Alma Mater Studiorum · Università di Bologna

**Scuola di Scienze**
**Dipartimento di Fisica e Astronomia**
**Corso di Laurea magistrale in Fisica: Curriculum Teorico Generale**

# Simplified worldline path integrals for p-forms and type-A trace anomalies

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**Sessione II**
**Anno Accademico 2017/2018**

Al mio “ingegnerino” Leonardo.
Con l’augurio che nella vita possa realizzare
tutti i suoi sogni e progetti

# Abstract

In this work we study a simplified version of the path integral for a particle on a sphere, and more generally on maximally symmetric spaces, in the case of $N = 2$ supersymmetries on the worldline.
This quantum mechanics is generically that of a nonlinear sigma model in one dimension with two supersymmetries ($N = 2$ supersymmetric quantum mechanics), and it is mostly used for describing spin 1 fields and $p$-forms in first quantization. Here, we conjecture a simplified path integral defined in terms of a linear sigma model, rather than a nonlinear one. The use of a quadratic kinetic term in the bosonic part of the particle action should be allowed by the use of Riemann normal coordinates, while a scalar effective potential is expected to reproduce the effects of the curvature. Such simplifications have already been proven to be possible for the cases of $N = 0$ and $N = 1$ supersymmetric quantum mechanics.
As a particular application, we employ our construction to give a simplified worldline representation of the one-loop effective action of gauge $p$-forms on maximally symmetric spaces. We use it to compute the first three Seeley-DeWitt coefficients, denoted by $a_{p+1}(d, p)$, namely $a_1(2, 0)$, $a_2(4, 1)$ and $a_3(6, 2)$, that appear in the calculation of the type-A trace anomalies of conformally invariant $p$-form gauge potentials in $d = 2p + 2$ dimensions.
The simplified model describes correctly the first two coefficients, while it seems to fail to reproduce the third one. One possible reason could be that the model is based on a conjecture about the effective potential that has been oversimplified in our analysis.
Future work could improve our construction, in order to give a correct description to all orders, or alternatively disprove the possibility of having such a simplification in the full $N = 2$ quantum mechanics.

# Sommario

In questo lavoro di tesi si è studiata una versione semplificata del path integral per una particella su una sfera e, in generale, su spazi massimamente simmetrici, in presenza di supersimmetria (SUSY $N = 2$) in un approccio di tipo “wordline”. La meccanica quantistica è la stessa di un modello sigma non lineare in una dimensione con due cariche di supersimmetria ed è principalmente usata nella descrizione di campi di spin 1 e, in generale, di $p$-forme in prima quantizzazione.
La congettura alla base di questo lavoro consiste nel definire il path integral in termini di un modello sigma lineare. L’utilizzo di una metrica piatta nel termine cinetico nella parte bosonica dell’azione è reso possibile dall’utilizzo di un particolare sistema di coordinate, dette coordinate normali di Riemann, mentre ci si aspetta che gli effetti della curvatura dello spazio siano riprodotti da un potenziale scalare effettivo.
Questa semplificazione è già stata testata, e la sua validità verificata, nei casi di supersimmetria con $N = 0$ e $N = 1$.
Questa costruzione è stata applicata per rappresentare in maniera semplificata l’azione effettiva one-loop di $p$-forme su spazi a massima simmetria ed è stata utilizzata per calcolare i primi tre coefficienti di Seeley-DeWitt, indicati con $a_{p+1}(d, p)$ ($a_1(2, 0)$, $a_2(4, 1)$ e $a_3(6, 2)$), i quali compaiono nel calcolo delle anomalie di traccia di tipo A per $p$-forme in $d = 2p + 2$ dimensioni.
Il modello semplificato descrive correttamente i primi due coefficienti, mentre sembra non riprodurre il terzo. Una possibile spiegazione consiste nel fatto che la congettura sul potenziale effettivo potrebbe essere non del tutto corretta nella nostra analisi.
Possibili sviluppi futuri potrebbero essere volti a migliorare questo modello, per raggiungere una corretta descrizione a tutti gli ordini perturbativi, oppure in alternativa, escludere la possibilità di applicare questo approccio alla meccanica supersimmetrica con $N = 2$.

# Contents

# Introduction

Quantum Field Theory provides the language that best reconciles quantum mechanics and special relativity (QED, Standard Model, etc..). The worldline formalism is a first quantized approach: coordinates of each relativistic "particle" are quantized. It is an old formalism. In fact, in "Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction" (1950) R.P. Feynman [1] showed a description, in Lagrangian Quantum Mechanical form, of particles satisfying the Klein-Gordon equation. Such a description involved the use of an extra parameter analogous to proper time to describe the trajectory of the particle in four dimensions. In the appendix of that paper Feynman wrote "The formulation given here [...] is given only for its own interest as an alternative to the formulation of second quantization."

The worldline formalism has been used in many contests, where it appears to be simpler than standard QFT Feynman rules, producing efficient tools for perturbative computations of Feynman diagrams of standard quantum field theories [2].

At the quantum mechanical level supersymmetry appears in worldline models for particles with spin. Supersymmetry is a particular symmetry which relates bosons and fermions. It is largely used to conjecture extensions of the Standard Model for the description of the elementary particles.

Recently, also gravitationally interacting field theories have been discussed in this framework by considering the path integral quantization of worldlines of particles of spin 0, 1/2 and 1 ($p$-forms, more generally) embedded in a curved space.

When the path integral formulation of quantum mechanics [3] is applied to particles on curved background, it implies some faintness that are the analogue of the ordering ambiguities of canonical quantization. In order make sense of the path integral, at least perturbatively, one needs to specify a regularization scheme.

The action of a nonrelativistic particle takes the form of a nonlinear sigma model in one dimension and identifies a super-renormalizable one-dimensional quantum field theory.

It can be treated by choosing a regularization scheme that produces a corresponding counterterms, the latter being needed to match the renormalization conditions, i.e. to fix uniquely the theory under study.
While several regularization schemes have been worked out and tested [4], recently a simplified version of the path integral on maximally symmetric spaces (MSS) has been constructed [5]. It realizes an old issue [6] which suggests a peculiar use of Riemann normal coordinates (RNC).
It assumes that in such a coordinate system an auxiliary flat metric can be used in the kinetic terms, while a suitable effective potential is supposed to reproduce the effects of the curvature. This construction turns the non linear sigma-model into a linear one.
The simplifications expected are rather interesting and motivate a further investigation of this issue. If the addition of a effective scalar potential to a kinetic term with a flat metric works for MSS, its validity can't be proved on arbitrary geometries, at this stage.
By using Riemann normal coordinates on MSS, the Schrödinger equation (the heat equation in our euclidean convention) for the transition amplitude can be simplified, and this gives us a corresponding simplified version on the path integral that generates its solutions. This works in the cases of $N = 0$ and $N = 1$ supersymmetric quantum mechanics [7, 8], which are relevant in the first quantization of particles of spin 0 and 1/2.
The aim of this work is to extend this approach to $N = 2$ supersymmetric quantum mechanics, so that the resulting path integral can be used to describe $p$-forms and, in particular, spin 1 fields in first quantization. While a direct approach through the Schrödinger equation seems unmanageable, we have tried to construct the simplified path integral by conjecturing the simplified version with a linear sigma model, and fixing the ensuing effective potential to reproduce the leading terms of the path integral.
The path integral for $p$-forms will be studied by making a perturbative expansion in the proper time $\beta$. In particular, we compute the first few Seeley-DeWitt coefficients for a $p$-form in arbitrary dimensions by considering the transition amplitude for the $N = 2$ quantum mechanics at coinciding points, as needed for identifying the one-loop effective action of $p$-forms through worldlines.
These Seeley-DeWitt coefficients can be related to the type-A trace anomalies for $p$-forms and we check if they reproduce known results.
Our construction of the simplified path integral works only at the first few orders and this will have to be improved in future extensions.

Thus, in chapter 1 we introduce the path integral formalism for a particle on a curved space, discussing the nonlinear sigma model and the simplification in terms of a linear one. Riemann normal coordinates are here introduced. In chapter 2 we discuss the $N = 1$ supersymmetric version and related computation of transition amplitude and trace anomalies for spin 1/2 relativistic particles. In chapter 3 we try to extend the simplified construction to the case $N = 2$ by computing the transition amplitude up to order $\beta^3$ and checking if the first three Seeley-DeWitt coefficients reproduce known results. This coefficients are related to the type-A trace anomalies of $p$-forms.

# Chapter 1

# Particles on curved spaces

In this chapter we start from the path integral description of a scalar particle on a curved space, based on a nonlinear sigma model, and we will treat the different regularization schemes that have been tested (time slicing, mode and dimensional regularization). Then we will consider a simplified path integral on a maximally symmetric space, as for example a sphere, by making use of the Riemann normal coordinates that allow us to turn the nonlinear sigma model into a linear one. At the end, we will show how to compute the transition amplitude and the type-A trace anomalies for the scalar field.

## 1.1 A nonlinear sigma model

A nonrelativistic particle of unit mass in a curved $d$-dimensional space has a lagrangian containing only the kinetic term

$$L(x,\dot{x}) = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j, \tag{1.1}$$

where $g_{ij}(x)$ is the metric in an arbitrary coordinate system. It is the action of a nonlinear sigma model in one dimension, and the corresponding equations of motion are written in terms of the affine parameter $t$, the time used in the definition of the velocity $\dot{x}^i = \frac{dx^i}{dt}$. The corresponding Hamiltonian reads

$$H(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j \tag{1.2}$$

where $p_i$ are the momenta conjugated to $x^i$. Upon canonical quantization it carries ordering ambiguities, which consist in terms containing one or two derivatives acting on the metric. In the coordinate representation the hermitian momentum acting on a scalar wave function takes the form $p_i = -ig^{-\frac{1}{4}}\partial_i g^{\frac{1}{4}}$. Further details may be found in the book [4].

These ambiguities are greatly reduced by requiring background general coordinate invariance. Since the only tensor that can be constructed with one and two derivative on the metric is the curvature tensor, the most general diffeomorphism invariant quantum hamiltonian that can be constructed reads

$$\hat{H} = -\frac{1}{2}\nabla^2 + \frac{\xi}{2}R \tag{1.3}$$

where $\nabla^2$ is the covariant laplacian acting on scalar wave functions, and $\xi$ is an arbitrary coupling to the scalar curvature $R$ (defined to be positive on a sphere) that parametrizes remaining ordering ambiguities. The value $\xi = 0$ defines the minimal coupling, while the value $\xi = \frac{d-2}{4(d-1)}$ is the conformally invariant coupling in $d$ dimensions.

For definiteness let us review the theory with the minimal coupling $\xi = 0$. Other values can be obtained by simply adding a scalar potential $V = \frac{\xi}{2}R$. The transition amplitude in euclidean time $\beta$ (the heat kernel)

$$K(x, x'; \beta) \equiv \langle x|e^{-\beta\hat{H}(\hat{x},\hat{p})}|x'\rangle \tag{1.4}$$

contains the covariant hamiltonian

$$\hat{H}(\hat{x}, \hat{p}) = \frac{1}{2}g^{-\frac{1}{4}}(\hat{x})\hat{p}_i g^{ij}(\hat{x})g^{\frac{1}{2}}(\hat{x})\hat{p}_j g^{-\frac{1}{4}}(\hat{x}). \tag{1.5}$$

We choose position eigenstates normalized as scalars, $\hat{x}|x\rangle = x^i|x\rangle$,$\langle x|x'\rangle = \frac{\delta^{(d)}(x-x')}{\sqrt{g(x)}}$, $\mathbb{1} = \int d^d x\sqrt{g(x)}|x\rangle\langle x|$, so the amplitude $K(x, x'; \beta)$ is a biscalar.

It solves the Schrödinger equation in euclidean time (heat equation)

$$-\frac{\partial}{\partial\beta}K(x, x'; \beta) = -\frac{1}{2}\nabla^2_x K(x, x'; \beta) \tag{1.6}$$

and satisfies the boundary conditions at $\beta \to 0$

$$K(x, x'; 0) = \frac{\delta^{(d)}(x - x')}{\sqrt{g(x)}}. \tag{1.7}$$

In (1.6) $\nabla_x^2$ indicates the covariant scalar laplacian acting on coordinates $x$.
The transition amplitude $K(x, x'; \beta)$ can be given a path integral representation.

## 1.2 Regularization schemes

Using different regularization schemes the action acquires finite and different counterterms. Let's see some regularization schemes.

### 1.2.1 Time slicing regularization

With a Weyl reordering of the quantum Hamiltonian $\hat{H}(\hat{x}, \hat{p})$ one can derive a discretized phase-space path integral containing the classical phase-space action suitably discretized by the midpoint rule [9]. The action acquires a finite counterterm $V_{TS}$ of quantum origin, arising from the Weyl reordering of the specific hamiltonian in (1.5), originally performed in [10] (the subscript TS reminds of the time slicing discretization of the time variable). The perturbative evaluation of the phase space path integral can be performed directly in the continuum limit [11]

$$\int \mathcal{D}x\mathcal{D}p \; e^{-S[x,p]} \tag{1.8}$$

with the phase-space euclidean action taking the form

$$\begin{aligned} S[x,p] &= \int_0^\beta dt(-ip_i\dot{x}^i + H(x,p)) \\ H(x,p) &= \frac{1}{2}g^{ij}(x)p_ip_j + V_{TS}(x) \\ V_{TS}(x) &= -\frac{1}{8}R(x) + \frac{1}{8}g^{ij}(x)\Gamma^l_{ik}(x)\Gamma^k_{jl}(x). \end{aligned} \tag{1.9}$$

To generate the amplitude $K(x, x'; \beta)$ the paths $x(t)$ must satisfy the boundary conditions $x(0) = x'$ and $x(\beta) = x$, while the paths $p(t)$ are unconstrained. We recall that perturbative corrections are finite in phase space. The presence of the noncovariant part of the counterterm $V_{TS}$ corrects the noncovariance of the midpoint discretization, and it ensures the covariance of the final result. These noncovariant counterterms were also derived in [12] by considering point transformations (i.e arbitrary changes of coordinates) in flat space.
The definition of the corresponding path integral in configuration space carries more subtle problems. The classical action takes the form of a non linear sigma model in one

dimension

$$S[x] = \int_0^\beta dt\ \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j \tag{1.10}$$

and power counting indicates that, in a perturbative expansion about flat space, it is a super-renormalizable model, with superficial degree of divergence $D = 2 - L$ where $L$ counts the number of loop [4]. Thus, considering quantum mechanics as a particular QFT in one euclidean dimension, one can find possible divergences at one- and two-loops. Therefore, just like in generic QFTs, one must define a regularization scheme, needed to cancel divergences, and a finite part, needed to match the renormalization conditions. In the present case the counterterms are finite if one includes the local terms arising from the general coordinate invariant path integral measure.

Three well-defined regularization schemes have been studied in the literature, required by the effort of computing QFT trace anomalies with quantum mechanical path integrals [13, 14]. The latter extended to trace anomalies the quantum mechanical method used for chiral anomalies in [15, 16].

To recall the various regularization schemes let us first notice that in configuration space the formally covariant measure can be related to a translational invariant measure by using ghost fields $a^i$,$b^i$ and $c^i$ à la Faddeev-Popov

$$\mathcal{D}x = \prod_{0<t<\beta} d^dx(t)\sqrt{g(x(t))} = \prod_{0<t<\beta} d^dx(t) \int DaDbDc\ e^{-S_{gh}[x,a,b,c]} \tag{1.11}$$

where

$$S_{gh}[x,a,b,c,d] = \int_0^\beta dt \frac{1}{2} g_{ij}(x)(a^ia^j + b^ic^j). \tag{1.12}$$

Considering $a^i$ bosonic variable and $b^i$,$c^i$ fermionic variables allows to reproduce the factor $\frac{g(x(t))}{\sqrt{g(x(t))}} = \sqrt{g(x(t))}$ in the measure. By $Dx$,$Da$,$Db$ and $Dc$ we indicate the translational invariant measure, useful for generating the perturbative expansion (e.g $Dx = \prod_{0<t<\beta} d^dx(t)$, and so on). Thus, the path integral for the nonlinear sigma model in configuration space can be written as

$$\int DxDaDbDc\ e^{-S[x,a,b,c]} \tag{1.13}$$

with the full action taking the form

$$S[x, a, b, c, d] = \int_0^\beta dt \left( \frac{1}{2} g_{ij}(x)(\dot{x}^i \dot{x}^j + a^i a^j + b^i c^j) + V_{CT} \right). \quad (1.14)$$

and with $V_{CT}$ is the counterterm associated to the chosen regularization. To generate the amplitude $K(x, x'; \beta)$ the paths $x(t)$ must of course satisfy the boundary conditions $x(0) = x'$ and $x(\beta) = x$.
The time slicing regularization (TS) in configuration space was studied in [17, 18], by deriving it from the phase space path integral, and studying carefully the continuum limit of the propagators together with the rules that must be used in evaluating their products. Recalling that the perturbative propagators are distributions, how to multiply them and their derivative together is the problem one faces in regulating the perturbative expression. This regularization leads to the insertion of the counterterm $V_{TS}$ in (1.9).

### 1.2.2 Mode regularization

Mode regularization (MR) was already applied in curved space in [13, 14]. The complete counterterm was identified in [19] and arises from the necessity of addressing mismatches originally found between TS and MR. Those mismatches disappear with the correct counterterm

$$V_{MR} = -\frac{1}{8} R - \frac{1}{24} g_{ij} g^{kl} g^{mn} \Gamma^i_{km} \Gamma^j_{ln}. \quad (1.15)$$

The rule how to define the products of distributions in this regularization scheme follows from the expansion of the quantum fluctuations in a Fourier series truncated by a cut-off, which eventually is removed in the continuum limit. Including the vertices originating from the counterterm produces the covariant final answer.

### 1.2.3 Dimensional regularization

Finally, dimensional regularization (DR) was introduced in the quantum mechanical context in [20–22]. It needs the counterterm

$$V_{DR} = -\frac{1}{8} R \quad (1.16)$$

which has the useful property of being covariant.

All these regularization schemes have been extensively tested and compared, see e.g [23,24]. Extensions to supersymmetric models have been recently discussed again in [25], where the counterterms in all the previous regularization schemes were identified for the supersymmetric nonlinear sigma model with $N$ supersymmetries at arbitrary $N$. Additional details may be found in the book [4].
The case of trace anomalies provided a precise observable on which one can test and verify the construction of the quantum mechanical path integrals in curved spaces, clearing the somewhat confusing status of the subject present in previous literature.
This tool allows more general applications of the path integral, in particular in the first quantized approach to quantum fields [26] coupled to gravitational background, such as the worldline description of field of spin 0,1/2 and 1 coupled to gravity [27–30], and many more.

## 1.3 A linear sigma model

In the previous sections we have reviewed the quantum mechanical path integral for a nonlinear sigma model, that describes a particle moving in a curved space by using arbitrary coordinates. In this section we wish to take up in a critical way an old proposal, put forward by Guven in [6], of constructing the path integral in curved spaces by using Riemann normal coordinates. The proposal assumes that in Riemann normal coordinates (RNC) an auxiliary flat metric can be used in the kinetic term, while an effective potential reproduces the effects of the curved space. This construction aims at transforming the original nonlinear sigma model into a linear one. If correct, it carries several simplifications, making perturbative calculations simpler and more efficient. It may also improve its use in the worldline applications mentioned earlier. Correctness has been proved for the $N = 0$ and $N = 1$ SUSY models. The case $N = 2$ is studied in this thesis.

