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Acoustic black holes and de Laval nozzle

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Che cosa pensa allora il fanciullo
mentre fa queste scoperte?
Innanzi tutto si meraviglia.
Questo sentimento di stupore
è la sorgente e la fonte inestinguibile
della sua sete di conoscere.

Max Planck,
Significato e limiti della scienza esatta
(saggio datato 1941 e 1947)

Sommario

I buchi neri acustici sono l'analogo sonoro dei buchi neri gravitazionali. La loro base teorica è l'analogia formale tra due tipi di equazione d'onda: quella per la perturbazione del potenziale della velocità in un fluido non viscoso e inomogeneo e quella per un campo scalare in uno spazio-tempo curvo. Nel presente lavoro, in primo luogo vengono introdotti gli elementi fondamentali della Relatività Generale. In secondo luogo, vengono mostrate la metrica di Schwarzschild e la sua rappresentazione nelle coordinate di Painlevé-Gullstrand. Successivamente, la parte principale di questa tesi è dedicata alla derivazione della *metrica acustica* e allo studio di buchi neri acustici a simmetria sferica o unidimensionali. Per quanto riguarda questi ultimi, viene illustrato l'*ugello di Laval*, essendo un possibile modello per realizzare sperimentalmente buchi neri acustici.

Abstract

Acoustic black holes are the sonic analogue of gravitational black holes. Their theoretical basis is the formal analogy between two types of wave equation: the one for the velocity potential perturbation in a non-homogeneous inviscid fluid and the one for a scalar field in a curved space-time. In the present work, firstly the basics of General Relativity are introduced. Secondly, the Schwarzschild metric and its representation in the Painlevé-Gullstrand coordinates are shown. Then, the main part of this thesis is dedicated to the derivation of the *acoustic metric* and to the study of spherically-symmetric or one-dimensional acoustic black holes. Concerning these last ones, the *de Laval nozzle* is depicted, being a possible model to realize acoustic black holes experimentally.

Contents

Introduction

The aim of this bachelor thesis is to describe the fundamental aspects of black holes hydrodynamics. This branch has its foundations on the formal analogy between the behaviour of acoustical disturbances in a non-homogeneous inviscid fluid and the wave propagation in a curved space-time. First of all, this requires an overview of the basic elements of Einstein's General Relativity, to which Chapter 1 is dedicated. It begins with a discussion about the Equivalence Principle, whose one of the main consequences is the geodesics equation. We then deal with two postulates of General Relativity, that are the Principles of General Relativity and General Covariance, which both lead to the necessity of general tensorial physical equations. The first chapter ends with further geometrical considerations about space-time curvature and with the derivation of Einstein field equations.

In Chapter 2 we shall concentrate on a specific solution to Einstein field equations, that is the Schwarzschild metric for a spherically-symmetric gravitational source. After its derivation, we will study the physical phenomena occuring at the *Schwarzschild radius*, which results to be a coordinate singularity. For this reason, in the following section we will introduce the Painlevé-Gullstrand coordinate system, which permits to write the Schwarzschild metric in such a way to eliminate this coordinate singularity. Chapter 1 and 2 will provide a solid background for the last chapter.

Chapter 3 is the core of this thesis. We will start by recalling the fundamental equations of fluid dynamics, that are the continuity equation and Euler's equation. In the second section, the first order perturbative expansion of these equations will be described. This leads to their *linearization*, which will result in the wave equation for the velocity potential perturbation in a non-homogeneous inviscid fluid. This equation will end up determining the propagation of sound in such fluid and presenting the same formal structure as the wave equation in curved space-time. In the following, this correspondence will be deepened up to obtain the so-called *acoustic metric*. Theoretically, this represents the only metric to which sound waves couple, thus showing an analogous behaviour to a scalar field propagating in a Lorentzian space-time manifold. Furthermore, by means of the Painlevé-Gullstrand coordinates, we will observe how the acoustic metric *conformally* achieves reproducing the Schwarzschild space-time for a spherically-symmetric massive source.

The last section of Chapter 3 plays a significant *experimental* role. We will provide an example of one-dimensional acoustic black hole realization, known as *de Laval nozzle*. It is a converging-diverging nozzle in which sound waves behave analogously to electromagnetic waves when subject to an hypothetical one-dimensional gravitational black hole. This confirms the theoretical framework depicted in the previous sections. Finally, this thesis concludes with an appendix where a mathematical proof of the wave equation in curved space-time is given.

Chapter 1

Introduction to General Relativity

In this chapter we shall provide an overview of the basic concepts of Einstein's General Theory of Relativity. This will give us a sufficient base to introduce the Schwarzschild solution to Einstein field equations and its representation in Painlevé-Gullstrand coordinates in the next chapter.

1.1 Equivalence Principle and geodesics equation

Einstein's General Theory of Relativity was invented as an attempt to extend Newton's theory of gravity in the framework of Special Relativity. However, this theoretical project appeared impossible quite immediately. Indeed, Newtonian gravity is described by the Poisson's law

$$\nabla^2\phi = 4\pi G\rho, \tag{1.1.1}$$

where ϕ denotes the gravitational potential, ρ the matter density and G the Newtonian constant of gravitation. Being a non-tensorial equation, this equation can be shown not to be covariant under Lorentz transformation. Furthermore, let us consider the relativistic tensorial form of Maxwell's equations in terms of the 4-potential A^μ and the 4-current J^μ in the *Lorentz gauge*

$$\Box A^\mu = \frac{4\pi}{c} J^\mu. \tag{1.1.2}$$

Unlike this, Poisson's equation for gravity (1.1.1) contains the Laplace operator ∇^2, which is only a *spatial* differential operator. Therefore, this implies an *instant* interaction, thus not leading to an interaction propagating at a finite speed, as required by Special Relativity. Such problems motivated Einstein to develop a completely new field theory for gravity, no longer inside the framework

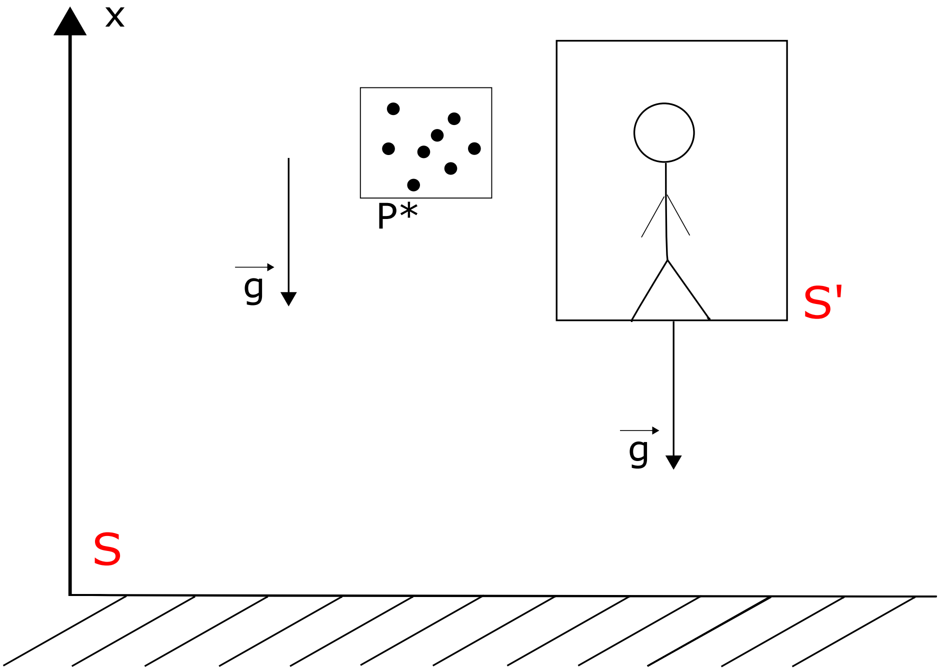

Figure 1.1.1: A system P^* of free-falling particles in the Earth gravitational field. S denotes a reference frame comoving with the Earth and can be assumed to be inertial. On the other hand, S' stands for a free-falling reference frame. S' is not inertial in Newtonian gravity, while it is considered inertial in General Relativity.

of Special Relativity. Finally, he came up with an innovative and more general description of space-time itself.

A core element of General Relativity is the *Equivalence Principle*. Its base is the experimental equality between inertial mass $m^{(i)}$ and gravitational mass $m^{(g)}$. Let us consider the Earth gravitational field as homogeneous and a system of free-falling particles (Figure 1.1.1) . The reference frame comoving with the Earth can be assumed to be inertial, neglecting rotational effects. Then, Newton's equation of motion for the N-th particle is

$$m_N^{(i)} \frac{d^2\vec{x}_N}{dt^2} = m_N^{(g)}\,\vec{g}. \tag{1.1.3}$$

We shall now perform the *non-galileian* coordinate transformation

$$\begin{cases} \vec{x}' = \vec{x} - \frac{1}{2}\,\vec{g}\,t^2 \\ t' = t \end{cases} . \tag{1.1.4}$$

Then, the equation of motion becomes

$$m_N^{(i)} \frac{d^2 \vec{x}'_N}{dt^2} + \cancel{m_N^{(i)} \vec{g}} = \cancel{m_N^{(g)} \vec{g}}, \tag{1.1.5}$$

being $m_N^{(i)} = m_N^{(g)}$. This means that gravity cannot be detected in a free-falling reference frame, since everything in the frame is subject to the same *homogeneous* gravitational force.

Let us now consider a parallel situation. We assume no gravity and an inertial reference frame K where some particles are at rest. We now define a reference frame K' through the same *non-galileian* coordinate transformation (1.1.4) (Figure 1.1.2)

$$K \to K' \qquad \Rightarrow \qquad \begin{cases} \vec{x} = \vec{x}' + \frac{1}{2}\,\vec{g}\,t^2 \\ t = t' \end{cases} ,$$

where now $\vec{g}$ is the acceleration owed by K' with respect to K.

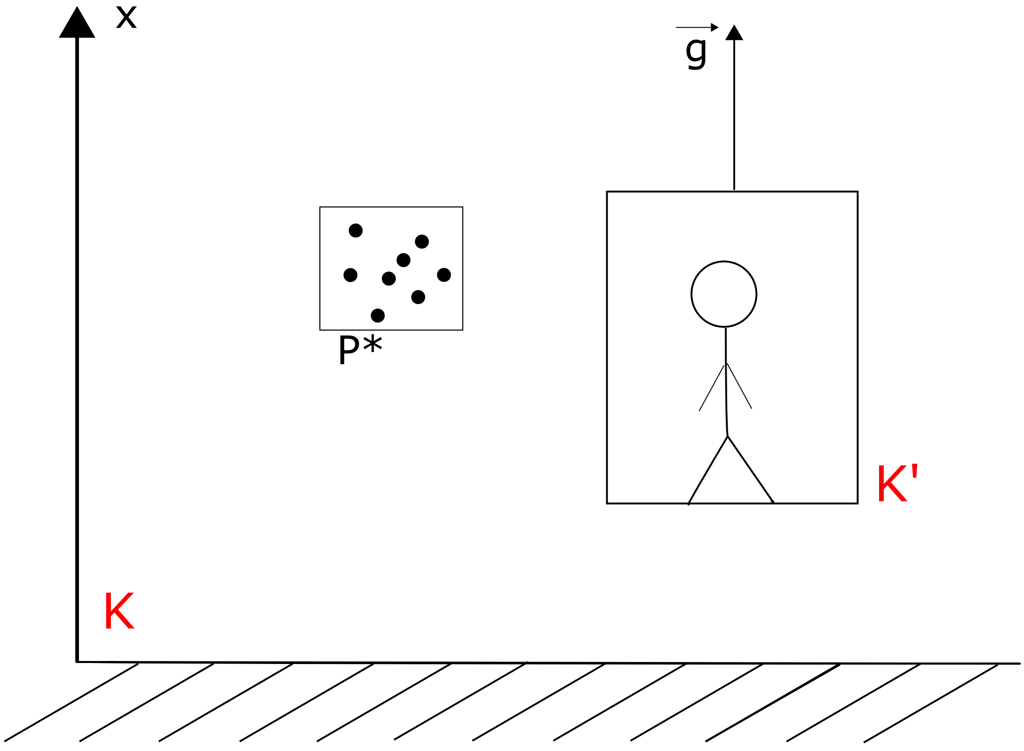

Figure 1.1.2: A system $P*$ of particles in the absence of gravity. K denotes an inertial reference frame in which the particles are at rest. On the other hand, K' stands for a reference frame moving upward with acceleration $\vec{g}$ with respect to K. K' is not inertial in Newtonian gravity but it conveys the same physical description as the inertial reference frame S in Figure 1.1.1.

Since for the N-th particle $\frac{d^2\vec{x}_N}{dt^2} = 0$ holds (it is at rest in K), in K' we have $\frac{d^2\vec{x}'_N}{dt^2} = -\vec{g}$. This means in that frame the particles are perceived as being accelerated downward, just as though they were subject to gravity. As a consequence, we infer the following duality between the two systems we have shown

- the inertial frame in the first case is equivalent to the non-inertial frame in the second case,
- the non-inertial frame in the first case is equivalent to the inertial frame in the second case.

