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Bethe Ansatz techniques for quasilocal charges in quench problems of integrable models

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Abstract

We give an introduction to the two fundamental methods to solve one-dimensional integrable systems: the Coordinate Bethe Ansatz and the Algebraic Bethe Ansatz. We focus particularly on two widely studied Heisenberg chains: the XXX model and the XXZ model.

After that we present a brief overview on the quench scenario, which has led in the last decade to a great progress in the field of out-of-equilibrium physics.

The fundamental techniques devoted to implement the quench approach are the Generalized Gibbs Ensemble (GGE) and the Quench Action (QA).

We treat in more details the GGE approach, based on the infinite set of local conserved charges of an integrable system. Recently the GGE technique has been improved, adding to the set of local conserved charges, the so called quasilocal conserved charges.

We finally propose a way to construct a set of quasilocal charges for the XXZ model in the gapless regime, through the so called Y-system, a hierarchic system of equations, which usually appears in the context of integrable theories.

Abstract

Diamo un' introduzione sui due metodi fondamentali per risolvere sistemi integrabili in una dimensione: il Coordinate Bethe Ansatz e l'Algebraic Bethe Ansatz. Concentriamo particolarmente la nostra attenzione su due catene di Heisenberg ampiamente studiate: il modello XXX e il modello XXZ.

Successivamente presentiamo una breve panoramica nell'ambito del quench (in inglese il verbo to quench significa spegnere), che ha portato negli ultimi dieci anni a grandi progressi nel campo della fisica del non-equilibrio.

Le tecniche fondamentali usate per implementare l'idea del quench sono l' Ensemble di Gibbs Generalizzato e l'Azione Quench.

Affrontiamo con più dettagli la tecnica dell' Ensemble di Gibbs Generalizzato, costruito attraverso l' insieme infinito di cariche conservate locali di un sistema integrabile.

Recentemente la tecnica dell' Ensemble di Gibbs Generalizzato è stata migliorata, aggiungendo all'insieme di cariche conservate locali, le cosiddette cariche conservate quasilocali.

Proponiamo infine un procedimento per costruire un insieme di cariche quasilocali per il modello XXZ nel regime critico, detto anche senza gap (salto), attraverso il cosiddetto Y-system, ovvero un sistema gerarchico di equazioni, che appare spesso nel contesto delle teorie integrabili.

Chapter 1

Introduction

From the exact solution of the Ising model by Onsager in 1944 up to that of the hard hexagon model by Baxter in 1980, the statistical mechanics of two-dimensional systems has been enriched by a number of exact results. One speaks of exact models once a convenient mathematical expression has been obtained for a physical quantity such as the free energy, correlation functions or other features of the physical model. Linked to two-dimensional classical models, one-dimensional quantum models such as the linear magnetic chain and Bethe's famous solution have certainly contributed to the understanding of fundamental excitations in many-body systems and have signed the beginning of a prolific branch of Statistical Mechanics: Integrable Models and the technique used to solve them, such as the various Bethe Ansatz approaches (Coordinate Bethe Ansatz, Algebraic Bethe Ansatz, Thermodynamic Bethe Ansatz, Asymptotic Bethe Ansatz, Nested Bethe Ansatz). Related to Bethe Ansatz a wide field of mathematical physics has been also influenced, i.e. group theory. With the Bethe Ansatz, many-body dynamics can be reduced to two-body dynamics. The many-particle scattering matrix is equal to the product of two-particle ones. This leads to the self consistency relation for the two-particle scattering matrix, which is the Yang-Baxter equation, the central concept of exactly solvable models. The role of Yang-Baxter equation goes beyond the theory of dynamical systems. It is very important in group theory, particularly in quantum group theory, as we stated above.

Recently the study of the thermodynamics of integrable systems has attracted a lot of attention, particularly the analysis of the time evolution of the physical model. The quantum quench is a modern application of these issues.

A quantum quench is an instantaneous change in the parameters that determine the dynamics of an isolated quantum system e.g. the masses or coupling constants of its Hamiltonian. From an experimental point of view

this is a feasible way to bring the system out-of-equilibrium and study its evolution under the quantum mechanical laws, in isolation from the environment. In particular, the scientific interest in quantum quenches started growing after the experimental realization of global sudden changes of the interaction in cold atom systems, a novel technology where quantum statistical physics can be experimentally demonstrated and probed.

Due to the confluence of various features, these quantum systems are in many ways near-ideal systems for the study of nonequilibrium quantum phenomena. Firstly, quantum gases can exhibit a remarkably high degree of isolation from environmental sources of decoherence and dissipation. Thus, to an excellent approximation, during duration of experiments they can be regarded as closed quantum systems. Further, the dilute nature of these gases and exceptionally low temperatures result in long timescales of dynamical effects (typically on the order of milliseconds or longer) allowing for timeresolved studies of nonequilibrium processes resulting from phase-coherent many-body dynamics. Secondly, an array of techniques have been developed to dynamically tune various parameters of the Hamiltonian governing these quantum gases. This has made it possible to realize various prototypical nonequilibrium processes such as quantum quenches discussed above.

From a theoretical point of view the problem consists in preparing the system in a particular trial state, which is typically the ground state of some Hamiltonian, and study its evolution under a different Hamiltonian. Apart from being one of the simplest and well-posed ways to study out-of-equilibrium quantum physics, quantum quenches also give rise to a fundamental long-standing open question of central importance in statistical physics, the question of thermalization: how do extended quantum physical systems tend to thermal equilibrium starting from an arbitrary initial state? Of particular interest is the case of (1+1)-dimensions where a discrimination between integrable and non-integrable systems is possible.

For generic systems it is expected that after a sufficiently long time they reach a steady state in which the expectation values of some class of relevant observables are described by a thermal Gibbs ensemble (canonical ensemble). The choice of the class of observables generally follows the idea that they are supported on subsystems which in the thermodynamic limit are infinitely smaller than the rest of the system. The rest of the system can then act as a heat bath, leading to thermalization.

Thermalization, however, is only expected to hold for systems with generic, i.e. nonintegrable dynamics. Integrable models are models that exhibit factorization of the scattering matrix and can be solved exactly. Their classical counterparts possess as many integrals of motion as their degrees of freedom

and this fact prevents thermalization of an arbitrary initial state, as not all of the micro-states of equal energy respect the conservation of all other integrals of motion. This property is also expected to hold at the quantum level. In a seminal experiment it was observed that a trapped (1+1)d Bose gas, initially prepared in a non-equilibrium state, does not thermalize but tends instead to a nonthermal momentum distribution. The absence of thermalization suggests as a possible reason the integrability of the system which approximates a homogeneous (1+1)d Bose gas with point-like collisional interactions, a typical integrable model, even though the confining potential used in the experiment breaks the homogeneity and therefore integrability of the system. This experiment triggered an intense discussion about the role of non-integrability in the thermalization process. It was soon conjectured that in an integrable case the system does exhibit stationary behavior for long times, described however not by the usual Gibbs ensemble but a Generalized Gibbs Ensemble (GGE) where new Lagrange multipliers, in addition to temperature related to energy, are introduced into the density matrix, one for each integral of motion, for accounting their conservation.

The GGE can be derived by applying the maximum entropy principle under the constraint provided by the conserved local-charges, therefore the idea is very natural in the framework of statistical mechanics. Most initial studies of GGE were carried out in theories equivalent to free fermions or by numerical studies of relatively small systems. More recently it became possible to examine interacting integrable systems such as the 1D Bose gas, the XXZ Heisenberg spin chain or field theories. The validity of the GGE for interacting theories has been called into question by a series of recent studies. A crucial step in this direction was the development of the Quench Action approach, which provided an alternative way to study the time evolution using the overlaps of the initial state with the eigenstates of the post-quench Hamiltonian.

The different way of study quantum quench has suggested that the GGE can describe better the out-of-equilibrium dynamics if it is enriched by a set of quasilocal charges, in addition to the above mentioned local charges. In this thesis we want to construct a set of quasilocal charges for the gapless regime of the XXZ model, taking advantage of the relation (see papers of Destri and de Vega written in the early nineties) between this model and the sine-Gordon model. The point of contact of the two theories (observe that one is a field theory and the other is a lattice model) is the Y-system, i.e. a set of functional equations peculiar to integrable models.

In the first two following chapter we give a brief introduction to spin chains and to Coordinate and Algebraic Bethe Ansatz, used to solve them. In the fourth chapter we study with more details the technique of quantum

quench and we give some new recent advances in this field. The fifth chapter is devoted to the description of how to define quasilocal charges and we give some example on how to construct them. The last chapter contains our attempt to construct a set of quasilocal charges for the XXZ model in the gapless regime.

Chapter 2

Spin chains

2.1 A simple Heisenberg chain: the XXX model

2.1.1 Introduction to the model

The Hamiltonian of the Heisenberg spin-1/2 chain with N sites and periodic boundary condition $\mathbf{S}_{j+N} = \mathbf{S}_j$ is

$$H = -J\sum_{n=1}^{N} \mathbf{S}_n \cdot \mathbf{S}_{n+1} = -J\sum_{n=1}^{N} \left[\frac{1}{2}\left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+\right) + S_n^z S_{n+1}^z\right], \quad (2.1)$$

where $\mathbf{S}_n = 1/2\,(\sigma_n^x, \sigma_n^y, \sigma_n^z)$, σ_n^α are Pauli matrices acting on the nth sites and $S_n^\pm \equiv S_n^x \pm iS_n^y$ are spin flip operators. Since for each sites the possible spin configurations are only up and down and each S_n acts on the nth site, the Hamiltonian acts on the tensor product of N two dimensions Hilbert spaces, which gives a 2^N dimensions total Hilbert space. The basis vectors which span the latter space are $|\sigma_1 ... \sigma_N\rangle$, where $\sigma_n = \uparrow$ represents an up spin and $\sigma_n = \downarrow$ a down spin at sites n. For subsequent calculations it is useful to see how $S_n^\pm$ operates on each site. Knowing that

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (2.2)$$

if we have two basis state $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then

$$S^+|\uparrow\rangle = 0, \qquad S^-|\uparrow\rangle = |\downarrow\rangle, \qquad S^z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle,$$
$$S^-|\downarrow\rangle = 0, \qquad S^+|\downarrow\rangle = |\uparrow\rangle, \qquad S^z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle. \quad (2.3a,b)$$

The fundamental commutation relations (with $\hbar = 1$) are

$$\left[S_n^z, S_m^\pm\right] = \pm S_n^\pm \delta_{nm}, \qquad \left[S_n^+, S_m^-\right] = 2S_n^z \delta_{nm}. \tag{2.3}$$

The Heisenberg Hamiltonian enjoys a rotational symmetry about the z-axis, so that the projection of the total spin on this axis $S^z = \sum_{n1}^N S_n^z$ is conserved and $[H, S^z] = 0$. Since the magnetization is conserved it is useful to consider separately sectors defined by the number of $S^z = N/2 - R$, where R is the number of down spins.

With these tools we can begin studying the model, starting from the vacuum state, defined by all spins up and so $R = 0$. Giving $|0\rangle = |\uparrow\uparrow ... \uparrow\rangle$ and observing that from the (2.3a,b) the operators $S_n^+ S_{n+1}^-$ and $S_n^- S_{n+1}^+$ give zero and that $S_n^z S_{n+1}^z$ gives $-1/4$ acting on the vacuum state, we have

$$H|0\rangle = E_0|0\rangle, \qquad E_0 = -\frac{J}{4}N. \tag{2.4}$$

The states in the sector $R = 1$, i.e. all spins up except than one, are constructed from the vacuum state, using the lowing operator:

$$|n\rangle = S_n^-|0\rangle, \qquad n = 1, ..., N. \tag{2.5}$$

However these are not eigenstates of the Hamiltonian, but we can obtain an eigenstate by a linear combination of the latter states:

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{ikn}|n\rangle, \tag{2.6}$$

for wave numbers $k = 2\pi m/N, m = 0, ..., N-1$.

We proceed with the calculation of the eigenvalues of (2.6). We have

$$-\frac{J}{\sqrt{N}} \sum_{n=1}^N \left[\frac{1}{2}\left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+\right) + S_n^z S_{n+1}^z\right] \sum_{j=1}^N e^{ikj} S_j^-|0\rangle = \frac{E}{\sqrt{N}} \sum_{j=1}^N e^{ikj} S_j^-|0\rangle. \tag{2.7}$$

We calculate separately each summand, using the fundamental commutation relations:

$$\sum_{n,j=1}^N S_n^+ S_{n+1}^- S_j^- e^{ikj}|0\rangle = \sum_{n,j=1}^N S_j^- S_n^+ S_{n+1}^- e^{ikj}|0\rangle + 2\sum_{n=1}^N S_n^z S_{n+1}^- e^{ikn}|0\rangle, \tag{2.8}$$

$$\sum_{n,j=1}^N S_n^- S_{n+1}^+ S_j^- e^{ikj}|0\rangle = \sum_{n,j=1}^N S_j^- S_n^- S_{n+1}^+ e^{ikj}|0\rangle + 2\sum_{n=1}^N S_n^- S_{n+1}^z e^{ik(n+1)}|0\rangle, \tag{2.9}$$

$$\sum_{n,j=1}^{N} S_n^z S_{n+1}^z S_j^- e^{ikj}|0\rangle = \sum_{n,j=1}^{N} S_j^- S_n^z S_{n+1}^z e^{ikj}|0\rangle - \sum_{n=1}^{N} S_n^- S_{n+1}^z e^{ikn}|0\rangle - \sum_{n=1}^{N} S_n^z S_{n+1}^- e^{ik(n+1)}|0\rangle. \tag{2.10}$$

Collecting all members we have

$$\sum_{n,j=1}^{N} S_j^- e^{ikj} \left(-\frac{J}{\sqrt{N}}\right) \left[\frac{1}{2}\left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+\right) + S_n^z S_{n+1}^z\right] |0\rangle$$
$$+\left(-\frac{J}{\sqrt{N}}\right) \sum_{n=1}^{N} \left[S_n^z S_{n+1}^- e^{ikn} + S_n^- S_{n+1}^z e^{ik(n+1)} - S_n^- S_{n+1}^z e^{ikn} - S_n^z S_{n+1}^- e^{ik(n+1)}\right] |0\rangle =$$
$$E\frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{ikj} S_j^-|0\rangle. \tag{2.11}$$

Observing that S_{n+1}^z and S_n^- always commute, that $S_n^z|0\rangle = 1/2|0\rangle$, using the periodicity of the chain and shifting the sums where it is necessary, we obtain

$$E_0|\psi\rangle + \frac{-J}{2}\left(e^{ik} + e^{-ik} - 2\right)|\psi\rangle = E|\psi\rangle, \tag{2.12}$$

so that $E = E_0 + J(1 - \cos k)$. The vectors (2.6) represent the so called *magnon* excitations.

We now turn to the $R = 2$ sector, where a generic state can be written as

$$|\psi\rangle = \sum_{1 \le n_1 < n_2 \le N} f(n_1, n_2)|n_1, n_2\rangle, \tag{2.13}$$

where $|n_1, n_2\rangle \equiv S_{n_1}^- S_{n_2}^-|0\rangle$, i.e. there are two down spins respect to the vacuum state. In order to $|\psi\rangle$ being an eigenstate of the Hamiltonian we will find a relation between the eigenvalue E and $f(n_1, n_2)$. To this purpose we don't use directly the commutation relations, but we calculate how each operator acts on the basis state. First of all we have to distinguish the case in which the down spins are not adjacent and case in which they are. Taking into account the (2.3a,b) we have that

$$\sum_{n=1}^{N} S_n^+ S_{n+1}^-|m_1, m_2\rangle = \begin{cases} |m_1, m_1 + 2\rangle & \text{if } m_2 = m_1 \pm 1, \\ |m_1 + 1, m_2\rangle + |m_1, m_2 + 1\rangle & \text{otherwise.} \end{cases} \tag{2.14}$$

To justify the latter equation we see that the operator $\sum_n S_n^+ S_{n+1}^-$ acts non-trivially on $|m_1, m_2\rangle$ only when $n = m_1$ and when $n = m_2$, if m_1 and m_2 are not consecutive and only when $n = m_1 + 1$ in the other cases.

With similar argument we have

$$\sum_{n=1}^{N} S_n^- S_{n+1}^+ |m_1, m_2\rangle = \begin{cases} |m_1 - 1, m_1 + 1\rangle & \text{if } m_2 = m_1 \pm 1, \\ |m_1 - 1, m_2\rangle + |m_1, m_2 - 1\rangle & \text{otherwise.} \end{cases} . \tag{2.15}$$

The $\sum_n S_n^z S_{n+1}^z$ operator gives a contribution $-1/4$ for each pair of misaligned adjacent spins and a contribution $1/4$ for all other adjacent pairs spins. If the two down spins are not adjacent, there are four pairs of misaligned spins; if the down spins are adjacent there are two such pairs. In this way we deduced that

$$\sum_{n=1}^{N} S_n^z S_{n+1}^z |m_1, m_2\rangle = \begin{cases} \left(\frac{N-2}{4} - \frac{1}{2}\right) |m_1, m_1 + 1\rangle & \text{if } m_2 = m_1 \pm 1, \\ \left(\frac{N-4}{4} - 1\right) |m_1, m_2\rangle & \text{otherwise.} \end{cases} \tag{2.16}$$

Now we want to find a relation between the eigenvalue E of $|\psi\rangle$ and $f(n_1, n_2)$, so that $|\psi\rangle$ is an eigenstate of H. To this aim it is convenient to split the case of adjacent down spins from the others:

$$H|\psi\rangle = \sum_{n_2 > n_1 + 1} \alpha(n_1, n_2) |n_1, n_2\rangle + \sum_{n=1}^{N} \beta(n_1, n_2) |n, n + 1\rangle. \tag{2.17}$$

Demanding $|\psi\rangle$ to be an eigenstate with eigenvalue E is equivalent to have:

$$\alpha(n_1, n_2) = E f(n_1, n_2) \quad \text{for} \quad n_2 > n_1 + 1 \tag{2.18}$$

$$\beta(n, n + 1) = E f(n, n + 1). \tag{2.19}$$

It follows that

$$
\begin{aligned}
-\frac{2}{J}H|\psi\rangle = \sum_{n_2>n_1} f(n_1,n_2) \sum_{m=1}^{N} \left(S_m^+ S_{m+1}^- + S_m^- S_{m+1}^+ + 2S_m^z S_{m+1}^z\right)|n_1,n_2\rangle = \\
\sum_{n_2>n_1+1} f(n_1,n_2)\left(|n_1+1,n_2\rangle + |n_1,n_2+1\rangle + |n_1-1,n_2\rangle\right. \\
\left. +|n_1,n_2-1\rangle + \frac{N-8}{2}|n_1,n_2\rangle\right) \\
+\sum_{n=1}^{N} f(n,n+1)\left(|n,n+2\rangle + |n-1,n+1\rangle + \frac{L-4}{2}|n,n+1\rangle\right) \\
= \sum_{n_2>n_1} f(n_1-1,n_2)|n_1,n_2\rangle + \sum_{n_2>n_1+2} f(n_1,n_2-1)|n_1,n_2\rangle + \\
\sum_{n_2>n_1+2} f(n_1+1,n_2)|n_1,n_2\rangle + \sum_{n_2>n_1} f(n_1,n_2+1)|n_1,n_2\rangle \\
+ \sum_{n_2>n_1+2} \frac{N-8}{2}|n_1,n_2\rangle \\
+\sum_{n=1}^{N} f(n,n+1)\left(|n,n+2\rangle + |n-1,n+1\rangle + \frac{L-4}{2}|n,n+1\rangle\right), \quad (2.20)
\end{aligned}
$$

where in the last step we shifted the first sum so as to get $|n_1, n_2\rangle$ in every term. We now change the sums over n_1 and n_2 such that all these sums all run over $n_2 > n_1 + 1$, remembering that it is necessary to subtract the terms that we add in doing so and vice-versa.

$$
\begin{aligned}
-\frac{2}{J}H|\psi\rangle = & \sum_{n_2>n_1+1}\Big(f(n_1-1,n_2)+f(n_1,n_2-1)+f(n_1+1,n_2)+ \\
& f(n_1,n_2+1)+\frac{N-8}{2}f(n_1,n_2)\Big)|n_1,n_2\rangle \\
& +\sum_{n=1}^{N}\big(f(n-1,n+1)|n,n+1\rangle - f(n,n+1)|n,n+2\rangle \\
& -f(n+1,n+2)|n,n+2\rangle + f(n,n+2)|n,n+1\rangle\big) \\
& +\sum_{n=1}^{N} f(n,n+1)\left(|n,n+2\rangle + |n-1,n+1\rangle + \frac{L-4}{2}|n,n+1\rangle\right) \\
= & \sum_{n_2>n_1+1}\Big(f(n_1-1,n_2)+f(n_1,n_2-1)+f(n_1+1,n_2)+ \\
& f(n_1,n_2+1)+\frac{N-8}{2}f(n_1,n_2)\Big)|n_1,n_2\rangle \\
& +\sum_{n=1}^{N}\left(f(n-1,n+1)+f(n,n+2)+\frac{L-4}{2}+f(n,n+1)\right)|n,n+1\rangle.
\end{aligned}
\tag{2.21}
$$

Comparing the coefficients of this equation with $\alpha(n_1,n_2)$ and $\beta(n,n+1)$ and remembering the values of E_0 we obtain

$$
\begin{aligned}
2(E-E_0)f(n_1,n_2) = J(&4f(n_1,n_2)-f(n_1-1,n_2)-f(n_1,n_2-1)- \\
&f(n_1+1,n_2)-f(n_1,n_2+1)) \quad for \quad n_2>n_1+1
\end{aligned}
\tag{2.22}
$$

and

$$
2(E-E_0)f(n,n+1) = J(2f(n,n+1)-f(n-1,n+1)-f(n,n+2)).
\tag{2.23}
$$

The trick to determine the coefficients $f(n_1,n_2)$ is due to Bethe and is the first step to implement the so called coordinate Bethe Ansatz technique. The idea is to make an attempt with

$$
f(n_1,n_2) = Ae^{i(k_1n_1+k_2n_2)} + A'e^{i(k_1n_2+k_2n_1)}.
\tag{2.24}
$$

With this ansatz the first condition on the eigenvalue E and $f(n_1, n_2)$ is automatically satisfied by

$$E = E_0 + J \sum_{j=1,2} (1 - \cos k_j) \tag{2.25}$$

which also gives a relation between the coefficients A and $A^{'}$:

$$\frac{A}{A'} \equiv e^{i\theta} = -\frac{e^{i(k_1+k_2)} + 1 - 2e^{ik_1}}{e^{i(k_1+k_2)} + 1 - 2e^{ik_2}}. \tag{2.26}$$

This relation can be put in the real form

$$2 \cot \frac{\theta}{2} = \cot \frac{k_1}{2} - \cot \frac{k_2}{2}, \tag{2.27}$$

as is shown in Appendix A.

Imposing the periodic boundary condition $f(n_1, n_2) = f(n_2, n_1 + N)$, we derive the relation between the phase θ and the momentum k_1 and k_2:

$$e^{ik_1 N} = e^{i\theta}, \qquad e^{ik_2 N} = e^{-i\theta}. \tag{2.28}$$

Taking the logarithm of the last equations and taking into account the polydromicity of complex logarithm, we obtain the fundamental form of the Bethe equations:

$$Nk_1 = 2\pi I_1 + \theta, \qquad Nk_2 = 2\pi I_2 - \theta, \tag{2.29}$$

where $I_j = \{0, 1, ..., N-1\}$.

