect fluid minimally coupled to gravity we have

$$G_\mu{}^\nu = \frac{1}{M_{pl}^2}\left(T_\mu{}^\nu + \partial_\mu\phi\partial^\nu\phi - \frac{1}{2}\delta_\mu^\nu(\partial\phi)^2 + \delta_\mu^\nu V(\phi)\right)$$

where ϕ is the inflaton which satisfies the (1.52) and $T_{\mu}{}^{\nu}$ is the stress-energy tensor of the perfect fluid which is the source in the subsequent evolution of the Universe. The unperturbed homogeneous isotropic and flat geometry is given in terms of the conformal time by

$$dS^2 = a^2(\eta)(d\eta^2 - |d\vec{x}|^2) \tag{1.74}$$

In order to obtain perturbations we must add spacetime dependent fluctuations to the metric and the sources.

$$\begin{gathered} g_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \vec{x}) \\ \phi(\eta) + \delta\phi(\eta, \vec{x}) \qquad T_{\mu}{}^{\nu}(\eta) + \delta T_{\mu}{}^{\nu}(\eta, \vec{x}) \end{gathered} \tag{1.75}$$

These fluctuations are distinguished in scalar, vector and tensor degrees of freedom. As long as we are in the linear approximation perturbations evolve independently and therefore we can study them separately. We are interested on the scalar ones because they have the necessary gravitational instability for the formation of the structures and for the generation of the thermal fluctuations of the CMB. The tensor perturbations are responsible for the generation of gravitational waves while the vector ones are rapidly decreasing with the expansion of the Universe and therefore can be neglected. Since in our case we are interested on the inflationary regime, we can consider as the only sources for the Einstein Equations the scalar field and its perturbations fixing $T_{\mu}{}^{\nu} = 0$ and $\delta T_{\mu}{}^{\nu} = 0$. Once all perturbations are introduced, we have to do some redefinitions through gauge transformations so that they correspond to the same point of the space-time. There are particular combinations that result automatically gauge invariant the most important of which is the Curvature Perturbation $\mathcal{R}$. Through a suitable gauge choice we obtain the inflaton's perturbation in terms of the inflaton itself and of the scalar perturbations of the metric, called Bardeen Potentials. To normalize the Bardeen perturbation in order to describe the evolution of quantum fluctuations we consider the scalar and gauge invariant variable v which diagonalizes the perturbed action. In fact, if we expand the unperturbed action

$$S = -\frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}R + \frac{1}{2}\int d^4x\sqrt{-g}(\partial_\mu\phi\partial^\mu\phi - 2V) \tag{1.76}$$

until the orders $(\delta g)^2$, $(\delta\phi)^2$ we obtain

$$\delta S = \frac{1}{2}\int d\mathbf{x}d\eta\; z^2(\eta)(\mathcal{R}'^2 + \mathcal{R}\nabla^2\mathcal{R}) \tag{1.77}$$

where z is the pump field, i.e. the external cosmological field that amplifies fluctuations. The prime in this context represents the derivative with respect to conformal time η. The previous action is equivalent to the

$$\delta S = \frac{1}{2}\int d\mathbf{x}d\eta\; z^2(\eta)(\mathcal{R}'^2 + \mathcal{R}\nabla^2\mathcal{R}) + \frac{d}{d\eta}(zz'\mathcal{R}^2) \tag{1.78}$$

and through the canonical variable $v = z\mathcal{R}$, it becomes

$$\delta S = \frac{1}{2}\int d\mathbf{x}d\eta\,(v'^2 + v\nabla^2 v + \frac{z''}{z}v^2) \tag{1.79}$$

The equation of motion that follows from (1.79) is

$$v'' - \left(\nabla^2 + \frac{z''}{z}\right)v = 0 \tag{1.80}$$

which gives a canonical description of the evolution of scalar perturbations. The normalization of v fixes the normalization of $\mathcal{R}$ which is a very useful quantity, as it links the evolution of a mode from the Horizon exit during inflation to its re-entry during the standard cosmological evolution. In the slow roll approximation where $\phi' = 0$, the pump field is equivalent to the scale factor. In order to solve (1.80) we expand the field in a Fourier expansion

$$v(\eta, \mathbf{x}) = \frac{\sqrt{V}}{(2\pi)^3}\int d\mathbf{k}\, v_k\, e^{i\mathbf{kx}} \tag{1.81}$$

where v_k satisfies the

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0 \tag{1.82}$$

The effective potential is $\frac{z''}{z}$ and the pump field during inflation acts to increase the amplitude of the Fourier components present in an appropriate sector by transferring energy from unperturbed external fields to individual fluctuations. During the inflationary epoch

$$a(\eta) \sim (-\eta)^\alpha \quad \text{with} \quad \eta \to 0_- \tag{1.83}$$

Equation (1.82), therefore, assumes the form

$$v_k'' + \left(k^2 - \frac{\alpha(\alpha-1)}{\eta^2}\right)v_k = 0 \tag{1.84}$$

For $k \gg |\eta|^{-1}$ the effective potential is negligible and therefore the modes oscillate without any amplification. For $k \ll |\eta|^{-1}$, on the other hand, the equation becomes

$$\frac{v_k''}{v_k} \simeq \frac{a''}{a} \quad \Rightarrow \quad v_k \sim a \tag{1.85}$$

so they follow the evolution of the scale factor and therefore their amplitude grows in a accelerated way. In conclusion modes with wavelength bigger than the Hubble scale (which we often call with a certain abuse Horizon), i.e. the ones with $k \ll |\eta|^{-1}$ are sensible to the geometry and are therefore amplified. For wavelength inside the Hubble Horizon $k \gg |\eta|^{-1}$, instead, there is no amplification. In the interval $\eta_{ex} < \eta < 0$ where η_{ex} is the moment of Horizon exit of the mode k, ($k|\eta_{ex}| \sim 1$), the (1.85) becomes

$$v_k(\eta) \simeq z(\eta)\alpha_k + z(\eta)\beta_k \int_{\eta_{ex}}^{\eta} \frac{d\eta'}{z^2(\eta')} + \cdots \tag{1.86}$$

where the constants α_k and β_k are fixed by imposing the initial conditions at the time $\eta = \eta_{ex}$. In the case of $z \sim a \sim (-\eta)^\alpha$, v_k assumes the explicit form

$$v_k(\eta) = \tilde{\alpha}_k(-\eta)^\alpha + \tilde{\beta}_k(-\eta)^{1-\alpha} \tag{1.87}$$

For accelerated expansion the first term is always dominating for $\eta \to 0_-$ and we actually find that outside the Horizon the amplitude of the mode v_k grows in a inflationary way with respect to the corresponding amplitude inside the Horizon. In this case therefore the potential term is the dominant one.
The general solution of (1.84) is expressed in terms of the Hankel functions of first and second kind

$$v_k(\eta) = \eta^{1/2}(A_k H_\nu^{(2)}(k\eta) + B_k H_\nu^{(1)}(k\eta)) \tag{1.88}$$

where $\nu = 1/2 - \alpha$, and A_k, B_k are constants of integration that are fixed through appropriate initial conditions. In order to fix the initial conditions we must keep in mind that the quantum fluctuations produce microscopic perturbations of the homogeneous and isotropic standard model.
The action (1.79) can be viewed as the action of a free scalar field with time-dependent mass $m = z'/z$ in Minkowski space. The time dependence of the mass is caused by the interaction of the perturbations with the homogeneous expanding background. It can therefore be quantized in the canonical way following the ordinary procedure through creation and annihilation operators. The associated momentum is

$$\pi = \frac{\delta \mathcal{L}}{\delta v} \tag{1.89}$$

Fields become operators satisfying the commutation relations

$$[\hat{v}(\eta, \mathbf{x}), \hat{v}(\eta, \mathbf{y})] = 0 \qquad [\hat{\pi}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{y})] = 0 \qquad [\hat{v}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}) \tag{1.90}$$

The modes v_k satisfy the relation

$$v_k v'^{\star}_k - v'_k v^{\star}_k = i \tag{1.91}$$

For $\eta \to -\infty$, equation (1.84) becomes

$$v''_k + k^2 v_k = 0 \Rightarrow v'_k = -ikv_k \tag{1.92}$$

and from the (1.91) we find the correct normalization of the vacuum state

$$v_k = \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad \text{for} \qquad \eta \to -\infty \tag{1.93}$$

This asymptotic state is called Bunch-Davies vacuum. By imposing this initial condition we can normalize the general solution (1.87) and using the limit of Hankel Functions for large arguments $k|\eta| \gg 1$

$$H_\nu^{(1)}(k\eta) \to \sqrt{\frac{2}{\pi k\eta}} e^{ik\eta + i\epsilon_\nu} \qquad H_\nu^{(2)}(k\eta) \to \sqrt{\frac{2}{\pi k\eta}} e^{-ik\eta - i\epsilon_\nu} \tag{1.94}$$

with ϵ_ν a factor of constant phase, we find

$$A_k = \sqrt{\frac{\pi}{4}} \qquad\qquad B_k = 0 \tag{1.95}$$

In conclusion the solution becomes

$$v_k(\eta) = \left(\frac{\pi\eta}{4}\right)^{1/2} H_\nu^{(2)}(k\eta) \tag{1.96}$$

Once we know the correct normalization for the Fourier components of the canonical coordinate, we can compute the normalised solution for any other scalar perturbation. To this end from a dimensional analysis we can parameterize the pump field using the convenient Planck units:

$$z(\eta) = \frac{M_{pl}}{\sqrt{2}}\left(-\frac{\eta}{\eta_1}\right)^\alpha \qquad\qquad \eta < 0 \tag{1.97}$$

where η_1 is an arbitrary time scale, for example the time when inflation ends. From the definition of the curvature perturbation we have

$$\mathcal{R}_k(\eta) = \frac{v_k}{z} = \frac{1}{M_{pl}}\left(\frac{\pi\eta_1}{2}\right)^{1/2}\left(-\frac{\eta}{\eta_1}\right)^{\frac{1}{2}-\alpha} H_\nu^{(2)}(k\eta) \tag{1.98}$$

For a generic scalar perturbation ψ we can find the power spectrum for the various modes k by calculating the Fourier transformation of the two point correlation function

$$\xi_\psi(\mathbf{r}) = \langle\psi(\mathbf{x},t),\psi(\mathbf{x}',t)\rangle \tag{1.99}$$

with $\mathbf{x}' = \mathbf{x}+\mathbf{r}$, and by valuating this function at scales equal to the comoving wavelength of each mode. If

$$\psi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\psi_k e^{i\mathbf{kx}}$$

then

$$\xi_\psi(\mathbf{r}) = \frac{1}{V}\int d^3x \;\; \psi(\mathbf{x})\psi(\mathbf{x}+\mathbf{r}) = \int d^3x \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\psi_k\psi_{k'}e^{i(\mathbf{k}+\mathbf{k}')\mathbf{x}+i\mathbf{k}'\mathbf{r}}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\psi_k\psi_{k'}e^{i\mathbf{k}'\mathbf{r}}\delta(\mathbf{k}+\mathbf{k}') = \int \frac{d^3k}{(2\pi)^3}|\psi_k|^2 e^{-i\mathbf{kr}} =$$

$$\int_0^\infty \frac{k^2dk}{(2\pi)^3}\int_0^\pi 2\pi\sin\theta d\theta e^{-ikr\cos\theta}|\psi_k|^2 = \int_0^\infty \frac{dk}{2\pi^2 k}\frac{\sin(kr)}{kr}k^3|\psi_k|^2 = \int_0^\infty \frac{dk}{k}\frac{\sin(kr)}{kr}\Delta_\psi^2$$

$$\Delta_\psi^2(k) = \frac{k^3}{2\pi^2}|\psi_k|^2 \tag{1.100}$$

is the power spectrum of the perturbation. Its square root controls the relative amplitude of the various modes and the typical amplitude of the fluctuations on length scales

$r \sim k^{-1}$.
Outside the Horizon, i.e. for $k|\eta| \ll 1$, we can use the limit of small arguments of the $H_\nu^{(2)}(k\eta)$:

$$H_\nu^{(2)} \underset{k|\eta|\ll 1}{\longrightarrow} p_\nu^\star (k\eta)^\nu - iq_\nu (k\eta)^{-\nu} + \cdots$$

where the complex coefficients q, p depend on ν and they have unitary modulus. In an expanding universe $\alpha < 0$, and so the expansion is dominated by the second term. The power spectrum of $\mathcal{R}$ for $k|\eta| \ll 1$, from (1.98) becomes

$$\Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2 = |q_\nu|^2 \frac{k^3\eta_1}{4\pi M_{pl}^2}(k\eta_1)^{-2\nu} = \frac{|q_\nu|^2}{4\pi}\left(\frac{k_1}{M_{pl}}\right)^2 \left(\frac{k}{k_1}\right)^{3-2\nu} \tag{1.101}$$

where $k_1 = 1/\eta_1$. This spectrum is time independent and therefore it remains constant for all modes outside the Horizon until the time of their possible re-entry. It is convenient to express this constant spectrum outside the Horizon in terms of the parameters of the unperturbed model valuated at the moment when each mode crosses the Horizon

$$\mathcal{R} = -i\frac{q_\nu}{M_{pl}}\left(\frac{\pi\eta_1}{2}\right)^{1/2}(k\eta_1)^{-\frac{1}{2}+\alpha} = -iq_\nu\left(\frac{\pi}{4k}\right)^{1/2}\left(\frac{1}{z}\right)_{hc} \tag{1.102}$$

where we have used the (1.97) and the pump field is valuated at Horizon crossing (hc). The corresponding spectrum assumes the form

$$\Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2 = \frac{|q_\nu|^2}{8\pi}\left(\frac{k}{z}\right)_{hc}^2 = \frac{|q_\nu|^2}{8\pi}\left(\frac{1}{z\eta}\right)_{hc}^2 = \frac{|q_\nu|^2}{8\pi\alpha^2}\left(\frac{H^2}{\dot{\phi}}\right)_{hc}^2 \tag{1.103}$$

From this expression we can see that the amplitude tends to be more amplified the more slowly the scalar field evolves in time. The slow roll inflation models are therefore particularly efficient in the process of amplifying scalar perturbations.

In the slow roll approximation $z''/z = (2 + 9\epsilon - 3\eta)/\eta^2$, where ϵ and η are the slow roll parameters and so the (1.82) becomes

$$v_k^{''} + (k^2 - \frac{2 + 9\epsilon - 3\eta}{\eta^2})v_k = 0 \tag{1.104}$$

Comparing this with the (1.84) we find

$$\nu^2 = \frac{9}{4} + 9\epsilon - 3\eta \Rightarrow \nu \simeq \frac{3}{2} + 3\epsilon - \eta \tag{1.105}$$

In order to obtain the explicit spectral amplitude for the slow roll we observe that for $\nu = 3/2$ we have $q_\nu = i\sqrt{\pi/2}$ and $\alpha = -1$. In consequence,

$$\Delta_{\mathcal{R}}^2(k) \simeq \frac{1}{8\pi^2 M_{pl}^2}\left(\frac{H^2}{\epsilon}\right)_{hc} \simeq \frac{1}{24\pi^2}\left(\frac{V}{\epsilon}\right)_{hc} \tag{1.106}$$

The dependence on k is parameterized by the spectral index n_s

$$n_s = 1 + \frac{d\ln\Delta^2(k)}{d\ln(k)} \equiv 1 + \frac{k}{\Delta^2(k)}\frac{d\Delta^2(k)}{dk} \tag{1.107}$$

For the slow-roll model we have

$$\Delta^2_{\mathcal{R}}(k) \sim \left(\frac{k}{k_1}\right)^{3-2\nu} = \left(\frac{k}{k_1}\right)^{-6\epsilon+2\eta} \tag{1.108}$$

and so

$$n_s = 1 - 6\epsilon + 2\eta \tag{1.109}$$

The spectral index is a very useful quantity because relates the theoretical model to observations. From the WMAP observations on CMB anisotropies in fact, it turns out $n_s = 0.96 \pm 0.01$. Another important quantity that links theory with observations is the tensor-to-scalar ratio r related to the presence of primordial gravity ways and therefore to tensor perturbations. For this we have

$$r = 16\epsilon \tag{1.110}$$

For the study of the CMB anisotropies the most relevant quantity is the gauge invariant Bardeen potential. The amplitude of the modes that exit the Horizon in an inflationary phase are freezing not only for the curvature perturbations but also for the Bardeen Potential. In fact, the scalar perturbation outside the Horizon is constant with a value proportional to this curvature perturbation:

$$P = \frac{1}{150\pi^2}\left(\frac{V}{\epsilon}\right)_{hc} \tag{1.111}$$

The modes of the Bardeen potential remain constant until the eventually re-entry in the Horizon at late times. Its values will be therefore the initial conditions for the subsequently evolution responsible for the CMB anisotropies. Another important aspect is that through scalar perturbations we can make a selection of the inflationary models in Ultraviolet theories. This quantity, in fact, is extremely useful to constrain inflation models embedded in String Theory because it depends on the microscopic parameters that come from the underlying supersymmetric theory.

Chapter 2

Supersymmetry

Many of the major developments in physics of the past century arose from identifying and overcoming contradictions between existing ideas. Today, we are facing the incompatibility between General Relativity and Quantum Field Theory since any straightforward attempt to quantize gravity is non renormalizable. Particle physics is described successfully in the context of the Standard Model, augmented by neutrino masses, which is in excellent agreement with the experimental data but unfortunately it is considered only an effective theory that fails at high energies and left a lot of open questions. The search for a complete theory of elementary particles and their interactions unifying the forces of Nature, including gravity, in a single quantum mechanical framework is focused today more and more in String Theory. It was developed for the first time in the late 1960s in order to explain the strong nuclear force and even if, eventually, it was realized that the correct context for describing the properties of hadrons is Quantum Chromodynamics, it turns out that String Theory solves many of the problems emerging in the Standard Model. Today, it is considered a highly promising area of research in Theoretical Physics with important applications also in Cosmology.
One of the fundamental aims of research is to investigate the physical motivation for the number and nature of different particles and interactions.
More precisely, in the Standard Model, the fact that there are three families of fermions and four interactions as well as the value of more than 20 free parameters are fixed by experiments without understanding their theoretical origin.
In this context there are also problems connected to naturalness as a result of loop corrections to the masses of scalar fields which are not protected by any symmetry and so are automatically brought to the biggest scale of the theory. It is, therefore, unclear why there are totally different energy scales like the electroweak scale with respect to the Planck scale, a problem known as the Hierarchy Problem, or why the cosmological constant is experimentally so small.
Moreover, in the Standard Model besides the kinetic electromagnetic term $F_{\mu\nu}F^{\mu\nu}$ one can add another coupling of the form $\theta F_{\mu\nu}\tilde{F}^{\mu\nu}$ where $\tilde{F}^{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}$ is the dual electromagnetic tensor and θ is a parameter. This term is usually overlooked because it can

be written as a total derivative of fields. In QCD, however, there are some fields, the istantons, that are not nulls at infinity and as a result the boundary term is non-zero. The introduction of this term implies the violation of CP symmetry and the production of an electric dipole moment for the electron which have never been observed experimentally. As a consequence, it emerges a problem of naturalness because while we expect a natural value for θ the experimental limits constrain the angle to be $\theta \leq 10^{-11}$. One possible explanation lies in the existence of a global symmetry $U(1)_{PQ}$ which sets the angle to the desired null value. When this symmetry is spontaneously broken the parameter becomes the pseudo-Goldstone Boson of the symmetry which called QCD axion $a(x)$ and non perturbative effects to the potential induce a minimum at $a(x) = 0$ and consequently $\theta = 0$.

The biggest problem, however, as mentioned above, is that any attempt to unify the Standard Model with gravity is to be considered in the context of effective theories since the coupling strength of General Relativity has negative mass dimension and so the interaction becomes stronger at higher energies making the theory non renormalizale. For energies above the Planck scale, in fact, it violates unitarity and stops making sense.

More precisely, gravity is the gauge theory of spacetime transformations. If we want to make the theory symmetric under the respective local transformations we have to change the action by introducing the corresponding gauge boson, i.e the graviton. The resulting field equations are nonlinear and the theory is non renormalizable. In general a theory with more symmetries is more convergent. In order to improve the divergent behavior of the interaction it is, therefore, useful to introduce new symmetries like Supersymmetry. The gauge theory of Supersymmetry is Supergravity (SUGRA), which actually is more convergent, but still not renormalizable. There is some theoretical evidence that $N = 8$ SUGRA might actually be free of divergences. The ultimate solution has to be searched in the context of String Theory which solves definitely this incompatibility introducing a cut-off on lengths by substituting point particles with one-dimensional extended objects, the strings.

In the context of an Ultraviolet Completion of the Standard Model it is possible to solve also all the other problems mentioned above. String Theory is, therefore, one of the best candidate for this purpose, even if there is no experimental evidence that it is the correct description of our world.

Supersymmetry is a symmetry that links bosons with fermions providing a solution to the Hierarchy problem through the mutual cancellation of the loop contributions associated to the two types of particles. Its introduction ensures that strings are consistently defined in ten dimensions. The limit at low energies of String Theory is Supergravity. More precisely SUGRA in ten dimensions is the effective theory of superstrings at energies lower that the string mass. To get meaningful results from the phenomenological point of view we need to compactify six of the ten dimensions in a complex manifold with particular properties, the Calabi-Yau, leaving only four extended dimensions through a dimensional reduction. Besides having incorporated the Gravitation and having solved the Hierarchy Problem through the miraculous cancellation, we can also derive all the parameters of the Standard Model in a natural way from a single unknown parameter

which is the string tension. Supersymmetry, finally, has opened the way for the advance of Grand Unification Theories (GUT) because in its context the running of the couplings lead to their coincidence at high energies, $M_{GUT} \sim 10^{16}\ GeV$.
For all these reasons, today it is thought that the unification of all fundamental forces is based on Supersymmetry and on the existence of extra dimensions in the context of M-Theory which is the more advanced version of String Theory. This direction is explored not only by theoretical physicists but also by experimental ones which search for supersymmetric particles in colliders as an important confirmation of the existence of strings.

2.1 Basics of Supersymmetry

Supersymmetry is the spacetime symmetry under the exchange of fields with integer spin and fields with semi-integer spin. The generators of such transformations are anticommutating spinors Q such that

$$Q|Boson\rangle = |Fermion\rangle$$
$$Q|Fermion\rangle = |Boson\rangle$$

The theorem that restricts the possible forms for such symmetries is the Haag-Lopuszanski-Sohnius extension of the Coleman-Mandula theorem. According to Coleman-Mandula's theorem the most general symmetry Lie group in field theories with mass gap is given by the product of Poincarè group and an internal symmetry group G whose Lie algebra is

$$\begin{gathered}
\left[P_\mu, P_\nu\right] = 0 \qquad \left[M_{\mu\nu}, P_\lambda\right] = -i\eta_{\nu\lambda}P_\mu + i\eta_{\mu\lambda}P_\nu \\
\left[M_{\mu\nu}, M_{\lambda\rho}\right] = -i\eta_{\nu\lambda}M_{\mu\rho} + i\eta_{\mu\lambda}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\lambda} - i\eta_{\mu\rho}M_{\nu\lambda} \\
\left[B^a, B^b\right] = if^{ab}{}_c B^c \qquad \left[B^a, P_\mu\right] = 0 \qquad \left[B^a, M_{\mu\nu}\right] = 0
\end{gathered} \tag{2.1}$$

where P_μ and $M_{\mu\nu}$ are the generators of the Poincaré and B^a are those of G group. We extend this algebra introducing fermionic generators which together with the bosonic ones form two sets of generators $G = G_0 \oplus G_1$, $G_0 = \{B^a\}$, $G_1 = \{F^\alpha\}$. The Lie superalgebras are Z_2 graduated algebras containing a set of bosonic generators with parity +1 under Z_2 and a set of fermionic generators with parity -1 under Z_2 satisfying

$$\begin{aligned}
\left[B^a, B^b\right] &= if^{ab}{}_c B^c \\
\left[B^a, F^\alpha\right] &= ig^{a\alpha}{}_\beta F^\beta \\
\{F^\alpha, F^\beta\} &= h^{\alpha\beta}{}_a B^a
\end{aligned} \tag{2.2}$$

The generalization of Coleman Mandula theorem states that admitting the use of anticommutators the most general Lie superalgebras are the supersymmetry algebras which are extensions of the Poincaré algebra including fermionic generators. To have a realistic theory, like the Standard Model, fermions must be chiral and this results in a constraint for the algebra satisfied by the generators Q and $Q^\dagger$. In a general supersymmetric theory we can also have more than one pair of fermionic generators up to a maximum of 4 for normal gauge theories and a maximum of 8 for the theories involving gravitation. Theories that contain more than one distinct copy of the generators Q and $Q^\dagger$ are mathematically interesting but do not have any phenomenological prospects because they cannot allow the existence of chiral fermions. We will therefore focus on the case of a single pair of fermionic generators (simple SUSY) for which we have:

$$\begin{gathered}
\{Q, Q^\dagger\} = 2\sigma^\mu P_\mu \qquad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \qquad \left[P^\mu, Q\right] = \left[P^\mu, Q^\dagger\right] = 0 \\
\left[Q_\alpha, M^{\mu\nu}\right] = \left(\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta
\end{gathered} \tag{2.3}$$

where P^μ is the four-momentum generator of spacetime translations, $Q^\dagger$ is the hermitian conjugate of Q and

$$\sigma^0 = \bar{\sigma}^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma^1 = -\bar{\sigma}^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma^2 = -\bar{\sigma}^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma^3 = -\bar{\sigma}^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.4}$$

$$\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} = \frac{i}{4}\left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\right)_\alpha^{\ \beta}$$

In the supersymmetric world instead of having single-particle states we have multiplets of particles states, called supermultiplets, which are irreducible representations of the supersymmetry algebra. From the expressions written above it is clear that particles in the same supermultiplet must have equal masses and different spins. Moreover they must have the same internal charges because the supersymmetry generators Q and $Q^\dagger$ commute with the generators of gauge transformations.

It can be shown that each supermultiplet contains an equal number of fermionic and bosonic degrees of freedom. This property allows us to distinguish the fundamentals particles into two supermultiplets: the chiral supermultiplet with one Weyl fermion and two real scalar fields, and the gauge or vector supermultiplet containing one massless gauge boson of spin-1, at least as long as the gauge symmetry is not spontaneously broken, and one Weyl fermion. The chiral fermions of the Standard Model belong to the chiral supermultiplet and not to the gauge one. The reason is that in the second case the gauge bosons are transformed in the adjoint representation and consequently since the particles in the same supermultiplet must transform according to the same representation also fermions transform in the adjoint representation and this implies having no chiral fermions but only supersymmetric partners of the gauge bosons, called gauginos. Supergravity is the gauge theory of supersymmetry. In fact, when SUSY becomes a local symmetry we introduce a Weyl spinor of spin-3/2 called gravitino. In that case we have also the gravitational supermultiplet which contains the graviton and the gravitino.

There are also other possible supermultiplets but these are always reducible to combinations of chiral and gauge supermultiplets. In a supersymmetric extension of the Standard Model each of the known particles belong in either a chiral or a gauge supermultiplet and must have a superpartner with spin differing by 1/2 unit.

The matter particles, as we have already seen, must be contained in the chiral supermultiplet and therefore their bosonic partner must be spin-0 vector bosons, i.e the sleptons and the squarks. The Higgs Boson, being a scalar particle, has to belong to the same supermultiplet. It turns out that it is not enough to have a single supermultiplet containing the Higgs particle but it necessary to have two of this. The vector bosons of the Standard Model must obviously belong to gauge supermultiplet together with gauginos.

The particles of the Standard Model together with their supersymmetric partners give rise to the most basic model phenomenologically supported called Minimal Supersymmetric Standard Model (MSSM).
It is interesting that while the SM particles have been found experimentally their supersymmetric partners have never been observed yet although they have the same mass. It turns out that SUSY must be a broken symmetry in the vacuum state in which we live. From the theoretical point of view we expect that the symmetry is spontaneously broken so that SUSY is hidden at low energies but the underlying theory is still supersymmetric. However, since it is not clear the mechanism through which this happens we parameterize our ignorance introducing terms which break explicitly the symmetry. These terms shall be such that it is still valid the miraculous cancellation leading to the resolution of the Hierarchy Problem. More precisely the contribution to the Higgs mass due to the coupling of Higgs H with the scalar particle s: $\lambda_s H^2 s^2$, is given by

$$\Delta m_H^2 = \frac{\lambda_s^2}{16\pi^2}\Lambda_{UV}^2 + \cdots \tag{2.5}$$

while the contribution due to the coupling with the fermion f: $-\lambda_f H f \bar{f}$, is

$$\Delta m_H^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{UV}^2 + \cdots \tag{2.6}$$

where Λ_{UV} is the highest scale of the theory. Supersymmetry implies the introduction of two complex scalar fields for each Dirac fermion and so, requiring a specific relationship between the two coupling constants, we get the mutual cancellation of the loop corrections due to the coupling with the two particles. The SUSY breaking terms that ensure the validity of the relationship between the coupling constants are called soft terms. The total Lagrangian in this case is

$$\mathcal{L} = \mathcal{L}_{SUSY} + \mathcal{L}_{soft} \tag{2.7}$$

Soft terms carry a correction of the form

$$\Delta m_H^2 = m_{soft}^2\Big[\frac{\lambda}{16\pi^2}\ln\big(\frac{\Lambda_{UV}}{m_{soft}}\big) + \cdots\Big] \tag{2.8}$$

where m_{soft}^2 is the the larger mass scale contained in the Lagrangian L_{soft} and the non supersymmetric corrections must vanish in the limit $m_{soft} \to 0$. Since the splitting between the masses of the particles and their partners depends on m_{soft} their difference will not be too big and therefore there is hope that they will be found also experimentally over the next years.

