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Functional Renormalization Group: higher order flows and pseudo-spectral methods

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Sommario

Diamo una rapida introduzione alle idee su cui si basa il Gruppo di Rinormalizzazione Funzionale e deriviamo l'equazione di flusso esatta per l'azione effettiva media nella formulazione proposta da Wetterich e Morris. Lavorando in tale contesto, discutiamo i più compuni troncamenti LPA e LPA' e li applichiamo a una teoria con un campo scalare reale in D dimensioni. Discutiamo la struttura dei punti fissi della teoria in dimensioni arbitrarie tramite un metodo numerico di shooting e concentrandoci sul caso $D = 3$, che corrisponde alla classe di universalità del modello di Ising, studiamo le soluzioni di punto fissi e gli esponenti critici tramite espansioni polinomiali. Come risultato originale deriviamo una nuova equazione di flusso esatta al secondo ordine nel tempo RG e consideriamo analoghi schemi di approssimazione LPA e LPA', applicandoli ad una teoria scalare in $D = 3$. Per risolvere le nuove e più complesse equazioni diventa necessaria una tecnica numerica più potente, pertanto i metodi psuedo-spettrali vengono introdotti e applicati con successo al problema. Infine, il confronto tra i risultati trovati con i diversi schemi di approssimazione permette di mettere in luce l'affidabilità e i limiti dei diversi troncamenti e di identificare promettenti ulteriori sviluppi.

Abstract

We give a brief introduction to the ideas underlying the non-perturbative Functional Renormalization Group and we derive the exact flow equation for the effective average action, in the formulation firstly proposed by Wetterich and Morris. Working in that framework, we discuss the standard LPA and LPA' truncations and we apply them to a theory with one real scalar field in D dimensions. We discuss the fixed point structure of the theory in arbitrary dimensions via numerical shooting method and focusing on the $D = 3$ case, corresponding to Ising universality class, we study fixed point solutions and critical exponents via polynomial expansions. As a novel result, we derive a new exact flow equation at second order in RG time and we consider corresponding LPA and LPA' truncations, applying them to a scalar theory in $D = 3$. In order to solve the resulting more complex equations a more powerful numerical technique is needed, thus pseudo-spectral methods are introduced and successfully applied to the problem. Finally, comparison between results found with different approximation schemes enables to shed light on the reliability and the limitations of different kinds of truncations and to identify future promising developments.

Contents

Introduction

Since its birth, the quantum theory of fields has been one of the most fruitful and troublesome tool for theoretical physicists. Even if one of the first successful examples of quantum description was given by the quantized electromagnetic field in the works of Planck on black-body radiation (1900), it was not until half of XX century that physicists were reasonably confident that a consistent quantum theory of fields was actually realizable.

Indeed, in the first three decades of 1900, the general principles of quantum theory were established[1] and, after the great successes that quantum mechanics was having in atomic physics, it was natural to extend them to field theories and to a relativistic framework. Nevertheless, this extension turned out to be not trivial at all; even if extremely important and far reaching concepts, such as anti-matter and spin-statistics, stemmed from the researches of that period.

As far as fields and relativistic particle were not interacting everything was clear, but it seemed impossible to develop a consistent theory of interactions: according to standard techniques of perturbative analysis, second order corrections were afflicted by divergent quantities, whose physical meaning was completely ignored. There was quite a discomfort about the idea of a quantum field theory.

However, around mid '900, experimental results (Lamb shift) and the development of new theoretical techniques capable to easily handle relativistic perturbative expansions (Feynman diagrams) and to make sense of divergences (regularization and the first drafts of renormalization) convinced the physicist community that a consistent and predictive quantum theory of electromagnetic interactions (QED) was realizable.

Despite the success of QED and other field theories that were used to describe interactions among elementary particles, a global understanding of divergences, renormalizable and non-renormalizable interactions and renormalization group transformations was still missing. Such understanding was favoured in the seventies by works in condensed matter physics, an area in which fields are emerging concepts while the underlying theory is often a lattice theory. In fact, it became clear that physical systems at critical points

[1]It is indeed quite a remarkable fact that the same principles have not gone outdate: all modern discoveries do not seem to go in the direction of modifying them, but rather in applying them to broader and broader contexts

are characterized by such a large correlation length that thermal fluctuations of their individual constituents can be safely approximated by a fluctuating continuous field. The analogy with the dynamic of a quantum field becomes even more stringent as soon one recognizes that the mathematical description of quantum field theories and statistical systems are equivalent, modulo Wick rotations.

In the theory of critical phenomena, renormalization group transformations acquire a very transparent meaning as a description of the changes of the physics with the scale. Moreover, together with the concept of universality, they describe clearly why only some interactions, the renormalizable ones (or the relevant ones, depending on the preferred vocabulary), emerge in the long distance physics. Thus the understanding of elementary particle physics was greatly boosted by works on condensed matter physics and, at the same time, Feynamn diagrams successfully spread in condensed matter community.

Nevertheless, in both area of research, perturbative approach soon showed its own limit, since it can be meaningfully applied only to weakly interacting systems and in many physical models such condition is simply not true (e.g. low energy QCD). Hence the need to develop techniques capable to describe quantum field theories beyond the weak coupling hypothesis have become mature and is, nowadays, urgent. One of such techniques, the non-perturbative functional renormalization group, will be the main subject of this work. Finally let us remark that a deeper comprehension of non-perturbative renormalization may shed light, in the paradigm of asymptotic safety, on fundamental questions concerning the quantum description of gravitational field or the high energy behaviour of the interactions now encompassed in the Standard Model.

The work is organized as follow. In chapter 1 we will quickly review the basis of functional formulation of quantum field theories (QFTs), then we will describe the ideas behind the functional renormalization group (FRG) and we will report a detailed derivation of Wetterich equation, the equation we will use to implement the ideas of non-perturbative FRG. Since in its exact form Wetterich equation can not be solved, different approximation schemes (which, at variance with perturbative analysis, do not rely on weak-coupling hypothesis) have been proposed in the literature, we will review them in chapter 2 together with the standard numerical techniques used to analyse them. For all concrete implementation will use scalar models. In chapter 3 we will derive a new exact flow equation which, at variance with Wetterich equation, is second order in RG time, we will discuss its general structure first and hence implementation of standard LPA and LPA' truncations in this new framework. A detailed analytic derivation will be given for scalar models. Since the equations derived with this new scheme will have a more complex structure, an analysis performed with standard numerical techniques will be useless. Following a recent suggestion in the literature, we will then introduce pseudo-spectral models in chapter 4 and show how they are very well suited for our problem. Having in hand this powerful numerical technique, we will be able to compare results obtained with different approximation schemes and have a clue about their reliability, limitations and possible improvements.

Chapter 1

Functional formulation of QFT and RG

This whole thesis project deals with quantum field theories (QFTs) and, in particular, with the functional approach to the subject. Thus we will now review the basics of the functional formulation of QFTs and the concepts and techniques of the functional renormalization group (FRG). We will work in natural units (i.e. $c = \hbar = 1$) and in euclidean formulation.

1.1 Basics of QFTs in functional formulation

In this section we will only state the main steps of the functional formulation of QFTs, for a more complete treatment of the subject we refer to the lecture notes [1] [2] [3] and the bibliography reported therein.

Even if they may appear quite abstract, *correlation functions* can be considered among the most important objects in a QFT and many physical predictions are stored inside them. Using the hyper-condensed notation[1] the n-points correlation function is defined as:

$$G^{(n)\ A_1 A_2 \ldots A_n} \equiv \langle \phi^{A_1} \phi^{A_2} \ldots \phi^{A_n} \rangle \tag{1.1}$$

$$\equiv \frac{1}{Z} \int D[\phi]\, P[\phi]\, \phi^{A_1} \phi^{A_2} \ldots \phi^{A_n} \tag{1.2}$$

where:

[1] In this work we will deal mainly with scalar field theories and the notation follows thereafter. However the definitions we give can be extend to spinor fields, vector fields or multiplets by giving an appropriate explicit form to the hyper-condensed notation

- the indexes A_i can be thought of as position coordinates for scalar fields or position and internal coordinates for more complex fields. Their contraction corresponds to summation and integration, e. g.:

$$J_A \phi^A = \sum_a \int d^D x \, J_a(x) \, \phi^a(x) \tag{1.3}$$

- $P[\phi]$ is the un-normalized probability density, which for an euclidean field theory takes the form

$$P[\phi] \equiv e^{-S[\phi]} \tag{1.4}$$

where $S[\phi]$ is the *euclidean action*, which is usually used to define the theory under study.

- $Z \equiv \int D[\phi] \, P[\phi]$ is the normalization factor and is called *partition function*.

The entire set of correlation functions can be stored inside a *generating functional* defined as:

$$Z[J] \equiv \frac{1}{Z} \int D[\phi] \, e^{-S[\phi]} \, e^{J_A \phi^A} \tag{1.5}$$

$$= \langle e^{J_A \phi^A} \rangle \tag{1.6}$$

where:

- J can be thought physically as an external *classical source* (of an appropriate nature) that couples with the field under study and mathematically as a useful device in order to take derivatives and obtain correlation functions.

Actually once the generating functional is know we can derive the correlation functions straightforwardly as we notice that:

$$G^{(n)\; A_1 A_2 \ldots A_n} = \lim_{J \to 0} \frac{\delta^n Z[J]}{\delta J_{A_n} \ldots \delta J_{A_1}} \tag{1.7}$$

Among the correlation functions it is possible to define a special class, that of the so-called *connected correlation functions*. As the general n-point correlation function can be decomposed into an algebra of connected pieces the latter ones are more useful objects to study. It can be shown that the functional

$$W[J] \equiv \ln Z[J] \tag{1.8}$$

is the generator of the connected correlation functions, in the sense that:

$$G_c^{(n)\; A_1 A_2 \ldots A_n} = \lim_{J \to 0} \frac{\delta^n W[J]}{\delta J_{A_n} \ldots \delta J_{A_1}} \tag{1.9}$$

where the subscript c identifies the n-points *connected* correlation function.

Nevertheless, notice how the content of information inside $Z[J]$ and $W[J]$ must be exactly the same, as they are in a one-to-one correspondence, therefore working with the connected correlation functions, and their generator, is just a better way to manage information.

Finally we are going to define the *effective action*, which will turn out to be the most important functional in the development of the work. In order to define it we first need to define the so-called *classical field*[2] as:

$$\phi_{cl}^A[J] \equiv \langle \phi^A \rangle_J \tag{1.10}$$

$$\equiv \frac{1}{Z[J]} \int D[\phi]\, \phi^A\, e^{-S[\phi]+J_B\phi^B} \tag{1.11}$$

$$= \frac{\delta W[J]}{\delta J_A} \tag{1.12}$$

Then, supposing that (hypothesis which we will always assume true in the following of the work):

$$\frac{\delta^2 W[J]}{\delta J_{A_1} \delta J_{A_2}} \quad \text{is positive definite} \tag{1.13}$$

the relation $\phi_{cl}^A = \phi_{cl}^A[J]$ can be inverted as $J_A = J_A[\phi_{cl}]$ and the effective action $\Gamma[\phi_{cl}]$ is defined as the functional Legendre transform of the generator of connected correlation functions:

$$\Gamma[\phi_{cl}] \equiv \underset{J}{\text{extr}} \left(\phi_{cl}^A J_A - W[J]\right) \tag{1.14}$$

$$= \phi_{cl}^A J_A[\phi_{cl}] - W[J[\phi_{cl}]] \tag{1.15}$$

In general the functional Legendre transform is supposed to be involutive and the following property to hold:

$$\frac{\delta^2 \Gamma[\phi_{cl}]}{\delta \phi_{cl}^{A_1} \delta \phi_{cl}^{A_2}} \quad \text{is positive definite} \tag{1.16}$$

which implies that the effective action is convex. Other properties which can be derived by the definition are:

$$\frac{\delta \Gamma[\phi_{cl}]}{\delta J_A} = 0 \tag{1.17}$$

$$\frac{\delta \Gamma[\phi_{cl}]}{\delta \phi_{cl}^A} = J_A \tag{1.18}$$

[2]this name can be misleading because, as it can be seen by its definition, this object is the functional average of the quantum field, therefore it is not classical in nature (besides it does not satisfy, in general, the classical equation of motion). However, along with really classical fields, it takes a well defined value in each point of space, as all the quantum fluctuations have been integrated out

where (1.17) states that the effective action does not depend on the classical source, while (1.18) can be thought of as the quantum equation of motion for the averaged quantum field, in the spirit of the classical equation of motion:

$$\frac{\delta S[\phi]}{\delta \phi^A} = J_A \tag{1.19}$$

From the definition of ϕ_{cl} it can be noticed that:

$$\phi_{cl}[0] = \langle \phi \rangle \tag{1.20}$$

and, if translational invariance is guaranteed, we can assume, for real scalar fields, that $\langle \phi \rangle$ is a constant value $\phi_0 \in \mathbb{R}$ throughout space. Physical systems usually takes $\phi_0 = 0$ while if $\phi_0 \neq 0$ the phenomenon of spontaneous symmetry breaking is occurring.

Since the relation between ϕ_{cl} and J is invertible when we turn off the classical source equation (1.18) reads:

$$\left.\frac{\delta \Gamma[\phi_{cl}]}{\delta \phi_{cl}^A}\right|_{\phi_{cl}=\phi_0} = 0 \tag{1.21}$$

which means that ϕ_0 is a stationary value for the effective action in absence of external source.

Moreover it can be shown that the effective action is the generator of the (1PI) *proper vertices*:

$$\Gamma^{(n)}_{A_1 \ldots A_n} = \lim_{\phi_{cl} \to \phi_0} \frac{\delta^n \Gamma[\phi_{cl}]}{\delta \phi_{cl}^{A_n} \ldots \delta \phi_{cl}^{A_1}} \tag{1.22}$$

and finally, by its own definition, it is possible to write the integro-differential equation:

$$e^{-\Gamma[\phi_{cl}]} = \int D[\phi]\, e^{-S[\phi] + \frac{\delta \Gamma[\phi_{cl}]}{\delta \phi_{cl}^A}(\phi^A - \phi_{cl}^A)} \tag{1.23}$$

which describes how the effective action for ϕ_{cl} is obtained by the classical one after all fluctuations around ϕ_{cl} have been integrated out. This idea will play a very important role in the formalism we will use throughout this thesis project.

1.1.1 Example 1: free theory

A free theory is, by definition, described by a quadratic classical action:

$$S_0[\phi] = \frac{1}{2} \phi^A \mathcal{K}_{AB} \phi^B \tag{1.24}$$

where $\mathcal{K}_{AB} = \mathcal{K}_{BA}$ is called *kinetic operator*. In this particular case the generator of correlation function can be exactly computed through Gaussian integration and it becomes:

$$\begin{aligned} Z_0[J] &= \frac{1}{Z_0}\int D[\phi]\, e^{-S_0[\phi]+J_A\phi^A} \\ &= \frac{1}{Z_0}\left(\int D[\phi]\, e^{-S_0[\phi-\mathcal{K}^{-1}J]}\right) e^{\frac{1}{2}J_A\mathcal{K}^{AB}J_B} \\ &= \frac{1}{Z_0}\left(\int D[\phi']\, e^{-S_0[\phi']}\right) e^{\frac{1}{2}J_A\mathcal{K}^{AB}J_B} \\ &= e^{\frac{1}{2}J_A\mathcal{K}^{AB}J_B} \end{aligned} \tag{1.25}$$

where $\mathcal{K}^{AB}$ is the inverse of the kinetic operator (i.e. $\mathcal{K}^{AB}\mathcal{K}_{BC} = \delta^A_C$). Consequently the generator of the connected correlation functions becomes:

$$W_0[J] = \frac{1}{2}J_A\mathcal{K}^{AB}J_B \tag{1.26}$$

and the relations among the classical field and the classical source in the free case are:

$$\phi^A_{cl}[J] = \mathcal{K}^{AB}J_B \quad \Leftrightarrow \quad J_A[\phi_{cl}] = \mathcal{K}_{AB}\phi^B_{cl} \tag{1.27}$$

finally the effective action becomes:

$$\begin{aligned} \Gamma_0[\phi_{cl}] &= \phi^A_{cl}J_A[\phi_{cl}] - W_0[J[\phi_{cl}]] \\ &= \frac{1}{2}\phi^A_{cl}\mathcal{K}_{AB}\phi^B_{cl} \end{aligned} \tag{1.28}$$

therefore we notice that in the free case the effective action has the same functional form of the classical action.

1.1.2 Example 2: interacting theory in perturbative expansion

A local interacting theory (in euclidean formulation) can be described in general by a classical action of the following form:

$$S[\phi] = S_0[\phi] + V[\phi] \tag{1.29}$$

where $S_0[\phi]$ is the action of a free theory and $V[\phi]$ is a local functional of the field, which encompasses the interactions, each of which is characterized by an appropriate coupling constant. The generator of correlation functions can not be straightforwardly computed in this case and the perturbative approach tries to compute it as a series expansion around the free theory:

$$
\begin{aligned}
Z[J] &= \frac{1}{Z}\int D[\phi]\, e^{-S_0[\phi]-V[\phi]+J_A\phi^A} \\
&= \frac{1}{Z}\int D[\phi]\, e^{-V\left[\frac{\delta}{\delta J_A}\right]}\left(e^{-S_0[\phi]+J_A\phi^A}\right) \\
&= \frac{1}{Z}e^{-V\left[\frac{\delta}{\delta J_A}\right]}\int D[\phi]\, e^{-S_0[\phi]+J_A\phi^A} \\
&= \frac{Z_0}{Z}e^{-V\left[\frac{\delta}{\delta J_A}\right]}e^{\frac{1}{2}J_A\mathcal{K}^{AB}J_B} \\
&= \frac{Z_0}{Z}\left(1-V\left[\frac{\delta}{\delta J_A}\right]+\frac{1}{2}V\left[\frac{\delta}{\delta J_A}\right]^2+\dots\right)e^{\frac{1}{2}J_A\mathcal{K}^{AB}J_B}
\end{aligned}
\tag{1.30}
$$

Once the generator of correlation function is defined all other functionals can be computed thereafter.

In order for the perturbative expansion to give meaningful result the coupling constant must be necessarily 'small', actually the philosophy is the following:

- start with non-interacting fields
- turn on some 'small' couplings between the fields
- organize the corrections in powers of the 'small' coupling constants
- for each order compute the corrections due to all possible fluctuations, at any length scale
- sum up the contributions of different orders

Now a few comments are due:

- from the second order onward, taking into account all length scale at once usually produces divergent integrals
- however, for *perturbative renormalizable theories*, one can introduce the concepts of bare couplings, counter terms and renormalized couplings and store all divergent contributions inside the bare quantities thus getting finite renormalized results.
- as a consequence of the above operation, the renormalized couplings can no longer be constant and have to become functions of the length scale of the process taken into account, i.e. we need to define their values at a given length scale and if we vary the defining scale their values change consequently, as described by the *perturbative renormalization group*.

1.2 Functional Renormalization Group

The scheme described above was initially developed in the context of high energy physics, in order to make sense of the divergences appearing in the QFTs used to describe particle physics, and it was implemented in the perturbative approach (see [3] for an introduction to the subject and citations of the original works). However very important insights came also from statistical physics and from a different formulation of the problem which helped to shed light on the meaning of divergences, renormalization and running constants. The starting point of this new approach is usually identified with the works of Kadanoff and Wilson in the sixties and seventies (e.g. [4] [5]). Then its modern implementation, developed in the eighties and nineties, has taken two main formulations, which are are usually called *Wilson effective action* and *effective average action* and are associated respectively to Wilson-Polchinski (e.g. [6]) and to Wetterich-Morris (e.g. [7]).

We will now give a brief overview of the original ideas of Wilson and Kadanoff and the picture of the RG that emerged thereafter. Then we will concentrate on the Wetterich-Morris approach which will be the starting point of this thesis project. For a more complete introduction to the subject, comprehensive citations of the original works, historical developments and applications of this formalism to many different areas of current research (e.g. statistical field theory, critical phenomena, fermionic systems, gauge theories, quantum gravity, ...) we refer to the lecture notes [8] [9] [10] and the monograph [11].

1.2.1 Kadanoff-Wilson approach and the RG picture

As discussed above, when facing interacting QFTs, an exact evaluation of generating functionals is usually an impossible task, therefore we are forced to sum up quantum fluctuations with approximate methods. From this point of view, the main idea of Kadanoff and Wilson is to tackle the problem with a different and, to some extent, more clever procedure with respect to the perturbative approach. They include since the beginning the idea of running couplings and they treat differently fluctuations of different length scales. Actually, if we are interested in long distance physics, the short distance details are averaged out, therefore their proposal is to iteratively build an effective theory for long distance degrees of freedom, which we are interested in, by integrating out the short distance ones and concurrently adjusting the couplings. The procedure, if stated in real space, is:

- begin with a *regularized* theory defined at a given minimal characteristic length (i.e. a lattice spacing)
- integrate fluctuations with 'short' (i.e. similar to the lattice spacing) length scales
- rescale all quantities of the system in order to express them with respect to the new minimal characteristic length (i.e. the new lattice spacing)

- adjust the couplings in order to preserve the generating functional and then start with a new iteration

The ultimate result of this procedure should be a complete integration of all quantum fluctuations up to the desired length scale and an appropriate description of long distance physics. Besides, the variations of couplings with the length scale can be safely followed along the progressive integration of fluctuations. Let's now see how these procedure is mathematically implemented.

Suppose to work with a scalar field and to expand it in momentum space (the method has been describe above in real space, however we shall now implement it in momentum space as it is the easiest formulation to start with):

$$\phi(x) = \int \frac{d^D q}{(2\pi)^D}\, \tilde{\phi}(q)\, e^{-iq\cdot x} \tag{1.31}$$

the functional integration can be defined as:

$$D[\phi] = \prod_{q\leq\Lambda} d\tilde{\phi}_q \tag{1.32}$$

where Λ is the UV cut-off (related to the lattice spacing a by: $\Lambda \sim a^{-1}$) at which the starting theory is defined, and the partition function[3] is therefore:

$$Z = \prod_{q\leq\Lambda} \int d\tilde{\phi}_q\, e^{-S_\Lambda[\phi,\{g\}]} \tag{1.33}$$

where it has been emphasized that the action is defined at a momentum scale of order Λ and is identified by the set of coupling $\{g\}$ (it is formally convenient to always consider a theory with all possible interactions permitted by symmetries and to identify different physical systems by simply setting to the appropriate value, eventually zero, the couplings). Then, taking a factor c slightly less than 1, we can write:

$$Z = \prod_{q\leq c\Lambda} \int d\tilde{\phi}_q \underbrace{\prod_{c\Lambda\leq q\leq\Lambda} \int d\tilde{\phi}_q\, e^{-S_\Lambda[\phi,\{g\}]}}_{\equiv e^{-S_{c\Lambda}[\phi,\{g\}]}} \tag{1.34}$$

$$= \prod_{q\leq c\Lambda} \int d\tilde{\phi}_q\, e^{-S_{c\Lambda}[\phi,\{g\}]} \tag{1.35}$$

$$\tag{1.36}$$

[3]Instead of considering the generating functional we restrict our study to the partition function, i.e. setting to zero the external source, for simplicity. However, to gain more information about the system, the formulation can be extend to the general case, with a few more difficulties.

and taking a new momentum unit such that:

$$\underbrace{c\Lambda}_{\text{old unit}} \to \underbrace{\Lambda'}_{\text{new unit}} \tag{1.37}$$

we need to correspondingly redefine the field:

$$\underbrace{\phi}_{\text{old unit}} \to \underbrace{\phi'}_{\text{new unit}} \tag{1.38}$$

and we finally arrive at:

$$Z = \prod_{q \leq \Lambda'} \int d\tilde{\phi}'_q \, e^{-S_{c\Lambda}[\phi(\phi'),\{g\}]} \tag{1.39}$$

$$= \prod_{q \leq \Lambda'} \int d\tilde{\phi}'_q \, e^{-S_{\Lambda'}[\phi',\{g'\}]} \tag{1.40}$$

where in the last equation all changes introduced by the above procedure of integration and rescaling have been converted into variations of the couplings, while leaving the formal structure of the action equal to the starting one. From this last viewpoint we can start to visualize the change of a quantum field theory with its defining scale as a *flow of its representing point in the theory space*, where each point represents a particular theory, identified by its set of couplings.

Actually, in this formalism, as well as in other formalisms which we will develop in more detail later, the study of this so called *Renormalization Group flow* brings important informations about the theory. The picture that emerged from it has had a great impact in the understanding of field theories and it has brought its own vocabulary, often borrowed by the theory of dynamical systems. So let us now state the conventions and the fundamental concepts:

- *Basis of operators*: in the jargon of the RG flow an *operator* is any possible kind of interaction and therefore it constitutes one of the (usually) infinite directions of the theory space, whose coordinate are the corresponding couplings. In local field theories an operator is thus any possible monomial realized with powers of fields and their derivatives, up to any order (i.e. quantum operators on Fock space).

- *RG-time*: the role of time in the RG-flow is played by the minimal characteristic length scale of the theory. As the most used formulation will be in momentum space, we will adopt from now on the convention that $t = -\infty$ corresponds to low momenta and long distance physics, while $t = +\infty$ corresponds to high momenta and short distance physics. The proper definition will be:

$$t = \ln\left(\frac{k}{\Lambda}\right) \tag{1.41}$$

  where k represents the momentum scale.

- *β-functions*: the RG-flow is usually formulated as:

$$\frac{dg^i}{dt} = \beta^i(g) \tag{1.42}$$

where t is the RG time and the $\beta_i(g)$ are the so called β-functions. They are the most important objects of the RG-flow, as they contain all the dynamic, and the goal of any formulation of RG-flow is to provide a reliable expression for them.

- *Fixed points*: there may be points of the theory space which do not flow with time. Corresponding to such points there are theories for which the combined operations of integration (often called *coarse graining*) and rescaling leaves the couplings unchanged. Such theories do not vary with the scale, therefore their cut-off can be pushed up to any arbitrarily high value. In terms of β-functions, fixed points g^* can be identified by $\beta(g^*) = 0$.

- *Limit cicles*: it may happen that the RG-flow admits isolated closed orbits which can represent the limit of a family of flow-trajectories. In such cases, once the limiting orbit is reached, the flow will be stuck forever to such closed orbit, realizing a periodical motion. We will not consider these exotic situations in this work.

- *Stability matrix*: supposing a regular flow, we can linearise the problem near a given fixed point and we can write the flow equation for the perturbations as:

$$\frac{d}{dt}\delta g_i = \left.\frac{\partial \beta_i(g)}{\partial g_j}\right|_{g^*} \delta g_j \tag{1.43}$$

where $\left.\frac{\partial\beta}{\partial g}\right|_{g^*}$ is called stability matrix and describes the dynamical property of the fixed point.

- Given a fixed point g^* and its corresponding stability matrix, we can diagonalize the latter[4] and classify the eigenperturbations δg_i^* according to their eigenvalues λ_i; with the above convention on the orientation of the RG-time:

  - δg_i^* is called a *relevant operator* if $\mathfrak{Re}\{\lambda_i\} < 0$
  - δg_i^* is called a *irrelevant operator* if $\mathfrak{Re}\{\lambda_i\} > 0$
  - δg_i^* is called a *marginal operator* if $\mathfrak{Re}\{\lambda_i\} = 0$

  Independently on the convention on the versus of time, the definitions above, which originated in the context of statistical physics, always refer to the behaviour of the eigenperturbations in the low-momentum/long-distance limit.

