Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Fisica

Gravitational Redshift in a Stochastic Black Hole Background

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Sessione V
Anno Accademico 2017/2018

Sommario

In questa tesi è stato analizzato il moto di un campo scalare non massivo in caduta libera radiale verso un buco nero di Schwarzschild. Tramite l'utilizzo di un modello reticolare, si ottiene un'equazione di moto estremamente complessa la cui soluzione viene data in modo approssimativo. Pertanto, è necessario lo studio del redshift di radiazione e la sua distribuzione, a partire da una configurazione di tipo cinetico, studiando la caduta di un singolo modo di radiazione. Tale trattatazione risulta semplificata, considerando lo spaziotempo statico ed il buco nero come oggetto completamente assorbitore della radiazione entrante. Se lo spaziotempo non è statico, le fluttuazioni quantistiche dei campi diventano consistenti. Inoltre, è stato dimostrato da Stephen Hawking (1974) che il buco nero può emettere radiazione, fino alla totale evaporazione, generando diverse problematiche che tutt'ora sono in fase di studio.

Abstract

In this thesis we have analyzed the motion of a massless scalar field radially falling towards a Schwarzschild Black Hole. Through a lattice configuration, we have obtained an extremely complex equation of motion, for which no solution can be found but an approximate one. Consequently, we have to study the redshift through a kinetic analysis of a single falling radiation mode. However, the problem has been simplified, as the spacetime is considered as static and the black hole as a completely radiation absorber. If the spacetime is non static, quantum fluctuations occur and it has been demonstrated by Stephen Hawking (1974) that a black hole can emit radiation till the total evaporation, showing up several issues still ongoing.

Contents

Introduction

Black Holes are some of the most fascinating and mysterious objects in the Universe. Despite their representations as black circles, black holes cannot be either seen or observed directly, but they are usually analyzed through their effects on spacetime and interactions with external physical objects, such as light, waves of different nature, fields, particles and so on. The concept of black hole comes out from a particular solution of Einstein field equations. One of the easiest known black holes is the Schwarzschild Black Hole, a simple model characterized by spherical geometry and static spacetime metric. The Schwarzschild metric shows two singularities in the radial positions $r = 0$ and $r = 2M$, the former being a curvature singularity where the Gravitational field becomes infinitely strong, the latter describing the event horizon, i.e. the position at which any object in the Universe is no more detectable from a far away observer as it is captured by the black hole with no chances to escape. In this thesis, we want to study the gravitational redshift of infalling radiation. As first, we have analyzed the motion of a massless scalar field radially falling towards the black hole event horizon. In order to describe such motion, we need to make a change in the coordinate system due to the Schwarzschild metric becoming singular on the event horizon, opting for the so-called Painlevé-Gullstrand metric. Moreover, a more proper configuration for the freely falling scalar field is represented by the Lattice Model in two dimensions (the motion is radial so no contribution is given by angles), through the introduction of a new radial coordinate y, constant along the geodesics, and the time coordinate defined in the Painlevé-Gullstrand configuration. Then, we shall use this two dimensional metric to compute the equation of motion, starting from the Euler Lagrange equation from Scalar Field Theory. As we will see, the obtained equation is very complex and not analytically solvable, thus we will have to make some approximations, such as the Wentzel-Kramers-Brillouin approximation, in order to obtain a solution. Nonetheless, this solution is not accurate and we cannot use it for our purpose, the study of the redshift. This effect is due to the strong gravitational field of the black hole that acts on the wavelenght of the radiation such that a distant observer will measure a reddened radiation. The redshift has been computed starting from a kinetic analysis of a single infalling mode, comparing the proper time of the mode with the time as measured from an asymptotic observer. From the redshift formula, we obtain the characteristic of the black hole event horizon stated above: as approaching

the event horizon, the signal becomes less detectable from the asymptotic observer, until disappearing.
Usually in physics we do not have precise values of physical quantities, in particular there may be fluctuations on the value of the black hole mass. Therefore, we have imposed the Schwarzschild radius to follow a Gaussian distribution and evaluated the response of the redshift to such oscillations. Despite this analysis looks simple, we will have to make many assumptions and simplifications in order to obtain self-consistent results. Indeed, near horizon physics can not be fully described by classical physics and when the system is quantized, i.e. when we consider the black hole described as a quantum state $|\psi\rangle$, many issues show up. This is what we want to explain in detail in the last part of the thesis. As we will see, the description of a freely falling scalar field towards the black hole event horizon can actually be done through semiclassical gravity, i.e. Quantum Field Theory in classical spacetimes, though it will manifest some problems regarding violations of the unitarity principle in Quantum Mechanics, appearance of quantities below Plank scale, where we are still not able to describe any physics, and violation of the Information Conservation Principle, questions that yet remain unanswered.

List of symbols and conventions

$;$ Covariant derivative such that $V^{\alpha}_{;\beta} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} + V^{\mu}\Gamma^{\alpha}_{\beta\mu}$; p. 29

$\Box$ D'Alambertian symbol equal to $\frac{\partial^2}{\partial t^2} + \nabla^2$; p. 34

c Speed of light in natural units, i.e c=1; p. 68

$d\Omega$ Infinitesimal angle such that $d\Omega^2 = d\theta^{2+}\sin^2\theta d\phi^2$; p. 16, 30, 31, 68, 84

$\eta_{\mu\nu}$ Metric tensor of flat spacetime equal to $diag(-1, 1, 1, 1)$; p. 9, 27

G Gravitational constant in natural units, i.e. G=1; p. 68

g Determinant of the metric tensor $g_{\mu\nu}$; p. 33–35

$\hbar$ Planck constant in natural units, i.e. $\hbar = 1$; p. 68

l_P Planck length with value $(1.616229 \pm 0.000038) \times 10^{-35} m$; p. 48

$(\mathcal{M}, g)$ Manifold with metric tensor g; p. 12

m_P Planck length with value $(2.176470 \pm 0.000051) \times 10^{-8} kg$; p. 48, 49

PG Abbreviation for Painlevé-Gullstrand; p. 25, 28–32, 35

$\mathcal{R}^n$ n-dimensional set of all real numbers; p. 10, 11

WKB Wentzel-Kramers-Brillouin approximation method; p. 41, 42

Part I

Black holes in general relativity

Chapter 1

Geometry and relativity

In this chapter, we will expose some useful concepts from differential geometry regarding the properties and the mathematical description of black holes. All the details are taken from [13].

1.1 Main features of general relativity

Einstein's theory of General Relativity dramatically changed our understanding of the concept of space and time. In particular, the Equivalence Principle of Gravitation and the Principle of General Covariance built up a connection between the gravitational field and the geometry of space and time. Thus, the language of general relativity mainly consists of differential geometry. Specifically the description of space and time is based on the intrinsic property of local distance between two points, i.e. the distance between two points that are infinitely close. The locality is a fundamental property to require, because it gives us the possibility to find a point whose neighborhood can be considered locally euclidean, namely it can be described by an euclidean geometry. In order to describe the metric of this local space, we calculate the distance between a point $P = (\xi^1, \xi^2)$ and $P' = (\xi^1 + d\xi^1, \xi^2 + d\xi^2)$ through the Pythagoras law:

$$ds^2 = \left(d\xi^1\right)^2 + (d\xi^2)^2 \tag{1.1}$$

We have thus stated that the geometry is locally euclidean, i.e. locally flat, so the above metric is valid only locally, unless the space is globally euclidean. Consider now another coordinate system, whose point is expressed as (x^1, x^2). There is a relation between the two systems and it can be calculated by computing the differential of P and then substituting into the metric such as it is expressed in (x^1, x^2), i.e.:

$$\begin{aligned} \xi^1 = \xi^1\left(x^1, x^2\right) &\Longrightarrow d\xi^1 = \frac{\partial \xi^1}{\partial x^1} dx^1 + \frac{\partial \xi^1}{\partial x^2} dx^2 \\ \xi^2 = \xi^2\left(x^1, x^2\right) &\Longrightarrow d\xi^2 = \frac{\partial \xi^2}{\partial x^1} dx^1 + \frac{\partial \xi^2}{\partial x^2} dx^2 \end{aligned} \tag{1.2}$$

substituting in the metric:

$$\begin{aligned} ds^2 = &\left[\left(\frac{\partial \xi^1}{\partial x^1}\right)^2 + \left(\frac{\partial \xi^1}{\partial x^2}\right)^2\right]\left(dx^1\right)^2 + \left[\left(\frac{\partial \xi^2}{\partial x^1}\right)^2 + \left(\frac{\partial \xi^2}{\partial x^2}\right)^2\right]\left(dx^2\right)^2 \\ &+ 2\left(\frac{\partial \xi^1}{\partial x^1}\frac{\partial \xi^1}{\partial x^2} + \frac{\partial \xi^2}{\partial x^1}\frac{\partial \xi^2}{\partial x^2}\right) dx^1 dx^2 \end{aligned} \tag{1.3}$$

The terms in squared brackets are components of the metric tensor $g_{\alpha\beta}$, such that the metric can be written as:

$$ds^2 = g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + 2g_{12}dx^1dx^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{1.4}$$

Thus the metric tensor is symmetric and it allows us to compute the distance in any coordinate system. In this way, the notion of metric associated to a space is straightforward. Moreover, the metric tensor allows us to determine the intrinsic property of a metric space, as it expresses the so-called Gaussian curvature defined as:

$$\begin{aligned} k\left(x^1, x^2\right) = &\frac{1}{2g}\left(2\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{11}}{\partial \left(x^2\right)^2} - \frac{\partial^2 g_{22}}{\partial \left(x^1\right)^2}\right) \\ &- \frac{g_{22}}{4g}\left[\left(\frac{\partial g_{11}}{\partial x^1}\right)\left(2\frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) - \left(\frac{\partial g_{11}}{\partial x^2}\right)^2\right] \\ &+ \frac{g_{12}}{4g^2}\left[\left(\frac{\partial g_{11}}{\partial x^1}\right)\left(\frac{\partial g_{22}}{\partial x^2}\right) - 2\left(\frac{\partial g_{11}}{\partial x^2}\right)\left(\frac{\partial g_{22}}{\partial x^1}\right)\right] \\ &+ \frac{g_{12}}{4g^2}\left[\left(2\frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right)\left(\frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right)\right] \\ &- \frac{g_{11}}{4g^2}\left[\left(\frac{\partial g_{22}}{\partial x^2}\right)\left(2\frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right) - \left(\frac{\partial g_{22}}{\partial x^1}\right)^2\right] \end{aligned} \tag{1.5}$$

If the space is flat, the curvature is $k = 0$. No matter which coordinate system we choose, $g_{\alpha\beta}$ changes but k remains the same.
Notice that we have exposed the properties for two-dimensional space. All of them can be extended to a D-dimensional space. For example, in General Relativity we use to deal with four-dimensional spaces, where we have one time coordinate (0-component) and

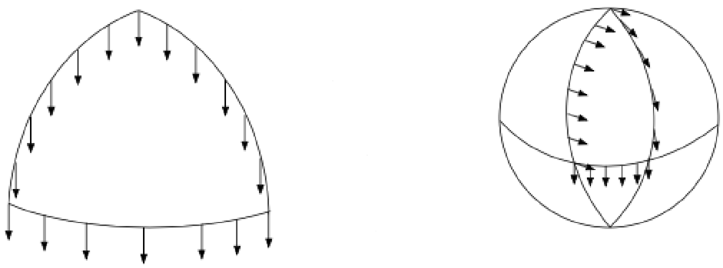

Figure 1.1: Parallel transport of a vector in a flat space (left) and in a curved one (right).

three space coordinates. We shall refer to this space as spacetime. In a four-dimensional spacetime, the flat metric is:

$$ds^2 = -\left(d\xi^0\right)^2 + \left(d\xi^1\right)^2 + \left(d\xi^1\right)^2 + \left(d\xi^3\right)^2 \quad (1.6)$$

This is known as Minkowski spacetime. We usually deal with asymptotically flat spacetimes, i.e spacetimes that very far away from the gravitational source can be considered as flat. The concept of curved and flat spacetime can be geometrically visualized through the parallel transport of a vector, i.e. the transport for which at any infinitesimal displacement, the displaced vector must be parallel to the original one, and must have the same length. Now, if the path belongs to a flat space, the transported vector keeps its direction and length during the transport, so the displaced vector back to point A coincides with the original vector in A. This is not the case for the curved spacetime, represented as a sphere, where the displaced vector back in A is rotated 90°, as the vector must always be tangent to the sphere. Thus, on a curved manifold it is not possible to define a globally parallel vector field. This is shown in Fig. 1.1. To conclude, we want now to find the equation of motion of a particle that moves only under the action of a gravitational field. First we start analyzing the motion in a locally inertial frame, the one freely falling with the particle. The distance between two neighboring points in this frame is given by the metric:

$$ds^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu \quad (1.7)$$

According to the Principle of Equivalence, "in an arbitrary gravitational field, at any given spacetime point, we can choose a locally inertial reference frame such that, in a sufficiently small region surrounding that point, all physical laws take the same form they would take in absence of gravity, namely the form prescribed by Special Relativity". That is, choosing as the time coordinate the proper time of the particle, its equation of motion is simply given by:

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{1.8}$$

so no gravity is felt as we chose a locally inertial frame. If we want to change frame through a new coordinate system $x^\alpha = x^\alpha(\xi^\alpha)$, the new metric will be:

$$ds^2 = \eta_{\mu\nu}\frac{\partial\xi^\mu}{\partial x^\alpha}dx^\alpha\frac{\partial\xi^\nu}{\partial x^\beta}dx^\beta = g_{\alpha\beta}dx^\alpha dx^\beta, \;\; g_{\alpha\beta} = \frac{\partial\xi^\mu}{\partial x^\alpha}\frac{\partial\xi^\nu}{\partial x^\beta}\eta_{\mu\nu} \tag{1.9}$$

Eq. (1.8) becomes:

$$\frac{d^2\xi^\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\beta\partial x^\nu}\frac{dx^\beta}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1.10}$$

Multiplying by $\frac{\partial x^\sigma}{\partial\xi^\alpha}$ and remembering that:

$$\frac{\partial\xi^\alpha}{\partial x^\nu}\frac{\partial x^\sigma}{\partial\xi^\alpha} = \delta^\sigma_\nu \tag{1.11}$$

we obtain the final equation, known as the geodetic equation:

$$\frac{d^2x^\sigma}{d\tau^2} + \left(\frac{\partial x^\sigma}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\beta\partial x^\nu}\right)\frac{dx^\beta}{d\tau}\frac{dx^\nu}{d\tau} = \frac{d^2x^\sigma}{d\tau^2} + \Gamma^\alpha_{\beta\nu}\frac{dx^\beta}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1.12}$$

where $\Gamma^\alpha_{\beta\nu}$ are the Christoffel's symbols, which are related to the metric tensor in the following way:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(\frac{\partial g_{\nu\lambda}}{\partial x^\mu} + \frac{\partial g_{\mu\lambda}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right) \tag{1.13}$$

Eq. (1.12) is the equation of motion of a freely falling particle as seen by a different reference frame than the local inertial one. The second term of the equation tells us that changing frame, we feel a gravitational force, specifically it express the gravitational force per unit mass that acts on the particle. If we were in Newton mechanics, that term would have simply been $\vec{g}$ plus additional apparent accelerations. Thus, the Christoffel's symbols are the generalization of the Newtonian gravitational field and the metric tensor is the generalization of the Newtonian gravitational potential.

