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Corso di Laurea Magistrale in Matematica

A COMBINATORIAL DESCRIPTION OF THE GOOD $\mathbb{Z}$-GRADINGS OF THE SYMPLECTIC LIE ALGEBRA

Tesi di Laurea in Algebra

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Introduzione

In questa tesi viene studiato il concetto di $\mathbb{Z}$-graduazione buona di un'algebra di Lie finito dimensionale su un campo $\mathbb{F}$ algebricamente chiuso di caratteristica 0 e vengono analizzate in maniera particolare le $\mathbb{Z}$-graduazioni buone dell'algebra di Lie semplice $\mathfrak{sp}_{2n}(\mathbb{F})$. Il lavoro è basato sull'articolo [3] contenente la classificazione delle $\mathbb{Z}$-graduazioni buone delle algebre di Lie semplici di dimensione finita su un campo algebricamente chiuso di caratteristica zero. Nel caso dei risultati relativi ad $\mathfrak{sp}_{2n}$, tuttavia, l'articolo [3] contiene soltanto gli enunciati e con questa tesi ci siamo prefissi l'obiettivo di ricostruire tutte le dimostrazioni.
Una $\mathbb{Z}$-graduazione di un'algebra di Lie $\mathfrak{g}$ è definita come una decomposizione del tipo

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k, \tag{1}$$

dove $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ per ogni $i, j \in \mathbb{Z}$. La $\mathbb{Z}$-graduazione (1) si dice *buona* se esiste un elemento $e \in \mathfrak{g}_2$ tale che:

1. $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ è iniettiva per ogni $j \leq -1$;

2. $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ è suriettiva per ogni $j \geq -1$.

Nel primo capitolo abbiamo richiamato i principali risultati della teoria delle algebre di Lie semisemplici di dimensione finita su un campo algebricamente chiuso di caratteristica 0.
Nel secondo capitolo si è dato risalto al concetto di algebra di Lie riduttiva, analizzandone le proprietà e caratterizzazioni più importanti, con l'obiettivo principale di dimostrare il teorema di Jacobson-Morozov. Questo teorema

afferma che, dati un'algebra di Lie semisemplice $\mathfrak{g}$ su un campo algebricamente chiuso di caratteristica 0 e un qualunque elemento nilpotente $e \in \mathfrak{g}$, allora e può essere immerso in una $\mathfrak{sl}_2$-tripla $\{e, h, f\}$, cioè in una terna tale che $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.
Nel terzo capitolo si è infine iniziata l'analisi delle buone graduazioni di un'algebra di Lie $\mathfrak{g}$. Si è data particolare importanza al caso in cui $\mathfrak{g}$ sia semisemplice; in questo caso, infatti, si può dimostrare che, data una qualunque $\mathbb{Z}$-graduazione $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$, allora esiste un elemento $H \in \mathfrak{g}_0$ che la definisce, cioè tale che $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid [H, x] = kx\}$. Inoltre si è dato un esempio fondamentale di buona graduazione: il Dynkin grading. Dato un qualunque elemento nilpotente $e \in \mathfrak{g}$, il Dynkin grading associato ad e è la $\mathbb{Z}$-graduazione data dalla decomposizione in autospazi di $\mathrm{ad}(h)$, dove $\{e, h, f\}$ è una $\mathfrak{sl}_2$-tripla la cui esistenza è garantita dal teorema di Jacobson-Morozov. Grazie alla teoria delle rappresentazioni irriducibili di $\mathfrak{sl}_2$ si riesce a dimostrare che tale graduazione è buona. Vengono poi analizzate le principali proprietà delle $\mathbb{Z}$-graduazioni buone, dimostrando in particolare che:

a. La condizione 1. è equivalente al fatto che $C_{\mathfrak{g}}(e) \subseteq \bigoplus_{j\geq 0} \mathfrak{g}_j$;

b. Le condizioni 1. e 2. della definizione di buona graduazione sono equivalenti;

c. Se $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ è una buona graduazione con buon elemento $e \in \mathfrak{g}_2$, $H \in \mathfrak{g}_0$ è un elemento che la definisce e $\mathfrak{s} = \{e, h, f\}$ è una $\mathfrak{sl}_2$-tripla contenente e, allora $H - h$ appartiene al centro di $C_{\mathfrak{g}}(\mathfrak{s})$.

Il quarto capitolo, infine, è dedicato alla classificazione delle $\mathbb{Z}$-graduazioni buone di $\mathfrak{sp}_{2n}$. A tal fine viene introdotto il concetto di partizione simplettica. In particolare vengono associati ad ogni partizione simplettica p un oggetto combinatorio $SP(p)$, detto piramide simplettica, e due endomorfismi simplettici legati a $SP(p)$:

- un endomorfismo nilpotente $e(p)$ (che descrive l'orbita nilpotente associata alla partizione p) (si veda la Definizione 4.2.1);

- un endomorfismo diagonale $h(p)$ (si veda la Definizione 4.2.5).

In particolare si dimostra che si può immergere $e(p)$ in una $\mathfrak{sl}_2$-tripla $\mathfrak{s} = \{e(p), h(p), f(p)\}$ contenente $h(p)$ e che quindi la $\mathbb{Z}$-graduazione indotta da $\mathrm{ad}(h(p))$ risulta essere il Dynkin grading. Si studiano infine tutte le graduazioni buone di $\mathfrak{sp}_{2n}$ con buon elemento $e(p)$, cioè le cosiddette "coppie buone" $(h(p) + h, e(p))$, dove $e(p)$ è un buon elemento per la $\mathbb{Z}$-graduazione indotta da $\mathrm{ad}(h(p) + h)$ e h è un endomorfismo diagonale. Una volta data una descrizione esplicita di $Z(C_{\mathfrak{g}}(\mathfrak{s}))$ per $\mathfrak{g} = \mathfrak{sp}_{2n}$, si sfrutta la proprietà c. per stabilire quando $h = (h(p) + h) - h(p) \in Z(C_{\mathfrak{g}}(\mathfrak{s}))$; in questo modo si è in grado di ridurre i casi in cui $(h(p) + h, e(p))$ può essere una buona coppia. Viene fornita una dettagliata descrizione di $C_{\mathfrak{sp}_{2n}}(e(p))$ grazie alla quale è possibile determinare una caratterizzazione completa delle coppie $(h(p) + h, e(p))$ buone.

Questa analisi di tipo combinatorio è stata generalizzata in [4] al caso delle cosiddette superalgebre di Lie *basic*. Ci proponiamo in futuro di studiare le graduazioni buone di un'altra classe di superalgebre di Lie di dimensione finita: le cosiddette superalgebre di Cartan (si veda [8]).

Introduction

In this thesis we study the concept of good $\mathbb{Z}$-grading of a finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0, and in particular we analyze the good $\mathbb{Z}$-gradings of the simple Lie algebra $\mathfrak{sp}_{2n}(\mathbb{F})$. Our work is based on the paper [3], containing the classification of good $\mathbb{Z}$-gradings of finite dimensional simple Lie algebras over an algebraically closed field of characteristic zero. However, in the case of $\mathfrak{sp}_{2n}$, the proofs of the results are omitted in the paper [3]. Thus, in this thesis, we have set the target to reconstruct all the proofs.

A $\mathbb{Z}$-grading of a Lie algebra $\mathfrak{g}$ is defined as a decomposition of the type

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k, \tag{2}$$

where $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}$. The $\mathbb{Z}$-grading (2) is called *good* if there exists an element $e \in \mathfrak{g}_2$ such that:

1. $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for all $j \leq -1$;

2. $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective for all $j \geq -1$.

In the first chapter we recall the main results of the theory of finite dimensional semisimple Lie algebras over an algebraically closed field of characteristic 0.

In the second chapter a big emphasis is put on the concept of reductive Lie algebra, analyzing the main properties and characterizations, with the principal goal to prove the Jacobson-Morozov theorem. This theorem states that, if $\mathfrak{g}$ is a semisimple Lie algebra over an algebraically closed field of characteristic 0 and $e \in \mathfrak{g}$ is any nilpotent element, then e can be embedded into an $\mathfrak{sl}_2$-triple $\{e, h, f\}$, i.e., a triple such that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. In the third chapter we finally study of the good $\mathbb{Z}$-gradings of a Lie algebra $\mathfrak{g}$. In particular, we focus our attention on a semisimple Lie algebra $\mathfrak{g}$ is semisimple. Indeed, in this case, it can be shown that, given any $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$, then there exists an element $H \in \mathfrak{g}_0$ defining it, i.e., such that $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid [H, x] = kx\}$ for all $k \in \mathbb{Z}$. Furthermore, we give a fundamental example of good $\mathbb{Z}$-grading: the Dynkin one. If $e \in \mathfrak{g}$ is any nilpotent element, the Dynkin grading associated to e is the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(h)$, where $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple whose existence is ensured by the Jacobson-Morozov theorem. Thanks to the theory on irreducible representations of $\mathfrak{sl}_2$ we prove that such grading is good. Then we analyze the main properties of good $\mathbb{Z}$-gradings, showing in particular that:

a. Condition 1. is equivalent to $C_{\mathfrak{g}}(e) \subseteq \bigoplus_{j\geq 0} \mathfrak{g}_j$;

b. The conditions 1. and 2. in the definition of good element are equivalent;

c. If $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ is a good $\mathbb{Z}$-grading with good element $e \in \mathfrak{g}_2$, $H \in \mathfrak{g}_0$ is an element defining it and $\mathfrak{s} = \{e, h, f\}$ is an $\mathfrak{sl}_2$-triple containing e, then $H - h$ belongs to the center of $C_{\mathfrak{g}}(\mathfrak{s})$.

Finally, the fourth chapter is dedicated to the classification of the good $\mathbb{Z}$-gradings of $\mathfrak{sp}_{2n}$. In order to do this, we introduce the concept of symplectic partition. In particular, we associate to any such partition a combinatorial object $SP(p)$, called symplectic pyramid, and two symplectic endomorphisms related to $SP(p)$:

- a nilpotent endomorphism $e(p)$ (that describes the nilpotent orbit associated to the partition p) (see Definition 4.2.1);

- a diagonal endomorphism $h(p)$ (see Definition 4.2.5).

Then, we show that $e(p)$ can be embedded in an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e(p), h(p), f(p)\}$ containing $h(p)$ and thus, that the $\mathbb{Z}$-grading induced by $\mathrm{ad}(h(p))$ is the Dynkin grading. Afterwords, we study all the $\mathbb{Z}$-gradings of $\mathfrak{sp}_{2n}$ with good element $e(p)$, i.e., the so called "good pairs" $(h(p) + h, e(p))$, where $e(p)$ is a good element for the $\mathbb{Z}$-grading induced by $\mathrm{ad}(h(p) + h)$ and h is a diagonal endomorphism. Once given an explicit description of $Z(C_{\mathfrak{g}}(\mathfrak{s}))$ for $\mathfrak{g} = \mathfrak{sp}_{2n}$, we use property c. to establish when $h = (h(p) + h) - h(p) \in Z(C_{\mathfrak{g}}(\mathfrak{s}))$; in this way we are able to reduce the cases in which $(h(p) + h, e(p))$ can be a good pair. We give also a detailed description of $C_{\mathfrak{sp}_{2n}}$, thanks to which we are able to give a complete characterization of the good pairs $(h(p)+h, e(p))$.

This combinatorial type analysis in generalized in [4] to the case of the so called *basic* Lie superalgebras. In the future, we intend to carry out the good gradings of another class of finite dimensional Lie superalgebras: the so called Cartan superalgebras (see [8]).

Contents

Chapter 1

Basic results on Lie algebras

1.1 Lie algebras

Definition 1.1.1. A vector space $\mathfrak{g}$ over a field $\mathbb{F}$, with a bilinear operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$ and called the *bracket* or *commutator* of x and y, is called a *Lie algebra* over $\mathbb{F}$ if the following axioms are satisfied:

(L1) $[x, x] = 0$ for all x in $\mathfrak{g}$.

(L2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for $x, y, z \in \mathfrak{g}$ (*Jacobi identity*).

Remark 1.1.2. Notice that (L1), applied to $[x + y, x + y]$, implies anticommutativity: $[x, y] = -[y, x]$.
Conversely, if char $\mathbb{F} \neq 2$, it is clear that anticommutativity implies (L1).

There is a standard way to associate a Lie algebra to an associative algebra.

Proposition 1.1.3. *Let $(A, \cdot)$ be an associative algebra. Then $(A, [,])$, with $[x, y] := x \cdot y - y \cdot x$ for $x, y \in A$, is a Lie algebra.*

Proof. It is easy to see that axioms (L1) and (L2) are satisfied. $\square$

Example 1.1.4.

- Let $M_n(\mathbb{F})$ be the set of matrices of order n over $\mathbb{F}$ and let $\cdot$ be the usual product of matrices. Then the Lie algebra

associated to the algebra $(M_n(\mathbb{F}), \cdot)$ is denoted by $\mathfrak{gl}(n, \mathbb{F}) \equiv \mathfrak{gl}_n$.
We write down the multiplication table for $\mathfrak{gl}_n$ relative to the standard basis consisting of the matrices E_{ij} (having 1 in the (i, j) position and 0 elsewhere). Since $E_{ij}E_{kl} = \delta_{jk}E_{il}$, we have:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}.$$

- If V is a finite dimensional vector space over $\mathbb{F}$, denote by $\mathrm{End}(V)$ the set of linear transformations $V \to V$. Define the bracket as in Proposition 1.1.3, so that $\mathrm{End}(V)$ becomes a Lie algebra over $\mathbb{F}$. In order to distinguish this new algebra structure from the old associative one, we write $\mathfrak{gl}(V)$ for $\mathrm{End}(V)$ viewed as a Lie algebra and call it the *general linear algebra.*

Definition 1.1.5. Let $\mathfrak{g}, \mathfrak{g}'$ be two Lie algebras. A linear map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ such that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$ is called a *Lie algebra homomorphism.* If φ is also bijective, it is called *Lie algebra isomorphism.*

Remark 1.1.6. Let V be an n-dimensional vector space over $\mathbb{F}$. The Lie algebras $\mathfrak{gl}(V)$ and $\mathfrak{gl}_n$ are isomorphic; hence in the following we will identify them.

Definition 1.1.7. Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a *(Lie) subalgebra* of $\mathfrak{g}$ if $[x, y] \in \mathfrak{h}$ whenever $x, y \in \mathfrak{h}$.

Example 1.1.8. Let $\mathfrak{sl}_n \equiv \mathfrak{sl}_n(\mathbb{F}) := \{X \in \mathfrak{gl}_n \mid \mathrm{tr}X = 0\}$. Then $\mathfrak{sl}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$ since, if $X, Y \in \mathfrak{sl}_n$, then also $[X, Y] \in \mathfrak{sl}_n$.

Example 1.1.9. Let V be a vector space, $\mathrm{dim}V = 2n$, with basis $\{v_1, \ldots, v_{2n}\}$. Consider the non-degenerate skew-symmetric bilinear form f on V given by the matrix $J = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$. Define the *symplectic algebra* as

$$\mathfrak{sp}(V) \equiv \mathfrak{sp}_{2n} := \{x \in \mathrm{End}V \mid f(x(v), w) = -f(v, x(w)) \text{ for all } v, w \in V\}.$$

It can be easily shown that $\mathfrak{sp}_{2n}$ is closed under the bracket operation, and thus it is a Lie subalgebra of $\mathfrak{gl}_{2n}$. In matrix terms, the condition for

$x = \left(\begin{array}{c|c} M & N \\ \hline P & Q \end{array}\right)$ $(M, N, P, Q \in \mathfrak{gl}_n)$ to be symplectic is that $Jx = -x^T J$, i.e., that $N^T = N$, $P^T = P$, and $Q = -M^T$ (this last condition forces $\mathrm{tr}(x) = 0$ and thus we can regard $\mathfrak{sp}_{2n}$ as a Lie subalgebra of $\mathfrak{sl}_{2n}$).
A basis for $\mathfrak{sp}_{2n}$ is given by the following matrices:

- $E_{ii} - E_{n+i,n+i}$, $1 \leq i \leq n$;
- $E_{ij} - E_{n+j,n+i}$, $1 \leq i \neq j \leq n$;
- $E_{i,n+i}$, $1 \leq i \leq n$;
- $E_{i,n+j} + E_{j,n+i}$, $1 \leq i < j \leq n$;
- $E_{n+i,i}$, $1 \leq i \leq n$;
- $E_{n+i,j} + E_{n+j,i}$, $1 \leq i < j \leq n$.

Definition 1.1.10. Let $\mathfrak{g}$ be a Lie algebra. A subset $I \subseteq \mathfrak{g}$ is called *ideal* if $[x, y] \in I$ for every $x \in I, y \in \mathfrak{g}$.

Definition 1.1.11. Let $\mathfrak{g}$ be a Lie algebra. We define the following ideals of $\mathfrak{g}$:

- the *derived algebra* of $\mathfrak{g}$ as $[\mathfrak{g}, \mathfrak{g}] := \langle [x, y] \mid x, y \in \mathfrak{g} \rangle$
- the *center* of $\mathfrak{g}$ as $Z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x, z] = 0 \text{ for all } z \in \mathfrak{g}\}$

Definition 1.1.12. A Lie algebra $\mathfrak{g}$ is called *simple* if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and it has no proper ideals.

Example 1.1.13. Let us consider the basis of $\mathfrak{sl}_2(\mathbb{F})$ given by

$$\{e, h, f\} = \{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}.$$

Then:

- $[h, e] = 2e$;

- $[e, f] = h$;
- $[h, f] = -2f$.

If char$\mathbb{F} \neq 2$, then $\mathfrak{sl}_2(\mathbb{F})$ is simple, and the basis $\{e, h, f\}$ is usually called the standard basis of $\mathfrak{sl}_2$.
Indeed, if $I \subseteq \mathfrak{sl}_2(\mathbb{F})$ is a nonzero ideal, take $0 \neq x \in I$. Then $x = \alpha e + \beta h + \gamma f$ for some $\alpha, \beta, \gamma \in \mathbb{F}$. Now, $[x, e] = 2\beta e - \gamma h \in I$ and $[2\beta e - \gamma h, e] = -2\gamma e \in I$ by definition of ideal.

- If $\gamma \neq 0$, then $e \in I$. Hence $[e, f] = h \in I$ and also $[h, f] = -2f \in I$. So $e, h, f \in I$, i.e., $I = \mathfrak{sl}_2(\mathbb{F})$
- If $\gamma = 0$, then $x = \alpha e + \beta h$. Then $[x, e] = 2\beta e \in I$. If $\beta \neq 0$, then $e \in I$ and we can conclude as before that $I = \mathfrak{sl}_2(\mathbb{F})$. Otherwise $x = \alpha e$, $\alpha \neq 0$, and hence $I = \mathfrak{sl}_2(\mathbb{F})$.

Example 1.1.14. $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \langle I_n \rangle$, where $\langle I_n \rangle = Z(\mathfrak{gl}_n)$. In particular $\mathfrak{gl}_n$ is not simple.

1.2 Lie algebras of derivations

Definition 1.2.1. Let $(A, \cdot)$ be an algebra over the field $\mathbb{F}$. A *derivation* of A is a linear map $\delta : A \to A$ satisfying the familiar product rule

$$\delta(ab) = a\delta(b) + \delta(a)b,$$

called the *Leibniz rule.*

It is easily checked that the collection Der(A) of all derivations of A is a vector subspace of End(A) and also that the commutator $[\delta, \delta']$ of two derivations is again a derivation. So Der(A) is a Lie subalgebra of $\mathfrak{gl}(A)$. Note that, on the contrary, Der(A) is not a subalgebra of the associative algebra End(A).
Since a Lie algebra $\mathfrak{g}$ is an $\mathbb{F}$-algebra, Der$(\mathfrak{g})$ is defined. Certain derivations arise quite naturally, as follows.

Definition 1.2.2. Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$. Define the endomorphism of $\mathfrak{g}$

$$\begin{array}{rccc} \mathrm{ad}_x : & \mathfrak{g} & \to & \mathfrak{g} \\ & y & \mapsto & [x,y] \end{array}$$

Remark 1.2.3. $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$, because ad_x satisfies the Leibniz rule with respect to the bracket. Indeed, by the Jacobi identity we have:

$$\mathrm{ad}_x([y,z]) = [x,[y,z]] = [[x,y],z] + [y,[x,z]] = [\mathrm{ad}_x(y),z] + [y,\mathrm{ad}_x(z)].$$

Derivations of this form are called *inner*, all others *outer*.

1.3 Representations and modules

Definition 1.3.1. A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V over $\mathbb{F}$ ($\dim V < \infty$) is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$.

An important example to keep in mind is the *adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ which sends x to ad_x, where $\mathrm{ad}_x(y) = [x,y]$. It is clear that ad is a linear transformation. To see that it preserves the bracket, we calculate: $\mathrm{ad}_{[x,y]}(z) = [[x,y],z] = [x,[y,z]] - [y,[x,z]] = \mathrm{ad}_x(\mathrm{ad}_y(z)) - \mathrm{ad}_y(\mathrm{ad}_x(z)) = [\mathrm{ad}_x, \mathrm{ad}_y](z)$.

Definition 1.3.2. A representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called *irreducible* if there does not exist a non-zero subspace $W \subsetneq V$ such that $\varphi(\mathfrak{g})(W) \subseteq W$.

Example 1.3.3. Consider the standard representation of $\mathfrak{gl}_n$ on $\mathbb{F}^n$, i.e.,

$$\begin{gathered} \varphi : \mathfrak{gl}_n \to \mathfrak{gl}(\mathbb{F}^n) \\ X \mapsto X \end{gathered}$$

where $X : \mathbb{F}^n \to \mathbb{F}^n$ is such that $v \mapsto Xv$. Then φ is irreducible since, given any proper subspace $W \subset \mathbb{F}^n$, for $0 \neq w \in W$ and $z \in \mathbb{F}^n \setminus W$, one can always find an element $X \in \mathfrak{gl}_n$ such that $X(w) = z$.

It is often convenient to use the language of modules along with the (equivalent) language of representations. As in other algebraic theories, there is a natural definition.

Definition 1.3.4. Let $\mathfrak{g}$ be a Lie algebra. A vector space V, endowed with a bilinear operation $\mathfrak{g} \times V \to V$ (denoted $(x, v) \mapsto x.v$) is called a $\mathfrak{g}$-*module* if the following condition is satisfied:

$$[x, y].v = x.y.v - y.x.v \text{ for all } x, y \in \mathfrak{g} \text{ and } v \in V.$$

Definition 1.3.5. Let V be a $\mathfrak{g}$-module. Then we can equip V^* with a structure of $\mathfrak{g}$-module setting, for $x \in \mathfrak{g}$, $f \in V^*$ and $v \in V$:

$$x.f(v) := -f(x.v).$$

Definition 1.3.6. Let V, W be $\mathfrak{g}$-modules. A $\mathfrak{g}$-*module homomorphism* between V and W is a homomorphism of vector spaces $\varphi : V \to W$ such that $\varphi(x.v) = x.\varphi(v)$ for every $x \in \mathfrak{g}, v \in V$.

Remark 1.3.7. If $\varphi : V \to W$ is a homomorphism of $\mathfrak{g}$-modules, then $\ker\varphi$ is a $\mathfrak{g}$-submodule of V, i.e., $\mathfrak{g}.\ker\varphi \subseteq \ker\varphi$

Proof. If $x \in \mathfrak{g}$ and $v \in \ker\varphi$, then $x.v \in \ker\varphi$ since $\varphi(x.v) = x.\varphi(v) = 0$. □

Remark 1.3.8. The concept of $\mathfrak{g}$-module is equivalent to the one of representation.

Proof. If $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, then V may be viewed as a $\mathfrak{g}$-module via the action $x.v = \varphi(x)(v)$. Conversely, given a $\mathfrak{g}$-module V, this equation defines a representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$. □

Definition 1.3.9. A $\mathfrak{g}$-module V is called *irreducible* if it has precisely two $\mathfrak{g}$-submodules (itself and 0), i.e., if it does not exist a non-zero $\mathfrak{g}$-submodule $W \subsetneq V$ such that $\mathfrak{g}.W \subseteq W$.

Definition 1.3.10. A $\mathfrak{g}$-module V is called *completely reducible* if V is a direct sum of irreducible $\mathfrak{g}$-submodules.

1.4 Finite irreducible representations of $\mathfrak{sl}_2$

In this section, we want to study the finite irreducible representations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of characteristic zero.
Let $\varphi : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ be an irreducible representation of finite dimension. Consider the standard basis of $\mathfrak{sl}_2$, given by $\{e, h, f\}$, where $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Since $\mathbb{F}$ is algebraically closed, then there exists a nonzero $v \in V$ such that $\varphi(h)(v) \equiv h.v = \lambda v$, $\lambda \in \mathbb{F}$. Then

$$h.e^r.v = (\lambda + 2r)e^r.v \text{ for all } r \in \mathbb{Z}^+.$$

Indeed, by induction on r:

- if $r = 0$, then $h.v = \lambda v$;
  if $r = 1$, then $h.e.v = [h, e].v + e.h.v = 2e.v + \lambda e.v = (2 + \lambda)e.v$.
- if $r > 1$, then $h.e^{r+1}.v = h.e.e^r.v = [h, e].e^r.v + e.h.e^r.v = 2e^{r+1}.v + (\lambda + 2r)e^{r+1}.v = (\lambda + 2(r + 1))e^{r+1}.v$.

This implies that the $e^r.v$'s (if nonzero) are linearly independent, because they are eigenvectors corresponding to distinct eigenvalues. But, as V is finite dimensional, there must be a $k \in \mathbb{Z}^+$ such that $e^k.v \neq 0$ and $e^{k+1}.v = 0$. Denote by $w = e^k.v \neq 0$. Then $h.w = \lambda' w$ and $e.w = 0$. We call w *vector of the highest weight* λ'.
As before, one can prove by induction that:

$$h.f^r.w = (\lambda' - 2r)f^r.w \text{ for all } r \in \mathbb{Z}^+. \tag{1.1}$$

Since V is finite dimensional, we can consider $s \in \mathbb{Z}^+$ such that $f^s.w \neq 0$ and $f^{s+1}.w = 0$. Now, consider the linear subspace $W = \langle w, f.w, \ldots, f^s.w\rangle$ of V, whose dimension is $s + 1$. Notice that W is stable with respect to the action of $\mathfrak{sl}_2$. Indeed:

- $f.W \subseteq W$ by construction;
- $h.W \subseteq W$ by (1.1);

- $e.W \subseteq W$ because, by induction, one can prove that

$$e.f^k.w = k(\lambda' - k + 1)f^{k-1}.w \text{ for all } k \in \mathbb{Z}^+. \tag{1.2}$$

But V is irreducible and hence, since $\dim W \geq 1$, then $W = V$.
Note that, by (1.2) with $k = s + 1$, we have:

$$\underbrace{e.f^{s+1}.w}_{=0} = \underbrace{(s+1)}_{>0}(\lambda' - s)\underbrace{f^s.w}_{\neq 0}.$$

Hence $\lambda' = s \in \mathbb{Z}^+$. This implies that the weight of the vector of highest weight is $\dim V - 1$. Therefore this weight uniquely determines the dimension of the module, and vice versa. By (1.1), we can write

$$V = \bigoplus_{k=0}^{s} V_{\lambda = s-2k},$$

where $V_\lambda = \{v \in V \mid h.v = \lambda v\}$, called *weight space*. So the eigenvalues of $\varphi(h)$ are integers and form an arithmetic progression with difference 2, $-s, -s+2, \ldots, s-2, s$. Note that these eigenvalues are either all even or all odd.
Finally we also have that V has (up to nonzero scalar multiples) a unique vector of highest weight, whose weight is s.

1.5 Nilpotency

Definition 1.5.1. Let $\mathfrak{g}$ be a Lie algebra. The sequence of ideals of $\mathfrak{g}$ defined by

$$\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1], \ldots, \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}]$$

is called the *descending central sequence* of $\mathfrak{g}$.

Definition 1.5.2. $\mathfrak{g}$ is called *nilpotent* if there exists $n \in \mathbb{Z}_+$ such that $\mathfrak{g}^n = 0$.

Example 1.5.3. • Any abelian Lie algebra is nilpotent.

- Let $\mathfrak{n}(n, \mathbb{F})$ be the Lie algebra of all strictly upper triangular matrices. Then $\mathfrak{n}(n, \mathbb{F})$ is nilpotent.

- Let $\mathfrak{t}(n, \mathbb{F})$ be the Lie algebra of all upper triangular matrices. Then $\mathfrak{t}(n, \mathbb{F})$ is not nilpotent since $\mathfrak{t}(n, \mathbb{F})^1 = [\mathfrak{t}(n, \mathbb{F}), \mathfrak{t}(n, \mathbb{F})] = \mathfrak{n}(n, \mathbb{F})$ and $\mathfrak{t}(n, \mathbb{F})^2 = [\mathfrak{t}(n, \mathbb{F}), \mathfrak{n}(n, \mathbb{F})] = \mathfrak{n}(n, \mathbb{F}) = \mathfrak{t}(n, \mathbb{F})^k$ for every $k \geq 1$.

Proposition 1.5.4. *1. Every homomorphic image of a nilpotent Lie algebra is nilpotent.*

2. If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then also $\mathfrak{g}$ is nilpotent.

Proof. 1. Let $\varphi : \mathfrak{g} \to \mathfrak{m}$ be a Lie algebra homomorphism. Then for every $k \in \mathbb{Z}^+$, $\varphi(\mathfrak{g}^k) = (\varphi(\mathfrak{g}))^k$. Hence, if $\mathfrak{g}^r = 0$ for some $r \in \mathbb{Z}^+$, also $(\varphi(\mathfrak{g}))^r = 0$.

2. Let $\pi : \mathfrak{g} \to \mathfrak{g}/Z(\mathfrak{g})$ be the canonical projection. As $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, we know that $(\mathfrak{g}/Z(\mathfrak{g}))^k = 0$ for some $k \in \mathbb{Z}^+$. But $\pi(\mathfrak{g}^k) = (\pi(\mathfrak{g}))^k = (\mathfrak{g}/Z(\mathfrak{g}))^k = 0$, i.e., $\mathfrak{g}^k \subseteq Z(\mathfrak{g})$. Hence $\mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k] = 0$.

□

Remark 1.5.5. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then ad_x is a nilpotent endomorphism for every $x \in \mathfrak{g}$. We will say that x is *ad-nilpotent*.

