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Gravitational collapse and Hawking radiation

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Abstract

This thesis is organized as follows:

- In chapter 1 the theory of a massless scalar field in curved spacetime is introduced. Furthermore the Bogolubov transformation are discussed.
- In chapter 2 the Hawking effect is derived with the necessary approximations. Then it is briefly mentioned the problem of the information loss paradox
- In chapter 3 the contribution of the Hawking radiation to the bidimensional stress energy tensor is analyzed.
- In chapter 4 a generic model of gravitational collapse is analyzed with emphasis on the quantum effects. Then we restrict to the case of the collapse of a thin shell, and the accuracy of the Unruh state approximation is examined.
- In chapter 5 the proposals that the emission of pre-Hawking radiation could prevent the formation of an event horizon for a collapsing object are analyzed.
- In chapter 6 the arguments which aim to disprove the possibility that pre-Hawking radiation could prevent the formation of a black hole are discussed. Then is questioned the existence of pre-Hawking radiation.

La tesi è organizzata come segue:

- nel capitolo 1 viene introdotta la teoria di un campo scalare masless in spazio tempo curvo. Vengono inoltre trattate le trasformazioni di Bogolubov
- nel capitolo 2 viene derivato con le dovute approssimazioni l'effetto Hawking. Viene poi brevemente citato il problema del paradosso dell'informazione
- nel capitolo 3 vengono analizzati i contributi della radiazione di Hawking al tensore energia impulso bidimensionale
- nel capitolo 4 viene analizzato un generico modello di collasso gravitazionale e ne vengono esaminati gli effetti quantistici. Ci si restringe poi al caso di una shell sottile che collassa, e viene esaminata la bontà dell'approssimazione dello stato di Unruh.
- nel capitolo 5 si analizzano le porposte che asseriscono che l'emissione della pre-Hawking radiation possa prevenire la formazione di un buco nero per un oggetto collassante.
- nel capitolo 6 si analizzano gli argomenti che vogliono confutare la possibilità che la pre-Hawking radiation possa causare l'impossibilità di formare buchi neri nel processo di collasso. Viene poi messa in dubbio la stessa esistenza della pre-Hawking radiation.

Introduction

Black holes are one of the most astonishing consequences of General Relativity. We know that matter produces a gravitational field and that this field can be interpreted as the curvature of spacetime. Even light is affected by this curvature, and therefore we can state that gravity modify the causal structure of spacetime since the spacetime curvature bends the lightcones. Matter can reach such high densities and therefore produce such strong gravitational fields that the distortion of the lightcones is so big to create regions of spacetime where even light is trapped. Since nothing can move faster than light, nothing can escape from these regions. We are dealing with a balck hole. General Relativity though, remain a classical theory. Among the firsts attempt to unify the quantum theory and gravity, an area of physics has been developed, the quantum field theory in curved spacetime. It is a semiclassical approach to the study of quantum phenomena in curved spacetime. In fact, matter fields are quantized while the gravitational field is treated classically. Inside this subject some surprising results have been achieved. Among the most intresting ones there is the result of Stephen Hawking which states that black holes are not completely "black". Infact they emitt a thermal spectrum of radiation. Emitting this radiation the black hole lose energy and therefore mass, until it evaporates completely. This result is completely quantomechanical. This discovery has given rise to a long dispute realated to the problem called "the information loss paradox". This paradox state that an initial pure state evolve into a mixed state in the presence of an evaporting black hole. This fact is in strong contrast with the quantum mechanical principle of unitary evolution. As an attempt to solve this and other paradoxes (we know that for a distant observer the time in which a black hole forms is infinite while the time in which it evaporates is finite) recently it was hipotyzed the existence of a semiclassical effects called pre-Hawking radiation. During the gravitational collapse, a star would emitt an indefinite kind of radiation. The result of this emission would be the one to prevent the star to contract to the point to form a black hole. This prespective did not convince some physicists, in particoular William Unruh who, in a series of paper tried to disprove the possibility that this pre-Hawking radiation could effectively prevent the formation of black holes, even questioning the existence of the radiation itself.

In this thesis the sign convenction is $(-,+,+,+)$ and unitl chapter 4.2 we will use geometrized units $c = G_N = 1$ and we mantain explicit $\hbar$ to emphatize the quantum aspects of the theory. Then we will set $\hbar = 1$ for simplify the formulae.

Chapter 1

Quantum Field Theory in Curved Spacetime

We will briefly review the quantizzation of a massless scalar field in Minkowski spacetime, then we will generalize the procedure to curved spacetime. Finally we will introduce the Bogolubov transformation and the concept of particle production.

1.1 Scalar field in Minkowski space-time

Given the action for a massless scalar field

$$S = -\frac{1}{2}\int d^4x \partial_\mu \phi \partial^\mu \phi \tag{1.1}$$

The equation of motion results

$$\partial_\mu \partial^\mu \phi = 0 \tag{1.2}$$

Now, splitting a generic solution of the Klein-Gordon Equation into positive and negative frequencies with respect a global inertial time "t" of Minkowski space-time:

$$\phi(t, \vec{x}) = \sum_i [a_i u_i(t, \vec{x}) + a_i^\dagger u_i^*(t, \vec{x})] \tag{1.3}$$

And this imply that

$$\frac{\partial}{\partial_t}u_j(t,\vec{x}) = -iw_j u_j(t,\vec{x}), w_j > 0. \tag{1.4}$$

The solution $u_i(t,\vec{x})$ are chosen to form a complete orthonormal basis with respect the conserved Klein-Gordon product

$$(\phi_1,\phi_2) = -i\int d^3\vec{x}\phi_1 \overleftrightarrow{\partial_t} \phi_2 \tag{1.5}$$

The usual choice for the orthonormal basis is given by the plane wave modes

$$u_{\vec{k}} = \frac{1}{\sqrt{16\pi^3 w}}e^{-iwt+i\vec{k}\vec{x}} \tag{1.6}$$

We are now ready to quantize the system treating ϕ as an operator and imposing the equal time commutation relation

$$[\phi(t,\vec{x}),\pi(t,\vec{x})] = i\hbar\delta^3(\vec{x}-\vec{x}') \tag{1.7}$$

where $\pi = \partial_t\phi$ is the conjugate momentum of ϕ. This implies the following commutation relations for the ladder operators

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \hbar\delta^3(\vec{k}-\vec{k'}) \tag{1.8}$$

All the other relations are equal to zero. We can now build the many particles Fock space. The vacuum state $|0\rangle$ is annihilated by the destruction operator $a_{\vec{k}}$

$$a_{\vec{k}}|0\rangle = 0, \forall\vec{k} \tag{1.9}$$

The one particle state is generated by the action of the creation operator $a^\dagger_{\vec{k}}$

$$a^\dagger_{\vec{k}}|0\rangle = |1_{\vec{k}}\rangle \tag{1.10}$$

and so on for the manyparticle states.

1.2 Scalar field in curved spacetime

Transposing the previous discussion to a curved background (endowed with metric $g_{\mu\nu}$) require some peculiar considerations.
We start by easily generalizing the massless Klein-Gordon equation by means of the generally covariant d'Alambertian operator $\Box$

$$\Box \equiv \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \phi] \tag{1.11}$$

and by extending the Klein-Gordon inner product

$$(\phi_1, \phi_2) = -i \int_\Sigma d\Sigma^\mu \sqrt{g_\Sigma} \phi_1 \overleftrightarrow{\partial}_\mu \phi_2^* \tag{1.12}$$

where Σ is a Cauchy hypersurface and $d\Sigma^\mu = d\Sigma n^\mu$ with $d\Sigma$ being the volume element, n^μ a future directed unit normal vector and g_Σ is the determinant of the induced metric on Σ. Using the Gauss theorem it is possible to show that the inner product is indipendent of the choice of Σ.
Now the discussion become more subtle.
The Poincare' symmetry is lost and the spacetime is usually non stationary (as it happens for example in the process of gravitational collapse), and this imply that there is not an unique way of splitting modes in positive and negative frequencies. Depending on the way one define the positive frequency modes, one is lead to different definitions of vacuum state and subsequently of the corresponding Fock space. We can therefore find two or more different set of orthonormal mode solution, with their own ladder operator and vacuum states (in the next section we will see how to relate these modes), the unambiguos minkowskian concept of particle though, disappear. Nevertheless, it is still possible to recover a particle interpretation for those spacetimes which possess asymptotic stationary regions in the past and in the future (respectively "in" and "out" region). One can construct an orthonormal set of modes such that they have positive frequencies with respect the inertial time in the past. These solutions will be called u^{in}. The same procedure can be done in the future region, finding the orthonormal modes u^{out}. Assuming that the quantum field is in the "in" vacuum state $|in\rangle$, one will find that the

effect of the dynamical gravitational field at late time, when the geometry has settled down to a stationary configuration, will impy the fact that the $|in\rangle$ state will be not percieved as vaccum in the "out" region.

1.3 Bogolubov tansformations

The following results are valid for each choice of modes we decide to expand the field ϕ, but we will focus on the solution u^{in}, u^{out}, which turns out to be important in the next chapters. Let us expand the field in terms of the positive frequency solution u^{in} in the "in" region

$$\phi = \sum_i [a_i^{in} u_i^{in} + a_i^{in\dagger} u_i^{in*}] \tag{1.13}$$

and let's do the same in the "out" region

$$\phi = \sum_i [a_i^{out} u_i^{out} + a_i^{out\dagger} u_i^{out*}] \tag{1.14}$$

The following relations hold

$$(u_i^{in}, u_j^{in}) = \delta_{ij}, \quad (u_i^{in*}, u_j^{in*}) = -\delta_{ij}, \quad (u_i^{in}, u_j^{in*}) = 0, \tag{1.15}$$

and

$$[a_i^{in}, a_j^{in\dagger}] = \hbar \delta_{ij} \tag{1.16}$$

the other commutators are equal to zero.
Similar relations hold for the u^{out} modes and the corresponding ladder operators. Since both sets are complete one can expand one set in terms of the other, so in general we have:

$$u_j^{out} = \sum_i (\alpha_{ji} u_i^{in} + \beta_{ji} u_i^{in*}) \tag{1.17}$$

These relations are the Bogolubov transformations and the matrices α_{ij}, β_{ij} are called Bogolubov coefficients. Taking into account the relations (1.15), we find that

$$\alpha_{ij} = (u_i^{out}, u_j^{in}), \quad \beta_{ij} = -(u_i^{out}, u_j^{in*}) \tag{1.18}$$

they also imply

$$\sum_k (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*) = \delta_{ij} \tag{1.19}$$

and

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0. \tag{1.20}$$

Now inverting (1.17) one obtain

$$u_i^{in} = \sum_i (\alpha_{ji}^* u_j^{out} - \beta_{ji} u_j^{out*}) \tag{1.21}$$

and using the fact that $a_i^{in} = (\phi, u_i^{in})$ and $a_i^{out} = (\phi, u_i^{out})$ we can expand one of the two sets of ladder operators in term of the other

$$a_i^{in} = \sum_j (\alpha_{ji} a_j^{out} + \beta_{ji}^* a_j^{out\dagger}) \tag{1.22}$$

or

$$a_i^{out} = \sum_j (\alpha_{ji}^* a_j^{in} - \beta_{ij}^* a_j^{in\dagger}) \tag{1.23}$$

It is clear that if the coefficients $\beta_{ij} \neq 0$ the vacuum states $|in\rangle$ and $|out\rangle$, which satisfies

$$a_i^{in} |in\rangle = 0, \quad a_i^{out} |out\rangle = 0 \tag{1.24}$$

are different. This can be made explicit evaluating the expectation value of the "out" particle number operator for the i^{th} mode $N_i^{out} \equiv \hbar^{-1} a_i^{out\dagger} a_i^{out}$ in the state $|in\rangle$, giving

$$\langle in| N_i^{out} |in\rangle = \sum_j |\beta_{ij}|^2 \tag{1.25}$$

and thus the vacuum state of the "out" region contains $\sum_j |\beta_{ij}|^2$ particles in the "in" modes. It is interesting to write the vacuum state $|in\rangle$ in the $|out\rangle$

basis. Acting with (1.22) on $|in\rangle$ and multiplyng for $\sum_i \alpha^{-1}_{ik}$ we find

$$(a_k^{out} - \sum_j V_{jk} a_j^{out\dagger})\, |in\rangle = 0 \tag{1.26}$$

where $V_{jk} \equiv -\sum_i \beta^*_{ji}\alpha^{-1}_{ik}$. The solution of this equation is proportional to

$$|in\rangle \propto exp(\frac{1}{2\hbar}\sum_{ij} V_{ij} a_i^{out\dagger} a_j^{out\dagger})\, |out\rangle \tag{1.27}$$

This result is remarkable. It says that the $|in\rangle$ state describes a state with an even nuber of "out" particles.

Chapter 2

The Hawking effect

In this chapter we will derive the Hawking result that black holes quantum mechanically emitt a thermal spectrum of particles. We will use some approximations to simplify the treatment but these will not change the key result of Hawking's work. Then we will briefly outline one of the main controversial aspects of this construction: the information loss paradox

2.1 Collapse in Vadya spacetime

We will analyze the model of a spherical shell which collapse at the speed of light. In this situation the spacetime is described by the Vadya metric

$$ds^2 = \left(1 - \frac{2M(v)}{r}\right)dv^2 - 2dvdr - r^2 d\Omega^2 \tag{2.1}$$

with stress energy tensor given by

$$T_{vv} = \frac{L(v)}{4\pi r^2}, \quad \frac{dM}{dv} = L(v) \tag{2.2}$$

Where

$$L(v) = m\delta(v - v_0) \tag{2.3}$$

and this gives

$$M(v) = m\theta(v - v_0) \tag{2.4}$$

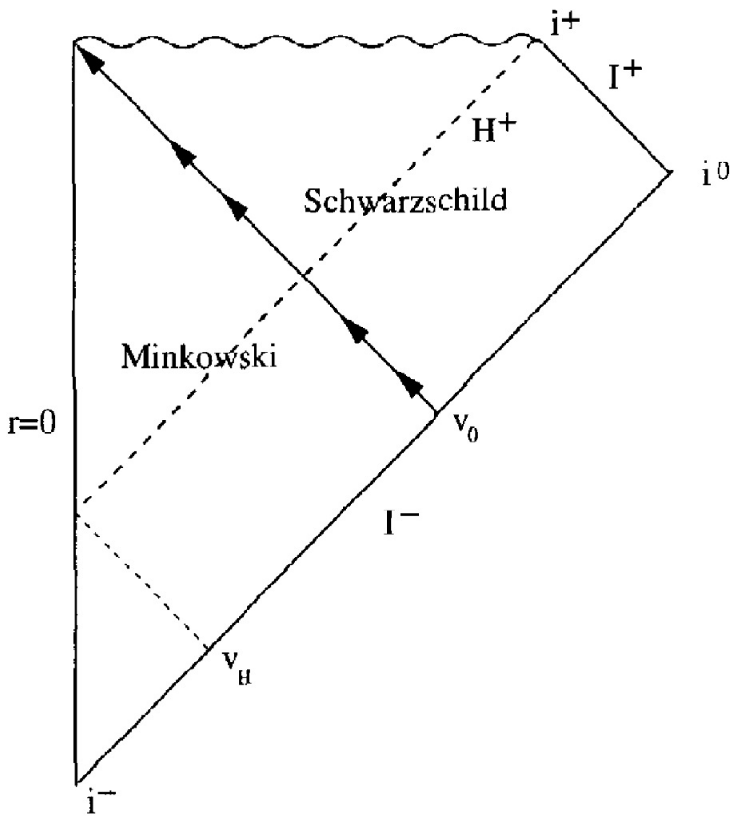

Figure 2.1: A schwarzschild black hole produced by the collapse of a spherical null shell

So the spacetime can be divided in two regions (see Fig. 2.1):

1. $v < v_0$ A Minkowski region described by the metric written in the double null form

$$ds^2_M = -du_{in}dv - r^2(u_{in}, v)d\Omega^2 \tag{2.5}$$

where $u_{in} = t_{in} - r$ and $v = t_{in} + r$.
The massless Klein-Gordon equation is given by

$$\partial_\mu \partial^\mu \phi = 0 \tag{2.6}$$

since the background is spherically symmetric we can expand the field

ϕ in the following way

$$\phi(x^\mu) = \sum_{l,m} \frac{f_l(t,r)}{r} Y_{lm}(\theta, \phi) \tag{2.7}$$

where Y_{lm} are the spherical armonics. For each angoular momentum l, the Klein-Gordon equation for ϕ is then converted in a two-dimensional wave equation for f_l, with a non-vanishing potential:

$$\left(-\frac{\partial^2}{\partial t_{in}^2} + \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} \right) f_l(t,r) = 0 \tag{2.8}$$

2. The Schwarzschild black hole region The double null metric is given by

$$ds_S^2 = -\left(1 - \frac{2m}{r}\right) dudv + r^2(u,v) d\Omega^2 \tag{2.9}$$

Where $u = t - r^*$, $v = t + r^*$ and r^* is given by

$$r^* = \int \frac{dr}{1 - 2m/r} = r + 2m\, ln \left| \frac{r}{2m} - 1 \right| \tag{2.10}$$

The Klein-Gordon equation reads

$$\Box \phi = 0 \tag{2.11}$$

We can still expand ϕ as in eq.(2.7), and for every l, we find

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}} - V_l(r) \right) f_l(t,r) = 0 \tag{2.12}$$

where

$$V_l(r) = \left(1 - \frac{2m}{r}\right) \left[\frac{l(l+1)}{r^2} + \frac{2m}{r^3} \right] \tag{2.13}$$

We will work with in the high frequency limit and in the s-wave approximation. So we can discard the potentials in eqs.(2.8) and (2.12). This approximation simplifies the discussion without changing the key ideas. In particoular having discarted the potential term, imply that there is no backscattering of the modes caused by the curvature of spacetime, and the ingoing modes fall directly into the black hole without being reflected. With this approximation, and passing in null coordinates we find that the wave equation becomes

$$\Big(- \frac{\partial^2}{\partial t_{in}^2} + \frac{\partial^2}{\partial r^2}\Big) f(t,r) = 0 \Rightarrow \partial_{u_{in}} \partial_v f = 0 \tag{2.14}$$

in the Minkowski region and

$$\Big(- \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}}\Big) f(t,r) = 0 \Rightarrow \partial_u \partial_v f = 0 \tag{2.15}$$

in the Schwarzschild region. The solution of the above wave equations will be given by modes proportional to ingoing and outgoing waves

$$\{e^{-i\omega v}, e^{-i\omega u}\}, \quad \{e^{-i\omega v}, e^{-i\omega u_{in}}\} \tag{2.16}$$

respectively in the Schwarzschild region and in the Minkowski region.

2.2 The matching condition

We will now match the metrics along $v = v_0$, i.e. we require that the metric must be the same on both side:

$$ds_M^2|_{v=v_0} = ds_S^2|_{v=v_0} \Rightarrow r(u_{in}, v) = r(u, v) \tag{2.17}$$

using relation (2.10), given that $r^* = \frac{v-u}{2}$,we find

$$\frac{v_0 - u}{2} = r(u_{in}, v_0) + 2m\, ln\Big|\frac{r(u, v_0)}{2m} - 1\Big| \tag{2.18}$$

But $r(u_{in}, v_0) = \frac{v_0 - u_{in}}{2}$, and so

$$u = u_{in} - 4m\, ln\Big|\frac{v_H}{4m} - \frac{u_{in}}{4m}\Big| \tag{2.19}$$

where $v_H = v_0 - 4m$, (the meaning of v_H is illustrated in figure (2.1)). On the horizon

$$r(u_{in}, v_0) = 2m = \frac{v_0 - u_{in}(H)}{2} \Rightarrow \frac{v_0}{4m} - 1 = \frac{u_{in}(H)}{4m} = \frac{v_H}{4m} \tag{2.20}$$

now we chose v_0 such that $v_H = 0$. This imply $u_{in}(H) = 0$ and

$$u = u_{in} - 4m\, ln\left| -\frac{u_{in}}{4m}\right| \tag{2.21}$$

which for $u_{in} \to 0$, i.e. near the horizon, which also imply at late time ($u \to \infty$) we have

$$u \sim -4m\, ln\left| -\frac{u_{in}}{4m}\right| \Rightarrow \frac{u_{in}}{4m} = -e^{-\kappa u} = U \tag{2.22}$$

where U is the outgoing Kruskal coordinate and $\kappa = 1/4m$ is the surface gravity of the black hole. This imply that in the late time collapse scenario near the horizon the outgoing modes behave like

$$e^{-i\omega U} \tag{2.23}$$

We remark that this behaviour is indipendet of the model of collapse.