General Relativity usually deals with the concept of **Manifold**. In order to understand what a manifold is, we need to define the topological space. Before introducing the general definition of a topological space, we recall some properties of a particular topological space, $\mathcal{R}^n$. We start from defining the **open set**, where a system of coordinate is settled and the **open ball**. An **open ball** is the set of all points $x \in \mathcal{R}^n$ such that $|x - y| < r$, for some fixed $y \in \mathcal{R}^n$ and $r \in \mathcal{R}^n$, where $|x - y| = \sqrt{\left[\sum_i (x^i - y^i)^2\right]}$.

A set $V \subset \mathcal{R}^n$ is an **open set** if, for any $y \in V$ there is an open ball centered in y that is completely inside V. Consider now a general set T and be $O = \{O_i\}$ a collection of subsets of T. Then, the couple (T, O) is a **Topological Space** if it satisfies the following properties:

- If two specific subsets O_1 and O_2 of O are open sets, their intersection is also an open set.
- The union of any collection of open sets (also an infinite one) is an open set.

Linked to the concept of manifold is the map. A map ϕ from a space M to a space N, i.e. $\phi : M \to N$, is a rule which associates an element $x \in M$ to a unique element $y = \phi(x) \in N$. Spaces M and N must not be different. Thus, if ϕ maps M to N, then for every set $S \in M$ there is an image $T \in N$ made by all the mapped points from S to N. Moreover, a map gives a unique value of $\phi(x)$ for every x, but not necessarily a unique x for every $\phi(x)$. If there is a unique value of x for every $\phi(x)$, then the map is said to be a one-to-one map. Only one-to-one maps allow inverse mapping $S = \phi^{-1}(T)$. It is also possible to make compositions of maps, such that given two maps $\phi : M \to N$ and $\theta : N \to P$, there exists a composed map that maps M to P, namely $\phi \circ \theta : M \to P$. Given two topological spaces M and N, a map $\phi : M \to N$ is a continuous map at $x \in M$ if an open set of N containing $\phi(x)$ contains the image of an open set of M. We can now give the definition of a Manifold: a **Manifold** $\mathcal{M}$ is a topological, continuous space such that an open neighborhood of each point of $\mathcal{M}$ has a continuous one-to-one map onto an open set of $\mathcal{R}^n$, being n the dimension of the manifold.
In general relativity we usually deal with coordinate system in order to describe the physics. A **coordinate system** is a subset U in a manifold $\mathcal{M}$ such that, given a one-to-one map $\phi : U \longrightarrow \mathcal{R}^n$, the image $\phi(U)$ is open in $\mathcal{R}^n$. So U is an open set in $\mathcal{M}$. We can then define a collection of coordinate system $\{(U_\alpha, \phi_\alpha)\}$ in a manifold $\mathcal{M}$ such that:

- $\mathcal{M} = \cup_\alpha U_\alpha$;
- If $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathcal{R}^n \longrightarrow \phi_\beta(U_\alpha \cap U_\beta) \subset \mathcal{R}^n$ is differentiable.

What characterizes a Manifold is that there can be many choices of ϕ_α, so many possible coordinate systems for the same point and its neighborhood, that do not have to cover the whole manifold. Besides, if two coordinate systems overlap, the transformation between the two is smooth. Examples of manifolds are the set $\mathcal{R}^n$ and the n-sphere S^n.

Chapter 2

Causality and black holes

In a given manifold $(\mathcal{M}, g)$ we can define different types of vectors, curves and hypersurfaces. A vector in a manifold $(\mathcal{M}, g)$ can be timelike, spacelike or null. Since the guiding principle of General Relativity concerns the fact that no signal can travel faster than the speed of light, any information will follow a timelike or null trajectory. Vectors defining timelike or null trajectories are said to be **causal vectors**. Then, a **causal curve** is a curve with tangent causal vectors. Such a manifold that admits causal vectors field is said to be **time-orientable**. Time orientation is given by defining a causal vector V^μ with a future orientation or a past orientation. All the causal vectors lying in the same light cone as V^μ will be future-directed, while all the causal vectors lying in the opposite light cone will be past-directed. Therefore, a curve with a future(past)-directed tangent vector will be a **future(past)-directed** curve.
We can now give some more general definitions.

Definition 2.0.1 *Given any subset $S \subset \mathcal{M}$, the* ***causal future(past)*** *of S, denoted as $\mathcal{J}^+(S)(\mathcal{J}^-(S))$, is the set of points that can be reached from S following a future(past)-directed timelike curve.*

Definition 2.0.2 *Given any subset $S \subset \mathcal{M}$, the* ***chronological future(past)*** *of S, denoted as $I^+(S)(I^-(S))$, is the set of all points which can be connected with a point of S following a future(past)-directed timelike curve.*

Definition 2.0.3 *A subset $S \subset \mathcal{M}$ is said to be* ***achronal*** *if there are no points in S connected by a timelike curve.*

Definition 2.0.4 *Given a closed achronal set S, the* ***future domain*** *of dependence of S, denoted as $D^+(S)$, is the set of all points p such that every pastmoving inextendible[1] causal curve through p must intersect S.*

[1]Inextendible means that the curve does not end at some finite point.

Similarly, we can define the **past domain** of dependence $D^-(S)$.
The boundaries of $D^+(S)$ and $D^-(S)$ are called **future Cauchy horizon**, denoted by $H^+(S)$, and **past Cauchy horizon**, denoted by $H^-(S)$, respectively.

Definition 2.0.5 *A closed achronal surface Σ is said to be a* ***Cauchy surface*** *if the domain of dependence $D(\Sigma)$ is the entire manifold.*

If a spacetime manifold has a Cauchy surface, it is said to be **globally hyperbolic**.
The causal structure of spacetimes can be visualized through the so-called Penrose diagrams. What they actually represent is a conformal transformation which brings an entire manifold onto a compact region such that the spacetime (i.e. its infinities) can be fitted on a finite 2-dimensional diagram, known as Penrose diagram. For example, consider the Minkowski spacetime in radial coordinates:

$$ds^2 = dt^2 - dr^2 - (angles) \tag{2.1}$$

The light, for which the metric is null, propagates on the light cones $dt = \pm dr$. The light cone will preserve its form under any transformation of the form $F = F(t \pm r)$. For example, we can choose transformations that map the infinite space $0 \leq r \leq \infty$,$-\infty \leq t \leq \infty$ to a finite portion of the plane, which can be written on the form:

$$\begin{aligned} Y^+ &\equiv F(t+r) = \tanh(t+r) \\ Y^- &\equiv F(t-r) = \tanh(t-r) \end{aligned} \tag{2.2}$$

For such transformations, the entire spacetime is mapped to a finite triangle with boundaries:

$$Y^+ = +1, \qquad (t+r) \to \infty \tag{2.3}$$

$$Y^- = -1, \qquad (t-r) \to \infty \tag{2.4}$$

$$Y^+ - Y^- = 0 \tag{2.5}$$

The Penrose diagram for Minkowski spacetime is shown in Fig. 2.1. The lines $t = \pm\infty$ are the future and past timelike infinities and they represent the beginnings and ends of timelike trajectories. Thus, timelike geodesics begin at $t = -\infty$ and end at $t = +\infty$. The line $r = \infty$ is the spacelike infinity and it is where all the spacelike surfaces end. All spacelike geodesics both begin and end at $r = \infty$. Radial null geodesics are at the $\pm 45°$. All null geodesics begin at Y^- and end at Y^+. Y^+ and Y^- are also called future and past null infinity, and they are usually marked as $\mathcal{I}^+$ and $\mathcal{I}^-$. We will explain their features in the next sections.
In the sketch we have also represented curves for some constant values of r and t. Moreover, there can be non-geodesic timelike curves that end at null infinity (if they become asymptotically null).
Penrose diagrams can be used to show more elaborate geometries too, such as black holes, in particular their general causal structure and the formation after gravitational collapse.

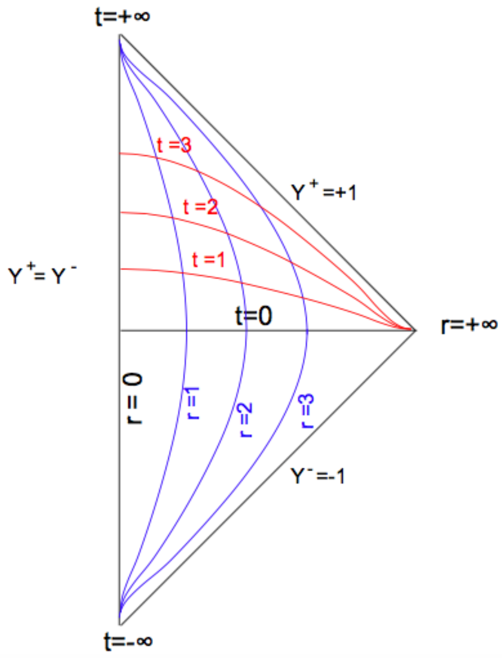

Figure 2.1: Penrose diagram for Minkowski spacetime.

2.1 Spacelike, timelike and null hypersufaces

A generic surface can be described as:

$$\Sigma\left(x^{\alpha}\right)=0 \tag{2.6}$$

Let $\vec{n}$ be the normal vector such that, if t^{μ} is the tangent vector, $t^{\mu}n_{\mu}=0$. We then introduce a local inertial frame in any point of the hypersurface and let us rotate it such that $n^{\mu}=\left(n^{0},n^{1},0,0\right)$ and $t^{\mu}n_{\mu}=0=-n_{0}t^{0}+n_{1}t^{1}\implies\frac{t^{1}}{t^{0}}=\frac{n_{0}}{n_{1}}$. So $t^{\mu}=\Lambda\left(n^{1},n^{0},y,z\right)$, where Λ, y and z are arbitrary constants. The norm of t^{μ} is then:

$$t^{\mu}t_{\mu}=\Lambda^{2}\left[-\left(n^{1}\right)^{2}+\left(n^{0}\right)^{2}+y^{2}+z^{2}\right]=\Lambda^{2}\left(-n_{\mu}n^{\mu}+y^{2}+z^{2}\right) \tag{2.7}$$

Since we want to study the disposition of the light-cone with respect to the hypersurfaces, we analyze the directions of the tangent and normal vectors. There can be three possibilities:

1. $n_{\mu}n^{\mu}<0$, so n^{μ} is a timelike vector, t^{μ} is a spacelike vector and Σ is a spacelike hypersurface. Calling P the vertex between the past and the future lightcones, no tangent vector to the hypersurface Σ in P lies in or on the lightcone through P. Since a massive(massless) particle starting from P moves in(on) the lightcone, a spacelike hypersurface can be crossed only in one direction.

2. $n_\mu n^\mu > 0$, so n^μ is a spacelike vector and Σ is a timelike hypersurface. In this case, t^μ can be spacelike, timelike or null, so some tangent vectors will lie in the lightcone. Therefore, a timelike surface can be crossed inward and outward.

3. $n_\mu n^\mu = 0$, so n^μ is a null vector and Σ is a null hypersurface. Vector t^μ can be spacelike or null, consequently there will be only one tangent vector in P lying on the hypersurface and on the lightcone.