Proof. Let $k \in \mathbb{Z}^+$ such that $\mathfrak{g}^k = 0$. Let $x \in \mathfrak{g}$. Then $\mathrm{ad}_x(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^1$, $\mathrm{ad}_x^2(\mathfrak{g}) \subseteq \mathfrak{g}^2, \ldots, \mathrm{ad}_x^k(\mathfrak{g}) \subseteq \mathfrak{g}^k = 0$. □

Lemma 1.5.6. *Let $x \in \mathfrak{gl}(V)$. If x is nilpotent, then x is ad-nilpotent.*

Proof. Let $y \in \mathfrak{gl}(V)$. Then $\mathrm{ad}_x(y) = xy - yx$. We want to show that for every $k \in \mathbb{Z}$ it holds:

$$\mathrm{ad}_x^k(y) = \sum_{i=0}^{k} \alpha_i x^i y x^{k-i} \text{ for some } \alpha_i \in \mathbb{F}.$$

$\cdot$ $k = 1$: $\mathrm{ad}_x(y) = xy - yx = \alpha_0 yx + \alpha_1 xy$.
$\cdot$ $k > 1$:

$$\mathrm{ad}_x^{k+1}(y) = \mathrm{ad}_x(\sum_{i=0}^{k} \alpha_i x^i y x^{k-i}) = \sum_{i=0}^{k} \alpha_i x^{i+1} y x^{k-i} + \sum_{i=0}^{k} \alpha_i x^i y x^{k+1-i} = \sum_{i=0}^{k+1} \beta_i x^i y x^{k+1-i}.$$

Hence, by the nilpotency of x, if we choose k sufficiently large, we have $\mathrm{ad}_x^k(y) = 0$. □

Theorem 1.5.7. *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra consisting of nilpotent endomorphisms. Then there exists a nonzero $v \in V$ such that $\mathfrak{g}.v = 0$, i.e., such that v is an eigenvector common to all endomorphisms in $\mathfrak{g}$, relative to the eigenvalue 0.*

Proof. We show this by induction on $\dim\mathfrak{g}$.
$\cdot$ $\dim\mathfrak{g} = 1$: $\mathfrak{g} = \langle x \rangle$. Then there exists $k \in \mathbb{Z}^+$ such that $x^k = 0$ and $x^{k-1}(v) \neq 0$ for some $v \in V$. Hence an eigenvector relative to the eigenvalue 0 is $x^{k-1}(v)$.
$\cdot$ $\dim\mathfrak{g} > 1$: Let $K \subsetneq \mathfrak{g}$ be a proper subalgebra (it exists, we can just consider $K = \langle y \rangle$ for some $y \in \mathfrak{g}$) and $x \in K$. Then ad_x induces an action on the vector space $\mathfrak{g}/K$, i.e., we can consider $\mathrm{ad}_x : \mathfrak{g}/K \to \mathfrak{g}/K$ such that $\mathrm{ad}_x(y + K) = [x, y] + K$, which results to be well defined.
Since $x \in K \subsetneq \mathfrak{g}$ is nilpotent, by Lemma 1.5.6 we can say that ad_x is nilpotent. Hence also $\mathrm{ad}_x : \mathfrak{g}/K \to \mathfrak{g}/K$ is nilpotent as endomorphism of $\mathfrak{g}/K$. So $\mathrm{ad}(K) \subseteq \mathfrak{gl}(\mathfrak{g}/K)$ consists of nilpotent endomorphisms. Furthermore $\dim(\mathrm{ad}(K)) \leq \dim K < \dim\mathfrak{g}$; hence, by induction hypothesis applied to $\mathrm{ad}(K)$, we can say that there exists $0 \neq (y + K) \in \mathfrak{g}/K$ such that $\mathrm{ad}(K)(y + K) = 0$, i.e., there exists $y \notin K$ such that $[K, y] \subseteq K$.
Set $N_\mathfrak{g}(K) := \{z \in \mathfrak{g} \mid [z, K] \subseteq K\} \supsetneq K$. It is easy to see that $N_\mathfrak{g}(K)$ is a Lie subalgebra of $\mathfrak{g}$. Hence, if we choose a maximal (proper) subalgebra of $\mathfrak{g}$, we can say that $N_\mathfrak{g}(K) = \mathfrak{g}$. This leads us to say that K is an ideal. So

$\mathfrak{g}/K$ has a structure of Lie algebra, and $\pi : \mathfrak{g} \to \mathfrak{g}/K$ is a Lie algebra homomorphism. Furthermore, since K is maximal, necessarily $\mathrm{codim}_{\mathfrak{g}}(K) = 1$; indeed, if not, we could choose a proper subalgebra $\langle x + K\rangle \subsetneq \mathfrak{g}/K$ and consequently $\pi^{-1}(\langle x + K\rangle) = K \oplus \langle x\rangle$ would be a proper subalgebra of $\mathfrak{g}$ containing K.
Hence $\mathfrak{g} = K \oplus \langle z\rangle$. Applying the induction hypothesis to K, we have that $W := \{v \in V \mid K.v = 0\} \neq 0$. If we show that $z(W) \subseteq W$ we are done; indeed, if this happens to be true, since z is nilpotent by assumption, also $z_{|W} : W \to W$ would be nilpotent, and hence there would exist $0 \neq w \in W$ such that $z.w = 0$. But $w \in W$ implies $K.w = 0$ and (since $\mathfrak{g} = K \oplus \langle z\rangle$) then $\mathfrak{g}.w = 0$.
So, let $v \in W$, $k \in K$. Then $k.z.v = [k, z].v + z.k.v = 0$ because $k.v = 0$ ($v \in W$) and $[k, z].v = 0$ ($[k, z] \in K$ because K is an ideal). Hence $z.v \in W$. □

Theorem 1.5.8 (Engel)**.** *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if x is ad-nilpotent for every $x \in \mathfrak{g}$.*

Proof. By Remark 1.5.5 we already know that if $\mathfrak{g}$ is nilpotent, then every $x \in \mathfrak{g}$ is ad-nilpotent.
Conversely, suppose that every $x \in \mathfrak{g}$ is ad-nilpotent. Consider the Lie subalgebra $\mathrm{ad}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$. By assumption, $\mathrm{ad}(\mathfrak{g})$ consists of nilpotent endomorphisms. Hence, by Theorem 1.5.7, we know that there exists $0 \neq x \in \mathfrak{g}$ such that $\mathrm{ad}(\mathfrak{g})(x) = 0$, i.e., $x \in Z(\mathfrak{g})$. Hence $Z(\mathfrak{g}) \neq 0$. Now, by induction on $\dim\mathfrak{g}$:
· if $\dim\mathfrak{g} = 1$, then $\mathfrak{g} = Z(\mathfrak{g})$; so $\mathfrak{g}$ is commutative and hence nilpotent;
· if $\dim\mathfrak{g} > 1$ then, since $Z(\mathfrak{g}) \neq 0$, $\dim(\mathfrak{g}/Z(\mathfrak{g})) < \dim(\mathfrak{g})$. Furthermore every element of $\mathfrak{g}/Z(\mathfrak{g})$ is ad-nilpotent as endomorphism of $\mathfrak{g}/Z(\mathfrak{g})$. Hence, by induction hypothesis, $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent; so, by point 2. of Proposition 1.5.4, also $\mathfrak{g}$ is nilpotent. □

1.6 Solvability

Definition 1.6.1. Let $\mathfrak{g}$ be a Lie algebra. The sequence of ideals of $\mathfrak{g}$ defined by

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \dots, \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$$

is called the *derived series* of $\mathfrak{g}$.

Definition 1.6.2. $\mathfrak{g}$ is called *solvable* if there exists $n \in \mathbb{Z}^+$ such that $\mathfrak{g}^{(n)} = 0$.

Example 1.6.3.

- If a Lie algebra $\mathfrak{g}$ is commutative, then it is solvable.
- If a Lie algebra $\mathfrak{g}$ is nilpotent, then it is solvable (since $\mathfrak{g}^{(k)} \subset \mathfrak{g}^k$ for every k).
- $\mathfrak{t}(n, \mathbb{F})$ is solvable (but not nilpotent).
- If a Lie algebra $\mathfrak{g}$ is simple, then it is not solvable, since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Next we assemble a few simple observations about solvability.

Proposition 1.6.4. *Let $\mathfrak{g}$ be a Lie algebra.*

1. *If $\mathfrak{g}$ is solvable, then so are all subalgebras and homomorphic images of $\mathfrak{g}$.*
2. *If I is a solvable ideal of $\mathfrak{g}$ such that $\mathfrak{g}/I$ is solvable, then $\mathfrak{g}$ itself is solvable.*
3. *If I, J are solvable ideals of $\mathfrak{g}$, then so is $I + J$.*

Proof. 1. Let $S \subseteq \mathfrak{g}$ be a subalgebra. Then $S^{(1)} = [S, S] \subseteq [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^{(1)}$, and analogously $S^{(k)} \subseteq \mathfrak{g}^{(k)}$ for every $k \in \mathbb{Z}^+$.
If $\varphi : \mathfrak{g} \to \mathfrak{m}$ is a Lie algebra homomorphism, then φ preserves the commutators, i.e., $\varphi(\mathfrak{g}^{(k)}) = (\varphi(\mathfrak{g}))^{(k)}$ for every $k \in \mathbb{Z}^+$. So, if $\mathfrak{g}^{(k)} = 0$, then also $(\varphi(\mathfrak{g}))^{(k)} = 0$.

2. Consider the projection $\pi : \mathfrak{g} \to \mathfrak{g}/I$. If $(\mathfrak{g}/I)^{(k)} = 0$, then $\pi(\mathfrak{g}^{(k)}) = (\pi(\mathfrak{g}))^{(k)} = (\mathfrak{g}/I)^{(k)} = 0$, i.e., $\mathfrak{g}^{(k)} \subseteq I$. Hence $\mathfrak{g}^{(k+m)} = I^{(m)}$ for every $m \in \mathbb{Z}^+$ and, since I is solvable, $\mathfrak{g}^{(k+m)} = 0$ for some m.

3. By a standard isomorphism theorem, we have $(I + J)/J \cong I/(I \cap J)$. Note that $I/(I \cap J)$ is solvable as homomorphic image of the solvable I. Hence $(I + J)/J$ is solvable and, by point 2., $I + J$ is solvable.

$\square$

Remark 1.6.5. Let $\mathfrak{g}$ be an arbitrary Lie algebra and let S be a maximal (with respect to inclusion) solvable ideal. If I is any other solvable ideal of $\mathfrak{g}$ then, by point 3. of Proposition 1.6.4, we know that $S + I = S$, i.e., $I \subseteq S$. This proves the existence of a unique maximal solvable ideal of $\mathfrak{g}$, called the *radical* of $\mathfrak{g}$ and denoted by $\mathrm{Rad}(\mathfrak{g})$.

Definition 1.6.6. $\mathfrak{g}$ is called *semisimple* if $\mathrm{Rad}(\mathfrak{g}) = 0$.

Example 1.6.7. A simple Lie algebra $\mathfrak{g}$ is semisimple.

Theorem 1.6.8 (Lie)**.** *Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$ ($\dim V < \infty$) over an algebraically closed field $\mathbb{F}$ of characteristic 0. If $V \neq 0$, then V contains a common eigenvector for all the endomorphisms of $\mathfrak{g}$.*

Proof. We use induction on $\dim\mathfrak{g}$.
· $\dim\mathfrak{g} = 1$: $\mathfrak{g} = \langle x \rangle$, where $x \in \mathfrak{gl}(V)$. Since $\mathbb{F}$ is algebraically closed, then x has an eigenvalue, which is eigenvalue of all scalar multiples of x.
· $\dim\mathfrak{g} > 1$: Since $\mathfrak{g}$ is solvable, $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Hence $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a nonzero commutative Lie algebra. Let $\pi : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ be the projection and $S \subseteq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ be a linear subspace of codimension 1. Then S is an ideal since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is commutative. Set $K := \pi^{-1}(S)$. Then K is an ideal of $\mathfrak{g}$ of codimension 1. Hence we can write $\mathfrak{g} = K \oplus \langle z \rangle$.
By induction hypothesis, we know that

$$W := \{v \in V \mid k(v) = \lambda(k)v \text{ for all } k \in K\}$$

is non empty. Let $0 \neq v \in W$. We want to show that $z(v) \in W$ (indeed, if this happens to be true, then $g(v) = \lambda(g)v$ for every $g \in \mathfrak{g}$). For $k \in K$, we have:

$$k(z(v)) = [k, z](v) + z(k(v)) = \lambda([k, z])v + \lambda(k)z(v).$$

In order to show that $z(v) \in W$, we need to show that $\lambda([k, z]) = 0$. Now, consider $W_n := \langle v, z(v), \ldots, z^{n-1}(v)\rangle$, where n is the minimum such that $\{v, z(v), \ldots z^n(v)\}$ are linearly dependent. Set $W_i = \langle v, z(v), \ldots, z^{i-1}(v)\rangle$ for every $i = 1, \ldots, n$. We state that $k(z^r(v)) = \lambda(k)z^r(v) + \omega_r$, where $\omega_r \in W_r$ for every $r = 1, \ldots, n-1$. Indeed, by induction on r:

- if $r = 1$, then $k(z(v)) = \underbrace{\lambda([k, z])v}_{\in W_1} + \lambda(k)z(v)$;
- if $r > 1$, then $k(z^{r+1}(v)) = k(z(z^r(v))) = [k, z](z^r(v)) + z(k(z^r(v))) = \lambda([k, z])z^r(v) + \underbrace{\omega_r}_{\in W_r} + \lambda(k)z^{r+1}(v) + \underbrace{z(\omega'_r)}_{\in W_{r+1}} = \lambda(k)z^{r+1}(v) + \underbrace{\omega_{r+1}}_{\in W_{r+1}}$.

Consider $k_{|W_n} : W_n \to W_n$ with respect to the basis $\{v, z(v), \ldots, z^{n-1}(v)\}$. Then the associated matrix is:

$$\begin{pmatrix} \lambda(k) & * & \ldots & * \\ 0 & \lambda(k) & \ldots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda(k) \end{pmatrix}$$

Hence its trace is $n\lambda(k)$. In particular every element of K of the form $[k, z]$ has trace $n\lambda([k, z])$. But the trace of a commutator is 0, and hence $n\lambda([k, z]) = 0$, i.e., $\lambda([k, z]) = 0$. □

Corollary 1.6.9. *Let $\mathfrak{g}$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then there exists a flag of V (i.e., a sequence of vector spaces $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = V$ with $\dim V_i = i$ and such that $x(V_i) \subseteq V_i$ for all $i = 0, \ldots, n$)*

Remark 1.6.10. Let $\mathfrak{g}$ be a solvable Lie algebra, $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V. Then $\varphi(\mathfrak{g})$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Hence, by the Corollary above, there exists a flag on V stabilized by $\varphi(\mathfrak{g})$. In particular, if we take $\varphi = \mathrm{ad}$, we obtain the following Corollary.

Corollary 1.6.11. *If $\mathfrak{g}$ is solvable, then there exists a chain of ideals in $\mathfrak{g}$:*

$$0 = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_n = \mathfrak{g},$$

with $\mathrm{dim} I_i = i$ *for all* $i = 0, \ldots, n$.

Corollary 1.6.12. *A Lie algebra $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. Obviously, if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, then $\mathfrak{g}$ is solvable.
Conversely, suppose that $\mathfrak{g}$ is solvable. By Engel's Theorem (Theorem 1.5.8) we just need to prove that every element of $[\mathfrak{g}, \mathfrak{g}]$ is ad-nilpotent. Fix a chain of ideals in $\mathfrak{g}$ as in Corollary 1.6.11. Fix a basis of $\mathfrak{g}$ obtained completing a basis of I_i to a basis of I_{i+1} for $i = 0, \ldots, n-1$. Then, with respect to this basis, for every $x \in \mathfrak{g}$, the matrix of ad_x is upper triangular. Now, if we take a generator $[x, y]$ of $[\mathfrak{g}, \mathfrak{g}]$, then $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$ which has, in the fixed basis, a matrix that is strictly upper triangular, and hence nilpotent. □

1.7 Jordan-Chevalley decomposition

We recall a linear algebra result, which will be very helpful in the following.

Proposition 1.7.1. *1. Let $x \in \mathrm{End}(V)$ be diagonalizable. Let $W \subsetneq V$ be a vector subspace such that $x(W) \subseteq W$. Then $x_{|W}$ is diagonalizable.*

2. Let $x, y \in \mathrm{End}(V)$ be diagonalizable such that $[x, y] = 0$. Then x, y are simultaneously diagonalizable, i.e. they have the same eigenvectors.

Proof. 1. Let $V = \bigoplus_{i=1}^k V_i$, with V_i eigenspace of x associated to the eigenvalue λ_i for all $i = 1, \ldots, k$ ($\lambda_i \neq \lambda_j$ if $i \neq j$). Let $w \in W \subsetneq V$. Then $w = v_1 + \ldots + v_k$, with $v_i \in V_i$. We want to show that $v_i \in V_i \cap W$, so that we get a generating set of W that consists of eigenvectors of

$x_{|W}$. By induction on k:
$\cdot$ $k = 1$: the thesis is true by construction.
$\cdot$ $k > 1$: Since $x(w) = \lambda_1 v_1 + \ldots + \lambda_k v_k \in W$ and $W \ni \lambda_1 w = \lambda_1 v_1 + \ldots + \lambda_k v_k$, then $W \ni x(w) - \lambda_1 w = (\lambda_2 - \lambda_1)v_2 + \ldots + (\lambda_k - \lambda_1)v_k$. By the induction hypothesis, we have that $v_i \in V_i \cap W$ for $i = 2, \ldots, k$. Hence also $v_1 \in V_1 \cap W$. So $v_i \in V_i \cap W$ for all $i = 1, \ldots, k$.

2. With the same notations as before, we want to show that $y(V_i) \subseteq V_i$ for all i. Let $v \in V_i$, i.e., $x(v) = \lambda_i v$. Then $x(y(v)) = y(x(v)) = \lambda_i y(v)$; therefore $y(v) \in V_i$. Hence, by point 1., we can conclude because $y_{|V_i}$ is diagonalizable, i.e., we have a basis of V_i consisting of eigenvectors of y.

□

Definition 1.7.2. An element $x \in \mathfrak{gl}(V)$ is called *semisimple* if it is diagonalizable.

Remark 1.7.3. Let $x, y \in \mathfrak{gl}(V)$ be semisimple (resp. nilpotent) such that $[x, y] = 0$. Then $x + y$ is semisimple (resp. nilpotent).

Proof. By point 2. of Proposition 1.7.1, if x, y are semisimple, we can find a basis of eigenvectors of V common both to x and to y. Hence, if v is such an eigenvector, then $(x + y)(v) = x(v) + y(v) = \lambda v + \sigma v = (\lambda + \sigma)v$. Hence $x + y$ is semisimple.
Furthermore, since x and y commute, $(x + y)^k = \sum_{i=0}^{k} \alpha_i x^i y^{k-i}$. Hence, if x and y are nilpotent, so is $x + y$. □

Proposition 1.7.4. *Let $x \in \mathfrak{gl}(V)$. Then:*

1. *there exist unique $x_s, x_n \in \mathfrak{gl}(V)$ with x_s semisimple and x_n nilpotent such that $[x_s, x_n] = 0$ and $x = x_s + x_n$.*

2. *There exist two polynomials $p(\lambda), q(\lambda) \in \mathbb{F}(\lambda)$ such that $p(0) = 0 = q(0)$ and $p(x) = x_s$ and $q(x) = x_n$. In particular x_s and x_n commute with all the endomorphisms commuting with x.*

3. *If $A \subset B \subset V$ are linear subspaces such that $x(B) \subseteq A$, then $x_s(B) \subseteq A$ and $x_n(B) \subseteq A$.*

Proof. Let $V = \bigoplus_{i=1}^{k} V_i$, where $V_i = \ker(x - a_i)^{m_i}$, be the generalized eigenspace decomposition, where $a_1, \dots, a_k$ are the eigenvalues of x and $m_1, \dots, m_k$ are the corresponding multiplicities. Notice that $(\lambda - a_i)^{m_i}$ is coprime with $(\lambda - a_j)^{m_j}$ for every $i \neq j$; hence, by the Chinese remainder theorem, there exists a polynomial $f(\lambda) \in \mathbb{F}(\lambda)$ that satisfies:

$$\begin{cases} p(\lambda) \equiv a_i & \mod (\lambda - a_i)^{m_i} \text{ for } i = 1, \dots, k \\ p(\lambda) \equiv 0 & \mod \lambda \end{cases}$$

Set $q(\lambda) := \lambda - p(\lambda)$. Then $p(0) = 0 = q(0)$.
Now, set $x_s := p(x)$, $x_n = q(x)$. Then $x = x_s + x_n$. Furthermore x_s and x_n commute, as polynomials in x. Analogously, since they are polynomials in x without constant term, x_s and x_n commute with every endomorphism commuting with x.
Now, if $x(B) \subseteq A$, then every polynomial without constant term sends B in A (for example $x^2(B) = x(x(B)) \subseteq x(A) \subseteq x(B) \subseteq A$), i.e., $x_s(B) \subseteq A$ and $x_n(B) \subseteq A$.
Notice that for every i, we have $p(\lambda) = a_i + \mu(\lambda - a_i)^{m_i}$; hence $x_{s|V_i} = a_i \mathrm{Id}_{V_i}$. So V_i is an eigenspace of x_s relative to the eigenvalue a_i. This tells us that x_s is semisimple. Furthermore, since $x_n = q(x) = x - x_s$, then $x_{n|V_i} = x_{|V_i} + x_{s|V_i} = \begin{pmatrix} a_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & a_i \end{pmatrix} + \begin{pmatrix} a_i & 0 & & \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & a_i \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$. Hence x_n is nilpotent.
Now, suppose that this decomposition is not unique, i.e., that $x = x_s + x_n = s + n$, with x_s, s semisimple and x_n, n nilpotent satisfying conditions 1, 2, 3. Then $x_s - s = x_n - n$. But $[x_s, s] = 0 = [n, x_n]$ because s and n are endomorphisms that commute with x (hence we can apply the condition 2. to x_s and x_n). So, by Remark 1.7.3, $x_s - s$ is semisimple and $x_n - n$ is

nilpotent. But an endomorphism that is both semisimple and nilpotent is necessarily 0, i.e., $x_s - s = 0 = x_n - n$; hence $x_s = s$ and $x_n = n$. $\square$

Definition 1.7.5. Let $x \in \mathfrak{gl}(V)$. Then the decomposition $x = x_s + x_n$ described in Proposition 1.7.4 is called the *Jordan-Chevalley decomposition* of x.

Proposition 1.7.6. *If $x \in \mathfrak{gl}(V)$ is semisimple, then so is ad_x. Furthermore, if $v_1, \ldots, v_n$ is a basis of eigenvectors of V relative to the eigenvalues $\lambda_1, \ldots, \lambda_n$, then the eigenvectors of ad_x are the standard basis of $\mathfrak{gl}(V)$ relative to $\{v_1, \ldots, v_n\}$ with eigenvalues $\lambda_i - \lambda_j$.*

Proof. The standard basis of $\mathfrak{gl}(V)$ relative to $\{v_1, \ldots, v_n\}$ is given by $\{e_{ij}\}$ satisfying $e_{ij}(v_k) = \delta_{jk} v_i$ for every k. Then, for every k we have:

$$\begin{aligned}\mathrm{ad}_x(e_{ij})(v_k) =& x(e_{ij}(v_k)) - e_{ij}(x(v_k)) = \delta_{jk} x(v_i) - \lambda_k e_{ij}(v_k) \\ =& \delta_{jk}\lambda_i v_i - \delta_{jk}\lambda_k v_i = (\lambda_i - \lambda_j)\delta_{jk} v_i = (\lambda_i - \lambda_j) e_{ij}(v_k).\end{aligned}$$

Hence $\mathrm{ad}_x(e_{ij}) = (\lambda_i - \lambda_j) e_{ij}$. $\square$

Remark 1.7.7. Let $x = x_s + x_n$ be the Jordan-Chevalley decomposition of $x \in \mathfrak{gl}(V)$. Then $\mathrm{ad}_x = \mathrm{ad}_{x_s} + \mathrm{ad}_{x_n}$, where ad_{x_s} is semisimple and ad_{x_n} is nilpotent. Furthermore $[\mathrm{ad}_{x_s}, \mathrm{ad}_{x_n}] = \mathrm{ad}_{[x_s, x_n]} = 0$. Hence, by the uniqueness of the Jordan-Chevalley decomposition of x, we can say that $\mathrm{ad}_x = \mathrm{ad}_{x_s} + \mathrm{ad}_{x_n}$ is the Jordan-Chevalley decomposition of ad_x.

1.8 Cartan's criterion

Remark 1.8.1. Let $b_1, \ldots, b_k \in \mathbb{F}$, and $f : \mathbb{F} \to \mathbb{F}$ any function. Then

$$r(x) = \sum_{i=1}^{k} f(b_i) \prod_{j \neq i} \frac{x - b_j}{b_i - b_j}$$

is called the *Lagrange interpolation polynomial*, and takes value $f(b_i)$ in the point b_i for all $i = 1, \ldots, k$.

Lemma 1.8.2. *Let $A \subseteq B \subseteq \mathfrak{gl}(V)$ be linear subspaces.*
Set $M := \{x \in \mathfrak{gl}(V) \mid [x, B] \subseteq A\}$. Suppose that $x \in M$ satisfies $\mathrm{tr}(xy) = 0$ *for any $y \in M$. Then x is nilpotent.*

Proof. Consider the Jordan-Chevalley decomposition of x, i.e., $x = x_s + x_n$. We want to show that $x_s = 0$.
Let $\mathcal{B} = \{v_1, \ldots, v_m\}$ be a basis of V relative to which x_s has diagonal matrix $x_s = \mathrm{diag}(a_1, \ldots, a_m)$. Since the field $\mathbb{F}$ has characteristic zero, it contains a subfield that is isomorphic to $\mathbb{Q}$. Set $E := \mathrm{span}_{\mathbb{Q}}\{a_1, \ldots, a_m\}$. We have to show that $E = 0$. Since E is finite dimensional over $\mathbb{Q}$, we just need to show that $E^* = \{f : E \to \mathbb{Q} \mid f \ \mathbb{Q}\text{-linear}\} = 0$, i.e., that for any $f \in E^*$, $f(a_i) = 0$ for all $i = 1, \ldots, m$.
Now, fix $f \in E^*$. By Remark 1.8.1, there exists a polynomial $r(\lambda) \in \mathbb{F}(\lambda)$ such that $r(a_i - a_j) = f(a_i) - f(a_j)$, with $r(0) = 0$. Let $y \in \mathfrak{gl}(V)$ be the element with matrix $\mathrm{diag}(f(a_1), \ldots, f(a_m))$ with respect to $\mathcal{B}$. By Proposition 1.7.6, we know that $\mathrm{ad}_{x_s}(e_{ij}) = (a_i - a_j)e_{ij}$ and then $\mathrm{ad}_y(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$, so $\mathrm{ad}_y = r(\mathrm{ad}_{x_s})$, that is a polynomial in ad_x with no constant terms. Hence, also ad_y is a polynomial in ad_x with no constant terms.
Now, since $x \in M$, we know that $\mathrm{ad}_x(B) \subseteq A$. Furthermore, since ad_y is a polynomial in ad_x with no constant terms, $\mathrm{ad}_y(B) \subseteq A$; hence $y \in M$. By assumption $0 = \mathrm{tr}(xy) = \sum_{i=1}^m a_i f(a_i)$. Hence $0 = f(\sum_{i=1}^m a_i f(a_i)) = \sum_{i=1}^m (f(a_i))^2$. So $f(a_i) = 0$ for all $i = 1, \ldots, m$. □

Theorem 1.8.3 (Cartan's criterion)**.** *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Suppose that* $\mathrm{tr}(xy) = 0$ *for all $x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.*

Proof. We just need to show that any element of $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Indeed, if this happens to be true then, by Proposition 1.7.6, every element in $[\mathfrak{g}, \mathfrak{g}]$ is ad-nilpotent, i.e., by Engel's Theorem (Theorem 1.5.8), $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, i.e., $\mathfrak{g}$ is solvable by Corollary 1.6.12. Hence we only need to prove that the generators of $[\mathfrak{g}, \mathfrak{g}]$, i.e., all the elements of the form $[z_1, z_2]$ with $z_1, z_2 \in \mathfrak{g}$, are nilpotent.

Set $A = [\mathfrak{g}, \mathfrak{g}]$, $B = \mathfrak{g}$, $M = \{x \in \mathfrak{gl}(V) \mid [x, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]\}$. Clearly, $M \supseteq \mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}]$. We want to use Lemma 1.8.2, i.e., we want to prove that $\mathrm{tr}([z_1, z_2]y) = 0$ for all $y \in M$.
Now, notice that $\mathrm{tr}([z_1, z_2]y) = \mathrm{tr}(z_1 z_2 y - z_2 z_1 y) = \mathrm{tr}(z_1 z_2 y - z_1 y z_2) = \mathrm{tr}(z_1[z_2, y]) = \mathrm{tr}([z_2, y]z_1)$. But $[z_2, y] \in [\mathfrak{g}, \mathfrak{g}]$ by definition of M. Hence $\mathrm{tr}([z_1, z_2], y) = \mathrm{tr}([z_2, y]z_1) = 0$ by assumption. So, using Lemma 1.8.2, we can conclude that $[z_1, z_2]$ is nilpotent. □

Remark 1.8.4. In particular, if $\mathfrak{g} \subset \mathfrak{gl}(V)$ satisfies $\mathrm{tr}(xy) = 0$ for all $x, y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.

Corollary 1.8.5. *Let $\mathfrak{g}$ be a Lie algebra such that $\mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.*

Proof. Using the Cartan's criterion, we can say that $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(V)$ is solvable. But $\mathrm{ad}(\mathfrak{g}) \cong \mathfrak{g}/Z(\mathfrak{g})$. Since $Z(\mathfrak{g})$ is solvable (it is abelian), so also is $\mathfrak{g}$ by point 2. of Proposition 1.6.4. □

1.9 Semisimple Lie algebras

Definition 1.9.1. Let $\mathfrak{g}$ be any Lie algebra. The bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ defined by $K(x, y) = \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y)$ for $x, y \in \mathfrak{g}$ is called the *Killing form* of $\mathfrak{g}$.

Remark 1.9.2. The Killing form K of $\mathfrak{g}$ is:

1. symmetric;

2. associative, i.e., $K([x, y], z) = K(x, [y, z])$.

Proof. 1. $K(x, y) = K(y, x)$ because, in general, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for A, B matrices.

$$\begin{aligned}
2. \quad K([x, y], z) &= \mathrm{tr}(\mathrm{ad}_{[x,y]} \mathrm{ad}_z) = \mathrm{tr}([\mathrm{ad}_x, \mathrm{ad}_y]\mathrm{ad}_z) \\
&= \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_z - \mathrm{ad}_y \mathrm{ad}_x \mathrm{ad}_z) = \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_z - \mathrm{ad}_x \mathrm{ad}_z \mathrm{ad}_y) \\
&= \mathrm{tr}(\mathrm{ad}_x [\mathrm{ad}_y, \mathrm{ad}_z]) = \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_{[y,z]}) = K(x, [y, z]).
\end{aligned}$$

□

Remark 1.9.3. We recall that the *radical* rad_K of the Killing form K is defined as follows:

$$\mathrm{rad}_K = \{x \in \mathfrak{g} \mid K(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}.$$

Then K is non-degenerate if and only if $\mathrm{rad}_K = 0$.
Notice also that rad_K is an ideal of $\mathfrak{g}$. Indeed, if $x \in \mathrm{rad}_K$, $y \in \mathfrak{g}$, then $K([x, z], y) = K(x, [z, y]) = 0$ for all $y \in \mathfrak{g}$.

Lemma 1.9.4. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it does not contain any nonzero abelian ideals.*

Proof. Any abelian ideal in a Lie algebra is solvable. Hence, if $\mathfrak{g}$ is semisimple, it has no nonzero abelian ideals.
Conversely, if by contradiction $\mathfrak{g}$ is not semisimple, then consider $k \in \mathbb{Z}^+$ such that $\mathrm{Rad}(\mathfrak{g})^{(k)} = 0$ with $\mathrm{Rad}(\mathfrak{g})^{(k-1)} \neq 0$. Then $\mathrm{Rad}(\mathfrak{g})^{(k-1)}$ is a nonzero abelian ideal in $\mathfrak{g}$. □

Theorem 1.9.5. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate.*

Proof. Suppose that $\mathfrak{g}$ is semisimple. Then we can apply the Cartan's criterion on rad_K (Theorem 1.8.3) to conclude that rad_K is solvable. But, since $\mathfrak{g}$ is semisimple, then necessarily $\mathrm{rad}_K = 0$.
Conversely, suppose that K is non-degenerate. We want to use Lemma 1.9.4, i.e., we want to prove that $\mathfrak{g}$ has no nonzero abelian ideals. By contradiction, if $I \neq 0$ is an abelian ideal of $\mathfrak{g}$, then we can consider a nonzero element $x \in I$. For $y \in \mathfrak{g}$, consider $\mathrm{ad}_x\mathrm{ad}_y : \mathfrak{g} \to I$. Then $(\mathrm{ad}_x\mathrm{ad}_y)^2 : \mathfrak{g} \to [I, I] = 0$, i.e., $\mathrm{ad}_x\mathrm{ad}_y$ is nilpotent. Hence $K(x, y) = \mathrm{tr}(\mathrm{ad}_x\mathrm{ad}_y) = 0$ for any $y \in \mathfrak{g}$, i.e., $x \in \mathrm{rad}_K$. □

We now recall two basic results that we will use in the following.

Remark 1.9.6. Let V be a finite dimensional $\mathbb{F}$-vector space, $\alpha : V \times V \to \mathbb{F}$ a bilinear form on V, and $U \subseteq V$ a linear subspace.

1. If α is non-degenerate then $\dim U + \dim U^{\perp} = \dim V$.
2. $V = U \oplus U^{\perp}$ if and only if α_U is non-degenerate.