The total momentum of this state is

$$K = k_1 + k_2 = \frac{2\pi}{N}(I_1 + I_2). \tag{2.30}$$

This state can be interpreted as the interaction of two magnons with momenta k_1 and k_2 and phase shift θ. Since the momentum can have real or complex solutions, the magnons either scatter off each other or form bound states.

2.1.2 The generic solution

The case with generic R overturned spins is a generalization of the case $R = 2$, so the first step is writing the state of the system as

$$|\psi\rangle = \sum_{1 \le n_1 < ... < n_R \le N} f(n_1, ..., n_R) |n_1, ..., n_R\rangle. \tag{2.31}$$

Knowing the ansatz for two overturned spins, we can use the following expression for the coefficients $f(n_1, ..., n_R)$:

$$f(n_1, ..., n_R) = \sum_{\mathcal{P}} e^{i\sum_{j=1}^{R} k_{\mathcal{P}j} n_j + \frac{i}{2}\sum_{l<j}\theta(k_{\mathcal{P}l}, k_{\mathcal{P}j})}, \tag{2.32}$$

where the sum extends over all $R!$ permutations $\mathcal{P}$ of the assignments of momenta to each overturned spin and where we introduce the antisymmetric phase shift $\theta(k_l, k_j) = -\theta(k_j, k_l)$.

The energy eigenvalue becomes

$$E = E_0 + J\sum_{j=1}^{R}(1 - \cos k_j). \tag{2.33}$$

From the consistency equations for the coefficients $f(n_1, ..., n_R)$ the phase shift is related to momenta as

$$e^{i\theta(k_j, k_l)} = -\frac{e^{i(k_j+k_l)} + 1 - 2e^{ik_j}}{e^{i(k_j+k_l)} + 1 - 2e^{ik_l}} \tag{2.34}$$

or equivalently

$$2\cot\frac{\theta(k_j, k_l)}{2} = \cot\frac{k_j}{2} - \cot\frac{k_l}{2}, \qquad j, l = 1, ..., R. \tag{2.35}$$

The periodic boundary condition $f(n_1, ..., n_R) = f(n_2, ..., n_R, n_1 + N)$ allows to write the Bethe equations

$$Nk_j = 2\pi I_j + \sum_{l\neq j}\theta(k_j, k_l), \qquad j = 1, ..., R, \tag{2.36}$$

where $I_j \in \{0, 1, ..., N-1\}$.

Now it is convenient to parametrize the momenta in a different way, introducing the *rapidities* λ_j:

$$\cot\frac{k_j}{2} = \lambda_j \tag{2.37}$$

or using the complex representation of cotangent and the identity $\arctan(x) + \operatorname{arcot}(x) = \pi/2$

$$k_j = \frac{1}{i}\ln\frac{\lambda_j + i}{\lambda_j - i} = \pi - \theta_1(\lambda_j), \tag{2.38}$$

where

$$\theta_n(\lambda) \equiv 2 \arctan \frac{\lambda}{n}. \tag{2.39}$$

The energy and the momentum k of an individual magnon are

$$p_0(\lambda) = \frac{1}{i} \ln \frac{\lambda + i}{\lambda - i} = k, \tag{2.40}$$

$$\epsilon_0(\lambda) = -J \frac{dk}{d\lambda} = \frac{2J}{\lambda^2 + 1} = J(1 - \cos k). \tag{2.41}$$

The subscript 0 reflects what we will call *0-complex* in the next subsection.

2.1.3 String hypothesis

Now we want to analyze the interesting advantages of using the rapidities parametrization. Using the complex representation of cotangent and knowing that $\cot(\theta/2) = (\lambda_1 - \lambda_2)/2$, in the $R = 2$ case the (2.29) become

$$\left(\frac{\lambda_1 + i}{\lambda_1 - i} \right)^N = \frac{\lambda_1 - \lambda_2 + 2i}{\lambda_1 - \lambda_2 - 2i}, \tag{2.42}$$

$$\left(\frac{\lambda_2 + i}{\lambda_2 - i} \right)^N = \frac{\lambda_2 - \lambda_1 + 2i}{\lambda_2 - \lambda_1 - 2i}. \tag{2.43}$$

If $Im(\lambda_1) \neq 0$, the left hand side (LHS) in the first of the preceding equations grows (or decrease) exponentially in N. Therefore in the thermodynamic limit the LHS is strictly zero or infinity and the right hand side (RHS) has to do the same. So we must have

$$\lambda_1 - \lambda_2 = \pm 2i \implies \lambda_{1,2} = \lambda \pm i. \tag{2.44}$$

The energy and momentum of this state are:

$$p_{1/2}(\lambda) = p_0(\lambda + i) + p_0(\lambda - i) = \frac{1}{i} \ln \frac{\lambda + 2i}{\lambda - 2i} = k, \tag{2.45}$$

$$\epsilon_{1/2}(\lambda) = \epsilon_0(\lambda + i) + \epsilon_0(\lambda - i) = -J \frac{dk}{d\lambda} = \frac{4J}{\lambda^2 + 4} = \frac{J}{2}(1 - \cos p_{1/2}). \tag{2.46}$$

For $R > 2$ the Bethe equations become in terms of rapidities

$$\left(\frac{\lambda_j + i}{\lambda_j - i} \right)^N = \prod_{k \neq j}^{R} \frac{\lambda_j - \lambda_k + 2i}{\lambda_j - \lambda_k - 2i}. \tag{2.47}$$

Now we assume that the so called string hypothesis is valid, i.e. the complexes solutions of the Bethe equations are organized into *complexes* (or *strings*) of $2M+1$ rapidities characterized by the same real value λ_M and different, equidistant, imaginary parts. M can assume half-integer values $0, 1/2, 1, ...$ and rapidities have the structure

$$\lambda_m^{(M)} = \lambda_M + 2im, \qquad m = -M, -M+1, ..., M-1, M. \tag{2.48}$$

Denoting by ν_M the number of comlexes of length M, a state with a given magnetization satisfies

$$R = \sum_M (2M+1)\nu_M. \tag{2.49}$$

The energy and momentum of a M-complex are obtained by summing over all rapidities within one string. By construction there are a lot of cancellations and the two quantities have a simple form:

$$p_M(\lambda_M) = \sum_{m=-M}^{M} p_0(\lambda_M + 2im) = \frac{1}{i}\ln\frac{\lambda_M + i(2M+1)}{\lambda_M - i(2M+1)} = \pi - \theta_{2M+1}(\lambda_M), \tag{2.50}$$

$$\epsilon_M(\lambda_M) = \sum_{m=-M}^{M} \epsilon_0(\lambda_M + 2im) = \frac{2J(2M+1)}{\lambda_M^2 + (2M+1)^2} = \frac{J}{2M+1}(1 - \cos p_M). \tag{2.51}$$

Since we can interpret each rapidity as a particle and since rapidities are strictly connected with phase shift we can obtain scattering matrix elements through simple products. The scattering matrix element of a M-complex with a single magnon (0-complex) can be obtained considering the scattering of each particle with the magnon in order to fall in the case $R = 2$ and use both value of $\lambda_{1,2}$ as in (2.44). So we have

$$\begin{aligned} S_{0,M}(\lambda_0 - \lambda_M) = S_{0,M}(\lambda) &= \prod_{m=-M}^{M} \frac{(\lambda+i) + 2im + i}{(\lambda+i) + 2im - i} \prod_{m=-M}^{M} \frac{(\lambda-i) + 2im + i}{(\lambda-i) + 2im - i} \\ &= \frac{\lambda + 2iM}{\lambda - 2iM}\frac{\lambda + 2i(M+1)}{\lambda - 2i(M+1)}. \end{aligned} \tag{2.52}$$

For the scattering of two complexes of length M and M' as written in the lectures of Faddeev [5] we have

$$S_{M,M'}(\lambda) = \prod_{L=|M-M'|}^{M+M'} S_{0,L}(\lambda). \tag{2.53}$$

Now we want to describe the system in terms of the number of complexes ν_M for each type M and by the real part of the j-th complex of length M $\lambda_{M,j}$, where $j = 1, ..., \nu_M$. Rearranging the (2.47) in terms of complexes we can write

$$e^{ip_M(\lambda_{M,j})N} = \prod_{M'} \prod_{\substack{j' \\ (M',j')\neq(M,j)}}^{\nu_{M'}} S_{M,M'}(\lambda_{M,j} - \lambda_{M',j'}), \qquad \forall M; j = 1, ..., \nu_M. \tag{2.54}$$

Taking the logarithm of the latest equation, introducing the (half-)integer (according to the value of M) quantum number $I_{M,j}$ due to the polidromicity of logarithm and making use of the already known identity

$$\frac{1}{i} \ln \frac{\lambda + in}{\lambda - in} = \pi - 2 \arctan \frac{\lambda}{n} = \pi - \theta_n(\lambda), \tag{2.55}$$

we have

$$N\theta_{2M+1}(\lambda_{M,j}) = 2\pi I_{M,j} + \sum_{(M',j')\neq(M,j)} \theta_{M,M'}(\lambda_{M,j} - \lambda_{M',j'}), \tag{2.56}$$

where

$$\theta_{M,M'}(\lambda) = \sum_{L=|M-M'|}^{M+M'} \left(\theta_{2L}(\lambda) + \theta_{2L+2}(\lambda)\right), \tag{2.57}$$

and $L = 0$ is intended to be excluded.

In string hypothesis each state is thus characterised by the number of complexes ν_M and the Bethe numbers $I_{M,j}$. However not all quantum numbers are allowed. In fact since the chain is limited and is a one-dimensional lattice the momenta are constrained within a Brillouin zone. In particular an infinite rapidity $\lambda_M^\infty = \infty$ corresponds to a momentum at the edge of Brillouin zone and defines a natural bound for the quantum numbers. Since $\arctan(\pm\infty) = \pm\pi/2$, inverting (2.56) we have

$$I_M^{(\infty)} = -\sum_{M' \neq M} \left(2\min(M', M) + 1\right) \nu_M - \left(2M + \frac{1}{2}\right)(\nu_M - 1) + \frac{N}{2}. \quad (2.58)$$

The presence of $\min(M', M)$ is due to the absolute value in the subscript of the sum in (2.57). The second term in RHS is obtained by considering the sum on j with $M = M'$ $(1/(2\pi)\theta_{M,M}(\infty) = 1/2\pi \sum_{L=1}^{2M}(\pi + \pi) + \pi = 2M + 1/2)$ and since $j \neq j'$ we have $\nu_M - 1$ terms. Since adding a M-complex shifts this boundary by $1/(2\pi)\theta_{M,M}(\infty)$, the maximum quantum number that characterizes a finite rapidity, before it joins the rapidities at the edge, is

$$I_M^{max} = I_M^{(\infty)} - \left(2M + \frac{1}{2} - \frac{1}{2}\right) = \frac{N-1}{2} - \sum_{M'} J(M, M')\nu_{M'}, \quad (2.59)$$

where

$$J(M, M') = \begin{cases} 2\min(M', M) + 1 & \text{if } M \neq M', \\ 2M + \frac{1}{2}, & \text{if } M = M'. \end{cases} \quad (2.60)$$

and where the additional shift $1/2$ in (2.59) takes into account that adding a rapidity shifts the Bethe numbers from integer to half-integer and vice-versa.

Since the inverse of the tangent is an odd function, we have that

$$I_M^{max} = -I_M^{min}, \quad (2.61)$$

which implies that the allowed Bethe numbers (we call them vacancies) for a M-complexes are

$$P_M = 2I_M^{max} + 1 = N - 2\sum_{M'} J(M, M')\nu_{M'} \quad (2.62)$$

2.1.4 The Anti-ferromagnetic case: $J = -1$

To study the anti-ferromagnetic regime we have to find firstly the ground state which is intuitively different from the completely polarized ground state of section 2.1.1. The ground state is obtained by putting the spins alternatively up and down, so that $S^z = 0$ and $R = N/2$. From (2.49) the ground state configuration must be composed only by 0-type complexes, i.e.

$$\nu_0 = \frac{N}{2}, \qquad \nu_M = 0, M \geq \frac{1}{2}. \quad (2.63)$$

The number of vacancies are

$$P_0 = N - 2J(0,0)\nu_0 = \frac{N}{2}. \tag{2.64}$$

which equals the number of particles. Hence

$$-\frac{N}{4} + \frac{1}{2} \leq I_{0,k} \leq \frac{N}{4} - \frac{1}{2}. \tag{2.65}$$

Excited states over ground state are constructed by progressively taking away particles from it and moving them into complexes, i.e. we will characterised the excited states by κ with

$$\nu_0 = \frac{N}{2} - \kappa. \tag{2.66}$$

For $\kappa = 1$ we cannot add any complexes (we can not overcome $R = N/2$) and so we have $R = N/2 - 1$, which implies $S^z = 1$. The number of vacancies in this case is

$$P_0 = N - 2 \cdot \frac{1}{2}\left(\frac{N}{2} - 1\right) = \frac{N}{2} + 1, \tag{2.67}$$

which exceeds the number of particles by two. This means that Bethe numbers of a state in this sector are all the allowed quantum numbers but two and the choice of this two holes characterizes the state.

For $\kappa = 2$ we have two possibilities: we can keep $\nu_M = 0$ for $M \geq 1/2$ and have a state with magnetization $S^z = 2$. The second possibility is to have $\nu_{1/2} = 1$ and $\nu_M = 0$ for $M \geq 1$, so that $R = N/2$ and $S^z = 0$. The vacancies are

$$P_0 = N - 2 \cdot \frac{1}{2}\left(\frac{N}{2} - 2\right) - 2J\left(0, \frac{1}{2}\right) = \frac{N}{2}, \tag{2.68}$$

$$P_{1/2} = N - 2 \cdot \frac{1}{2}\left(\frac{N}{2} - 2\right) J\left(0, \frac{1}{2}\right) - 2J\left(\frac{1}{2}, \frac{1}{2}\right) = 4 - 3 = 1. \tag{2.69}$$

The number of vacancies for quantum numbers allows for two holes, while there is no freedom for the $1/2$-complex, whose state is therefore fixed.

For generic κ, we can have configurations with

$$\nu_0 = \frac{N}{2} - \kappa, \qquad \nu_M = 0, M \geq \frac{1}{2} \implies R = \frac{N}{2} - \kappa, \tag{2.70}$$

with $P_0 = N/2 + \kappa$ vacancies, which give rise to 2κ holes and a total spin $S^z = \kappa$. In addition to these solutions, we can have states with smaller magnetization and an increasing of complexes.

We now analyze the states with $\kappa = 0, 1$.

Starting with $\kappa = 0$ and assuming that $N/2$ is odd for we have for the ground state

$$I_{0,j} = j, \qquad j = --\frac{N}{4} + \frac{1}{2}, -\frac{N}{4} + \frac{3}{2}, ..., \frac{N}{4} - \frac{1}{2}. \tag{2.71}$$

The equation (2.56) becomes

$$\arctan \lambda_j = \pi \frac{j}{N} + \frac{1}{N} \sum_k \arctan \left(\frac{\lambda_j - \lambda_k}{2} \right). \tag{2.72}$$

Taking the thermodynamic limit $N \to \infty$, the variable $x = j/N$ becomes continuous and limited in the range $-1/4 \leq x \leq 1/4$. Thus turning the sums into integrals and the roots λ_j into functions of x the preceding equation becomes

$$\arctan \lambda(x) = \pi x + \int_{-1/4}^{1/4} \arctan \left(\frac{\lambda(x) - \lambda(y)}{2} \right) dy. \tag{2.73}$$

Performing a change of variables $x \to \lambda(x)$, so that the (2.73) depends directly on rapidities and their density $\rho_0(\lambda) = \frac{dx}{d\lambda}$ and differentiating (2.73) with respect to λ we obtain a linear integral equation for the density $\rho_0(\lambda)$:

$$\rho_0(\lambda) = \frac{1}{\pi} \frac{1}{\lambda^2 + 1} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2}{(\lambda - \mu)^2 + 4} \rho_0(\mu) d\mu. \tag{2.74}$$

This integral equation can be solved by Fourier transform:

$$\tilde{\rho}_0(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\lambda} \rho_0(\lambda) d\lambda. \tag{2.75}$$

Using

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i\omega\lambda} \frac{n}{\lambda^2 + n^2} d\lambda = e^{-n|\omega|}, \tag{2.76}$$

which is obtained by residues theorem and the appropriate choice of the path integration according to the sign of ω, the integral equation reduces to

$$\tilde{\rho}_0(\omega)(1 + e^{-2|\omega|}) = e^{-|\omega|} \implies \tilde{\rho}_0(\omega) = \frac{1}{2} \text{sech}(\omega). \tag{2.77}$$

This yields

$$\rho_0(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\lambda} \tilde{\rho}_0(\omega) d\omega = \frac{1}{4 \cosh(\frac{\pi\lambda}{2})}. \tag{2.78}$$

The momentum and the energy of the ground state are then

$$K = N\int_{-\infty}^{\infty} p_0(\lambda)\rho_0(\lambda)d\lambda = \frac{\pi}{2}N, \tag{2.79}$$

$$E = E_0 + N\int_{-\infty}^{\infty} \epsilon_0(\lambda)\rho_0(\lambda) = N\left(\frac{1}{4} - \ln 2\right). \tag{2.80}$$

For the case $\kappa = 1$ we use again the continuous form of the Bethe equations, but we have to take into account that there are two holes. We can suppose that the empty quantum numbers are j_1 and j_2:

$$I_{0,j} = j + \theta_H(j - j_1) + \theta_H(j - j_2), \tag{2.81}$$

where

$$\theta_H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}. \tag{2.82}$$

The integral equation for the density, which we call ρ_1 for this case becomes

$$\rho_1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2}{(\lambda-\mu)^2+4}\rho_1(\mu)d\mu = \frac{1}{\pi}\frac{1}{\lambda^2+1} - \frac{1}{N}\left(\delta(\lambda-\lambda_1) + \delta(\lambda-\lambda_2)\right), \tag{2.83}$$

where $\lambda_{1,2}$ are the images of $x_1 = j_1/N$ $x_2 = j_2/N$ under the map $x \to \lambda(x)$. Using once more Fourier transform we can obtain the momentum and the energy of the state:

$$K = N\int_{-\infty}^{\infty} p_0(\lambda)\rho_1(\lambda)d\lambda = \frac{\pi}{2}N + k(\lambda_1) + k(\lambda_2), \tag{2.84}$$

$$E = E_0 + N\int_{-\infty}^{\infty} \epsilon_0(\lambda)\rho_1(\lambda) = N\left(\frac{1}{4} - \ln 2\right) + \epsilon(\lambda_1) + \epsilon(\lambda_2), \tag{2.85}$$

where

$$k(\lambda) \equiv \arctan\ \sinh(\frac{\pi\lambda}{2}), \qquad \epsilon(\lambda) \equiv \frac{\pi}{2\cosh(\frac{\pi\lambda}{2})}. \tag{2.86}$$

Combining the expressions of $\epsilon(\lambda)$ and $k(\lambda)$, we obtain the dispersion relation of these excitations from the ground state:

$$\epsilon(k) = \frac{\pi}{2}\cos k, \qquad -\frac{\pi}{2} \leq k \leq \frac{\pi}{2}. \tag{2.87}$$

2.2 XXZ chain

The Hamiltonian of the XXZ model is similar to XXX Hamiltonian, but for the introduction of a degree Δ of anisotropy along the z-axes. We also add a term, which takes into account the presence of an external magnetic field h. We have

$$H = -J\sum_{n=1}^{N}\left[\frac{1}{2}\left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+\right) + \Delta S_n^z S_{n+1}^z\right] - 2h\sum_{n=1}^{N} S_n^z, \tag{2.88}$$

with the analogous conventions and symbols of the XXX chain.

We proceed following the steps as for the preceding model. We define a magnetization $S^z \equiv \sum_{n=1}^{N} S_n^z$, which is conserved since it commutes with the Hamiltonian. The reference state $|0\rangle$ with all spins up ($S^z = N/2$) is an eigenstate of H with eigenvalue

$$E_0 = -\left(\frac{J\Delta}{4} + h\right)N. \tag{2.89}$$

As already done we can write the generic state with R spin-flips as

$$|\psi\rangle = \sum_{1\le n_1<\ldots<n_R\le N} f(n_1,\ldots,n_R)|n_1,\ldots,n_R\rangle. \tag{2.90}$$

with

$$|n_1,\ldots,n_R\rangle \equiv S_{n_1}^-\ldots S_{n_1}^-|0\rangle. \tag{2.91}$$

The ansatz for the coefficients $f(n_1,\ldots,n_R)$ is again

$$f(n_1,\ldots,n_R) = \sum_{\mathcal{P}} e^{i\sum_{j=1}^{R} k_{\mathcal{P}j} n_j + \frac{i}{2}\sum_{l<j}\tilde{\Theta}(k_{\mathcal{P}l},k_{\mathcal{P}j})}, \tag{2.92}$$

where the sum is over the $R!$ permutation $\mathcal{P}$ of momenta k_j.

The wave function with these coefficients is an eigenfuntion of H with eigenvalue

$$E = E_0 + (J\Delta + 2h)R - \sum_{j=1}^{R}\cos k_j, \tag{2.93}$$

if the phase shift $\tilde{\Theta}(k_j,k_l)$ satisfies

$$e^{i\tilde{\Theta}(k_j,k_l)} = -\frac{e^{i(k_j+k_l)} + 1 - 2\Delta e^{ik_j}}{e^{i(k_j+k_l)} + 1 - \Delta 2e^{ik_l}}. \tag{2.94}$$

We can redefine the phase shift for later convenience as

$$\Theta(k_j, k_l) \equiv \tilde{\Theta}(k_j, k_l) - \pi. \tag{2.95}$$

By imposing boundary conditions we obtain the Bethe equations

$$k_j N = 2\pi \tilde{I}_j - \sum_{l \neq j}^{R} \Theta(k_j, k_l), \qquad j = 1, ...R, \tag{2.96}$$

where $\tilde{I}_j$ are integer or half-integer.
Now we introduce the rapidities $\tilde{\lambda_j}$ to parametrize the momentum k_j:

$$e^{ik_j} = \frac{\sin\frac{\phi}{2}\left(\tilde{\lambda_j} + i\right)}{\sin\frac{\phi}{2}\left(\tilde{\lambda_j} - i\right)}, \qquad \text{or} \qquad \cot\frac{k_j}{2} = \coth\frac{\phi}{2}\tan\left(\frac{\phi\tilde{\lambda_j}}{2}\right). \tag{2.97}$$

The parameter ϕ (which is renamed γ in a particular case as we see right now) depends on the value of Δ:

- Uni-axial ferromagnet (gapped regime) $\Delta > 1$: $\Delta = \cosh\phi$ with $0 < \phi < \infty$;
- Planar paramagnet (gapless regime) $|\Delta| < 1$: $\Delta = -\cos\gamma$ with $0 < \gamma < \pi$;
- Uni-axial anti-ferromagnet $\Delta < -1$: $\Delta = -\cosh\phi$ with $0 < \phi < \infty$.