2.1.1 Supersymmetric Lagrangians in 4 dimensions

The simplest supersymmetric model is the one that describes a free chiral supermultiplet. The action contains the scalar part that describes a complex scalar field ϕ and the fermionic part that contains a single left-handed two component Weyl fermion ψ.

$$S = \int d^4x \quad \big(\mathcal{L}_{scalar} + \mathcal{L}_{fermion}\big) \tag{2.9}$$

where

$$\mathcal{L}_{scalar} = -\partial^\mu \phi^\star \, \partial_\mu \phi \qquad \mathcal{L}_{fermion} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi \tag{2.10}$$

A supersymmetry transformation transforms the scalar boson field ϕ into something involving the fermion field ψ_α. The simplest possibility is

$$\delta\phi = \epsilon\psi \qquad \delta\phi^\star = \epsilon^\dagger \psi^\dagger \tag{2.11}$$

where ϵ^α is an infinitesimal, anti-commuting, two-component Weyl fermion constant that parameterizes the supersymmetry transformation. For the fermionic field, instead, we have

$$\delta\psi_\alpha = -i\big(\sigma^\mu \epsilon^\dagger\big)_\alpha \partial_\mu \phi \qquad \delta\psi^\dagger_{\dot\alpha} = i\big(\epsilon\sigma^\mu\big)_{\dot\alpha} \partial_\mu \phi^\star \tag{2.12}$$

With these transformations we eventually get

$$\begin{aligned}\delta\mathcal{L}_{scalar} + \delta\mathcal{L}_{fermion} = &-\epsilon^\dagger \partial_\mu \psi^\dagger \partial^\mu \phi - \epsilon \partial_\mu \phi^\star \partial^\mu \psi + \epsilon \partial_\mu \phi^\star \partial^\mu \psi + \epsilon^\dagger \partial_\mu \psi^\dagger \partial^\mu \phi \\ &-\partial_\mu \big(\epsilon \sigma^\nu \bar{\sigma}^\mu \psi \partial_\nu \psi^\star + \epsilon\psi \partial^\mu \phi^\star + \epsilon^\dagger \psi^\dagger \partial^\mu \phi\big)\end{aligned}$$

and so

$$\delta S = \int d^4x \quad \big(\delta\mathcal{L}_{scalar} + \delta\mathcal{L}_{fermion}\big) \;\doteq\; 0$$

To prove that the above action is supersymmetric we must also show that SUSY algebra is closed, i.e. that the commutator of two supersymmetry transformations parameterized by two different spinors ϵ_1 and ϵ_2 gives another symmetry of the theory. In fact, we have

$$\big(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2}\big)\; \phi = i\big(- \epsilon_1 \sigma^\mu \epsilon_2^\dagger + \epsilon_2 \sigma^\mu \epsilon_1^\dagger \big) \partial_\mu \phi$$

where $-i\partial_\mu$ corresponds to the generator of spacetime translations P_μ in the Heisenberg picture. In the fermionic case, however, emerges an additional term which is canceled only on-shell using the classic equations of motion. In fact, if we look at the following expression

$$\big(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2}\big)\; \psi_\alpha = i\big(- \epsilon_1 \sigma^\mu \epsilon_2^\dagger + \epsilon_2 \sigma^\mu \epsilon_1^\dagger \big)\; \partial_\mu \psi_\alpha + i\epsilon_{1\alpha} \epsilon_2^\dagger \bar{\sigma}^\mu \partial_\mu \psi - i\epsilon_{2\alpha} \epsilon_1^\dagger \bar{\sigma}^\mu \partial_\mu \psi$$

we note that the last two terms cancel each other out only if the equation of motion $\bar{\sigma}\partial_\mu \psi = 0$ is valid. To ensure that symmetry holds even off shell we introduce an auxiliary field, i.e. a new complex scalar field F which does not have a kinetic term.

$$\mathcal{L}_{auxiliary} = F^\star F \tag{2.13}$$

Only the off-shell Lagrangian depends on the auxiliary fields while using the equations of motion they are canceled out. In the modified theory

$$\mathcal{L} = \mathcal{L}_{scalar} + \mathcal{L}_{fermion} + \mathcal{L}_{auxiliary} \tag{2.14}$$

where

$$\delta F = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi + i\partial_\mu \psi \qquad \delta F^\star = i\partial_\mu \psi^\dagger \bar{\sigma}^\mu \epsilon \tag{2.15}$$

$$\delta \mathcal{L}_{auxiliary} = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi + i\partial_\mu \psi F^\star + i\partial_\mu \psi^\dagger \bar{\sigma}^\mu \epsilon F \tag{2.16}$$

and

$$\delta \psi_\alpha = -i\big(\sigma^\mu \epsilon^\dagger\big)_\alpha \partial_\mu \phi + \epsilon_\alpha F \qquad \delta \psi^\dagger_{\dot{\alpha}} = i\big(\epsilon \sigma^\mu\big)_{\dot{\alpha}} \partial_\mu \phi^\star + \epsilon^\dagger_{\dot{\alpha}} F^\star \tag{2.17}$$

we have the closure of SUSY algebra also off-shell. Applying the Noether's theorem we obtain the conserved four-current called supercurrent which is an anti-commuting four-vector and its hermitian conjugate

$$J^\mu_\alpha = \big(\sigma^\nu \bar{\sigma}^\mu \psi\big)_\alpha \partial_\nu \phi^\star \qquad J^{\dagger\mu}_{\dot{\alpha}} = \big(\psi^\dagger \bar{\sigma}^\mu \sigma^\nu\big) \partial_\nu \phi \tag{2.18}$$

that are separately conserved

$$\partial_\mu J^\mu_\alpha = 0 \qquad \partial_\mu J^{\dagger\mu}_{\dot{\alpha}} = 0 \tag{2.19}$$

giving rise to the conserved charges

$$Q_\alpha = \sqrt{2} \int d^3x\, J^0_\alpha \qquad Q^\dagger_{\dot{\alpha}} = \sqrt{2} \int d^3x\, J^{\dagger 0}_{\dot{\alpha}} \tag{2.20}$$

These are the generators of supersymmetry transformations satisfying the algebra (2.3).

Interactions

To have a realistic theory with many chiral supermultiplets we must add to the free Lagrangian

$$\mathcal{L} = -\partial^\mu \phi^\star_i \partial_\mu \phi_i + i\psi^{\dagger i} \bar{\sigma}^\mu \partial_\mu \psi_i + F^{\star i} F_i \tag{2.21}$$

the most general interaction Lagrangian which contains renormalizable gauge and non-gauge interactions

$$\mathcal{L}_{int} = \left(-\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i + x^{ij} F_i F_j \right) + c.c. - U \tag{2.22}$$

where W^{ij},W^i,x^{ij} and U are polynomials in the scalar fields ϕ_i, $\phi^{\star i}$, with degrees 1,2,0 and 4 respectively. The Lagrangian must be supersymmetric and so eventually we get

$$\mathcal{L}_{int} = \big(-\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i \big) + c.c. \tag{2.23}$$

with W^{ij} a holomorphic function of ϕ_i. Therefore, we can write

$$W^{ij} = M^{ij} + y^{ijk}\phi_k \tag{2.24}$$

where M^{ij} is a symmetric mass matrix for the fermion fields, and y^{ijk} is the Yukawa coupling of a scalar field ϕ_k with two fermions ψ_i, ψ_j, that must be totally symmetric under interchange of i, j, k. It is therefore convenient to express

$$W^{ij} = \frac{\delta^2}{\delta\phi_i\delta\phi_j}W \tag{2.25}$$

where

$$W = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k \tag{2.26}$$

is a holomorphic function of the scalar fields ϕ_i treated as complex variables, called superpotential. In a similar way we obtain

$$W^i = \frac{\delta W}{\delta\phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k \tag{2.27}$$

We can therefore conclude that the more general non-gauge interactions for chiral supermultiplets are determined by the superpotential W. The full Lagrangian $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$ lead to the equations of motion

$$F_i = -W_i^\star \qquad F^{\star i} = -W^i \tag{2.28}$$

and therefore

$$\mathcal{L} = \underbrace{-\partial^\mu\phi^{\star i}\partial_\mu\phi_i + i\psi^{\dagger i}\bar{\sigma}^\mu\partial_\mu\psi_i}_{\text{kinetic terms}} - \underbrace{\frac{1}{2}\left(W^{ij}\psi_i\psi_j + W_{ij}^\star\psi^{\dagger i}\psi^{\dagger j}\right)}_{\text{interactions}} - \underbrace{W^iW_i^\star}_{\text{scalar potential}} \tag{2.29}$$

which clearly shows that the scalar potential is

$$V(\phi,\phi^\star) = W^kW_k^\star = F^{\star k}F_k = M_{ik}^\star M^{kj}\phi^{\star i}\phi_j + \frac{1}{2}M^{in}y_{jkn}^\star\phi_i\phi^{\star j}\phi^{\star k} + $$

$$+\frac{1}{2}M_{in}^\star y^{jkn}\phi^{\star i}\phi_j\phi_k + \frac{1}{4}y^{ijn}y_{kln}^\star\phi_i\phi_j\phi^{\star k}\phi^{\star l} \tag{2.30}$$

If we have a gauge supermultiplet, i.e a massless gauge boson A_μ^a and a two-component Weyl fermion gaugino λ^a, we have the gauge transformations

$$A_\mu^a \to A_\mu^a + \partial_\mu\Lambda^a + gf^{abc}A_\mu^b\Lambda^c \tag{2.31}$$

$$\lambda^a \to \lambda^a + gf^{abc}\lambda^b\Lambda^c \tag{2.32}$$

where a, b, c run over the adjoint representation of the gauge group, Λ^a is an infinitesimal gauge transformation parameter, g is the gauge coupling and f^{abc} are the totally antisymmetric structure constants that define the gauge group. Again, it is necessary to introduce an auxiliary real and bosonic field traditionally called D in order to have a consistent supersymmetry also off-shell. The Lagrangian density for a gauge supermultiplet turns out to be

$$\mathcal{L}_{gauge} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + i\lambda^{\dagger a}\bar{\sigma}^\mu\nabla_\mu\lambda^a + \frac{1}{2}D^aD^a \tag{2.33}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{2.34}$$

and

$$\nabla_\mu\lambda^a = \partial_\mu\lambda^a + gf^{abc}A^b_\mu\lambda^c \tag{2.35}$$

is the covariant derivative of the gaugino field.
The auxiliary fields satisfy the trivial equation of motion $D^a = 0$, which however changes when we introduce the coupling between chiral and gauge supermultiplets. Let us suppose that chiral supermultiplets transform under the gauge group in a representation with hermitian matrices $\left(T^a\right)_i^{\ j}$ satisfying $[T^a, T^b] = if^{abc}T^c$. The scalar, fermion and auxiliary fields must be in the same representation because supersymmetric transformations commute with gauge transformations and therefore $X_i \to X_i + ig\Lambda^a(T^aX)_i$, for $X_i = \phi_i, \psi_i, F_i$. To have a gauge-invariant Lagrangian we have to replace the ordinary derivatives with the covariant ones

$$\nabla_\mu\phi_i = \partial_\mu\phi_i - igA^a_\mu(T^a\phi)_i \tag{2.36}$$

$$\nabla_\mu\psi_i = \partial_\mu\psi_i - igA^a_\mu(T^a\psi)_i \tag{2.37}$$

In this way, the vector bosons of gauge supermultiplets are coupled with the fermions of chiral supermultiplets. In conclusion the most general Lagrangian is

$$\mathcal{L} = \mathcal{L}_{chiral} + \mathcal{L}_{gauge} \underbrace{-\sqrt{2}g(\phi^\star T^a\psi)\lambda^a - \sqrt{2}g\lambda^{\dagger a}(\psi^\dagger T^a\phi) + g(\phi^\star T^a\phi)D^a}_{\text{interactions}} \tag{2.38}$$

where $\mathcal{L}_{chiral}$ is the fermionic Lagrangian in which we have replaced the normal derivatives with the covariant ones and $\mathcal{L}_{gauge}$ is that of expression (2.33).
The scalar potential in this case turns out to be

$$V(\phi, \phi^\star) = F^{\star i}F_i + \frac{1}{2}\sum_a D^aD^a \tag{2.39}$$

where D^a in the modified Lagrangian has the equation of motion $D^a = -g(\phi^\star T^a\phi)$. The (2.39), thus, becomes

$$V(\phi, \phi^\star) = F^{\star i}F_i + \frac{1}{2}\sum_a g^2(\phi^\star T^a\phi)^2 \tag{2.40}$$

The sum of squares is always non negative for any field configuration.
The two types of contributions that compose the scalar potential are known as F-term and D-term.

Superspace formalism

All the above results can be obtained in a very elegant way through the method of the superspace that allows to find actions manifestly invariant under SUSY through a geometric approach. More precisely, the superspace is built adding to the ordinary spatio-temporal x^μ coordinates the anti-commuting ones, $\{x^\mu, \theta^\alpha, \theta^\dagger_{\dot{\alpha}}\}$, and the translations along the latter give rise to the supersymmetry transformations. In this method the components of a supermultiplet are joined together in a single object called superfield which is a function of the superspace coordinates.
The generators of the translations in the directions of the superspace are defined as

$$\begin{aligned} \hat{Q}_\alpha &= i\frac{\partial}{\partial\theta^\alpha} - \left(\sigma^\mu\theta^\dagger\right)_\alpha \partial_\mu & \hat{Q}^\alpha &= -i\frac{\partial}{\partial\theta_\alpha} + \left(\theta^\dagger\bar{\sigma}^\mu\right)^\alpha \partial_\mu \\ \hat{Q}^{\dagger\dot{\alpha}} &= i\frac{\partial}{\partial\theta^\dagger_{\dot{\alpha}}} - \left(\bar{\sigma}^\mu\theta\right)^{\dot{\alpha}} \partial_\mu & \hat{Q}^\dagger_{\dot{\alpha}} &= -i\frac{\partial}{\partial\theta^{\dagger\dot{\alpha}}} + \left(\theta\sigma^\mu\right)_{\dot{\alpha}} \partial_\mu \end{aligned} \tag{2.41}$$

These operators satisfy the same commuting and anti-commuting rules of the generators of SUSY but they act on different spaces: the former act on the functions of the superspace while the latter on the Hilbert space. A function S of the superspace is a superfield if it is transformed as

$$\delta_\epsilon S = -i\left(\epsilon\hat{Q} + \epsilon^\dagger\hat{Q}^\dagger\right) S \tag{2.42}$$

It is useful to define the chiral covariant derivatives so that the derivative of a superfield with respect to θ_α and $\theta^\dagger_{\dot{\alpha}}$ it is also a superfield

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\left(\sigma^\mu\theta^\dagger\right)_\alpha \partial_\mu & D^\alpha &= -\frac{\partial}{\partial\theta_\alpha} + i\left(\theta^\dagger\bar{\sigma}^\mu\right)^\alpha \partial_\mu \\ \bar{D}^{\dot{\alpha}} &= \frac{\partial}{\partial\theta^\dagger_{\dot{\alpha}}} - i\left(\bar{\sigma}^\mu\theta\right)^{\dot{\alpha}} \partial_\mu & \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\theta^{\dagger\dot{\alpha}}} + i\left(\theta\sigma^\mu\right)_{\dot{\alpha}} \partial_\mu \end{aligned} \tag{2.43}$$

which satisfy the relationships

$$\begin{aligned} \{D_\alpha, \bar{D}_{\dot{\beta}}\} &= 2i\sigma^\mu{}_{\alpha\dot{\beta}}\,\partial_\mu \\ \{D_\alpha, D_\beta\} &= \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \end{aligned} \tag{2.44}$$

The chiral superfield $\Phi\left(x, \theta, \theta^\dagger\right)$ which contains the components of the chiral supermultiplets is defined through the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{2.45}$$

and it has the form

$$\begin{aligned} \Phi\left(x, \theta, \theta^\dagger\right) = \phi(x) + i\theta^\dagger\sigma^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\theta^\dagger\theta^\dagger\partial_\mu\partial^\mu\phi(x) \\ + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\theta^\dagger\bar{\sigma}^\mu\partial_\mu\psi(x) + \theta\theta F(x) \end{aligned} \tag{2.46}$$

Similarly, the anti-chiral superfield is such that

$$D_\alpha \Phi^\star = 0 \tag{2.47}$$

The vector or real superfield $V\big(x,\theta,\theta^\dagger\big)$, instead, it is obtained through the constraint

$$V = V^\star \tag{2.48}$$

and therefore

$$\begin{aligned} V\big(x,\theta,\theta^\dagger\big) &= a + \theta\xi + \theta^\dagger\xi^\dagger + \theta\theta b + \theta^\dagger\theta^\dagger b^\star + \theta^\dagger\bar{\sigma}^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\big(\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\xi^\dagger\big) \\ &\quad + \theta\theta\theta^\dagger\big(\lambda^\dagger - \frac{i}{2}\bar{\sigma}\partial_\mu\xi\big) + \theta\theta\theta^\dagger\theta^\dagger\big(\frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a\big) \end{aligned} \tag{2.49}$$

where in addition to the usual components of a gauge multiplet we have the bosonic fields a, b and the fermionic one ξ. The additional auxiliary fields can be supergauged away through the transformation

$$V \to V + i\big(\Omega^\star - \Omega\big) \tag{2.50}$$

and, in the so-called Wess-Zumino gauge, we obtain

$$V_{\mathrm{WZ}} = \theta^\dagger\bar{\sigma}^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\lambda + \theta\theta\theta^\dagger\lambda^\dagger + \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger D \tag{2.51}$$

The advantage of working with this method is that by integrating any superfield with respect to the coordinates of the superspace we automatically obtain a manifestly SUSY invariant action. In fact, since the supersymmetric variation of a superfield is written as total derivative of the coordinates of integration, an action of the form

$$A = \int d^4x \int d^2\theta \int d^2\theta^\dagger \;\; S\big(x,\theta,\theta^\dagger\big) \tag{2.52}$$

where S is a superfield, is such that $\delta_\epsilon A = 0$.
In order to have a real action, the superfield must be real and so we can have either a real superfield or the sum of a chiral superfield and its complex conjugated. To obtain the Lagrangian density we integrate on the fermionic coordinates. We have two possibilities called F- and D-term contributions:

$$\int d^2\theta \int d^2\theta^\dagger \;\; V\big(x,\theta,\theta^\dagger\big) \equiv [V]_D = \frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a \tag{2.53}$$

where it should be noted that the term $\partial_\mu\partial^\mu a$ is canceled when we integrate on the bosonic spatio-temporal coordinates.

$$\int d^2\theta \;\; \Phi\big(x,\theta,\theta^\dagger\big)\Big|_{\theta^\dagger=0} \equiv [\Phi]_F = F \tag{2.54}$$

In conclusion, to create a supersymmetric Lagrangian it is necessary the D-term of a real superfield and the F-term of a chiral superfield together with its complex conjugated. The supersymmetric fermionic Lagrangian is obtained choosing the D-term of the real combination $\Phi^{\star i}\Phi_i$. To introduce interactions and mass terms we use the superpotential that in this context is a holomorphic function of the chiral superfields and then it is reduced to a function of simple scalar fields by integrating on the anti-commuting coordinates.

$$W = \frac{1}{2}M^{ij}\Phi_i\Phi_j + \frac{1}{2}y^{ijk}\Phi_i\Phi_j\Phi_k \tag{2.55}$$

$$\mathcal{L} = [\Phi^{\star i}\Phi_i]_D + \big([W\big(\Phi_i\big)]_F + c.c.\big) \tag{2.56}$$

In order to treat an Abelian gauge theory, i.e. a theory with a $U(1)$ symmetry, we must introduce chiral and anti-chiral field strength superfields:

$$\mathcal{W}_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \qquad \mathcal{W}^\dagger_{\dot\alpha} = -\frac{1}{4}DD\bar{D}_{\dot\alpha}V \tag{2.57}$$

In fact under the transformation (2.50) these superfields remain unchanged. The ordinary gauge Lagrangian (2.33) is obtained with a change of variable from the combination

$$\int d^4x\mathcal{L}_{gauge} = \int d^4x\frac{1}{4}\big([\mathcal{W}^\alpha\mathcal{W}_\alpha]_F + c.c.\big) \tag{2.58}$$

$$\text{where} \quad [\mathcal{W}^\alpha\mathcal{W}_\alpha]_F = D^2 + 2i\lambda\sigma^\mu\partial_\mu\lambda^\dagger - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{2.59}$$

in the Wess-Zumino gauge.

We can add an additional term, known as Fayet-Iliopoulos term, of the form

$$\mathcal{L}_{FI} = -2\kappa[V]_D = -\kappa D \tag{2.60}$$

which is important for the SUSY breaking.
If we consider the coupling of the gauge Abelian field with a set of chiral superfields Φ_i carrying a $U(1)$ charge q_i, we have the gauge transformations

$$\begin{aligned}\Phi_i &\to \exp\big(2igq_i\Omega\big)\Phi_i\\ \Phi^{i\star} &\to \exp\big(-2igq_i\Omega\big)\Phi^{i\star}\end{aligned} \tag{2.61}$$

With these transformations the kinetic term $[\Phi^{\star i}\Phi_i]_D$ is no longer gauge invariant and so must change to

$$[\Phi^{\star i}\exp\big(2gq_iV\big)\Phi_i]_D \tag{2.62}$$

The presence of V in the exponential is possible because V is dimensionless but there is the possibility that it may not be renormalizable. However, thanks to the gauge dependence of V the higher order terms can be supergauged away.
The total Lagrangian eventually has the form

$$\mathcal{L} = [\Phi^{\star i}\exp\big(2gq_iV\big)\Phi_i]_D + \big([W\big(\Phi_i\big)]_F + c.c.\big) + \frac{1}{4}\big([\mathcal{W}^\alpha\mathcal{W}_\alpha]_F + c.c.\big) - 2\kappa[V]_D \tag{2.63}$$

where $[\Phi^{\star i}\exp\big(2gq_iV\big)\Phi_i]_D =$

$$F^{\star i}F_i - \nabla_\mu\phi^{\star i}\nabla^\mu\phi_i + i\psi^{\dagger i}\bar{\sigma}^\mu\nabla_\mu\psi_i - \sqrt{2}gq_i\big(\phi^{\star i}\psi_i\lambda + \lambda^\dagger\psi^{\dagger i}\phi_i\big) \quad (2.64)$$

From the (2.59) and (2.64) we obtain the contribution of the D-term to the scalar potential

$$V_D\big(\Phi\big) = \kappa D - \frac{1}{2}D^2 - gq_iD\phi^{\star i}\phi_i \quad (2.65)$$

while the contribution of the F-term is the same of (2.30).

Non-renormalizable supersymmetric Lagrangians

The non-renormalizable interactions can be neglected for most phenomenological applications as they involve couplings with negative mass size proportional to $1/M_{pl}$. In fact, for energies of order $E \leq 1TeV$ the influence of the non-renormalizable interactions is too small to be interesting. There are, however, particular contexts, for example those in which gravitation cannot be treated classically, in which such effects become important. A non-renormalizable Lagrangian that involves chiral and gauge supermultiplets (superfields) it is written as

$$\mathcal{L} = [K\big(\Phi_i,\tilde{\Phi}^{\star j}\big)]_D + \big([\frac{1}{4}f_{ab}\big(\Phi_i\big)\hat{\mathcal{W}}^{a\alpha}\hat{\mathcal{W}}^b_\alpha + W\big(\Phi_i\big)]_F + c.c\big) \quad (2.66)$$

where $\tilde{\Phi}^{\star j} = \big(\Phi^\star\exp\big(2g_aT^aV^a\big)\big)^j$.
The density Lagrangian depends on three functions of superfields. The Superpotential $W(\Phi_i)$ is a gauge invariant holomorphic function of chiral superfields treated as complex variables. It has dimension $[mass]^3$. The Gauge Kinetic Function $f_{ab}\big(\Phi_i\big)$ is a dimensionless function of chiral superfields treated as complex variables. It is symmetric under the interchange of its two indexes which run over the adjoint representation of the simple and Abelian component gauge groups of the model. For the non-Abelian components of the gauge group, it is just proportional to δ_{ab}, but if there are two or more Abelian components, the gauge invariance of the field-strength superfields allows kinetic mixing so that f_{ab} is not proportional to δ_{ab} in general. The Kähler Potential $K\big(\Phi_i,\tilde{\Phi}^{\star j}\big)$ is a real function of both chiral and non-chiral superfields and has dimension $[mass]^2$. In renormalizable theories it is simply reduced to $\Phi_i\tilde{\Phi}^{\star i}$. The part of the Lagrangian coming from the superpotential is

$$[W\big(\Phi_i\big)]_F = W^iF_i - \frac{1}{2}W^{ij}\psi_i\psi_j \quad (2.67)$$

with $W^i = \frac{\delta W}{\delta\Phi_i}\Big|_{\Phi_i\to\phi_i}$ and $W^{ij} = \frac{\delta^2 W}{\delta\Phi_i\delta\Phi_j}\Big|_{\Phi_i\to\phi_i}$.

After integrating out the auxiliary fields the contribution of the superpotential to the scalar potential is

$$V_F = W^iW^\star_{\bar{j}}\big(K^{-1}\big)^{\bar{j}}{}_i \quad (2.68)$$

where K^{-1} is the inverse of the Kähler metric

$$K^i{}_{\bar{j}} = \frac{\delta^2 K}{\delta\Phi_i \delta\tilde{\Phi}^{\star j}}\Big|_{\Phi_i \to \phi_i, \tilde{\Phi}^{\star i} \to \phi^{\star i}} \tag{2.69}$$

which characterizes the complex Kähler manifolds. As we will see in more detail later, these are complex and symplectic manifolds for which the complex and symplectic structures are compatible.

More generally, the whole field content of the Lagrangian after integrating out the auxiliary fields is determined by the functions W, K, f_{ab} and their derivatives with respect to the chiral superfields while the remaining chiral superfields are replaced by their scalar components. It is, therefore, important to study the behavior of these functions under quantum corrections: The Kähler Potential gets corrections order by order in perturbation theory. The Gauge Kinetic Function receives corrections only at one loop while the superpotential and the Fayet Iliopoulos term κ are not renormalized in perturbation theory.

2.2 Supersymmetry Breaking

As we have already said the soft terms are introduced in order to parameterize our ignorance about the mechanism that governs the spontaneous SUSY breaking, i.e the situation in which the Lagrangian is supersymmetric while the vacuum state is not invariant under the supersymmetry transformations

$$Q_\alpha|0\rangle \neq 0 \qquad\qquad Q^\dagger_{\dot\alpha}|0\rangle \neq 0 \tag{2.70}$$

Considering the $\{Q_\alpha, Q^\dagger_{\dot\beta}\} = 2\big(\sigma^\mu\big)_{\alpha\dot\beta}P_\mu$, and contracting it with $\big(\bar\sigma^\nu\big)^{\dot\beta\alpha}$, we have

$$\sum_{\alpha=1}^{2}\big(Q_\alpha Q^\dagger_\alpha + Q^\dagger_\alpha Q_\alpha\big) = 4E \tag{2.71}$$

From (2.70) and (2.71) it follows that the state where SUSY is spontaneously broken has strictly positive energy. If there are no spacetime dependent effects and fermion condensates we have $\langle 0|E|0\rangle = \langle 0|V|0\rangle$ and therefore SUSY breaking can only take place if the VEVs of F_i and D^a are not canceled out on the same vacuum state. Spontaneous SUSY breaking always implies the presence of a massless Nambu-Goldstone mode with the same quantum numbers as the broken generators. These neutral massless Weyl fermions are called goldstini. Its components among the various fermions of the theory are proportional to the VEVs of the corresponding auxiliary fields. Therefore, SUSY breaking occurs through the auxiliary fields and more precisely the F-term gives the breaking in the case of chiral superfields while the D-term gives the breaking in case of vector superfields.
In the case of chiral superfield, in particular, as we have said, the transformations are

$$\delta\phi = \epsilon\psi \qquad \delta\psi = \epsilon F - i\sigma^\mu\epsilon^\dagger\partial_\mu\phi \qquad \delta F = i\epsilon^\dagger\bar\sigma^\mu\partial_\mu\psi$$

SUSY breaking requires that at least one of them must be non zero. Since Lorentz invariance implies that $\langle\psi\rangle = \langle\partial_\mu\psi\rangle = 0$, the only possibility to having SUSY breaking is

$$\langle F\rangle \neq 0 \rightarrow \langle\delta\psi\rangle \neq 0 \tag{2.72}$$

and so the goldstino is ψ and we have $\langle V_F\rangle > 0$.
If, on the other hand, we have a vector superfield $V = \big(\lambda, A_\mu, D\big)$ then $\delta\lambda \propto \epsilon D$ is the only that can have a non-zero contribution if $\langle D\rangle \neq 0$. So in this case the goldstino is λ.