### 1.3.1 Geometry of maximally symmetric spaces and Riemann normal coordinates

As a preliminary, we discuss the geometry of maximally symmetric spaces in Riemann normal coordinates (RNC), which will be used extensively later on.

On maximally symmetric spaces the Riemann tensor is related to the metric tensor by

$$R_{mnab} = M^2(g_{ma}g_{nb} - g_{mb}g_{na}) \tag{1.17}$$

where $M^2 = 1/a^2$ is a constant that can be either positive for a sphere of radius $a$, negative for a real hyperbolic space, or vanishing for a flat space. For simplicity we consider at first spheres, as real hyperbolic spaces can be obtained by analytic continuation.
The Ricci tensor are then defined by

$$\begin{aligned} R_{mn} &= R_{am}{}^{m}{}_{n} = M^2(d-1)g_{mn} \\ R &= R_m{}^m = M^2(d-1)d \end{aligned} \tag{1.18}$$

and the constant $M^2$ is related to the constant Ricci scalar $R$ by

$$M^2 = \frac{R}{(d-1)d}. \tag{1.19}$$

We want to use RNC, where the metric can be expressed as

$$\begin{aligned} g_{ij}(x) &= \delta_{ij} + f(r)P_{ij} \\ &= \delta_{ij} + \sum_{l=1}^{\infty} c_l M^{2l}(-1)^l(x^2)^l P_{ij}, \end{aligned} \tag{1.20}$$

where $x^i$ are the RNC centered at a point (the origin), $P_{ij}$ is a projector given by

$$P_{ij} = \delta_{ij} - \hat{x}_i\hat{x}_j, \qquad \hat{x}^i = \frac{x^i}{r}, \qquad r = \sqrt{\delta_{ij}x^ix^j}, \tag{1.21}$$

and

$$c_l = \frac{1}{2}\frac{4^{l+1}}{(2l+2)!}. \tag{1.22}$$

The series can be summed up to give

$$f(r) = \frac{1 - 2(Mr)^2 - \cos(2Mr)}{2(Mr)^2}. \tag{1.23}$$

The function $f(r)$ does not have poles and is even in $r$ and, because of the projector $P_{ij}$, one has the equality $r^2 = \vec{x}^2 = \delta_{ij}x^ix^j = g_{ij}(x)x^ix^j$.

The inverse metric $g^{ij}(x)$ and metric determinant $g(x)$ are given by

$$g^{ij} = \delta^{ij} + h(r)P^{ij}, \qquad g(x) = (1 + f(r))^{d-1}, \tag{1.24}$$

where

$$h(r) = -\frac{f(r)}{1 + f(r)}. \tag{1.25}$$

On the right hand side of these formulae indices are raised and lowered by the flat metric $\delta_{ij}$.

For completeness, we discuss the case of real hyperbolic spaces as well. Now the sectional curvature is negative, $M^2 < 0$. It can be obtained from the previous case by the analytic continuation $M \to i|M|$, with the imaginary unit $i$ giving rise to the negative sign of the sectional curvature, and $|M| = \sqrt{-M^2}$. Performing this analytic continuation in (1.20) we find that in the sum the minus from $(-1)^l$ get canceled

$$g_{ij} = \delta_{ij} + \sum_{l=1}^{\infty} c_l |M|^{2l} (x^2)^l P_{ij} = \delta_{ij} + f(x)P_{ij}, \tag{1.26}$$

and the sum now converges to the function

$$f(r) = \frac{-1 - 2(|M|r)^2 + \cosh(2|M|r)}{2(|M|r)^2}. \tag{1.27}$$

Finally the function $f(r)$ vanishes in the flat space case, where Riemann normal coordinates are just the standard cartesian coordinates. It may also be obtained as a smooth limit of the curved cases, as $f(r) \to 0$ for $M \to 0$.

This metric in (1.20) can be generated by the following choice of vielbein

$$e_i^a(x) = \delta_i^a + l(r)P_i^a(x) \tag{1.28}$$

where $x^a = \delta_i^a x^i$ and

$$l(r) = -1 + \sqrt{1 + f(r)} = -1 + \frac{\sin(Mr)}{Mr}. \tag{1.29}$$

A priori, there are two independent solutions $l_\pm = -1 \pm \sqrt{1 + f(r)}$ of the quadratic equation that follows from $g_{ij} = \eta_{ab} e_i^a e_j^b$. However, only with the upper solution does the vielbein reduce to the flat vielbein when $M^2 \to 0$.

The inverse vielbein reads

$$e^{ai}(x) = \delta^{ai} + \left(-1 + \frac{1}{\sqrt{1+f(r)}}\right) P^{ai}(x). \tag{1.30}$$

Thus, by using the relation

$$\omega_i{}^{ab}(x) = \frac{1}{2} e^{aj}(\partial_i e_j^b - \partial_j e_i^b) - \frac{1}{2} e^{bj}(\partial_i e_j^a - \partial_j e_i^a) - \frac{1}{2} e_i^c e^{aj} e^{bk}(\partial_j e_{ck} - \partial_k e_{cj}) \tag{1.31}$$

one can find that the associated spin connection is given by

$$\omega_i{}^{ab}(x) = \Omega(r) \frac{1}{2} x^j (\delta_j^a \delta_i^b - \delta_j^b \delta_i^a), \tag{1.32}$$

with

$$\Omega(r) = -\frac{2}{r}\left(l'(r) + \frac{l(r)}{r}\right) = 2M^2 \frac{1-\cos(Mr)}{(Mr)^2}, \tag{1.33}$$

where the prime denotes the derivative with respect to the radial coordinate $r$. Equivalently we can write the spin connection in the form

$$\omega_i{}^{ab}(x) = \frac{1}{M^2} \Omega(r) \frac{1}{2} x^j R_{ij}{}^{ab}(0). \tag{1.34}$$

where the prefactor reads

$$\begin{aligned} \frac{1}{M^2}\Omega(r) = 2\frac{1-\cos(Mr)}{(Mr)^2} &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n+2)!}(Mr)^{2n} \\ &= 1 - \frac{(Mr)^2}{12} + \frac{(Mr)^4}{360} - \frac{(Mr)^6}{20160} + ... \end{aligned} \tag{1.35}$$

and a power of $M^2$ is absorbed by $R_{ij}^{ab}(0)$. Note that the vielbein (1.28) with (1.29) and the spin connection (1.34) satisfy the Fock-Schwinger gauge conditions

$$\begin{aligned} e_i^a(x) x^i &= \delta_i^a x^i \\ x^i \omega_i{}^{ab}(x) &= 0. \end{aligned} \tag{1.36}$$

### 1.3.2 Simplified path integral in maximally symmetric spaces

We wish to compute the path integral in RNC using the linear sigma model of (1.1) and considering maximally symmetric spaces in order to obtain the transition amplitude at

coinciding points $K(x, x'; \beta)$.
Let us review the considerations put forward in [6]. First of all it is convenient to consider the transition amplitude as a bidensity by defining

$$\overline{K}(x, x'; \beta) = g^{\frac{1}{4}}(x) K(x, x'; \beta) g^{\frac{1}{4}}(x') \tag{1.37}$$

so that, from (1.6), $\overline{K}$ is seen to satisfy the equation

$$-\frac{\partial}{\partial \beta} \overline{K}(x, x'; \beta) = -\frac{1}{2} g^{\frac{1}{4}}(x) \nabla_x^2 g^{-\frac{1}{4}}(x) \overline{K}(x, x'; \beta) \tag{1.38}$$

with the boundary condition

$$\overline{K}(x, x'; 0) = \delta^{(d)}(x - x') \tag{1.39}$$

where $\nabla_x^2$ is the scalar laplacian $\nabla^2 = \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j$ acting on the $x$ coordinates, with $\sqrt{g} = \sqrt{|\det(g)|}$.
By direct computation the differential operator appearing in the right hand side of (1.38) can be rewritten as

$$-\frac{1}{2} g^{\frac{1}{4}} \nabla^2 g^{-\frac{1}{4}} = -\frac{1}{2} \partial_i g^{ij} \partial_j + V_0 \tag{1.40}$$

where derivatives act through and with the effective potential given by

$$V_0 = -\frac{1}{2} g^{-\frac{1}{4}} \partial_i \sqrt{g} g^{ij} \partial_j g^{-\frac{1}{4}} \tag{1.41}$$

where all derivatives now stop after acting on the last function. The heat equation (1.6) now reads more explicitly as

$$-\frac{\partial}{\partial \beta} \overline{K}(x, x'; \beta) = \left( -\frac{1}{2} g^{ij} \partial_i \partial_j + V_0(x) \right) \overline{K}(x, x'; \beta) \tag{1.42}$$

The Lorentz invariance (rotational invariance in euclidean conventions) of the momentum-space representation of $\overline{K}$ written in Riemann normal coordinates implies that the $g^{ij}$ in the $\partial_i g^{ij} \partial_j$ operator of (1.40) can be replaced by the constant $\delta^{ij}$, as seen in [31]. Indeed, in the momentum-space representation of $\overline{K}$ previously studied in [32] by using Riemann normal coordinates, it was found that in an adiabatic expansion of $\overline{K}$ the first few terms depended on certain scalar functions, which were functions of $\delta_{ij} x^i x^j$ only. However it is not obvious why such a property should hold to all orders. In a curved space Lorentz

invariance obviously cannot hold, for example scalar terms proportional to $R_{ij}x^i x^j$ may also arise (we denote by $R_{ij}$ the Ricci tensor evaluated at the origin of the Riemann coordinates and by $x^i$ the Riemann normal coordinates themselves).
Let's consider the euclidean Schrödinger equation

$$-\frac{\partial}{\partial\beta}\overline{K}(x,x';\beta) = \left(-\frac{1}{2}\delta^{ij}\partial_i\partial_j + V_0(x)\right)\overline{K}(x,x';\beta). \tag{1.43}$$

The heat kernel equation (1.43) contains now an hamiltonian operator

$$H = -\frac{1}{2}\delta^{ij}\partial_i\partial_j + V_0(x) \tag{1.44}$$

which is interpreted as that of a particle on a flat space (in cartesian coordinates) interacting with an effective potential $V_0$ of quantum origin (in fact, it would be proportional to $\hbar^2$ in arbitrary units). To be sure that the replacement of $g^{ij}(x)$ with $\delta^{ij}$ is valid, one must show that

$$\left(\partial_i g^{ij}(x)\partial_j - \delta^{ij}\partial_i\partial_j\right)\overline{K}(x,x';\beta) = 0. \tag{1.45}$$

where $x' = 0$ is the chosen origin of RNC. Using (1.21) and (1.24) we find that the equation that we must verify takes the form

$$\left(h(r)P^{ij}(x)\partial_i\partial_j + \partial_i(h(r)P^{ij}(x))\partial_j\right)\overline{K}(x,0;\beta) = 0, \tag{1.46}$$

where the projector $P^{ij}$ and the function $h(r)$ are given by (1.21) and (1.25) respectively. The function $h(r)$ is a function of only $r^2 = \delta_{ij}x^i x^j$ and it is even in $r = \sqrt{\delta_{ij}x^i x^j}$. This is a consequence of the maximal symmetry of the sphere. The explicit evaluation of the derivatives appearing in (1.46) produces (recalling the orthogonality condition $P^{ij}x_j = 0$)

$$\begin{aligned} h(r)P^{ij}(x)\partial_i\partial_j\overline{K}(x,0;\beta) &= 2h(x)\delta_{ij}P^{ij}\frac{\partial}{\partial x^2}\overline{K}(x,0;\beta) \\ &= 2(d-1)h(r)\frac{\partial}{\partial x^2}\overline{K}(x,0;\beta), \end{aligned} \tag{1.47}$$

and

$$\partial_i(h(r)P^{ij}(x))\partial_j\overline{K}(x,0;\beta) = -2(d-1)h(r)\frac{\partial}{\partial x^2}\overline{K}(x,0;\beta). \tag{1.48}$$

The two terms cancel each other out, so that we have indeed verified (1.45) and the correctness of the heat kernel equation (1.43), that can be solved by a standard path integral for our linear sigma model.

## 1.4 Transition amplitude for a scalar field

The path integral discussed in section 1.1 gives the transition amplitude for a particle on curved spaces, and one can restrict to maximally symmetric spaces (MSS) such as spheres. The latter can be used to extract the type-A trace anomalies for scalar fields (spin 0 particles), as shown in [5].
As we have seen the path integral contains the nonlinear sigma model and must be regulated carefully. However, on MSS we can introduce a simplification that allows to turn the nonlinear sigma model into a linear one. This arises at the expense of adding an effective potential on top of using RNC

$$\overline{K}(x,x';\beta) \sim \int_{x(0)=x'}^{x(\beta)=x} \mathcal{D}x \;\, e^{-S_0[x]}, \qquad S_0[x] = \int_0^\beta dt \left(\frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + V_0(x)\right) \tag{1.49}$$

as follows from (1.44). We wish to evaluate the transition amplitude at coinciding points $x = x' = 0$ (taken to be the origin of the Riemann coordinates) in a perturbative expansion in terms of the propagation time $\beta$. To control the $\beta$ expansion it is useful to rescale the time $t \to \tau = \frac{t}{\beta}$ so that $\tau \in [0,1]$ and the action takes the form

$$S_0[x] = \int_0^\beta dt \left(\frac{1}{2\beta}\delta_{ij}\dot{x}^i\dot{x}^j + \beta V_0(x)\right). \tag{1.50}$$

$V_0$ is the effective potential given by

$$\begin{aligned} V_0(x) &= -\frac{1}{2} g^{-\frac{1}{4}} \partial_i \sqrt{g} g^{ij} \partial_j g^{-\frac{1}{4}} \\ &= \frac{(d-1)}{8}\left[\frac{(d-5)}{4}\left(\frac{f'(r)}{1+f'(x)}\right)^2 + \frac{1}{1+f(x)}\left(\frac{(d-1)}{x} f'(x) + f''(x)\right)\right] \\ &= \frac{d(d-1)}{12} M^2 + \frac{(d-1)(d-3)}{48}\frac{(5(Mr)^2 - 3 + ((Mr)^2+3)\cos(2Mr))}{r^2 \sin^2(Mr)}. \end{aligned} \tag{1.51}$$

with expansion

$$V_0(x) = -\frac{d-d^2}{12} + (d-1)(d-3)\bigg(\frac{x^2}{120} + \frac{x^4}{756} + \frac{x^6}{5400} + \frac{x^8}{41580} + \frac{691x^{10}}{232186500} + \frac{x^{12}}{2806650} + O(x^{14})\bigg). \tag{1.52}$$

### 1.4.1 Perturbative expansion for a scalar field

The perturbative expansion of the path integral is obtained by setting

$$S_0[x] = S_{free}[x] + S_{int}[x] \tag{1.53}$$

with

$$S_{free}[x] = \frac{1}{\beta}\int_0^1 d\tau \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j, \qquad S_{int}[x] = \beta\int_0^1 d\tau V_0(x) \tag{1.54}$$

so that (1.49) reduces to

$$\overline{K}(0,0;\beta) = \frac{\langle e^{-S_{int}}\rangle}{(2\pi\beta)^{\frac{d}{2}}} \tag{1.55}$$

where $\langle ... \rangle$ denotes normalized correlation function with the free path integral. The propagator associated to the free kinetic term is given by

$$\langle x^i(\tau)x^j(\tau')\rangle = -\beta\delta^{ij}\Delta(\tau,\tau'), \qquad \Delta(\tau,\tau') = \frac{1}{2}|\tau-\tau'| - \frac{1}{2}(\tau+\tau') + \tau\tau', \tag{1.56}$$

The couplings $k_{2l}$ needed for a description of $\overline{K}(0,0;\beta)$ are read off (1.52). Let's write them explicitly up to $k_{10}$ included

$$\begin{aligned} k_0 &= -\frac{d(d-1)}{12} \\ k_2 &= \frac{(d-1)(d-3)}{120} \\ k_4 &= \frac{(d-1)(d-3)}{756} \\ k_6 &= \frac{(d-1)(d-3)}{5400} \\ k_8 &= \frac{(d-1)(d-3)}{41580} \\ k_{10} &= \frac{691(d-1)(d-3)}{232186500}. \end{aligned} \tag{1.57}$$

Thus, by using Wick contractions one finds

$$\begin{aligned}\overline{K}(0,0;\beta) = \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp\Bigg[&\frac{\beta R}{12} - \frac{(\beta R)^2}{6!}\frac{(d-3)}{d(d-1)} - \frac{(\beta R)^3}{9!}\frac{16(d-3)(d+2)}{d^2(d-1)^2} \\ &- \frac{(\beta R)^4}{10!}\frac{2(d-3)(d^2+20d+15)}{d^3(d-1)^3} \\ &+ \frac{(\beta R)^5}{11!}\frac{8(d-3)(d+2)(d^2-12d-9)}{3d^4(d-1)^4} \\ &+ \frac{(\beta R)^6}{13!}\frac{8(d-3)(1623d^4-716d^3-65930d^2-123572d-60165)}{315d^5(d-1)^5} \\ &+ O(\beta^7)\Bigg] .\end{aligned} \tag{1.58}$$

Computational details will be shown in chapter 3. The exponential in (1.58) can be expanded to identity the first six heat kernel coefficients (also known as Seeley-DeWitt coefficients), defined as the coefficients $a_n$ in the expansion

$$\overline{K}(0,0;\beta) = \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \sum_n a_n \beta^n . \tag{1.59}$$

### 1.4.2 Type-A trace anomaly of a scalar field

A further test of the previous formulas is to use them to compute the type-A trace anomaly of a conformal scalar field. Trace anomalies characterize conformal field theories. In fact, the trace of the energy-momentum tensor for conformal fields vanishes at the classical level but it acquires anomalous terms at the quantum level. These terms depend on the background geometry of the spacetime which the conformal fields are coupled to, and they are related to the appropriate Seeley-DeWitt coefficient sitting in the heat kernel expansion of the associated conformal operator.