The cases $\Delta = \pm 1$ recover respectively the ferromagnetic and anti-ferromagnetic regimes in XXX model.
With $\Delta = \cosh\phi$ the Bethe equations become

$$N\theta_1(\lambda_j) = 2\pi I_j + \sum_{l \neq j}^{R} \theta_1(\lambda_j - \lambda_l), \qquad j = 1, ..., R, \tag{2.98}$$

where $\lambda = \phi\tilde{\lambda}$ and

$$\theta_n(\lambda) \equiv 2\arctan\left(\coth\frac{n\phi}{2}\tan\frac{\lambda}{2}\right). \tag{2.99}$$

In terms of rapidities the energy is given by

$$E = E_0 + 2hR + \sum_{j=1}^{R} \epsilon(\lambda_j), \tag{2.100}$$

with

$$\epsilon(\lambda) \equiv -\frac{J\sinh^2\phi}{\cosh\phi - \cos(\lambda)} = -J\sinh\phi\frac{d}{d\lambda}p(\lambda) \tag{2.101}$$

and

$$p(\lambda) \equiv \theta_1(\lambda) = k. \tag{2.102}$$

2.2.1 Uni-axial ferromagnet

For this regime we can introduce again the string hypothesis to group complex rapidities. So we assume that each solution of Bethe equations belongs to a M-type complex of rapidities:

$$\lambda_{M,j} = \lambda_M + 2i(M-j)\phi, \qquad j = 0, ..., 2M, \tag{2.103}$$

with $\lambda_M \in [-\pi, \pi]$.

Regarding each complex as an elementary excitation, its momentum and energy are:

$$p_M(\lambda_M) = \frac{1}{i}\sum_{j=0}^{2M}\ln\frac{\sin\frac{1}{2}(\lambda_{M,j}+i\phi)}{\sin\frac{1}{2}(\lambda_{M,j}-i\phi)} = \frac{1}{i}\ln\frac{\sin\frac{1}{2}(\lambda_M+i(2M+1)\phi)}{\sin\frac{1}{2}(\lambda_M-i(2M+1)\phi)}, \tag{2.104}$$

$$\epsilon_M(\lambda_M) = J\sum_{j=0}^{2M}\frac{\sinh^2\phi}{\cosh\phi-\cos\lambda_{M,j}} = J\frac{\sinh\phi\sinh[(2M+1)\phi]}{\cosh[(2M+1)\phi]-\cos\lambda_M}, \tag{2.105}$$

which combined give the dispersion relation

$$\epsilon_M(p_M) = J\frac{\sinh\phi}{\sinh[(2M+1)\phi]}\left(\cosh[(2M+1)\phi]-\cos p_M\right). \tag{2.106}$$

In the thermodynamic limit, the centers (real part) of each type of complex become dense on the interval $[-\pi/2, \pi/2]$ (with another convention for the interval of λ_M) and their distribution can be described by a set of continous densities $\rho_m(\lambda)$, where m denotes the M-complex, together with a set of densities for the corresponding holes $\rho_m^h(\lambda)$. It can be seen (as exposed in the famous monograph of Takahashi [26], that these densities satisfy the equations (known also as Thermodynamic Bethe Ansatz TBA equations)

$$\rho_m + \rho_m^h = a_m(\lambda) - \sum_{n=1}^{\infty}(a_{mn} \star \rho_n)(\lambda), \tag{2.107}$$

for $m \geq 1$, where the convolution is defined by

$$(f \star g)(\lambda) = \int_{-\pi/2}^{\pi/2} d\mu f(\lambda - \mu) g(\mu). \tag{2.108}$$

The kernels are

$$a_{mn}(\lambda) = (1 - \delta_{mn}) a_{|m-n|}(\lambda) + 2a_{|m-n|+2}(\lambda) + ... + 2a_{m+n-2}(\lambda) + a_{m+n}(\lambda), \tag{2.109}$$

with

$$a_m(\lambda) = \frac{1}{2\pi} \frac{2\sinh(m\phi)}{\cosh(m\phi) - \cos(2\lambda)}. \tag{2.110}$$

A convenient rewriting of (2.107) is in the form

$$\rho_m(1 + \eta_m) = s \star (\eta_{m-1}\rho_{m-1} + \eta_{m+1}\rho_{m+1}), \tag{2.111}$$

for $m \geq 1$ and where $\eta_m \equiv \rho_m^h/\rho_m$. As convention we put $\eta_0 = 1$ and $\rho_0(\lambda) = \delta(\lambda)$.

The kernel $s(\lambda)$ is

$$s(\lambda) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \frac{e^{-2ik\lambda}}{\cosh(k\phi)}. \tag{2.112}$$

Taking into account the definition of $\eta_m(\lambda)$ and using the Euler formula for trigonometric functions in order to use only positive integers in the sum of $s(\lambda)$, we obtain another useful form of (2.107), i.e.

$$\rho_m(1 + \eta_m) = \delta_{k,1} d + d \star (\rho_{m-1}^h + \rho_{m+1}^h), \tag{2.113}$$

where

$$d(\lambda) = 1 + 2\sum_{n=1}^{\infty} \frac{\cos(2n\lambda)}{\cosh(\phi n)}. \tag{2.114}$$

2.2.2 Planar paramagnet

For this regime the parametrization of momenta is
$k(\lambda) = \theta_1(\lambda) = 2\arctan\left(\cot\left(\frac{n\gamma}{2}\right)\tanh\left(\frac{\lambda}{2}\right)\right)$, so that real rapidities generate real momenta constrained on the interval $[-(\pi-\gamma), \pi-\gamma]$. Momenta outside of this interval are allowed, provided for rapidities of the form $\lambda + i\pi$, with λ real.

Proceeding as in XXX model, for the ground state we obtain an integral equation

$$\rho_0(\lambda) + \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} \mathcal{K}(\lambda-\mu)\rho_0(\mu)d\mu = \frac{1}{2\pi}\theta_1'(\lambda), \tag{2.115}$$

where

$$\mathcal{K}(\lambda) \equiv \frac{d}{d\lambda}\theta_2(\lambda) = \frac{\sin(2\gamma)}{\cosh\lambda - \cos 2\gamma}, \tag{2.116}$$

and Λ is a parameter which tends to infinity if the external magnetic field h is zero. Therefore we have

$$S_z = \frac{N}{2} - N\int_{-\Lambda}^{\Lambda}\rho_0(\lambda)d\lambda, \tag{2.117}$$

$$E = E_0 - N\int_{-\Lambda}^{\Lambda}\epsilon_0(\lambda)\rho_0(\lambda)d\lambda. \tag{2.118}$$

At $h = 0$

$$\rho_0(\lambda) = \frac{1}{8\gamma}\mathrm{sech}\left(\frac{\pi\lambda}{2\gamma}\right). \tag{2.119}$$

The excitations from the ground state are created by removing κ real rapidities from the ground state itself, generating two holes (*spinons*) among the allowed vacancies. At $h = 0$ the contributions of spinons to total momentum and energy are

$$k(\lambda) = \arctan\frac{\pi\lambda}{2\gamma}, \qquad \epsilon(\lambda) = J\frac{\pi\sin(\gamma)}{2\gamma}\frac{1}{\cosh\frac{\pi\lambda}{2\gamma}}, \tag{2.120}$$

which give the dispersion relation

$$\epsilon(k) = J\frac{\pi\sin(\gamma)}{2\gamma}\cos k. \tag{2.121}$$

Additional excitation can be generated by placing some of the κ rapidities removed from real axes on the $i\pi$ axis. We do not enter into details for this case.

String solutions are also allowed for paramagnetic phase, but their treatment is a bit tricky, so we give only the principal results. The rapidities are

$$\lambda_{M,j} = \lambda_M + \frac{1-\eta}{2}\pi + 2i(M-j)\gamma, \qquad j = 0, ..., 2M, \tag{2.122}$$

where $\eta = \pm 1$ is called the parity of the string. The energy and the momentum of an M-complexes are

$$p_M(\lambda_M) = \frac{1}{i}\ln\frac{\sinh\frac{1}{2}\left(i(2M+1)\gamma - \lambda_M - i\frac{1-\eta}{2}\pi\right)}{\sinh\frac{1}{2}\left(i(2M+1)\gamma - \lambda_M + i\frac{1-\eta}{2}\pi\right)}, \tag{2.123}$$

$$\epsilon_M(\lambda_M) = -J\frac{\sin\gamma\sin[(2M+1)\gamma]}{\eta\cosh\lambda_M - \cos[(2M+1)\gamma]}. \tag{2.124}$$

2.2.3 Uni-axial anti-ferromagnet

For $h = 0$ the ground state has zero magnetization and is given by $N/2$ real magnons satisfying the Bethe equations. Energy excitations are 2κ spinons generated by removing κ rapidities from ground state. Each spinon contributes with energy

$$\epsilon(k) = h + J\frac{\sinh\phi}{\pi}I(k)\sqrt{1-k^2\cos^2 k}, \tag{2.125}$$

where $I(k)$ is the complete elliptic integral.

Chapter 3

Algebraic Bethe Ansatz

In this chapter we analyze the Algebraic Bethe Ansatz (ABA), one of the tools of the so-called Quantum Inverse Scattering Method, which is used to solve integrable systems. We proceed constructing the fundamental elements of ABA, which resemble some aspects of classical inverse scattering, taking as a constant example the XXX model. From now on we use the following Hamiltonian for the XXX model, which differs from that used in the preceding chapter, only for a factor 1/4 and for imposing $J = -1$:

$$H = \sum_{k=1}^{N} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right), \tag{3.1}$$

where

$$\sigma_k^\alpha = \mathbb{I} \otimes ... \otimes \sigma^\alpha \otimes ... \otimes \mathbb{I} \tag{3.2}$$

are $2^N \times 2^N$ matrices and σ^α are 2×2 Pauli matrices.

3.1 Monodromy matrix and transfer matrix

Let be $T(u)$ a 2×2 matrix, whose entries are operators which act on some Hilbert space $\mathcal{H}$. We call it *monodromy matrix*. It is usually denoted as

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{3.3}$$

We demand that $T(u)$ satisfies the following relation

$$R_{12}(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u-v), \tag{3.4}$$

where $R(u-v)$ is the solution of the YB equation found in appendix B :

$$R(u-v) = \begin{pmatrix} u-v+c & 0 & 0 & 0 \\ 0 & u-v & c & 0 \\ 0 & c & u-v & 0 \\ 0 & 0 & 0 & u-v+c \end{pmatrix}. \tag{3.5}$$

The relation above takes sense in the tensor product $V_1 \otimes V_2 \otimes \mathcal{H}$, where $V_k \sim \mathbb{C}^2$. $R_{12}(u-v)$ acts non-trivially in $V_1 \otimes V_2$, while T_k acts non-trivially in $V_k \otimes \mathcal{H}$. In this way the (3.4) can be seen as a relation between 4×4 matrices:

$$R(u-v)(T(u) \otimes \mathbb{I})(\mathbb{I} \otimes T(u)) = (\mathbb{I} \otimes T(u))(T(u) \otimes \mathbb{I})R(u-v). \tag{3.6}$$

Writing explicitly the components of matrices in (3.6), i.e.

$$R(u-v)^{jk,\alpha\beta} = (u-v)\delta^{jk}\delta^{\alpha\beta} + c\delta^{j\beta}\delta^{k\alpha}, \tag{3.7}$$

$$(T_1(u))^{jk,\alpha\beta} = T^{jk}(u)\delta^{\alpha\beta}, \tag{3.8}$$

$$(T_2(v))^{jk,\alpha\beta} = T^{\alpha\beta}(v)\delta^{jk}, \tag{3.9}$$

we obtain the commutation relations for the monodromy matrix's entries:

$$[T^{ij}(u), T^{kl}(v)] = \frac{c}{u-v}(T^{kj}(v)T^{il}(u) - T^{kj}(u)T^{il}(v)). \tag{3.10}$$

We deduce immediately that $[T^{ij}(u), T^{ij}(v)] = 0$, i.e. that each operator commutes with itself. This property will arise again in the frame of Algebraic Bethe Ansatz, as we will see later on.

Turning to (3.4), multiplying it for $R_{12}^{-1}(u-v)$ from the right, taking the trace with respect to $V_1 \otimes V_2$ and using the cyclic property of trace we obtain

$$tr_{12}T_1(u)T_2(v) = tr_{12}T_2(v)T_1(u). \tag{3.11}$$

Since the trace of a tensor product is the product of the traces, we have

$$\mathcal{T}(u)\mathcal{T}(v) = \mathcal{T}(v)\mathcal{T}(u), \tag{3.12}$$

where

$$\mathcal{T}(u) = trT(u) = A(u) + D(u). \tag{3.13}$$

$\mathcal{T}(u)$ is called *transfer matrix* and should be intended in the sense of an operator.

Now the question on which is the form of $T(u)$ is essential. Remembering the YB equation, we note that R-matrix itself satisfies (3.4). Indeed from

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v), \quad (3.14)$$

we can say that $T(u) = R(u-v)$. In this case we consider $\mathcal{H} \sim \mathbb{C}^2$ and as usual $V_k \sim \mathbb{C}^2$. Setting $w = c/2$

$$R(u-\frac{c}{2}) = \begin{pmatrix} u-\frac{c}{2} & 0 & 0 & 0 \\ 0 & u-\frac{c}{2} & c & 0 \\ 0 & c & u-\frac{c}{2} & 0 \\ 0 & 0 & 0 & u+\frac{c}{2} \end{pmatrix} = \begin{pmatrix} u+\frac{c}{2}\sigma^z & c\sigma^- \\ c\sigma^+ & u-\frac{c}{2} \end{pmatrix}, \quad (3.15)$$

where $2\sigma^{\pm} = \sigma^x \pm i\sigma^y$. Thus we can set

$$T(u) = \begin{pmatrix} u+\frac{c}{2}\sigma^z & c\sigma^- \\ c\sigma^+ & u-\frac{c}{2}\sigma^z \end{pmatrix}. \quad (3.16)$$

This observation indicates a way to construct a more general form of monodromy matrix.

Let $V_1 \otimes V_2 \otimes \mathcal{H}_a$ and $V_1 \otimes V_2 \otimes \mathcal{H}_b$ two triple tensor product spaces, where $V_k \sim \mathbb{C}^2$, $k = 1, 2$. Let be A_k and B_k two matrices, acting non-trivially only on $V_k \otimes \mathcal{H}_a$ and $V_k \otimes \mathcal{H}_b$ respectively. With these notations we demonstrate a useful proposition.

Proposition 3.1.1. *If both matrices A and B satisfies* (3.4)

$$R_{12}(u_1-u_2)A_1(u_1)A_2(u_2) = A_2(u_2)A_1(u_1)R_{12}(u_1-u_2), \quad (3.17)$$
$$R_{12}(u_1-u_2)B_1(u_1)B_2(u_2) = B_2(u_2)B_1(u_1)R_{12}(u_1-u_2), \quad (3.18)$$

then their product AB also satisfies (3.4).

Proof. Substituting the product AB in the lhs of (3.4) and knowing that matrices which act in different spaces commute we have

$$R_{12}(u_1-u_2)A_1(u_1)B_1(u_1)A_2(u_2)B_2(u_2) = R_{12}(u_1-u_2)A_1(u_1)A_2(u_2)B_1(u_2)B_2(u_2). \quad (3.19)$$

Thanks to (3.17) we can move $R_{12}(u_1-u_2)$ to the right, so that

$$R_{12}(u_1-u_2)A_1(u_1)B_1(u_1)A_2(u_2)B_2(u_2) = A_2(u_2)A_1(u_1)B_2(u_2)B_1(u_2)R_{12}(u_1-u_2). \quad (3.20)$$

Swapping $A_1(u_1)$ and $B_2(u_2)$, we obtain

$$R_{12}(u_1-u_2)A_1(u_1)B_1(u_1)A_2(u_2)B_2(u_2) = A_2(u_2)B_2(u_2)A_1(u_1)B_1(u_1)R_{12}(u_1-u_2). \tag{3.21}$$

□

Consider now the space $\mathcal{H}$ as the tensor product $V_1 \otimes ... \otimes V_n \otimes ... \otimes V_N$, where $V_n \sim \mathbb{C}^2$.

We can construct the *Lax-operator* (L-operator) $L_n(u)$ as follows:

$$L_n(u) = \begin{pmatrix} u + \frac{c}{2}\sigma_n^z & c\sigma_n^- \\ c\sigma_n^+ & u - \frac{c}{2}\sigma_n^z \end{pmatrix}, \tag{3.22}$$

where the σ_n^α are defined as in (3.2). We note that each entry of L-operator acts non-trivially only on V_n. Since $L_n(u)$ has the same form of $T(u)$ in (3.16), its matrix elements certainly satisfy the commutation relation (3.10), so that for each $L_n(u)$ (3.6) is valid

$$R(u-v)(L_n(u)\otimes\mathbb{I})(\mathbb{I}\otimes L_n(v)) = (\mathbb{I}\otimes L_n(v))(L_n(u)\otimes\mathbb{I})R(u-v). \tag{3.23}$$

We define the monodromy matrix as follows:

$$T(u) = L_N(u)...L_1(u). \tag{3.24}$$

Applying several times the proposition 3.1.1, we deduce that $T(u)$ enjoys equation (3.6).

Now we show that from (3.24) it is possible to derive the XXX model's Hamiltonian. First of all it is convenient to write the Hamiltonian in the following equivalent form:

$$H = 2\sum_{k=1}^{N} P_{kk+1} - N\mathbb{I}, \tag{3.25}$$

where $\mathbb{I} = \mathbf{1}\otimes ...\otimes \mathbf{1}$, $\mathbf{1}$ is the 2×2 identity and P is the permutation operator defined in Appendix B. This can be shown, noting that the operator

$$h_{kk+1} = \sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \sigma_k^z\sigma_{k+1}^z \tag{3.26}$$

acts non-trivially only in $V_k \otimes V_{k+1}$, so that it can be express as a 4×4 matrix

$$h_{kk+1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 2P_{kk+1} - \mathbf{1}\otimes\mathbf{1}. \tag{3.27}$$

We prove that the Hamiltonian of XXX model can be obtained from the transfer matrix of the monodromy matrix (3.24) by the following formula:

$$H = 2c\frac{d\mathcal{T}(u)}{du}\mathcal{T}^{-1}(u)\Big|_{u=\frac{c}{2}} - N. \tag{3.28}$$

From the definition it is clear that each matrix element of $T(u)$ acts in $\mathcal{H} = V_1 \otimes ... \otimes V_N$. But the monodromy matrix is itself a 2×2 matrix, which acts in a space $\mathbb{C}^2$, which we indicate with V_0 and call *auxiliary space*. So $T(u)$ acts in $V_0 \otimes V_1 \otimes ... \otimes V_N$ and the operators $L_n(u)$ acts nontrivially in $V_0 \otimes V_n$. Since $L_n(u) = R_{0n}(u - \frac{c}{2})$, then

$$T(u) = R_{0N}\left(u - \frac{c}{2}\right)...R_{01}\left(u - \frac{c}{2}\right) \tag{3.29}$$

and so

$$\mathcal{T}(u) = tr_0\left(R_{0N}\left(u - \frac{c}{2}\right)...R_{01}\left(u - \frac{c}{2}\right)\right). \tag{3.30}$$

Taking into account that $R_{0k}(0) = cP_{0k}$, we have

$$\begin{aligned}\frac{d\mathcal{T}(u)}{du}\Big|_{u=\frac{c}{2}} &= \frac{d}{du}\, tr_0\left(R_{0N}\left(u - \frac{c}{2}\right)...R_{01}\left(u - \frac{c}{2}\right)\right)\Big|_{u=\frac{c}{2}} \\ &= c^{N-1}\sum_{k=1}^{N} tr_0\left(P_{0N}...R'_{0k}(0)...P_{01}\right) \\ &= c^{N-1}\sum_{k=1}^{N} tr_0\left(R'_{0k}(0)P_{0k-1}...P_{0k+1}\right). \end{aligned} \tag{3.31}$$

Here we used the cyclicity of the trace and we intend $P_{00} = P_{0N}$ and $P_{0N+1} = P_{01}$. This means that trace operation automatically provides with periodic boundary condition.

Multiplying from the right the product in the trace for P^2_{0k+1} and using the fundamental property of permutation operator we obtain

$$\frac{d\mathcal{T}(u)}{du}\Big|_{u=\frac{c}{2}} = c^{N-1}\sum_{k=1}^{N} tr_0\left(R'_{kk+1}(0)P_{k+1k-1}...P_{k+1k+2}P_{0k+1}\right). \tag{3.32}$$

Since only P_{0k+1} acts in V_0 and its trace on it gives one (the identity in the space V_{k+1}), we finally obtain ($R'_{kk+1}(0) = \mathbf{1}_{4\times4}$)

$$\left.\frac{d\mathcal{T}(u)}{du}\right|_{u=\frac{c}{2}} = c^{N-1}\sum_{k=1}^{N} P_{k+1k-1}...P_{k+1k+2}. \tag{3.33}$$

To calculate the transfer matrix at $u = c/2$ we repeat the same calculation made above. We have

$$\mathcal{T}\left(\frac{c}{2}\right) = c^{N} P_{k+1k}P_{k+1k-1}...P_{k+1k+2}, \tag{3.34}$$

where the k, i.e the space V_k we have chosen, is uninfluential, because P_{ab} is always idempotent for all a and b. The idempotency and the relation $(AB)^{-1} = B^{-1}A^{-1}$ allows also to write

$$\mathcal{T}^{-1}\left(\frac{c}{2}\right) = c^{-N} P_{k+1k+2}...P_{k+1k-1}P_{k+1k}. \tag{3.35}$$

Multiplying the derivative of the transfer matrix by the inverse of transfer matrix and using the same index k (it is arbitrary) between each term of the sum in (3.33) and the (3.35), we finally obtain

$$H = 2c\left.\frac{d\mathcal{T}(u)}{du}\mathcal{T}^{-1}(u)\right|_{u=\frac{c}{2}} = 2\sum_{k=1}^{N} P_{kk+1}, \tag{3.36}$$

which is the Hamiltonian of XXX model up to a constant proportional to the identity.

3.2 Implementation of Algebraic Bethe Ansatz

The first step to define the ABA method is to determine the Hilbert space $\mathcal{H}$ on which the entries of the monodromy matrix act. The fundamental and physical requirement is that it is provided with a vacuum vector $|0\rangle$ so that:

$$A(u)|0\rangle = a(u)|0\rangle, \qquad D(u)|0\rangle = d(u)|0\rangle, \qquad C(u)|0\rangle = 0, \tag{3.37}$$

where the eigenvalues $a(u)$ and $d(u)$ are general functions of u. The action of operator $B(u)$ on the vacuum allows to generate all the space $\mathcal{H}$.

Now a proposition is very useful.