2.3 Vacuum states

2.3.1 The Boulware vacuum

Let us consider the metric described by the isual Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.24}$$

We are looking for normalized solution of the wave equation in this background. To do so we have to properly define the scalar product. Outside the black hole region ($r < 2m$), we can chose the surface $t = const.$ as a Cauchy surface. The scalar product in this region therefore reads

$$(u_1, u_2) = -i \int_{2m}^{\infty} dr^* \int d\theta d\varphi r^2 sen\theta (u_1 \overleftrightarrow{\partial}_t u_2^*) \tag{2.25}$$

The normalized mode solution of the wave equation are therefore given by

$$u^R = \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega u}}{r} \qquad u^I = \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega v}}{r} \tag{2.26}$$

This basis has positive norm and positive frequency with respect to the Schwarzschild time t. In the black hole region the roles of t and r are inverted. Now a surface with r=const. is a Cauchy surface, therefore the scalar product is given by

$$(u_1, u_2) = -i \int_{-\infty}^{+\infty} dt \int d\theta d\varphi r^2 sen\theta (u_1 \overleftrightarrow{\partial}_{r^*} u_2^*) \tag{2.27}$$

and the positive norm modes are

$$u^L = \frac{1}{4\pi\sqrt{\omega}} \frac{e^{+i\omega u}}{r} \qquad u^I = \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega v}}{r} \tag{2.28}$$

But now u^L is a negative frequency mode. This means that an outgoing excitation of the field created inside the black hole region decreases the energy of the system. The massless scalar field can be expand in this basis as

$$\phi(r,t) = \sum_{\omega} a_\omega^I u^I(x) + a_\omega^R u^R(x) + a_\omega^L u^L(x) + h.c. \tag{2.29}$$

The vacuum state associated with these modes is the Boulware vacuum $|B\rangle$.

$$a_\omega^i |B\rangle = 0 \qquad i = I, R, L \tag{2.30}$$

By the structure of the mode we can see that for an observer at infinity, the Boulware vacuum reduces to the standard Minkowski one, i.e. we will find absence of particles at infinity. Physically $|B\rangle$ describes corectly the vacuum outside a static star.

2.3.2 The Unruh vacuum

We consider now the situation of a gravitational collapse at late time. We have seen that in this case the outgoing modes of the field behaves like $e^{-i\omega U}$ motivated by the fact that near the horizon $u_{in} \to U$. In this approximation scheme, we want to find the normalized modes of the field which this time are defined on the maximally extended spacetime. In order to normalize these modes we look for a Cauchy surface to define the scalar product. In this case we will choose the past event horizon for the outgoing modes, i.e. the surface defined by $V = 0$, where V is the advanced Kruskal coordinate. The scalar product now reads

$$(u_1, u_2) = -i \int_{-\infty}^{+\infty} \int d\theta d\varphi r^2 sen\theta (u_1 \overleftrightarrow{\partial}_U u_2^*) \tag{2.31}$$

The normalized modes for this scenario are therefore given by

$$u_K = \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega U}}{r} \qquad u^I = \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega v}}{r} \tag{2.32}$$

And the field can be expanded in this basis

$$\phi(r,t) = \sum_{\omega_K} (a_{\omega_K} u_K + h.c.) + \sum_{\omega} (a^I_\omega u^I + h.c.) \tag{2.33}$$

We call the vacuum state associated with these modes, the Unruh vacuum $|U\rangle$, and

$$a_{\omega_K} |U\rangle = a^I_\omega |U\rangle = 0 \tag{2.34}$$

2.4 Bogolubov transformations and particle creation

Following the procedure of chapter one, we want to see if during the process of gravitational collapse to form a black hole, particle production takes place. Now we can expand the mode u_K in the $u^{L/R/I}$ basis. We get

$$\begin{aligned} u_K = & \int_0^\infty d\omega[\alpha^L_{\omega_K\omega}u^L(\omega,x) + \beta^L_{\omega_K\omega}u^{*L}(\omega,x)] + \\ & \int_0^\infty d\omega[\alpha^R_{\omega_K\omega}u^R(\omega,x) + \beta^R_{\omega_K\omega}u^{*R}(\omega,x)] \end{aligned} \tag{2.35}$$

Let us now compute the Bogolubov coefficients. We have

$$\begin{aligned} \alpha^R_{\omega_K\omega} &= (u_K(\omega_K,x), u^R(\omega,x)) \\ \beta^R_{\omega_K\omega} &= -(u_K(\omega_K,x), u^{*R}(\omega,x)) \\ \alpha^L_{\omega_K\omega} &= (u_K(\omega_K,x), u^L(\omega,x)) \\ \beta^L_{\omega_K\omega} &= -(u_K(\omega_K,x), u^{*L}(\omega,x)) \end{aligned} \tag{2.36}$$

We will chose the past horizon as Cauchy surface to compute the scalar product. Using the fact that $u = -\frac{1}{\kappa}ln(-\kappa U)$ we can write

$$\alpha^R_{\omega_K\omega} = -i\int_{-\infty}^{+\infty} dU \int d\theta d\varphi r^2 sen\theta\; u_K \overleftrightarrow{\partial_U}\; u^{*R} =$$

$$= -\frac{i}{4\pi\sqrt{\omega_K\omega}}\int_{-\infty}^{0} dU\; exp[-i\omega_K U - \frac{i\omega}{\kappa}ln(-\kappa U)][\frac{i\omega}{-\kappa U} + i\omega_K]]$$

Now, changing variable $x = -\kappa U$,

$$\alpha^R_{\omega_K\omega} = -\frac{i}{\kappa 4\pi\sqrt{\omega_K\omega}}\int_0^\infty dx\; e^{i\frac{\omega_K}{\kappa}x}x^{-i\frac{\omega}{\kappa}}[i\frac{\omega}{x} + i\omega_K] =$$

$$\begin{aligned} =& \frac{1}{4\pi\kappa}\sqrt{\frac{\omega}{\omega_K}}\int_0^\infty dx\; e^{i\frac{\omega_K}{\kappa}x}x^{-i\frac{\omega}{\kappa}-1} \\ &+ \frac{1}{4\pi\kappa}\sqrt{\frac{\omega_K}{\omega}}\int_0^\infty dx\; e^{i\frac{\omega_K}{\kappa}x}x^{-i\frac{\omega}{\kappa}} = \end{aligned}$$

$$= \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_K}}\Big(-i\frac{\omega_K}{\kappa}\Big)^{i\frac{\omega}{\kappa}}\Gamma\Big(-i\frac{\omega}{\kappa}\Big) \tag{2.37}$$

Where we used the propriety of the Gamma function $\int_0^\infty dx x^a e^{-bx} = b^{-1-a} a\Gamma(a)$. Similarly we find

$$\begin{aligned}
\beta^R_{\omega_K\omega} &= \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_K}}\Big(-i\frac{\omega_K}{\kappa}\Big)^{i\frac{\omega}{\kappa}}\Gamma\Big(\frac{\omega}{\kappa}\Big) \\
\alpha^L_{\omega_K\omega} &= \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_K}}\Big(\frac{\omega_K}{\kappa}\Big)^{i\frac{\omega}{\kappa}}\Gamma\Big(\frac{\omega}{\kappa}\Big) \\
\beta^L_{\omega_K\omega} &= \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_K}}\Big(\frac{\omega_K}{\kappa}\Big)^{i\frac{\omega}{\kappa}}\Gamma\Big(-i\frac{\omega}{\kappa}\Big)
\end{aligned} \tag{2.38}$$

Now we find that $|\beta^R_{\omega_K\omega}|^2 = |\beta^L_{\omega_K\omega}|^2$ and therefore in virtue of (1.25) we find

$$N_\omega = \langle U| a^{R\dagger}_\omega a^R_\omega |U\rangle = \langle U| a^{L\dagger}_\omega a^L_\omega |U\rangle \tag{2.39}$$

Therefore we have particle pair production. From a naif point of view we can say that one particle fall inside the horizon, with negative frequency decreasing the mass of the black hole leading to black hole evaporation (see ref []), the other escape at infinity.
Now we can note that

$$|\alpha^R_{\omega_K\omega}|^2 = e^{2\pi\frac{\omega}{\kappa}}|\beta_{\omega_K\omega}|^2 \tag{2.40}$$

Now, from (1.19), it follows that

$$\sum_{\omega_K}|\alpha^R_{\omega_K\omega}|^2 - |\beta_{\omega_K\omega}|^2 = 1 \tag{2.41}$$

we find

$$\sum_{\omega_K}|\beta_{\omega_K\omega}|^2(e^{8\pi m\omega} - 1) = 1 \tag{2.42}$$

and

$$N_\omega = \frac{1}{e^{8\pi m\omega} - 1} = \frac{1}{e^{\frac{\hbar\omega}{K_B T_H}} - 1} \tag{2.43}$$

This distribution rappresent a thermal distribution of particles at the temperature T_H, given by

$$T_H \equiv \frac{\hbar}{8\pi m K_B} \tag{2.44}$$

2.5 The information loss paradox

We will briefly outline one of the most controversial aspect of the Hawking effect and black hole evaporation, namely the information loss paradox. In a classical background the presence of an event horizon prevents an external observer to know the full detail of the star from which the black hole has benn formed. The only parameters that are accessible to the observer are the mass, the charge and the angular momentum of the black hole. However the external observer can argue that the remaining information is not lost since it still lies inside the black hole.
When we consider quantum effect the situation is more complicated. Let us first analyze the evolution of a quantum system in Minkowsi spacetime. A state $|in\rangle$ in an initial Cauchy surface Σ_i is mapped by an unitary operator into a final state $|f\rangle$, at a final Cauchy surface Σ_f. Writing the initial state in terms of a fixed basis $|in\rangle = \sum_j c_j^{in} |\psi_j\rangle$ one can determine exactly the complex coefficients c_k^f of the final state $|f\rangle = \sum_k c_j^f |\psi_k\rangle$, from the coefficients of the $|in\rangle$ state. Things change if the causal structure of spacetime is not as simple as Minkowski. Consider for example the Schwarzschild black hole metric. The final Cauchy surface Σ_f can be split into two surfaces $\Sigma_f = \Sigma_{int} \cup \Sigma_{ext}$ where Σ_{int} is placed inside the black hole (see Fig. 2.2). We can now repeat the above procedure. A state $|in\rangle$ evolves in a final state $|f\rangle$ in a unitary way. But considering measurement performed by an observer placed outside the black hole, a new phenomenon emerges. The $|in\rangle$ state can be regarded as a flux of pairs of entangled particles (see ref[1]), one falling into the black hole, and the other emitted at future null infinity and the outgoing flux forms at late time a thermal radiation flux. The final quantum state can no longer be described as a pure quantum state of the form $|f\rangle = \sum_k c_j^f |\psi_k\rangle$. We cannot know the value of the coefficients c_k^f. We can only know the probabilities of finding a state $|\psi_k\rangle$. Thus the only thing we can compute is the probability distribution $P(N)$ of finding N particles in the region outside the black hole and we cannot define a pure final state for the exterior region. It is the causal structure of the black hole the ultimate responsible of this effective breakdown of quantum predictability.

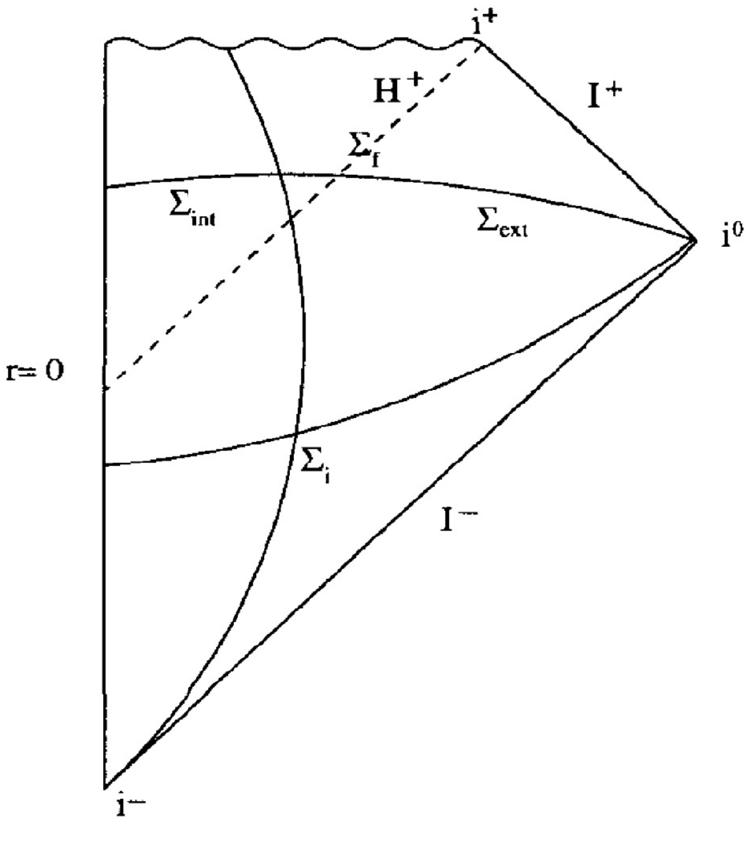

Figure 2.2: Initial and final Cauchy surfaces

Chapter 3

The renormalized stress tensor in 2D

The Hawking radiation contributes to the stress-energy tensor, modifying the classical one describing the collapsing matter, and therefore the classical gravity-matter equations should be modified accordingly. This is the motivation underlying the introduction of a modified set of equations, called the semiclassical Einstein equation

$$G_{\mu\nu}(g_{\mu\nu}) = 8\pi \langle T_{\mu\nu}(g_{\mu\nu}) \rangle \tag{3.1}$$

where the right hand side represents the expectation value of the stress-energy tensor operator of the matter fields propagating on a spacetime with metric $g_{\mu\nu}$. Since the left hand side is a conserved tensor, $\nabla_\mu G^{\mu\nu} = 0$, consistency implies that

$$\nabla_\mu \langle T_{\mu\nu} \rangle = 0 \tag{3.2}$$

This require that the quantization of the matter field in the curved spacetime should be compatible with general covariance. The problem of finding an expression for $\langle T_{\mu\nu} \rangle$ is in general very difficult, except when the degree of symmetry of the spacetime considered is sufficently high. In particoular for black holes, only in two dimension and for conformally invariant matter fields (which means that the classical action is ivariant under conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$), we are able to find an expression for $\langle T_{\mu\nu}(g_{\mu\nu})\rangle$. This is a consequence of the fact that in two dimension every spacetime is (locally) conformally flat. Up to diffeomorphism, every metric can be written as

$$ds^2 = e^{2\rho} ds^2_{(0)} \tag{3.3}$$

Where $ds^2_{(0)}$ represents a flat metric. The classical invariance under conformal transformation manifest itself in the existence of a traceless stress-energy tensor $g^{\mu\nu}T_{\mu\nu} = 0$. At quantum level things are different though. The requirement of compatibility of the regoularizzation procedure with general covariance forces to produces a quantum non vanishing trace that it's indipendent of the state in which the expectation value is taken, which in two dimension is proportional to the two dimensional Ricci scalar, i.e.

$$\langle T \rangle = \frac{\hbar}{24\pi} R \tag{3.4}$$

In the next sections we will derive an expression for the expectation value of the stress energy tensor avoiding the technicalities of any covariant regoularizzation scheme. We will first directly introduce the so called Polyakov effective action which can be functional differentiated to derive the componenst of the stress-energy tensor. These components will present some ambiguities deriving by the introduction of an auxiliary field to make the action local and so we will compare these results with the results deriving from the quantum formulation to eliminate these ambiguities. As a final remark we will compute the component of the stress-energy tensor in the states relevant for black holes, i.e. the Boulware and the Unruh states.

3.1 The Polyakov effective action

The expression for the expectation value of the stress-energy tensor can be derived by functional differentiation of an effective action S_P called the

Polyakov effective action.

$$S_P = -\frac{\hbar}{96\pi}\int dx^2\sqrt{-g}R\frac{1}{\Box}R \tag{3.5}$$

This expression is clearly non local. We can convert it into a local action by introducing an auxiliary scalar field φ constrained to obey the equation

$$\Box\varphi = R \tag{3.6}$$

The local action

$$S_P = -\frac{\hbar}{96\pi}\int dx^2\sqrt{-g}(-\varphi\Box\varphi + 2\varphi R) \tag{3.7}$$

is equivalent to (3.5) if φ satisfies (3.6).
Now to obtain the stress-energy tensor we have to vary the above action with respect to the metric. We will use for this porpouse some useful relations. First we can use the fact that $\delta\sqrt{-g} = -1/2\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}$. Then another useful formula is the Palatini identity for the variation of the Ricci tensor

$$\delta R_{\alpha\beta} = \frac{1}{2}(-\nabla^\gamma\nabla_\gamma\delta g_{\alpha\beta} + \nabla^\gamma\nabla_\alpha\delta g_{\gamma\beta} + \nabla^\gamma\nabla_\beta\delta g_{\gamma\alpha} - \nabla_\alpha\nabla_\beta g^{\gamma\delta}\delta g_{\gamma\delta}) \tag{3.8}$$

Knowing that $\delta g_{\alpha\beta} = -g_{\alpha\gamma}g_{\beta\delta}\delta g^{\gamma\delta}$, the above formula reduces to

$$g^{\alpha\beta}\delta R_{\alpha\beta} = g_{\alpha\beta}\Box\delta g^{\alpha\beta} - \nabla_\alpha\nabla_\beta\delta g^{\alpha\beta} \tag{3.9}$$

Then negletting total derivatives we can vary (3.7) obtaining

$$\begin{aligned}\delta S_P = &-\frac{\hbar}{96\pi}\int dx^2\sqrt{-g}[-\frac{1}{2}(\nabla\varphi)^2 g_{\alpha\beta}\delta g^{\alpha\beta} + \nabla_\alpha\varphi\nabla_\beta\varphi\delta g^{\alpha\beta} \\ &- 2\varphi(\frac{1}{2}Rg_{\alpha\beta} - R_{\alpha\beta})\delta g^{\alpha\beta} + 2\varphi(g_{\alpha\beta}\Box\delta g^{\alpha\beta} - \nabla_\alpha\nabla_\beta\delta g^{\alpha\beta})]\end{aligned} \tag{3.10}$$

The term proportional to the two dimensional Einstein tensor vanishes identically and after integrating by part we are left with

$$\delta S_P = -\frac{\hbar}{96\pi}\int dx^2\sqrt{-g}[-\frac{1}{2}(\nabla\varphi)^2 g_{\alpha\beta}\delta g^{\alpha\beta} + \nabla_\alpha\varphi\nabla_\beta\varphi\delta g^{\alpha\beta} + 2g_{\alpha\beta}\Box\varphi\delta g^{\alpha\beta} - 2\nabla_\alpha\nabla_\beta\varphi\delta g^{\alpha\beta}] \tag{3.11}$$

From this we obtain the following stress tensor

$$\langle\Psi| T_{\alpha\beta}|\Psi\rangle = -\frac{2}{\sqrt{-g}}\frac{\delta S_P}{\delta g^{\alpha\beta}} = \frac{\hbar}{48\pi}\left[\nabla_\alpha\varphi\nabla_\beta\varphi - \frac{1}{2}g_{\alpha\beta}(\nabla\varphi)^2\right] - \frac{\hbar}{24\pi}\left[\nabla_\alpha\nabla_\beta\varphi - \frac{1}{2}g_{\alpha\beta}\Box\varphi\right] + \frac{\hbar}{48\pi}g_{\alpha\beta}R \tag{3.12}$$

From this expression we can recover immediatly the trace anomaly, i.e.

$$-\frac{2}{\sqrt{-g}}g^{\alpha\beta}\frac{\delta S_P}{\delta g^{\alpha\beta}} = \frac{\hbar}{24\pi}R \tag{3.13}$$