[4] we suppose the stability matrix to be diagonalizable, however this must be checked for each case

- In the jargon of high energy physics it is more common a classification based on the behaviour in the high-momentum/short-distance limit; with the above convention on the orientation of the RG-time:
  - δg_i^* is called a *super-renormalizable operator* if $\mathfrak{Re}\{\lambda_i\} < 0$
  - δg_i^* is called a *non-renormalizable operator* if $\mathfrak{Re}\{\lambda_i\} > 0$
  - δg_i^* is called a *renormalizable operator* if $\mathfrak{Re}\{\lambda_i\} = 0$

  Actually, in the perturbative renormalization approach, the above classification is usually based on simple dimensional arguments, i.e. taking into account only the classical scaling property of the coupling. However a more careful analysis of the behaviour of couplings with the scale requires the complete study of the RG-flow.

- *Basin of attraction of a fixed point*: near a fixed point, according to the spectrum of its stability matrix (and the chosen conventional versus of the RG-flow), there are directions in the theory space along which the couplings flow into the fixed point (*attractive directions*) or out of the fixed point (*repulsive directions*). Independently on the convention on the versus of RG-time, relevant directions are UV-attractive and IR-repulsive, while irrelevant directions are UV-repulsive and IR-attractive. The domain of the theory space spanned by all the attractive directions constitutes the basin of attraction of the fixed point.

- *Universality class*: since the correlation length tends to diverge, a statistical system near criticality (i.e. near a second order phase transition) can be safely described by an appropriate field theory close to a fixed point in the theory space. Besides, its critical behaviour is totally determined by the property of the fixed point. Therefore, any statistical system, whose representative field theory at criticality belongs to the IR-attractive basin of a given fixed point, shares the same critical properties of all the other theories in the IR-attractive basin, even if the nature of the systems are very different away from criticality. The IR-attractive basin of a fixed point thus determines an universality class of critical systems.

- *Critical exponents*: because of the above relation between fixed points and critical systems, the eigenvalues of the stability matrix are often called critical exponents, as many properties of critical systems are described by power law behaviours whose exponents are determined by the eigenvalues of the stability matrix.

- *Asymptotic safety*: fixed points are of great interest also for high energy physics because, as we have already seen, they correspond to field theories whose UV cut-off can be safely sent to infinity. Therefore, a renormalizable theory able to describe physics up to any length scale must flow into a UV-attractive fixed point with a finite number of relevant directions. This last condition is needed if we want our theory to be predictive, otherwise an infinite number of couplings would be needed to properly describe the flow of the theory while approaching the fixed point or moving away from it in the IR limit. A field theory whose RG-flow satisfies the above conditions is said asymptotically safe[5]. *Asymptotic freedom* is a special case of asymptotic safety characterized by a free fixed point.

The picture described above is really charming, however we have completely ignored the problems that afflict it. The main one is that we still have to perform a functional integration, even if restricted to a thin momentum shell (see (1.34)), and such shell integration is technically as difficult as a complete integration. In non trivial models we are usually unable to perform it exactly, therefore it does not look like we have gone much further in tackling the problem. However, the above formulation has a great advantage: it admits a full variety of approximation schemes that are reliable even when couplings are not small. This is its main power, besides the qualitative understanding of the RG-flow it depicts. Indeed all the perturbative results can also be re-derived if the approximation scheme used in computing the integration is the usual perturbative approach. Nevertheless we will show and study other approximation schemes that do not rely on weak couplings.

Let us conclude this section with two final remarks:

- It is possible to derive from the above integral formulation a differential flow equation for the Wilsonian effective action. This was performed for the first time by Polchinski and the equation is therefore often called after his and Wilson's name.

- The above procedure has a hidden scheme dependence in the definition of the momentum shell over which to perform the integration, i.e. in the definition of the cut-off regulator. As this scheme dependence is common to any kind of FRG technique, we will discuss it in more detail for the following approach.

1.2.2 Wetterich-Morris approach

The ideas and the technical formulation of this approach are almost identical to the ones involved in the Wilson-Polchinski approach. What is different is the functional followed in its flow with the scale. In fact we will study a scale dependent effective action rather than a scale dependent Wilsoniaan action. Or more precisely, we will study a functional that interpolates between the two.

[5]Asymptotic safety was introduced by Steve Weinberg as a possible scenario for a consistent theory of quantum gravity; a standard reference may be found in [12]

In formal terms, the object we will be dealing with is a family of scale dependent functionals $\hat{\Gamma}_k[\phi_{cl}, \{g\}_k]$, where the subscript k identifies the momentum scale, such that:

$$\begin{cases} \hat{\Gamma}[\phi_{cl}, \{g\}_k] \xrightarrow[k\to\Lambda]{} S_\Lambda[\phi_{cl}, \{g\}_\Lambda] \\ \hat{\Gamma}[\phi_{cl}, \{g\}_k] \xrightarrow[k\to 0]{} \Gamma[\phi_{cl}, \{g\}_0] \end{cases} \tag{1.44}$$

where:

- $S_\Lambda[\phi_{cl}, \{g\}_\Lambda]$ is the classical action of the theory, defined at the cut-off scale Λ and characterized by a given set of couplings $\{g\}_\Lambda$
- $\Gamma[\phi_{cl}, \{g\}_0]$ is the full effective action of the theory, as defined in (1.15), and is characterized by its own set of couplings $\{g\}_0$
- $\hat{\Gamma}[\phi_{cl}, \{g\}_k]$ is called *effective average action*

In order to make notation simpler, we will no more write explicitly the sets of couplings that characterize the functionals, just bearing in mind the RG picture of flowing couplings.

Let us now construct the effective average action. The key idea is to modify the generating functional of correlation functions by adding an IR cut-off to the classical action. This will kill the propagation of long-distance modes while leaving unaltered the propagation of short-distance modes. Thus we define:

$$Z_k[J] \equiv \int D[\phi]\, e^{-S_\Lambda[\phi] + J_A\phi^A - \Delta S_k[\phi]} \tag{1.45}$$

where the IR cut-off $\Delta S_k[\phi]$ has to satisfy an appropriate set of conditions:

- we assume it to be quadratic:

$$\Delta S_k[\phi] = \frac{1}{2}\phi^A (R_k)_{AB}\phi^B \tag{1.46}$$

- we assume R_k to be a function of the kinetic operator $\mathcal{K}$ of the classical action such that, once in diagonal form:

$$\mathcal{K} + R_k[K] \to \lambda_i^2 + R_k[\lambda_i^2] \tag{1.47}$$

  where λ_i^2 are the eigenvalues of the kinetic operator (which is supposed to be positive definite)

- we assume $R_k[\lambda^2]$ to satisfy:
  - $R_0[\lambda^2] = 0 \quad \forall \lambda^2$ so that $Z_{k=0}[J] = Z[J]$

– $R_\Lambda[\lambda^2] = +\infty \quad \forall\lambda^2$ so that fluctuations of any kind do not propagate at the scale Λ
– $R_k[\lambda^2] > 0$ for $\lambda^2 < k^2$ so that the propagation of low momentum fluctuations is suppressed by a mass-like term
– $R_k[\lambda^2] \simeq 0$ for $\lambda^2 > k^2$ so that the propagation of high momentum fluctuations is unaffected

Once a scale dependent generator of correlation functions is defined, we can go on and define a scale dependent generator of connected correlation functions:

$$W_k[J] \equiv \ln Z_k[J] \tag{1.48}$$

and a scale dependent classical field:

$$\begin{aligned}
\phi^A_{cl\,k}[J] &\equiv \langle \phi^A \rangle_{J\,k} && (1.49)\\
&\equiv \frac{1}{Z_k[J]} \int D[\phi]\, \phi^A\, e^{-S[\phi]+J_B\phi^B-\Delta S_k[\phi]} && (1.50)\\
&= \frac{\delta W_k[J]}{\delta J_A} && (1.51)
\end{aligned}$$

Then we invert the functional part of the above relation and we get a scale dependent classical source:

$$\phi_{cl\,k}[J] = \phi_{cl}[J;k] \quad \Rightarrow \quad J_k[\phi_{cl}] = J[\phi_{cl};k] \tag{1.52}$$

Finally, we can define the effective average action as a modified Legendre transform of $W_k[J]$:

$$\hat{\Gamma}_k[\phi_{cl}] \equiv \underbrace{\phi^A_{cl} J_{k\,A}[\phi_{cl}] - W_k[J_k[\phi_{cl}]]}_{\equiv \Gamma_k[\phi]} - \Delta S_k[\phi_{cl}] \tag{1.53}$$

where $\Delta S_k[\phi_{cl}]$ has been introduced in order to balance its presence in $Z_k[J]$ and recover the appropriate limit:

$$\hat{\Gamma}_k[\phi_{cl}] \xrightarrow[k\to\Lambda]{} S_\Lambda[\phi_{cl}] \tag{1.54}$$

while at the same time preserving the limit:

$$\hat{\Gamma}_k[\phi_{cl}] \xrightarrow[k\to 0]{} \Gamma[\phi_{cl}] \tag{1.55}$$

Having defined the effective average action we now want to derive the flow equation it satisfies. We start with:

$$\begin{aligned}
\frac{\partial Z_k[J]}{\partial k} &= \int D[\phi] e^{-S[\phi]+J_A\phi^A-\Delta S_k[\phi]} \left(-\frac{\partial}{\partial k} \Delta S_k[\phi] \right) && (1.56)\\
&= \int D[\phi] e^{-S[\phi]+J_A\phi^A-\Delta S_k[\phi]} \left(-\frac{1}{2} \phi^A \left[\partial_k(R_k)\right]_{AB} \phi^B \right) && (1.57)\\
&= -\frac{1}{2} \left\{ \frac{\delta}{\delta J_A} \left[\partial_k(R_k)\right]_{AB} \frac{\delta}{\delta J_B} \right\} Z_k[J] && (1.58)
\end{aligned}$$

and:

$$\frac{\partial W_k[J]}{\partial k} = \frac{1}{Z_k[J]}\partial_k Z_k[J] \tag{1.59}$$

$$= -\frac{1}{2}\left(\frac{\delta W_k[J]}{\delta J_A}\left[\partial_k(R_k)\right]_{AB}\frac{\delta W_k[J]}{\delta J_B} + \left[\partial_k(R_k)_{AB}\right]\frac{\delta^2 W_k[J]}{\delta J_A \delta J_B}\right) \tag{1.60}$$

$$= -\frac{1}{2}\left[\partial_k(R_k)\right]_{AB}\left(\frac{\delta W_k[J]}{\delta J_A}\frac{\delta W_k[J]}{\delta J_B} + \frac{\delta^2 W_k[J]}{\delta J_A \delta J_B}\right) \tag{1.61}$$

Then we check that eq.(1.18) holds true even with a sliding scale:

$$\begin{aligned}\frac{\delta \Gamma_k[\phi_{cl}]}{\delta \phi_{cl}^A} &= \frac{\delta J_{k\,B}[\phi_{cl}]}{\delta \phi_{cl}^A}\phi_{cl}^B + J_{k\,A}[\phi_{cl}] - \left.\frac{\delta W_k[J_k]}{\delta J_{k\,B}}\right|_{J_k[\phi_{cl}]}\frac{\delta J_{k\,B}[\phi_{cl}]}{\delta \phi_{cl}^A} \\ &= \frac{\delta J_{k\,B}[\phi_{cl}]}{\delta \phi_{cl}^A}\phi_{cl}^B + J_{k\,A}[\phi_{cl}] - \phi_{cl}^B\frac{\delta J_{k\,B}[\phi_{cl}]}{\delta \phi_{cl}^A} \\ &= J_{k\,A}[\phi_{cl}]\end{aligned} \tag{1.62}$$

Besides we notice that:

$$\frac{\delta \Gamma_k[\phi_{cl}]}{\delta \phi_{cl}^A} = \frac{\delta \hat{\Gamma}_k[\phi_{cl}]}{\delta \phi_{cl}^A} + \frac{\delta \Delta S_k[\phi_{cl}]}{\delta \phi_{cl}^A} \tag{1.63}$$

$$= \frac{\delta \hat{\Gamma}_k[\phi_{cl}]}{\delta \phi_{cl}^A} + (R_k)_{AB}\,\phi_{cl}^B \tag{1.64}$$

$$\Rightarrow \quad \frac{\delta \hat{\Gamma}_k[\phi_{cl}]}{\delta \phi_{cl}^A} = J_{k\,A}[\phi_{cl}] - (R_k)_{AB}\,\phi_{cl}^B \tag{1.65}$$

and we can also switch the functional dependence for later convenience:

$$\left.\frac{\delta \hat{\Gamma}_k[\phi_{cl}]}{\delta \phi_{cl}^A}\right|_{\phi_{cl}[J;k]} = J_A - (R_k)_{AB}\,\phi_{cl}^B\big|_{\phi_{cl}[J;k]} \tag{1.66}$$

Finally, if we want to compute the k-derivative of the effective average action at fixed J we need to consider:

$$\hat{\Gamma}_k[\phi_{cl}] \quad \Rightarrow \quad \hat{\Gamma}[\phi_{cl}[J;k];k] \tag{1.67}$$

and consequently, from the above functional structure, we get:

$$\left.\frac{\partial \hat{\Gamma}_k[\phi_{cl}]}{\partial k}\right|_J \equiv \left.\frac{\partial \hat{\Gamma}[\phi_{cl}[J;k];k]}{\partial k}\right|_J \tag{1.68}$$

$$= \left.\frac{\partial \hat{\Gamma}[\phi_{cl};k]}{\partial k}\right|_{\phi_{cl}[J;k]} + \left.\frac{\partial\,\phi_{cl}^A[J;k]}{\partial k}\right|_J \left.\frac{\delta \hat{\Gamma}[\phi_{cl};k]}{\delta \phi_{cl}^A}\right|_{\phi_{cl}[J;k]} \tag{1.69}$$

$$\Rightarrow \quad \left.\frac{\partial \hat{\Gamma}[\phi_{cl};k]}{\partial k}\right|_{\phi_{cl}[J;k]} = \left.\frac{\partial \hat{\Gamma}_k[\phi_{cl}]}{\partial k}\right|_J - \left.\frac{\partial\,\phi_{cl}^A[J;k]}{\partial k}\right|_J \left.\frac{\delta \hat{\Gamma}[\phi_{cl};k]}{\delta \phi_{cl}^A}\right|_{\phi_{cl}[J;k]} \tag{1.70}$$

while, according to the definition of the effective average action (1.53), we can also write:

$$\left.\frac{\partial\hat{\Gamma}_k[\phi_{cl}]}{\partial k}\right|_J = \left.\frac{\partial\,\phi^A_{cl}[J;k]}{\partial k}\right|_J J_A - \left.\frac{\partial W_k[J]}{\partial k}\right|_J - \left.\frac{\partial\Delta S_k[\phi_{cl}]}{\partial k}\right|_J \tag{1.71}$$

therefore, putting all pieces together, we arrive at:

$$\begin{aligned}\left.\frac{\partial\hat{\Gamma}[\phi_{cl};k]}{\partial k}\right|_{\phi_{cl}[J;k]} &= \left.\frac{\partial\,\phi^A_{cl}[J;k]}{\partial k}\right|_J J_A - \left.\frac{\partial W_k[J]}{\partial k}\right|_J - \left.\frac{\partial\Delta S_k[\phi_{cl}]}{\partial k}\right|_J - \left.\frac{\partial\,\phi^A_{cl}[J;k]}{\partial k}\right|_J \left.\frac{\delta\hat{\Gamma}[\phi_{cl};k]}{\delta\phi^A_{cl}}\right|_{\phi_{cl}[J;k]} \\ &= \underbrace{-\left.\frac{\partial W_k[J]}{\partial k}\right|_J}_{a}\underbrace{-\left.\frac{\partial\Delta S_k[\phi_{cl}]}{\partial k}\right|_J}_{b}\underbrace{+\left.\frac{\partial\,\phi^A_{cl}[J;k]}{\partial k}\right|_J\left(J_A - \left.\frac{\delta\hat{\Gamma}[\phi_{cl};k]}{\delta\phi^A_{cl}}\right|_{\phi_{cl}[J;k]}\right)}_{c}\end{aligned} \tag{1.72}$$

We now need to split, as in (1.69), the k-derivative at fixed J of $\Delta S_k[\phi_{cl}]$ in two contributions:

$$\left.\frac{\partial\Delta S_k[\phi_{cl}]}{\partial k}\right|_J \equiv \left.\frac{\partial\Delta S[\phi_{cl}[J;k];k]}{\partial k}\right|_J \tag{1.73}$$

$$= \left.\frac{\partial\Delta S[\phi_{cl};k]}{\partial k}\right|_{\phi_{cl}[J;k]} + \left.\frac{\partial\,\phi^A_{cl}[J;k]}{\partial k}\right|_J \left.\frac{\partial\Delta S[\phi_{cl};k]}{\delta\phi^A_{cl}}\right|_{\phi_{cl}[J;k]} \tag{1.74}$$

then we need to use the previous results, reported in (1.61) and (1.66), and the definitions of $\Delta S_k[\phi_{cl}]$ and ϕ_{cl}, reported in (1.46) and (1.51), to simplify the above result to:

$$\begin{aligned}\left.\frac{\partial\hat{\Gamma}[\phi_{cl};k]}{\partial k}\right|_{\phi_{cl}[J;k]} &= \underbrace{\frac{1}{2}\left[\partial_k(R_k)\right]_{AB}\left(\phi^A_{cl}[J;k]\phi^B_{cl}[J;k] + \frac{\delta^2 W_k[J]}{\delta J_A\delta J_B}\right)}_{a} \\ &\quad \underbrace{-\frac{1}{2}\phi^A_{cl}[J;k]\left[\partial_k(R_k)\right]_{AB}\phi^B_{cl}[J;k] - \left.\frac{\partial\,\phi^A_{cl}[J;k]}{\partial k}\right|_J (R_k)_{AB}\,\phi^B_{cl}[J;k]}_{b} \\ &\quad \underbrace{+\left.\frac{\partial\,\phi^A_{cl}[J;k]}{\partial k}\right|_J (R_k)_{AB}\,\phi^B_{cl}[J;k]}_{c} \\ &= \frac{1}{2}\left[\partial_k(R_k)\right]_{AB}\frac{\delta^2 W_k[J]}{\delta J_A\delta J_B}\end{aligned} \tag{1.75}$$

The final step of the derivation is to make use of the relation between the Hessians of $W_k[J]$ and $\hat{\Gamma}_k[\phi_{cl}]$:

$$\frac{\delta^2 W_k}{\delta J_A\delta J_B} = \frac{\delta\phi^A_{cl}}{\delta J_B} = \left(\frac{\delta J_B}{\delta\phi^A_{cl}}\right)^{-1} = \left(\frac{\delta^2\Gamma_k}{\delta\phi^A_{cl}\delta\phi^B_{cl}}\right)^{-1} = \left(\frac{\delta^2\,\hat{\Gamma}_k + \Delta S_k}{\delta\phi^A_{cl}\,\delta\phi^B_{cl}}\right)^{-1} \tag{1.76}$$

hence we arrive at:

$$\frac{\partial \hat{\Gamma}_k[\phi_{cl}]}{\partial k} = \frac{1}{2}\Big[\partial_k(R_k)\Big]_{AB}\left[\left(\hat{\Gamma}_k^{(2)}[\phi_{cl}] + R_k\right)^{-1}\right]^{AB} \tag{1.77}$$

If we define the RG-time as:

$$t = \ln \frac{k}{\Lambda} \tag{1.78}$$

we can finally write down the *exact* renormalization group equation for the effective average action as:

$$\partial_t \hat{\Gamma}_k[\phi_{cl}] = \frac{1}{2}\Big[\partial_t(R_k)\Big]_{AB}\left[\left(\hat{\Gamma}_k^{(2)}[\phi_{cl}] + R_k\right)^{-1}\right]^{AB} \tag{1.79}$$

or, in a more compact notation, as:

$$\partial_t \hat{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left[(\partial_t R_k)\left(\hat{\Gamma}_k^{(2)} + R_k\right)^{-1}\right] \tag{1.80}$$

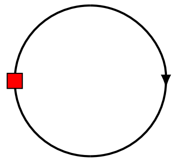
We will refer to the above result as to the *Wetterich equation* and it will be the starting point of this thesis project.

A few comments are now due:

- The Wetterich equation is *exact* and it involves the *exact* scale-dependent and field-dependent propagator $G_k^{(2)}[\phi_{cl}] = \left(\hat{\Gamma}_k^{(2)}[\phi_{cl}] + R_k\right)^{-1}$
- As for all FRG techniques, we are not able to solve the equation exactly and we will have to introduce approximation schemes. As for the Kadanoff-Wilson approach we discussed in the previous section, if we choose the perturbative expansion we can re-derive all the perturbative RG results. However we will show in the next chapters how to work with approximation schemes that do not rely on weak couplings.
- As for all FRG techniques, there is a scheme dependence hidden in the choice of the regulator. However, as far as the regulator satisfies the restrictions described above, any result found in the limit $k \to 0$ is independent on the chosen regulator.
- If the functional measure *and* the regulator preserve the symmetries of the the classical action, such symmetries will be preserved in the effective average action as well.
- Thanks to the requirements $\lim_{\frac{q^2}{k^2}\to 0} R_k(q^2) > 0$ and $\lim_{\frac{k^2}{q^2}\to 0} R_k(q^2) = 0$, while the regulator R_k acts as a IR cut-off, his time derivative $\partial_t R_k$ acts as a UV cut-off. Therefore all divergences are avoided during calculations.

- We can notice that the above flow equation has a 1-loop structure:

$$\partial_t \hat{\Gamma}_k = \frac{1}{2}$$



where the red square represents $\partial_t R_k$ and the line represents $G_k^{(2)} = \left(\hat{\Gamma}_k^{(2)} + R_k\right)^{-1}$. However we shall not confuse the *exact* propagator involved in this equation with the Feynman propagators involved in standard perturbative calculations.

Chapter 2

Classical approximation schemes for Wetterich equation

In this chapter we will briefly review the two most common non-perturbative approximation schemes used for solving Wetterich equation, they are known as *vertex expansion* and *operator expansion*. They both consist in solving the equation in a *restricted functional space* and they do not rely on any kind of weak-coupling hypothesis. We will then show how to obtain non trivial results for scalar models using a simple form of operator expansion and appropriate numerical techniques.

We will restrict from now on to systems involving just one scalar field and, for notational simplicity, we will write the effective average action as:

$$\Gamma_k[\phi] \tag{2.1}$$

bearing in mind that the exact notation would be:

$$\Gamma_k \to \hat{\Gamma}_k \qquad\qquad \phi \to \phi_{cl} \tag{2.2}$$

Besides we will use a dot to represent a RG-time derivative[1], hence Wetterich equation will now read:

$$\dot{\Gamma}_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\dot{R}_k\left(\Gamma_k^{(2)} + R_k\right)^{-1}\right] \tag{2.3}$$

[1]Keep in mind that, at this level, all RG-time dependence is stored inside the functional form of $\Gamma_k[\phi]$, i.e. in the couplings, ϕ being time-independent.

2.1 Vertex expansion

This approximation scheme was introduced by Morris (e.g. [13]) and is based on the following observation. If we expand the effective average action in powers of the field we obtain (we work in the hypothesis on no SSB):

$$\Gamma_k[\phi] = \sum_{n=0}^{+\infty} \frac{1}{n!} \int d^D x_1 \dots \int d^D k_n \, \Gamma_k^{(n)}(x_1, \dots, x_n) \, \phi(x_1) \dots \phi(x_n) \tag{2.4}$$

where $\Gamma_k^{(n)}(x_1, \dots, x_n)$ is the scale dependent n-point (1PI) proper vertex. Then if we study the expansion of the exact flow equation we obtain an infinite tower of coupled equations for the proper vertices:

$$\big(\dot{\Gamma}_k^{(1)}\big)_x = -\frac{1}{2}\big(\dot{R}_k\big)_{ab}\big(G_k^{(2)}\big)^{bc}\big(\Gamma_k^{(3)}\big)_{cxd}\big(G_k^{(2)}\big)^{da} \tag{2.5}$$

$$\begin{aligned}\big(\dot{\Gamma}_k^{(2)}\big)_{xy} &= -\frac{1}{2}\big(\dot{R}_k\big)_{ab}\big(G_k^{(2)}\big)^{bc}\big(\Gamma_k^{(4)}\big)_{cxyd}\big(G_k^{(2)}\big)^{da} \\ &\quad + \big(\dot{R}_k\big)_{ab}\big(G_k^{(2)}\big)^{bc}\big(\Gamma_k^{(3)}\big)_{cxd}\big(G_k^{(2)}\big)^{de}\big(\Gamma_k^{(3)}\big)_{eyf}\big(G_k^{(2)}\big)^{fa}\end{aligned} \tag{2.6}$$

$$\begin{aligned}\big(\dot{\Gamma}_k^{(3)}\big)_{xyz} &= -\frac{1}{2}\big(\dot{R}_k\big)_{ab}\big(G_k^{(2)}\big)^{bc}\big(\Gamma_k^{(5)}\big)_{cxyzd}\big(G_k^{(2)}\big)^{da} \\ &\quad + 3\big(\dot{R}_k\big)_{ab}\big(G_k^{(2)}\big)^{bc}\big(\Gamma_k^{(4)}\big)_{cxyd}\big(G_k^{(2)}\big)^{de}\big(\Gamma_k^{(3)}\big)_{ezf}\big(G_k^{(2)}\big)^{fa} \\ &\quad - 3\big(\dot{R}_k\big)_{ab}\big(G_k^{(2)}\big)^{bc}\big(\Gamma_k^{(3)}\big)_{cxd}\big(G_k^{(2)}\big)^{de}\big(\Gamma_k^{(3)}\big)_{eyf}\big(G_k^{(2)}\big)^{fg}\big(\Gamma_k^{(3)}\big)_{gzh}\big(G_k^{(2)}\big)^{ha}\end{aligned} \tag{2.7}$$

$$\vdots$$

where $\big(G_k^{(2)}\big) = \Big(\Gamma_k^{(2)} + R_k\Big)^{-1}$. By a simple generalization it can be argued that the n-th equation reads:

$$\dot{\Gamma}_k^{(n)} = \mathcal{F}_n[\Gamma_k^{(2)}, \dots, \Gamma_k^{(n+2)}] \tag{2.8}$$

and what this result is telling us is that the exact flow equation is equivalent to an *infinite hierarchy* of flow equations for the proper vertices.

Such infinite set of equations can not be solved exactly, we need to truncate it at some point. The *vertex expansion* approximation scheme is thus based on solving the above system with an appropriate truncation. Examples of truncation consist, for instance, in neglecting the contributions of higher order vertices, in writing them as functions of lower order ones, according to a previously defined ansatz, or taking them as contact terms. Up to now, no general rule has been found for the best approximation to be made. Depending on the problem, one choice can be better than another and some physical insight is needed in order to find the correct truncation for a given system. However, let us notice that this method can, in principle, be used to keep track of the momentum dependence of the vertices while truncating the field dependence of $\Gamma_k[\phi]$.