2.2 Spherical symmetry

Einstein field equations tell us how the curvature of space-time reacts to the presence of energy-momentum[2]:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.8}$$

$G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature, $g_{\mu\nu}$ the metric of the space-time and $T_{\mu\nu}$ is the energy-momentum tensor, which describes the density and the flux of a matter distribution in a given region. They are a set of ten second order differential independent equations for the tensor $g_{\mu\nu}$. Being an extremely difficult equation to solve, not only for its non-linearity, but also because it involves products and derivatives of Christoffel symbols, it is not fair to solve it in any sort of generality. Therefore, we need to make some simplifying assumptions, such as particular symmetries of the metric. A remarkable simplification is the assumption of spherical symmetry. Taking into account the vacuum Einstein equation, regarding the metric outside the spherical body:

$$R_{\mu\nu} = 0 \tag{2.9}$$

the only spherically symmetric vacuum solution is the Schwarzschild metric. Using the spherical coordinates (t, r, θ, ϕ), the Schwarzschild metric is defined as:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2.10}$$

Where M is the Newtonian mass of the gravitational spherical body. Another assumption made by Schwarzschild is that the spherical body is static and uncharged. We notice that for $M = 0$, the Schwarzschild metric reduces to the Minkowskian one, as it must be for a flat space-time, i.e for non gravitating object curving the space-time. In addition, for $r \to +\infty$, the metric becomes flat so that the Schwarzschild solution is said to be asymptotically flat.
Written in this form, the metric becomes singular at $r = 0$ and $r = 2M$. The first is a

[2]We recall that we use the notation in which $G = c = 1$.

curvature singularity, the latter is a coordinate singularity, so due to an unappropriated choice of the coordinate. The coordinate singularity $r = 2M$ is called Schwarzschild radius, and it defines the event horizon of a black hole, a null hypersurface causally disconnected with the exterior region. We will describe its main features in the next chapters.
Let us explore now the causal structure of the Schwarzschild geometry using the Penrose diagram. We shall consider radial null curves, i.e. curves with θ and ϕ constant and null metric:

$$ds^2 = 0 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \Longrightarrow \frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1} \tag{2.11}$$

This measures the slope of the light cones on a spacetime diagram of the $t-r$ plane. For large r, the slope is ± 1, as it would be in flat space, while as approaching $r = 2M$ we get $\frac{dt}{dr} \to \infty$ and the light cones close up. However, as $r = 2M$ is a coordinate singularity, we need to change coordinate system and choose a proper one. We may rewrite the two-dimensional Schwarzschild metric, i.e. neglecting the contribution given by the angular part, as:

$$ds^2 = \left(1 - \frac{2M}{r}\right) \left[-dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2}\right] \tag{2.12}$$

The idea is to find a coordinate such that:

$$dr^{*2} = \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} \Longrightarrow dr^* = \frac{dr}{1 - \frac{2M}{r}} \tag{2.13}$$

Integrating between 0 to a general value of r, we obtained the so-called tortoise coordinate:

$$r^* = r + 2M \log\left(\frac{r}{2M} - 1\right) \tag{2.14}$$

Thus the metric (2.10) becomes:

$$ds^2 = \left(1 - \frac{2M}{r}\right) \left(-dt^2 + dr^{*2}\right) + r^2 dd\Omega^2 \tag{2.15}$$

Move now to the null ingoing and outgoing coordinates:

$$u = t + r^* \qquad v = t - r^* \tag{2.16}$$

This brings to:

$$ds^2 = \left(1 - \frac{2M}{r}\right) (-dudv) + r^2 d\Omega^2 \tag{2.17}$$

Let us now analyze the range of these coordinates. The range $r \in (2M, +\infty)$ maps $r^* \in (-\infty, +\infty)$. Considering a null geodesic radially falling towards the black hole, θ,

ϕ are constant and the worldline of the geodesic can be found from $ds^2 = 0$, obtaining that at infinity, where the metric is flat, $t \pm r = const$. In tortoise coordinates:

$$t + r^* = u = u_0 \qquad t - r^* = v = v_0 \tag{2.18}$$

We see that taking different values of t, the null coordinate u maps the whole range $u \in (-\infty, +\infty)$ for points outside the horizon. The same happens for v, mapped in $v \in (-\infty, +\infty)$ taking different values of t. However, as v null geodesic approaches the event horizon, we get:

$$v = t - r^* = u_0 - 2r^* \to +\infty \tag{2.19}$$

Thus,the event horizon in which the geodesic will fall is given by:

$$-\infty < u < +\infty \quad v \to \infty \tag{2.20}$$

We notice that (u, v) coordinates end at the horizon. But we wish to see the horizon as a regular region on our manifold, therefore we need a further coordinate change such that these new coordinates smoothly bring us towards the event horizon. A strongly regular function in $\mathcal{R}$ is the exponential one, so in order to have the horizon to be at finite values of our coordinates, we can choose:

$$U = e^{\alpha u}, \;\; V = -e^{-\alpha u} \tag{2.21}$$

assuming $\alpha > 0$. We see that:

$$U > 0, \;\; V < 0 \tag{2.22}$$

thus the event horizon is at:

$$0 < U < \infty \quad V = 0 \tag{2.23}$$

We have brought the horizon to a finite position through our new coordinate system. The metric can be rewritten as:

$$dU = \alpha e^{\alpha u} du, \;\; dV = \alpha e^{-\alpha v} \Longrightarrow ds^2 = -\left(1 - \frac{2M}{r}\right) \frac{e^{-\alpha(u-v)}}{\alpha^2} dU dV + r^2 d\Omega^2 \tag{2.24}$$

Notice that:

$$e^{-\alpha(u-v)} = e^{-2\alpha r^*} = e^{-2\alpha} \left[r + 2M \log\left(\frac{r}{2M} - 1\right)\right] = e^{-2\alpha r}(2M)^{4\alpha M} (r - 2M)^{-4\alpha M} \tag{2.25}$$

Choosing $\alpha = \frac{1}{4M}$ and substituting in (2.24), we obtain the final metric:

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU dV + r^2 d\Omega^2 \tag{2.26}$$

where:

$$U = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} e^{\frac{t}{4M}} \qquad V = -\left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} e^{-\frac{t}{4M}} \tag{2.27}$$

and the function $r(U,V)$ is given implicitly by:

$$UV = -\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} \tag{2.28}$$

The metric (2.26) is the Schwarzschild metric in the so-called Kruskal-Szekeres coordinates (U, V, θ, ϕ). As written in (2.26), the metric is non singular at $r = 2M$. There is still the singularity at $r = 2M$, which can not be removed through a change of coordinate system, as it is a curvature singularity. From (2.28), we see that for $r = 0$, $UV = 1$. But for $r < 2M$, we get $UV > 1$ which is not compatible with the range (2.22). However, we consider the initial definitions of U and V in terms of the Schwarzschild coordinates for now as only holding for $r > 2M$ and we can extend the range (2.22) to all values such that $U > -\infty$ and $V < \infty$. This is called extended Schwarzschild spacetime (Kruskal diagram) and it can be represented as in Fig. 2.2. The diagram is interpreted as the causal structure of radial motion at fixed angles (θ, ϕ). Null lines are conventionally plotted at $45°$, while the axis are $T = U + V$ and $X = T - V$. At the horizon $r = 2M$, $UV = 0$, which means either $U = 0$ (negative slope) or $V = 0$ (positive slope), which corresponds to the solid diagonals. This means that regions are not causally connected to each others, as they are separated through event horizons. Specifically, $U = 0$ is past horizon, while $V = 0$ is future horizon. The singularity $r = 0$ corresponds to the two branches of the hyperbola $UV = 1$, which is conventionally represented by a dashed line (red colored in the figure). In particular, the future singularity is at $U, V > 0$, $UV = 1$(region II), thus a radially falling observer will hit the singularity at some time in the future, while at $U, V < 0$, $UV = 1$ (region III) there is the past singularity. In general, surfaces $r = const$ correspond to hyperbola $UV = const$ with $UV < 1$, as shown in blue on the diagram. Spatial sections with $t = const$ have $U/V = const$ and $|U/V| < 1$, which corresponds to straight lines through the origin with slope between -1 and $+1$. Finally, ingoing and outgoing null geodesics are respectively given by $U = const$ and $V = const$. Regions I and II are due to the geometrical extension through the use of Kruskal-Szekeres coordinates, as well as region IV. In particular, region IV is isometric to region I and in general relativity, two regions of spacetime (or two spacetimes) that are isometric are physically identical. This diagram is the first step for the construction of the Penrose diagram of the black hole. Indeed, U, V coordinates cover the entire spacetime, so they do not have a bounded range. When we draw such diagram, we need to cut off the coordinates at some fixed values. If we want to see explicitly how the points at infinity border the spacetime, we need to make a conformal rescaling of the metric[3] such that the angles between different directions at some point do not change, and in particular null directions remain null directions. This is what we basically did in the construction of the Penrose diagram for Minkowski spacetime in Section 2, where $t+r = U$, $t-r = V$

[3] The conformal transformation on a manifold $(\mathcal{M}, g)$ is expressed as $\bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$, where $x \in \mathcal{M}$ and Ω function in $\mathcal{M}$ and it preserves the causal structure of a spacetime.

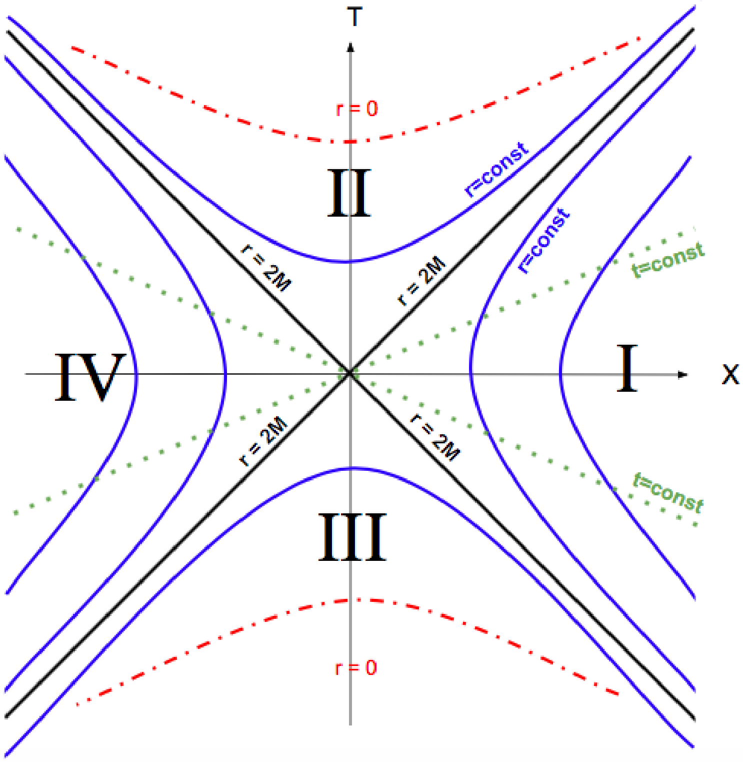

Figure 2.2: Kruskal diagram for a Schwarzschild black hole.

and $Y^+ = \tanh U$, $Y^- = \tanh V$, which substituting in the flat metric gives a conformal transformation of the form:

$$\tilde{g}_{\mu\nu} = \left(1 - \left(Y^+\right)^2\right)\left(1 - \left(Y^-\right)^2\right) g_{\mu\nu} \tag{2.29}$$

We carry on a similar transformation for the Schwarzschild metric, renaming:

$$\tilde{U} = \tanh U \qquad \tilde{V} = \tanh V \tag{2.30}$$

such that the new ranges are $-1 < \tilde{U} < 1$, $-1 < \tilde{V} < 1$ and $\tilde{U} \geq \tilde{V}$.
The metric is rescaled as (2.29), obtaining

$$\tilde{ds}^2 = -\frac{32M^3}{r^2} e^{-\frac{r}{2M}} d\tilde{U} d\tilde{V} \tag{2.31}$$

The singularity $r = 0$ is now given by

$$\operatorname{arctanh} \tilde{U} \operatorname{arctanh} \tilde{V} = 1 \tag{2.32}$$

which is a curve on the $(\tilde{U}, \tilde{V})$ space and all the points beyond this curve are not represented in the Penrose diagram, since they lie past the singularity. The singularity is

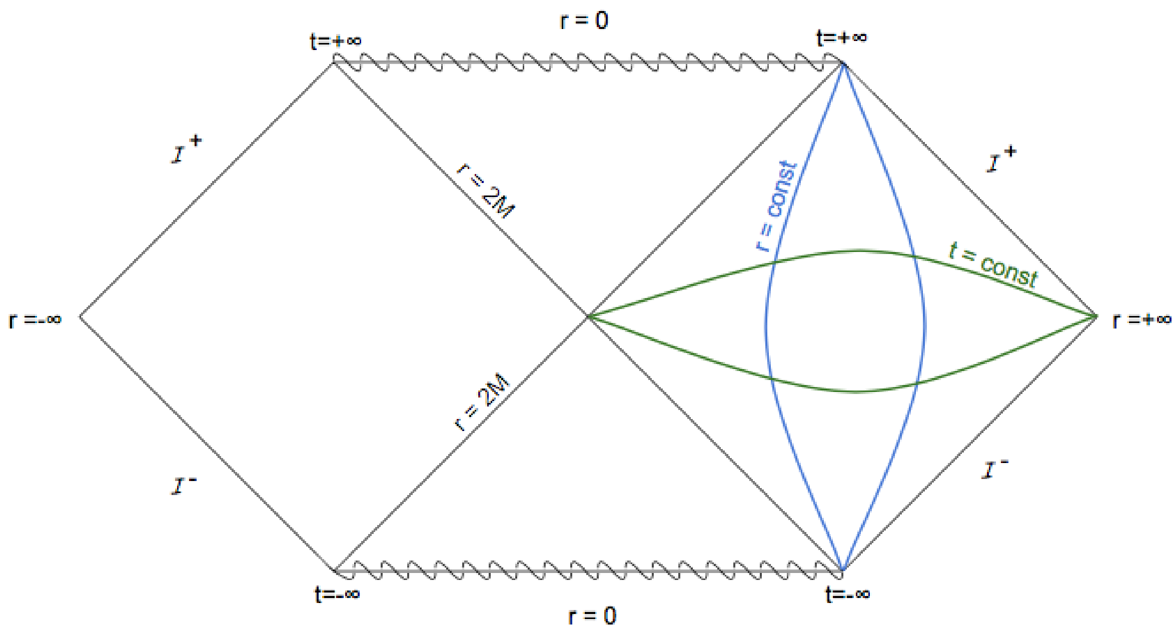

Figure 2.3: Penrose diagram for a Schwarzschild black hole.

along a curve that runs from $\tilde{U} = 0$, $\tilde{V} = 1$ to $\tilde{U} = 1$, $\tilde{V} = 0$. We can make a further rescaling in which the singularity is a straight line from $\tilde{U} = 0$, $\tilde{V} = 1$ to $\tilde{U} = 1$, $\tilde{V} = 0$. The Penrose diagram of a Schwarzschild black hole is shown in Fig. 2.3. Notice that we have indicated the future null infinity as $\mathcal{I}^{\pm}$. The lines $r = 2M$ are the future horizon (positive slope) and the past horizon (negative slope), $r = \pm\infty$ is the spacelike infinity and $t = \pm\infty$ the past and future timelike infinity.