In a generic conformal coordinate system $x^{\pm}$, the metric is given by

$$ds^2 = -e^{2\rho}dx^+dx^- \tag{3.14}$$

In this conformal gauge we have

$$R = 8e^{-2\rho}\partial_+\partial_-\rho \tag{3.15}$$

and

$$\Box = -4e^{-2\rho}\partial_+\partial_- \tag{3.16}$$

So eq. (3.6) reads

$$\partial_+\partial_-\varphi = -2\partial_+\partial_-\rho \tag{3.17}$$

and therefore

$$\varphi = -2\rho + 2(\varphi_+(x^+) + \varphi_-(x^-)) \tag{3.18}$$

where $\varphi_\pm(x^\pm)$ are arbitrary chiral functions generating the general solution of the corresponding homogeneus equation.
Now we can compute the components of the stress-energy tensor, they will be given by

$$\langle\Psi|\, T_{+-}\, |\Psi\rangle = -\frac{2}{\sqrt{-g}}\frac{\delta S_P}{\delta g^{+-}} = -\frac{\hbar}{12\pi}\partial_+\partial-\rho \tag{3.19}$$

$$\langle\Psi|\, T_{\pm\pm}\, |\Psi\rangle = -\frac{2}{\sqrt{-g}}\frac{\delta S_P}{\delta g^{\pm\pm}} = -\frac{\hbar}{12\pi}((\partial_\pm\rho)^2 - \partial_\pm^2\rho) + \frac{\hbar}{12\pi}((\partial_\pm\varphi_\pm)^2 - \partial_\pm^2\varphi) \tag{3.20}$$

3.2 The $t_\pm$ functions

We see that the components $\langle T_{\pm\pm}\rangle$ presents an ambiguity given by the arbitraritness of the $\varphi_\pm$ functions.
To solve this ambiguity we now compare this result with the one which can be found in litterature (see ref [1], [5] ,[16]).
It is infact known that the components $\langle T_{\pm\pm}\rangle$ are given by

$$\langle\Psi|\, T_{\pm\pm}\, |\Psi\rangle = -\frac{\hbar}{12\pi}((\partial_\pm\rho)^2 - \partial_\pm^2\rho + t_\pm(x^\pm)) \tag{3.21}$$

Where $t_\pm$ are functions which depend on the choiche of the vacuum state respect to which we decide to compute the expectation value of the stress energy tensor. In particoular taking the metric (3.14) we can define a quantizzation associated to the choiche of the positive frequency modes

$$(4\pi\omega)^{-1/2}e^{-i\omega x^+}, \qquad (4\pi\omega)^{-1/2}e^{-i\omega x^-} \tag{3.22}$$

and we can define $|x^\pm\rangle$ the vacuum state with respect to these modes. One find that if $|\Psi\rangle$ is chosen to be $|x^\pm\rangle$, the function $t_\pm$ vanish and expression (3.21) turns out to be

$$\langle x^\pm|\, T_{\pm\pm}\, |x^\pm\rangle = -\frac{\hbar}{12\pi}((\partial_\pm\rho)^2 - \partial_\pm^2\rho) \tag{3.23}$$

It is clear that by denoting

$$-(\partial_{\pm}\varphi_{\pm})^2 + \partial_{\pm}^2\varphi \equiv t_{\pm}(x^{\pm}) \tag{3.24}$$

We can identify eq.(3.20) with eq. (3.21) It is important to note that the function $\varphi_{\pm}$ are not scalars. In order to make φ a scalar field, they must transform under conformal transformations as

$$\varphi_{\pm}(x^{\pm}) \to \varphi_{\pm}(y^{\pm}) = \varphi_{\pm}(y^{\pm}(x^{\pm}) + \frac{1}{2}ln\frac{dx^{\pm}}{dy^{\pm}} \tag{3.25}$$

to compensate the transformation law of ρ

$$\rho(x^{\pm}) \to \rho(y^{\pm}) = \rho(x^{\pm}) + \frac{1}{2}ln\frac{dx^{+}}{dy^{+}}\frac{dx^{-}}{dy^{-}} \tag{3.26}$$

All this means that $t_{\pm}(x^{\pm})$ transform as

$$t_{\pm}(y^{\pm}) = \left(\frac{dx^{\pm}}{dy^{\pm}}\right)^2 t_{\pm}(x^{\pm}) + \frac{1}{2}\{x^{\pm}, y^{\pm}\} \tag{3.27}$$

Where

$$\{x^{\pm}, y^{\pm}\} = \frac{d^3x^{\pm}}{dy^{\pm 3}}/\frac{dx^{\pm}}{dy^{\pm}} - \frac{3}{2}\left(\frac{d^2x^{\pm}}{dy^{\pm 2}}/\frac{dx^{\pm}}{dy^{\pm}}\right)^2 \tag{3.28}$$

is the Schwarzian derivative. From this we can notice that a change of vacuum state from $|x^{\pm}\rangle$ to $|\tilde{x}^{\pm}\rangle$ produces a change in the expectation value of the $T_{\pm\pm}(x^{\pm})$ components of the stress tensor of the form

$$\langle\tilde{x}^{\pm}| T_{\pm\pm} |\tilde{x}^{\pm}\rangle = \langle x^{\pm}| T_{\pm\pm} |x^{\pm}\rangle - \frac{\hbar}{24\pi}\{\tilde{x}^{\pm}, x^{\pm}\} \tag{3.29}$$

We can conclude that the nonlocality of the Polyakov effective action (3.5) is associated to the auxiliary field in (3.7) and physically reflects the dependence on the state $|\Psi\rangle$ of $\langle\Psi| T_{\alpha\beta} |\Psi\rangle$. In practice the state dependence is contained in the functions $t_{\pm}$ and, therefore in $\varphi_{\pm}$.

3.3 Quantum states

We will now apply the results of the previous section to the 2D Schwarzschild black hole.
The radial part of the Schwarzschild metric in the Eddington-Finkelstein coordinate is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dudv \tag{3.30}$$

We will then analize two cases. The static case where the vacuum state is described by the Boulware state, and the dynamical situation of the formation of a black hole by gravitational collapse in the Unruh state approssimation.

3.3.1 The Boulware state

We will now compute the stress energy tensor outside a spherical body, in a static configuration.

As we have already seen the vacuum state associated to this configuration is the Boulware state $|B\rangle$. As already stressed in chapter 2, the boulware states is defined by the modes

$$(4\pi\omega)^{-1/2} e^{-i\omega u} \qquad (4\pi\omega)^{-1/2} e^{-i\omega v} \tag{3.31}$$

outside the event horizon. Straightforward application of eqs. (3.19) and (3.23) gives

$$\begin{aligned}
\langle B | T_{uu} | B \rangle &= \frac{\hbar}{24\pi} \left[\frac{3}{2} \frac{m^2}{r^4} - \frac{M}{r^3} \right] \\
\langle B | T_{vv} | B \rangle &= \frac{\hbar}{24\pi} \left[\frac{3}{2} \frac{m^2}{r^4} - \frac{M}{r^3} \right] \\
\langle B | T_{uv} | B \rangle &= -\frac{\hbar}{24\pi} (1 - \frac{2m}{r}) \frac{m}{r^3}
\end{aligned} \tag{3.32}$$

As $r \to \infty$ the modes (3.31) reduces to the usual Minkowski plane waves and there $\langle B | T_{uu} | B \rangle = 0$. So $|B\rangle$ rappresent asintoticaly the usual Minkowski

vacuum as expected.
The situation is different for $r \to 2m$. Let us compute the components of the stress-energy tensor in this limit. We find

$$\begin{aligned} \langle B| T_{uu} |B\rangle &= \langle B| T_{vv} |B\rangle \sim -\frac{\hbar}{768\pi m^2} \\ \langle B| T_{uv} |B\rangle &\sim 0 \end{aligned} \tag{3.33}$$

These expression are finite and we expect that a freely falling observer must detect a non singoular energy density. Let us rewrite the expression for $\langle B| T_{\alpha\beta} |B\rangle$ in(t, r) coordinates.
We find

$$\begin{aligned} \langle B| T_{tt} |B\rangle &= \frac{\hbar}{24\pi}\left[\frac{7m^2}{r^4} - \frac{4m}{r^3}\right] \\ \langle B| T_{rr} |B\rangle &= -\frac{\hbar}{24\pi}(1 - \frac{2m}{r})^{-2}\frac{m^2}{r^4} \\ \langle B| T_{tr} |B\rangle &= 0 \end{aligned} \tag{3.34}$$

Now consider a freely falling observer in this 2-D model.

$$u^\alpha = \left(\frac{E}{1 - 2m/r}, -\sqrt{E^2 - \left(1 - \frac{2m}{r}\right)}\right) \tag{3.35}$$

This observer will measure an energy density

$$\rho = T_{\alpha\beta}u^\alpha u^\beta \propto \left(1 - \frac{2m}{r}\right)^{-2} \to -\infty \tag{3.36}$$

This behaviour is connected to the fact that the state $|B\rangle$ is defined in terms of the modes (u, v) in eq. (3.31) which are ill defined at the horizon (i.e. they oscillate infinetely).
Therefore we can interpret the state $|B\rangle$ as describing the vacuum polarization outside a static spherical body whose radius is bigger than $2m$.
In this way the physical portion of the Schwarzschild spacetime does not contain any horizon.

3.3.2 The Unruh state

The Unruh state $|U\rangle$ was introduced to describe late time thermal radiation in the configuration of a dynamical gravitational collapse. It is constructed from the modes

$$(4\pi\omega)^{-1/2}e^{-i\omega v} \qquad (4\pi\omega)^{-1/2}e^{-i\omega U} \tag{3.37}$$

Where, we rimember, U is the retarded Kruskal coordinate. Applying eq. (3.29) we find

$$\begin{aligned}
\langle U|\, T_{uu}\, |U\rangle &= \frac{\hbar}{786\pi m^2}(1-\frac{2m}{r})^2\left[1+\frac{4m}{r}+\frac{12m^2}{r^2}\right] \\
\langle U|\, T_{vv}\, |U\rangle &= \frac{\hbar}{24\pi}\left[\frac{3}{2}\frac{m^2}{r^4}-\frac{M}{r^3}\right] \\
\langle U|\, T_{uv}\, |U\rangle &= -\frac{\hbar}{24\pi}(1-\frac{2m}{r})\frac{m}{r^3}
\end{aligned} \tag{3.38}$$

As $r \to \infty$ we find that the only non-vanishing component is

$$\langle U|\, T_{uu}\, |U\rangle = \frac{\hbar}{786\pi m^2} \tag{3.39}$$

which rappresent the Hawking thermal outgoing flux at the temperature

$$T_H = \frac{\hbar}{8\pi k_B m} \tag{3.40}$$

Moreover at the event horizon we find

$$\langle U|\, T_{vv}\, |U\rangle = -\frac{\hbar}{786\pi m^2} \tag{3.41}$$

rappresenting a flux of negative energy entering the black hole making the black hole to shrink due to evaporation.
It is easy to check that the stress energy tensor is regular on the future horizon H^+.
Changing coordinates from Eddington Finkelstein to Kruskal (coordinates which are regular on the horizon), we find that the change of coordinates

between u and U is singular, namely $\frac{du}{dU} \sim \frac{1}{U} \sim \frac{1}{(r-2m)}$, while $\frac{dv}{dV}$ is regular (the situation is the opposite on the past horizon where $\frac{du}{dU}$ is regular and $\frac{dv}{dV} \sim \frac{1}{(r-2m)}$).
This imply that $\langle T_{UU} \rangle \sim \frac{\langle T_{uu} \rangle}{(r-2m)^2}$, $\langle T_{UV} \rangle \sim \frac{\langle T_{uv} \rangle}{r-2m}$, $\langle T_{VV} \rangle \sim \langle T_{vv} \rangle$. Therefore we find the conditions that the stress energy tensor must satisfy at the future horizon in order to be regular there:

$$\begin{aligned} |\,\langle T_{vv} \rangle\,| &< \infty \\ (r-2m)^{-1}|\,\langle T_{uv} \rangle\,| &< \infty \\ (r-2m)^{-2}|\,\langle T_{uu} \rangle\,| &< \infty \end{aligned} \tag{3.42}$$

The regularity conditions on the past horizon are expressed by similar inequalities but with u and v interchanged. It is easy to see that for the Unruh state the regularity conditions on the future horizon are satisfied while T_{VV} is singular on H^-. However this behaviour on the past horizon is not physically relevant if we consider black holes created by gravitational collapse, which do not admitt a past horizon.

Chapter 4

Quantum effects in gravitational collapse

In this chapter we wish to examine the situation of the two dimensional gravitational collapse withouth any apporximation (if we exclude backscattering) and how this affects the components of the expectation value of the stress energy tensor. In the first section we will analyze the general case, with a spherically symmetric collapsing body described by a general unknown metric collapsing in empty space. In the second section we will analyze the general case of the collapse of a null thin shell and we will confront the result obtained in this section with the ones derived by the Unruh approximation that we have already seen in the previous chapters.

4.1 Generical collapsing ball of matter

4.1.1 The model

We will now analyze the process of a ball of matter undergoing a complete gravitational collapse to a black hole. Outside the ball the line element is taken to be

$$ds^2 = -C(r)dudv \tag{4.1}$$

where

$$\begin{cases} u = t - r^* + R_0^* \\ v = t + r^* - R_0^* \end{cases} \tag{4.2}$$

$$r^* = \int C^{-1} dr \tag{4.3}$$

and R_0^* =constant. Inside the ball we take the general line element

$$ds^2 = -A(U, V) dU dV \tag{4.4}$$

where A is an arbitrary smooth, non singoular function and

$$\begin{cases} U = \tau - r + R_0 \\ V = \tau + r - R_0 \end{cases} \tag{4.5}$$

The relation between R_0 and R_0^* is the same as that between r and r^* given by (4.3). The exterior metric (4.1) is choosen so that $C \to 1$ and $\partial C/\partial r \to 0$ as $r \to \infty$

There is an event horizon for some value of r for which $C = 0$.

For example, for the two dimensional Schwarzschild black hole, $C = 1 - 2m/r$ has an event horizon at $r = 2m$.

Before $\tau = 0$ the ball is at rest with its surface at $r = R_0$.

For $\tau > 0$ we assume that the sufrace shrinks along the worldline $r = R(\tau)$.

We shall find that for late times at $\mathcal{I}^+$, i.e. large u, the precise form of both $A(U, V)$ and $R(\tau)$ are irrelevant.

The coordinates (4.2) and (4.5) have been choosen so that at $\tau = t = 0$, the onset of collapse, $u = U = v = V = 0$ at the surface of the ball.

We restirct our treatment to the region $r \geq 0$, and thus reflect null rays at $r = 0$. Such reflection can be achieved by imposing the boundary condition $\phi = 0$ at $r = 0$.

Denote the transformation between interior and exterior coordinates by

$$U = \alpha(u) \tag{4.6}$$

and

$$V = \beta(v) \tag{4.7}$$

Where we have ignored any reflection at the surface of the ball.
The center of radial coordinates is the line

$$V = U - 2R_0 \tag{4.8}$$

We look for a solution of the two dimensional wave equation

$$\Box\phi = 0 \tag{4.9}$$

that vanish along (4.8) and reduce to standard exponentials on $\mathcal{I}^-$ Noting from (4.8) that at r=0

$$v = \beta(v) = \beta(U - 2R_0) = \beta[\alpha(u) - 2R_0] \tag{4.10}$$

one is led to the mode solution

$$i(4\pi\omega)^{-1/2}(e^{-i\omega v} - e^{-i\omega\beta[\alpha(u)-2R_0]}) \tag{4.11}$$

Thus we see that the simple incoming wave $e^{i\omega v}$ is converted by the collapsing ball to the complicated outgoing wave $e^{-iw\beta[\alpha(u)-2R_0]}$.

4.1.2 Stress energy tensor

Let us first calcoulate the stress-energy tensor for the static case, i.e. the ball is at rest. The wave equation may be solved in terms of the modes $e^{-i\omega u}$ and $e^{-i\omega v}$ and we denote the vacuum state with respect these modes $|0\rangle$. Now taking into account that $\rho = \frac{1}{2}lnC$ applying formulas (3.19) and (3.23) we find:

$$\langle 0| T_{uu} |0\rangle = \langle 0| T_{vv} |0\rangle = \frac{\hbar}{192\pi}[2CC' - C'^2] \tag{4.12}$$

$$\langle 0| T_{uv} |0\rangle = \frac{\hbar}{96\pi}CC'' \tag{4.13}$$

where a prime denotes differentiation with resoect to r.
We can now examine the collapse scenario.
Althoug the metric (4.1) still correctly describe the geometry in the exterior of the ball, the coordinates (u, v) are no longer appropriate for the solution of the wave equation, because of the degeneracy of the outgoing modes.
Instead we wish to choose coordinates such that the incoming modes, and hence the vacuum state (denoted by $|\hat{0}\rangle$), are of the standard Minkowski form on $\mathcal{I}^-$.
This means using in place of u and v as defined above, the coordinates

$$\hat{u} = \beta[\alpha(u) - 2R_0] \tag{4.14}$$

$$\hat{v} = v \tag{4.15}$$

Appliyng eq.(3.) we find that the components of the stress energy tensor are

$$\langle\hat{0}| T_{uu} |\hat{0}\rangle = \langle 0| T_{uu} |0\rangle - \frac{\hbar}{24\pi}\{\hat{u}, u\} \tag{4.16}$$

$$\langle\hat{0}| T_{vv} |\hat{0}\rangle = \langle 0| T_{vv} |0\rangle \tag{4.17}$$

$$\langle\hat{0}| T_{uv} |\hat{0}\rangle = \langle 0| T_{uv} |0\rangle \tag{4.18}$$

In the interior region, described by the metric (4.4), we still denote as $|0\rangle$ the vacuum state with respect the modes $e^{-i\omega U}$ and $e^{-i\omega V}$. We have

$$\langle\hat{0}| T_{UU} |\hat{0}\rangle = \langle 0| T_{UU} |0\rangle - \frac{\hbar}{24\pi}\{\hat{u}, U\} \tag{4.19}$$

$$\langle\hat{0}| T_{VV} |\hat{0}\rangle = \langle 0| T_{VV} |0\rangle - \frac{\hbar}{24\pi}\{\hat{u}, V\} \tag{4.20}$$

4.1.3 Physical aspects

Defining F as the functional

$$F_x(y) = \frac{\hbar}{12\pi} y^{1/2} \frac{\partial^2}{\partial x^2}(y^{-1/2}) \tag{4.21}$$

With some algebraic manipulation, noting that $d\hat{u}/du = \beta'\alpha'$ we can rewrite (4.16) as

$$\langle\hat{0}|\, T_{uu}\, |\hat{0}\rangle = \langle 0|\, T_{uu}\, |0\rangle + (\alpha')^2 F_U(\beta') + F_u(\alpha') \tag{4.22}$$

where $\alpha = \alpha(u)$, $\beta = \beta(\alpha(u) - 2R_0)$ and primes denote differentiation with respect the argoument of a function.

In the same way for the interior metric we obtain

$$\langle\hat{0}|\, T_{UU}\, |\hat{0}\rangle = F_U(\beta') - F_U(A) \tag{4.23}$$

$$\langle\hat{0}|\, T_{VV}\, |\hat{0}\rangle = F_V(\beta') - F_V(A) \tag{4.24}$$

where the terms $F_U(\beta')$ and $F_V(\beta')$ are given by the Schwarzian derivative in (4.19) (4.20) respectively, and where $\beta = \beta(U - 2R_0)$, $\beta = \beta(V)$, respectively. The terms $-F_U(A)$,$-F_V(A)$, come from application of the formula (3.23), taking into account that $\rho = ln\sqrt{A}$. Having writing the expression of the stress-energy tensor this way helps us to grasp the physical meaning of each term.

Analyzing eq.(4.22) the results show that the effect of collapse is to augment the static vacuum energy (4.12) with an outgoing flux of radiation. In the interior of the body, both ingoing and outgoing radiation fluxes occur that depend in a complicated way upon the interior metric A and the collapse trajectory $R(\tau)$. This represent particles created inside the material of the collapsing ball, and at its surface. For Example, $F_V(\beta')$ in (4.24) describes radiation created at the surface that propagates towards $r = 0$, while $F_U(\beta')$ in (4.23) describes the same radiation propagating outwards after passage through the center of the body. When this outgoing flux emerges from the body, it is described by the term proportional to $(\alpha')^2$ in (4.22). In addition to this outgoing flux, there will be the contribution of the final term in (4.22), which also can be traced back to the surface but which is outgoing from the outset.