2.2 Operator expansion

In this scheme we realize the truncation by constructing the effective action with a given set of operators and restricting the flow to the chosen ansatz. This means in general that:

$$\Gamma_k[\phi] = \sum_i g_i(k)\, O_i \tag{2.9}$$

where the number of couplings may be finite as well as infinite and organized into functions. Different kinds of ansatz can been tested, the most used expression consists in a derivative expansion (e.g. [14]). In increasing level of complexity, we can mention:

- LPA (Local Potential Approximation): the effective action is characterized by an unknown local (and scale dependent) potential $V_k(\phi)$:

$$\Gamma_k[\phi] = \int d^D x \left(\frac{1}{2}\partial_\mu \phi(x)\, \partial^\mu \phi(x) + V_k(\phi(x)) \right) \tag{2.10}$$

- LPA': a (scale-dependent) field strength renormalization constant Z_k is introduced:

$$\Gamma_k[\phi] = \int d^D x \left(\frac{1}{2} Z_k\, \partial_\mu \phi(x)\, \partial^\mu \phi(x) + V_k(\phi(x)) \right) \tag{2.11}$$

  with this ansatz an assumption on the field configuration is needed in order to define the flow of Z_k and hence the anomalous dimension. We will discuss it in more details in chapter 4.

- Leading order in derivative expansion: the effective action depends on two unknown (scale dependent) local functions:

$$\Gamma_k[\phi] = \int d^D x \left(\frac{1}{2} Z_k(\phi(x))\, \partial_\mu \phi(x)\, \partial^\mu \phi(x) + V_k(\phi(x)) \right) \tag{2.12}$$

The above truncations will give an RG flow for $V_k(\phi)$, $\big(Z_k, V_k(\phi)\big)$ and $\big(Z_k(\phi), V_k(\phi)\big)$ respectively. The underlying idea is that we want to accurately describe the long distance physics (we keep only the lowest order of the expansion in the derivative, which is the leading order in the zero momentum limit) and many important informations are already stored in the effective potential $V_{k=0}$. In the next sections we will actually show how, even in the simple LPA scheme, interesting results can be obtained.

2.3 LPA for scalar models

As a paradigmatic example, let us study the flow equation in LPA for a scalar field in D dimensions with a $\mathbb{Z}_2$ symmetry.

In order to follow the flow of the effective potential V_k (i.e. the flow of the infinite set of couplings it contains) we evaluate Wetterich equation on a constant field configuration $\phi(x) = \phi$, in such a way that:

$$\dot{\Gamma}_k(\phi) = \int_x \dot{V}_k(\phi) \tag{2.13}$$
$$= L^D \dot{V}_k(\phi) \tag{2.14}$$

where $\int_x = \int d^d x$ and $\delta^D(0) = L^D$ is the volume of the hypercube used to regularize the theory. At the same time, the right and side of Wetterich equation reads:

$$\frac{1}{2}\int_x\int_y (\dot{R}_k)_{xy}(\Gamma^{(2)}_k(\phi) + R_k)^{-1}_{yx} = \frac{1}{2}\int_x\int_y \dot{R}_k(-\Box_x)\delta^D_{xy}\frac{\delta^D_{yx}}{[-\Box_y + V''_k(\phi) + R_k(-\Box_y)]} \tag{2.15}$$

$$= L^D \frac{1}{2}\int_x \frac{\dot{R}_k(-\Box_x)}{[-\Box_x + V''_k(\phi) + R_k(-\Box_x)]} \tag{2.16}$$

and switching to momentum space we get:

$$\int_x \frac{\dot{R}_k(-\Box_x)}{[-\Box_x + V''_k(\phi) + R_k(-\Box_x)]} = \int_q \frac{\dot{R}_k(q^2)}{[q^2 + V''_k(\phi) + R_k(q^2)]} \tag{2.17}$$

where $\int_q = \int \frac{d^D q}{(2\pi)^D}$.

Finally we can write down the flow equation for the effective potential as:

$$\dot{V}_k(\phi) = \frac{1}{2}\int_q \frac{\dot{R}_k(q^2)}{[q^2 + V''_k(\phi) + R_k(q^2)]} \tag{2.18}$$

In order to compute the integral we now have to write explicitly the cut-off function. It has been shown [15] that the optimized cut-off is:

$$R_k(q^2) \equiv (k^2 - q^2)\Theta(k^2 - q^2) \tag{2.19}$$
$$\Rightarrow \dot{R}_k(q^2) = 2k^2\Theta(k^2 - q^2) + 2k^2(k^2 - q^2)\delta(k^2 - q^2) \tag{2.20}$$
$$\simeq 2k^2\Theta(k^2 - q^2) \tag{2.21}$$

where the symbol $\simeq$ is used to express equivalence, once the expressions are inserted inside integrals.

Using the above regulator the integral can be easily computed, thus getting:

$$\dot{V}_k(\phi) = \frac{\Omega_D}{(2\pi)^D}\frac{k^D}{D}\frac{1}{\left[1 + \frac{V''_k(\phi)}{k^2}\right]} \tag{2.22}$$

where $\Omega_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$ is the solid angle in D dimensions. It is thus convenient to write the equation as:

$$\dot{V}_k(\phi) = C_D \frac{1}{\left[1 + \frac{V_k''(\phi)}{k^2}\right]} k^D \tag{2.23}$$

where:

$$C_D = \frac{1}{(4\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2}+1)} \tag{2.24}$$

In order to study the fixed point structure of the flow equation, we need to write it completely in terms of dimensionless variables (i.e. write all quantities with respect to the sliding scale). This is the implementation of the *rescaling* procedure described in the first chapter.

In a D dimensional space we know that (in natural units):

$$\begin{cases} [\phi] = [\text{mass}]^{\frac{D-2}{2}} \\ [V_k] = [\text{mass}]^D \end{cases} \tag{2.25}$$

therefore we need to define the following dimensionless quantities:

$$\begin{cases} \varphi \equiv k^{-\frac{D-2}{2}}\phi \equiv k^{-d_\phi}\phi \\ v_k(\varphi) \equiv k^{-D} V_k(\phi(\varphi)) \end{cases} \tag{2.26}$$

where $d_\phi = \frac{D-2}{2}$ is the mass-dimension of the field, and to write the flow equation for this dimensionless effective potential.

Switching to dimensionless variables we have introduced a RG-time dependence in the argument of the effective potential, besides its usual RG-time dependence of the functional form. Actually we notice that:

$$\partial_t V_k(\phi) = \frac{d}{dt} V_k(\phi) \tag{2.27}$$

$$= \frac{d}{dt}\{k^D v_k[\varphi(\phi)]\} \tag{2.28}$$

$$= D k^D\, v_k[\varphi(\phi)] + k^D\, \frac{d}{dt}\{v_k[\varphi(\phi)]\} \tag{2.29}$$

$$= D k^D\, v_k[\varphi(\phi)] + k^D\, (\dot{v}_k)[\varphi(\phi)] + k^D\, v_k'[\varphi(\phi)] \frac{d}{dt}\{\varphi(\phi)\} \tag{2.30}$$

$$= D k^D\, v_k[\varphi(\phi)] + k^D\, (\dot{v}_k)[\varphi(\phi)] - k^D (v_k')[\varphi(\phi)]\, d_\phi\, [\varphi(\phi)] \tag{2.31}$$

$$= k^D \left\{ (\dot{v}_k)[\varphi(\phi)] - \left(d_\phi\, [\varphi(\phi)]\, (v_k')[\varphi(\phi)] - D\, v_k[\varphi(\phi)] \right) \right\} \tag{2.32}$$

where a dot is used to express a partial RG-time derivative (i.e. acting only on the functional form) while $\frac{d}{dt}$ expresses a total RG-time derivative (i.e. acting both on the functional form and the argument). Moreover:

$$V_k''(\phi) = \frac{\partial^2}{\partial^2\phi}\{k^D v_k[\varphi(\phi)]\} \tag{2.33}$$
$$= k^D\, v_k''[\varphi(\phi)]\, k^{-(D-2)} \tag{2.34}$$
$$= k^2\, v_k''[\varphi(\phi)] \tag{2.35}$$

therefore the dimensionless flow equation becomes:

$$\big[\partial_t - \big(d_\phi\, \varphi\, \partial_\varphi - D\big)\big]\, v_k(\varphi) = C_D \frac{1}{\big[1 + v_k''(\varphi)\big]} \tag{2.36}$$

where it is evident that the classical scaling property of the potential would require a vanishing right hand side, which in turn is non zero and generates the non-trivial flow.

As a last step before numerical analysis, if we exploit the $\mathbb{Z}_2$ symmetry we can define:

$$\begin{cases} \rho = \phi^2 \\ u_k(\rho) = v_k(\varphi(\rho)) \end{cases} \tag{2.37}$$

in such a way that:

$$\begin{cases} v_k' = \sqrt{\rho}\,(2u_k') \\ v_k'' = (4\,\rho\, u_k'' + 2u_k') \end{cases} \tag{2.38}$$

and the flow equation becomes:

$$\big[\partial_t - \big(d_\phi\, 2\rho\, \partial_\rho - D\big)\big]\, u_k(\rho) = C_D \frac{1}{\big[1 + 4\,\rho\, u_k''(\rho) + 2u'(\rho)\big]} \tag{2.39}$$

2.4 Numerical solution of LPA equation

Having defined the LPA flow equation for a $\mathbb{Z}_2$ invariant[2] scalar model, we now want to extract numerical results about fixed points and critical exponents. Because of the non-linear part of the equation we are not able to find exact analytic solutions, except for trivial cases, and we need to rely on numerical integrations. The most used numerical methods involve *shooting methods* and *polynomial expansions* and we will now apply both to our problem.

[2]Actually, equation (2.36), at variance with equation (2.39), does not rely on $\mathbb{Z}_2$ symmetry, hence it is totally general.

2.4.1 Shooting methods

We are now going to use shooting methods in order to gain informations about the global properties of the problem.

Shooting method from the origin

First of all, we would like to know if the above flow equation predicts the existence of fixed points. In LPA a fixed point is identified by an effective dimensionless potential $v_*(\varphi)$ which does not flow with time:

$$\dot{v}_*(\varphi) = 0 \quad \Leftrightarrow \quad d_\phi \, \varphi \, v_*'(\varphi) - D \, v_*(\varphi) + C_D \frac{1}{\left[1 + v_*''(\varphi)\right]} = 0 \tag{2.40}$$

The search for a fixed point $v_*(\varphi)$ thus seems to be equivalent to the search for a $\mathbb{Z}_2$ invariant solution of the above second order non-linear ODE. This result can be quite puzzling as we know that a second order ODE admits a solution for any given couple of initial conditions. Thanks to the $\mathbb{Z}_2$ invariance, if we start the integration from $\varphi = 0$, we can easily reduce the arbitrary parameters to one, as we need to impose $v_*'(0) = 0$. However, there would still be an infinite number of fixed points, identified by one real parameter, for any arbitrary D; and this would be in contrast to physical results. Nevertheless, thanks to the non linear structure of the equation, we soon find out that arbitrary solutions can not be infinitely extended, as some singularity in encountered. Therefore, as we need a potential defined for any[3] value of φ, what we are really looking for are global solutions to the above equation. This global problem is non trivial and we will soon show that the answer brings a very interesting predictive result: for any $D > 2$ the number of fixed points is finite and is consistent with the results obtained by power counting renormalizability.

Actually, in a D dimensional space, the mass dimension of the operator ϕ^{2n} is equal to $n(D-2)$ therefore its coupling is relevant by power counting[4] as far as:

$$D - n(D-2) > 0 \quad \Leftrightarrow \quad n < \frac{D}{D-2} \tag{2.41}$$

which means that for any given g_{2n} it exists a critical dimension:

$$d_c(2n) = \frac{2n}{n-1} \tag{2.42}$$

[3]Since $\phi = k^{d_\phi}\varphi$, where $d_\phi > 0$, we need $\varphi \to +\infty$ in order to be able to describe finite values of the dimensionful field in the long distance limit $k \to 0$

[4]Remember that a coupling is relevant by power counting if its mass dimension is positive

at which g_{2n} becomes marginal. Then we can easily evaluate:

$$d_c(2) = +\infty \qquad d_c(4) = 4 \qquad d_c(6) = 3 \qquad d_c(8) = \frac{8}{3} \qquad d_c(10) = \frac{10}{4} \quad \dots \tag{2.43}$$

and consequently, as it is necessary to have $D < d_c(n)$ in order for g_{2n} to be relevant, we can classify:

$$\begin{cases} D \geq 4 & \Rightarrow \quad \text{Relevant couplings:} \quad g_2 \\ 3 \leq D < 4 & \Rightarrow \quad \text{Relevant couplings:} \quad g_2\,,\,g_4 \\ \frac{8}{3} \leq D < 3 & \Rightarrow \quad \text{Relevant couplings:} \quad g_2\,,\,g_4\,,\,g_6 \\ \frac{5}{2} \leq D < \frac{8}{3} & \Rightarrow \quad \text{Relevant couplings:} \quad g_2\,,\,g_4\,,\,g_6\,,\,g_8 \\ \quad\vdots & \\ D = 2 & \Rightarrow \quad \text{Relevant couplings:} \quad g_2\,,\,g_4\,,\,g_6\,,\,g_8\,,\,\dots\,,\,g_{+\infty} \end{cases} \tag{2.44}$$

Thus we see that, by lowering the dimension of space, an increasing number of relevant couplings appear. For any relevant coupling $g_{n>2}$ we expect that, switching on the corresponding interaction, the RG-flow can escape from the trivial fixed point (i.e. the free fixed point) and, eventually, reach a new non-trivial fixed point. Hence, by these power counting arguments, we may guess the number of non-trivial fixed points to be equal, for any given D, to the corresponding number of relevant couplings $g_{n>2}$ reported in (2.44). Which means that for $D \geq 4$ we expect only the trivial fixed point, while if $d_c(2n+2) \leq D < d_c(2n)$ we expect 1 trivial fixed point and $n-1$ non-trivial fixed points. As a limiting case, in $D = 2$ we expect an infinite number of fixed points and, actually, a 1-to-1 correspondence with the infinite number of conformal minimal unitary models can be realized. Finally, let us remark that, working with dimensionless quantities and a massive regulator, all couplings will be involved in the definition of the dimensionless fixed point. However, switching back to dimensionful quantities and taking the long distance limit, it may be showed that, for any D, $V_0(\phi)$ contains only marginal couplings.

Let us now show how the above predictions are stored inside the LPA fixed point equation. As already noticed, the number of integration parameters for a $\mathbb{Z}_2$ invariant solution is just one, therefore, for any given D we can numerically integrate the following Cauchy problem[5]:

$$\begin{cases} d_\phi\, \varphi\, v_*'(\varphi) - D\, v_*(\varphi) + C_D \frac{1}{\left[1+v_*''(\varphi)\right]} = 0 \\ v_*(0) = \frac{C_D}{D}\frac{1}{1+\sigma} \\ v_*'(0) = 0 \end{cases} \tag{2.45}$$

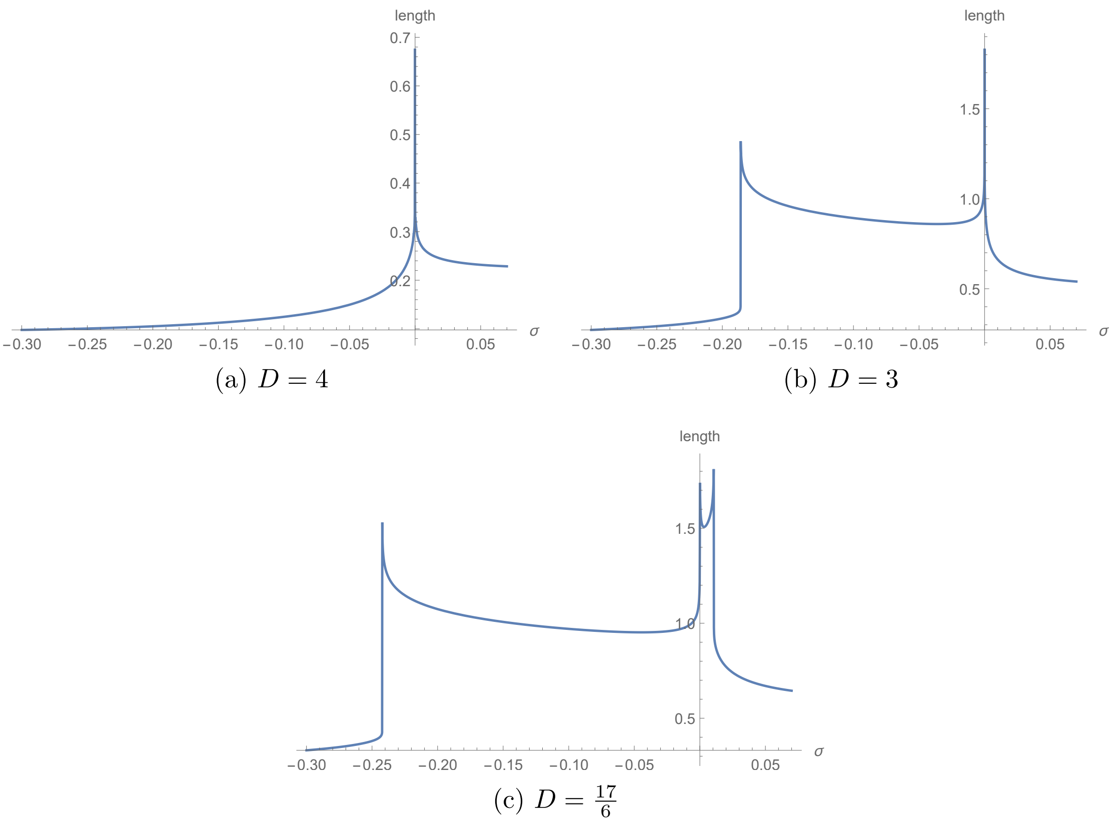
[5]The choice made for $v_*(0)$ enables us to identify σ with the value of $v_*''(0)$

and study how the domain of the solution varies with σ.

What we find out is that, for any given D, only a finite number of values of σ (consistent with the power counting analysis) produce global solutions, i.e. solutions whose domain can be arbitrarily extended refining the value of σ and the numerical precision of the algorithm.

In figure 2.1 we show plots of the length of the domain as a function of σ for:

- $D = 4$: only one (trivial) fixed point appears
- $D = 3$: the above trivial fixed point appears together with one non-trivial fixed point, the so-called *Wilson-Fisher fixed point*, which will be thoroughly studied later
- $D = \frac{17}{6}$: besides the trivial and the Wilson-Fisher fixed point, one more non-trivial fixed point appears (such fixed point, together with all the other fixed points appearing for $2 < D < 3$, are often referred to as *multicritical Ising fixed points*)



(a) $D = 4$ (b) $D = 3$ (c) $D = \frac{17}{6}$

Figure 2.1: Fixed point structure at different space dimensions

Shooting method from the asymptotic region

Having found the existence of a non-trivial fixed point in $D = 3$, we are now going to study it in detail. In this section we will show how a shooting method from the asymptotic region can be used to find a global description of the corresponding effective potential $v_*(\varphi)$.

In $D = 3$, we can write the LPA flow equation as:

$$\left[\partial_t - \left(\frac{1}{2}\,\varphi\,\partial_\varphi - 3\right)\right] v_k(\varphi) = \frac{1}{6\pi^2}\frac{1}{\left[1 + v_k''(\varphi)\right]} \tag{2.46}$$

and the corresponding fixed point equation becomes:

$$\left(\frac{1}{2}\,\varphi\,\partial_\varphi - 3\right) v_*(\varphi) + \frac{1}{6\pi^2}\frac{1}{\left[1 + v_*''(\varphi)\right]} = 0 \tag{2.47}$$

If we suppose (as required by its physical meaning) that the global non-trivial[6] solution of the equation is such that:

$$\lim_{\varphi\to+\infty} \frac{v_*(\varphi)}{\varphi^2} = +\infty \tag{2.48}$$

we can neglect the non linear part of the equation in the asymptotic region:

$$\left(\frac{1}{2}\,\varphi\,\partial_\varphi - 3\right) v_{*\,as}(\varphi) \simeq 0 \tag{2.49}$$

and find the following (consistent) asymptotic behaviour:

$$v_{*\,as}(\varphi) \sim \varphi^6 \tag{2.50}$$

Therefore we can write the asymptotic expansion for the solution we are looking for as:

$$v_{*\,as}(\varphi) = a\,\varphi^6(1 + \epsilon(\varphi)) \tag{2.51}$$

where a is, up to now, a free parameter and $\epsilon(\varphi)$ is a small correction, which we approximate with a finite polynomial:

$$\epsilon(\varphi) = \sum_{n=-N}^{-1} \lambda_{2n}\,\varphi^{2n} \tag{2.52}$$

[6]The constant potential $v_*(\varphi) = \frac{1}{18\pi^2}$ is a global solution to the problem corresponding to the trivial fixed point

If we insert the above expansion inside the complete fixed point equation and we request the first N coefficients of the Laurent expansion around $+\infty$ to be vanishing, we can write all coefficients λ_{2n} as functions of a. Doing this for $N = 12$ we get:

$$
\begin{aligned}
v_{*\,as}(\varphi\,;\,a) = a\varphi^6 &+ \frac{1}{900\,a\,\pi^2}\frac{1}{\varphi^4} - \frac{1}{37800\,a^2\,\pi^2}\frac{1}{\varphi^8} + \frac{1}{145800000\,a^3\,\pi^2}\frac{1}{\varphi^{12}} \\
&- \frac{1}{243000000\,a^3\,\pi^4}\frac{1}{\varphi^{14}} - \frac{1}{53460000\,a^4\,\pi^2}\frac{1}{\varphi^{16}} + \frac{1}{19136250\,a^4\,\pi^4}\frac{1}{\varphi^{18}} \\
&+ \frac{1}{19136250\,a^5\,\pi^4}\frac{1}{\varphi^{20}} - \frac{1}{32148900000\,a^5\,\pi^4}\frac{1}{\varphi^{22}} \\
&+ \frac{250a - 3\pi^4}{196830000000\,a^6\,\pi^6}\frac{1}{\varphi^{24}} + O\left(\frac{1}{\varphi^{26}}\right)
\end{aligned}
\tag{2.53}
$$

Once the asymptotic expansion is defined as a function of only parameter a, we can use it to generate the starting conditions for a numerical integration from the asymptotic region towards the origin. Requesting the $\mathbb{Z}_2$ symmetry to be satisfied (i.e. a vanishing first derivative in the origin) we can constraint a to the appropriate value. In practice this means that:

- we choose an asymptotic value φ_{as} from which to start our numerical shooting towards the origin. Such value must be big enough in order for the asymptotic expansion to be accurate and small enough in order for the future numerical integration to be reliable near the origin. For our example we choose $\varphi_{as} = 3$.
- we numerically solve the Cauchy problem:

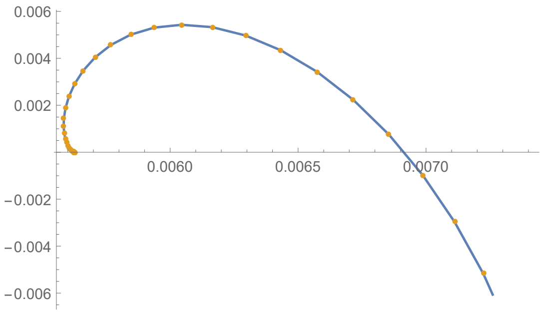
$$
\begin{cases}
d_\phi\,\varphi\,v_*'(\varphi) - D\,v_*(\varphi) + C_D\frac{1}{\left[1+v_*''(\varphi)\right]} = 0 \\
v_*(\varphi_{as}) = v_{*\,as}(\varphi_{as}\,;\,a) \\
v_*'(\varphi_{as}) = v_{*\,as}'(\varphi_{as}\,;\,a)
\end{cases}
\tag{2.54}
$$

  for different values of a.
- we vary a in such a way that:
  - the numerical integration reaches the origin. For our problem we find that $\forall a \in [10^{-3},10]$ this property is satisfied
  - we fine-tune the value of a in order to find $v_*'(0) = 0$. For our problem we find that the correct value is $a \simeq 3.50759$

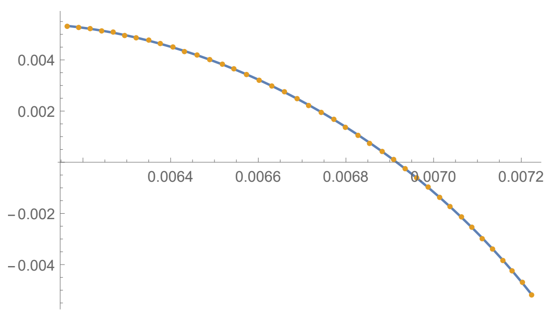
In figure 2.2 we show the results of the above procedure, while in figure 2.3 we compare the fixed point solutions found with the two shooting methods, from the origin and from the asymptotic region. Even if the solution shot from the origin breaks down, for numerical difficulties, at a finite value a φ, near the origin they perfectly agree.

Finally we show in figure 2.4 the shape of the dimensionful effective potential corresponding to the dimensionless Wilson-Fisher fixed point just found.

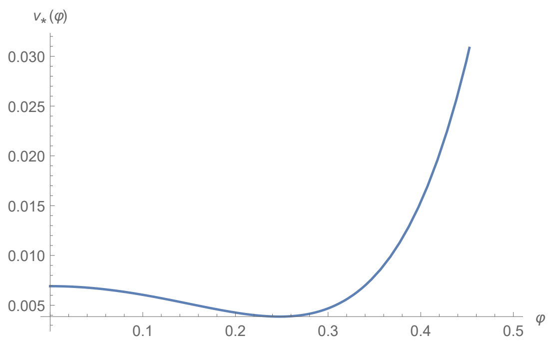
The shape of the dimensionless fixed point can be misleading, because it can be interpreted as the potential of a system with a $\mathbb{Z}_2$ symmetry spontaneously broken. However this is not correct: in order to make physical predictions we first need to switch to dimensionful quantities and the dimensionful potential, in the limit $k \to 0$, clearly shows that the Wilson-Fisher fixed point describes a macroscopic system at criticality (i.e. at a critical point where the $\mathbb{Z}_2$ symmetry is still not broken). Moreover it can be noticed that $V_0(\phi) \sim \phi^6$, hence the dimensionful effective potential in $D = 3$ is fully characterized by the only marginal coupling g_6.