Therefore, from an operational point of view, the best definition of *inertial* reference frame is a *freely falling* frame in a *homogeneous* gravitational field, in that there are no other reliable ways to screen gravity. Finally, we are now able to state the *Equivalence Principle*:

Equivalence Principle *In an arbitrary gravitational field, for every space-time point there exists a sufficiently small neighbourhood where*

- *the field can be approximated as homogeneous,*
- *a freely falling reference frame can be defined that locally acts as an inertial frame where Special Relativity holds.*

From this discussion, we deduce that Special Relativity holds only *locally*, since a *global* inertial reference frame cannot be defined if the field is non-homogeneous. Indeed, Special Relativity must be regarded as a valid framework only within inertial reference frames in the absence of gravity.

Secondly, gravity acts as an inertial force and emerges in the transition from a free-falling (i.e. inertial) reference frame to a non-inertial reference frame[1]. As concerns this aspect, the *Equivalence Principle* has the remarkable property to convey an algorithm to describe gravitational forces in a certain non-inertial reference frame. We first need to write the physical equations for a specific system in an inertial reference frame according to Special Relativity. Thereafter, we perform the general coordinate transformation to the non-inertial reference frame and we obtain the correct equations of motion in the presence of gravity.

Let us now see a significant example of it, which is a system of free-falling particles in a gravitational field. $\{\xi^\alpha\}$ denote the space-time coordinates of the

[1] The only gravitational effects a free-falling observer can measure are tidal forces.

locally inertial reference frame, and $\{x^\alpha\}$ the space-time coordinates of the laboratory reference frame. In the freely falling frame the relativistic tensorial form of Newton's dynamical equations is

$$\frac{dP^\alpha}{ds} = K^\alpha = 0, \tag{1.1.6}$$

where P^α denotes the 4-momentum, K^α the 4-force and s the invariant proper time under Lorentz transformation ($ds^2 = \eta_{\mu\nu}\, d\xi^\mu\, d\xi^\nu$, $\eta_{\mu\nu} \equiv \mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric). If a massive particle has vanishing 4-momentum, its 4-acceleration will be vanishing as well

$$\frac{dP^\alpha}{ds} = 0 \quad \Rightarrow \quad \frac{d^2\xi^\alpha}{ds^2} = 0. \tag{1.1.7}$$

Now we perform the coordinate transformation to the laboratory frame

$$\frac{d}{ds}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{ds}\right) = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{ds^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0, \tag{1.1.8}$$

and multiplying both hand sides by $\frac{\partial x^\lambda}{\partial \xi^\alpha}$ we obtain the following equation of motion

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{1.1.9}$$

$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$ is defined *affine connection* and depends only on coordinate transformation. That is why $\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$ is an acceleration term that encodes the inertial forces from gravity acting on the particle.

If we were supposed to deal with massless particles (i.e. light propagation), their trajectories would have $ds^2 = 0$ identically. Therefore, in the freely falling reference frame they would have to be parametrized by a different parameter q

$$\frac{d^2\xi^\alpha}{dq^2} = 0. \tag{1.1.10}$$

Analogously, by performing the same coordinate transformation as previously, we obtain

$$\frac{d^2 x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dq}\frac{dx^\nu}{dq} = 0. \tag{1.1.11}$$

The trajectories which are solutions to equation (1.1.9) (or (1.1.11)) are called *geodesics* and the corresponding equation (1.1.9) (or (1.1.11)) is then called *geodesics equation*. This entails that in a gravitational field both massive particles and light travel along *geodesics*, which are not straight lines in general. This fact has also an impact on the causal structure of space-time, since light cones are deformed by gravity. The metric tensor $\eta_{\alpha\beta}$ itself changes. It undergoes a general coordinate transformation:

$$ds^2 = \eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta = \eta_{\alpha\beta}\, \frac{\partial \xi^\alpha}{\partial x^\mu}\, dx^\mu\, \frac{\partial \xi^\beta}{\partial x^\nu}\, dx^\nu = g_{\mu\nu}\, dx^\mu\, dx^\nu, \qquad (1.1.12)$$

where $g_{\mu\nu} \equiv \eta_{\alpha\beta}\, \frac{\partial \xi^\alpha}{\partial x^\mu}\, \frac{\partial \xi^\beta}{\partial x^\nu}$ is the metric tensor in the general coordinates frame and is a (0,2) tensor. It can be proven such definition of ds^2 preserves its property of being a scalar under general coordinate transformations as well.

By looking at the form of the *affine connection* $\Gamma^\lambda_{\mu\nu}$ and the general metric $g_{\mu\nu}$, we observe they are given in terms of second derivatives of $\{\xi^\alpha\}$ and first derivatives of $\{\xi^\alpha\}$ respectively. Therefore, we may grasp the general metric can be expressed as derivatives of the *affine connection*, which encodes the gravitational force once multiplied for a mass. As a consequence, the general metric $g_{\mu\nu}$ plays the role of potential for gravity in General Relativity. Moreover, in a metric manifold the *affine connection* can be expressed in terms of the metric itself in such a way

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right). \qquad (1.1.13)$$

1.2 Principle of General Relativity and Principle of General Covariance

One of the postulates General Relativity is based on is the *Principle of General Relativity*

Principle of General Relativity *All reference frames are equivalent, that is, physical equations must be covariant under general coordinate transformations.*

It is an extension of the *Principle of Special Relativity*, according to which *inertial* observers were a kind of preferred reference frames. In order to accomplish this new principle, physical equations then need to be written in *tensorial* form. Unlike Special Relativity, tensors definition in General Relativity has to deal with *general* coordinate transformations and not only *Lorentz* ones. Once an equation is formulated in a *specific* frame by means of *general* tensors, then that equation holds for *any* observer and the *Principle of General Relativity* is fulfilled.

Despite its notation, the *affine connection* is not a tensor. Indeed, we saw in a local inertial frame $\Gamma^{\lambda}_{\mu\nu} = 0$ and gravity is screened, whereas in a general coordinate frame $\Gamma^{\lambda}_{\mu\nu} \neq 0$ and gravitational forces arise. Nevertheless, the *geodesics equation* can be shown to be tensorial, even though it contains the *affine connection*. We then infer that in the *geodesics equation* there appears a particular combination of differential operators such that, once applied to a tensor, the result is again a tensor. On the one hand, by computing $\Gamma'^{\lambda}_{\mu\nu}$ as associated to the general coordinate transformation $\xi^{\alpha} \leftrightarrow x'^{\alpha}$, we obtain

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}}. \tag{1.2.1}$$

If we contract $\Gamma'^{\lambda}_{\mu\nu}$ with a transformed contravariant 4-vector V'^{μ}, we have

$$\begin{aligned}
\Gamma'^{\lambda}_{\mu\nu} V'^{\mu} &= \left[\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}}\right] \frac{\partial x'^{\mu}}{\partial x^{\eta}} V^{\eta} = \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} \delta^{\tau}_{\eta} V^{\eta} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} \delta^{\sigma}_{\eta} V^{\eta} = \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} V^{\tau} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} V^{\sigma}
\end{aligned} \tag{1.2.2}$$

On the other hand, the derivative of a transformed contravariant 4-vector V'^{μ} results in

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \frac{\partial}{\partial x'^{\lambda}} \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}\right) = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}, \tag{1.2.3}$$

which shows that partial derivatives of tensors are not tensors in general. We observe that the second addend in equation (1.2.2) and the one in (1.2.3) are the same with opposite sign. That addend is precisely the term preventing both $\Gamma^{\lambda}_{\mu\nu} V^{\mu}$ and $\frac{\partial V^{\mu}}{\partial x^{\lambda}}$ from being tensors. As a consequence, by summing (1.2.2) and (1.2.3), we obtain

$$\begin{aligned}
\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} + \Gamma'^{\mu}_{k\lambda} V'^{k} &= \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}} + \cancel{\frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}}} V^{\nu} + \\
&+ \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \Gamma^{\nu}_{\tau\rho} V^{\tau} - \cancel{\frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}}} V^{\nu} = \\
&= \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left[\frac{\partial V^{\nu}}{\partial x^{\rho}} + \Gamma^{\nu}_{\tau\rho} V^{\tau}\right],
\end{aligned} \tag{1.2.4}$$

which implies $\frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{k\lambda} V^{k}$ is a (1,1) tensor. We define *covariant derivative* of a contravariant vector V^{μ} the linear map

$$D : V^{\mu} \longmapsto V^{\mu}_{;\lambda} \equiv \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{k\lambda} V^{k}. \tag{1.2.5}$$

We note that this differential operation contains a combination of partial derivatives and *affine connection*, that encodes gravity. This suggests the *covariant derivative* be the correct differential operation in the framework of General Relativity, since it presents a coupling between the ordinary partial differentiation and the gravitational field.

Analogously, we can define the *covariant derivative* of a contravariant vector A^μ *along a curve* $x^\mu = x^\mu(s)$

$$\frac{DA^\mu}{Ds} \equiv \frac{dA^\mu}{ds} + \Gamma^\mu_{k\lambda}\frac{dx^\lambda}{ds}A^k, \tag{1.2.6}$$

which yields

$$\frac{DA^\mu}{Ds} \equiv \frac{dA^\mu}{ds} + \Gamma^\mu_{k\lambda}\frac{dx^\lambda}{ds}A^k = \left[\frac{\partial A^\mu}{\partial x^\lambda} + \Gamma^\mu_{k\lambda}A^k\right]\frac{dx^\lambda}{ds} = A^\mu_{;\lambda}\frac{dx^\lambda}{ds}. \tag{1.2.7}$$

Therefore, the covariant derivative of a contravariant vector along a curve is given by the contraction of its covariant derivative with the tangent vector to the curve $\frac{dx^\lambda}{ds}$. From a purely geometrical point of view, a vector field whose covariant derivative along a curve $x^\mu = x^\mu(s)$ vanishes is said *parallel transported* along the curve. If this field is the tangent vector to the curve itself $u^\mu = \frac{dx^\mu}{ds}$, then we have

$$\frac{Du^\mu}{Ds} = \frac{du^\mu}{ds} + \Gamma^\mu_{k\lambda}\frac{dx^\lambda}{ds}u^k = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{k\lambda}\frac{dx^\lambda}{ds}\frac{dx^k}{ds} = 0, \tag{1.2.8}$$

which is exactly the *geodesics equation* (1.1.9). As a result, the *geodesics equation* can be interpreted as the parallel transport equation for the tangent vector along a space-time curve, that is, the 4-velocity $u^\mu = \frac{dx^\mu}{ds}$. In addition, we have just demonstrated the *geodesics equation* is indeed a tensorial equation.

We can now state the *Principle of General Covariance*

Principle of General Covariance *The equations that describe a physical system immersed in a gravitational field have to satisfy the following requirements:*

- *they must be tensorial;*
- *they have to correctly describe the system in the absence of gravity.*

Once an equation fulfils these constraints, then it will correctly describe the system when gravity is present as well. The reason resides in that gravity is an inertial force appearing when a change of reference frame is performed. The

Principle of General Covariance then allows us to construct the correct *tensorial* equation in the locally inertial reference frame in order to obtain a physical equation holding for *all* reference frames.

An applying example of this principle is again the *geodesics equation* for a particle. A particle immersed in a gravitational field becomes a free particle in a locally inertial reference frame, due to the *Equivalence Principle.* In the absence of gravity, its equation of motion is

$$\frac{du^\alpha}{ds} = 0, \tag{1.2.9}$$

where $u^\alpha = \frac{d\xi^\alpha}{ds}$ and $ds^2 = \eta_{\mu\nu}\, d\xi^\mu\, d\xi^\nu$. In a general coordinates frame, we have

$$\tilde{u}^\alpha = \frac{dx^\alpha}{ds} = \frac{\partial x^\alpha}{\partial \xi^\beta} u^\beta, \quad ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \eta_{\mu\nu}\, d\xi^\mu\, d\xi^\nu.$$

Applying the *Principle of General Covariance*, we find the correct equation in this frame is again

$$\frac{Du^\alpha}{Ds} = 0,$$

since

- in the absence of gravity, covariant derivative and general metric respectively reduce to partial derivative and Minkowski metric, therefore we obtain equation (1.2.9);
- this equation is tensorial.

Practically speaking, once we have described a system in the absence of gravity according to Special Relativity, we replace ordinary partial differentiation with covariant differentiation and Minkowski metric $\eta_{\mu\nu}$ with general metric $g_{\mu\nu}$. Thereby, we obtain the correct equation in the presence of gravity.

1.3 Space-time curvature and Einstein field equations

When we discussed the consequences of the *Equivalence Principle*, we found that in a general space-time coordinate frame the metric is no longer constant but can change from point to point ($g_{\mu\nu} = g_{\mu\nu}(x^\mu)$). On the basis of equation (1.1.12), the metric tensor $g_{\mu\nu}$ encodes the geometrical properties of the manifold on which it is defined, that is space-time itself in General Relativity. Hence, if $g_{\mu\nu}$ depends on the coordinates, the space-time will be possibly *curved*, in differential geometry language. Indeed, the *Equivalence Principle* ensures the space-time manifold is *differentiable*, by stating that in an arbitrary gravitational field a *locally* inertial reference frame where Special Relativity holds can always be found. This means space-time is a general *curved* manifold which is *locally flat*, in that locally it can be approximated by its tangent space, whose framework is within Special Relativity. That is why the *Equivalence Principle* leads to the *geometrization* of space-time in the presence of gravity.