Proposition 3.2.1. *Let the monodromy matrix $T(u)$ be given as in* (3.24)*, and every Lax-operator have the form*

$$L_n(u) = \begin{pmatrix} \alpha_n(u) & \beta_n(u) \\ \gamma_n(u) & \delta_n(u) \end{pmatrix}, \qquad n = 1, ..., N. \tag{3.38}$$

Assume that there exists a vacuum vector $|0\rangle$ *such that*

$$\alpha_n(u)|0\rangle = \tilde{\alpha}_n(u)|0\rangle, \qquad \delta_n(u)|0\rangle = \tilde{\delta}_n(u)|0\rangle, \qquad \gamma_n(u)|0\rangle = 0, \qquad n = 1, ..., N, \tag{3.39}$$

where $\tilde{\alpha}_n(u)$ *and* $\tilde{\delta}_n(u)$ *are functions of* u*. Then the eigenvalues of the monodromy matrix diagonal entries are*

$$a(u) = \prod_{n=1}^{N} \tilde{\alpha}_n(u), \qquad d(u) = \prod_{n=1}^{N} \tilde{\delta}_n(u). \tag{3.40}$$

Proof. In order to prove this statement, it is convenient to decompose the Lax-operator as the sum of two terms: $L_n = L_n^+ + L_n^-$, where

$$L_n^+(u) = \begin{pmatrix} \alpha_n(u) & \beta_n(u) \\ 0 & \delta_n(u) \end{pmatrix}, \qquad L_n^-(u) = \begin{pmatrix} 0 & 0 \\ \gamma_n(u) & 0 \end{pmatrix}. \tag{3.41}$$

Using the standard product between matrix, the monodromy matrix takes the form

$$T = (L_N^+ + L_N^-)...(L_1^+ + L_1^-) = L_N^+...L_1^+ + K = T^+ + K. \tag{3.42}$$

The term K collects the sum of all terms, that contains at least one L_n^-. The action of any matrix elements of K on the vacuum vector is zero. Indeed, it is easy to see that due to the form of L_n^- each entries of K is a linear combination of products which contains at least one $\gamma_n(u)$. Since the entries of Lax-operators with different index n commute with each other, it is always possible to move a $\gamma_n(u)$ to right, so that acting on the vacuum it gives zero.

It follows that if

$$T^+(u) = \begin{pmatrix} A^+(u) & B^+(u) \\ C^+(u) & D^+(u), \end{pmatrix} \tag{3.43}$$

then

$$\begin{aligned} A(u)|0\rangle &= A^+(u)|0\rangle, \qquad & B(u)|0\rangle &= B^+(u)|0\rangle, \\ C(u)|0\rangle &= C^+(u)|0\rangle, \qquad & D(u)|0\rangle &= D^+(u)|0\rangle. \end{aligned} \tag{3.44}$$

Using induction and the fact that L_n^+ is an upper triangular matrix, we obtain the matrix elements of $T^+(u)$:

$$A^+(u) = \prod_{n=1}^{N} \alpha_n(u), \quad (3.45)$$

$$D^+(u) = \prod_{n=1}^{N} \delta_n(u), \quad (3.46)$$

$$C^+(u) = 0, \quad (3.47)$$

$$B^+(u) = \sum_{k=1}^{N} \left(\prod_{n=k+1}^{N} \alpha_n(u) \right) \beta_k(u) \left(\prod_{n=1}^{k-1} \delta_n(u) \right), \quad (3.48)$$

where in the last expression when the index n is $N+1$ or 0 it is intended that the relative factor is absent.

So the proposition is proved.

□

Now our purpose is to study the eigenvectors of the transfer matrix, in order to find also the eigenvectors of the Hamiltonian, since, the transfer matrix generates the integral of motion of the model, included the Hamiltonian. This can be seen in an easy way.

Expand $\mathcal{T}(u)$ in power series over some point u_0

$$\mathcal{T}(u) = \sum_k (u - u_0)^k I_k, \quad (3.49)$$

where I_k are some operators acting on $\mathcal{H}$. Then taking the kth derivative over u and the nth-derivative over v at $u, v = u_0$ in the equation (3.12), we find that all the operators I_k commute

$$[I_k, I_n] = 0, \qquad \forall k, n. \quad (3.50)$$

So we a have an infinite set of operators which commute and fixing one of them to be the Hamiltonian, we can construct an integrable model.

Returning to the ABA we now set some notations and conventions. First of all we introduce the functions

$$f(u,v) = \frac{u - v + c}{u - v} \qquad \text{and} \qquad g(u,v) = \frac{c}{u - v}, \quad (3.51)$$

and we write the R-matrix of XXX model in the form

$$R(u,v) = \mathbb{I} + \frac{c}{u - v} P = \begin{pmatrix} f(u,v) & 0 & 0 & 0 \\ 0 & 1 & g(u,v) & 0 \\ 0 & g(u,v) & 1 & 0 \\ 0 & 0 & 0 & f(u,v) \end{pmatrix}. \quad (3.52)$$

This expression of R-matrix differs from that written before for a factor $u - v$, but it is evident that the YB equation remains valid if we multiply both members for the same factor. In addition in what follows we will see that all steps do not depend on the explicit form of functions $f(u, v)$ and $g(u, v)$.

We denote sets of variables by a bar: $\bar{u} = (u_1, ..., u_n)$. In this way we can adopt a shorthand notation for products. If an operator or a function depend on more variables, we write their arguments with a bar above. For example

$$A(\bar{u}) = \prod_{u_j \in \bar{u}} A(u_j), \qquad a(\bar{v}) = \prod_{v_j \in \bar{v}} A(v_j). \tag{3.53}$$

If we want to exclude a variable from the set we write $\bar{u}_k = (u_1, ..., u_n) \backslash u_k$. So for the products we have for example

$$B(\bar{u}_k) = \prod_{\substack{u_j \in \bar{u} \\ j \neq k}} B(u_j), \qquad f(v_k, \bar{v}_k) = \prod_{\substack{v_i \in \bar{v} \\ i \neq k}} f(v_k, v_i). \tag{3.54}$$

Finally a product over the empty set is intended to be equal to one.

Now we write the sixteen commutation relations between monodromy matrix entries (some of them are related to each others), which derives directly from equation (3.6) and using the expression (3.52) for R-Matrix:

$$\begin{aligned} &[A(u), A(v)] = 0, \qquad &&[B(u), B(v)] = 0 \\ &[C(u), C(v)] = 0, \qquad &&[D(u), D(v)] = 0, \end{aligned} \tag{3.55}$$

$$[A(u), D(v)] = g(u, v)(C(v)B(u) - C(u)B(v)), \tag{3.56}$$

$$[D(u), A(v)] = g(u, v)(B(v)C(u) - B(u)C(v)), \tag{3.57}$$

$$[C(u), B(v)] = g(u, v)(A(v)D(u) - A(u)D(v)), \tag{3.58}$$

$$[B(u), C(v)] = g(u, v)(D(v)A(u) - D(u)A(v)), \tag{3.59}$$

$$A(v)B(u) = f(u, v)B(u)A(v) + g(v, u)B(v)A(u), \tag{3.60}$$

$$B(v)A(u) = f(u, v)A(u)B(v) + g(v, u)A(v)B(u), \tag{3.61}$$

$$D(v)C(u) = f(u, v)C(u)D(v) + g(v, u)C(v)D(u), \tag{3.62}$$

$$C(v)D(u) = f(u, v)D(u)C(v) + g(v, u)D(v)C(u), \tag{3.63}$$

$$A(u)C(v) = f(u, v)C(v)A(u) + g(v, u)C(u)A(v), \tag{3.64}$$

$$C(u)A(v) = f(u,v)A(v)C(u) + g(v,u)A(u)C(v), \tag{3.65}$$

$$B(u)D(v) = f(u,v)D(v)B(u) + g(v,u)D(u)B(v), \tag{3.66}$$

$$D(u)B(v) = f(u,v)B(v)D(u) + g(v,u)B(u)D(v). \tag{3.67}$$

Now we are ready to develop the main idea of Algebraic Bethe Ansatz, i.e. looking for the eigenvectors of transfer matrix in the form $B(\bar{u})|0\rangle$. Since $\mathcal{T}(u) = A(u) + D(u)$ we have to analyze the action of $A(u)$ and $D(u)$ on $B(\bar{u})|0\rangle$. Let's start calculating

$$A(v)B(\bar{u})|0\rangle. \tag{3.68}$$

Using the commutation relation (3.60) we can move the operator A until the extreme right, where it acts on the vacuum vector. If the set $\bar{u}$ consists of n variables, then applying n times the (3.60) we obtain 2^n terms. This is apparently a great obstacle to overcome, but if we pay more attention on the structure of the commutation relation the problem is easier. In fact, we see that the rhs of (3.60) consists of two terms: the first preserves the operators' arguments as in the lhs, the second exchange them. In this way moving $A(v)$ to the right, the operator $A(v)$ preserves its argument or exchange it with the argument of $B(u_j)$. In the last case the operator B takes the argument v. When A reaches the vacuum, produces a factor (it is its eigenvalue) $a(u_k)$ or $a(v)$, where the index k indicates that a $B(u_k)$ is disappeared and a $B(v)$ is appeared instead. So the most general expression for $A(v)B(\bar{u})|0\rangle$ is

$$A(v)B(\bar{u})|0\rangle = a(v)\Lambda(v,\bar{u})B(\bar{u})|0\rangle + \sum_{k=1}^{n} a(u_k)\Lambda_k(v,\bar{u})B(v)B(\bar{u}_k)|0\rangle. \tag{3.69}$$

We note that the last expression contains only $n+1$ terms instead of 2^N. Here $\Lambda(v,\bar{u})$ and $\Lambda_k(v,\bar{u})$ are coefficients which depend on $f(u,v)$ and $g(u,v)$. Let's now calculate them.

The first term in the rhs of equation (3.69) is obtained, if only the first term of (3.60) is used. Otherwise using only one time the second term on the rhs of (3.60) A exchanges the argument v with one $B(u_k)$ and when moving it further to the right, it can no longer accept v and so the factor $a(v)$ does not emerge. So using n times only the first term in rhs of (3.60) we obtain a product of $f(u_k,v)$ over u_k and therefore

$$\Lambda(v,\bar{u}) = f(\bar{u},v). \tag{3.70}$$

In order to find the coefficients $\Lambda_k(v,\bar{u})$, we take advantage of the (3.55), which allows to put any $B(u_k)$ in the extreme left position, i.e.

$$A(v)B(\bar{u})|0\rangle = A(v)B(u_k)B(\bar{u}_k)|0\rangle. \tag{3.71}$$

So if we choose $k = 1$ we are interested in the term of (3.69) proportional to $a(u_1)$ and to $\Lambda_1(v, \bar{u})$. The operatorial part of this term does not contain $B(u_1)$. The only possibility to "eliminate" $B(u_1)$ is to use the second term in the rhs of (3.60):

$$A(v)B(u_1)B(\bar{u}_1) \;\rightarrow\; g(v, u_1)B(v)A(u_1)B(\bar{u}_1). \tag{3.72}$$

In commuting the operator $A(u_1)$ with all the other operators $B(u_j)$ with $j = 2, 3, ..., n$, we have to use only the first term in rhs side of (3.60), otherwise $B(u_1)$ appears again and it is absent in the term of (3.69) with $\Lambda_1(v, \bar{u})$. Hence,

$$\Lambda_1(v, \bar{u}) = g(v, u_1)f(\bar{u}_1, u_1). \tag{3.73}$$

Due to the symmetry of the state $B(\bar{u})|0\rangle$, $\Lambda_k(v, \bar{u})$ is obtained simply exchanging u_1 with u_k, i.e.

$$\Lambda_k(v, \bar{u}) = g(v, u_k)f(\bar{u}_k, u_k). \tag{3.74}$$

The action of the operator $D(v)$ on $B(\bar{u})|0\rangle$ can be found exactly in the same way, but with a smaller effort than before, using for convenience the explicit expressions of function $f(u, v)$ and $g(u, v)$. Indeed, the commutation relation (3.67) between D and B can be obtained from the (3.60) under the replacement of the constant c with $-c$, noting that

$$f(u, v)|_{c\to -c} = f(v, u), \qquad g(u, v)|_{c\to -c} = g(v, u). \tag{3.75}$$

Thus, the action of $D(v)$ on the vector $B(\bar{u})|0\rangle$ is given by (3.69), where one should replace c with $-c$ and the function a with the function d:

$$D(v)B(\bar{u})|0\rangle = d(v)\hat{\Lambda}(v, \bar{u})B(\bar{u})|0\rangle + \sum_{k=1}^{n} d(u_k)\hat{\Lambda}_k(v, \bar{u})B(v)B(\bar{u}_k)|0\rangle, \tag{3.76}$$

where

$$\hat{\Lambda}(v, \bar{u}) = f(v, \bar{u}), \qquad \hat{\Lambda}_k(v, \bar{u}) = g(u_k, v)f(u_k, \bar{u}_k). \tag{3.77}$$

Remark. *The idea of writing the action of A or D operators on $B(\bar{u})|0\rangle$ as in* (3.69) *is due to Faddeev and Takhtadzhan [27] and allows us to reduce the terms in the final expression from 2^n to $n+1$. But one can ask how all these 2^n terms collect together in order to give only $n+1$ terms. An example could give a hint to explain this fact.*

If the string of operator $B(\bar{u})$ consists of only two operator $B(u_1)$ and $B(u_2)$, using twice the (3.60) *we obtain*

$$\begin{aligned} A(v)B(u_1)B(u_2) = {} & f(u_1,v)f(u_2,v)B(u_1)B(u_2)A(v) \\ & + g(v,u_1)f(u_2,u_1)B(v)B(u_2)A(u_1) \\ & + \left(g(v,u_2)f(u_1,v) + g(v,u_1)g(u_1,u_2)\right) B(v)B(u_1)A(u_2). \end{aligned} \tag{3.78}$$

Since $B(u_1)$ and $B(u_2)$ commute, therefore

$$A(v)B(u_1)B(u_2) = A(v)B(u_2)B(u_1). \tag{3.79}$$

Thus the expression (3.78) *does not change, if we replace u_1 with u_2 and vice versa:*

$$\begin{aligned} A(v)B(u_1)B(u_2) = {} & f(u_1,v)f(u_2,v)B(u_1)B(u_2)A(v) \\ & + g(v,u_2)f(u_1,u_2)B(v)B(u_1)A(u_2) \\ & + \left(g(v,u_1)f(u_2,v) + g(v,u_2)g(u_2,u_1)\right) B(v)B(u_2)A(u_1). \end{aligned} \tag{3.80}$$

Comparing (3.78) *with* (3.80) *we see that they are equivalent if*

$$g(v,u_2)f(u_1,v) + g(v,u_1)g(u_1,u_2) = g(v,u_2)f(u_1,u_2) \tag{3.81}$$

or

$$g(v,u_1)f(u_2,v) + g(v,u_2)g(u_2,u_1) = g(v,u_1)f(u_2,u_1). \tag{3.82}$$

Substituting the explicit expressions of f and g for the XXX model the last two equations become identities.

This kind of relations between f and g are emerged as a compatibility relation, obtained reordering the operators in a triple product. Combining other operators we can obtain other relations, so it is convenient to use directly a triple product between three monodromy matrices. Using the conventions as in (3.4)*, we want to reorder the product $T_1(u_1)T_2(u_2)T_3(u_3)$ in order to obtain $T_3(u_3)T_2(u_2)T_1(u_1)$. This can be done in two different ways (we indicate with a number the respective monodromy matrix):*

$$123 \to 213 \to 231 \to 321, \tag{3.83}$$
$$123 \to 132 \to 312 \to 321. \tag{3.84}$$

Each permutation of two monodromy matrices can be done using (3.4)*:*

$$R_{jk}(u_j,u_k)T_j(u_j)T_k(u_k)R_{jk}^{-1}(u_j,u_k) = T_k(u_k)T_j(u_j), \qquad j,k = 1,2,3. \tag{3.85}$$

The first way of reordering the matrices T_j *leads us to*

$$T_1(u_1)T_2(u_2)T_3(u_3) = R_{12}^{-1}(u_1,u_2)R_{13}^{-1}(u_1,u_3)R_{23}^{-1}(u_2,u_3)T_3(u_3)T_2(u_2)T_1(u_1)R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2). \tag{3.86}$$

The second way gives

$$T_1(u_1)T_2(u_2)T_3(u_3) = R_{23}^{-1}(u_2,u_3)R_{13}^{-1}(u_1,u_3)R_{12}^{-1}(u_1,u_2)T_3(u_3)T_2(u_2)T_1(u_1)R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3). \tag{3.87}$$

The last two equations are equivalent if

$$R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2) = R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3), \tag{3.88}$$

which is nothing else that the YB equation. As a consequence of this, we can claim that a lot of relations emerge between $f(u,v)$ *and* $g(u,v)$*, which in principle allow to collect the* 2^n *terms together in order to give the* $n+1$ *terms. The result that we obtained confirms the importance of YB equation in the field of integrable models.*

Now we are ready to obtain the action of the transfer matrix on $B(\bar{u})|0\rangle$:

$$\mathcal{T}(v)B(\bar{u})|0\rangle = \Big(a(v)\Lambda(v,\bar{u}) + d(v)\hat{\Lambda}(v,\bar{u})\Big)\, B(\bar{u})|0\rangle + \sum_{k=1}^{n}\Big(a(u_k)\Lambda_k(v,\bar{u}) + d(u_k)\hat{\Lambda}_k(v,\bar{u})\Big)\, B(v)B(\bar{u}_k)|0\rangle. \tag{3.89}$$

The vector $B(\bar{u})|0\rangle$ is an eigenvector of the transfer matrix if in the rhs of the last equation the coefficients of the terms in the sum vanish, i.e.

$$a(u_k)\Lambda_k(v,\bar{u}) + d(u_k)\hat{\Lambda}_k(v,\bar{u}) = 0, \qquad k = 1,...,n. \tag{3.90}$$

The eigenvalues relative to $B(\bar{u})|0\rangle$ is

$$a(v)\Lambda(v,\bar{u}) + d(v)\hat{\Lambda}(v,\bar{u}) = a(v)f(\bar{u},u) + d(v)f(v,\bar{u}) \tag{3.91}$$

and (3.90) is usually written as

$$\frac{a(u_k)}{d(u_k)} = \frac{f(u_k,\bar{u}_k)}{f(\bar{u}_k,u_k)}, \qquad k = 1,...,n. \tag{3.92}$$

The last system of equations is called the system of Bethe equations. For the specific case of XXX model the Bethe equations are

$$\left(\frac{u_k + \frac{c}{2}}{u_k - \frac{c}{2}}\right)^N = \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{u_k - u_j + c}{u_k - u_j - c}, \qquad k = 1,...,n, \tag{3.93}$$

where we have used the proposition 3.2.1 and the Lax-operator (3.22), knowing that $\sigma_n^z|0\rangle = |0\rangle$.

Now we proceed with the XXZ-model.
The R-matrix of XXZ-model is

$$R(u,v) = \begin{pmatrix} f(u,v) & 0 & 0 & 0 \\ 0 & 1 & g(u,v) & 0 \\ 0 & g(u,v) & 1 & 0 \\ 0 & 0 & 0 & f(u,v) \end{pmatrix}, \tag{3.94}$$

where

$$f(u,v) = \frac{\sinh(u-v+\eta)}{\sinh(u-v)} \quad \text{and} \quad g(u,v) = \frac{\sinh\eta}{\sinh(u-v)} \tag{3.95}$$

and $\Delta = \cosh\eta$ (Δ is the anisotropy constant).
The Lax-operator is

$$L_n(u) = \begin{pmatrix} \sinh\left(u + \frac{\eta}{2}\sigma_n^z\right) & \sigma_n^- \sinh\eta \\ \sigma_n^+ \sinh\eta & \sinh\left(u - \frac{\eta}{2}\sigma_n^z\right) \end{pmatrix}, \tag{3.96}$$

which generates through the monodromy matrix and then the transfer matrix the XXZ-Hamiltonian:

$$H = 2\sinh\eta \left.\frac{d\mathcal{T}(u)}{du}\mathcal{T}^{-1}(u)\right|_{u=\frac{\eta}{2}} - N\Delta$$
$$= \sum_{k=1}^{N} \left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \Delta\sigma_k^z\sigma_{k+1}^z\right). \quad (3.97)$$

The Bethe equations for the XXZ-model are

$$\left(\frac{\sinh\left(u_j + \frac{\eta}{2}\right)}{\sinh\left(u_j - \frac{\eta}{2}\right)}\right)^N = \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{\sinh\left(u_j - u_k + \eta\right)}{\sinh\left(u_j - u_k - \eta\right)}. \quad (3.98)$$

3.3 Inhomogeneous XXZ model

In this section we introduce the Inhomogeneous-twisted XXZ spin chain [3] and the related Algebraic Bethe Ansatz formalism, which will be useful in the following chapters. The construction of this model is very similar to that of the XXZ model with $\Delta = \cos\gamma$. As usual we associated to each site of the chain with N sites a triplet of Pauli matrices with the standard commutation rules

$$\left[\sigma_m^\alpha, \sigma_n^\beta\right] = 2i\delta_{mn}\epsilon^{\alpha\beta\gamma}\sigma_n^\gamma, \quad (3.99)$$

with the greek letters being x, y or z and $n = 1, 2, ..., N$. The two-dimensional auxiliary space is denoted with $n = 0$. For any complex spectral parameter λ and an arbitrary set of inhomogeneities $\theta_1, \theta_2, ..., \theta_N$, the Lax-operators L_n are written as

$$L_n = R_{0n}(\lambda + \theta_n)P_{0n}, \quad (3.100)$$

where the R-matrices are

$$R_{kn}(\theta) = \frac{a+c}{2} + \frac{a-c}{2}\sigma_k^z\sigma_n^z + \frac{b}{2}\left(\sigma_k^x\sigma_n^x + \sigma_k^y\sigma_n^y\right), \quad (3.101)$$

with $P_{kn} = 1/2\left(1 + \vec{\sigma_k}\cdot\vec{\sigma_n}\right)$ the usual permutation operator and

$$a = a(\theta) \equiv \sin(\gamma - \theta), \qquad b = b(\theta) \equiv \sin\theta, \qquad c = \sin\gamma, \quad (3.102)$$

where γ is the anisotropy parameter, which belongs to $[0, \pi]$.

The inhomogeneous and twisted transfer matrix associated to the model is

$$t(\lambda, \theta_1, \theta_2, ..., \theta_N, \omega) = e^{i\omega\sigma_0^z} tr_0[L_1 L_2 ... L_N]. \tag{3.103}$$

The twist angle ω defines the relation between spin operators after a translation after N sites:

$$\sigma_{n+N}^{\pm} = e^{\pm i\omega}\sigma_n^{\pm}, \qquad \sigma_{n+N}^z = \sigma_n^z. \tag{3.104}$$

The eigenstates of the transfer matrix related to the $S^z = N/2 - M$ sector with $M = 0, 1, 2, ..., [N/2]$, are labelled by M distinct Bethe roots $\lambda_1, \lambda_2, ..., \lambda_M$, which obey the Bethe Ansatz Equations

$$\prod_{n=1}^{N} \frac{\sinh(\lambda_j + i\theta_n + i\gamma/2)}{\sinh(\lambda_j + i\theta_n - i\gamma/2)} = -e^{2i\omega} \prod_{l=1}^{M} \frac{\sinh(\lambda_j - \lambda_l + i\gamma)}{\sinh(\lambda_j - \lambda_l - i\gamma)}. \tag{3.105}$$

If we choose N even and $\theta_n = (-1)^{n-1}\theta/2$, we obtain the twisted alternating transfer matrix

$$t_{2N}(\lambda, \theta, \omega) = t(\lambda, \theta_1 = \theta/2, \theta_2 = -\theta/2, ..., \theta_{2N} = -\theta/2, \omega). \tag{3.106}$$

Chapter 4

Quantum quench

4.1 Matrix density

The most important device to study quantum many particle systems is the density matrix (statistical distribution in a classical context), which allows to obtain the mean value and the probability of any eigenvalue of an observable. It is known in quantum mechanics that the state of a system is described by a superposition of wave functions which are the eigenvectors of an Hamiltonian and which represent a complete set, i.e. a base through which one can construct any function by means of a linear combination of the base elements. So if we indicate with ψ_n the elements of the complete set, where n stands for a collection of quantum numbers characterising the system, the state ψ of a physical system can be written as

$$\psi = \sum_n c_n \psi_n. \tag{4.1}$$

The mean value of an observable f in a particular state could be obtained by

$$\bar{f} = \sum_{nm} c_n^* c_m f_{nm} \tag{4.2}$$

where

$$f_{nm} = \int \psi_n^* \hat{f} \psi_m dq \tag{4.3}$$

are the matrix elements of f ($\hat{f}$ is the corresponding operator).