- F-term SUSY breaking

  The models in which the F-term is responsible for the SUSY breaking are called Raifeartaigh's models. The idea of these models is to choose the superpotential so that the equations $F_i = -\frac{\delta W^\star}{\delta\phi^{i\star}} = 0$ do not have simultaneous solutions within a compact domain. In this way the potential will be positive in the minimum ensuring the breaking of the symmetry. In this case it emerges a massless scalar field, i.e a flat direction in the scalar potential which can only be raised by quantum corrections $V_{eff} = V_{\text{tree level}} + V_{\text{1-loop}} + \cdots$. The fermionic superpartner, however, remains massless because it is constitutes the goldstino.

- D-term SUSY breaking

  SUSY breaking with a non-zero D-term can occur through the Fayet-Iliopoulos mechanism which works only in the case of a $U(1)$ gauge symmetry. It is, therefore, complicated to use this method in the supersymmetric extensions of the Standard Model.

Supersymmetry breaking occurs in a hidden sector of particles that have no or only very small direct couplings to the visible sector chiral supermultiplets of the MSSM. The two sectors share some interactions that are responsible for mediating supersymmetry breaking represented by the MSSM soft terms. The mechanisms that can be used for this purpose are *gravity mediation* and the *gauge mediation.* In the first case the mediators fields are coupled to the SM particles through gravitational interactions. The couplings are suppressed by the inverse of Planck mass. From the dimensional analysis we have

$$m_{soft} \sim \frac{\langle F \rangle}{M_{pl}} \tag{2.73}$$

For $m_{soft} \sim O(100GeV)$ its results $\sqrt{F} \sim o(10^{10} - 10^{11}\ GeV)$. In the second case, instead, the ordinary gauge interactions are responsible for the presence of soft terms. The idea is to introduce new chiral supermultiplets which are coupled not only with the source of SUSY breaking, i.e a non zero $\langle F \rangle$, but also with the particles of MSSM. The soft terms arise from loop diagrams that involve messenger particles. Again, from the dimensional analysis we have

$$m_{soft} \sim \frac{\alpha_a}{4\pi} \frac{\langle F \rangle}{M_{mess}} \tag{2.74}$$

where $\alpha_a/4\pi$ is the loop factor for Feynman diagrams and M_{mess} is the characteristic mass scales of messenger fields. If M_{mess} and $\sqrt{F}$ are comparable, then the SUSY breaking scale is $\sqrt{F} \sim O(10^4\ GeV)$

2.3 $\mathcal{N} = 1$ Supergravity

In quantum field theories, in order to have an invariant action under a gauge group, for example the $U(1)$ local symmetry group we must introduce a spin-1 gauge field A_μ. If, instead, we want to ensure the invariance under local spatio-temporal Poincarè transformations, the gauge field which must be introduced is a massless spin-2 field, i.e the graviton. So gravity originates in a natural way when we have general coordinate transformations. The resulting theory is not renormalizable and since theories with more symmetries are more convergent, it is useful to apply in this context the ideas of supersymmetry. Supergravity is the gauge theory of global supersymmetry. In the superspace formalism, as we have seen, a superfield is transformed under supersymmetry as $\delta\Phi = i\big(\epsilon Q + \epsilon^\dagger Q^\dagger\big)\Phi$. If ϵ becomes a function of spacetime $\epsilon\big(x\big)$ then SUSY is extended to a local symmetry. The resulting theory is the $\mathcal{N} = 1$ supergravity. The gauge field of local supersymmetry transformations consists in the spin-3/2 gravitino Ψ^μ_α. The inclusion of gravity, therefore, can be described by the gravity supermultiplet.
The scalar potential of global SUSY, V_F, is modified in supergravity as

$$V_F = \exp\left(\frac{K}{M_{pl}^2}\right)\left\{(K^{-1})^{i\bar{j}} D_i W D_{\bar{j}} \bar{W}^\star - 3\frac{|W|^2}{M_{pl}^2}\right\} \tag{2.75}$$

where $D_i W = \partial_i W + \frac{1}{M_{pl}^2}\big(\partial_i K\big)W$. For finite values of the Planck mass the potential V_F is no longer semi-positive definite. The global supersymmetric scalar potential $V_F = \big(K^{-1}\big)^{i\bar{j}}\partial_i W \partial_{\bar{j}} W^\star$ is restored in the limit $M_{pl} \to \infty$.
Similarly to the case of global supersymmetry, also the local one has to be spontaneously broken in order to match observations. As long as supersymmetry is unbroken, the graviton and the gravitino are both massless, each of them with two helicity states. Once supersymmetry is spontaneously broken the gravitino acquires a mass by absorbing the goldstino which becomes its longitudinal component in the so-called super-Higgs effect. The massive spin-3/2 gravitino has now four helicity states and it is traditionally indicated as $m_{3/2}$. Since $m_{3/2}$ must vanish when SUSY is restored $\big(\langle F\rangle \to 0\big)$ and when gravity is turned off $\big(M_{pl} \to 0\big)$, it turns out that

$$m_{3/2} \sim \langle F\rangle / M_{pl} \tag{2.76}$$

It follows that the mass of the gravitino is very different in the two methods of SUSY breaking mediation, i.e the gravity and the gauge one, since they usually make very different predictions for $\langle F\rangle$. From a phenomenological point of view it is interesting notice that local SUSY-breaking automatically generates soft-terms in the global supersymmetric Lagrangian, realizing the so-called gravity mediated SUSY-breaking. Usually, the SUSY-breaking scale is roughly given by the gravitino mass as we can see by comparing the (2.73) and (2.76).

Chapter 3

String Compactifications

The biggest success of String Theory is that it unifies General Relativity with QFT in a single theoretical context containing all the ingredients of the Standard Model, i.e. gauge interactions and chirality. It also has all the advantages of a supersymmetric theory since supersymmetry must be intrinsically included to remove tachyonic instability. Since it is defined in a consistent way in ten dimensions in order to have informations about physics in four dimensions, interesting from the point of view of phenomenology, we have to compactify the six extra dimensions in a suitable complex manifold called Calabi-Yau. An important consequence of the compactifications and the subsequent dimensional reduction, is the emergence of a large number of massless scalar fields called moduli. These generate unobserved long range fifth forces which can cause phenomenological problems and have to be stabilised by assigning them a potential. Moduli stabilisation is necessary also because many of the couplings of other fields depend on their vacuum expectation values. One of the most promising mechanisms within perturbative String Theory that generate a non trivial potential for the massless scalar fields is via fluxes which, however, have a backreaction on the geometry of the compactification. For this reason we have focused on Type IIB flux compactifications in which the backreaction is well-controlled and gives rise only to warped Calabi-Yaus. In this chapter we explain the basic elements of string theory which allow to understand the procedure of compactification used in two specific cases in the next chapter.

3.1 Strings

String theory is a quantum theory of 1-dimensional extended objects that are moving in a D-dimensional spacetime. In their motion the strings sweep a 2-dimensional surface, the worldsheet labeled by the coordinates σ along the string and τ. σ takes the values $0 \leq \sigma \leq \pi$ for an open string and $0 \leq \sigma \leq 2\pi$ for a closed one. The embedding of the worldsheet in the D-dimensional spacetime is defined by the set of functions $X^M(\tau, \sigma)$ where $M = 0, \ldots, D-1$. The evolution of the worldsheet is given by the Polyakov action

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma \int d\tau \, \sqrt{\gamma} \, \gamma^{\alpha\beta} \, \eta_{MN} \, \partial_\alpha X^M \partial_\beta X^N \qquad \alpha = \tau, \sigma \tag{3.1}$$

where $X^M\big(\tau, \sigma\big)$ describes the position of the string, η_{MN} is the spacetime Minkowski metric in ten dimensions, $\gamma^{\alpha\beta}$ is the worldsheet metric, $\gamma = -\det \gamma_{\alpha\beta}$ and α' is the *Regge slope*, i.e. a free parameter related to the string tension by $T = 1/(2\pi\alpha') = 1/(2\pi l_s^2)$ with l_s is the string scale. By varying the Polyakov action with respect to X^M we obtain the equation

$$\big(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial}{\partial\tau^2}\big) X^M(\tau, \sigma) = 0 \tag{3.2}$$

and introducing left- and right-moving worldsheet coordinates $\sigma^{\pm} = \tau \pm \sigma$, the equation becomes

$$\frac{\partial}{\partial\sigma^+} \frac{\partial}{\partial\sigma^-} X^M(\tau, \sigma) = 0 \tag{3.3}$$

which means that X^M is the sum of left- and right-moving degrees of freedom

$$X^M(\tau, \sigma) = X_R^M(\sigma^-) + X_L^M(\sigma^+) \tag{3.4}$$

In the case of closed strings $X^M(\tau, 0) = X^M(\tau, 2\pi)$, $X'^M(\tau, 0) = X'^M(\tau, 2\pi)$ the mode decomposition is obtained introducing a pair of left and right creation and annihilation operators which are respectively α_n^M, $\tilde{\alpha}_n^M$ and α_{-n}^M, $\tilde{\alpha}_{-n}^M$ with $n > 0$. Each mode carries an energy proportional to the level. The mass of a state is obtained using the operator

$$M^2 = \frac{2}{\alpha'} \, \big(\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n - 2\big) \tag{3.5}$$

The vanishing condition for the energy-momentum tensor is translated in the vanishing of the Virasoro operators on the physical spectrum. The most important of these constraints tell us that the operator

$$\hat{L}_0 = \frac{1}{2} \big(\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - \tilde{\alpha}_{-n}\tilde{\alpha}_n\big) \tag{3.6}$$

when it is applied on physical states must be zero. This condition implies that the number of the excited left oscillator levels it is equal to the number of the excited right oscillator levels. The massless states have, therefore, one left-moving and one right-moving excitation, namely

$$|\xi_{MN}\rangle = \xi_M \tilde{\xi}_N \alpha_1^M \tilde{\alpha}_1^N |0\rangle \tag{3.7}$$

The tensor $\xi_{MN} = \xi_M \tilde{\xi}_N$ is decomposed into $\xi_{MN} = \xi^s_{MN} + \xi_t \eta_{MN} + \xi^a_{MN}$. The state corresponding to the polarization ξ^s_{MN} is a massless state of spin-2, i.e. the graviton. The state corresponding to the scalar ξ_t is the dilaton, while the antisymmetric tensor (2-form) ξ^a_{MN} describes the B-field. These fields form the massless closed string spectrum and by their equations of motion and in particular from that of the dilaton we can fix the dimensions of the spacetime to 26. We notice that in the massless closed string spectrum it is absent a gauge field (one-form). This is obtained in the spectrum of the open strings where the massless states are given by

$$|\xi_M\rangle = \xi_M \alpha_1^M |0\rangle \tag{3.8}$$

The string coupling g_s is not an unknown arbitrary parameter but it is given by the VEV of the dilaton which is always present in the massless string spectrum.

$$g_s = e^{\langle \Phi \rangle} \tag{3.9}$$

The ground state of Hilbert space $|0\rangle$ has a negative mass square. This tachyonic instability means that the bosonic string is unstable and will condensate to the true vacuum of the theory. In order to remove the tachyon from the spectrum we need supersymmetry. In fact by introducing the superpartners of X^M: Ψ^M,$\tilde{\Psi}^M$, which are Majorana fermions on the worldsheet, we obtain the superstring theory. The action is modified according to

$$S = \frac{1}{4\pi} \int d^2\sigma \;\; \eta_{MN} \left(\frac{1}{\alpha'} \partial X^M \bar{\partial} X^N + \Psi^M \bar{\partial} \Psi^N + \tilde{\Psi}^M \bar{\partial} \tilde{\Psi}^N \right) \tag{3.10}$$

We have two different boundary conditions for the equations of Ψ and $\tilde{\Psi}$, i.e. those of Ramond (R):

$$\Psi^M(\tau, 0) = \Psi^M(\tau, 2\pi) \tag{3.11}$$

and those of Neveu-Schwarz (NS):

$$\Psi^M(\tau, 0) = -\Psi^M(\tau, 2\pi) \tag{3.12}$$

In the normal mode expansion the first type of conditions give rise to integer modes while the second ones give semi-integer modes. The cancellation of the conformal anomalies implies that the Superstring Theory is defined consistently in ten dimensions. The mass operator is modified accordingly to

$$M^2 = \frac{1}{\alpha'} \Bigg(\sum_{n=1}^{\infty} \alpha_n \alpha_{-n} + \sum_r r \psi_r \psi_{-r} - a + \{\text{the same with tilde}\} \Bigg) \tag{3.13}$$

where the constant a come out from the normal ordering and it is equal to 0 in the first case and to 1/2 in the second one. ψ_r and $\tilde{\psi}_r$ are respectively the right and left creation operators while ψ_{-r} and $\tilde{\psi}_{-r}$ are the corresponding annihilations operators which satisfy $\{\psi_r^M, \psi_s^N\} = \{\tilde{\psi}_r^M, \tilde{\psi}_s^N\} = \eta^{MN}\delta_{r+s}$. The physical states are obtained from the cancellation of the operator

$$\tilde{L}_0 = \left(\sum_{n=1}^{\infty} \alpha_n \alpha_{-n} + \sum_r r\psi_r \psi_{-r} - a + \{\text{the same with tilde}\} \right) \tag{3.14}$$

At the end we have the massless states

$$\begin{array}{llll} \text{R-R} & \xi_{MN}\psi_0^M \tilde{\psi}_0^N |0\rangle & \text{NS-R} & \xi_{MN}\psi_{1/2}^M \tilde{\psi}_0^N |0\rangle \\ \text{NS-NS} & \xi_{MN}\psi_{1/2}^M \tilde{\psi}_{1/2}^N |0\rangle & \text{R-NS} & \xi_{MN}\psi_0^M \tilde{\psi}_{1/2}^N |0\rangle \end{array} \tag{3.15}$$

Based on how the GSO projection is done we get two different theories : Type IIA and Type IIB linked together by $T - duality$ a symmetry under the exchange of winding and vibration modes with the contemporary substitution of R with $1/R$, where R is the radius of the compactification. In this thesis we have used the Type IIB String Theory and consequently from now and then we will focus only on it. The only important thing to note in this regard is that in addition to these two theories there are also the $E_8 \times E_8$ heterotic, $SO(32)$ heterotic and Type I and all together they are constitute low-energy limits of the same complete theory. They are not independent but are connected by dualities and we can move from one to another by varying the VEVs of the massless scalar fields inside the moduli space. For the aim of this research we are interested in the closed string bosonic spectrum and so we focus only on the NS-NS and R-R sectors of Type IIB theory. In addition, since the involved energies are quite low, $E \ll M_s$ where M_s is the string mass, we can ignore the massive string modes and we consider only the massless spectrum. More precisely from the R-R sector we obtain a zero-form C_0, a two-form C_2 and a four-form C_4 while from the NS-NS sector we have the same content of the closed bosonic string, i.e. the dilaton Φ, the graviton G_{MN} and the two-form known as B-field. When we will focused on four dimensions the corresponding R-R 4-form C_4 integrated on the 4-cycles of the Calabi-Yau will give the axionic partner of the Kähler moduli, and the 3-form fluxes of the two sectors will be used as background fluxes in order to stabilise the scalar potential at tree level.
In open strings from the R and NS sectors we get respectively a fermion and a gauge field A_M. With N D-branes on top of each other, the open strings transform in adjoint representations of $U(N)$. The excitations along the brane represent a $U(N)$ gauge field and gaugino, while the excitations orthogonal to the branes are bosons and fermions in the adjoint representation of $U(N)$. With stacks of D-branes intersecting at angles, or D-branes placed at special singularities, the $U(N)$ symmetry can be broken to the ordinary gauge group of the Standard Model. This is related with the two methods that we have to embed the visible sector in the Calabi-Yau manifold.

3.2 Extra dimensions and compactifications

The idea of extra dimensions has not emerged in the context of String Theory but had already been formulated around 1920 by Kaluza in his attempt to unify gravitation and electromagnetism. Since in the Universe the visible dimensions are only four, three spatial and one temporal, if we add some dimensions they must necessarily be very small and imperceptible for the ordinary experiments: they are compactified. In the simplest case we have a massless scalar field in five dimensions $\phi\big(x^M\big)$ with $M = 0, 1, 2, 3, 4$.
The action is

$$S_{5D} = \int d^5x\, \partial^M \phi \partial_M \phi^\star \tag{3.16}$$

We choose the extra dimension $x^4 = y$ to be circular, i.e. $y \equiv y + 2\pi R$.
The total spacetime in this case is $M_4 \times S^1$, where M_4 is the Minkowski spacetime and S^1 is a circumference. Then, the five-dimensional field can be expanded in Fourier modes

$$\phi\big(x^\mu, y\big) = \sum_{n=-\infty}^{\infty} \phi_n\big(x^\mu\big) \exp\left(\frac{iny}{R}\right) \tag{3.17}$$

The five-dimensional equation of motion is

$$\partial_M \partial^M \phi\big(x^M\big) = 0 \Rightarrow \big(\partial_\mu \partial^\mu + \partial_4 \partial^4\big) \sum_{n=-\infty}^{\infty} \phi_n\big(x^\mu\big) \exp\left(\frac{iny}{R}\right) = 0$$

and therefore

$$\left(\partial_\mu \partial^\mu - \frac{n^2}{R^2}\right) \phi_n\big(x^\mu\big) = 0 \tag{3.18}$$

At the end we have a massless scalar field and a "tower" of massive scalar fields with masses which increase with n

$$m_n^2 = \frac{n^2}{R^2} \tag{3.19}$$

The action for the 5-dimensional massless scalar field is reduced to the sum of the action in four dimensions for a massless scalar field and an infinite number of massive scalar actions also in four dimensions. If we are interested on energy scales lower than the compactification scale $1/R$, known as Kaluza-Klein scale, then we can neglect all the massive modes and we can focus our attention only on the massless one. This dimensional reduction, it is responsible for the big number of massless scalar fields that appears when we move from a theory with more dimensions to an effective one with less dimensions.
Let us focus now on the case of String Theory. The theory of superstrings, as we have already said, is defined in a consistent way in 10 dimensions. Hence it requires six spatial extra dimensions described by a six-dimensional Calabi-Yau manifold. The main purpose of string phenomenology is to find a compactification whose low energy EFT reproduces a suitable extension of the Standard Model and more precisely $\mathcal{N} = 1$, $D = 4$

supergravity. In particular, if we are at energies above the string mass M_s we have a ten-dimensional String Theory where both massive and massless states are excited. Below the M_s scale, instead, we still have a ten-dimensional theory but this time the massive modes are not excited and we can focus only on the massless ten-dimensional spectrum as we have done in the previous section. When, finally, we go below the Kaluza-Klein scale, integrating out the six extra dimensions, we get the four dimensional effective supersymmetric theory with a large number of massless scalar fields, the moduli, while we neglect the tower of the massive ones. There is a huge number of possible Calabi-Yau three-folds each of which leads to a different effective theory. In the next section we describe the general features of these manifolds.

3.3 Calabi-Yau manifolds

Taking into account the compactification in S^1, previously seen, the immediate generalization in the case of six extra dimensions is the toroidal one where the total manifold is decomposed as

$$\mathcal{M}_{10} = \mathcal{M}_{10-d} \times T^d$$

with T^d the d-dimensional torus, defined by the identification

$$x^m \equiv x^m + 2\pi R^m$$

These compactifications preserve $\mathcal{N} = 8$ SUSY, and so they do not allow to reproduce most of the physical results that we observe. For this reason we are looking for more sophisticated manifolds that support less supersymmetry: the Calabi-Yau, $\mathcal{M}_6$:

$$\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{M}_6$$

In the absence of fluxes these manifolds must be Ricci-flat

$$R_{mn} = 0 \tag{3.20}$$

We also have constraints related to the vacuum state:
Since $\mathcal{M}_4$ is the four dimensional Minkowski spacetime in order to respect Poincaré symmetry only scalar fields can have non-zero vacuum expectation values. The other constraint come from supersymmetry because a supersymmetric vacuum where only bosonic fields have non-vanishing expectation values should obey $\langle Q_\epsilon \chi \rangle = \langle \delta_\epsilon \chi \rangle = 0$. In Type II theories the fermionic fields are two gravitini and two dilatini. When no fluxes are present, requiring zero VEV for the gravitino variation is equivalent to demand the existence of a covariantly constant spinor on the ten-dimensional manifold, i.e. $\nabla_M \, \epsilon = 0$. Splitting the spinors into four-dimensional and six-dimensional ones and considering that the four dimensional spinors are just constants, since $\mathcal{M}_4$ is the Minkowski spacetime, at the end it is requested the existence of a covarianty constant spinor in the six dimensional compactification manifold, i.e. a spinor such that

$$\nabla_m \, \eta = 0 \tag{3.21}$$

This condition implies an algebraic constrain, i.e. that there should exist an everywhere non-vanishing Weyl spinor and a differential constrain related to the integrability of the covariantly constant spinor. The first implies that the structure group is reduced to $SU(3)$, while the second implies that the manifold should have $SU(3)$ holonomy.

Alternatively, these properties are obtained from the fact that the Calabi-Yau is a Kähler manifold, i.e. a complex and symplectic manifold with compatible respective structures
.
A complex manifold of dimension $2n$ is an integrable almost-complex manifold that allows to decompose any n-form

$$A = \frac{1}{n!} A_{m_1,\ldots,m_n} dx^{m_1} \wedge \cdots \wedge dx^{m_n} \tag{3.22}$$

in an holomorphic and an anti-holomorphic part:

$$A_{p,q} = \frac{1}{p!q!} A_{i_1,\dots,i_p,\bar{j}_1\dots,\bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \dots \wedge d\bar{z}^{\bar{j}_q} \tag{3.23}$$

where z^i is a set of holomorphic one-forms, $\bar{z}^i$ a set of anti-holomorphic one-forms and $p + q = n$. Thanks to the integrability also the exterior derivative can split in an holomorphic and an anti-holomorphic component with result

$$\begin{gathered} d = \partial + \bar{\partial} \\ \partial : (p,q) \to (p+1,q) \qquad \bar{\partial} : (p,q) \to (p,q+1) \end{gathered} \tag{3.24}$$

Such manifolds have holonomy $GL(n, \mathbb{C})$.
A $2n$-dimensional manifold is symplectic if there is a globally-defined nowhere vanishing non-degenerate two-form J such that

$$dJ = 0 \tag{3.25}$$

Similarly to complex coordinates z^i for complex manifolds, in symplectic manifolds we can define *Darboux* coordinates (x^i, y^i) such that

$$J = \sum_{i=1}^{n} dx^i \wedge dy^i \tag{3.26}$$

Such structure has holonomy $Sp(2n)$. A complex and symplectic manifold with compatible complex and symplectic structures have

$$J = j_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \tag{3.27}$$

and the holonomy of the manifold must be contained in the intersection of the holonomies of the two structures i.e. $U(3)$. The complex structure I and the symplectic two-form together define the metric of the Kähler manifold which at the end is

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$$

with K the Kähler potential.
On complex manifolds one can define cohomology classes for (p, q) forms, $H^{p,q}$. The dimensions of these cohomology classes are denoted by the so called Hodge numbers $h^{(p,q)}$, which satisfy the

$$\sum_{k=0}^{p} h^{(k,p-k)} = b_p \qquad h^{(p,q)} = h^{(q,p)} = h^{(n-p,n-q)} \tag{3.28}$$

where b_p are the Betty numbers that define the dimension of cohomology. A particularly important cohomology class is the first Chern class where lives the Ricci two-form. A Calabi-Yau manifold always admit a Ricci-flat metric and so it is a Kähler manifold with non trivial first Chern class.

In conclusion, the Calabi-Yau is a manifold with $h^{(1,0)} = h^{(2,0)} = 0$ and $h^{(3,0)} = 1$. A representative of this cohomology class is the holomorphic three-form Ω. Using these properties we can arrange the Hodge numbers in a diamond which in the six-dimensional case is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Calabi-Yau compactifications preserve the $N = 2$ SUSY. This amount of supersymmetry, however, does not give rise to the necessary chiral interactions needed by phenomenological applications and so in order to have a realistic theory half of the SUSY generators must be projected out. This is obtained through the orientifold projection which implies the presence of O_7/O_3 planes that carry RR charges and the presence of D_7/D_3 branes in different places of the compact space, necessary to cancel tadpoles.
In order to stabilise part of the moduli we use background fluxes. Calabi-Yau manifolds are both complex and symplectic with the additional constrain $c_1 = 0$. For general flux compactifications, however, this is not true anymore and both J and Ω can have a non-trivial exterior derivative. Moreover, the two supersymmetry parameters ϵ^i in the absence of fluxes have been expanded in terms of the same internal spinor while in presence of fluxes the symmetry between left- and right-movers can be broken and consequently we need two different internal spinors. In conclusion general flux backgrounds can be understood as complex or symplectic manifolds in a generalized sense. The only case in which the backreaction is under control it is in the Type IIB theory where gives rise only to warped Calabi-Yaus.

In the next sections we focus on the Large Volume Scenario which emerges naturally in the context of Type IIB flux compactifications on Calabi-Yau orientifolds in presence of spacetime filling D_3/D_7 branes and O_3/O_7 planes. An excellent description of these techniques can be found in the articles [2],[6],[7],[9],[12],[13],[11],[5],[14].

3.4 Type IIB compactifications on Calabi Yau three-folds

The moduli fields that arise from the Kaluza-Klein reduction parametrize the shape and the size of the compactified extra dimensions and compose the hidden sector of the effective theory that corresponds to the $N = 2$, $D = 4$ supergravity. The effective action arises from expanding all the ten-dimensional fields in a basis of massless forms. As we have seen in the simple case of the scalar field in 5 dimensions we are interested to the harmonic forms on $Calabi - Yau$ which give the massless spectrum of the effective four dimensional theory. To this end, it is useful to note the isomorphism between the space of harmonic p-forms and the p-th cohomology which implies that each cohomology class has exactly one harmonic form which is taken to be a representative of it. The four dimensional low energy action it is eventually obtained by expanding all the 10-dimensional fields in harmonic forms. From the NS-NS sector we obtain a total of $2\big(h^{(1,1)} + h^{(2,1)} + 1\big)$ moduli. More precisely, we have $h^{(2,1)}$ complex structure moduli U_α, $\alpha = 1, \ldots, h^{(2,1)}$, parameterizing the deformations of the complex structure, i.e. of the shape of the extra dimensions and $h^{(1,1)}$ of τ_i, i.e. the real parts of Kähler moduli characterizing the volume of the divisors D_i, which are dual to the elements of the base of $H^{1,1}$. Finally we have the axio-dilaton whose VEV fix the string scale. If we compose the τ_i with its axionic partner coming by the integration of the R-R sector C_4 form along D_i, $\int_{D_i} C_4 = \theta_i$, we obtain the Kähler moduli $T_i = \tau_i + i\theta_i$, which parametrize the deformations in size of the Calabi-Yau. The Kähler potential at tree level assumes the form

$$K_{tree} = -2\ln(\mathcal{V}) - \ln(S + \bar{S}) - \ln(-i \int_{\mathcal{M}} \Omega(U) \wedge \bar{\Omega}(\bar{U})) \tag{3.29}$$

where $\mathcal{V}$ is the volume of the Calabi-Yau manifold $\mathcal{M}$ expressed in string units $l_s = 2\pi\sqrt{\alpha'}$ and Ω is a holomorphic $(3,0)$-form of $\mathcal{M}$. The volume can be expressed in terms of the Kähler form J, once this is expanded in the base of $H^{1,1}(\mathcal{M}, \mathbb{Z})$ as

$$J = \sum_{i=1}^{h^{(1,1)}} t^i \hat{D}_i \tag{3.30}$$

where t_i are the 2-cycle volumes

$$\mathcal{V} = \frac{1}{6}\int_{\mathcal{V}} J \wedge J \wedge J = \frac{1}{6} k_{ijk} t^i t^j t^k \tag{3.31}$$

with k_{ijk} the triple intersection numbers of $\mathcal{M}$.

$$\tau_i = \frac{\partial \mathcal{V}}{\partial t_i} = \frac{1}{2}\int_{\mathcal{M}} \hat{D}_i \wedge J \wedge J = \frac{1}{2} k_{ijk} t^j t^k \tag{3.32}$$

Once the intersections number are known, using equations (3.31) and (3.32), we can compute the explicit forms for the volumes of the compactifications. The superpotential at tree level is given by the 3-form fluxes F_3 and H_3 coming respectively from the $R-R$

and the $NS - NS$ sector. Setting $G_3 = F_3 - SH_3$, in fact, the superpotential assumes the form

$$W_{tree} = \int_{\mathcal{M}} G_3 \wedge \Omega \tag{3.33}$$

Chapter 4

Inflationary String Models

The inflationary paradigm is very successful in resolving the most important puzzles of modern cosmology. Any inflation model is sensitive to Planck scale physics and so any attempt to understand this period of cosmic evolution should be embedded in an ultraviolet complete theory like String Theory. The dimensional reduction followed in the context of compactifications gives rise to a large number of massless scalar fields with flat potential which can be the ideal candidates to drive inflation. Since their presence is in contrast with phenomenological observations they must be stabilised using non zero background fluxes. Recent developments in techniques of moduli stabilisation have led to a large number of new inflationary scenarios based on either open or closed string modes. In this thesis we have been focused on two very promising models of string inflation coming from the large volume limit of the scalar potential in Calabi-Yau Type IIB flux compactification in which the inflaton is a closed string mode: K"ahler Inflation and Fibre Inflation. More precisely, in the first case the inflaton is the size of a blow-up mode yielding a small field inflationary model which is in good agreement with current observational data. In the second model, instead, the inflaton is the size of a K3 fibre over a $\mathbb{C}P^1$ base producing a large field inflationary model, which is particularly suitable for the production of primordial gravity waves.