Theorem 1.9.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exist ideals $I_1, \ldots, I_k$ of $\mathfrak{g}$ which are simple (as Lie algebras) such that $\mathfrak{g} = I_1 \oplus \ldots \oplus I_k$. Moreover, if I is a simple ideal of $\mathfrak{g}$, then $I = I_j$ for some j.*

Proof. As a first step, let I be an arbitrary ideal of $\mathfrak{g}$.
Then $I^{\perp} = \{x \in \mathfrak{g} \mid K(x,y) = 0 \text{ for all } y \in I\}$ is also an ideal since, if $x \in I^{\perp}$ and $z \in \mathfrak{g}$, then $K([x,z],y) = K(x,[z,y]) = 0$ for any $y \in I$. Cartan's Criterion (Theorem 1.8.3), applied to the Lie algebra I, shows that the ideal $I \cap I^{\perp}$ of $\mathfrak{g}$ is solvable, so $I \cap I^{\perp} \subseteq \mathrm{Rad}(\mathfrak{g}) = 0$ since $\mathfrak{g}$ is semisimple. Hence, $I \cap I^{\perp} = 0$. Therefore, since $\dim I + \dim I^{\perp} = \dim \mathfrak{g}$ by point 1. of Remark 1.9.6, we must have $I \oplus I^{\perp} = \mathfrak{g}$. Now proceed by induction on $\dim \mathfrak{g}$ to obtain the desired decomposition into direct sum of simple ideals.
Now, if $I \neq 0$ is a simple ideal of $\mathfrak{g}$, then $[I, \mathfrak{g}] \neq 0$ (otherwise $I \subseteq Z(\mathfrak{g})$, that is not possible since $\mathfrak{g}$ is semisimple). Furthermore $[I, \mathfrak{g}] \subseteq I$ is an ideal of $\mathfrak{g}$. Hence, by the semplicity of I, $I = [I, \mathfrak{g}] = [I, I_1] \oplus \ldots [I, I_k]$. But $[I, I_r]$ are ideals both of I and of I_r; hence, since I_r are simple, $[I, I_r] = 0$ or $[I, I_r] = I_r$. Analogously, $[I, I_r] = 0$ or $[I, I_r] = I$. Hence only one of the commutators is nonzero, i.e., there exists $r \in \{1, \ldots, k\}$ such that $I = [I, I_r] = I_r$. $\square$

Corollary 1.9.8. *If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

Proof. By Theorem 1.9.7 $\mathfrak{g} = I_1 \oplus \ldots \oplus I_k$. So:

$$[\mathfrak{g}, \mathfrak{g}] = [I_1, I_1] \oplus \ldots \oplus [I_k, I_k] = I_1 \oplus \ldots \oplus I_k = \mathfrak{g}$$

because $[I_i, I_j] \subseteq I_i \cap I_j = 0$ and I_j are simple. $\square$

Lemma 1.9.9. *Let $\mathfrak{g}$ be any Lie algebra. Then* $\mathrm{ad}(\mathfrak{g})$ *is an ideal of* $\mathrm{Der}(\mathfrak{g})$.

Proof. Let $\delta \in \mathrm{Der}(\mathfrak{g}), x \in \mathfrak{g}$. Then for any $y \in \mathfrak{g}$ we have:

$$\begin{aligned}[\delta, \mathrm{ad}_x](y) &= \delta(\mathrm{ad}_x(y)) - \mathrm{ad}_x(\delta(y)) = \delta([x,y]) - [x, \delta(y)] \\ &= [\delta(x), y] + [x, \delta(y)] - [x, \delta(y)] = [\delta(x), y] = \mathrm{ad}_{\delta(x)}(y).\end{aligned}$$

Hence $[\delta, \mathrm{ad}_x] = \mathrm{ad}_{\delta(x)}$, i.e., $\mathrm{ad}(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$. $\square$

Theorem 1.9.10. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Then* $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$*.*

Proof. Let us consider the map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.
Then $\ker \mathrm{ad} = \{x \in \mathfrak{g} \mid [x, z] = 0 \text{ for all } z \in \mathfrak{g}\} = Z(\mathfrak{g})$. But $\mathfrak{g}$ is semisimple, and so $Z(\mathfrak{g}) = 0$. So $\mathrm{ad} : \mathfrak{g} \xrightarrow{\sim} \mathrm{ad}(\mathfrak{g})$, i.e., $\mathrm{ad}(\mathfrak{g})$ is semisimple. Hence $\mathrm{ad}(\mathfrak{g})$ has a Killing form that is non-degenerate (which is the restriction of the Killing form of $\mathrm{Der}(\mathfrak{g})$ to $\mathrm{ad}(\mathfrak{g})$). Therefore, by point 2. of Remark 1.9.6, $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) \oplus (\mathrm{ad}(\mathfrak{g}))^{\perp}$.
We need to show that $(\mathrm{ad}(\mathfrak{g}))^{\perp} = 0$. Take $\delta \in (\mathrm{ad}(\mathfrak{g}))^{\perp}$, $x \in \mathfrak{g}$; then, by the proof of Proposition 1.9.9, $[\delta, \mathrm{ad}_x] = \mathrm{ad}_{\delta(x)}$. But $[\delta, \mathrm{ad}_x] \in \mathrm{ad}(\mathfrak{g}) \cap (\mathrm{ad}(\mathfrak{g}))^{\perp} = \{0\}$; hence $\mathrm{ad}_{\delta(x)} = 0$. Since ad is injective we have $\delta(x) = 0$ and, by the arbitrariness of $x \in \mathfrak{g}$, $\delta = 0$. □

Theorem 1.9.11. *(Schur's Lemma) Let* φ *be an irreducible representation of* $\mathfrak{g}$ *on* V*. Let* $f \in \mathfrak{gl}(V)$ *such that* $[\varphi(x), f] = 0$ *for every* $x \in \mathfrak{g}$*. Then* $f = \lambda \mathrm{id}_V$ *for some* $\lambda \in \mathbb{F}$*.*

Proof. Let $V = \oplus V_\lambda$ be the generalized eigenspace decomposition with respect to f, where $V_\lambda = \{v \in V \mid (f - \lambda \mathrm{id}_V)^k(v) = 0 \text{ for some } k\}$.
Notice that $\varphi(x)(V_\lambda) \subseteq V_\lambda$ for every $x \in \mathfrak{g}$. Indeed, since f and $\varphi(x)$ commute by hypothesis, we have $(f - \lambda \mathrm{id}_V)^k(\varphi(x)(v)) = \varphi(x)(f - \lambda \mathrm{id}_V)^k(v) = 0$.
Hence V_λ is a $\mathfrak{g}$-submodule of V. So, by the irreducibility of V, we have $V = V_\lambda$.
Now, let $V_0 := \{v \in V \mid (f - \lambda \mathrm{id}_V)(v) = 0\} \neq 0$. Then, for every $x \in \mathfrak{g}$, $\varphi(x)(V_0) \subseteq V_0$. Hence, by the irreducibility of V, $V = V_0$. □

Definition 1.9.12. Let $\mathfrak{g}$ be semisimple and let $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful finite dimensional representation of $\mathfrak{g}$. The bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ such that $\beta(x, y) = \mathrm{tr}(\Phi(x)\Phi(y))$ for $x, y \in \mathfrak{g}$ is called the *trace form* of $\mathfrak{g}$ associated to Φ.

Example 1.9.13. If $\mathfrak{g}$ is a Lie algebra, then the trace form associated to the adjoint representation is the Killing form.

Remark 1.9.14. As for the Killing form, one can show that β is symmetric and associative (i.e. $\beta([x,y],z) = \beta(x,[y,z])$ for every $x,y,z \in \mathfrak{g}$). Furthermore, if $\mathfrak{g}$ is semisimple then, by the Cartan's criterion, β is non-degenerate.

Remark 1.9.15. Let $\mathfrak{g}$ be a Lie algebra, β the trace form associated to a representation Φ of $\mathfrak{g}$, $\mathcal{B} = \{x_1,\dots,x_n\}$ a basis of $\mathfrak{g}$ and $\mathcal{B}' = \{y_1,\dots,y_n\}$ the dual basis of $\mathcal{B}$ with respect to β, i.e., such that $\beta(x_i,y_j) = \delta_{ij}$ for every $i,j = 1,\dots,n$. Then, for $x \in \mathfrak{g}$, we can write:

- $[x,x_i] = \sum_{j=1}^n a_{ij}x_j$, where $a_{ij} \in \mathbb{F}$;
- $[x,y_i] = \sum_{j=1}^n b_{ij}y_j$, where $b_{ij} \in \mathbb{F}$;

Notice that $\beta([x,x_i],y_k) = \sum_{j=1}^n a_{ij}\beta(x_j,y_k) = \sum_{j=1}^n a_{ij}\delta_{jk} = a_{ik}$. But $\beta([x,x_i],y_k) = -\beta([x_i,x],y_k) = -\beta(x_i,[x,y_k]) = -\sum_{j=1}^n b_{kj}\beta(x_i,y_j) = -b_{ki}$. Hence $a_{ik} = -b_{ki}$ for every $i,k = 1,\dots,n$.

Definition 1.9.16. Let $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation of $\mathfrak{g}$. In the notations of Remark 1.9.15, we call the *Casimir element* associated to Φ the following element of $\mathfrak{gl}(V)$:

$$C_\Phi := \sum_{i=1}^n \Phi(x_i)\Phi(y_i).$$

Lemma 1.9.17. *Let $a,b,c \in \mathfrak{gl}(V)$. Then $[a,bc] = [a,b]c + b[a,c]$.*

Proof. $\cdot$ $[a,bc] = abc - bca$
$\cdot$ $[a,b]c + b[a,c] = abc - bac + bac - bca = abc - bca.$ □

Proposition 1.9.18. *C_Φ is a $\mathfrak{g}$-module homomorphism. Furthermore, if V is an irreducible $\mathfrak{g}$-module, then $C_\Phi = \dfrac{\dim\mathfrak{g}}{\dim V}\mathrm{id}_V$.*

Proof. Let $x \in \mathfrak{g}$. Then

$$[\Phi(x), C_\Phi] = \sum_{i=1}^n [\Phi(x), \Phi(x_i)\Phi(y_i)] \overset{1.9.17}{=} \sum_{i=1}^n [\Phi(x),\Phi(x_i)]\Phi(y_i) + \sum_{i=1}^n \Phi(x_i)[\Phi(x),\Phi(y_i)]$$
$$\overset{1.9.15}{=} \sum_{i,j=1}^n a_{ij}\Phi(x_j)\Phi(y_i) + \sum_{i,j=1}^n b_{ji}\Phi(x_j)\Phi(y_i) = 0$$

since $a_{ij} = -b_{ji}$ by Remark 1.9.15.
Hence $\Phi(x)(C_\Phi(v)) = C_\Phi(\Phi(x)(v))$, i.e., C_Φ is a homomorphism of $\mathfrak{g}$-modules.
Now, if V is irreducible, by Schur's Lemma (Theorem 1.9.11), $C_\Phi = \lambda \mathrm{id}_V$ for some $\lambda \in \mathbb{F}$. Hence $\mathrm{tr}C_\Phi = \mathrm{tr}(\lambda \mathrm{id}_V) = \lambda \dim V$.
But, by definition of Casimir element, we have that $\mathrm{tr}C_\Phi = \sum_{i=1}^n \mathrm{tr}\Phi(x_i)\Phi(y_i) = \sum_{i=1}^n \beta(x_i, y_i) = \sum_{i=1}^n \delta_{ii} = n = \dim\mathfrak{g}$. So $\lambda = \dfrac{\dim\mathfrak{g}}{\dim V}$. □

Definition 1.9.19. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ any finite dimensional representation of $\mathfrak{g}$ (not necessarily faithful). Then $\ker\Phi$ is an ideal of $\mathfrak{g}$. But since $\mathfrak{g}$ is semisimple, by point 2. of Remark 1.9.6, we know that $\mathfrak{g} = \ker\Phi \oplus (\ker\Phi)^\perp$, where both $\mathrm{Ker}\Phi$ and $(\mathrm{Ker}\Phi)^\perp$ are semisimple. Then $\Phi_{|(\ker\Phi)^\perp}$ is faithful.
We define the *Casimir element* associated to Φ as the Casimir element associated to $\Phi_{|(\ker\Phi)^\perp}$.

Lemma 1.9.20. *Let $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra $\mathfrak{g}$. Then $\Phi(\mathfrak{g}) \subset \mathfrak{sl}(V)$. In particular, Φ acts trivially on any one dimensional $\mathfrak{g}$-module.*

Proof. Since $\mathfrak{g}$ is semisimple, by Corollary 1.9.8 we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Hence, $\Phi(\mathfrak{g}) = \Phi([\mathfrak{g}, \mathfrak{g}]) = [\Phi(\mathfrak{g}), \Phi(\mathfrak{g})]$, then $\mathrm{tr}\Phi(\mathfrak{g}) = \mathrm{tr}[\Phi(\mathfrak{g}), \Phi(\mathfrak{g})] = 0$ because the trace of any commutator is zero. Therefore, $\Phi(\mathfrak{g}) \subset \mathfrak{sl}(V)$.
In particular, if $\dim(V) = 1$ we have the thesis because $\mathfrak{sl}(V) = 0$. □

Lemma 1.9.21. *A $\mathfrak{g}$-module V is completely reducible if and only if every proper $\mathfrak{g}$-submodule of V has a direct complement that is itself a $\mathfrak{g}$-submodule of V, i.e., if for every submodule $W \subset V$ there exists a submodule $W' \subset V$ such that $V = W \oplus W'$.*

Proof. Suppose that V is a completely reducible $\mathfrak{g}$-module, i.e., $V = \oplus_{i \in I} V_i$ where I is a finite set of indices and the V_i's are irreducible $\mathfrak{g}$-submodule of V. Let W' be a $\mathfrak{g}$-submodule of V and W be the maximal $\mathfrak{g}$-submodule such that $W' \oplus W = \{0\}$.
Suppose by contradiction that there exists $j \in I$ such that $V_j \not\subseteq W' \oplus W$;

then $V_j \cap (W' \oplus W)$ is a $\mathfrak{g}$-submodule of V_j. Therefore, $V_j \cap (W' \oplus W) = \{0\}$ because V_j is irreducible and $V_j \not\subseteq V_j \cap (W' \oplus W)$.
Now, if we consider $W \oplus V_j$, then $W \oplus V_j \supsetneq W$ and $W' \cap (W \oplus V_j) = \{0\}$. This contradicts the maximality of W, so for all $j \in I$ we have $V_j \subseteq W' \oplus W$. Hence, $V = W' \oplus W$.
Conversely, we proceed by induction on $\dim V = n$.

- If $n = 1$, then V is irreducible for dimensional reasons.

- If V is irreducible, there is nothing to prove. Hence, suppose that V is not irreducible. If we take a nonzero $\mathfrak{g}$-submodule $W \subsetneq V$ then, by assumption, there exists a $\mathfrak{g}$-submodule $W' \subset V$ such that $V = W \oplus W'$. Let $S \subset W$ be a $\mathfrak{g}$-submodule of W. Then S is $\mathfrak{g}$-submodule of V, so by assumption there exists a $\mathfrak{g}$-submodule $\tilde{S}$ of V such that $V = S \oplus \tilde{S}$. Hence, $W = V \cap W = (S \cap W) \oplus (\tilde{S} \cap W) = S \oplus (\tilde{S} \cap W)$. Now, notice that $\tilde{S} \cap W$ is a $\mathfrak{g}$-module since both $\tilde{S}$ and W are $\mathfrak{g}$-modules.
  Thus, by induction hypothesis, W is completely reducible; analogously also W' is completely reducible. Therefore $V = W \oplus W'$ is completely reducible.

$\square$

Theorem 1.9.22 (Weyl). *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ a finite dimensional representation of $\mathfrak{g}$. Then Φ is completely reducible.*

Proof. In order to prove the theorem we will use Lemma 1.9.21, i.e., we will to show that for every submodule $W \subset V$ there exists a submodule $W' \subset V$ such that $V = W \oplus W'$.
We can suppose that Φ is faithful. Indeed, if not, then $\mathfrak{g} = I \oplus J$, where $I = \ker\Phi$, I and J are semisimple, and $\Phi_{|J}$ is faithful. So we just need to prove that Φ is completely reducible on J, since I does not alter the irreducible components of V, because $\Phi(I) = 0$.
STEP 1. Suppose that $\mathrm{codim}_V(W) = 1$.

· Case 1: W irreducible. We want to show that $V = W \oplus \ker C_\Phi$ (i.e., we want to show that dim $\ker C_\Phi = 1$ and that $\ker C_\Phi \cap W = \{0\}$).
We can define on V/W a structure of $\mathfrak{g}$-module, given by $x.(v + W) = x.v + W$. By Lemma 1.9.20, since V/W is one dimensional, $\mathfrak{g}$ acts trivially on V/W, i.e., $\Phi(\mathfrak{g})(V) \subseteq W$. Hence $C_\Phi(V) \subseteq W$. But W is irreducible and so, by Proposition 1.9.18, we have that $C_{\Phi|W} : W \to W$ is such that $C_{\Phi|W} = \dfrac{\dim\mathfrak{g}}{\dim W} W$. Hence $\ker C_\Phi \cap W = \{0\}$ and, by the dimension theorem, dim $\ker C_\Phi = 1$.
· Case 2: If W is any $\mathfrak{g}$- submodule of V of codimension 1, we proceed by induction on $\dim W$.

- If $\dim W = 1$, then W is necessarily irreducible for dimensional reasons. Hence we can apply Case 1 to conclude.

- If $\dim W > 1$ and W is irreducible, we are in Case 1. Otherwise, let W' be a proper $\mathfrak{g}$-submodule of W, i.e., such that $0 \neq W' \subsetneq W \subseteq V$. Then $\mathrm{codim}_{V/W'}(W/W') = 1$. Since W' is a proper submodule of W, then $\dim W/W' < \dim W$. Hence, by induction hypothesis, we can say that there exists $\tilde{X}$ such that $V/W = W/W' \oplus \tilde{X}/W'$, with $\dim(\tilde{X}/W') = 1$. Hence W' is a submodule of codimension 1 in $\tilde{X}$. Furthermore $\dim W' < \dim W$; so, by induction hypothesis $\tilde{X} = W' \oplus Z$. We state that $V = W \oplus Z$.
  Indeed, Z is a $\mathfrak{g}$-submodule of V of dimension 1 by construction. So we just need to prove that $W \cap Z = \{0\}$. By contradiction, take $0 \neq x \in W \cap Z$. Then $x \notin W'$ since $Z \cap W' = \{0\}$; hence $(\tilde{X}/W') \ni \bar{x} \neq 0$. Analogously $(W/W') \ni \bar{x} \neq 0$. So $0 \neq \bar{x} \in (W/W') \cap (\tilde{X}/W')$, that is absurd since W/W' and $\tilde{X}/W'$ are in direct sum.

Step 2. Let $W \subset V$ be any $\mathfrak{g}$-submodule of V. We can define of $\mathrm{Hom}(V, W)$ a structure of $\mathfrak{g}$-module setting, for $x \in \mathfrak{g}, f \in \mathrm{Hom}(V, W)$ and $v \in V$, $(x.f)(v) := x.(f(v)) - f(x.v)$.

Now, consider the following linear subspaces of $\mathrm{Hom}(V, W)$:

$$\mathcal{V} := \{f \in \mathrm{Hom}(V, W) \mid f_{|W} = a\mathrm{id}_W,\ a \in \mathbb{F}\}$$
$$\mathcal{W} := \{f \in \mathrm{Hom}(V, W) \mid f_{|W} = 0\} \subset \mathcal{V}$$

Notice that $\mathcal{V}$ and $\mathcal{W}$ are $\mathfrak{g}$-submodule of $\mathrm{Hom}(V, W)$. Indeed, if $f \in \mathcal{V}$ and $x \in \mathfrak{g}$, for any $w \in W$ we have $(x.f)(w) = x.(f(w)) - f(x.w) = x.(aw) - a(x.w) = 0$. Furthermore $\mathrm{codim}_{\mathcal{V}}\mathcal{W}$=1. Indeed, if we take a basis $\{w_1, \dots, w_k\}$ of W and complete it to a basis $\mathcal{B} = \{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$ of V then, for $f \in \mathcal{V}$, the matrix of f with respect to $\mathcal{B}$ is $f = \left(\begin{array}{c|c} aI_k & * \\ \hline 0 & 0 \end{array}\right)$.

Instead, for $g \in \mathcal{W}$, the matrix relative to $\mathcal{B}$ is $f = \left(\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right)$. So, by Step 1 we know that $\mathcal{V} = \mathcal{W} \oplus \langle f \rangle$, where $\langle f \rangle$ is a $\mathfrak{g}$-submodule of dimension 1. Since $f \in \mathcal{V} \setminus \mathcal{W}$, then $f_{|W} = a\mathrm{id}_W$, with $a \neq 0$. In particular we can suppose that $a = 1$.

We state that $V = W \oplus \ker f$. Indeed:

- $f : V \to W$ is such that $f_{|W} = \mathrm{id}_W$. Hence $\mathrm{Ker} f \cap W = \{0\}$.
- $\dim V = \dim W + \dim \ker f$ by the dimension theorem.
- $\ker f$ is a $\mathfrak{g}$-submodule. In fact, since $\langle f \rangle$ is a $\mathfrak{g}$-submodule of dimension 1 and $\mathfrak{g}$ is semisimple, by Lemma 1.9.20, $\mathfrak{g}$ acts trivially on $\langle f \rangle$, i.e., $x.f = 0$ for all $x \in \mathfrak{g}$. Hence $0 = (x.f)(v) = x.(f(v)) - f(x.v)$, i.e., f is a $\mathfrak{g}$-module homomorphism. So, by Remark 1.3.7, $\ker f$ is a $\mathfrak{g}$-submodule of V.

□

1.10 Cartan decomposition

Definition 1.10.1. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We define the *normalizer* of $\mathfrak{h}$ in $\mathfrak{g}$ as the following subalgebra:

$$N_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] \subseteq \mathfrak{g}\}.$$

Definition 1.10.2. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. $\mathfrak{h}$ is called a *Cartan subalgebra* of $\mathfrak{g}$ if it is nilpotent and $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Definition 1.10.3. A *toral subalgebra* of a Lie algebra $\mathfrak{g}$ is a non-zero subalgebra of $\mathfrak{g}$ consisting of semisimple elements.

We recall that if $\mathfrak{g}$ is a semisimple Lie algebra, then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if it is a maximal toral subalgebra of $\mathfrak{g}$ (see [5]). This holds also more in general for reductive Lie algebras, which will be introduced in Chapter 2.

In this section, $\mathfrak{g}$ will denote a (non-zero) semisimple Lie algebra. Here, we want to recall the Cartan decomposition of a semisimple Lie algebra.

Remark 1.10.4. Every semisimple Lie algebra $\mathfrak{g}$ contains at least one semisimple element. Hence $\mathfrak{g}$ always contains a non-zero toral subalgebra.

Proof. We can find $x \neq 0$, $x \in \mathfrak{g}$, whose semisimple part s in the abstract Jordan decomposition is non-zero. Indeed if not, $\mathfrak{g}$ would consist entirely of ad-nilpotent elements, then $\mathfrak{g}$ would be nilpotent by Engel's Theorem. So $\langle s \rangle$ is a non-zero toral subalgebra of $\mathfrak{g}$. □

Lemma 1.10.5. *A toral subalgebra T of $\mathfrak{g}$ is abelian.*

Proof. We want to show that $[x, y] = 0$ for every $x, y \in \mathfrak{g}$. This is equivalent to showing that, for every $x \in \mathfrak{g}$, $\mathrm{ad}_x : T \to T$ has all the eigenvalues equal to 0. Let $x \in \mathfrak{g}$. By contradiction, suppose that there exists a nonzero $y \in \mathfrak{g}$ such that $\mathrm{ad}_x(y) = ay$, with $a \neq 0$. But $\mathrm{ad}_x(y) = [x, y] = -[y, x]$; hence $[y, x] = -ax$. Since ad_y is diagonalizable by assumption, there exists a basis of T that consists of eigenvectors of ad_y. Therefore $x = x_1 + \ldots + x_k$, where $\mathrm{ad}_y(x) = \lambda_i x_i$ for some $\lambda_i \in \mathbb{F}$. Hence $\mathrm{ad}_y(x) = \lambda_1 x_1 + \ldots + \lambda_k x_k$, that is a linear combination of eigenvectors of ad_y relative to nonzero eigenvalues (if $\lambda_i = 0$, then the corresponding eigenvector does not appear in the sum). But $\mathrm{ad}_y(x) = -ay \neq 0$, that is an eigenvector of ad_y relative to the eigenvalue 0. □

Now fix a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Since $\mathfrak{h}$ is abelian by Lemma 1.10.5, $\{\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g} \mid h \in \mathfrak{h}\}$ is a commuting family of semisimple endomorphisms. Then, according to point 2. of Proposition 1.7.1, the family above is simultaneously diagonalizable.
Hence we can write

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$.
Notice that $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h}) \supseteq \mathfrak{h} \neq 0$, i.e. $\mathfrak{g}_0 \neq 0$. So we can decompose $\mathfrak{g}$ as follows.

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\Phi = \{\alpha \in \mathfrak{h}^*, \alpha \neq 0 \mid \mathfrak{g}_\alpha \neq 0\}$. This decomposition is known as the *Cartan decomposition* of $\mathfrak{g}$ and Φ is called the *root system of* $\mathfrak{g}$ relative to $\mathfrak{h}$.

We begin with a few simple observations about the root space decomposition of a semisimple Lie algebra.

Remark 1.10.6. $\#\Phi < \infty$ (because $\mathfrak{g}$ is finite dimensional).

Proposition 1.10.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be the Cartan decomposition of $\mathfrak{g}$. Then, for all $\alpha, \beta \in \mathfrak{h}^*$:*

1. *$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$;*

2. *if $\alpha, \beta \in \mathfrak{h}^*$ such that $\alpha + \beta \neq 0$, then $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.*

Proof. 1. This assertion follows from the Jacobi identity. Indeed, for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and $h \in \mathfrak{h}$, we have:

$$\begin{aligned} \mathrm{ad}_h([x, y]) = [h, [x, y]] &= [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] \\ &= (\alpha + \beta)(h)[x, y]. \end{aligned}$$

This means that $[x, y] \in \mathfrak{g}_{\alpha+\beta}$.

2. Let $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$. We have:
   $\cdot$ $K([h,x],y) = \alpha(h)K(x,y)$,
   $\cdot$ $K([h,x],y) = -K([x,h],y) = -K(x,[h,y]) = -\beta(h)K(x,y)$, by anti-symmetry of commutator and associativity of the Killing form.
   Then $\alpha(h)K(x,y) = -\beta(h)K(x,y) \Rightarrow (\alpha+\beta)(h)K(x,y) = 0$. This forces $K(x,y) = 0$ because $\alpha + \beta \neq 0$ that means that there exists $h \in \mathfrak{h}$ such that $(\alpha+\beta)(h) \neq 0$.

□

Corollary 1.10.8. *The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$ is non-degenerate.*

Proposition 1.10.9. *Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$.*

Using Proposition 1.10.9, we can rewrite the Cartan decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

Remark 1.10.10. Corollary 1.10.8 combined with Proposition 1.10.9 allows us to say that the restriction of the Killing form to $\mathfrak{h}$ is non-degenerate. Hence we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via the isomorphism given by:

$$\begin{array}{ccccc} h & \mapsto & \varphi_h(t) = K(h,t) & & \\ t_\alpha & \leftarrow\!\shortmid & \alpha & \text{if } \alpha(h) = K(t_\alpha, h) & (*) \end{array}$$

Theorem 1.10.11. *Let $\mathfrak{g}$ be a semisimple Lie algebra and Φ be the root system of $\mathfrak{g}$.*

1. *Φ spans $\mathfrak{h}^*$.*
2. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*
3. *Let $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$. Then $[x,y] = K(x,y)t_\alpha$ (t_α as in $(*)$).*
4. *If $\alpha \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \langle t_\alpha \rangle$.*

5. *If* $\alpha \in \Phi$*, then* $K(t_\alpha, t_\alpha) \neq 0$.

6. *If* $\alpha \in \Phi$ *and* x_α *is a nonzero element of* $\mathfrak{g}_\alpha$*, then there exists* $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ *such that* $\langle x_\alpha, x_{-\alpha}, h_\alpha := [x_\alpha, x_{-\alpha}]\rangle \cong \mathfrak{sl}_2$.

7. $h_\alpha = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$ *and* $h_{-\alpha} = -h_\alpha$.

Proof.

1. If Φ fails to span $\mathfrak{h}^*$, then (by duality) there exists nonzero $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. This means that $[h, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in \Phi$. Since $[h, \mathfrak{h}] = 0$, this in turn forces $[h, \mathfrak{g}] = 0 \Leftrightarrow h \in Z(\mathfrak{g})$. But $Z(\mathfrak{g}) = 0$ since $\mathfrak{g}$ is semisimple, which is absurd because $h \neq 0$.

2. Let $\alpha \in \Phi$. If $-\alpha \notin \Phi$, then for all $\beta \in \Phi \quad \alpha + \beta \neq 0$. Therefore $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ by point 2. of Proposition 1.10.7. Moreover $K(\mathfrak{g}_\alpha, \mathfrak{h}) = K(\mathfrak{g}_\alpha, \mathfrak{g}_0) = 0$, then $K(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$, contradicting the non-degeneracy of K.

3. Let $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$. Let $h \in \mathfrak{h}$ be arbitrary. The associativity of K implies:

$$\begin{aligned} K(h, [x, y]) &= K([h, x], y) = K(\alpha(h)x, y) = \alpha(h)K(x, y) \\ &= K(h, t_\alpha)K(x, y) = K(h, K(x, y)t_\alpha). \end{aligned}$$

   Since K is non-degenerate and the last relation is true for all $h \in \mathfrak{h}$, we have that:
   $[x, y] = K(x, y)t_\alpha$.

4. Point 3. tells us that if suffices to prove that if $\alpha \in \Phi$ and $x_\alpha \in \mathfrak{g}_\alpha$, then there exists $y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) \neq 0$. Otherwise, we would have $K(x, \mathfrak{g}_\alpha) = 0$. On the other hand, we know that:

   - $K(x, \mathfrak{g}_\beta) = 0$ for all β such that $\alpha + \beta \neq 0$ by point 2. of Proposition 1.10.7,

   - $K(x, \mathfrak{h}) = K(x, \mathfrak{g}_0) = 0$.

It follows that $K(x, \mathfrak{g}) = 0$, which is absurd since K is non-degenerate. Therefore we can find $0 \neq y \in \mathfrak{g}_{-\alpha}$ for which $K(x, y) \neq 0$.

5. Suppose $K(t_\alpha, t_\alpha) = 0 \Leftrightarrow \alpha(t_\alpha) = 0$, so that $[t_\alpha, x] = 0 = [t_\alpha, y]$ for all $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$.
As in 4., we can find such x_α, y_α satisfying $K(x_\alpha, y_\alpha) \neq 0$. Modifying one or the other by a scalar, we may as well assume that $K(x_\alpha, y_\alpha) = 1$. Then $[x_\alpha, y_\alpha] = t_\alpha$, by 3. It follows that the subspace S of $\mathfrak{g}$ spanned by $x_\alpha, y_\alpha, t_\alpha$ is a three dimensional solvable algebra. Since S is solvable, $\mathrm{ad}(S) \in \mathfrak{gl}(\mathfrak{g})$ is solvable because it is the homomorphic image of S with respect to ad: $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Then there exists a basis of $\mathfrak{g}$ such that the elements ad_s, with $s \in S$, have upper triangular matrix. In particular, since $\mathrm{ad}_{t_\alpha} = [\mathrm{ad}_{x_\alpha}, \mathrm{ad}_{y_\alpha}]$, ad_{t_α} has a strictly upper triangular matrix, i.e., ad_{t_α} is nilpotent. But $t_\alpha \in \mathfrak{h}$, then ad_{t_α} is semisimple.
So ad_{t_α} is both semisimple and nilpotent, i.e., $\mathrm{ad}_{t_\alpha} = 0$. By injectivity of ad_{t_α}, this says that $t_\alpha = 0$, contrary to choice of t_α.

6. We want to prove that the following linear map:

$$\begin{array}{rccc} f\colon & \langle x_\alpha, x_{-\alpha}, h_\alpha \rangle & \to & \mathfrak{sl}_2 \\ & x_\alpha & \mapsto & e \\ & x_{-\alpha} & \mapsto & f \\ & h_\alpha & \mapsto & h \end{array}$$

where $\{e,f,h\}$ is the standard basis of $\mathfrak{sl}_2$, is an isomorphism of Lie algebras.
Given $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, by 4. and 5. we can find $y \in \mathfrak{g}_{-\alpha}$ such that $K(x_\alpha, y_\alpha) = \dfrac{2}{K(t_\alpha, t_\alpha)}$. Set $h_\alpha := \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$, then $[x_\alpha, y_\alpha] = h_\alpha$, by 3. Moreover,

$$[h_\alpha, x_\alpha] = \frac{2}{K(t_\alpha, t_\alpha)}[t_\alpha, x_\alpha] = \frac{2}{K(t_\alpha, t_\alpha)}\alpha(t_\alpha)x_\alpha = \frac{2}{\alpha(t_\alpha)}\alpha(t_\alpha)x_\alpha = 2x_\alpha$$

$$[h_\alpha, y_\alpha] = \frac{2}{K(t_\alpha, t_\alpha)}[t_\alpha, y_\alpha] = -\frac{2}{K(t_\alpha, t_\alpha)}\alpha(t_\alpha)y_\alpha = -\frac{2}{\alpha(t_\alpha)}\alpha(t_\alpha)y_\alpha = -2y_\alpha$$

7. Point 6. shows that $h_\alpha = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$. So we only have to prove that $h_{-\alpha} = -h_\alpha$.
   Recall that t_α is defined by $K(t_\alpha, h) = \alpha(h)$ $(h \in \mathfrak{h})$. This shows that $t_{-\alpha} = -t_\alpha$ and in view of the way h_α is defined, the assertion follows.