2.3 The future null infinity

We briefly describe a particular null hypersurface, the **Future Null Infinity** $\mathcal{I}^+$, that we represented on the Kruskal and Penrose diagrams. We can firstly apply a conformal transformations to a globally hyperbolic, time-orientable manifold $(\mathcal{M}, g)$. The resulting manifold $(\bar{\mathcal{M}}, \bar{g})$ is non-physical, but it keeps the causal structure of the previous one. Let us now consider the conformal factor Ω^2 satisfying the following properties:

- $\Omega > 0$ on $\mathcal{M}$;
- $\Omega|_{\partial\mathcal{M}} = 0$;

where $\partial\mathcal{M}$ is the future boundary of the conformal compactification[4] of $(\mathcal{M}, g)$. Then, the future boundary $\partial\mathcal{M}$ is a null hypersurface which we shall call **Future Null Infinity** $\mathcal{I}^+$. Therefore, $\mathcal{I}^+$ is an incoming null hypersurface and so it represents the future boundary of a set.
Any asymptotically flat space-time admits a null infinity. In particular, a time-orientable space time $(\mathcal{M}, g)$ is **asymptotically flat** if we can apply a conformal compactification on the conformal extended space-time $(\bar{\mathcal{M}}, \bar{g})$ such that:

- $\mathcal{M} \cup \partial\mathcal{M} \subset \bar{\mathcal{M}}$ is a manifold with boundaries;
- the conformal factor Ω is extended to be a function of $\bar{\mathcal{M}}$ such that $\Omega|_{\partial\mathcal{M}} = 0$;
- the boundary $\partial\mathcal{M}$ is characterized by $\mathcal{I}^- \cap \mathcal{I}^+$;
- $\mathcal{I}^-(\mathcal{I}^+)$ never intersect with any future (past) causal curve;
- $\mathcal{I}^-$ and $\mathcal{I}^+$ are complete, so the affine parameter of their generators[5] can assume any value in $\mathcal{R}$;

$\mathcal{I}^-$ is called **Past Null Infinity** and it represents the past boundary of a set. Thus Past and Future null infinities act as the beginning of the incoming light ray and the end of the outgoing light ray trajectories.

[4] Conformal compactification brings points at infinity on a non-compact manifold to a finite distance by a conformal transformation of the metric, so that the non-compact manifold can be embedded in a new, non-physical compact manifold keeping the causal structure.

[5] A null hypersurface is generated by null geodesics whose tangent is its normal.

2.4 Black hole and event horizon

Let us now define a black hole region from a geometrical point of view.

Definition 2.4.1 *In an asymptotically flat space-time* $(\mathcal{M}, g)$*, a* ***Black Hole*** *region is defined as* $\mathcal{B} = \mathcal{M} \setminus (\mathcal{M} \cap \mathcal{J}^-(\mathcal{I}^+))$*. The* ***future event horizon*** *is the boundary* $\mathcal{H}^+ = \partial\mathcal{B}$ *of this region.*

In other words, the black hole region in an asymptotically flat space-time is the non-empty complement of the past of the future null infinity. The common boundary with the complementary region communicating with $\mathcal{I}^+$ is called future event horizon. The future event horizon must be a null hypersurface, as it is the boundary of the past of a set.

Analogously, we can define the white hole region.

Definition 2.4.2 *In an asymptotically flat space-time* $\mathcal{M}$*, a* ***White Hole*** *region is defined as* $\mathcal{W} = \mathcal{M}\setminus(\mathcal{M}\cap\mathcal{J}^+(\mathcal{I}^-))$*. The* ***past event horizon*** *is the boundary* $\mathcal{H}^- = \partial\mathcal{W}$ *of this region.*

From a physical point of view, we can describe the effects of a black hole region as follow. Consider a hypersurface in the Schwarzschild geometry described in Eq. (2.10):

$$\Sigma = r - const = 0 \tag{2.33}$$

The norm of the normal vector is then:

$$n_\mu n^\mu = g^{\mu\nu} n_\mu n_\nu = g^{\mu\nu} \frac{\partial\Sigma}{\partial x^\mu}\frac{\partial\Sigma}{\partial x^\nu} = \left(1 - \frac{2M}{r}\right) \tag{2.34}$$

There are three possibilities:

- $r > 2M \Longrightarrow n_\mu n^\mu > 0 \Longrightarrow \Sigma$ is timelike;
- $r = 2M \Longrightarrow n_\mu n^\mu = 0 \Longrightarrow \Sigma$ is null;
- $r < 2M \Longrightarrow n_\mu n^\mu < 0 \Longrightarrow \Sigma$ is spacelike;

Consider now two hypersurfaces $\mathcal{S}_2$ and $\mathcal{S}_1$, in and out from the event horizon, respectively. From the characteristics of the hypersurfaces explained in Section 2.1, since $\mathcal{S}_1$ is timelike, any signal starting from a point on $\mathcal{S}_1$ can be sent both towards the singularity in $r = 0$ and towards infinity. Conversely, since $\mathcal{S}_2$ is spacelike, any signal starting from a point on $\mathcal{S}_2$ can be sent only towards the singularity. The null hypersurface $r = 2M$ is the transition between spacelike and timelike hypersurfaces, so it physically represents the event horizon of the black hole: once crossed in it, there is no way for the signal to escape and it will be captured by the singularity in $r = 0$.

Going back to the geometrical description of a black hole region, we can state that the physics is predictable outside and on $\mathcal{H}^+$. Formally, an asimptotically flat spacetime manifold $(\mathcal{M}, g)$ is said to be **strongly asymptotic predictable** if there is an open region $V \subset \mathcal{M}$ such that $\mathcal{M} \cap \mathcal{J}^-(\mathcal{I}^+) \subset V$ and the manifold (V, g) is globally hyperbolic.

Part II

Scalar field and gravitational redshift

Chapter 3

Radial infalling lattice model

In order study the dynamic of the of a scalar field falling towards the black hole, we may need to analyze the redshift of the field. In other words, we want to see how the frequency emitted behaves as seen from a distant observer. In fact, as we will show in these further chapters, finding a direct solution of dynamic is not possible either analytically or numerically. However, before focusing our attention on the study of the redshift, we need to introduce a new model to simplify our dynamic analysis, starting from the definition of a new coordinate system.

3.1 Painlevé-Gullstrand coordinate system

As we have seen in Section 2.2, Schwarzschild coordinates show difficulties on the event horizon, being $r = 2M$ a singularity for the metric (2.10). Therefore, Schwarzschild metric can not be used to describe a freely falling object towards the singularity, as we would encounter issues once the object reaches the event horizon. A useful alternative is represented by the **Painlevé-Gullstand coordinates** (PG) as shown in [9].
Let us consider an observer that moves along a radial ingoing timelike geodesic in a Schwarzschild space-time, i.e. $\dot{r} < 0$, θ and ϕ are constant. The dot is the differentiation with respect to the observer's proper time. In order to find the geodesic, we may use the variational principle. The Lagrangian of the freely falling observer can be written as:

$$\mathcal{L} = \mathcal{L}\left(x^\alpha, \frac{dx^\alpha}{d\lambda}\right) = \frac{1}{2} g_{\mu\nu}\left(x^\alpha\right) \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{3.1}$$

where $\lambda \in [\lambda_1.\lambda_2]$ parametrized the curve $x^\mu(\lambda)$ and it can be considered as the proper time of the observer in a timelike trajectory. The action will then be:

$$S = \int d\lambda \mathcal{L}\left(x^\alpha, \dot{x}^\alpha\right), \;\; \dot{x}^\alpha = \frac{dx^\alpha}{d\lambda} \tag{3.2}$$

We let the curve vary of an infinitesimal value such that:

$$x^{\mu}(\lambda) \longrightarrow x^{\mu}(\lambda) + \delta x^{\mu}(\lambda) \tag{3.3}$$

with $\delta x^{\mu}(\lambda_1) = \delta x^{\mu}(\lambda_2)$. The action varies as:

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial x^{\sigma}} \delta x^{\sigma} + \frac{\partial \mathcal{L}}{\partial \dot{x}^{\sigma}} \delta \dot{x}^{\sigma} \right) d\lambda \tag{3.4}$$

The last term can be rewritten as:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^{\sigma}} \delta \dot{x}^{\sigma} = \frac{\partial \mathcal{L}}{\partial \dot{x}^{\sigma}} \frac{d\delta x^{\sigma}}{d\lambda} = \frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\sigma}} \delta x^{\sigma} \right) - \delta x^{\sigma} \frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\sigma}} \right) \tag{3.5}$$

Integrating between λ_1 and λ_2, the first term vanishes as $\delta x^{\mu}(\lambda_1) = \delta x^{\mu}(\lambda_2)$. Therefore, (3.4) becomes:

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial x^{\sigma}} \delta x^{\sigma} - \frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\sigma}} \right) \delta x^{\sigma} \right] d\lambda \tag{3.6}$$

For the variational principle, S must keep constant in the infinitesimal variation, thus we want δS to vanish, and this happens when:

$$\frac{\partial \mathcal{L}}{\partial x^{\sigma}} - \frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\sigma}} \right) = 0 \tag{3.7}$$

These are known as the Euler-Lagrange equations for a freely moving massive observer. Considering a two dimensional Schwarzschild metric, i.e. a radial falling with θ and ϕ constant, the Lagrangian is:

$$\mathcal{L} = \frac{1}{2} \left[- \left(1 - \frac{2M}{r} \right) \dot{t}^2 + \left(1 - \frac{2M}{r} \right)^{-1} \dot{r}^2 \right] \tag{3.8}$$

Then, we may write the equations for t:

$$\frac{\partial \mathcal{L}}{\partial t} - \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{t}} \Longrightarrow \frac{d}{d\lambda} \left(1 - \frac{2M}{r} \right) \dot{t} = 0 \tag{3.9}$$

for which we obtain:

$$\dot{t} = const \left(1 - \frac{2M}{r} \right)^{-1} \tag{3.10}$$

The constant is simply the energy of the observer. The reason is due to the Killing vectors definition. A Killing vector is a vector that identifies a symmetry in the space. We can visualize it as a vector $\vec{\xi}(x^{\mu})$ defined at every point x^{μ} of a spacetime region such that an infinitesimal translation along $\vec{\xi}$ leaves the metric unchanged. Thus, $\vec{\xi}$ is the tangent vector to some curve $x^\mu(\lambda)$, i.e $\xi^\mu = \frac{dx^\mu}{d\lambda}$ such that making an infinitesimal translation along the curve in the direction of $\vec{\xi}$, the metric keeps invariant. We can choose a timelike Killing vector such that the time coordinate lines coincide with the worldline to which $\vec{\xi}$ is tangent, thus we can write the Killing vector as:

$$\xi^\mu = (1, 0, 0, 0) \tag{3.11}$$

An important property of the Killing vectors is the fact that they define conserved quantities in geodesic motion. In fact, multiplying the geodesic equation (1.12) written in terms of the four velocity $u^\mu = \frac{dx^\mu}{d\tau}$ for a Killing vector ξ^μ, we obtain the important property for which:

$$g_{\mu\nu}\xi^\mu u^\nu = const \tag{3.12}$$

Now, if the metric is asymptotically flat, at radial infinity $g_{\mu\nu} = \eta_{\mu\nu}$ and choosing a timelike Killing vector such that the geodesic admits a Killing vector of the form $\xi^\mu = (1, 0, 0, 0)$, (3.12) becomes at infinity:

$$g_{00}u^0 = -u^0 = const \tag{3.13}$$

remembering that in a flat spacetime the energy-momentum of a massive particle is

$$p^\mu = mu^\mu = \left(E, \frac{mv^i}{\sqrt{1-v^2}}\right) \tag{3.14}$$

Eq. (3.13) is simply:

$$\frac{E}{m} = const \tag{3.15}$$

Therefore, the constant in Eq. (3.10) is the energy per unit mass $\tilde{E}$ of the observer at infinity:

$$\dot{t} = \frac{\tilde{E}}{1 - \frac{2M}{r}} \tag{3.16}$$

Now, in order to find the equation for r, it is more convenient and less elaborated to consider the property of a massive particle:

$$u_\mu u^\mu = -1 \tag{3.17}$$

for which we obtain:

$$\dot{r}^2 + 1 - \frac{2M}{r} = \tilde{E}^2 \tag{3.18}$$

From (3.14), the energy per unit rest mass can be expressed in terms of the observer's initial speed $v(r = \infty) = v_\infty$:

$$\tilde{E} = \frac{1}{\sqrt{1 - v_\infty^2}} \tag{3.19}$$

Specifically, we first consider the case in which $\tilde{E} = 1$, i.e. when the observer starts the motion with a zero initial speed, $v_\infty = 0$. Equations (3.16) and (3.18) reduce to:

$$\dot{t} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad \dot{r}^2 + 1 - \frac{2M}{r} = 1 \tag{3.20}$$

The speed of the freely falling observer in Schwarzschild coordinates with respect to r can be calculated from (3.20):

$$\begin{aligned} \dot{r} \equiv \frac{dr}{d\tau} &= -\sqrt{\frac{2M}{r}} \\ \frac{dr}{dt}\frac{dt}{d\tau} &= -\sqrt{\frac{2M}{r}} \end{aligned} \tag{3.21}$$

And this implies that:

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\sqrt{\frac{2M}{r}} \tag{3.22}$$