(a) $(v''_*(0), v'_*(0))$ for $10^{-3} < a < 10$



(b) Zoom of $(v''_*(0), v'_*(0))$ around the correct value of a

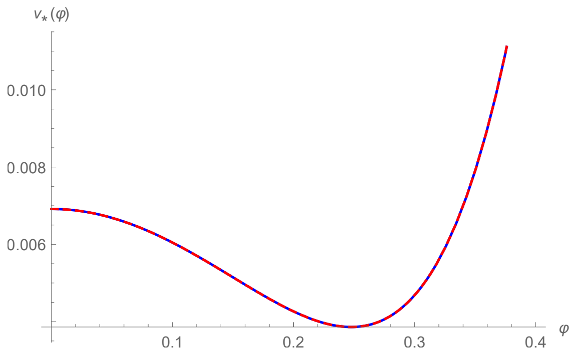


(c) The Wilson-Fisher fixed point found with asymptotic analysis

Figure 2.2

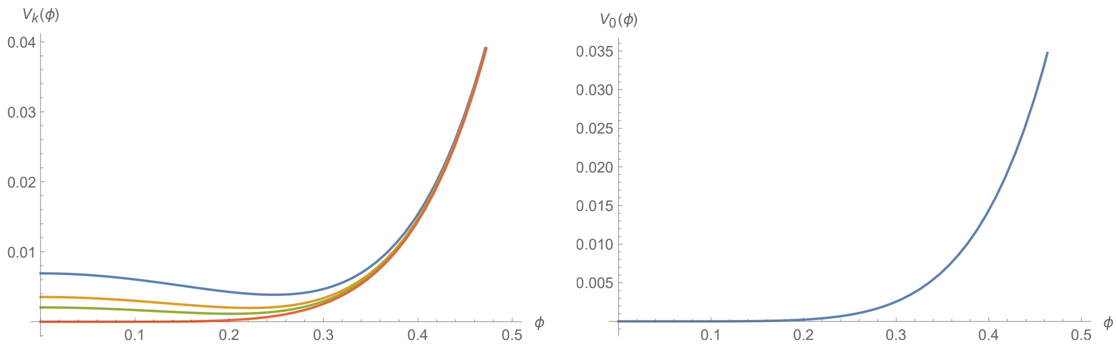
2.4.2 Polynomial expansions

Thanks to shooting methods we have been able to find the fixed point structure of the LPA equation ad different space dimensions and a global description of Wilson-Fisher



(a) Blue is the fixed point found shooting from the origin, dashed-red is the fixed point found shooting from the asymptotic region

Figure 2.3



(a) The dimensionful potential, corresponding to Wilson-Fisher fixed point, evaluated for some $k \leq 1$

(b) The dimensionful potential, corresponding to Wilson-Fisher fixed point, in the limit $k \to 0$

Figure 2.4

fixed point in $D = 3$. We will now continue to investigate Wilson-Fisher fixed point via polynomial expansions in order to extract informations about critical exponents.

For these methods it is more efficient to exploit the $\mathbb{Z}_2$ symmetry since the beginning and work with the variable $\rho = \varphi^2$ and the corresponding flow equation (see (2.39)):

$$\left[\partial_t - \Big(\rho\,\partial_\rho - 3\Big)\right] u_k(\rho) = \frac{1}{6\pi^2}\frac{1}{\left[1 + 4\,\rho\,u_k''(\rho) + 2u_k'(\rho)\right]}$$

Polynomial expansion around $\rho = 0$

If we suppose the effective potential to be analytic around $\rho = 0$:

$$u_k(\rho) = \sum_{n=0}^{+\infty} g_n(k)\rho^n \tag{2.55}$$

we can write the flow equation as:

$$\dot{u}_k(\rho) = \underbrace{\rho u_k'(\rho) - 3u_k(\rho) + \frac{1}{6\pi^2}\frac{1}{1 + 4\rho u_k''(\rho) + 2u_k'(\rho)}}_{\mathcal{F}[u_(\rho)]} \tag{2.56}$$

then we can Taylor expand both sides:

$$\sum_{n=0}^{+\infty} \dot{g}_n(k)\,\rho^n = \sum_{n=0}^{+\infty} \frac{1}{n!}\frac{\partial^n}{\partial^n \rho}\mathcal{F}[u_k(\rho)]\Big|_{\rho=0}\rho^n \tag{2.57}$$

and consequently, according to the definition of the β-functions, we get:

$$\beta_n = \frac{1}{n!}\frac{\partial^n}{\partial^n \rho}\mathcal{F}[u_k(\rho)]\Big|_{\rho=0} \tag{2.58}$$

$$= \beta_n(g) \tag{2.59}$$

Thus a fixed point would be identified by a set of coupling $g^* = \{g_n^*\}$ such that:

$$\beta_n(g^*) = 0 \quad \forall n \tag{2.60}$$

and the corresponding stability matrix would be:

$$M_{nm} = \frac{\partial \beta_n(g)}{\partial g_m}\Big|_{g^*} \tag{2.61}$$

We can try to solve numerically the above problem by using polynomial approximations of increasing order. Actually we find quite a good convergence in this case and if we expand the effective potential up to order ρ^8 we find the following expression for the non-trivial fixed point[7]:

$$u_*(\rho) \simeq -9(3)\rho^8 - 4(0)\rho^7 - 4.(8)\rho^6 + 3.(4)\rho^5 + 2.8(0)\rho^4 \tag{2.62}$$

$$+ 1.3(8)\rho^3 + 0.607(5)\rho^2 - 0.092(8)\rho^1 + 0.00691(2) \tag{2.63}$$

[7]Digits affected by uncertainness are reported in parenthesis

moreover the stability matrix can be diagonalized and the only relevant eigenvalue is $\lambda_1 = -1.54(1)$.

Since the Ising model is the most known system whose critical behaviour belongs to the universality class of the Wilson-Fisher fixed point it is customary to refer all results to the jargon of the Ising model. Therefore it is customary to write the above eigenvalue as $\nu = -\frac{1}{\lambda_1} = 0.64(9)$.

Finally, if we compare our result with the most precise result up to date $\nu = 0.62997(1)$ (obtained by conformal bootstrap [16]) we can conclude that the simple LPA is able to describe the relevant behaviour near criticality with an accuracy of 3%. Notice, however, that LPA intrinsically works in the hypothesis of vanishing anomalous dimension, therefore it can be reliable as far as this hypothesis is quite accurate. Conformal bootstrap predicts $\eta = 0.03629(8)$ for Wilson-Fisher fixed point and hence, as we have seen, LPA can be consistently applied.

Polynomial expansion around $\rho = \rho_0$

Since we know that the fixed point potential develops a non-trivial minimum, we can decide to repeat the above analysis expanding the potential around it. In fact, as a general rule of thumb for these problems, the best convergence is reached expanding the potential around the minima. Actually, as we will see, this expansion has a better convergence for the fixed point potential and the critical exponents.

If we expand the potential around the minimum we get:

$$u_k(\rho) = g_0(k) + \sum_{n=2}^{+\infty} g_n(k)(\rho - \rho_0(k))^n \tag{2.64}$$

where $\rho_0(k)$ is the minimum, whose value depends on the RG-time and has to be considered on the same footing of the other unknown coupling $g_n(k)$. On the other hand we can assume $g_1(k) \equiv 0$ because we request the potential to identically satisfy:

$$\frac{\partial u_k(\rho)}{\partial \rho}\Big|_{\rho=\rho_0(k)} = 0 \qquad \forall\, k \tag{2.65}$$

If we now write the flow equation as (see (2.56)):

$$\dot{u}_k(\rho) = \mathcal{F}[u_k(\rho)] \tag{2.66}$$

and we explicitly write the time derivative in the l.h.s. we get:

$$\dot{g}_0(k) + \sum_{n=2}^{+\infty} \dot{g}_n(k)(\rho - \rho_0(k))^n - \sum_{n=2}^{+\infty} g_n(k)\, n\, (\rho - \rho_0(k))^{n-1} \dot{\rho}_0(k) = \mathcal{F}[u_k(\rho)] \tag{2.67}$$

Then from (2.65) we find the evolution low for the minimum:

$$0 = \frac{d}{dt}\left(\frac{\partial u_k(\rho)}{\partial \rho}\Big|_{\rho=\rho_0(k)}\right) = \frac{\partial \dot{u}_k(\rho)}{\partial \rho}\Big|_{\rho=\rho_0(k)} + \frac{\partial^2 u_k(\rho)}{\partial^2 \rho}\Big|_{\rho=\rho_0(k)}\dot{\rho}_0(k) \quad (2.68)$$

$$\Rightarrow \qquad \dot{\rho}_0(k) = -\frac{\frac{\partial \dot{u}_k(\rho)}{\partial \rho}\Big|_{\rho=\rho_0(k)}}{\frac{\partial^2 u_k(\rho)}{\partial^2 \rho}\Big|_{\rho=\rho_0(k)}} = -\frac{\frac{\partial \mathcal{F}[u_k(\rho)]}{\partial \rho}\Big|_{\rho=\rho_0(k)}}{2g_2(k)} \quad (2.69)$$

Hence, substituting it in the flow equation and expanding the r.h.s as well we get:

$$\dot{g}_0(k) + \sum_{n=2}^{+\infty} \dot{g}_n(k)(\rho-\rho_0(k))^n + \sum_{n=2}^{+\infty} \frac{n\, g_n(k)\, \frac{\partial \mathcal{F}[u_k(\rho)]}{\partial \rho}\Big|_{\rho=\rho_0(k)}}{2g_2(k)}(\rho-\rho_0(k))^{n-1}$$
$$= \sum_{n=0}^{+\infty} \frac{1}{n!}\frac{\partial^n}{\partial^n \rho}\mathcal{F}[u_k(\rho)]\Big|_{\rho=\rho_0(k)}(\rho-\rho_0(k))^n \quad (2.70)$$

(2.71)

and re-arranging all the terms we arrive at:

$$\dot{g}_0(k) + \sum_{n=2}^{+\infty} \dot{g}_n(k)(\rho-\rho_0(k))^n = \mathcal{F}[u_k(\rho)]\Big|_{\rho=\rho_0(k)}$$
$$+ \sum_{n=2}^{+\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial^n \rho}\mathcal{F}[u_k(\rho)]\Big|_{\rho=\rho_0(k)} - \frac{(n+1)}{2}\frac{g_{n+1}(k)}{g_2(k)}\frac{\partial \mathcal{F}[u_k(\rho)]}{\partial \rho}\Big|_{\rho=\rho_0(k)}\right)(\rho-\rho_0(k))^n \quad (2.72)$$

Therefore the complete set of beta functions is:

$$\begin{cases} \beta_0(g;\rho_0) = \mathcal{F}[u_k(\rho)]\Big|_{\rho=\rho_0(k)} \\ \beta_{\rho_0}(g;\rho_0) = -\frac{\frac{\partial \mathcal{F}[u_k(\rho)]}{\partial \rho}\Big|_{\rho=\rho_0(k)}}{2g_2(k)} \\ \beta_n(g;\rho_0) = \frac{1}{n!}\frac{\partial^n}{\partial^n \rho}\mathcal{F}[u_k(\rho)]\Big|_{\rho=\rho_0(k)} - \frac{(n+1)}{2}\frac{g_{n+1}(k)}{g_2(k)}\frac{\partial \mathcal{F}[u_k(\rho)]}{\partial \rho}\Big|_{\rho=\rho_0(k)} \qquad \text{for } n \geq 2 \end{cases} \quad (2.73)$$

and the stability matrix can be defined as usual:

$$M_{nm} = \frac{\partial \beta_n(g)}{\partial g_m}\Big|_{g^*} \quad (2.74)$$

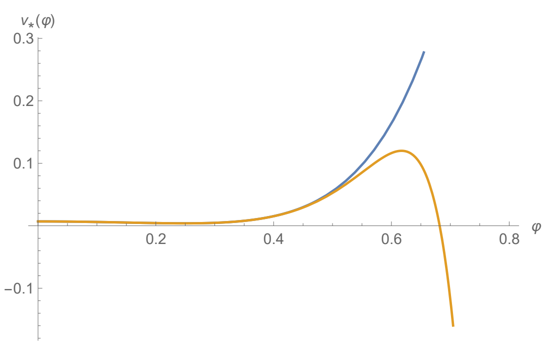
simply making the substitutions: $\beta_1 \to \beta_{\rho_0}$ and $g_1 \to \rho_0$.

Working with a polynomial of order 8 we find the following fixed point potential:

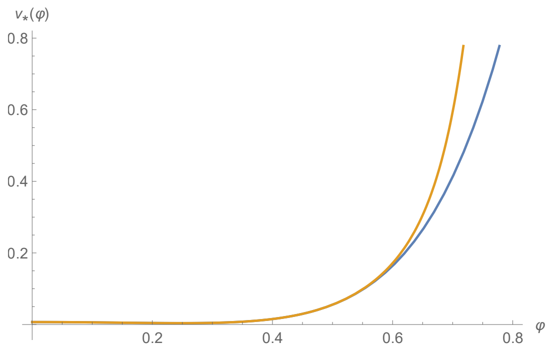
$$\begin{aligned} u_*(\rho) = &+ 2(40)\Big(\rho - 0.06129(3)\Big)^8 - (5)\Big(\rho - 0.06129(3)\Big)^7 \\ &- 17.(7)\Big(\rho - 0.06129(3)\Big)^6 - 0.8(9)\Big(\rho - 0.06129(3)\Big)^5 \\ &+ 3.388(3)\Big(\rho - 0.06129(3)\Big)^4 + 2.178(3)\Big(\rho - 0.06129(3)\Big)^3 \\ &+ 0.9339(8)\Big(\rho - 0.06129(3)\Big)^2 + 0.0038608(1) \end{aligned} \tag{2.75}$$

and diagonalizing the stability matrix we find $\lambda_1 = -1.539(7)$ which corresponds to $\nu = 0.649(5)$.

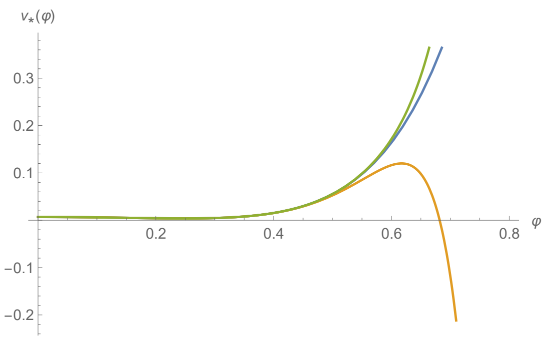
In figure 2.5 we show the fixed point solution found with the two polynomial expansions and we compare them with the global solution found with the asymptotic shooting method. It can be noticed that the expansion around the minimum has a greater radius of convergence.



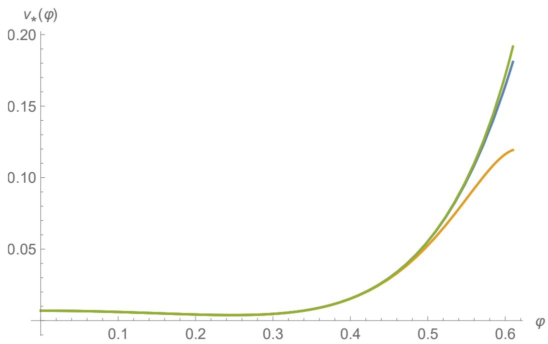
(a) In blue the global solution found with the asymptotic shooting method, in yellow the polynomial expansion around the origin



(b) In blue the global solution found with the asymptotic shooting method, in yellow the polynomial expansion around the minimum



(c) In blue the global solution, in yellow the polynomial expansion around the origin, in green the polynomial expansion around the minimum



(d) Near the origin all solutions coincide

Figure 2.5

Chapter 3

A new approach: higher order flows

In this chapter we would like to introduce a new approach to Wetterich equation, which will enable us to implement novel approximation schemes. We will present the general idea first and then we will perform a complete analytic calculation for a theory with a single scalar field in LPA' truncation. The result of the calculation will be a new approximate flow equation which, at variance with the standard LPA equation, is second order in time and preserves the momentum structure of the vertices.

3.1 General idea

If we denote the exact scale-dependent 2-point vertex as:

$$(G_k)_{AB} = (\Gamma_k^{(2)} + R_k)_{AB} \tag{3.1}$$

its inverse, the exact scale-dependent propagator, will be denoted:

$$(G_k)^{AB} \ : \quad (G_k)^{AB}(G_k)_{BC} = \delta^A_C \tag{3.2}$$

and Wetterich equation will read:

$$\dot{\Gamma}_k = \frac{1}{2}(G_k)^{AB}(\dot{R}_k)_{BA} \tag{3.3}$$

If we now take a RG-time derivative of the equation we get:

$$\ddot{\Gamma}_k = \frac{1}{2}(\dot{G}_k)^{AB}(\dot{R}_k)_{BA} + \frac{1}{2}(G_k)^{AB}(\ddot{R}_k)_{BA} \tag{3.4}$$

where[1]:

$$(\dot{G}_k)^{AB} = -\big(G_k\big)^{AC}\big(\dot{G}_k\big)_{CD}\big(G_k\big)^{DB} \tag{3.5}$$

$$= -\big(G_k\big)^{AC}(\dot{\Gamma}_k^{(2)} + \dot{R}_k)_{CD}\big(G_k\big)^{DB} \tag{3.6}$$

[1]We exploit the relation: $0 = \frac{d}{dt}[\mathbb{I}] = \frac{d}{dt}\left[O_t O_t^{-1}\right] = \left[\frac{d}{dt}O_t\right]O_t^{-1} + O_t\left[\frac{d}{dt}O_t^{-1}\right]$, hence: $\left[\frac{d}{dt}O_t^{-1}\right] = -O_t^{-1}\left[\frac{d}{dt}O_t\right]O_t^{-1}$

and $(\dot{\Gamma}_k^{(2)})_{CD}$ is computed by taking two functional derivatives of the starting flow equation (as done in the vertex expansion scheme, see section 2.1):

$$(\dot{\Gamma}_k^{(1)})_C = -\frac{1}{2}(\dot{R}_k)_{MN}(G_k^{(2)})^{NL}(\Gamma_k^{(3)})_{LCK}(G_k^{(2)})^{KM} \tag{3.7}$$

$$\begin{aligned}(\dot{\Gamma}_k^{(2)})_{CD} = &-\frac{1}{2}(\dot{R}_k)_{MN}(G_k^{(2)})^{NL}(\Gamma_k^{(4)})_{LCDK}(G_k^{(2)})^{KM} \\ &+(\dot{R}_k)_{MN}(G_k^{(2)})^{NL}(\Gamma_k^{(3)})_{LCK}(G_k^{(2)})^{KJ}(\Gamma_k^{(3)})_{JDH}(G_k^{(2)})^{HM}\end{aligned} \tag{3.8}$$

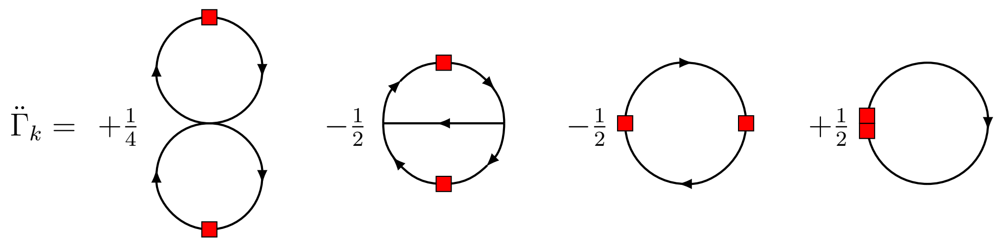
If we now insert the above expressions into the equation for $\ddot{\Gamma}_k$ we arrive at:

$$\begin{aligned}\ddot{\Gamma}_k = &-\frac{1}{2}(G_k)^{AC}\Big[-\frac{1}{2}(\dot{R}_k)_{MN}(G_k^{(2)})^{NL}(\Gamma_k^{(4)})_{LCDK}(G_k^{(2)})^{KM}\Big](G_k)^{DB}(\dot{R}_k)_{BA} \\ &-\frac{1}{2}(G_k)^{AC}\Big[+(\dot{R}_k)_{MN}(G_k^{(2)})^{NL}(\Gamma_k^{(3)})_{LCK}(G_k^{(2)})^{KJ}(\Gamma_k^{(3)})_{JDH}(G_k^{(2)})^{HM}\Big](G_k)^{DB}(\dot{R}_k)_{BA} \\ &-\frac{1}{2}(G_k)^{AC}\Big[+(\dot{R}_k)_{CD}\Big](G_k)^{DB}(\dot{R}_k)_{BA} \\ &+\frac{1}{2}(G_k)^{AB}(\ddot{R}_k)_{BA}\end{aligned} \tag{3.9}$$

and finally:

$$\begin{aligned}\ddot{\Gamma}_k = &+\frac{1}{4}(G_k)^{AC}(\dot{R}_k)_{MN}(G_k^{(2)})^{NL}(\Gamma_k^{(4)})_{LCDK}(G_k^{(2)})^{KM}(\dot{R}_k)_{CD}(\dot{R}_k)_{BA} \\ &-\frac{1}{2}(G_k)^{AC}(\dot{R}_k)_{MN}(G_k^{(2)})^{NL}(\Gamma_k^{(3)})_{LCK}(G_k^{(2)})^{KJ}(\Gamma_k^{(3)})_{JDH}(G_k^{(2)})^{HM}(G_k)^{DB}(\dot{R}_k)_{BA} \\ &-\frac{1}{2}(G_k)^{AC}(\dot{R}_k)_{CD}(G_k)^{DB}(\dot{R}_k)_{BA} \\ &+\frac{1}{2}(G_k)^{AB}(\ddot{R}_k)_{BA}\end{aligned} \tag{3.10}$$

It order to make the above equation more intelligible we can write it pictorially as:



where:

- lines represent the exact propagator G_k
- single red squares represent $\dot{R}_k$

- coupled red squares represent $\ddot{R}_k$.

Hence we can easily recognize the appearance of a two-loops structure governed by the proper vertices $\Gamma_k^{(3)}$ and $\Gamma_k^{(4)}$. Clearly it is possible to push further this idea and derive new flow equations at any arbitrary order, all *equivalent* to the original first order equation as far as we work with an *exact* formulation.

As we just said, the above second order equation is, in its exact form, equivalent to Wetterich equation. However, as soon as some truncation is introduced, the equivalence is lost. Therefore the founding idea for the new approximation scheme we would like to try is to insert truncations only at second order in RG-time, in order to preserve a more complete description of the momentum dependence of the vertices. In the next section we will show how this is realized for a theory with a single scalar field.

3.2 Approximation schemes: an application to scalar models

Suppose to have a D-dimensional theory with a single scalar field, the goal of this section is to compare the flow equations that result from the two following procedures:

- 1-LPA':
  - start with *first order exact* Wetterich equation
  - insert LPA'
- 2-LPA':
  - start with *second order exact* Wetterich equation
  - insert LPA'

As equation 1-LPA' will be first order in time while 2-LPA' will be second order in time, we will actually define a third equation, which will be just the time derivative of 1-LPA and will be called 1^2-LPA'.

To obtain the corresponding LPA cases one can just set $Z_k \equiv 1$, i.e. $\eta_k = \dot{\eta}_k = 0$.

3.2.1 1-LPA' flow equation

Let us start with 1-LPA'. The derivation is very similar to the one performed in 2.3, hence we will state only the main results. We start with the ansatz:

$$\Gamma_k[\phi] = \int d^D x \left(\frac{1}{2} Z_k \, \partial_\mu \phi(x) \, \partial^\mu \phi(x) + V_k(\phi(x)) \right) \tag{3.11}$$

then, in order to follow the flow of the effective potential, we evaluate Wetterich equation on a constant field configuration $\phi(x) = \phi$, in such a way that:

$$\begin{aligned} \dot{\Gamma}_k(\phi) &= \int_x \dot{V}_k(\phi) & (3.12) \\ &= L^D \dot{V}_k(\phi) & (3.13) \end{aligned}$$

At the same time, the right and side of Wetterich equation reads:

$$\frac{1}{2} \int_x \int_y (\dot{R}_k)_{xy} (\Gamma_k^{(2)}(\phi) + R_k)^{-1}_{yx} = L^D \frac{1}{2} \int_x \frac{\dot{R}_k(-\Box_x)}{[Z_k \, (-\Box_x) + V_k''(\phi) + R_k(-\Box_x)]} \tag{3.14}$$

and switching to momentum space we arrive at:

$$\dot{V}_k(\phi) = \frac{1}{2} \int_q \frac{\dot{R}_k(q^2)}{[Z_k \, q^2 + V_k''(\phi) + R_k(q^2)]} \tag{3.15}$$

We now have to define the regulator, in order to include the field strength renormalization constant, we make a slightly difference choice:

$$\begin{aligned} R_k(q^2) &\equiv Z_k \, \mathcal{R}_k(q^2) & (3.16) \\ &\equiv Z_k \, (k^2 - q^2) \Theta(k^2 - q^2) & (3.17) \\ \Rightarrow \quad \dot{R}_k(q^2) &= Z_k \, \dot{\mathcal{R}}_k(q^2) + \dot{Z}_k \, \mathcal{R}_k(q^2) & (3.18) \\ &= Z_k \left[2k^2 \Theta(k^2 - q^2) + 2k^2 (k^2 - q^2) \delta(k^2 - q^2) \right] & \\ &\quad + \dot{Z}_k \left[(k^2 - q^2) \Theta(k^2 - q^2) \right] & (3.19) \\ &\simeq Z_k \, 2k^2 \Theta(k^2 - q^2) + \dot{Z}_k \, (k^2 - q^2) \Theta(k^2 - q^2) & (3.20) \end{aligned}$$

Inserting it into flow equation we get:

$$\begin{aligned} \dot{V}_k(\phi) &= \frac{1}{2} \int_q \frac{Z_k \dot{\mathcal{R}}_k(q^2)}{[Z_k \, q^2 + V_k''(\phi) + Z_k \, \mathcal{R}_k(q^2)]} + \frac{1}{2} \int_q \frac{\dot{Z}_k \mathcal{R}_k(q^2)}{[Z_k \, q^2 + V_k''(\phi) + Z_k \, \mathcal{R}_k(q^2)]} & (3.21) \\ &= \frac{\Omega_D}{(2\pi)^D} \frac{k^D}{D} \frac{1}{\left[1 + \frac{V_k''(\phi)}{Z_k \, k^2} \right]} - \frac{\Omega_D}{(2\pi)^D} \frac{k^D}{D(D+2)} \frac{\eta_k}{\left[1 + \frac{V_k''(\phi)}{Z_k \, k^2} \right]} & (3.22) \end{aligned}$$

where $\eta_k \equiv -\frac{\dot{Z}_k}{Z_k}$ represents a scale-dependent anomalous dimension (this definition will be clear soon, when we will switch to dimensionless renormalized variables). Finally we can write the dimensionsful 1-LPA' equation as:

$$\dot{V}_k(\phi) = C_D \frac{1 - \frac{\eta_k}{D+2}}{\left[1 + \frac{V_k''(\phi)}{Z_k\, k^2}\right]}\, k^D \tag{3.23}$$

where $C_D = \frac{1}{(4\pi)^{\frac{D}{2}}\,\Gamma(\frac{D}{2}+1)}$. If we now define the dimensionless renormalized[2] variables:

$$\begin{cases} \varphi \equiv k^{-\frac{D-2}{2}} \sqrt{Z_k}\phi \\ v_k(\varphi) \equiv k^{-D} V_k(\phi(\varphi)) \end{cases} \tag{3.24}$$

we notice that:

$$\partial_t\, \varphi = -\frac{D-2}{2}\,\varphi + \frac{1}{2}\frac{\dot{Z}_k}{Z_k}\,\varphi \tag{3.25}$$

$$= -\frac{D-2+\eta_k}{2}\,\varphi \tag{3.26}$$

$$\equiv -d_\phi(k)\,\varphi \tag{3.27}$$

therefore $d_\phi(k) = \frac{D-2+\eta_k}{2}$ becomes the scale-dependent mass dimension of the field and η_k acquires a perfectly consistent meaning. Moreover we find out that the equation:

$$\dot{V}_k(\phi)\big|_{\phi(\varphi)} = k^D \left\{ \dot{v}_k(\varphi) - \Big(d_\phi(k)\,\varphi\, v_k'(\varphi) - D\, v_k(\varphi) \Big) \right\} \tag{3.28}$$

holds true even in this case, simply applying the above definition of $d_\phi(k)$; a consistent equation for $V_k''(\phi)$ can be found as well:

$$V_k''(\phi)\big|_{\phi(\varphi)} = k^2\, Z_k\, v_k''(\varphi) \tag{3.29}$$

therefore the dimensionless 1-LPA' flow equation reads:

$$\Big[\partial_t - \Big(d_\phi(k)\,\varphi\,\partial_\varphi - D\Big)\Big] v_k(\varphi) = C_D \frac{1 - \frac{\eta_k}{D+2}}{\left[1 + v_k''(\varphi)\right]} \tag{3.30}$$

It can be noticed that we have derived a flow equation for $v_k(\varphi)$ which depends on a scale dependent anomalous dimension η_k. Yet we have not derived any flow equation for η_k itself. Hence the problem is not closed. This is an intrinsic difficulty of the LPA' scheme and we will discuss in chapter 4 how it can be overcome only by making some arbitrary assumptions on the definition of η_k.