A simple way to obtain this relation is to consider the trace anomaly as due to the QFT path integral measure, so that it is computed by the regulated Jacobian arising from the Weyl transformation of the QFT path integral measure [33]. For a scalar field the infinitesimal Weyl transformation $\delta_\sigma g_{mn} = \sigma(x) g_{mn}(x)$, applied to the one-loop effective action, yields

$$\int d^dx \sqrt{g}\ \sigma(x) \langle T^m{}_m(x)\rangle = \lim_{\beta\to 0} \mathrm{Tr}\Big\{\sigma e^{-\beta\mathcal{R}}\Big\} \tag{1.60}$$

where the consistent regulator $\mathcal{R}$ appearing in the exponent is just the conformal operator associated to the scalar field, and reads

$$\mathcal{R} = -\frac{1}{2}\nabla^2 + \frac{\xi}{2}R. \tag{1.61}$$

It can be identified as the hamiltonian operator (1.3) for a nonrelativistic particle in curved space. Therefore, one identifies the trace anomaly in terms of a particle path integral by

$$\langle T^m{}_m(x)\rangle = \lim_{\beta\to 0} K(x,x;\beta) \tag{1.62}$$

where the limit picks up just the $\beta$-independent term, while divergent terms are removed by QFT renormalization. This procedure selects the appropriate Seeley-DeWitt coefficient sitting in the expansion of $K(x,x';\beta)$. Trace anomalies have been classified

| $d$ | $\langle T^\mu{}_\mu\rangle$ | $\langle T^\mu{}_\mu\rangle$ |
|---|---|---|
| 2 | $\frac{R}{24\pi}$ | $\frac{1}{12\pi a}$ |
| 4 | $-\frac{R^2}{34560\pi^2}$ | $-\frac{1}{240\pi^2 a^4}$ |
| 6 | $\frac{R^3}{21772800\pi^3}$ | $\frac{5}{4032\pi^3 a^6}$ |
| 8 | $-\frac{23R^4}{339880181760\pi^4}$ | $-\frac{23}{2334560\pi^4 a^8}$ |
| 10 | $\frac{263R^5}{2993075712000000\pi^5}$ | $\frac{263}{506880\pi^5 a^{10}}$ |
| 12 | $-\frac{133787R^6}{1330910037208675123200\pi^6}$ | $-\frac{133787}{251596800\pi^6 a^{12}}$ |

**Table 1.1:** The type-A trace anomaly of a scalar field from [5].

as type-A, type-B and trivial anomalies in [34]. On conformally flat space the type-B anomalies vanish so only the type-A anomaly survives. It is proportional to the topological Euler density, and its coefficient enters the so-called $c$-theorem of 2 dimensions [35] and $a$-theorem of 4 dimensions [36] at fixed points. These theorems capture the irreversibility of the renormalization group flow in 2 and 4 dimensions. Their extension to arbitrary even dimensions has been conjectured, but not proven (see [37] for a more general conjecture).

The sphere is a conformally flat space, so we can calculate the type-A anomaly for a scalar field in arbitrary dimensions using the expansion obtained in the previous section. Choosing $x$ as the origin of the RNC system we have by definition $g_{mn}(x) = \delta_{mn}$, and

$$\langle T^m{}_m(x)\rangle = \lim_{\beta\to 0} \overline{K}(x,x;\beta) \tag{1.63}$$

so that expanding (1.58) and picking the $\beta^0$ term in the chosen dimension $d$, we obtain the trace anomalies for a conformal scalar field in $d$ dimensions reported in Table 1.1, where the second form is written in terms of $a^2 = \frac{1}{M^2} = \frac{d(d-1)}{R}$ to directly compare with the results tabulated in [38].

By way of example, let's compute explicitly the first three Seeley-DeWitt coefficients. The expansion up to order $\beta^3$ of (1.58) with the addition of $V_\xi$ reads

$$\begin{aligned}\overline{K}(0,0;\beta) = \frac{1}{(2\pi\beta)^{\frac{d}{2}}}\Bigg[1 &+ \frac{4-d}{24(d-1)}\beta R - \frac{(d-3)}{720d(d-1)}(\beta R)^2 + \frac{(4-d)^2}{576(d-1)^2}(\beta R)^2 \\ &+ \frac{1}{6}\frac{(4-d)^3}{24^3(d-1)^2}(\beta R)^3 - \frac{16(d-3)(d+2)}{9!d^2(d-1)^2}(\beta R)^3 - \frac{(d-3)(d-4)}{24\cdot 720d(d-1)^2}(\beta R)^3 + O(\beta^4)\Bigg].\end{aligned} \tag{1.64}$$

Using (1.63) we have

$$d=2 \quad \to \quad \frac{1}{2\pi\beta}\frac{2}{24}\beta R = \frac{R}{24\pi} \tag{1.65}$$

$$d=4 \quad \to \quad \frac{1}{4\pi^2\beta^2}\left[-\frac{\beta^2R^2}{8640}\right] = -\frac{R^2}{34560\pi^2} \tag{1.66}$$

$$d=6 \quad \to \quad \frac{1}{8\pi^3\beta^3}\left[-\frac{8}{6\cdot 24^3\cdot 125} - \frac{16\cdot 24}{9!\cdot 900} + \frac{6}{24\cdot 720\cdot 150}\right]\beta^3R^3 \tag{1.67}$$

$$= \frac{R^3}{8\pi^3}\left[-\frac{1}{1296000} - \frac{1}{850500} + \frac{1}{432000}\right] = \frac{R^3}{21772800\pi^3} \tag{1.68}$$

The comparison is successful, except at $d = 12$, where the coefficients in Table 1.1 differ by a number of the order of $10^{-13}$. This result is correct since a zeta function approach employed in [38, 39] allows us to reproduce the results shown in Table 1.1.

# Chapter 2

# Path integral for supersymmetric quantum mechanics

In the previous chapter we treated the heat equation for a scalar particle on MSS and a simplified path integral for it, that we used to extract the type-A trace anomalies of a scalar particle. We now repeat the same procedure with the presence of supersymmetry. We discuss the case $N = 1$ studied in [8], which is relevant for the description of a spin 1/2 particle.

## 2.1 Transition amplitude

Let us discuss the $N = 1$ supersymmetric version of the particle mechanics, identified by the euclidean lagrangian

$$L = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j + \frac{1}{2}\psi^a(\dot{\psi}_a + \dot{x}^i\omega_{iab}(x)\psi^b) \tag{2.1}$$

where $\psi^a$ are real Grassmann variables with flat indices, and $\omega_{iab}$ is the spin connection built from the vielbein $e_i^a$. The fermionic variables $\psi^a$ are the supersymmetric partners of the coordinates $x^i$. Upon quantization they lead to operators that satisfy the anticommutation relations

$$\{\hat{\psi}^a, \hat{\psi}^b\} = \delta^{ab} \tag{2.2}$$

a Clifford algebra which we can either represent by the usual Dirac gamma matrices ($\hat{\psi}^a = \frac{1}{\sqrt{2}}\gamma^a$, with $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$), or treat by a fermionic path integral.
The conserved quantum supersymmetric charge of the model is proportional to the Dirac

operator, and reads

$$\hat{Q} = -\frac{i}{\sqrt{2}}\nabla\!\!\!/(\omega) = -\frac{i}{\sqrt{2}}\gamma^a e_a^i(x)\left(\partial_i + \frac{1}{4}\omega_{iab}\gamma^a\gamma^b\right) \tag{2.3}$$

while the related quantum hamiltonian becomes

$$\hat{H} = \hat{Q}^2 = -\frac{1}{2}\nabla\!\!\!/^2 = -\frac{1}{2}g^{ij}\nabla_i(\omega,\Gamma)\nabla_j(\omega) + \frac{1}{8}R, \tag{2.4}$$

where we have indicated the connections present in the various covariant derivatives. All these operators act on a spinorial wave function (a Dirac spinor).
The heat kernel associated to this hamiltonian

$$\mathbb{K} = e^{-\beta\hat{H}} \tag{2.5}$$

has quantum mechanical matrix elements

$$\mathbb{K}_{\alpha\alpha'}(x,0;\beta) = \langle x,\alpha|e^{-\beta\hat{H}}|0,\alpha'\rangle \tag{2.6}$$

where $\alpha$,$\alpha'$ are spinorial indices. In the following we will not show the spinorial indices explicitly and just remember that $\mathbb{K}$ is matrix-valued. Now, from the maximal symmetry of the space one deduces that the heat kernel $\mathbb{K}(x,0;\beta)$ can only be a function of $x^2$, $\gamma_a\gamma^a \sim \mathbb{1}$ and $\delta_{ia}x^i\gamma^a$. In addition, as the gamma matrices appear only in even combinations (they are contained quadratically in the spin connections inside the hamiltonian (2.4)), one finds that the dependence on $\delta_{ia}x^i\gamma^a$ arises only through its square

$$(\delta_{ia}x^i\gamma^a)^2 = \mathbb{1}x^2 \tag{2.7}$$

which is again proportional to the identity matrix. Thus, the full heat kernel is proportional to the identity, and must be a function of $r = \sqrt{\delta_{ij}x^ix^j}$ only,

$$\mathbb{K}(x,0;\beta) = \mathbb{1}U(r;\beta). \tag{2.8}$$

Now let us analyze the heat equation satisfied by the bidensity

$$\overline{\mathbb{K}}(x,0;\beta) = g^{\frac{1}{4}}(x)\mathbb{K}(x,0;\beta)g^{\frac{1}{4}}(0), \tag{2.9}$$

(the value $g(0) = 1$ is actually irrelevant) which is

$$-\partial_\beta \overline{\mathbb{K}}(x, 0; \beta) = g^{\frac{1}{4}}(x)\hat{H}g^{-\frac{1}{4}}(x)\overline{\mathbb{K}}(x, 0; \beta). \tag{2.10}$$

By expanding out the expression of the hamiltonian given in equation (2.4), we write

$$\begin{aligned} g^{\frac{1}{4}}\hat{H}g^{-\frac{1}{4}} &= -\frac{1}{2}g^{\frac{1}{4}}\nabla^2 g^{-\frac{1}{4}} \\ &- \frac{1}{8}(\partial_i \omega^i{}_{ab})\gamma^{ab} - \frac{1}{4}\omega^i{}_{ab}\gamma^{ab}\partial_i \\ &- \frac{1}{32}\omega_{iab}\omega^i{}_{cd}\gamma^{ab}\gamma^{cd} + \frac{1}{8}R. \end{aligned}$$

Using the explicit expression of the spin connection (1.34), which satisfies the Fock-Schwinger gauge (1.36), one can prove that the terms in the second line do not contribute when applied to $\mathbb{K}$ (they give rise to terms proportional to $x^i\omega_{iab} \sim 0$ because of the Fock-Schwinger gauge), while the terms in the third line give expressions proportional to $\mathbb{1}$. In particular, we find

$$-\frac{1}{32}\omega_{iab}\omega^i{}_{cd}\gamma^{ab}\gamma^{cd} = \frac{d-1}{8}M^2\left(\frac{1-\cos(Mr)}{\sin(Mr)}\right)^2 \mathbb{1}. \tag{2.11}$$

Recalling (1.40) we find that a simplified heat equation for the case $N = 1$ holds

$$-\partial_\beta \overline{\mathbb{K}}(x, 0; \beta) = \left(-\frac{1}{2}\delta^{ij}\partial_i\partial_j + V_{\frac{1}{2}}(x)\right)\overline{\mathbb{K}}(x, 0; \beta) \tag{2.12}$$

with

$$V_{\frac{1}{2}}(x) = V_0(x) + \frac{d(d-1)M^2}{8} + \frac{d-1}{8}M^2\left(\frac{1-\cos(Mr)}{\sin(Mr)}\right)^2 \tag{2.13}$$

where the second addendum is just $\frac{1}{8}R$, and $V_0(x)$ has given in (1.51). Equations (2.12) and (2.13) allow to find the following simplified path integral

$$\overline{\mathbb{K}}(x, 0; \beta) \sim \mathbb{1}\int_{x(0)=0}^{x(\beta)=x} \mathcal{D}x e^{-S_{\frac{1}{2}}[x]}, \qquad S_{\frac{1}{2}}[x] = \int_0^\beta dt(\frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + V_{\frac{1}{2}}(x)) \tag{2.14}$$

with the effective potential $V_{\frac{1}{2}}$ is given explicitly as a function of $r = \sqrt{\delta_{ij}x^i x^j}$ by

$$V_{\frac{1}{2}}(x) = \frac{d(d-1)}{24}M^2 + \frac{(d-1)(d-3)}{48}\frac{\Big(5(Mr)^2 - 3 + ((Mr)^2+3)\cos(2Mr)\Big)}{r^2\sin^2(Mr)} + \frac{d-1}{8}M^2\left(\frac{1-\cos(Mr)}{\sin(Mr)}\right)^2. \tag{2.15}$$

Of course, one may reintroduce free worldline fermions $\psi^a$ to represent the identity with a Grassmann path integral, so to have the full linear sigma model lagrangian

$$L = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \frac{1}{2}\psi_a\dot{\psi}^a + V_{\frac{1}{2}}(x) \tag{2.16}$$

instead of the original nonlinear sigma model we started with in (2.1).
One could then use antiperiodic boundary conditions on the $\psi$'s to produce the trace on the spinor indices, periodic boundary conditions to produce the trace with an insertion of $\gamma^5$, or more generally leave open boundary conditions.
The heat kernel remains trivial on the spinor indices (in particular, traces are trivially computed). This simplified path integral was tested in [8] by computing its perturbative expansion and used to obtain the type-A trace anomalies of a Dirac field coupled to gravity in dimensions $d \leq 12$.

### 2.1.1 Perturbative expansion for Dirac field

The short-time perturbative expansion of the kernel (2.14) can be formally written as a power series in $\beta$

$$\overline{\mathbb{K}}(x,0;\beta) = g^{\frac{1}{4}}(x)\frac{e^{-\frac{x^2}{2\beta}}}{(2\pi\beta)^{\frac{d}{2}}}\sum_{n=0}^{\infty}a_n(x,0)\beta^n, \tag{2.17}$$

where $a_n$ are the Seeley-DeWitt coefficients. In general they are matrix-valued, but as we have discussed they are proportional to the identity matrix on maximally symmetric spaces.
As in the previous case, in order to compute perturbatively the expansion we use a rescaled time $\tau = \frac{t}{\beta}$, so that

$$S_{\frac{1}{2}}[x] = \int_0^1 d\tau\left(\frac{1}{2\beta}\delta_{ij}\dot{x}^i\dot{x}^j + \beta V_{\frac{1}{2}}(x(\tau))\right), \tag{2.18}$$

where the dot now indicates derivative with respect to $\tau$. Then we Taylor expand the potential about the origin of the Riemann coordinates

$$V_{\frac{1}{2}}(x) = M^2 \sum_{l=0}^{\infty} k_{2l}(Mr)^{2l}, \Rightarrow S_{\frac{1}{2}}[x] = \int_0^1 d\tau \frac{1}{2\beta}\delta_{ij}\dot{x}^i\dot{x}^j + \sum_{l=0}^{\infty} S_{2l}[x] \tag{2.19}$$

and retain only the relevant "coupling constants" $k_{2l}$ needed to carry out the expansion at the desired order. Explicitly,

$$S_{2l}[x] = \beta M^{2+2l} k_{2l} \int_0^1 d\tau (\delta_{ij}x^i x^j)^l. \tag{2.20}$$

The perturbative expansion can be obtained as seen before. In order to carry out an expansion say to order $\beta^m$, one needs to retain couplings up to $k_{2(m-1)}$, so that to reach order $\beta^6$ we need the couplings from $V_{\frac{1}{2}}$ up to $k_{10}$

$$\begin{aligned}
k_0 &= d(d-1)\left(-\frac{1}{12}+\frac{1}{8}\right) = \frac{d(d-1)}{24} \\
k_2 &= (d-1)\left((d-3)\frac{1}{120}+\frac{1}{32}\right) = \frac{(d-1)(d-3)}{480} \\
k_4 &= (d-1)\left((d-3)\frac{1}{756}+\frac{1}{192}\right) = \frac{(d-1)(16d+15)}{12096} \\
k_6 &= (d-1)\left((d-3)\frac{1}{5400}+\frac{17}{23040}\right) = \frac{(d-1)(64d+63)}{345600} \\
k_8 &= (d-1)\left((d-3)\frac{1}{41580}+\frac{31}{322560}\right) = \frac{(d-1)(256d+255)}{10644480} \\
k_{10} &= (d-1)\left((d-3)\frac{691}{232186500}+\frac{691}{58060800}\right) = \frac{691(d-1)(1024d+1023)}{237758976000}
\end{aligned} \tag{2.21}$$

For simplicity, we consider the diagonal part of the heat kernel only by setting $x=0$, which is relevant to obtain the trace anomalies or to compute the one-loop effective action of a Dirac spinor. This involves the following correlators

$$\begin{aligned}
\overline{\mathbb{K}}(0,0;\beta) = \frac{\mathbb{1}}{(2\pi\beta)^{\frac{1}{2}}} e^{-S_0} \exp\Big[ &- \underbrace{\langle S_2\rangle}_{O(\beta^2)} - \underbrace{\langle S_4\rangle}_{O(\beta^3)} \underbrace{-\langle S_6\rangle + \frac{1}{2}\langle S_2^2\rangle_c}_{O(\beta^4)} \underbrace{-\langle S_8\rangle + \langle S_4 S_2\rangle_c}_{O(\beta^5)} \\
&\underbrace{-\langle S_{10}\rangle + \langle S_6 S_2\rangle_c + \frac{1}{2} - \langle S_4^2\rangle_c - \frac{1}{3!}\langle S_2^3\rangle_c}_{O(\beta^6)} + O(\beta^7)\Big]
\end{aligned} \tag{2.22}$$

where the subscript "c" stands for "connected" correlation functions.
The expression for the kernel (2.22) differs from that obtained in the scalar case only in the coupling constants, now given by (2.21). Hence, by inserting the new coupling constants into the expression of the scalar heat kernel we can obtain the final results for the fermion heat kernel at coinciding points

$$\begin{aligned}\overline{\mathbb{K}}(0,0;\beta) = \frac{\mathbb{1}}{(2\pi\beta)^{\frac{d}{2}}}\exp\Bigg[&-d(d-1)\frac{\beta M^2}{24} + d(d-1)\bigg(-\frac{(\beta M^2)^2}{6!}\frac{4d+3}{4}\\ &-\frac{(\beta M^2)^3}{9!}(d+2)(16d+15)\\ &-\frac{(\beta M^2)^4}{10!}\frac{(16d^3+257d^2+555d+315)}{8}\\ &+\frac{(\beta M^2)^5}{11!}\frac{(d+2)(64d^3-333d^2-1341d-945)}{24}\\ &+\frac{(\beta M^2)^6}{13!}\frac{207744d^5+943595d^4-2652226d^3-18403426d^2-29381262d-14365890}{5040}\\ &+O(\beta^7)\bigg)\Bigg]\end{aligned} \tag{2.23}$$

which we can write in terms of the constant scalar curvature $R$. Computational details will be shown in chapter 3. In this expression the exponential must be expanded, keeping terms up to order $O(\beta^7)$ excluded. This allows to read off the diagonal coefficients $a_n(0,0)$, with integer $n$ up to $n = 6$, in order to identify the type-A trace of a Dirac fermion in various dimensions.