$\square$

Proposition 1.10.12. *1. $\alpha \in \Phi$ implies dim $\mathfrak{g}_\alpha = 1$.*

2. If $\alpha \in \Phi$, the only scalar multiples of α which are roots are α and $-\alpha$.

3. If $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$, where h_α is the element introduced in Theorem 1.10.11. The numbers $\beta(h_\alpha)$ are called Cartan integers.

4. If $\alpha, \beta, \alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

5. Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Let r, q be (respectively) the largest integers for which $\beta - r\alpha, \beta + q\alpha$ are roots. Then $\beta + k\alpha \in \Phi$ for all $k = -r, \ldots, q$, and $\beta(h_\alpha) = r - q$.

Proof. Fix $\alpha \in \Phi$. Consider the subspace M of $\mathfrak{g}$ spanned by $\mathfrak{h}$ along with all root spaces of the form $\mathfrak{g}_{c\alpha}$ $(c \in \mathbb{F}^*)$, i.e.:

$$M = \mathfrak{h} \oplus \bigoplus_{\substack{c\alpha \in \Phi \\ c \in \mathbb{F}^*}} \mathfrak{g}_{c\alpha} \text{ with } \alpha \in \Phi.$$

For sure $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are in this sum by point 2. of Proposition 1.10.11. Now, take $x_\alpha \in \mathfrak{g}$ and consider $S_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle \cong \mathfrak{sl}_2$ the subalgebra of $\mathfrak{g}$ constructed as in point 6. of Proposition 1.10.11. Note that M is an S_α-submodule of $\mathfrak{g}$ with respect to the adjoint action. Indeed, by point 1. of Proposition 1.10.7:

- $[h_\alpha, \mathfrak{h}] = 0$ since $\mathfrak{h}$ is abelian and $[h_\alpha, \mathfrak{g}_{c\alpha}] \subseteq (c\alpha)(h_\alpha)\mathfrak{g}_{c\alpha}$.
  Hence $\mathrm{ad}_{h_\alpha}(M) \subseteq M$.

- $[x_\alpha, h] = -\alpha(h)x_\alpha$ for all $h \in \mathfrak{h}$ and $[x_\alpha, \mathfrak{g}_{c\alpha}] \subseteq \mathfrak{g}_{(c+1)\alpha}$.
  Hence $\mathrm{ad}_{x_\alpha}(M) \subseteq M$.

- $[y_\alpha, h] = \alpha(h)y_\alpha$ for all $h \in \mathfrak{h}$ and $[y_\alpha, \mathfrak{g}_{c\alpha}] \subseteq \mathfrak{g}_{(c-1)\alpha}$.
  Hence $\mathrm{ad}_{y_\alpha}(M) \subseteq M$.

Therefore M is a finite dimensional $(S_\alpha \cong \mathfrak{sl}_2)$-module. The weights of h_α on M are 0 and $2c = c\alpha(h_\alpha)$ for nonzero c's such that $\mathfrak{g}_{c\alpha} \neq 0$. In particular, since all weights of an $\mathfrak{sl}_2$-module are integers, $2c \in \mathbb{Z}$, i.e., c must be an integral multiple of $\frac{1}{2}$. On the other hand, $\alpha \in \Phi \subset \mathfrak{h}^*$ is such that $\alpha : \mathfrak{h} \to \mathbb{F}$ with $\alpha(h_\alpha) = 2$ and $\alpha \neq 0$. Hence $\mathfrak{h} = \ker\alpha \oplus \langle h_\alpha \rangle$. Note that S_α acts trivially on $\ker\alpha$. Indeed, for $z \in \ker\alpha \subset \mathfrak{h}$:

- $[h_\alpha, z] = 0$ since $\mathfrak{h}$ is abelian;
- $[x_\alpha, z] = -[z, x_\alpha] = -\alpha(z)x_\alpha = 0$ since $z \in \ker\alpha$;
- $[y_\alpha, z] = -[z, y_\alpha] = \alpha(z)y_\alpha = 0$ since $z \in \ker\alpha$;

Hence $\ker\alpha$ is a trivial S_α-submodule of M. Moreover, also S_α is an S_α-submodule of M. Therefore, thanks to Weyl's Theorem (Theorem 1.9.22) we have:

$$M = \ker\alpha \oplus S_\alpha \oplus T,$$

where $T \subseteq \bigoplus_{\substack{c\alpha \in \Phi \\ c \neq 0}} \mathfrak{g}_{c\alpha}$ is an S_α-submodule.

Recall that the weights of an irreducible $\mathfrak{sl}_2$-module form an arithmetic progression with difference 2 from $-s$ to s for some $s \in \mathbb{Z}^+$. Hence in this progression either 0 or 1 must appear. Now, the weights on T are $2c \neq 0$. This means that even weights can not appear in T, since 0 does not appear. In particular $c \neq 2$, i.e., $2\alpha \notin \Phi$. So $\frac{\alpha}{2} \notin \Phi$ (because otherwise $\alpha \notin \Phi$ against assumption). Then $c \neq \frac{1}{2}$. Hence, also 1 can not be a weight of T. Therefore $T = 0$.

In this way we proved that $M = \ker\alpha \oplus S_\alpha$. This proves points 1. and 2.

Next we examine how S_α acts on root spaces $\mathfrak{g}_\beta$, $\beta \neq \pm\alpha$. Set

$$K = \bigoplus_{\substack{\beta + k\alpha \in \Phi \\ k \in \mathbb{Z}}} \mathfrak{g}_{\beta + k\alpha}.$$

As before, one can show that K is a S_α-submodule of $\mathfrak{g}$ with respect to the adjoint action. The weights of h_α on K are $(\beta + k\alpha)(h_\alpha) = \beta(h_\alpha) + 2k$. On the other hand $\beta(h_\alpha) + 2k \in \mathbb{Z}$ since all the weights of an $\mathfrak{sl}_2$-module are integers; hence $\beta(h_\alpha) \in \mathbb{Z}$. Using Weyl's Theorem we can write $K = \oplus V_i$, where V_i are irreducible $\mathfrak{sl}_2$-modules, and $\#\{V_i\} = \mu(0) + \mu(1)$, where $\mu(i)$ is the number of times i appears as weight of K. But the weights $\beta(h_\alpha) + 2k$ are all even or all odd and 0 or 1 appears only once. Hence $K = V_i$ for some i, i.e., K is irreducible.
Denote by q the maximum non-negative integer such that $\beta + q\alpha \in \Phi$, and by r the maximum non-negative integer such that $\beta - r\alpha \in \Phi$. Hence the weights of K form an arithmetic progression with difference 2:

$$\beta(h_\alpha) - 2r, \dots, \beta(h_\alpha) + 2q.$$

This implies that the roots $\beta + k\alpha$ form a string $\beta - r\alpha, \dots, \beta, \dots, \beta + q\alpha$. Notice also that $(\beta - r\alpha)(h_\alpha) = (-\beta + q\alpha)(h_\alpha)$; this implies that $\beta(h_\alpha) = r - q$.
Finally, observe that if $\alpha, \beta, \alpha + \beta \in \Phi$, then $\mathrm{ad}_{\mathfrak{g}_\alpha}$ maps $\mathfrak{g}_\beta$ onto $\mathfrak{g}_{\alpha+\beta}$. Otherwise, we could assume that $[x_\alpha, \mathfrak{g}_\beta] = 0$. But then $\bigoplus_{k \leq 0} \mathfrak{g}_{\beta+k\alpha}$ would be a proper S_α-submodule of K, against its irreducibility. Indeed for all $k \leq 0$: $[x_\alpha, \mathfrak{g}_{\beta+k\alpha}] \subseteq \mathfrak{g}_{\beta+(k+1)\alpha} \subseteq \bigoplus_{k \leq 0} \mathfrak{g}_{\beta+k\alpha}$ because, if $k < 0$, then $k + 1 \leq 0$ and $[x_\alpha, \mathfrak{g}_\beta] = 0$ for $k = 0$.
Analogously, we can prove that also h_α and y_α stabilize $\bigoplus_{k \leq 0} \mathfrak{g}_{\beta+k\alpha}$. □

Remark 1.10.13. Since the restriction to $\mathfrak{h}$ of the Killing form is non-degenerate by Remark 1.10.10, we may transfer the form to $\mathfrak{h}^*$, letting $(\alpha, \beta) := K(t_\alpha, t_\beta)$ for all $\alpha, \beta \in \mathfrak{h}^*$. This is a non-degenerate bilinear form on $\mathfrak{h}^*$, with $(\alpha, \beta) \in \mathbb{Q}$ for every $\alpha, \beta \in \mathfrak{h}^*$. We know that Φ spans $\mathfrak{h}^*$ by point 1. of Theorem 1.10.11, so we can choose a basis $\{\alpha_1, \dots, \alpha_l\}$ of $\mathfrak{h}^*$ consisting of roots. If we set $E_\mathbb{Q} := \mathrm{span}_\mathbb{Q}\{\alpha_1, \dots, \alpha_l\}$, we can prove that $(.,.)$ is a positive definite form on $E_\mathbb{Q}$. Now, let $E_\mathbb{R}$ be the real vector space obtained by extending the base field from $\mathbb{Q}$ to $\mathbb{R}$ (i.e., $E_\mathbb{R} := \mathbb{R} \otimes E_\mathbb{Q}$); the form extends canonically to $E_\mathbb{R}$ and is positive definite.

1.11 Root systems

Throughout this section, E will denote a Euclidean space, i.e., a finite dimensional vector space over $\mathbb{R}$ endowed with a positive definite symmetric bilinear form $(.,.)$. Moreover, α will denote a nonzero element in E.

Definition 1.11.1. Define the *reflecting hyperplane* of α as follows:

$$P_\alpha = \{v \in E \mid (v, \alpha) = 0\}.$$

Definition 1.11.2. The invertible linear transformation $\sigma_\alpha : E \to E$ such that $\sigma_\alpha(v) = v$ for all $v \in P_\alpha$ and $\sigma_\alpha(\alpha) = -\alpha$ is called the *reflecion* with respect to P_α.

Remark 1.11.3. It is easy to write down an explicit formula for σ_α:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

Definition 1.11.4. Let $\alpha, \beta \in E$. We define

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Definition 1.11.5. A subset Φ of the Euclidean space E is called an *abstract root system* in E if the following axioms are satisfied:

(R1) Φ is finite, spans E and does not contain 0.

(R2) If $\alpha \in \Phi$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$.

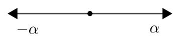
(R3) For all $\alpha, \beta \in \Phi$, $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in \Phi$.

(R4) For all $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

Example 1.11.6. The root system of a semisimple Lie algebra is an abstract root system in $E_{\mathbb{R}}$.

Definition 1.11.7. Call $\ell := \dim E$ the *rank* of the root system Φ.

Example 1.11.8. Let us describe the root systems of rank 1.
For $\ell = 1$, we have $E \cong \mathbb{R} \supseteq \Phi$. Thus, by property (R2) of Definition 1.11.5, $\Phi = \{\pm\alpha\}$, with $\alpha \neq 0$.
We can represent it with the following diagram, labeled by A_1:

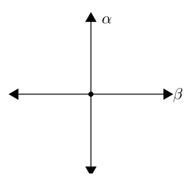


Example 1.11.9. Let us describe the root systems of rank 2.
For $\ell = 2$, we have $E \cong \mathbb{R}^2 \supseteq \Phi$. Thus, by property (R1) of Definition 1.11.5, we can consider $\alpha, \beta \in \Phi$, with $\beta \neq \pm\alpha$.
Recall that $(\alpha, \beta) = \|\alpha\|\|\beta\| \cos \widehat{\alpha\beta}$. Thus, $\langle \alpha, \beta \rangle = \dfrac{2(\alpha, \beta)}{(\beta, \beta)} = \dfrac{2\|\alpha\|}{\|\beta\|} \cos \widehat{\alpha\beta}$.
So, since $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ have like sign and $\langle \alpha, \beta \rangle \in \mathbb{Z}$ by property (R4) of Definition 1.11.5,

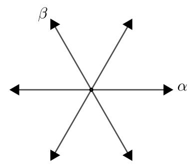
$$0 \leq \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \cos^2 \widehat{\alpha\beta} \leq 3,$$

where the last inequality holds because $\beta \neq \pm\alpha$.
From now on, suppose that $\|\beta\| \geq \|\alpha\|$. Then, according to the choice of $\cos^2 \widehat{\alpha\beta} \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, we have the following possibilities that can be obtained using the axioms of the abstract root systems.

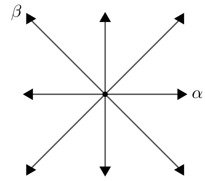
- If $\cos^2 \widehat{\alpha\beta} = 0$, then $\Phi = \{\pm\alpha, \pm\beta\}$ and it can be represented by the following diagram, labeled by $A_1 \times A_1$:



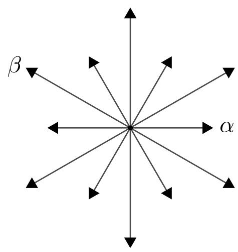
- If $\cos^2 \widehat{\alpha\beta} = \frac{1}{4}$, then $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ and it can be represented by the following diagram, labeled by A_2:



- If $\cos^2 \widehat{\alpha\beta} = \frac{1}{2}$, then $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$ and it can be represented by the following diagram, labeled by B_2:



- If $\cos^2 \widehat{\alpha\beta} = \frac{3}{4}$, then $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$ and it can be represented by the following diagram, labeled by G_2:



Therefore, the following possibilities are the only ones when $\beta \neq \pm\alpha$ and $\|\beta\| \geq \|\alpha\|$.

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$\widehat{\alpha\beta}$	$\dfrac{\|\beta\|^2}{\|\alpha\|^2}$
0	0	$\frac{\pi}{2}$	undetermined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2}{3}\pi$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3}{4}\pi$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5}{6}\pi$	3

Table 1.

Proposition 1.11.10. *Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Then:*

1. *if $(\alpha, \beta) > 0$, $\alpha - \beta \in \Phi$;*
2. *if $(\alpha, \beta) < 0$, $\alpha + \beta \in \Phi$.*

Proof. 1. Since $(\alpha, \beta) > 0$ if and only if $\langle \alpha, \beta \rangle > 0$, Table 1 shows that either $\langle \alpha, \beta \rangle = 1$ (if $\|\beta\| \geq \|\alpha\|$) or $\langle \beta, \alpha \rangle = 1$ (if $\|\alpha\| \geq \|\beta\|$).
In the first case $\sigma_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta = \alpha - \beta \in \Phi$ by property (R3).
In the second case $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta - \alpha \in \Phi$ by property (R3).
Thus $\alpha - \beta \in \Phi$ by property (R2).

2. The second assertion follows from the first applied to $-\beta$ in place of β.

□

Definition 1.11.11. A subset Δ of Φ is called a *base* if:

(B1) Δ is a basis of E;

(B2) each root $\beta \in \Phi$ can be written as $\beta = \sum_{\gamma \in \Delta} n_\gamma \gamma$, with integral coefficients n_γ all non-negative or all non-positive.

The roots in Δ are called *simple*. We call $\beta \in \Phi$ *positive* (resp. *negative*) if $n_\gamma \geq 0$ (resp. $n_\gamma \leq 0$) for every $\gamma \in \Delta$.

Theorem 1.11.12. *Every root system Φ has a base.*

Proof. See [5], 10.1. □

Definition 1.11.13. Let $(\Phi, E_{\mathbb{R}})$ be a root system and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a base of Φ. The *Coxeter graph* associated to (Φ, Δ) is defined as a graph having l vertices in which the i^{th} and the j^{th} vertix ($i \neq j$) are linked by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges.

Definition 1.11.14. In the same setting as before, we call *Dynkin diagram* of Φ the Coxeter graph of Φ in which we add an arrow from the i^{th} to the j^{th} vertix if $(\alpha_i, \alpha_i) > (\alpha_j, \alpha_j)$.

Chapter 2

Reductive Lie algebras

2.1 Basic results on reductive Lie algebras

Definition 2.1.1. A Lie algebra $\mathfrak{g}$ for which $\mathrm{Rad}(\mathfrak{g}) = Z(\mathfrak{g})$ is called *reductive.*

Example 2.1.2.

1. A commutative Lie algebra $\mathfrak{g}$ is reductive, since $\mathrm{Rad}(\mathfrak{g}) = \mathfrak{g} = Z(\mathfrak{g})$.

2. If $\mathfrak{g}$ is a semisimple Lie algebra, then it is reductive since $\mathrm{Rad}(\mathfrak{g}) = \{0\} = Z(\mathfrak{g})$.

3. $\mathfrak{gl}_n$ is reductive, since $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \langle I_n \rangle$, and $Z(\mathfrak{gl}_n) = \langle I_n \rangle = \mathrm{Rad}(\mathfrak{gl}_n)$.

Proposition 2.1.3.

1. *$\mathfrak{g}$ is a reductive Lie algebra if and only if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$, with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.*

2. *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a nonzero Lie algebra acting irreducibly on V (via the natural action). Then $\mathfrak{g}$ is reductive, with $dim Z(\mathfrak{g}) \leq 1$. If in addition $\mathfrak{g} \subseteq \mathfrak{sl}(V)$, then $\mathfrak{g}$ is semisimple.*

Proof. 1. By definition of reductive Lie algebra we have $\mathrm{Rad}(\mathfrak{g}) = Z(\mathfrak{g})$; hence $\mathfrak{g}' := \mathfrak{g}/Z(\mathfrak{g})$ is semisimple.

The adjoint action induces an action of $\mathfrak{g}'$ on $\mathfrak{g}$: for $\bar{x} := x + Z(\mathfrak{g}) \in \mathfrak{g}'$, we consider $\mathrm{ad}_{\bar{x}} : \mathfrak{g} \to \mathfrak{g}$ such that $y \mapsto [x, y]$ (it is easy to see that it is well defined).

Since $\mathfrak{g}'$ is semisimple, $\mathfrak{g}$ is a completely reducible $\mathfrak{g}'$-module by Weyl's theorem (Theorem 1.9.22). Hence we can write $\mathfrak{g} = Z(\mathfrak{g}) \oplus M$, where $Z(\mathfrak{g})$ is a trivial $\mathfrak{g}'$-submodule of $\mathfrak{g}$ because it is an ideal of $\mathfrak{g}$, and M is an ideal of $\mathfrak{g}$ by definition of $\mathrm{ad}_{\bar{x}}$. Moreover, $[\mathfrak{g}, \mathfrak{g}] = [M, M] = M$ by Corollary 1.9.8 because $M \cong \mathfrak{g}'$, that is semisimple.

The converse is true since, if we consider any solvable ideal $I \subset [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$, then it must be $I \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$ since otherwise $I \cap [\mathfrak{g}, \mathfrak{g}]$ would be a nonzero solvable ideal in $[\mathfrak{g}, \mathfrak{g}]$, that is semisimple. So $I \subseteq Z(\mathfrak{g})$ and hence $\mathrm{Rad}(\mathfrak{g}) \subseteq Z(\mathfrak{g})$.

2. Let $S = \mathrm{Rad}\mathfrak{g}$. By Lie's theorem (Theorem 1.6.8), there exists an eigenvector $v \in V$ common to all the elements of S, i.e., such that $s.v = \lambda(s)v$ for all $s \in S$. Now, if $x \in \mathfrak{g}$ then $[x, s] \in S$; thus

$$s.(x.v) = x.(s.v) - [s, x].v = \lambda(s)x.v - \lambda([s, x])v. \tag{2.1}$$

Since $\mathfrak{g}$ acts irreducibly on V, all vectors in V are obtainable by repeated applications of elements of $\mathfrak{g}$ to v and formation of linear combinations. It therefore follows from (2.1) that the matrices of all $s \in S$ (relative to a suitable basis of V) are be upper triangular, with $\lambda(s)$ the only diagonal entry. However, the commutators $[s, x] \in S$ ($s \in S$, $x \in \mathfrak{g}$) have trace 0, so this condition forces λ to vanish on $[S, \mathfrak{g}]$. Referring back to (2.1), we now conclude that $s \in S$ acts diagonally on V as the scalars $\lambda(s)$. In particular, $S = Z(\mathfrak{g})$; so $\mathfrak{g}$ is reductive and $\dim S \leq 1$.

Finally, if $\mathfrak{g} \subseteq \mathfrak{sl}(V)$, since $\mathfrak{sl}(V)$ contains no scalars except 0, $S = 0$ and thus $\mathfrak{g}$ is semisimple.

□

Proposition 2.1.4. *Let $n \in \mathbb{N}$. Then:*

1. *$\mathfrak{sl}_n$ is semisimple;*

2. *$\mathfrak{sp}_{2n}$ is semisimple.*

Proof. 1. Let V be an n-dimensional vector space. Since $\mathfrak{gl}(V) = \mathfrak{sl}(V) \oplus \langle I_V \rangle$ and since $\mathfrak{gl}(V)$ acts irreducibly on V (see Example 1.3.3), then it is clear that $\mathfrak{sl}(V)$ acts irreducibly as well. Thus, by point 2. of Proposition 2.1.3, $\mathfrak{sl}_n$ is semisimple.

2. Let V be a $2n$-dimensional vector space. Notice that any subspace W of V which is invariant under the action of a subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is also invariant under the action of the (associative) subalgebra of End$V$ generated by I_V and $\mathfrak{g}$. Indeed, if $w \in W$ then $(\alpha I_V + \sum_i \beta_i x_i)(w) = \alpha w + \sum_i \beta_i x_i(w) \in W$ for all $x_i \in \mathfrak{g}$, $\alpha, \beta_i \in \mathbb{F}$. We now want to prove that all the endomorphisms in V are obtainable from I_V and $\mathfrak{sp}_{2n}$ using addiction, scalar multiplication and ordinary multiplication. From I_V we get all scalars. Now, $E_{ii} = \frac{1}{2}((E_{ii} - E_{i+n,i+n}) + I_{2n})(E_{ii} - E_{i+n,i+n})$ for all $i = 1, \dots, n$ and similarly for $i = n+1, \dots, 2n$. Therefore we get all possible diagonal matrices. Now, multiplying various other basis elements (such as $E_{ij} - E_{ji}$) by suitable E_{ii} yields all the possible off-diagonal matrices E_{ij}.
Thus, using Example 1.3.3 combined with point 2. of Proposition 2.1.3, we get that $\mathfrak{sp}_{2n}$ is semisimple.

$\square$

Proposition 2.1.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $x \in \mathfrak{g}$ a semisimple element. Then $C_{\mathfrak{g}}(x)$ is reductive. Furthermore, if $\mathfrak{h} \subset \mathfrak{g}$ is a maximal toral subalgebra containing x, $C_{\mathfrak{g}}(x) = \mathfrak{h} \oplus \oplus_{\alpha \in \Phi_x} \mathfrak{g}_\alpha$, where $\Phi_x = \{\alpha \in \Phi \mid \alpha(x) = 0\}$ and Φ is the root system of $\mathfrak{g}$.*

Proof. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. On one hand, if $w \in C_{\mathfrak{g}}(x)$ lies in $\mathfrak{g}_\alpha$ for some $\alpha \in \Phi$ then, by definition, $[h, w] = \alpha(h)w$ for all $h \in \mathfrak{h}$. But the element $x \in \mathfrak{h}$ must centralize w, and so we get $\alpha(x)w = 0$. Let $\Phi_x = \{\alpha \in \Phi \mid \alpha(x) = 0\}$. These remarks show that

$$C_{\mathfrak{g}}(x) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_x} \mathfrak{g}_\alpha.$$

One can easily show that Φ_x satisfies the axioms of a root system; hence we can consider a base of simple roots Δ_x for Φ_x.
In order to show that $C_{\mathfrak{g}}(x)$ is reductive, we may show (using point 1. of Proposition 2.1.3) the decomposition $[C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)] \oplus Z(C_{\mathfrak{g}}(x))$. Begin by defining

$$\mathfrak{h}_1 := \bigcap_{\alpha \in \Phi_x} \ker\alpha.$$

Since the span of the roots in Φ_x has dimension equal to the rank of the root system Φ_x by Definition 1.11.7 and Remark 1.10.13, we get that $\dim(\mathfrak{h}_1) = \dim(\mathfrak{h}) - \mathrm{rank}(\Phi_x)$. For each $\alpha \in \Phi$, pick elements $h_\alpha \in \mathfrak{h}$ as in point 6. of Theorem 1.10.11. Now define

$$\mathfrak{h}_2 := \mathrm{span}_{\mathbb{F}}\{h_\alpha \mid \alpha \in \Delta_x\}.$$

Clearly $\dim(\mathfrak{h}_2) = \mathrm{rank}(\Phi_x)$. We have now the following refined decomposition of $C_{\mathfrak{g}}(x)$:

$$C_{\mathfrak{g}}(x) = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \bigoplus_{\alpha \in \Phi_x} \mathfrak{g}_\alpha.$$

Finally we can say that:

- $Z(C_{\mathfrak{g}}(x)) = \mathfrak{h}_1$. Indeed, if we take $h_1 \in \mathfrak{h}_1$ then $[h_1, \mathfrak{h}_1] = [h_1, \mathfrak{h}_2] = 0$ since toral subalgebras are commutative by Lemma 1.10.5. Furthermore, if $z_\alpha \in \mathfrak{g}_\alpha$, then $[h_1, z_\alpha] = \alpha(h_1)z_\alpha = 0$. Hence $\mathfrak{h}_1 \subseteq Z(C_{\mathfrak{g}}(x))$. The converse follows using standard properties of root systems.

- $[C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)] = \mathfrak{h}_2 \oplus \oplus_{\alpha \in \Phi_x} \mathfrak{g}_\alpha$. Indeed we know that $[\mathfrak{h}_1, C_{\mathfrak{g}}(x)] = 0$. Furthermore $[\mathfrak{h}_2, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \langle h_\alpha \rangle$ for all $\alpha \in \Delta_x$.

Notice that $K([C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)], \mathfrak{h}_1) = K(C_{\mathfrak{g}}(x), [C_{\mathfrak{g}}(x), \mathfrak{h}_1]) = 0$. Hence the Killing form of $[C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)]$ is non-degenerate because otherwise it would result $K([C_{\mathfrak{g}}(x), C_{\mathfrak{g}}(x)], \mathfrak{g}) = 0$, against the semisemplicity of $\mathfrak{g}$ (see Theorem 1.9.5). Thus, by point 1. of Proposition 2.1.3, $C_{\mathfrak{g}}(x)$ is reductive. $\square$

Lemma 2.1.6. *Let $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible representation of $\mathfrak{g}$ on V (with $\dim V < \infty$). Then $\Phi([\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]) = 0$.*

Proof. We know that $\mathrm{Rad}(\mathfrak{g})$ is solvable; then $\Phi(\mathrm{Rad}(\mathfrak{g})) \subset \mathfrak{gl}(V)$ is solvable by Proposition 1.6.4. Hence, by Lie's Theorem, there exists a nonzero $v \in V$ such that $\Phi(h)(v) = \lambda(h)v$ for every $h \in \mathrm{Rad}(\mathfrak{g})$, where $\lambda \in \mathrm{Rad}(\mathfrak{g})^*$.
Now, let $V_\lambda = \{w \in V : \Phi(h)(w) = h.w = \lambda(h)w \text{ for all } h \in \mathrm{Rad}(\mathfrak{g})\} \neq 0$. We want to show that $\mathfrak{g}.V_\lambda \subset V_\lambda$. If $x \in \mathfrak{g}$, $h \in \mathrm{Rad}(\mathfrak{g})$, $w \in V_\lambda$, then:

$$h.x.w = [h,x].w + x.h.w = \lambda([h,x])w + \lambda(h)x.w \tag{2.2}$$

Now we want to show that $\lambda([h,x]) = 0$. Let $W_n = \mathrm{span}_{\mathbb{F}}\{v, x.v, \dots, x^{n-1}.v\}$, where $n \in \mathbb{N}$ is the minimum such that $\{v, x.v, \dots, x^n.v\}$ are linearly dependent. Denote $W_i = \mathrm{span}_{\mathbb{F}}\{v, x.v, \dots, x^{i-1}.v\}$, for $i = 1, \dots, n$. We claim that $h.x^r.v = \lambda(h)x^r.v + \omega_r$, where $\omega_r \in W_r$. Indeed, by induction on r:

- if $r = 1$ we are in the case of (2.2);
- if $r > 1$,
$h.(x^{r+1}.v) = h.(x.(x^r.v)) = [h,x].(x^r.v) + x.(h.(x^r.v)) = \lambda([h,x])x^r.v + \omega_r + \lambda(h)x^{r+1}.v + x.\omega'_r$, where $\omega_r, \omega'_r \in W_r$. Hence $h.(x^{r+1}.v) = \lambda(h)x^{r+1}.v + \omega_{r+1}$, where $\omega_{r+1} \in W_{r+1}$.

Now, let $h \in \mathrm{Rad}(\mathfrak{g})$. Consider $h_{|W_n} : W_n \to W_n$; its matrix, with respect to the basis $\{v, x.v, \dots, x^{n-1}.v\}$ is:

$$\begin{pmatrix} \lambda(h) & * & \dots & * \\ 0 & \lambda(h) & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda(h) \end{pmatrix}$$

Hence the trace of $h_{|W_n}$ is $n\lambda(h)$. In particular, every element of $\mathfrak{g}$ of the form $[h,x] \in \mathfrak{g}$ has trace $n\lambda([h,x])$; but the trace of a commutator is 0, and so $\lambda([h,x]) = 0$.
Hence, by (2.2) we obtain that $h.x.v = \lambda(h)x.v$. So V_λ is a submodule of V and, since $V_\lambda \neq 0$ and V is irreducible, it holds $V = V_\lambda$, that implies $\Phi(h) = \lambda(h)\mathrm{id}_V$ for every $h \in \mathrm{Rad}(\mathfrak{g})$.
Now, if $h = [x, h'] \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$, then $\Phi(h)v = \Phi(x)\Phi(h')v - \Phi(h')\Phi(x)v = \lambda(h')\Phi(x)v - \lambda(h')\Phi(x)v = 0$ for every $v \in V$. Hence $\Phi([\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]) = 0$. $\square$

Theorem 2.1.7. *Let $\Phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V (with* $\dim V < \infty$*), and let β_Φ be the associated trace form on $\mathfrak{g}$. If β_Φ is non-degenerate, then $\mathfrak{g}$ is reductive.*

Proof. We can construct a sequence of $\mathfrak{g}$-submodules of V:

$$\{0\} = V_0 \subset V_1 \subset \dots V_t = V$$

such that V_i/V_{i-1} is irreducible. Indeed:

- if V is irreducible, we take $V_0 = \{0\}$, $V_1 = V$;
- if V is not irreducible, we take W to be a maximal $\mathfrak{g}$-submodule of V. Then V/W is irreducible and we can construct the sequence of $\mathfrak{g}$-submodules of V above iterating this procedure.

By Lemma 2.1.6, $[\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$ acts trivially on V_i/V_{i-1}, i.e., for every $x \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$ we have $x.V_i \subseteq V_{i-1}$. But the V_i's are $\mathfrak{g}$-modules, and hence we can say that $y.x.V_i \subseteq V_{i-1}$ for every $y \in \mathfrak{g}$. So, if we take a basis of V obtained by completing a basis of V_{i-1} to a basis of V_i for every $i = 1, \dots, t$, we have that the matrix associated to $\Phi(y)\Phi(x)$ is strictly upper triangular, and hence $\beta_\Phi(y, x) = \mathrm{tr}(\Phi(y)\Phi(x)) = 0$ for every $y \in \mathfrak{g}$, $x \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$. But β_Φ is non-degenerate by assumption, hence $[\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})] = 0$. So $\mathrm{Rad}(\mathfrak{g}) \subseteq Z(\mathfrak{g})$ and, since the converse is always true, $\mathfrak{g}$ is reductive. □

2.2 The Jacobson Morozov theorem

Remark 2.2.1. Let $A \in \mathfrak{gl}(V)$, where V is a finite dimensional vector space. Then A is nilpotent if and only if $\mathrm{tr}(A^k) = 0$ for all $k \in \mathbb{Z}^+$.