We see that the speed tends to zero as the observer approaches the event horizon. Thus, another observer outside the event horizon will never see the freely falling observer crossing the event horizon, as she will measure an infinite redshift frequency.
Now, as the the four-velocity contravariant components are $u^\mu = (\dot{t}, \dot{r}, 0, 0)$, so that the covariant components are $u_\mu = \left(-1, -\frac{\sqrt{\frac{2M}{r}}}{1-\frac{2M}{r}}, 0, 0\right)$, u_μ can be written as the gradient of a function T:

$$u_\mu = -\partial_\mu T \tag{3.23}$$

where T will be:

$$T = t + \int \frac{\sqrt{\frac{2M}{r}}}{1 - \frac{2M}{r}} dr \tag{3.24}$$

Integrating, we obtain the new temporal coordinate:

$$T = t + 4M\left(\sqrt{\frac{r}{2M}} + \frac{1}{2}\log\left|\frac{\sqrt{\frac{r}{2M}} - 1}{\sqrt{\frac{r}{2M}} + 1}\right|\right) \tag{3.25}$$

which will define a new set of coordinates (T, r, θ, ϕ), known as Panlievé-Gullstrand coordinates. Differentiating:

$$dT = dt + \frac{\sqrt{\frac{2M}{r}}}{1 - \frac{2M}{r}} dr \tag{3.26}$$

and substituting into the Schwarzschild metric (2.10), gives the *PG* metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dT^2 + 2\sqrt{\frac{2M}{r}}dTdr + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{3.27}$$

which is manifestly regular at $r = 2M$, but, of course, still has a curvature singularity at $r = 0$. The main difference with the Schwarzschild metric is the non-diagonal form. In particular, setting $dT = 0$ brings us back to the flat three-dimensional metric in polar coordinates. Therefore, surfaces $T = const$ are intrinsically flat and this means that in PG coordinates the information about the curvature is given by the off-diagonal term in (3.27). As before, we can calculate the speed of the freely falling observer in the PG coordinates, starting from the differentiation (3.26), obtaining:

$$\frac{dr}{dT} = -\sqrt{\frac{2M}{r}} \tag{3.28}$$

In this case, the speed increases as it approaches the event horizon, where it has the maximum value 1. Inside the black hole, the speed keeps on increasing till infinite at the singularity. However, the formula above may not be correct near the singularity since the true solution may be quite different when quantum mechanics is incorporated near the singularity. The comparison between the speed in Schwarzschild coordinates and the one in PG coordinates is shown in Fig. 3.1.

Our construction is limited to a radial, freely moving observer a in static, spherically-symmetric space-time. This may not work for more general space-times, due to the particular statement expressed by Eq. (3.23) which, from differential geometry, is valid if the following two equations hold:

$$u^{\mu}_{;\nu} u^{\nu} = 0 \qquad\qquad u_{[\mu;\nu} u_{\delta]} = 0 \tag{3.29}$$

where the square brackets express the anti-symmetrization of the indices. Physically, this means that we have to impose that the motion is geodetic and the world lines are in every point orthogonal to the spacelike hypersurfaces of constant T. In order to be coherent with these requirements, in our analysis we have considered a freely moving radial observer on geodetics. We can generalize PG coordinates in Eq. (3.27) considering the same geometrical configuration as before, but with $\tilde{E} \neq 1$. In particular, we analyze a set of observers such that $\tilde{E}$ is equal for all of them.

Proceeding similarly as above, we define a new parameter $p = 1/\tilde{E}^2 = 1 - v_\infty^2$, with $0 < p \leq 1$, such that the new geodetics are:

$$\dot{t} = \frac{1}{\sqrt{p}\left(1 - \frac{2M}{r}\right)} \qquad\qquad \dot{r} = -\sqrt{\frac{1 - p\left(1 - \frac{2M}{r}\right)}{p}} \tag{3.30}$$

and the property (3.23) becomes:

$$u_\mu = -\frac{1}{\sqrt{p}} \partial_\mu T \tag{3.31}$$

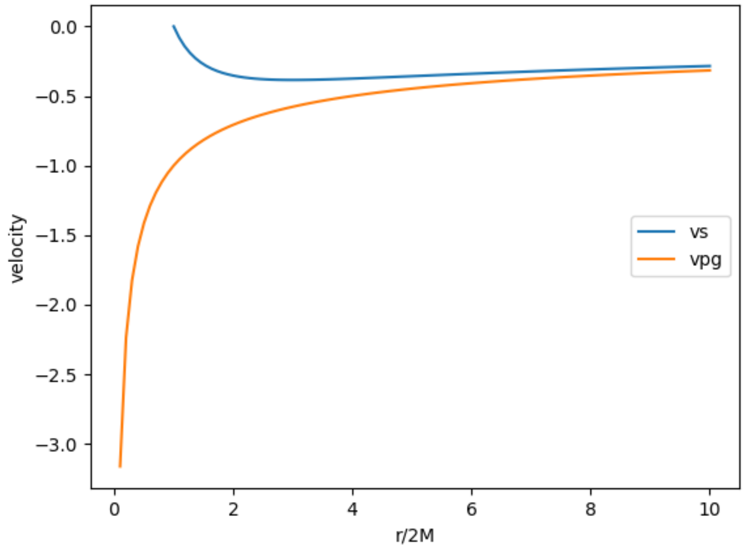

Figure 3.1: Speeds of the freely falling observer towards the horizon, expressed in natural units. In the legend, v_s is the speed in Schwarzschild coordinates, v_{pg} is the speed in PG coordinates.

with T, given now by the integral:

$$T = t + \int \frac{\sqrt{1 - p\left(1 - \frac{2M}{r}\right)}}{1 - \frac{2M}{r}} dr \tag{3.32}$$

which assumes the more complicated form:

$$\begin{aligned} T = t + 2M \Bigg\{ & \frac{r}{2M} \left[1 - p\left(1 - \frac{2M}{r}\right) \right] + \log \left| \frac{1 - \sqrt{1 - p(1 - 2M/r)}}{1 + \sqrt{1 - p(1 - 2M/r)}} \right| \\ & - \frac{1 - p/2}{\sqrt{1 - p}} \log \left| \frac{\sqrt{1 - p(1 - 2M/r)} - \sqrt{1 - p}}{\sqrt{1 - p(1 - 2M/r)} + \sqrt{1 - p}} \right| \Bigg\} \end{aligned} \tag{3.33}$$

Differentiating, we find the more general PG metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dT^2 + 2\sqrt{1 - p\left(1 - \frac{2M}{r}\right)} dT dr + p dr^2 + r^2 d\Omega \tag{3.34}$$

which is still regular in $r = 2M$. We notice that the constant surfaces $T = const$ are no more intrinsically flat, as there is a term p which generates a curvature and when $p \to 1$,

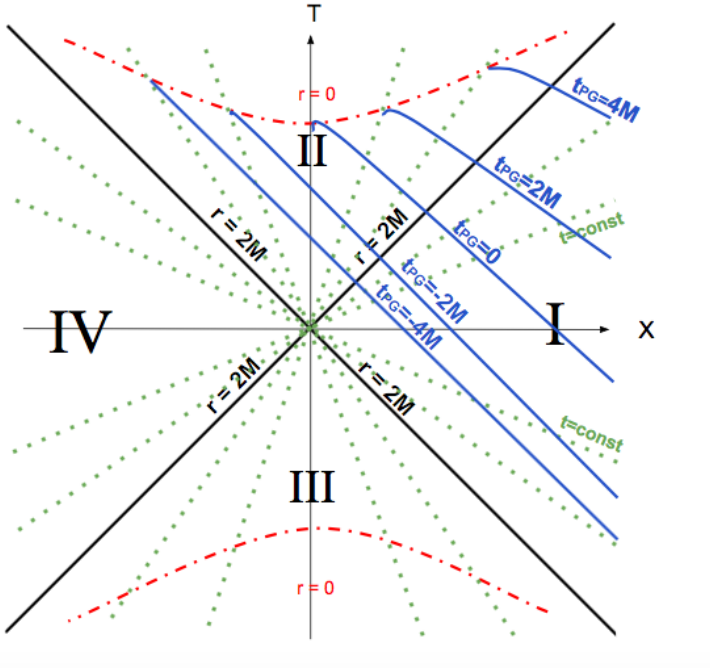

Figure 3.2: Kruskal diagram for a Schwarzschild black hole expressed in PG coordinates. In the sketch, t_{PG} is the PG time labeled as T in the main text.

the metric (3.34) reduces to the one in (3.27).

We can make one last generalization to other static and spherically symmetric spacetimes. Carrying on the same calculations as above, the generalized PG metric can be written as:

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)e^{2\psi}dT^2 + 2e^{2\psi}\sqrt{e^{-2\psi} - p\left(1-\frac{2M}{r}\right)}dTdr + \\ &\quad pe^{2\psi}dr^2 + r^2 d\Omega \end{aligned} \tag{3.35}$$

where ψ is an arbitrary function of r such that $\psi \longrightarrow 0$ when $r \longrightarrow \infty$ (asymptotically flat spacetime) and ψ non singular $\forall r \neq 0$.
The Kruskal diagram of the Schwarzschild black hole in PG coordinates is shown in Fig. 3.2. In the sketch, curves of constant r do not penetrate the horizon, only approaching it asymptotically. This does not happen in the PG case.

3.2 Radial infalling lattice model

We now proceed with a further modeling as regard the configuration of the freely falling scalar field. Namely, we treat the radial direction as a freely falling lattice [1].
Let us consider a Schwarzschild black hole in an asymptotically flat, static[1] space-time. Since we are dealing with a freely falling model, we take into account the Schwarzschild geometry in PG metric (3.27), rewritten in terms of $v = -\sqrt{\frac{2M}{r}}$, which represents the speed of the freely falling particle, i.e.:

$$ds^2 = -\left(1 - v^2(r)\right) dt^2 - 2v(r)dtdr + r^2 d\Omega^2 \tag{3.36}$$

being $\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and we have used the notation for which the PG coordinate time is $T \equiv t$. We recall that this construction is made from geodetics with $\tilde{E}^2 = 1$, at rest at infinity. We then introduce a new radial coordinate:

$$y = t - \int_{2M}^{r} \frac{dr'}{v\left(r'\right)} \tag{3.37}$$

which is constant along the geodetics, so it is constant along the worldlines $dr = v\left(r\right) dt$. Therefore, the metric becomes:

$$ds^2 = -dt^2 + v^2\left(r\right) dy^2 + r^2 d\Omega^2 \tag{3.38}$$

with:

$$r\left(y, t\right) = 2M \left[1 + \frac{3}{4M}\left(y - t\right)\right]^{\frac{2}{3}} \tag{3.39}$$

In this model, the horizon is at $y = t$ and the curvature singularity is at $y = t - \frac{4}{3}M$. The lattice is obtained by a discretization of y coordinate, in particular choosing a Plank scale lattice near the horizon. Far away from the horizon, the proper spacing falls below the Planck length, where we are not able to describe any physics, limiting the region of spacetime where the effective field theory description we have applied will be valid.
With this assumption, we are able to construct a theory with a local Lagrangian that can be treated in terms of semi-classical expansions, ignoring gravitational quantum effects. Usually, near the black hole horizon, an arbitrary frame is characterized by trans-Planckian energies with respect to the black hole rest frame, as we will discuss in the third part of the thesis. However, in a freely falling frame outside the black hole frequencies are typically much smaller than Planck scale, so we are able to construct a local field theory. In other words, we can build up quasi-local operators on scales λ in this frame, such that $M \gg \lambda \gg l_{planck}$. Thus, we will focus on near-horizon physics in the simple case of a Schwarzschild black hole in asymptotically flat spacetime. Furthermore, we will consider timescales short compared to the black hole lifetime, so that we model the background geometry as a static Schwarzschild spacetime.

[1] This is due to the fact that the considered timescales are short compared to the lifetime of a black hole.

Chapter 4

Scalar field dynamics

Given the mathematical configuration in the previous chapter, we are now able to expose the Classical Field Theory for a Klein-Gordon real, massive scalar field, as regard General Relativity.