We then infer that what distinguishes the presence or the absence of gravity is the impossibility to define a metric tensor which is *globally* equivalent to $\eta_{\mu\nu}$. If a coordinate transformation can be globally defined that maps a general metric $g_{\mu\nu}$ into the Minkowski flat metric $\eta_{\mu\nu}$, then there is not a gravitational field. Here the geometrical structure of space-time comes in. In fact, we are facing the problem to distinguish *mathematically* between a curved and a flat manifold separately from the metric alone, which is coordinate dependent. The mathematical object which fulfils this request is the *Riemann tensor*

$$R^{\lambda}_{\mu\nu k} = -\frac{\partial}{\partial x^k}\Gamma^{\lambda}_{\mu\nu} + \frac{\partial}{\partial x^{\nu}}\Gamma^{\lambda}_{\mu k} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{k\alpha} + \Gamma^{\alpha}_{\mu k}\Gamma^{\lambda}_{\nu\alpha}, \tag{1.3.1}$$

since it is related to the following theorem:

Theorem *A necessary and sufficient condition for the metric $g_{\mu\nu}$ to be equivalent to $\eta_{\mu\nu}$ - that is, there exists a coordinate transformation mapping $g_{\mu\nu}$ into $\eta_{\mu\nu}$ globally - is $R^{\lambda}_{\mu\nu k} = 0$.*

Therefore, a manifold is curved if and only if $R^{\lambda}_{\mu\nu k} \neq 0$.

We shall now present two of *Riemann tensor*'s contractions that will play a fundamental role in Einstein field equations:

- the *Ricci tensor*
$$R_{\mu k} = R^{\lambda}_{\mu\lambda k} = g^{\lambda\nu}R_{\lambda\mu\nu k}, \tag{1.3.2}$$
where $R_{\lambda\mu\nu k} \equiv g_{\lambda\sigma}R^{\sigma}_{\mu\nu k}$. The *Ricci tensor* results symmetric ($R_{\mu k} = R_{k\mu}$);
- the *Ricci scalar*
$$R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}. \tag{1.3.3}$$

The *Riemann tensor* also satisfies the so-called *Bianchi identities*

$$R_{\lambda\mu\nu k\,;\,\eta} + R_{\lambda\mu\eta\nu\,;\,k} + R_{\lambda\mu k\eta\,;\,\nu} = 0. \tag{1.3.4}$$

A useful contracted form of these identities is the following

$$g^{\lambda\nu}R_{\lambda\mu\nu k\,;\,\eta} + g^{\lambda\nu}R_{\lambda\mu\eta\nu\,;\,k} + g^{\lambda\nu}R_{\lambda\mu k\eta\,;\,\nu} = 0, \tag{1.3.5}$$

which yields

$$R_{\mu k\,;\,\eta} - R_{\mu\eta\,;\,k} + R^{\nu}_{\mu k\eta\,;\,\nu} = 0. \tag{1.3.6}$$

After performing a second contraction with $g^{\mu k}$, we obtain

$$R_{;\,\eta} - 2R^{\mu}_{\eta\,;\,\mu} = 0, \tag{1.3.7}$$

which reduces to

$$\left(R^{\mu}_{\eta} - \frac{1}{2}\delta^{\mu}_{\eta}R\right)_{;\,\mu} = 0 \tag{1.3.8}$$

by means of algebraic manipulations. We finally compute the contraction with $g^{\eta\nu}$ and we obtain the *contracted Bianchi identities*

$$G^{\mu\nu}_{;\,\mu} = \left(R^{\mu\nu} - \frac{1}{2}\,R\,g^{\mu\nu}\right)_{;\,\mu} = 0, \tag{1.3.9}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}$ is named *Einstein tensor.*

We are now ready to derive the field equations for gravity in General Relativity. Taking Maxwell equations (1.1.2) as a model, we expect the gravitational field equations to have a form such that the derivative (or a sort of) of the field is related to the gravitational source. Then, we may first attempt to evaluate the covariant derivative of the *potential* $g_{\mu\nu}$ to construct the field equation. This would end up being identically null though. We shall now try and find the *sources* of gravitational field. By Poisson's equation (1.1.1) (which must be the Newtonian limit of General Relativity field equations), sources are matter densities. By evaluating the Newtonian limit of *geodesics equations*, we would find that Poisson's equation (1.1.1) can be expressed as

$$\nabla^2 g_{00} = 8\pi G\,T_{00}, \tag{1.3.10}$$

where T_{00} is the 00-component of the energy-momentum tensor $T_{\alpha\beta}$. In order to have *tensorial* equations, that relation has to be coherently extended to all of the energy-momentum tensor components and we finally expect the source to be the *complete* energy-momentum tensor itself.

The field equation form must then be

$$H_{\alpha\beta} = 8\pi G\,T_{\alpha\beta}, \tag{1.3.11}$$

where $H_{\alpha\beta}$ should be a (0,2) symmetric tensor constructed with the metric and its first and second derivatives. Since $T_{\alpha\beta}$ satisfies the continuity equation $\partial_\mu T^{\mu}_{\nu} = 0$ without gravity, we expect $T^{\mu}_{\nu\,;\,\mu} = 0$ holds with gravity. It is known that the

Riemann tensor is the only tensor that can be obtained by the metric and its first and second derivatives assuming linearity in the second derivative. We can then think about making use of the *Riemann tensor* to construct the field equations. In order to satisfy the covariant continuity equation, we suppose $H_{\alpha\beta}$ is given by a combination of the *Ricci tensor* and the *Ricci scalar*

$$H_{\alpha\beta} = c_1\, R_{\alpha\beta} + c_2\, g_{\alpha\beta}\, R, \tag{1.3.12}$$

where c_1 and c_2 are constants to be defined. Now, by raising the index μ and recalling the *contracted Bianchi identities* (1.3.9), we have the following constraint

$$\left(\frac{c_1}{2} + c_2\right) R_{;\mu} = 0. \tag{1.3.13}$$

We assume the equation is satisfied by requiring $c_2 = -\frac{c_1}{2}$, since the alternative $R_{;\mu} = 0$ identically would impose other constraints on the theory. The candidate field equations are then

$$c_1 \left(R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu}\right) = 8\pi G\, T_{\mu\nu}. \tag{1.3.14}$$

To evaluate c_1, we study the non-relativistic limit

$$c_1 \left(R_{00} - \frac{1}{2}\, R\, g_{00}\right) = 8\pi G\, T_{00}. \tag{1.3.15}$$

In that case, this equation reduces to Poisson's law as expressed in (1.3.10) if and only if $c_1 = 1$. As a conclusion, we have now achieved the formulation of *Einstein*[2] *field equations*[3]

$$R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{1.3.16}$$

It is a non-linear (in the first derivative of the metric $g_{\mu\nu}$) system of 10 coupled differential equations. For this reason, they are very complicated to solve[4] and very few exact solutions are known, only for systems with strong symmetries. In the next chapter we shall examine Schwarzschild solution, valid in the exterior vacuum of a spherically-symmetric massive body.

[2] They are also called *Einstein-Hilbert field equations* due to their alternative derivation by Hilbert.

[3] In modern cosmological models, the source right-hand side presents the further "dark energy" addend $\Lambda g_{\mu\nu}$ - where Λ stands for the so-called *cosmological constant* - in order to take the accelerated expansion of the Universe into account. In actual fact, Einstein himself first introduced that term so as to obtain *static* Universe solutions, as was expected at that time.

[4] Their non-linearity conveys the fact that a gravitational field carries its own source, that is energy, and causes a sort of back-reaction that invalidates the superposition principle.

Chapter 2

Schwarzschild metric and Painlevé-Gullstrand coordinates

This chapter will be dedicated to the derivation of the Schwarzschild metric. We shall also provide the Painlevé-Gullstrand set of coordinates, which has the propriety to appear regular at the event horizon.

2.1 Schwarzschild metric

The Schwarzschild solution to Einstein field equations was found by K. Schwarzschild in 1916. It describes the space-time in the exterior vacuum of a spherically-symmetric massive source.

In this region we have $T_{\mu\nu} = 0$ (in the absence of electric charge), which implies

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0, \\ &g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \right) = 0, \\ &R - \frac{1}{2} \cdot 4R = 0 \;\Rightarrow\; R = 0. \end{aligned} \tag{2.1.1}$$

This leads to the *vacuum Einstein equation*

$$R_{\mu\nu} = 0. \tag{2.1.2}$$

Therefore, given a source with a certain boundary Σ, we will have two solutions:

- inside the source: $T_{\mu\nu} \neq 0$ and $g^{-}_{\mu\nu}$ is determined by the *standard* Einstein field equations.

- outside the source: $T_{\mu\nu} = 0$ and $g^{+}_{\mu\nu}$ is determined by the *vacuum Einstein equation* (2.1.2).

Then, the two solutions must join continuously:

$$ds^2\Big|_{\Sigma} = g^{-}_{\mu\nu}\, dx^\mu\, dx^\nu\Big|_{\Sigma^-} = g^{+}_{\mu\nu}\, dx^\mu\, dx^\nu\Big|_{\Sigma^+}. \tag{2.1.3}$$

In case of a spherically-symmetric source, it will not be necessary to check this boundary condition. As S. Weinberg points out in [7], in that case the Newtonian limit ensures the resulting metric will depend only on the mass of the source, not on its details.

Due to the spherical symmetry of the system, we are looking for a rotation-invariant line element in four dimensions. Its most general form is

$$ds^2 = F(r,t)dt^2 + C(r,t)d\vec{x}^{\,2} + E(r,t)dt\,\vec{x}\cdot d\vec{x} + D(r,t)(\vec{x}\cdot d\vec{x})^2, \tag{2.1.4}$$

where $r = \sqrt{\vec{x}\cdot\vec{x}}$ and all the differentials are rotation-invariant. This reduces the 10 independent components of $g_{\mu\nu}$ to 4. By introducing spherical polar coordinates

$$\begin{cases} x^1 = r\sin\theta\cos\phi \\ x^2 = r\sin\theta\sin\phi \\ x^3 = r\cos\theta \end{cases}, \tag{2.1.5}$$

we obtain

$$ds^2 = \gamma(r,t)\, dt^2 + \delta(r,t)\, dr\, dt + \alpha(r,t)\, dr^2 + \beta^2(r,t)(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.1.6}$$

Since on a 2-sphere S^2 with radius R the line element is expressed by

$$dl^2\Big|_{S^2} = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\Big|_{r=R} = R^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.1.7}$$

we observe that the last term in (2.1.6) represents the line element of a 2-sphere with radius $\beta(r,t)$ *depending* on r. Then, r is not the distance from the gravitational source, but the 2-sphere is identified by $r = R = const$ analogously.

If we perform a change of coordinates for (r,t), the spherical symmetry is maintained. Hence, we define

$$\begin{cases} \tilde{r} = \beta^2(r,t) \\ \tilde{t} = t \end{cases}, \tag{2.1.8}$$

which entails

$$\begin{cases} d\tilde{r} = \frac{1}{2\sqrt{\beta}}\left[\frac{\partial\beta}{\partial r}dr + \frac{\partial\beta}{\partial t}dt\right] \\ d\tilde{t} = dt \end{cases} \Rightarrow dr = \left[2\sqrt{\beta}\, d\tilde{r} - \frac{\partial\beta}{\partial t}d\tilde{t}\right]\left(\frac{\partial\beta}{\partial r}\right)^{-1}. \tag{2.1.9}$$

It is physically reasonable to expect that the gravitational field tends towards zero at infinity. This condition is called *asymptotic flatness* and implies $g_{\mu\nu} \longrightarrow \eta_{\mu\nu}$ for $r \longrightarrow +\infty$ in spherical coordinates. For that purpose, $\beta(r,t)$ cannot be constant in r. In the new coordinates, we have

$$ds^2 = \tilde{\gamma}(\tilde{r},\tilde{t})\, d\tilde{t}^2 + 2\tilde{\delta}(\tilde{r},\tilde{t})\, d\tilde{r}\, d\tilde{t} + \tilde{\alpha}(\tilde{r},\tilde{t})\, d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.1.10}$$

As concerns the time-like coordinate, it is convenient to choose one that diagonalizes the metric

$$\begin{cases} \hat{r} = \tilde{r} \\ \hat{t} = \phi(\tilde{r},\tilde{t}), \quad \text{with } \frac{\partial \phi}{\partial \tilde{t}} \neq 0 \end{cases}. \tag{2.1.11}$$

In this new set of $(\tilde{r},\tilde{t})$ coordinates, a mixed term of the following form appears

$$2\left(\frac{\partial \phi}{\partial t}\right)^{-1}\left[\delta - \gamma\left(\frac{\partial \phi}{\partial t}\right)^{-1}\frac{\partial \phi}{\partial r}\right] d\hat{r}\, d\hat{t}, \tag{2.1.12}$$

where we called $\tilde{r}$ and $\tilde{t}$ again r and t, respectively. We are free to make the mixed term vanish by choosing a function ϕ such that $\delta = \gamma\left(\frac{\partial \phi}{\partial t}\right)^{-1}\frac{\partial \phi}{\partial r}$. Finally, the metric can be expressed as

$$ds^2 = -e^{\nu(r,t)}dt^2 + e^{\mu(r,t)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.1.13}$$

where we used a specific notation known as *Schwarzschild gauge*. This shows that all spherically-symmetric metrics have only 2 independent components. The two functions $\nu(r,t)$ and $\mu(r,t)$ are found from the *Bianchi identities* (1.3.4) and from the *vacuum Einstein equation* (2.1.2), which further constrains them to be time-independent. Eventually, we are left with the solution

$$\mu + \nu = \lambda, \quad \text{for } r \geq R, \tag{2.1.14}$$

where λ is a constant, and

$$e^{-\mu(r,t)} = -\left(1 - \frac{2m}{r}\right), \tag{2.1.15}$$

where $2m$ is another constant. Then, the metric (2.1.13) becomes

$$ds^2 = -e^{\lambda}\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.1.16}$$

and only the constants λ and m remain to be fixed. By rescaling $t \longrightarrow e^{-\lambda/2}\, t$, the new time-like coordinate $e^{-\lambda/2}\, t$ still diagonalizes the metric and leads to $e^{\lambda}dt^2 \longrightarrow dt^2$. This means we can choose $\lambda = 0$.