4.1 Moduli Stabilisation

The process of moduli stabilisation is the first step towards string phenomenology as the moduli vacuum expectation values determine the string scale and the gauge couplings. The axio-dilaton and the complex structure moduli can be fixed to their values at their supersymmetric minimum by turning on background fluxes on the internal manifold. These fluxes could in general have a backreaction on the geometry of the Calabi-Yau modifying the internal space. In the case of Type IIB compactifications however, as we have already said, fluxes induce just a warp factor and this is why this compactifications are so suitable for the cosmological applications. On the other hand, the Kähler moduli are stabilised in a second time through non perturbative corrections to the superpotential. Using the expressions at tree level for the Kähler potential and superpotential (3.29),(3.33) the F-scalar potential (2.75), in Planck units, becomes

$$V_F = e^K \big(\sum_T K^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} + \sum_{U,S} K^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - 3|W|^2 \big) =$$

$$e^K \big(|W|^2 (K^{T\bar{T}} \partial_T K \partial_{\bar{T}} K - 3) + \sum_{U,S} K^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - 3|W|^2 \big) =$$

$$= e^K \big(K^{U\bar{U}} D_U W D_{\bar{U}} \bar{W} + K^{S\bar{S}} D_S W D_{\bar{S}} \bar{W} \big) \tag{4.1}$$

where we have used the no-scale property of the Kähler potential

$$K^{T\bar{T}} \partial_T K \partial_{\bar{T}} K = 3 \tag{4.2}$$

Because $V_F \geq 0$, we have the minimum of the axio-dilaton $< S >$ and of the complex structure moduli $< U >$, for $D_U W = 0$ and $D_S W = 0$. In the simplest case we have $\mathcal{V} = \tau^{3/2}$, where τ is the real part of the single Kähler moduli $T = \tau + i\theta$. The Kähler potential $K = -2\ln(\mathcal{V})$, assumes the form

$$K = -3\ln(\tau) \tag{4.3}$$

In order to stabilise the Kähler modulus we introduce non-perturbative corrections on the superpotential coming from the coupling with D_3 and D_7 branes

$$W = W_0 + Ae^{-aT} \tag{4.4}$$

where W_0 is the constant value of the superpotential obtained from the stabilisation of the axio-dilaton and of the complex structure moduli. From the explicit computation

$$K_T = \frac{1}{2}\frac{\partial K}{\partial \tau} = -\frac{3}{2\tau} \qquad K_{T\bar{T}} = \frac{1}{4}\frac{\partial^2 K}{\partial \tau^2} = \frac{3}{4\tau^2} \tag{4.5}$$

we have

$$V_F = e^K \big(K^{T\bar{T}}(W_T \bar{W}_{\bar{T}} + W_T \bar{W} K_{\bar{T}} + W \bar{W}_{\bar{T}} K_T + |W|^2 K_T K_{\bar{T}}) - 3|W|^2\big) =$$

$$\frac{1}{\tau^3}\big(\frac{4\tau^2}{3} a^2 A^2 e^{-2a\tau} + 4\tau a A^2 e^{-2a\tau} - 4\tau a A W_0 e^{-a\tau}\cos(a\theta)\big) \tag{4.6}$$

and minimizing with respect to the axionic partner of the Kähler moduli θ, at the end we find

$$V_F = \frac{4}{3\tau} A^2 a^2 e^{-2a\tau} + \frac{4}{\tau^2} a A^2 e^{-2a\tau} - \frac{4a}{\tau^2} A W_0 e^{-a\tau} \tag{4.7}$$

Minimizing the potential (4.7) with respect to τ, $\frac{\partial V_F}{\partial \tau} = 0$, we have

$$-\frac{4}{3} A^2 a^2 \big(\tau e^{-2a\tau} + 2\tau^2 a e^{-2a\tau}\big) - 8aA^2\big(e^{-2a\tau} a\tau + e^{-2a\tau}\big) + 4aAW_0\big(2e^{-a\tau} + a\tau e^{-a\tau}\big) = 0$$

which in the limit of large volume $a\tau \gg 1$, is simplified to

$$-\frac{8}{3} A^2 a^3 \tau^2 e^{-a\tau} + 4a^2 A\tau W_0 = 0$$

from which it follows that in the minimum

$$e^{-a\tau} = \frac{3}{2aA}\frac{W_0}{\tau} \sim \frac{W_0}{\tau} \tag{4.8}$$

In order to have a large volume limit, it is necessary to have an exponentially small value of the tree level superpotential, $|W_0| < 10^{-4}$. In this model, known as the $KKLT$ scenario, the presence of the minimum is not guaranteed for natural values of W_0. Moreover, the vacuum solution corresponding to an AdS supersymmetric minimum has negative energy and it is not suitable for a realistic cosmological theory. To uplift the initial AdS minimum it is necessary to add $\bar{D}3$-branes which, however, break SUSY explicitly and this is in contrast with the fact that we are working in a supersymmetric context.
These problems can be solved in the Large Volume Scenario framework in which we take into account perturbative α' corrections to the Kähler potential but we neglect string loop corrections.

$$K = -2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2g_s^{3/2}}\right) \quad \text{with} \quad \hat{\xi} = \frac{h^{(1,2)} - h^{(1,1)}}{(2\pi)^3}\zeta(3) \quad \textit{and} \quad \xi = \hat{\xi}/g_s^{3/2} \tag{4.9}$$

where $\zeta(3) \sim 1.2$ and g_s is the string coupling.
In this case moduli stabilisation is performed without fine tuning of the internal fluxes and the compactification volume is fixed at an exponentially large value (in string units l_s). Moreover the minimum is AdS but, contrary to KKLT, breaks SUSY spontaneously giving rise to the super-Higgs effect. The LVS in conclusion can be viewed as a generalization of the KKLT scenario which allows to introduce more complicated topologies. In the following we describe two models embedded in the Large Volume Scenario very promising for the realization of the inflationary slow roll dynamic, in which the inflaton is a closed string mode of the hidden sector: Kähler Inflation [14],[5] and Fibre Inflation [7],[12],[13].

4.2 Kähler Inflation

In Kähler Inflation we use a Swiss cheese type volume of the form

$$\mathcal{V} = \alpha(\tau_1^{3/2} - \sum_i \lambda_i \tau_i^{3/2}) \tag{4.10}$$

where all the τ_i are rigid cycles, blow ups of pointlike singularities of the CY which are fixed small by non perturbative corrections while τ_1 is stabilised exponentially large due to α' and non-perturbative effects. The parameters α and λ_i are model dependent constants that can be computed once we have identified a particular compactification geometry. From the (4.9) we can compute the Kähler metric for an arbitrary number of moduli.

$$\frac{\partial K}{\partial \tau_1} = -\frac{6\alpha\tau_1^{1/2}}{2\mathcal{V}+\xi} \qquad \frac{\partial^2 K}{\partial \tau_1^2} = \frac{18\alpha^2\tau_1^{3/2} - 3\alpha(2\mathcal{V}+\xi)}{(2\mathcal{V}+\xi)^2\tau_1^{1/2}} = \frac{3\alpha^{4/3}(4\mathcal{V}-\xi+6\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2})}{(2\mathcal{V}+\xi)^2(\mathcal{V}+\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2})^{1/3}}$$

$$\frac{\partial K}{\partial \tau_j} = \frac{6\alpha\lambda_j\tau_j^{1/2}}{2\mathcal{V}+\xi} \qquad \frac{\partial^2 K}{\partial \tau_i \partial \tau_j} = \frac{18\alpha^2\lambda_i\lambda_j\tau_i^{3/2}\tau_j^{3/2}}{(2\mathcal{V}+\xi)^2}$$

$$\frac{\partial^2 K}{\partial \tau_1 \partial \tau_j} = -\frac{18\alpha^2\lambda_j\tau_j^{1/2}\tau_1^{1/2}}{(2\mathcal{V}+\xi)^2} = -\frac{18\alpha^{5/3}\lambda_j\tau_j^{1/2}(\mathcal{V}+\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2})^{1/3}}{(2\mathcal{V}+\xi)^2}$$

$$\frac{\partial^2 K}{\partial \tau_i^2} = \frac{3\alpha\lambda_i(2\mathcal{V}+\xi+6\alpha\tau_i^{3/2}\lambda_i)}{(2\mathcal{V}+\xi)^2\sqrt{\tau_i}} \tag{4.11}$$

where we have substitute $\tau_1 = (\frac{1}{\alpha})^{2/3}(\mathcal{V}+\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2})^{2/3}$.

Considering that $K_{i\bar{j}} = \frac{1}{4}\frac{\partial^2 K}{\partial\tau_i\partial\tau_j}$, we have

$$K_{1\bar{1}} = \frac{3\alpha^{4/3}}{4}\frac{4\mathcal{V}-\xi+6\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2}}{(2\mathcal{V}+\xi)^2(\mathcal{V}+\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2})^{1/3}} \qquad K_{i\bar{j}} = \frac{9\alpha^2}{2}\frac{\lambda_i\lambda_j\sqrt{\tau_i}\sqrt{\tau_j}}{(2\mathcal{V}+\xi)^2} \tag{4.12}$$

$$K_{i\bar{i}} = \frac{3\alpha}{4}\frac{\lambda_i(2\mathcal{V}+\xi+6\alpha\lambda_i\tau_i^{3/2})}{(2\mathcal{V}+\xi)^2\sqrt{\tau_i}} \qquad K_{1\bar{j}} = -\frac{9\alpha^{5/3}}{2}\frac{\lambda_j\tau_j^{1/2}(\mathcal{V}+\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2})^{1/3}}{(2\mathcal{V}+\xi)^2}$$

which can be inverted to give

$$K^{1\bar{1}} = \frac{4(2\mathcal{V}+\xi)(\mathcal{V}+\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2})^{1/3}(2\mathcal{V}+\xi+6\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2})}{3\alpha^{4/3}(4\mathcal{V}-\xi)}$$

$$K^{i\bar{j}} = \frac{8(2\mathcal{V}+\xi)\tau_i\tau_j}{4\mathcal{V}-\xi} \qquad K^{1\bar{j}} = \frac{8(2\mathcal{V}+\xi)\tau_j(\mathcal{V}+\alpha\sum_{i=2}^n \lambda_i\tau_i^{3/2})^{2/3}}{\alpha^{2/3}(4\mathcal{V}-\xi)} \tag{4.13}$$

$$K^{i\bar{i}} = \frac{4(2\mathcal{V}+\xi)\sqrt{\tau_i}(4\mathcal{V}-\xi+6\alpha\lambda_i\tau_i^{3/2})}{3\alpha(4\mathcal{V}-\xi)\lambda_i}$$

Using a superpotential of the form

$$W = W_0 + \sum_{i=2}^{n} A_i e^{-a_i T_i} \tag{4.14}$$

we eventually have

$$V_F = \frac{4}{(2\mathcal{V}+\xi)^2} \Bigg(K^{1\bar{1}} D_1 W D_{\bar{1}} \bar{W} + \sum_{i=2}^{n} K^{1\bar{i}} (D_1 W D_{\bar{i}} \bar{W} + D_i W D_{\bar{1}} \bar{W})$$

$$+ \sum_{i=2}^{n} K^{i\bar{i}} D_i W D_{\bar{i}} \bar{W} + \sum_{\substack{i<j \\ i,j=2}}^{n} K^{i\bar{j}} (D_i W D_{\bar{j}} \bar{W} + D_j W D_{\bar{j}} \bar{W}) - 3|W|^2 \Bigg) \tag{4.15}$$

$$K^{1\bar{1}} D_1 W D_{\bar{1}} \bar{W} = \frac{12(\mathcal{V} + \alpha \sum_{k=2}^{n} \lambda_k \tau_k^{3/2})(2\mathcal{V} + \xi + 6\alpha \sum_{k=2}^{n} \lambda_k \tau_k^{3/2})}{(4\mathcal{V}-\xi)(2\mathcal{V}+\xi)} \times$$

$$\Bigg(W_0^2 + 2W_0 \sum_{k=2}^{n} A_k e^{-a_k \tau_k} \cos(a_k \theta_k) + \sum_{k,l=2}^{n} A_k A_l e^{-a_k T_k - a_l \bar{T} l} \Bigg) \tag{4.16}$$

$$\sum_{i=2}^{n} K^{1\bar{i}} (D_1 W D_{\bar{i}} \bar{W} + D_i W D_{\bar{1}} \bar{W}) = \sum_{i=2}^{n} \frac{24\tau_i (\mathcal{V} + \alpha \sum_{k=2}^{n} \lambda_k \tau_k^{3/2})}{4\mathcal{V}-\xi} \times$$

$$\Bigg(2W_0 a_i A_i e^{-a_i \tau_i} \cos(a_i \theta_i) + 2a_i A_i \sum_{k=2}^{n} A_k e^{-(a_i \tau_i + a_k \tau_k)} \cos(a_i \theta_i - a_k \theta_k) - \frac{6\alpha W_0^2 \lambda_i \tau_i^{1/2}}{2\mathcal{V}+\xi}$$

$$- \frac{12 W_0 \alpha \lambda_i \tau_i^{1/2}}{2\mathcal{V}+\xi} \sum_{k=2}^{n} A_k e^{-a_k \tau_k} \cos(a_k \theta_k) - \frac{6\alpha \lambda_i \tau_i^{1/2}}{2\mathcal{V}+\xi} \sum_{k,l=2}^{n} A_k A_l e^{-a_k \bar{T}_k - a_l T_l} \Bigg) \tag{4.17}$$

$$\sum_{i=2}^{n} K^{i\bar{i}} D_i W D_{\bar{i}} \bar{W} = \sum_{i=2}^{n} \frac{4(2\mathcal{V}+\xi)\sqrt{\tau_i}(4\mathcal{V} - \xi + 6\alpha \lambda_k \tau_k^{3/2})}{3\alpha \lambda_i (4\mathcal{V}-\xi)} \times$$

$$\Bigg(a_i^2 A_i^2 e^{-2a_i \tau_i} - \frac{6 W_0 \alpha \lambda_i \tau_i^{1/2}}{2\mathcal{V}+\xi} a_i A_i e^{a_i \tau_i} \cos(a_i \theta_i) + \frac{9 W_0^2 \alpha^2 \lambda_i^2 \tau_i}{(2\mathcal{V}+\xi)^2}$$

$$+ \frac{18 W_0 \alpha^2 \lambda_i^2 \tau_i}{(2\mathcal{V}+\xi)^2} \sum_{k=2}^{n} A_k e^{-a_k \tau_k} \cos(a_k \theta_k) + \frac{9\alpha^2 \lambda_i^2 \tau_i}{2\mathcal{V}+\xi} \sum_{k,l} A_k A_l e^{-a_k T_k - a_l \bar{T}_l}$$

$$- \frac{6 a_i A_i \alpha \lambda_i \tau_i^{1/2}}{2\mathcal{V}+\xi} \sum_{k=2}^{n} A_k e^{-(a_k \tau_k + a_i \tau_i)} \cos(a_k \theta_k - a_i \theta_i) \Bigg) \tag{4.18}$$

$$\sum_{\substack{i<j\\i,j=2}}^{n} K^{i\bar{j}}(D_iWD_{\bar{j}}\bar{W}+D_jWD_{\bar{j}}\bar{W})=\sum_{i,j=2,i<j}^{n}\frac{8(2\mathcal{V}+\xi)\tau_i\tau_j}{4\mathcal{V}-\xi}\times$$

$$\Big(2a_iA_ia_jA_je^{-(a_i\tau_i+a_j\tau_j)}\cos(a_i\theta-a_j\theta_j)+$$

$$\frac{18\lambda_i\lambda_j\tau_i^{1/2}\tau_j^{1/2}\alpha^2}{(2\mathcal{V}+\xi)^2}(W_0^2+2W_0\sum_{k=2}^{n}A_ke^{-a_kA_k}\cos(a_k\theta_k)+\sum_{k,l=2}^{n}A_kA_le^{-a_kT_k-a_l\bar{T}_l})$$

$$-\frac{6\alpha\lambda_j\tau_j^{1/2}}{2\mathcal{V}+\xi}a_iA_iW_0e^{-2a_iA_i}\cos(a_i\theta_i)-\frac{6\alpha\lambda_i\tau_i^{1/2}}{2\mathcal{V}+\xi}a_jA_je^{-2a_j\tau_j}\cos(a_j\theta_j)$$

$$-\frac{6a_iA_i\alpha}{2\mathcal{V}+\xi}\lambda_i\tau_i^{1/2}\sum_{k=2}^{n}A_ke^{-(a_k\tau_k-a_i\tau_i)}\cos(a_i\theta_i-a_k\theta_k)$$

$$-\frac{6a_jA_j\alpha}{2\mathcal{V}+\xi}\lambda_j\tau_j^{1/2}\sum_{k=2}^{n}A_ke^{-(a_k\tau_k-a_j\tau_j)}\cos(a_j\theta_j-a_k\theta_k)\Big) \tag{4.19}$$

Adding all terms together at the end we have

$$V_F=\sum_{\substack{i<j\\i,j=2}}^{n}\frac{A_iA_jcos(a_i\theta_i-a_j\theta_j)}{(4\mathcal{V}-\xi)(2\mathcal{V}+\xi)^2}e^{-(a_i\tau_i+a_j\tau_j)}(32(2\mathcal{V}+\xi)(a_i\tau_i+\tau_ja_j+2a_ia_j\tau_i\tau_j)+24\xi)$$

$$+\frac{12W_0^2\xi}{(4\mathcal{V}-\xi)(2\mathcal{V}+\xi)^2}+\sum_{i=2}^{n}\Big(\frac{12e^{-2a_i\tau_i}\xi A_i^2}{(4\mathcal{V}-\xi)(2\mathcal{V}+\xi)^2}+\frac{16(a_iA_i)^2\sqrt{\tau_i}e^{-2a_i\tau_i}}{3\alpha\lambda_i(2\mathcal{V}+\xi)}$$

$$+\frac{32e^{-2a_i\tau_i}a_i\tau_iA_i^2(1+a_i\tau_i)}{(4\mathcal{V}-\xi)(2\mathcal{V}+\xi)}+\frac{8W_0A_ie^{-a_i\tau_i}\cos(a_i\theta_i)}{(4\mathcal{V}-\xi)(2\mathcal{V}+\xi)}(\frac{3\xi}{2\mathcal{V}+\xi}+4a_i\tau_i)\Big) \tag{4.20}$$

The vacua that emerge from stabilisation in the Large Volume Scenario, like in the KKLT, have negative energy. In order to describe inflation we need a dS minimum with positive energy which however is more difficult to obtain than a stable AdS minimum. The reason why the AdS configuration is more stable is because it is protected by SUSY in the KKLT and thanks to the large volume limit in the case of LVS. In order to obtain de Sitter vacuum configurations we must add an uplifting term at the background solution derived from SUSY breaking. Since this uplifting term cannot be perturbatively small it is important to break SUSY in a parametrically controlled way in order not to invalidate the stabilisation method which is based on an effective supersymmetric action. One way could be through multiple D-branes in the singular vertex of a cone of the Calabi-Yau which supports gauge theories in four dimensions, some of which will give metastable vacua where SUSY is dynamically broken. The problem is that it is not guaranteed that this metastability will survive under the subsequent stabilisation. The idea, therefore, it is that the stabilisation in the AdS minimum and the SUSY breaking responsible for the uplifting of this minimum occur in two different regions of the compactification. The uplifting term in our case will assume the form

$$V_{uplift} = \frac{\beta}{\mathcal{V}^2} \tag{4.21}$$

and must be added to the potential (4.20).

In the limit $\mathcal{V} \to \infty$ with $\tau_1 \gg \tau_i$, the potential becomes

$$V_L = \sum_{i=2}^{n} \frac{8(a_i A_i)^2}{3\alpha\lambda_i \mathcal{V}} \sqrt{\tau_i} e^{-2a_i\tau_i} - \sum_{i=2}^{n} \frac{4W_0 a_i A_i \tau_i}{\mathcal{V}^2} e^{-a_i\tau_i} + \frac{3\xi W_0^2}{4\mathcal{V}^3} + \frac{\beta}{\mathcal{V}^2} \tag{4.22}$$

where we have minimize with respect to the C_4-axions θ_i. We find the minimum of τ_i

$$\frac{\partial V_L}{\partial \tau_i} = 0 \ \Rightarrow \ a_i A_i e^{-a_i\tau_i} = \frac{3\alpha\lambda_i W_0 (1 - a_i\tau_i)}{2\mathcal{V}(\frac{1}{2} - 2a_i\tau_i)} \tag{4.23}$$

which in the large volume limit $a_i\tau_i \gg 1$, becomes

$$e^{a_i\tau_i} \sim \frac{4\mathcal{V} a_i A_i}{3\alpha\lambda_i W_0 \sqrt{\tau_i}} \Rightarrow \tau_i \sim \frac{1}{a_i}\left(\ln(\mathcal{V}) - \ln\left(\frac{3\alpha\lambda_i W_0}{2a_i A_i \tau_i}\right)\right) \tag{4.24}$$

The second term in the bracket is tend to zero and so at the minimum we have $a_i\tau_i \sim \ln(\mathcal{V})$. If we substitute the minimum in the potential we obtain

$$V_L^{min} = -\frac{3}{2}\sum_{i=2}^{n} \frac{W_0^2 \alpha\lambda_i \tau_i^{3/2}}{\mathcal{V}^3} + \frac{3\xi W_0^2}{4\mathcal{V}^3} + \frac{\beta}{\mathcal{V}^2}$$

and considering $a_i\tau_i \sim \ln(\mathcal{V})$

$$V_L^{min} = -\frac{3W_0^2}{2\mathcal{V}^3}\left(\sum_{i=2}^{n} \frac{\lambda_i\alpha}{a_i^{3/2}} \ln(\mathcal{V})^{3/2} - \frac{\xi}{2}\right) + \frac{\beta}{\mathcal{V}^2} \tag{4.25}$$

In order to have inflation a flat enough potential must exists. This can be obtained by displacing one of the small modulus τ_i, with $i = 1 \dots n$, away from its minimum value keeping the others fixed at their global minima. This modulus will be the inflaton since at constant volume the potential is exponentially flat along this direction and we obtain a successful inflationary fashion when it is rolls back to its minimum. Inflation ends when all moduli take their minimum values. If we choose the inflaton to be τ_n then after displacing it, the potential (4.25) becomes

$$V_L^{min} = -\frac{3W_0^2}{2\mathcal{V}^3}\left(\sum_{i=2}^{n-1} \frac{\lambda_i\alpha}{a_i^{3/2}} \ln(\mathcal{V})^{3/2} - \frac{\xi}{2}\right) + \frac{\beta}{\mathcal{V}^2} \tag{4.26}$$

To ensure that the minimum of the potential along the other directions remains unchanged if we displace τ_n from its minimum there should be little difference between (4.25) and (4.26). In conclusion, all the fields except τ_n will remain unchanged during inflation if

$$\rho = \frac{\lambda_n}{a_n^{3/2}} / \sum_{i=2}^{n} \frac{\lambda_i}{a_i^{3/2}} \ll 1 \tag{4.27}$$

The potential along the inflationary direction τ_n, if we neglect the double exponential is

$$V_{inf}(\tau_n) = \frac{BW_0^2}{\mathcal{V}^3} - \frac{4W_0 a_n A_n \tau_n}{\mathcal{V}^2} e^{a_n \tau_n} \tag{4.28}$$

where B includes several terms of (4.22). In terms of the canonically normalised field $\frac{\psi}{M_{pl}} = \sqrt{\frac{4\alpha\lambda_n}{3\mathcal{V}}} \tau_n^{3/4}$, the inflationary potential, in Plank units, is

$$V_{inf}(\psi) = \frac{BW_0^2}{\mathcal{V}^3} - \frac{4W_0 a_n A_n}{\mathcal{V}^2} \left(\frac{3\mathcal{V}}{4\alpha\lambda_n} \right)^{2/3} \psi^{4/3} \exp \left(-a_n \left(\frac{3\mathcal{V}}{4\alpha\lambda_n} \right)^{2/3} \psi^{4/3} \right) \tag{4.29}$$

$$V'_{inf}(\psi) = -\frac{16W_0 a_n A_n}{3\mathcal{V}^2} \left(\frac{3\mathcal{V}}{4\alpha\lambda_n} \right)^{2/3} \psi^{1/3} \exp \left(-a_n \left(\frac{3\mathcal{V}}{4\alpha\lambda_n} \right)^{2/3} \psi^{4/3} \right) \times \left(1 - \psi^{4/3} a_n \left(\frac{3\mathcal{V}}{4\alpha\lambda_n} \right)^{2/3} \right) \tag{4.30}$$

$$V''_{inf}(\psi) = \frac{16W_0 a_n A_n}{9\mathcal{V}^2} \left(\frac{3\mathcal{V}}{4\alpha\lambda_n} \right) e^{-a_n \tau_n} (9a_n \tau_n^{1/2} - 4a_n^2 \tau_n^{3/2} - \tau_n^{-1/2}) \tag{4.31}$$

Substituting these expressions into (1.62) and (1.63), we find

$$\epsilon \simeq \frac{32\mathcal{V}^3 a_n^2 A_n^2 \sqrt{\tau_n} (1 - a_n \tau_n)^2 e^{-2a_n \tau_n}}{3B^2 \alpha \lambda_n W_0^2} \tag{4.32}$$

$$\eta \simeq -\frac{4a_n A_n \mathcal{V}^2}{3\alpha\lambda_n \sqrt{\tau_n} B W_0} (1 - 9a_n \tau_n + (2a_n \tau_n)^2) e^{-a_n \tau_n} \tag{4.33}$$

The number of e-foldings is

$$N_e = \int_{\psi_{end}}^{\psi} \frac{1}{\sqrt{2\epsilon}} d\psi = -\frac{3BW_0 \lambda_n \alpha}{16\mathcal{V}^2 a_n A_n} \int_{\tau_n^{end}}^{\tau_n} \frac{e^{a_n A_n}}{\sqrt{\tau_n}(1 - a_n \tau_n)} d\tau_n \tag{4.34}$$

where τ_n^{end} is the point in field space where $\epsilon \sim \eta \sim 1$. In order to obtain a large enough number of e-foldings, at the beginning we must have large values of τ_n so that $\mathcal{V}^2 e^{-2a_n \tau_n} \ll 1$.

In the limit of slow roll we have

$$n_s - 1 = 2\eta - 6\epsilon \qquad\qquad r = 16\epsilon \tag{4.35}$$

and, since the slow roll parameters at the time of Horizon exit are related to the number of e-foldings, we can express n_s and r in terms of N_e. By expressing primordial perturbations in terms of the empirical parameters such as the scalar tilt n_s and the tensor to scalar ratio r, we are able to constrain models of inflation. In particular the most likely values of the empirical parameters are computed as a function of model parameters. They are determined by evolve the initial fluctuations and comparing them with the observed CMB fluctuations. In this way, we have a reliable way to constrain all the model dependent microscopic parameters of the theory. The COBE normalization for the density fluctuations $\delta_H = 1.92 \times 10^{-5}$ requires

$$\left(\frac{g_s^4}{8\pi}\right)\frac{3\lambda\beta^3 W_0^2}{64\sqrt{\tau_n}(1-a_n\tau_n)^2}\left(\frac{W_0}{a_n A_n}\right)^2\frac{e^{2a_n\tau_n}}{\mathcal{V}^6} = 2.7\times 10^{-7} \tag{4.36}$$

where we have included a factor of $\frac{g_s^4}{8\pi}$ as an overall normalization in V. It turns out that the internal volume must take the values $10^5 < \mathcal{V} < 10^7$.