[2] In this contest "renormalized" means that the kinetic term acquires the canonical pre-factor $\frac{1}{2}$ once written as a function of φ. At the same time, the switch to dimensionless variables corresponds to rescaling, i.e. the second step of the Wilsonian renormalization process

3.2.2 1^2-LPA' flow equation

Before deriving equation 2-LPA', let us study the RG-time derivative of 1-LPA', i.e. let us define 1^2-LPA'. We can work with dimensionful quantities and study:

$$\dot{V}_k(\phi) \quad \rightarrow \quad \ddot{V}_k(\phi) \tag{3.31}$$

otherwise we can exploit equation (3.28) and work straight ahead with dimensionless quantities:

$$\Big[\partial_t - \Big(d_\phi(k)\,\varphi\,\partial_\varphi - D\Big)\Big] v_k(\varphi) \quad \rightarrow \quad \Big[\partial_t - \Big(d_\phi(k)\,\varphi\,\partial_\varphi - D\Big)\Big]^2 v_k(\varphi) \tag{3.32}$$

Since the first path is more transparent we will follow it and show at the end the equivalence with the other. Hence we start with the approximate first order Wetterich equation with an arbitrary regulator:

$$\dot{V}_k(\phi) = \frac{1}{2}\int_q \dot{R}_k(q^2)\tilde{G}_k(q^2) \tag{3.33}$$

where $\tilde{G}_k(q^2) = [Z_k q^2 + R_k(q^2) + V_k''(\phi)]^{-1}$ is the approximate propagator in momentum space, and we take one more time derivative[3]:

$$\ddot{V}_k(\phi) = \frac{1}{2}\int_q \ddot{R}_k(q^2)\tilde{G}_k(q^2) + \frac{1}{2}\int_q \dot{R}_k(q^2)\dot{\tilde{G}}_k(q^2) \tag{3.34}$$

$$= \frac{1}{2}\int_q \ddot{R}_k(q^2)\tilde{G}_k(q^2) - \frac{1}{2}\int_q \dot{R}_k(q^2)\big[\tilde{G}_k(q^2)\big]^2\big[\dot{Z}_k\, q^2 + \dot{R}_k(q^2) + \dot{V}_k''(\phi)\big] \tag{3.35}$$

$$\tag{3.36}$$

In order to find a consistent expression for $\dot{V}_k''(\phi)$ we take two field derivatives of the approximate first order equation:

$$\dot{V}_k'(\phi) = -\frac{1}{2}\int_p \dot{R}_k(q^2)\big[\tilde{G}_k(q^2)\big]^2 V_k^{(3)}(\phi) \tag{3.37}$$

$$\dot{V}_k''(\phi) = +\int_p \dot{R}_k(q^2)\big[\tilde{G}_k(q^2)\big]^3\big[V_k^{(3)}(\phi)\big]^2 - \frac{1}{2}\int_q \dot{R}_k(q^2)\big[\tilde{G}_k(q^2)\big]^2 V_k^{(4)}(\phi) \tag{3.38}$$

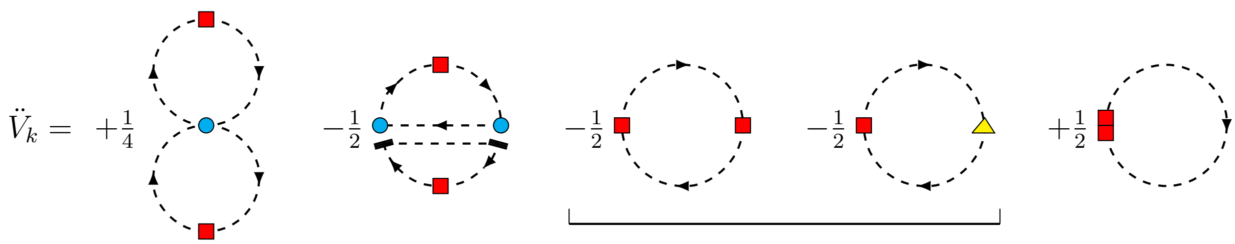
$$\tag{3.39}$$

[3]The derivation is in the same spirit of the one performed in section 3.1 above, however it is important to remember that we are now working with an approximate equation evaluated at a constant field configuration

and we finally get:

$$\begin{aligned}\ddot{V}_k(\phi) = &+ \frac{1}{2}\int_q \ddot{R}_k(q^2)\tilde{G}_k(q^2) \\ &- \frac{1}{2}\int_q \dot{R}_k(q^2)\left[\tilde{G}_k(q^2)\right]^2\left[\dot{Z}_k\, q^2\right] \\ &- \frac{1}{2}\int_q \left[\dot{R}_k(q^2)\tilde{G}_k(q^2)\right]^2 \\ &+ \frac{1}{4}\left[\int_q \dot{R}_k(q^2)\tilde{G}_k^2(q^2)\right]^2 V_k^{(4)}(\phi) \\ &- \frac{1}{2}\left[\int_q\int_p \dot{R}_k(q^2)\tilde{G}_k^2(q^2)\dot{R}_k(p^2)\tilde{G}_k^3(p^2)\right]\left[V_k^{(3)}(\phi)\right]^2\end{aligned} \tag{3.40}$$

As we did for the exact second order equation, we can write the above equation pictorially in order to visualize some similarities:



where:

- dashed lines represent the approximate propagator $\tilde{G}_k$
- single red squares represent $\dot{R}_k$
- coupled red squares represent $\ddot{R}_k$
- cyan circles represent approximate (momentum-independent) vertices
- yellow triangles represent the kinetic contribution $\dot{Z}_k\, q^2$ of a scale dependent field strength renormalization constant
- thick cuts represent "momentum non-conservation" (we will explain in detail this concept a few lines below)

and contributions with the same structure have been grouped together

Thanks to this pictorial representation we can easily visualize how the insertion of LPA' in the exact first order equation has effected its second order structure, in particular we notice that:

- vertices have become momentum independent
- in the second diagram the propagator shared by the two loops carries only momentum coming from one of the loops, hence such loops become actually independent in this approximation. We can equivalently say that, due to approximations, momentum is not conserved in the 3-points interactions. We will show in the next section how 2-LPA' will restore momentum conservation in this diagram.

In order to perform integrations we need to write an explicit form for the regulator. As before we take:

$$\begin{aligned} R_k(q^2) &\equiv Z_k\,\mathcal{R}_k(q^2) && (3.41)\\ &\equiv Z_k\,(k^2-q^2)\Theta(k^2-q^2) && (3.42)\end{aligned}$$

$$\begin{aligned} \Rightarrow\quad \dot{R}_k(q^2) &= Z_k\,\dot{\mathcal{R}}_k(q^2)+\dot{Z}_k\,\mathcal{R}_k(q^2) && (3.43)\\ &\simeq Z_k\,2k^2\Theta(k^2-q^2)+\dot{Z}_k\,(k^2-q^2)\Theta(k^2-q^2) && (3.44)\end{aligned}$$

$$\begin{aligned} \Rightarrow\quad \ddot{R}_k(q^2) &= \ddot{Z}_k\mathcal{R}_k(q^2)+2\dot{Z}_k\dot{\mathcal{R}}_k(q^2)+Z_k\ddot{\mathcal{R}}_k(q^2) && (3.45)\\ &\simeq \ddot{Z}_k\,(k^2-q^2)\Theta(k^2-q^2)\\ &+\dot{Z}_k\,4k^2\Theta(k^2-q^2)\\ &+Z_k\left[4k^2\Theta(k^2-q^2)+8k^4\delta(k^2-q^2)+4k^4(k^2-q^2)\frac{d\,\delta(k^2-q^2)}{d\,k^2}\right] && (3.46)\end{aligned}$$

and 1^2-LPA' flow equation becomes:

$$\begin{aligned}\ddot{V}_k(\phi) &= \frac{1}{2}\int_q \frac{\frac{\ddot{Z}_k}{Z_k}\mathcal{R}_k(q^2)+2\frac{\dot{Z}_k}{Z_k}\dot{\mathcal{R}}_k(q^2)+\ddot{\mathcal{R}}_k(q^2)}{\left[q^2+\mathcal{R}_k(q^2)+\frac{V_k''(\phi)}{Z_k}\right]}\\ &-\frac{1}{2}\int_q \frac{\left[\frac{\dot{Z}_k}{Z_k}\mathcal{R}_k(q^2)+\dot{\mathcal{R}}_k(q^2)\right]^2}{\left[q^2+\mathcal{R}_k(q^2)+\frac{V_k''(\phi)}{Z_k}\right]^2}-\frac{1}{2}\int_q \frac{\left[\frac{\dot{Z}_k}{Z_k}\mathcal{R}_k(q^2)+\dot{\mathcal{R}}_k(q^2)\right]\left[\frac{\dot{Z}_k}{Z_k}q^2\right]}{\left[q^2+\mathcal{R}_k(q^2)+\frac{V_k''(\phi)}{Z_k}\right]^2}\\ &+\frac{1}{4}\left[\int_q \frac{\frac{\dot{Z}_k}{Z_k}\mathcal{R}_k(q^2)+\dot{\mathcal{R}}_k(q^2)}{\left[q^2+\mathcal{R}_k(q^2)+\frac{V_k''(\phi)}{Z_k}\right]^2}\right]^2\frac{V_k^{(4)}(\phi)}{\left[Z_k\right]^2}\\ &-\frac{1}{2}\left[\int_q \frac{\frac{\dot{Z}_k}{Z_k}\mathcal{R}_k(q^2)+\dot{\mathcal{R}}_k(q^2)}{\left[q^2+\mathcal{R}_k(q^2)+\frac{V_k''(\phi)}{Z_k}\right]^2}\int_p \frac{\frac{\dot{Z}_k}{Z_k}\mathcal{R}_k(p^2)+\dot{\mathcal{R}}_k(p^2)}{\left[p^2+\mathcal{R}_k(p^2)+\frac{V_k''(\phi)}{Z_k}\right]^3}\right]\frac{\left[V_k^{(3)}(\phi)\right]^2}{\left[Z_k\right]^3} \qquad (3.47)\end{aligned}$$

hence, setting:

$$\eta_k=-\frac{\dot{Z}_k}{Z_k}\quad\Rightarrow\quad\frac{\ddot{Z}_k}{Z_k}=\eta_k^2-\dot{\eta}_k \qquad (3.48)$$

and:

$$D_k(q^2) = \left[q^2 + \mathcal{R}_k(q^2) + \frac{V_k''(\phi)}{Z_k}\right] \tag{3.49}$$

we get:

$$\begin{aligned}
\ddot{V}_k(\phi) = &+\frac{1}{2}\left[\int_q \frac{\ddot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]} - 2\eta_k \int_q \frac{\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]} + (\eta_k^2 - \dot{\eta}_k)\int_q \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]}\right] \\
&- \frac{1}{2}\left[\int_q \frac{\big[\dot{\mathcal{R}}_k(q^2)\big]^2}{\big[D_k(q^2)\big]^2} - 2\eta_k \int_q \frac{\mathcal{R}_k(q^2)\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]^2} + \eta_k^2 \int_q \frac{\big[\mathcal{R}_k(q^2)\big]^2}{\big[D_k(q^2)\big]^2}\right] \\
&- \frac{1}{2}\left[-\eta_k \int_q \frac{\dot{\mathcal{R}}_k(q^2)\, q^2}{\big[D_k(q^2)\big]^2} + \eta_k^2 \int_q \frac{\mathcal{R}_k(q^2)\, q^2}{\big[D_k(q^2)\big]^2}\right] \\
&+ \frac{1}{4}\left[\left(\int_q \frac{\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]^2}\right)^2 - 2\eta_k \int_q \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]^2}\int_q \frac{\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]^2} + \eta_k^2\left(\int_q \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]^2}\right)^2\right]\frac{V_k^{(4)}(\phi)}{\big[Z_k\big]^2} \\
&- \frac{1}{2}\left[\int_q \frac{\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]^2}\int_p \frac{\dot{\mathcal{R}}_k(p^2)}{\big[D_k(p^2)\big]^3} - 2\eta_k \int_q \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]^2}\int_p \frac{\dot{\mathcal{R}}_k(p^2)}{\big[D_k(p^2)\big]^3}\right. \\
&\qquad \left. + \eta_k^2 \int_q \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]^2}\int_p \frac{\mathcal{R}_k(p^2)}{\big[D_k(p^2)\big]^3}\right]\frac{\big[V_k^{(3)}(\phi)\big]^2}{\big[Z_k\big]^3}
\end{aligned} \tag{3.50}$$

Thanks to the regulator defined above the integrals can be easily computed:

$$\int_q \frac{\ddot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]} = \frac{\Omega_D}{(2\pi)^D}\frac{2(D+2)}{D}\left[\frac{1}{1+v_k''}\right] k^D \tag{3.51}$$

$$\int_q \frac{\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]} = \frac{\Omega_D}{(2\pi)^D}\frac{2}{D}\left[\frac{1}{1+v_k''}\right] k^D \tag{3.52}$$

$$\int_q \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]} = \frac{\Omega_D}{(2\pi)^D}\frac{2}{D(D+2)}\left[\frac{1}{1+v_k''}\right] k^D \tag{3.53}$$

$$\int_q \frac{\big[\dot{\mathcal{R}}_k(q^2)\big]^2}{\big[D_k(q^2)\big]^2} = \frac{\Omega_D}{(2\pi)^D}\frac{4}{D}\left[\frac{1}{1+v_k''}\right]^2 k^D \tag{3.54}$$

$$\int_q \frac{\mathcal{R}_k(q^2)\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]^2} = \frac{\Omega_D}{(2\pi)^D}\frac{4}{D(D+2)}\left[\frac{1}{1+v_k''}\right]^2 k^D \tag{3.55}$$

$$\int_q \frac{\big[\mathcal{R}_k(q^2)\big]^2}{\big[D_k(q^2)\big]^2} = \frac{\Omega_D}{(2\pi)^D}\frac{8}{D(D+2)(D+4)}\left[\frac{1}{1+v_k''}\right]^2 k^D \tag{3.56}$$

$$\int_q \frac{\dot{\mathcal{R}}_k(q^2)\, q^2}{\big[D_k(q^2)\big]^2} = \frac{\Omega_D}{(2\pi)^D}\frac{2}{D+2}\left[\frac{1}{1+v_k''}\right]^2 k^D \tag{3.57}$$

$$\int_q \frac{\mathcal{R}_k(q^2)\,q^2}{\big[D_k(q^2)\big]^2} = \frac{\Omega_D}{(2\pi)^D}\frac{2}{(D+2)(D+4)}\left[\frac{1}{1+v_k''}\right]^2 k^D \tag{3.58}$$

$$\int_q \frac{\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]^2} = \frac{\Omega_D}{(2\pi)^D}\frac{2}{D}\left[\frac{1}{1+v_k''}\right]^2 k^{D-2} \tag{3.59}$$

$$\int_q \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]^2} = \frac{\Omega_D}{(2\pi)^D}\frac{2}{D(D+2)}\left[\frac{1}{1+v_k''}\right]^2 k^{D-2} \tag{3.60}$$

$$\int_q \frac{\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]^3} = \frac{\Omega_D}{(2\pi)^D}\frac{2}{D}\left[\frac{1}{1+v_k''}\right]^3 k^{D-4} \tag{3.61}$$

$$\int_q \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]^3} = \frac{\Omega_D}{(2\pi)^D}\frac{2}{D(D+2)}\left[\frac{1}{1+v_k''}\right]^3 k^{D-4} \tag{3.62}$$

and we can write 1^2-LPA' flow equation in a closed form:

$$\begin{aligned}
\ddot{V}_k(\phi) = &+\frac{1}{2}\left[\frac{2(D+2)}{D} - 2\eta_k\,\frac{2}{D} + (\eta_k^2 - \dot{\eta}_k)\,\frac{2}{D(D+2)}\right]\frac{\Omega_D}{(2\pi)^D}\left[\frac{1}{1+v_k''}\right] k^D \\
&-\frac{1}{2}\left[\frac{4}{D} - 2\eta_k\,\frac{4}{D(D+2)} + \eta_k^2\,\frac{8}{D(D+2)(D+4)}\right]\frac{\Omega_D}{(2\pi)^D}\left[\frac{1}{1+v_k''}\right]^2 k^D \\
&-\frac{1}{2}\left[-\eta_k\frac{2}{(D+2)} + \eta_k^2\frac{2}{(D+2)(D+4)}\right]\frac{\Omega_D}{(2\pi)^D}\left[\frac{1}{1+v_k''}\right]^2 k^D \\
&+\frac{1}{4}\left[\left(\frac{2}{D}\right)^2 - 2\eta_k\left(\frac{2}{D}\right)^2\left(\frac{1}{D+2}\right) + \eta_k^2\left(\frac{2}{D}\right)^2\left(\frac{1}{D+2}\right)^2\right]\left(\frac{\Omega_D}{(2\pi)^D}\right)^2 \\
&\qquad\qquad\left[\frac{1}{1+v_k''}\right]^4 k^{2D-4}\,\frac{V_k^{(4)}(\phi)}{\big[Z_k\big]^2} \\
&-\frac{1}{2}\left[\left(\frac{2}{D}\right)^2 - 2\eta_k\left(\frac{2}{D}\right)^2\left(\frac{1}{D+2}\right) + \eta_k^2\left(\frac{2}{D}\right)^2\left(\frac{1}{D+2}\right)^2\right]\left(\frac{\Omega_D}{(2\pi)^D}\right)^2 \\
&\qquad\qquad\left[\frac{1}{1+v_k''}\right]^5 k^{2D-6}\,\frac{\big[V_k^{(3)}(\phi)\big]^2}{\big[Z_k\big]^3}
\end{aligned} \tag{3.63}$$

and re-arranging the contributions we arrive at the final form:

$$\begin{aligned}
\frac{\ddot{V}_k(\phi)}{k^D} = &+\frac{1}{2}\left[2(D+2) - 4\eta_k + (\eta_k^2 - \dot{\eta}_k)\,\frac{2}{(D+2)}\right] C_D\,\left[\frac{1}{1+v_k''}\right] \\
&-\frac{1}{2}\left[4 - 2\eta_k\,\frac{4+D}{(D+2)} + \eta_k^2\,\frac{8+2D}{(D+2)(D+4)}\right] C_D\,\left[\frac{1}{1+v_k''}\right]^2
\end{aligned}$$

$$+\frac{1}{4}\left[4-8\eta_k\left(\frac{1}{D+2}\right)+4\eta_k^2\left(\frac{1}{D+2}\right)^2\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^4\frac{V_k^{(4)}(\phi)}{k^{4-D}\big[Z_k\big]^2}$$
$$-\frac{1}{2}\left[4-8\eta_k\left(\frac{1}{D+2}\right)+4\eta_k^2\left(\frac{1}{D+2}\right)^2\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^5\frac{\big[V_k^{(3)}(\phi)\big]^2}{k^{6-D}\big[Z_k\big]^3} \quad (3.64)$$

where $C_D = \frac{1}{(4\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2}+1)}$.

We can now switch to dimensionless variables:

$$\begin{cases}\varphi = k^{-\frac{D-2}{2}}\sqrt{Z_k}\phi \\ v_k(\varphi) = k^{-D}V_k(\phi(\varphi))\end{cases}$$

and we notice that:

$$\frac{V_k^{(n)}(\phi)|_{\phi(\varphi)}}{Z_k^{\frac{n}{2}}k^{D-n\frac{D-2}{2}}} = v_k^{(n)}(\varphi) \quad (3.65)$$

Moreover:

$$\begin{aligned}\dot{V}_k(\phi) &= \frac{d}{dt}V_k(\phi)\\ &= k^D\left(Dv_k\big[\varphi(\phi)\big]+\frac{d}{dt}v_k\big[\varphi(\phi)\big]\right)\end{aligned} \quad (3.66)$$

$$\begin{aligned}\Rightarrow\quad \ddot{V}_k(\phi) &= \frac{d^2}{d^2t}V_k(\phi)\\ &= k^D\left(D^2v_k\big[\varphi(\phi)\big]+2D\frac{d}{dt}v_k\big[\varphi(\phi)\big]+\frac{d^2}{d^2t}v_k\big[\varphi(\phi)\big]\right)\end{aligned} \quad (3.67)$$

where:

$$\frac{d}{dt}v_k\big[\varphi(\phi)\big] = \dot{v}_k(\varphi)+v_k'(\varphi_k)|_{\varphi(\phi)}\,\frac{d}{dt}\big[\varphi(\phi)\big] \quad (3.68)$$

$$\begin{aligned}\Rightarrow\quad \frac{d^2}{d^2t}v_k\big[\varphi(\phi)\big] &= \ddot{v}_k(\varphi)+2\dot{v}_k'(\varphi)|_{\varphi(\phi)}\,\frac{d}{dt}\big[\varphi(\phi)\big]\\ &\quad + v_k''(\varphi)|_{\varphi(\phi)}\left(\frac{d}{dt}\big[\varphi(\phi)\big]\right)^2+v_k'(\varphi)|_{\varphi(\phi)}\,\frac{d^2}{d^2t}\big[\varphi(\phi)\big]\end{aligned} \quad (3.69)$$

and:

$$\frac{d}{dt}\big[\varphi(\phi)\big] = -d_\phi(k)\varphi_k(\phi) \quad (3.70)$$

$$\Rightarrow\quad \frac{d^2}{d^2t}\big[\varphi(\phi)\big] = -\dot{d}_\phi(k)\,\varphi(\phi)+d_\phi^2(k)\,\varphi(\phi) \quad (3.71)$$

Therefore we find:

$$
\begin{aligned}
\frac{\ddot{V}_k(\phi)}{k^D}|_{\phi(\varphi)} &= \ddot{v}_k(\varphi) - 2d_\phi(k)\,\varphi\,\dot{v}_k'(\varphi) + d_\phi^2(k)\,\varphi^2\,v_k''(\varphi) + (d_\phi^2(k) - \dot{d}_\phi(k))\varphi\,v_k'(\varphi) \\
&\quad + 2D\,[\dot{v}_k(\varphi) - d_\phi(k)\varphi\,v_k'(\varphi)] + D^2\,v_k(\varphi) && (3.72) \\
&= \left[\frac{\partial}{\partial t} - \left(d_\phi(k)\,\varphi\,\frac{\partial}{\partial\varphi} - D\right)\right]^2 v_k(\varphi) && (3.73)
\end{aligned}
$$

where $[\partial_t, \partial_\varphi] = 0$ while $\partial_t\varphi = 0$ and $\partial_t d_\phi(k) = \dot{d}_\phi(k)$.

Thanks to the above identity we can finally write the dimensionless 1^2-LPA' flow equation as:

$$
\begin{aligned}
\left[\frac{\partial}{\partial t} - \left(d_\phi(k)\,\varphi\,\frac{\partial}{\partial\varphi} - D\right)\right]^2 v_k(\varphi) &= +\frac{1}{2}\left[2(D+2) - 4\eta_k + (\eta_k^2 - \dot{\eta}_k)\,\frac{2}{(D+2)}\right] C_D\,\left[\frac{1}{1+v_k''}\right] \\
&\quad - \frac{1}{2}\left[4 - 2\eta_k\,\frac{4+D}{(D+2)} + \eta_k^2\,\frac{8+2D}{(D+2)(D+4)}\right] C_D\,\left[\frac{1}{1+v_k''}\right]^2 \\
&\quad + \frac{1}{4}\left[4 - 8\eta_k\left(\frac{1}{D+2}\right) + 4\eta_k^2\left(\frac{1}{D+2}\right)^2\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^4 v_k^{(4)} \\
&\quad - \frac{1}{2}\left[4 - 8\eta_k\left(\frac{1}{D+2}\right) + 4\eta_k^2\left(\frac{1}{D+2}\right)^2\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^5 \left[v_k^{(3)}\right]^2 \\
&&& (3.74)
\end{aligned}
$$

As we said at the beginning of the section, we could derive the same result by simply applying the linear operator:

$$
\left[\frac{\partial}{\partial t} - \left(d_\phi(k)\,\varphi\,\frac{\partial}{\partial\varphi} - D\right)\right] \tag{3.75}
$$

to the dimensionless 1-LPA' equation reported in (3.30). Thus we can easily argue that any solution of (3.30) must satisfy equation (3.74) as well. The converse is not true because of arbitrary spurious terms belonging to the kernel of the operator.