4.1 Classical field theory

As stated from Classical field theory, the equation of motion for a real, massive scalar field $\Phi(x)$ can be obtained by the Action Principle. Here we use the notation in which $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. The Lagrangian is written as:

$$\mathcal{L}\left(\Phi, \partial_\mu \Phi\right) = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi - m^2\Phi^2\right) \tag{4.1}$$

being $g^{\mu\nu}$ the tensor of a manifold $\mathcal{M}$. The action is defined as:

$$\mathcal{S} = \int_{\mathcal{M}} d^4x \mathcal{L} \tag{4.2}$$

The Action Principle requires that the Action is unchanged under small variations of the field:

$$\begin{aligned} \Phi &\longrightarrow \Phi + \delta\Phi \\ \partial_\mu \Phi &\longrightarrow \partial_\mu \Phi + \delta\left(\partial_\mu \Phi\right) = \partial_\mu \Phi + \partial_\mu\left(\delta\Phi\right) \end{aligned} \tag{4.3}$$

The Lagrangian thus varies as:

$$\mathcal{L}\left(\Phi, \partial_\mu \Phi\right) \longrightarrow \mathcal{L}\left(\Phi + \delta\Phi, \partial_\mu \Phi + \partial_\mu \delta\Phi\right) \tag{4.4}$$

Since $\delta\Phi$ is assumed to be small, we can Taylor-expand the Lagrangian as follows:

$$\mathcal{L}\left(\Phi, \partial_\mu \Phi\right) + \frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)}\partial_\mu\left(\delta\Phi\right) \tag{4.5}$$

So the variation of the Action is:

$$\delta\mathcal{S} = \int_{\mathcal{M}} d^4x \left[\frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)}\partial_\mu\left(\delta\Phi\right)\right] \tag{4.6}$$

Integrating the second term by parts, $\delta\Phi$ factors out so that:

$$\begin{aligned}\int_{\mathcal{M}} d^4x \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)}\partial_\mu\left(\delta\Phi\right) = &-\int_{\mathcal{M}} d^4x \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)}\right]\delta\Phi \\ &+\int_{\mathcal{M}} d^4x \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)}\delta\Phi\right]\end{aligned} \tag{4.7}$$

The final term can be converted into a surface integral by Stokes's theorem, being a total derivative. Since we are dealing with a variational problem, we can choose to consider variations that vanish at the boundary, along with their derivatives, so ignore the last term as a boundary contribution. Therefore, Eq. (4.6) becomes:

$$\delta\mathcal{S} = \int_{\mathcal{M}} d^4x \left[\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)}\right)\right]\delta\Phi \tag{4.8}$$

Defining the variation of the action in terms of the functional derivative $\frac{\delta\mathcal{S}}{\delta\Phi}$:

$$\delta\mathcal{S} = \int_{\mathcal{M}} d^4x \frac{\delta\mathcal{S}}{\delta\Phi}\delta\Phi \tag{4.9}$$

and applying the Action Principle, we can state that the functional derivative vanishes, so that we obtain the Euler-Lagrange equation for a real scalar field:

$$\frac{\delta\mathcal{S}}{\delta\Phi} = \frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)}\right] = 0 \tag{4.10}$$

Applying the Euler-Lagrange equation to (4.1), we obtain the Klein-Gordon equation of motion for a real, massive scalar field in General Relativity:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi\right) - m^2\Phi = 0 \tag{4.11}$$

Notice that for a flat space-time, the Klein-Gordon equation reduces to the simpler form:

$$\left(\Box + m^2\right)\Phi = 0 \tag{4.12}$$

4.2 Massive scalar field equation

We can now write the equation of motion for a freely falling massive scalar field towards a black hole singularity. Considering the configuration of the radial infalling lattice model explained in Section 3.2, we develop the Euler-Lagrange equations (4.10), obtaining the Klein-Gordon equation of motion (4.11) for the particular geometric configuration of the lattice model described by the metric (3.38). The Lagrangian of the freely falling scalar field $\psi(t, y)$ in the lattice model is then:

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left[-\left(\partial_t\psi\right)^2 + \frac{1}{v^2\left(r\right)}\left(\partial_y\psi\right)^2 - m^2\psi^2\right] \tag{4.13}$$

where g is the determinant of the metric (3.38), $v(r) = \sqrt{\frac{2M}{r(t,y)}}$ so that v depends on (t, y) through r due to Eq. (3.39). The Euler-Lagrange equations of motion will then be:

$$\partial_t\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_t\psi\right)}\right] + \partial_y\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_y\psi\right)}\right] + \sqrt{-g}m^2\psi = 0 \tag{4.14}$$

i.e.:

$$\sqrt{-g}\left(\partial_t^2\psi\right) - \partial_t\left(\sqrt{-g}\right)\partial_t\psi + \frac{\sqrt{-g}}{v^2(r)}\partial_y^2\psi + \partial_y\left(\frac{\sqrt{-g}}{v^2(r)}\right)\partial_y\psi + \sqrt{-g}m^2\psi = 0 \tag{4.15}$$

which, explicitly, becomes:

$$\begin{aligned}-\partial_t^2\psi + \frac{1}{v^2(r)}\partial_y^2\psi + m^2\psi = &\left(\frac{1}{v(r)}\partial_t v(r) + \frac{2}{r(t,y)}\partial_t r(t,y)\right)\partial_t\psi \\ &+\left(\frac{1}{v(r)}\partial_y v(r) - \frac{2}{r(t,y)}\partial_y r(t,y)\right)\partial_y\psi\end{aligned} \tag{4.16}$$

Calculating explicitly the derivatives of v and r, we get to the final equation of motion in terms of t and y:

$$\left[-\partial_t^2 + \frac{1}{v^2\left(r(t,y)\right)}\partial_y^2 + m^2\right]\psi(t,y) = \frac{v\left(r(t,y)\right)}{2r(t,y)}\left[3\partial_t + 5\partial_y\right]\psi(t,y) \tag{4.17}$$

where we recall that t is the *PG* time This equation corresponds to the explicit form of Klein-Gordon equation of motion (4.11) for the metric (3.38).
The difficulty of the equation is represented not only by the presence of first order derivatives, but also by the term v which depends on (t, y) through r. Therefore, no solution can be found but an approximate scalar field which assumes the exponential form of some other fields.
We will focus our analysis on massless scalar fields. Solutions for Eq. (4.17) can be found

only approximately through the **Wentzel-Kramers-Brillouin(WKB)** method.
The scalar field solution of Eq. (4.17) can be written as the exponential form of a generic field $S(t, y)$, i.e.:

$$\psi(t, y) \sim e^{\frac{i}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(t,y)} \tag{4.18}$$

Substituting in (4.17):

$$\left[\left(\sum_{n=0}^{\infty}\delta^n\partial_t S_n\right)^2 + \delta\sum_{n=0}^{\infty}\delta^n\partial_t^2 S_n\right] + \frac{1}{v^2}\left[\left(\sum_{n=0}^{\infty}\delta^n\partial_y S_n\right)^2 + \delta\sum_{n=0}^{\infty}\delta^n\partial_y^2 S_n\right]$$
$$= \frac{\delta v}{2r}\left(3\sum_{n=0}^{\infty}\delta^n\partial_t S_n + 5\sum_{n=0}^{\infty}\delta^n\partial_y S_n\right) \tag{4.19}$$

Expressing the sum explicitly, we obtain:

$$-\left(\partial_t S_0 + \delta\partial_t S_1 + ...\right)^2 + \delta\partial_t^2 S_0 + \frac{1}{v^2}\left(\partial_y S_0 + \delta\partial_y S_1 + ...\right)^2 + \frac{\delta}{v^2}\partial_y^2 S_0$$
$$= \frac{\delta v}{2r}\left(3\partial_t S_0 + 3\delta\partial_t S_1 + 5\partial_y S_0 + 5\delta\partial_y S_1 + ...\right) \tag{4.20}$$

We shall now consider just the leading order, neglecting $o\left(\delta^2\right)$ terms, therefore:

$$-\left[\left(\partial_t S_0\right)^2 + 2\delta\partial_t S_0\partial_t S_1\right] + \delta\partial_t^2 S_0 + \frac{1}{v^2}\left[\left(\partial_y S_0\right)^2 + 2\delta\partial_y S_0\partial_y S_1\right] + \frac{\delta}{v^2}\partial_y^2 S_0$$
$$= \frac{\delta v}{2r}\left(3\partial_t S_0 + 5\partial_y S_0\right) \tag{4.21}$$

In the limit $\delta \to 0$, we obtain the well-known equation of transport:

$$\left(\partial_t - \frac{1}{v}\partial_y\right)S_0 = 0 \tag{4.22}$$

Although this equation may seem easy to solve, it is not possible to find an analytic solution as the term $v = v\left(r(t, y)\right)$ depends on (t, y) coordinates. However, we can make an additional approximation, considering a starting, fixed point $x_0^\mu = (t_0, y_0)$ from which the radiation falls towards the black hole. Thus the term $\frac{1}{v(t_0,y_0)} = \frac{1}{v_0}$ is constant and the equation:

$$\left(\partial_t - \frac{1}{v_0}\partial_y\right)S_0 = 0 \tag{4.23}$$

has a solution of the form:

$$S_0 = S_0\left(y \pm \frac{1}{v_0}t\right) \tag{4.24}$$

Defining the four vector $k^\mu_{(0)} = (\omega_0, k_0, 0, 0)$, so that $k_{(0)\mu} = g_{\mu\nu} k^\nu_{(0)} = (-\omega_0, v_0^2 k_0)$, we can write the action as:

$$S_0(t, y) = k_{(0)\mu} x^\mu = \left(-\omega_0 t \mp v_0^2 k_0 y\right) \tag{4.25}$$

being $x^\mu = (t, \mp y, 0, 0)$, where the $\mp$ defines if the radiation is outside (-) or inside (+) the horizon. Substituting (4.25) in (4.23), we obtain the dispersion relation:

$$\omega_0 = \pm v_0 k_0 \tag{4.26}$$

for which the frequency is positive outside the horizon, negative inside the horizon. Notice that (4.26) can be obtained also from the property of k^μ_0 to be light-like.

We can now suppose that the field makes an infinitesimal shift such that it reaches the point $x^\mu = x^\mu_0 + \delta x^\mu = (t_0 + \delta t, y_0 + \delta y)$. Therefore, $v = v\left(r_0 + \delta r\right)$ is no more constant and can be expressed in terms of δt and δy:

$$v\left(r_0 + \delta r\right) = v(t_0, y_0) + \partial_r v(t, y)\Big|_{(t_0, y_0)} \delta r\left(\delta t, \delta y\right) \tag{4.27}$$

Remembering that $v(t, y) = \sqrt{\frac{2M}{r(t,y)}}$, the derivative in the above equation is:

$$\partial_r v(t, y)\Big|_{(t_0, y_0)} = -\frac{1}{2r_0}\sqrt{\frac{2M}{r_0}} \tag{4.28}$$

from Eq. (3.39):

$$\delta r = \left[1 + \frac{3}{4M}\left(y_0 - t_0\right)\right]^{-1/3} \left(\delta y - \delta t\right) = \sqrt{\frac{2M}{r(t_0, y_0)}}\left[\delta y - \delta t\right] \tag{4.29}$$

The condition of δx^μ to be light-like gives the relation between δy and δt:

$$\delta y = \pm\sqrt{\frac{r_0}{2M}}\delta t \tag{4.30}$$

So, Eq. (4.27) becomes:

$$v\left(r_0 + \delta r\right) = \sqrt{\frac{2M}{r_0}} - \frac{M}{r_0^2}\left(-\sqrt{\frac{r_0}{2M}} \pm 1\right)\delta t \tag{4.31}$$

Making the same calculations used to find Eq (4.22), the new equation of transport is:

$$\left(\partial_t - \frac{1}{v\left(r_0 + \delta r\right)}\partial_y\right) S = 0 \tag{4.32}$$

The solution can be written in terms of the four-vector $k^\mu = k_0^\mu + \delta k^\mu = (\omega_0 + \delta\omega, k_0 + \delta k)$, i.e.:

$$S(t, y) = k_\mu x^\mu = S_0(t, y) + k_{(0)\mu}\delta x^\mu + \delta k_\mu x^\mu_{(0)} + o\left(\delta^2\right) = S_0(t, y) + S\left(\delta t, \delta y\right) \tag{4.33}$$

where S_0 is the solution of Eq. (4.23).

Therefore, the scalar field, approximate solution of Eq (4.17), is:

$$\psi(t, y) \sim e^{\frac{i}{\delta}(S_0(t_0, y_0) + \delta S(\delta t, \delta y))} \tag{4.34}$$

We can calculate the dispersion relation from the property of k^μ to be light-like:

$$k_\mu k^\mu = 0 = -\left(\omega_0 + \delta\omega\right)^2 + v^2\left(r_0 + \delta r\right)\left(k_0 + \delta k\right) = 0 \tag{4.35}$$

$$\left(\omega_0 + \delta\omega\right) = \pm v\left(r_0 + \delta r\right)\left(k_0 + \delta k\right) \tag{4.36}$$

Supposing that k varies much slower than ω in δt and using Eq. (4.26), Eq. (4.30) and Eq. (4.31), the final dispersion relation for an infinitesimal variation of ω_0 is:

$$\delta\omega = \frac{\omega_0}{2r_0}\left(1 \mp v_0\right)\delta t \tag{4.37}$$

However, (4.25) makes sense only if the radius value is considered very far away from the black hole horizon. Indeed, in a flat spacetime, (4.11) reduces to (4.12), which admits plane waves solution. This is no longer true near the horizon.

4.3 Numerical analysis

We focus the numerical analysis on Eq. (4.22). As we have already seen, the difficulty of this equation is mainly due to the term $v\left(r(t, y)\right) = -\sqrt{\frac{r_s}{r(t,y)}}$, depending on (t, y) through r. Specifically:

$$r(t, y) = r_s\left[1 + \frac{3}{2r_s}\left(y - t\right)\right]^{\frac{2}{3}} \Rightarrow v(t, y) = -\left[1 + \frac{3}{2r_s}\left(y - t\right)\right]^{-\frac{1}{3}} \tag{4.38}$$

Thus, if we want to carry on a numerical analysis with (t, y), we need a simplification of $y = y(t, r)$ in terms of $y = y(t)$. In other words, we suppose that y varies just with t. We may for example consider r in multiples of the Schwarzschild radius, i.e. $r = \alpha r_s$, with $\alpha \geq 1$, with $\alpha = 1$ meaning the object has reached the black hole event horizon. In this case:

$$y(t, r) = t + \frac{2r_s}{3}\left[\left(\frac{r}{r_s}\right)^{\frac{3}{2}} - 1\right] \Rightarrow y(t) = t + \frac{2r_s}{3}\left(\alpha^{\frac{3}{2}} - 1\right) \tag{4.39}$$

Therefore, $y \in [t, y(t)]$, being $y = t$ the event horizon. Substituting in (4.38), $v = -\alpha^{\frac{3}{2}}$. Nonetheless, this simplification means considering the whole problem as just varying in t. Indeed, the equation of transport reduces to:

$$\partial_t S = 0 \tag{4.40}$$

for which S does not depend upon t, i.e. $S(t, y) = constant$.
For a less straightforward analysis, we need to consider both variation in y and t. Let us bring the problem back to the usual Schwarzschild coordinates t_s, r. Remembering that the Painlevé-Gullstrand time is linked to the Schwarzschild time by (3.25)[1], we can calculate the differential in terms of r:

$$dy = dt + \sqrt{\frac{r}{r_s}} dr = dt_s + \sqrt{\frac{r}{r_s}} \frac{r}{r - r_s} dr \tag{4.41}$$

Therefore, the equation to solve becomes:

$$\left[\partial_{t_s} + \frac{r_s \sqrt{r} + r\sqrt{r_s}}{\left(\sqrt{r_s} + \sqrt{r}\right)(r - r_s)} \right] S = 0 \tag{4.42}$$

with $r \in [r_s, \infty]$. Thus, the equation to solve is of the form:

$$\partial_t U(r, t) - f(r)\partial_r U = 0 \tag{4.43}$$

In order to solve it numerically, we shall discretize it both in r and t:

$$\partial_t U = \frac{U_j^{k+1} - U_j^k}{\Delta t} \tag{4.44}$$

$$\partial_r U = \frac{U_{j+1}^k - U_j^k}{\Delta r} \tag{4.45}$$

where the index k runs over all the values of the t coordinate, while j runs over all the values of r. The discretized equation of transport then becomes:

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} - f(r)\frac{U_{j+1}^k - U_j^k}{\Delta r} = 0 \Longrightarrow U_j^{k+1} = U_j^k + \frac{\Delta t}{\Delta r} f(r) \left(U_{j+1}^k - U_j^k\right) \tag{4.46}$$

and calling $\frac{\Delta t}{\Delta r} f(r) = s$:

$$U_j^{k+1} = (1 - s)U_j^k + sU_{j+1}^k \tag{4.47}$$

[1] There we used the notation for which T was the PG time, t the Schwarzschild time, while in the actual notation t is the PG time, t_s the Schwarzschild time.