To conclude the discussion on Kähler inflation it is worth noting that in general the displacement of the inflaton from its global minimum affects the potential experienced by the other moduli fields [10]. Thus, in general, the minimum values for the moduli during inflation differs from the minimum during the post-inflationary epoch and light moduli at the end of inflation are typically displaced from their post inflationary minimum. When the Hubble scale becomes smaller than the post inflationary mass of the moduli the energy density of the Universe is dominated by their coherent oscillations. This vacuum misalignment is useful to constrain the masses of moduli fields, requiring that the reheating temperature is suitable for the successive Big Bang Nucleosinthesis. On the other hand it can affects the number of e-foldings which is dependent on the post inflationary history of the Universe. In the case of Kähler inflation we are interested on the misalignment of the volume modulus during inflation. By determining the difference between its inflationary and post inflationary minima we obtain the post inflationary history of the Universe and how it affects the number of e-foldings. After the end of inflation we have coherent oscillations of the inflaton and the volume modulus. The energy is dominated by matter until the inflaton decay. Inflaton quanta behaves as radiation while the energy associated to the volume modulus behaves as matter. Since, as we have seen in the first chapter, radiation energy dilutes faster than matter energy, at the end we have a second phase of matter dominance that ends when also the volume modulus decays. Once the precise post inflationary history is understood it is possible to compute the precise preferred range of number of e-foldings which turns out to be smaller than the corresponding number in the standard cosmological timeline. The reduction of the number of e-foldings between the Horizon exit of the pivot mode and the end of inflation, due to the presence of cold particles of the volume modulus has a significant effect on the inflationary predictions.

4.3 Fibre Inflation

Fibre Inflation is a large field inflationary model in which the inflaton is the size of a $K3$ fibre over a $\mathbb{C}P^1$ base. These models are realised for $K3$ fibrered Calabi-Yau three-folds with almost three Kähler moduli. In my case i have used exactly three Kähler moduli T_i, $i = 1, 2, 3$. The real part of T_1 parameterises the volume of a K_3 or T^4 divisor D_1 fibred over a P^1 base and will play the role of the inflaton. This field is stabilised at subleading order due to string loop corrections to K. The real part of T_2, instead, parametrizes the volume of the base D_2 of the fibration and it controls the overall volume $\mathcal{V}$ which is stabilised at leading order via α' corrections. The corresponding axions of these two moduli, as we have already said, come from the reduction of the ten-dimensional bulk form C_4 over D_1 and D_2 respectively and they are fixed small through non-perturbative corrections to the superpotential. The real part of T_3 finally is a blow up mode which is fixed small by non perturbative corrections and it is necessary for the full stabilisation of the volume at leading order. By expanding the Kähler form in the basis of dual forms we find

$$\mathcal{V} = \frac{1}{6}(3k_{122}t_1t_2^2 - k_{333}t_3^3) \tag{4.37}$$

where t_1 is the volume of the P^1 base and t_2^2 is the size of the K^3 or T^4 fibre. Using the relation (3.32) we have

$$\tau_1 = \frac{1}{2}k_{122}t_2^2 \qquad \tau_2 = k_{122}t_1t_2 \qquad \tau_3 = \frac{1}{2}k_{333}t_3^2 \tag{4.38}$$

and finally

$$\mathcal{V} = \alpha(\sqrt{\tau_1}\tau_2 - \gamma\tau_3^{3/2}) \tag{4.39}$$

with $\alpha = 1/\sqrt{2k_{122}}$ and $\gamma = 2\sqrt{k_{122}}/(3\sqrt{k_{333}})$.

The tree level Kähler potential with the leading α' and string loop perturbative corrections reads

$$K = K_{tree} + K_{\alpha'} + K_{g_s} = -2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2g_s^{3/2}}\right) + K_{g_s} \tag{4.40}$$

Considering only the α' contribution in the previous expression and including non-perturbative corrections to the superpotential

$$W = W_0 + \sum_{i=1}^{3} A_k e^{-a_kT_k} \simeq W_0 + A_3e^{-a_3T_3} \tag{4.41}$$

we have

$$\frac{\partial K}{\partial \tau_1} = \left(-\frac{2}{\mathcal{V}} + \frac{\xi}{\mathcal{V}^2}\right)\frac{\alpha\tau_2}{2\sqrt{\tau_1}} \qquad \frac{\partial^2 K}{\partial \tau_1^2} = -\frac{\alpha\tau_2}{4\tau_1^{3/2}}\left(-\frac{2}{\mathcal{V}} + \frac{\xi}{\mathcal{V}^2}\right) + \frac{\alpha^2\tau_2^2}{4\tau_1}\left(\frac{2}{\mathcal{V}^2} - 2\frac{\xi}{\mathcal{V}^3}\right)$$

$$\frac{\partial K}{\partial \tau_2} = \left(-\frac{2}{\mathcal{V}} + \frac{\xi}{\mathcal{V}^2}\right)\alpha\sqrt{\tau_1} \qquad \frac{\partial^2 K}{\partial \tau_2^2} = \alpha^2\tau_1\left(\frac{2}{\mathcal{V}^2} - 2\frac{\xi}{\mathcal{V}^3}\right)$$

$$\frac{\partial K}{\partial \tau_3} = -\left(-\frac{2}{\mathcal{V}} + \frac{\xi}{\mathcal{V}^2}\right)\frac{3}{2}\alpha\gamma\sqrt{\tau_3} \qquad \frac{\partial^2 K}{\partial \tau_3^2} = \frac{9}{4}\alpha^2\gamma^2\tau_3\left(\frac{2}{\mathcal{V}^2} - 2\frac{\xi}{\mathcal{V}^3}\right) - \frac{3}{4}\frac{\alpha\gamma}{\sqrt{\tau_3}}\left(-\frac{2}{\mathcal{V}} + \frac{\xi}{\mathcal{V}^2}\right)$$

$$\frac{\partial^2 K}{\partial \tau_1 \partial \tau_3} = -\frac{3\alpha^2\gamma\sqrt{\tau_3}\tau_2}{4\sqrt{\tau_1}}\left(\frac{2}{\mathcal{V}^2} - 2\frac{\xi}{\mathcal{V}^3}\right) \qquad \frac{\partial^2 K}{\partial \tau_2 \partial \tau_3} = -\frac{3}{2}\alpha^2\gamma\sqrt{\tau_1}\sqrt{\tau_3}\left(\frac{2}{\mathcal{V}^2} - 2\frac{\xi}{\mathcal{V}^3}\right)$$

$$\frac{\partial K}{\partial \tau_1 \partial \tau_2} = \frac{\alpha}{2\sqrt{\tau_1}}\left(-\frac{2}{\mathcal{V}} + \frac{\xi}{\mathcal{V}^2}\right) + \frac{\alpha^2\tau_2}{2}\left(\frac{2}{\mathcal{V}^2} - 2\frac{\xi}{\mathcal{V}^3}\right) \tag{4.42}$$

These expressions can be approximated neglecting α' corrections. If we write the volume as $\mathcal{V} \sim \alpha\sqrt{\tau_1}\tau_2$ the Kähler metric at tree level it turns out to be

$$K^0_{i\bar{j}} = \frac{1}{4\tau_2^2}\begin{bmatrix} \frac{\tau_2^2}{\tau_1^2} & \gamma(\frac{\tau_3}{\tau_1})^{3/2} & -\frac{3\gamma}{2}\frac{\sqrt{\tau_3}\tau_2}{\tau_1^{3/2}} \\ \gamma(\frac{\tau_3}{\tau_1})^{3/2} & 2 & -3\gamma\frac{\sqrt{\tau_3}}{\sqrt{\tau_1}} \\ -\frac{3\gamma}{2}\frac{\sqrt{\tau_3}\tau_2}{\tau_1^{3/2}} & -3\gamma\frac{\sqrt{\tau_3}}{\sqrt{\tau_1}} & \frac{3\alpha\gamma}{2\mathcal{V}}\frac{\tau_2^2}{\sqrt{\tau_3}} \end{bmatrix} \tag{4.43}$$

and its inverse:
$$K_0^{\bar{i}j} = 4\begin{bmatrix} \tau_1^2 & \gamma\sqrt{\tau_1}\tau_3^{3/2} & \tau_1\tau_3 \\ \gamma\sqrt{\tau_1}\tau_3^{3/2} & \frac{1}{2}\tau_2^2 & \tau_2\tau_3 \\ \tau_1\tau_3 & \tau_2\tau_3 & \frac{2\mathcal{V}}{3\alpha\gamma}\sqrt{\tau_3} \end{bmatrix} \tag{4.44}$$

In the light of $K_{i\bar{j}} = \frac{1}{4}\frac{\partial K}{\partial\tau_i\partial\tau_j}$ the scalar potential, without string loop corrections, becomes

$$V = \frac{8a_3^2A_3^2\sqrt{\tau_3}}{3\alpha\gamma\mathcal{V}}e^{-2a_3\tau_3} + \frac{4W_0a_3A_3\tau_3\cos(a_3\theta_3)}{\mathcal{V}^2}e^{-a_3\tau_3} + \frac{3\xi W_0^2}{4\mathcal{V}^3} \tag{4.45}$$

and after minimizing with respect to the axionic partner θ_3

$$V = \frac{8a_3^2A_3^2\sqrt{\tau_3}}{3\alpha\gamma\mathcal{V}}e^{-2a_3\tau_3} - \frac{4W_0a_3A_3\tau_3}{\mathcal{V}^2}e^{-a_3\tau_3} + \frac{3\xi W_0^2}{4\mathcal{V}^3} \tag{4.46}$$

In the (4.41) we have chosen τ_2 and τ_3 big enough to remove the dependence of the superpotential on these two moduli. We must also add the uplifting term at the scalar potential in order to find the correct minimum, with a zero or a tiny positive value, produced by the tension of an $\bar{D}3$-brane in a warped region of the extra dimensions. This term will not depend on τ_1 once $\mathcal{V}$ is fixed. As a consequence, the scalar potential at this level depends only on the overall volume $\mathcal{V}$ and on the mode τ_3 which it results completely stabilised. In fact we can rewrite for simplicity

$$V = \frac{\lambda\sqrt{\tau_3}}{\mathcal{V}}e^{-2a_3\tau_3} - \frac{\mu\tau_3e^{-a_3\tau_3}}{\mathcal{V}^2} + \frac{\nu}{\mathcal{V}^3} \tag{4.47}$$

with

$$\lambda = \frac{8a_3^2A_3^2}{3\alpha\gamma} \qquad \mu = 4W_0A_3a_3 \qquad \nu = \frac{3\xi}{4}W_0^2 \tag{4.48}$$

$$\frac{\partial V}{\partial \tau_3} = 0 \quad \Rightarrow \quad \frac{\lambda \mathcal{V}}{\sqrt{\tau_3}} e^{-a_3\tau_3}\left(\frac{1}{2} - 2a_3\tau_3\right) = \mu(1 - a_3\tau_3) \tag{4.49}$$

$$\frac{\partial V}{\partial \mathcal{V}} = 0 \quad \Rightarrow \quad -\lambda\sqrt{\tau_3}e^{-2a_3\tau_3}\mathcal{V}^2 + 2\mu\tau_3\mathcal{V}e^{-a_3\tau_3} - 3\nu = 0 \tag{4.50}$$

From the second equation we find

$$\mathcal{V} = \frac{\mu\sqrt{\tau_3}e^{a_3\tau_3}}{\lambda}\left(1 + \sqrt{1 - \frac{3\nu\lambda}{\mu^2\tau_3^{3/2}}}\right) \tag{4.51}$$

and substituting in (4.49) in the the approximation $a_3\tau_3 \gg 1$, we get

$$\langle\tau_3\rangle = \left(\frac{\hat{\xi}}{2\alpha\gamma}\right)^{2/3} \qquad \langle\mathcal{V}\rangle = \left(\frac{3\alpha\gamma}{4a_3A^3}\right)W_0\sqrt{<\tau_3>}e^{a_3<\tau_3>} \tag{4.52}$$

There is a combination of τ_1 and τ_2 that has not yet been stabilised, that is, a direction in the moduli space along which the potential is still completely flat. This combination therefore will gives rise to the inflaton field.

String loop corrections give the necessary contribution in order to stabilise τ_1 and τ_2 at large values. They are the result of two contributions, i.e. the exchange between D_3 and D_7-branes of closed strings, which carry Kaluza-Klein momentum $\delta K^{KK}_{(g_s)}$ and the exchange of winding strings, between intersecting stacks of D_7 branes $\delta K^{W}_{(g_s)}$.

$$\delta K^{KK}_{(g_s)} = -\frac{1}{128\pi^4}\sum_{i=1}^{3}\frac{\mathcal{E}^{KK}_i(U,\bar{U})}{Re(S)\tau_i} \qquad \delta K^{W}_{(g_s)} = -\frac{1}{128\pi^4}\sum_{i=1}^{3}\frac{\mathcal{E}^{W}_i(U,\bar{U})}{\tau_j\tau_k}\bigg|_{i\neq j\neq k} \tag{4.53}$$

In the first correction we assume that all the three 4-cycles of the torus are wrapped by D_7-branes and τ_i denotes the volume of the 4-cycle wrapped by the i-th $D7$-brane while in the second expression τ_i and τ_j denote the volume of the 4-cycle wrapped by the i-th and the j-th intersecting $D7$-branes. These expressions are significantly simplified since they contain a very complicated dependence on the complex structure moduli which however are fixed by fluxes at tree level and a very simple dependence on T moduli. The expressions (4.53) have been calculated in the case of $\mathcal{N} = 1$ compactifications on $T^6/(\mathbb{Z}\times\mathbb{Z})$. On general Calabi-Yau manifolds the corrections are generalized as

$$\delta K^{KK}_{(g_s)} \sim \sum_{i=1}^{h^{(1,1)}}\frac{C^{KK}_i(U,\bar{U})}{Re(S)\mathcal{V}}(a_{il}t^l) \tag{4.54}$$

where $a_{il}t^l$ is a linear combination of the basis of 2-cycle volume t^l that it is transverse to the 4-cycle wrapped by the i-th D_7 brane, while

$$\delta K^{W}_{(g_s)} \sim \sum_{i}\frac{C^{W}_i(U,\bar{U})}{(a_{il}t^l)\mathcal{V}} \tag{4.55}$$

where $a_{il}t^l$ now is the 2-cycle where the two D_7-branes intersect. The two functions $C_i^{KK}(U,\bar{U})$, $C_i^W(U,\bar{U})$ are unknown, but fortunately are simply constants once the complex structure moduli have fixed. In terms of the tree level Kähler metric these corrections give a contribution to the scalar potential of the form

$$\delta V_{(g_s)}^{1-loop} = \left(\frac{(C_i^{KK})^2}{Re(S)^2} a_{ik} a_{ij} K^0_{k\bar{j}} - 2 \sum_i \delta K^W_{(g_s),t_i} \right) \frac{W_0^2}{\mathcal{V}^2} \tag{4.56}$$

In our case branes are wrapped only around the basis 4-cycles and the contribution (4.56) is simplified because $a_{ik}a_{ij}K^0_{kj} = K_{i\bar{i}}$.
The KK string loop correction associated to a stack of D_7-branes that warp τ_3 does not depend on the combination of fields which represent the inflaton and consequently it is not important for the general inflationary characteristics. Moreover from the precise form of the volume (4.39), it is clear that the blow up mode τ_3 does not intersect any other cycle and so we can neglect also the winding string corrections for it. The relevant contributions, therefore, are related to τ_1 and τ_2:

$$\delta V_{(g_s)} = \delta V^{KK}_{(g_s),\tau_1} + \delta V^{KK}_{(g_s),\tau_2} + V^W_{(g_s),\tau_1\tau_2} \tag{4.57}$$

$$\delta V^{KK}_{(g_s),\tau_1} = g_s^2 \frac{(C_1^{KK})^2}{\tau_1^2} \frac{W_0^2}{\mathcal{V}^2} \qquad \delta V^{KK}_{(g_s),\tau_2} = 2g_s^2 \frac{(C_2^{KK})^2}{\tau_2^2} \frac{W_0^2}{\mathcal{V}^2} \qquad V^W_{(g_s),\tau_1\tau_2} = -\left(\frac{2C^W_{12}}{t_{int}} \right) \frac{W_0^2}{\mathcal{V}^3}$$

where t_{int} is the 2-cycle denoting the intersection locus of the 4-cycles τ_1 and τ_2.

$$\frac{\partial \mathcal{V}}{\partial t_1} = \frac{1}{2} k_{122} t_2^2 = \left(\frac{1}{2} k_{122} t_2 \right) t_2 \qquad \frac{\partial \mathcal{V}}{\partial t_2} = k_{122} t_1 t_2 = \left(\frac{1}{2} k_{122} t_2 \right) 2t_1$$

$$t_{int} = \left(\frac{1}{2} k_{122} t_2 \right) = \sqrt{(\frac{1}{2} k_{122} t_2)\tau_1} \equiv \sqrt{\lambda_1 \tau_1} \tag{4.58}$$

The string loop correction (4.57), eventually assumes the form

$$\delta V_{(g_s)} = \left(\frac{A}{\tau_1^2} - \frac{B}{\mathcal{V}\sqrt{\tau_1}} + \frac{C\tau_1}{\mathcal{V}^2} \right) \frac{W_0^2}{\mathcal{V}^2} \tag{4.59}$$

where $A = (g_s C_1^{KK})^2$, $B = 2C^W_{12}/\sqrt{\lambda} = 4\alpha C^W_{12}$, since as we have seen $\alpha = 1/\sqrt{2k_{122}}$ and $C = 2(\alpha g_s C_2^{KK})^2$. It is clear that A and B are positive while the sign of B is undetermined. The form of the corrections suggest us to use only τ_1 as the parameter along which the potential is flat and therefore it is represent the inflaton field. Inflation will correspond to an initial situation, with the $K3$ fibre much larger than the base and a final situation with the base larger than the K3 fibre. The inflaton field can be stabilised through the string loop corrections

$$\frac{d(\delta V_{(g_s)})}{d\tau_1} = 0 \Rightarrow \left(-2A\tau_1^{-3} + \frac{B}{2\mathcal{V}} \tau_1^{-3/2} + \frac{C}{\mathcal{V}^2} \right) \frac{W_0^2}{\mathcal{V}^2} = 0 \tag{4.60}$$

We solve the equation

$$-2Ax^2+\frac{B}{2\mathcal{V}}x+\frac{C}{\mathcal{V}^2}=0 \quad \text{for} \quad x=\frac{1}{\tau_1^{3/2}}$$

and we find

$$\frac{1}{\tau_1^{3/2}}=\frac{B}{8A\mathcal{V}}\left(1+\frac{B}{|B|}\sqrt{1+\frac{32AC}{B^2}}\right) \tag{4.61}$$

For $B>0$:

$$\frac{1}{\tau_1^{3/2}}=\frac{B}{4A\mathcal{V}}\left(\frac{16AC}{B^2}+2\right)\sim\frac{B}{4A\mathcal{V}}$$

For $B<0$:

$$\frac{1}{\tau_1^{3/2}}=-\frac{B}{8A\mathcal{V}}\frac{16AC}{B^2}=-\frac{2C}{B\mathcal{V}}$$

$$\tau_1\simeq\left(-\frac{B\mathcal{V}}{2C}\right)^{2/3} \text{ for } B<0 \qquad\qquad \tau_1\simeq\left(\frac{4A\mathcal{V}}{B}\right)^{2/3} \text{ for } B>0 \tag{4.62}$$

The kinetic term of the Lagrangian, at leading order, is

$$-\mathcal{L}_{kin}=K^0_{i\bar{j}}(\partial_\mu T_i\partial^\mu \bar{T}_{\bar{j}})=\frac{\partial^2 K}{4\partial\tau_i\partial\tau_j}(\partial_\mu\tau_i+i\partial_\mu\theta_i)(\partial^\mu\tau_i-i\partial^\mu\theta_j)$$

$$=\frac{\partial^2 K}{4\partial\tau_i\partial\tau_j}(\partial_\mu\tau_i\partial^\mu\tau_j+\partial_\mu\theta_i\partial^\mu\theta^j)=\frac{1}{4\tau_1^2}\partial_\mu\tau_1\partial^\mu\tau_1+\frac{1}{2\tau_2^2}\partial_\mu\tau_2\partial^\mu\tau_2+\cdots \tag{4.63}$$

By writing $\tau_2=2\mathcal{V}\left(\frac{\lambda_1}{\tau_1}\right)^{1/2}$ with $\lambda_1=1$ and so $\alpha^2=1/4\lambda_1^2=1/4$, we have

$$-\mathcal{L}_{kin}=\frac{1}{4\tau_1^2}\partial_\mu\tau_1\partial^\mu\tau_1+\frac{2}{\tau_1\tau_2^2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V}-\frac{2\mathcal{V}}{\tau_1^2\tau_2^2}\partial_\mu\mathcal{V}\partial^\mu\tau_1+\frac{\mathcal{V}^2}{2\tau_2^2\tau_1^3}\partial_\mu\tau_1\partial^\mu\tau_1$$

$$-\mathcal{L}_{kin}=\frac{3}{8\tau_1^2}\partial_\mu\tau_1\partial^\mu\tau_1+\frac{1}{2\mathcal{V}^2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V}-\frac{1}{2\mathcal{V}\tau_1}\partial_\mu\mathcal{V}\partial^\mu\tau_1+\cdots \tag{4.64}$$

where we have used the $\mathcal{V}\sim\alpha\sqrt{\tau_1}\tau_2$.
From (4.64) it is clear that the fields τ_1 and $\mathcal{V}$ are not canonically normalised. In first place we can rewrite the expression as

$$-\mathcal{L}_{kin}=\frac{3}{8}\left(\frac{1}{\tau_1}\partial_\mu\tau_1\right)\left(\frac{1}{\tau_1}\partial^\mu\tau_1\right)+\frac{1}{2}\left(\frac{1}{\mathcal{V}}\partial_\mu\mathcal{V}\right)\left(\frac{1}{\mathcal{V}}\partial^\mu\mathcal{V}\right)-\frac{1}{2}\left(\frac{1}{\mathcal{V}}\partial_\mu\mathcal{V}\right)\left(\frac{1}{\tau_1}\partial^\mu\tau_1\right)$$

$$=\frac{3}{8}\partial_\mu\chi_1\partial^\mu\chi_1+\frac{1}{2}\partial_\mu\chi_v\partial^\mu\chi_v-\frac{1}{2}\partial_\mu\chi_v\partial^\mu\chi_1 \tag{4.65}$$

where we have substitute $\chi_v=\ln(\mathcal{V})$ and $\chi_1=\ln(\tau_1)$. At this point it is easy to canonically normalised the two fields by

$$\begin{pmatrix}\partial_\mu\chi_1\\ \partial_\mu\chi_v\end{pmatrix}=\mathcal{M}\begin{pmatrix}\partial_\mu\phi_1\\ \partial_\mu\phi_v\end{pmatrix} \tag{4.66}$$

with $\mathcal{M}$ satisfying the

$$\mathcal{M}^T \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \mathcal{M} = 1_{2\times 2} \tag{4.67}$$

to obtain the kinetic Lagrangian in terms of the mass eigenstates $-\mathcal{L}_{kin} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2$. In order to study the possibility of having inflation we must displace one of the fields from its minimum similarly to what we did in the case of Kähler. Since the potential for τ_1 is systematically flat in the absence of string loop corrections we expect this very field to be the best candidate for representing the inflaton. The inflationary Lagrangian with all the other field fixed at their minima will be

$$\mathcal{L}_{inf} = -\frac{3}{8}\left(\frac{\partial_\mu\tau_1\partial^\mu\tau_1}{\tau_1^2}\right) - V_{inf}(\tau_1) \tag{4.68}$$

and the inflationary potential

$$V_{inf}(\tau_1) = V_0 + \left(\frac{A}{\tau_1^2} - \frac{B}{\mathcal{V}\sqrt{\tau_1}} + \frac{C\tau_1}{\mathcal{V}^2}\right)\frac{W_0^2}{\mathcal{V}^2} \tag{4.69}$$

where V_0 comes from the (4.46) with τ_3 replaced by its minimum value $\langle\tau_3\rangle$. We obtain therefore a single field inflation model where $\mathcal{V}$ and τ_3 remain fixed while the inflaton τ_1 rolls back to its minimum. In terms of the canonically normalised field

$$\phi = \frac{\sqrt{3}}{2}\ln(\tau_1) \equiv \frac{1}{k}\ln(\tau_1) \tag{4.70}$$

the potential (4.69) becomes

$$\begin{aligned} V_{inf}(\phi) &= V_0 + \frac{W_0^2}{\mathcal{V}^2}\left(Ae^{-2k\phi} - \frac{B}{\mathcal{V}}e^{\frac{-k\phi}{2}} + \frac{C}{\mathcal{V}^2}e^{k\phi}\right) \\ &= \frac{1}{\mathcal{V}^{10/3}}(C_0 e^{k\hat{\phi}} - C_1 e^{-\frac{k\hat{\phi}}{2}} + C_2 e^{-2k\hat{\phi}} + C_{up}) \end{aligned} \tag{4.71}$$

where we have shifted $\phi = \langle\phi\rangle + \hat{\phi}$ by its vacuum and we have adjusted $V_0 = C_{up}/\langle\mathcal{V}\rangle^{10/3}$ to ensure that $V_{inf}(\langle\phi\rangle) = 0$.

In the case of $A, C \ll B$ and $B > 0$, the (4.71) is approximated by

$$V_{inf} \simeq \frac{C_2}{\langle\mathcal{V}\rangle^{3/2}}\left(3 - 4e^{\frac{-k\hat{\phi}}{2}} + e^{-2k\hat{\phi}} + Re^{k\hat{\phi}}\right) \tag{4.72}$$

with $C_2 = W_0^2\left(\frac{B^4}{256 g_s^2 A}\right)^{1/3}$ and $R = 4g_s^4\frac{AC}{B}$.

$$\frac{\partial V_{inf}}{\partial\hat{\phi}} = \frac{C_2}{\langle\mathcal{V}\rangle^{3/2}}\,2k\left(e^{-\frac{k\hat{\phi}}{2}} - e^{-2k\hat{\phi}} + \frac{1}{2}Re^{k\hat{\phi}}\right) \tag{4.73}$$

It follows that

$$\epsilon = \frac{1}{2}4k^2\left(\frac{e^{-\frac{k\hat{\phi}}{2}} - e^{-2k\hat{\phi}} + \frac{1}{2}Re^{k\hat{\phi}}}{3 - 4e^{-\frac{k\hat{\phi}}{2}} + e^{-2k\hat{\phi}} + Re^{k\hat{\phi}}}\right)^2 = \frac{8}{3}\left(\frac{e^{-\frac{k\hat{\phi}}{2}} - e^{-2k\hat{\phi}} + \frac{1}{2}Re^{k\hat{\phi}}}{3 - 4e^{-\frac{k\hat{\phi}}{2}} + e^{-2k\hat{\phi}} + Re^{k\hat{\phi}}}\right)^2 \tag{4.74}$$

and

$$\eta = -\frac{4}{3}\,\frac{e^{-\frac{k\hat{\phi}}{2}} - 4e^{-2k\hat{\phi}} - Re^{k\hat{\phi}}}{3 - 4e^{-\frac{k\hat{\phi}}{2}} + e^{-2k\hat{\phi}} + Re^{k\hat{\phi}}} \tag{4.75}$$

In the slow roll regime $R^{1/3} \ll e^{-\frac{\hat{\phi}}{2}} \ll 1$ the potential is

$$V_{inf}(\hat{\phi}) \simeq \frac{C_2}{\langle\mathcal{V}\rangle^{10/3}}(3 - 4e^{\frac{-k\hat{\phi}}{2}}) \tag{4.76}$$

and the slow roll parameters become

$$\epsilon \simeq \frac{8}{3}\,\frac{1}{\left(3e^{\frac{k\hat{\phi}}{2}} - 4\right)^2} \qquad \eta \simeq -\frac{4}{3}\frac{1}{\left(3e^{\frac{k\hat{\phi}}{2}} - 4\right)} \qquad \epsilon \simeq \frac{3\eta^2}{2} \tag{4.77}$$

$$N_e = \int_{\hat{\phi}_{end}}^{\hat{\phi}} \frac{1}{\sqrt{2\epsilon}} = \int_{\hat{\phi}_{end}}^{\hat{\phi}} \frac{\sqrt{3}}{4}\,(3e^{\frac{k\hat{\phi}}{2}} - 4) = \left(\frac{9}{4}e^{\frac{k\hat{\phi}}{2}} - \sqrt{3}\hat{\phi}\right)\Bigg|_{\hat{\phi}_{end}}^{\hat{\phi}} \tag{4.78}$$

Slow roll conditions break down once $\hat{\phi}$ is small enough. We have the end of inflation around the point $\hat{\phi}_{end} \sim 1$. For bigger values of $\hat{\phi}$ we have the slow roll regime, where the potential presents the necessary for inflation, plateau. The number of e-foldings depends mostly on the initial value of $\hat{\phi}$ while it is independent from the precise value of the inflaton at the end of inflation. It is therefore model dependent and so sensible on the parameters of the underlying supergravity theory. It depends also on R and in fact the request of having almost 60 e-foldings constraints $R \leq 5 \times 10^{-5}$.
The slow roll parameters, instead, are independent on the precise normalization of the potential. Using the $n_s = 1 + 2\eta_\star - 6\epsilon_\star \sim 1 + 2\eta_\star$, $r = 16\epsilon_\star$ and the (4.77) we get

$$r \simeq 6(n_s - 1)^2 \tag{4.79}$$

As we have done also for Kähler inflation in order to reproduce the COBE normalization for primordial scalar perturbations $\delta_H = 1.92 \times 10^{-5}$ we have to impose

$$\left(\frac{g_s e^{K_{g_s}}}{8\pi}\right)\left(\frac{V^{3/2}}{V'}\right) \simeq 2.7 \times 10^{-7} \tag{4.80}$$

where the prefactor $g_s e^{K_{g_s}}/8\pi$ is the correct overall normalization of the potential.