As regards m, we shall take the Newtonian limit into account. In the Newtonian limit we have $g_{00} = -(1+2\phi)$, where $\phi = -\frac{GM}{r}$ is the Newtonian potential. On the other hand, in the Schwarzschild metric $g_{00} = -\left(1 - \frac{2m}{r}\right)$. As a consequence - by reintroducing the speed of light c - there must be

$$2m = \frac{2GM}{c^2}.$$

Finally, the *Schwarzschild solution* to Einstein field equation is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (2.1.17)$$

As we mentioned previously, ν and μ result to be time-independent. This means the staticity of the metric comes as a consequence and does not need to be assumed *a priori*. This remarkable outcome constitutes the

Birkhoff Theorem *The gravitational field generated by any spherically-symmetric source is static, even though the source itself is not static.*

The Schwarzschild metric presents two singularities, at $r = 0$ and $r = 2m = \frac{2GM}{c^2} \equiv R_H$ respectively. R_H is called *Schwarzschild radius*. It is important to mark that the coordinate r does not measure distances directly, but it is a quantity *related* to space position. In fact, the distance l at a fixed instant between two points labelled by r_1 and r_2 is given by

$$l = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{2m}{r}}}. \quad (2.1.18)$$

Analogously, the coordinate t is a quantity *related* to time measures. In fact, the time interval τ between two events labelled by t_1 and t_2, at a fixed space point denoted by r_0, is given by

$$\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{2m}{r_0}}\, dt = \sqrt{1 - \frac{2m}{r_0}}\, (t_2 - t_1). \quad (2.1.19)$$

This means the coordinate t can be interpreted as the proper time of a static observer in the asymptotic limit $r_0 \longrightarrow +\infty$. If the radius R of the source is greater than its *Schwarzschild radius* R_H, the singularity in R_H occurs in a region where the metric has not the (2.1.17) definition. However, there exist astrophysical sources for which $R < R_H$, like black holes. In that case, the singularity at the *Schwarzschild radius* is known as *Schwarzschild singularity*.

Let us consider the motion of a test particle in a Schwarzschild gravitational field. The geodesics equations imply that the particle cannot leave the plane identified by the centre of the gravitational source and the test particle's velocity. Then, we shall make use of the following theorem:

Theorem *If a metric tensor $g_{\mu\nu}$ is independent on a certain coordinate x^α, the corresponding component u_α of the cotangent vector is constant along the trajectory*

$$\frac{\partial g_{\beta\sigma}}{\partial x^\alpha} = 0 \quad \Rightarrow \quad \frac{du_\alpha}{ds} = 0.$$

Since in the Schwarzschild metric we have symmetry with respect to both t and ϕ coordinates, two first integrals of motion are obtained, just as happens in classical mechanics

$$\begin{array}{lll} \dfrac{\partial g_{\mu\nu}}{\partial t} = 0 & \Rightarrow & u_0 \equiv \tilde{E} \text{ (energy) is constant} \\ \dfrac{\partial g_{\mu\nu}}{\partial \phi} = 0 & \Rightarrow & u_3 \equiv \tilde{L} \text{ (angular momentum) is constant} \end{array} .$$

Upon the constraint

$$g_{\mu\nu}\, u^\mu\, u^\nu = -1 \qquad \text{with } u^2 = \frac{d\theta}{ds} = 0, \tag{2.1.20}$$

we have

$$-\frac{\tilde{E}^2}{1-\frac{2m}{r(s)}} + \frac{\left(\frac{dr}{ds}\right)^2}{1-\frac{2m}{r(s)}} + \frac{\tilde{L}^2}{r^2(s)} = -1, \tag{2.1.21}$$

which finally leads to the *radial* equation of motion for the test particle:

$$\left(\frac{dr}{ds}\right)^2 = \left(-1 + \frac{\tilde{E}^2}{1-\frac{2m}{r(s)}} - \frac{\tilde{L}^2}{r^2(s)}\right)\left(1 - \frac{2m}{r(s)}\right). \tag{2.1.22}$$

If a test particle is moving along a radial time-like geodesic (that is, $u_3 = \tilde{L} = 0$), from equation (2.1.22) we have

$$\left(\frac{dr}{ds}\right)^2 = \left(-1 + \frac{\tilde{E}^2}{1-\frac{2m}{r(s)}}\right)\left(1 - \frac{2m}{r(s)}\right) = \tilde{E}^2 - \left(1 - \frac{2m}{r(s)}\right). \tag{2.1.23}$$

Therefore, a freely falling observer measures a *finite* time interval

$$\tau = -\int_{r_1}^{r_2} \frac{1}{\sqrt{\left(1-\frac{2m}{r(s)}\right) - \tilde{E}^2}}\, dr, \tag{2.1.24}$$

even if $r_1 < R_H < r_2$. The integral in equation (2.1.18) is also well defined if $r_1 < R_H < r_2$. In conclusion, spatial distances and time intervals are *finite* across the *Schwarzschild singularity*, which does not result a boundary of space-time. The regularity of physical quantities around the *Schwarzschild singularity* suggests it might be just a *coordinate* singularity. In the next section we will introduce a different set of coordinates in which there is no such singularity. The gravitational field of the *Schwarzschild singularity* can be shown to be finite indeed. By computing a scalar from the *Riemann tensor*, we have information on the nature of gravitational field in a coordinate-independent way. For instance,

$$R_{\mu\nu\lambda k}\, R^{\mu\nu\lambda k} = 48\frac{m^2}{r^6}, \tag{2.1.25}$$

which is regular for $r = R_H$. This also conveys the singularity at $r = 0$ is truly physical.

Nevertheless, unusual physical phenomena occur due to the *Schwarzschild singularity*. For a test particle moving along a radial time-like geodesic, from equation (2.1.23) we have

$$\tilde{E}^2 - \left(1 - \frac{2m}{r(s)}\right) = \left(\frac{dr}{dt}\frac{dt}{ds}\right)^2 = \left(\frac{dr}{dt}\right)^2 \frac{\tilde{E}^2}{\left(1 - \frac{2m}{r(s)}\right)^2}, \tag{2.1.26}$$

since

$$\frac{dt}{ds} = u^0 = g^{00}\, u_0 = -\frac{\tilde{E}}{1 - \frac{2m}{r(s)}}. \tag{2.1.27}$$

Therefore, a static observer asymptotically far from the gravitational source would measure the following proper time interval

$$\Delta t = -\int_{r_1}^{r_2} \frac{\tilde{E}}{\sqrt{\tilde{E}^2 - \left(1 - \frac{2m}{r(s)}\right)}\left(1 - \frac{2m}{r(s)}\right)}\, dr, \tag{2.1.28}$$

which results *divergent* if $r_1 < R_H < r_2$.

As concerns the interior of a black hole ($0 < r < R_H$), the Schwarzschild metric is regular but its signature becomes $(+,-,+,+)$. This means r and t exchange their roles: r is now a time-like coordinate, while t is space-like. The metric is no longer static and the t-symmetry turns into invariance under spatial-transaltions. Furthermore, since time is characterized by flowing without interruption, we may infer that a particle inside a black hole will necessarily continue falling towards the singularity. For a static observer inside a black hole we would have

$$ds^2 = -\left(1 - \frac{2m}{r_0}\right) dt^2 > 0, \tag{2.1.29}$$

which is not physically admitted. Even light cannot be stopped once inside a black hole, since light is characterized by having $ds^2 = 0$. For this reason, in Schwarzschild space-time $r = R_H$ determines the *event horizon*, which is defined as the boundary of the region from which null geodesics cannot escape. There can be static photons only in the *Schwarzschild singularity*, where

$$ds^2 = -\left(1 - \frac{2m}{r_0}\right)\bigg|_{r_0 = 2m} dt^2 = 0. \tag{2.1.30}$$

For example, when a photon is radially emitted at $r = 2m$ during a star collapse, it continues staying there.

2.2 Painlevé-Gullstrand coordinate system

We shall now introduce a coordinate system in which the Schwarzschild metric (2.1.17) results regular across the event horizon. These coordinates are named after P. Painlevé and A. Gullstrand, who discovered them independently one on the other in the early 1920's. The Painlevé-Gullstrand coordinates will also play a significant role when we deal with spherically-symmetric acoustic black holes in the next chapter.

Let us consider an observer moving along in-going, radial, time-like geodesics of the Schwarzschild space-time and starting from $r = +\infty$. Within the set of standard coordinates (t, r, θ, ϕ) we introduced in the previous section, we define

$$f \equiv 1 - \frac{2m}{r}. \tag{2.2.1}$$

Then, the geodesics equations can be expressed as

$$\frac{dt}{ds} = -\frac{\tilde{E}}{f}, \qquad \left(\frac{dr}{ds}\right)^2 + f = \tilde{E}^2, \tag{2.2.2}$$

as we already saw in equations (2.1.27) and (2.1.23). Due to the asymptotic flatness, the energy parameter $\tilde{E}$ is related to the observer's initial speed v_∞ by

$$\tilde{E} = \frac{E(r = +\infty)}{m} = \gamma(r = +\infty) = \frac{1}{\sqrt{1 - v_\infty^2}}, \tag{2.2.3}$$

where $E(r = +\infty)$ is the relativistic energy at $r = +\infty$, m is the proper mass, $v_\infty \equiv v(r = +\infty) = \frac{dr}{dt}\big|_{r=+\infty}$ and c is set equal to unity.

Firstly, we shall concentrate on the case $v_\infty = 0$, which entails $\tilde{E} = 1$. The geodesics equations then reduce to ($\frac{dr}{ds}$ is negative since r decreases)

$$\frac{dt}{ds} = -\frac{1}{f}, \qquad \frac{dr}{ds} = -\sqrt{1-f}. \tag{2.2.4}$$

The covariant components u_α

$$u_\alpha = \left(-1, -\frac{\sqrt{1-f}}{f}, 0, 0\right) \tag{2.2.5}$$

of the observer's 4-velocity are also equal to the gradient of a time function T:

$$u_\alpha = -\partial_\alpha T, \tag{2.2.6}$$

where

$$T = t + \int_{r_1}^{r} \frac{\sqrt{1-f(r^*)}}{f(r^*)}\, dr^*. \tag{2.2.7}$$

Except for an arbitrary integration constant, this yields

$$T = t + 4M\left(\sqrt{\frac{r}{2M}} + \frac{1}{2}\ln\left|\frac{\sqrt{\frac{r}{2M}} - 1}{\sqrt{\frac{r}{2M}} + 1}\right|\right), \tag{2.2.8}$$

where c and G are set equal to unity. This shall be the new time-like coordinate, thus defining the *Painlevé-Gullstrand coordinates* as (T, r, θ, ϕ). Now we have

$$dt = dT - f^{-1}(r)\sqrt{\frac{2M}{r}}\, dr,$$

and the *Schwarzschild metric in the Painlevé-Gullstrand coordinates* becomes

$$ds^2 = -f dT^2 + 2\sqrt{\frac{2M}{r}}\, dT\, dr + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.2.9}$$

or equivalently

$$ds^2 = -dT^2 + \left(dr + \sqrt{\frac{2M}{r}} dT\right)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.2.10}$$

We observe that this metric is manifestly regular at the *event horizon* ($r = 2M$), and still singular at the centre of the source ($r = 0$). Another remarkable feature of this metric is the property that surfaces with constant T are intrinsically *flat*:

$$ds^2 = \left[-dT^2 + \left(dr + \sqrt{\frac{2M}{r}} dT\right)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]_{dT=0} =$$
$$= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

which is the metric of flat, three-dimensional space in spherical polar coordinates. All the space-time curvature is therefore entirely encoded in the so-called *shift vector*, the off-diagonal component of the metric tensor.