### 2.1.2 The type-A trace anomalies for a Dirac fermion

In general, the trace anomaly of a Dirac fermion can be related to the transition amplitude of a $N = 1$ spinning particle in a curved space by

$$\langle T^m{}_m(x)\rangle_{QFT} = -\lim_{\beta\to 0}\operatorname{Tr}\mathbb{K}(x,x;\beta), \tag{2.24}$$

where on the left hand side $T^m{}_m(x)$ is the trace of the stress tensor of the Dirac spinor in a curved background, obtained from the appropriate Dirac action $S_D$ by $T_{ma}(x) = \frac{1}{e}\frac{\delta S_D}{\delta e^{ma}(x)}$ where $e_m^a(x)$ is the vielbein of the curved spacetime. The expectation value is performed in the corresponding quantum field theory. We can view the right hand side as the anomalous contribution arising from QFT path integral measure, regulated à la Fujikawa [33], with the minus sign due to the fermionic measure, as usual, and the trace being the trace on spinor indices. The regulator corresponds to the square of the Dirac operator, and is identified with the quantum hamiltonian $\hat{H}$ of the $N = 1$ spinning particle in a curved space

$$\hat{H} = -\frac{1}{2}(\nabla\!\!\!/)^2, \tag{2.25}$$

which appears in the heat kernel at coinciding points $\mathbb{K}(x, x; \beta)$. As in the previous case, it is understood that the $\beta \to 0$ limit in (2.24) picks up just the $\beta$-independent term, as divergent terms are removed by the QFT renormalization. This procedure selects the appropriate heat kernel coefficient $a_n(x, x)$ sitting in the expansion of $\mathbb{K}(x, x; \beta)$. It may be interpreted as the contribution to the anomaly of the regularized particle making its virtual loop, see for example [40], where a Pauli-Villars regularization gives rise to the Fujikawa regulator used above.

Expanding $\mathbb{K}(x, x; \beta)$ at the required order one can read off the trace anomalies in even $d$ dimensions (odd dimensions support no anomaly if the space is boundaryless)

$$\langle T^m{}_m(x)\rangle_{QFT} = \frac{\mathrm{Tr}\, a_{\frac{d}{2}}(x)}{(2\pi)^{\frac{d}{2}}} \tag{2.26}$$

that is, for even $d = 2n$ dimensions, the relevant coefficient is precisely $a_n(x, x)$. Of course, one may use Riemann normal coordinates centered at x, so that $\sqrt{g(x)} = 1$ and $\overline{\mathbb{K}}(x, x; \beta) = \mathbb{K}(x, x; \beta)$. This formula holds on a generic space. In the present maximally symmetric case, due to translational invariance, the choice of which point is the origin of the Riemann coordinates becomes irrelevant. Hence, $\overline{\mathbb{K}}(x, x; \beta) = \overline{\mathbb{K}}(0, 0; \beta)$, and the result obtained in the previous subsection is directly applicable. The trace in (2.26) reduces to the trace of the identity matrix, and counts the dimension of the spinor space, $2^{\frac{d}{2}}$ for even $d$ dimensions.

This calculations are similar to the one performed in the $N = 0$ model and they reproduce again the trace anomalies in $d = 2, 4, 6....$ Now we report the calculation for the first three coefficients. Let's consider the expansion of (2.23) up to order $\beta^3$

$$\begin{aligned}\overline{\mathbb{K}}(0, 0; \beta) = \frac{\mathbb{1}}{(2\pi\beta)^{\frac{d}{2}}}\bigg[1 - \frac{\beta R}{24} - \frac{(\beta R)^2}{720}\frac{4d + 3}{4d(d - 1)} + \frac{1}{2}\frac{(\beta R)^2}{576} \\ + \frac{(\beta R)^3}{24 \cdot 720}\frac{(4d + 3)}{4d(d - 1)} - \frac{(\beta R)^3}{9!}(d + 2)(16d + 15) - \frac{(\beta R)^3}{82944}\bigg].\end{aligned} \tag{2.27}$$

| $d$ | $\langle T^\mu{}_\mu \rangle$ | $\langle T^\mu{}_\mu \rangle$ |
|---|---|---|
| 2 | $\frac{R}{24\pi}$ | $\frac{1}{12\pi a}$ |
| 4 | $-\frac{11R^2}{34560\pi^2}$ | $-\frac{11}{240\pi^2 a^4}$ |
| 6 | $\frac{191R^3}{108864000\pi^3}$ | $\frac{191}{4032\pi^3 a^6}$ |
| 8 | $-\frac{2497R^4}{339880181760\pi^4}$ | $-\frac{2497}{34560\pi^4 a^8}$ |
| 10 | $\frac{14797R^5}{598615142400000\pi^5}$ | $\frac{14797}{101376\pi^5 a^{10}}$ |
| 12 | $-\frac{92427157R^6}{1330910037208675123200\pi^6}$ | $-\frac{92427157}{251596800\pi^6 a^{12}}$ |

**Table 2.1:** *The type-A trace anomaly of a Dirac spinor in terms of the curvature scalar R and in terms of the radius a, in various dimensions reported in [8].*

The first three coefficients read

$$d = 2 \quad \to \quad -\,\mathrm{Tr}\left\{ \frac{\mathbb{1}}{2\pi\beta}\left(-\frac{\beta R}{24}\right)\right\} = \frac{R}{24\pi} \tag{2.28}$$

$$d = 4 \quad \to \quad -\,\mathrm{Tr}\left\{ \frac{\mathbb{1}}{(2\pi\beta)^2}\left[-\frac{19}{34560} + \frac{1}{1152}\right](\beta R)^2\right\} = -\frac{11}{34560\pi^2}R^2 \tag{2.29}$$

$$d = 6 \quad \to \quad -\,\mathrm{Tr}\left\{ \frac{\mathbb{1}}{(2\pi\beta)^3}\left[\frac{1}{76800} - \frac{1}{82944} - \frac{111}{40824000}\right](\beta R)^3\right\} = \frac{191}{108864000\pi^3}R^3 \tag{2.30}$$

The other anomalies obtained from the expansion (2.23) up to order $\beta^6$ have been reported in Table 2.1. The type-A trace anomaly can also be obtained using the Riemann zeta-function associated to the differential operator (2.25)

$$\langle T^m{}_m(x)\rangle_{QFT} = -\frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d+1}{2}}a^d}\zeta_{\nabla^2(0)}, \tag{2.31}$$

as discussed in [38, 39].

# Chapter 3

# Path integral for $N = 2$ supersymmetric quantum mechanics

## 3.1 Transition amplitude for $p$-forms

We have considered up to now the worldline path integral for the $N = 0$ and $N = 1$ supersymmetric quantum mechanics in maximally symmetric curved spaces, and we have described a simplified path integral by making use of Riemann normal coordinates.

Now we wish to investigate if this construction can be extended to the $N = 2$ supersymmetric quantum mechanics, which is useful for treating the quantum field theory of spin 1 fields, and more generally of $p$-forms, in first quantization.
So, we want to study the path integral

$$\int \mathcal{D}X e^{-S[X;g_{ij}]} \tag{3.1}$$

where $X = (x^i, \psi^a, \bar{\psi}^a)$, with $x^i$ the bosonic coordinates of a particle on a sphere (or, more generally, on a maximally symmetric space) and $\psi^a$,$\bar{\psi}^a$ related fermionic variables (the $N = 2$ supersymmetric partners of $x^i$). We denote by $i$,$j$... curved indices and by $a$,$b$,... flat indices, i.e. coordinate and frame indices, respectively.
The precise nonlinear sigma model of interest is given by

$$S[X; g_{ij}] = \int_0^\beta dt \bigg( \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j + \bar{\psi}_a(\dot{\psi}^a + \dot{x}^i {\omega_i}^a{}_b \psi^b) - \frac{1}{2} R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d \bigg), \tag{3.2}$$

where $\beta$ is the total propagating euclidean time, $g_{ij}$ the metric of the sphere, ${\omega_i}^a{}_b$ the

spin connection related to the vielbein $e_i^a$, and $R_{abcd}$ is the Riemann tensor.
On generic curved spaces the path integral must be regulated carefully. However on maximally symmetric spaces such as spheres, Riemann normal coordinates may be helpful to turn the nonlinear sigma model into a linear one, and possibly find simplifications as seen in the $N = 0$ (bosonic) and $N = 1$ supersymmetric sigma model.
However, a direct study of the transition amplitude shows now that in general it does not only depend on $r^2 = \delta_{ij}x^i x^j$, but there could be more general structures allowed by the symmetry, such as $(x_\mu \gamma_1^\mu)(x_\nu \gamma_2^\nu)$, where the gamma matrices are two anticommuting set of gamma matrices realizing the real and imaginary part of the worldline fermions (and acting on a bispinorial wave function on the first and on the second index, respectively). Thus, a direct splitting into noninteracting bosonic and fermionic parts does not seem achievable.
We can still assume that some simplifications will occur in the path integral. The simplest conjecture is to assume a suitable effective potential, that linearizes the kinetic term of the bosonic part of the sigma model, and test if this ansatz can reproduce the known results at order $\beta^2$ given in [29]. This matching leads us to our conjecture, based on the use of RNC, which consists in introducing an effective potential

$$V_1(x) = V_0(x) + \frac{1}{8}R = V_0(x) + \frac{d(d-1)M^2}{8} \tag{3.3}$$

and a linear sigma model action

$$\begin{aligned} S[x,\psi,\bar\psi] &= S_b[x] + S_f[x,\psi,\bar\psi] \\ S_b[x] &= \int_0^\beta dt \left(\frac{1}{2}\delta_{ij}\dot x^i \dot x^j + V_1(x)\right) \\ S_f[x,\psi,\bar\psi] &= \int_0^\beta dt \left(\bar\psi_a \frac{D\psi^a}{dt} - \frac{1}{2}R_{abcd}\bar\psi^a\psi^b\bar\psi^c\psi^d\right) \end{aligned} \tag{3.4}$$

where the coupling to the Riemann tensor reduces to

$$R_{abcd}\bar\psi^a\psi^b\bar\psi^c\psi^d = M^2(\bar\psi_a\psi^a)^2. \tag{3.5}$$

Thus, the purely bosonic part is semplified to a linear sigma model augmented by the scalar effective potential $V_1$, while the fermionic part is left untouched, but to be treated in RNC. The fermionic path integral requires the additional use of a worldline regularization, that we choose to be dimensional regularization (DR). Indeed we find that the perturbative expansion reproduces the results in [29], where the nonlinear sigma model was used to compute the transition amplitude up to order $\beta^2$. As this matching was at the basis of our conjecture, a crucial test is to check whether the order $\beta^3$ is correct. While we do not know how to check the full transition amplitude at that order, we use it to see if it correctly reproduces the trace anomaly of a 2-form in 6 dimensions.

### 3.1.1 Perturbative expansion for p-forms

With our linear sigma model the transition amplitude in RNC is computed by

$$\overline{\mathcal{K}}(x,\bar{\eta},0,\eta;\beta) = \int_{x(0)=0}^{x(\beta)=x} \mathcal{D}x \int_{\psi(0)=\eta}^{\psi(\beta)=\bar{\eta}} \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S[x,\psi,\bar{\psi}]} \tag{3.6}$$

where the Grassmann variables $\eta$ and $\bar{\eta}$ (not conjugated to each other) label the external states associated to the fermionic variables (coherent states).
We will not compute it in full generality but content ourselves to compute perturbatively the transition amplitude at coinciding points ($x = x' = 0$) using twisted boundary conditions on the fermions

$$\overline{\mathcal{K}}(0,0;\beta,\phi) = \int_{x(0)=0}^{x(\beta)=0} \mathcal{D}x \int_T \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S[x,\psi,\bar{\psi}]} \tag{3.7}$$

where the subscript $T$ denotes the antiperiodic boundary conditions twisted by an angle $\phi$, that is $\psi(\beta) = -e^{i\phi}\psi(0)$ and $\bar{\psi}(\beta) = e^{-i\phi}\bar{\psi}(0)$. This amplitude is enough to study the one-loop effective action of $p$-forms, and, in particular, to extract the trace anomalies of the conformal $p$-forms in $d = 2p+2$ dimensions. At the same time this calculation offers a test of the previous construction.
At first, we rescale the time $t \to \tau = \frac{t}{\beta}$ (with the parameter $\tau \in [0,1]$) to control the perturbative $\beta$ expansion and the bosonic part of the action, that we denote with $S_b[x]$, takes the form

$$S_b[x] = \int_0^1 d\tau \left( \frac{1}{2\beta}\delta_{ij}\dot{x}^i\dot{x}^j + \beta V_1(x) \right). \tag{3.8}$$

From the kinetic term we identify the free propagator

$$\langle x^i(\tau)x^j(\tau')\rangle = -\beta\delta^{ij}\Delta(\tau,\tau') \tag{3.9}$$

with

$$\Delta(\tau, \tau') = (\tau - 1)\tau'\theta(\tau - \tau') + (\tau' - 1)\tau\theta(\tau' - \tau) \tag{3.10}$$

where $\theta(x)$ is the step function with $\theta(0) = \frac{1}{2}$. The propagator implements the correct boundary conditions ($x(0) = x(1) = 0$) and at coinciding points reduces to

$$\Delta(\tau, \tau) = \tau^2 - \tau. \tag{3.11}$$

Then, by Wick contractions one computes the correlation functions arising from the interactions depending on the effective potential $V_1$ and given by

$$S_{b,int} = \beta \int_0^1 d\tau V_1(x) = \sum_{m=0}^{\infty} S_{2m} = \sum_{m=0}^{\infty} \beta M^{2+2m} k_{2m} \int_0^1 d\tau (x^2)^m, \tag{3.12}$$

where the coefficients $K_{2m}$ are obtained from the Taylor expansion of the potential $V_1$. The first few ones are

$$\begin{aligned} k_0 &= d(d-1)\left(-\frac{1}{12} + \frac{1}{8}\right) = \frac{d(d-1)}{24} \\ k_2 &= (d-1)(d-3)\frac{1}{120} \\ k_4 &= (d-1)(d-3)\frac{1}{756} \end{aligned} \tag{3.13}$$

which contribute to expansion of the bosonic part of (3.7) to order $\beta^3$

$$\overline{\mathcal{K}}_b(0, 0; \beta) = \frac{e^{-S_0}}{(2\pi\beta)^{\frac{d}{2}}} \exp\left[-\underbrace{\langle S_2 \rangle}_{O(\beta^2)} - \underbrace{\langle S_4 \rangle}_{O(\beta^3)} + O(\beta^4)\right] \tag{3.14}$$

Let's set $M = 1$ for simplicity.
At order $\beta$ there is only a constant term that does not require any Wick contraction

$$-S_0 = -\beta k_0 = -\beta \frac{d(d-1)}{24} \tag{3.15}$$

**Figure 3.1:** *Diagram for* $S_2$

**Figure 3.2:** *Diagram for* $S_4$

At order $\beta^2$ we have to perform the Wick contraction of bosonic lines in the diagram in figure 3.1, that gives:

$$-\langle S_2\rangle = -\beta k_2 \int_0^1 d\tau \langle \delta_{ij} x^i(\tau) x^j(\tau)\rangle = \beta^2 k_2 d \int_0^1 d\tau \Delta(\tau,\tau) = \tag{3.16}$$

$$\beta^2 \frac{d(d-1)(d-3)}{120} \underbrace{\int_0^1 d\tau(\tau^2-\tau)}_{-\frac{1}{6}} = -\beta^2 \frac{d(d-1)(d-3)}{6!} \tag{3.17}$$

At order $\beta^3$ we have to consider the diagram in fig 3.2

$$-\langle S_4\rangle = -\beta k_4 \int_0^1 d\tau \langle \delta_{ij} x^i(\tau) x^j(\tau) \delta_{lm} x^l(\tau) x^m(\tau)\rangle \tag{3.18}$$

$$= -\beta^3 d(d+2) k_4 \int_0^1 d\tau \Delta(\tau,\tau)^2 \tag{3.19}$$

$$= -\beta^3 \frac{d(d+2)(d-1)(d-3)}{756} \underbrace{\int_0^1 d\tau(\tau^2-\tau)^2}_{\frac{1}{30}} = -\frac{16d(d-1)(d-3)(d+2)}{9!} \tag{3.20}$$

The bosonic transition amplitude reads

$$\overline{\mathcal{K}}_b(0,0;\beta) = \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp\left[d(d-1)\left(-\frac{\beta}{24} - \frac{\beta^2}{6!}(d-3) - \frac{\beta^3}{9!}16(d-3)(d+2) + O(\beta^4)\right)\right]. \tag{3.21}$$

Now we consider the part of the action containing the fermionic variables:

$$\overline{\mathcal{K}}_f(\beta,\phi) = \int_{\mathcal{T}} \mathcal{D}\psi\mathcal{D}\bar{\psi}\langle e^{-S_f[\psi,\bar{\psi},x]}\rangle. \tag{3.22}$$

where $\langle ...\rangle$ denotes averaging on the $x$ variables. Again, we rescale the time $t \to \tau = \frac{t}{\beta}$ to find

$$S_f[\psi,\bar{\psi},x] = \int_0^1 d\tau \left[\bar{\psi}_a(\dot{\psi}^a + \dot{x}^i\omega_i{}^a{}_b\psi^b) - \frac{\beta}{2}(\bar{\psi}_a\psi^a)^2\right]. \tag{3.23}$$

From the kinetic term we identify the free fermionic propagator with twisted boundary conditions

$$\langle\psi^a(\tau)\bar{\psi}_b(\tau')\rangle = \delta^a_b\Delta_f(\tau-\tau',\phi) \tag{3.24}$$

where the function $\Delta_f(x,\phi)$ is given for $x \in (-1,1)$ by

$$\Delta_f(x,\phi) = \frac{1}{2\cos\frac{\phi}{2}}\left[e^{i\frac{\phi}{2}}\theta(x) - e^{-i\frac{\phi}{2}}\theta(-x)\right] \tag{3.25}$$

with $\theta(x)$ the step function. Note that for $x \neq 0$

$$\Delta_f(x,\phi)\Delta_f(-x,\phi) = -\frac{1}{4}\cos^2\frac{\phi}{2} \tag{3.26}$$

while at coinciding points ($\tau = \tau'$, i.e. $x = 0$) it takes the regulated value

$$\Delta_f(0,\phi) = \frac{i}{2}\tan\frac{\phi}{2} \tag{3.27}$$

so that

$$\Delta_f^2(0,\phi) = \frac{1}{4}\tan^2\frac{\phi}{2} = -\frac{1}{4}\cos^{-2}\frac{\phi}{2} + \frac{1}{4} \tag{3.28}$$

This regulated value is responsible for the counterterm mentioned earlier in constructing the effective potential $V_1$. More details are shown in Appendix A. Then, by Wick contractions on the diagrams shown in figure 3.2, one computes the correlation functions

$$\langle S_{4f}\rangle = -\frac{\beta}{2}\int_0^1 d\tau\langle\bar{\psi}_a\psi^a\bar{\psi}_b\psi^b\rangle. \tag{3.29}$$

and

$$\frac{1}{2}\langle S_{4f}^2\rangle_c = \frac{\beta^2 d(d-1)}{8}\left[\frac{1}{8}\cos^{-4}\frac{\phi}{2} - \frac{(d-1)}{4}\cos^{-2}\frac{\phi}{2}\tan^2\frac{\phi}{2}\right] \tag{3.30}$$