4.1.4 Explicit form of α' and β'

As a final remark, we wish to determine the form of α', β' (where now the prime denotes differentiation with respect the argoument of a function). To do so we have to match the interior and the exterior metrics across the collapsing surface $r = R(\tau)$.
Let us begin rewriting the metrics (4.1) and (4.4) in terms of (u, r) and (U, r) respectively.
We get

$$ds^2 = -Cdu^2 - 2dudr \tag{4.25}$$

$$ds^2 = -AdU^2 - 2AdUdr \tag{4.26}$$

Computing the tangent vector to the surface $r = R(\tau)$ for the exterior and the interior respectively, we get

$$e^{\mu}_{\tau+} = (\dot{u}, \dot{R}) \tag{4.27}$$

$$e^{\mu}_{\tau-} = (\dot{U}, \dot{R}) \tag{4.28}$$

where the subscript + and − are referred to the exterior and the interior of the ball respectively, and the dot rappresent differentiation with respect to τ.
Imposing that the induced metric must be the same on both side, we get

$$h_{ab} = g^{-}_{\mu\nu} e^{\mu}_{a-} e^{\nu}_{b-} = g^{+}_{\mu\nu} e^{\mu}_{a+} e^{\nu}_{b+} \tag{4.29}$$

and so

$$C\dot{u}^2 + 2\dot{u}\dot{R} = A\dot{U}^2 + 2A\dot{U}\dot{R} \tag{4.30}$$

Where U,V,C are evaluated at $r = R(\tau)$.
Taking into account that $\dot{U} = 1 - \dot{R}$, we find

$$\alpha'(u) = \frac{dU}{du} = (1 - \dot{R})C\{[AC(1 - \dot{R}^2) + \dot{R}^2]^{1/2} - \dot{R}\}^{-1} \tag{4.31}$$

Now, rewriting the metrics in terms of (v, r) and (V, r) respectively, we find

$$\beta'(V) = \frac{dv}{dV} = C^{-1}(1 + \dot{R})^{-1}\{[AC(1 - \dot{R}^2) + \dot{R}^2]^{1/2} + \dot{R}\} \tag{4.32}$$

4.2 Null shell collapse: a comparison with the stress energy tensor constructed from the Unruh state

4.2.1 The model

The spacetime we consider is equipped with the ingoing Vaidya metric

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right)dv^2 + 2dvdr + r^2 d\Omega \qquad (4.33)$$

with

$$m(v) = \begin{cases} 0, & \text{if } v < 0, \\ m, & \text{if } v \geq 0, \end{cases} \qquad (4.34)$$

The diagram in figure 4.1 describes the region of spacetime given this metric.
Region 1 describe the exterior of the black hole to the past of the shell not including the shell.
Region 2 describe the interior black hole region to the past of the shell not including the shell.
Region 3 describe the exterior black hole region in the causal future of the shell, and finally we call region 4 the interior black hole region in the causal future of the shell .

The shell is located at $v = 0$ along a null surface S.
For $v \geq 0$ the spacetime is isometric to a portion of Schwarzschild spacetime with length parameter M, with v playing the role of the advanced Eddington-Finkelstein coordinate (region 3 and 4). For $v < 0$ the spacetime is isometric to a portion of Minkowski spacetime with v playing the role of a Minkowski ingoing light-cone coordinate (region 1 and 2).
From now on we restrict our model to a 2-dimensional version of the spacetime by suppressing the angular coordinates.

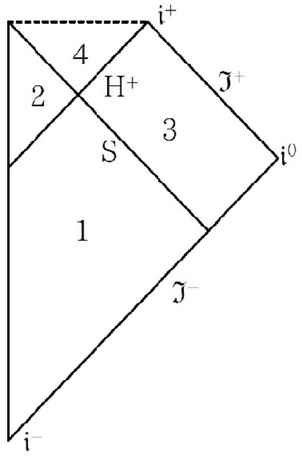

Figure 4.1: Conformal diagram of the Vaidya spacetime describing a collapsing null shell that forms a black hole

Consider now a massless scalar field ϕ propagatin in this spacetime, satisfying the equation

$$\Box\phi = 0 \tag{4.35}$$

We consider its solution as incoming modes from infinity, which are of positive frequency with respect to v, appropriately defined in region 1 and 3, which in addition satisfies a Dirichlet boundary condition at r=0 in regions 1 and 2, i.e. $\phi(t, 0) = 0$.
In regions 1 and 2, the mode solutions are

$$\Phi_\omega^{1,2} = -\frac{i}{(4\pi\omega)^{1/2}}(e^{-i\omega u} - e^{-i\omega v}) \tag{4.36}$$

Where $u = t - r$, while in regions 3 and 4

$$\Phi_\omega^{3,4} = -\frac{i}{(4\pi\omega)^{1/2}}(G(U) - e^{-i\omega v}) \tag{4.37}$$

Where G(U) is a function of the kruskal U-coordinate. In order to determine G(U) we match the mode functions along the shell at $v = 0$.

We have $v = 0 \Rightarrow u = -2r$ and so

$$G(U)_{|v=0} = e^{-i\omega u} = e^{i2\omega r} \tag{4.38}$$

Now, in $v = 0$, $V = e^{v/4m} = 1$, where V is the Kruskal V-coordinate. Then we can rewrite the foundamental relation which links r, U and V as

$$\left(\frac{r}{2m} - 1\right)e^{r/2m} = -U \tag{4.39}$$

Then calling $x = \frac{r}{2m} - 1$, we find

$$xe^x = -\frac{U}{e} \tag{4.40}$$

Now we introduce the Lambert W-function (see ref [17]) which is defined as the inverse of the function

$$f(W) = We^W \tag{4.41}$$

and hence we find

$$x = W(\frac{-U}{e}) \Rightarrow r = 2mW(\frac{-U}{e}) + 2m \tag{4.42}$$

and this imply

$$G(U) = e^{i2\omega r} = \exp[i4\omega M(1 + W(-U/e))] \tag{4.43}$$

We mention that W is strictly increasing and defined on the domain $(-1/e, \infty)$ (recall that $-\infty < U < 1$ in the region of interest)
Finally we have that

$$\Phi^{3,4}_{\omega} = -\frac{i}{(4\pi\omega)^{1/2}}(e^{i4\omega M(1+W(-U/e))} - e^{-i\omega v}) \tag{4.44}$$

4.2.2 Computation of the stress energy tensor

In this section, we wish to compute the renormalized stress-energy tensor in region 3 and 4, in the state that we have constructed in the collapsing null shell spacetime, with state vector $|\Omega\rangle$ constructed from the mode function

$$\Phi_\omega^{3,4} = -\frac{i}{(4\pi\omega)^{1/2}}(e^{-i\omega\hat{u}} - e^{-i\omega v}) \tag{4.45}$$

Where $\hat{u}$ can be related to the U Kruskal coordinate by

$$\hat{u} = -4m(1 + W(-U/e)) \tag{4.46}$$

We will rewrite the 2-D Schwarzschild metric in (U, v) coordinate, i.e.

$$ds^2 = -\frac{8m^2 e^{-r/(2m)+(v/4m)}}{r} dU dv \tag{4.47}$$

Now we want to obtain the expression of the metric in $(\hat{u}, v)$ coordinate. Thus we need an expression for $\frac{dU}{d\hat{u}}$. The derivative of W can be written as $W'(x) = \frac{W}{x(1+W)}$, and this leads to

$$\frac{dU}{d\hat{u}} = -\frac{U\hat{u}}{4m(\hat{u} + 4m)} \tag{4.48}$$

The metric then becomes

$$ds^2 = \frac{2me^{-r/(2m)+(v/4m)}U\hat{u}}{(\hat{u} + 4m)r} d\hat{u}dv = -\frac{\hat{u}(1 - 2m/r)}{\hat{u} + 4m} d\hat{u}dv \tag{4.49}$$

and r is viewed as a function of $\hat{u}$ and v, and U as a function of $\hat{u}$. In the last step we have used the relation

$$(\frac{r}{2m} - 1)e^{r/2m} = -Ue^{v/4m} \tag{4.50}$$

Now we can compute the components of the stress energy tensor applying the formulas (3.23) and (3.19). This yelds

$$\begin{aligned} \langle\Omega| T_{\hat{u}\hat{u}} |\Omega\rangle &= \frac{m(-16(3m+\hat{u})r^4 - 2\hat{u}r + 3m\hat{u}^4)}{48\pi\hat{u}^2(4m+\hat{u})^2r^4} \\ \langle\Omega| T_{vv} |\Omega\rangle &= \frac{1}{24\pi}\left(\frac{3m^2}{2r^4} - \frac{m}{r^3}\right) \\ \langle\Omega| T_{\hat{u}v} |\Omega\rangle &= \langle\Omega| T_{v\hat{u}} |\Omega\rangle = -\frac{m\hat{u}(1-2m/r)}{24\pi(4m+\hat{u})r^3} \end{aligned} \tag{4.51}$$

4.2.3 Comparison of the stress energy tensors

In this section we want to compare the components of the stress energy tensor (4.45) with the renormalized stress energy tensor in the Unruh state $|U\rangle$ found in chapter 3. This is always possible because, while the two states are defined on different spacetimes, it can be proved that there exists a region of the Kruskal spacetime that is isometric to the union of region 3 and 4 in the collapsing null shell spacetime.
In order to compare the components of the two stress energy tensors we rewrite the formulas (4.45) in Eddington Finkelstein coordinates.
We obtain

$$\begin{aligned} \langle\Omega| T_{uu} |\Omega\rangle &= \frac{m(-16(3m+\hat{u})r^4 - 2\hat{u}r + 3m\hat{u}^4)}{48\pi\hat{u}^4r^4} \\ \langle\Omega| T_{vv} |\Omega\rangle &= \frac{1}{24\pi}\left(\frac{3m^2}{2r^4} - \frac{m}{r^3}\right) \\ \langle\Omega| T_{uv} |\Omega\rangle &= \langle\Omega| T_{vu} |\Omega\rangle = -\frac{1}{24\pi}\left(1 - \frac{2m}{r}\right)\frac{m}{r^3} \end{aligned} \tag{4.52}$$

While the component of the stress energy tensor in the Unruh state are given by

$$
\begin{aligned}
\langle U| T_{uu} |U\rangle &= (786\pi m^2)^{-1}(1-\frac{2m}{r})^2\left[1+\frac{4m}{r}+\frac{12m^2}{r^2}\right] \\
\langle U| T_{vv} |U\rangle &= (24\pi)^{-1}\left[\frac{3}{2}\frac{m^2}{r^4}-\frac{M}{r^3}\right] \\
\langle U| T_{uv} |U\rangle &= \langle U| T_{vu} |U\rangle = -(24\pi)^{-1}\left(1-\frac{2m}{r}\right)\frac{m}{r^3}
\end{aligned}
\tag{4.53}
$$

Near-horizon behaviour

We can readily compare the behaviour of the stress-energy tensor in the near horizon region. For both the Unruh state and for the state produced out of the shell collapse we have that

$$
\begin{aligned}
\langle \Omega| T_{vv} |\Omega\rangle = \langle U| T_{vv} |U\rangle &= -\frac{1}{768\pi m^2}+\frac{(r-2m)^2}{512\pi m^4}-\frac{5(r-2m)^3}{1536\pi m^5}+\mathcal{O}((r-2m)^4) \\
\langle \Omega| T_{uv} |\Omega\rangle = \langle U| T_{vu} |U\rangle &= \frac{r-2m}{384\pi m^3}-\frac{(r-2m)^2}{192\pi m^4}+\frac{5(r-2m)^3}{768\pi m^5}+\mathcal{O}((r-2m)^4)
\end{aligned}
\tag{4.54}
$$

On the other hand for the Unruh state,

$$
\langle U| T_{uu} |U\rangle = \frac{(r-2m)^2}{512\pi m^4}-\frac{5(r-2m)^3}{1536\pi m^5}+\mathcal{O}((r-2m)^4) \tag{4.55}
$$

while for the collapsing shell state, knowing that (see ref[17]) $W(x) = x - x^2 + \frac{3}{2}x^3 + \mathcal{O}(x^4)$ for $x \to 0$, (infact $-U/e \to 0$ when $r \to 2m$), we have

$$
\langle \Omega| T_{uu} |\Omega\rangle = \frac{(r-2m)^2}{512\pi m^4}(1-e^{-v/2m})-\frac{(r-2m)^3(5+3e^{-v/4m}-8e^{-3v/4m})}{1536\pi m^5}+\mathcal{O}((r-2m)^4) \tag{4.56}
$$

Thus we can see from eq. (4.48),(4.49) and (4.50) that the near horizon behaviour of the stress energy tensor of a Klein-Gordon field during stellar collapse is captured very precisely by the Unruh state, with deviation of order $\mathcal{O}((r-2m)^2)$.

In particular, the negative flux of energy that gives rise to black hole radiation, first of the equations (4.48), is the dominant contribution of the stress-energy tensor in this regime.

Near-future-infinity behaviour

Near the future null infinity, at fixed $\hat{u}$ and as $r \to \infty$ (of $v \to \infty$), we have on the one hand,

$$\begin{aligned} \langle\Omega| T_{vv} |\Omega\rangle &= \langle U| T_{vv} |U\rangle = -\frac{m}{24\pi r^3} + \frac{m^2}{16\pi r^4} + \mathcal{O}(r^{-5}) \\ \langle\Omega| T_{uv} |\Omega\rangle &= \langle U| T_{vu} |U\rangle = \frac{m}{24\pi r^3} - \frac{m^2}{12\pi r^4} + \mathcal{O}(r^{-5}) \end{aligned} \tag{4.57}$$

On the other hand, for the Unruh state

$$\langle U| T_{uu} |U\rangle = \frac{1}{768\pi m^2} - \frac{m}{23\pi r^3} + \frac{m^2}{16\pi r^4} + \mathcal{O}(r^{-5}) \tag{4.58}$$

while in the collapsing shell spacetime

$$\langle\Omega| T_{uu} |\Omega\rangle = \frac{-3m^2 - m\hat{u}}{3\pi\hat{u}^4} - \frac{m}{23\pi r^3} + \frac{m^2}{16\pi r^4} + \mathcal{O}(r^{-5}) \tag{4.59}$$

From eq. (4.59) it is clear that the radiation output to infinity is positive, but has a richer form compared to the output radiated in the Unruh state. We also note that as one apporaches the future timelike infinity ($t \to \infty$), i.e. as $\hat{u} \to -4m$ along the future null infinity, the leading terms of eq.(4.58) and (4.59) coincide, i.e.

$$\langle\Omega| T_{uu} |\Omega\rangle |_{\mathcal{J}^+} - \langle U| T_{uu} |U\rangle |_{\mathcal{J}^+} = \mathcal{O}((\hat{u} + 4m)^2). \tag{4.60}$$

Moreover, the leading term of eq.(4.59) has non.decreasing derivative with respect to $\hat{u}$, and therefore $\partial_u \langle\Omega| T_{uu} |\Omega\rangle \geq 0$ for $\hat{u} \in (-\infty, -4m]$ indicating that at late u-time the output of radiation at $\mathcal{J}^+$ increases. Hence we find that

$$0 \leq \langle\Omega| T_{uu} |\Omega\rangle |_{\mathcal{J}^+} \leq \frac{1}{768\pi m^2} \tag{4.61}$$

The upper bound in eq.(4.61) is attained exponentially fast as $u \to \infty$. This can be verified analysing the asymptotic behaviour of

$$F(u) = \frac{1}{768\pi m^2} + \frac{3m^2 + m\hat{u}}{3\pi\hat{u}^4} = \frac{1}{768\pi m^2}\left(1 - \frac{1 - 4W(e^{-u/(4m)-1})}{(1 + W(e^{-u/(4m)-1}))^4}\right) \quad (4.62)$$

with $(1 - 4W(x))/(1 + 4W(x))^4 = 1 - 8x + \mathcal{O}(x^2)$ Our findings are that the Unruh state provides an excellent estimate in the near-horizon region of the spacetime. The behaviour of the radiation at future null infinity is not completely captured by the Unruh state, but near future timelike infinity the outgoing flux in the collapsing shell spacetime is well characterized by the Unruh state. Moreover we find that at every point at $\mathcal{J}^+$ the outgoing flux of radiation in the Unruh state dominates the radiation output in the collapsing null shell spacetime, i.e. $0 \leq \langle\Omega|\, T_{uu}\, |\Omega\rangle\,|_{\mathcal{J}^+} \leq \langle U|\, T_{uu}\, |U\rangle\,|_{\mathcal{J}^+}$. The energy output at infinity predicted by using calcoulation based on the Unruh state is however approached exponentially fast in u-time.

Chapter 5

Pre-Hawking radiation and horizon avoidance

The goal of the papers analized in this chapter (Ref. [8],[9],[10]) is to show that, due to an evaporation phenomenon during the gravitational collapse, the collapsing objects never reach its Schwarzschild radius. As a consequence an event horizon never forms. Such evaporation is caused by the emanation of an indefinite kind of radiation by the collapsing body, called pre-Hawking radiation. The treatment is based on a semiclassical aproach and on the following assumptions:

1. The classical spacetime structure is meaningful and described by a metric $g_{\mu\nu}$
2. Classical concepts such as trajectory and hevent horizon can be used
3. the collapse leads to a pre-Hawking radiation
4. The metric is modified by quantum effects. The resulting curvature satisfies the semiclassical equation $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle$

The horizon avoidance prespective wuold give solution to some paradox of the semiclassical theory of black holes such as the fact that for an asymptotic observer the collapse of matter takes an infinite amount of time to reach the Schwarzschild radius while the evaporation time of the resulting black hole is finite. Furthermore the information loss paradox treated in chapter 2 would be avoided.

In the following section a wide variety of evaporating metrics will be analized showing that with these assumtion horizon avoidance seems to be an universal feature of the gravitational collapse.

5.1 Classical shell dynamics

We will consider the collapse of a massive spherically-symmetric thin shell Σ. This configuration divides the spacetime in two regions. The interior of the shell where the spacetime is flat and the exterior, where in absence of Hawking radiation the metric is given by the Schwarzschild solution. In the following the subscript "-" will be reffered to the interior geometry, while the subscript "+" to the exterior. We have therefore

$$ds_-^2 = -dt_-^2 + dr_-^2 + r_-^2 d\Omega = -du_-^2 - 2du_- dr_- + r_-^2 d\Omega \tag{5.1}$$

and

$$ds_+^2 = -f(r_+)dt_+^2 + f(r_+)^{-1}dr_+^2 + r_+^2 d\Omega = -f(r_+)du_+^2 - 2du_+ dr_+ + r_+^2 d\Omega \tag{5.2}$$

where $f(r) = 1 - 2M/r$ and $u_- = t_- - r_-$, $du_+ = dt_+ - dr_+/f(r_+)$ are the outgoing Eddington Finkelstein coordinate, respectively in Minkowski and Schwarzschild spacetime.
The shell trajectory is parametrized by its proper time τ, the time of a comoving observer, as

$$u_\pm = U_\pm(\tau), \quad r_\pm = R_\pm(\tau), \quad \theta = \theta, \quad \phi = \phi \tag{5.3}$$

The basis vectors tangent to Σ are

$$e_\tau^\alpha = (\dot{U}_\pm, \dot{R}_\pm, 0, 0) \tag{5.4}$$

which coincide with the four-velocity u^α of the shell and the dot denotes a τ

derivatives, and

$$e^{\alpha}_{\theta} = (0,0,1,0), \quad e^{\alpha}_{\phi} = (0,0,0,1) \tag{5.5}$$

Now, the first junction condition (see Appendix A), impose that the induced metric h_{ab} must be the same on both sides of Σ. We find

$$ds^2_{\Sigma} = -d\tau^2 + R^2 d\Omega \tag{5.6}$$

And this leads to the identification $R_+ = R_- \equiv R(\tau)$.
The shell is timelike so the four-velocity must satisfy $u^{\alpha}u_{\alpha} = -1$, where u^{α} is given by eq. (5.4) and so

$$\dot{U}_+ = \frac{-\dot{R} + \sqrt{F + \dot{R}^2}}{F} \tag{5.7}$$

The equation of motion of the shell can be written as

$$\mathcal{D}(R) := \frac{2\ddot{R} + F'}{2\sqrt{F + \dot{R}^2}} - \frac{\ddot{R}}{\sqrt{1 + \dot{R}^2}} + \frac{\sqrt{F + \dot{R}^2} - \sqrt{1 + \dot{R}^2}}{R} = 0 \tag{5.8}$$

where $A' = \partial A / \partial r|_{\Sigma}$. This equation is obtained imposing the second junction condition (see appendix A), and will be formally derived in the next section.