The latest arguments are used in standard Quantum Mechanics, but in Statistical Mechanics it is often more convenient to make use of the density

matrix, defined as $\rho_{mn} = c_n^* c_m$, which can be thought as the matrix elements of an operator $\hat{\rho}$:

$$\hat{f} = \sum_{nm} \rho_{mn} f_{nm}. \tag{4.4}$$

The latest equation expresses formally the trace of the operator product $\hat{\rho}\hat{f}$ and so

$$\hat{f} = Tr(\hat{\rho}\hat{f}). \tag{4.5}$$

4.2 Time evolution

It is known that in Quantum Mechanics the time evolution of a state is expressed through the Schrödinger picture or the Heisenberg picture. In the first case the physical objects which evolve in time are the eigenstates of the Hamiltonian H of the system or a superposition of them, i.e.:

$$|\psi_S(t)\rangle = U(t, t_0)|\psi_S(t_0)\rangle, \qquad U = e^{-iH(t-t_0)}.^{1} \tag{4.6}$$

In the second case the objects which evolve are the observables, represented by an operator O, i.e.:

$$O_H = U^\dagger(t, t_0) O_S U(t, t_0), \tag{4.7}$$

where O_S is the operator in the Schrödinger picture. The Heisenberg picture allows to write the Heisenberg equation of motion:

$$i\frac{dA_H}{dt} = [A_H, H_H] + i\frac{\partial A_H}{\partial t}. \tag{4.8}$$

Both pictures are equivalent, so in what follows we make use of both of them, without adding any label to kets or to operators.

4.3 Global quantum quench

Statistical Mechanics studies systems at equilibrium. An isolated system, i.e. a system which is characterised by the absence of any coupling to its environment, is at equilibrium if the physical observables are equal to their mean values with a good approximation. Some examples of system at equilibrium are a Bose gas, governed by Bose-Einstein distribution, a fermionic

[1] We put $\hbar = 1$.

gas, governed by Fermi-Dirac distribution and a perfect gas, governed by Maxwell-Boltzmann distribution. All these distributions have a common feature: they are independent of time. The time dependence (through a unitary transformation performed by the time evolution operator $U(t, t_0)$) of the statistical distribution or of the density matrix is the first step to analyze system away from equilibrium. If a system is not at equilibrium, the system necessarily evolves towards the same or another state of equilibrium in a time interval, called relaxation time. The phenomena of non equilibrium need more techniques than Statistical Physics. One of these is the quantum quench, which we are going to analyze. If we have an Hamiltonian $H(h)$, depending on a parameter h, such that the system is prepared initially at $t = 0$ in the ground state $|\psi(0)\rangle$ of $H(h_0)$ and then we suddenly change (from now we will use the term "quench") the parameter h_0 to a new value h, then we are "quenching" the system. After a quench the system evolves with the usual unitary time evolution operator with the Hamiltonian $H(h)$. As the change of the parameter is instantaneous, the systems remains in the state $|\psi(0)\rangle$, so that at $t > 0$

$$|\psi(t)\rangle = e^{-iH(h)t}|\psi(0)\rangle. \tag{4.9}$$

Through the Hamiltonian eigenstates

$$H(h)|n\rangle = E_n|n\rangle, \tag{4.10}$$

it is possible to express the state of the system at time t as

$$|\psi(t)\rangle = \sum_n \langle n|\psi(0)\rangle e^{-iE_n t}|n\rangle. \tag{4.11}$$

Using the latest expression we can write the expectation value of an operator O in the state $|\psi(t)\rangle$ as

$$\langle\psi(t)|O|\psi(t)\rangle = \sum_{nm} \langle\psi(0)|n\rangle\langle m|\psi(0)\rangle\langle n|O|m\rangle e^{-i(E_m - E_n)t}. \tag{4.12}$$

An important property of a quantum quench is that the energy is conserved during time ($t > 0$) and if we are working with a lattice the post-quench energy density is larger than the ground state energy per site

$$e = \lim_{L\to\infty} \frac{1}{L}\langle\psi(t)|H(h)|\psi(t)\rangle > \lim_{L\to\infty} \frac{E_0}{L}, \tag{4.13}$$

where L is the number of sites and E_0 is the energy of the ground state.

Some definitions are now useful. In lattice models an operator O is said *local* if in the thermodynamic limit O acts non-trivially only on a finite number of sites separated by a finite distance. For example in a quantum spin-1/2 chain with L sites the following operators are local

$$\sigma_j^\alpha, \qquad\qquad \sigma_j^\alpha \sigma_{j+k}^\beta, \tag{4.14}$$

where σ_j^α ($\alpha = x, y, z$) are Pauli matrices acting on site j. We remember that σ_j^α is expressed as

$$\sigma_j^\alpha = \overset{1}{\underset{\downarrow}{\mathbb{I}}} \otimes ... \otimes \overset{j}{\underset{\downarrow}{\mathbb{I}}} \otimes \overset{j}{\underset{\downarrow}{\sigma^\alpha}} \otimes \overset{j+1}{\underset{\downarrow}{\mathbb{I}}} \otimes ... \otimes \overset{L}{\underset{\downarrow}{\mathbb{I}}} \tag{4.15}$$

and acts on a Hilbert space

$$\overset{1}{\underset{\downarrow}{V}} \otimes ... \otimes \overset{j}{\underset{\downarrow}{V}} \otimes ... \otimes \overset{L}{\underset{\downarrow}{V}}. \tag{4.16}$$

An example of non local operator is $\sigma_1^x \sigma_{L/2}^x$, where it is evident the dependence on the "dimension" L of the lattice.

The range of a local operator O is the size of the largest interval on which it acts non-trivially. For example we have that $\text{range}(\sigma_j^\alpha \sigma_{j+k}^\beta) = k + 1$.

Now it is necessary to analyze some concepts related to the phase of relaxation.

Suppose that the physical system taken into exam is divided into an arbitrary but finite subsystem B and its complement A. Then we take the thermodynamic limit while keeping B fixed. The system is prepared at $t = 0$ in a state with density matrix $\rho(0)$, which evolves at later times according to the law

$$\rho(t) = e^{iHt} \rho(0) e^{-iHt}. \tag{4.17}$$

The *reduced density matrix* of the subsystem B is obtained by taking the trace on the degrees of freedom of A

$$\rho_B(t) = Tr_A \rho(t). \tag{4.18}$$

We say that our system relaxes locally if the limit

$$\lim_{t\to\infty} \lim_{L\to\infty} \rho_B(t) = \rho_B(\infty) \tag{4.19}$$

exists for any finite subsystem B (L represents the dimensions of the system, like the number of sites in a spin chain for example).

If a system relaxes locally, then its stationary state is defined as a time-**independent** density matrix ρ^{SS} for the entire system such that for any finite subsystem B

$$\lim_{L\to\infty} Tr_A(\rho^{SS}) = \rho_B(\infty), \tag{4.20}$$

where A is the complement of B.

Let ρ_1 and ρ_2 be two density matrices. The corresponding two ensembles are said locally equivalent $\rho_1 =_{loc} \rho_2$, if in the thermodynamic limit the reduced density matrices for any finite subsystem B coincide, i.e.

$$\lim_{|A|\to\infty} Tr_A(\rho_1) = \lim_{|A|\to\infty} Tr_A(\rho_2), \tag{4.21}$$

where A is as usual the complement of B and $|A|$ denotes its volume.

As we are dealing with isolated systems, energy is always conserved, so that

$$E = Tr(\rho(t)H) = Tr(e^{iHt}\rho(0)e^{-iHt}H) = Tr(\rho(0)H), \tag{4.22}$$

where we have used the cyclic property of the trace and the fact that H commutes with e^{-iHt}.

In absence of other conserved quantities isolated system are believed to locally relax to equilibrium. This is known as *thermalization*. With the preceding definitions a stationary state is described by a Gibbs ensemble

$$\rho^{SS} =_{loc} \rho^{Gibbs} = \frac{e^{-\beta_{eff}H}}{Tr(e^{-\beta_{eff}H})}. \tag{4.23}$$

The inverse effective temperature β_{eff} is fixed by the initial value of the energy density

$$e \equiv \lim_{L\to\infty} \frac{Tr(\rho(0)H)}{L} = \lim_{L\to\infty} \frac{Tr(\rho^{Gibbs}H)}{L}. \tag{4.24}$$

The physical idea which is behind the latest definition is that the infinite complement A behaves like a thermal bath with an effective inverse temperature β_{eff}.

If the system has other conserved quantities (called charges similarly) other than the Hamiltonian (we remember that a physical model is integrable if has an infinite set of conserved charges) the Gibbs distribution has to be

generalized. The operators $I^{(n)}$ [2] corresponding to local conservation laws commute with Hamiltonian H and with each other

$$[H, I^{(n)}] = 0, \qquad [I^{(n)}, I^{(m)}] = 0. \tag{4.25}$$

Considering the (4.17) and the fact that the Hamiltonian commutes with the conserved charges we obtain that the expectation values of $I^{(n)}$ are independent of time

$$\frac{1}{L} Tr\left(\rho(t) I^{(n)}\right) = \frac{1}{L} Tr\left(\rho(0) I^{(n)}\right) \equiv \frac{E^{(n)}}{L}. \tag{4.26}$$

As a consequence of this it is necessary to modified the stationary state density matrix in the generalized Gibbs ensemble (GGE) density given by

$$\rho^{GGE} = \frac{e^{-\sum_n \lambda_n I^{(n)}}}{Tr\left(e^{-\sum_n \lambda_n I^{(n)}}\right)}. \tag{4.27}$$

Here λ_n are Lagrange multipliers that are fixed by the initial conditions (4.26), if we require that

$$\lim_{L\to\infty} \frac{E^{(n)}}{L} = \lim_{L\to\infty} \frac{1}{L} Tr\left(\rho^{GGE} I^{(n)}\right). \tag{4.28}$$

The presence of symmetries in the Hamiltonian of a system is the key to solve integrable models as well as to deduce interesting properties after a quantum quench. If we denote the symmetry operator by U and suppose that the initial state $|\psi(0)\rangle$ after the quench is not invariant under U we have

$$[H, U] = 0, \qquad U|\psi(0)\rangle \neq |\psi(0)\rangle. \tag{4.29}$$

Although this property of the initial state, it could happen that in the stationary state the symmetry is restored. In fact if

$$[I^{(n)}, U] = 0, \qquad \forall n, \tag{4.30}$$

we have that

$$[\rho^{SS}, U] = 0, \tag{4.31}$$

with

$$\rho^{SS} =_{loc} \rho^{GGE} = \frac{e^{-\sum_n \lambda_n I^{(n)}}}{Tr\left(e^{-\sum_n \lambda_n I^{(n)}}\right)}. \tag{4.32}$$

[2] In our cases of interest $I^{(n)}$ are usually expressed as a sum of densities, i.e. $I^{(n)} = \sum_j \mathcal{I}_j^{(n)}$.

Conversely, if there is at least one conservation law for which $[I^{(n)}, U] \neq 0$, than the symmetry will remain broken in the stationary state.

Another interesting observation could be done on the conserved quantities $I^{(n)}$: one could ask if every of them is equally important to describe the stationary state. We could select a finite subset $\{J_m | m = 1, ..., y\} \subset \{I^{(n)}\}$ and verify how well the partial density matrices

$$\rho^{pGGE,y} = \frac{e^{-\sum_{n=1}^{y} \lambda_n^{(y)} J_n}}{Tr\left(e^{-\sum_{n=1}^{y} \lambda_n^{(y)} J_n}\right)} \tag{4.33}$$

approximate the original ρ^{GGE}. In this case the Lagrange multipliers $\lambda_n^{(y)}$ are fixed by an appropriate initial condition

$$\lim_{L\to\infty} = \frac{\langle\psi(0)|J_n|\psi(0)\rangle}{L} = \lim_{L\to\infty} \frac{Tr\left(\rho^{pGGE,y} J_n\right)}{L}, \qquad n = 1, ...y. \tag{4.34}$$

As showed in [6] a method to compare ρ^{GGE} and $\rho^{pGGE,y}$ is through a distance $D(\rho_l, \rho_l^{'})$ on the space of reduced density matrices on a interval of length l. If this distance could be made arbitrarily small

$$D(\rho_l^{GGE}, \rho_l^{pGGE,y}) < \epsilon, \qquad \forall l, \tag{4.35}$$

$\rho^{pGGE,y}$ is a good approximation of ρ^{GGE}. In the cited work it is found that if a single conservation law with small degrees of locality, i.e. with a small number of sites on which each conserved density $\mathcal{I}_j^{(n)}$ acts non-trivially, the approximation (4.33) is rough, so that an expression like (4.33) works properly if we retain all conserved charged with the smallest degrees of locality , which assures that (4.35) is satisfied. This operation of selection the proper $I^{(n)}$ is called "truncation", and the density $\rho^{pGGE,y}$ is now denoted as $\rho^{tGGE,y}$, where y is chosen, so that (4.35) is satisfied.

As a consequence of these results we have that

$$\rho^{GGE} \equiv \lim_{y\to\infty} \rho^{tGGE,y}. \tag{4.36}$$

4.3.1 Quench Action Approach

There is an alternative approach that not rely on the GGE assumption and that, besides predicting the steady state after a quantum quench, also gives access to the time evolution. This is the so-called Quench Action approach

and it start from the idea to overcome the double sum in (4.12), which we rewrite in terms of Bethe states and the initial state (before quench) $|\Psi_0\rangle$:

$$\langle\Psi(t)|O|\Psi(t)\rangle = \sum_{\lambda,\lambda'} e^{-S^*_\lambda - S_{\lambda'}} e^{i(E_\lambda - E_{\lambda'})t} \langle\lambda|O|\lambda'\rangle, \tag{4.37}$$

where $S_\lambda = -\ln\langle\lambda|\Psi_0\rangle$ are called overlap coefficients.

The first step is to replace the above double sum by a functional integral over Bethe root densities $\boldsymbol{\rho} = \{\rho_n\}_{n=1}^{\infty}$:

$$\sum_\alpha \to \int \prod_{n=1}^{\infty} D\rho_n(\lambda) e^{S_{YY}(\boldsymbol{\rho})}, \tag{4.38}$$

where $S_{YY}(\boldsymbol{\rho})$ is the so-called Yang-Yang entropy and have the following expression

$$S_{YY}(\boldsymbol{\rho}) = N \sum_{n=1}^{\infty} \int_{-\pi/2}^{\pi/2} d\lambda [(\rho_n(\lambda) + \rho_n^h(\lambda) - \rho_n(\lambda)\ln\rho_n(\lambda) - \rho_n^h(\lambda)\ln\rho_n^h(\lambda)]. \tag{4.39}$$

The expectation value of an observable becomes

$$\langle\Psi(t)|O|\Psi(t)\rangle = \int \prod_{n=1}^{\infty} D\rho_n(\lambda) e^{-2Re(\ln S_{\boldsymbol{\rho}}) - S_{YY}(\boldsymbol{\rho})} \langle\boldsymbol{\rho}|O|\boldsymbol{\rho}\rangle, \tag{4.40}$$

where we have identified the Bethe states $|\lambda\rangle$ with the collection of densities $\boldsymbol{\rho}$.

The Quench Action is defined as

$$S_{QA}(\boldsymbol{\rho}) = 2Re(\ln S_{\boldsymbol{\rho}}) - S_{YY}(\boldsymbol{\rho}). \tag{4.41}$$

It is analogous to the free energy in thermodynamic systems. Minimizing the Quench Action, through a variational equation

$$\frac{\delta S_{QA}(\boldsymbol{\rho})}{\delta\rho_n} = 0, \tag{4.42}$$

we obtain the so-called saddle point string densities $\boldsymbol{\rho}^{sp}(\lambda)$, which obey a set of equation, called Generalized Thermodynamic Bethe Ansatz equations.

Knowing the $\boldsymbol{\rho}^{sp}(\lambda)$ we have that

$$\lim_{t\to\infty} \lim_{th} \langle\Psi(t)|O|\Psi(t)\rangle = \langle\boldsymbol{\rho}^{sp}(\lambda)|O|\boldsymbol{\rho}^{sp}(\lambda)\rangle. \tag{4.43}$$

We give an example of this procedure in the next section for a particular case of initial state.

4.4 An example: quenching XXZ model with $\Delta > 1$

We illustrate the main steps for quenching XXZ model in the two approach: the GGE and the Quench Action. We refer to the specific reference for the explicit derivation of some results.

4.4.1 Generalized Gibbs Ensemble

First of all we write the conserved charges, which are defined by the ubiquitous transfer matrix $t(\lambda)$:

$$Q_{m+1} = i\frac{\sinh^m(\phi)}{2^m}\frac{\partial^m}{\partial\lambda^m}\ln t(\lambda)\Big|_{\lambda=i\phi/2}, \qquad m \geq 0. \tag{4.44}$$

We note that the total momentum is $-Q_1$ and the Hamiltonian is JQ_2.
In the thermodynamic limit [3] the eigenvalues of charge Q_{m+1} is given by

$$\lim_{th}\langle\lambda|\frac{Q_{m+1}}{N}|\lambda\rangle = \sum_{n=1}^{\infty}\int_{-\pi/2}^{\pi/2} d\lambda\rho_n(\lambda)c_{m+1}^{(n)}(\lambda), \qquad m \geq 0, \tag{4.45}$$

where $|\lambda\rangle$ represents a Bethe state defined by the M Bethe roots, which characterize a state with M down spins, the densities ρ_n are obtained by (2.107) and the coefficients $c_{m+1}^{(n)}(\lambda)$ are given by

$$c_{m+1}^{(n)}(\lambda) = i(-1)^m\frac{\sinh^m(\phi)}{2^m}\frac{\partial^m}{\partial\lambda^m}\ln\frac{\sin(\lambda+\frac{i\phi}{2})}{\sin(\lambda-\frac{i\phi}{2})}. \tag{4.46}$$

In the paper of *Mossel and Caux* [18] the so-called generalized Thermodynamic Bethe Ansatz equations are obtained:

$$\ln\eta_n = -\delta_{n,1}(s\star d) + s\star[\ln(1+\eta_{n-1}) + \ln(1+\eta_{n+1})], \qquad n \geq 1 \tag{4.47}$$

where $\eta_0 = 0$ and s is given by (2.112). In this case d is given by

$$d(\lambda) = \sum_{k\in\mathbb{Z}} e^{-2ik\lambda}\sum_{m=2}^{\infty}\beta^m\sinh^{m-1}(\phi)(ik)^{m-2}, \tag{4.48}$$

[3] The thermodynamic limit is indicated with subscript th.

where the Lagrange multipliers or generalized chemical potentials β_m are obtained by imposing the initial conditions on the conserved charges Q_{m+1} like in (4.28):

$$\lim_{th} \frac{1}{N}\langle\Psi_0|Q_{m+1}|\Psi_0\rangle = \lim_{n\to\infty}\lim_{th}\frac{1}{N}\frac{Tr(Q_{m+1}\rho^{GGE})}{Tr\rho^{GGE}}, \tag{4.49}$$

where $|\Psi_0\rangle$ is the initial state of the system before the quench. It is important to note that $|\Psi_0\rangle$ is a generic state and after the quench the expectation value of an obsarvable O is given by

$$\lim_{t\to\infty}\lim_{th}\langle\Psi(t)|O|\Psi(t)\rangle = \lim_{t\to\infty}\lim_{th}\langle\Psi_0|e^{iHt}Oe^{-iHt}|\Psi_0\rangle = \langle\boldsymbol{\rho}^{\Psi_0}|O|\boldsymbol{\rho}^{\Psi_0}\rangle, \tag{4.50}$$

where $|\boldsymbol{\rho}^{\Psi_0}\rangle$ is the postquench steady state. Therefore, $\boldsymbol{\rho}^{\Psi_0}$ represents a set $\{\rho_n^{\Psi_0}\}_{n=1}^{\infty}$ of densities, which have to reproduce the initial values of all conserved charges. To proceed on we state that the claim of the GGE is that for any local operator this set of densities reproduces the steady state expectation values, i.e.:

$$\langle\boldsymbol{\rho}^{\Psi_0}|O|\boldsymbol{\rho}^{\Psi_0}\rangle = \langle\boldsymbol{\rho}^{GGE}|O|\boldsymbol{\rho}^{GGE}\rangle, \tag{4.51}$$

where $\boldsymbol{\rho}^{GGE} = \{\rho_n^{GGE}\}_{n=1}^{\infty}$ are obtained combining the generalized TBA equations with (2.111).

The first step to implement the GGE ideas is to impose the constraints

$$\lim_{th}\frac{1}{N}\langle\Psi_0|Q_{m+1}|\Psi_0\rangle = \sum_{n=1}^{\infty}\int_{-\pi/2}^{\pi/2}d\lambda\rho_n^{\Psi_0}(\lambda)c_{m+1}^{(n)}(\lambda), \qquad m\geq 0. \tag{4.52}$$

Now we will see that the initial expectation values of local conserved charges $\{Q_m\}_{m=2}^{\infty}$ depend only on the 1-string holes density. Observe that the momentum charge is excluded (see for details [2]).