This equation can be rewritten in terms of a matrix whose indexes span the whole grid of (k, j). Suppose to consider a 3×3 grid, so with $k, j = 0, 1, 2$ thus a 3×3 matrix. The above equation takes the matrix form:

$$\begin{pmatrix} U_0^{k+1} \\ U_0^{k+1} \\ U_0^{k+1} \end{pmatrix} = \begin{pmatrix} (1-s) & s & 0 \\ 0 & (1-s) & s \\ 0 & 0 & (1-s) \end{pmatrix} \begin{pmatrix} U_0^k \\ U_1^k \\ U_2^k \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ U_3^k \end{pmatrix} \tag{4.48}$$

The final equation to compute has the following vectorial form:

$$\vec{U}^{k+1} = A\vec{U}^k + \vec{b} \tag{4.49}$$

If we left (t, y) as variables, the result would be exactly the same, with the only difference that $s = \frac{1}{v(t,y)} \frac{\Delta t}{\Delta y}$.

In order to implement the code, we need now to give the initial conditions $U(r, 0)$, $U(r_\infty, t)$ and $U(r_s, t)$. If we know the form of the field at infinity, i.e. a plane wave so that $U(r_\infty, t) = (-\omega t + kr)$, we can not give any consistent form of the field near the event horizon. In fact, if at infinity the vacuum state is well defined and it is represented as a plane wave, it is not possible to define a unique vacuum state near the event horizon. We will underline this important feature in the next chapters.

Therefore, the numerical analysis is limited by the unknown initial conditions. Without any consistent form on the field, we are not able to give any trend of the action, so of the final field ψ.

4.4 Lattice discretization

A more consistent analysis can be done if we discretize the lattice [8]. We consider a massless scalar field falling on the lattice. The action of the field in terms of t, y is given by:

$$S = \frac{1}{2} \int dydt\sqrt{-g} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi = \frac{1}{2} \int dtdy \left[|v(r)| (\partial_t \psi)^2 - \frac{1}{|v(r)|} (\partial_y \psi)^2 \right] \tag{4.50}$$

Applying once again the Euler-Lagrangian equation and using the fact that $\partial_t v = v \partial_r v$ and $\partial_y v = -v \partial_r v$, we obtain:

$$-\partial_t^2 \psi + \frac{1}{v^2} \partial_y^2 \psi - (\partial_r v) \partial_t \psi + \frac{\partial_r v}{|v|} \partial_y \psi = 0 \tag{4.51}$$

We shall now discretize the y coordinate such that the action is rewritten as:

$$\begin{aligned} S &= 2\pi \sum_y \int dt r^2(t, y) \left[|v(r(t, y))| (\partial_t \psi)^2 - \frac{2 (\psi(t, y+1) - \psi(t, y))^2}{|v(r(t, y+1)) + v(r(t, y))| \delta} \right] = \\ & \quad 2\pi \sum_y \int dt r^2(t, y) \left[|v(r(t, y))| (\partial_t \psi)^2 - \frac{2 (D_y \psi)^2}{|v(r(t, y+1)) + v(r(t, y))|} \right] \end{aligned} \tag{4.52}$$

being δ the lattice spacing that we may normalize to 1 and $D_y\psi = \psi(t, y+1) - \psi(t, y)$. Differentiating the action, i.e. applying the variational principle, we obtain the equation of motion:

$$\partial_t(v(r(t,y))\partial_t\psi(t,y)) - D_y\left[\frac{2D_y\psi}{v(r(t,y)) + v(r(t,y-1))}\right] = 0 \tag{4.53}$$

The particular feature of this discretization is that the action shows a symmetry of the form:

$$(t, y) \longrightarrow (t+1, y+1) \tag{4.54}$$

Thus, we can write an exponential form of the scalar field separating the temporal part with the radial part:

$$\psi(t, y) = e^{-i\omega t} f(y - t) \tag{4.55}$$

where $\omega \in \mathcal{R}$, while k is such that the function is invariant under $k \to k + 2n\pi$, $\omega \to \omega - 2n\pi$, being n an integer. This invariance allows us to build up a one-to-one map of (ω, k) such that:

$$-\pi < k \leq \pi \qquad -\infty < \omega < +\infty \tag{4.56}$$

Using once again the WKB approximation, the field follows an exponential function:

$$\psi(t, y) = e^{-i(\omega + k(r))t} e^{ik(r)y} \tag{4.57}$$

where we have assumed the wave vector k to depend on the radial position. If we plug this field into (4.53) and assuming that $k(r)$ varies slowly in the chosen scale, we finally obtain the dispersion relation:

$$|v|(\omega + k) = \pm 2\sin(k/2) \tag{4.58}$$

where the signs determine whether the mode is right(+) or left(-) moving. This is shown in Fig. 4.1. In the graphic, straight lines are the left hand side of ∼(4.58) for different values of r, i.e. for different values of v and ω chosen with a positive value. Specifically, the green line stands for deep inside the horizon, the violet one on the horizon and the red one outside the horizon. The intersection of lines and the sinusoidal curve are the solutions of (4.58) with positive frequency ω. We note that far away from the horizon(red line), there are four intersections, thus four solutions, which we shall indicate as k_+, k_- for the solution out of the lattice cutoff defined in (4.56), k_{-s}, k_{+s} for those in the domain close to the axes origin. In particular, k_{-s} gives rise to left moving while the other three give rise to right moving modes. From this analysis we see that this lattice discretization gives rise to infinite modes in (ω, k) space, including modes behind the horizon. This happens because we used a semiclassical field theory, while using a complete quantum theory there would have been some kind of discretization resulting in a finite number of states behind the horizon. However, this is not relevant for a freely falling observer behind the horizon, because her lifetime will be maximum of the order of the light-crossing

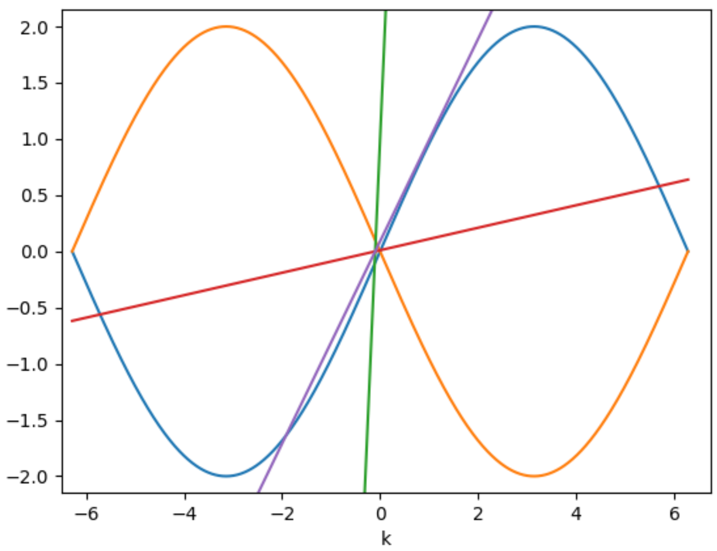

Figure 4.1: Dispersion relation for the scalar field.

time of the black hole.
Consider now an outgoing wavepacket centered on a small positive wavevector, k_{+s} represented in Fig. 4.1. In that dispersion curve, following this wavepacket back in time, we see that it reaches a tangent point to the sinusoidal curve, which we may call k_{tp}. In this point, the group velocity of the wavetrain (4.57)[8]:

$$v_g = \frac{d\omega}{dk} = \pm\frac{\cos(k/2)}{|v|} - 1 \tag{4.59}$$

vanishes, which means that our WKB approximation is no longer valid. This is also called the classical turning point. We notice that in the figure, the wavepacket tunnels beyond the classical turning point, but it is not propagating there. Moreover, for $\omega << 1$, the straight line is very close to the sine curve for many k values, also the negative ones. The positive and negative wavevectors both propagate away from the horizon, evolving into the modes k_+ and k_- respectively. Thus, we can see this in the prospective for which ingoing wavepackets with high wavevectors bounce off the horizon, creating outgoing modes with low wavevectors. The negative wavevectors k_{-s} determine the so-called Hawking radiation, which we will study in detail in the next part.

Chapter 5

Gravitational redshift

In the previous chapter, we have analyzed the motion of a freely falling scalar field towards the black hole. We underlined the difficulties in obtaining a direct form of the field from the equation of motion. As we have not been able to solve the dynamic of the field either analytically or numerically and in order to understand how the field actually behaves approaching the event horizon, we will discuss the property of the Redshift for a freely falling scalar field on the exterior region of the black hole, namely $r > 2M$, through a kinetic analysis of a single mode motion in a classical, static spacetime. The kinetic analysis simplifies our problem for which we can not extrapolate the frequency from the field.
Most experimental tests of general relativity involve the motion of test particles in the solar system, and hence geodesics of the Schwarzschild metric. The exception is the gravitational redshift of emission lines, which is measured in distant objects. In particular, Einstein suggested three tests of general relativity:

- The gravitational redshift of spectral lines;
- Deflection of light by the Sun;
- Precession of the perihelia of the inner planets;

In our work we will focus on the analysis of the Gravitational Redshift. Before starting our discussion on the specific case of an infalling massless scalar field, we need to introduce the concept of Gravitational Redshift. We start from the definition of proper time in general relativity [14]:

$$dT = \sqrt{-ds^2} \equiv \sqrt{-g_{\mu\nu}\left(x^\mu\right) dx^\mu dx^\nu} \tag{5.1}$$

where the metric is calculated in the space-time position of the considered object, such as a clock measuring time. Considering the simpler case when the clock is at rest with respect to a chosen reference frame, (5.1), which can in this configuration be considered as the proper time interval between two clock ticks, assumes the form:

$$dT = \sqrt{-g_{00}\left(x^{\mu}\right)}dx^0 = \sqrt{-g_{00}\left(x^{\mu}\right)}dt \tag{5.2}$$

dt being the coordinate time between two ticks.
Let us consider a source emitting light in a strong gravitational field like that of a neutron star or a black hole from the reference frame of a different observer. The interval of the proper time ΔT_{em} is the period of the light wave, i.e. the interval between the emission of two successive wave crests, so for Eq. (5.2):

$$\Delta T_{em} = \sqrt{-g_{00}\left(x^{\mu}_{em}\right)}\Delta t_{em} \tag{5.3}$$

So the emission frequency is:

$$\nu_{em} = \frac{1}{\Delta T_{em}} = \frac{1}{\sqrt{-g_{00}\left(x^{\mu}_{em}\right)}\Delta t_{em}} \tag{5.4}$$

The proper time of the observer, ΔT_{obs}, is the interval between the detection of two successive wave crests, so:

$$\Delta T_{obs} = \sqrt{-g_{00}\left(x^{\mu}_{obs}\right)}\Delta t_{obs} \tag{5.5}$$

and the frequency:

$$\nu_{obs} = \frac{1}{\Delta T_{obs}} = \frac{1}{\sqrt{-g_{00}\left(x^{\mu}_{obs}\right)}\Delta t_{obs}} \tag{5.6}$$

Now, if the metric is time independent and if the source and the observer are at rest with respect to each others, the interval of time Δt_{em} between two successive emitted wave crests will be the same as the observed one. Therefore:

$$\frac{\nu_{obs}}{\nu_{em}} = \sqrt{\frac{g_{00}\left(x^{\mu}_{em}\right)}{g_{00}\left(x^{\mu}_{obs}\right)}} \tag{5.7}$$

If the metric is that of a Schwarzschild black hole described in Eq. (2.10) and assuming the observer is at infinity, the explicit form of (5.7) is:

$$\frac{\nu_{obs}}{\nu_{em}} = \sqrt{\frac{1-2M/r_{em}}{1-2M/r_{obs}}} \sim \sqrt{1-\frac{2M}{r_{em}}} \tag{5.8}$$

The typical trend is shown in Fig. 5.1. The explicit shift between the observed and the emitted frequencies is usually written as the following ratio:

$$\frac{\Delta\nu}{\nu} = \frac{\nu_{obs}-\nu_{em}}{\nu_{em}} \tag{5.9}$$

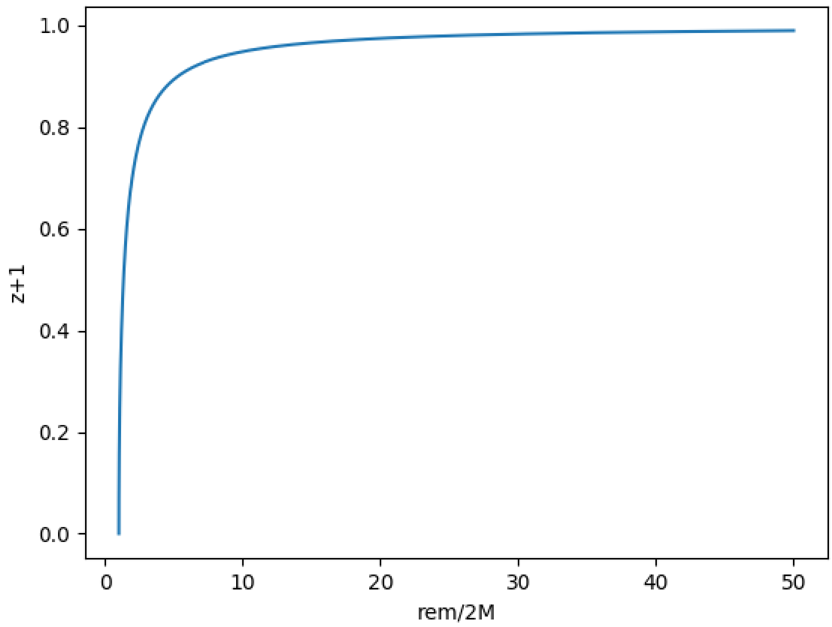

Figure 5.1: Redshift of the light falling towards a Schwarzschild black hole.