4.4 General properties of inflationary string models

We conclude this chapter with a discussion on general properties emerging in the contest of Type IIB string inflationary models such as those described in the previous sections. In slow roll inflation the parameters ϵ and η must be sufficiently small during all the inflationary phase, i.e. while the inflaton rolls back to its minimum. However the η parameter is related to the mass of the inflaton which is a scalar and therefore features the general problem due to the absence of symmetries that protect scalar masses against any kind of quantum corrections.
More precisely

$$\eta = \frac{V''(\phi)}{V(\phi)} M_{pl}^2 \sim \frac{m_\phi^2}{H^2 M_{pl}^2} M_{pl}^2 \sim \frac{m_\phi^2}{H^2} \ll 1$$

To avoid fine tuning, we solve the η problem by giving a mass to the inflaton only via α' and non perturbative corrections. This means that at tree level all the τ-directions are flat. This feature is expressed in the no scale structure of the Kähler potential $K_0^{i\bar{j}} K_{0i} K_{0\bar{j}} = 3$.

A very important moment in the context of inflation is reheating, i.e. when the inflaton decays into ordinary matter degrees of freedom. In our case, where the inflaton is a closed string mode and therefore it belongs to the hidden sector, it is highly non trivial to understand how the coupling with the visible sector takes place. In general there are two different ways to introduce the visible sector in the inflationary string models in the LVS: the first is based on the presence of a rigid 4-cycle which does not intersect other cycles and can shrink down at a singularity. In that case the visible sector is built via $D3$-branes at the singularity. In the second way the visible sector is built via magnetized intersecting $D7$-branes wrapping a blow up 4-cycle which is stabilised at a volume larger than the string scale. In both methods SUSY is broken due to non vanishing background fluxes by the $F-$term of the Kähler moduli. An important difference is that in the case of $D3$-branes at singularity there is no local SUSY breaking and the visible sector is sequestered. This feature implies that the main inflaton decay channels are just the axionic particles leading to an overproduction of relativistic degrees of freedom whose number is however constrain by observations. We are therefore forced to consider models where the visible sector lives on $D7$-branes wrapped around the inflaton $4-$cycle in order to maximize the inflaton branching ratio into SM fields. In the next chapter there is the explicit computation of the coupling functions with the gauge bosons for the two models of inflation that we have seen.

Finally, an essential characteristic to consider is related to the range of allowed inflaton values [8]. In the context of Calabi-Yau compactifications with background fluxes in the Large Volume Scenario it emerges that the inflaton field range could be bounded from above. Moduli fields belong to the moduli space described by the Kähler metric. The directions of the overall volume as well as these of the small blow up modes that resolve singularities are uplifted with the inclusion of leading α' and non perturbative effects. This stabilisation imposes the constraints $\mathcal{V} = \langle \mathcal{V} \rangle$ and $\tau_s = \langle \tau_s \rangle$, on the moduli space which is reduced in that of the remaining flat directions. The latter therefore is a subspace of the full Kähler moduli space and contains the unstabilised moduli fields that in general represent the inflaton. In the case of $h^{(1,1)} = 3$ the reduced space is one-dimensional and compact. More precisely it can be parameterized in terms of the canonically normalised field ϕ and its volume turns out to be limited from above

$$\Delta\phi \leq M_{pl} \ln(\mathcal{V}) \tag{4.81}$$

This implies, therefore, a constrain on the inflaton field range. For an intuitive way to understand this limitation, we can focus our attention on the case of $K3$ fibred geometries where $\mathcal{V} \sim \sqrt{\tau_1}\tau_2 - \gamma\tau_3^{3/2}$. In this case τ_1 and τ_2 can be seen pictorially as the two sides of the rectangle representing the volume. When one of the two gets smaller, it causes the corresponding growth of the other. If the volume is fixed and contains also a fixed small blow up cycle within it then the variations of τ_1 and τ_2 are constrain not only by the fact that the volume must remain constant but also are limited by the cycle inside the rectangle. In other worlds the mode that are shrinking can get smaller only until it meets the rigid fixed 4-cycle and then it stops causing the arrest in the growing of the other. The size of the reduced moduli space is dependent on the fixed overall volume and in a more moderate way on the fixed value of the small blow up mode. In general there is an increasing in the allowed range of inflaton values at larger volumes but this variation is significant only in the case of $K3$ fibred geometries. These are also the only cases in which we can obtain trans-Planckian size of the moduli space and inflation is even more UV sensitive. This fact makes the Fibre inflation models particularly suitable for studying primordial gravity waves. Kähler moduli inflation on the other hand, belong to small field models of inflation with sub-Planckian inflaton field range. The implications of these purely geometric considerations combined with the particular shape of the inflationary potential which determine the inflaton range must be compared with the range needed to obtain enough e-foldings of inflation in order to solve the flatness and the Horizon problem. In this thesis, as it is described in the next chapter, we have focused mainly on trying to overcome these limitations by slowing-down the inflaton with backreaction effects, in order not to invalidate the theory of inflation as a good candidate to explain primordial cosmology.

Chapter 5

Numerical analysis of backreaction from magnetogenesis

The aim of this chapter is the numerical study of the inflationary evolution in the presence of backreaction effects due to the coupling of the inflaton with the electromagnetic field. For this purpose we have to modify the standard electromagnetic Lagrangian term by introducing a coupling function dependent on time through the inflaton. During inflation the inflaton changes monotonously and therefore it is natural to assume that the coupling function is decreasing. This leads to a large electric energy density which could influence the standard inflationary evolution.
In this direction we have used the Kähler (KI) and Fibre (FI) models of Inflation. We have first studied the evolution of the inflaton in the absence of backreaction and after we have introduced the coupling function in order to compare the results obtained in the two different regimes.
The numerical analysis has pointed out that the coupling of the inflaton with the electromagnetic field can cause a slowdown of the inflationary dynamics. The result is the increase of the number of e-foldings which is very useful from the point of view of inflationary string models where the limits related on the allowed values of the inflaton could in principle invalidate their reliability.

5.1 The necessity of more e-foldings of Inflation

In general in the geometric moduli space of a compact Kähler manifold there is at least one universal flat direction given by the modulus describing the total volume of the internal space. In order to find phenomenologically viable vacua where the effective field theory approximation is holding we are interested in the moduli space of Kähler deformations in the context of Type IIB Calabi-Yau orientifolds with background fluxes in the Large Volume Scenario. In these models, as we have already said, leading α' and non-perturbative corrections can lift the overall volume modulus as well as any blow up mode that resolving point-like singularities reducing the total moduli space in that of the, at least one, flat direction representing the inflaton field. The reduced moduli space is described by the reduced Kähler metric and, if n_s is the number of the stabilised 4-cycles, it contains $(h^{(1,1)} - n_s - 1)$ 4-cycles flat directions and $(h^{(1,1)} - n_s)$ axionic flat directions. The axionic directions cannot give rise to a non-compact moduli space as they are periodic and so can be neglected in this discussion. In the special case of three Kähler moduli, $h^{(1,1)} = 3$, the reduced moduli space turns out to be unidimensional. In fact, it is obtained as the intersection between the two hypersurfaces inside the Kähler cone defined fixing the volume modulus and the small blow up mode at their minima

$$\mathcal{V}(\tau_i) = \langle \mathcal{V} \rangle \qquad \tau_s = \langle \tau_s \rangle \tag{5.1}$$

as it can be seen from the Figure (5.1).

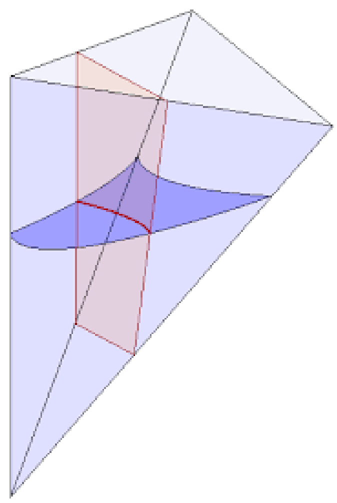


Figure 5.1: Unidimensional reduced moduli space in the Kähler cone defined by the intersection of the two hypersurfaces coming from the stabilisation of the volume and the blow up modes.

The crucial point is that the reduced moduli space is found to be compact [8]. In fact, if we parameterize it in terms of the canonically normalised inflaton field ϕ, its volume is bounded from above

$$\frac{\Delta\phi}{M_{pl}} \leq c \ln(\mathcal{V}) \tag{5.2}$$

This limit in the range of the allowed values of the inflaton, puts a restriction on the number of e-foldings during inflation. In order, however, to solve the Horizon and flatness problem, we need at least 50-60 e-foldings of inflation and consequently this upper bound sets strong constraints on inflationary models and it could in principle invalidate the success of inflation in describing primordial cosmology. Moreover, a geometrical constrain on $\Delta\phi$ causes a theoretical upper bound in the observed tensor to scalar ratio. In order to have observable primordial tensor fluctuations in string constructions we need, therefore, a transplanckian range for the inflaton field $\Delta\phi \geq M_{pl}$, and this can be obtained only in large field inflationary models.

After these considerations, it becomes clear that we need a mechanism which is able to slowdown the inflaton allowing in this way to increase the number of e-foldings.

In this chapter we present the attempt to find such mechanism in the context of Kähler and Fibre inflation through the coupling of the inflaton to the electromagnetic field.

5.2 System of equations

We consider a spatially flat Friedmann-Robertson-Walker Universe with metric $g_{\mu\nu} = (1, -a^2, -a^2, -a^2)$ and $\sqrt{-g} = a^3$. The action for the inflaton coupled to the electromagnetic field is

$$S = \int d^4x\sqrt{-g}\mathcal{L}_0 = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{4}f^2(\phi)g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\right) \quad (5.3)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ the vector potential.
From the field equations for the scalar field

$$\partial_\alpha\frac{\delta\mathcal{L}}{\delta\partial_\alpha\phi} - \frac{\delta\mathcal{L}}{\delta\phi} = 0$$

we have

$$\partial_\alpha(\sqrt{-g}g^{\alpha\nu}\,\partial_\nu\phi) + \frac{dV(\phi)}{d\phi}\sqrt{-g} + \frac{1}{2}f(\phi)f'(\phi)g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\sqrt{-g} = 0$$

and so

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\nu}\,\partial_\nu\phi) + \frac{dV(\phi)}{d\phi} = -\frac{1}{2}f(\phi)f'(\phi)F_{\mu\nu}F^{\mu\nu} \quad (5.4)$$

When it is explicitly reported the argument of a function the prime represent the derivative with respect to that argument.
In equation (5.4) the left hand side is the ordinary equation for the inflaton field while the new term appearing in the right hand side is responsible for the backreaction effect.
From the field equations for the electromagnetic field

$$\partial_\alpha\frac{\delta\mathcal{L}}{\delta\partial_\alpha A_\mu} - \frac{\delta\mathcal{L}}{\delta A_\mu} = 0$$

we have

$$\partial_\alpha(\sqrt{-g}\,f^2(\phi)\,g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}) = 0 \quad (5.5)$$

The stress energy tensor is

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{-g}\mathcal{L}_0\right) = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}}\mathcal{L}_0 + \frac{2}{\sqrt{-g}}\sqrt{-g}\frac{\delta\mathcal{L}_0}{\delta g^{\mu\nu}}$$

$$T_{\mu\nu} = -g_{\mu\nu}\mathcal{L}_0 + \partial_\mu\phi\partial_\nu\phi - f^2(\phi)g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} \quad (5.6)$$

where we have use the fact that

$$\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu} \quad (5.7)$$

If the coupling function $f(\phi)$ is decreasing in time then the electric component of the electromagnetic field dominates the magnetic one and so we can neglect it

$$F_{ij} = a^2(t)\epsilon_{ijk}B_k(t) \simeq 0 \qquad F_{0i} = a(t)E_i(t) \tag{5.8}$$

In this approximation we have

$$\rho = T_{00} = \partial_0\phi\partial_0\phi - f^2(\phi)g^{\nu\kappa}F_{0\nu}F_{0k} - \mathcal{L}_0 = \partial_0\phi\partial_0\phi - f^2(\phi)g^{ii}F_{0i}F_{0i} - \mathcal{L}_0$$

In addition we consider a spatially homogeneous inflaton field, so

$$\mathcal{L}_0 = \frac{1}{2}\partial_0\phi\partial_0\phi - V(\phi) - \frac{1}{2}f^2(\phi)g^{00}g^{ii}F_{0i}F_{0i}$$

and consequently

$$\rho = \frac{1}{2}\partial_0\phi\partial_0\phi + V(\phi) - \frac{1}{2}f^2(\phi)g^{ii}F_{0i}F_{0i} = \frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{1}{2}f^2(\phi)\left(-\frac{1}{a^2}\right)a^2E_iE_i$$

$$\rho = \underbrace{\frac{1}{2}\dot{\phi}^2 + V(\phi)}_{\rho_{inf}} + \underbrace{\frac{1}{2}f^2(\phi)E^2}_{\rho_E} \tag{5.9}$$

where we have denoted with ρ_{inf} the energy density of the inflaton field and with ρ_E the electric energy density.
From the Friedman equation for a flat Universe we have

$$H^2 = \frac{1}{3M_{pl}^2}(\rho_{inf} + \rho_E) = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_E\right) \tag{5.10}$$

The equation (5.4) becomes

$$\frac{1}{a^3}\partial_0(a^3\partial_0\phi) + \frac{dV(\phi)}{d\phi} = -\frac{1}{2}f(\phi)f'(\phi)2g^{00}g^{ii}(-E^2)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = f(\phi)f'(\phi)E^2 = 2\frac{f'(\phi)}{f(\phi)}\rho_E \tag{5.11}$$

and the (5.5) becomes

$$\partial_0(a^3f^2(\phi)\frac{1}{a^2}aE_i) = 0 \Rightarrow \partial_0(a^2f^2(\phi)E_i) = 0$$

$$2\dot{a}f^2(\phi)E_i + 2af(\phi)f'(\phi)E_i + af^2(\phi)\dot{E}_i = 0$$

By multiplying by E_i/a at the end we find

$$\dot{\rho}_E + 4H\rho_E + 2\rho_E\frac{\dot{f}(\phi)}{f(\phi)} = 0 \tag{5.12}$$

which gives the evolution of the electric energy density.
Inside the Horizon the modes of the electric energy density oscillate in time and have to be treated as quantum fluctuations. When they cross the Horizon, however, start to behave monotonically and become classical. These are the modes which contribute to the electric energy density. Since their wavenumber is larger than the Hubble scale the corresponding electric field can be considered spatially homogeneous. In the equation (5.12), the classical part of the electric energy density at a certain moment of time is determined by the modes which crossed the Horizon from the beginning of inflation till the moment under consideration

$$\rho_E = \int_{k_i}^{k_H(t)} \frac{d\rho_E}{dk} dk \tag{5.13}$$

where k_i is the moment of the mode which crosses the Horizon at the beginning of inflation and $k_H(t) = H(t)a(t)$. It is important to note that the number of relevant modes with wavelength larger than Hubble radius changes in time. In order to keep this fact in consideration in the equation (5.12) we must introduce an additional boundary term describing the mode which cross the Horizon at a given time t

$$(\dot{\rho}_E)_H = \left.\frac{d\rho_E}{dk}\right|_{k=k_H} \times \frac{dk_H}{dt} \tag{5.14}$$

In order to calculate $(\dot{\rho}_E)_H$ we need the electric power spectrum $\frac{d\rho_E}{dk}$ and so it is useful to study the mode decomposition of the electric field.
At this end it is convenient to use the conformal time in terms of which the metric becomes

$$dS^2 = a^2(\eta)(d\eta^2 - \delta_{ik}dx^i dx^k) \tag{5.15}$$

The action for the electric field in a curved background suitably modified in order to contain the coupling function, dependent on time through the inflaton, is

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}f^2(\eta)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^4x\sqrt{-g}f^2(\eta)g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}$$

$$S = \frac{1}{4}\int d^4x f^2(\eta)(2F_{0i}F_{0i} - F_{ij}F_{ij}) \tag{5.16}$$

We want to write the (5.16) in terms of the vector potential $A_\alpha = (A_0, A_i)$. It is convenient to decompose its spatial part in its transverse and longitudinal components as $A_i = A_i^T + \partial_i\chi$, with $\partial_i A_i^T = 0$.

$$F_{0i}F_{0i} = (\partial_0 A_i - \partial_i A_0)^2 = (\partial_0(A_i^T + \partial_i\chi) - \partial_i A_0)^2$$

$$= \partial_0 A_i^T \partial_0 A_i^T + \partial_0 A_i^T \partial_0\partial_i\chi + \partial_0\partial_i\chi\partial_0 A_i^T + \partial_0\partial_i\chi\partial_0\partial_i\chi - 2\partial_0 A_i^T \partial_i A_0$$

$$- 2\partial_0\partial_i\chi\partial_i A_0 + \partial_i A_0\partial_i A_0 = A_i'^T A_i'^T - \chi'\Delta\chi' + 2A_0\Delta\chi' - A_0\Delta A_0 \tag{5.17}$$

where we have reject the terms containing $\partial_i A_i^T = 0$. In this context the prime denotes derivative with respect to conformal time η.

$$F_{ij}F_{ij} = (\partial_i A_j - \partial_j A_i)^2 = 2\partial_i A_j^T \partial_i A_j^T - 2\partial_i A_j^T \partial_j A_i^T = -2A_j^T \Delta A_j^T \tag{5.18}$$

Using these expressions in the (5.16) we have

$$S = \frac{1}{2}\int d^4x f^2(\eta)(A_i'^T A_i'^T - \chi'\Delta\chi' + 2A_0\Delta\chi' - A_0\Delta A_0 + A_j^T \Delta A_j^T) \tag{5.19}$$

By varying the (5.19) with respect to χ' we find $\chi' = A_0$ and so the action becomes

$$S = \frac{1}{2}\int d^4x f^2(\eta)\,(A_i'^T A_i'^T + A_i^T \Delta A_i^T) \equiv \frac{1}{2}\int d\eta \int d\mathbf{x} f^2(\eta)\,(A_i'^T A_i'^T + A_i^T \Delta A_i^T) \tag{5.20}$$

At this point we expand the transverse component of the vector field in the momentum space

$$A_i^T(\mathbf{x},\eta) = \sum_{\sigma=1,2}\int \frac{d\mathbf{k}}{(2\pi)^{3/2}} A_{\mathbf{k}}^\sigma(\eta)\epsilon_i^\sigma(\mathbf{k})e^{i\mathbf{kx}} \tag{5.21}$$

where $\epsilon_i^\sigma, \sigma = 1,2$ are two orthogonal polarization vectors.

$$A_i'^T(\mathbf{x},\eta) = \sum_{\sigma=1,2}\int \frac{d\mathbf{k}}{(2\pi)^{3/2}} A_{\mathbf{k}}'^\sigma(\eta)\epsilon_i^\sigma(\mathbf{k})e^{i\mathbf{kx}} \tag{5.22}$$

$$\Delta A_i^T(\mathbf{x},\eta) = \sum_{\sigma=1,2}\int \frac{d\mathbf{k}}{(2\pi)^{3/2}} A_{\mathbf{k}}^\sigma(\eta)\epsilon_i^\sigma(\mathbf{k})(-\mathbf{k}^2)e^{i\mathbf{kx}} \tag{5.23}$$

$$\begin{aligned}
&\int d\mathbf{x} A_i'^T A_i'^T \\
&= \int d\mathbf{x} \sum_{\sigma,\sigma'=1,2}\int \frac{d\mathbf{k}}{(2\pi)^{3/2}}\int \frac{d\mathbf{k}'}{(2\pi)^{3/2}} A_{\mathbf{k}'}'^{\sigma'}(\eta)A_{\mathbf{k}}'^\sigma(\eta)e^{i\mathbf{kx}+i\mathbf{k}'\mathbf{x}'}\epsilon_i^\sigma(\mathbf{k})\epsilon_i^{\sigma'}(\mathbf{k}') \\
&= \sum_{\sigma=1,2}\int d\mathbf{k} A_{\mathbf{k}}'^\sigma(\eta)A_{-\mathbf{k}}'^\sigma(\eta)\epsilon_i^\sigma(\mathbf{k})\epsilon_i^{\sigma'}(-\mathbf{k})
\end{aligned} \tag{5.24}$$

In the same way we find

$$\int d\mathbf{x} A_i^T \Delta A_i^T = \sum_{\sigma=1,2}\int d\mathbf{k} A_{\mathbf{k}}^\sigma(\eta)A_{-\mathbf{k}}^\sigma(\eta)\epsilon_i^\sigma(\mathbf{k})\epsilon_i^\sigma(-\mathbf{k})(-\mathbf{k}^2) \tag{5.25}$$

Using (5.24), (5.25) in (5.20) we obtain

$$S = \frac{1}{2}\sum_{\sigma=1,2}\int d\eta \int d\mathbf{k}\, f^2(\eta)\epsilon_i^\sigma(\mathbf{k})\epsilon_i^\sigma(-\mathbf{k})\Big(A_{\mathbf{k}}'^\sigma(\eta)A_{-\mathbf{k}}'^\sigma(\eta) - \mathbf{k}^2 A_{\mathbf{k}}^\sigma(\eta)A_{-\mathbf{k}}^\sigma(\eta)\Big) \tag{5.26}$$

Rewritten in terms of the new variable

$$v_{\mathbf{k}}^{\sigma}(\eta) = \sqrt{\epsilon_i^{\sigma}(\mathbf{k})\epsilon_i^{\sigma}(\mathbf{k})}\, f(\eta) A_{\mathbf{k}}^{\sigma}(\eta) \tag{5.27}$$

Using the

$$v_{\mathbf{k}}^{\sigma}(\eta) v_{-\mathbf{k}}^{\sigma}(\eta) = \epsilon_i^{\sigma}(\mathbf{k})\epsilon_i^{\sigma}(-\mathbf{k}) f^2(\eta) A_{\mathbf{k}}^{\sigma}(\eta) A_{-\mathbf{k}}^{\sigma}(\eta) \tag{5.28}$$

and

$$f^2(\eta)\epsilon_i^{\sigma}(\mathbf{k})\epsilon_i^{\sigma}(-\mathbf{k}) A_{\mathbf{k}}^{'\sigma}(\eta) A_{-\mathbf{k}}^{'\sigma}(\eta) = v_{\mathbf{k}}^{'\sigma}(\eta) v_{-\mathbf{k}}^{'\sigma}(\eta) + \frac{f''(\eta)}{f(\eta)} v_{\mathbf{k}}^{\sigma}(\eta) v_{-\mathbf{k}}^{\sigma}(\eta) \tag{5.29}$$

at the end we have the action

$$S = \frac{1}{2}\sum_{\sigma=1,2}\int d\eta \int d\mathbf{k} \left(v_{\mathbf{k}}^{\sigma'}(\eta) v_{-\mathbf{k}}^{'\sigma}(\eta) - \left(\mathbf{k}^2 - \frac{f''(\eta)}{f(\eta)}\right) v_{\mathbf{k}}^{\sigma}(\eta) v_{-\mathbf{k}}^{\sigma}(\eta) \right) \tag{5.30}$$

This action describes two real scalar fields with time dependent effective masses in terms of their Fourier Components. From (5.30) it follows that

$$v_{\mathbf{k}}^{''\sigma}(\eta) + \left(\mathbf{k}^2 - \frac{f''(\eta)}{f(\eta)}\right) v_{\mathbf{k}}^{\sigma}(\eta) = 0 \tag{5.31}$$

The initial condition for this equation is the analogue of the Bunch-Davis vacuum

$$v_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta(t)} \qquad \text{for} \qquad k\eta(t) \to -\infty \tag{5.32}$$

The quantization of the system is very simple because it is reduced to the classical procedure that is followed in the case of a scalar field in the curved space. After the quantization we have

$$\hat{A}_i^T(\mathbf{x},\eta) = \frac{1}{f(\eta)}\sum_{\sigma=1,2}\int \frac{d\mathbf{k}}{(2\pi)^{3/2}\sqrt{2}} \left(e^{i\mathbf{k}\mathbf{x}} v_{\mathbf{k}}^{\star}(\eta)\hat{a}_{\mathbf{k}}^{-} + e^{-i\mathbf{k}\mathbf{x}} v_{\mathbf{k}}(\eta)\hat{a}_{\mathbf{k}}^{+} \right) \tag{5.33}$$

In order to find the energy density we compute the stress-energy tensor

$$T^{\mu}{}_{\nu} = \frac{\delta\mathcal{L}}{\delta\partial_{\mu}A_{\rho}}\partial_{\nu}A_{\rho} - \delta_{\nu}^{\mu}\mathcal{L} = -f^2(\eta)F^{\mu\rho}F_{\nu\rho} - \delta_{\nu}^{\mu}\mathcal{L} \tag{5.34}$$

and then

$$T_0^0 = f^2(\eta)\left(\frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}F_{0i}F^{0i}\right) = \frac{f^2(\eta)}{2a^4}\left(\partial_i A_j^T \partial_i A_j^T + A_i^{'T} A_i^{'T}\right) \tag{5.35}$$

We are now interesting in finding the electric energy density

$$\rho_E = \langle 0|\hat{T}_0^0|0\rangle \tag{5.36}$$

where we consider only the second term in the bracket of (5.35), since we are interested only in the homogeneous classical part which is the one given by the modes outside the Horizon.