It is convenient to work with Fourier transform, with conventions

$$\hat{f}(k) = \int_{-\pi/2}^{\pi/2}d\lambda e^{2ik\lambda}f(\lambda), \qquad k\in\mathbb{Z}, \tag{4.53}$$

$$f(k) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}}e^{-2ik\lambda}\hat{f}(k), \qquad \lambda\in[-\pi/2,\pi/2)\,. \tag{4.54}$$

For $m\geq 1$, $m-1$ integrations by parts gives a simple expression for the Fourier transform of $c_{m+1}^{(n)}$:

$$\hat{c}^{(n)}_{m+1}(k) = -2\pi \frac{\sinh^m(\phi)}{2^m}(2ik)^{m-1}\int_{-\pi/2}^{\pi/2} d\lambda e^{2ik\lambda} a_n(\lambda)$$
$$= -\pi \sinh^m(\phi)(ik)^{m-1} e^{-|k|\phi n}, \quad (4.55)$$

where the Fourier transform of $a_n(\lambda)$ is obtained in Appendix D.
The eigenvalue of charge Q_{m+1} can then be rewritten as

$$\sum_{n=1}^{\infty}\int_{-\pi/2}^{\pi/2} d\lambda \rho_n^{\Psi_0}(\lambda) c^{(n)}_{m+1}(\lambda) = \frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{k\in\mathbb{Z}} \hat{\rho}_n^{\Psi_0}(k)\hat{c}^{(n)}_{m+1}(k)$$
$$= -\sinh^m(\phi)\sum_{k\in\mathbb{Z}}(ik)^{m-1}\sum_{n=1}^{\infty}\hat{\rho}_n^{\Psi_0}(k)e^{-|k|\phi n}. \quad (4.56)$$

Rewriting the sum over all string densities in terms of $\hat{\rho}_1^{h\Psi_0}(k)$ (see [2]), i.e.

$$\sum_{n=1}^{\infty}\hat{\rho}_n^{\Psi_0}(k)e^{-|k|\phi n} = \frac{e^{-|k|\phi} - \hat{\rho}_1^{h\Psi_0}(k)}{2\cosh(k\phi)}, \quad (4.57)$$

we obtain that

$$\lim_{th}\frac{\langle\Psi_0|Q_{m+1}|\Psi_0\rangle}{N\sinh^m(\phi)} = \sum_{k\in\mathbb{Z}}\frac{-e^{-|k|\phi} + \hat{\rho}_1^{h\Psi_0}(k)}{2\cosh(k\phi)}(ik)^{m-1}. \quad (4.58)$$

Knowing the initial state $|\Psi_0\rangle$, the method used in [7] gives an explicit expression for $\rho_1^{h\Psi_0}$:

$$\rho_1^{h\Psi_0}(\lambda) = a_1(\lambda) + \frac{1}{2\pi}\left(\Omega_{\Psi_0}\left(\lambda + \frac{i\phi}{2}\right) + \Omega_{\Psi_0}\left(\lambda - \frac{i\phi}{2}\right)\right), \quad (4.59)$$

where

$$\Omega_{\Psi_0}(\lambda) = \lim_{th}\frac{i}{N}\langle\Psi_0|t^{-1}\left(\lambda + \frac{i\phi}{2}\right)\partial_\lambda t\left(\lambda - \frac{i\phi}{2}\right)|\Psi_0\rangle. \quad (4.60)$$

The densities of the GGE can be found by solving (using numerical algorithms) the generalized TBA equations for $n \geq 2$, combined with (2.111) and the fundamental constraint $\rho_1^{hGGE} = \rho_1^{h\Psi_0}$.

4.4.2 Quench Action for the Néel state-to-gapped XXZ quench

We take as initial state the so-called Néel state, i.e.

$$|\Psi_0^N\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle^{\otimes N/2} + |\downarrow\uparrow\rangle^{\otimes N/2}\right) = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow ...\rangle + |\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow ...\rangle\right). \tag{4.61}$$

We only consider Bethe states with $M = N/2$ flipped spins since the Néel state lies in this sector of the XXZ chain. For the sake of simplicity we choose N divisible by four such that M is even. We consider parity-invariant Bethe states $|\{\pm\lambda_j\}_{j=1}^{M/2}\rangle$ which have non-vanishing overlap $\langle\Psi_0^N|\{\pm\lambda_j\}_{j=1}^{M/2}\rangle$. λ_j are Bethe roots, which can be collected in such a manner that they form the set of n-strings. Obviously $\sum_n nM_n = M$, where M_n is the number of n-strings and the sum goes to infinity in theory.

The overlap is given in [17] and in references therein. We have[4]

$$\ln\frac{\langle\Psi_0^N|\{\pm\lambda_j\}_{j=1}^{M/2}\rangle}{\sqrt{\langle\{\pm\lambda_j\}_{j=1}^{M/2}|\{\pm\lambda_j\}_{j=1}^{M/2}\rangle}} = \sum_{j=1}^{M/2}\ln\left(\frac{\sqrt{\tan(\lambda_j+i\phi/2)\tan(\lambda_j-i\phi/2)}}{2\sin(2\lambda_j)}\right) + \frac{1}{2}\ln\frac{2\det_{N/4}G_{jk}^{+}}{\det_{N/4}G_{jk}^{-}}, \tag{4.62}$$

where

$$G_{jk}^{\pm} = \delta_{jk}\left(NK(\lambda_j) - \sum_{l=1}^{N/4}K(\lambda_j,\lambda_l)\right) + K^{\pm}(\lambda_j,\lambda_k), \qquad j,k = 1,...,N/4 \tag{4.63}$$

$$K^{\pm}(\lambda,\mu) = K(\lambda-\mu) \pm K(\lambda+\mu), \tag{4.64}$$

$$K(\lambda) = \frac{\sinh(2\phi)}{\sinh(\lambda+i\phi)\sinh(\lambda-i\phi)}. \tag{4.65}$$

Knowing the overlap coefficients, the variation of the Quench Action gives the generalized Thermodynamic Bethe Ansatz equation for the Néel-to-XXZ quench:

[4] $\det_{N/4}$ indicates that the determinant is taken on a $N/4 \times N/4$ matrix.

$$\ln \eta_n(\lambda) = d_n(\lambda) + s(\lambda) \star \left[\ln(1+\eta_{n-1}(\lambda)) + \ln\left(1+\eta_{n+1}(\lambda)\right)\right], \qquad (4.66)$$

where

$$d_n(\lambda) = \sum_{k\in\mathbb{Z}} e^{-2ik\lambda} \frac{\tanh(k\phi)}{k} [(-1)^n - (-1)^k]. \qquad (4.67)$$

This equation combined with (2.111), can be solved numerically. The solution gives the $\boldsymbol{\rho}^{sp}$.

Chapter 5

Quasilocal charges

We have already stated that a physical system is integrable, if it is endowed with an infinite set of conserved charges, which generate an infinite set of conservation laws. Dealing with lattice models or spin chains the conserved charges gain a fundametal feature, called locality, which means that the densities of these charges act non-trivially only on a finite number of adjacent lattice sites. Local charges cover an important role in the analysis of the thermodynamics of such systems: we have seen that a generalization of the canonical ensemble, called Generalized Gibbs Ensemble, which takes into account the presence of the conserved charges in addition to the Hamiltonian, is a useful tool to describe thermalization after a quantum quench.

But it has been recently shown that the GGE sometimes fails to describe the correct quantum Quench Action, like in the gapped XXZ model ($\Delta = \cosh\eta > 1, \eta > 0$. In this chapter we use η instead of ϕ, because the last letter is used for other quantities). Some researchers have proposed a solution in order to solve the problem: the family of local conserved charges has to be extended incorporating *quasilocal* conserved charges, whose meaning we are going to analyze.

5.1 Definitions

The total Hilbert space, formed by a tensor product of d-dimensional single-site Hilbert spaces, will be denoted by $\mathcal{H}$. The Hilbert space of a lattice subinterval between sites x and $x^{'}$, $x \leq x^{'}$, will be denoted by $\mathcal{H}_{[x,x']}$ and the corresponding operator subalgebra by $\mathcal{U}_{[x,x']}$. An observable represented by an operator $a \in \mathcal{U}$ is local if it acts non-trivially only on a finite subinterval $[x, x^{'}]$:

$$a = a_{[x,x']} \otimes \mathbb{I}_{\mathbb{Z}\setminus[x,x']}, \qquad a_{[x,x']} \in \mathcal{U}_{[x,x']}, \tag{5.1}$$

where $\mathbb{Z}$ denotes the set of integers, i.e. a one-dimensional lattice.

Denoting by $Tr_{[x,x']}$ the trace over $\mathcal{H}_{[x,x']}$one defines the *tracial state* ω_0 as

$$\omega_0(a) = \frac{Tr_{[x,x']} a_{[x,x']}}{Tr_{[x,x']} \mathbb{I}_{[x,x']}}. \tag{5.2}$$

We define the Hilbert-Schmidt (HS) inner product as

$$(a, b) = \omega_0(a^\dagger b) - \omega_0(a^\dagger)\omega_0(b), \tag{5.3}$$

and denote the corresponding norm by $\| a \|_{HS} \equiv \sqrt{(a, a)}$.

We also define a lattice shift automorphism by $\hat{\mathcal{S}}^y(a_{[x,x']}) = a_{[x-y,x'-y]}$ and associate to each element $a \in \mathcal{U}$ a translationally invariant sum

$$A = \sum_x \hat{\mathcal{S}}^x(a), \tag{5.4}$$

which represents an extensive observable of a translationally invariant infinite quantum spin chain. The operator a is called a density of the observable A. Any operator sequence satisfying extensivity, i.e.

$$0 < \lim_{N\to\infty} \frac{1}{N}(A, A) < \infty \tag{5.5}$$

and the finite overlap criterion, i.e. $\lim_{N\to\infty}(b, A)$, with at least a one local operator b , is called *pseudolocal*.

If the density a can be written as a sum of mutually orthogonal terms $a_{[1,r]}$

$$a = \sum_{r=1}^{N} a_{[1,r]}, \tag{5.6}$$

for which a condition, known **quasilocality**

$$\| a_{[1,r]} \|_{HS} < Ce^{-\xi r}, \qquad \xi > 0 \tag{5.7}$$

holds, A is automatically pseudolocal. In other words a quasi-local operator has support on all lattice sites, though with an exponentially decaying norm.

5.2 Quantum Hirota equations

In the previous chapters we defined and used the fundamental tool of transfer matrix on a two-dimensional auxiliary space. For what follows it is necessary to extend the dimension of auxiliary space in order to treat with higher-dimensional irreducible unitary representation of auxiliary space ($s > 1/2$). These transfer operators are constructed from Lax operator $L_s(\lambda)$ associated with (2s+1)-dimensional auxiliary spaces $\mathcal{H}_a = V_s \cong \mathbb{C}^{2s+1}$ and satisfy

$$[T_s(\lambda), T_{s'}(\mu)] = 0, \qquad \forall s, s^{'} \in \frac{1}{2}\mathbb{Z}_+ \qquad \text{and} \qquad \lambda, \mu \in \mathbb{C}. \tag{5.8}$$

For $s = 1/2$ the standard set of local charges of XXZ model is generated by an expansion of $\log T_{\frac{1}{2}}(\lambda)$ around $\lambda = 0$:

$$H^{(k)} = -i\partial_\lambda^{k-1} \log T_{\frac{1}{2}} \left(\lambda + \frac{i\eta}{2}\right) |_{\lambda=0}, \tag{5.9}$$

where $H^{(2)}$ is the Hamiltonian. The locality of conserved operators $H^{(k)}$ lies in the fact that each $H^{(k)}$ admits an expansion in terms of a sum of local densities $h^{(k)}$ of order k, i.e.

$$H^{(k)} = \sum_{x=0}^{N-1} \hat{\mathcal{S}}^x(h^{(k)}) \equiv \sum_{x=0}^{N-1} h_x^{(k)}, \tag{5.10}$$

for any finite length N.

The higher-dimensional transfer operator satisfies an equation, known as quantum Hirota equation or T-system, which is a bilinear equation which takes the form

$$T_s\left(\lambda + \frac{i\eta}{2}\right) T_s\left(\lambda - \frac{i\eta}{2}\right) = \phi\left(\lambda + s\frac{i\eta}{2}\right) \bar{\phi}\left(\lambda + s\frac{i\eta}{2}\right) + T_{s-\frac{1}{2}}(\lambda) T_{s+\frac{1}{2}}(\lambda), \qquad s = \frac{1}{2}\mathbb{Z}_+, \tag{5.11}$$

with bar denoting complex conjugation and $\phi(\lambda) = T_0\left(\lambda + \frac{i\eta}{2}\right)$, where $T_0(\lambda) = (\sin(\lambda)/\sinh(\eta))^N$.

Higher-spin transfer operators T_s represent the canonical solution to the Hirota equation. However, because of there is a gauge freedom in choosing the operators T_s, it is convenient to define a gauge-invariant combinations known as Y-operators, which are

$$Y_{2s} = \frac{T_{s-\frac{1}{2}} T_{s+\frac{1}{2}}}{T_0^{[2s+1]} T_0^{[-2s-1]}} = \frac{T_s^+ T_s^-}{T_0^{[2s+1]} T_0^{[-2s-1]}} - \mathbb{I}, \qquad s = \frac{1}{2}\mathbb{Z}_+, \tag{5.12}$$

where the following notation is introduced: $f^{[\pm k]}(\lambda) \equiv f\left(\lambda \pm k\frac{i\eta}{2} \mp i0^+\right)$ for $\eta \neq 0$ (gapped and gapless regimes) and $f^{[\pm k]}(\lambda) \equiv f\left(\lambda \pm k\frac{i}{2} \mp i0^+\right)$ in the isotropic case ($\Delta = 1$). For the sake of clearness we write $f^{\pm}(\lambda) \equiv f^{[\pm 1]}(\lambda)$.

The Y-operators obey the so-called Y-system functional relation, which is an ubiquitous structure of integrability:

$$Y_j^+ Y_j^- = (\mathbb{I} + Y_{j-1})(\mathbb{I} + Y_{j+1}), \qquad j = 1, 2, ..., \tag{5.13}$$

where it is assumed that $Y_0 = 0$.

Another important relation which involves the transfer operator is the Baxter's TQ-equation:

$$T_{\frac{1}{2}} Q = T_0^+ Q^{[-2]} + T_0^- Q^{[+2]}, \tag{5.14}$$

where Q stands for the Baxter's Q-operator.

The Q-operator allows to linearize the bilinear equation (5.11), i.e.

$$\frac{T_s^+}{T_0^{[2s+1]}} = Q^{[2s+2]} Q^{[-2s]} \sum_{k=0}^{2s} \frac{\zeta_{2s,k}^N}{Q^{[2(k-s)]} Q^{[2(k-s+1)]}}, \tag{5.15}$$

where the scalars are expressed by

$$\zeta_{2s,k}(\lambda) = \frac{T_0^{[2(k-s)+1]}(\lambda)}{T_0^{[2s+1]\cdot}(\lambda)}. \tag{5.16}$$

The TQ-equation (5.14) admits two linearly independent solutions, Q and $\tilde{Q}$, whose independence is guaranteed if the Wronskian determinant is non-degenerate, i.e.

$$T_0 = Q^+ \tilde{Q}^- - Q^- \tilde{Q}^+. \tag{5.17}$$

Combining the equation (5.14) and the condition (5.17), it is possible to obtain an equation for the eigenvalues $\mathcal{Q}$ of Q:

$$\frac{T_0^-(\lambda_j)\mathcal{Q}^{[+2]}(\lambda_j)}{T_0^+(\lambda_j)\mathcal{Q}^{[-2]}(\lambda_j)} = -1. \tag{5.18}$$

where the λ_j are the Bethe roots of the system, which are the zeros of the eigenvalues $\mathcal{Q}$.

5.3 Quasilocal charges from unitary representation

Ilievski, Medenjak and Prosen [10] have proved that the gapped regime of XXZ model has an infinite set of conserved operators

$$X_s(\lambda) = -i\partial_\lambda \log \frac{T_s^+(\lambda)}{T_0^{[2s+1]}(\lambda)}. \qquad \lambda \in \mathbb{R}, \qquad s = \frac{1}{2}\mathbb{Z}_+, \tag{5.19}$$

generated from the higher-spin transfer operators T_s and which are quasilocal conserved charges. The proof of this statement starts from the inversion formula

$$\frac{T_s^+(\lambda)T_s^-(\lambda)}{T_0^{[-2s-1]}(\lambda)T_0^{[2s+1]}(\lambda)} \xrightarrow{N\to\infty} \mathbb{I}, \tag{5.20}$$

which allows to write an alternative definition of the charges (5.19) in a more convenient product form

$$X_s(\lambda) = -i\,\partial_\mu \frac{T_s^-(\lambda)}{T_0^{[-2s-1]}(\lambda)} \frac{T_s^+(\mu)}{T_0^{[2s+1]}(\mu)}\bigg|_{\mu=\lambda}, \qquad \lambda \in \mathbb{R}. \tag{5.21}$$

The details of the proof are explained in [10], but one of the most important achievement of the proof's steps is exactly the inversion formula, which can be deduced in an analogous form by (5.15), taking the dominating term at index $k = 2s$ in the thermodynamic limit:

$$\frac{T_s^+(\lambda)}{T^{[2s+1]}(\lambda)} \xrightarrow{N\to\infty} \frac{Q^{[-2s]}(\lambda)}{Q^{[2s]}(\lambda)}. \tag{5.22}$$

As a consequence of (5.22), the general version (for arbitrary anisotropy Δ) of the unitary quasilocal charges admits a useful representation in terms of Q-operator

$$X_s(\lambda) = -i\partial_\lambda \log \frac{Q^{[-2s]}}{Q^{[2s]}}, \qquad \lambda \in I_\eta = \left\{\lambda \in \mathbb{C}; |Im(\lambda)| < \frac{\eta}{2}\right\}. \tag{5.23}$$

The charges $X_s(\lambda)$ can now be diagonalized using the fact that the eigenvalues of Baxter's Q-operator are polynomials with the zeroes coinciding with the set of Bethe roots $\{\lambda_j\}$,

$$\mathcal{Q}(\lambda) = c\prod_{j=1}^{M}\sin(\lambda-\lambda_j), \tag{5.24}$$

where c is a constant.

5.4 Quantum quench and quasilocal charges

We have stated before that GGE fails to describe thermalization after a quantum quench, if one does not also take into account quasilocal charges. In what follows we explain the connection between the spectra of quasilocal charges X_s and distributions of Bethe strings. To this aim we briefly review the main results of Bethe Ansatz for gapped XXZ model with some new conventions.

The first important tool for an integrable model is the single-particle S-matrix S_1, which for XXZ chain takes the form

$$S_1(\lambda,\mu) \equiv S_1(\lambda-\mu) = \frac{\sin(\lambda-\mu-\frac{i\eta}{2})}{\sin(\lambda-\mu+\frac{i\eta}{2})}. \tag{5.25}$$

For states which consist of j excitations (j-strings) a set of fused scattering matrices S_j are introduced

$$S_j(\lambda) = \frac{\sin(\lambda-j\frac{i\eta}{2})}{\sin(\lambda+j\frac{i\eta}{2})}, \qquad j=1,2,.... \tag{5.26}$$

Scattering among two different types of strings is governed by string-to-string scattering matrices

$$S_{j,k}(\lambda) = S_{|j-k|}(\lambda)S_{j+k}(\lambda)\prod_{i=1}^{\min(j,k)-1} S^2_{|j-k|+2i}(\lambda). \tag{5.27}$$

The quantization condition in a periodic system for the rapidities λ_j takes the form

$$e^{ip(\lambda_j)N}\prod_{k=1}^{M} S_{1,1}(\lambda_j-\lambda_k) = -1, \qquad j=1,2,...,M, \tag{5.28}$$

where M is the number of Bethe roots related to the magnetization state and $p(\lambda)$ is the momentum of an elementary excitation of a ferromagnetic vacuum state

$$e^{ip(\lambda)} = \frac{\sin(\lambda + \frac{i\eta}{2})}{\sin(\lambda - \frac{i\eta}{2})}. \tag{5.29}$$

The string hypothesis states that in the large-N limit the Bethe roots for an eigenstate become equidistantly displaced in the imaginary axes in the rapidity complex plane:

$$\{\lambda_\alpha^{k,j}\} \equiv \{\lambda_\alpha^k + (k+1-2j)\frac{i\eta}{2} | j = 1, 2, ..., k\}. \tag{5.30}$$

Such string configuration corresponds to bound states of magnons.

In the continuum limit it is more convenient to use densities $\rho_j(\lambda)$, which describe the distribution of the string centers and obey the following non-linear coupled integral equations

$$\rho_j(\lambda) + \bar{\rho}_j(\lambda) = a_j(\lambda) - \sum_k \int_{-\pi/2}^{\pi/2} \frac{d\mu}{2\pi} a_{j,k}(\lambda - \mu)\rho_k(\mu), \tag{5.31}$$

where

$$a_j(\lambda) = -i\partial_\lambda \log S_j(\lambda), \qquad a_{j,k}(\lambda) = -i\partial_\lambda \log S_{j,k}(\lambda) \tag{5.32}$$

and $\bar{\rho}_j(\lambda)$ parametrize distributions of Bethe holes.

To obtain the spectra of charges X_s we make use of equation (5.23):

$$\langle\{\lambda_j\}|X_s(\lambda)|\{\lambda_j\}\rangle = -i\partial_\lambda \log \frac{\mathcal{Q}^{[-2s]}(\lambda)}{\mathcal{Q}^{[2s]}(\lambda)}, \tag{5.33}$$

where $|\{\lambda_j\}\rangle$ denote a Bethe eigenstate parametrized by a set of roots λ_j.

Working under the string hypothesis, the spectra of quasilocal charges $\mathcal{X}_s$

$$\mathcal{X}_s = \lim_{N\to\infty} \frac{1}{N} \langle\{\lambda_j\}|X_s(\lambda)|\{\lambda_j\}\rangle \tag{5.34}$$

can be expressed in terms of densities of string centers $\rho_j(\lambda)$. Explicitly we have

$$\mathcal{X}_s(\lambda) = \sum_k \int_{-\pi/2}^{\pi/2} \frac{d\mu}{2\pi} G_{2s,k}(\lambda - \mu)\rho_k(\mu), \tag{5.35}$$

where

$$G_{2s,k}(\lambda) = \sum_{j=1}^{k} -i\partial_\lambda \log S_{2s}\left(\lambda + (k+1-2j)\frac{i\eta}{2}\right) = \sum_{j=1}^{\min(2s,k)} a_{|2s-k|-1+2j}(\lambda). \tag{5.36}$$

To the aim of dealing with the quench operation it is also important the following result. In section 5.2 it has been shown that higher-spin T-operators constitute the canonical solution of Hirota equations. However, Hirota equations admit a class of non-canonical solutions, which are related to a class of initial conditions, which relax to equilibrium steady states. Examples of these states are the spin-singlet dimerized state $|D\rangle = 1/\sqrt{2}(|\uparrow\downarrow\rangle) - |\downarrow\uparrow\rangle))^{\otimes N/2}$ and Néel state $|N\rangle = |\uparrow\downarrow\rangle)^{\otimes N/2}$.

Indicating with $t_s(\lambda)$ and $q(\lambda)$ the non-canonical functions of Hirota equations, as the counterparts of the canonical operators $T_s(\lambda)$ and $Q(\lambda)$ and relaxing the constraint $t_0 = \phi^-$ Hirota equations take the form

$$t_{s+\frac{1}{2}} q^{[2s]} - t_s^- q^{[2s+2]} = \phi^{[2s]} \bar{q}^{[-2s-2]} \tag{5.37}$$

whose solution is

$$t_s = t_0^{[-2s]} \frac{q^{[2s+1]}}{q^{[-2s+1]}} + q^{[2s+1]} \bar{q}^{[-2s-1]} \sum_{k=1}^{2s} \frac{\phi^{[2(k-s)-1]}}{q^{[2(k-s)-1]} q^{[2(k-s)+1]}}. \tag{5.38}$$

With these tools in [12] it has been found explicitly the expectation values of quasilocal charges X_s for the dimerized state and the Néel state.

5.5 Explicit calculation of quasilocal charges for the XXZ model

In what follows we refer to the paper [21]. For the conventions adopted here and some formalism related to quantum group we refer to appendix C.