Secondly, it is possible to generalize these coordinates to other families of freely falling observers. Each family will be labelled by a different value of $\tilde{E}$, that is, by a different value of v_∞. By defining

$$p = \frac{1}{\tilde{E}^2} = 1 - v_\infty^2, \tag{2.2.11}$$

the geodesics equations now become

$$\frac{dt}{ds} = -\frac{1}{\sqrt{p}f}, \qquad \frac{dr}{ds} = -\sqrt{\frac{1-pf}{p}}. \tag{2.2.12}$$

The new parameter p results being $0 < p \leq 1$. Analogously to the previous case, the covariant components u_α of the observer's 4-velocity are proportional to the gradient of a time function T:

$$u_\alpha = -\frac{1}{\sqrt{p}}\partial_\alpha T, \tag{2.2.13}$$

where

$$T = t + \int_{r_1}^{r} \frac{\sqrt{1-pf(r^*)}}{f(r^*)} dr^*. \tag{2.2.14}$$

Except for an arbitrary integration constant, this yields

$$T = t + 2M\left(\frac{1-pf}{1-f} + \ln\left|\frac{1-\sqrt{1-pf}}{1+\sqrt{1-pf}}\right| - \frac{1-\frac{p}{2}}{\sqrt{1-p}}\ln\left|\frac{\sqrt{1-pf}-\sqrt{1-p}}{\sqrt{1-pf}+\sqrt{1-p}}\right|\right), \tag{2.2.15}$$

which shall be the new time-like coordinate. We have then

$$dt = dT - f^{-1}(r)\sqrt{1-pf(r)}\, dr,$$

and the Schwarzschild metric takes the form

$$ds^2 = -f dT^2 + 2\sqrt{1-pf}\, dT\, dr + p\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.2.16}$$

or equivalently

$$ds^2 = -\frac{1}{p}dT^2 + p\left(dr + \frac{1}{p}\sqrt{1-pf}dT\right)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.2.17}$$

This metric is still regular at the *event horizon* ($r = 2M$), but surfaces with constant T are no longer intrinsically flat. Indeed, the induced metric

$$ds^2 = p\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

is associated to a nonzero Riemann tensor just because of the factor p in front of dr^2, as can be proven.

If we compute the limits of T for p tending towards the extreme values of its range, we obtain

$$\lim_{p\to 1} T = t + 2M\left(\frac{2}{\sqrt{1-f}} + \ln\left|\frac{1-\sqrt{1-f}}{1+\sqrt{1-f}}\right|\right), \tag{2.2.18a}$$

$$\lim_{p\to 0} T = t + 2M\left(\frac{1}{1-f} + \ln\left|\frac{f}{1-f}\right|\right). \tag{2.2.18b}$$

They result to be exactly the time-like variables of Painlevé-Gullstrand and Eddington-Finkelstein coordinates[1] respectively. This means that p defines a one-parameter family of (Painlevé-Gullstrand)-like coordinates for Schwarzschild space-time. This one-parameter family is shown to go "smoothly" from Painlevé-Gullstrand ($p = 1$) to Eddington-Finkelstein ($p = 0$) system.

[1]The *Eddington-Finkelstein* coordinates are another set of regular coordinates $(\tilde{v}, r, \theta, \phi)$ – where $\tilde{v} = t + \int_{r_1}^{r} \frac{dr^*}{f(r^*)}$ – and describe radial null geodesics.

Chapter 3

Black holes hydrodynamics

In this chapter we shall show how to deal with acoustic disturbances propagation in a non-homogeneous flowing fluid by invoking the tools of Lorentzian differential geometry. The analogy between these apparently distant frameworks is based on the following theorem:

Theorem *If a fluid is barotropic and inviscid, and the flow is irrotational (though possibly time-dependent) then the equation of motion for the velocity potential ψ describing an acoustic disturbance is identical to the d'Alembertian equation of motion for a minimally coupled massless scalar field propagating in a (3+1)-dimensional Lorentzian geometry*

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\psi) = 0, \tag{3.0.1}$$

where $g_{\mu\nu}(t, \vec{x})$ stands for the so called acoustic metric *and depends on the density, flow velocity and local speed of sound.*

Therefore, upon proving this theorem, we will find a remarkable connection between classical Newtonian fluid physics and the differential geometry of curved (3+1)-dimensional Lorentzian space-times. This will provide a fruitful analogy between the black holes of Einstein gravity and supersonic fluid flows. It will finally lead to think of several concrete non-relativistic laboratory size systems to form acoustic black holes e.g. *de Laval nozzle.*

3.1 Fundamental equations of fluid dynamics

To begin with, we recall that for a static homogeneous inviscid (zero viscosity) fluid the propagation of sound waves is governed by the d'Alembert equation

$$\partial_t^2 \psi = c^2 \nabla^2 \psi, \tag{3.1.1}$$

where ψ indicates the sound wave function and c the speed of sound. Then, if the fluid is in motion, it is necessary to take the fundamental equations of fluid dynamics into account. They are the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0, \tag{3.1.2}$$

and Euler's equation

$$\rho \frac{d\vec{v}}{dt} \equiv \rho[\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v}] = \vec{F}, \tag{3.1.3}$$

where

- $\rho \equiv$ fluid volumetric mass density
- $\vec{v} \equiv$ fluid velocity vector
- $\vec{F} \equiv$ force on the fluid per unit volume

Let us assume the viscosity is negligible, with the only forces present being those due to pressure, Newtonian gravity and an arbitrary gradient-derived and possibly even time-dependent externally-imposed body force. In this case, the expression for the force per unit volume takes the following form:

$$\vec{F} = -\nabla p - \rho \nabla \phi - \rho \nabla \Phi, \tag{3.1.4}$$

where

- $p \equiv$ pressure on the fluid
- $\phi \equiv$ Newtonian gravitational potential
- $\Phi \equiv$ potential of the external driving force (which may in fact be zero)

By manipulating Euler's equation with standard algebra, it can be rewritten as

$$\partial_t \vec{v} = \vec{v} \times (\nabla \times \vec{v}) - \frac{1}{\rho} \nabla p - \nabla \left(\frac{1}{2} v^2 + \phi + \Phi \right) \tag{3.1.5}$$

We will now take the flow to be *vorticity free*, that is *locally irrotational* ($\nabla \times \vec{v} = 0$). This is quite common for normal fluids, especially in situations of high symmetry. Provided that, we can introduce the velocity potential ψ such that $\vec{v} = -\nabla\psi$, with ψ not necessarily being globally defined (it is sufficient that the flow be vorticity free for velocity potentials existing on an atlas of open patches). We will further assume the fluid is *barotropic*, which means that ρ is a function of p only. On the other hand, in [5] W. G. Unruh makes the stronger assumption that the fluid is *isentropic* - that is, the specific entropy density is constant throughout the fluid - but this is not required. It is now possible to define the specific enthalpy *h(p)* as a function of p only:

$$h(p) = \int_0^p \frac{dp'}{\rho(p')}; \tag{3.1.6}$$

so that

$$\nabla h = \frac{1}{\rho}\nabla p. \tag{3.1.7}$$

By integration, now Euler's equation reduces to

$$-\partial_t\psi + h + \frac{1}{2}(\nabla\psi)^2 + \phi + \Phi = constant, \tag{3.1.8}$$

which is a version of Bernoulli's equation in the presence of external driving-forces.

3.2 Perturbative approach

At this point, we could proceed by solving the complete equations of motion for the fluid variables (ρ, p, ψ). However, in practice it appears more convenient to separate the *exact* motion into two terms: some average bulk motion plus low amplitude fluctuations. That is to say, we demand to *linearize* these equations of motion around a certain *background* in order to deal with acoustical disturbances. This means we shall set the functions (ρ, p, ψ) to be given by a definite bulk triad (ρ_0, p_0, ψ_0) plus a first order perturbation (ρ_1, p_1, ψ_1):

$$\rho = \rho_0 + \epsilon\rho_1 + O(\epsilon^2),$$

$$p = p_0 + \epsilon p_1 + O(\epsilon^2),$$

$$\psi = \psi_0 + \epsilon\psi_1 + O(\epsilon^2),$$

with the gravitational potential ϕ and the driving potential Φ taken to be *fixed* and external. Here *fixed* means neither is back-reaction allowed to modify the potentials nor *time-independent*, as we expect the external driving forces can be time-dependent indeed.

Sound - and more generally acoustical disturbances - is *defined* to be these linearized fluctuations in the dynamical quantities. It can be useful to clarify the subtle difference between *wind gusts* and sound waves, which is to some extent a matter of convention. The former are sufficiently low-frequency long-wavelength disturbances and are conventionally lumped in with the average bulk motion. On the other hand, the latter are higher-frequency shorter-wavelength disturbances. Underlying our linearization programme there is the requirement that the amplitude of such high-frequency short-wavelength disturbances be small. Sufficiently high-amplitude sound waves then need direct solution of the full equations of fluid dynamics.

Let us start by linearizing the continuity equation:

$$\partial_t(\rho_0 + \epsilon\rho_1 + O(\epsilon^2)) + \nabla\cdot[(\rho_0 + \epsilon\rho_1 + O(\epsilon^2))\,(\vec{v}_0 + \epsilon\vec{v}_1 + O(\epsilon^2))] = 0. \quad (3.2.1)$$

Neglecting second order terms, this yields

$$\partial_t\rho_0 + \nabla\cdot(\rho_0\,\vec{v}_0) + \epsilon[\partial_t\rho_1 + \nabla\cdot(\rho_0\,\vec{v}_1 + \rho_1\,\vec{v}_0)] = 0. \quad (3.2.2)$$

We shall assume the background functions (ρ_0, p_0, ψ_0) satisfy the continuity equation. This means that the coefficient of the first-degree term in ϵ in (3.2.2) must vanish in order that (ρ, p, ψ) satisfy the continuity equation as well. The result is the pair of equations

$$\partial_t\rho_0 + \nabla\cdot(\rho_0\,\vec{v}_0) = 0, \quad (3.2.3a)$$
$$\partial_t\rho_1 + \nabla\cdot(\rho_0\,\vec{v}_1 + \rho_1\,\vec{v}_0) = 0. \quad (3.2.3b)$$

The barotropic condition implies

$$\begin{aligned} h(p) &= h(p_0 + \epsilon p_1 + O(\epsilon^2)) = h(p_0) + \left(\left.\frac{dh(p)}{dp}\right|_{p=p_0}\right)\epsilon p_1 + O(\epsilon^2) = \\ &= h_0 + \epsilon\,\frac{p_1}{\rho_0} + O(\epsilon^2), \end{aligned} \quad (3.2.4)$$

where $h_0 \equiv h(p_0)$ and $\rho_0 = \rho(p_0)$.

We will now linearize Euler's equation, taking what (3.2.4) entails into consideration:

$$\begin{aligned} &-\partial_t(\psi_0 + \epsilon\psi_1 + O(\epsilon^2)) + \left(h_0 + \epsilon\frac{p_1}{\rho_0} + O(\epsilon^2)\right) + \frac{1}{2}[\nabla(\psi_0 + \epsilon\psi_1 + O(\epsilon^2))]^2 + \\ &+ \phi + \Phi = constant \end{aligned} \quad (3.2.5)$$

Neglecting second order terms, this yields

$$-\partial_t\psi_0+h_0+\frac{1}{2}(\nabla\psi_0)^2+\phi+\Phi+\epsilon\left[-\partial_t\psi_1+\frac{p_1}{\rho_0}-\vec{v}_0\cdot(\nabla\psi_1)\right]=constant. \quad (3.2.6)$$

We shall assume the background functions (ρ_0, p_0, ψ_0) also satisfy Euler's equation. This means that the coefficient of the first-degree term in ϵ in (3.2.2) must vanish in order that (ρ, p, ψ) satisfy Euler's equation as well. The result is the pair of equations

$$-\partial_t\psi_0+h_0+\frac{1}{2}(\nabla\psi_0)^2+\phi+\Phi=constant, \quad (3.2.7a)$$

$$-\partial_t\psi_1+\frac{p_1}{\rho_0}-\vec{v}_0\cdot(\nabla\psi_1)=0. \quad (3.2.7b)$$

This last equation may be rearranged in the following way

$$p_1=\rho_0(\partial_t\psi_1+\vec{v}_0\cdot\nabla\psi_1). \quad (3.2.8)$$

Due to the barotropic assumption, we have

$$\rho=\rho(p)=\rho(p_0)+\left(\left.\frac{d\rho}{dp}\right|_{p=p_0}\right)dp+O(dp^2)=\rho_0+\left(\left.\frac{d\rho}{dp}\right|_{p=p_0}\right)\epsilon p_1+O(\epsilon^2), \quad (3.2.9)$$

and we had already constrained the expression of ρ:

$$\rho=\rho_0+\epsilon\rho_1+O(\epsilon^2). \quad (3.2.10)$$

Consequently,

$$\epsilon\rho_1=\left(\left.\frac{d\rho}{dp}\right|_{p=p_0}\right)\epsilon p_1 \qquad \Rightarrow \qquad \rho_1=\left(\left.\frac{d\rho}{dp}\right|_{p=p_0}\right)p_1, \quad (3.2.11)$$

and substituting (3.2.8) into (3.2.11) we obtain

$$\rho_1=\left(\left.\frac{d\rho}{dp}\right|_{p=p_0}\right)\rho_0(\partial_t\psi_1+\vec{v}_0\cdot\nabla\psi_1). \quad (3.2.12)$$