5.2 The Vaidya metric

To model the evaporation of the shell during the process of collapse we will use the outgoing Vaidya metric to describe the exterior geometry of Σ. This metric is given by eq.(5.2) but with $f \mapsto f(u,r) = 1 - C(u)/r$, and we assume that $f(u,r) > 0$ for $r > r_g(u) \geq 0$ and $dC/du \leq 0$, where $r_g(u) = C(u)$ is the analogous of the Schwarzschild radius, solution of $f(u,r) = 0$. Now we are interested in the form of the equation of motion of the shell. This can be obtained via the second junction condition which equates the discontinuity of the extrinsic curvature whith the surface energy momentum tensor (see appendix A)

$$S^{ab} = -([K_{ab}] - [K]h_{ab})/8\pi \tag{5.9}$$

where $K := K^a_a$ and $[K] := K|_{\Sigma^+} - K|_{\Sigma^-}$ The components of the estrinsic curvature are given by

$$K_{\tau\tau} := -n_\alpha u^\alpha{}_{;\beta} u^\beta = -n_\alpha a^\alpha, \quad K_{\theta\theta} := n_{\theta;\theta}, \quad K_{\phi\phi} := n_{\phi;\phi} \tag{5.10}$$

Where u^α is the four velocity of Σ defined in (5.4) and n_α is its outward pointing unit vector obtained from the relations $n_\alpha u^\alpha = 0$ and $n_\alpha n^\alpha = 1$, so

$$n_\alpha = (-\dot{R}, \dot{U}, 0, 0) \tag{5.11}$$

Now, let us compute the component of K_{ab} in the "+" sector. The non zero components of the four-acceleration are

$$a^0 = \ddot{U} - \frac{1}{2}F'\dot{U}^2, \quad a^1 = \ddot{R} + \frac{1}{2}(F_U + FF')\dot{U}^2 + F'\dot{U}\dot{R} \tag{5.12}$$

Where $A_u = dA/du$

Hence

$$K^+_{\tau\tau} = \dot{R}\ddot{U} - \dot{U}\ddot{R} - \frac{3}{2}F'\dot{U}^2\dot{R} - \frac{1}{2}(F_U + FF')\dot{U}^3 \tag{5.13}$$

Calculating $\ddot{U}$ deriving equation (5.7) with respect to τ, and after some algebra we find

$$K^+_{\tau\tau} = -\frac{2\ddot{R} + F'}{2\sqrt{\dot{R}^2 + F}} + F_U\dot{U}\left(\frac{\dot{R}}{2F\sqrt{F + \dot{R}^2}} - \frac{1}{2F}\right) \tag{5.14}$$

Now using the second and the third relation in (5.10) we find the angoular component of the extrinsic curvature

$$K^+_{\theta\theta} = K^+_{\phi\phi} = (\dot{R}R + F\dot{U}R) = R\sqrt{F + \dot{R}^2} \tag{5.15}$$

where we used (5.7) as a definition for $\dot{U}$ and hence

$$K^{\theta}_{+\theta} = K^{\phi}_{+\phi} = \frac{\sqrt{F + \dot{R}^2}}{R} \tag{5.16}$$

Now, analogous calcoulations for the "-" sector give

$$K^{-}_{\tau\tau} = \dot{R}\ddot{U} - \dot{U}\ddot{R} = -\frac{\ddot{R}}{\sqrt{1+\dot{R}^2}} \tag{5.17}$$

and

$$K^{\theta}_{-\theta} = K^{\phi}_{-\phi} = \frac{\sqrt{1+\dot{R}^2}}{R} \tag{5.18}$$

Then from (5.9) we have

$$S^{\tau}_{\tau} = \frac{1}{4\pi}(\frac{\sqrt{F+\dot{R}^2} - \sqrt{1+\dot{R}^2}}{R}) = -\sigma \tag{5.19}$$

$$\begin{aligned} S^{\theta}_{\theta} = S^{\phi}_{\phi} = & \frac{1}{8\pi}[\frac{2\ddot{R}+F'}{2\sqrt{\dot{R}^2+F}} - \frac{\ddot{R}}{\sqrt{1+\dot{R}^2}} - F_U\dot{U}(\frac{\dot{R}}{2F\sqrt{F+\dot{R}^2}} - \frac{1}{2F}) \\ & + (\frac{\sqrt{F+\dot{R}^2} - \sqrt{1+\dot{R}^2}}{R})] = 0 \end{aligned} \tag{5.20}$$

This last condition gives the equation of motion of the shell, namely

$$\mathcal{D}(R) - F_U\dot{U}(\frac{\dot{R}}{2F\sqrt{F+\dot{R}^2}} - \frac{1}{2F}) = 0 \tag{5.21}$$

And suppressin the u-dipendency of the metric we can recover the classical equation of motion (5.8). Now we define the quantity

$$x := R - r_g \tag{5.22}$$

Close to the Schwarzschild radius

$$F \approx \frac{x}{C} \equiv \frac{x}{r_g} \tag{5.23}$$

Then we can solve eq.(5.21) for $\ddot{R}$, and expanding it using (5.23) we obtain

$$\ddot{R} = \frac{2C\dot{R}^2\sqrt{1+\dot{R}^2}}{(\dot{R}+\sqrt{1+\dot{R}^2})x^2}\frac{dC}{dU} + \mathcal{O}(x^{-1}) \tag{5.24}$$

Note that the coefficient of dC/dU is positive so, $\ddot{R} < 0$, this means that near the Schwarzschild radius the collapse always accelerates. Now, we do an approximation. At later stage, when $|\dot{R}| \geq 2$, we have

$$\ddot{R} \approx \frac{4C\dot{R}^4}{x^2}\frac{dC}{dU} < 0 \tag{5.25}$$

Where we had to use the fact that $\sqrt{\dot{R}^2} = -\dot{R}$ because the shell is collapsing. Now we evaluate the shell's rate of approach to r_g. We first note that near to the Schwarzschild radius eq.(5.7) reduces to

$$\dot{U} \approx -2\dot{R}/F \tag{5.26}$$

Now we can differentiate $\dot{x} = \dot{R} - \dot{r}_g$. Using (5.26) and (5.23) in the evaluation of $\dot{r}_g$, we find

$$\dot{x} = \dot{R} - \dot{r}_g = \dot{R} - \frac{dr_g}{dC}|\frac{dC}{dU}|\dot{U} = \dot{R}(1 - \frac{2r_g}{x}\frac{dr_g}{dC}|\frac{dC}{dU}|) \tag{5.27}$$

where $dr_g/dC = 1$. So we can rewrite (5.27) as

$$\dot{x} = \dot{R}(1 - \frac{2C}{x}|\frac{dC}{dU}|) \tag{5.28}$$

Hence, defining the τ-dependent parameter

$$\epsilon_* := 2C|\frac{dC}{dU}| \tag{5.29}$$

we find

$$\dot{x} = \dot{R}(1 - \epsilon_*(\tau)/x(\tau)) \tag{5.30}$$

Here we find that the gap decreases only as long as $\epsilon_* < x$. If this is true for the entire duration of evaporation, we have $R > r_g$ until the evaporation is complete. Otherwise once x is reduced to ϵ_* it cannot decrease any further. From the point of view of the observer comoving with the shell the collapse accelerates, while from the point of view of the distant observer the shell is still stuck within a slowly changing coordinate distance ϵ_* from the receding

r_g. Using (5.26) and (5.23) it is easy to show that close to r_g

$$\dot{C} \approx 2C\frac{dC}{dU}\frac{|\dot{R}|}{x} \tag{5.31}$$

so the comoving observer will see the evaporation rate vanish iff the distant one (at spatial infinity) does. If $dC/dU \to 0$ then (5.21) reduces to (5.8), $\epsilon_* \to 0$ and an horizon forms. If $dC/dU < 0$ then since the gap x never vanishes, $\dot{R}$ increases but rimane finite for finite x. We know from chapter 2 that the only non-vanishing component of the stress-energy tensor for Vaidya spacetime is

$$8\pi T_{uu} = -\frac{1}{r^2}\frac{dC}{du} \tag{5.32}$$

and the energy density in the frame of an observer moving with four-velocity u^μ is

$$\rho = T_{\mu\nu}u^\mu u^\nu \tag{5.33}$$

Then on the outer shell surface, where u^μ is given by (5.4) we have

$$\rho = \frac{1}{8\pi}(-\frac{1}{R^2})\frac{dC}{dU}\dot{U}^2 \approx \frac{1}{2\pi}|\frac{dC}{dU}|\frac{\dot{R}^2}{x^2} \tag{5.34}$$

Where in the last step we have used (5.26) and (5.23). The energy flux in this frame is given by

$$j_n := T_{\mu\nu}u^\mu n^\nu \tag{5.35}$$

where n^ν is defined by eq. (5.11). Approximating $x \sim \epsilon_* = 2C|dC/dU|$ the asymtotic form of the equation of motion becomes

$$\ddot{R} \approx \frac{\dot{R}^4}{CC_U} \tag{5.36}$$

These considerations make possible too see that the comoving observer will experience an enormously encreased energy density flux in the late time, with the radiation flux growing to some maximum value depending on the proprieties of the shell, acting as a tame firewall.

5.3 The retarded Schwarzschild metric

The analogous of eq.(5.7) if the classical shell dynamic is parametrized as $(T_\pm, R_\pm)$ (i.e. using t, r coordinates) is

$$\dot{T}_+ = \sqrt{F + \dot{R}^2}/F \tag{5.37}$$

In fact in these coordinates the four-velocity of the shell is given by

$$u^\alpha = (\dot{T}, \dot{R}, 0, 0) \tag{5.38}$$

These facts will be useful later.
Now we introduce the retarded Schwarzschild metric which allows to describe the evaporation from the point of view of the distant observer.

$$ds_+^2 = -\tilde{f}(t,r)dt^2 + \tilde{f}(t,r)^{-1}dr^2 + r^2 d\Omega \tag{5.39}$$

This is given by a modification of eq.(5.2) consistent with the assumption that an outgoing massless particle propagates in the Schwarzschild spacetime which is frozen in the moment of the emission

$$\frac{dr_{out}}{dt} = 1 - C(\tilde{t})/r_{out} \tag{5.40}$$

where $C(t)$ is the mass function as described by the distant observer, and the retarded time $\tilde{t}$ is given implicitly by the equation

$$t - \tilde{t} = \int_{R(\tilde{t})}^{r} \frac{dr'}{f(\tilde{t}, r')} \equiv r_* - R_*(\tilde{t}) \tag{5.41}$$

and finally $\tilde{f}(t,r) = 1 - C(\tilde{t})/r \equiv f(\tilde{t}, r)$.
In this model the shell follows the classical trajectory $(T(\tau), R(\tau))$ until a certain coordinate distance ϵ from r_g, when the evaporation starts. We will set $\tau = t = 0$ at this event for convenience.
We should compute now the equation of motion of the evaporating shell following an equivalent procedure to the previous section, writin $F := \tilde{f}(T, R)$.

The only component of the extrinsic curvature which is worth calculating is $K^+_{\tau\tau}$, being the only one affected by the evaporation. The outward pointin unit vector is given by

$$n_\mu = (-\dot{R}, \dot{T}, 0, 0) \tag{5.42}$$

while the four-velocity is given by (5.38). From the first one of the (5.10) we find

$$\begin{aligned} K^+_{\tau\tau} =& \dot{R}\ddot{T} - \dot{T}\ddot{R} + \frac{3}{2}F^{-1}F'\dot{T}\dot{R}^2 - \frac{1}{2}FF'\dot{T}^3 \\ &+ \frac{1}{2}F^{-1}F_t\dot{R}\dot{T}^2 + \frac{1}{2}F^{-1}(F^{-1})_t\dot{R}^3 - F(F^{-1})_t\dot{R}\dot{T}^2 \end{aligned} \tag{5.43}$$

Calcoulating $\ddot{T}$ from (5.37), sobstituting in the equation above and semplificating one find

$$K^+_{\tau\tau} = -\frac{2\ddot{R} + F'}{2\sqrt{F + \dot{R}^2}} + \frac{F_t\dot{R}}{F^2} \tag{5.44}$$

and the equation of motion results

$$\mathcal{D}(R) - \frac{F_t\dot{R}}{F^2} \tag{5.45}$$

Where

$$F' = \frac{C}{R^2} - \frac{dC}{d\tilde{t}}\frac{\partial\tilde{t}}{\partial r}\Big|_\Sigma \frac{1}{R} \tag{5.46}$$

$$F_T = -\frac{dC}{d\tilde{t}}\frac{\partial\tilde{t}}{\partial t}\Big|_\Sigma \frac{1}{R} \tag{5.47}$$

We need now to evaluate the derivatives of the retarded time on the shell $\frac{\partial\tilde{t}}{\partial r}\big|_\Sigma$ and $\frac{\partial\tilde{t}}{\partial t}\big|_\Sigma$.
These can be found deriving (5.41)

$$\begin{aligned} \frac{\partial}{\partial t}(t - \tilde{t}) = 1 - \frac{\partial\tilde{t}}{\partial t} &= \frac{\partial}{\partial t}\int_{R(\tilde{t})}^r \frac{dr'}{f(\tilde{t}, r')} \\ &= -\frac{1}{F(\tilde{t}, R(\tilde{t}))}\frac{dR}{d\tilde{t}}\frac{\partial\tilde{t}}{\partial t} - \int_{R(\tilde{t})}^r \frac{dr'}{f^2(\tilde{t}, r)}\frac{\partial f}{\partial\tilde{t}}\frac{\partial\tilde{t}}{\partial t} \end{aligned} \tag{5.48}$$

In the limit $r \to R(\tilde{t})$ (and so $T = t \to \tilde{t}$)

$$1 - \frac{\partial \tilde{t}}{\partial t} = -\frac{1}{F(\tilde{t}, R(\tilde{t}))} \frac{dR}{d\tilde{t}} \frac{\partial \tilde{t}}{\partial t} \tag{5.49}$$

hence

$$\frac{\partial \tilde{t}}{\partial t}\Big|_{\Sigma} = -\frac{1}{1 - R_T/F} \tag{5.50}$$

In the same way we find

$$\frac{\partial \tilde{t}}{\partial r}\Big|_{\Sigma} = -\frac{1}{F - R_T} \tag{5.51}$$

Equation (5.23) holds from the last stage of the evaporation and thus (5.37) becomes

$$\dot{T}_+ \approx -\frac{\dot{R}}{F} \approx \frac{r_g}{x}|\dot{R}| \tag{5.52}$$

and

$$\dot{R} = \frac{dR}{dt}\dot{T} \approx -\frac{dR}{dt}\frac{\dot{R}}{F} \tag{5.53}$$

This implies the equality $R_{\tilde{t}}|_{\Sigma} = R_t \equiv R_T \approx -F$ So near r_g we have

$$\frac{\partial \tilde{t}}{\partial t}\Big|_{\Sigma} = \frac{1}{2} \tag{5.54}$$

and

$$\frac{\partial \tilde{t}}{\partial r}\Big|_{\Sigma} = -\frac{1}{2F} \tag{5.55}$$

We can now calcoulate the evaporation parametrized by the shell's proper time

$$\frac{dC}{d\tau} = \frac{dC}{d\tilde{t}}\left(\frac{\partial \tilde{t}}{\partial t}\Big|_{\Sigma}\dot{T} + \frac{\partial \tilde{t}}{\partial r}\Big|_{\Sigma}\dot{R}\right) \approx -\frac{dC}{d\tilde{t}}\frac{\dot{R}}{F} = -\frac{dC(T)}{dT}\Big|_{T=T(\tau)}\frac{\dot{R}}{F} \tag{5.56}$$

Solving (5.45) equation for $\ddot{R}$ and using (5.23) to perform the expansion near r_g we find

$$\ddot{R} = \frac{C\dot{R}^2\sqrt{1+\dot{R}^2}}{2(\dot{R} + \sqrt{1+\dot{R}^2})x^2}\frac{dC}{dT} + \mathcal{O}(x^{-1}) \tag{5.57}$$

Now we evaluate the approach to the shell $\dot{x} = \dot{R} - \dot{r}_g$

$$\dot{x} = \dot{R} - \frac{dr_g}{dC}\frac{dC}{d\tau} = \dot{R}(1 - \frac{C}{x}|\frac{dC}{dT}|) \tag{5.58}$$

Where use has been made of eq.(5.56) to evaluate $\frac{dC}{d\tau}$ So we find again

$$\dot{x} \approx \dot{R}(1 - \epsilon_*(\tau)/x(\tau)) \tag{5.59}$$

where

$$\epsilon_*(\tau) = C|\frac{dC}{dT}| \tag{5.60}$$

Now, considering the late-time evaporation law using the Page's formula see ref.[10]

$$\frac{dC}{dt} = -\frac{1}{3\mathcal{K}}\frac{1}{C^2} \tag{5.61}$$

Starting from the initial value C_0 we see that the evaporation lasts a finite time t_E

$$C(t) = C_0(1 - t/t_E)^{1/3}, \quad t_E = \mathcal{K}C_0^3 \tag{5.62}$$

Evaluating the equation of motion for $|\dot{R}| \geq 2$ and using (5.61), we find

$$\ddot{R} \approx -\frac{\dot{R}^4}{3\mathcal{K}Cx^2} \tag{5.63}$$

And using again (5.61) and (5.56) we find also

$$\dot{C} \approx \frac{1}{3\mathcal{K}C}\frac{\dot{R}}{x} \tag{5.64}$$

From these consideration we can say that the effect of evaporation is negligible until about $\tau_1 \sim \epsilon_*/|\dot{R}(0)|$, when the distance to r_g reaches $\mathcal{O}(\epsilon_*)$. From this point we are guaranteed that $\dot{x} > 0$ and so the shell do not cross r_g despite the rapidly increasing acceleration.