Now we want to show how to obtain a quasilocal charge from the two-parameter conserved quantity which is odd under parity

$$\mathcal{I}(z,u) = \sum_{\{\alpha_j\}} tr(A_{\alpha 1}...A_{\alpha N} - A^t_{\alpha 1}...A^t_{\alpha N}) \prod_{j=1}^{N} \sigma_j^{\alpha_j}. \tag{5.39}$$

Using the inner product defined in section 5.1 we want to compute $(\mathcal{I}^\dagger, \mathcal{I}) = 2^{-N} Tr(\mathcal{I}^\dagger \mathcal{I})$, where Tr denotes the trace over the quantum space $V^{\otimes N}$. It turns out that

$$(\mathcal{I}^\dagger, \mathcal{I}) = 2 tr_{\mathcal{A}\otimes\mathcal{A}}(T_1(z,u,\bar{u})^N - T_2(z,u,\bar{u})^N). \tag{5.40}$$

Here $T_1(z,u,\bar{u})^N$ and $T_2(z,u,\bar{u})^N$ are transfer matrices in $\mathcal{A}\otimes\mathcal{A}$

$$T_1(z,u,\bar{u}) = \sum_{\{\alpha=0,z,\pm\}} C_\alpha A^*_\alpha(z,u) \otimes A_\alpha(z,\bar{u}), \tag{5.41}$$

$$T_2(z,u,\bar{u}) = \sum_{\{\alpha=0,z,\pm\}} C_\alpha A^*_\alpha(z,u) \otimes A^t_\alpha(z,\bar{u}), \tag{5.42}$$

where

$$C_\alpha = \frac{1}{2} Tr(\sigma_\alpha (\sigma_\alpha)^\dagger). \tag{5.43}$$

We focus on the analysis of the operator

$$\mathcal{I}_0 \equiv \mathcal{I}(z = i, u = 1). \tag{5.44}$$

With these values of the parameters z and u the operators, which construct the Lax operator become

$$A_0(1) \equiv A_0(z = i, u = 1) = \sum_{r=0}^{m-1} \cos(\lambda r) |r\rangle\langle r|, \tag{5.45}$$

$$A_z = 0 \tag{5.46}$$

$$A_+(1) \equiv A_+(z = i, u = 1) = i \sum_{r=0}^{m-2} \sin(\lambda(r+1)) |r\rangle\langle r+1|, \tag{5.47}$$

$$A_-(1) \equiv A_-(z = i, u = 1) = -i \sum_{r=0}^{m-2} \sin(\lambda r) |r+1\rangle\langle r|. \tag{5.48}$$

The transfer matrices (5.41) and (5.42) become

$$T_1(1) \equiv T_2(z=i, u=\bar{u}=1) = \sum_{r,s=0}^{m-1} \cos(\lambda r)\cos(\lambda s)|r,s\rangle\langle r,s| + \frac{1}{2}\sum_{r,s=0}^{m-2} \sin(\lambda(r+1))\sin(\lambda(s+1))|r,s\rangle\langle r+1,s+1| - \frac{1}{2}\sum_{r,s=0}^{m-2} \sin(\lambda r)\sin(\lambda s)|r+1,s+1\rangle\langle r,s|, \quad (5.49)$$

$$T_2(1) \equiv T_2(z=i, u=\bar{u}=1) = \sum_{r,s=0}^{m-1} \cos(\lambda r)\cos(\lambda s)|r,s\rangle\langle r,s| + \frac{1}{2}\sum_{r,s=0}^{m-2} \sin(\lambda(r+1))\sin(\lambda(s+1))|r,s+1\rangle\langle r+1,s| - \frac{1}{2}\sum_{r,s=0}^{m-2} \sin(\lambda r)\sin(\lambda s)|r+1,s\rangle\langle r,s+1|. \quad (5.50)$$

Since we are interested in the thermodynamic limit, we restrict to the subspace in which the transfer matrices have their largest eigenvalue. Denoting $|r,\pm r\rangle$ with $|r\rangle$ we obtain

$$\mathcal{T}_1 = \sum_{r=0}^{m-1} \cos^2(\lambda r)|r\rangle\langle r| + \frac{1}{2}\sum_{r=0}^{m-2} \sin^2(\lambda(r+1))|r\rangle\langle r+1| + \frac{1}{2}\sum_{r=0}^{m-2} \sin^2(\lambda r)|r+1\rangle\langle r|, \quad (5.51)$$

$$\mathcal{T}_2 = \sum_{r=0}^{m-1} \cos^2(\lambda r)|r\rangle\langle r| - \frac{1}{2}\sum_{r=0}^{m-2} \sin(\lambda r)\sin(\lambda(r+1))\times \times (|r\rangle\langle r+1| + |r+1\rangle\langle r|). \quad (5.52)$$

The matrix $\mathcal{T}_2$ is symmetric and since $\mathcal{T}_2|r=0\rangle = |r=0\rangle = |0\rangle$, $|0\rangle$ is an eigenvector of $\mathcal{T}_2$ with eigenvalue (the largest one) 1. $\mathcal{T}_1$ is not symmetric instead and so we have to distinguish the right eigenvector $|0\rangle$, which coincides with that of $\mathcal{T}_2$, from the left eigenvector $\langle 0_L|$, which has the form

$$\langle 0_L| = \sum_{r=0}^{m-1}(1-\frac{r}{m})\langle r|. \tag{5.53}$$

In order to calculate the thermodynamic limit, the following relations are useful:

$$\lim_{n\to\infty} \mathcal{T}_1^n = |0\rangle\langle 0_L|, \tag{5.54}$$

$$\lim_{n\to\infty} \mathcal{T}_2^n = |0\rangle\langle 0|. \tag{5.55}$$

The norm $(\mathcal{I}_0^\dagger, \mathcal{I}_0)$ is zero, because it can be shown that $\mathcal{T}_1$ and $\mathcal{T}_2$ are related by a similarity transformation. But we can find a quasilocal operator by expanding $\mathcal{I}(z=i,u)$ about $u=1$, i.e.

$$\mathcal{I}(z=i,u=1+\epsilon) = \epsilon\mathcal{I}_1 + O(\epsilon^2), \tag{5.56}$$

where

$$\mathcal{I}_1 = \left.\frac{\partial\mathcal{I}(z,u)}{\partial u}\right|_{z=i,u=1}. \tag{5.57}$$

From (5.41) and (5.42) the norm of $\mathcal{I}_1$ is

$$\begin{aligned}(\mathcal{I}_1^\dagger, \mathcal{I}_1) = 2\ tr\left(\frac{\partial^2}{\partial u\partial\bar{u}} T_1(z,u,\bar{u})^N\right)\Big|_{z=i,u=\bar{u}=1} \\ - 2\ tr\left(\frac{\partial^2}{\partial u\partial\bar{u}} T_2(z,u,\bar{u})^N\right)\Big|_{z=i,u=\bar{u}=1}\end{aligned} \tag{5.58}$$

Using the following notation for the derivatives of the reduced transfer matrices

$$\mathcal{T}_x^{(n,l)} \equiv \frac{\partial^n}{\partial u^n}\frac{\partial^l}{\partial\bar{u}^l}\, T_x(z=i,u,\bar{u})|_{u=\bar{u}=1} \tag{5.59}$$

with $x=1,2$, the (5.58) becomes

$$\frac{(\mathcal{I}_1^\dagger,\mathcal{I}_1)}{N} = 2tr\left(\mathcal{T}_1^{N-1}\mathcal{T}_1^{(1,1)} + \mathcal{T}_2^{N-1}\mathcal{T}_2^{(1,1)}\right) + 2\sum_{n=0}^{N-2} tr\left(\mathcal{T}_1^{(1,0)}\mathcal{T}_1^{n}\mathcal{T}_1^{(0,1)}\mathcal{T}_1^{N-2-n}\right) + 2\sum_{n=0}^{N-2} tr\left(\mathcal{T}_2^{(1,0)}\mathcal{T}_2^{n}\mathcal{T}_2^{(0,1)}\mathcal{T}_2^{N-2-n}\right). \quad (5.60)$$

In order for $\mathcal{I}_1$ to be quasilocal, the rhs of (5.60) must approach a finite value in the limit $N \to \infty$.

First of all we explicit the expression of the derivatives of the reduced transfer matrices:

$$\mathcal{T}_1^{(1,0)} = \mathcal{T}_1^{(0,1)} = \frac{i}{2}\sum_{r=0}^{m-1}\sin(2\lambda r)|r\rangle\langle r| + \frac{i}{2}\sum_{r=0}^{m-2}\sin(2\lambda r)|r+1\rangle\langle r|, \quad (5.61)$$

$$\mathcal{T}_2^{(0,1)} = i\sum_{r=0}^{m-1}\sin(\lambda r)\cos(\lambda r)|r\rangle\langle r| + i\sum_{r=0}^{m-2}\sin(\lambda r)\cos(\lambda(r+1))|r+1\rangle\langle r|, \quad (5.62)$$

$$\mathcal{T}_2^{(1,0)} = i\sum_{r=0}^{m-1}\sin(\lambda r)\cos(\lambda r)|r\rangle\langle r| - i\sum_{r=0}^{m-2}\sin(\lambda(r+1))\cos(\lambda r)|r+1\rangle\langle r|, \quad (5.63)$$

$$\mathcal{T}_1^{(1,1)} = \sum_{r=0}^{m-1}\sin^2(\lambda r)|r\rangle\langle r| + 2\sum_{r=0}^{m-2}\cos^2(\lambda r)|r+1\rangle\langle r|, \quad (5.64)$$

$$\mathcal{T}_2^{(1,1)} = \sum_{r=0}^{m-1}\sin^2(\lambda r)|r\rangle\langle r| + 2\sum_{r=0}^{m-2}\cos(\lambda r)\cos(\lambda(r+1))|r+1\rangle\langle r|. \quad (5.65)$$

Noticing that

$$\mathcal{T}_1^{(1,0)}|0\rangle = \mathcal{T}_1^{(0,1)}|0\rangle = 0, \tag{5.66}$$

$$\mathcal{T}_2^{(1,0)}|0\rangle = \langle 0|\mathcal{T}_2^{(0,1)} = 0, \tag{5.67}$$

and recalling the projection (5.54), we are left only with the first term in the rhs of (5.60):

$$\lim_{N\to N} \frac{(\mathcal{I}_1^\dagger, \mathcal{I}_1)}{N} = 2\langle 0_L|\mathcal{T}_1^{(1,1)}|0\rangle + 2\langle 0|\mathcal{T}_2^{(1,1)}|0\rangle. \tag{5.68}$$

Using the (5.53) we finally obtain

$$\lim_{N\to N} \frac{(\mathcal{I}_1^\dagger, \mathcal{I}_1)}{N} = 4\left(1 - \frac{1}{m}\right), \tag{5.69}$$

proving that $\mathcal{I}_1$ is quasilocal for $m > 1$.

Chapter 6

Quasilocal charges in the gapless XXZ model and relation to SG model

6.1 A brief overview on Sine-Gordon (SG) model

The SG model is an integrable field theory with the following Lagrangian:

$$\mathscr{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + g\frac{\sqrt{a\pi}}{\beta}\cos\left(\frac{\beta}{\sqrt{4\pi}}\varphi\right). \tag{6.1}$$

The SG model is well defined within the interval $0 \leq \beta < \sqrt{8\pi}$. The dynamics of the model depends on the parameter β. Therefore we have to distinguish two cases:

1. $0 \leq \beta < \sqrt{4\pi}$: it is the so called attractive regime. The dynamics is described by the scattering of particles (or quasi-particles), i.e. the soliton-antisoliton scattering, the soliton-breather scattering and the breather-breather scattering. The breathers are considered as bound states. The mass of a breather M_j is related to the mass M_{sol} of a soliton: $M_j = 2M_{sol}\sin(\pi\xi j/2)$, where

$$\xi = \frac{\frac{\beta^2}{8\pi}}{1-\frac{\beta^2}{8\pi}}. \tag{6.2}$$

2. $\sqrt{4\pi} \leq \beta < \sqrt{8\pi}$: this is the so called repulsive regime. The dynamics is described by a scattering of a soliton and an antisoliton, a soliton and a soliton or an antisoliton and an antisoliton.

For more details on the SG model see the book of G. Mussardo *Statistical Field Theory* [19] and the literature therein.

6.2 SG model and its Y-system

In the paper of Vernier and Cubero [29], using the approach of Destri and de Vega [3], it is shown that the inhomogeneous XXZ chain (with $\omega = 0$) in the gapless regime (see section 3.3) becomes with an appropriate scaling limit ($\theta \to \infty$) the repulsive SG model. This procedure is called light-cone regularization. The connection between these two models allows to write a relation between the respective densities, which constitutes the TBA equations. As hinted in the preceding chapter, there is a relation between the quasilocal charges and the TBA densities. In addition since the quasilocal charges are defined through the higher-spin transfer matrices, it possible to make a link between the quasilocal charges and the Y-system (the Y-operator is obtained by a gauge transformation on T_s).

Following [29], the parameter of the SG model and the inhomogeneous XXZ chain are related as follows:

$$\frac{\beta^2}{8\pi} = \frac{p}{p+1} = 1 - \frac{\gamma}{\pi} \qquad \longrightarrow \qquad \gamma = \frac{\pi}{p+1}, \tag{6.3}$$

where p is a positive integer, and it is used to number the densities.

In what follows we attempt to generalize the results of the just cited paper to rational values of p. To this aim we take advantage of the work of R. Tateo in [28] in order to construct the Y-system for the attractive regime. Thanks to the so called duality property of this system, we obtain also the Y-system for the repulsive regime and so for the inhomogeneous XXZ chain.

Tateo [28] works directly on the ξ parameter, which is expressed through the continued fraction formalism. In the attractive regime $\xi < 1$. We have that

$$\xi = \frac{p}{q-p} = \hat{\xi}(n_1, n_2, ..., n_F) \stackrel{def}{=} \frac{1}{n_1 + \frac{1}{n_2 + ... + \frac{1}{n_F - 1}}}, \tag{6.4}$$

where p, q are integers, which satisfies the inequality $q > 2p$ in order to remain in the attractive regime (p has not any relation between the p used in [29]).

The TBA equation at this point contains n_1 breathers, a soliton and $\sum_{i=2}^{F} n_i$ magnons (these are quasi-particles introduced to describe the colour interchange of the soliton) for a total of $n_T = \sum_{i=1}^{F} n_i$ (quasi-)particles. Defining the shifts

$$s_1 = i\pi\frac{\xi_1}{2}, \qquad s_2 = i\pi\frac{\xi_1\xi_2}{2}, \qquad ..., \qquad s_F = i\pi\frac{\xi_1\xi_2...\xi_F}{2} \tag{6.5}$$

with

$$\xi_i = \hat{\xi}(n_i, n_{i+1}, ..., n_F), \tag{6.6}$$

we associate to any (quasi-)particle a node in a D_{n_T} Dynkin diagram. An example of this kind of diagrams is shown in Fig. 6.1.

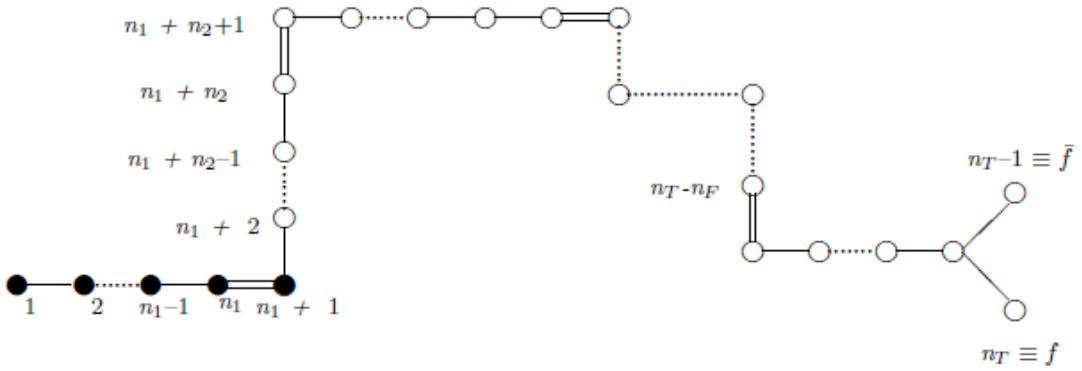

Figure 6.1: The TBA graph associated to the SG model in a rational point. The black or white spots are called nodes. There are n_1 black nodes associated to the breathers, the node n_1 corresponds to the soliton, all the other nodes are magnons. The links on the bifurcation are horizontal if F is even or vertical if F is odd. The double-link defines the concatenation of an horizontal(vertical) with a vertical (horizontal) sub-diagram with a change of the shift S_i. For conventions we indicate as $f \equiv n_T$ and $\bar{f} \equiv n_T - 1$ the two nodes on the bifurcation of the D_{n_T} diagram.

We define for any nodes a shift

$$S_i = s_a, \qquad \sum_{k=1}^{a-1} n_k < i \leq \sum_{k=1}^{a} n_k, \tag{6.7}$$

so that i runs on the same horizontal or vertical sub-diagram.
In addition we define for any pairs of nodes (i, j) an exponent

$$c_{i,j} = c_{j,i} = (-1)^{a-1}, \qquad \sum_{k=1}^{a-1} n_k < i,j \leq 1 + \sum_{k=1}^{a} n_k, \tag{6.8}$$

with $c_{i,j} = 0$ if i and j are not adjacent. We finally put $\tilde{c}_j = c_{j,j+1}$. The Y-system for the nodes in the set $\{n_1, n_1+n_2, ..., n_T - n_F - n_{F-1}\}$, i.e. the index k of the Y-operators is $\sum_{l=1}^{a} n_l$ and $a < F-1$, is

$$\begin{aligned} Y_k(\lambda+S_k)Y_k(\lambda-S_k) &= (1+Y_{k-1}(\lambda)^{c_{k,k-1}})^{c_{k,k-1}}(1+Y_{k+n_{a+1}+1}(\lambda)^{\tilde{c}_k})^{\tilde{c}_k} \\ &\times \prod_{j=k+1}^{k+n_{a+1}} (1+Y_j(\lambda+(k+n_{a+1}-j)S_j + S_{k+n_{a+1}+1})^{\tilde{c}_k})^{\tilde{c}_k} \\ &\times \prod_{j=k+1}^{k+n_{a+1}} (1+Y_j(\lambda-(k+n_{a+1}-j)S_j - S_{k+n_{a+1}+1})^{\tilde{c}_k})^{\tilde{c}_k}. \end{aligned} \tag{6.9}$$

For the node $k = n_T - n_F$ we have

$$\begin{aligned} Y_k(\lambda+S_k)Y_k(\lambda-S_k) &= (1+Y_{k-1}(\lambda)^{c_{k,k-1}})^{c_{k,k-1}}(1+Y_f(\lambda)^{\tilde{c}_k})^{\tilde{c}_k}(1+Y_{\bar{f}}(\lambda)^{\tilde{c}_k})^{\tilde{c}_k} \\ &\times \prod_{j=k+1}^{n_T-2} (1+Y_j(\lambda+(n_T-1-j)S_j)^{\tilde{c}_k})^{\tilde{c}_k} \\ &\times \prod_{j=k+1}^{n_T-2} (1+Y_j(\lambda-(n_T-1-j)S_j)^{\tilde{c}_k})^{\tilde{c}_k}, \end{aligned} \tag{6.10}$$

and for the remaining nodes

$$Y_k(\lambda+S_k)Y_k(\lambda-S_k) = \prod_{j\in adj} (1+Y_j^{c_{i,j}})^{c_{i,j}}, \tag{6.11}$$

where the product runs over all adjacent nodes in the Dynkin diagram. A crucial properties for our purposes is that the Y-systems at ξ^{-1} in the repulsive region can be obtained from those at ξ changing $c_{i,j}$ with $-c_{i,j}$ and s_i with $\xi^{-1}s_i$. This is the so-called TBA-duality in the SG model.

Now we have to link the notations of [28] and that of [29]. From the definition of ξ in the (6.4) we have that $\beta^2/8\pi = p/q$ and it is obvious that $\beta^2/8\pi < 1/2$. Defining $\xi^{-1} = \bar{\xi}$ we have that

$$\bar{\xi} = \frac{q-p}{p}, \tag{6.12}$$

and so

$$\frac{q-p}{p} = \frac{\bar{\beta}^2/8\pi}{1-\bar{\beta}^2/8\pi} \qquad \longrightarrow \qquad \frac{\bar{\beta}^2}{8\pi} = \frac{q-p}{q} = 1 - \frac{p}{q} = 1 - \frac{\gamma}{\pi}. \tag{6.13}$$

This implies that

$$\gamma = \frac{\pi p}{q} \qquad \text{and} \qquad \gamma < \frac{\pi}{2}. \tag{6.14}$$

In this way we have the Y-system for the repulsive regime (thanks to the duality property) and the corresponding Y-system for the inhomogeneous XXZ model, which in turn is related to the gapless XXZ model by setting the inhomogeneity θ equal to zero.

In the TBA context the Y-operators are related to the TBA densities through the relation

$$Y_j = \frac{\rho_j^h}{\rho_j}. \tag{6.15}$$

This fact establish a deep contact between the integrable structure (Y-system) and the thermodynamics (densities ρ) of a system.

Since the Y-operators are related to the higher-spin transfer matrices T_j (now we label the transfer matrices with j-th representation of the auxiliary space, while before we labeled them with the spin s) through a gauge transformation (see (5.12)), we can take advantage of the relation found in [12]

$$X_j(\mu) = \frac{1}{N}\frac{1}{2\pi i}\partial_\mu \ln \frac{T_j^+(\mu)}{T_0^{[j+1]}(\mu)} \tag{6.16}$$

to connect the indices of the Y-operators, and so the indices of the densities, to the indices of the quasilocal charges (here $T_j^{[\pm k]}(\mu) = T_j(\mu \pm ik\gamma/2)$). This is the so called string-charge duality, where the string content, referred to the Bethe roots densities, is now governed by the continued fractions decomposition. The eigenvalues of X_j could be calculated once Y-operators and higher-spin transfer matrix T_j are known, for example solving numerically the TBA equations.

We conclude with the expression that could have the GGE density matrix, once that the quasilocal charges are added to the set of local charges:

$$\rho^{GGE} = \frac{1}{Z} e^{-\sum_j \sum_n \beta_{j,n} X_{j,n}(\lambda)}, \tag{6.17}$$

where the $\beta_{j,n}$ are the generalized chemical potentials, Z is chosen so that the trace of the density is one and the double indices follows from the definition

$$X_{j,n} = -\left(\partial_\lambda\right)^{n-1} X_j(\lambda)\big|_{\lambda=0}\,. \tag{6.18}$$

With $j = 1$ we have the usual local conserved charges.

6.3 Conclusions and perspectives

In this thesis we have proposed a construction of quasilocal charges for the gapless XXZ model, using the functional relations of Y-system, which are related to the thermodynamics of the model. This method has allowed to connect the gapless XXZ model with SG model, which is an interacting, integrable field theory.

For further developments in this line of research one can envisages to apply this same reasonings to the restricted SG theories, i.e. the minimal models of conformal fields theory perturbed by $\phi_{1,3}$ (the least relevant operator), as well as to many other integrable models, where TBA technique is known.