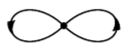


**Figure 3.3:** *Diagram for* $\langle S_{4f} \rangle$

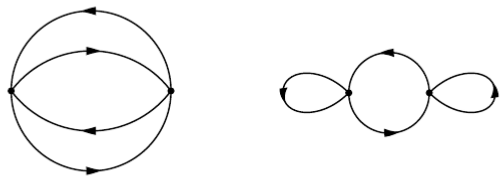


**Figure 3.4:** *Diagram for* $\langle S_{4f}^2 \rangle_c$

It is not necessary to compute the non connected correlation function $\frac{1}{2}\langle S_{4f}^2 \rangle_{nc}$ because it will arise from the term of order $\beta$ in the exponential expansion.
Let's consider now the part of the action containing the spin connection

$$\dot{x}^i \omega_{iab}(x) \bar{\psi}^a \psi^b, \tag{3.31}$$

We want to compute the correlation function

$$\langle S_{SC} \rangle = \int_0^1 d\tau \langle \bar{\psi}_a \dot{x}^i \omega_i{}^a{}_b(x) \psi^b \rangle. \tag{3.32}$$

Let's consider the relations

$$\omega_i{}^{ab}(x) = \frac{1}{M^2} \Omega(r) \frac{1}{2} x^j R_{ij}{}^{ab}(0) \tag{3.33}$$

and

$$\Omega(x) = -\frac{2}{r}\left( l'(r) + \frac{l(r)}{r} \right) = 2M^2 \frac{1 - \cos(Mr)}{(Mr)^2}, \tag{3.34}$$

We can rewrite $\Omega(x)$ by Taylor expanding it around the point $x = 0$ (we set $M = 1$)

$$\Omega(x) = \Omega_0(x) + \Omega_1(x) + ... = 2(\frac{1}{2} - \frac{1}{24} r^2 + ...) = 1 - \frac{1}{12} r^2 + ... \tag{3.35}$$

Let's use the first term of the Taylor expansion of $\Omega(x)$ in (3.35) and perform the Wick contractions between bosonic and fermionic separately. Then, the integral vanishes because of the antisymmetry of the Riemann tensor. Appendix B.1 shows how to compute the quadratic contribute of this term by making use of the dimensional regularization, as shown in [29], that reads

$$\frac{1}{2}\langle S_{SC}^2\rangle = -\frac{\beta^2}{192}R_{abcd}^2\cos^{-2}\frac{\phi}{2}. \tag{3.36}$$

At this perturbative order there are other diagrams that involve the interaction of the spin connection with the bosonic expansion of the potential $V_1$ and with the fermionic part of the action $S_{4f}$, but they vanish. Now we put all these results together in the exponential expansion of the full transition amplitude up to order $\beta^2$

$$\begin{aligned}\overline{\mathcal{K}}(0,0;\beta) &= \\ &= \frac{(2\cos\frac{\phi}{2})^d}{(2\pi\beta)^{\frac{d}{2}}}\Big[1 - S_0 - \langle S_2\rangle - \langle S_4\rangle - \langle S_{4f}\rangle - \langle S_{SC}\rangle + \frac{1}{2}S_0^2 + \frac{1}{2}\langle S_{SC}^2\rangle + \langle S_0\rangle\langle S_{4f}\rangle + O(\beta^3)\Big]\end{aligned} \tag{3.37}$$

that reads

$$\begin{aligned}\overline{\mathcal{K}}(0,0;\beta) = \frac{(2\cos\frac{\phi}{2})^d}{(2\pi\beta)^{\frac{d}{2}}}\Big\{&1 + \beta d(d-1)\left(\frac{1}{12} - \frac{1}{8}\cos^{-2}\frac{\phi}{2}\right) \\ &+ \beta^2 d(d-1)\left[\frac{5d^2-7d+6}{1440} - \frac{(d-2)^2}{96}\cos^{-2}\frac{\phi}{2} + \frac{(d-2)(d-3)}{128}\cos^{-4}\frac{\phi}{2}\right] + O(\beta^3)\Big\}.\end{aligned} \tag{3.38}$$

where the factor $(2\cos\frac{\phi}{2})^d$ comes from the normalization of the fermionic part of the path integral as shown in [29]. At the orders $\beta$ and $\beta^2$ this transition amplitude seems to check the correctness of the effective potential $V_1$.

### Simplified fermionic model (without spin connection)

For computing $\overline{\mathcal{K}}_f(\beta,\phi) = \int_T D\psi D\bar{\psi}\, e^{-S_f[\psi,\bar{\psi}]}$ we may also use a different action with an auxiliary bosonic field $\phi$. Rather then

$$S_f[\psi,\bar{\psi}] = \int_0^1 d\tau\left(\bar{\psi}_a\dot{\psi}^a - \frac{\beta}{2}(\bar{\psi}_a\psi^a)^2\right) \tag{3.39}$$

we consider

$$S_f[\psi,\bar\psi,\phi] = \int_0^1 d\tau \left( \bar\psi_a \dot\psi^a + \frac{1}{2}\phi^2 - \sqrt{\beta}\phi\bar\psi_a\psi^a \right) \tag{3.40}$$

which is classically equivalent (we use $M = 1$).
At the quantum level we have the propagators

$$\langle \phi(\tau)\phi(\tau')\rangle = \delta(\tau-\tau') \tag{3.41}$$

$$\langle \psi^a(\tau)\bar\psi_b(\tau')\rangle = \delta^a_b\, \Delta_f(\tau-\tau',\phi) \tag{3.42}$$

where the function $\Delta_f(x,\phi)$ is given for $x \in (-1,1)$ by

$$\Delta_f(x,\phi) = \frac{1}{2\cos\frac{\phi}{2}}\left[ e^{i\frac{\phi}{2}}\theta(x) - e^{-i\frac{\phi}{2}}\theta(-x)\right] \tag{3.43}$$

with $\theta(x)$ the step function. Note that for $x \neq 0$

$$\Delta_f(x,\phi)\Delta_f(-x,\phi) = -\frac{1}{4}\cos^{-2}\frac{\phi}{2} \tag{3.44}$$

while at coinciding points ($\tau = \tau'$, i.e. $x = 0$) it takes the regulated value

$$\Delta_f(0,\phi) = \frac{i}{2}\tan\frac{\phi}{2} \tag{3.45}$$

so that

$$\Delta_f^2(0,\phi) = -\frac{1}{4}\tan^2\frac{\phi}{2} = -\frac{1}{4}\cos^{-2}\frac{\phi}{2} + \frac{1}{4}\ . \tag{3.46}$$

This regulated value is responsible for the counterterm mentioned earlier in constructing the effective potential $V_1$.
By Wick contractions one computes perturbatively the transition amplitude

$$\overline{\mathcal{K}}_f(\beta,\phi) = \big(2\cos\frac{\phi}{2}\big)^d \langle e^{-S_{int}}\rangle \tag{3.47}$$

where $\langle ...\rangle$ denotes normalized correlation functions. Using the formulation without the auxiliary field

$$S_{int} = S_{4f} = -\frac{\beta}{2}\int_0^1 d\tau\, \bar\psi_a\psi^a\bar\psi_b\psi^b \tag{3.48}$$

and at the leading order one finds

$$\langle e^{-S_{4f}} \rangle = 1 - \langle S_{4f} \rangle + ... \tag{3.49}$$

$$= 1 + \frac{\beta}{2} d(d-1) \Delta_f^2(0, \phi) + ... \tag{3.50}$$

$$= 1 - \frac{\beta}{8} d(d-1) \tan^2 \frac{\phi}{2} + ... \tag{3.51}$$

$$= 1 - \frac{\beta}{8} d(d-1) \left( \cos^{-2} \frac{\phi}{2} - 1 \right) + ... \tag{3.52}$$

where we have used the regulated value (3.46) for $\Delta_f^2(0, \phi)$ (maybe one could use a different regularization). Using the formulation with the auxiliary field we have

$$S_{int} = -\sqrt{\beta} \int_0^1 d\tau \phi \bar{\psi}_a \psi^a \tag{3.53}$$

and at the leading nonvanishing order

$$\langle e^{-S_{4f}} \rangle = 1 - \langle S_{int} \rangle + \frac{1}{2} \langle S_{int}^2 \rangle + ... \tag{3.54}$$

$$= 1 + 0 + \frac{\beta}{2} \int_0^1 d\tau \int_0^1 d\tau' \, \delta(\tau - \tau') \left( d^2 \Delta_f^2(0, \phi) - d \Delta_f(\tau - \tau', \phi) \Delta_f(\tau' - \tau, \phi) \right) \tag{3.55}$$

$$= 1 + \frac{\beta}{2} d(d-1) \Delta_f^2(0, \phi) + ... \tag{3.56}$$

$$= 1 - \frac{\beta}{8} d(d-1) \tan^2 \frac{\phi}{2} + ... \tag{3.57}$$

$$= 1 - \frac{\beta}{8} d(d-1) \left( \cos^{-2} \frac{\phi}{2} - 1 \right) + ... \tag{3.58}$$

This reproduces indeed what we have found earlier. This is an alternative set-up that can be studied in future works.

**Transition amplitude up to order $\beta^3$**

Now let's extend the computation of the original transition amplitude in (3.7) with the action (3.2) up to order $\beta^3$, that includes the following terms:

$$\begin{aligned} O(\beta^3) \to & -\frac{1}{3!}(\langle S_0\rangle)^3 - \langle S_4\rangle + \langle S_0\rangle\langle S_2\rangle - \frac{1}{3!}\langle S_{4f}^3\rangle - \frac{1}{3!}\langle S_{SC}^3\rangle \\ & + \langle S_2\rangle\langle S_{4f}\rangle - \frac{1}{2}\langle S_{SC}^2 S_{4f}\rangle - \frac{1}{2}\langle S_0\rangle\langle S_{4f}^2\rangle - \frac{1}{2}\langle S_0\rangle\langle S_{SC}^2\rangle. \end{aligned} \tag{3.59}$$

$\langle S_4\rangle$ is given by (3.20) and by a direct computation one can obtain:

$$-\frac{1}{3!}(\langle S_0\rangle)^3 = -\frac{\beta^3 d^3(d-1)^3}{3!\cdot 24^3} \tag{3.60}$$

$$\langle S_0\rangle\langle S_2\rangle = \frac{\beta^3 d^2(d-1)^2(d-3)}{17280} \tag{3.61}$$

$$\langle S_2\rangle\langle S_{4f}\rangle = \frac{\beta^3 d^2(d-1)^2(d-3)}{5760}\left(\cos^{-2}\frac{\phi}{2} - 1\right) \tag{3.62}$$

$$-\frac{1}{2}\langle S_0\rangle\langle S_{4f}^2\rangle = -\frac{\beta^3 d^2(d-1)^2}{192}\left[\frac{(d-2)(d-3)}{16}\cos^{-4}\frac{\phi}{2} - \frac{(d-1)(d-2)}{8}\cos^{-2}\frac{\phi}{2} + \frac{d(d-1)}{16}\right] \tag{3.63}$$

$$-\frac{1}{2}\langle S_0\rangle\langle S_{SC}^2\rangle = \frac{\beta^3 d^2(d-1)^2}{2304}\cos^{-2}\frac{\phi}{2}. \tag{3.64}$$

Let's consider the contribution of the diagrams with three vertices corresponding to

$$-\frac{1}{3!}\langle S_{4f}^3\rangle = \frac{\beta^3}{3!\cdot 8}\iiint d\tau d\sigma d\xi\, \langle \bar\psi_a\psi^a\bar\psi_b\psi^b(\tau)\,\bar\psi_c\psi^c\bar\psi_d\psi^d(\sigma)\,\bar\psi_e\psi^e\bar\psi_f\psi^f(\xi)\rangle. \tag{3.65}$$

To compute the correlation function $\langle S_{2nf}\rangle$ we have to perform Wick contractions between $n$ fermions and $n$ antifermions, and it can be done in $n!$ different ways, i.e 2! for $\langle S_{4f}\rangle$ and $4! = 24$ for $\langle S_{4f}^2\rangle$. The latter takes into account non connected ($2!\cdot 2! = 4$ ways) and connected diagrams (20 ways).
If we consider the contribution of $\langle S_{4f}^3\rangle$, we have $6! = 720$ possible different ways to contract fermions, taking into account not connected (128 ways) and connected (592 ways) correlation functions. We are just interested in 5 connected diagrams for $\langle S_{4f}^3\rangle_c$, that are

shown in Figure 3.5 and give

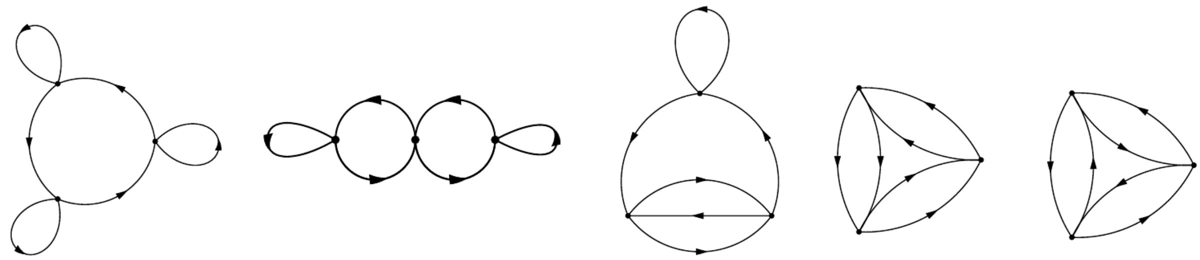
$$
\begin{array}{llc}
(a) & \left(\frac{\beta^3}{8}\right)16d(d-1)^3 I_a & (128)\\
(b) & \left(\frac{\beta^3}{8}\right)24d(d-1)^3 I_b & (192)\\
(c) & \left(\frac{\beta^3}{8}\right)96d(d-1) I_c & (192)\\
(d) & \left(\frac{\beta^3}{8}\right)8d(d-1) I_d & (16)\\
(e) & \left(\frac{\beta^3}{8}\right)(-16d(d-1)(d-3)) I_e, & (128)
\end{array}
\tag{3.66}
$$

The computation of the integrals in (3.66) gives:

$$
\begin{aligned}
I_a &= \iiint d\tau d\sigma d\xi \Delta^3(0)\Delta_{\tau\sigma}\Delta_{\sigma\xi}\Delta_{\xi\tau} = -\frac{1}{64}(\cos^{-6}\frac{\phi}{2} - 2\cos^{-4}\frac{\phi}{2} + \cos^{-2}\frac{\phi}{2})\\
I_b &= \iiint d\tau d\sigma d\xi \Delta^2(0)\Delta_{\tau\sigma}\Delta_{\sigma\tau}\Delta_{\sigma\xi}\Delta_{\xi\sigma} = -\frac{1}{64}(\cos^{-6}\frac{\phi}{2} - \cos^{-4}\frac{\phi}{2})\\
I_c &= \iiint d\tau d\sigma d\xi \Delta(0)\Delta_{\tau\sigma}\Delta_{\sigma\xi}\Delta_{\xi\sigma}\Delta_{\sigma\xi}\Delta_{\xi\tau} = -\frac{1}{64}(\cos^{-6}\frac{\phi}{2} - \cos^{-4}\frac{\phi}{2})\\
I_d &= \iiint d\tau d\sigma d\xi \Delta_{\tau\sigma}\Delta_{\tau\sigma}\Delta_{\sigma\xi}\Delta_{\sigma\xi}\Delta_{\xi\tau}\Delta_{\xi\tau} = -\frac{1}{64}(\cos^{-6}\frac{\phi}{2} - 2\cos^{-4}\frac{\phi}{2})\\
I_e &= \iiint d\tau d\sigma d\xi \Delta_{\tau\sigma}\Delta_{\sigma\tau}\Delta_{\sigma\xi}\Delta_{\xi\sigma}\Delta_{\xi\tau}\Delta_{\tau\xi} = -\frac{1}{64}\cos^{-6}\frac{\phi}{2}.
\end{aligned}
\tag{3.67}
$$

where we use, for example, $\Delta_{\tau\sigma}$ instead of $\Delta_{AF}(\tau-\sigma)$. We obtain

$$
\begin{aligned}
&-\frac{1}{3!}\langle S_{4f}^3\rangle_C =\\
&= \beta^3 d(d-1)\left[-\frac{5d^2-12d+24}{384}\cos^{-6}\frac{\phi}{2} + \frac{7d^2-14d+21}{384}\cos^{-4}\frac{\phi}{2} - \frac{(d-1)^2}{192}\cos^{-2}\frac{\phi}{2}\right].
\end{aligned}
\tag{3.68}
$$



**(a)** **(b)** **(c)** **(d)** **(e)**

**Figure 3.5:** *Connected fermionic diagrams at three vertices*

Not connected diagrams are shown in Figure 3.6 and one can easily compute their contributions:

$$\begin{aligned}
&(a) \quad -\frac{\beta^3 d^3 (d-1)^3}{3072} \cos^{-6}\frac{\phi}{2} \\
&(b) \quad \frac{\beta^3 d^2 (d-1)^3}{256}\left(\cos^{-6}\frac{\phi}{2} - 2\cos^{-4}\frac{\phi}{2} + \cos^{-2}\frac{\phi}{2}\right) \\
&(c) \quad -\frac{\beta^3 d^2 (d-1)^2}{512}\left(\cos^{-6}\frac{\phi}{2} - \cos^{-4}\frac{\phi}{2}\right)
\end{aligned} \tag{3.69}$$

If we put together all the contributions in (3.69) we obtain

$$\begin{aligned}
-\frac{1}{3!}\langle S_{4f}^3\rangle_{NC} = &-\frac{\beta^3 d^2 (d-1)^2}{3072}\Big[\big(d^2 - 13d + 18\big)\cos^{-6}\frac{\phi}{2} \\
&- 3\big(d^2 - 9d + 10\big)\cos^{-4}\frac{\phi}{2} + 3(d-4)(d-1)\cos^{-2}\frac{\phi}{2} - d(d-1)\Big].
\end{aligned} \tag{3.70}$$

Let's consider now terms involving the spin connection at order $\beta^3$. Calculation details are shown in Appendix B.2 and lead to

$$-\frac{1}{2}\langle S_{SC}^2 S_{4f}\rangle = -\frac{\beta^3 d(d-1)}{192}\left((d-1)\cos^{-4}\frac{\phi}{2} - (d-3)\cos^{-2}\frac{\phi}{2}\right), \tag{3.71}$$

$$-\frac{1}{2}\langle S_{SC}^2 S_{4f}\rangle_{NC} = \frac{\beta^3 d^2 (d-1)^2}{768}\left(\cos^{-4}\frac{\phi}{2} - \cos^{-2}\frac{\phi}{2}\right), \tag{3.72}$$

and

$$-\frac{1}{3!}\langle S_{SC}^3\rangle = -\frac{\beta^3 d(d-1)(d-2)}{2880}\cos^{-2}\frac{\phi}{2}. \tag{3.73}$$