5.3.1 Numerical results

For the retarded Schwarzschild scenario, a numerical simulation has been performed, solving eq.(5.45) and (5.56) using the evaporation model of eq.(5.62) with $\mathcal{K} = \frac{1}{8}(5120\pi/8)$. Rewriting the equation in terms of the time of the

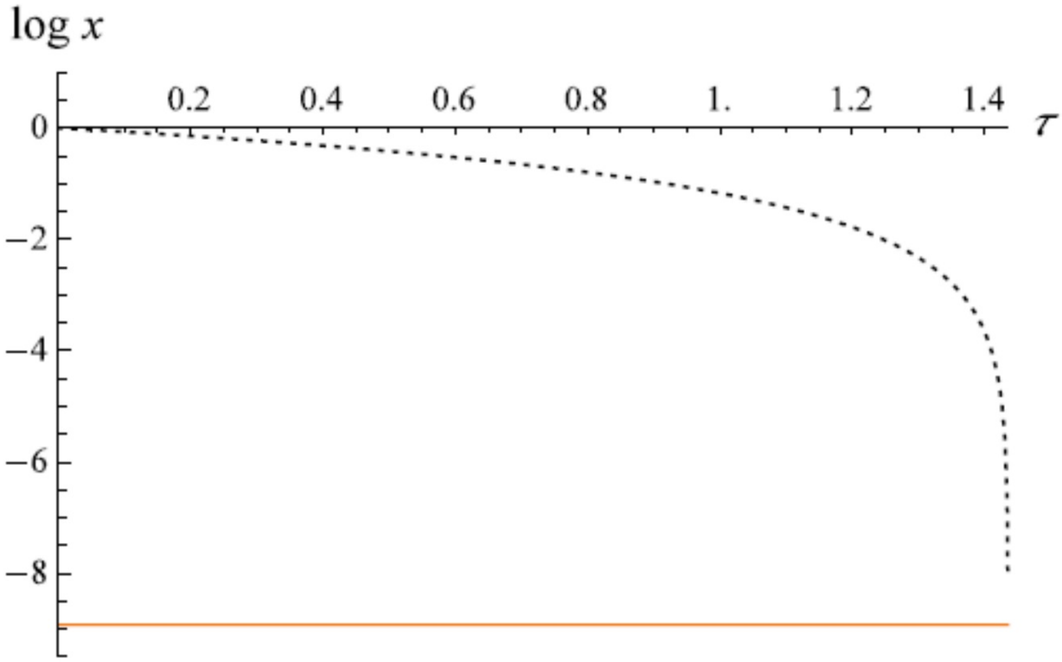

Figure 5.1: Here was chosen $C_0 = 10, R_0 = 100$. The black dotted line rappresent the exact solution of $x(\tau)$ up to $\tau = 1.4348..$ where the numerical integration breaks down. The orange line rappresent the steady value of $\epsilon_* = 1/3\mathcal{K}C$

distant observer the brakdown of numerical integration is avoided. We need to compute

$$\dot{R} = R_T\dot{T}, \quad \ddot{R} = R_{TT}\dot{T}^2 + R_T\ddot{T} \tag{5.65}$$

Obtaining $\dot{T}$ and $\ddot{T}$ from eq.(5.2), namely we can write

$$-d\tau^2 = -FdT^2 + R_T^2 dT^2/F \tag{5.66}$$

and so

$$\dot{T} = \sqrt{\frac{F}{F^2 - R_T^2}} \tag{5.67}$$

and

$$\ddot{T} = \frac{d\dot{T}}{dT}\dot{T} \tag{5.68}$$

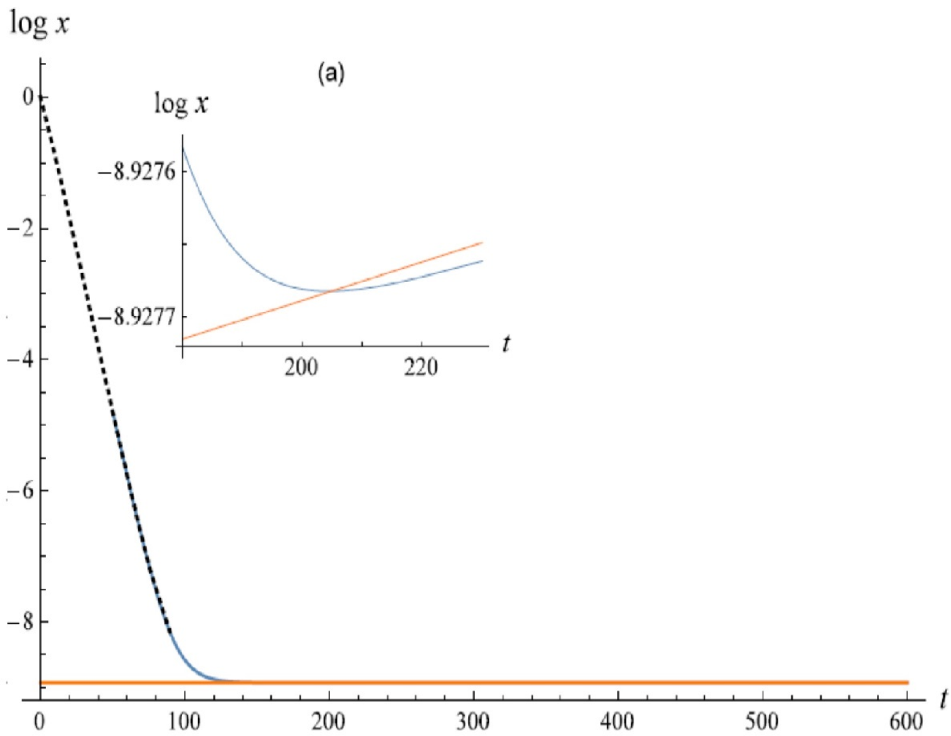

Figure 5.2: approach of $x(t)$ to $\epsilon_* \approx$ const. on this time scale. The inset (a) shows the approach of $x(t)$ to $\epsilon_*(t)$. We see that after t_* when $x(t_*) = \epsilon_*(t_*)$ the distance from r_g begins to increase.

As we stated above, near the horizon $R_T \approx -F$ so we can write

$$x_T = R_T - \frac{dr_g}{dT} \approx -\frac{x}{r_g} - \frac{dr_g}{dT} \tag{5.69}$$

which for $\epsilon_* = 1/3\mathcal{K}C$ becomes

$$x_T = -\frac{x}{C} + \frac{1}{3\mathcal{K}C^2} \tag{5.70}$$

Then we sobstitute $C(t)$ using (5.62)

5.4 The core and the shell model

In this scenario we analize the collapse of a thin shell on a massive spherically-symmetric core of radius R_c and with Schwarzschild radius $r_g^c = C_c$. The shell collapse form rest at some $R(0)$, so we have

$$r_g^c < R_c < r_g < R \tag{5.71}$$

We assume the core is large ($r_g^c << R_c$) so it behaves classically. The shell in this case is the only source of Hawking radiation. We can easely generalize the equation of motion. The metric of the inner region (between the core and the shell) is Schwarzschild, with $f_{-1} = 1 - C_c/r$. Combining the results of the preceding cases we find the component of the surface stress-energy tensor, and in particoular for the angoular components we have

$$\frac{1}{8\pi}[\frac{2\ddot{R} + F'}{2\sqrt{F + \dot{R}^2}} - \frac{2\ddot{R} + F'_-}{2\sqrt{F_- + \dot{R}^2}} - \frac{F_t\dot{R}}{F^2} + \frac{\sqrt{F + \dot{R}^2} - \sqrt{F_- + \dot{R}^2}}{R}] = 0 \tag{5.72}$$

Here F' is given by (5.46) , F_t by (5.47) and F'_- is simply

$$F'_- = \frac{C_c}{R^2} \tag{5.73}$$

We can still use all the approximation of the previous chapter, and so performing the usual expansion and using (5.61) we find

$$\ddot{R} \approx \frac{\dot{R}^2\sqrt{\dot{R}^2 + 1}}{6\mathcal{K}x^2C(\dot{R} + \sqrt{1 + \dot{R}^2})} + \mathcal{O}(x^{-1}) \tag{5.74}$$

and once $|\dot{R}| \geq 2$

$$\ddot{R} \approx -\frac{\dot{R}^4}{3\mathcal{K}Cx^2} \tag{5.75}$$

Now from (5.56) and (5.61)

$$\dot{C} \approx \frac{\dot{R}}{3\mathcal{K}C} \tag{5.76}$$

And since $\dot{r}_g = \dot{C}$ we have

$$\dot{r}_g = \frac{\dot{R}}{3\mathcal{K}C} \tag{5.77}$$

So the gap evolves according to(5.28), (5.59) with

$$\epsilon_*(\tau) = \frac{1}{3\mathcal{K}C(\tau)} \tag{5.78}$$

This guarantee $\dot{x} > 0$, and the shell do not cross the Schwarzschild radius. For this scenario we have two possible outcomes: or the shell crashes in to the core, or it evaporates beforehand .

5.5 General spherically-symmetric metric

In this case we do not assume any particoular form of the expectation value of the stress.energy tensor, except for spherical symmetry. The most general spherically-symmetric metric in (u, r) coordinates is given by

$$ds^2 = -e^{2h(u,r)} f(u, r) du^2 - 2e^{h(u,r)} dudr + r^2 d\Omega \tag{5.79}$$

and describe the spacetime outside the shell. Introducing the mass function $C(u, r)$ as

$$f := 1 - C(u, r)/r \tag{5.80}$$

we see that the outgoing Vaydia metric correspond to $C(u, r) = C(u) = r_g(u)$ and $h(u, r) = 0$. We will do the following assumption:

1. $0 \leq C < \infty$, with $C(u, r) > 0$ for $u < u_E < \infty$ and $\partial C/\partial u < 0$ as long as $C > 0$, to ensure a finite positive gravitational mass, positive energy density and flux at infinity.

2. h(u,r) is continuous

3. the metric has only one coordinate singularity, namely a surface $f(u,r) = 0$. This surface is located at the Schwarzschild radius $r_g(u)$, given by

$$r_g \equiv C(u, r_g) \tag{5.81}$$

From 3. follows

$$C(u,r) = r_g(u) + w(u,r)(r - r_g(u)) \tag{5.82}$$

for some function $w(u,r)$ and 1. implies that $w(u, r_g) \leq 1$.
Assumption 1. also imply tha for large r, $f \to 1$ and since by 3. it has the same sign for $r > r_g$, we find

$$f(u,r) > 0, \quad r > r_g \tag{5.83}$$

Which is equivalent to say $C(u,r) < r$, and thus $w(u,r) < 1$ for $r > r_g$. Now, deriving r_g

$$\frac{dr_g}{du} = \frac{\partial C}{\partial u}\Big|_{r=r_g(u)} + \frac{\partial C(u,r)}{\partial r}\Big|_{r=r_g(u)}\frac{dr_g}{du} \tag{5.84}$$

with $\frac{\partial C(u,r)}{\partial r}\big|_{r=r_g(u)} = w(u, r_g)$ and from 1. $\frac{\partial C}{\partial u}\big|_{r=r_g(u)} < 0$ so we obtain

$$(1 - w(u, r_g))\frac{dr_g}{du} < 0 \tag{5.85}$$

sharpening the bound on w to $W(u, r_g) < 1$ and ensuring $dr_g/du < 0$.
The shell trajectory is given by as usual $(U_\pm(\tau), R_\pm(\tau))$. The first junction condition impose again $R_+ = R_- := R$. The four-velocity is still given by (5.4), and with this metric the condition $u_\mu u^\mu = -1$ leads to

$$\dot{U} = \frac{-\dot{R} + \sqrt{\dot{R}^2 + F}}{EF} \tag{5.86}$$

Where $E := exp(h(U,R))$ and the continuity of h (2.) ensure that for finite U, E is finite.
This time we will not impose the second junction condition and subsequently no use will be made of the equation of motion. From (5.80) we note that $\dot{U} > 2|\dot{R}|/EF$, and we can write F as $F \approx \frac{x(1-W)}{r_g}$ near r_g, where $W(U) :=$

$w(U, r_g(U))$. Hence

$$\dot{U} > \frac{2r_g}{E(1-W)} \frac{|\dot{R}|}{x} \tag{5.87}$$

Now

$$\dot{x} = \dot{R} - \dot{r}_g = \dot{R} - \dot{U}|\frac{dr_g}{dU}| > \dot{R}(1 - \frac{2r_g}{Ex(1-W)}|\frac{dr_g}{dU}|) \tag{5.88}$$

and so

$$\dot{x} > \dot{R}(1 - \epsilon/x) \tag{5.89}$$

with

$$\epsilon_* = \frac{2r_g}{E(1-W)}|\frac{dr_g}{dU}| \tag{5.90}$$

we conclude that $\epsilon(\tau) > 0$ and the Schwarzschild radius is not reached in a finite proper time.

5.5.1 Oppenehimer Snyder collapse

We start considering the same problem of the previous chapter, but this time we write the metric outside the shell using the Schwarzschild coordinate (t,r). The metric then become

$$ds^2 = -k(t,r)^2 f(t,r) dt^2 + f(t,r)^{-1} dr^2 + r^2 d\Omega \tag{5.91}$$

Where $k(t, r)$ is some function and

$$f(t,r) = 1 - C(t,r)/r \tag{5.92}$$

We assume, similarly:

1. $0 \leq C < \infty$, with $C(u, r) > 0$ for $0 < t < t_E$ and $\partial C/\partial t < 0$ as long as $C > 0$;
2. k(t,r) is continuous
3. the metric has only one coordinate singularity, namely a surface $f(t, r) = 0$. This surface is located at the Schwarzschild radius $r_g(t)$, given by

$$r_g \equiv C(t, r_g) \tag{5.93}$$

and

$$C(t,r) = r_g(t) + w(t,r)(r - r_g(t)) \tag{5.94}$$

The shell trajectory is parametrized as $(T(\tau), R(\tau))$. The condition that the shell is timelike gives

$$\dot{T} = \frac{\sqrt{F + \dot{R}^2}}{KF} \tag{5.95}$$

with $K = k(T(\tau), R(\tau))$. As in the previous section we find that the horizon is avoided, according the equation (5.38), with the scale ϵ_* defined as

$$\epsilon_* = \frac{1}{K}\frac{r_g}{1-W}\left|\frac{dr_g}{dT}\right| \tag{5.96}$$

with $W(T) := w(T, r_g(T))$. Now we can treat the case of a spherically-symmetric collapse of a ball of dust, the so called Oppenehimer-Snyder collapse. The metric outside is given by (5.85) while the geometry inside the dust is given by the renowned FLRW metric for a closed universe.

$$ds_-^2 = -d\tau + a^2(\tau)(d\chi^2 + sin^2\chi d\Omega) \tag{5.97}$$

with τ the comoving time, and χ the comoving radial coordinate. This metric can be transormed into the (t,r) coordinates of eq.(5.85) by first transorming to the Painlevee-Gullstrand coordinates (see ref.[6]). We label each layer of dust by χ, so their motion is parametrized as $(T_\chi(\tau), R_\chi(\tau))$, and we assume that the dynamics modified by pre-Hawking radiation is such that individual layers of dust do not cross. So, each particle of dust moves along a radial geodesic, and at each layer the metric is given by (5.85) but with the modified parameters

$$ds_-^2 = -k_\chi(t,r)^2 f_\chi(t,r)dt^2 + f_\chi(t,r)^{-1}dr^2 + r^2 d\Omega \tag{5.98}$$

The Schwarzschild radius that any layer could potentially cross is $r_g^\chi(T_\chi(\tau)) \equiv C_\chi(T_\chi(\tau), r_g^\chi(T_\chi(\tau)))$. The previous discussion ensure that this will not happen.

Chapter 6

Arguments against horizon avoidance

6.1 Conditions on the energy radiated by a collapsing shell

A first agoument against horizon avoidance, is to investigate what would happen assuming that a collapsing shell could radiate part of its energy during the collapse. This works will demonstrate that prehawking radiation cannot carry away all the energy of the infalling matter. Some must remain to form a black hole. We will see in the next subsections that radiation must be turned off at some point, otherwise the shell becomes spacelike, a situation which is believed to be unphysical. Furthermore the maximum amount of radiation that a timelike or null shell could emit, is evaluated.

6.1.1 Null radiation from a plane in flat spacetime

With this introductive chapter we want to show some fundamental proprieties that a radiating body must hold even without gravity. We consider a radiating plan in flat spacetime, a sheet of pressureless matter with density

$\sigma(u)$ following the path $x = X(u)$ in a flat spacetime with metric given by

$$ds^2 = du^2 + 2dudx \tag{6.1}$$

We assume that the sheet emits radiation of massless particle in the positive x direction with positive intensity $\lambda(u)$. The energy mommentum tensor of the system is therefore given by the one for a pressureless fluid plus the radiated energy component

$$T^{\mu\nu} = \sigma\delta(x - X)u^\mu u^\nu + \lambda\Theta(x - X)\delta_{x\mu}\delta_{x\nu} \tag{6.2}$$

where $\Theta(z)$ is the Heaviside step function, defined as $\Theta(z) = 1$ for $z \geq 0$ and $\Theta(z) = 0$ for $z < 0$ u^μ is the four-velocity of the sheet, given by

$$u^\mu = (1, \dot{X}) \tag{6.3}$$

where $\dot{X}$ is dX/du, while $\delta_{\mu\nu}$ is the four velocity of the massless radiation. We can now evaluate the components of $T^{\mu\nu}$.

$$T^{uu} = \sigma\delta(x - X) \tag{6.4}$$

$$T^{ux} = T^{xu} = \sigma\dot{X}\delta(x - X) \tag{6.5}$$

$$T^{xx} = \sigma\dot{X}^2\delta(x - X) + \lambda(u)\Theta(x - X) \tag{6.6}$$

From the conservation of $T^{\mu\nu}$: $T^{\mu\nu}{}_{,\nu} = 0$, we obtain

$$T^{uu}{}_{,u} + T^{ux}{}_{,x} = 0 \quad \rightarrow \quad \dot{\sigma} = 0 \tag{6.7}$$

$$T^{xu}{}_{,u} + T^{xx}{}_{,x} = 0 \quad \rightarrow \quad \ddot{X} = -\frac{\lambda}{\sigma} \tag{6.8}$$

Now, if the sheet is null, we have, imposing $u_\mu u^\mu = 0$

$$\dot{X} = -1/2 \tag{6.9}$$

Then we see from (6.8), because λ is positive, $\dot{X}$ will becom less than $-1/2$, i.e. the sheet become spacelike. So the sheet could not radiate indefinitely. It has to stop before becoming spacelike.

6.1.2 Null radiation from a gravitational collapsing shell

Now we consider a timelike shell which radiates energy in the outgoing direction. The spacetime inside the shell is flat, while the exterior geometry is described by the outgoing Vaidya metric, respectively:

$$ds_-^2 = -du^2 - 2dudr + r^2 d\Omega \tag{6.10}$$

$$ds_+^2 = -(1 - \frac{2m(U)}{r})dU^2 - 2dUdr + r^2 d\Omega \tag{6.11}$$

where as usual $u = t - r$, $U = t - r_*$, and $m(U)$ is the mass function for outside the shell. We express now ds_+ in terms of u, which is a coordinate finite and regoular at the horizon, unlike U. Using u as coordinate will permit to describe the behavior of the shell even at the horizon. The metric outside becomes

$$ds_+ = -(1 - \frac{2m}{r})U'^2 du^2 - 2U' drdu + r^2 d\Omega \tag{6.12}$$

where $U' = (\frac{dU}{du})$. Now we have to impose that the induced metric must be the same on both side of the shell, knowing that the shell is defined by its path $r = R(U)$. We can now calcoulate the tangent vector to the shell, to compute the induced metric.
We have

$$e_u^\mu = (1, R', 0, 0) \quad e_\theta^\mu = (0, 0, 1, 0) \quad e_\phi^\mu = (0, 0, 0, 1) \tag{6.13}$$

Where $R' = dR/du$.
We must impose therefore

$$h_{ab} = g_{\mu\nu}^- e_a^\mu e_b^\nu = g_{\mu\nu}^+ e_a^\mu e_b^\nu \tag{6.14}$$

And this requires

$$1 + 2R' = U'^2(1 - \frac{2m}{r}) + 2U'R' \tag{6.15}$$

from which

$$R' = \frac{1 - U'^2(1 - 2m/R)}{2(U' - 1)} \tag{6.16}$$

Note that if the shell is a null ingoing shell,we must have

$$R' = -1/2 \tag{6.17}$$

This result can be achieved imposing, as in the previous section, $u_\mu u^\mu = 0$, where $u^\mu = (1, R', 0, 0)$ is the shell four-velocity within this choiche of coordinates.
We now replace r by a new coordiante z such that the shell is located at $z = 0$, in order to make the full metric continuous across the shell.

$$r = \begin{cases} R + \frac{z}{U'}, & (z > 0) \\ R + z, & (z \leq 0) \end{cases} \tag{6.18}$$

and

$$dr = \begin{cases} R'du + \frac{dz}{U'} - \frac{z}{U'^2}U'', & (z > 0) \\ R'du + dz, & (z \leq 0) \end{cases} \tag{6.19}$$

We obtain a metric for the full spacetime, namely

$$\begin{aligned} ds^2 = & -\left[\left(\left(1 - \frac{2m}{R + z/U'}\right)U'^2 + 2U'R' - \frac{2zU''}{U'}\right)\Theta(z) + (1 + 2R')(1 - \Theta(z))\right]du^2 \\ & - 2dudz + \left(R + \frac{z}{U'}\Theta(z) + z(1 - \Theta(z))\right)^2 d\Omega \end{aligned} \tag{6.20}$$

For which all componenets are continuous across the shell. Now we can solve the Einstein equation of motion of the shell. The Einstein tensor will be given by

$$G^{\mu\nu} = G^{\mu\nu}_{bulk} + G^{\mu\nu}_{shell}\delta(z) \tag{6.21}$$

Due to spherical symmetry the only non vanishing components of $G^{\mu\nu}_{shell}$ are G^{uu}_{shell} ,$G^{\theta\theta}_{shell}$ and $G^{\phi\phi}_{shell}$.
By assuming that the shell is composed of dust, then the angoular component must be zero.