If we now express the linearized subequation of continuity (3.2.3b) in terms of this result, we find

$$\partial_t\left[\left(\left.\frac{d\rho}{dp}\right|_{p=p_0}\right)\rho_0(\partial_t\psi_1+\vec{v}_0\cdot\nabla\psi_1)\right] + \nabla\cdot\left\{\rho_0\,\vec{v}_1+\left[\left(\left.\frac{d\rho}{dp}\right|_{p=p_0}\right)\rho_0(\partial_t\psi_1+\vec{v}_0\cdot\nabla\psi_1)\right]\vec{v}_0\right\}=0, \quad (3.2.13)$$

$$\partial_t \left[\left(\frac{d\rho}{dp}\bigg|_{p=p_0} \right) \rho_0 (\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1) \right] \\ - \nabla \cdot \left\{ \rho_0 \, \nabla \psi_1 - \left[\left(\frac{d\rho}{dp}\bigg|_{p=p_0} \right) \rho_0 (\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1) \right] \vec{v}_0 \right\} = 0, \quad (3.2.14)$$

$$- \partial_t \left[\left(\frac{d\rho}{dp}\bigg|_{p=p_0} \right) \rho_0 (\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1) \right] \\ + \nabla \cdot \left\{ \rho_0 \, \nabla \psi_1 - \left[\left(\frac{d\rho}{dp}\bigg|_{p=p_0} \right) \rho_0 (\partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1) \right] \vec{v}_0 \right\} = 0, \quad (3.2.15)$$

The final result (3.2.15) is a wave equation describing the propagation of the linearized scalar potential ψ_1 in a non-homogeneous inviscid fluid. Once ψ_1 is determined, p_1 and ρ_1 can be computed respectively from equations (3.2.8) and (3.2.12). Thus, the wave equation (3.2.15) completely determines the propagation of acoustical disturbances (ρ_1, p_1, ψ_1). Furthermore, we can observe that the local speed of sound is defined by

$$c^{-2} \equiv \left(\frac{d\rho}{dp}\bigg|_{p=p_0} \right). \quad (3.2.16)$$

For the purpose of this dissertation, it is useful to render our wave equation in a different shape. Let us introduce (3+1)-dimensional space-time coordinates $x^\mu \equiv (t; x^i)$ - Greek indices run from 0 to 3, while Roman indices run from 1 to 3. Once constructed the following symmetric 4×4 matrix depending on these coordinates x^μ

$$f^{\mu\nu}(t, \vec{x}) \equiv \frac{\rho_0}{c^2} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots & & \cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2 \delta^{ij} - v_0^i v_0^j) \end{bmatrix}, \quad (3.2.17)$$

the wave equation (3.2.15) can be rewritten as

$$\partial_\mu (f^{\mu\nu} \, \partial_\nu \psi_1) = 0. \quad (3.2.18)$$

This remarkably compact formulation shows a clear resemblance to the covariant wave equation in curved space-time (see in the Appendix) and suggests employing the techniques of curved space-time (3+1)-dimensional Lorentzian geometry.

3.3 The acoustic metric

Let us start this section with a pair of definitions from differential geometry. A pseudo-Riemannian manifold is a manifold on which a non-degenerate scalar product from a metric tensor is defined. A Lorentzian manifold is a pseudo-Riemannian manifold whose metric tensor has signature $(n-1,1)$ (or equivalently $(1,n-1)$), where n stands for the number of dimensions of the manifold itself. As proven in the Appendix, in any Lorentzian manifold the curved space-time covariant d'Alembertian operator is given in terms of the metric $g_{\mu\nu}(t,\vec{x})$ and partial derivatives only by

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\psi), \tag{3.3.1}$$

where ψ is a scalar field. Therefore, the covariant wave equation in curved space-time is written as

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\psi) = 0. \tag{3.3.2}$$

The inverse metric $g^{\mu\nu}(t,\vec{x})$ is pointwise the inverse matrix of $g_{\mu\nu}(t,\vec{x})$, and $g \equiv \det(g_{\mu\nu})$. Let us compare the wave operator in (3.2.18) for the propagation of the linearized scalar velocity potential ψ_1 with the one in (3.3.2) for the propagation of a scalar field ψ in a curved space-time

$$\Delta \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu) \tag{3.3.3a}$$

$$\partial_\mu(f^{\mu\nu}\,\partial_\nu). \tag{3.3.3b}$$

Then, we infer that we may identify

$$\sqrt{-g}\,g^{\mu\nu} = f^{\mu\nu}. \tag{3.3.4}$$

On the one hand, this implies

$$\begin{aligned}\det(f^{\mu\nu}) &= \det\big(\sqrt{-g}\,g^{\mu\nu}\big) = (\sqrt{-g})^4\det(g^{\mu\nu}) = (\sqrt{-g})^4[\det(g_{\mu\nu})]^{-1} = \\ &= (\sqrt{-g})^4 g^{-1} = g.\end{aligned} \tag{3.3.5}$$

On the other hand, from the explicit expression (3.2.17),

$$\begin{aligned}\det(f^{\mu\nu}) &= \det\left(\frac{\rho_0}{c^2}\begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots & & \cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\delta^{ij} - v_0^i v_0^j)\end{bmatrix}\right) = \left(\frac{\rho_0}{c^2}\right)^4 \begin{vmatrix} -1 & \vdots & -v_0^j \\ \cdots & & \cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\delta^{ij} - v_0^i v_0^j)\end{vmatrix} = \\ &= \left(\frac{\rho_0}{c^2}\right)^4 \cdot [(-1)\cdot(c^2 - v_0^2) - (-v_0)^2]\cdot[c^2]\cdot[c^2] = -\frac{\rho_0^4}{c^2},\end{aligned} \tag{3.3.6}$$

where we expanded the determinant in minors by means of the Laplace theorem. Hence,

$$g = -\frac{\rho_0^4}{c^2} \qquad \Rightarrow \qquad \sqrt{-g} = \frac{\rho_0^2}{c}. \tag{3.3.7}$$

By the hypothesized identification (3.3.4), $g^{\mu\nu}(t, \vec{x})$ becomes

$$\begin{aligned} g^{\mu\nu}(t, \vec{x}) = \frac{f^{\mu\nu}(t, \vec{x})}{\sqrt{-g}} &= \frac{c}{\rho_0^2} \left(\frac{\rho_0}{c^2} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots & & \cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\delta^{ij} - v_0^i v_0^j) \end{bmatrix} \right) = \\ &= \frac{1}{\rho_0 c} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots & & \cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\delta^{ij} - v_0^i v_0^j) \end{bmatrix}. \end{aligned} \tag{3.3.8}$$

We can now determine the metric $g_{\mu\nu}(t, \vec{x})$ itself by inverting this 4×4 matrix. Finally, we obtain the so-called *acoustic metric*

$$g_{\mu\nu}(t, \vec{x}) \equiv \frac{\rho_0}{c} \begin{bmatrix} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots & & \cdots \\ -v_0^i & \vdots & \delta_{ij} \end{bmatrix}, \tag{3.3.9}$$

and equivalently the *acoustic interval* can be expressed as

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu\, dx^\nu = \frac{\rho_0}{c} \left[-c^2 dt^2 + (dx^i - v_0^i\, dt)\, \delta_{ij}\, (dx^j - v_0^j\, dt) \right]. \tag{3.3.10}$$

We point out that the signature of this metric is indeed $(-, +, +, +)$ - or equivalently (3,1) - which allows us to regard it as Lorentzian. Moreover, we can deduce that there are two distinct metrics relevant to our physical system and they play two different roles:

- The *physical space-time metric* is just the usual flat Minkowski metric

  $$\eta_{\mu\nu} \equiv (\text{diag}[-c_{light}^2, 1, 1, 1])_{\mu\nu}, \tag{3.3.11}$$

  where c_{light} = speed of light. This is the only metric to which the fluid particles couple - in fact, we assume that their motion is completely non-relativistic ($\|\vec{v}_0\| \ll c_{light}$).

- Conversely, sound waves couple only to the *acoustic metric* $g_{\mu\nu}$ without "perceiving" the physical metric $\eta_{\mu\nu}$ at all.

Before proceeding with the next section, we shall add another pair of comments. While in a completely general (3+1)-dimensional Lorentzian geometry the metric has 6 degrees of freedom per point in space-time, the *acoustic metric* is more constrained. The former is described by a 4×4 symmetric matrix, which leaves 10 independent components, then the 4 Bianchi identities lead to 6 degrees of freedom in whole. The latter, in addition, is specified completely by the 3 scalars $\psi_0(t, \vec{x})$, $\rho_0(t, \vec{x})$ and $c(t, \vec{x})$, and the continuity equation adds another constraint. In the end, the *acoustic metric* has at most 2 degrees of freedom per point in space-time.

Another difference between the two kinds of metric is the relation to the distribution of matter. For the former it is determined by the non-linear Einstein-Hilbert differential field equations, whereas the equation for the latter (3.3.9) is linear in the volumetric mass density ρ_0.

3.4 The acoustic metric in Painlevé-Gullstrand coordinates

We shall now see how close the *acoustic metric* achieves reproducing the Schwarzschild geometry we depicted in Chapter 2. This appears in a clearer way if we introduce the Painlevé-Gullstrand coordinates to represent Schwarzschild space-time (see Chapter 2). As showed in Chapter 2, the Schwarzschild metric in the Painlevé-Gullstrand coordinates may be written as:

$$ds^2 = -dT^2 + \left(dr + \sqrt{\frac{2M}{r}} dT \right)^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{3.4.1}$$

where the gravitational constant G and the speed of light c_{light} have been conventionally set equal to 1. Given such Painlevé-Gullstrand line element, we could try to force the *acoustic interval* (3.3.10) in the specific case of a fluid flow surrounding a point sink (i.e. a spherically-symmetric fluid flow) into this form. Once performed the summation over the indexes i and j, the *acoustic interval* (3.3.10) is equivalent to

$$\begin{aligned}
ds^2 = {} & \frac{\rho_0}{c}\left[-c^2dt^2 + (dx^1 - v_0^1\,dt)^2 + (dx^2 - v_0^2\,dt)^2 + (dx^3 - v_0^3\,dt)^2\right] = \\
= {} & \frac{\rho_0}{c}[-c^2dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (v_0^1\,dt)^2 + (v_0^2\,dt)^2 + (v_0^3\,dt)^2 + \\
& - 2\,dx^1 v_0^1\,dt - 2\,dx^2 v_0^2\,dt - 2\,dx^3 v_0^3\,dt] = \\
= {} & \frac{\rho_0}{c}[-c^2dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + ((v_0^1)^2 + (v_0^2)^2 + (v_0^3)^2)\,dt^2 + \\
& - 2\,dt(v_0^1\,dx^1 + v_0^2\,dx^2 + v_0^3\,dx^3) = \\
= {} & \frac{\rho_0}{c}[-c^2dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + v_0^2\,dt^2 + \\
& - 2\,dt(v_0^1\,dx^1 + v_0^2\,dx^2 + v_0^3\,dx^3).
\end{aligned} \tag{3.4.2}$$

We shall now move to spherical polar coordinates and the above becomes

$$\begin{aligned}
ds^2 = {} & \frac{\rho_0}{c}[-c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2 + v_0^2\,dt^2 + \\
& - 2\,dt(v_0^{\hat{r}}\,dr + v_0^{\hat{\theta}}\,r\,d\theta + v_0^{\hat{\phi}}\,r\,\sin\theta\,d\phi)].
\end{aligned} \tag{3.4.3}$$

For a fluid flow surrounding a point sink (i.e. a spherically-symmetric fluid flow), the $\vec{v}_0$ components along the $\hat{\theta}$ and $\hat{\phi}$ directions vanish, hence the equation (3.4.3) reduces to

$$\begin{aligned}
ds^2 = {} & \frac{\rho_0}{c}[-c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2 + (v_0^{\hat{r}})^2\,dt^2 - 2\,dt\,v_0^{\hat{r}}\,dr] = \\
= {} & \frac{\rho_0}{c}[-c^2dt^2 + (dr - v_0^{\hat{r}}\,dt)^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)].
\end{aligned} \tag{3.4.4}$$

If we compare this result to the Painlevé-Gullstrand line element (3.4.1), we immediately observe a remarkable analogy (T and t play the same role of "time" coordinate).

As a consequence, we infer that the analogy is completely fulfilled if the following constraints can be compelled:

- the local speed of sound c is picked to be a time and position independent constant, which we normilize to unity,
- the density ρ_0 is picked to be time independent,
- $v_0 = v_0^{\hat{r}}$ is set equal to $-\sqrt{\frac{2M}{r}}$ (the minus sign indicates that the velocity vector $\vec{v}_0$ points towards the sink).