Proof. If A is nilpotent, then $\mathrm{tr}(A^k) = 0$ for all $k \in \mathbb{Z}^+$ because all eigenvalues of A are 0, and hence so are all eigenvalues of A^k.
Conversely, suppose that $\mathrm{tr}(A^k) = 0$ for all $k \in \mathbb{Z}^+$. By contradiction, suppose that A is not nilpotent, with nonzero eigenvalues $\lambda_1, \dots, \lambda_r$ and

corresponding multiplicities $m_1, \ldots, m_r$. Then $\mathrm{tr}(A^k) = m_1\lambda_1^k + \ldots + m_r\lambda_r^k$ for every k. Hence, we have:

$$\begin{cases} m_1\lambda_1 + \ldots + m_r\lambda_r = 0 \\ \qquad\vdots \\ m_1\lambda_1^r + \ldots + m_r\lambda_r^r = 0 \end{cases} \tag{2.3}$$

i.e.,

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

But

$$\det \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{pmatrix} = \lambda_1 \cdot \ldots \cdot \lambda_r \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \cdots & \lambda_r^{r-1} \end{pmatrix}.$$

This is the determinant of the Vandermonde matrix, that is nonzero. Hence the unique solution of (2.3) is $m_1 = \ldots = m_r = 0$, that is absurd. □

Lemma 2.2.2. *Let $C \in \mathfrak{gl}(V)$, where V is a finite-dimensional vector space. Suppose that $C = \sum_{i=1}^{r}[A_i, B_i]$ (with $A_i, B_i \in \mathfrak{gl}(V)$) and that $[C, A_i] = 0$ for $i = 1, 2, \ldots r$. Then C is nilpotent.*

Proof. For $i \in \{1, \ldots r\}$, we have $[C^{k-1}, A_i] = 0$ for $k \geq 1$ where $C^0 = Id_V$, indeed:

$$\begin{aligned} [C^{k-1}, A_i] &= [\underbrace{C \cdot \ldots \cdot C}_{k-1}, A_i] = \underbrace{C \cdot \ldots \cdot C}_{k-1} A_i - A_i \underbrace{C \cdot \ldots \cdot C}_{k-1} = \\ &= \underbrace{C \cdot \ldots \cdot C}_{k-2} A_i C - A_i \underbrace{C \cdot \ldots \cdot C}_{k-1} = \\ &= \ldots = A_i \underbrace{C \cdot \ldots \cdot C}_{k-1} - A_i \underbrace{C \cdot \ldots \cdot C}_{k-1} = 0 \end{aligned}$$

Hence
$C^k = C^{k-1}C = \sum_{i=1}^{r} C^{k-1}[A_i, B_i] = \sum_{i=1}^{r} C^{k-1}(A_iB_i - B_iA_i) = \sum_{i=1}^{r}(A_i(C^{k-1}B_i) - (C^{k-1}B_i)A_i) = \sum_{i=1}^{r}[A_i, C^{k-1}B_i]$.
Since the trace of any commutator is 0, this gives $\mathrm{tr}(C^k) = 0$ for $k \geq 1$. Hence C is nilpotent by Remark 2.2.1. □

Lemma 2.2.3 (Morozov). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0. Suppose that there exist $e, h \in \mathfrak{g}$ such that $[h, e] = 2e$ and $h \in [e, \mathfrak{g}]$. Then there exists $f \in \mathfrak{g}$ such that $[h, f] = -2f$, $[e, f] = h$ (and $[h, e] = 2e$).*

Proof. By assumption $h \in [e, \mathfrak{g}]$, then there exists an element $z \in \mathfrak{g}$ such that $h = [e, z]$. Moreover, since ad is a homomorphism of Lie algebras, we have that:

- $[\mathrm{ad}_h, \mathrm{ad}_e] = \mathrm{ad}_{[h,e]} = \mathrm{ad}_{2e} = 2\mathrm{ad}_e$;
- $[\mathrm{ad}_e, \mathrm{ad}_z] = \mathrm{ad}_{[e,z]} = \mathrm{ad}_h$.

The first of these relations together with Lemma 2.2.2 implies that ad_e is nilpotent. Besides,

$$\begin{aligned}[e, [h, z] + 2z] &= [e, [h, z]] + 2[e, z] = [[e, h], z] - [h, [e, z]] + 2h \\ &= [-2e, z] - [h, h] + 2h = -2h + 2h = 0\end{aligned}$$

Hence $[h, z] = -2z + x_1$, where $x_1 \in C_{\mathfrak{g}}(e)$, the centralizer of e in $\mathfrak{g}$.
Since $[\mathrm{ad}_h, \mathrm{ad}_e] = 2\mathrm{ad}_e$, if $b \in C_{\mathfrak{g}}(e)$, then:

$$\mathrm{ad}_e\mathrm{ad}_h(b) = \mathrm{ad}_h\mathrm{ad}_e(b) - 2\mathrm{ad}_e(b) = 0 \tag{2.4}$$

Hence $\mathrm{ad}_h(b) \in C_{\mathfrak{g}}(e)$; therefore, $\mathrm{ad}_h(C_{\mathfrak{g}}(e)) \subseteq C_{\mathfrak{g}}(e)$.
Moreover we notice that the following relation holds:

$$\begin{aligned}[\mathrm{ad}_e^i, \mathrm{ad}_z] &\underset{(*)}{=} \mathrm{ad}_e^{i-1}[\mathrm{ad}_e, \mathrm{ad}_z] + \mathrm{ad}_e^{i-2}[\mathrm{ad}_e, \mathrm{ad}_z]\mathrm{ad}_e + \cdots + [\mathrm{ad}_e, \mathrm{ad}_z]\mathrm{ad}_e^{i-1} \\ &\underset{(**)}{=} \mathrm{ad}_e^{i-1}\mathrm{ad}_h + \mathrm{ad}_e^{i-2}\mathrm{ad}_h\mathrm{ad}_e + \cdots + \mathrm{ad}_h\mathrm{ad}_e^{i-1}\end{aligned}$$

(the right hand side of $(*)$ is $(\mathrm{ad}_e^i\mathrm{ad}_z - \mathrm{ad}_e^{i-1}\mathrm{ad}_z\mathrm{ad}_e) + (\mathrm{ad}_e^{i-1}\mathrm{ad}_z\mathrm{ad}_e - \mathrm{ad}_e^{i-2}\mathrm{ad}_z\mathrm{ad}_e^2) + (\mathrm{ad}_e^{i-2}\mathrm{ad}_z\mathrm{ad}_e^2 - \mathrm{ad}_e^{i-3}\mathrm{ad}_z\mathrm{ad}_e^3) + \cdots + (\mathrm{ad}_e\mathrm{ad}_z\mathrm{ad}_e^{i-1} - \mathrm{ad}_z\mathrm{ad}_e^i)$; thus only the first and the last term of the sum survive, that is exactly $[\mathrm{ad}_e^i, \mathrm{ad}_z]$).

By induction on $k \in \mathbb{N}$, we can also prove that:

$$\mathrm{ad}_e^k\mathrm{ad}_h = \mathrm{ad}_h\mathrm{ad}_e^k - 2k\mathrm{ad}_e^k. \tag{2.5}$$

Indeed:

- if $k = 1$
  $\mathrm{ad}_e\mathrm{ad}_h - \mathrm{ad}_h\mathrm{ad}_e = [\mathrm{ad}_e, \mathrm{ad}_h] = -2\mathrm{ad}_e$.

- if $k > 1$

$$\begin{aligned}
\mathrm{ad}_e^k\mathrm{ad}_h &= \mathrm{ad}_e\mathrm{ad}_e^{k-1}\mathrm{ad}_h = \mathrm{ad}_e(\mathrm{ad}_h\mathrm{ad}_e^{k-1} - 2(k-1)\mathrm{ad}_e^{k-1}) \\
&= (\mathrm{ad}_e\mathrm{ad}_h)\mathrm{ad}_e^{k-1} - 2(k-1)\mathrm{ad}_e^k \\
&= (\mathrm{ad}_h\mathrm{ad}_e - 2\mathrm{ad}_e)\mathrm{ad}_e^{k-1} - 2(k-1)\mathrm{ad}_e^k \\
&= \mathrm{ad}_h\mathrm{ad}_e^k - 2\mathrm{ad}_e - 2(k-1)\mathrm{ad}_e^k \\
&= \mathrm{ad}_h\mathrm{ad}_e^k - 2k\mathrm{ad}_e^k.
\end{aligned}$$

Now, applying relation (2.5) in the equality $(**)$, we get:

$$[\mathrm{ad}_e^i, \mathrm{ad}_z] = i(\mathrm{ad}_h - (i-1))\mathrm{ad}_e^{i-1}.$$

Let $b \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^{i-1})$. Then there exists $a \in \mathfrak{g}$ such that $b = \mathrm{ad}_e^{i-1}(a)$ and $\mathrm{ad}_e(b) = \mathrm{ad}_e(\mathrm{ad}_e^{i-1}(a)) = \mathrm{ad}_e^i(a) = 0$. Hence:

$$i(\mathrm{ad}_h - (i-1))\mathrm{ad}_e^{i-1}(a) = [\mathrm{ad}_e^i, \mathrm{ad}_z](a) = \mathrm{ad}_e^i\mathrm{ad}_z(a) - \mathrm{ad}_z\mathrm{ad}_e^i(a) = \mathrm{ad}_e^i(\mathrm{ad}_z(a)),$$

meaning that $i(\mathrm{ad}_h - (i-1))\mathrm{ad}_e^{i-1}(a) \in \mathrm{Im}(\mathrm{ad}_e^i)$.
Thus, by this and since $\mathrm{ad}_h(C_{\mathfrak{g}}(e)) \subseteq C_{\mathfrak{g}}(e)$, we have

$$i(\mathrm{ad}_h - (i-1))(b) \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^i). \tag{2.6}$$

It follows from this relation and the nilpotency of ad_e that, if b is any element of $C_{\mathfrak{g}}(e)$, there exists a positive integer m such that:

$$(\mathrm{ad}_h - m)(\mathrm{ad}_h - (m-1)) \cdot \ldots \cdot (\mathrm{ad}_h - 1)\mathrm{ad}_h(b) = 0.$$

In fact, by (2.6) we have that:

$$\begin{aligned}
& b \in C_{\mathfrak{g}}(e) = C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^0) \\
& \Rightarrow \mathrm{ad}_h(b) \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^1) \\
& \Rightarrow (\mathrm{ad}_h - 1)\mathrm{ad}_h(b) \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^2) \\
& \vdots \\
& \Rightarrow (\mathrm{ad}_h - m)(\mathrm{ad}_h - (m-1)) \cdot \ldots \cdot (\mathrm{ad}_h - 1)\mathrm{ad}_h(b) \in C_{\mathfrak{g}}(e) \cap \mathrm{Im}(\mathrm{ad}_e^{m+1}) = \{0\}.
\end{aligned}$$

This tells us that the characteristic roots of $\mathrm{ad}_{h|C_{\mathfrak{g}}(e)} : C_{\mathfrak{g}}(e) \to C_{\mathfrak{g}}(e)$ are non-negative integers. Hence $\mathrm{ad}_h + 2$ induces a non-singular linear transformation in $C_{\mathfrak{g}}(e)$ and consequently there exists $y_1 \in C_{\mathfrak{g}}(e)$ such that $(\mathrm{ad}_h + 2)(y_1) = x_1$, where $x_1 \in C_{\mathfrak{g}}(e)$ is the element such that $[h, z] = 2z + x_1$. Then $[h, y_1] = -2y_1 + x_1$. Hence, if we set $f = z - y_1$, we have $[h, f] = [h, z] - [h, y_1] = -2z + x_1 + 2y_1 - x_1 = -2(z - y_1) = -2f$. Also, thanks to the fact that $y_1 \in C_{\mathfrak{g}}(e)$, we have $[e, f] = [e, z] - [e, y_1] = [e, z] = h$. Hence the thesis holds. □

Lemma 2.2.4. *Let $e \in \mathfrak{g}$ be a nilpotent element and K be the Killing form on $\mathfrak{g}$. Then $K(e, C_{\mathfrak{g}}(e)) = 0$.*

Proof. Take $y \in C_{\mathfrak{g}}(e)$. Then $\mathrm{ad}_{[e,y]} = 0$, i.e., $[\mathrm{ad}_e, \mathrm{ad}_y] = 0$. Therefore $\mathrm{ad}_e \mathrm{ad}_y = \mathrm{ad}_y \mathrm{ad}_e$. This means that, for arbitrary $k \in \mathbb{Z}^+$, $(\mathrm{ad}_e \mathrm{ad}_y)^k = \mathrm{ad}_e^k \mathrm{ad}_y^k$. By the nilpotency of e we can take $k >> 1$ such that $\mathrm{ad}_e^k = 0$. Then $(\mathrm{ad}_e \mathrm{ad}_y)^k = 0$, i.e., $\mathrm{ad}_e \mathrm{ad}_y$ is nilpotent and hence its trace is zero. Therefore $K(e, y) = \mathrm{tr}(\mathrm{ad}_e \mathrm{ad}_y) = 0$. □

Now we prove a strengthened version of point 6. of Theorem 1.10.11, the so called Jacobson-Morozov Theorem.

Theorem 2.2.5 (Jacobson-Morozov)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field of charactesristic 0. If e is a nonzero nilpotent element of $\mathfrak{g}$, then there exists a standard triple $\{e, h, f\}$ for $\mathfrak{g}$.*

Proof. We will argue by induction on the dimension of $\mathfrak{g}$.
If this is 3 (the smallest dimension for a semisimple Lie algebra), then $\mathfrak{g}$ must be isomorphic to $\mathfrak{sl}_2$. Indeed, if $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is the Cartan decomposition of $\mathfrak{g}$ then, by point 1. of Theorem 1.10.11, $\#\Phi > 0$. Thus by point 6. of Theorem 1.10.11 we know that $\mathfrak{g}$ contains an $\mathfrak{sl}_2$-triple. But, since $\dim\mathfrak{g} = 3$, then $\mathfrak{g} \cong \mathfrak{sl}_2$. Now, take $z \in \mathfrak{sl}_2 \cong \mathfrak{g}$ nilpotent and denote by $\{e, h, f\}$ the standard basis of $\mathfrak{sl}_2$. Then $z = ae + bf$ (h can not appear because it is semisimple); but $\det(z - \lambda I_2) = \det \begin{pmatrix} -\lambda & a \\ b & -\lambda \end{pmatrix} = \lambda^2 - ab$. So z is nilpotent if and only if either $a = 0$ or $b = 0$. Therefore,

- if $z = ae$, then $\{z, h, \frac{1}{a}f\}$ is an $\mathfrak{sl}_2$-triple containing z;
- if $z = bf$, then $\{z, -h, \frac{1}{b}e\}$ is an $\mathfrak{sl}_2$-triple containing z.

Assume $\dim(\mathfrak{g}) > 3$. If e lies in a proper semisimple Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, then by induction we can find an $\mathfrak{sl}_2$-triple in $\mathfrak{a}$, that is an $\mathfrak{sl}_2$-triple also in $\mathfrak{g}$.
Thus we may assume for the remainder of the proof that e does not lie in any proper semisimple Lie subalgebra of $\mathfrak{g}$.
Let K be the Killing form on $\mathfrak{g}$. First of all, notice that $(C_\mathfrak{g}(e))^\perp = [\mathfrak{g}, e]$ where the orthogonal complement is take relative to the Killing form. Indeed:

- $[\mathfrak{g}, e] \subseteq (C_\mathfrak{g}(e))^\perp$ because, if $x = [z, e] \in [\mathfrak{g}, e]$, then $K(x, C_\mathfrak{g}(e)) = K([z, e], C_\mathfrak{g}(e)) = K(z, [e, C_\mathfrak{g}(e)]) = 0$;
- $\dim(C_\mathfrak{g}(e))^\perp = \dim\mathfrak{g} - \dim(C_\mathfrak{g}(e))$ and, considering $\mathrm{ad}_e : \mathfrak{g} \to \mathfrak{g}$, we can say that $\dim\mathfrak{g} = \dim[e, \mathfrak{g}] + \dim C_\mathfrak{g}(e)$.

By Lemma 2.2.4 we can say that $K(e, C_\mathfrak{g}(e)) = 0$ and so $e \in (C_\mathfrak{g}(e))^\perp = [\mathfrak{g}, e]$. Thus there exists $h' \in \mathfrak{g}$ such that $[h', e] = 2e$.

Claim 1. There exists a semisimple element h such that $[h, e] = 2e$.
To see this, let $h' = h'_s + h'_n$ be the Jordan-Chevalley decomposition of h' in $\mathfrak{g}$. By point 3. of Proposition 1.7.4 we know that h'_s and h'_n stabilize every subspace that h' stabilizes. So h'_s acts semisimply and h'_n act nilpotently on the subspace $\langle e \rangle$; hence $[h'_s, e] = 2e$, $[h'_n, e] = 0$. Thus we may take $h = h'_s$.
Claim 2. If h is as in Claim 1, then $h \in [\mathfrak{g}, e]$
By contradiction, suppose that $h \notin [\mathfrak{g}, e]$. Then, as $[\mathfrak{g}, e] = (C_{\mathfrak{g}}(e))^{\perp}$, we must have

$$K(h, C_{\mathfrak{g}}(e)) \neq 0. \tag{2.7}$$

By an easy calculation with the Jacobi identity we see that ad_h leaves $C_{\mathfrak{g}}(e)$ invariant. Hence ad_h must act semisimply on $C_{\mathfrak{g}}(e)$, so we may decompose $C_{\mathfrak{g}}(e)$ into ad_h eigenspaces:

$$C_{\mathfrak{g}}(e) = \bigoplus_{\tau_i \in \mathbb{F}} C_{\mathfrak{g}}(e)_{\tau_i}.$$

Note that $C_{\mathfrak{g}}(e)_0 = \{z \in C_{\mathfrak{g}}(e) \mid [h, z] = 0\} = C_{C_{\mathfrak{g}}(e)}(h)$. So we have:

$$C_{\mathfrak{g}}(e) = C_{C_{\mathfrak{g}}(e)}(h) \oplus \bigoplus_{\tau_i \neq 0} C_{\mathfrak{g}}(e)_{\tau_i}. \tag{2.8}$$

By the invariance of the Killing form $K(h, [h, C_{\mathfrak{g}}(e)]) = K([h, h], C_{\mathfrak{g}}(e)) = 0$. Thus, if z is a nonzero element of $C_{\mathfrak{g}}(e)_{\tau_i}$ with $\tau_i \neq 0$, then $0 = K(h, [h, z]) = K(h, \tau_i z) = \tau_i K(h, z)$. This shows that

$$h \in (C_{\mathfrak{g}}(e)_{\tau_i})^{\perp} \text{ for all } \tau_i \neq 0. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we can say that there exists $z \in C_{C_{\mathfrak{g}}(e)}(h)$ such that $K(h, z) \neq 0$. If z is nilpotent then, by Lemma 2.2.4 we can say that $K(h, z) = 0$, a contradiction. Hence $z_s \neq 0$. By point 3. of Proposition 1.7.4, we can say that z_s is a nonzero semisimple element in $C_{C_{\mathfrak{g}}(e)}(f)$. By Proposition 2.1.5 we know that $C_{\mathfrak{g}}(z_s)$ is reductive, whence $[C_{\mathfrak{g}}(z_s), C_{\mathfrak{g}}(z_s)]$ is a semisimple Lie subalgebra of $\mathfrak{g}$. It is a proper subalgebra, since $C_{\mathfrak{g}}(z_s) = \mathfrak{g}$ only if $z_s = 0$. We have now shown that $h \in C_{\mathfrak{g}}(z_s)$ and $e \in C_{\mathfrak{g}}(z_s)$. Hence $2e = [h, e] \in [C_{\mathfrak{g}}(z_s), C_{\mathfrak{g}}(z_s)]$. Thus our nilpotent element e belongs to a proper semisimple subalgebra of $\mathfrak{g}$, in contradiction to our assumption. Hence, by Lemma 2.2.3, we conclude that there exists $f \in \mathfrak{g}$ such that $\{e, h, f\}$ in an $\mathfrak{sl}_2$-triple. $\square$

Chapter 3

Good $\mathbb{Z}$-gradings

3.1 Basic definitions

From now on, we shall assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0.

Definition 3.1.1. Let $\mathfrak{g}$ be a Lie algebra. A *$\mathbb{Z}$-grading* of $\mathfrak{g}$ is a decomposition:

$$\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$$

where the $\mathfrak{g}_j$'s are vector subspaces of $\mathfrak{g}$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

Remark 3.1.2. If $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ is semisimple, then there exists an element $H \in \mathfrak{g}_0$ defining the $\mathbb{Z}$-grading, i.e., such that $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid [H, x] = kx\}$ for all $k \in \mathbb{Z}$.

Proof. Define $\phi : \mathfrak{g} \to \mathfrak{g}$ such that, for $x \in \mathfrak{g}_k$, $\phi(x) = kx$, and extend it on $\mathfrak{g}$ by linearity. This endomorphism is a derivation. Indeed, if $x \in \mathfrak{g}_k$ and $y \in \mathfrak{g}_j$, then

$$\phi([x, y]) = (k + j)[x, y] \text{ since } [x, y] \in \mathfrak{g}_{k+j}$$
$$[\phi(x), y] + [x, \phi(y)] = k[x, y] + j[x, y] = (k + j)[x, y].$$

Since all derivations of $\mathfrak{g}$ are inner by Theorem 1.9.10, i.e., $\mathrm{Der}\mathfrak{g} = \mathrm{ad}\mathfrak{g}$, there exists $H \in \mathfrak{g}$ such that $\phi = \mathrm{ad}_H$. So if $x \in \mathfrak{g}_k$, we have that $\phi(x) = \mathrm{ad}_H(x) \Leftrightarrow kx = [H, x]$. Hence $\mathfrak{g}_k = \{x \in \mathfrak{g} : [H, x] = kx\}$. $\square$

Definition 3.1.3. A $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ is called *even* if $\dim\mathfrak{g}_j = 0$ for all j odd. Otherwise it is called *odd.*

Proposition 3.1.4. *Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be a semisimple Lie algebra. Then $K(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i + j \neq 0$.*

Proof. Take $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$ $(i + j \neq 0)$ and $H \in \mathfrak{g}_0$ defining the $\mathbb{Z}$-grading. Then:

$$-iK(x, y) = K([x, H], y) = K(x, [H, y]) = jK(x, y).$$

Hence $(i + j)K(x, y) = 0$ and, as $i + j \neq 0$, $K(x, y) = 0$. □

Proposition 3.1.5. *Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be a semisimple Lie algebra. Then $\mathfrak{g}_0$ is reductive.*

Proof. By Proposition 3.1.4, we have $K(\mathfrak{g}_0, \mathfrak{g}_i) = 0$ for every $i \neq 0$. Hence $K_{|\mathfrak{g}_0\times\mathfrak{g}_0}$ is non-degenerate. Indeed, if we take $z \in \mathfrak{g}_0$ such that $K(z, \mathfrak{g}_0) = 0$ then, since $K(z, \mathfrak{g}_i) = 0$ for every $i \neq 0$, $K(z, \mathfrak{g}) = 0$. But K is non-degenerate on $\mathfrak{g}$ because $\mathfrak{g}$ is semisimple, and so $z = 0$. Hence, by Theorem 2.1.7, $\mathfrak{g}_0$ is reductive. □

Definition 3.1.6. Let $\mathfrak{g}$ be a Lie algebra and $S \subset \mathfrak{g}$. The *centralizer* of S in $\mathfrak{g}$ is defined as follows:

$$C_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} \mid [x, S] = 0\}.$$

Definition 3.1.7. An element $e \in \mathfrak{g}_2$ is called *good* if the following properties hold:

a) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for $j \leq -1$;

b) $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$.

Remark 3.1.8. Given the definition of good element, we can immediately observe that:

1. e is a nonzero ad-nilpotent element of $\mathfrak{g}$;

2. Point a) of Definition 3.1.7 is equivalent to the fact that the centralizer $C_{\mathfrak{g}}(e)$ of e lies in $\oplus_{j\geq 0}\mathfrak{g}_j$;

3. $\mathrm{ad}_e : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is bijective;

4. $[\mathfrak{g}_0, \mathfrak{g}_2] = \mathfrak{g}_2$.

Proof. 1. If $e = 0$, then $\mathrm{ad}_e(x) = 0$ for all $x \in \mathfrak{g}_j$. But this contradicts point a) of Definition 3.1.7.
Moreover $\mathrm{ad}_e^k \in \mathfrak{g}_{2k} = 0$ for $k >> 1$ since $\mathfrak{g}$ is finite-dimensional.

2. 3.1.7a) $\Rightarrow C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$.
Suppose by contradiction that $x \in C_{\mathfrak{g}}(e)$, $x \neq 0$ such that $x \in \bigoplus_{j\leq -1} \mathfrak{g}_j$.
Write $x = \sum_{j\leq -1} x_j$, with $x_j \in \mathfrak{g}_j$. Then $0 = \mathrm{ad}_e(x) = \sum_{j\leq -1} \mathrm{ad}_e(x_j)$.
Since every summand lies in a different homogeneous component of the $\mathbb{Z}$-grading, then $\mathrm{ad}_e(x_j) = 0$ for all $j \leq -1$. But ad_e is injective for $j \leq -1$, i.e., $x_j = 0$ for all j. Hence $x = 0$.
$C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j \Rightarrow$ 3.1.7a).
Fix $j \leq -1$ and let $x, y \in \mathfrak{g}_j$ with $x \neq y$, such that $\mathrm{ad}_e(x) = \mathrm{ad}_e(y)$, then
$$[e, x] = [e, y] \Leftrightarrow [e, x - y] = 0 \Leftrightarrow \mathrm{ad}_e(x - y) = 0.$$
Since $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$, $x - y \in \oplus_{j\geq 0}\mathfrak{g}_j$. This is a contradiction because, by assumption, $0 \neq x - y \in \mathfrak{g}_j$, with $j \leq -1$.

3. It follows from a) and b) for $j = -1$.

4. Obvious by property b) of Definition 3.1.7 for $j = 0$.

□

Definition 3.1.9. A $\mathbb{Z}$-grading of $\mathfrak{g}$ is called *good* if it admits a good element.

3.2 Dynkin $\mathbb{Z}$-gradings

The most important examples of good $\mathbb{Z}$-gradings of $\mathfrak{g}$ correspond to $\mathfrak{sl}_2$-triples $\{e, h, f\}$, where $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. We call the good

$\mathbb{Z}$-gradings thus obtained the *Dynkin $\mathbb{Z}$-gradings*. In this section we show more precisely what a Dynkin $\mathbb{Z}$-grading is and why it is good.

Let $e \in \mathfrak{g}$ be a nonzero nilpotent element. By the Jacobson-Morozov Theorem (Theorem 2.2.5), e embeds into a $\mathfrak{sl}_2$-triple $\{e, h, f\}$, i.e., $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Since ad_h acts semisimply on $\mathfrak{g}$, we can decompose $\mathfrak{g}$ into the direct sum of its eigenspaces:

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}_\lambda,$$

where $\mathfrak{g}_\lambda = \{z \in \mathfrak{g} \mid \mathrm{ad}_h(z) = \lambda z\}$.
Let $\mathfrak{a} := \langle e, h, f\rangle$ and consider the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ such that $x \mapsto \mathrm{ad}_x$). Then, by Weyl's Theorem (Theorem 1.9.22), since $\mathfrak{a}$ is semisimple, $\mathfrak{g}$ decomposes as a direct sum of irreducible finite-dimensional ($\mathfrak{a} \cong \mathfrak{sl}_2$)-modules $\mathfrak{g}_{s_k}$:

$$\mathfrak{g} = \bigoplus_{k=1}^{r} \mathfrak{g}_{s_k} \tag{3.1}$$

where $\mathfrak{g}_{s_k} = \langle w_k, f.w_k, \dots, f^{s_k}.w_k\rangle$ with $h.w_k = s_k w_k$ and $e.w_k = 0$ (by $x.z$ we denote $\mathrm{ad}_x(z)$).
Now, since the weights of h on $\mathfrak{g}_{s_k}$ are integers for every k, we can write:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \tag{3.2}$$

where $\mathfrak{g}_i = \{z \in \mathfrak{g} \mid [h, z] = ix\}$.
This decomposition of $\mathfrak{g}$ is called *Dynkin $\mathbb{Z}$-grading* associated to the nilpotent element e, and does not depend on the choice of the $\mathfrak{sl}_2$-triple containing e (see Chapter x. in [9]).

Remark 3.2.1. The decomposition $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ introduced in (3.2) is a $\mathbb{Z}$-grading.

Proof. Let $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, then $[h, [x, y]] = [[h, x], y] + [x, [h, y]] = i[x, y] + j[x, y] = (i + j)[x, y]$, i.e., $[x, y] \in \mathfrak{g}_{i+j}$. $\square$

Proposition 3.2.2. *The $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ introduced in* (3.2) *is good with good element e.*

Proof.
- $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for $j \leq -1$.
  By point 2. of Remark 3.1.8, it is enough to show that $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$. Thanks to the decomposition (3.1) of $\mathfrak{g}$ as sum of irreducible $\mathfrak{sl}_2$-modules, we can say that $C_{\mathfrak{g}}(e) = \langle w_1, \dots, w_r\rangle$, with $h.w_i = s_i w_i$ and $s_i \geq 0$ for all $i = 1, \dots, r$. Hence $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$.

- $\mathrm{ad}_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$.
  Fix $j \geq -1$. Thanks to the decomposition (3.1), we can find a basis of $\mathfrak{g}_{j+2}$ consisting of elements of the form $f^k.w_i$, where $h.f^k.w_i = (j+2)f^k.w_i$ for some $k \in \{0, \dots, s_i\}$ and $i \in \{1, \dots, r\}$. By the representation theory of $\mathfrak{sl}_2$ (in particular equation (1.2)), we know that:

$$e.f^l.w_i = l(s_i - l + 1)f^{l-1}.w_i \text{ for all } l \in \mathbb{Z}^+.$$

  Then, for $l = k+1$, we have $e.f^{k+1}.w_i = (k+1)(s_i - k)f^k.w_i$. Hence ad_e is surjective on $\mathfrak{g}_j$ because:
  · $f^{k+1}.w_i \in \mathfrak{g}_j$ since $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ is a $\mathbb{Z}$-grading;
  · $s_i \neq k$ because if not, we would have $h.f^{s_i}.w_i = (j+2)f^{s_i}.w_i$. But, by representation theory of $\mathfrak{sl}_2$ we know that $h.f^{s_i}.w_i = -s_i f^{s_i}.w_i$. So we would get $j + 2 = -s_i$, which can not happen because $j + 2 \geq 1$ and $-s_i \leq 0$.
  · $f^{k+1}.w_i \neq 0$ because we have seen that $k < s_i$ in the previous point.

□

Example 3.2.3 ($\mathfrak{sp}_4$)**.** Consider the Cartan decomposition of $\mathfrak{sp}_4$

$$\mathfrak{sp}_4 = H \oplus L_{\pm\alpha} \oplus L_{\pm\beta} \oplus L_{\pm(\alpha+\beta)} \oplus L_{\pm(2\alpha+\beta)}.$$

Taken $x_\alpha \in L_\alpha$, we know by point 6. of Proposition 1.10.11 that there exist $x_{-\alpha} \in L_{-\alpha}$ such that $\langle x_\alpha, x_{-\alpha}, h_\alpha := [x_\alpha, x_{-\alpha}]\rangle \cong \mathfrak{sl}_2$.