Computing the Einstein tensor from the metric (6.20) we find

$$G^{uu}_{shell} = \frac{2(U'-1)}{RU'} \tag{6.22}$$

$$G^{\theta\theta}_{shell} = \frac{U'^2 m + U'' R^2 - RU'2 + RU'}{R^4 U'} \tag{6.23}$$

The dust condition then implies that

$$U'' = -\frac{U'(U'm - RU' + R)}{R^2} \tag{6.24}$$

Now differentiating both side of eq.(6.16) and simplifying R', U'' by using (6.16), (6.24) we find

$$R'' = \frac{U'}{2R(U'-1)}\left\{2m'U' - \left[1 - (1 - \frac{2m}{R})U'\right]^2\right\} \tag{6.25}$$

The prefactor in the right hand side is always positive, infact this is linked to $\sigma > 0$, i.e. taking the four-velocity of the shell v^μ in (u, z) coordinates, we have

$$v^\mu = (1, z'_{|shell}, 0, 0) = (1, 0, 0, 0) \tag{6.26}$$

so we have

$$G^{uu} = 8\pi T^{uu} = 8\pi\sigma v^u v^u = 8\pi\sigma \tag{6.27}$$

Then using (6.22) we see that the prefactor in (6.25) is positve. This implies that if $m' < 0$, then R'' is always negative in (6.25). Now defining

$$\rho \equiv R'(u) + \frac{1}{2} = -\frac{U'}{2(1-U')}\left[1 - U'\left(1 - \frac{2m}{r}\right)\right] \tag{6.28}$$

We see that as $R' \to -1/2$, ρ goes to zero.
We can rephrase R'' in the following way:

$$\rho' = R'' = \frac{m'U'^2}{R(U'-1)} - \frac{2(U'-1)}{RU'}\rho^2 \tag{6.29}$$

As the shell approach a null shell ($R' = -1/2$), eq.(6.16) implies

$$U' \to (1 - 2m/R) \tag{6.30}$$

Hence, (6.29) become

$$\rho' = \frac{m'}{2m(1 - 2m/R)} - \frac{4m}{R^2}\rho^2 \tag{6.31}$$

Therefore, if ρ goes to zero while the shell continue to radiate ($m' < 0$), then ρ will become negative, which implies $R' < -1/2$, and the shell is spacelike. We conclude that the shell could not radiate an indefinite amount of energy, at a certain point m' should approach zero.

6.1.3 Numerical evaluations

In order to solve equations (6.24), (6.25) for U and R respectively, we need to know the form of the function $m(u)$. Following the model of the previous chapter, passing from t to u coordinate, we assume that $m(u)$ satisfies eq. (5.61):

$$\frac{dm}{du} = -U' \frac{1}{3\mathcal{K}} \frac{1}{m^2} \tag{6.32}$$

For numerical calcoulation we choose $3\mathcal{K} = 1$.
For initial condition we choose arbitrary constants for $R(0)$, $m(0)$, $U(0)$, with the obvious condition that $R(0) > 2m(0)$. $R'(0)$ is also arbitrary as long as $R'(0) > -1/2$. Once these initial condition are chosen, U' is determined by (6.16), where the positive solution is taken for consistency ($U' \to 1$ when $m \to 0$).
In the following graphs, we allow the shell to become spacelike. The dashed line rapresent the behaviour of the shell, when m' goes to zero as the shell become null.

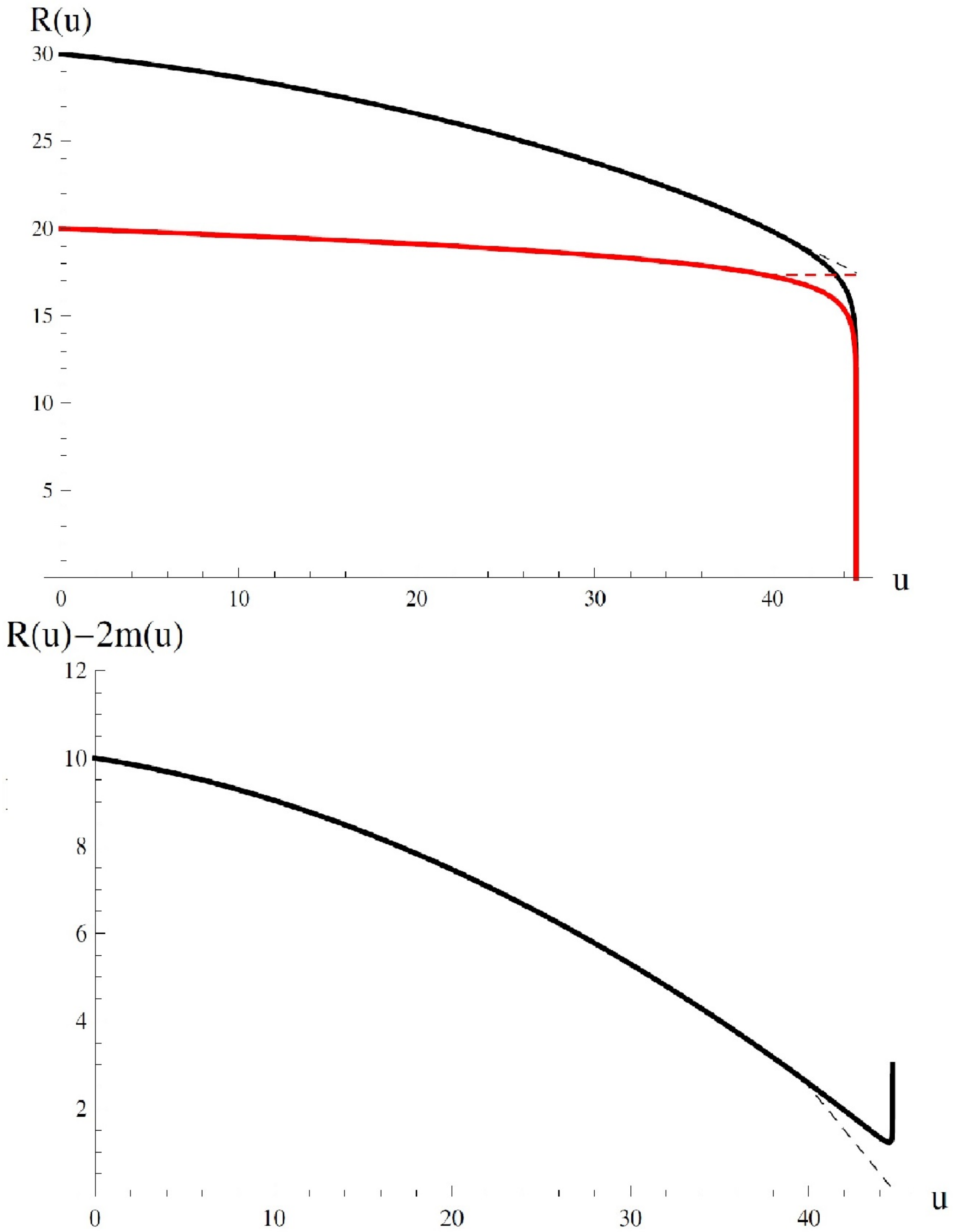

Figure 6.1: Initial conditions: $R(0) = 30$,$R'(0) = -0,1$, $m(0) = 10$, $U(0) = 10$. In the upper graph $R(u)$ is given by the black line and $2m(u)$ by the red one. We see that the shell does not cross the apparent horizon. The dashed line, though, does cross: $R'(u)$ remains equal to $-1/2$ since the shell become null, and will rapidliy equal $2m(u)$ and continues to $R(u) = 0$

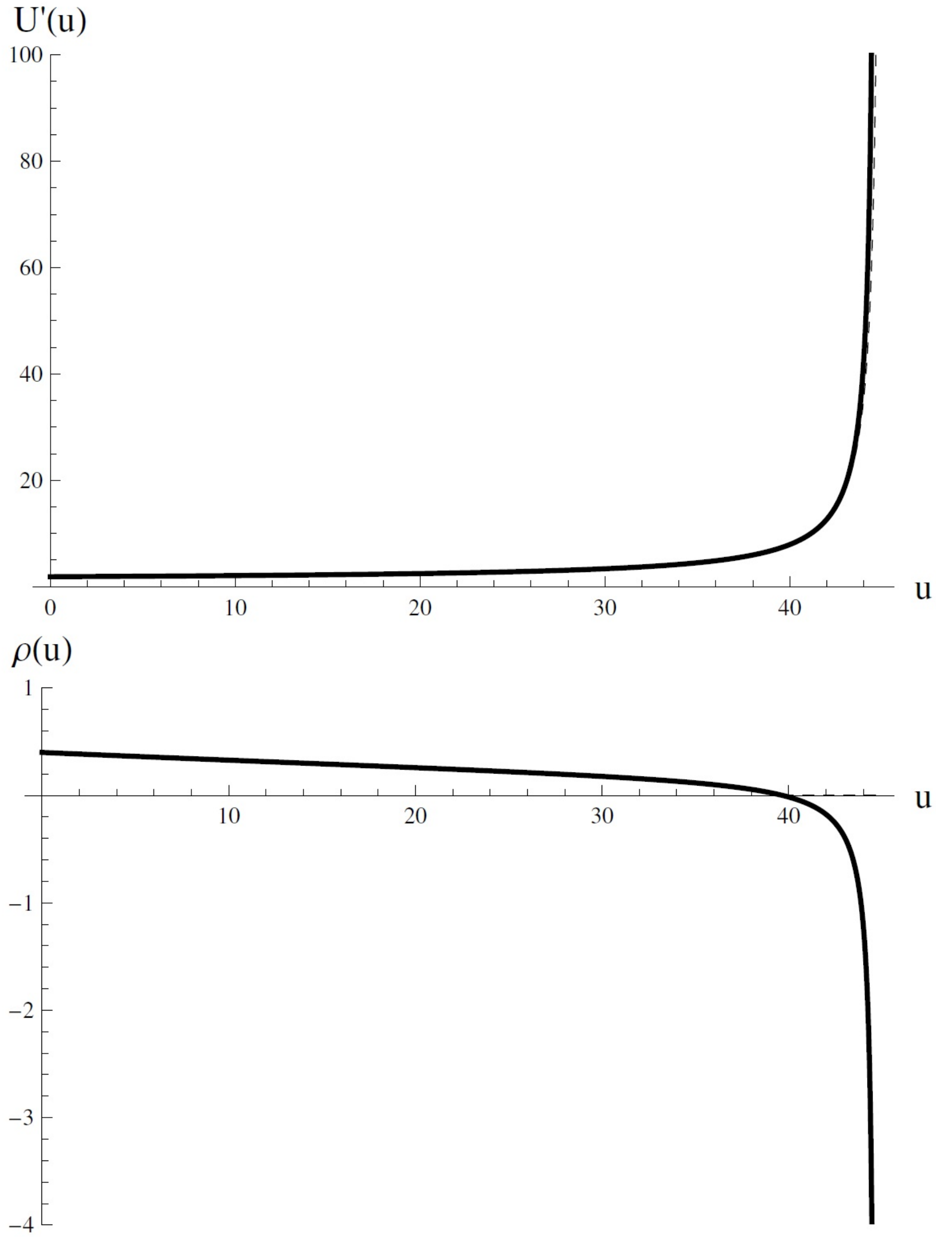

Figure 6.2: As the shell collapse U' increase rapidly. $\rho(u)$ goes large and negative indicating a strongly spacelike shell

6.1.4 Maximum emitted radiation

Being

$$ds^2_{|shell} = -(1+2R')du^2 + R^2 d\Omega \qquad (6.33)$$

the induced metric on the shell, and given

$$\sigma = \frac{1}{8\pi} G^{uu} \qquad (6.34)$$

we define the energy density on the shell for timelike matter as

$$\sqrt{S} = \sqrt{h}\sigma \qquad (6.35)$$

Where h is the determinant of the induced metric on the shell, where the angoular variables are being integrated.
Using (6.28) we obtain then the quantity

$$S \equiv 4\frac{U'-1}{U'}R^2\left[1-U'\left(1-\frac{2m}{R}\right)\right] \qquad (6.36)$$

which is the square of the energy density defined above and it would be expected to be conserved if $m'=0$. From equations (6.24), (6.28), we obtain

$$S' = 8m'(U'-1)R \qquad (6.37)$$

And, in fact, we see that S is conserved if $m'=0$.
If the shell is null, we see from eq.(6.30) that $S=0$, and negative for spacelike matter.
Now let us consider the situation where the shell is located at R_0 is at rest when it begin to collapse.
Then $R'=0$ at the startin point so from eq.(6.16) we find

$$U'_0 = \frac{1}{\sqrt{1-2m_0/R_0}} \qquad (6.38)$$

where the subscript 0 detones the intial condition of the shell.
This imply

$$S_0 = 4R_0^2 \left(\frac{U_0' - 1}{U_0'} \right)^2 = \frac{16m_0^2}{\left(1 + \sqrt{1 - 2m_0/R_0}\right)} \tag{6.39}$$

If $R_0 >> 2m_0$, which means that the shell begin to collapse far away from the putative horizon $2m_0$, (6.39) reduces to

$$S_0 = 4m_0^2 \tag{6.40}$$

Defining $\Delta S = \int S' du$ for a given integration domain and assuming that the minimum value of S is 0 (if S become negative, the shell turns spacelike) we find

$$\frac{16m_0^2}{\left(1 + \sqrt{1 - 2m_0/R_0}\right)} \geq -\Delta S = -8 \int (U' - 1) R \frac{dm}{du} du \tag{6.41}$$

Where m' is assumed to be negative. Now using eq.(6.16), (6.24) we find the condition

$$\frac{d(U' - 1)R}{du} = \frac{(U' - 1)^2}{2} \geq 0 \tag{6.42}$$

This means that the multiplier of m' is increasing and reaches its minimum at R_0.
Therefore

$$-\int (U' - 1) R dm \geq -(U_0' - 1) R_0 \int dm \tag{6.43}$$

and using (6.38)

$$-\Delta m \leq \frac{m_0}{2} \frac{2\sqrt{1 - 2m_0/R_0}}{1 + \sqrt{1 - 2m_0/R_0}} \leq \frac{m_0}{2} \tag{6.44}$$

This means that that the maximum amount of mass that the shell can be radiated must be less than 1/2 that its initial value. Evidently the shell cannot radiate away its entire gravitational mass to prevent the formation of

the horizon.
Now we consider the case of a shell with a nonzero initial inward velocity. From eq.(6.16)

$$R_0' = \frac{1 - U_0'^2(1 - 2m_0/R_0)}{2(U_0' - 1)} \tag{6.45}$$

Now, we assume $m_0 << R_0$, so we can expand U_0' as

$$U_0' = 1 + \alpha m_0/R_0 + \mathcal{O}(m_0/R_0)^2 \tag{6.46}$$

Sobstituting in (6.45) we find

$$R_0' = \frac{1-\alpha}{\alpha} \Rightarrow \alpha = \frac{1}{1+R_0'} \tag{6.47}$$

Now sobstituting (6.46) in (6.36) where all the parameters are evaluated in their initial value we find

$$S_0 = 4m_0^2\alpha(2-\alpha) = 4m_0^2\frac{1+2R_0'}{(1+R_0')^2} \tag{6.48}$$

Therefore we have again the following inequalities

$$S_0 \geq -8\int (U'-1)R\frac{dm}{du}du \tag{6.49}$$

and

$$-8\int (U'-1)R\frac{dm}{du}du \geq -8(U_0'-1)R_0\Delta m \tag{6.50}$$

and sobstituting (6.64)

$$-8(U_0'-1)R_0\Delta m = -\left(\frac{8m_0}{1+R_0'}\right)\Delta m \tag{6.51}$$

Therefore in the end we find from (6.49)

$$-\Delta m \leq \frac{m_0}{2}\left(\frac{1+2R_0'}{1+R_0}\right) \leq \frac{m_0}{2} \tag{6.52}$$

since $-1/2 \leq R_0' \leq 0$, and the shell cannot radiate more than half of its initial mass.

6.2 Analysis of the stress-energy tensor

In this section we analyze the behaviour of the stress energy tensor near a collapsing null shell which stops collapsing just outside its putative horizon, as the models of chapter five would expect. We will chose the mass function to be a constant. If there is trace of prehawking radiation it should emerge as a quantum effect in the analysis of the stress energy tensor.
To model the collapse we will use the metric (6.10) for $r < R(u)$ and the metric (6.11) for $r > R(u)$.
Equation (6.16) express the continuity of the metric across the surface r=R(u). As we already stated we take m(u) to be a constant and the shell to be a null shell so that $R = -1/2$ and

$$R(u) = 2m + \epsilon - \frac{u}{2} \tag{6.53}$$

This until R(u) is within ϵ of $2m$ (ie, at $u = 0$) where $\epsilon << 2m$.
At that point we stop the shell so $R(u) = 2m + \epsilon$ thereafter, and the shell becomes timelike.
For a null shell U' is given by eq.(6.30) so we have

$$U' = \frac{1}{\left(1 - \frac{2m}{(2m+\epsilon-u/2)}\right)} = \frac{2m + \epsilon - u/2}{\epsilon - u/2} \approx \frac{2m}{\epsilon - u/2} \tag{6.54}$$

Which is valid near the point where the shell stops, and so

$$U(u) \approx 4mln\left(\frac{\epsilon - u/2}{m}\right) \tag{6.55}$$

Where we chose as initial condition $u = -2m$.
To get an estimate of the energy momentum tensor we use the 2-dimensional formalism treated in chapter 3.

We therefore have to rewrite the metric in the following form

$$ds^2 = -e^{2\rho(u,v)} dudv \tag{6.56}$$

We then write the metric in a two dimensional form

$$ds^2 = \begin{cases} -du^2 - 2dudr & r < R(u) \\ -U'(u)^2(1 - \frac{2m}{r})du^2 - 2U'(dudr) & r > R(u) \end{cases} \tag{6.57}$$

and then we rewrite it in the double null form, performing the usual coordinate tranformation

$$r^* = \frac{V - U}{2} \tag{6.58}$$

we find

$$ds^2 = \begin{cases} -v'(V)dudV & v(V) - u < 2R(u) \\ -U'(u)(1 - \frac{2m}{r})dudV & v(V) - u > 2R(u) \end{cases} \tag{6.59}$$

We remember that for a null shell U' is given by eq.(6.30) so the conformal factor become

$$e^{2\rho} = \frac{1 - \frac{2m}{r}}{1 - \frac{2m}{R(u)}} \tag{6.60}$$

We can now evaluate the stress energy tensor in the various regions of spacetime. The equation of the shell during the collapse is $v = 0$. The vacuum state $|0\rangle$ is chosen such that the incoming modes V and hence $|0\rangle$ are of the standard Minkowsi form on $\mathcal{I}^-$.

- We see that along the shell, $r = R(u)$ and hence, $\rho(u, 0) = 0$. Thus along the shell

$$\langle 0| T_{UU} |0\rangle = 0 \tag{6.61}$$

  and there is no quantum emission from the shell into the region outside the shell.

- For $v < 0$, $V = v$, therefore we are in flat spacetime and the expectation value of the stress energy tensor is zero.

- For v>0, the v coordinate cross the shell after it has stopped. We look for a relation for v and V. Choosing $r = 2m + \epsilon$, and imposing the junction condition of the metrics (6.57) we find

$$U'^2 \frac{\epsilon}{2m} = 1 \tag{6.62}$$

and therefore

$$U = \sqrt{\frac{2m}{\epsilon}} u + const. \tag{6.63}$$

Rewritng the metrics (6.57) in advanced coordinates, and repeating the procedure, we find an analogous realtion for V and v, ie,

$$v(V) = \sqrt{\frac{\epsilon}{2m}} V + const. \tag{6.64}$$

Now, application of the formula (4.24), ie $\langle 0| T_{vv} |0\rangle = F_v(\beta') - F_v(A)$ gives

$$\langle 0| T_{vv} |0\rangle = 0 \tag{6.65}$$

Being β a linear function of v and $A = -1$. Therefore there is no flux of radiation inwards beyond the shell.