In that case, the continuity subequation (3.2.3a) reduces to

$$\nabla \cdot (\rho_0 \vec{v}_0) = 0, \tag{3.4.5}$$

which can be used to deduce the position-dependency of ρ_0. Since $\vec{v}_0 = v_0^{\hat{r}}\,\hat{r}$, the reduced continuity subequation (3.4.5) yields

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_0 v_0\right) = 0. \tag{3.4.6}$$

Thereafter,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_0 v_0\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_0 \sqrt{\frac{2M}{r}}\right) = 0, \tag{3.4.7}$$

and avoiding the singularity for $r = 0$, we obtain

$$\begin{aligned} \frac{\partial}{\partial r}\left(r^2 \rho_0 \sqrt{\frac{2M}{r}}\right) = 0 \quad &\Rightarrow \quad r^2 \rho_0 \sqrt{\frac{2M}{r}} = \text{constant (in } r) \quad \Rightarrow \\ &\Rightarrow \quad \rho_0 \propto r^{-3/2}. \end{aligned} \tag{3.4.8}$$

Overall, the *acoustic metric* (3.4.4) is now

$$ds^2 \propto r^{-3/2}\left[-dT^2 + \left(dr + \sqrt{\frac{2M}{r}}dT\right)^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{3.4.9}$$

thus achieving a *conformal* net result to the Painlevé-Gullstrand form of the Schwarzschild metric. The relationship between the *acoustic metric* and the Schwarzschild geometry is not *exact* though. However, if our attention were focused either on analyzing basic features of the Hawking radiation process or on the behaviour in the immediate region of the event horizon, the conformal factor would not be influential and could be neglected. In actual fact, this matter lies outside the purpose of our dissertation and we shall rather move to an example of acoustic black hole realization.

3.5 De Laval nozzle

Unlike gravitational black holes, which are formed by the collapse of very massive objects, acoustic (or, equivalently, sonic) black holes do not require such extreme conditions for their realization. A simple example is provided by a converging-diverging nozzle, called a *de Laval nozzle* (Figure 3.5.1).

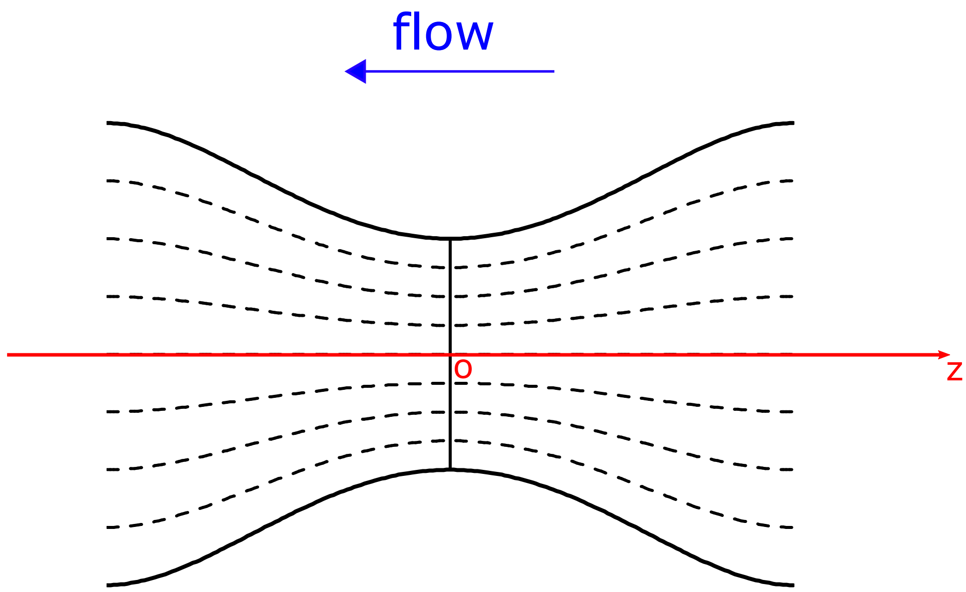

Figure 3.5.1: A de Laval nozzle. As explained in the text, it is possible to set the system to have a regular flow that is subsonic on one side of the waist and supersonic on the other. The one-dimensional reference frame is fixed along the axis of the nozzle and points to the right. The origin is located at the waist of the nozzle, where the sonic horizon ($v_0(z) = c$) is formed. The *acoustic metric* associated with this flow is that of a one-dimensional acoustic black hole $ds^2 = \frac{\rho_0}{c}\left[-c^2dt^2 + (dz - v_0(z)\,dt)^2\right]$.

Let us fix a reference frame such that the nozzle points along the z-axis and the origin is set at the waist of the nozzle. The fluid is made to flow from right to left with transverse velocities (i.e. in the x and y directions) negligible with respect to the velocity along the z-axis. We also make the further assumption that this quasi one-dimensional flow is stationary. This means that the volumetric mass density ρ_0 does not depend on time and the continuity equation (3.2.3a) yields

$$\nabla \cdot (\rho_0\,\vec{v}_0) = 0. \tag{3.5.1}$$

Let us now integrate this equation on the volume V of a flux tube in the nozzle starting at coordinate z_1 and finishing at coordinate z_2 (Figure 3.5.2). Upon applying the divergence theorem, we have:

$$\int_V d^3x\,\nabla \cdot (\rho_0\,\vec{v}_0) = \oint_{\partial V} d\sigma\,\hat{n} \cdot \rho_0\,\vec{v}_0, \tag{3.5.2}$$

where

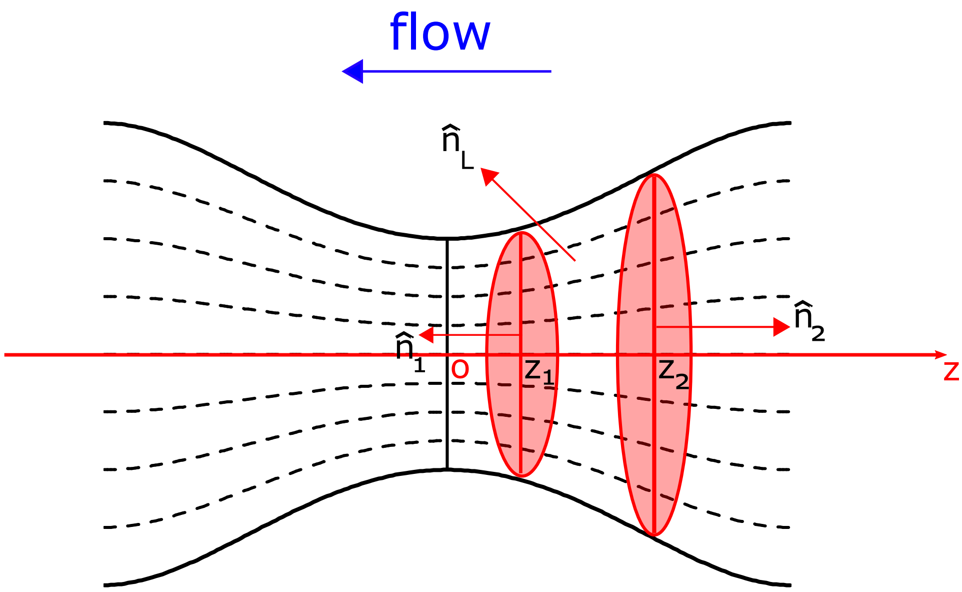

Figure 3.5.2: The flux tube of integration. $\hat{n}_1$ and $\hat{n}_2$ denote the unit vectors orthogonal to the base and to the top of the flux tube respectively, whereas $\hat{n}_L$ stands for the unit vector orthogonal to the lateral surface of the flux tube.

- $\partial V \equiv$ boundary of the volume V,
- $d\sigma \equiv$ infinitesimal surface element of ∂V,
- $\hat{n} \equiv$ unit vector perpendicular to ∂V.

Since the velocity $\vec{v}_0$ points along streamlines, $\vec{v}_0$ is orthogonal to $\hat{n}$ on the lateral surface of the flux tube and their scalar product will be null. On the other hand, the only non-vanishing contributions to the surface integral in (3.5.2) will be given by the scalar products on the base and the top of the flux tube. In the physical system that we are considering – being quasi one-dimensional – ρ_0 and $\vec{v}_0$ are set to depend only on the z-coordinate. Therefore, (3.5.2) becomes ($\hat{n}_2 = -\hat{n}_1$)

$$\begin{aligned}\int_{A(z_1)} d\sigma\, \hat{n}_1 \cdot \rho_0\, \vec{v}_0 - \int_{A(z_2)} d\sigma\, \hat{n}_1 \cdot \rho_0\, \vec{v}_0 = \\ = \rho_0(z_1)\, v_0(z_1)\, A(z_1) - \rho_0(z_2)\, v_0(z_2)\, A(z_2),\end{aligned} \tag{3.5.3}$$

where $A(z)$ indicates the cross section area at coordinate z. The stationary condition (3.5.1) holding, the integral (3.5.2) will be identically null, thus yielding

$$\rho_0(z_1)\, v_0(z_1)\, A(z_1) = \rho_0(z_2)\, v_0(z_2)\, A(z_2). \tag{3.5.4}$$

Without changing this result, we can ideally reduce the lateral surface as much as we want. In the limit of the lateral surface area tending towards zero, we obtain

$$\rho_0(z)\, v_0(z)\, A(z) = constant. \tag{3.5.5}$$

Assuming no influence of either gravity or external forces, the reduced Euler's subequation (3.2.7a) of our system is

$$\partial_t \vec{v}_0 = -\frac{1}{\rho}\nabla p_0 - \nabla\left(\frac{1}{2}v_0^2\right). \tag{3.5.6}$$

Since for a stationary flow the velocity does not depend explicitly on time, we have

$$\begin{aligned} &-\frac{1}{\rho}\nabla p_0 - \nabla\left(\frac{1}{2}v_0^2\right) = 0 \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial z}p_0 + \frac{\partial}{\partial z}\left(\frac{1}{2}v_0^2\right) = 0 \quad \Rightarrow \\ &\Rightarrow \quad \frac{1}{\rho}\frac{\partial p_0}{\partial \rho}\frac{\partial \rho}{\partial z} + v_0(z)\frac{\partial v_0(z)}{\partial z} = 0. \end{aligned} \tag{3.5.7}$$

As we are not considering perturbations in this case, $\rho = \rho_0 = \rho_0(z)$. Moreover, due to the barotropic assumption,

$$\frac{\partial p_0}{\partial \rho} = \frac{dp_0}{d\rho} = \left(\left.\frac{d\rho}{dp}\right|_{p=p_0}\right)^{-1} = c^2, \tag{3.5.8}$$

and we obtain

$$\frac{c^2}{\rho_0(z)}\frac{d\rho_0(z)}{dz} + v_0(z)\frac{dv_0(z)}{dz} = 0. \tag{3.5.9}$$

By differentiating equation (3.5.5), we have

$$\rho_0'\, v_0\, A + \rho_0\, v_0'\, A + \rho_0\, v_0\, A' = 0 \quad \Rightarrow \quad \rho_0' = -\frac{\rho_0}{A\, v_0}(A' v_0 + A\, v_0'), \tag{3.5.10}$$

where $'$ stands for derivative with respect to z. Now, substituting (3.5.10) into (3.5.9), this yields

$$\begin{aligned} &\frac{c^2}{\rho_0}\left[-\frac{\rho_0}{A\, v_0}(A' v_0 + A\, v_0')\right] + v_0\, v_0' = 0 \quad \Rightarrow \quad -\frac{c^2 A'}{A} - \frac{c^2 v_0'}{v_0} + v_0\, v_0' = 0 \quad \Rightarrow \\ &\Rightarrow \quad \frac{A'}{A} = -\frac{1}{c^2}\left(\frac{c^2 v_0'}{v_0} - v_0\, v_0'\right) = \left(\frac{v_0\, v_0'}{c^2} - \frac{v_0'}{v_0}\right) = \frac{v_0'}{v_0}\left(\frac{v_0^2 - c^2}{c^2}\right) = \lambda(v_0)\,\frac{v_0'}{v_0}, \end{aligned} \tag{3.5.11}$$

where

$$\lambda(v_0) \equiv \frac{v_0^2 - c^2}{c^2} \tag{3.5.12}$$

This equation tells us a difference in behaviour between subsonic and supersonic flows.

- For the former (that is, $v_0 < c$), $\lambda(v_0) < 0$ therefore the speed decreases as the area increases and vice versa. Let us suppose the fluid starts from $z = +\infty$ with a speed $v_{+\infty}$. We have

$$A'/A < 0 \;\Rightarrow\; v_0'/v_0 > 0$$

through the converging nozzle – until the waist of the nozzle, where

$$A'/A = 0 \;\Rightarrow\; v_0'/v_0 = 0.$$

Then,

$$A'/A > 0 \;\Rightarrow\; v_0'/v_0 < 0$$

along the diverging part.

Consequently, we infer that the speed increases till a maximum value $v_0^* < c$ in correspondence with the waist of the nozzle, and then decreases to a value $v_{-\infty} \leq v_{+\infty}$.

- For the latter (that is, $v_0 > c$), $\lambda(v_0) > 0$ and the picture changes completely. Let us consider the same situation in the previous point with an exception.