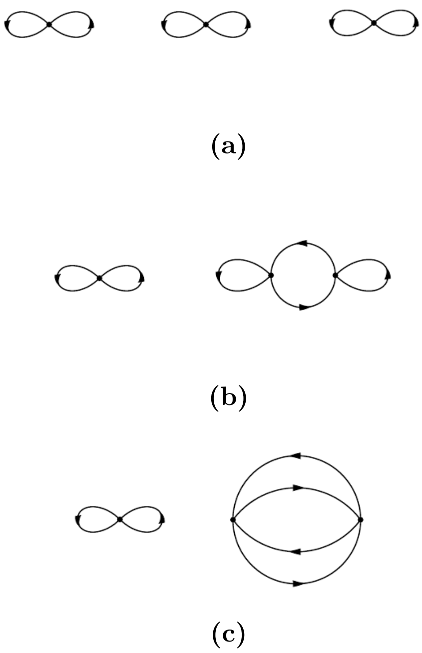



**Figure 3.6:** *Not connected fermionic diagrams at three vertices*

Finally, the transition amplitude at order $\beta^3$ reads

$$
\begin{aligned}
\overline{\mathcal{K}}(0,0;\beta) = \frac{(2\cos\frac{\phi}{2})^d}{(2\pi\beta)^{\frac{d}{2}}}\Bigg\{ & 1 + \beta d(d-1)\left(\frac{1}{12} - \frac{1}{8}\cos^{-2}\frac{\phi}{2}\right) \\
& + \beta^2 d(d-1)\left[\frac{5d^2-7d+6}{1440} - \frac{(d-2)^2}{96}\cos^{-2}\frac{\phi}{2} + \frac{(d-2)(d-3)}{128}\cos^{-4}\frac{\phi}{2}\right] \\
& + \beta^3 d(d-1)\Bigg[-\frac{d^4-14d^3+71d^2-114d+192}{3072}\cos^{-6}\frac{\phi}{2} \\
& + \frac{2d^4-24d^3+106d^2-156d+184}{3072}\cos^{-4}\frac{\phi}{2} \\
& - \frac{15d^4-158d^3+567d^2+720d+928}{46080}\cos^{-2}\frac{\phi}{2} \\
& - \frac{35d^4+266d^3-1181d^2+880d-768}{2903040}\Bigg] + O(\beta^4)\Bigg\}.
\end{aligned}
\tag{3.74}
$$

### 3.1.2 One-loop effective action for $p$-forms and trace anomalies

The transition amplitude at coinciding points is a constant (any choice of the origin of RNC is equivalent on MSS). Denoting $\overline{\mathcal{K}}(\beta,\phi) = \overline{\mathcal{K}}(0,0;\beta,\phi)$, we represent the one-loop effective action of a $p$-form, as a functional of the metric, by

$$\Gamma_p^{QFT}[g] = \int d^dx\sqrt{g}\Big[-\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{e^{iq\phi}}{(2\cos\frac{\phi}{2})^2}\,\overline{\mathcal{K}}(\beta,\phi)\Big] \tag{3.75}$$

where $q = \frac{d}{2} - p - 1$, as discussed in [29] and reviewed in appendix A. The crucial input is the integral over the angle $\phi$ which implements a projector on the degrees of freedom of the $p$-form . One may write 3.75 in the form

$$\Gamma_p^{QFT}[g] = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta} Z_p(\beta) \tag{3.76}$$

$$Z_p(\beta) = \int \frac{d^dx\sqrt{g}}{(2\pi\beta)^{\frac{d}{2}}}\underbrace{\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{iq\phi}\left(2\cos\frac{\phi}{2}\right)^{d-2}[1+...]}_{\left(a_0+a_1\beta+a_2\beta^2+a_3\beta^3+...\right)} \tag{3.77}$$

where the coefficients $a_n$ are the Seeley-DeWitt coefficients. They characterize the theory. For example, $a_1$ gives the trace anomaly for a scalar field in two dimensions (with $p = 0$) and $a_2$ gives the trace anomaly for a spin 1 field in four dimensions (with $p = 1$). Similarly, $a_3$ for $p = 2$ should give the trace anomaly of a gauge 2-form field $B_{\mu\nu}$ in six dimensions. On MSS $a_n = b_n R^n$ for certain numbers $b_n$, which we now compute (they depend on $d$ and $p$, and are more conveniently denoted by $b_n(d, p)$).
One should check $b_1(2, 0)$, $b_2(4, 1)$, and $b_3(6, 2)$, to see if they reproduce the trace anomalies.
At the end, $b_1(2, 0)$ and $b_3(6, 2)$ should arise from $c$ coefficients shown in Table 3.1 while an explicit calculation of $b_2(4, 1)$ is reported in [29]. Trace anomalies read

$$\left\langle T^m{}_m(x)\right\rangle = (-1)^{n+1}c_{2n}\frac{E_{2n}}{(2\pi)^n} \tag{3.78}$$

where $E_{2n}$ is the Euler density in $2n$ dimension, that on $2n$-dimensional spheres is given by

$$E_{2n} = \frac{(2n)!}{(2n(2n-1))^n}R^n \tag{3.79}$$

and the stress tensor normalized is

$$T_{mn} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{mn}}c\,. \tag{3.80}$$

So our $b_n$ coefficients should be given by

| $d = 2n$ | $(2n+1)!\ c(scalar)$ | $(2n+1)!\ c(fermion)$ | $(2n+1)!\ c(AT)$ |
|---|---|---|---|
| 2 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 4 | $\frac{1}{12}$ | $\frac{11}{12}$ | $\frac{124}{19}$ |
| 6 | $\frac{5}{72}$ | $\frac{191}{72}$ | $\frac{221}{4}$ |

**Table 3.1:** *The c coefficients of the type-A trace anomaly of a real conformal scalar, Dirac fermion, conformal antisymmetric tensor (AT), i.e. $(n-1)$-form gauge potential. We have multiplied them by $(2n+1)!$ to make the numbers more readable.*

$$b_n = \frac{(2n)!}{(2n(2n-1))^n} c_{2n} \tag{3.81}$$

Using (3.77) we can compute

$$\begin{aligned}
a_1(2,0) &= \int_0^{2\pi} \frac{d\phi}{2\pi} \beta R \left[ \frac{1}{12} - \frac{1}{8} \cos^{-2} \frac{\phi}{2} \right] = \frac{1}{12} \beta R \\
a_2(4,1) &= \int_0^{2\pi} \frac{d\phi}{2\pi} \big(2\cos\frac{\phi}{2}\big)^2 \beta^2 \frac{R^2}{12} \left[ \frac{1}{64} \cos^{-4} \frac{\phi}{2} - \frac{1}{24} \cos^{-2} \frac{\phi}{2} + \frac{29}{720} \right] = \frac{-31}{4320} \beta^2 R^2 \\
a_3(6,2) &= \int_0^{2\pi} \frac{d\phi}{2\pi} \beta^3 \frac{R^3}{900} \big(2\cos\frac{\phi}{2}\big)^4 \left[ -\frac{7}{64} \cos^{-6} \frac{\phi}{2} + \frac{59}{384} \cos^{-4} \frac{\phi}{2} - \frac{5486}{23040} \cos^{-2} \frac{\phi}{2} - \frac{2159}{241920} \right] \\
&= \frac{6719}{2400 \cdot 7!} \beta^3 R^3,
\end{aligned} \tag{3.82}$$

that give

$$\begin{aligned}
b_1(2,0) &= \frac{1}{12} \\
b_2(4,1) &= -\frac{31}{4320} \\
b_3(6,2) &= \frac{6719}{2400 \cdot 7!}.
\end{aligned} \tag{3.83}$$

The first two coefficient are correctly reproduced by our model while the third isn't since the expected value, which can be extracted from [24], is

$$b_3(6,2)_{exp} = \frac{221}{150 \cdot 7!} \quad . \tag{3.84}$$

Thus it seems that our ansatz for a simplified path integral for the $N = 2$ model fails after the first 2 perturbative orders. Perhaps our assumption have been too naive, or perhaps one should test other regularization schemes. We plan to test these assumptions in future works.

# Conclusions

In this work we have studied the worldline path integral formulation for the $N = 2$ supersymmetric quantum mechanics in curved space, which is characterized by a supersymmetric nonlinear sigma model action. One of its most important applications is to describe in a worldline approach the one-loop effects due to the propagation of $p$-forms coupled to gravity, including as particular cases a gauge field $A_\mu$ and a gauge 2-form field $B_{\mu\nu}$.

In particular, we have addressed the possibility that a simplified path integral, in terms of a linear sigma model, might be applicable to maximally symmetric spaces. We have tested a particular method of computing the path integral in Riemann normal coordinates on maximally symmetric spaces, i.e. spheres, that turns the nonlinear sigma model into a linear one in which the curvature effects are taken care of by a suitable effective scalar potential, as already seen for the $N = 0$ and $N = 1$ cases.

Lacking a direct proof, we have made an ansatz for a simplified path integral, and computed the transition amplitude at coinciding points up to order $\beta^3$. We had to take into account the part of the action containing the spin connection, that was crucial to reproduce the correct transition amplitude at order $\beta^2$. However this made the computation more complicated at order $\beta^3$. We obtained the first few Seeley-DeWitt coefficients, that characterize the theory, and computed the type-A trace anomalies in order to verify that our model reproduces known results. $a_1$ gives the trace anomaly for a scalar field in two dimensions (with $p = 0$) and $a_2$ gives the trace anomaly for a spin 1 field in four dimensions (with $p = 1$). Similarly, $a_3$ for $p = 2$ should give the trace anomaly of a gauge 2-form field $B_{\mu\nu}$ in six dimensions. On MSS $a_n = b_n R^n$ for certain numbers $b_n$ (they depend on $d$ and $p$, and are more conveniently denoted by $b_n(d, p)$).

We have computed $b_1(2, 0)$, $b_2(4, 1)$, and $b_3(6, 2)$. The first two coefficients are correctly reproduced by our model, as said, while the last one turns out not to be correct. This fact deserves more attentions. One reason for this failure may be that the model has been oversimplified. Thus, the main purpose of a future work will be to improve on it in order to give a correct description at order $\beta^3$, and possibly to all orders.

# Appendix A

# $N = 2$ model, propagators and determinants

In this appendix we discuss the theory of a $N = 2$ supersymmetric sigma model used in [29], from which we obtain the propagators and the determinants that we used in treating our $N = 2$ linear sigma model.

## A.1 A brief review of the $N = 2$ spinning particle

Let us review the particle action characterized by a $N = 2$ extended supergravity on the worldline. It was analyzed in [41], where it was shown that its quantization produces the equation of motion of a $p$-form in flat spacetime. The action in phase space contains gauge symmetries generated by certain constraints $H$,$Q$,$\bar{Q}$,$J$ (to be discussed shortly). The coupling of the $N = 2$ spinning particle to spacetime gravity can be achieved by suitably covariantizing the constraints $H$,$Q$,$\bar{Q}$,$J$. It is convenient to use flat indices for the worldline fermions by introducing the vielbein $e_\mu{}^a$ and the corrisponding spin connection $\omega_\mu{}^{ab}$. The action reads

$$S = \int dt \, [p_\mu \dot{x}^\mu + i\bar{\chi}_a \dot{\psi}^a - eH - i\bar{\chi} Q - i\chi\bar{Q} - aJ] \tag{A.1}$$

with covariantized constraints (including a Chern-Simons coupling $q$ in $J$)

$$
\begin{aligned}
J &= \bar\psi^a \psi_a - q \\
Q &= \psi^a e_a{}^\mu \pi_\mu \\
\bar Q &= \bar\psi^a e_a{}^\mu \pi_\mu \\
H &= \frac{1}{2} g^{\mu\nu}\pi_\mu\pi_\nu - \frac{1}{2} R_{abcd}\bar\psi^a\psi^b\bar\psi^c\psi^d .
\end{aligned}
\tag{A.2}
$$

where $\pi_\mu$ is the"covariant" momentum

$$
\pi_i \equiv p_\mu - i\omega_{\mu ab}\bar\psi^a\psi^b \tag{A.3}
$$

which becomes the Lorentz covariant derivative upon canonical quantization. The covariantizations of $Q$ and $\bar Q$ are easy to guess and the algebra identifies $H$. One must also check that the full constraint algebra remains unchanged, of course. For example, $\{Q,Q\}_{PB} = 0$ is verified using the cyclic identity satisfied by the Riemann tensor. Elimination of the momentum $p_i$ produces the configuration space action

$$
\begin{aligned}
S = \int dt \bigg[ &\frac{1}{2} e^{-1} g_{\mu\nu}(\dot x^\mu - i\bar\chi\psi^\mu - i\chi\bar\psi^\mu)(\dot x^\nu - i\bar\chi\psi^\nu - i\chi\bar\psi^\nu) \\
&+ i\bar\psi_a(\dot\psi^a + \dot x^\mu \omega_\mu{}^a{}_b\psi^b + ia\psi^a) + \frac{e}{2} R_{abcd}\bar\psi^a\psi^b\bar\psi^c\psi^d + qa \bigg]
\end{aligned}
\tag{A.4}
$$

which has been quantized using path integrals on a closed worldline (to study one-loop effects) [29]. To use euclidean conventions, we perform a Wick rotation to euclidean time ($t \to -i\tau$, and also $a \to ia$ to keep the gauge group $U(1)$ compact) which produces the euclidean action ($S_E = -iS$)

$$
\begin{aligned}
S = \int_0^1 d\tau \bigg[ &\frac{1}{2} e^{-1} g_{\mu\nu}(\dot x^\mu - \bar\chi\psi^\mu - \chi\bar\psi^\mu)(\dot x^\nu - \bar\chi\psi^\nu - \chi\bar\psi^\nu) \\
&+ \bar\psi_a(\dot\psi^a + \dot x^\mu \omega_\mu{}^a{}_b\psi^b + ia\psi^a) - \frac{e}{2} R_{abcd}\bar\psi^a\psi^b\bar\psi^c\psi^d - iqa \bigg]
\end{aligned}
\tag{A.5}
$$

where $\tau \in [0,1]$ parametrizes the closed loop. From now on we will drop the subscript on $S_E$ as no confusion should arise. The gauge transformations of the supergravity multiplet in euclidean time are needed to study the gauge fixing and are given by

$$\begin{aligned}\delta e &= \dot{\xi} + 2\bar{\chi}\epsilon + 2\chi\bar{\epsilon} \\ \delta\chi &= \dot{\epsilon} + ia\epsilon - i\alpha\chi \\ \delta\bar{\chi} &= \dot{\bar{\epsilon}} - ia\bar{\epsilon} + i\alpha\bar{\chi} \\ \delta a &= \dot{\alpha}\end{aligned} \tag{A.6}$$

## A.2 Quantization of a $p$-form on a curved space

The path integral quantization of the action of the particle of the $p$-form in a background metric $g_{\mu\nu}$ gives the following one-loop effective action $\Gamma_p^{QFT}[g_{\mu\nu}]$, as a functional of the metric

$$\Gamma_p^{QFT}[g_{\mu\nu}] \equiv Z[g_{\mu\nu}] = \int_{T^1} \frac{\mathcal{D}G\mathcal{D}X}{Vol(Gauge)}\, e^{-S[X,G;g_{\mu\nu}]} \tag{A.7}$$

where $G = (e, \chi, \bar{\chi}, a)$ and $X = (x^\mu, \psi^\mu, \bar{\psi}^\mu)$ are the fields that must be path integrated over, and $S[X, G; g_{\mu\nu}]$ is the action in (A.5). Division by the volume of the gauge group reflects the necessity of fixing the gauge symmetries.
We impose periodic boundary conditions on the bosonic fields $x^\mu$ and $e$ (the gauge field $a$ is instead treated as a connection) and antiperiodic boundary conditions for the fermions.
We can use gauge symmetries to fix the supergravity multiplet to $\hat{G} = (\beta, 0, 0, \phi)$, where $\beta$ and $\phi$ are the leftover bosonic moduli that must be integrated over. The parameter $\beta$ is the usual proper time and the parameter $\phi$ is a phase that corresponds to the only modular parameter that the gauge field $a$ can have on the torus. Gravitinos $\chi$ and $\bar{\chi}$ are antiperiodic and can be completely gauged away using (A.6), leaving no moduli.
One can discuss more extensively how the modular parameter $\phi$ arises.
The action for the fermions in (A.5) is of the standard form (the target space geometry is essential for this particular gauge fixing, and one can take it flat)

$$S \sim \int_0^1 d\tau \bar{\psi}(\dot{\psi} + ia\psi). \tag{A.8}$$

Finite gauge transformations are given by

$$\begin{aligned} \psi(\tau) &\to \psi'(\tau) = e^{-i\alpha(\tau)}\psi(\tau) \\ \bar{\psi}(\tau) &\to \bar{\psi}'(\tau) = e^{i\alpha(\tau)}\bar{\psi}(\tau) \\ a(\tau) &\to a'(\tau) = a(\tau) + \dot{\alpha}(\tau) \end{aligned} \tag{A.9}$$

where the gauge transformations $e^{-i\alpha(\tau)}$ are required to be periodic functions on $[0, 1]$. In one dimension the only gauge invariant quantity is the Wilson loop

$$w = e^{i\int_0^1 d\tau a(\tau)}. \tag{A.10}$$

Using "small" gauge transformations, for example those continuously connected to the identity, one can bring $a(\tau)$ to a constant value $\phi$

$$\phi = \int_0^1 d\tau a(\tau). \tag{A.11}$$

Then "large" gauge transformations with $\alpha(\tau) = 2\pi n\tau$ allow to identify

$$\phi \sim \phi + 2\pi n, \tag{A.12}$$

with $n$ integer. Therefore one can take $\phi \in [0, 2\pi]$ as the fundamental region of the moduli space.
The value of the Wilson loop is given by the phase $w = e^{i\phi}$, and once again one can $\phi$ to be an angle.
In the gauge $a(\tau) = \phi$ the action (A.8) becomes

$$S \sim \int_0^1 d\tau\bar{\psi}(\dot{\psi} + i\phi\psi). \tag{A.13}$$

One may now redefine the fermion by $\psi' = e^{i\phi\tau}\psi$, in order to eliminate the gauge field from the action

$$S \sim \int_0^1 d\tau\bar{\psi}'\dot{\psi}'. \tag{A.14}$$

The new field acquires twisted boundary conditions

$$\psi'(1) = -e^{i\phi}\psi'(0) \tag{A.15}$$

so that the modulus $\phi \in [0, 2\pi]$ interpolates between all possible boundary conditions specified by a phase. With the original assumption of antiperiodic boundary conditions we did not loose generality, in fact the integration over the $U(1)$ modulus automatically takes into account the sum over spin structures.
For $\phi = \pi$ one obtain periodic boundary conditions. Thus, the fermions acquire zero modes (and the gravitinos develop corresponding moduli). We now discuss the gauge fixing of (A.7). If we choose the gauge $\hat{G} = (\beta, 0, 0, \phi)$, insert the Faddeev-Popov determinant to eliminate the volume of gauge group, and integrate over the moduli, we obtain

$$\Gamma_p^{QFT} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{e^{iq\phi}}{(2\cos\frac{\phi}{2})^2}\int_{T^1}\mathcal{D}X e^{-S[X,\hat{G},g_{\mu\nu}]}, \tag{A.16}$$

where the measure over the proper time $\beta$ takes into account the effect of the symmetry generated by the Killing vector on the torus. The Faddeev-Popov determinants from the commuting SUSY ghosts are given by $\det^{-1}(\partial_\tau + i\phi)\det^{-1}(\partial_\tau + i\phi) = (2\cos\frac{\phi}{2})^{-2}$. These are the inverse of the fermionic determinant arising from (A.13) which is easily computed: antiperiodic boundary conditions produce a trace over the corresponding two-dimensional Hilbert space and thus $det(\partial_\tau + i\phi) = e^{-i\frac{\phi}{2}} + e^{i\frac{\phi}{2}} = 2cos\frac{\phi}{2}$. The other Faddeev-Popov determinants do not give rise to any moduli dependent term. The overall normalization $-1/2$ has been inserted to match QFT results. Up to the overall sign, one could argue that this factor is due to the fact that one is considering a real field rather that a complex one.
Up to the final integration over the moduli, one is left with a standard path integral for a nonlinear $N = 2$ susy sigma model. This path integral cannot be evaluated exactly for arbitrary background metrics $g_{\mu\nu}$, but it is the starting point of various approximations schemes.
One can consider here an expansion in terms of the proper time $\beta$ which leads to the local heat-kernel expansion of the effective action. It is a derivative expansion depending on the Seeley-DeWitt coefficients.