3.2.3 2-LPA' flow equation

It is now quite a easy task to derive equation 2-LPA', as there are only a few pieces left to evaluate. Actually, if we start from the second order exact flow equation:

$$\begin{aligned}\ddot{\Gamma}_k = &+ \frac{1}{4}(G_k)^{AC}(\dot{R}_k)_{MN}(G_k^{(2)})^{NL}(\Gamma_k^{(4)})_{LCDK}(G_k^{(2)})^{KM}(\dot{R}_k)_{CD}(\dot{R}_k)_{BA} \\ &- \frac{1}{2}(G_k)^{AC}(\dot{R}_k)_{MN}(G_k^{(2)})^{NL}(\Gamma_k^{(3)})_{LCK}(G_k^{(2)})^{KJ}(\Gamma_k^{(3)})_{JDH}(G_k^{(2)})^{HM}(G_k)^{DB}(\dot{R}_k)_{BA} \\ &- \frac{1}{2}(G_k)^{AC}(\dot{R}_k)_{CD}(G_k)^{DB}(\dot{R}_k)_{BA} \\ &+ \frac{1}{2}(G_k)^{AB}(\ddot{R}_k)_{BA}\end{aligned}$$

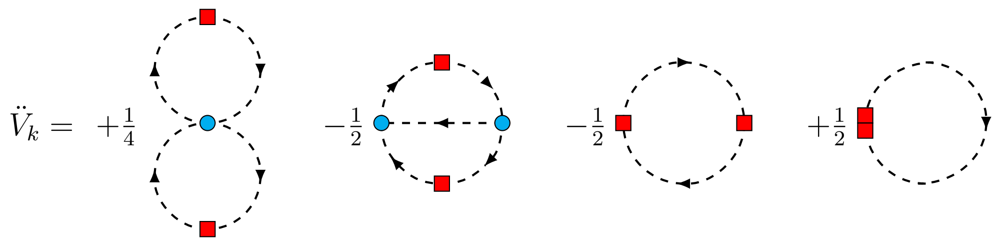
we insert the LPA' ansatz:

$$\Gamma_k[\phi] = \int d^D x \left(\frac{1}{2} Z_k\, \partial_\mu \phi(x)\, \partial^\mu \phi(x) + V_k(\phi(x))\right)$$

and we evaluate the equation on a constant field configuration, we get:

$$\begin{aligned}\ddot{V}_k(\phi) = &\frac{1}{2}\int_q \ddot{R}_k(q^2)\tilde{G}_k(q^2) \\ &- \frac{1}{2}\int_q \left[\dot{R}_k(q^2)\tilde{G}_k(q^2)\right]^2 \\ &+ \frac{1}{4}\left[\int_q \dot{R}_k(q^2)\tilde{G}_k^2(q^2)\right]^2 V_k^{(4)}(\phi) \\ &- \frac{1}{2}\left[\int_q\int_p \dot{R}_k(q^2)\tilde{G}_k^2(q^2)\dot{R}_k(p^2)\tilde{G}_k^2(p^2)\tilde{G}_k((q-p)^2)\right]\left[V_k^{(3)}(\phi)\right]^2 \end{aligned} \tag{3.76}$$

As done before, we can write the above equation pictorially:

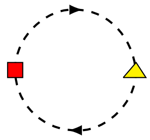


and we can make a few comments:

- as already anticipated, if we insert LPA' at second order we preserve the momentum structure of the 3-points vertices, hence the two loops in the second diagram are no more independent. This is the main difference with the above equations and the evaluation of this new non-trivial diagram will be the main task of this section.

- it may be argued that, in order to consistently follow the flow of the field strength renormalization constant, we should add to the above equation the contribution coming from:

$$-\tfrac{1}{2}\int_q \dot{R}_k(q^2)\big[\tilde{G}_k(q^2)\big]^2\big[\dot{Z}_k\, q^2\big] = -\tfrac{1}{2}$$



however, as we have already computed such diagram, we can evaluate 2-LPA' flow equation without it and eventually add it at the end.

We now need to evaluate the new contribution proportional to $\big[V_k^{(3)}(\phi)\big]^2$:

$$-\frac{1}{2}\left[\int_q\int_p \dot{R}_k(q^2)\tilde{G}_k^2(q^2)\dot{R}_k(p^2)\tilde{G}_k^2(p^2)\tilde{G}_k((q-p)^2)\right]\big[V_k^{(3)}(\phi)\big]^2$$

$$= -\frac{1}{2}\Bigg[\int_q\int_p \frac{\frac{\dot{Z}_k}{Z_k}\mathcal{R}_k(q^2)+\dot{\mathcal{R}}_k(q^2)}{\big[q^2+\mathcal{R}_k(q^2)+\frac{V_k''(\phi)}{Z_k}\big]^2}\frac{\frac{\dot{Z}_k}{Z_k}\mathcal{R}_k(p^2)+\dot{\mathcal{R}}_k(p^2)}{\big[p^2+\mathcal{R}_k(p^2)+\frac{V_k''(\phi)}{Z_k}\big]^2}$$

$$\frac{1}{\big[(q-p)^2+\mathcal{R}_k((q-p)^2)+\frac{V_k''(\phi)}{Z_k}\big]}\Bigg]\frac{\big[V_k^{(3)}(\phi)\big]^2}{\big[Z_k\big]^3} \tag{3.77}$$

$$= -\frac{1}{2}\Bigg[\underbrace{\int_q\int_p \frac{\dot{\mathcal{R}}_k(q^2)}{\big[D_k(q^2)\big]^2}\frac{\dot{\mathcal{R}}_k(p^2)}{\big[D_k(p^2)\big]^2}\frac{1}{\big[D_k((q-p)^2)\big]}}_{I_1^D(v_k'')}$$

$$-2\eta_k\underbrace{\int_q\int_p \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]^2}\frac{\dot{\mathcal{R}}_k(p^2)}{\big[D_k(p^2)\big]^2}\frac{1}{\big[D_k((q-p)^2)\big]}}_{I_\eta^D(v_k'')}$$

$$+\eta_k^2\underbrace{\int_q\int_p \frac{\mathcal{R}_k(q^2)}{\big[D_k(q^2)\big]^2}\frac{\mathcal{R}_k(p^2)}{\big[D_k(p^2)\big]^2}\frac{1}{\big[D_k((q-p)^2)\big]}}_{I_{\eta^2}^D(v_k'')}\Bigg]\frac{\big[V_k^{(3)}(\phi)\big]^2}{\big[Z_k\big]^3} \tag{3.78}$$

We notice the appearance of 3 non-trivial integrals, hence we need to study them one by one. A detailed description of this evaluation may be found in the Appendix A, here we just mention that, thanks symmetry properties, we can simplify their structure to:

$$I_1^D(v_k'') = \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\frac{k^{2D-6}}{\left[1+v_k''\right]^4}\int_0^1 dl \int_0^1 dm \int_{-1}^1 dx$$
$$\frac{4\,(1-x^2)^{\frac{D-3}{2}}\, m^{D-1}\, l^{D-1}}{(l^2+m^2-2lmx)+(1-l^2-m^2+2lmx)\Theta(1-l^2-m^2+2lmx)+v_k''}$$
$$= \left(C_D\right)^2 \frac{k^{2D-6}}{\left[1+v_k''\right]^4}\int_0^1 dl \int_0^1 dm \int_{-1}^1 dx$$
$$\frac{4D^2\,\frac{\Omega_{D-1}}{\Omega_D}\,(1-x^2)^{\frac{D-3}{2}}\, m^{D-1}\, l^{D-1}}{(l^2+m^2-2lmx)+(1-l^2-m^2+2lmx)\Theta(1-l^2-m^2+2lmx)+v_k''}$$
$$\equiv \left(C_D\right)^2 \frac{k^{2D-6}}{\left[1+v_k''\right]^4}\, F_1^D(v_k'') \tag{3.79}$$

where $F_1^D(v_k'')$ can not be written in terms of elementary functions for arbitrary D. Nevertheless a closed form exists for $D=3$ and we will exploit it in the next chapter.

The same results can be found for:

$$I_\eta^D(v_k'') = \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\frac{k^{2D-6}}{\left[1+v_k''\right]^4}\int_0^1 dl \int_0^1 dm \int_{-1}^1 dx$$
$$\frac{2(1-x^2)^{\frac{D-3}{2}}\, l^{D-1}\, m^{D-1}\,(1-l^2)}{(l^2+m^2-2lmx)+(1-l^2-m^2+2lmx)\Theta(1-l^2-m^2+2lmx)+v_k''}$$
$$= \left(C_D\right)^2 \frac{k^{2D-6}}{\left[1+v_k''\right]^4}\int_0^1 dl \int_0^1 dm \int_{-1}^1 dx$$
$$\frac{2D^2\,\frac{\Omega_{D-1}}{\Omega_D}(1-x^2)^{\frac{D-3}{2}}\, l^{D-1}\, m^{D-1}\,(1-l^2)}{(l^2+m^2-2lmx)+(1-l^2-m^2+2lmx)\Theta(1-l^2-m^2+2lmx)+v_k''}$$
$$\equiv \left(C_D\right)^2 \frac{k^{2D-6}}{\left[1+v_k''\right]^4}\, F_\eta^D(v_k'') \tag{3.80}$$

and:

$$I_{\eta^2}^D(v_k'') = \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\frac{k^{2D-6}}{\big[1+v_k''\big]^4}\int_0^1 dl\int_0^1 dm\int_{-1}^1 dx$$
$$\frac{(1-x^2)^{\frac{D-3}{2}}\, l^{D-1}\, m^{D-1}\,(1-l^2)\,(1-m^2)}{(l^2+m^2-2lmx)+(1-l^2-m^2+2lmx)\Theta(1-l^2-m^2+2lmx)+v_k''}$$

$$= \Big(C_D\Big)^2\frac{k^{2D-6}}{\big[1+v_k''\big]^4}\int_0^1 dl\int_0^1 dm\int_{-1}^1 dx$$
$$\frac{D^2\,\frac{\Omega_{D-1}}{\Omega_D}(1-x^2)^{\frac{D-3}{2}}\, l^{D-1}\, m^{D-1}\,(1-l^2)\,(1-m^2)}{(l^2+m^2-2lmx)+(1-l^2-m^2+2lmx)\Theta(1-l^2-m^2+2lmx)+v_k''}$$

$$\equiv \Big(C_D\Big)^2\frac{k^{2D-6}}{\big[1+v_k''\big]^4}\,F_{\eta^2}^D(v_k'') \tag{3.81}$$

Since we now know the structure of all the pieces involved, we can easily write down the 2-LPA' flow equation *without* the kinetic corrections due to scale-dependent field strength:

$$\begin{aligned}\frac{\ddot V_k(\phi)}{k^D} =& +\frac{1}{2}\left[2(D+2)-4\eta_k+(\eta_k^2-\dot\eta_k)\,\frac{2}{(D+2)}\right]C_D\left[\frac{1}{1+v_k''}\right]\\ &-\frac{1}{2}\left[4-2\eta_k\,\frac{4}{(D+2)}+\eta_k^2\,\frac{8}{(D+2)(D+4)}\right]C_D\left[\frac{1}{1+v_k''}\right]^2\\ &+\frac{1}{4}\left[4-8\eta_k\left(\frac{1}{D+2}\right)+4\eta_k^2\left(\frac{1}{D+2}\right)^2\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^4\frac{V_k^{(4)}(\phi)}{k^{4-D}\big[Z_k\big]^2}\\ &-\frac{1}{2}\left[F_1^D(v_k'')-2\eta_k\,F_\eta^D(v_k'')+\eta_k^2\,F_{\eta^2}^D(v_k'')\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^4\frac{\big[V_k^{(3)}(\phi)\big]^2}{k^{6-D}\big[Z_k\big]^3}\end{aligned} \tag{3.82}$$

or *with* the kinetic corrections due to scale-dependent field strength:

$$\begin{aligned}\frac{\ddot V_k(\phi)}{k^D} =& +\frac{1}{2}\left[2(D+2)-4\eta_k+(\eta_k^2-\dot\eta_k)\,\frac{2}{(D+2)}\right]C_D\left[\frac{1}{1+v_k''}\right]\\ &-\frac{1}{2}\left[4-2\eta_k\,\frac{4+D}{(D+2)}+\eta_k^2\,\frac{8+2D}{(D+2)(D+4)}\right]C_D\left[\frac{1}{1+v_k''}\right]^2\\ &+\frac{1}{4}\left[4-8\eta_k\left(\frac{1}{D+2}\right)+4\eta_k^2\left(\frac{1}{D+2}\right)^2\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^4\frac{V_k^{(4)}(\phi)}{k^{4-D}\big[Z_k\big]^2}\end{aligned}$$

$$-\frac{1}{2}\left[F_1^D(v_k'') - 2\eta_k\, F_\eta^D(v_k'') + \eta_k^2\, F_{\eta^2}^D(v_k'')\right]\Big(C_D\Big)^2 \left[\frac{1}{1+v_k''}\right]^4 \frac{\left[V_k^{(3)}(\phi)\right]^2}{k^{6-D}\left[Z_k\right]^3} \tag{3.83}$$

Then, thanks to the results of the previous section, dimensionless equations can be written straightforwardly as:

$$\begin{aligned}\left[\frac{\partial}{\partial t} - \Big(d_\phi(k)\,\varphi\,\frac{\partial}{\partial\varphi} - D\Big)\right]^2 v_k(\varphi) = &+\frac{1}{2}\left[2(D+2) - 4\eta_k + (\eta_k^2 - \dot{\eta}_k)\,\frac{2}{(D+2)}\right] C_D \left[\frac{1}{1+v_k''}\right] \\ &-\frac{1}{2}\left[4 - 2\eta_k\,\frac{4}{(D+2)} + \eta_k^2\,\frac{8}{(D+2)(D+4)}\right] C_D \left[\frac{1}{1+v_k''}\right]^2 \\ &+\frac{1}{4}\left[4 - 8\eta_k\Big(\frac{1}{D+2}\Big) + 4\eta_k^2\Big(\frac{1}{D+2}\Big)^2\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^4 v_k^{(4)} \\ &-\frac{1}{2}\left[F_1^D(v_k'') - 2\eta_k\, F_\eta^D(v_k'') + \eta_k^2\, F_{\eta^2}^D(v_k'')\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^4 \left[v_k^{(3)}\right]^2\end{aligned} \tag{3.84}$$

or:

$$\begin{aligned}\left[\frac{\partial}{\partial t} - \Big(d_\phi(k)\,\varphi\,\frac{\partial}{\partial\varphi} - D\Big)\right]^2 v_k(\varphi) = &+\frac{1}{2}\left[2(D+2) - 4\eta_k + (\eta_k^2 - \dot{\eta}_k)\,\frac{2}{(D+2)}\right] C_D \left[\frac{1}{1+v_k''}\right] \\ &-\frac{1}{2}\left[4 - 2\eta_k\,\frac{4+D}{(D+2)} + \eta_k^2\,\frac{8+2D}{(D+2)(D+4)}\right] C_D \left[\frac{1}{1+v_k''}\right]^2 \\ &+\frac{1}{4}\left[4 - 8\eta_k\Big(\frac{1}{D+2}\Big) + 4\eta_k^2\Big(\frac{1}{D+2}\Big)^2\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^4 v_k^{(4)} \\ &-\frac{1}{2}\left[F_1^D(v_k'') - 2\eta_k\, F_\eta^D(v_k'') + \eta_k^2\, F_{\eta^2}^D(v_k'')\right]\Big(C_D\Big)^2\left[\frac{1}{1+v_k''}\right]^4 \left[v_k^{(3)}\right]^2\end{aligned} \tag{3.85}$$

The former without and the latter with the above mentioned kinetic corrections.

We have finally written down all the equations we would like to study. It is evident that the structure of second order equations is much more complex and, actually, even working in $D = 3$, we soon find out that the classical numerical techniques that were described in the previous chapter are not powerful enough to handle them. Thus we will need to develop a new numerical technique and we will describe it in the next chapter.

Chapter 4

Application of pseudo-spectral methods to LPA' flow equations

In the previous chapter we derived three approximate flow equations for a scalar theory in D dimensions. In this chapter we will restrict to Z_2-symmetric theories in $D = 3$ in order to study Wilson-Fisher fixed point. As a starting point for numerical calculations, we write down explicitly how the equations look like in $D = 3$ (the corresponding LPA equations can be recovered by simply setting $\eta_k = \dot{\eta}_k = 0$):

- 1-LPA'

$$\left[\partial_t - \left(\frac{1+\eta_k}{2}\,\varphi\,\partial_\varphi - 3\right)\right] v_k(\varphi) = \frac{1}{6\pi^2}\left[1 - \frac{1}{5}\eta_k\right]\left[\frac{1}{1+v_k''}\right] \tag{4.1}$$

- 1^2-LPA'

$$\begin{aligned}\left[\partial_t - \left(\frac{1+\eta_k}{2}\,\varphi\,\partial_\varphi - 3\right)\right]^2 v_k(\varphi) = &+\frac{1}{6\pi^2}\left[5 - 2\eta_k + \frac{1}{5}(\eta_k^2 - \dot{\eta}_k)\right]\left[\frac{1}{1+v_k''}\right] \\ &- \frac{1}{6\pi^2}\left[2 - \frac{7}{5}\eta_k + \frac{1}{5}\eta_k^2\right]\left[\frac{1}{1+v_k''}\right]^2 \\ &+ \left(\frac{1}{6\pi^2}\right)^2\left[1 - \frac{2}{5}\eta_k + \frac{1}{25}\eta_k^2\right]\left[\frac{1}{1+v_k''}\right]^4 v_k^{(4)} \\ &- \left(\frac{1}{6\pi^2}\right)^2\left[2 - \frac{4}{5}\eta_k + \frac{2}{25}\eta_k^2\right]\left[\frac{1}{1+v_k''}\right]^5 \left[v_k^{(3)}\right]^2\end{aligned} \tag{4.2}$$

It is important to remark that physical results from 1^2-LPA', as a second order flow, should be strictly equivalent to the ones obtained from 1-LPA', which is a first order flow. Therefore this constitutes a useful playground to test numerical algorithms used to solve second order flows.

- 2-LPA' *without* kinetic corrections due to $\dot{Z}_k$

$$\left[\partial_t - \left(\frac{1+\eta_k}{2}\,\varphi\,\partial_\varphi - 3\right)\right]^2 v_k(\varphi) = +\frac{1}{6\pi^2}\left[5 - 2\eta_k + \frac{1}{5}(\eta_k^2 - \dot{\eta}_k)\right]\left[\frac{1}{1+v_k''}\right] - \frac{1}{6\pi^2}\left[2 - \frac{4}{5}\eta_k + \frac{4}{35}\eta_k^2\right]\left[\frac{1}{1+v_k''}\right]^2 + \left(\frac{1}{6\pi^2}\right)^2\left[1 - \frac{2}{5}\eta_k + \frac{1}{25}\eta_k^2\right]\left[\frac{1}{1+v_k''}\right]^4 v_k^{(4)} - \left(\frac{1}{6\pi^2}\right)^2\left[\frac{1}{2}\,F_1^3(v_k'') - \eta_k\,F_\eta^3(v_k'') + \frac{1}{2}\,\eta_k^2\,F_{\eta^2}^3(v_k'')\right]\left[\frac{1}{1+v_k''}\right]^4\left[v_k^{(3)}\right]^2 \tag{4.3}$$

- 2-LPA' *with* kinetic corrections due to $\dot{Z}_k$

$$\left[\partial_t - \left(\frac{1+\eta_k}{2}\,\varphi\,\partial_\varphi - 3\right)\right]^2 v_k(\varphi) = +\frac{1}{6\pi^2}\left[5 - 2\eta_k + \frac{1}{5}(\eta_k^2 - \dot{\eta}_k)\right]\left[\frac{1}{1+v_k''}\right] - \frac{1}{6\pi^2}\left[2 - \frac{7}{5}\eta_k + \frac{1}{5}\eta_k^2\right]\left[\frac{1}{1+v_k''}\right]^2 + \left(\frac{1}{6\pi^2}\right)^2\left[1 - \frac{2}{5}\eta_k + \frac{1}{25}\eta_k^2\right]\left[\frac{1}{1+v_k''}\right]^4 v_k^{(4)} - \left(\frac{1}{6\pi^2}\right)^2\left[\frac{1}{2}\,F_1^3(v_k'') - \eta_k\,F_\eta^3(v_k'') + \frac{1}{2}\,\eta_k^2\,F_{\eta^2}^3(v_k'')\right]\left[\frac{1}{1+v_k''}\right]^4\left[v_k^{(3)}\right]^2 \tag{4.4}$$

where (see Appendix A):

$$F_1^3(v_k'') = \frac{3}{8}\left\{5\,\frac{1}{1+v_k''} + \left(\frac{7}{2} - 3v_k''\right) - 32\sqrt{v_k''}\arctan\left(\frac{\sqrt{v_k''}}{2+v_k''}\right) + v_k''(12+v_k'')\log\left(\frac{4+v_k''}{1+v_k''}\right)\right\} \tag{4.5}$$

$$F_\eta^3(v_k'') = \frac{1}{160}\left\{\frac{141}{2}\frac{1}{1+v_k''} + \left(38 + 165\,v_k'' - 18\,(v_k'')^2\right)\right.$$

$$
\begin{aligned}
& - (384 + 480\, v_k'')\sqrt{v_k''}\arctan\left(\frac{\sqrt{v_k''}}{2+v_k''}\right) \\
& + 6\,(v_k'')^2\Big(20 + \; v_k''\Big)\log\left(\frac{4+v_k''}{1+v_k''}\right)\Bigg\} \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
F^3_{\eta^2}(v_k'') = \frac{3}{4480}\Bigg\{ & \frac{781}{5}\frac{1}{1+v_k''} + \frac{1}{12}\Big(695 + 5628\, v_k'' - 1746\,(v_k'')^2 - 108\,(v_k'')^3\Big) \\
& - 256(4 + 7\, v_k'')\sqrt{v_k''}\arctan\left(\frac{\sqrt{v_k''}}{2+v_k''}\right) \\
& + (v_k'')^2\Big(560 + 56\, v_k'' + 3\,(v_k'')^2\Big)\log\left(\frac{4+v_k''}{1+v_k''}\right)\Bigg\} \qquad (4.7)
\end{aligned}
$$

As already anticipated, shooting methods and polynomial expansions are not sophisticated enough to tackle the above second order equations. Polynomial expansions, either around the origin or the minimum, are not stable, the radius of convergence simply being not enough to find an extended solution. At the same time shooting methods, either from the origin or the asymptotic region, have to tackle a problem made much harder by two main facts: besides the difficulties of a more complex non-linear structure of the equations, they now need to sample at least a 2 dimensional parameter space[1]. Thus starting from the origin, they are not able to find any global solution different from the trivial fixed point, while, starting from the asymptotic region, they are not able to reach the origin for a finite domain of the parameter space. In conclusion, in order to extract meaningful results about fixed points and critical exponents we need to use a more powerful technique, the so-called pseudo-spectral method.

In the first section we are going to describe this technique, then we will show how it can reproduce the LPA results with a much higher accuracy; after that we will apply it to 1-LPA' in order to find new results about anomalous dimension and critical exponents. We will then use 1^2-LPA' as a bridge between first order and second order equations since it contains the same physical informations of 1-LPA', but it shares the same criticality of 2-LPA'. Finally we will be able to tackle 2-LPA' equation, first setting to zero all contributions due to anomalous dimension (i.e. restricting to LPA) and then in its complete form. We will let to the final chapter a complete discussion of all the results.

---

[1] Notice that first order flow equations correspond to second order ODEs for fixed points, while second order flow equations correspond to fourth order ODEs for fixed points. Thanks to Z_2 symmetry, if we integrate from the origin we need one initial condition in the first case and two initial conditions in the second case.

4.1 Pseudo-spectral methods

Pseudo-spectral methods consist in approximating an unknown solution f of a given differential problem[2]:

$$\mathcal{L}[f] = 0 \tag{4.8}$$

thanks to the following procedure:

- write the function as an expansion on some basis set $\{e_n\}$:

$$f \simeq \sum_{n=0}^{N} c_n \, e_n \tag{4.9}$$

- determine the $N+1$ coefficients $\{c_n\}_{n=0}^{N}$ by requiring equation (4.8) to be satisfied on $N+1$ points (the so called *collocation points*)

Obviously, a great care is necessary in the choice of the basis set *and* the collocation points, as they will completely determine the outcomes of the method.

The general theory of pseudo-spectral methods is well established and a good and comprehensive reference may be found in [17]. However, their application to the study of RG fixed points is not well known and it has been proposed only very recently (see [18]). Moreover it has been used so far only to increase the accuracy of problems already treatable with other numerical techniques. On the contrary, pseudo-spectral methods become a necessary tool if we want to find fixed points and critical exponents using our second order equations.

Let us now quickly review the method and its applications to our problem. First, let us make a few comments concerning the choice of the basis set:

- on general ground:
  - we would like the expansions to be accurate and numerically treatable
  - during the implementation of the method, we will need to evaluate the function and its derivatives at arbitrary points
- for our specific problems:
  - we are looking for solutions defined on the unbounded domain $[0, +\infty[$

[2]Since these methods will be applied only to ODEs we can focus on such situation, however the general idea can be generalized to PDEs and integro-differential equations

– we will always know the asymptotic behaviour of the solutions from a previous analytic study, since it will be described by the linear part of the equations, the rest being subleading in the large field region.

Because of the first two points, which are common to any implementation of pseudo spectral methods, the natural candidates as basis functions are, in general, the classical orthogonal polynomials. Then, because of the last two points, we will restrict our attention (as suggested also in [18]) to Chebyshev polynomials of the first kind, defined by:

$$T_n(\cos(x)) = \cos(nx) \tag{4.10}$$

and to rational Chebyshev polynomials, defined by:

$$R_n^L(x) = T_n\left(\frac{x-L}{x+L}\right) \tag{4.11}$$

which provides a "compactified" description for the asymptotic region.

The main reason for this choice is that Chebyshev polynomials, at variance with other orthogonal polynomials (e.g. Hermite, Laguerre), are defined on a bounded interval (which canonically is defined as [−1,1]), therefore we will be able to split our global problems into two parts and keep easily under control the already known asymptotic behaviour. For instance, since we already know that in LPA equations in $D = 3$ the behaviour is $v_{as}(\varphi) \sim \varphi^6$, fixed point solutions will be written as:

$$\begin{cases} v_*(\varphi) \simeq \sum_{n=0}^{N_0} c_i T_i(\frac{2\varphi}{\varphi_0} - 1) \quad \text{for } \varphi \in [0, \varphi_0] \\ \\ v_*(\varphi) \simeq \varphi^6 \sum_{n=0}^{N_{as}} r_i R_i^L(\varphi - \varphi_0) \quad \text{for } \varphi \in [\varphi_0, +\infty[\end{cases} \tag{4.12}$$

where the arbitrary parameters φ_0 and L will have to be specified consistently with the asymptotic behaviour of the solution. Thanks to the above splitting the Chebyshev expansions will be used to describe only "small corrections", hence assuming finite values over their entire domain. Another important reason for choosing Chebyshev polynomials is that they have very good convergence properties, better than, for instance, Legendre polynomials (see [17] for details).

Regarding collocation points in one dimensional problems, there are only two optimized choices (see [17]). With reference to the canonical interval [−1,1], the $N + 1$ optimized collocation points are given:

- the roots of $T_{N+1}(x)$ (Gauss-Chebyshev grid):

$$\cos\left[\frac{(2i-1)\pi}{2N+2}\right] \text{ for } i = 1, \ldots, N+1 \tag{4.13}$$

- the extrema and the end-points of $T_{N+1}(x)$ (Gauss-Lobatto grid):

$$\cos\left[\frac{i\pi}{N}\right] \text{ for } i = 0, \ldots, N \tag{4.14}$$

We will use the first choice in all future calculations, however this choice should only slightly affect the speed of convergence.