- If $v > 0$, $u < 0$ and $r > 2m + \epsilon$ this is the case of the where the stress energy tensor describe the process of collapse and as we have seen in chapter 3 this is modelized by the Unruh vacuum states. The components of the stress energy tensor are therefore given by

$$\begin{aligned}
\langle U| T_{UU} |U\rangle &= (786\pi m^2)^{-1}(1 - \frac{2m}{r})^2 \left[1 + \frac{4m}{r} + \frac{12m^2}{r^2}\right] \\
\langle U| T_{VV} |U\rangle &= (24\pi)^{-1} \left[\frac{3}{2}\frac{m^2}{r^4} - \frac{M}{r^3}\right] \\
\langle U| T_{UV} |U\rangle &= -(24\pi)^{-1}(1 - \frac{2m}{r})\frac{m}{r^3}
\end{aligned} \tag{6.66}$$

There is a positive energy flux travelling out to infinity and a negative energy flux heading toward the shell. In these limits these fluxex are very small, ie of order $1/m^2$ in Planck units (about one photon of frequency the inverse of black hole time $\approx 1/m$ emitted per black hole timescale $\approx 1/m$) They cannot have an effect on the metric (except for extremely long time scales of order of m^3 in Planck units, ie, 10^{53} ages of the current universe) and on the behaviour of the shell.

- For $u > 0$, $v > 0$, when the shell has stopped but $u < u_0$ where $u_0 > 0$ is the value of u where the null ingoing line from the turnaround reflects off $r = 0$, the zero ingoing $\langle T_{vv} \rangle$ radiation produces zero outgoing radiation through the shell.

- For $u > 0$, $v > 0$ and $u > u_0$ this is the case of eternal Schwarzschild spacetime. As we have seen in chapter 3 the vacuum state is described by the Boulware vacuum and the component of the energy momentum tensor are

$$\begin{aligned}
\langle B| T_{UU} |B\rangle &= (24\pi)^{-1}\left[\frac{3}{2}\frac{m^2}{r^4} - \frac{M}{r^3}\right] \\
\langle B| T_{VV} |B\rangle &= (24\pi)^{-1}\left[\frac{3}{2}\frac{m^2}{r^4} - \frac{M}{r^3}\right] \\
\langle B| T_{UV} |B\rangle &= -(24\pi)^{-1}(1 - \frac{2m}{r})\frac{m}{r^3}
\end{aligned} \tag{6.67}$$

  This gives an energy momentum tensor with negative energy density near $r = 2m$ going as $1/m^2$ in Planck units.
  We remember, as we have already seen in chapter 3, that in the free falling frame the components of this stress-energy tensor become singoular at the horizon (i.e. when $\epsilon \to 0$) .

We conclude therefore that there is no prehawking radiation. The radiation is just the Hawking radiation for $u < 0$ and the static Schwarzschild radiation for $u > u_0$ and these are both very tiny. In the following graph the various regions of spacetime are described.

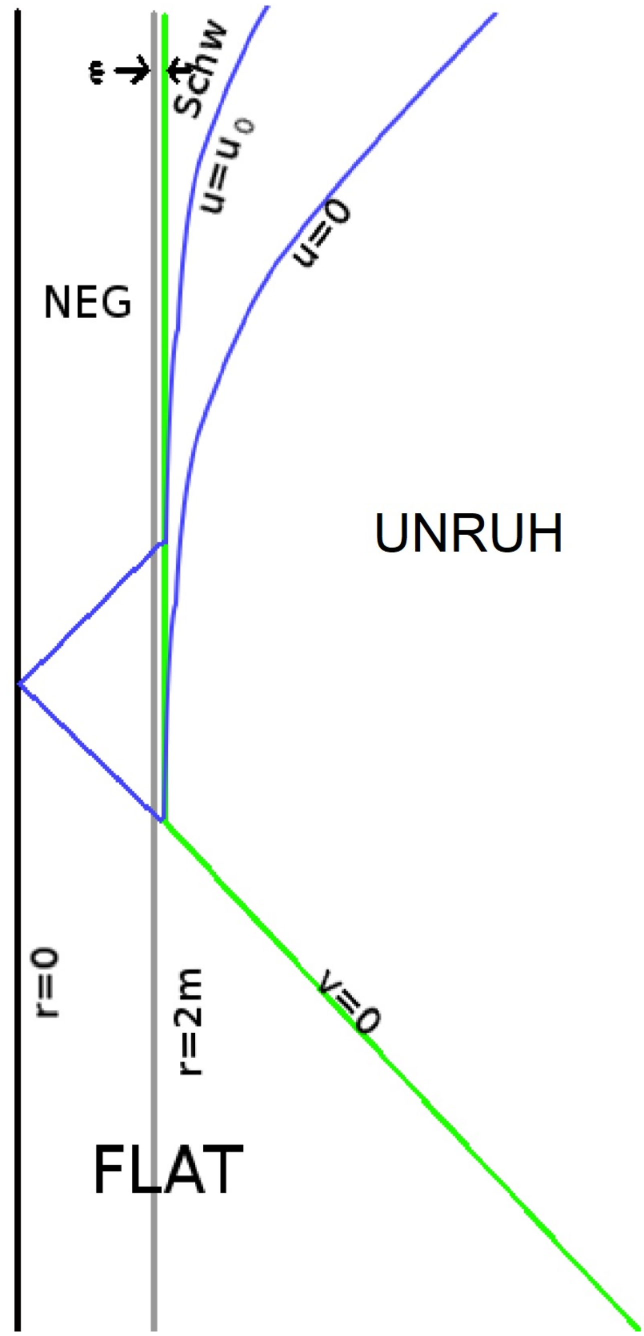

Figure 6.3: This figure, in Eddington Finkelstein coordinates (ingoing null rays are at 45 degrees), shows the various region of spacetime. The green line designate the ingoing matter. The blue lines designate null surfaces. The region labelled Unruh is where the energy momentum tensor is equal to the one calcoulated in the Unruh vacuum and it is identical to what it would be if the collapse had actually produced a black hole. The region labelled Schwarzschild is where the energy momentum tensor is what it would be in the Boulware vacuum of a black hole. The area labelled flat is flat spacetime, where the energy momentum tensor expectation value would be expected to be 0

Conclusion

In the first four chapter of this thesis we have analized some well known results of the semiclassical theory of gravitational collapse. In particoular, after having enounced the principles of a quantum field theory in curved spacetime, we have derived the Hawking effect and analized one of its most dramatic consequencies, the information loss paradox. We have then seen how the Hawking radiation contributes to the expectation values of the stress energy tensor and in the end we have analyzed a realistic process of spherically symmetryc collapse, and we move on to analyze the collapse of a thin shell and come to the conclusion that the Unruh state, which is foundamental to describe the Hawking effect, is a very good approximation of the real process. In the following chapters we dealt with the problem of pre-Hawking radiation. In the fifth chapter we have analized some model of collapse in which the metrics outside the collapsing body presented a time-dependent mass-function, designed to model the mass loss caused by the emission by the collapsing body of an undefined kind of radiation, the so called pre-Hawking radiation. We have seen that this imply, studiyng the equation of motion of the body derived from the Israel's junction conditions, that the radius of the collapsing body get close to its Schwarzschild radius but never crosses it. And thus a black hole never forms. This perspective wuold be a solution of a certain number of problems such as the fact that for a distant observer the time that the body takes to form a black hole is infinte while the evaporation time of the black hole is finite, and of course the information loss paradox. Then we have analized some work aimed at disprove this argoumentations. In particular it is noted how the collapsing body, a shell of matter, cannot emitt an indefinite quantity of radiation without becoming spacelike, i.e. composed of unphysical matter. To stay in to a physical range, we see that the maximum amount of radiation the shell can radiate without becoming spacelike it must be less than a half of its total energy. It seems that black holes should form afterall. As a final attempt to attack the hypotesis of pre-Hawking radiation it is performed the analysis of the stress energy tensor near a shell which stop collapsing near its putative horizon. The results of this analisys show how the only kind of radiation emitted is the Hawking radiation, a radiation which is to weak to have a consistent effect on the metric and on the behaviour of the shell, and there is no trace of exotic radiation such as the pre-Hawking radiation. We need to remark the fact that during this analysis the mass function in the external metric was choosen to be constant. In fact we are able to compute the stress energy tensor only for the Schwarzschild metric. We don't know the form of the metric corrected by the backreaction, and so we don't know the real stress energy tensor (this would require to solve exactly the Einstein field equation for the collapse process). This let the discussion about the existence of pre-Hawking radiation still open although, for what we have seen in the previous chapters, it seem unlikely that it could prevent the formation of black holes.

Appendix A

Israel's junction conditions

We will now present the Israel's junction conditions for a timelike thin shell.
A hypersurface Σ divides the spacetime in two regions $\mathcal{V}^+$ and $\mathcal{V}^-$. The metrics in these regions are respectively $g^+_{\alpha\beta}$ and $g^-_{\alpha\beta}$. $g^+_{\alpha\beta}$ is expressed in coordinates x^α_+ while $g^-_{\alpha\beta}$ is expressed in coordinates x^α_-. We want to find what condition must satisfy these metric to ensure that $\mathcal{V}^+$ and $\mathcal{V}^-$ are joined smoothly at Σ, so that the union of $g^+_{\alpha\beta}$ and $g^-_{\alpha\beta}$ forms a valid solution to the Einstein field equation. We will assume that the same coordinates y^a can be installed on both sides of Σ. We choose n^α, the unit normal to the hypersurface, to point from $\mathcal{V}^-$ to $\mathcal{V}^+$. We suppose that we can introduce on both sides of Σ a continuos coordinate system x^α which overlaps with x^α_+ in an open region of $\mathcal{V}^+$ that contains Σ, and the same can be said for a open region of $\mathcal{V}^-$ for the coordinates x^α_-.
We immagine Σ to be intersect orthogonally by a congruence of geodesics. We denote the proper distance along the geodesics by l, and we choose the parametrization such that $l = 0$ when the geodesics cross the hypesurface, with l negative in $\mathcal{V}^-$ and positive in $\mathcal{V}^+$. The displacement away from Σ along one of the geodesics is described by $dx^\alpha = n^\alpha dl$ and that

$$n_\alpha = \epsilon \partial_\alpha l \tag{A.1}$$

and also $n^\alpha n_\alpha = \epsilon$.
For a quantity A defined on both sides of the hypersurface we will use the following notation:

$$[A] \equiv A(\mathcal{V}^+)|_\Sigma - A(\mathcal{V}^-)|_\Sigma \tag{A.2}$$

A.1 First junction condition

We can express the metric of the whole spacetime in the coordinate x^α as a distribution valued tensor:

$$g_{\alpha\beta} = \Theta(l) g^+_{\alpha\beta} + \Theta(-l) g^-_{\alpha\beta} \tag{A.3}$$

Differentiating (A.3) yields

$$g_{\alpha\beta,\gamma} = \Theta(l) g^+_{\alpha\beta,\gamma} + \Theta(-l) g^-_{\alpha\beta,\gamma} + \epsilon\delta(l)[g_{\alpha\beta}]n_\gamma \tag{A.4}$$

where we used (A.1). The last term is singular and it causes problems when we try to compute the Christoffel symbols. To eliminate this term we impose the continuity of the metric across Σ: $[g_{\alpha\beta}] = 0$.
This statement holds only in the coordinate system x^α. By noting that for $e^\alpha_a = \partial x^\alpha / \partial y^a$ holds

$$[e^\alpha_a] = 0 \tag{A.5}$$

Because the coordinates y^a are the same on both sides of the hypersurface, we can find a coordinate invariant version of the statement above: $0 = [g_{\alpha\beta}]e^\alpha_a e^\beta_b = [g_{\alpha\beta} e^\alpha_a e^\beta_b]$. Calling h the induced metric on the hypersurface, we have obtained the first junction condition

$$[h_{ab}] = 0 \tag{A.6}$$

The induced metric must be the same on both side of the hypersurface. This condition is independent from the coordinate systems x^α or $x^\alpha_\pm$

A.2 Riemann tensor

To enunciate the second junction condition we must calculate the distribution-valued Riemann tensor. The Christoffel symbols are

$$\Gamma^{\alpha}_{\beta\gamma} = \Theta(l)\Gamma^{+\alpha}_{\beta\gamma} + \Theta(-l)\Gamma^{-\alpha}_{\beta\gamma} \tag{A.7}$$

Where $\Gamma^{\pm\alpha}_{\beta\gamma}$ are constructed from $g^{\pm}_{\alpha\beta}$.
Now

$$\Gamma^{\alpha}_{\beta\gamma,\delta} = \Theta(l)\Gamma^{+\alpha}_{\beta\gamma,\delta} + \Theta(-l)\Gamma^{-\alpha}_{\beta\gamma,\delta} + \epsilon\delta(l)[\Gamma^{\alpha}_{\beta\gamma}]n_{\delta} \tag{A.8}$$

And from this follows the Riemann tensor:

$$R^{\alpha}_{\beta\gamma\delta} = \Theta(l)R^{+\alpha}_{\beta\gamma\delta} + \Theta(-l)R^{-\alpha}_{\beta\gamma\delta} + \delta(l)A^{\alpha}_{\beta\gamma\delta} \tag{A.9}$$

where

$$A^{\alpha}_{\beta\gamma\delta} = \epsilon([\Gamma^{\alpha}_{\beta\delta}]n_{\gamma} - [\Gamma^{\alpha}_{\beta\gamma}]n_{\delta}) \tag{A.10}$$

We see that the delta function term rapresent a curvature singoularity at Σ. In the next sections we will give physical interpretation to this singoularity.

A.3 Surface stress-energy tensor

We note that $A^{\alpha}_{\beta\gamma\delta}$ is a tensor because it is constructed from the difference between two sets of Christoffel symbols. We are looking for an explicit expression for this tensor.
The fact that the metric is continuos across Σ in the coordinates x^{α} implies that its tangential derivatives must be continuos. This means that if $g_{\alpha\beta,\gamma}$ is discontinuos, the discontinuity must be directed along n^{α}. Therefore there must exist a tensor field $\kappa_{\alpha\beta}$ such that

$$[g_{\alpha\beta,\gamma}] = \kappa_{\alpha\beta}n_{\gamma} \tag{A.11}$$

this tensor is explicitly given by

$$\kappa_{\alpha\beta} = \epsilon[g_{\alpha\beta,\gamma}]n^{\gamma} \tag{A.12}$$

Eq. (A.11) implies

$$[\Gamma^{\alpha}_{\beta\gamma}] = \frac{1}{2}(\kappa^{\alpha}_{\beta}n_{\gamma} + \kappa^{\alpha}_{\gamma}n_{\beta} - \kappa_{\beta\gamma}n^{\alpha}) \tag{A.13}$$

now we can obtain the δ-function part of the Riemann tensor

$$A^{\alpha}_{\beta\gamma\delta} = \frac{\epsilon}{2}(\kappa^{\alpha}_{\delta}n_{\beta}n_{\gamma} - \kappa^{\alpha}_{\gamma}n_{\beta}n_{\delta} - \kappa_{\beta\delta}n^{\alpha}n_{\gamma} + \kappa_{\beta\gamma}n^{\alpha}n_{\delta}) \tag{A.14}$$

Contracting over the first and the third indices gives the δ-function part of the Ricci tensor:

$$A_{\alpha\beta} \equiv A^{\mu}_{\alpha\mu\beta} = \frac{\epsilon}{2}(\kappa_{\mu\alpha}n^{\mu}n_{\beta} + \kappa_{\mu\beta}n^{\mu}n_{\alpha} - \kappa n_{\alpha}n_{\beta} - \epsilon\kappa_{\alpha\beta}) \tag{A.15}$$

where $\kappa \equiv \kappa^{\alpha}_{\alpha}$. After an additional contraction we obtain the δ-function part of the Ricci scalar

$$A \equiv A^{\alpha}_{\alpha} = \epsilon(\kappa_{\mu\nu}n^{\mu}n^{\nu} - \epsilon\kappa) \tag{A.16}$$

We can now form the δ-function part of the Einstein tensor, and from the Einstein field equation we obtain an expression for the stress-energy tensor:

$$T_{\alpha\beta} = \Theta(l)T^{+}_{\alpha\beta} + \Theta(-l)T^{-}_{\alpha\beta} + \delta(l)S_{\alpha\beta} \tag{A.17}$$

Where $8\pi S_{\alpha\beta} \equiv A_{\alpha\beta} - \frac{1}{2}Ag_{\alpha\beta}$. The first two terms on the rhs of Eq. (A.17) represent the stress energy tensor of regions $\mathcal{V}^{+}$ and $\mathcal{V}^{-}$, respectively. The interpretation of the δ-function term is to be associated with the presence of a thin distribution of matter, a thin shell, at Σ.

A.4 Second junction condition

The explicit form of the surface stress-energy tensor is given by

$$16\pi\epsilon S_{\alpha\beta} = \kappa_{\mu\alpha} n^\mu n_\beta + \kappa_{\mu\beta} n^\mu n_\alpha - \kappa n_\alpha n_\beta - \epsilon\kappa_{\alpha\beta} - (\kappa_{\mu\nu} n^\mu n^\nu - \epsilon\kappa) g_{\alpha\beta} \quad (A.18)$$

We notice then that $S_{\alpha\beta}$ is tangent to the hypersurface: $S_{\alpha\beta} n^\beta = 0$. Therefore it can be decomposed as

$$S^{\alpha\beta} = S^{ab} e^\alpha_a e^\beta_b \quad (A.19)$$

where $S_{ab} = S_{\alpha\beta} e^\alpha_a e^\beta_b$ is a symmetric three-tensor. This is evaluated as follows:

$$\begin{aligned} 16\pi S_{ab} &= -\kappa_{\alpha\beta} e^\alpha_a e^\beta_b - \epsilon(\kappa_{\mu\nu} n^\mu n^\nu - \epsilon\kappa) h_{ab} \\ &= -\kappa_{\alpha\beta} e^\alpha_a e^\beta_b - \kappa_{\mu\nu}(g^{\mu\nu} - h^{mn} e^\mu_m e^\nu_n) h_{ab} + \kappa h_{ab} \\ &= -\kappa_{\alpha\beta} e^\alpha_a e^\beta_b + h^{mn} \kappa_{\mu\nu} e^\mu_m e^\nu_n h_{ab} \end{aligned} \quad (A.20)$$

We recall now that the extrinsic curvature of an hypersurface is a three-tensor defined as

$$K_{ab} \equiv n_{\alpha;\beta} e^\alpha_a e^\beta_b \quad (A.21)$$

Now we have

$$\begin{aligned} [n_{\alpha;\beta}] &= -[\Gamma^\gamma_{\alpha\beta}] n_\gamma \\ &= -\frac{1}{2}(\kappa_{\gamma\alpha} n_\beta + \kappa_{\gamma\beta} n_\alpha - \kappa_{\alpha\beta} n_\gamma) n^\gamma \\ &= \frac{1}{2}(\epsilon\kappa_{\alpha\beta} - \kappa_{\gamma\alpha} n_\beta n^\gamma - \kappa_{\gamma\beta} n_\alpha n^\gamma) \end{aligned} \quad (A.22)$$

and we can thus write

$$[K_{ab}] = [n_{\alpha;\beta}] e^\alpha_a e^\beta_b = \frac{\epsilon}{2} \kappa_{\alpha\beta} e^\alpha_a e^\beta_b \quad (A.23)$$

From (A.20) it follows that

$$S_{ab} = -\frac{\epsilon}{8\pi}([K_{ab}] - [K] h_{ab}) \quad (A.24)$$

This formula relates the surface stress energy tensor to the jump in the extrinsic curvature from on side of Σ to another. The complete stress-energy tensor of the surface layer is

$$T_{\Sigma}^{\alpha\beta} = \delta(l) S^{ab} e_a^{\alpha} e_b^{\beta} \tag{A.25}$$

A smooth transition across Σ requires

$$[K_{ab}] = 0 \tag{A.26}$$

This is the second junction condition. If this condition is violated the space-time is singular at Σ, but this singularity has a physical interpretation. It means that a surface layer with stress energy tensor given by $T_{\Sigma}^{\alpha\beta}$ is present at the hypersurface.

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