$\langle 0|\hat{A}_i^{\prime T}(\mathbf{x},\eta)\hat{A}_i^{\prime T}(\mathbf{x},\eta)|0\rangle =$

$$4\int \frac{d\mathbf{k}}{2(2\pi)^3}\left(\frac{f'^2(\eta)}{f^4(\eta)}|v_{\mathbf{k}}(\eta)|^2 - 2\frac{f'(\eta)}{f^3(\eta)}v^{\star}_{\mathbf{k}}(\eta)v'_{\mathbf{k}}(\eta) + \frac{1}{f^2(\phi)}|v'_{\mathbf{k}}(\eta)|^2\right)$$

$$= 2\int \frac{d\mathbf{k}}{(2\pi)^3}\left|\frac{\partial}{\partial\eta}\left(\frac{v_{\mathbf{k}}(\eta)}{f(\eta)}\right)\right|^2 = \frac{1}{\pi^2}\int_0^{\infty} dk k^2\left|\frac{\partial}{\partial\eta}\left(\frac{v_{\mathbf{k}}(\eta)}{f(\eta)}\right)\right|^2 \tag{5.37}$$

$$\rho_E = \langle 0|\hat{T}_0^0|0\rangle = \int_0^{\infty} dk\frac{k^2 f^2(\eta)}{2a^4\pi^2}\left|\frac{\partial}{\partial\eta}\left(\frac{v_{\mathbf{k}}(\eta)}{f(\eta)}\right)\right|^2 \tag{5.38}$$

The electric power spectrum, therefore, results

$$\frac{d\rho_E}{dk} = \frac{f^2(\eta)}{2\pi^2}\frac{k^2}{a^4}\left|\frac{\partial}{\partial\eta}\left(\frac{v_{\mathbf{k}}(\eta)}{f(\eta)}\right)\right|^2 = \frac{f^2(t)}{2\pi^2}\frac{k^2}{a^2}\left|\frac{\partial}{\partial t}\left(\frac{v_{\mathbf{k}}(t)}{f(t)}\right)\right|^2 \tag{5.39}$$

where in the last expression we have returned to the cosmic time. Once we have determined the electric power spectrum we can calculate the boundary term which has to be added at the equation (5.12). At the moment of Horizon crossing the evolution of each mode changes from oscillatory to monotonous. We can assume that at the moment of Horizon crossing its behavior is still approximated by the Bunch-Davied vacuum. Then

$$\left|\frac{\partial}{\partial t}\left(\frac{v_{\mathbf{k}}}{f}\right)\right|^2 = \left|\frac{-ik}{\sqrt{2k}}\frac{1}{f}e^{-ikt/a} - \frac{\dot{f}}{f^2}\frac{1}{\sqrt{2k}}e^{-ikt/a}\right|^2$$

$$= \frac{1}{2kf^2(t)}\left(\frac{k^2}{a^2} + \left(\frac{\dot{f}}{f}\right)^2\right) \tag{5.40}$$

If we calculate this at Horizon exit at the end we have

$$\left.\frac{d\rho_E}{dk}\right|_H \times \frac{dk_H}{dt} = \frac{k}{2\pi^2 a^2}\left(\frac{k^2}{a^2} + \left(\frac{\dot{f}}{f}\right)^2\right)\Bigg|_{k=k_H=a(t)H} \times \frac{dk_H}{dt}$$

$$(\dot{\rho}_E)_H = \frac{H^3}{4\pi^2}\left(H^2 + \left(\frac{\dot{f}}{f}\right)^2\right) \tag{5.41}$$

And so the equation (5.12) becomes,

$$\dot{\rho}_E + 4H\rho_E + 2\rho_E\frac{\dot{f}}{f} = \frac{H^3}{4\pi^2}\left(H^2 + \left(\frac{\dot{f}}{f}\right)^2\right) \tag{5.42}$$

The system of equations that we are needed for the numerical calculation is (5.10), (5.11), (5.42). In order to have the equations in Planck units we must use the redefinitions

$$\phi' = \frac{\phi}{M_{pl}},\ H' = \frac{H}{M_{pl}},\ V' = \frac{V}{M_{pl}^4},\ \rho' = \frac{\rho}{M_{pl}^4},\ t' = t\cdot M_{pl} \tag{5.43}$$

It is convenient to express the evolution of the inflaton field in terms of the number of e-foldings rather than of the cosmic time t. At this end we appropriately modify the (5.10), (5.11) and (5.42) using the fact that

$$dN = H dt \tag{5.44}$$

and it is straightforward to find the equations

$$\frac{d^2\phi}{dN^2} + \left(3 + \frac{1}{H}\frac{dH}{dN}\right)\frac{d\phi}{dN} + \frac{V'(\phi)}{H^2} = \frac{2}{H^2}\frac{f'(\phi)}{f(\phi)}\rho_E \tag{5.45}$$

$$3H^2 = \frac{H^2}{2}\left(\frac{d\phi}{dN}\right)^2 + V(\phi) + \rho_E \tag{5.46}$$

$$\frac{d\rho_E}{dN} + 4\rho_E + 2\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\rho_E = \frac{H^4}{4\pi^2}\left(1 + \left(\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\right)^2\right) \tag{5.47}$$

In the absence of backreaction the terms related to the electric energy density and to the coupling function are missing. We further manipulate the equations in order to make them more compact, in these two cases with and without backreaction. We take the derivative with respect to the number of e-foldings of the (5.46) initially without the contribution of the electric energy density.

$$6H\frac{dH}{dN} = H\left(\frac{d\phi}{dN}\right)^2\frac{dH}{dN} + H^2\frac{d\phi}{dN}\frac{d^2\phi}{dN^2} + V'(\phi)\frac{d\phi}{dN}$$

$$\frac{d^2\phi}{dN^2} = 6\frac{1}{H}\frac{dH}{dN}\frac{1}{\frac{d\phi}{dN}} - \frac{1}{H}\frac{dH}{dN}\frac{d\phi}{dN} - \frac{V'(\phi)}{H^2} \tag{5.48}$$

we substitute (5.48) in the (5.45) where for now we neglect the term in the right hand side.

$$6\frac{1}{H}\frac{dH}{dN}\frac{1}{\frac{d\phi}{dN}} + \left(3 + \frac{1}{H}\frac{dH}{dN}\right)\frac{d\phi}{dN} - \frac{1}{H}\frac{dH}{dN}\frac{d\phi}{dN} - \frac{V'(\phi)}{H^2} + \frac{V'(\phi)}{H^2} = 0$$

from which follows that

$$\frac{1}{H}\frac{dH}{dN} = -\frac{1}{2}\left(\frac{d\phi}{dN}\right)^2 = \frac{V(\phi)}{H^2} - 3 \tag{5.49}$$

and so

$$\frac{d^2\phi}{dN^2} + \left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)\frac{d\phi}{dN} + \frac{V'(\phi)}{V(\phi)}\left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right) = 0$$

$$\frac{d^2\phi}{dN^2} + \left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)\left(\frac{d\phi}{dN} + \frac{V'(\phi)}{V(\phi)}\right) = 0 \tag{5.50}$$

where we have used the fact that

$$\frac{1}{H^2} = \frac{1}{V(\phi)}\left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right) \tag{5.51}$$

We repeat the same computation in the presence of backreaction, i.e. considering the electric energy density and the coupling function in the equations. By taking the derivative of (5.46) with respect to the number of e-foldings we have

$$6H\frac{dH}{dN} = H\frac{dH}{dN}\left(\frac{d\phi}{dN}\right)^2 + H^2\left(\frac{d\phi}{dN}\right)\frac{d^2\phi}{dN^2} + V'(\phi)\frac{d\phi}{dN} + \frac{d\rho_E}{dN}$$

from which we find

$$\frac{d^2\phi}{dN^2} = 6\frac{1}{H}\frac{dH}{dN}\frac{1}{\frac{d\phi}{dN}} - \frac{1}{H}\frac{dH}{dN}\frac{d\phi}{dN} - \frac{V'(\phi)}{H^2} - \frac{1}{H^2}\frac{d\rho_E}{dN}\frac{1}{\frac{d\phi}{dN}} \tag{5.52}$$

Substituting this in (5.45), we eventually obtain

$$\frac{1}{H}\frac{dH}{dN} = \frac{1}{6}\frac{1}{H^2}\frac{d\rho_E}{dN} - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2 + \frac{1}{3}\frac{1}{H^2}\frac{d\phi}{dN}\rho_E\frac{f'(\phi)}{f(\phi)} \tag{5.53}$$

From (5.45) and (5.53) we have

$$\frac{d^2\phi}{dN^2} + \left(\frac{1}{6}\frac{1}{H^2}\frac{d\rho_E}{dN} + 3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2 + \frac{1}{3}\frac{1}{H^2}\frac{d\phi}{dN}\rho_E\frac{f'(\phi)}{f(\phi)}\right)\frac{d\phi}{dN} + \frac{V'(\phi)}{H^2} = \frac{2}{H^2}\frac{f'(\phi)}{f(\phi)}\rho_E$$

$$\frac{d^2\phi}{dN^2} + \left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)\left(\frac{1}{6}\frac{d\rho_E}{dN}\frac{d\phi}{dN} + \frac{d\phi}{dN}(V(\phi)+\rho_E) + \frac{1}{3}\left(\frac{d\phi}{dN}\right)^2\rho_E\frac{f'(\phi)}{f(\phi)} + V'(\phi) - 2\frac{f'(\phi)}{f(\phi)}\rho_E\right)\frac{1}{V(\phi)+\rho_E} = 0 \tag{5.54}$$

where we have used the fact that

$$\frac{1}{H^2} = \frac{1}{V(\phi)+\rho}\left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right) \tag{5.55}$$

From (5.47) and (5.55) we find the equation for the electric energy density

$$\frac{d\rho_E}{dN} + 4\rho_E + 2\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\rho_E = \left(V(\phi)+\rho_E\right)^2\frac{1}{4\pi^2}\left(1 + \left(\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\right)^2\right)\left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)\right)^{-2} \tag{5.56}$$

5.3 Coupling Functions

In the context of any successful model of inflation, we have to provide a method to excite the Standard Model degrees of freedom in order to link the inflationary phase with the subsequent standard Big Bang evolution of our Universe. Since in string inflationary models the inflaton belong to the hidden sector, it is highly non trivial to explain its decay into the particles of the visible sector. As we have explained in the previous sector, it is convenient to use magnetized D7-branes wrapping the inflaton 4-cycle.
In the context of our study we are interested on the coupling of the inflaton field with the massless gauge bosons. In order to find the coupling function once we have expanded the inflaton around its minimum, $\tau = \langle\tau\rangle + \hat{\tau}$, we express it in terms of the canonically normalised field. In general the couplings to the gauge bosons are obtained from the moduli dependence of the tree-level Gauge Kinetic Function.

$$\frac{1}{g^2} = Re(f_{D7}) = \frac{\langle\tau\rangle}{2\pi} \tag{5.57}$$

When the $D7$-branes have non-diagonal magnetic fluxes $F_{1,i}$ which break the gauge group to $SU(3) \times SU(2) \times U(1)$ the three gauge couplings are given by

$$\frac{1}{g_i^2} = \frac{\tau - h(F_{1,i})s}{2\pi} \tag{5.58}$$

where the inflaton is shifted by the real part of the dilaton with a coefficient dependent by $F_{1,i}$. The electromagnetic term, therefore, becomes

$$\mathcal{L}_{kin}^{gauge} = -\frac{1}{4}\frac{1}{g^2}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\left(\frac{\tau - hs}{2\pi}\right)F^{\mu\nu}F_{\mu\nu} \tag{5.59}$$

• <u>Kähler Inflation</u>

We expand the inflaton τ around the minimum in the (5.59)

$$\begin{aligned}\mathcal{L}_{kin}^{gauge} &= -\frac{1}{4}\left(\frac{\langle\tau\rangle + \hat{\tau} - h\langle s\rangle}{2\pi}\right)F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\left(\frac{\langle\tau\rangle - h\langle s\rangle}{2\pi} + \frac{\hat{\tau}}{2\pi}\right)F^{\mu\nu}F_{\mu\nu} \\ &= -\frac{1}{4}\left(\frac{\langle\tau\rangle - h\langle s\rangle}{\langle\tau\rangle}\frac{\langle\tau\rangle}{2\pi} + \frac{\hat{\tau}}{2\pi}\right)F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\frac{\langle\tau\rangle}{2\pi\gamma}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}\frac{\hat{\tau}}{2\pi}F^{\mu\nu}F_{\mu\nu}\end{aligned}$$

where we have set

$$\gamma = \frac{\langle\tau\rangle}{\langle\tau\rangle - h\langle s\rangle} \tag{5.60}$$

In terms of the canonically normalised field strength

$$\hat{F}^{\mu\nu} = \sqrt{\frac{\langle\tau\rangle}{2\pi\gamma}}F^{\mu\nu} \tag{5.61}$$

we have

$$\mathcal{L}_{kin}^{gauge} = -\frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} - \frac{1}{4}\gamma\frac{\hat{\tau}}{\langle\tau\rangle}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} = -\frac{1}{4}\left(1+\gamma\frac{\hat{\tau}}{\langle\tau\rangle}\right)\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} \tag{5.62}$$

Considering that $\tau = \left(\frac{3\mathcal{V}}{4\alpha\lambda}\right)^{2/3}\phi^{4/3}$ we have

$$\frac{\hat{\tau}}{\langle\tau\rangle} = \frac{\tau}{\langle\tau\rangle} - 1 = \frac{\phi^{4/3}}{\langle\phi\rangle^{4/3}} - 1 \tag{5.63}$$

and finally

$$\mathcal{L}_{kin}^{gauge} = -\frac{1}{4}\left(1+\gamma\left(\frac{\phi^{4/3}}{\langle\phi\rangle^{4/3}} - 1\right)\right)\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} \tag{5.64}$$

We can set, therefore, the coupling function of the inflaton field with the electromagnetic field

$$f_K(\phi) = \sqrt{1+\gamma\left(\frac{\phi^{4/3}}{\langle\phi\rangle^{4/3}} - 1\right)} \tag{5.65}$$

• Fibre Inflation

In the same way we find the coupling function in the case of the Fibre Inflation model.

From (5.62) considering that in this case

$$\frac{\hat{\tau}}{\langle\tau\rangle} = \frac{\tau}{\langle\tau\rangle} - 1 = \frac{e^{k\phi}}{e^{k\langle\phi\rangle}} - 1 \tag{5.66}$$

we have

$$f_F(\phi) = \sqrt{1+\gamma\left(\frac{e^{k\phi}}{e^{k\langle\phi\rangle}} - 1\right)} \tag{5.67}$$

5.4 Numerical evolution

For the numerical analysis of the inflationary evolution we study first the dynamic in the absence of backreaction. More precisely, using the expression for the slow roll parameter ϵ, we can identify the initial and final value of the inflaton by requiring a suitably amount of e-foldings. Next we solve numerically the (5.50) using the previously found initial condition and a zero initial velocity. Once the ordinary evolution is known it must be compared with the evolution in the backreaction regime. At this end we solve the (5.54), (5.56) with the same initial conditions and a zero initial electric energy density. In Fibre Inflation the slow roll is controlled by the large value of the moduli rather than on the detailed tunning of parameters in the scalar potential. In this case, therefore, the numerical research of the inflationary evolution is simpler with respect to the case of Kähler Inflation where instead the choice of a particular set of parameters is quite important. In both cases however, there could be a slowdown of the inflationary dynamic in the backreaction regime which, for a particular fixed number of e-foldings, reduces the corresponding range of the inflaton field.

5.4.1 Fibre Inflation

In the case of Fibre inflation we use the potential found in the previous chapter in which we neglect the term with R and the overall normalization.

$$V(\phi) = 3 - 4e^{-\frac{\phi}{\sqrt{3}}} + e^{-\frac{4\phi}{\sqrt{3}}} \tag{5.68}$$

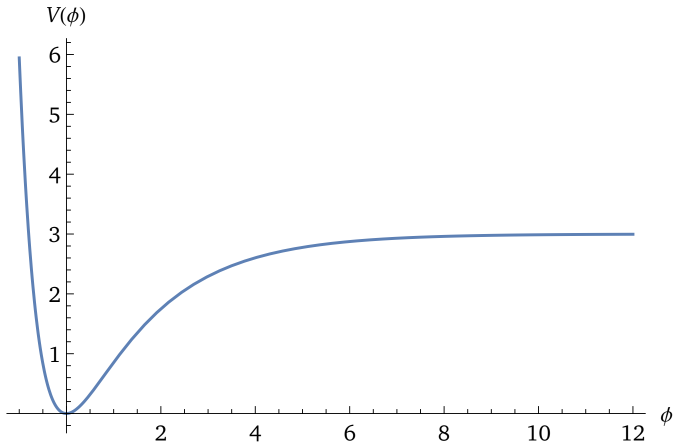


Figure 5.2: Potential in FI

The minimum of the potential is for $\phi_{vev} = 0$, while the value at the end of inflation, which is obtained from the $\epsilon(\phi) \simeq 1$ with ϵ given by (4.74) with $R = 0$, is $\phi_{end} = 0.92$. By requiring $N = 59.5$ number of e-foldings we find the initial value for the inflaton

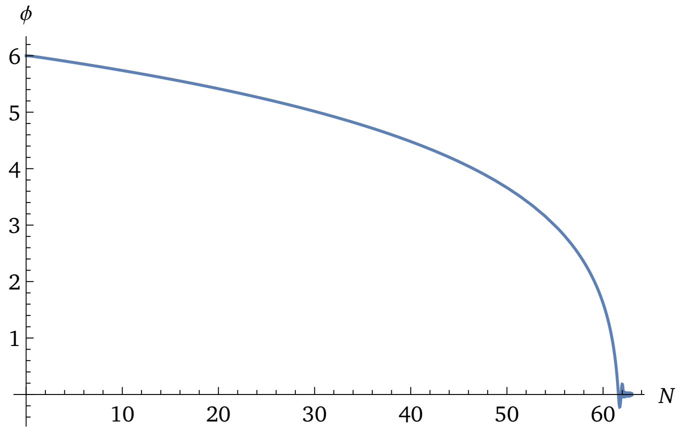


Figure 5.3: Evolution without BR in FI

$\phi_{in} = 6$. By numerically solving the (5.50), with $\phi(0) = \phi_{in}$ and $\phi'(0) = 0$ we find the detailed evolution presented in Figures (5.3), (5.4).

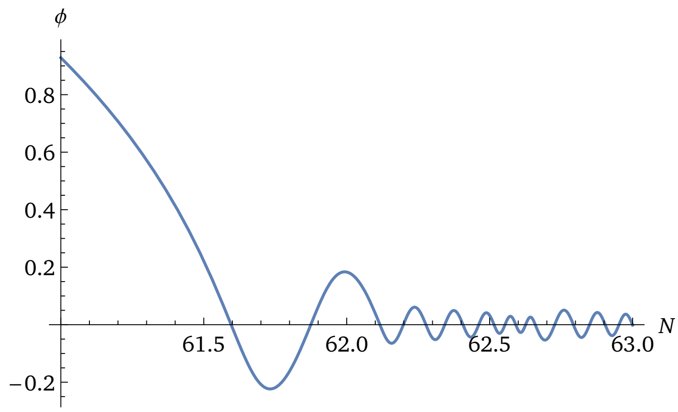


Figure 5.4: Evolution at the end of inflation without BR in FI

Next we consider the backreaction regime in which we numerically solve (5.54) and (5.55) with initial conditions $\phi(0) = \phi_{in}$, $\phi'(0) = 0$ and $\rho(0) = 0$. The coupling function is given by (5.67) with $\gamma = 1$ in all cases unless another choice is explicitly indicated. The result is presented in Figures (5.5) and (5.6). Looking at the Figures (5.4), (5.6), it is evident that the backreaction allow to earn a little amount of e-foldings. If we fix the number of e-foldings we have therefore a reduction of the inflaton range.
It is interesting to note that the result is quite independent from the precise value of γ. In fact, looking the form of the coupling function we can see that in the quantity $f'(\phi)/f(\phi)$ the dependence on γ is canceled when the latest take large values. In fact the slowdown increases slightly from $\gamma = 1$ at $\gamma = 100$ and after that remains unchanged.
The evolution of the electric energy density is represented in Figure (5.7).

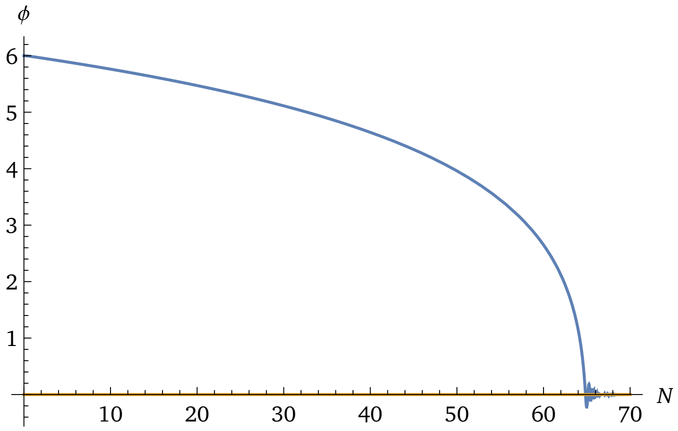


Figure 5.5: Evolution in presence of BR in FI

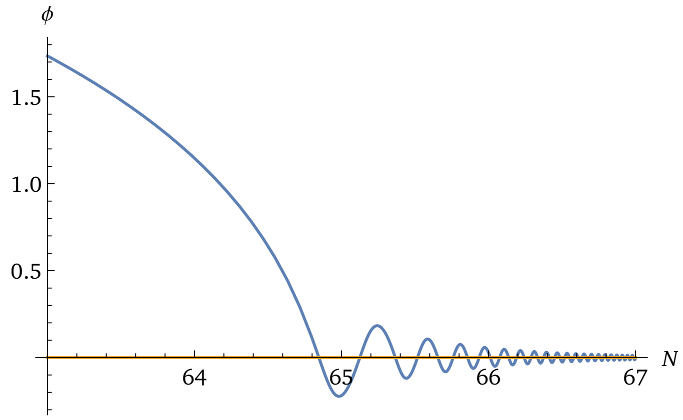


Figure 5.6: Evolution at the end of inflation in presence of BR in FI

If we change the initial condition for the electric energy density allowing a non zero initial value $\rho(0) = 0.7$, then we obtain a larger slowdown of the inflaton field (Figure 5.8).

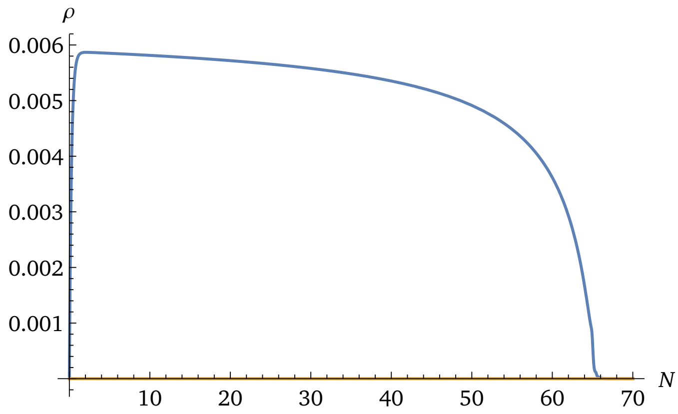


Figure 5.7: Evolution of the electric energy density in FI

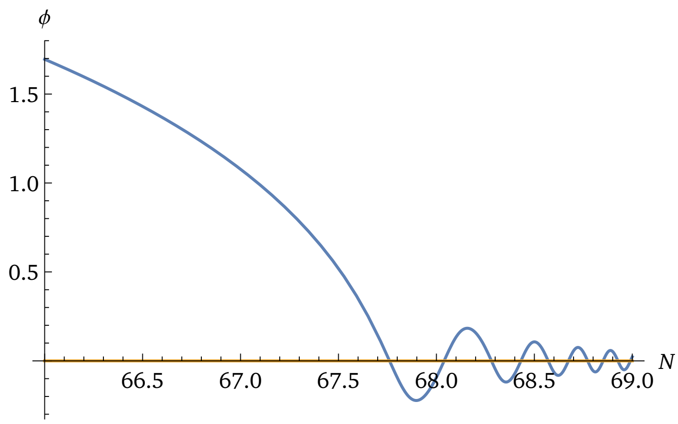


Figure 5.8: Evolution in presence of BR in FI with $\rho(0) = 0.7$

5.4.2 Kähler Inflation

In the case of Kähler Inflation the procedure is more complicated due to the dependence on the microscopic parameters. The scalar potential, in terms of the non canonically normalised field τ, is

$$V(\tau) = V_0 + \frac{8aA\sqrt{\tau}e^{-2a\tau}}{3\mathcal{V}\lambda\alpha} - \frac{4aAW_0\tau e^{-a\tau}}{\mathcal{V}^2} \tag{5.69}$$

with

$$V_0 = \frac{W_0^2\beta}{\mathcal{V}^3} \qquad \beta = \frac{3}{2}\lambda a^{-3/2}(\log(\mathcal{V}))^{3/2} \tag{5.70}$$

where V_0 is the term which dominates the scalar potential. In a similar way with that presented in Fibre Inflation, we express the potential in terms of the canonically normalised field, and we find from (4.32) the final value for the inflaton field and the initial one requiring a suitably number of e-foldings. We numerically solve the equation

(5.50) first and the system of (5.54) and (5.56) next in order to find the inflationary evolution in the two regimes. The initial conditions even in this case are $\phi(0) = \phi_{in}$, $\phi'(0) = 0$ and $\rho(0) = 0$. The coupling function is that of Kähler Inflation (5.65).
In this case we choose different sets of the underlying parameters to see which of them is more suitable for the slowdown. We present the results in the same graphics where the blue line always represent the solution in the backreaction regime, while the orange one corresponds to the ordinary inflationary evolution. A natural choice of the microscopic parameters which can be done following the article [4], is

$$\text{Case 1}: \qquad W_0 = 57 \quad \alpha = 1 \quad \lambda = 1 \quad A = 1.87 \quad a = 2\pi \tag{5.71}$$

With this set of parameters it is quite difficult to achieve the slowdown of the inflaton field. In fact, as we will see, the effect of the backreaction is related to the intensity of the scalar potential which for natural values of the compactification volume $10^5 \leq \mathcal{V} \leq 10^8$ is very small. For $\mathcal{V} = 10^8$ we find that in the backreaction regime the inflaton slow rolls in its minimum faster than in the ordinary evolution. By reducing the value of the volume the two evolutions become more and more similar until the case of $\mathcal{V} = 10^2 - 10^3$ in which we obtain the desired slowdown. These differences can be seen in Figures (5.9)-(5.11).

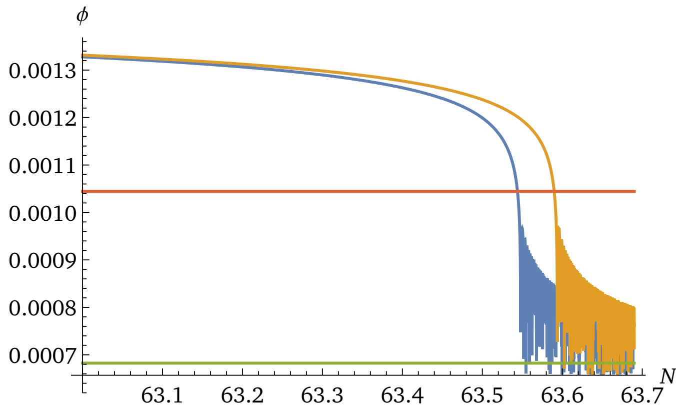


Figure 5.9: Case 1, $\mathcal{V} = 10^7$ in KI

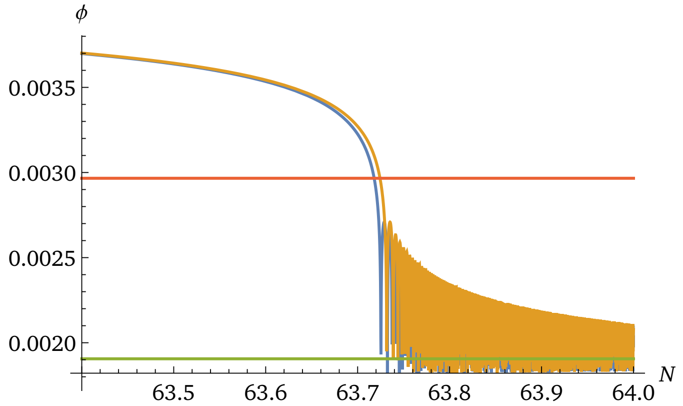


Figure 5.10: Case 1, $\mathcal{V} = 10^6$ in KI

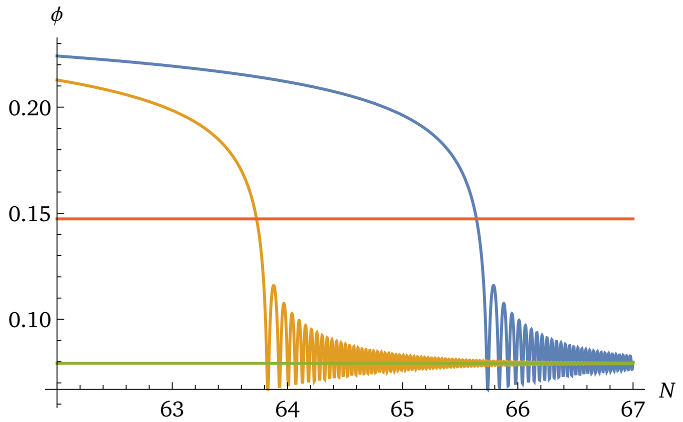


Figure 5.11: Case 1, $\mathcal{V} = 10^2$ in KI

In order to obtain the slowdown for bigger values of the compactification volume the most effective way is to increase the value of W_0 and reduce α and on a less important note by reducing a and incrementing λ and A. For

$$\text{Case 2}: \quad W_0 = 130 \quad \mathcal{V} = 10^4 \quad \alpha = 1/100 \quad \lambda = 1 \quad A = 10 \quad a = 2\pi/10 \tag{5.72}$$

we have a backreaction already for $\mathcal{V} = 10^4$ as can be seen in Figure (5.12). For

$$\text{Case 3}: \quad W_0 = 400 \quad \mathcal{V} = 10^4 \quad \alpha = 1/100 \quad \lambda = 1 \quad A = 1 \quad a = 2\pi \tag{5.73}$$

we find a valid difference in the evolution of the two regimes as can be seen from the Figure (5.13). For

$$\text{Case 4}: \quad W_0 = 400 \quad \mathcal{V} = 10^4 \quad \alpha = 1/100 \quad \lambda = 10 \quad A = 10 \quad a = 2\pi \tag{5.74}$$

we have the plot of Figure (5.14).