Before moving on to calculations we should spend a few words describing why we expect the above method to be suitable for our search of fixed points. Working with a first order equation, we have understood that fixed point solutions need to be global solutions. Moreover we have seen that in $D = 3$, thanks to the non linear structure of the equation, only two solutions, corresponding to trivial and Wilson-Fisher fixed point, can be globally extended. Finally, we already know the asymptotic behaviour of the global solution we are looking for (i.e. in LPA in $D = 3$ we know $v_{as}(\varphi) \sim \varphi^6$). These three points together suggest that pseudo-spectral methods are very suited for our problem because they build up solutions globally. Actually, if we try to solve the equations with finite-element methods, which work locally, we must be really accurate in choosing the correct initial conditions and we must push numerical integration to a very high precision, or we will soon end up in some singularity; and, in fact, such a precise fine tuning is not realizable in second order equations, due to their complex structure on the one hand and the enlarging of parameter space on the other hand. Conversely, if we work with pseudo-spectral methods and we constrain, by construction, the solution to be globally defined and to have the desired asymptotic behaviour, we do not need to specify any other parameter because, thanks to the intrinsic discrete number of global solutions of the problem, the algorithm will relax autonomously to the desired solution (provided we make it start by a reasonable seed). Indeed we will see that 1-LPA and 1^2-LPA will converge to the exact same fixed point by only specifying the same asymptotic behaviour, even if 1^2-LPA is a fourth order ODEs and it may be expected that some additional conditions are needed.

4.2 1-LPA equation via pseudo-spectral methods

In order to gain some experience with the pseudo-spectral method let us start with an already studied equation[3] (see section 2.4.2):

$$\left[\partial_t - (\rho\,\partial_\rho - 3)\right] u_k(\rho) = \frac{1}{6\pi^2}\frac{1}{\left[1 + 4\,\rho\, u_k''(\rho) + 2u_k'(\rho)\right]}$$

[3] At the beginning of this chapter we wrote down all the equations as functions of φ and $v_k(\varphi)$, since their meaning is more transparent in that form. However, all calculations are better to be performed working with $\rho = \varphi^2$ and $u_k(\rho) = v_k(\varphi(\rho))$, which is define on $[0, +\infty]$.

which at fixed point reduces to:

$$(\rho\,\partial_\rho - 3)\,u_*(\rho) + \frac{1}{6\pi^2}\frac{1}{\left[1 + 4\,\rho\,u_*''(\rho) + 2u_k'(\rho)\right]} = 0$$

We expand the fixed point potential in Chebyshev series, as described above:

$$\begin{cases} u_*(\rho) \simeq \sum_{n=0}^{N_0} c_i T_i(\frac{2\rho}{\rho_0} - 1) \quad \text{for } \rho \in [0, \rho_0] \\ \\ u_*(\rho) \simeq \rho^3 \sum_{n=0}^{N_{as}} r_i T_i(\frac{\rho-\rho_0-L}{\rho-\rho_0+L}) \quad \text{for } \rho \in [\rho_0, +\infty[\end{cases} \tag{4.15}$$

and we choose the following values for the parameters ρ_0 and L:

$$\rho_0 = \frac{1}{4} \qquad\qquad L = \frac{1}{2} \tag{4.16}$$

This choice, which will guarantee a good convergence and will be used for all other implementations, was suggested by the numerical results already found with standard techniques: actually we did find that the asymptotic description is already a good approximation for $\rho > \frac{1}{4}$. However a good convergence and the same numerical results can be found for other choices of the parameters, as far as they are consistent with the already known properties of the solution.

Then we need to define the collocation points and, as stated above, we choose the Gauss-Chebyshev grid for both expansions. Yet, at variance with standard pseudo-spectral methods defined on a single interval, we need to impose regularity conditions at ρ_0, hence we we will have to sacrifice two collocation conditions[4] and request the two additional conditions:

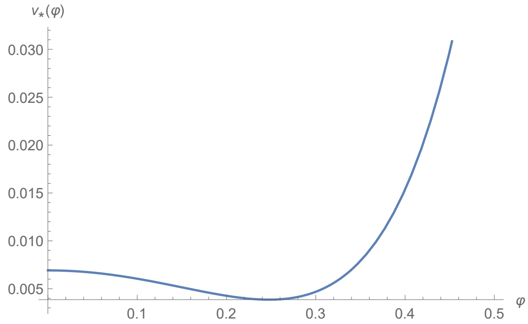
$$u_*(\rho_0^-) = u_*(\rho_0^+) \qquad\qquad u_*'(\rho_0^-) = u_*'(\rho_0^+) \tag{4.17}$$

required by the second order ODE.

Finally we need to define an appropriate seed for the algorithm and just a few terms of a polynomial expansion around the origin are sufficient in order to guarantee convergence.

Actually, the convergence to Wilson-Fisher fixed point is very quick and if we stop at polynomials of order 40 we find out that the corresponding u_* already satisfies the equation with a maximum error of $\sim 10^{-22}$ in the interval [0,1]. In figure 4.1 we report the result of pseudo-spectral analysis and we can easily recognize the shape of dimensionless Wilson-Fisher fixed point.

[4] Any time additional conditions has to be imposed at the expense of collocation conditions, the best convergence is obtained by evenly subtracting collocation conditions in the two regions, where $\rho \simeq 0$ and $\rho \simeq +\infty$, having the two respective descriptions given in eq. (4.15)



(a)

Figure 4.1: Wilson-Fisher solution found with pseudo-spectral methods

Having found a pseudo-spectral solution for Wilson-Fisher fixed point we can now look for critical exponents. With this powerful technique we can find critical exponents by directly solving the eigenvalue problem:

$$\lambda\,\delta u(\rho) = (\rho\,\partial_\rho - 3)\,\delta u(\rho) - \frac{1}{6\pi^2}\left[\frac{1}{1+4\,\rho\,u''_*(\rho)+2u'_*(\rho)}\right]^2 (4\,\rho\,\delta u''(\rho) + 2\,\delta u'(\rho)) \quad (4.18)$$

which corresponds to the linearized flow near the fixed point.

Actually, in this case we need to impose a different asymptotic behaviour for $\delta u(\rho)$, which depends on the unknown eigenvalue:

$$\begin{cases} \delta u(\rho) \simeq \sum_{n=0}^{N_0} c_i T_i(\frac{2\rho}{\rho_0} - 1) \quad \text{for } \rho \in [0, \rho_0] \\ \\ \delta u(\rho) \simeq \rho^{3+\lambda} \sum_{n=0}^{N_{as}} r_i T_i(\frac{\rho-\rho_0-L}{\rho-\rho_0+L}) \quad \text{for } \rho \in [\rho_0, +\infty[\end{cases} \quad (4.19)$$

thus, at variance with fixed point equation, the asymptotic behaviour is not exactly known *a priori*. This fact may seem to spoil convergence, yet it is internal consistency between the asymptotic behaviour and the equation that fixes the value of λ. On the other hand, we need to impose an additional (arbitrary) normalization condition, which we choose to be:

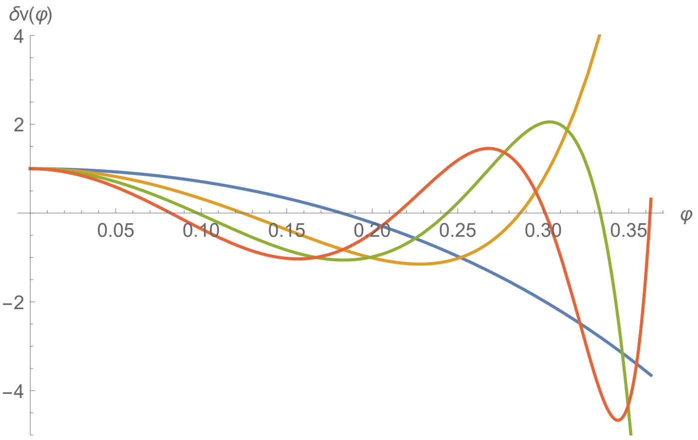
$$\delta u(0) = 1 \quad (4.20)$$

A part from the above details, the general idea is the same and with polynomials of order 40 we quickly find out a very accurate prediction for the only relevant exponent:

$$\lambda_1 = -1.5394994598061774374 20(4) \quad (4.21)$$

Such prediction is consistent with our previous results but its accuracy, obtained with a comparable computational time, is much higher. In fact this method can be used to find results up to any desired accuracy, in order to be sure that one is left only with theoretical errors due to truncation.

In figure 4.2 we show the first eigenpertubations corresponding to $\lambda_1 = -1.539$, $\lambda_2 = 0.6557$, $\lambda_3 = 3.180$ and $\lambda_4 = 5.912$ (we show only four digits, however, as we said for λ_1, the accuracy can be arbitrarily higher).



(a)

Figure 4.2: Eigenperturbations of Wilson-Fisher found with 1-LPA. In blue $\delta v_1(\varphi)$, in yellow $\delta v_2(\varphi)$, in green $\delta v_3(\varphi)$ and in red $\delta v_4(\varphi)$

4.3 1-LPA' equation via pseudo-spectral methods

In the above section we showed how to apply pseudo-spectral methods to fixed point equations and we showed how such methods overcome the results found with other more used techniques. Hence we can now turn to equation 1-LPA' and try to solve it directly via pseudo-spectra methods.

The equation reads:

$$\left[\partial_t - \Big((1+\eta_k)\,\rho\,\partial_\rho - 3\Big)\right] u_k(\rho) = \frac{1}{6\pi^2}\frac{1-\frac{\eta_k}{5}}{\left[1+4\,\rho\,u_k''(\rho)+2u_k'(\rho)\right]} \tag{4.22}$$

and for the first time we have to deal with a scale dependent anomalous dimension, thus we need to give it a proper definition as a function of the other couplings involved.

We stress that LPA' is not a totally consistent approximation as the unknown function $Z_k(\phi)$ that multiplies the kinetic term in the derivative expansion is truncated at zero order and only the flow of the effective potential is followed. Hence it is not possible to define a consistent independent flow for $\dot{Z}_k$ or, equivalently, for η_k, since it always depends on an arbitrary field configuration. Moreover, in the literature it has been argued that a better estimate is given by the anomalous dimension of a $O(N)$ model in the limit $N \to 1$, even if such procedure may involve Goldstone modes, which are absent for $N = 1$. As any definition has its degree of arbitrariness, we will simply stick to (see [19] [20] [21] [18]):

$$\eta_k[u_k] = \frac{1}{6\pi^2} 32 \frac{\rho_0(k)\left[u_k''(\rho_0(k))\right]^2}{\left[1 + 4\rho_0(k)u_k''(\rho_0(k))\right]^2} \tag{4.23}$$

where $\rho_0(k)$ is the minimum of the flowing effective point potential, bearing in mind that it can only be a rough, an not very reliable, estimate of the true anomalous dimension.

From a computational point of view, the inclusion of such a $\eta_k = \eta_k[u_k]$ requires two changes. The asymptotic behaviour needs to be modified:

$$\begin{cases} u_*(\rho) \simeq \sum_{n=0}^{N_0} c_i T_i(\frac{2\rho}{\rho_0} - 1) \quad \text{for } \rho \in [0, \rho_0] \\ \\ u_*(\rho) \simeq \rho^{\frac{3}{1+\eta_k}} \sum_{n=0}^{N_{as}} r_i T_i(\frac{\rho - \rho_0 - L}{\rho - \rho_0 + L}) \quad \text{for } \rho \in [\rho_0, +\infty[\end{cases} \tag{4.24}$$

and an iterative procedure is necessary:

- we start from $u_*^{LPA}(\rho)$
- we evaluate $\eta_k^{(0)} \equiv \eta_k[u_*^{LPA}]$
- we find from equation:

$$\left[\partial_t - \left((1 + \eta_k^{(0)})\,\rho\,\partial_\rho - 3\right)\right] u_k(\rho) = \frac{1}{6\pi^2} \frac{1 - \frac{\eta_k^{(0)}}{5}}{\left[1 + 4\,\rho\, u_k''(\rho) + 2u_k'(\rho)\right]}$$

  a new fixed point $u_*^{(1)}(\rho)$ and we use it as a new starting point

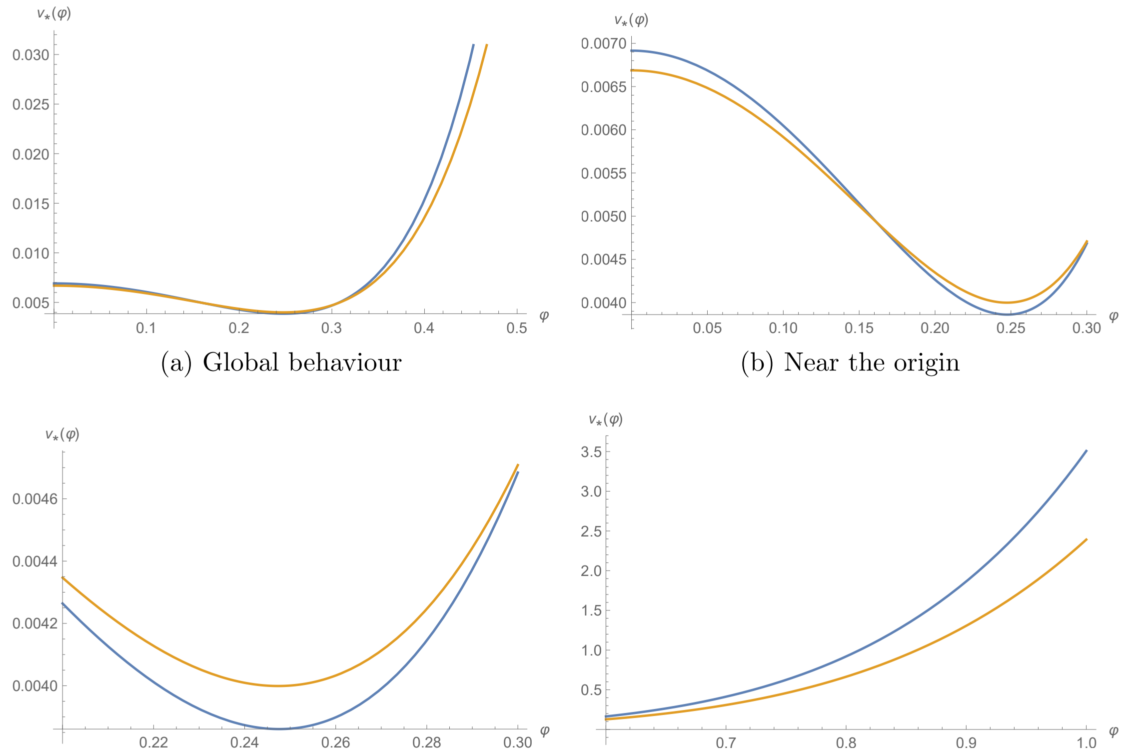
Actually we find out that the iterative method converges and, using polynomials of order 100, we arrive at:

$$\begin{cases} \eta_* \equiv \eta_k^{(+\infty)} = 0.04427233737031503 52(6) \\ u_*^{LPA'}(\rho) \equiv u_*^{(+\infty)}(\rho) \end{cases} \tag{4.25}$$

A fortiori we can notice that the chosen definition of η_k gives a result similar to the correct one: $\eta = 0.03629(8)$. In figure 4.3 we compare $v_*^{LPA}(\varphi)$ and $v_*^{LPA'}(\varphi)$ and the main differences, due to anomalous dimension, are visible in the asymptotic region.

Having found the LPA' fixed point we can now look for critical exponents. Because of anomalous dimension the linearized equation becomes:

$$\begin{aligned} \lambda\,\delta u(\rho) = &\left((1 + \eta_k)\,\rho\,\partial_\rho - 3\right)\delta u(\rho) + \rho\, v_*(\rho)\,\delta\eta[\delta u] \\ &- \frac{1}{6\pi^2}\left[\frac{1 - \frac{\eta_*}{5}}{1 + 4\,\rho\, u_*''(\rho) + 2u_*'(\rho)}\right]^2 (4\,\rho\,\delta u''(\rho) + 2\,\delta u'(\rho)) \\ &- \frac{1}{6\pi^2}\left[\frac{1}{1 + 4\,\rho\, u_*''(\rho) + 2u_*'(\rho)}\right]\frac{\delta\eta[\delta u]}{5} \end{aligned} \tag{4.26}$$



(a) Global behaviour (b) Near the origin

(c) Near the minimum (d) Asymptotic behaviour

Figure 4.3: Wilson-Fisher fixed point. In blue 1-LPA, in yellow 1-LPA’

hence we need to include the linear (local) operator:

$$\delta\eta[\delta u] = \frac{1}{6\pi^2} 32 \frac{u_*''(\rho_{0*})}{\left[1 + 4\rho_{0*}\, u_*''(\rho_{0*})\right]^2} \left[-\frac{\partial}{\partial\rho}\bigg|_{\rho_{0*}} + 2\rho_{0*} \left(+ \frac{\partial^2}{\partial^2\rho}\bigg|_{\rho_{0*}} - \frac{u_*'''(\rho_{0*})}{u_*''(\rho_{0*})} \frac{\partial}{\partial\rho}\bigg|_{\rho_{0*}} \right) \right. \\ \left. - \frac{2\rho_{0*}\, u_*''(\rho_{0*})}{\left[1 + 4\rho_{0*}\, u_*''(\rho_{0*})\right]} \left(4\rho_{0*} \frac{\partial^2}{\partial^2\rho}\bigg|_{\rho_{0*}} - 4\rho_{0*} \frac{u_*'''(\rho_{0*})}{u_*''(\rho_{0*})} \frac{\partial}{\partial\rho}\bigg|_{\rho_{0*}} - 4\frac{\partial}{\partial\rho}\bigg|_{\rho_{0*}} \right) \right] \delta u \tag{4.27}$$

where ρ_{0*} is the minimum of fixed point potential.

Concordantly the asymptotic behaviour needs to be slightly modified:

$$\begin{cases} \delta u(\rho) \simeq \sum_{n=0}^{N_0} c_i T_i(\frac{2\rho}{\rho_0} - 1) & \text{for } \rho \in [0, \rho_0] \\ \delta(\rho) \simeq \rho^{\frac{3+\lambda}{1+\eta_* + \delta\eta_*}} \sum_{n=0}^{N_{as}} r_i T_i(\frac{\rho - \rho_0 - L}{\rho - \rho_0 + L}) & \text{for } \rho \in [\rho_0, +\infty[\end{cases} \tag{4.28}$$

where $\delta\eta_*$ is an unknown coefficient, which must be determined by the equation, together with λ, $\{c_i\}$ and $\{r_i\}$. It is quite a remarkable fact that the algorithm can reach some convergence even if the asymptotic behaviour, whose internal consistency with the equation determines the eigenvalue, is modified by an unknown $\delta\eta_*$. As an additional hint about the problem it must be noticed that in 1-LPA polynomials of order 40 are sufficient to describe an eigenperturbation δu_1^{LPA} whose maximum error in the interval [0,1] is of order 10^{-15}, while in 1-LPA' polynomials of order 50 give an eigenperturbation $\delta u_1^{LPA'}$ whose maximum error, in the same interval, is of order 10^{-7}. Moreover the error becomes more and more significant in the asymptotic region. All these facts suggest that the corrections $\delta\eta[\delta u]$, which depend only on the behaviour of the δu in ρ_{0*} because of the LPA' definition of anomalous dimension, are not consistent with a global definition of the eigenperturbations.

Nevertheless, the relevant eigenvalue found with the above analysis is:

$$\lambda_1 = -1.570 \tag{4.29}$$

the others being: $\lambda_2 = 0.7268$, $\lambda_3 = 3.074$ and $\lambda_4 = 5.736$. It seems that we find a better estimate (consistent with the one found in [21]) with a polynomial expansion) for the relevant exponent $\nu = 0.637$, since it is now accurate up to 1%.

4.4 Second order equations via pseudo-spectral methods

4.4.1 1^2-LPA'

We can now turn to second order equation and, as a test for pseudo-spectral methods, we start by analysing 1^2-LPA'. The method is exactly the same one used for 1-LPA', the only difference are regularity conditions between the two domains, because in this case they are:

$$u_*(\rho_0^-) = u_*(\rho_0^+) \quad u_*'(\rho_0^-) = u_*''(\rho_0^+) \quad u_*''(\rho_0^-) = u_*''(\rho_0^+) \quad u_*'''(\rho_0^-) = u_*'''(\rho_0^+) \tag{4.30}$$

since the equation in now a fourth order ODE.

What we find out is that:

- if we restrict to 1^2-LPA we reproduce the exact same results of 1-LPA: same fixed point and same critical exponents
- studying the complete 1^2-LPA' equation we reproduce the exact same fixed point and anomalous dimension, however, studying critical exponents, we run into the problem noticed above, which, in this more complex formulation, the stability of the method is not able to overcome: a consistent global eigenperturbation for the relevant exponent can not be found, since convergence properties are poor.

Those problems, due to corrections in the asymptotic behaviour, that may be neglected at first sight in 1-LPA' appear evidently in this second order equation. Nevertheless the two equations are perfectly equivalent: if we insert the relevant eigenperturbation and the relevant eigenvalue found with 1-LPA' inside the linearized form of 1^2-LPA' the equation is as well satisfied as it was in the former case. Thus it seems that the results found with 1-LPA' descended by the stability of the solution in the finite region (and the polynomial analysis used in [21] is only based on the finite region), while it is doubtful that they describe a global consistent solution. But this should not be a real surprise since the LPA' scheme, even if used successfully, is not a consistent scheme.

4.4.2 2-LPA

Having already tested the pseudo-spectral methods, we can turn to equation 2-LPA, that is equation (4.3), or equivalently (4.4), without corrections due to anomalous dimension. Since we expect only small variations from the 1-LPA case, it seems reasonable to use the solution of 1-LPA as a seed for 2-LPA. Actually, despite the more complex equation, the method converges very well and with polynomials of order 60 we find a solution whose maximum error in the interval [0,1] is of order 10^{-17}. We remind the reader that it was not possible to construct numerically the global scaling solutions, i. e. the fixed point of the potential, using the standard techniques presented in chapter 2.

In figure 4.4 we compare 1-LPA fixed point with 2-LPA fixed point and we can notice only a slightly difference between the two.

More interesting than fixed points are critical exponents, in particular we are interested in the only relevant one. Solving the linearized equation we find out that it becomes:

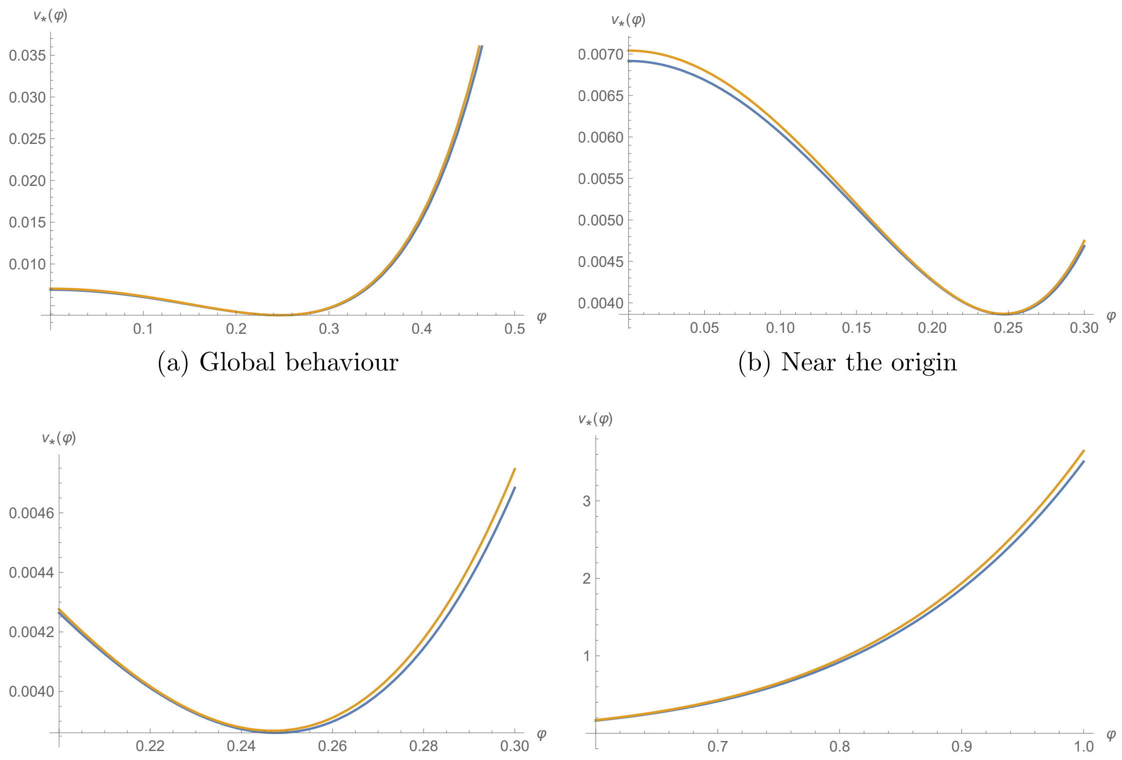
$$\lambda_1 = -1.504 \tag{4.31}$$

This means that 2-LPA gives a worse estimate of ν than 1-LPA. Also this should not be a surprise since in the second order scheme it is evident, at diagrammatic level, that contributions of structure and order similar to those describing anomalous dimension are present. Therefore in the second order flows anomalous dimension must be taken into account. As a first step we shall try to introduce a partial information of η using the "inconsistent" LPA' scheme.

4.4.3 2-LPA'

Finally we can study the whole 2-LPA' both with kinetic corrections (2LPA'+Kin) or without (2LPA' NoKin). As it was done for 1-LPA' we need to find the anomalous dimension and the fixed point by an iterative procedure. Thanks to pseudo-spectral methods we are able to perform such procedure. The anomalous dimensions become:

$$\text{2LPA'+Kin: } \eta_* = 0.0466 \qquad \text{2LPA' NoKin: } \eta_* = 0.0476 \tag{4.32}$$



(a) Global behaviour

(b) Near the origin

(c) Near the minimum

(d) Asymptotic behaviour

Figure 4.4: Wilson-Fisher fixed point. In blue 1-LPA, in yellow 2-LPA

and the fixed point potentials, compared to 1-LPA', are shown in figure 4.5, while a comparison with 2-LPA is given in figure 4.6. It can be noticed that the main differences are always between "primed" and "not primed" fixed point potentials, while the order of derivation has a weaker impact. Moreover the difference between 2-LPA' +Kin and 2LPA' NoKin is even weaker. Notice how, strictly speaking, the second order flow equations would lead to the 2-LPA' NoKIn version, but LPA' inconsistency leads to a better behaviour of the 2-LPA'+Kin scheme.

Once again we would like to find out critical exponents, in particular we focus on the relevant one. As done for 1-LPA' we linearize the flow equation including corrections due to $\delta\eta[\delta u]$, but as expected after the experience with 1^2-LPA', we find out no real convergence for the relevant eigenvalue.