Now we want to construct a $\mathbb{Z}$-grading of $\mathfrak{sp}_4$ given by the eigenspace decomposition of ad_{h_α}, so:

$$
\begin{aligned}
&[h_\alpha, H] = 0 \text{ (since } h_\alpha \in H) && \Rightarrow H \subseteq \mathfrak{g}_0,\\
&[h_\alpha, x_\beta] = \beta(h_\alpha)x_\beta = \langle\beta,\alpha\rangle x_\beta = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}x_\beta = -x_\beta && \Rightarrow x_\beta \in \mathfrak{g}_{-1},\\
&[h_\alpha, x_{-\beta}] = x_{-\beta} && \Rightarrow x_{-\beta} \in \mathfrak{g}_1,\\
&[h_\alpha, x_\alpha] = 2x_\alpha && \Rightarrow x_\alpha \in \mathfrak{g}_2,\\
&[h_\alpha, x_{-\alpha}] = -2x_{-\alpha} && \Rightarrow x_{-\alpha} \in \mathfrak{g}_{-2},\\
&[h_\alpha, x_{\alpha+\beta}] = (\alpha+\beta)(h_\alpha)x_{\alpha+\beta} = (2-1)x_{\alpha+\beta} = x_{\alpha+\beta} && \Rightarrow x_{\alpha+\beta} \in \mathfrak{g}_1,\\
&[h_\alpha, x_{-\alpha-\beta}] = -x_{-\alpha-\beta} && \Rightarrow x_{-\alpha-\beta} \in \mathfrak{g}_{-1},\\
&[h_\alpha, x_{2\alpha+\beta}] = (2\alpha+\beta)(h_\alpha)x_{2\alpha+\beta} = (4-1)x_{2\alpha+\beta} = 3x_{2\alpha+\beta} && \Rightarrow x_{\alpha+\beta} \in \mathfrak{g}_3,\\
&[h_\alpha, x_{-2\alpha-\beta}] = -3x_{-2\alpha-\beta} && \Rightarrow x_{-2\alpha-\beta} \in \mathfrak{g}_{-3}.
\end{aligned}
$$

Hence

$$\mathfrak{sp}_4 = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \tag{3.3}$$

is the eigenspace decomposition of ad_{h_α} of $\mathfrak{sp}_4$ where

- $\mathfrak{g}_0 = H$,
- $\mathfrak{g}_1 = \langle x_{-\beta}, x_{\alpha+\beta}\rangle$ and $\mathfrak{g}_{-1} = \langle x_\beta, x_{-\alpha-\beta}\rangle$,
- $\mathfrak{g}_2 = \langle x_\alpha\rangle$ and $\mathfrak{g}_{-2} = \langle x_{-\alpha}\rangle$,
- $\mathfrak{g}_3 = \langle x_{2\alpha+\beta}\rangle$ and $\mathfrak{g}_{-3} = \langle x_{-2\alpha-\beta}\rangle$.

This decomposition (3.3) is a good $\mathbb{Z}$-grading of $\mathfrak{sp}_4$. In fact:

1. (3.3) is a $\mathbb{Z}$-grading of $\mathfrak{sp}_4$ by Remark 3.2.1.
2. (3.3) is a good $\mathbb{Z}$-grading since it admits a good element that is $x_\alpha \in \mathfrak{g}_2$, in fact:
   - $\mathrm{ad}_{x_\alpha}\colon \mathfrak{g}_{-3} \to \mathfrak{g}_{-1}$ such that $x_{-2\alpha-\beta} \mapsto x_{-\alpha-\beta}$ is injective because $\dim\mathfrak{g}_{-3} = 1$ and $\mathrm{ad}_{x_\alpha}(x_{-2\alpha-\beta}) = x_{-\alpha-\beta} \neq 0$,

- $\mathrm{ad}_{x_\alpha} : \mathfrak{g}_{-2} \to \mathfrak{g}_0$ such that $x_\alpha \mapsto h_\alpha$ is injective because $\dim\mathfrak{g}_{-2} = 1$ and $\mathrm{ad}_{x_\alpha}(x_{-\alpha}) = h_\alpha \neq 0$,
- $\mathrm{ad}_{x_\alpha} : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ such that $x_\beta \mapsto x_{\alpha+\beta}$, $x_{-\alpha-\beta} \mapsto x_{-\beta}$, up to scalars, is bijective because it maps a basis into a basis and $ad_{x_\alpha}(x_\beta) \neq 0$ since $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ and $\dim L_{\alpha+\beta} = 1$ by points 4. and 1. of Proposition 1.10.12,
- $\mathrm{ad}_{x_\alpha} : \mathfrak{g}_0 \to \mathfrak{g}_2$ such that $h_\alpha \mapsto -2x_\alpha$ is surjective because $\dim\mathfrak{g}_2 = 1$,
- $\mathrm{ad}_{x_\alpha} : \mathfrak{g}_1 \to \mathfrak{g}_3$ such that $x_{\alpha+\beta} \mapsto x_{2\alpha+\beta}$, $x_{-\beta} \mapsto 0$ ($L_{\alpha-\beta} = 0$) is surjective because $\dim\mathfrak{g}_3 = 1$.

3.3 Properties of good gradings

From now on, we shall assume that $\mathfrak{g}$ is a semisimple Lie algebra. Fix a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j \tag{3.4}$$

Lemma 3.3.1. *Let $e \in \mathfrak{g}_2$, $e \neq 0$. Then there exists $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-2}$ such that $\{e,\ h,\ f\}$ forms an $\mathfrak{sl}_2$-triple, i.e., $[h,e] = 2e$, $[e,f] = h$, $[h,f] = -2f$.*

Proof. By the Jacobson Morozov Theorem (Theorem 2.2.5), there exist $h, f \in \mathfrak{g}$ such that $\{e,\ h,\ f\}$ is an $\mathfrak{sl}_2$-triple. We write $h = \sum_{j\in\mathbb{Z}} h_j$, $f = \sum_{j\in\mathbb{Z}} f_j$ according to the given $\mathbb{Z}$-grading of $\mathfrak{g}$. Then

- $[h_0, e] = 2e$ because $2e = [h,e] = [\sum_{j\in\mathbb{Z}} h_j, e] = \sum_{j\in\mathbb{Z}}[h_j, e]$. But $e \in \mathfrak{g}_2$ and $[h_j, e] \in \mathfrak{g}_{j+2}$, so $[h_j, e] = 0$ for $j \neq 0$ and $[h_0, e] = 2e$.
- $[e, \mathfrak{g}] \ni h_0$ since $[e, f_{-2}] = h_0$; in fact $\sum_{j\in\mathbb{Z}} h_j = h = [e,f] = [e, \sum_{j\in\mathbb{Z}} f_j] = \sum_{j\in\mathbb{Z}} [e, f_j] \Rightarrow h_{j+2} = [e, f_j]$ for $j \in \mathbb{Z}$.

Therefore, by Morozov's lemma (Lemma 2.2.3), there exists f' such that $\{e,\ h_0,\ f'\}$ is an $\mathfrak{sl}_2$-triple. But then $\{e,\ h_0,\ f'_{-2}\}$ is an $\mathfrak{sl}_2$-triple, in fact:

- $[h_0, e] = 2e$,

- $[e, f'] = h_0$, then $\mathfrak{g}_0 \ni h_0 = [e, f'] = [e, \sum_{j\in\mathbb{Z}} f'_j] = \sum_{j\in\mathbb{Z}} [e, f'_j]$ so, as above, $[e, f'_{-2}] = h_0$,

- $[h_0, f'] = -2f'$, but $-2f' = -2\sum_{j\in\mathbb{Z}} f'_j = \sum_{j\in\mathbb{Z}} -2f'_j]$ and $[h_0, f'] = [h_0, \sum_{j\in\mathbb{Z}} f'_j] = \sum_{j\in\mathbb{Z}} [h_0, f'_j]$, hence $-2f'_j = [h_0, f'_j]$ for $j \in \mathbb{Z}$.

$\square$

Lemma 3.3.2. *Let $e \in \mathfrak{g}$ be a nonzero nilpotent element, $\mathfrak{s} = \{e, h, f\}$ an $\mathfrak{sl}_2$-triple and $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ the Dynkin grading introduced in* (3.2). *Set $C_{\mathfrak{g}}(e)_i = C_{\mathfrak{g}}(e) \cap \mathfrak{g}_i$. Then:*

1. $C_{\mathfrak{g}}(e) = \bigoplus_{i\geq 0} C_{\mathfrak{g}}(e)_i$;

2. $C_{\mathfrak{g}}(e) \cap [\mathfrak{g}, e] = \bigoplus_{i>0} C_{\mathfrak{g}}(e)_i$;

3. $C_{\mathfrak{g}}(e)_0 = C_{\mathfrak{g}}(\mathfrak{s})$.

Proof. Thanks to the decomposition of $\mathfrak{g} = \oplus_{j=1}^{k}\mathfrak{g}_{s_k}$ introduced in (3.1), we can say that $C_{\mathfrak{g}}(e) = \langle w_1, \dots, w_r\rangle$, with $h.w_i = s_i w_i$ and $s_i \geq 0$ for all $i = 1, \dots, r$.

1. This implies that $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$. Thus $C_{\mathfrak{g}}(e) = \oplus_{j\geq 0} C_{\mathfrak{g}}(e)_j$.

2. In order to prove the second point, we want to show that $w_i \in [\mathfrak{g}, e]$ if and only if $s_i > 0$. Indeed $s_i > 0$ is equivalent to $\dim\mathfrak{g}_{s_i} > 1$. This means that $f.w_i \neq 0$. Thus, since $e.f^k.w_i = k(s_i - k + 1)f^{k-1}.w_i$ for all k (see equation (1.2)), we have that $e.f.w_i = s_i w_i$, i.e., $w_i \in [\mathfrak{g}, e]$.

3. $C_{\mathfrak{g}}(e)_0 = \{w_{i_1}, \dots, w_{i_k}\}$, with $h.w_{i_j} = 0$ (i.e., $s_{i_j} = 0$).
   $C_{\mathfrak{g}}(\mathfrak{s}) = \{z \in \mathfrak{g} \mid e.z = f.z = h.z = 0\}$; this means that $z \in C_{\mathfrak{g}}(\mathfrak{s})$ if and only if $z = w_i$ (because $e.z = 0$) with $s_i = 0$ (because $h.z = 0$). Thus $C_{\mathfrak{g}}(\mathfrak{s}) = \{w_{i_1}, \dots, w_{i_k}\}$.

$\square$

Proposition 3.3.3. *Let e be a non-zero nilpotent element of $\mathfrak{g}$ and let $\mathfrak{s} = \{e, h, f\}$ be an $\mathfrak{sl}_2$-triple. Then $C_{\mathfrak{g}}(\mathfrak{s})$ is a reductive subalgebra of $C_{\mathfrak{g}}(e)$, called the reductive part of $C_{\mathfrak{g}}(e)$.*

Proof. By point 3. in Lemma 3.3.2, we know that $C_{\mathfrak{g}}(\mathfrak{s}) = C_{\mathfrak{g}}(e)_0$; thus, using Proposition 3.1.5, we can say that $C_{\mathfrak{g}}(\mathfrak{s})$ is reductive. □

Theorem 3.3.4. *Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be a good $\mathbb{Z}$-grading and $e \in \mathfrak{g}_2$ a good element. Let $H \in \mathfrak{g}$ be the element defining the $\mathbb{Z}$-grading, and let $\mathfrak{s} = \{e, h, f\}$ be an $\mathfrak{sl}_2$-triple given by Lemma 3.3.1. Then $z := H - h$ lies in the center of $C_{\mathfrak{g}}(\mathfrak{s})$.*

Proof. The existence of H is guaranteed by Remark 3.1.2. The eigenvalues of ad_H on $C_{\mathfrak{g}}(e)$ are non-negative since, if $a \in C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$ is an eigenvector of ad_H, by point 2. of Remark 3.1.8 we have $C_{\mathfrak{g}}(e) \subseteq \oplus_{j\geq 0}\mathfrak{g}_j$. Thus there exists $\overline{j} \geq 0$ such that $a \in \mathfrak{g}_{\overline{j}}$, so $\mathrm{ad}_H(a) = [H, a] = \overline{j}a$ with $\overline{j} \geq 0$.
Hence the eigenvalues of ad_H on $C_{\mathfrak{g}}(\mathfrak{s})$ are non-negative because $C_{\mathfrak{g}}(\mathfrak{s}) = \{a \in \mathfrak{g} : [a, e] = 0, [a, h] = 0, [a, f] = 0\} \subseteq C_{\mathfrak{g}}(e)$. So we can write $C_{\mathfrak{g}}(\mathfrak{s}) = \oplus_{i\geq 0}C_{\mathfrak{g}}(\mathfrak{s})_i$. By Proposition 3.3.3, $C_{\mathfrak{g}}(\mathfrak{s})$ is reductive; thus we can say that $C_{\mathfrak{g}}(\mathfrak{s}) = [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})] \oplus Z(C_{\mathfrak{g}}(\mathfrak{s}))$, where $[C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]$ is semisimple thanks to Proposition 2.1.3. Notice the following facts.

1. $[H, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] = 0$. Indeed, since $[C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]$ is semisimple, the Killing form restricted to it is non-degenerate by Theorem 1.9.5. Since $K([H, [C_{\mathfrak{g}}(\mathfrak{s}), \oplus_{j>0}C_{\mathfrak{g}}(\mathfrak{s})_j]], [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]) = 0$ by Proposition 3.1.4 and $[H, [C_{\mathfrak{g}}(\mathfrak{s})_0, C_{\mathfrak{g}}(\mathfrak{s})_0]] = 0$ because H is the element defining the $\mathbb{Z}$-grading, then $[H, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] = 0$.

2. $[h, C_{\mathfrak{g}}(\mathfrak{s})] = 0$ by the definition of $C_{\mathfrak{g}}(\mathfrak{s})$.

3. $[H - h, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] = [H, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] - [h, [C_{\mathfrak{g}}(\mathfrak{s}), C_{\mathfrak{g}}(\mathfrak{s})]] = 0$ by points 1., 2. and the Jacobi identity.

4. $H - h \in C_{\mathfrak{g}}(\mathfrak{s})$ because H is the element defining the $\mathbb{Z}$-grading and $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple. Thus $[H - h, Z(C_{\mathfrak{g}}(\mathfrak{s}))] = 0$.

Therefore $[z, C_{\mathfrak{g}}(\mathfrak{s})] = 0$, i.e., $z \in Z(C_{\mathfrak{g}}(\mathfrak{s}))$. $\square$

Corollary 3.3.5. *If $\mathfrak{s} = \{e, h, f\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ and the center of $C_{\mathfrak{g}}(\mathfrak{s})$ is trivial, then the only good grading for which e is a good element is the Dynkin grading.*

Proof. If $Z(C_{\mathfrak{g}}(\mathfrak{s})) = 0$, then $z := H - h = 0$ because by Theorem 3.3.4 we have $z \in Z(C_{\mathfrak{g}}(\mathfrak{s}))$. Hence $H = h$. But $H \in \mathfrak{g}$ is the element defining the $\mathbb{Z}$-grading, i.e., $\mathfrak{g}_j = \{a \in \mathfrak{g} : [H, a] = ja\} = \{a \in \mathfrak{g} : [h, a] = ja\}$. Since $H = h$, the good $\mathbb{Z}$-grading of $\mathfrak{g}$ with good element e is the one obtained by the eigenspace decomposition of ad_h in $\mathfrak{g}$, that means the Dynkin $\mathbb{Z}$-grading. $\square$

Example 3.3.6 ($\mathfrak{sl}_2$)**.**
Consider $\mathfrak{g} = \mathfrak{sl}_2 = \langle \mathfrak{s} \rangle = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$.
Notice that $C_{\mathfrak{g}}(\mathfrak{s}) = Z(\mathfrak{sl}_2) = 0$ because $\mathfrak{sl}_2$ is semisimple. Hence $Z(C_{\mathfrak{g}}(\mathfrak{s})) = 0$ and, by Corollary 3.3.5, the Dyinkin grading is the only $\mathbb{Z}$-grading for which $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a good element.

Example 3.3.7 ($\mathfrak{sl}_3$)**.**
Consider $\mathfrak{g} = \mathfrak{sl}_3$. Up to conjugation, the only nilpotent elements of $\mathfrak{g}$ are

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We start analyzing the case of an $\mathfrak{sl}_2$-triple containing e_1.
With a simple calculation one can check that $\mathfrak{s}_1 = \{e_1, h_1, f_1\}$ is an $\mathfrak{sl}_2$-triple, with $h_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ and $f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. Now we want to compute

$C_{\mathfrak{g}}(\mathfrak{s}_1)$. Consider $x = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \in C_{\mathfrak{g}}(\mathfrak{s}_1)$. Then:

$$\begin{aligned} 0 = [x, h_1] &= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2b & -4c \\ 2d & 0 & -2f \\ 4g & 2h & 0 \end{pmatrix} \end{aligned}$$

if and only if $b = c = d = f = g = h = 0$. Thus $x = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix}$.

Moreover,

$$\begin{aligned} 0 = [x, e_1] &= \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix} \\ &= \begin{pmatrix} 0 & a+e & 0 \\ 0 & 0 & a+2e \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

if and only if $a = e = 0$.
Therefore $C_{\mathfrak{g}}(\mathfrak{s}_1) = 0$; hence $Z(C_{\mathfrak{g}}(\mathfrak{s}_1)) = 0$ and, by Corollary 3.3.5, the Dyinkin grading is the only $\mathbb{Z}$-grading for which e_1 is a good element.

Now we analyze the second case, in which we consider an $\mathfrak{sl}_2$-triple containing e_2.

It easy to see that $\mathfrak{s}_2 = \{e_2, h_2, f_2\}$ is an $\mathfrak{sl}_2$-triple, with $h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Now we want to compute $C_{\mathfrak{g}}(\mathfrak{s}_2)$.

Consider $x = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \in C_{\mathfrak{g}}(\mathfrak{s}_2)$. Then:

$$0 = [x, h_2] = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -2b & -c \\ 2d & 0 & f \\ g & -h & 0 \end{pmatrix}$$

if and only if $b = c = d = f = g = h = 0$. Thus $x = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix}$.

Moreover,

$$0 = [x, e_2] = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -a-e \end{pmatrix}$$
$$= \begin{pmatrix} 0 & a-e & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

if and only if $a = e$. Thus $x = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \in C_{\mathfrak{g}}(\mathfrak{s}_2)$ (it is easy to see that such x commutes with f_2).

Therefore $C_{\mathfrak{g}}(\mathfrak{s}_2) = \langle \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \rangle$. So $Z(C_{\mathfrak{g}}(\mathfrak{s}_2)) = C_{\mathfrak{g}}(\mathfrak{s}_2)$ because $C_{\mathfrak{g}}(\mathfrak{s}_2)$ is one-dimensional and hence commutative.

Hence, in this case we can not establish if the Dynkin grading is the only good $\mathbb{Z}$-grading with good element e_2.

Definition 3.3.8. The following construction can be found in [3].

By Proposition 3.1.5, we know that $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$. Furthermore, it can be proven that a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$ is a Cartan

subalgebra of $\mathfrak{g}$ (see [7]).
Let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_\alpha \mathfrak{g}_\alpha)$ be the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let Δ_0^+ be a system of positive roots of the subalgebra $\mathfrak{g}_0$. It is well known that $\Delta^+ = \Delta_0^+ \cup (\alpha \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_s,\ s > 0)$ is a set of positive roots of $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Delta^+$ be the set of the simple roots. Setting $\Pi_s = (\alpha \in \Pi | \mathfrak{g}_\alpha \subset \mathfrak{g}_s)$ we obtain a decomposition of Π into a disjoint union of subsets $\Pi = \cup_{s \geq 0} \Pi_s$. This decomposition is called the *characteristic* of the $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. So we obtain a bijection between all $\mathbb{Z}$-gradings up to conjugation and all characteristics.

Theorem 3.3.9. *If the $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is good, then $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2$.*

Proof. Let $e \in \mathfrak{g}_2$ be a good element. From the construction above, we can write $e = \sum_{\rho_j \in \Phi^+} e_{\rho_j}$, with $\rho_j = \alpha_{j_1} + \ldots + \alpha_{j_{k_j}}$ for some non-negative simple roots α_{j_i}. Suppose, by contradiction, that there exists a simple root $\alpha_j \notin \Pi_0 \cup \Pi_1 \cup \Pi_2$. Then e lies in the Lie subalgebra generated by e_{α_i}, $i \neq j$. Indeed, if not, we could find an addend e_{ρ_r} of e such that $\alpha_j \in \{\alpha_{r_1}, \ldots, \alpha_{r_{k_r}}\}$. But, since $\deg e_{\rho_r} = \sum_k \deg e_{\alpha_{r_k}}$, then e could not belong to $\mathfrak{g}_2$. Therefore $[e_{\rho_i}, e_{-\alpha_j}] \in \mathfrak{g}_{\rho_i - \alpha_j} = \{0\}$ for all i and hence $[e, e_{-\alpha_j}] = 0$. This contradicts property a) of Definition 3.1.7.

$\square$

Corollary 3.3.10. *All good $\mathbb{Z}$-gradings are among those defined by $\deg e_{\alpha_i} = -\deg e_{-\alpha_i} = 0, 1$ or 2, $i = 1, \ldots, r$.*

Lemma 3.3.11. *Let $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ be a $\mathbb{Z}$-grading, $e \in \mathfrak{g}_2$ and K the Killing form on $\mathfrak{g}$. Then $[e, \mathfrak{g}_j] \neq \mathfrak{g}_{j+2}$ if and only if there exists a non-zero $a \in \mathfrak{g}_{-j-2}$ such that $K([e, \mathfrak{g}_j], a) = 0$.*

Proof. Suppose that $K([e, \mathfrak{g}_j], a) = 0$ for some non-zero $a \in \mathfrak{g}_{-j-2}$. Suppose by contradiction that $[e, \mathfrak{g}_j] = \mathfrak{g}_{j+2}$ for some $j \geq -1$. Then $K(\mathfrak{g}_{j+2}, a) = 0$. Now, take $H \in \mathfrak{g}_0$ defining the grading. $K(\mathfrak{g}_k, a) = 0$ for all $k \in \mathbb{Z}$, $k \neq j + 2$ by Proposition 3.1.4. Hence $(\mathfrak{g}, a) = 0$. This is a contradiction because K is non-degenerate and $a \neq 0$.

Conversely, suppose that $[e, \mathfrak{g}_j] \subsetneq \mathfrak{g}_{j+2}$. Notice that $\mathfrak{g}_{j+2}^{\perp} = \oplus_{k \neq -j-2} \mathfrak{g}_k$ by the non-degeneracy of K (see Proposition 3.1.4). Then $[e, \mathfrak{g}_j]^{\perp} \supsetneq \mathfrak{g}_{j+2}^{\perp} = \oplus_{k \neq -j-2} \mathfrak{g}_k$. This implies that $[e, \mathfrak{g}_j]^{\perp} \cap \mathfrak{g}_{-j-2} \neq 0$, i.e., there exists a nonzero element $a \in \mathfrak{g}_{-j-2}$ such that $K([e, \mathfrak{g}_j], a) = 0$. $\square$

Theorem 3.3.12. *Properties a) and b) of the definition of good element (Definition 3.1.7) of a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ are equivalent.*

Proof. By Lemma 3.3.11 we know that the property $[e, \mathfrak{g}_j] \neq \mathfrak{g}_{j+2}$ for $j \geq -1$ is equivalent to the existence of a non-zero element $a \in \mathfrak{g}_{-j-2}$ such that $K([e, \mathfrak{g}_j], a) = 0$. But the latter equality is equivalent to $K([e, a], \mathfrak{g}_j) = 0$ by the invariance of K and this is equivalent to $[e, a] = 0$ by the non-degeneracy of K. Then $\mathrm{ad}_e : \mathfrak{g}_{-j-2} \to \mathfrak{g}_{-j}$ is not injective. $\square$

Theorem 3.3.13. *Let $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a good $\mathbb{Z}$-grading with good element e. Then $C_{\mathfrak{g}}(e) \cong \mathfrak{g}_0 + \mathfrak{g}_{-1}$ as $C_{\mathfrak{g}_0}(e)$-modules.*

Proof. Due to properties a) and b) of Definition 3.1.7 we have the following exact sequence of $C_{\mathfrak{g}_0}(e)$-modules:

$$0 \to C_{\mathfrak{g}}(e) \overset{\mathrm{id}}{\to} \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \overset{\mathrm{ad}_e}{\to} \mathfrak{g}_+ \to 0.$$

Indeed,

- $\mathrm{id} : C_{\mathfrak{g}}(e) \to \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+$ is injective because, by point 2. of Remark 3.1.8, $C_{\mathfrak{g}}(e) \subset \mathfrak{g}_{\geq}$;
- $\mathrm{ad}_e : \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \to \mathfrak{g}_+$ is surjective by property b) of Definition 3.1.7;
- $\ker(\mathrm{ad}_e) = \{x \in \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \mid [e, x] = 0\} = C_{\mathfrak{g}}(e) = \mathrm{Im}(\mathrm{id})$

Moreover, we can note that $C_{\mathfrak{g}}(e)$, $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$, and $\mathfrak{g}_+$ are $C_{\mathfrak{g}_0}(e)$-modules. We show that only for $C_{\mathfrak{g}}(e)$ (it will be analogue in the other cases).
Since the concept of $\mathfrak{g}$-module is equivalent to the concept of representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$, it is enough to consider the adjoint representation:

$$\begin{array}{rccc} \mathrm{ad} : & C_{\mathfrak{g}_0}(e) & \to & \mathfrak{gl}(C_{\mathfrak{g}}(e)) \\ & x & \mapsto & \mathrm{ad}_x : C_{\mathfrak{g}}(e) \to C_{\mathfrak{g}}(e) \end{array}$$

and verify that ad_x is well defined as $C_{\mathfrak{g}}(e)$-endomorphism, i.e., $\mathrm{ad}_x(C_{\mathfrak{g}}(e)) \subseteq C_{\mathfrak{g}}(e)$. Thus, let $x \in C_{\mathfrak{g}_0}(e)$ and $y \in C_{\mathfrak{g}}(e)$; then $[e,[x,y]] = [[e,x],y] + [x,[e,y]] = [0,y] + [x,0] = 0$, so $[x,y] \in C_{\mathfrak{g}}(e)$. Hence, $C_{\mathfrak{g}}(e) \cong \mathfrak{g}_0 + \mathfrak{g}_{-1}$ as $C_{\mathfrak{g}_0}(e)$-modules because for all $x \in C_{\mathfrak{g}_0}(e)$ and $y \in \mathfrak{g}_0 + \mathfrak{g}_{-1}$ we have that $\mathrm{ad}_e(\mathrm{ad}_x(y)) = \mathrm{ad}_x(\mathrm{ad}_e(y))$ since $[e,x] = 0$. $\square$

Corollary 3.3.14. *Let $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ be a $\mathbb{Z}$-grading and let $e \in \mathfrak{g}_2$. Then $\dim C_{\mathfrak{g}}(e) \geq \dim\mathfrak{g}_{-1} + \dim\mathfrak{g}_0$, and equality holds if and only if e is a good element.*

Proof. We have an exact sequence of vector spaces (the proof of its exactness is analogue to the one in Theorem 3.3.13):

$$0 \to C_{\mathfrak{g}}(e) \cap (\mathfrak{g}_{-1} + \mathfrak{g}_{\geq}) \stackrel{\mathrm{id}}{\to} \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_+ \stackrel{\mathrm{ad}_e}{\to} [e, \mathfrak{g}_{-1} + \mathfrak{g}_{\geq}] \to 0.$$

Hence $\dim C_{\mathfrak{g}}(e) + \dim[e, \mathfrak{g}_{-1} + \mathfrak{g}_{\geq}] \geq \dim(C_{\mathfrak{g}}(e) \cap (\mathfrak{g}_{-1} + \mathfrak{g}_{\geq})) = \dim\mathfrak{g}_{-1} + \dim\mathfrak{g}_0 + \dim\mathfrak{g}_+$. But, since $[e, \mathfrak{g}_{-1} + \mathfrak{g}_{\geq}] \subseteq \mathfrak{g}_+$ (and equality holds if and only if e is good), one has $\dim C_{\mathfrak{g}}(e) + \dim\mathfrak{g}_+ \geq \dim\mathfrak{g}_{-1} + \dim\mathfrak{g}_0 + \dim\mathfrak{g}_+$, i.e. $\dim C_{\mathfrak{g}}(e) \geq \dim\mathfrak{g}_{-1} + \dim\mathfrak{g}_0$, and hence the Corollary follows.

$\square$

Definition 3.3.15. Let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$-module. Then V is called *self-dual* if it is isomorphic to V^* as $\mathfrak{g}$-module.

Lemma 3.3.16. *Let $\mathfrak{g}$ be a Lie algebra, V be a $\mathfrak{g}$-module via the adjoint action. If there exists a non-degenerate $\mathfrak{g}$-invariant bilinear form $(.,.) : V \times V \to \mathbb{F}$, then V is self-dual.*

Proof. Set

$$\begin{array}{rccc} \varphi : & V & \to & V^* \\ & v & \mapsto & (w \mapsto (v,w)) \end{array}$$

Then:

- φ is bijective since $(.,.)$ is non-degenerate;

- $\varphi(x.v) = x.\varphi(v)$ for every $x \in \mathfrak{g}$ and $v \in V$. Indeed, if $w \in V$:
  $(\varphi(x.v))(w) = (x.v, w) = ([x, v], w)$
  $(x.\varphi(v))(w) = -\varphi(v)(x.w) = -(v, x.w) = -(v, [x, w]) = -([v, x], w) = ([x, v], w)$
  where we used the invariancy of $(.,.)$.

$\square$

Corollary 3.3.17. *Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be a good $\mathbb{Z}$-grading with good element e. Then the representation of $C_{\mathfrak{g}_0}(e)$ on $C_{\mathfrak{g}}(e)$ is self-dual.*

Proof. Consider the bilinear form on $\mathfrak{g}_{-1}$ given by $\langle a, b\rangle := K(e, [a, b])$. Note that $\langle ., .\rangle$ has the following properties:

1. It is $C_{\mathfrak{g}_0}(e)$-invariant. Indeed, if we take $c \in C_{\mathfrak{g}_0}(e)$, $a, b \in \mathfrak{g}_{-1}$, one has $[a, c], [b, c] \in \mathfrak{g}_{-1}$. Furthermore $\langle [a, c], b\rangle = K(e, [[a, c], b]) = K(e, [a, [c, b]]) - K(e, [c, [a, b]]) = \langle a, [c, b]\rangle - K([e, c], [a, b]) = \langle a, [c, b]\rangle$ because $c \in C_{\mathfrak{g}_0}(e)$.

2. It is non-degenerate. Indeed, if we take $a \in \mathfrak{g}_{-1}$ such that $\langle \mathfrak{g}_{-1}, a\rangle = 0$, then $K(e, [\mathfrak{g}_{-1}, a]) = 0$, i.e. $K([e, \mathfrak{g}_{-1}], a) = 0$. Using point 3. of Remark 3.1.8 we can say that the latter is equivalent to $K(\mathfrak{g}_1, a) = 0$. Moreover, by Proposition 3.1.4, $K(\mathfrak{g}_k, a) = 0$ for all $k \neq 1$. So $K(\mathfrak{g}, a) = 0$ and, by non-degeneracy of K, $a = 0$.

Hence the $C_{\mathfrak{g}_0}(e)$-module $\mathfrak{g}_{-1}$ is self-dual by Lemma 3.3.16.
Similarly, the $C_{\mathfrak{g}_0}(e)$-module $\mathfrak{g}_0$ is self-dual since the bilinear form K is non-degenerate on $\mathfrak{g}_0$. So we can conclude using Theorem 3.3.13. $\square$

Chapter 4

Good gradings of $\mathfrak{sp}_{2n}$

4.1 Symplectic partitions and symplectic pyramids

Definition 4.1.1. A *partition* of n is a tuple $p = (p_1, \ldots, p_s)$ with $p_i \in \mathbb{N}$, $p_i \geq p_{i+1}$ and $p_1 + \ldots + p_s = n$. We denote by $Par(n)$ the set of all the partitions of n.

Definition 4.1.2. We denote by $\mathrm{mult}_p(j)$ the *multiplicity* of the number j in the partition p, i.e.,

$$\mathrm{mult}_p(j) := \#\{i : p_i = j\}.$$

Definition 4.1.3. Let $p = (p_1, \ldots, p_s) \in \mathrm{Par}(n)$. Then $p^* = (p_1^*, p_2^*, \ldots)$, where $p_j^* := \#\{i : p_i \geq j\}$, $j = 1, 2, \ldots$, is called the *dual partition* of p.

From now on, given a partition p, we denote by $p_1 > p_2 > \ldots > p_s$ its distinct non-zero parts and use notation $p = (p_1^{m_1}, \ldots, p_s^{m_s})$, where m_i is the multiplicity of p_i in p.

Definition 4.1.4. A partition $p = (p_1^{m_1}, \ldots, p_s^{m_s})$ is called *symplectic* if m_i is even for odd p_i.

Example 4.1.5. The partition $p = (5, 5, 4, 3, 3, 3, 3) = (5^2, 4^1, 3^4)$ is symplectic.

Recall the following result that will be very useful in this section and whose proof can be found in [[2], Theorem 5.13].

Theorem 4.1.6. *Symplectic partitions of 2n correspond bijectively to nilpotent orbits in* $\mathfrak{sp}_{2n}$.