## A.3 Proper time expansion

One can evaluate perturbatively in $\beta$ the following path integral

$$\int_{T^1}\mathcal{D}X\, e^{-S[X,\hat{G};g_{\mu\nu}]+iq\phi} \tag{A.17}$$

where we have extracted the constant Chern-Simons term from the action. The sigma model action reads

$$S[X,\hat{G}] = \frac{1}{\beta}\int_0^1 d\tau[\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\mu + \bar{\psi}_a(\dot{\psi}^a + i\phi\psi^a + \dot{x}^\mu\omega_\mu{}^a{}_b\psi^b) - \frac{1}{2}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d]. \quad (A.18)$$

where the fermion has been scaled to extract a global factor $1/\beta$. One can extract the dependence on the zero modes $x_0^\mu$ of the coordinates and obtains

$$\Gamma_p^{QFT}[g] = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int \frac{d^dx\sqrt{g}}{(2\pi\beta)^{\frac{d}{2}}}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{iq\phi}\left(2\cos\frac{\phi}{2}\right)^{d-2}\int \frac{d^dx_0\sqrt{g(x_0)}}{(2\pi\beta)^{\frac{d}{2}}}\langle e^{-S_{int}}\rangle. \quad (A.19)$$

The constant zero modes $x_0^\mu$ can be factorized by setting $x^\mu(\tau) = x_0^\mu + y^\mu(\tau)$ and imposing the Dirichlet boundary conditions $y^\mu(0) = y^\mu(1) = 0$ on the quantum fields $y^\mu(\tau)$. This describes a loop with a fixed point $x_0$. The extra factor $(2\cos\frac{\phi}{2})^d$ comes from the normalization of the fermionic path integral and correspond to $\det^d(\partial_\tau + i\phi)$.

## A.4 Propagators ans determinants

The propagators are obtained from the quadratic part of the action which is obtained by Taylor expanding the metric around the point $x_0^i$

$$S = \frac{1}{\beta}\int_0^1 d\tau\left[\frac{1}{2}g_{\mu\nu}(x_0)\dot{y}^\mu\dot{y}^\nu + \bar{\psi}_a(\dot{\psi}^a + i\phi\psi^a)\right]. \quad (A.20)$$

The bosonic propagator is given by

$$\langle y^\mu(\tau)y^\nu(\sigma)\rangle = -\beta g^{\mu\nu}(x_0)\Delta(\tau,\sigma) \quad (A.21)$$

$$(A.22)$$

where

$$\begin{aligned}\Delta(\tau,\sigma) &= \sum_{m=1}^\infty\left[-\frac{2}{\pi^2m^2}\sin(\pi m\tau)\sin(\pi m\sigma)\right]\\ &= (\tau-1)\sigma\theta(\tau-\sigma) + (\sigma-1)\tau\theta(\sigma-\tau)\end{aligned} \quad (A.23)$$

with $\theta(\tau-\sigma)$ the standard step function.

The fermionic fields with antiperiodic boundary conditions can be expanded in half-integer modes

$$\psi^a(\tau) = \sum_{r \in Z+\frac{1}{2}} \psi^a_r e^{2\pi i r\tau}, \qquad \bar{\psi}^a(\tau) = \sum_{r \in Z+\frac{1}{2}} \bar{\psi}^a_r e^{-2\pi i r\tau} \tag{A.24}$$

and from the action (A.20) one finds the propagator (AF stands for antiperiodic fermions)

$$\langle \psi^a(\tau)\bar{\psi}_b(\sigma)\rangle = \beta\delta^b_a \Delta_{AF}(\tau-\sigma,\phi), \qquad \Delta_{AF}(x,\phi) = \sum_{r \in Z+\frac{1}{2}} \frac{-i}{2\pi r + \phi}\, e^{2\pi i r x} \tag{A.25}$$

which satisfies

$$(\partial_x + i\phi)\Delta_{AF}(x,\phi) = \sum_{r \in Z+\frac{1}{2}} e^{2\pi i r x} = \delta_{AF}(x) \tag{A.26}$$

where $\delta_{AF}$ is the delta function on the space of antiperiodic functions. For $x \in ]-1,1[$ the propagator can be summed up to yield

$$\Delta_{AF}(x,\phi) = \frac{e^{-i\phi x}}{2\cos\frac{\phi}{2}} \Big[ e^{i\frac{\phi}{2}}\theta(x) - e^{-i\frac{\phi}{2}}\theta(-x) \Big]. \tag{A.27}$$

For coinciding points ($\tau = \sigma$ i.e. at $x = 0$) it takes the regulated value

$$\Delta_{AF}(x,\phi) = \frac{i}{2}\tan\frac{\phi}{2} \tag{A.28}$$

which can be computed by "symmetric integration", i.e. symmetrically combining the modes $+r$ and $-r$ and then summing up the series. On the other hand for $x \neq 0$ one finds

$$\Delta_{AF}(x,\phi)\Delta_{AF}(-x,\phi) = -\frac{1}{4}\cos^{-2}\frac{\phi}{2} \tag{A.29}$$

which shows that this function has a discontinuity at $x = 0$. When multiplied by a distribution it necessitates a regularization. An example is discussed in Appendix B.
Let us now review the calculation of the fermionic determinant

$$\int_{ABC} \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-S} = \det{}^d(\partial_\tau + i\phi) \tag{A.30}$$

where ABC stands for antiperiodic boundary conditions and the action is the one in

(A.20). The easiest way to obtain the determinant is to use the operator formalism

$$\int_{ABC} \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-S} = \mathrm{Tr}\, e^{-\hat{H}_\phi} \tag{A.31}$$

where $\hat{H}_\phi$ is the hamiltonian operator of the system and equals

$$\hat{H}_\phi = i\phi\frac{1}{2}(\hat{\psi}^\dagger_a\hat{\psi}^a - \hat{\psi}^a\hat{\psi}^\dagger_a) = i\psi\left(\hat{\psi}^\dagger_a\hat{\psi}^a - \frac{d}{2}\right). \tag{A.32}$$

One can recognize in (A.32) the hamiltonian for a $d$ dimensional fermionic oscillator and easily compute the trace. In one dimension the eigenvalues of the fermionic number operator $\hat{\psi}^\dagger\hat{\psi}$ are either 0 or 1, and one gets

$$\det{}^d(\partial_\tau + i\phi) = \mathrm{Tr}\, e^{-i\phi(\hat{\psi}^\dagger_a\hat{\psi}^a - \frac{d}{2})} = e^{-i\phi\frac{d}{2}}(1 + e^{-i\phi})^d = (2\cos\frac{\phi}{2})^d. \tag{A.33}$$

An alternative way to compute the determinant directly from the path integral consists in expanding the fermions in antiperiodic modes and taking the infinite product of eigenvalues

$$\det{}^d(\partial_\tau + i\phi) = \det{}^d(\partial_\tau)\left[\frac{\det(\partial_\tau + i\phi)}{\det(\partial_\tau)}\right]^d \tag{A.34}$$

$$= 2^d \prod_{n=-\infty}^{+\infty}\left(1 + \frac{\phi}{2\pi(n+\frac{1}{2})}\right)^d = \left(2\cos\frac{\phi}{2}\right)^d \tag{A.35}$$

where we have used a standard representation of the cosine as an infinite product (after combining positive and negative frequencies together as part of our regularization prescription).

# Appendix B

# Dimensional regularization

Here we discuss the procedure of dimensional regularization (DR) applied to our $N = 2$ supersymmetric linear sigma model, extending the treatment presented in [22, 28] for $N = 0$ and $N = 1$ supersymmetric sigma models.
If the quantum hamiltonian for the $N = 0$ system is proportional to the covariant scalar laplacian $H = -\frac{1}{2}\nabla^2$, without any coupling to the scalar curvature $R$, then the rules of dimensional regularization requires a counterterm $V_{DR} = -\frac{1}{8}R$ in addition to the classical euclidean action, normalized as $\Delta S_{DR} = \int_0^1 d\tau \beta V_{DR}$. The quantum hamiltonian of the $N = 1$ model acts on a spinor space, and it is fixed by supersymmetry to be the square of the Dirac operator (the susy charge) $H = -\frac{1}{2}\nabla\!\!\!/\nabla\!\!\!/ = -\frac{1}{2}(\nabla^2 - \frac{1}{4}R)$. Now the total counterterm in dimensional regularization vanishes [28].
For the $N = 2$ model the counterterm vanishes as well [29].
Dimensional regularization requires the extension of the space $I = [0, 1]$ to $I \times \mathbb{R}^d$. The action can also be extended to $d + 1$ dimensions and reads

$$S = \frac{1}{\beta}\int_{I\times\mathbb{R}^d} d^{d+1}t\left[\frac{1}{2}g_{\mu\nu}\partial^\alpha x^\mu \partial_\alpha x^\nu + \bar{\psi}_a\gamma^\alpha(\partial_\alpha\psi^a + \partial_\alpha x^\mu {\omega_\mu}^{ab}\psi_b) + i\phi\bar{\psi}_a\psi^a - \frac{1}{2}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\right] \tag{B.1}$$

where $t^\alpha \equiv (\tau, \mathbf{t})$ are the coordinates in the extended space (bold face indicates vectors in the extra $d$ dimensions) and $\gamma^\alpha$ are the corresponding Dirac matrices. The propagators in $d + 1$ dimensions read

$$\Delta(t, s) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\sum_{m=1}^{\infty}\frac{-2}{(\pi m)^2 + \mathbf{k}^2}\sin(\pi m\tau)\sin(\pi m\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \tag{B.2}$$

$$\Delta_{AF}(t-s) = -i\int \frac{d^d\mathbf{k}}{(2\pi)^d} \sum_{r\in Z+\frac{1}{2}} \frac{2\pi r\gamma^0 + \mathbf{k}\cdot\vec{\gamma} - \phi}{(2\pi r)^2 + \mathbf{k}^2 - \phi^2} e^{2\pi i r(\tau-\sigma)} e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \tag{B.3}$$

and satisfy

$$\partial^\alpha\partial_\alpha\Delta(t,s) = \delta(\tau,\sigma)\delta^d(\mathbf{t}-\mathbf{s})$$
$$\left(\gamma^\alpha\frac{\partial}{\partial t^\alpha} + i\phi\right)\Delta_{AF}(t-s) = \Delta_{AF}(t-s)\left(-\gamma^\beta\frac{\overleftarrow{\partial}}{\partial s^\beta} + i\phi\right) = \delta_{AF}(\tau-\sigma)\delta^d(\mathbf{t}-\mathbf{s}) \tag{B.4}$$

where

$$\delta(\tau,\sigma)\delta^d(\mathbf{t}-\mathbf{s}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \sum_{m=1}^{\infty} 2\sin(\pi m\tau)\sin(\pi m\sigma) e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \tag{B.5}$$

and a similar equation for the one containing $\delta_{AF}(\tau-\sigma)$.

The index contractions in $d+1$ dimensions keeps track of which derivative can be contracted to which vertex to eventually produce the $(d+1)$-dimensional delta function. The delta functions in (B.4) are only to be used in $d+1$ dimensions, as one assumes that only in such a situation the regularization is taking place because of the extra dimensions. Then, by using partial integration one casts the various loop integrals in a form which can be directly computed in the $d \to 0$ limit. At this stage one can use the propagators in one dimensions, $\gamma^0 = 1$ and no extra factors arise from the Dirac algebra.

## B.1 Contribution of the spin connection

### B.1.1 Order $\beta^2$

Let us apply the procedure of DR explained before to one of the most relevant graphs of order $\beta^2$ coming from the vertex involving the spin connection

$$\dot{x}^\mu\omega_{\mu ab}(x)\bar{\psi}^a\psi^b \to \partial_\alpha x^\mu\omega_{\mu ab}(x)\bar{\psi}^a\gamma^\alpha\psi^b. \tag{B.6}$$

Expanding the vertex around $x_0^\mu = x^\mu(\tau) - y^\mu(\tau)$, and using Riemann normal coordinates and a Lorentz gauge such that $\omega_{\mu ab}(x_0) = 0$ and $\partial_{(\nu}\omega_{\mu)ab}(x_0) = 0$, one obtains a quartic

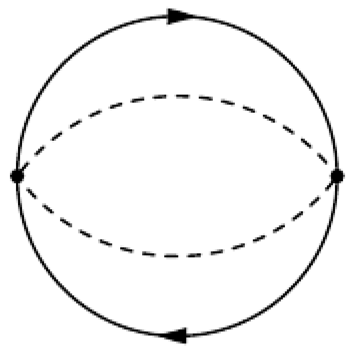


**Figure B.1:** *Diagram for* $\langle \Delta S_{SC}^2 \rangle$

vertex of the form

$$\Delta S_{SC} = \frac{1}{\beta}\int_0^1 d\tau \frac{1}{2}\dot{y}^\mu y^\nu R_{\nu\mu ab}(x_0)\bar{\psi}^a\psi^b \quad \rightarrow \quad \frac{1}{\beta}\int d^{d+1}t \frac{1}{2}\partial_\alpha y^\mu y^\nu R_{\nu\mu ab}(x_0)\bar{\psi}^a\gamma^\alpha\psi^b. \tag{B.7}$$

whose diagram is shown in Figure B.1. To compute the correlation function we must evaluate the Wick contractions, obtaining:

$$\frac{1}{2}\langle(\Delta S_{SC})^2\rangle = \frac{\beta^2}{8}R^2_{\mu\nu ab}(x_0) I \tag{B.8}$$

with

$$I = \int_0^1 d\tau \int_0^1 d\sigma \left[{}^\bullet\Delta^\bullet(\tau,\sigma)\Delta(\tau,\sigma) - {}^\bullet\Delta(\tau,\sigma)\Delta^\bullet(\tau,\sigma)\right]\Delta_{AF}(\tau-\sigma)\Delta_{AF}(\sigma-\tau)$$
$$\rightarrow \int d^{d+1}t \int d^{d+1}s \; [{}_\alpha\Delta_\beta(t,s)\Delta(t,s) - {}_\alpha\Delta(t,s)\Delta_\beta(t,s)] \times \mathrm{Tr}[\gamma^\alpha\Delta_{AF}(t-s)\gamma^\beta\Delta_{AF}(s-t)] \tag{B.9}$$

where dots and indices on the left/right of the propagators indicates derivatives with respect to the first/second variable. The function ${}^\bullet\Delta^\bullet$ contains a delta function multiplying the step functions contained in $\Delta_{AF}$, and these products of distributions are ambiguous and must be carefully regularized. Thus we extend the integrals in (B.9) to $d+1$ dimensions.
In DR we integrate by part the $\partial_\alpha$ from ${}_\alpha\Delta_\beta$. This produces a boundary term which

vanish, a term which doubles the other term in (B.9), and the following extra term

$$\int d^{d+1}t \int d^{d+1}s \Big\{ \Delta_\beta(t,s)\Delta(t,s)\,\mathrm{Tr}\Big[\Big(\gamma^\alpha \frac{\partial}{\partial t^\alpha}\Delta_{AF}(t,s)\Big)\gamma^\beta \Delta_{AF}(s,t) + \Delta_{AF}(t,s)\gamma^\beta\Big(\Delta_{AF}(s,t)\frac{\overleftarrow{\partial}}{\partial t^\alpha}\gamma^\alpha\Big)\Big]\Big\}. \tag{B.10}$$

We can add for free the "mass" term $i\phi$ in order to obtain the Dirac equations. By using the second line in (B.4) one can show that this extra contribution vanishes

$$2\int d^{d+1}t\Delta_\beta(t,t)\Delta(t,t)\,\mathrm{Tr}\big[\gamma^\beta\Delta_{AF}(0)\big] \to 2\Delta_{AF}(0)\int_0^1 d\tau \Delta^\bullet(\tau,\tau)\Delta(\tau,\tau) = 0. \tag{B.11}$$

We have used $\Delta^\bullet(\tau,\tau) = \tau - \frac{1}{2}$ and $\Delta(\tau,\tau) = \tau^2 - \tau$. The regularization has been removed only when it was obvious that the integral did not contain any dangerous product of distributions at $d = 0$. The remaining nonvanishing term is

$$\begin{aligned} I &= -2\int d^{d+1}t \int d^{d+1}s\; {}_\alpha\Delta(t,s)\Delta_\beta(t,s)\,\mathrm{Tr}\big[\gamma^\alpha\Delta_{AF}(t-s)\gamma^\beta\Delta_{AF}(s-t)\big] \\ &\to -2\int_0^1 d\tau \int_0^1 d\sigma\; {}^\bullet\Delta(\tau,\sigma)\Delta^\bullet(\tau,\sigma)\Delta_{AF}(\tau-\sigma)\Delta_{AF}(\sigma-\tau) \\ &= -\frac{1}{24}\cos^{-2}\frac{\phi}{2} \end{aligned} \tag{B.12}$$

where ${}^\bullet\Delta(\tau,\sigma) = \sigma - \theta(\sigma-\tau)$, $\Delta^\bullet(\tau,\sigma) = \tau - \theta(\tau-\sigma)$ and $\Delta_{AF}(\tau-\sigma)\Delta_{AF}(\sigma-\tau) = -\frac{1}{4}\cos^{-2}\frac{\phi}{2}$ (which are the correct limits of the Fourier sums up to a set of points of zero measure). Thus we have obtained

$$\frac{1}{2}\langle(\Delta S_{SC})^2\rangle = -\frac{\beta^2}{192}R^2_{\mu\nu ab}\cos^{-2}\frac{\phi}{2}. \tag{B.13}$$

### B.1.2 Order $\beta^3$

Let us now discuss the contribution of terms involving the spin connection at the order $\beta^3$. One possible contribute that we have to consider is:

$$-\frac{1}{2}\langle S^2_{SC}S_{4f}\rangle = \Big(\frac{\beta}{4}\Big)\iiint d\tau d\sigma d\xi\;\; \langle \dot{x}^i\omega_{iab}\bar{\psi}^a\psi^b(\tau)\; \dot{x}^j\omega_{jcd}\bar{\psi}^c\psi^d(\sigma)\; \bar{\psi}^e\psi_e\bar{\psi}^f\psi_f(\xi)\rangle. \tag{B.14}$$