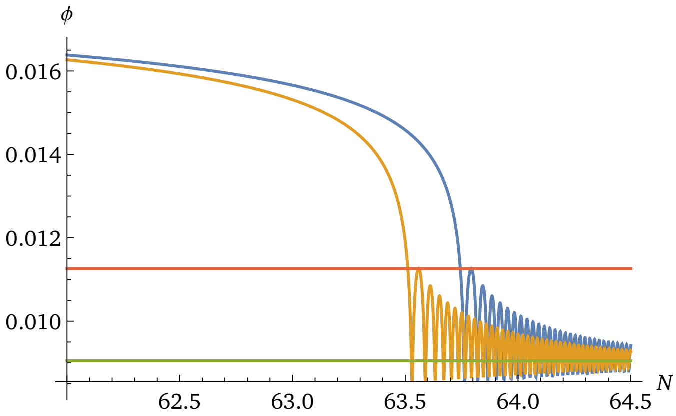


Figure 5.12: Case 2, $\mathcal{V} = 10^4$, in KI

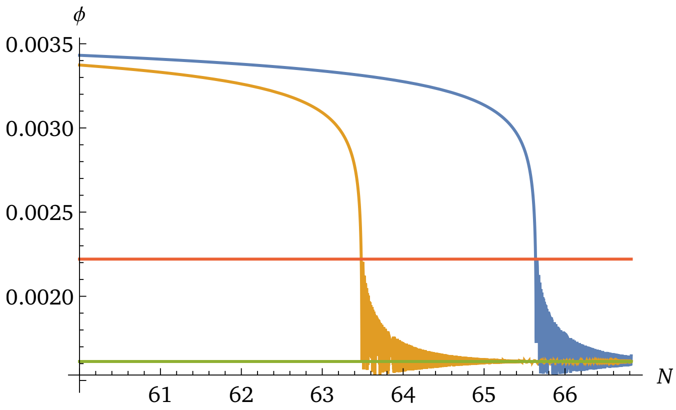


Figure 5.13: Case 3, $\mathcal{V} = 10^4$, in KI

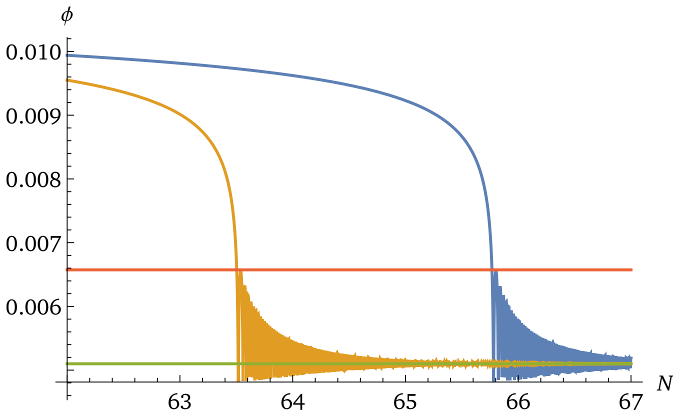


Figure 5.14: Case 4, $\mathcal{V} = 10^4$ in KI

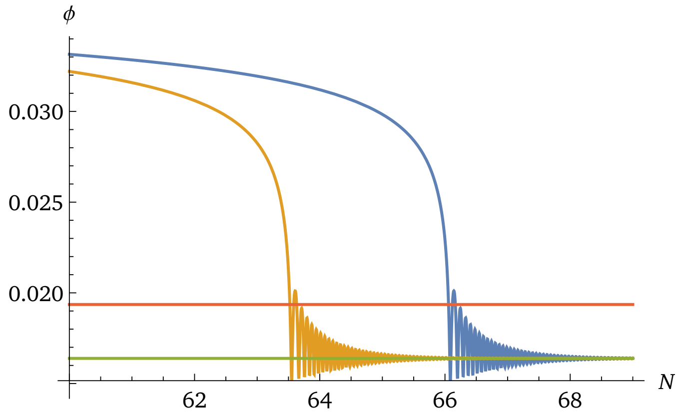


Figure 5.15: Case 5, $\mathcal{V} = 10^4$ in KI

For

$$\text{Case 5}: \quad W_0 = 400 \quad \mathcal{V} = 10^4 \quad \alpha = 1/100 \quad \lambda = 1 \quad A = 10 \quad a = 2\pi/30 \tag{5.75}$$

we have the plot of Figure (5.15).

For

$$\text{Case 6}: \quad W_0 = 500 \quad \mathcal{V} = 10^4 \quad \alpha = 1/100 \quad \lambda = 10 \quad A = 10 \quad a = 2\pi \tag{5.76}$$

we have the plot of Figure (5.16)

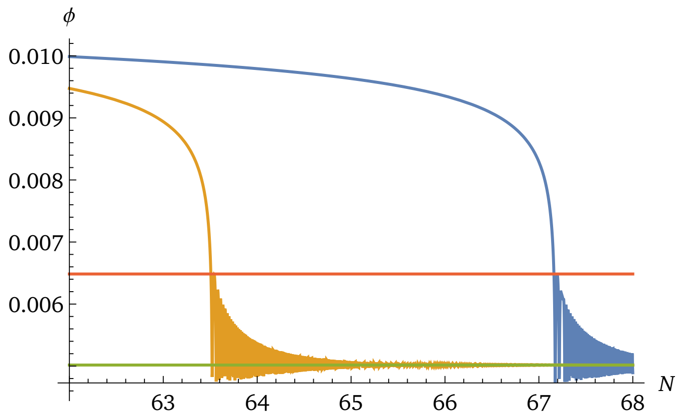


Figure 5.16: Case 6, $\mathcal{V} = 10^4$ in KI

For

$$\text{Case 7}: \quad W_0 = 200 \quad \mathcal{V} = 10^4 \quad \alpha = 1/100 \quad \lambda = 1 \quad A = 10 \quad a = 2\pi/30 \tag{5.77}$$

we have the plot of Figure (5.17). Finally, if we choose the biggest value of W_0, we

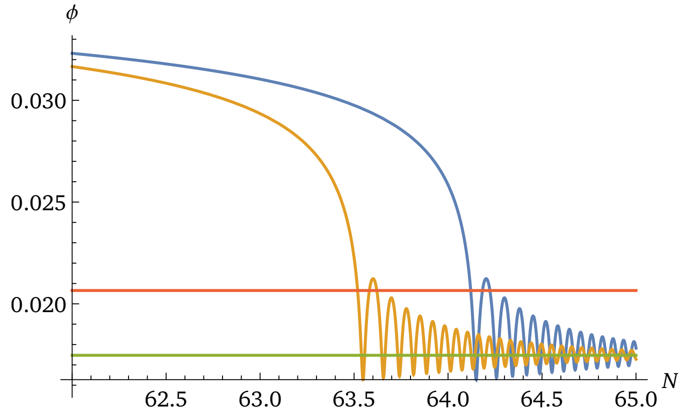


Figure 5.17: Case 7, $\mathcal{V} = 10^4$ in KI

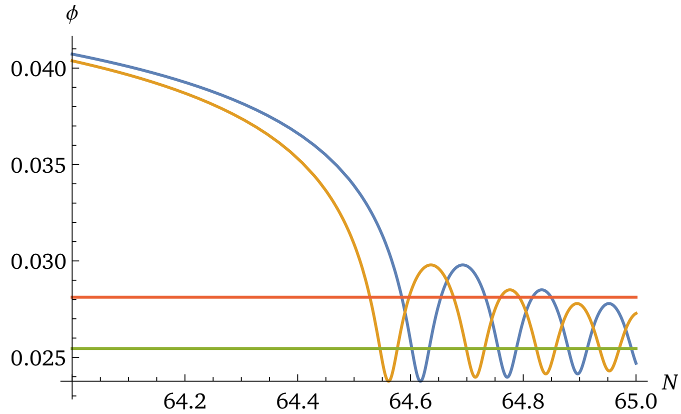


Figure 5.18: Case 8, $\mathcal{V} = 10^5$ in KI

increase A and we further reduce a we have a small slowdown even for the case of $\mathcal{V} = 10^5$. In fact, for

$$\text{Case 8}: \quad W_0 = 500 \quad \mathcal{V} = 10^5 \quad \alpha = 1/100 \quad \lambda = 10 \quad A = 50 \quad a = 2\pi/50 \tag{5.78}$$

we have the plot of Figure (5.18)

Even in this case if we choose a non zero initial condition for the electric energy density we obtain a higher number of e-foldings. In fact, if we take for example the Case 1, with $\mathcal{V} = 10^6$ and with initial condition $\rho(0) = 10^{-22}$ we obtain the plot of Figure (5.19).

The electric energy density in all these cases has the same evolution. At the beginning it is quite constant and very small because it is dumped by the dominating term in (5.56) which is the one containing the Hubble constant. When the term in the right hand side of the same equation becomes larger, the electric energy density starts growing and presents a peak. If the peak is high enough then we have a satisfactory slowdown. This evolution can be seen for the Case 4, for example, in Figure (5.20)

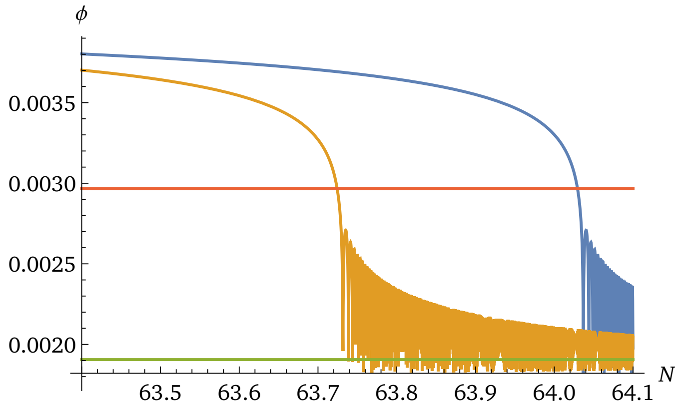


Figure 5.19: Caso 1, $\mathcal{V} = 10^6$, with $\rho(0) = 10^{-22}$ in KI

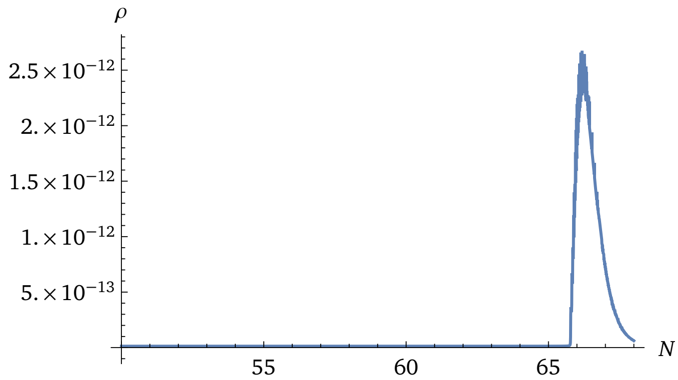
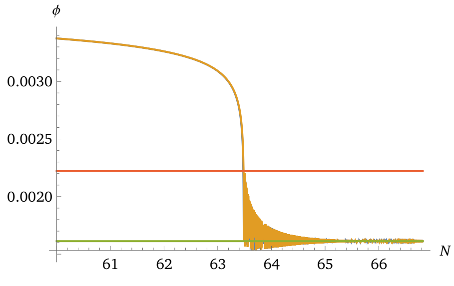


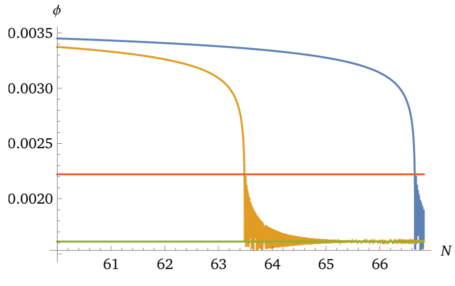
Figure 5.20: Case 4: Evolution of the electric energy density in KI

Comparing the plots it is clear that the most significant parameter is the stabilised at tree level superpotential, W_0, which depends on the complex structure moduli. For a constant value of W_0 a little more slowdown is obtained by reducing the model dependent constant $a = 2\pi/N$ where N is related to the number of the branes and take the values $1 \leq N \leq 100$, and the constant α dependent on the particular Calabi-Yau that we have chosen. In order to have more backreaction it is also useful to increase a little the model dependent parameters A and λ, without however ruining the slow roll dynamics.

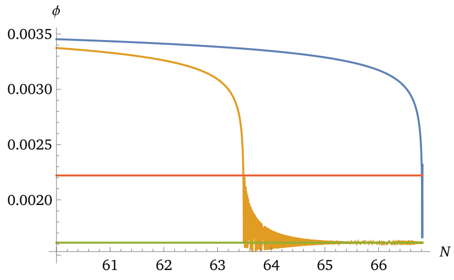
Finally, there is also a light dependence on $f'(\phi)/f(\phi)$. For $1 < \gamma < 1000$ the ratio increase with γ, giving a little more delay for the inflaton, while for $\gamma > 1000$ become independent from it. Obviously for $\gamma = 0$ the backreaction is turned off. In Figure (5.21) we present the inflationary evolution for $\gamma = 0, 10, 100$ corresponding to the set of parameters chosen in Case 3.



(a) $\gamma = 0$



(b) $\gamma = 10$



(c) $\gamma = 100$

Figure 5.21: Case 3: Plots with different values of γ in KI

The explanation of these differences can be understood by analyzing the form of the equations that we report below for convenience.

$$\frac{d^2\phi}{dN^2} = -\left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)\left(\frac{1}{6}\frac{d\rho_E}{dN}\frac{d\phi}{dN} + \frac{d\phi}{dN}(V(\phi) + \rho_E) + \frac{1}{3}\left(\frac{d\phi}{dN}\right)^2 \rho_E \frac{f'(\phi)}{f(\phi)} + V'(\phi) - 2\frac{f'(\phi)}{f(\phi)}\rho_E\right)\frac{1}{V(\phi) + \rho_E}$$

The most important term for the backreaction is the last one which contains the electric energy density. If we compare the above equation with that of the ordinary regime

$$\frac{d^2\phi}{dN^2} = -\left(3 - \frac{1}{2}\left(\frac{d\phi}{dN}\right)^2\right)\left(\frac{d\phi}{dN} + \frac{V'(\phi)}{V(\phi)}\right)$$

we see that the two evolutions are quite the same when

$$\frac{1}{6}\frac{d\rho}{dN}\frac{d\phi}{dN} + \frac{1}{3}\left(\frac{d\phi}{dN}\right)^2 \rho \frac{f'(\phi)}{f(\phi)} - 2\frac{f'(\phi)}{f(\phi)}\rho \simeq 0$$

When ρ remains small the expression is positive and the second derivative of the inflaton more negative causing the accelerated progress which can be seen in the cases of large compactification volumes with natural values (Figure (5.9)). In order to have the desired slowdown the electric energy density have to become quite large. In the equation which governs the evolution of the electric energy density:

$$\frac{d\rho_E}{dN}+4\rho_E+2\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\rho_E=\Big(V(\phi)+\rho_E\Big)^2\frac{1}{4\pi^2}\left(1+\left(\frac{df(\phi)}{dN}\frac{1}{f(\phi)}\right)^2\right)\left(3-\frac{1}{2}\left(\frac{d\phi}{dN}\right)\right)^{-2}$$

the growing of the density in the peak depends on the potential which is contained in the right hand side. So it is quite predictable that the choices of the microscopic parameters that lead to a bigger potential cause more backreaction and consequently more slowdown of the inflaton field.

In the case of Fibre Inflation the energy density was quite large from the beginning and the slowdown of the inflaton field it is more easy to obtain. Even this effect is dependent on the potential which is more stable and independent from the precise form of the underlying parameters.

Conclusions and outlook

The goal of this thesis was the study of the inflationary dynamic in presence of backreaction effects due to the coupling of the inflaton with electromagnetic fields. The numerical analysis was carried on in the context of two very promising inflationary string models, Kähler inflation and Fibre inflation, where the inflaton is a closed string mode.
The numerical analysis has pointed out that in general the backreaction cause a slowdown of the inflaton field with respect to its standard evolution. This effect is due to the extra terms that appears together with the coupling function in the modified equation which make the second derivative of the inflaton less negative and therefore the evolution less sharp.
In the case of Fibre inflation the slowdown of the inflaton is always present because the electric energy density, which is mostly responsible for the backreaction, is high enough. In Kähler inflation, however, this effect it in not so easy to obtain and in some circumstances we even get the opposite effect characterized by a more abrupt decrease of the inflaton field. We have to make the correct choice of the underlying parameters in order to allow more number of e-foldings of inflation. In this case the evolution of the electric energy density is quite constant at the beginning while towards the end of inflation it has a peak. In order to have the slowdown it is necessary a quite high peak which is obtained increasing the coefficients of the potential and in particular V_0 which is the dominating term.
This result is very important because it allows to reduce the range of the inflaton corresponding to a particular fixed number of e-foldings. In this way we could provide a solution to the incompatibility which emerge between the geometrical limits of the inflaton range and the phenomenological requirements related on the number of e-foldings. It is possible, therefore, to reconcile the inflationary models coming from string theory with the standard cosmological inflation and observations.

In this direction, it could be interesting in some future work to compute the precise amount by which the inflaton range is reduced due to the presence of the backreaction. The resulting range could be then compared with the limits emerging from the geometrical arguments presented in the article [8]. In this way it would be possible in principle to make a selection or a characterization of all possible Calabi-Yau compactifications and different models of string inflation in the context of which such limits emerge creating a point of contact between the two theories.

In the context of our analysis based on the coupling of the inflaton with the electromagnetic field, it is also possible to explain the origin of the magnetic fields carried by all celestial bodies which have been misused from planets to interstellar medium [20],[15],[1]. In fact, the mechanism based on the Biermann Battery proposed to generate the seed magnetic fields and the different types of dynamo which can enhance them cannot explain the presence of large scale magnetic fields with large correlation length into the cosmic voids. It seems, therefore, more likely that their origin occurred in the primordial Universe during inflation. In this context, the presence of a time dependent coupling function in the standard electromagnetic action break the conformal invariance which would not allow the enhancement of electromagnetic fluctuations during inflation and thus constitute an excellent way for their production.
From the analysis computed in section 5.2 we have found the equation satisfied by the modes of the electromagnetic field (5.31). It is convenient to rewrite this equation using instead of the conformal time the number of e-foldings. At this end it is useful to find the Hubble constant in conformal time

$$d\eta = \frac{dt}{a} \tag{5.79}$$

$$H = \frac{\dot{a}}{a} = \frac{a'}{a^2} = \frac{\mathcal{H}}{a} \tag{5.80}$$

where the dot represent the derivative respect to cosmic time, the prime represent the derivative respect to conformal time and $\mathcal{H}$ is the Hubble constant in conformal time. At this point from $dN = Hdt$ we find $dN = \mathcal{H}d\eta$.
The equation (5.31) thus becomes

$$\frac{\partial^2 v^\sigma_{\mathbf{k}}}{\partial N^2} + \frac{\partial v^\sigma_{\mathbf{k}}}{\partial N}\frac{1}{\mathcal{H}}\frac{\partial \mathcal{H}}{\partial N} + \left(\frac{k^2}{\mathcal{H}^2} - \frac{\partial^2 f}{dN^2} - \frac{1}{\mathcal{H}}\frac{\partial \mathcal{H}}{\partial N}\frac{\partial f}{dN}\right)v^\sigma_{\mathbf{k}} = 0 \tag{5.81}$$

In the presence of backreaction the equation (5.11) in terms of the conformal time η becomes

$$\frac{d^2\phi}{d\eta^2} + 2\frac{d\phi}{d\eta}\mathcal{H} + V'(\phi)a^2 = 2\frac{f'(\phi)}{f(\phi)}a^2\rho_E \tag{5.82}$$

and the (5.10) becomes

$$3\mathcal{H} = \frac{1}{2}\left(\frac{d\phi}{d\eta}\right) + V(\phi)a^2 + a^2\rho_E \tag{5.83}$$

In terms of the number of e-foldings these equations become

$$\frac{\partial^2\phi}{\partial N^2} + \left(2 + \frac{1}{\mathcal{H}}\frac{\partial \mathcal{H}}{\partial_N}\right)\frac{\partial\phi}{\partial N} + \frac{V'(\phi)a^2}{\mathcal{H}^2} = \frac{2}{\mathcal{H}^2}\frac{f'(\phi)}{f(\phi)}a^2\rho_E \tag{5.84}$$

and

$$\mathcal{H}^2 = \frac{1}{2}\left(\frac{\partial\phi}{\partial N}\right)^2\mathcal{H} + (V(\phi) + \rho_E)a^2 \tag{5.85}$$

If we take the derivative with respect to the number of e-foldings of the equation (5.85) we find

$$6\mathcal{H}\frac{\partial \mathcal{H}}{\partial N} = \mathcal{H}\frac{\partial \mathcal{H}}{dN}\left(\frac{\partial \phi}{\partial N}\right)^2 + \mathcal{H}^2\frac{\partial \phi}{\partial N}\frac{\partial^2 \phi}{\partial N^2} + V'(\phi)\frac{\partial \phi}{\partial N}a^2 + \frac{\partial \rho_E}{\partial N}a^2 + (V(\phi)+\rho_E)2a^2$$

and so

$$\frac{\partial^2 \phi}{\partial N^2} = 6\frac{1}{\mathcal{H}}\frac{\partial \mathcal{H}}{\partial N}\frac{1}{\frac{\partial \phi}{\partial N}} - \frac{1}{\mathcal{H}}\frac{\partial \mathcal{H}}{\partial N}\frac{\partial \phi}{\partial N} - \frac{V'(\phi)a^2}{\mathcal{H}^2} - \frac{1}{\mathcal{H}^2}\frac{1}{\frac{\partial \phi}{\partial N}}\frac{\partial \rho_E}{\partial N}a^2 - \frac{V(\phi)+\rho_E}{\mathcal{H}^2}2a^2\frac{1}{\frac{\partial \phi}{dN}} \quad (5.86)$$

Substituting this into (5.84) at the end we obtain

$$\frac{1}{\mathcal{H}}\frac{\partial \mathcal{H}}{\partial N} = -2 + \frac{1}{3}\rho_E\frac{\partial \phi}{\partial N}\frac{f'(\phi)}{f(\phi)}\frac{3-\frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2}{V(\phi)+\rho_E} + 3 - \frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2 + \frac{1}{6}\frac{3-\frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2}{V(\phi)+\rho_E}\frac{\partial \rho_E}{\partial N} \quad (5.87)$$

Using the (5.87) and the

$$\frac{1}{\mathcal{H}^2} = \frac{3-\frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2}{(V(\phi)+\rho_E)a^2} \quad (5.88)$$

the (5.81) becomes

$$\frac{\partial^2 v_{\mathbf{k}}^\sigma}{\partial N^2} + \frac{\partial v_{\mathbf{k}}^\sigma}{\partial N}\left(-2 + \frac{1}{3}\rho_E\frac{\partial \phi}{\partial N}\frac{f'(\phi)}{f(\phi)}\frac{3-\frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2}{V(\phi)+\rho_E} + 3 - \frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2 + \frac{1}{6}\frac{3-\frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2}{V(\phi)+\rho_E}\frac{\partial \rho_E}{\partial N}\right)$$
$$+\left(k^2\frac{3-\frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2}{(V(\phi)+\rho_E)a^2} - \frac{\partial^2 f}{dN^2} - \frac{\partial f}{dN} \times \left(-2 + \frac{1}{3}\rho_E\frac{\partial \phi}{\partial N}\frac{f'(\phi)}{f(\phi)}\frac{3-\frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2}{V(\phi)+\rho_E} + 3\right.\right.$$
$$\left.\left. - \frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2 + \frac{1}{6}\frac{3-\frac{1}{2}\left(\frac{\partial \phi}{\partial N}\right)^2}{V(\phi)+\rho_E}\frac{\partial \rho_E}{\partial N}\right)\right)v_{\mathbf{k}}^\sigma = 0 \quad (5.89)$$

This equation must be resolved numerically using as background the numerical solutions found in the previous section with initial conditions these of the Bunch-Davies vacuum. Once the $v_{\mathbf{k}}^\sigma$ are determined, it is easy to find the magnetic power spectrum from equation

$$\delta_B^2(k,\eta) = \sum_{\sigma=1,2}\frac{|v_{\mathbf{k}}^\sigma(\eta)|^2 k^5}{4\pi^2 a^4 f^2} \quad (5.90)$$

In fact the power spectrum of the vector potential $A = \sqrt{-A_i A^i}$ is found by the

$$\langle 0|\hat{A}_i^T(\mathbf{x},\eta)\hat{A}^{iT}(\mathbf{y},\eta)|0\rangle = -\frac{1}{a^2}\langle 0|\hat{A}_i^T(\mathbf{x},\eta)\hat{A}_i^T(\mathbf{y},\eta)|0\rangle$$

$$= -\frac{1}{a^2 f^2} \sum_{\sigma=1,2} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} |v^\sigma_\mathbf{k}(\eta)|^2 = \frac{1}{a^2 f^2} \sum_\sigma \int \frac{dk}{k} \frac{\sin(k|\mathbf{x}-\mathbf{y}|)}{k|\mathbf{x}-\mathbf{y}|)} \frac{k^3}{(2\pi^2)} |v^\sigma_\mathbf{k}(\eta)|^2$$

and so

$$\delta^2_A(k,\eta) = \sum_{\sigma=1,2} \frac{k^3 |v^\sigma_\mathbf{k}(\eta)|^2}{4\pi^2 a^2 f^2} \tag{5.91}$$

Considering that

$$B^2 = -B_i B^i = \frac{1}{2a^4} F_{ik} F_{ik} = \frac{1}{a^4} (\partial_i A_k \partial_i A_k - \partial_k A_i \partial_i A_k)$$

we have

$$\delta^2_B(k,\eta) = \delta^2_A(k,\eta) \frac{k^2}{a^2} \tag{5.92}$$

from which follows (5.90).

The study on the origin of the magnetic fields in the primordial Universe due to the presence of the coupling function that breaks the conformal invariance it is today of great interest. In fact, besides explaining their presence on large scales, it constitutes also another sector in which the predictions of string theory could in principle be compared with observations. It is therefore a very broad and important topic to focus future researches.

Bibliography

[1] Luis Aparicio and Anshuman Maharana. Inflating Kahler Moduli and Primordial Magnetic Fields. *Phys. Lett.*, B768:46–51, 2017.

[2] Vijay Balasubramanian, Per Berglund, Joseph P. Conlon, and Fernando Quevedo. Systematics of moduli stabilisation in Calabi-Yau flux compactifications. *JHEP*, 03:007, 2005.

[3] Daniel Baumann and Liam McAllister. *Inflation and String Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2015.

[4] Sukannya Bhattacharya, Koushik Dutta, Mayukh R. Gangopadhyay, and Anshuman Maharana. Confronting Kähler moduli inflation with CMB data. *Phys. Rev.*, D97(12):123533, 2018.

[5] Jose J. Blanco-Pillado, Duncan Buck, Edmund J. Copeland, Marta Gomez-Reino, and Nelson J. Nunes. Kahler Moduli Inflation Revisited. *JHEP*, 01:081, 2010.

[6] C. P. Burgess, M. Cicoli, M. Gomez-Reino, F. Quevedo, G. Tasinato, and I. Zavala. Non-standard primordial fluctuations and nongaussianity in string inflation. *JHEP*, 08:045, 2010.

[7] M. Cicoli, C. P. Burgess, and F. Quevedo. Fibre Inflation: Observable Gravity Waves from IIB String Compactifications. *JCAP*, 0903:013, 2009.

[8] Michele Cicoli, David Ciupke, Christoph Mayrhofer, and Pramod Shukla. A Geometrical Upper Bound on the Inflaton Range. *JHEP*, 05:001, 2018.

[9] Michele Cicoli, Joseph P. Conlon, and Fernando Quevedo. General Analysis of LARGE Volume Scenarios with String Loop Moduli Stabilisation. *JHEP*, 10:105, 2008.

[10] Michele Cicoli, Koushik Dutta, Anshuman Maharana, and Fernando Quevedo. Moduli Vacuum Misalignment and Precise Predictions in String Inflation. *JCAP*, 1608(08):006, 2016.

[11] Michele Cicoli, Iñaki Garcìa-Etxebarria, Christoph Mayrhofer, Fernando Quevedo, Pramod Shukla, and Roberto Valandro. Global Orientifolded Quivers with Inflation. *JHEP*, 11:134, 2017.

[12] Michele Cicoli and Anupam Mazumdar. Reheating for Closed String Inflation. *JCAP*, 1009(09):025, 2010.

[13] Michele Cicoli and Gabriel A. Piovano. Reheating and Dark Radiation after Fibre Inflation. 2018.

[14] Joseph P. Conlon and Fernando Quevedo. Kahler moduli inflation. *JHEP*, 01:146, 2006.

[15] Vittoria Demozzi, Viatcheslav Mukhanov, and Hector Rubinstein. Magnetic fields from inflation? *JCAP*, 0908:025, 2009.

[16] Andrew R Liddle and David H Lyth. *Cosmological Inflation and Large-Scale Structure*. Cambridge Univ. Press, Cambridge, 2000.

[17] Stephen P. Martin. A Supersymmetry primer. pages 1–98, 1997. [Adv. Ser. Direct. High Energy Phys.18,1(1998)].

[18] Viatcheslav Mukhanov. *Physical Foundations of Cosmology*. Cambridge Univ. Press, Cambridge, 2005.

[19] Fernando Quevedo, Sven Krippendorf, and Oliver Schlotterer. Cambridge Lectures on Supersymmetry and Extra Dimensions. 2010.

[20] O. O. Sobol, E. V. Gorbar, M. Kamarpour, and S. I. Vilchinskii. Influence of backreaction of electric fields and Schwinger effect on inflationary magnetogenesis. *Phys. Rev.*, D98(6):063534, 2018.