Definition 4.1.7. Let $p = (p_1^{m_1}, \dots, p_s^{m_s})$ be a symplectic partition of $2n$. We define the *symplectic pyramid* $SP(p)$ as follows. It is a centrally symmetric (around (0,0)) collection of $2n$ boxes of size 1×1 on the plane, centered at points with integer coordinates (called the coordinates of the corresponding boxes).

- If $m_1 = 2k_1 + 1$ is odd, then the 0^{th} row of $SP(p)$ is non-empty and the first coordinates of boxes in this row form an arithmetic progression $-p_1 + 1, -p_1 + 3, \dots, p_1 - 1$. The rows from 1^{st} to k_1^{th} consist of boxes with the first coordinates forming the same arithmetic progression.

- If $m_1 = 2k_1$ is even, then the 0^{th} row of $SP(p)$ is empty and the first coordinates of boxes in the rows from 1^{st} to k_1^{th} form an arithmetic progression $-p_1 + 1, -p_1 + 3, \dots, p_1 - 1$.

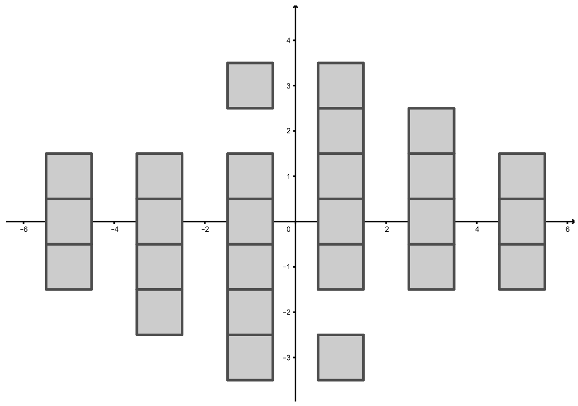
For the subsequent rows:

- If the multiplicity m_2 of p_2 is even, then the rows from the $(k_1 + 1)^{\text{th}}$ to the $(k_1 + \frac{m_2}{2})^{\text{th}}$ consist of boxes with first coordinates forming an arithmetic progression $-p_2 + 1, -p_2 + 3, \dots, p_2 - 1$.

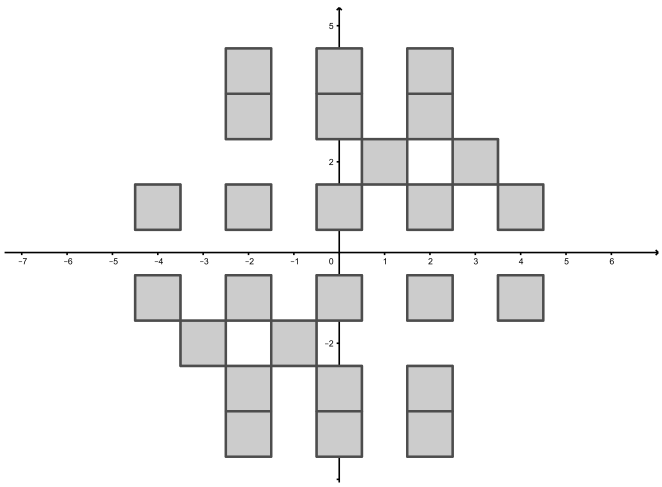
- If m_2 is odd, then the $(k_1 + 1)^{\text{th}}$ row consists of boxes with first coordinates forming the arithmetic progression $1, 3, \dots, p_2 - 1$ (recall that p_2 must be even if m_2 is odd) and the $\frac{m_2-1}{2}$ subsequent rows consist of boxes with first coordinates forming the arithmetic progression $-p_2 + 1, -p_2 + 3, \dots, p_2 - 1$.

All the subsequent parts of p are treated in the same way as p_2. The rows in the lower half-plane are obtained by the central symmetry.

Example 4.1.8. The symplectic pyramid $SP(p)$ associated to the symplectic partition $p = (6^3, 4^1, 2^2)$ is:



Example 4.1.9. The symplectic pyramid $SP(p)$ associated to the symplectic partition $p = (5^2, 4^1, 3^4)$ is:



4.2 The symplectic endomorphisms $e(p)$ and $h(p)$

Definition 4.2.1. The nilpotent endomorphism $e(p)$ of $\mathbb{F}^{2n}$ corresponding to a symplectic partition p is obtained by filling the boxes of $SP(p)$ by the standard basis vectors $v_1, \ldots, v_n, v_{-1}, \ldots, v_{-n}$ of $\mathbb{F}^{2n}$ such that vectors in boxes in the right half-plane ($x \geq 0$ and $y > 0$ if $x = 0$) have position indices

i and those in the centrally symmetric boxes have indices $-i$. Then $e(p)$ maps vectors in each box to its right neighbor (changed by sign if both the vectors involved are labeled with negative indices) and to 0 if there is no right neighbor, with the exception of the boxes with coordinates $(-1, -\ell)$ and no right neighbors; the vector in such a box is mapped by $e(p)$ to the vector in the $(1, \ell)$ box (which has no left neighbors).

Remark 4.2.2. The nilpotent endomorphism $e(p)$ is symplectic.

Proof. Recall that every symplectic endomorphism has matrix of this form:

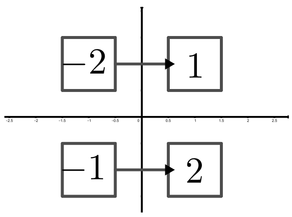
$$\left(\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array}\right)$$

where $A, B, C \in \mathfrak{gl}_n$, $B = B^T$, $C = C^T$.
Let $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be the matrix associated to $e(p)$. Then, by construction, $C = 0$ because the vectors $\{v_1, \dots, v_n\}$ are mapped to themselves. Furthermore, since $B = (b_{ij})$ is the matrix of the coordinates of the vectors $v_1, \dots, v_n$ in the images of $v_{-1}, \dots, v_{-n}$, B is symmetric because:

- if $v_{-i} \mapsto v_i$, then the i^{th} diagonal entry of B is 1;
- if $v_{-i} \mapsto v_j$ for $j \neq i$, by construction we have that $v_{-j} \mapsto v_i$, and so $b_{ij} = 1$ if and only if $b_{ji} = 1$.

Similarly, since by construction, if $v_i \mapsto v_j$, then $v_{-j} \mapsto -v_{-i}$, $D = -A^T$. $\square$

Example 4.2.3. Let $p = (2^2)$. The nilpotent symplectic endomorphism $e(p)$ can be graphically represented by the following collection of arrows.



The endomorphism $e(p)$ represented in figure corresponds to:

$v_{-2} \mapsto v_1 \mapsto 0$

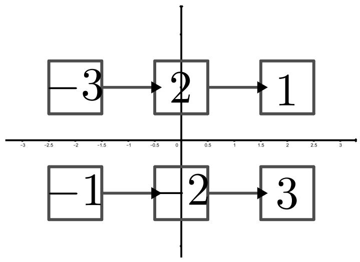
$v_{-1} \mapsto v_2 \mapsto 0$

where v_i is the i^{th} basis vector of the standard basis of $\mathbb{F}^4$ and v_{-i} is the $(n+i)^{\text{th}}$ basis vector of the standard basis of $\mathbb{F}^4$ $(n = 2)$.

The symplectic matrix associated to $e(p)$ is:

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Example 4.2.4. Let $p = (3^2)$. The nilpotent symplectic endomorphism $e(p)$ can be graphically represented by the following collection of arrows.



The endomorphism $e(p)$ represented in figure corresponds to:

$v_{-3} \mapsto v_2 \mapsto v_1 \mapsto 0$

$v_{-1} \mapsto v_{-2} \mapsto v_3 \mapsto 0.$

The symplectic matrix associated to $e(p)$ is:

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Definition 4.2.5. In the same setting as before, we define the diagonal matrix $h(p) \in \mathfrak{sp}_{2n}$ by letting its j^{th} diagonal entry equal to the first coordinate of the center of the j^{th} box of $SP(p)$.

Remark 4.2.6. The eigenspace decomposition of $\mathrm{ad}(h(p))$ is a $\mathbb{Z}$-grading of $\mathfrak{sp}_{2n}$.

Proof. By Proposition 1.7.6, we know that $\mathrm{ad}(h(p))$ is semisimple. Let $h(p) = \mathrm{diag}(h_1, \dots, h_n, -h_1, \dots, -h_n)$. In order to show that the eigenspace decomposition of $\mathrm{ad}(h(p))$ is a $\mathbb{Z}$-grading, we calculate the brackets of $h(p)$ with the elements of the basis of $\mathfrak{sp}_{2n}$ described in Example 1.1.9, using Proposition 1.7.6.

- $[h(p), E_{ii} - E_{n+i,n+i}] = (h_i - h_i)E_{ii} - (h_{n+i} - h_{n+i})E_{n+i,n+i} = 0$;
- $[h(p), E_{ij} - E_{n+j,n+i}] = (h_i - h_j)E_{ij} - (-h_j + h_i)E_{n+j,n+i}$
  $= (h_i - h_j)(E_{ij} - E_{n+j,n+i})$;
- $[h(p), E_{i,n+i}] = (h_i + h_i)E_{i,n+i} = 2h_iE_{i,n+i}$;
- $[h(p), E_{i,n+j} + E_{j,n+i}] = (h_i + h_j)E_{i,n+j} + (h_j + h_i)E_{j,n+i}$
  $= (h_i + h_j)(E_{i,n+j} + E_{j,n+i})$;
- $[h(p), E_{n+i,i}] = (-h_i - h_i)E_{n+i,i} = -2h_iE_{n+i,i}$;
- $[h(p), E_{n+i,j} + E_{n+j,i}] = (-h_i - h_j)E_{n+i,j} + (-h_j - h_i)E_{i,n+j}$
  $= -2(h_i + h_j)(E_{n+i,j} + E_{n+j,i})$.

Hence we can write the eigenspace decomposition of $\mathfrak{sp}_{2n}$ as

$$\mathfrak{sp}_{2n} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{sp}(\mathbb{F}^{2n})_k,$$

where $\mathfrak{sp}(\mathbb{F}^{2n})_k = \{x \in \mathfrak{sp}_{2n} \mid [h(p), x] = kx\}$.
One can easily show that if $x \in \mathfrak{sp}(\mathbb{F}^{2n})_i$ and $y \in \mathfrak{sp}(\mathbb{F}^{2n})_j$, then $[x, y] \in \mathfrak{sp}(\mathbb{F}^{2n})_{i+j}$. So the eigenspace decomposition of $\mathrm{ad}(h(p))$ is a $\mathbb{Z}$-grading of $\mathfrak{sp}_{2n}$. $\square$

Remark 4.2.7. Let $\mathfrak{sp}_{2n} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be the $\mathbb{Z}$-grading defined by $\mathrm{ad}(h(p))$. Then $e(p) \in \mathfrak{g}_2$.

Proof. By definition, $e(p)$ is a sum of elementary endomorphisms E_{st} which connect boxes with centers with first coordinate h_s with boxes with centers with first coordinate $h_s + 2$ (see figure in Example 4.2.4). Thus, by Proposition 1.7.6, $[h(p), E_{st}] = 2E_{st}$. So $E_{st} \in \mathfrak{g}_2$ and also $e(p) \in \mathfrak{g}_2$. □

Remark 4.2.8. The nilpotent endomorphism $e(p)$ can be embedded into an $\mathfrak{sl}_2$-triple $\{e(p), h(p), f(p)\}$ containing $h(p)$.

Proof. By Remark 4.2.7, $[h(p), e(p)] = 2e(p)$. Thus, by Claim 2. of the proof of the Jacobson-Morozov Theorem (Theorem 2.2.5), $h(p) \in [\mathfrak{sp}_{2n}, e(p)]$. So, by Lemma 2.2.3, there exists an $\mathfrak{sl}_2$-triple as requested. □

4.3 Classification of good gradings of $\mathfrak{sp}_{2n}$

Definition 4.3.1. Let $k \in \mathfrak{sp}_{2n}$ be a diagonal matrix and e be a nilpotent symplectic endomorphism. Then we say that the pair (k, e) is good if e is a good element in the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(k)$.

Remark 4.3.2. The diagonal matrix $h(p) \in \mathfrak{sp}_{2n}$ defines the Dynkin grading corresponding to the symplectic nilpotent endomorphism $e(p)$. Thus, the pair $(h(p), e(p))$ is good.

Proof. By Remark 4.2.8, we know that there exists an $\mathfrak{sl}_2$-triple containing $h(p)$ and $e(p)$; thus the grading induced by $\mathrm{ad}(h(p))$ is the Dynkin grading, which is good with good element $e(p)$ by Proposition 3.2.2. □

Theorem 4.3.3. *Let $e \in \mathfrak{sp}_{2n}$ be a nilpotent element, and $p = (p_1, \ldots, p_s)$ the partition of the Jordan canonical form of e. Let $(p_1^*, \ldots, p_r^*)$ be the dual partition of p. Then the dimension of the centralizer of e in $\mathfrak{sp}_{2n}$ is:*

$$\sum_{i=1}^{r} \frac{(p_i^*)^2}{2} + \frac{1}{2}\#\{j \in \{1, \ldots, s\} \mid p_j \textit{ is odd}\}.$$

Proof. See [10]. □

A description of the reductive parts of centralizers $C_{\mathfrak{g}}(e)$ for a nilpotent element e in the Lie algebras of classical type can be found in [1]. The following theorem follows from this description.

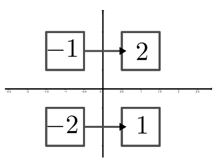
Theorem 4.3.4. *Let $e = e(p) \in \mathfrak{sp}_{2n}$ be the nilpotent element corresponding to a partition p, and let $c(p)$ be the dimension of the center of the reductive part of $C_{\mathfrak{sp}_{2n}}(e)$. Then:*

$$c(p) = \#\{\textit{even parts of the partition } p \textit{ with multiplicity } 2\}.$$

By Theorem 4.3.4, the dimension of the center of the reductive part of $C_{\mathfrak{sp}_{2n}}(e(p))$ is equal to $c(p)$, the number of even parts of the partition p having multiplicity 2. If $c(p) = 0$ then, by Corollary 3.3.5, the only good grading of $\mathfrak{sp}_{2n}$ with good element $e(p)$ is the Dynkin one. Thus, we may assume from now on that $c(p) > 0$.
An explicit description of the center of the reductive part of $C_{\mathfrak{sp}_{2n}}(e(p))$ is given by the following result, that can be found in [3].

Lemma 4.3.5. *Let $p_1, \dots, p_{c(p)}$ be all distinct even parts of a symplectic partition p, having multiplicity 2. Define diagonal matrices $z(t_1, \dots, t_{c(p)}) \in \mathfrak{sp}_{2n}$, $t_1, \dots, t_{c(p)} \in \mathbb{F}$, whose j^{th} diagonal entry is t_i if the j^{th} basis vector lies in a box of $SP(p)$ in the (strictly) upper half-plane in a row corresponding to the part p_i, and is $-t_i$ if the j^{th} basis vector lies in the centrally symmetric box, and all other entries are zero. Then the center of the reductive part of $C_{\mathfrak{sp}_{2n}}(e(p))$ consists of all these matrices.*

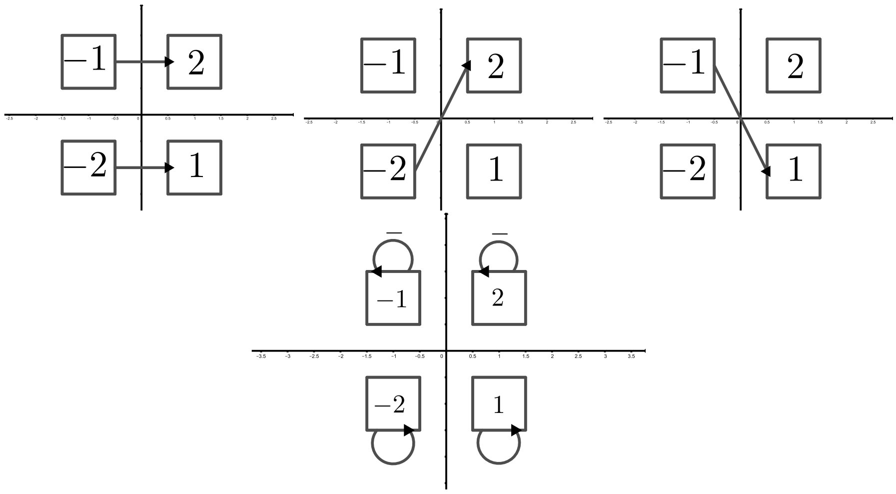
Example 4.3.6. Consider the partition $p = (2^2)$ of the number 4. Notice that, by Theorem 4.3.4, this is the only symplectic partition of the number 4 such that $c(p) > 0$ (in particular $c(p) = 1$).
Consider the symplectic endomorphism $e(p)$ associated to the following choice of the labeling of $SP(p)$.



whose associated matrix is $e(p) = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \end{array}\right)$.

By Lemma 4.3.5 the center of the reductive part of the centralizer of $e(p)$ consists of all the matrices of the form $z(t_1) = \mathrm{diag}(-t_1, t_1, t_1, -t_1)$. Now we want to show explicitly that the elements of the center of the reductive part of the centralizer of $e(p)$ are precisely those of this form.

First of all notice that, by Theorem 4.3.3, $\dim C_{\mathfrak{sp}_4}(e(p)) = 4$. A basis of $C_{\mathfrak{sp}_4}(e(p))$ is given by the following diagrams:



where the minus above an arrow connecting the i^{th} and the j^{th} basis vector indicates that $v_i \mapsto -v_j$.

So any element of $C_{\mathfrak{sp}_4}(e(p))$ is of the form:

$$x = \left(\begin{array}{cc|cc} d & 0 & b & a \\ 0 & -d & a & c \\ \hline 0 & 0 & -d & 0 \\ 0 & 0 & 0 & d \end{array}\right).$$

We now want to determine $f(p) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & a & \multirow{2}{*}{0} \\ b & 0 & \end{array}\right) \in \mathfrak{sp}_4$ such that $\mathfrak{s} = \{e(p), h(p), f(p)\}$ is an $\mathfrak{sl}_2$-triple, with $h(p) = \mathrm{diag}(1, 1, -1, -1)$. So, imposing that $[e(p), f(p)] = h(p)$, we get $a = b = 1$.

In order to describe $C_{\mathfrak{sp}_4}(\mathfrak{s})$ it is enough to see which conditions an element $x \in C_{\mathfrak{sp}_4}(e(p))$ has to satisfy so that it also belongs to $C_{\mathfrak{sp}_4}(f(p))$. Hence $x \in C_{\mathfrak{sp}_4}(f(p))$ if and only if:

$$0 = [f(p), x] = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & -d & a & c \\ d & 0 & b & a \end{array}\right) - \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & a & 0 & 0 \\ \hline 0 & -d & 0 & 0 \\ d & 0 & 0 & 0 \end{array}\right)$$

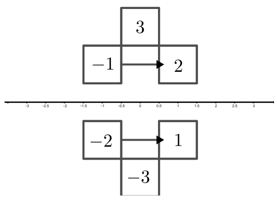
from which we get the conditions $a = b = c = 0$. So an element x belongs to $C_{\mathfrak{sp}_4}(\mathfrak{s})$ if and only if it is of the form

$$x = \left(\begin{array}{cc|cc} d & & & \\ & -d & & \\ \hline & & -d & \\ & & & d \end{array}\right).$$

Notice that such an element also belongs to $Z(C_{\mathfrak{sp}_4}(\mathfrak{s}))$ because it is diagonal. This agrees with the description of $Z(C_{\mathfrak{sp}_4}(\mathfrak{s}))$ given by Lemma 4.3.5.

Example 4.3.7. Consider the partition $p = (2^2, 1^2)$ of the number 6. Notice that this is the only symplectic partition of the number 6 such that $c(p) > 0$ (in particular $c(p) = 1$) by Theorem 4.3.4.

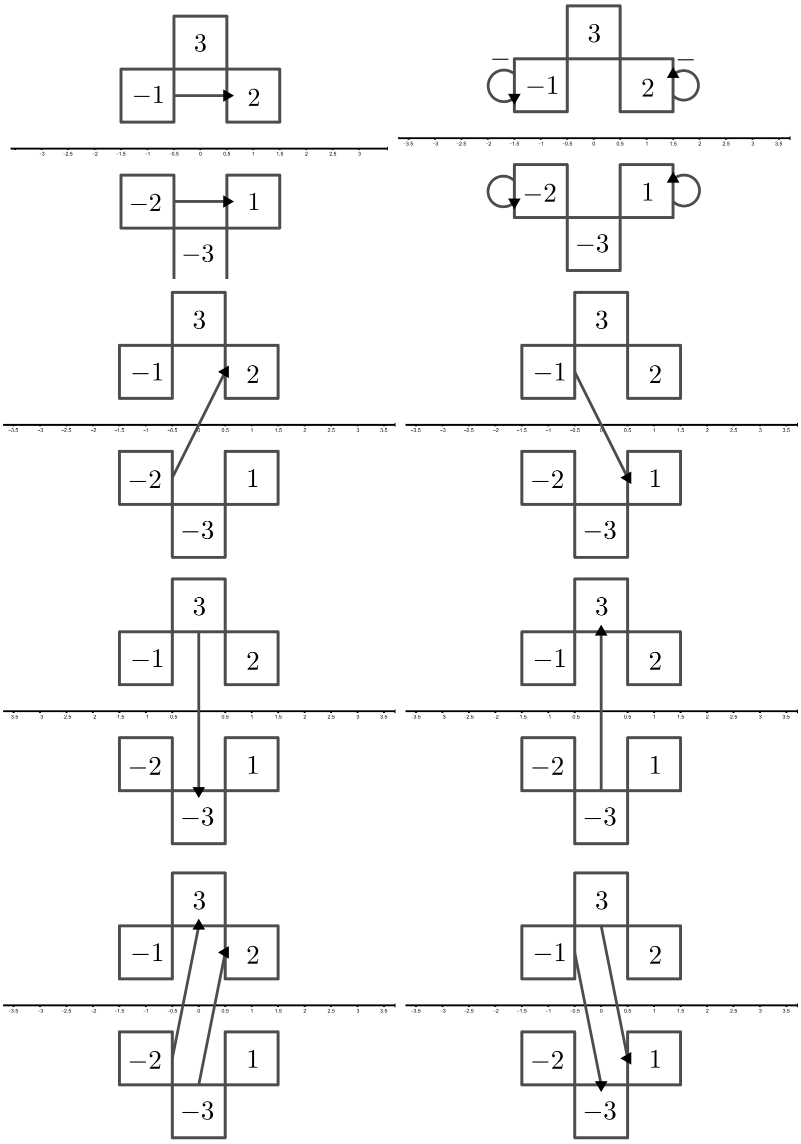
Consider the symplectic endomorphism $e(p)$ associated to the following choice of the labeling of $SP(p)$.

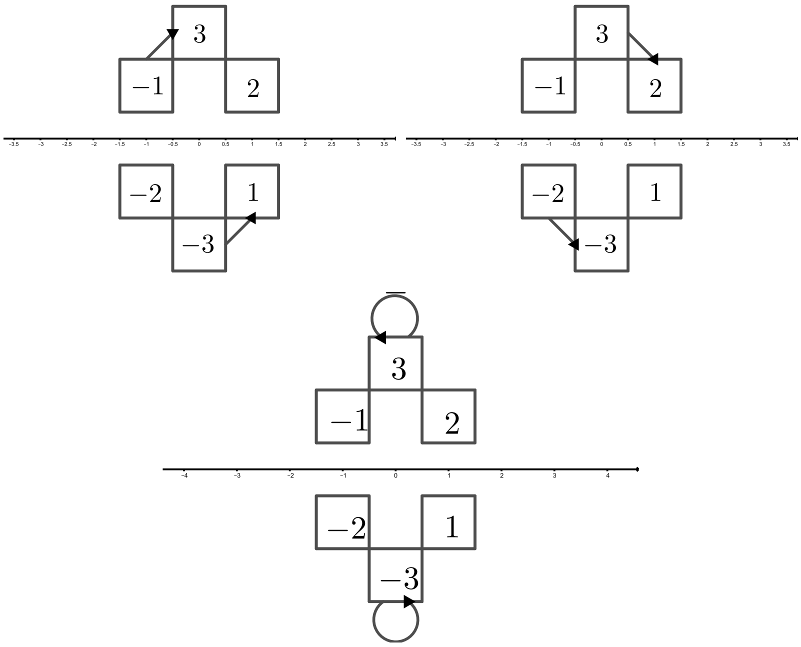


whose associated matrix is $e(p) = \left(\begin{array}{c|ccc} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ \hline 0 & & 0 & \end{array}\right)$.

By Lemma 4.3.5 the center of the reductive part of the centralizer of $e(p)$

consists of all the matrices of the form $z(t_1) = \text{diag}(-t_1, t_1, 0, t_1, -t_1, 0)$. Now we want to show explicitly that the elements of the center of the reductive part of the centralizer of $e(p)$ are precisely those of this form.
First of all notice that, by Theorem 4.3.3, $\dim C_{\mathfrak{sp}_4}(e(p)) = 11$. A basis is given by the following diagrams:





Thus any element of $C_{\mathfrak{sp}_6}(e(p))$ is of the form:

$$x = \left(\begin{array}{ccc|ccc} a & 0 & l & e & c & i \\ 0 & -a & m & c & d & h \\ 0 & 0 & -b & i & h & f \\ \hline 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & g & -l & -m & b \end{array}\right).$$

We now want to determine $f(p) = \left(\begin{array}{c|c} 0 & 0 \\ \hline \begin{array}{ccc} 0 & n & 0 \\ p & 0 & 0 \\ 0 & 0 & 0 \end{array} & 0 \end{array}\right) \in \mathfrak{sp}_6$ such that $\mathfrak{s} = \{e(p), h(p), f(p)\}$ is an $\mathfrak{sl}_2$-triple, with $h(p) = \operatorname{diag}(1,1,0,-1,-1,0)$. So, imposing that $[e(p), f(p)] = h(p)$, we get $n = p = 1$.

In order to describe $C_{\mathfrak{sp}_6}(\mathfrak{s})$ it is enough to see which conditions an element $x \in C_{\mathfrak{sp}_6}(e(p))$ has to satisfy so that it also belongs to $C_{\mathfrak{sp}_6}(f(p))$. Hence

$x \in C_{\mathfrak{sp}_6}(f(p))$ if and only if:

$$0 = [x, f(p)] = \left(\begin{array}{ccc|ccc} c & e & 0 & 0 & 0 & 0 \\ d & c & 0 & 0 & 0 & 0 \\ h & i & 0 & 0 & 0 & 0 \\ \hline 0 & -a & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ -m & -l & 0 & 0 & 0 & 0 \end{array}\right) - \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -a & m & c & d & h \\ a & 0 & l & e & c & i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

from which we get the conditions $c = d = e = h = i = l = m = 0$. So an element x belongs to $C_{\mathfrak{sp}_6}(\mathfrak{s})$ if and only if it is of the form

$$x = \left(\begin{array}{ccc|ccc} a & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 & 0 & f \\ \hline 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & g & 0 & 0 & b \end{array}\right).$$

So, imposing that $x \in Z(C_{\mathfrak{sp}_6}(\mathfrak{s}))$ we get the conditions $b = f = g = 0$. Thus a generic element of $Z(C_{\mathfrak{sp}_6}(\mathfrak{s}))$ is of the form

$$\left(\begin{array}{ccc|ccc} a & & & & & \\ & -a & & & & \\ & & 0 & & & \\ \hline & & & -a & & \\ & & & & a & \\ & & & & & 0 \end{array}\right).$$

Remark 4.3.8. In the next theorem we want to characterize all the good $\mathbb{Z}$-gradings with good element $e(p)$. Such gradings are induced by a pair $(h(p) + h, e(p))$, where $h \in \mathfrak{sp}_{2n}$ is a diagonal matrix. By Theorem 3.3.4, h belongs to the center of the reductive part of the centralizer of $e(p)$; thus, by Lemma 4.3.5, $h = z(t_1, \ldots, t_{c(p)})$ for some $t_1, \ldots, t_{c(p)} \in \mathbb{F}$.

Remark 4.3.9. In the same notation as in Remark 4.3.8 notice that, if we consider the "symplectic pyramid" $\tilde{P}$ obtained by $SP(p)$ by shifting by t_i to the right the row of length p_i in the upper half plane, by t_i to the left the row of length p_i in the lower half plane and leaving all the other rows fixed, then the grading determined by $\mathrm{ad}(h(p)+h)$ is the one induced by $h(\tilde{P})$. Recall that $(h(p)+h, e(p))$ is a good pair if and only if $C_{\mathfrak{sp}_{2n}}(e(p))$ is contained in the non-negative part of the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(h(p)+z(t_1,\ldots,t_{c(p)}))$. But the pyramid $\tilde{P}$ is obtained by $SP(p)$ moving only the rows of length $p_1,\ldots,p_{c(p)}$; thus we will need to see when the endomorphisms of $C_{\mathfrak{sp}_{2n}}(e(p))$ connecting rows with at least one of length p_i ($i \in \{1,\ldots,c(p)\}$) are contained in the non-negative part of the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(h(p)+z(t_1,\ldots,t_{c(p)}))$.

Remark 4.3.10. Let $k, h \in \mathbb{Z}$, $k > h$. consider the following partitions:

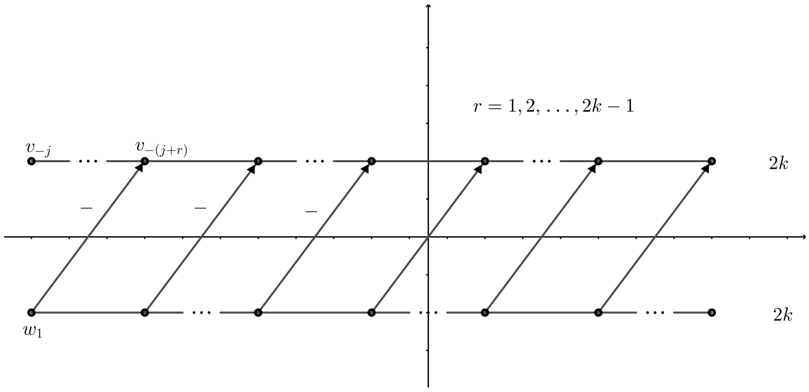
1. $p_1 = ((2k)^2)$;
2. $p_2 = ((2k)^2, 2h)$;
3. $p_3 = ((2k)^2, (2h+1)^2)$.
4. $p_4 = ((2k)^2, (2h)^2)$;

Then:

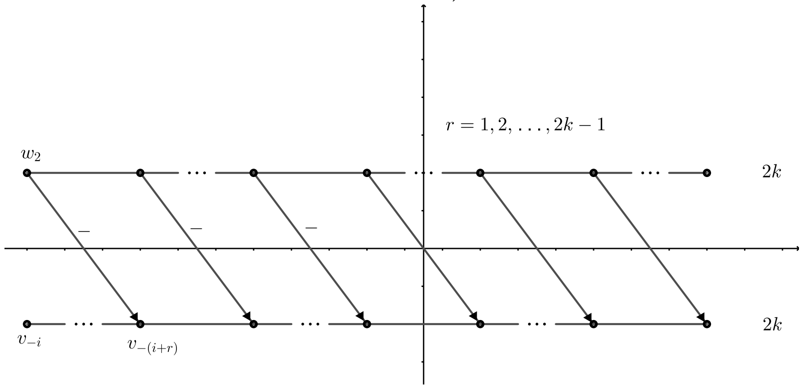
1. The dimension of the centralizer of the symplectic endomorphism $e(p_1)$ can be obtained by Theorem 4.3.3, and it turns out to be

$$\mathrm{dim} C_{\mathfrak{sp}_{2n}}(e(p_1)) = 4k.$$

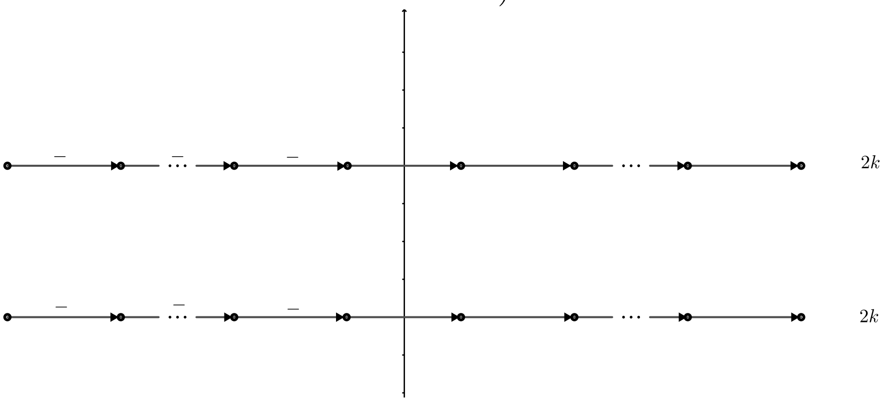
   A description can be given by the following diagrams commuting with $e(p_1)$ and describing symplectic endomorphisms because they are linearly independent since every such endomorphism f is a sum of elementary ones, which appear as summands only in f, and in no other endomorphism.