The fluid motion starts being subsonic but now it has been adjusted so as to increase its speed beyond the value c. According to (3.5.11), the transition from subsonic to supersonic flow can occur only at the waist of the nozzle. Unless the acceleration $\frac{dv_0}{dt} = \frac{dv_0}{dz}\frac{dz}{dt} = v_0'\, v_0$ diverges when v_0 tends towards c[1] (which we consider as unphysical), $v_0 = c$ implies $A'/A = 0$, which holds only at the origin indeed. Then, after passing the waist, $v_0 > c \;\Rightarrow\; \lambda(v_0 > c) > 0$, which entails A'/A and v_0'/v_0 have the same sign. This means that the fluid now increases its speed along the divergent part of the nozzle, unlike the subsonic case. In addition, from the continuity equation (3.5.5) we infer that in the supersonic region

[1]In that case,

$$\lim_{v_0\to c} \frac{A'}{A} = \lim_{v_0\to c} \frac{v_0'v_0}{v_0^2}\left(\frac{v_0^2 - c^2}{c^2}\right) = \lim_{v_0\to c} v_0' v_0 \left(\frac{v_0^2 - c^2}{v_0^2 c^2}\right)$$

is not defined, since

$$\lim_{v_0\to c} \left(\frac{v_0^2 - c^2}{v_0^2 c^2}\right) = 0.$$

($z < 0$), since both A and v_0 increase, the density ρ_0 (and the pressure p_0[2]) decreases.

In conclusion, the fluid configuration we have just depicted in the second point corresponds to that of a sonic black hole. Its *sonic horizon* is located at the waist of the nozzle $z = 0$ and its *sonic trapped region* is the supersonic region $z < 0$[3].

We shall now integrate the nozzle equation (3.5.11) from $z = 0$ to a generic coordinate z^*. If the velocity of sound c is constant, we have for the acoustic black hole configuration

$$\int_0^{z^*} \frac{1}{A}\frac{dA}{dz}\,dz = \int_0^{z^*} \frac{1}{v_0}\frac{dv_0}{dz}\left(\frac{v_0^2 - c^2}{c^2}\right)dz \;\Rightarrow\; A(z^*) = \frac{c}{v_0(z^*)}A_H e^{[v_0^2(z^*)-c^2]/2c^2}, \tag{3.5.13}$$

where $A_H \equiv A(0)$ is the area of the waist of the nozzle and $\vec{v}_0(0) \equiv -c\,\hat{z}$. On the other hand, the subsonic solution (v_0 always $< c$) is

$$A(z^*) = \frac{v_0(0)}{v_0(z^*)}A_H e^{[v_0^2(z^*)-v_0^2(0)]/2c^2}. \tag{3.5.14}$$

Starting from this initial subsonic configuration, an acoustic black hole can be formed by lowering the pressure in the exhaust region (i.e. large negative z), which entails a continuous deformation of the velocity profile.

Finally, let us see how an external force causes a sonic horizon translation. We have observed above that this is forced to form and remain at the waist of the nozzle, due to the fine tuning condition $A' = 0$ being in correspondence with the origin. Indeed, the flow will self-adjust to satisfy this fine tuning, which further keeps the acceleration finite at the acoustic horizon. On the contrary, in the presence of an external driving force $Q = Q(z) = -\rho_0\frac{\partial\Phi}{\partial z}$ ($\Phi \equiv$ potential), the situation slightly changes. The reduced Euler's subequation (3.2.7a) is now

$$\partial_t\vec{v}_0 = -\frac{1}{\rho}\frac{\partial p_0}{\partial z} - \frac{\partial}{\partial z}\left(\frac{1}{2}v_0^2 + \Phi\right). \tag{3.5.15}$$

[2]From equation (3.5.7), we have $v_0(z)\frac{dv_0(z)}{dz} = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}$. Since $\rho_0 > 0$ and the left-hand side turns out to be positive for $z < 0$, then $\frac{\partial p_0}{\partial z}$ must be negative for $z < 0$. This means the pressure p_0 decreases in the supersonic region.

[3]By analogy with General Relativity, the *sonic trapped region* is the region containing sonic outer trapped surfaces. They are surfaces for which the fluid velocity is inward-pointing and the normal component of the fluid velocity is greater than c. Then, any sound wave will be swept inward by the fluid flow, hence the appellative *trapped*. The *sonic event horizon* is defined, like in General Relativity, by demanding that it be the boundary of the region from which null geodesics (sound waves or, quantically, phonons) cannot escape. In a steady flow (the analogue of a stationary geometry in General Relativity), the *sonic event horizon* coincides with the boundary of the *sonic trapped region* (the so-called *sonic apparent horizon*).

Then, we shall make the same considerations and perform the same calculations as before but taking the potential Φ into account. At the end, this yields

$$\frac{A'}{A} = \frac{v_0'}{v_0}\left(\frac{v_0^2 - c^2}{c^2}\right) + \frac{Q}{\rho_0 c^2}. \tag{3.5.16}$$

Therefore, when $v_0 = c$ the above equation becomes

$$\left.\frac{A'}{A}\right|_H = \left.\frac{Q}{\rho_0 c^2}\right|_H, \tag{3.5.17}$$

with $\Big|_H$ meaning that this equation conveys the horizon location. As a result, in the presence of an external force the sonic horizon is not necessarily located at the waist of the nozzle.

After all this dissertation, the reader must have noticed that, in the context of black holes hydrodynamics, the local speed of sound plays the same role as the speed of light in General Relativity. That is why it has been conventionally denoted by the letter c.

Conclusions

In this work, we presented the basics of acoustic black holes and a model of its experimental realization. This shows that applying concepts and tools from General Relativity to acoustics permits to map gravitational (theoretical) issues into (possibly) experimental problems.

We first had an overview of the fundamentals of General Relativity and spherically-symmetric gravitational fields. Then, we focused on a purely formal analogy between two wave equations within apparently far Physics branches respectively. In the end, we were able to construct a fruitful framework in which to deal with some fluid systems in the same way as Einstein's theory of gravitation. In particular, we studied the spherically-symmetric and the one-dimensional sonic analogues of gravitational black holes, up to guess a possible realization as well.

In recent years, all of this led to simulate quantum black holes phenomena by means of analogous quantum condensed matter systems. In fact, what seems to be the sonic analogue of Hawking radiation has been detected by J. Steinhauer (see [4]). Consequently, it may be claimed − within reasonable limits − acoustic black holes are a valid alternative to investigate quantum phenomena related to gravitational systems.

Appendix A

Wave equation in curved space-time

We have already mentioned that the wave equation in a curved space-time takes the form

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \psi) = 0. \tag{A.0.1}$$

More technically, this is the d'Alembertian equation of motion for a minimally coupled massless scalar field ψ propagating in a (3+1)-dimensional Lorentzian geometry on which a metric tensor $g_{\mu\nu}(t, \vec{x})$ is defined.

Let us demonstrate this specific form for the covariant d'Alembertian operator. We will first start by computing the covariant divergence of a contravariant vector

$$\nabla_\mu V^\mu \equiv V^\mu_{;\mu} = \frac{\partial V^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda} V^\lambda, \tag{A.0.2}$$

where $\Gamma^\mu_{\mu\lambda}$ is given by

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\rho}\right) = \frac{1}{2} g^{\mu\rho}\, \frac{\partial g_{\rho\mu}}{\partial x^\lambda}. \tag{A.0.3}$$

To evaluate such $\Gamma^\mu_{\mu\lambda}$, we recall that for an arbitrary matrix M the following equation holds

$$\mathrm{tr}\left[M^{-1}(x)\, \frac{\partial M(x)}{\partial x^\lambda}\right] = \frac{\partial}{\partial x^\lambda} \ln|\det M(x)\,|, \tag{A.0.4}$$

where det denotes the determinant and tr the trace, that is, the sum of the diagonal elements. Indeed, for a slight variation dM owing to a slight variation dx^λ, we have

$$\begin{aligned} \ln|\det(M + dM)\,| &\approx \ln|\det M + d(\det M)\,| \approx \\ &\approx \ln|\det M\,| + \frac{\partial}{\partial x^\lambda}[\ln|\det M(x)\,|]dx^\lambda \;\Rightarrow \end{aligned}$$

$$\Rightarrow \ln|\det(M+dM)| - \ln|\det M| \approx \frac{\partial}{\partial x^\lambda}[\ln|\det M(x)|]dx^\lambda \Rightarrow$$
$$\Rightarrow \ln\left|\frac{\det(M+dM)}{\det M}\right| \approx \frac{\partial}{\partial x^\lambda}[\ln|\det M(x)|]dx^\lambda,$$
$$\ln\left|\frac{\det(M+dM)}{\det M}\right| = \ln|\det M^{-1}(M+dM)| = \ln|\det\big(I+M^{-1}dM\big)|.$$

For any $n \times n$ real matrix ε,

$$\det(I+\varepsilon) = 1 + \operatorname{tr}(\varepsilon) + O(\varepsilon^2)$$

holds.
Since $M^{-1}dM$ is an infinitesimal matrix, then

$$\ln|\det\big(I+M^{-1}dM\big)| \approx \ln|1+\operatorname{tr}(M^{-1}dM)| \approx \operatorname{tr}(M^{-1}dM),$$

where we used the approximation $\ln|1+x| \approx x$ for $x \ll 1$. Thereafter,

$$\operatorname{tr}(M^{-1}dM) = \operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial x^\lambda}dx^\lambda\right) = \operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial x^\lambda}\right)dx^\lambda.$$

Therefore,

$$\operatorname{tr}\left[M^{-1}(x)\frac{\partial M(x)}{\partial x^\lambda}\right]dx^\lambda = \frac{\partial}{\partial x^\lambda}[\ln|\det M(x)|]dx^\lambda,$$

neglecting higher orders. Finally, equaling the coefficients of each differential dx^λ, we have proven equation (A.0.4).

In particular, if M is the metric tensor $g_{\mu\nu}$, we have

$$\operatorname{tr}\left[g^{\mu\rho}\frac{\partial g_{\rho\nu}}{\partial x^\lambda}\right] = \frac{\partial}{\partial x^\lambda}\ln|\det g_{\mu\nu}|, \tag{A.0.5}$$

and from (A.0.3) (having Einstein's notation for indexes in mind)

$$\operatorname{tr}\left[g^{\mu\rho}\frac{\partial g_{\rho\nu}}{\partial x^\lambda}\right] = \sum_{\mu=\nu} g^{\mu\rho}\frac{\partial g_{\rho\nu}}{\partial x^\lambda} = g^{\mu\rho}\frac{\partial g_{\rho\mu}}{\partial x^\lambda} = 2\,\Gamma^\mu_{\mu\lambda} \Rightarrow$$

$$\Rightarrow \Gamma^\mu_{\mu\lambda} = \frac{1}{2}\frac{\partial}{\partial x^\lambda}\ln|g| = \frac{\partial}{\partial x^\lambda}\ln\left(\sqrt{-g}\right) = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\sqrt{-g}, \tag{A.0.6}$$

where $g \equiv \det g_{\mu\nu}$. The covariant divergence becomes

$$V^\mu_{;\mu} = \partial_\mu V^\mu + \frac{1}{\sqrt{-g}}\,\partial_\lambda(\sqrt{-g})\,V^\lambda. \tag{A.0.7}$$

Since $\partial_\lambda(\sqrt{-g})\, V^\lambda = \partial_\lambda(\sqrt{-g}\, V^\lambda) - \sqrt{-g}\, \partial_\lambda V^\lambda$, the above equation (A.0.7) reduces to

$$\begin{aligned} V^\mu_{;\mu} &= \partial_\mu V^\mu + \frac{1}{\sqrt{-g}}[\partial_\lambda(\sqrt{-g}\, V^\lambda) - \sqrt{-g}\, \partial_\lambda V^\lambda] = \\ &= \partial_\mu V^\mu + \frac{1}{\sqrt{-g}}\partial_\lambda(\sqrt{-g}\, V^\lambda) - \partial_\lambda V^\lambda = \\ &= \cancel{\partial_\mu V^\mu} + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, V^\mu) - \cancel{\partial_\mu V^\mu} = \\ &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, V^\mu), \end{aligned} \tag{A.0.8}$$

where we were allowed to change the index λ with the index μ, since λ appeared as a repeated index.

Let us now suppose V^μ is $\nabla^\mu\psi \equiv g^{\mu\nu}\, \nabla_\nu\psi$[1]. Since ψ is a scalar field, its covariant derivative coincides with the ordinary gradient

$$\nabla_\nu\psi = \partial_\nu\psi.$$

From equation (A.0.8), in the present case we obtain

$$\begin{aligned} \nabla_\mu V^\mu = \nabla_\mu\nabla^\mu\psi &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, \nabla^\mu\psi) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \nabla_\nu\psi) = \\ &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\psi). \end{aligned} \tag{A.0.9}$$

The expression that we have just derived shows many remarkable features. Firstly – by interpreting $\Delta \equiv \nabla_\mu\nabla^\mu$ as the covariant d'Alembertian operator – if (A.0.9) vanishes we have

$$\Delta\psi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\psi) = 0.$$

Indeed, this turns out to represent the wave equation in a curved space-time, as we wanted to prove. Secondly, we observe that the curved space-time covariant d'Alembertian operator $\Delta \equiv \nabla_\mu\nabla^\mu$ is given in terms of the metric tensor $g_{\mu\nu}(t, \vec{x})$ and ordinary partial derivatives only, without any covariant differentiation.

[1] $\nabla^\mu\psi \equiv g^{\mu\nu}\, \nabla_\nu\psi = g^{\mu\nu}\, \partial_\nu\psi \equiv \partial^\mu\psi$ is indeed a contravariant vector.

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