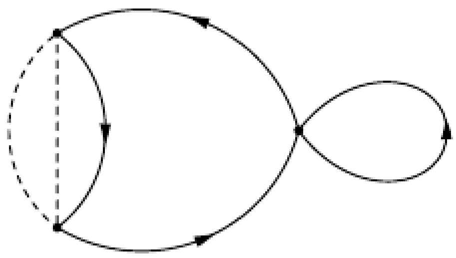


**Figure B.2:** *One possible diagram for* $\langle \Delta S_{SC}^2 \rangle$

The right hand side can be rewritten, by making use of (1.34), as

$$\left(\frac{\beta}{4}\right) \iiint d\tau\, d\sigma\, d\xi\ \langle \frac{1}{2}\dot{x}^i x^k \Omega_0 R_{kiab}\ \bar{\psi}^a \psi^b(\tau)\ \frac{1}{2}\dot{x}^j x^l \Omega_0 R_{ljcd} \bar{\psi}^c \psi^d(\sigma)\ \bar{\psi}^e \psi_e \bar{\psi}^f \psi_f(\xi) \rangle \tag{B.15}$$

where $\Omega_0$ is the term of lowest order in the Taylor expansion of the $\Omega$ function that does not contain other bosonic variables and allows us to obtain a contribute of order $\beta^3$. There are two non vanishing diagrams that arise from the different ways to perform the Wick contractions. Other contractions give vanishing contributes because of the antisymmetry of the Riemann tensor. The first one is shown in Figure B.2 and leads to

$$\beta^2 \left( \delta^i_j \delta^k_l\ {}^\bullet\Delta^\bullet(\tau - \sigma)\Delta(\tau - \sigma) + \delta^i_l \delta^k_j\ \ {}^\bullet\Delta(\tau - \sigma)\Delta^\bullet(\tau - \sigma) \right). \tag{B.16}$$

If we perform the Wick contractions of the fermionic variables, we obtain terms with products of fermionic propagators that contain step function. As discussed earlier, we need a regularization as ${}^\bullet\Delta^\bullet$ contains a delta function multiplying the step functions contained in $\Delta_{AF}$. Thus, we extend the integral in (B.15) to $d+1$ dimensions:

$$\begin{aligned} &-\frac{\beta^3}{16} R^2_{kiab} \iint d^{d+1}t\ d^{d+1}s\ d^{d+1}u\ \Big[ {}_\alpha\Delta_\beta(t,s)\Delta(t,s) - {}_\alpha\Delta(t,s)\Delta_\beta(t,s) \Big] 2(d-1) \\ &\times \mathrm{Tr}\{\gamma^\alpha \Delta_F(t-s)\gamma^\beta \Delta_F(s-u)\Delta_F(u-t) + \gamma^\alpha \Delta_F(s-t)\gamma^\beta \Delta_F(u-s)\Delta_F(t-u)\} \\ &\times \mathrm{Tr}\{\Delta_F(u-u)\} \end{aligned} \tag{B.17}$$

As in the previous case, we can integrate by parts the $\partial_\alpha$ from ${}_\alpha\Delta_\beta$. This produces a boundary term which vanishes, a term which doubles the other term in (B.17)

$$\begin{aligned}&\beta^3 R^2_{kiab}\frac{(d-1)}{4}\iint d^{d+1}t\ d^{d+1}s\ d^{d+1}u\ {}_\alpha\Delta(t,s)\Delta_\beta(t,s)\\ &\times \mathrm{Tr}\{\gamma^\alpha\Delta_F(t-s)\gamma^\beta\Delta_F(s-u)\Delta_F(u-t)+\gamma^\alpha\Delta_F(s-t)\gamma^\beta\Delta_F(u-s)\Delta_F(t-u)\}\\ &\times \mathrm{Tr}\{\Delta_F(u-u)\}\end{aligned} \tag{B.18}$$

and the extra term

$$\begin{aligned}&\beta^3 R^2_{kiab}\frac{(d-1)}{4}\iint d^{d+1}t\ d^{d+1}s\ d^{d+1}u\ \Delta(t,s)\Delta_\beta(t,s)\\ &\times \mathrm{Tr}\Bigg\{\left(\gamma^\alpha\frac{\partial}{\partial t^\alpha}\Delta_F(t-s)\right)\gamma^\beta\Delta_F(s-u)\Delta_F(u-t)\\ &+\Delta_F(t-s))\gamma^\beta\Delta_F(s-u)\left(\Delta_F(u-t)\frac{\overleftarrow{\partial}}{\partial t^\alpha}\gamma^\alpha\right)\\ &+\left(\Delta_F(s-t)\frac{\overleftarrow{\partial}}{\partial t^\alpha}\gamma^\alpha\right)\gamma^\beta\Delta_F(u-s)\Delta_F(t-u)\\ &+\Delta_F(s-t)\gamma^\beta\Delta_F(u-s)\left(\gamma^\alpha\frac{\partial}{\partial t^\alpha}\Delta_F(u-t)\right)\Bigg\}\\ &\times \mathrm{Tr}\{\Delta_F(u-u)\}\end{aligned} \tag{B.19}$$

Triple integration in the latter expression can be reduced at one integration over the variable $t$ after the application of the Dirac equation (B.4), that introduces the delta functions $\delta(t-s)$ and $\delta(t-u)$, so one obtains

$$\beta^3\frac{(d-1)}{4}R^2_{kiab}\int d^{d+1}t\,\Delta(t,t)\Delta_\beta(t,t)\,\mathrm{Tr}\Big\{2\gamma^\beta\Delta_F(t-t)\Delta_F(t-t)\Delta_F(t-t)\Big\}. \tag{B.20}$$

Now, we remove DR and obtain

$$\beta^3\frac{(d-1)}{2}R^2_{kiab}\,\Delta^3_F(0)\int_0^1 d\tau\,\Delta(\tau,\tau)\Delta^\bullet(\tau,\tau)=0 \tag{B.21}$$

where we have used that $\Delta(\tau,\tau) = \tau^2 - \tau$ and $\Delta^{\bullet}(\tau,\tau) = \tau - \frac{1}{2}$ and $\Delta_F(0) = \frac{i}{2}\tan\frac{\phi}{2}$. Thus, we are left with

$$\begin{aligned}
&-\frac{1}{2}\langle S_{SC}^2 S_{4f}\rangle_A = \beta^3 R_{kiab}^2 \frac{(d-1)}{4} \iint d^{d+1}t\, d^{d+1}s\, d^{d+1}u\ {}_{\alpha}\Delta(t,s)\Delta_{\beta}(t,s) \\
&\times \mathrm{Tr}\Big\{\gamma^{\alpha}\Delta_F(t-s)\gamma^{\beta}\Delta_F(s-u)\Delta_F(u-t) + \gamma^{\alpha}\Delta_F(s-t)\gamma^{\beta}\Delta_F(u-s)\Delta_F(t-u)\Big\} \\
&\times \mathrm{Tr}\{\Delta_F(u-u)\}
\end{aligned} \tag{B.22}$$

that gives, after sending $d$ to 0,

$$\begin{aligned}
&\beta^3 R_{kiab}^2 \frac{(d-1)}{4} \int_0^1\int_0^1\int_0^1 d\tau d\sigma d\xi\ {}^{\bullet}\Delta(\tau-\sigma)\Delta^{\bullet}(\tau-\sigma) \\
&\times \Big[\Delta_F(\tau-\sigma)\Delta_F(\sigma-\xi)\Delta_F(\xi-\tau)\Delta_F(0) + \Delta_F(\sigma-\tau)\Delta_F(\xi-\sigma)\Delta_F(\tau-\xi)\Delta_F(0)\Big].
\end{aligned} \tag{B.23}$$

The final result reads

$$\begin{aligned}
-\frac{1}{2}\langle S_{SC}^2 S_{4f}\rangle_A &= \frac{\beta^3(d-1)R_{kiab}^2}{4}\left(\cos^{-4}\frac{\phi}{2} - \cos^{-2}\frac{\phi}{2}\right)\left(-\frac{1}{12}\right) \\
&= -\frac{\beta^3 d(d-1)^2}{192}\left(\cos^{-4}\frac{\phi}{2} - \cos^{-2}\frac{\phi}{2}\right)
\end{aligned} \tag{B.24}$$

The other nonvanishing diagram is shown in Figure B.3. The bosonic factor is identical

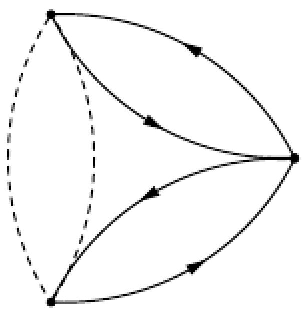


**Figure B.3:** *Another possible diagram for* $\langle \Delta S_{SC}^2\rangle$

to the one just analyzed. As in the previous case, if we perform the Wick contractions of the fermionic variables, we obtain terms with products of fermionic propagators that contain step functions. By performing DR one can obtain, without ambiguities, the following integral:

$$\frac{\beta^3}{8} R^2_{kiab} \iiint d^{d+1}t\, d^{d+1}s\, d^{d+1}\, u \Big[ {}_\alpha\Delta_\beta(t,s)\Delta(t,s) - {}_\alpha\Delta(t,s)\Delta_\beta(t,s) \Big] \\ \times \mathrm{Tr}\big\{ \gamma^\alpha \Delta_F(t-u)\Delta_F(u-s)\gamma^\beta\Delta_F(s-u)\Delta_F(u-t) \big\} \tag{B.25}$$

By proceeding as in the previous case, we integrate by parts, obtaining

$$\frac{\beta^3}{4} R^2_{kiab} \iiint d^{d+1}t\, d^{d+1}s\, d^{d+1}u\ {}_\alpha\Delta(t,s)\Delta_\beta(t,s) \\ \times \mathrm{Tr}\Big\{ \gamma^\alpha \Delta_F(t-u)\Delta_F(u-s)\gamma^\beta\Delta_F(s-u)\Delta_F(u-t) \Big\} \tag{B.26}$$

and

$$\frac{\beta^3}{8} R^2_{kiab} \iiint d^{d+1}t\, d^{d+1}s\, d^{d+1}u\ \Delta_\beta(t,s)\Delta(t,s) \\ \times \mathrm{Tr}\Big\{ \Big( \gamma^\alpha \frac{\partial}{\partial t^\alpha}\Delta_F(t-u) \Big) \gamma^\beta \Delta_F(u-s)\Delta_F(s-u)\Delta_F(u-t) \\ + \gamma^\beta \Delta_F(t-u)\Delta_F(s-u)\Delta_F(u-s) \Big( \Delta_F(u-t) \frac{\overleftarrow{\partial}}{\partial t^\alpha}\gamma^\alpha \Big) \Big\}. \tag{B.27}$$

If we use the Dirac equation in (B.27), we can reduce triple integration to a double one

$$\frac{\beta^3}{8} R^2_{kiab} \iint d^{d+1}t\ d^{d+1}s\ \Delta_\beta(t,s)\Delta(t,s) \\ \times \mathrm{Tr}\Big\{ \gamma^\beta \Delta_F(t-t)\Delta_F(s-t)\Delta_F(t-s) + \gamma^\beta \Delta_F(t-t)\Delta_F(s-t)\Delta_F(t-s) \Big\} \\ = \frac{\beta^3}{4} R^2_{kiab} \iint d^{d+1}t\ d^{d+1}s\ \Delta_\beta(t,s)\Delta(t,s)\, \mathrm{Tr}\Big\{ \gamma^\beta \Delta_F(t-t)\Delta_F(s-t)\Delta_F(t-s) \Big\} \tag{B.28}$$

If we set $d \to 0$ and compute the integral by inserting the expressions of the propagators, we obtain a vanishing result. Let's consider the contribute from (B.26)

$$\frac{\beta^3 R^2_{kiab}}{4} \int_0^1 d\tau \int_0^1 d\sigma \int_0^1 d\xi\ {}^\bullet\Delta(\tau,\sigma)\Delta^\bullet(\tau,\sigma)\Delta_F(\tau-\xi)\Delta_F(\sigma-\xi)\Delta_F(\xi-\sigma) \tag{B.29}$$

from which one can obtain the following final result for diagram in Figure B.3

$$-\frac{1}{2}\langle S_{SC}^2 S_{4f}\rangle_B = \frac{\beta^3 R_{kiab}^2}{16}\cos^{-2}\frac{\phi}{2}\left(-\frac{1}{12}\right) = -\frac{\beta^3 d(d-1)}{96}\cos^{-2}\frac{\phi}{2} \tag{B.30}$$

If we put together (B.24) and (B.30) we obtain

$$-\frac{1}{2}\langle S_{SC}^2 S_{4f}\rangle = -\frac{\beta^3 d(d-1)}{192}\left((d-1)\cos^{-4}\frac{\phi}{2} - (d-3)\cos^{-2}\frac{\phi}{2}\right) \tag{B.31}$$

Finally, let's calculate the contribution of the term with $\langle S_{SC}^3\rangle$ (whose diagram is not shown for reason of complexity)

$$-\frac{1}{3!}\int_0^1\int_0^1\int_0^1 d\tau d\sigma d\xi\ \langle\frac{1}{2}\dot{x}^i x^k R_{kiab}\bar{\psi}^a\psi^b(\tau)\frac{1}{2}\dot{x}^l x^m R_{mlcd}\bar{\psi}^c\psi^d(\sigma)\frac{1}{2}\dot{x}^\tau x^{\lambda\tau ef}R_{\lambda\tau}\bar{\psi}^e\psi^f(\xi)\rangle \tag{B.32}$$

One can extend the integral in (B.32) to $d+1$ dimensions

$$\begin{aligned}-\frac{1}{48}\iiint d^{d+1}t\,d^{d+1}s\,d^{d+1}u\ \langle\partial_\alpha x^i x^k R_{kiab}\bar{\psi}^a\gamma^\alpha\psi^b(t)\ \partial_\beta x^l x^m R_{mlcd}\bar{\psi}^c\gamma^\beta\psi^d(s)\\ \partial_\delta x^\tau x^\lambda R_{\lambda\tau ef}\bar{\psi}^e\gamma^\delta\psi^f(u)\rangle\end{aligned} \tag{B.33}$$

If we Wick contract bosonic and fermionic parts and perform DR we obtain:

$$\begin{aligned}&\frac{\beta^3}{24}d(d-1)(d-2)\iiint d^{d+1}t\ d^{d+1}s\ d^{d+1}u\\
&\times\Big[{}_\alpha\Delta_\beta(t,s)\Delta(t,u)\Delta_\alpha(s,u) + {}_\alpha\Delta_\beta(t,s)\Delta_\delta(t,u)\Delta_\alpha(s,u)\\
&+{}_\alpha\Delta_\delta(t,u)\Delta_\beta(t,s)\Delta(s,u) + {}_\alpha\Delta_\delta(t,u)\Delta(t,s)_\beta\Delta(s,u)\\
&+{}_\beta\Delta_\delta(s,u)_\alpha\Delta(t,s)\Delta(t,u) + {}_\beta\Delta_\delta(s,u)_\alpha\Delta(t,u)\Delta(t,s)\\
&+{}_\alpha\Delta(t,s)_\beta\Delta(s,u)\Delta_\delta(t,u) + {}_\alpha\Delta(t,u)\Delta_\beta(t,s)\Delta(s,u)\Big]\\
&\times \mathrm{Tr}\{\gamma^\alpha\Delta_F(t-s)\gamma^\beta\Delta_F(s-u)\gamma^\delta\Delta_F(u-t)\}.\end{aligned} \tag{B.34}$$

We can integrate by parts the $\partial_\alpha$ from ${}_\alpha\Delta_\beta$ and $\partial_\beta$ from ${}_\beta\Delta_\delta$, we obtain terms that cancel each others and we are left with

$$\begin{aligned}
&-\frac{\beta^3 d(d-1)(d-2)}{24}\iint d^{d+1}t\; d^{d+1}s\; d^{d+1}u \\
&\times\Big[\Delta_\delta(t,u)\Delta_\beta(t,s)\Delta(s,u)+\Delta_\delta(t,u)\Delta(t,s)_\beta\Delta(s,u)\Big] \\
&\times\,\mathrm{Tr}\bigg\{\Big(\gamma^\alpha\frac{\partial}{\partial t^\alpha}\Delta_F(t-s)\Big)\gamma^\beta\Delta_F(s-u)\gamma^\delta\Delta_F(u-t) \\
&+\Delta_F(t-s)\gamma^\beta\Delta_F(s-u)\gamma^\delta\Big(\Delta_F(u-t)\frac{\overleftarrow{\partial}}{\partial t^\alpha}\gamma^\alpha\Big)\bigg\}
\end{aligned} \tag{B.35}$$

and

$$\begin{aligned}
&-\frac{\beta^3 d(d-1)(d-2)}{24}\iint d^{d+1}t\; d^{d+1}s\; d^{d+1}u \\
&\times\Big[\Delta_\delta(s,u)_\alpha\Delta(t,s)\Delta(t,u)+\Delta_\delta(s,u)\Delta(t,s)_\alpha\Delta(t,u)\Big] \\
&\times\,\mathrm{Tr}\bigg\{\Big(\gamma^\beta\frac{\partial}{\partial s^\alpha}\Delta_F(s-u)\Big)\gamma^\delta\Delta_F(u-t)\gamma^\alpha\Delta_F(t-s) \\
&+\Delta_F(s-u)\gamma^\delta\Delta_F(u-t)\gamma^\alpha\Big(\Delta_F(t-s)\frac{\overleftarrow{\partial}}{\partial s^\beta}\gamma^\beta\Big)\bigg\}.
\end{aligned} \tag{B.36}$$

If we use Dirac equation and then remove DR, we obtain

$$\begin{aligned}
&-\frac{\beta^3 d(d-1)(d-2)}{48} \\
&\times\int_0^1\!\!\int_0^1 d\tau\, d\sigma\,\Big[3\Delta^\bullet(\tau,\tau)\Delta^\bullet(\tau,\sigma)\Delta(\sigma,\tau)+\Delta^\bullet(\tau,\sigma)\Delta(\tau,\tau)^\bullet\Delta(\tau,\sigma)\Big] \\
&\qquad\qquad\times\Delta_F(\tau-\sigma)\Delta_F(\sigma-\tau).
\end{aligned} \tag{B.37}$$

If we insert the expression of the propagators in (B.37), we obtain

$$\begin{aligned}
-\frac{1}{3!}\langle S_{SC}^3\rangle &= -\frac{\beta^3 d(d-1)(d-2)}{96}\left(\frac{1}{30}\right)\cos^{-2}\frac{\phi}{2} \\
&= -\frac{\beta^3 d(d-1)(d-2)}{2880}\cos^{-2}\frac{\phi}{2}.
\end{aligned} \tag{B.38}$$

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# Ringraziamenti

Ringrazio la mia famiglia, in particolare mio figlio, il quale mi ha dato la forza di rimettermi in gioco e di affrontare questa nuova sfida, mio marito, i miei genitori, i miei suoceri, i miei amici e parenti e tutte le persone che mi hanno sempre sostenuta e hanno sempre creduto in me.

Desidero ringraziare, inoltre, il professor Fiorenzo Bastianelli per avermi seguita in questo lavoro di tesi e per avermi insegnato molte cose sulla teoria dei campi.

Rivolgo un sentito grazie ai miei professori del liceo, in particolare Oscar Brandolini, docente di fisica, e Donatella Brunelli, insegnante di matematica, per avermi trasmesso la passione per queste splendide materie.

Volevo infine dedicare un pensiero ai gestori e dipendenti del Caffè Barriera Cavour di Cesena per avermi concesso il piacere di rendere la taverna del locale il mio principale luogo di studio.