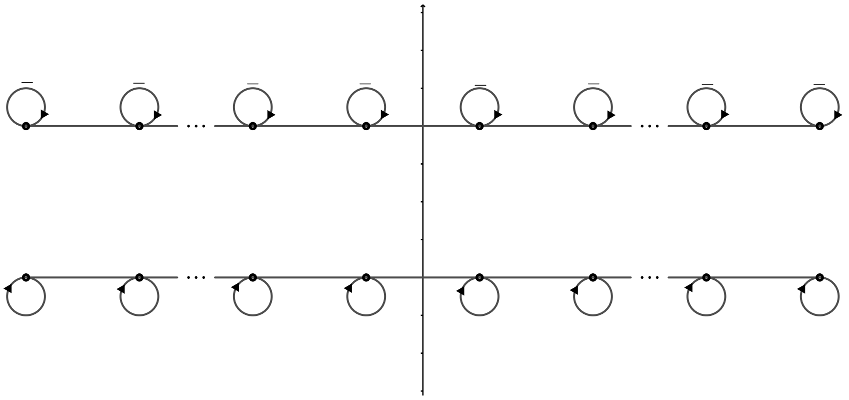
Type a)



Type b)



Type c)



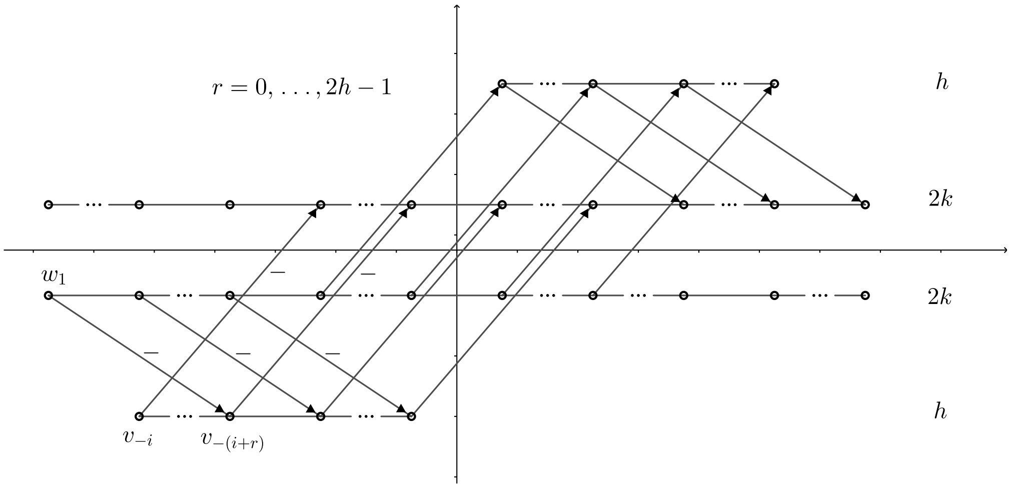
TYPE D)

Notice that, due to its symplecticity, every endomorphism of Type A) (resp. Type B)) is uniquely determined by the image of the vector w_1 (resp. w_2) indicated in the figure in the upper (resp. lower) half plane.

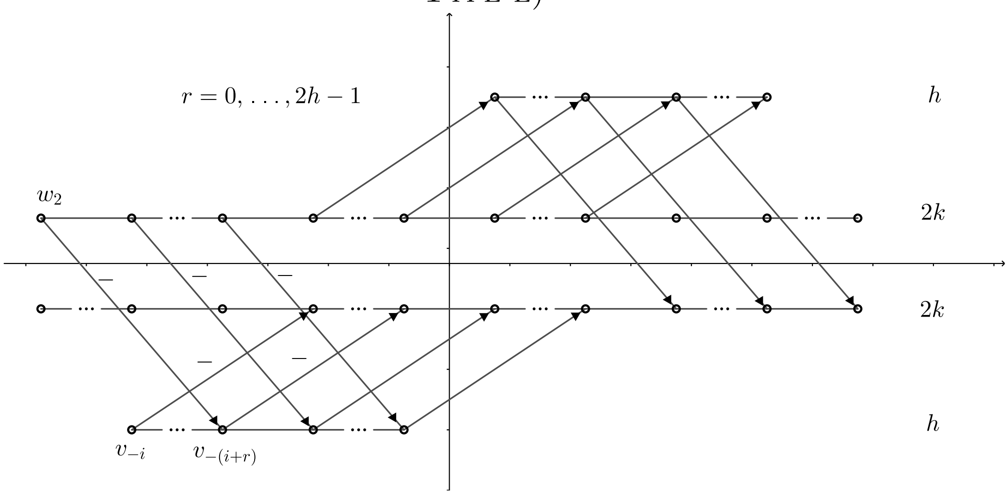
2. The dimension of the centralizer of the symplectic endomorphism $e(p_2)$ can be obtained by Theorem 4.3.3, and it turns out to be $\dim C_{\mathfrak{sp}_{2n}}(e(p_2)) = 5h + 4k$. Furthermore, Theorem 4.3.3 allows us to say that $\dim C_{\mathfrak{sp}_{2n}}(e((2k)^2)) = 4k$ and $\dim C_{\mathfrak{sp}_{2n}}(e((2h))) = h$; thus the number of diagrams commuting with $e(p_2)$ that describe symplectic endomorphisms and link rows of different lengths is

$$\dim C_{\mathfrak{sp}_{2n}}(e(p_2)) - \dim C_{\mathfrak{sp}_{2n}}(e((2k)^2)) - \dim C_{\mathfrak{sp}_{2n}}(e((2h))) = 4h.$$

A description can be given by the following diagrams, which are linearly independent for the same reason of 1.



TYPE E)



TYPE F)

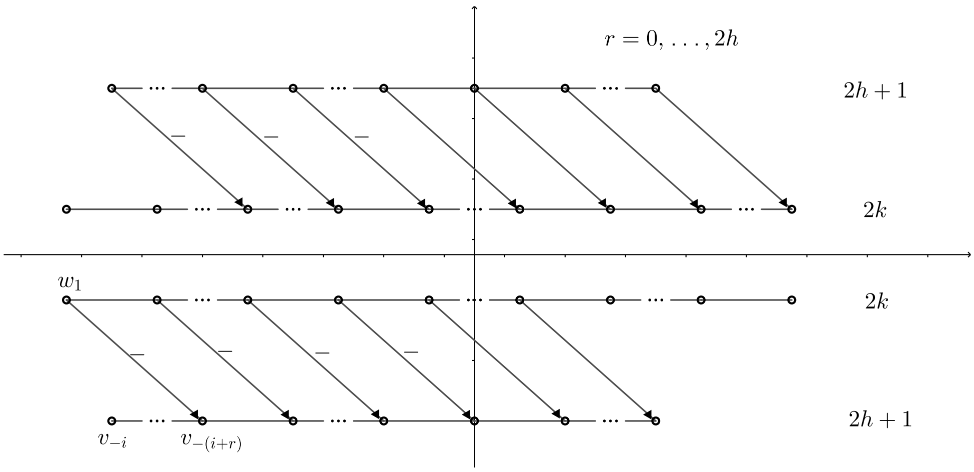
Notice that, due to its symplecticity, every endomorphism of Type E) (resp. Type F)) is uniquely determined by the image of the vector w_1 (resp. w_2) indicated in figure.

3. The dimension of the centralizer of the symplectic endomorphism $e(p_3)$ can be obtained by Theorem 4.3.3, and it turns out to be $\dim C_{\mathfrak{sp}_{2n}}(e(p_3)) = 12h + 4k + 7$. Furthermore, Theorem 4.3.3 allows us to say that $\dim C_{\mathfrak{sp}_{2n}}(e((2k)^2)) = 4k$ and $\dim C_{\mathfrak{sp}_{2n}}(e((2h+1)^2)) = 4h+3$; thus the number of diagrams commuting with $e(p_3)$ that describe

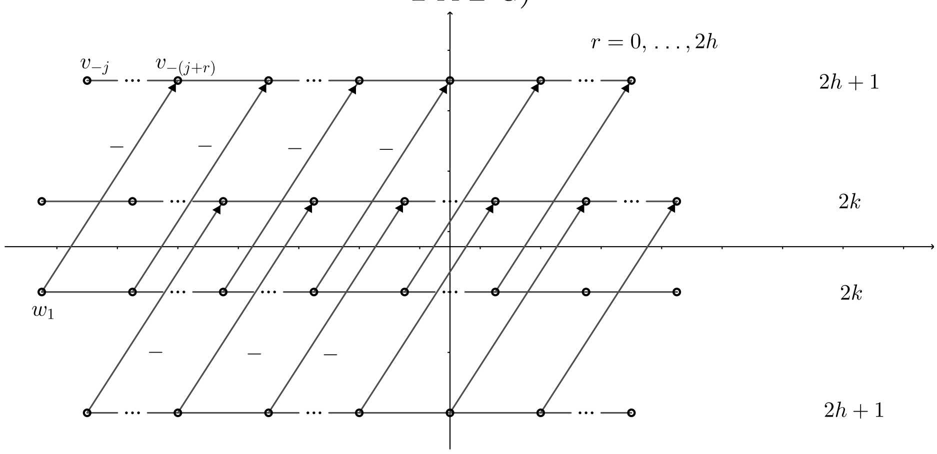
symplectic endomorphisms and link rows of different lengths is

$$\dim C_{\mathfrak{sp}_{2n}}(e(p_3)) - \dim C_{\mathfrak{sp}_{2n}}(e((2k)^2)) - \dim C_{\mathfrak{sp}_{2n}}(e((2h+1)^2)) = 8h+4.$$

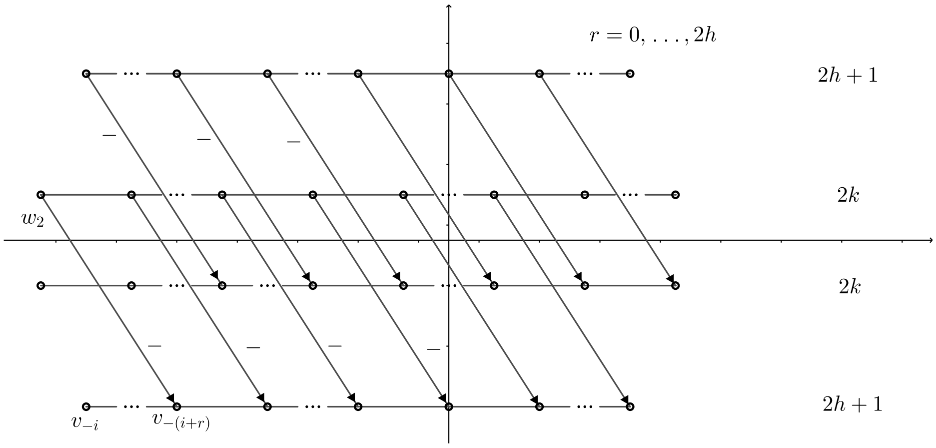
A description can be given by the following diagrams, which are linearly independent for the same reason of 1.



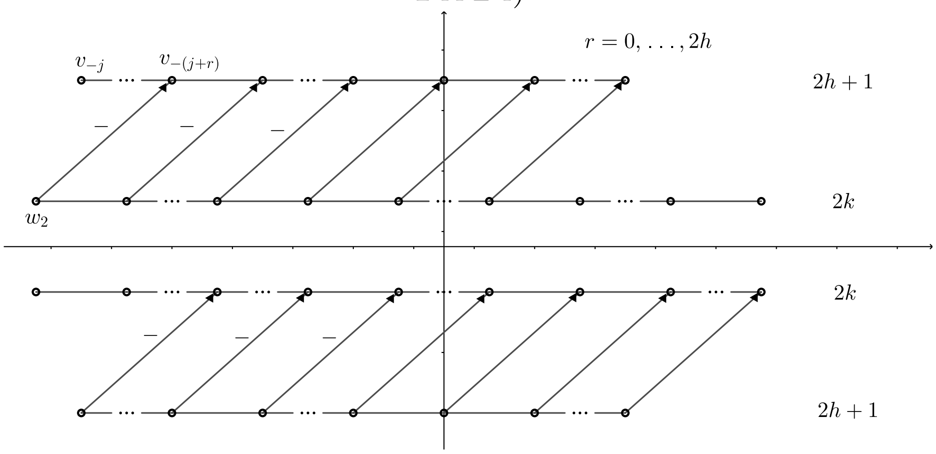
TYPE G)



TYPE H)



TYPE I)



TYPE L)

Notice that, due to its symplecticity, every endomorphism of Type G) (resp. Type H)) is uniquely determined by the image of the vector w_1 in the lower (resp. upper) half plane. Analogously, every endomorphism of Type I) (resp. Type L)) is uniquely determined by the image of the vector w_2 in the lower (resp. upper) half plane.

4. The dimension of the centralizer of the symplectic endomorphism $e(p_4)$ can be obtained by Theorem 4.3.3, and it turns out to be $\dim C_{\mathfrak{sp}_{2n}}(e(p_4)) = 12h + 4k$. Furthermore, Theorem 4.3.3 allows us to say that $\dim C_{\mathfrak{sp}_{2n}}(e((2k)^2)) = 4k$ and $\dim C_{\mathfrak{sp}_{2n}}(e((2h)^2)) = 4h$; thus the number of diagrams commuting with $e(p_4)$ that describe symplectic

endomorphisms and link rows of different lengths is

$$\dim C_{\mathfrak{sp}_{2n}}(e(p_4)) - \dim C_{\mathfrak{sp}_{2n}}(e((2k)^2)) - \dim C_{\mathfrak{sp}_{2n}}(e((2h)^2)) = 8h.$$

The diagrams describing such endomorphisms are of the same form of the ones commuting with $e(p_3)$ (see diagrams of Type G), H), I), L)), with the only difference that the step r ranges from 0 to $2h-1$.

Theorem 4.3.11. *The element $H(p) := h(p)+h$ defines a good $\mathbb{Z}$-grading of $\mathfrak{sp}_{2n}$ if and only if $h = z(t_1, \dots, t_{c(p)})$ for some $t_1, \dots, t_{c(p)} \in \mathbb{F}$ (as in Lemma 4.3.5) and one of the following cases holds:*

1. *all parts of p are even and have multiplicity 2, and either all $t_i \in \{-1, 0, 1\}$ or all $t_i \in \{\frac{1}{2}, -\frac{1}{2}\}$;*
2. *not all parts of p are even of multiplicity 2, and all $t_i \in \{-1, 0, 1\}$.*

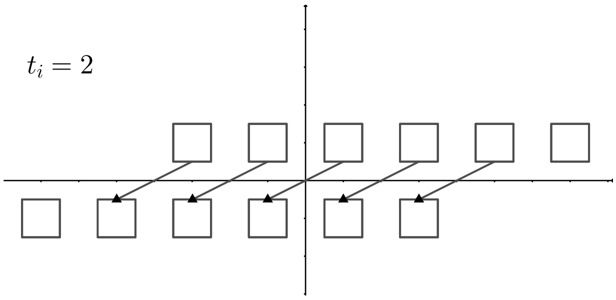
We will denote $H(p)$ by $H(p; t_1, \dots, t_{c(p)})$.
These $\mathbb{Z}$-gradings are the same if and only if the t_i's differ by signs. Furthermore, these are all good $\mathbb{Z}$-gradings of $\mathfrak{sp}_{2n}$ for which $e(p)$ is a good element (up to conjugation by the centralizer of $e(p)$ in Sp_{2n}).

Proof. Suppose that the pair $(H(p), e(p))$ is good. First of all notice that, by Remark 4.3.8, $h = z(t_1, \dots, t_{c(p)})$ for some $t_1, \dots, t_{c(p)} \in \mathbb{F}$.

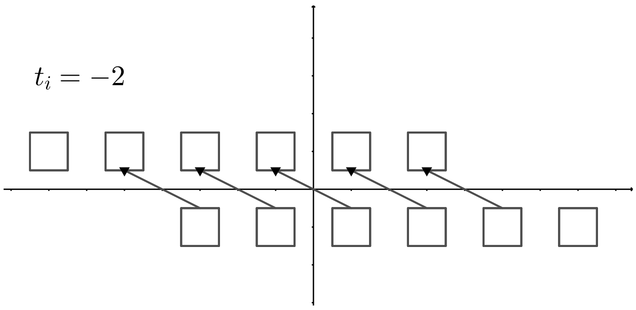
1. If all the parts of p are even with multiplicity 2, then $\mathrm{ad}(H(p))$ defines a $\mathbb{Z}$-grading if and only if all the differences of the diagonal elements of $z(t_1, \dots, t_{c(p)})$ are integers. But such differences are $\pm 2t_i$, $\pm(t_i - t_j)$ or $\pm(t_i + t_j) = \pm(t_i - t_j + 2t_j)$; thus $\mathrm{ad}(H(p))$ defines a $\mathbb{Z}$-grading if and only if $t_i \in \mathbb{Z}/2$ and $t_i - t_j \in \mathbb{Z}$ for all $i, j = 1, \dots, c(p)$.
2. If not all the parts of p are even with multiplicity 2, then $\mathrm{ad}(H(p))$ defines a $\mathbb{Z}$-grading if and only if all the differences of the diagonal elements of $z(t_1, \dots, t_{c(p)})$ are integers. But such differences are $\pm t_i$, $\pm(t_i - t_j)$ or $\pm(t_i + t_j) = \pm(t_i - t_j + 2t_j)$; thus $\mathrm{ad}(H(p))$ defines a $\mathbb{Z}$-grading if and only if $t_i \in \mathbb{Z}$ and $t_i - t_j \in \mathbb{Z}$ for all $i, j = 1, \dots, c(p)$.

Now, consider the "symplectic pyramid" $\tilde{P}$ obtained as in Remark 4.3.9. Then the grading determined by $\mathrm{ad}(H(p; t_1, \dots, t_{c(p)}))$ is the one induced by $h(\tilde{P})$.

By Remark 4.3.9, $(H(p; t_1, \dots, t_{c(p)}), e(p))$ is a good pair if and only if the endomorphisms of $C_{\mathfrak{sp}_{2n}}(e(p))$ connecting rows with at least one with length p_i ($i \in \{1, \dots, c(p)\}$) are contained in the non-negative part of the $\mathbb{Z}$-grading given by the eigenspace decomposition of $\mathrm{ad}(H(p; t_1, \dots, t_{c(p)}))$. If $t_i > 1$, then the diagram of the type of figure B) in Remark 4.3.10 relative to the two rows of lenght p_i with step $r = 1$ becomes, for example:



If $t_i < -1$, then the diagram of the type of figure A) in Remark 4.3.10 relative to the two rows of lenght p_i with step $r = 1$ becomes, for example:



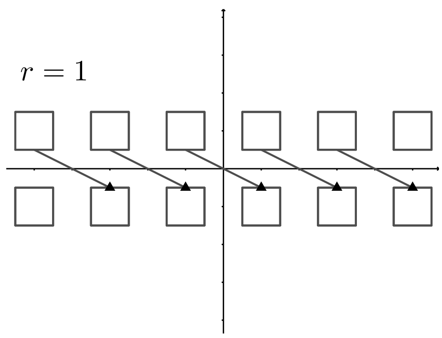
So, if $|t_i| > 1$, we can find an element $\varphi \in C_{\mathfrak{sp}_{2n}}(e(p))$ that belongs to the negative part of the grading induced by $\mathrm{ad}(H(p; t_1, \dots, t_{c(p)}))$; indeed the ending of the arrows in the diagrams above are located strictly to the left of their source. Hence $|t_i| \le 1$. By the conditions previously obtained, the only cases that can occur are the following:

1. if all the parts of p are even with multiplicity 2, then either all $t_i \in \{-1, 0, 1\}$ or all $t_i \in \{\frac{1}{2}, -\frac{1}{2}\}$;

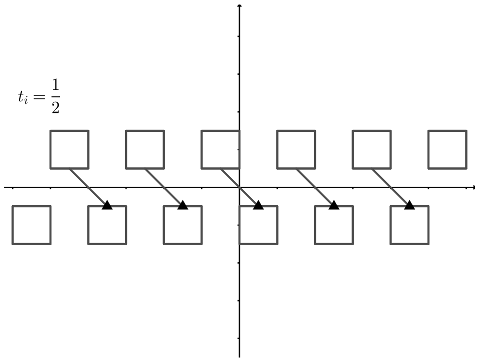
2. if not all parts of p are even of multiplicity 2, then all $t_i \in \{-1, 0, 1\}$.

Conversely, suppose $h = z(t_1, \ldots, t_{c(p)})$ for some $t_1, \ldots, t_{c(p)} \in \mathbb{F}$ and that either condition 1. or 2. holds. First of all, notice that condition 1. or 2. implies that the eigenspace decomposition of $\mathrm{ad}(H(p; t_1, \ldots, t_{c(p)}))$ is a $\mathbb{Z}$-grading. Let $\mathfrak{sp}_{2n} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ be the $\mathbb{Z}$-grading induced by $\mathrm{ad}(h(p))$ and $\mathfrak{sp}_{2n} = \bigoplus_{j\in\mathbb{Z}} \tilde{\mathfrak{g}}_j$ be the $\mathbb{Z}$-grading induced by $\mathrm{ad}(H(p; t_1, \ldots, t_{c(p)}))$. We want to show that, under conditions 1. or 2., $C_{\mathfrak{sp}_{2n}}(e(p)) \subseteq \bigoplus_{j\geq 0} \tilde{\mathfrak{g}}_j$. By Remark 4.3.9 we just need to prove that the diagrams described in Remark 4.3.10 are contained in $\bigoplus_{j\geq 0} \tilde{\mathfrak{g}}_j$. Furthermore it will be sufficient to prove just that the diagrams described in Remark 4.3.10 with the smallest step r are contained in $\bigoplus_{j\geq 0} \tilde{\mathfrak{g}}_j$ (indeed, if this happens to be true, this will hold also for the diagrams with bigger step r). Consider $i \in \{1, \ldots, c(p)\}$ and let $p_i = 2k$.

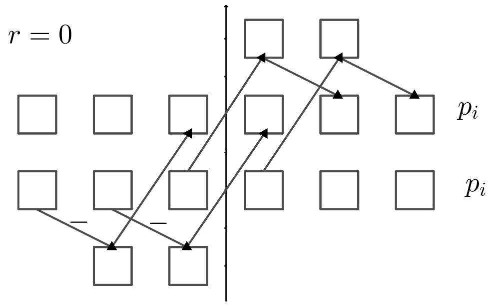
If we are in Case 1. of Remark 4.3.10 (i.e., we consider the partition $((2k)^2)$), then the minimum possible step is $r = 1$. Such an endomorphism φ can be represented in the following way:



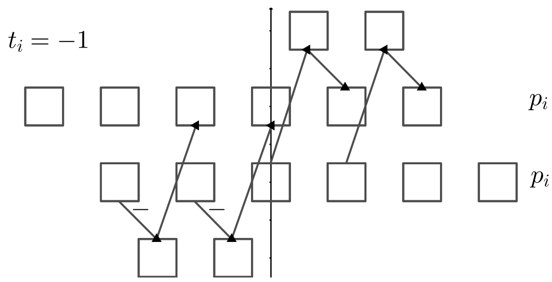
Therefore, since the difference of the coordinates of the columns between the ending of every row and its source in the diagram is 2, then $\varphi \in \mathfrak{g}_2$ and so the endomorphism φ will belong to $\tilde{\mathfrak{g}}_{2-2t_i}$ (see for example the figure below). But, as $|t_i| \leq 1$, $\varphi \in \tilde{\mathfrak{g}}_{\geq 0}$.



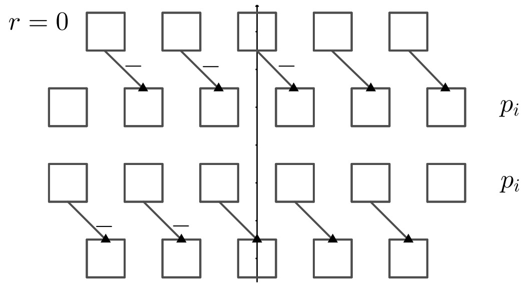
If we are in Case 2. of Remark 4.3.10 (i.e., we consider the partition $((2k)^2, 2h)$ with $k > h$), then the minimum possible step is $r = 0$. Such an endomorphism φ can be represented in the following way:



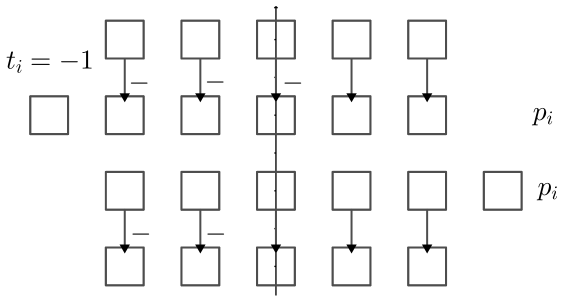
Therefore, since the difference of the coordinates of the columns between the ending of every row and its source in the diagram is 2, then $\varphi \in \mathfrak{g}_{2(k-h)}$ and so the endomorphism φ will belong to $\tilde{\mathfrak{g}}_{2(k-h)+t_i}$ (see for example the figure below). But, as $|t_i| \leq 1$ and $k > h$, $\varphi \in \tilde{\mathfrak{g}}_{\geq 0}$.



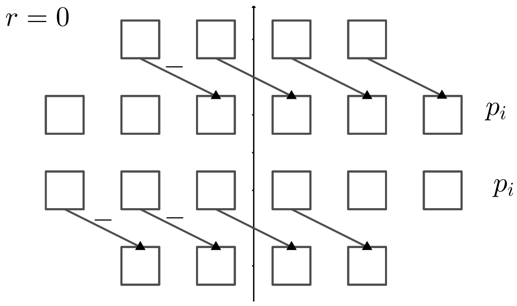
If we are in Case 3. of Remark 4.3.10 (i.e., we consider the partition $((2k)^2, (2h+1)^2)$ with $k > h$), then the minimum possible step is $r = 0$. Such an endomorphism φ can be represented in the following way:



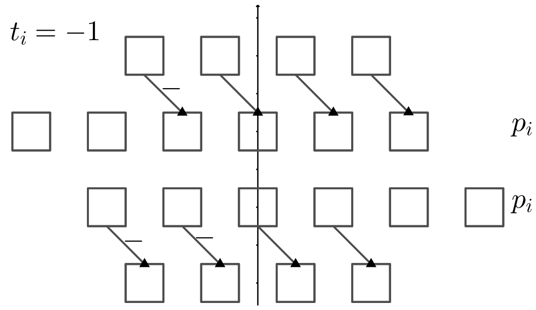
Therefore, since the difference of the coordinates of the columns between the ending of every row and its source in the diagram is 1, then $\varphi \in \mathfrak{g}_{2(k-h)-1}$ and so the endomorphism φ will belong to $\tilde{\mathfrak{g}}_{2(k-h)-1+t_i}$ (see for example the figure below). But, as $|t_i| \leq 1$, $\varphi \in \tilde{\mathfrak{g}}_{\geq 0}$.



If we are in Case 4. of Remark 4.3.10 (i.e., we consider the partition $((2k)^2, (2h)^2)$ with $k > h$), then the minimum possible step is $r = 0$. Such an endomorphism φ can be represented in the following way:



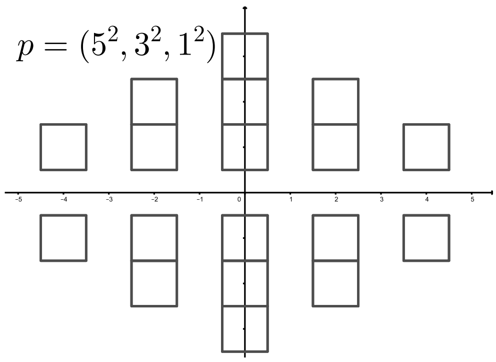
Therefore, since the difference of the coordinates of the columns between the ending of every row and its source in the diagram is 2, then $\varphi \in \mathfrak{g}_{2(k-h)}$ and so the endomorphism φ will belong to $\tilde{\mathfrak{g}}_{2(k-h)+t_i}$ (see for example the figure below). But, as $|t_i| \leq 1$ and $k > h$, $\varphi \in \tilde{\mathfrak{g}}_{\geq 0}$.



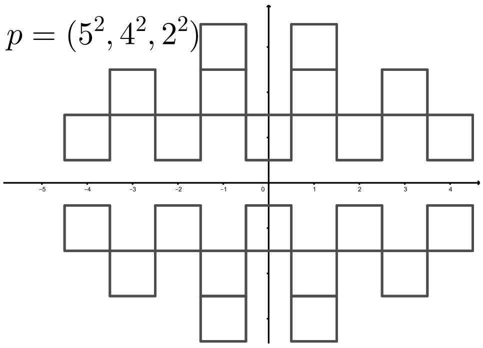
Hence, $C_{\mathfrak{sp}_{2n}}(e(p)) \subseteq \bigoplus_{j\geq 0} \tilde{\mathfrak{g}}_j$. Thus, the pair $(H(p; t_1, \ldots, t_{c(p)}), e(p))$ is good. $\square$

Corollary 4.3.12. *A nilpotent element $e(p)$ of $\mathfrak{sp}_{2n}$ is good for at least one even $\mathbb{Z}$-grading if and only if it is either even (i.e., its Dynkin grading is even), or it is odd and all even parts of p have multiplicity* 2.

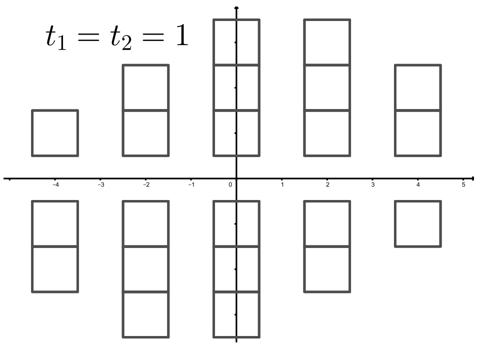
Proof. If p is even (i.e., all the parts of p have the same parity), then it is clear that the differences between the first coordinates of any two boxes in $SP(p)$ is even, as shown by the picture below.



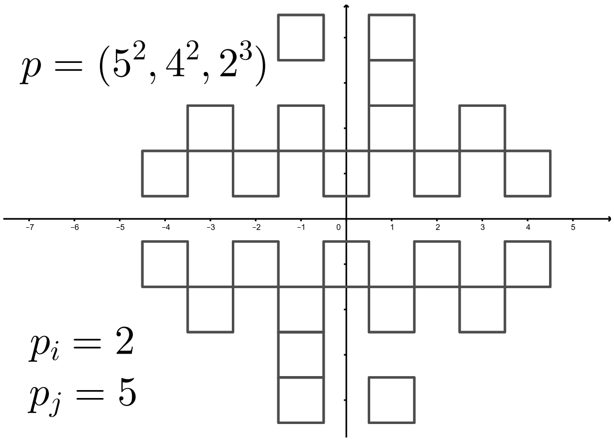
Thus the Dynkin grading with good element $e(p)$ is even.
Now, suppose that $p = (p_1^{m_1}, \ldots, p_s^{m_s})$ is odd (i.e., not all p_i's have the same parity) and all even parts of p have multiplicity 2. Then we are in the case given by the following figure.



If we consider the grading given by the eigenspace decomposition of $\mathrm{ad}(H(p, 1, \ldots, 1)) = \mathrm{ad}(h(p) + z(1, \ldots, 1))$, then it results to be even, as the figure below underlines.



But by Theorem 4.3.11, the pair $(H(p, 1, \ldots, 1), e(p))$ is good; thus we have also the second case.
Now, the only case we did not consider is the one when $p = (p_1^{m_1}, \ldots, p_s^{m_s})$ is odd but not all its even parts have multiplicity 2. In this case we can take an even p_i such that $m_i \neq 2$ and an odd p_j, like in the figure below.



The differences between the first coordinates of two rows with length p_i and p_j in $SP(p)$ is clearly odd. Furthermore, since adding $z(t_1, \ldots, t_{c(p)})$ to $h(p)$ does not change the position of the rows of length p_i and p_j in the "symplectic pyramid" associated to $H(p, t_1, \ldots, t_{c(p)})$, we can conclude that $(h(p) + z(t_1, \ldots, t_{c(p)}), e(p))$ is never an even pair. $\square$

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