4.4.4 Formulation without $\delta\eta$

Finally, as an intermediate formulation for "primed" equations we can solve the linearized flow without corrections due to $\delta\eta[\delta u]$, hence simply assuming $\eta_k = \eta_*$. In such formulation we find perfect convergence for all equations, moreover 1^2-LPA' behaves

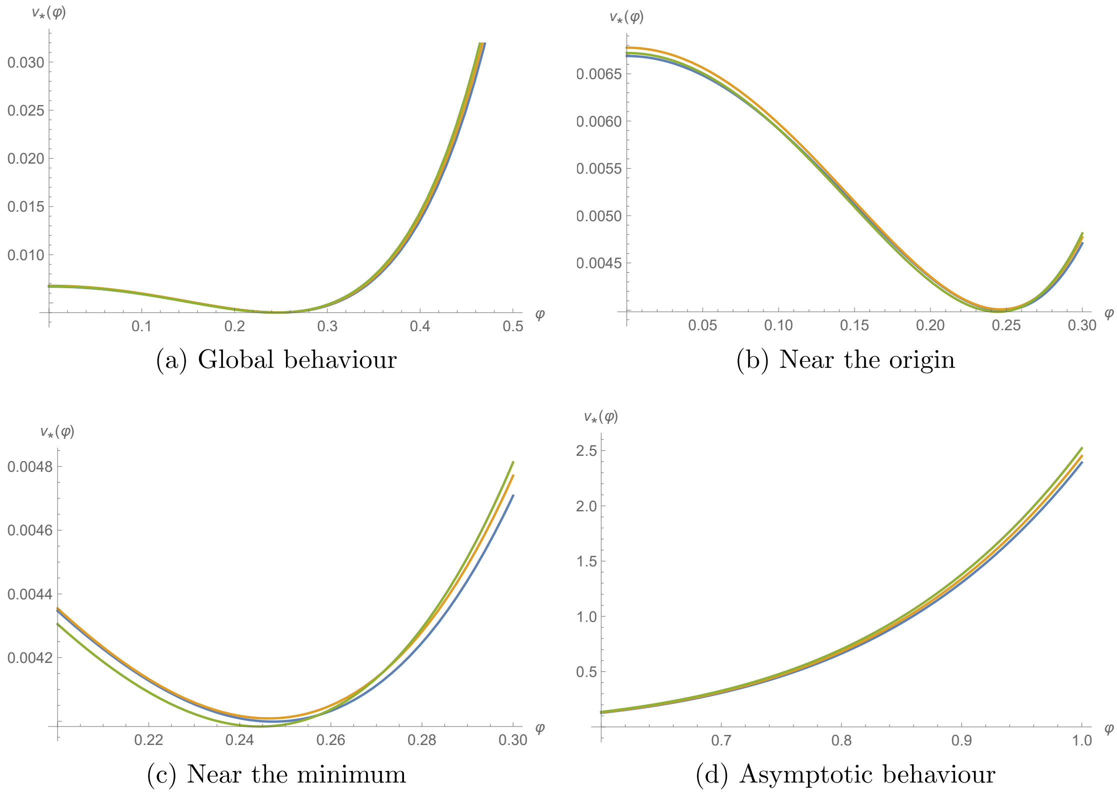
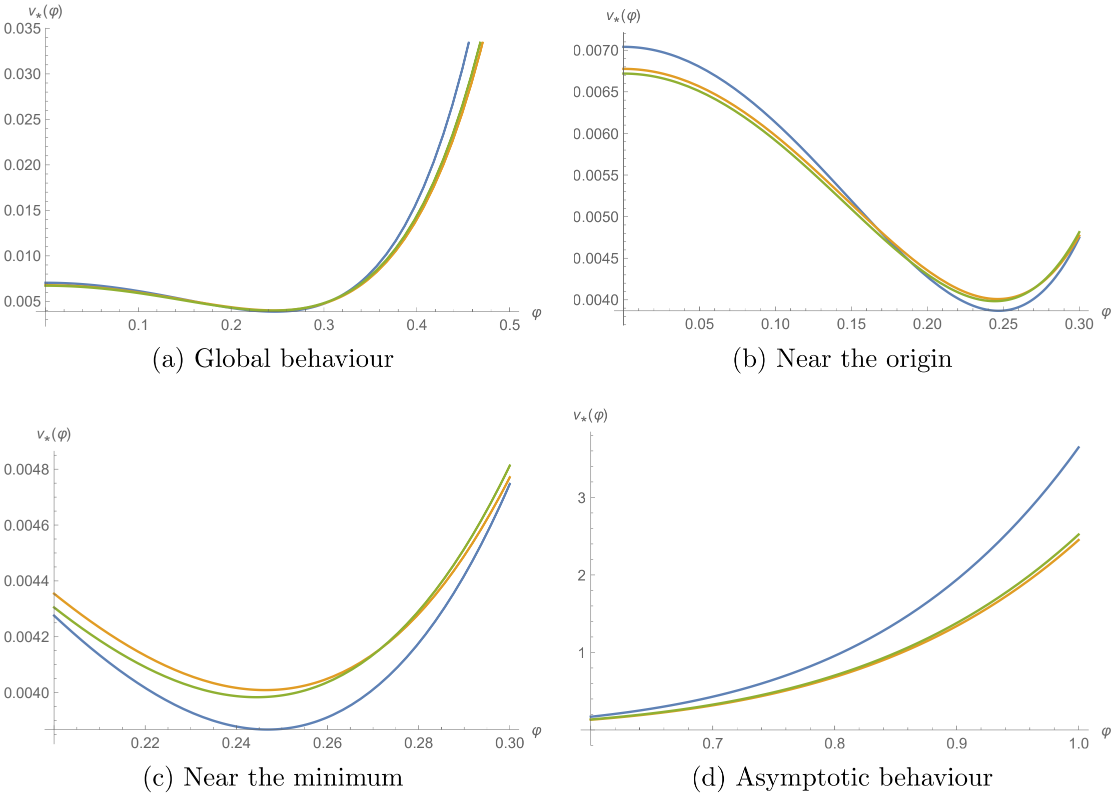


Figure 4.5: Wilson-Fisher fixed point. In blue 1-LPA', in yellow 2-LPA'+Kin, in green 2-LPA' NoKin

exactly as 1-LPA' and the relevant eigenvalue, in the different schemes, becomes:

$$\text{1LPA'}: \lambda_1 = -1.545 \quad \text{2LPA'+Kin}: \lambda_1 = -1.516 \quad \text{2LPA' NoKin}: \lambda_1 = -1.519 \quad (4.33)$$



(a) Global behaviour

(b) Near the origin

(c) Near the minimum

(d) Asymptotic behaviour

Figure 4.6: Wilson-Fisher fixed point. In blue 2-LPA, in yellow 2-LPA'+Kin, in green 2-LPA' NoKin

Chapter 5

Conclusions and acknowledgements

In the previous chapter we have found many different results about a scalar model in $D = 3$ from different approximation schemes, we now want to summarize them and try to find a rationale. Yet before moving to physical predictions, let us spend a few words about numerical methods. One of the most difficult aspect of non-perturbative FRG is to keep track of the errors, in principles any kind of truncation should be devised in such a way that enlarging the number of coupling involved the description becomes more and more accurate and the error decreases concurrently. However, if the truncated problem is *approximately* solved, it becomes hard to estimate if the error is intrinsic of the truncation or it is a spurious effect of further approximations. Hence we would like to emphasize that the numerical methods developed in this work, which are still not very much used in the FRG community, are extremely useful for this kind of problems, since they have been capable to solve *exactly* a truncated problem, hence addressing all the discrepancies of their predictions to truncation itself, even for highly non linear differential equations of non algebraic nature.

Having said that we can now concentrate on the physical scenario described by different approximations schemes. In particular we should focus on the relevant critical exponent and the anomalous dimension. It is quite remarkable that a simple LPA truncation is able to predict, with an accuracy of 3%, the relevant critical exponent of Ising model in D=3. Nevertheless, since LPA works in the hypothesis $\eta = 0$, its accuracy can be satisfying as far as anomalous dimension is very small. Hence it must be thought of as a first order approximation. If we want to push our estimates a little further, corrections of the same order of anomalous dimension must be included. The LPA' scheme is very used in the literature as a method to include such corrections. It is characterized by quite a high degree of arbitrariness, but indeed, in some realizations, it seems to predict better estimates for the relevant exponent, accurate up to 1%. Nevertheless, our analysis via pseudo-spectral methods suggests that such result should be taken with care: the corrections due to anomalous dimension that LPA' introduce in the linearized flow seem not to be consistent with global solutions, but rather the result of a local, hence *approximate*, analysis around the origin. Therefore to be consistent, one direction which can be taken, is to use the next order of the truncation in the derivative expansion.

As a novel result, we have introduced a new set of exact higher order flow equations in RG-time, equivalent to the original one at an exact level. They allow to consider, in the derivative expansion, completely new approximation schemes, which are not based on a different truncation, but rather to work directly at a second level structure, where corrections due to anomalous dimension are supposed to play a significant role. Indeed, if we insert LPA at this level, we find out that the absence of an appropriate flow for anomalous dimension becomes manifest, as predictions get slightly worse. Moreover the limitations of the standard LPA' become evident as well, as it is not able to consistently describe the flow of anomalous dimension. On the other hand this new approximation scheme can be used as a starting point for future promising analysis. Actually, working at second order, a different LPA' approach can be developed evaluating the flow of $\ddot{Z}_k$ at zero field configuration, where standard LPA' predicts a vanishing anomalous dimension. Anomalous dimension would be given by integrals of a similar form of the ones discussed in the appendix, as we find out that:

$$\begin{aligned}\ddot{Z}_k = &-\frac{3}{2}\left[V_k^{(4)}(0)\right]^2 \int_p \int_q \dot{R}_k(q^2)\tilde{G}_k^2(q^2)\dot{R}_k(p^2)\tilde{G}_k^2(p^2)\tilde{G}_k'((q-p)^2) \\ &-\frac{3}{D}\left[V_k^{(4)}(0)\right]^2 \int_p \int_q \dot{R}_k(q^2)\tilde{G}_k^2(q^2)\dot{R}_k(p^2)\tilde{G}_k^2(p^2)(q-p)^2\tilde{G}_k''((q-p)^2)\end{aligned} \tag{5.1}$$

Even more promising would be the study, in this new approximation scheme, of the first order derivative expansion, where the whole $Z_k(\phi)$ is retained. In such a case we could follow consistently the flow both of $Z_k(\phi)$ and $V_k(\phi)$ and anomalous dimension would become, as it should be, a global property of the solution, computable as a spectral parameter of the coupled system of differential equations.

In conclusion, thanks to the implementation of pseudo-spectral methods, we have been able to study *exactly* the behaviour of standard truncations and describe, without ambiguities, their limitations and inconsistencies. Moreover we have developed a new approximation scheme which is devised in such a way to include coherently corrections due to anomalous dimension and whose implementation with an appropriate truncation is expected to give results which go beyond nowadays first order predictions. Finally, two more interesting property of this new scheme are its systematic nature, as equations of higher orders can be straightforwardly derived, and its similarity with standard perturbative calculations, organized in loops, which may be used in order to link the two approaches.

Arrived at the end of this work, it is a pleasure for me to thank Dott. Gian Paolo Vacca for his constant support, for the passion showed in research and teaching, for his humanity and his always punctual and encouraging responses. I would also like to thank Prof. Michele Cicoli for his kind supervision of the work.

Appendix A

Evaluation of loops integrals

In section 3.2.3 we had to evaluate the following integrals:

$$
\begin{aligned}
&\int_q \int_p \frac{\frac{\dot{Z}_k}{Z_k}\mathcal{R}_k(q^2) + \dot{\mathcal{R}}_k(q^2)}{\left[q^2 + \mathcal{R}_k(q^2) + \frac{V_k''(\phi)}{Z_k}\right]^2} \frac{\frac{\dot{Z}_k}{Z_k}\mathcal{R}_k(p^2) + \dot{\mathcal{R}}_k(p^2)}{\left[p^2 + \mathcal{R}_k(p^2) + \frac{V_k''(\phi)}{Z_k}\right]^2} \frac{1}{\left[(q-p)^2 + \mathcal{R}_k((q-p)^2) + \frac{V_k''(\phi)}{Z_k}\right]} \\
&= \underbrace{\int_q \int_p \frac{\dot{\mathcal{R}}_k(q^2)}{\left[D_k(q^2)\right]^2} \frac{\dot{\mathcal{R}}_k(p^2)}{\left[D_k(p^2)\right]^2} \frac{1}{\left[D_k((q-p)^2)\right]}}_{I_1^D(v_k'')} \\
&\quad - 2\eta_k \underbrace{\int_q \int_p \frac{\mathcal{R}_k(q^2)}{\left[D_k(q^2)\right]^2} \frac{\dot{\mathcal{R}}_k(p^2)}{\left[D_k(p^2)\right]^2} \frac{1}{\left[D_k((q-p)^2)\right]}}_{I_\eta^D(v_k'')} \\
&\quad + \eta_k^2 \underbrace{\int_q \int_p \frac{\mathcal{R}_k(q^2)}{\left[D_k(q^2)\right]^2} \frac{\mathcal{R}_k(p^2)}{\left[D_k(p^2)\right]^2} \frac{1}{\left[D_k((q-p)^2)\right]}}_{I_{\eta^2}^D(v_k'')}
\end{aligned} \tag{A.1}
$$

where:

$$\mathcal{R}_k(q^2) = (k^2 - q^2)\Theta(k^2 - q^2) \tag{A.2}$$

$$\dot{\mathcal{R}}_k(q^2) \simeq 2k^2\Theta(k^2 - q^2) \tag{A.3}$$

Since the structure, and the consequent analysis, of the integrals is very similar we will focus our attention on $I_1^D(v_k'')$. We can first analyse its structure for a general regulator and then insert the above expression in order to find a closed form. Hence we start with:

$$I_1^D(v_k'') = \int_q \int_p \frac{\dot{\mathcal{R}}_k(q^2)}{\left[D_k(q^2)\right]^2} \frac{\dot{\mathcal{R}}_k(p^2)}{\left[D_k(p^2)\right]^2} \frac{1}{\left[D_k((q-p)^2)\right]} \tag{A.4}$$

then, if we turn the integral on p to D-dimensional spherical coordinates:

$$\begin{cases} p_D & = p\cos(\theta_{D-1}) \\ p_{D-1} & = p\sin(\theta_{D-1})\cos(\theta_{D-2}) \\ \vdots & \\ p_2 & = p\sin(\theta_{D-1})\sin(\theta_{D-2})\dots\cos(\theta_1) \\ p_1 & = p\sin(\theta_{D-1})\sin(\theta_{D-2})\dots\sin(\theta_1) \end{cases} \tag{A.5}$$

we can factorize a D-dimensional solid angle Ω_D, as the direction of p is totally arbitrary.

Thus, after setting the D direction of q along the D direction of p we can switch even the q-integration to D-dimensional spherical coordinates:

$$I_1^D(v_k'') = \frac{\Omega_D}{(2\pi)^D}\int_0^{+\infty} dp\, p^{D-1}\int_0^{+\infty}\frac{dq}{(2\pi)^D}q^{D-1}\int_0^{\pi} d\theta_{D-1}\sin^{D-2}(\theta_{D-1})\dots$$
$$\dots\int_0^{\pi} d\theta_2\sin^1(\theta_2)\int_0^{2\pi} d\theta_1\frac{\dot{\mathcal{R}}_k(q^2)}{\left[D_k(q^2)\right]^2}\frac{\dot{\mathcal{R}}_k(p^2)}{\left[D_k(p^2)\right]^2}\frac{1}{\left[D_k(p^2+q^2-2\,p\,q\cos\theta_{D-1})\right]} \tag{A.6}$$

and noticing that the integration over $\theta_{D-2},\dots,\theta_1$ is trivial we can factorize Ω_{D-1} and simplify the integral as:

$$I_1^D(v_k'') = \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\int_0^{+\infty} dp\, p^{D-1}\int_0^{+\infty} dq\, q^{D-1}\int_0^{\pi} d\theta\,\sin^{D-2}\theta$$
$$\frac{\dot{\mathcal{R}}_k(q^2)}{\left[D_k(q^2)\right]^2}\frac{\dot{\mathcal{R}}_k(p^2)}{\left[D_k(p^2)\right]^2}\frac{1}{\left[D_k(p^2+q^2-2\,p\,q\cos\theta_{D-1})\right]} \tag{A.7}$$

If we now insert the explicit form of the regulator we find:

$$I_1^D(v_k'') = \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\int_0^{k} dp\, p^{D-1}\int_0^{k} dq\, q^{D-1}\int_0^{\pi} d\theta\,\sin^{D-2}\theta\frac{2k^2}{\left[k^2+\frac{V_k''(\phi)}{Z_k}\right]^2}\frac{2k^2}{\left[k^2+\frac{V_k''(\phi)}{Z_k}\right]^2}$$
$$\frac{1}{(p^2+q^2-2p\,q\cos\theta)+[k^2-(p^2+q^2-2p\,q\cos\theta)]\Theta[k^2-(p^2+q^2-2\,p\,q\cos\theta)]+\frac{V_k''(\phi)}{Z_k}} \tag{A.8}$$

Then, defining the dimensionless variables $(l,m,v_k'') = (\frac{p}{k},\frac{q}{k},\frac{V_k''(\phi)}{Z_k\,k^2})$, we arrive at:

$$I_1^D(v_k'') = \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v_k'')^2}\right]^2\int_0^{1} dl\, l^{D-1}\int_0^{1} dm\, m^{D-1}\int_0^{\pi} d\theta\,\sin^{D-2}\theta$$
$$\frac{1}{(l^2+m^2-2l\,m\cos\theta)+[1-(l^2+m^2-2l\,m\cos\theta)]\Theta[1-(l^2+m^2-2\,l\,m\cos\theta)]+v_k''} \tag{A.9}$$

To proceed further we need to study the argument of the Θ function for $(l,h) \in [0,1]^2$:

$$1-(l^2+h^2+2l\,h\cos\theta) > 0 \qquad \Longleftrightarrow \qquad \cos\theta < \frac{1-(l^2+h^2)}{2hl}$$

and as we get:

$$\begin{cases} \frac{1-(l^2+h^2)}{2hl} < 1 \\ \frac{1-(l^2+h^2)}{2hl} > -1 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} (l+h)^2 > 1 \\ (l-h)^2 < 1 \end{cases}$$
$$\Longleftrightarrow \qquad \begin{cases} (l+h) > 1 \\ |l-h| < 1 \quad \forall (l,h) \in [0,1]^2 \end{cases}$$

we can safely assume that the Θ function is equal to 1 in the region

$$\{(l,h) \in [0,1]^2 : \, 0 \le l \le 1 \, , \, 0 \le h \le 1-l\}$$

while in the region

$$\{(l,h) \in [0,1]^2 : \, 0 \le l \le 1 \, , \, 1-l < h \le 1\}$$

the Θ function is equal to 1 iff

$$\cos\theta < t_*(l,h) \le 1$$

where $t_*(l,h) \equiv \frac{1-(l^2+h^2)}{2hl}$. Consequently it is convenient to split the integral into 3 sub-integrals:

$$\begin{aligned} I^D_{1A}(v''_k) &= \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v''_k)^2}\right]^2 \int_0^1 dl\, l^{D-1} \int_0^{1-l} dh\, h^{D-1} \int_0^\pi d\theta\, \sin^{D-2}\theta\, \frac{1}{1+v''_k} \\ &= \left(\frac{\Omega_D}{(2\pi)^D}\right)^2 \left[\frac{2k^{D-3}}{(1+v''_k)^2}\right]^2 \frac{1}{1+v''_k} \int_0^1 dl\, l^{D-1} \int_0^{1-l} dh\, h^{D-1} \\ &= \left(\frac{\Omega_D}{(2\pi)^D}\right)^2 \left[\frac{2k^{D-3}}{(1+v''_k)^2}\right]^2 \frac{1}{1+v''_k} \frac{\Gamma(D)^2}{\Gamma(2D+1)} \end{aligned} \tag{A.10}$$

$$\begin{aligned} I^D_{1B}(v''_k) &= \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v''_k)^2}\right]^2 \int_0^1 dl\, l^{D-1} \int_{1-l}^{1} dh\, h^{D-1} \int_0^{\arccos(t_*(l,h))} d\theta\, \sin^{D-2}\theta\, \frac{1}{1+v''_k} \\ &= \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v''_k)^2}\right]^2 \frac{1}{1+v''_k} \int_0^1 dl\, l^{D-1} \int_{1-l}^{1} dh\, h^{D-1} \int_{-1}^{t_*(l,h))} dt\, (1-t^2)^{\frac{D-3}{2}} \\ &= \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v''_k)^2}\right]^2 \frac{2}{1+v''_k} \int_{1/2}^1 dl\, l^{D-1} \int_{1-l}^{l} dh\, h^{D-1} \int_{-1}^{t_*(l,h))} dt\, (1-t^2)^{\frac{D-3}{2}} \end{aligned} \tag{A.11}$$

$$
\begin{aligned}
I_{1C}^{D}(v_k'') &= \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v_k'')^2}\right]^2 \int_0^1 dl\, l^{D-1}\int_{1-l}^{1} dh\, h^{D-1}\int_{\arccos(t_*(l,h))}^{1} d\theta\, \sin^{D-2}\theta \\
&\qquad \frac{1}{l^2+h^2+2\,l\,h\cos\theta+v_k''} \\
&= \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v_k'')^2}\right]^2 \int_0^1 dl\, l^{D-1}\int_{1-l}^{1} dh\, h^{D-1}\int_{t_*(l,h)}^{1} dt\,(1-t^2)^{\frac{D-3}{2}} \\
&\qquad \frac{1}{l^2+h^2+2\,l\,h\,t+v_k''} \\
&= \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v_k'')^2}\right]^2 \int_{1/2}^1 dl\, l^{D-2}\int_{1-l}^{l} dh\, h^{D-2}\int_{t_*(l,h)}^{1} dt\,(1-t^2)^{\frac{D-3}{2}} \\
&\qquad \frac{1}{\frac{l^2+h^2}{2\,l\,h}+\frac{v_k''}{2\,l\,h}+t}
\end{aligned}
\tag{A.12}
$$

where $t=\cos\theta$ and the symmetry $l\leftrightarrow h$ has been employed to halve the volume of integration.

For $D=3$ we can complete the evaluation of the integrals and we get:

$$
\begin{aligned}
I_{1A}^{3}(v_k'') &= \left(\frac{\Omega_D}{(2\pi)^D}\right)^2\left[\frac{2k^{D-3}}{(1+v_k'')^2}\right]^2\frac{1}{1+v_k''}\frac{\Gamma(D)^2}{\Gamma(2D+1)}\Big|_{D=3} \\
&= \left(\frac{4\pi}{(2\pi)^3}\right)^2\left[\frac{2}{(1+v_k'')^2}\right]^2\frac{1}{1+v_k''}\frac{4}{720} \\
&= \frac{1}{180}\frac{1}{\pi^4}\frac{1}{[1+v_k'']^5}
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
I_{1B}^{3}(v_k'') &= \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v_k'')^2}\right]^2\frac{2}{1+v_k''}\int_{1/2}^1 dl\, l^{D-1}\int_{1-l}^{l} dh\, h^{D-1}\int_{-1}^{t_*(l,h))} dt\,(1-t^2)^{\frac{D-3}{2}}\Big|_{D=3} \\
&= \frac{4\pi}{(2\pi)^3}\frac{2\pi}{(2\pi)^3}\left[\frac{2}{(1+v_k'')^2}\right]^2\frac{2}{1+v_k''}\int_{1/2}^1 dl\, l^2\int_{1-l}^{l} dh\, h^2\left[\frac{1-(l^2+h^2)}{2hl}+1\right] \\
&= \frac{67}{1440}\frac{1}{\pi^4}\frac{1}{[1+v_k'']^5}
\end{aligned}
\tag{A.14}
$$

$$
\begin{aligned}
I_{1C}^{3}(v_k'') &= \frac{\Omega_D}{(2\pi)^D}\frac{\Omega_{D-1}}{(2\pi)^D}\left[\frac{2k^{D-3}}{(1+v_k'')^2}\right]^2\int_{1/2}^1 dl\, l^{D-2}\int_{1-l}^{l} dh\, h^{D-2}\int_{t_*(l,h)}^{1} dt\,\frac{(1-t^2)^{\frac{D-3}{2}}}{\frac{l^2+h^2}{2\,l\,h}+\frac{v_k''}{2\,l\,h}+t}\Big|_{D=3} \\
&= \frac{4\pi}{(2\pi)^3}\frac{2\pi}{(2\pi)^3}\left[\frac{2}{(1+v_k'')^2}\right]^2\int_{1/2}^1 dl\, l\int_{1-l}^{l} dh\, h\log\left[\frac{v_k''+(l+h)^2}{v_k''+1}\right] \\
&= \frac{1}{96}\frac{1}{\pi^4}\frac{1}{[1+v_k'']^4}\left\{\frac{7}{2}-3v_k''-32\sqrt{v_k''}\arctan\left(\frac{\sqrt{v_k''}}{2+v_k''}\right)+\right.
\end{aligned}
$$

$$+ v_k''(12 + v_k'') \log\left(\frac{4 + v_k''}{1 + v_k''}\right)\Bigg\} \tag{A.15}$$

and finally:

$$\begin{aligned}
I_1^3(v_k'') &= I_{1A}^3(v_k'') + I_{1B}^3(v_k'') + I_{1C}^3(v_k'') \\
&= \frac{1}{96}\frac{1}{\pi^4}\frac{1}{[1 + v_k'']^4}\left\{\frac{5}{1 + v_k''} + \frac{7}{2} - 3v_k'' - 32\sqrt{v_k''}\arctan\left(\frac{\sqrt{v_k''}}{2 + v_k''}\right) + \right. \\
&\qquad\left. + v_k''(12 + v_k'')\log\left(\frac{4 + v_k''}{1 + v_k''}\right)\right\} \\
&= \left(\frac{1}{6\pi^2}\right)^2 \frac{1}{[1 + v_k'']^4}\frac{3}{8}\left\{\frac{5}{1 + v_k''} + \frac{7}{2} - 3v_k'' - 32\sqrt{v_k''}\arctan\left(\frac{\sqrt{v_k''}}{2 + v_k''}\right) + \right. \\
&\qquad\left. + v_k''(12 + v_k'')\log\left(\frac{4 + v_k''}{1 + v_k''}\right)\right\}
\end{aligned} \tag{A.16}$$

Along the same line we can find:

$$\begin{aligned}
I_\eta^3(v_k'') = \left(\frac{1}{6\pi^2}\right)^2 \frac{1}{[1 + v_k'']^4}\frac{1}{160}\Bigg\{&\frac{141}{2}\frac{1}{1 + v_k''} + \Big(38 + 165\, v_k'' - 18\,(v_k'')^2\Big) \\
&- (384 + 480\, v_k'')\sqrt{v_k''}\arctan\left(\frac{\sqrt{v_k''}}{2 + v_k''}\right) \\
&+ 6\,(v_k'')^2\Big(20 + v_k''\Big)\log\left(\frac{4 + v_k''}{1 + v_k''}\right)\Bigg\}
\end{aligned} \tag{A.17}$$

and:

$$\begin{aligned}
I_{\eta^2}^3(v_k'') = \left(\frac{1}{6\pi^2}\right)^2 \frac{1}{[1 + v_k'']^4}\frac{3}{4480}\Bigg\{&\frac{781}{5}\frac{1}{1 + v_k''} + \frac{1}{12}\Big(695 + 5628\, v_k'' - 1746\,(v_k'')^2 - 108\,(v_k'')^3\Big) \\
&- 256(4 + 7\, v_k'')\sqrt{v_k''}\arctan\left(\frac{\sqrt{v_k''}}{2 + v_k''}\right) \\
&+ (v_k'')^2\Big(560 + 56\, v_k'' + 3\,(v_k'')^2\Big)\log\left(\frac{4 + v_k''}{1 + v_k''}\right)\Bigg\}
\end{aligned} \tag{A.18}$$

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