Appendix A

Bethe equation

From the definition of cotangent we have that

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = i\frac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}} \tag{A.1}$$

and so

$$\cot\frac{\theta}{2} = i\frac{e^{i\theta}+1}{e^{i\theta}-1}. \tag{A.2}$$

Using (2.26) we have

$$e^{i\theta}+1 = \frac{2e^{ik_1}-2e^{ik_2}}{e^{i(k_1+k_2)}+1-2e^{ik_2}} \tag{A.3}$$

$$e^{i\theta}-1 = -\frac{2e^{i(k_1+k_2)}+2-2e^{ik_1}-2e^{ik_2}}{e^{i(k_1+k_2)}+1-2e^{ik_2}}. \tag{A.4}$$

From the ratio of the two last equations we obtain

$$\begin{aligned}\frac{e^{i\theta}+1}{e^{i\theta}-1} &= -\frac{e^{ik_1}-e^{ik_2}}{e^{i(k_1+k_2)}+1-e^{ik_1}-e^{ik_2}} \\ &= \frac{e^{ik_1}+1}{2(e^{ik_1}-1)} - \frac{e^{ik_2}+1}{2(e^{ik_2}-1)} \implies 2\cot\frac{\theta}{2} = \cot\frac{k_1}{2} - \cot\frac{k_2}{2}.\end{aligned} \tag{A.5}$$

Appendix B

Tensor product

Given two finite-dimensional linear spaces V_1 and V_2 with bases $\{e^k\}$ and $\{f^k\}$ respectively, their *tensor product* $V_1 \otimes V_2$ is the space generated by the basis elements $e^{ik} = e^i \otimes f^k$. It is clear that if the dimensions of V_1 and V_2 are respectively N_1 and N_2, the space $V_1 \otimes V_2$ have dimension $N_1 N_2$.

The operation of the tensor product is linear:

$$(ax_1 + bx_2) \otimes y = ax_1 \otimes y + bx_2 \otimes y, \qquad x_1, x_2 \in V_1,\ y \in V_2, \tag{B.1}$$

$$x \otimes (ay_1 + by_2) = ax \otimes y_1 + bx \otimes y_2, \qquad x \in V_1,\ y_1, y_2 \in V_2. \tag{B.2}$$

Then given two vectors $x \in V_1$ and $y \in V_2$ and considering their expansions on basis elements

$$x = \sum_{i=1}^{N_1} \lambda_i e^i, \qquad y = \sum_{k=1}^{N_2} \mu_k f^k, \tag{B.3}$$

we have

$$x \otimes y = \sum_{i=1}^{N_1} \sum_{k=1}^{N_2} \lambda_i \mu_k e^{ik}. \tag{B.4}$$

If the spaces V_1 and V_2 are $\mathbb{C}^2$, the basis of the space $V_1 \otimes V_2$ consists of four vectors

$$e^{11} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; e^{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$
$$e^{21} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; e^{22} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad \text{(B.5)}$$

In general the components of the tensor product $z = x \otimes y$ of two vectors $x, y \in \mathbb{C}^2$ are labeled by two indices $z^{ik} = x^i y^k$:

$$\begin{pmatrix} z^{11} \\ z^{12} \\ z^{21} \\ z^{22} \end{pmatrix} = \begin{pmatrix} x^1 y^1 \\ x^1 y^2 \\ x^2 y^1 \\ x^2 y^2 \end{pmatrix}. \quad \text{(B.6)}$$

We note that the components of z are grouped in a way so that the first two belong to a block labeled by 1 and the last two to a block labeled by 2.

This order of the components suggests a way to write the elements of a 4×4 matrix acting on $V_1 \otimes V_2$. If A is such a matrix, its elements are $A^{ab,jk}$, where each indices takes values 1, 2:

$$\begin{pmatrix} A^{11,11} & A^{11,12} A^{12,11} & A^{12,12} \\ A^{11,21} & A^{11,22} A^{12,21} & A^{12,22} \\ A^{21,11} & A^{21,12} A^{22,11} & A^{22,12} \\ A^{21,21} & A^{21,22} A^{22,21} & A^{22,22} \end{pmatrix}. \quad \text{(B.7)}$$

The matrix A can be seen as 2×2 block-matrix, where each block is a 2×2 matrix. The first pair of superscripts a and b corresponds to the space V_1 and specify the block number, while the second pair j and k corresponds to the space V_2 and specify the element number in the block. Therefore, if $z \in V_1 \otimes V_2$ the usual row/column product gives

$$(Az)^{aj} = A^{ab,jk} z^{bk}. \quad \text{(B.8)}$$

Analogously the standard product C of two matrices A and B acting on $V_1 \otimes V_2$ is

$$C^{ab,jk} = A^{ac,jl} B^{cb,lk}. \quad \text{(B.9)}$$

For our purposes the matrices acting in $V_1 \otimes V_2$ which act non-trivially only in one of the two spaces, while act as the identity in the other, are of particular interest.

Denoting by subscripts 1 or 2 the space in which a matrix acts non-trivially, if A is a 2×2 matrix we have

$$A_1(x \otimes y) = (Ax) \otimes y, \qquad A_2(x \otimes y) = x \otimes (Ay). \tag{B.10}$$

The 4×4 matrices A_1 and A_2 have entries

$$A_1^{ab,jk} = A^{ab}\delta^{jk}, \qquad A_2^{ab,jk} = \delta^{ab}A^{jk}, \tag{B.11}$$

or explicitly

$$A_1 = \begin{pmatrix} A^{11} & 0 & A^{12} & 0 \\ 0 & A^{11} & 0 & A^{12} \\ A^{21} & 0 & A^{22} & 0 \\ 0 & A^{21} & 0 & A^{22} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} A^{11} & A^{12} & 0 & 0 \\ A^{21} & A^{22} & 0 & 0 \\ 0 & 0 & A^{11} & A^{12} \\ 0 & 0 & A^{21} & A^{22} \end{pmatrix}. \tag{B.12}$$

Now we define the tensor product of two matrices. Given two 2×2 matrices A and B acting in V_1 and V_2 respectively, their tensor product $A \otimes B$ is

$$(A \otimes B)(x \otimes y) = (Ax) \otimes (By). \tag{B.13}$$

Using the notations above we have

$$A_1B_2(x \otimes y) = A_1(x \otimes (By)) = (Ax) \otimes (By), \tag{B.14}$$
$$B_2A_1(x \otimes y) = B_2((Ax) \otimes y) = (Ax) \otimes (By). \tag{B.15}$$

We note that

$$(A \otimes B) = A_1B_2 = B_2A_1, \tag{B.16}$$

which is proved by direct calculation:

$$(A_1B_2)^{ab,jk} = A_1^{ac,jl}B_2^{cb,lk} = A^{ac}\delta^{jl}\delta^{cb}B^{lk} = A^{ab}B^{jk}, \tag{B.17}$$
$$(B_2A_1)^{ab,jk} = B_2^{ac,jl}A_1^{cb,lk} = \delta^{ac}B^{jl}A^{cb}\delta^{lk} = A^{ab}B^{jk}. \tag{B.18}$$

The tensor product of the matrices A and B is 4×4 matrix having a block structure, whose blocks are 2×2 matrix B multiplied by the corresponding element of the matrix A:

$$(A \otimes B) = \begin{pmatrix} A^{11}B^{11} & A^{11}B^{12} & A^{12}B^{11} & A^{12}B^{12} \\ A^{11}B^{21} & A^{11}B^{22} & A^{12}B^{21} & A^{12}B^{22} \\ A^{21}B^{11} & A^{21}B^{12} & A^{22}B^{11} & A^{22}B^{12} \\ A^{21}B^{21} & A^{21}B^{22} & A^{22}B^{21} & A^{22}B^{22} \end{pmatrix}. \tag{B.19}$$

The tensor product of matrices of other dimensions could be easily generalized.

An important operator acting in the space $V_1 \otimes V_2$ is the permutation matrix P, which has the property

$$P(x \otimes y) = y \otimes x. \tag{B.20}$$

The entries of P are

$$P^{ab,jk} = \delta^{ak}\delta^{bj}, \tag{B.21}$$

i.e.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.22}$$

We immediately note that $P^2 = \mathbb{I}$. P acts similarly on matrices:

$$P(A \otimes B)P = B \otimes A. \tag{B.23}$$

In addition P swaps the pair of the indices of a matrix C acting in $V_1 \otimes V_2$:

$$(PCP)^{ab,jk} = P^{ac,jl}C^{cd,le}P^{db,ek} = \delta^{al}\delta^{cj}C^{cd,le}\delta^{dk}\delta^{be} = C^{jk,ab}. \tag{B.24}$$

This operation can be written as

$$PC_{12}P = C_{21}, \text{ where } C_{21}^{ab,jk} = C_{12}^{jk,ab}. \tag{B.25}$$

The subscripts indicate to which space the pairs of superscript refer to.

Working with the tensor product $V_1 \otimes V_2$, it is possible to take the trace of a matrix only over one space: $tr_1 A = A^{aa,jk}$ and $tr_2 A = A^{ab,jj}$. In this way the partial traces are 2×2 matrices, which act in V_2 and V_1 respectively.

Now we generalize the discussions above to the tensor product of several space. Let $V_1 \otimes ... \otimes V_N$, where every V_k is isomorphic to $\mathbb{C}^2$. Each vector has 2^N components, which are numbered by a sequence of superscripts $z^{a_1 ... a_N}$. Every index a_α corresponds to space V_α and takes two values $a_\alpha = 1, 2$. A matrix acting in $V_1 \otimes ... \otimes V_N$ is $2^N \times 2^N$, whose entries are numbered by N pairs of superscripts $A^{a_1 b_1, ..., a_n b_n}$ and its products for a vector or for a matrix B are

$$v = Az \to v^{a_1 ... a_N} = A^{a_1 b_1, ..., a_n b_n} z^{b_1 ... b_N}, \tag{B.26}$$

$$C = AB \to C^{a_1 b_1, ..., a_n b_n} = A^{a_1 c_1, ..., a_n c_n} B^{c_1 b_1, ..., c_n b_n}. \tag{B.27}$$

A matrix $2^N \times 2^N$ which acts non-trivially only in the V_k and in the V_n spaces is indicated by A_{kn} and its components are

$$A_{kn}^{a_1 b_1, ..., a_k b_k, ..., a_n b_n, ..., a_N b_N} = A^{a_k b_k, a_n b_n} \prod_{\substack{j=1 \\ j \neq k,n}}^{N} \delta^{a_j b_j}. \tag{B.28}$$

The permutation operator acts in the $V_1 \otimes ... \otimes V_N$ as:

$$P_{kn} V_1 \otimes ... \otimes V_k \otimes ... \otimes V_n \otimes ... \otimes V_N = V_1 \otimes ... \otimes V_n \otimes ... \otimes V_k \otimes ... \otimes V_N \tag{B.29}$$

and has entries

$$P_{kn}^{a_1 b_1, ..., a_N b_N} = \delta^{a_k b_n} \delta^{a_n b_k} \prod_{\substack{j=1 \\ j \neq k,n}}^{N} \delta^{a_j b_j}. \tag{B.30}$$

Remembering that the subscripts indicate the space in which an operator acts non-trivially, we have that

$$P_{kn} A_{1...k...n...N} P_{kn} = A_{1...n...k...N} \tag{B.31}$$

and

$$P_{kn} A_{1...k...n-1n+1...N} P_{kn} = A_{1...n...n-1n+1...N}. \tag{B.32}$$

For the partial trace we have

$$tr_k A = A^{a_1 b_1, ..., a_k a_k, ... a_N b_N} \tag{B.33}$$

B.1 Yang-Baxter equation

As an application of the tensor product rules we want to analyze a particular solution of the famous Yang-Baxter (YB) equation, which is of great importance in the field of integrable systems.

Let $R(u_1, u_2)$ a 4×4 matrix acting on $V_1 \otimes V_2$ and depending on two complex variable u_1 and u_2. Considering a space $V_1 \otimes V_2 \otimes V_3$ and using two subscripts to indicate the space in which an operator acts non-trivially, the YB equations is

$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2). \tag{B.34}$$

We try to find a particular solution of YB equation, which depends only on the difference $u_1 - u_2$ and that takes the form $R(u) = f(u)\mathbb{I} + cP$, where $f(u)$ is a smooth function and c is a constant. Changing the variables in $u_1 - u_2 = u$ and $u_2 - u_3 = v$, the YB equation becomes

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u). \tag{B.35}$$

Substituting $R(u) = f(u)\mathbb{I} + cP$ in the last equation and separating the terms according to the powers of c, we do not obtain an identity only for the c^2 power, i.e.

$$f(u)P_{12}P_{23}+f(v)P_{12}P_{13}+f(u+v)P_{12}P_{23} = f(u)P_{23}P_{13}+f(v)P_{13}P_{12}+f(u+v)P_{23}P_{12}. \tag{B.36}$$

Multiplying (B.36) from the left by P_{12}, then from the right by P_{13} and finally from the right by P_{23}, we obtain

$$f(u)P_{23} + f(v)P_{23} + f(u+v)P_{12} = f(u)P_{12} + f(v)P_{12} + f(u+v)P_{23}, \tag{B.37}$$

or equivalently

$$(f(u) + f(v) - f(u+v))P_{23} = (f(u) + f(v) - f(u+v))P_{12}. \tag{B.38}$$

Since P_{12} and P_{23} are different matrices the equality is possible only if

$$f(u) + f(v) = f(u+v), \qquad \forall u, v. \tag{B.39}$$

This is a functional equation, i.e. an equation in which the unknown is a function, whose only smooth solution is $f(u) = au$, where a is an arbitrary constant.

Including a in the constant c the matrix

$$R(u_1, u_2) = (u_1 - u_2)\mathbb{I} + cP \tag{B.40}$$

is a solution of YB equation.

Appendix C

XXZ model in quantum group formalism

In this appendix we describe briefly the way to construct the two parameter conserved charge $\mathcal{I}(z,u)$, from which we have derived a quasilocal charge. For this aim we need the help of some tools of quantum group formalism. The Lax operator of XXZ model for an arbitrary auxiliary space $\mathcal{Q}$ can be written as a matrix in V_j (the physical space, whose Nth tensor product generates the total quantum physical space of the system) with entries that act on $\mathcal{Q}$:

$$L_j(z) = \frac{1}{2}\begin{pmatrix} zK - z^{-1}K^{-1} & z(q-q^{-1})S^- \\ z^{-1}(q-q^{-1})S^+ & zK^{-1} - z^{-1}K \end{pmatrix}, \tag{C.1}$$

where $z \in \mathbb{C}$, q is related to the anisotopy parameter Δ, i.e. $\Delta = (q + q^{-1})/2$ and K, S^+, S^- are operators acting on $\mathcal{Q}$ and obeying the so called quantum group algebra $U_q[SU(2)]$:

$$KS^+ = qS^+K, \tag{C.2}$$

$$KS^- = q^{-1}S^-K, \tag{C.3}$$

$$[S^+, S^-] = \frac{K^2 - K^{-2}}{q - q^{-1}}. \tag{C.4}$$

For example choosing $\mathcal{Q} = \mathbb{C}^2$, we can use the spin-1/2 representation

$$K = q^{\frac{\sigma^z}{2}}, \qquad S^{\pm} = \sigma^{\pm}. \tag{C.5}$$

The monodromy matrix and the transfer matrix are as usual:

$$T_{\mathcal{Q}}(z) = L_N(z)L_{N-1}(z)...L_1(z), \tag{C.6}$$

$$t_{\mathcal{Q}}(z) = tr_{\mathcal{Q}}(T_{\mathcal{Q}}(z)), \tag{C.7}$$

where $tr_{\mathcal{Q}}$ denotes the trace over the auxiliary space $\mathcal{Q}$.

The quasilocal operators have the property that they can not be written as a linear combination of the local conserved charges because they have different symmetry properties, like spin inversion. The first step is to find conserved quantities which are not invariant under spin inversion. For this aim it is convenient to introduce another transfer matrix which commutes with $t_{\mathcal{Q}}(z)$ but employs a different representation of quantum group algebra. Let us consider an auxiliary space $\mathcal{A}$ with dimension $d_{\mathcal{A}}$. Denoting with $\mathcal{L}_j$ the Lax operator defined in $V_j \otimes \mathcal{A}$, we have

$$T_{\mathcal{A}}(z) = \mathcal{L}_N(z)\mathcal{L}_{N-1}(z)...\mathcal{L}_1(z), \tag{C.8}$$

$$t_{\mathcal{A}}(z) = tr_{\mathcal{A}}(T_{\mathcal{A}}(z)). \tag{C.9}$$

Using Yang-Baxter relation it can be shown that

$$[t_{\mathcal{Q}}(z), t_{\mathcal{A}}(w)] = 0. \qquad \forall z, w \in \mathbb{C}. \tag{C.10}$$

Since the XXZ Hamiltonian is among the operators generated by $t_{\mathcal{Q}}(z)$, we are allowed to use $t_{\mathcal{A}}(z)$ as a generating function of conserved quantities.

We shall work with the so called highest weight representation of $U_q[SU(2)]$:

$$K|r\rangle = uq^r|r\rangle, \tag{C.11}$$

$$S^+|r\rangle = -a_r|r+1\rangle, \tag{C.12}$$

$$S^-|r\rangle = b_r|r-1\rangle, \tag{C.13}$$

where $u \in \mathbb{C}$ is arbitrary and a_r and b_r are functions of q and u. The dimension of the auxiliary space depends on the value of the anisotropy parameter. We shall focus on the case in which q is a root of unity, i.e. $q = e^{i\lambda}$ with $\lambda = l\pi/m$ and $l, m \in \mathbb{Z}$ coprimes. In this case we can restrict the auxiliary space index r to $0 \leq r \leq m-1$, so that the representation has finite dimension $d_{\mathcal{A}} = m$. It is important to notice that for $q = e^{i\pi l/m}$, $\Delta = \cos(\pi l/m)$, therefore $|\Delta| \leq 1$, which corresponds to the gapless phase of the XXZ model.

In analogy with (C.1) the Lax operator $\mathcal{L}_j$ can be written as

$$\mathcal{L}_j(z,u) = i\left(\mathbb{I}_j \otimes A_0(z,u) + \sigma_j^z \otimes A_z(z,u) + \sigma_j^+ \otimes A_+(z,u) + \sigma_j^- \otimes A_-(z,u)\right), \quad \text{(C.14)}$$

where

$$A_0(z,u) = \frac{(z-z^{-1})}{4i}\left(K(u) + K^{-1}(u)\right), \quad \text{(C.15)}$$

$$A_z(z,u) = \frac{(z+z^{-1})}{4i}\left(K(u) - K^{-1}(u)\right), \quad \text{(C.16)}$$

$$A_+(z,u) = \frac{z}{2i}(q-q^{-1})S^-(u), \quad \text{(C.17)}$$

$$A_-(z,u) = \frac{z^{-1}}{2i}(q-q^{-1})S^+(u). \quad \text{(C.18)}$$

It follows that the transfer matrix can be written as

$$t_{\mathcal{A}}(z,u) = i^N \sum_{\{\alpha_j\}} tr_{\mathcal{A}}\left(A_{\alpha N}...A_{\alpha 2}A_{\alpha 1}\right)\prod_{j=1}^{N}\sigma_j^{\alpha_j}, \quad \text{(C.19)}$$

where $\alpha_j \in \{0, z, +, -\}$ and $\sigma_j^0 \equiv \mathbb{I}_j$.

Using parity transformation $\mathcal{P}$, i.e. $\mathcal{P}^{-1}\sigma_j^{\alpha_j}\mathcal{P} = \sigma_{N+1-j}^{\alpha_j}$, we can construct the two parameter conserved quantity, which is odd under parity as

$$\mathcal{I}(z,u) = (-i)^N\left(\mathcal{P}^{-1}t_{\mathcal{A}}(z,u)\mathcal{P} - t_{\mathcal{A}}(z,u)\right), \quad \text{(C.20)}$$

whose explicit expression is given in (5.39).

Appendix D

Fourier transform of coefficient $a_n(\lambda)$

We want to evaluate the integral

$$\int_{-\pi/2}^{\pi/2} d\lambda e^{2ik\lambda} a_n(\lambda) = \int_{-\pi/2}^{\pi/2} d\lambda e^{2ik\lambda} \frac{1}{2\pi} \frac{2\sinh(n\phi)}{\cosh(n\phi) - \cos(2\lambda)}. \tag{D.1}$$

After the change of variable $2\lambda = y$ the integral becomes

$$\int_{-\pi}^{\pi} dy e^{iky} \frac{1}{2\pi} \frac{\sinh(n\phi)}{\cosh(n\phi) - \cos y}. \tag{D.2}$$

We use the residues theorem to solve the integral. For $k > 0$ we choose a rectangular counterclockwise path $ABCD$, so that AB is the interval $[-\pi, \pi]$, BC is the interval $[\pi, \pi + i\infty]$, CD is the $[\pi + i\infty, -\pi + i\infty]$ and DA is the interval $[-\pi + i\infty, -\pi]$. So we have

$$\int_{ABCD} dy e^{iky} \frac{1}{2\pi} \frac{\sinh(n\phi)}{\cosh(n\phi) - \cos y} = 2\pi i \sum Res \left[e^{iky} \frac{1}{2\pi} \frac{\sinh(n\phi)}{\cosh(n\phi) - \cos y} \right]. \tag{D.3}$$

We note that the sum of the integrals on BC and DA is zero and the integral on CD annihilates thanks to exponential factor of the integrand and the fact that $k > 0$.

The poles of the integrand is of order one and they are given by the zeros of the denominator, i.e. $y = \pm in\phi + 2m\pi$ with $m \in \mathbb{Z}$. The only pole which is contained in the inner region of the path $ABCD$ is $y = in\phi$. Since the pole is a simple pole we can use the following formula:

$$Res\ f(z)|_{z=z_0} = \frac{p(z_0)}{q'(z_0)}, \tag{D.4}$$

where $f(z) = \frac{p(z)}{q(z)}$ and $q(z_0) = 0$, z_0 being a simple pole.
In this case we have (we remember that $k > 0$)

$$Res\left[e^{iky}\frac{1}{2\pi}\frac{\sinh(n\phi)}{\cosh(n\phi)-\cos y}\right]\Big|_{y=in\phi} = e^{-kn\phi}\frac{\sinh(n\phi)}{2\pi\sin in\phi} = \frac{-ie^{-kn\phi}}{2\pi}. \tag{D.5}$$

For $k < 0$ we use use the same path, but reflected on the real axes (it is worth to note that now the path is a clockwise curve and so a minus appears in the calculus of residues). The pole contained in the inner region of the path is $y = -in\phi$. The result is:

$$Res\left[e^{-i|k|y}\frac{1}{2\pi}\frac{\sinh(n\phi)}{\cosh(n\phi)-\cos y}\right]\Big|_{y=-in\phi} = e^{-|k|n\phi}\frac{\sinh(n\phi)}{2\pi\sin in\phi} = \frac{-ie^{-|k|n\phi}}{2\pi}. \tag{D.6}$$

Combining the results we finally obtain from (D.3)

$$\int_{-\pi}^{\pi} dy e^{iky}\frac{1}{2\pi}\frac{\sinh(n\phi)}{\cosh(n\phi)-\cos y} = e^{-|k|n\phi}. \tag{D.7}$$

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