For example, in case of a source on a neutron star, the shift is given by $\frac{\Delta\nu}{\nu} \cong -0.24$. As it is negative, the observer at infinity will measure a red-shifted frequency. In case of a black hole, we notice that the observed frequency approaches zero, as the emitting source approaches to the event horizon at $r = 2M$. This physically means, coherently with the property of the black holes information, that once the signal reaches the event horizon, there is no chance for the outside observer to catch it anymore.

5.1 Single radiation mode

We now analyze the field described by Eq. (4.17). As the form of the scalar field is unknown, and consequently the frequency emitted is unknown, which causes the equation not to be analytically solvable, we need to build up a simplified configuration in which one oscillation of the radiation is considered. The configuration can be visualized in Fig. 5.2. In the sketch, we see that in the radius axis, r_1 is the position at which the single oscillation starts. The emission time as measured from the mode reference frame is τ_1, i.e. the proper emission time. At a proper time τ_2, the oscillation reaches the position r_2, so that the proper period of the wave will be $\tau = \tau_2 - \tau_1$. An observer based at infinity,

r_∞ in the sketch, will measure a corresponding period $T = t_2 - t_1$. The two frames are related by the geodesic equation, namely we can express the proper period τ as a function of the period T measured by the asymptotic observer. It is convenient to express this

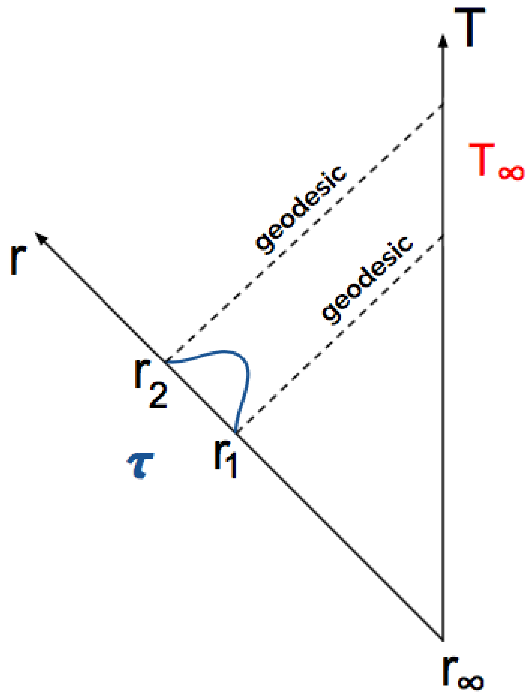

Figure 5.2: Configuration for the analysis of the redshift of a single mode.

relation in the Schwarzschild coordinates. Remembering that we are analyzing a massless scalar field, we can impose null-metric condition in the Schwarzschild coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 = 0 \Longrightarrow dt = dr\left(1 - \frac{2M}{r}\right)^{-1} \tag{5.10}$$

Integrating:

$$\int_{\tau_2}^{t_2} dt - \int_{\tau_1}^{t_1} dt = \int_{r_2}^{r_\infty} \frac{rdr}{r-2M} - \int_{r_1}^{r_\infty} \frac{rdr}{r-2M} \tag{5.11}$$

we obtain the relation between T and τ:

$$T = \tau + r_1 - r_2 + 2M \log\left(\frac{r_1 - 2M}{r_2 - 2M}\right) \tag{5.12}$$

We can implement this operation always shifting the interval $r_1 - r_2$ along the radial coordinate, approaching the event horizon and keeping the proper period constant. In

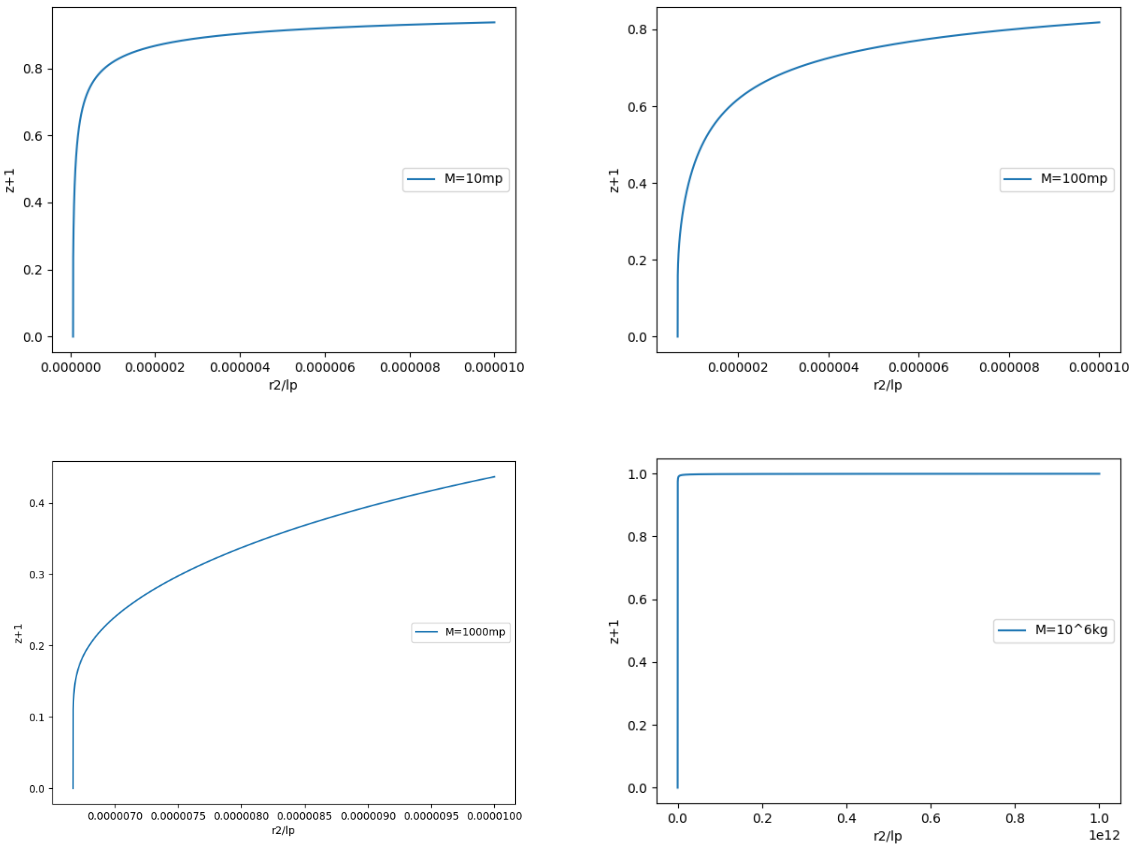

Figure 5.3: Redshift for different values of the black hole mass.

other words, we calculate the period T_i with r_{2i} and r_{2i} approaching the event horizon as i increases. Therefore:

$$T_i = \tau + r_{1i} - r_{2i} + 2M \log\left(\frac{r_{1i} - 2M}{r_{2i} - 2M}\right) \tag{5.13}$$

The frequency of the emitted radiation is $\nu^{em} = \frac{1}{\tau}$, while that of the observed is $\nu^{obs} = \frac{1}{T}$, so the implemented redshift will be:

$$z_i = \frac{\tau}{\tau + r_{1i} - r_{2i} + 2M \log\left(\frac{r_{1i}-2M}{r_{2i}-2M}\right)} - 1 \tag{5.14}$$

The proper period τ depends on r_1 and r_2. As $r_1 - r_2$ is the wave length of the mode, which falls towards the black hole with speed $v(r) = -\sqrt{\frac{2M}{r}}$, the infinitesimal variation of the wave length will be:

$$dr = v(r)d\tau \Longrightarrow \tau = -\int_{r_1}^{r_2} dr\sqrt{\frac{r}{2M}} = \frac{2}{3\sqrt{2M}}\left(r_1^{3/2} - r_2^{3/2}\right) \tag{5.15}$$

Being the Schwarzschild radius $r_s = 2M$, the final formula for the redshift is:

$$z_i = \frac{2}{3} \cdot \frac{r_{1i}^{3/2} - r_{2i}^{3/2}}{r_{1i}^{3/2} - r_{2i}^{3/2} + \sqrt{r_s}r_{1i} - \sqrt{r_s}r_{2i} + r_s\sqrt{r_s}\log\left(\frac{r_{1i}-r_s}{r_{2i}-r_s}\right)} - 1 \tag{5.16}$$

For i such that $r_2 = r_s$ and remembering that in the chosen configuration we can express r_1 as a multiple of r_2, i.e. $r_1 = \alpha r_2$, $\alpha > 1$, $z = -1$, so $\nu^{obs} = 0$, which means that when the signal approaches the event horizon, the outside observer does not receive signal anymore. In other words, the radiation has no problem to reach the event horizon. But an observer far away would never be able to tell. If we stayed outside while an observer dove into the black hole, sending back signals all the time, we would simply see the signals reach us more and more slowly. As infalling observer approaches $r = 2M$, any fixed interval of her proper time corresponds to a longer and longer interval from our point of view. This continues forever, as we would never see the observer cross the event horizon, we would just see her more and more slowly and become redder and redder. Fig. 5.3 shows the behavior of the redshift for different masses of the black hole.

5.2 Fluctuating horizon

In physics we do not deal with precise value of the analyzed quantities. Thus, we should consider that, instead of a unique value, the mass of the black hole is distributed along a fixed value which shall represent its expectation value. Specifically, we let the Schwarzschild radius follow a gaussian distribution with a certain standard deviation, for two different values of the mass, i.e. for two different values of the Schwarzschild radius. Besides, we suppose r_1 and r_2 to be fixed values, multiples of the expectation value of R_s.

Concurrently, we expect the redshift to follow a gaussian distribution too along its expectation value.

Our statistical analysis starts from a Planck rescaling the variables, i.e. we have expressed the radius and the mass both in Planck units, for which the rescaled Schwarzschild radius is:

$$R_s \equiv \frac{r_s}{l_P} = \frac{2M}{m_P} \tag{5.17}$$

The values of Planck length and Planck mass are reported in the List of Symbols at the beginning of the thesis[1].

[1] The values have been taken from `https://physics.nist.gov/cuu/Constants/index.html`

In order to analyze numerically the distribution of the Schwarzschild radius, we need to give as input the standard deviation of R_s, which can be computed from the errors propagation formula:

$$\sigma(f(x_i)) = \left[\sum_i \left(\frac{\partial f}{\partial x_i} \right)^2 \right]^{1/2} \tag{5.18}$$

which for the Schwarzschild radius takes the form:

$$\delta R_s = \sqrt{\left(\frac{\partial R_s}{\partial M} \delta M \right)^2 + \left(\frac{\partial R_s}{\partial m_P} \delta m_P \right)^2} \tag{5.19}$$

The first value of the mass we have chosen is $M_1 = (4.595 \pm 0.005) \times 10^{17}$ kg, such that $R_s(M_1) = (9.189 \pm 0.009) \times 10^{17}$. The second value of the mass is $M_2 = (1.000 \pm 0.001) \times 10^4$, so that the Schwarzschild radius is $R_s(M_2) = (2.000 \pm 0.002) \times 10^4$. All the values are expressed in Planck units.
Fig. 5.4 shows their trend for a mode near the event horizon. We see that in both cases, when the mode is near the event horizon, i.e. when r_2 approaches the event horizon, the signal begins to disappear for the observer at infinity. However, this characteristic is not so well verified, as the redshift is not dramatically close to zero value. This is obvious considering that while the horizon is fluctuating, the radial position of the single mode has a fixed value, specifically we must choose r_2 in the logarithm domain, so:

$$r_2 > max(R_s), \quad r_2 < r_1 \tag{5.20}$$

In the numerical analysis we chose r_1 at a distance $2R_s$ from the horizon, while r_2 infinitely close to the horizon. With this configuration, we have been able to build up a self-consistent theory, in which from a gaussian fluctuation of the black hole mass, we have obtained a gaussian fluctuation of the redshift, spanning over different values of the black hole mass. We actually observed that the signal decreases as approaching the event horizon, thus the above statistical analysis seems to be in accordance with the property stating that any signal is lost past the event horizon.
If, instead of a distribution of r_s, we consider a fluctuation of the wave length of the mode, letting r_2 follow a gaussian distribution, we expect the redshift to have a gaussian distribution too. This result is shown in Fig. 5.5 for the same mass values as the previous analysis. As in the previous case, the redshift is not dramatically close to zero once r_2 reaches the event horizon for the same numerical reason as for fluctuating horizon.

More general configurations can be constructed starting from our analysis. For example, we could have considered quantum fluctuation of the field in a classical spacetime. This is an approximation we are usually forced to make, as we still lack of a consistent quantum gravitational field theory dealing with quantum fields in quantum spacetimes. However, despite the classical spacetime allows the dynamics of both matter and spacetime to be solved self-consistently, quantum fields bring a modification in the above analysis, as the

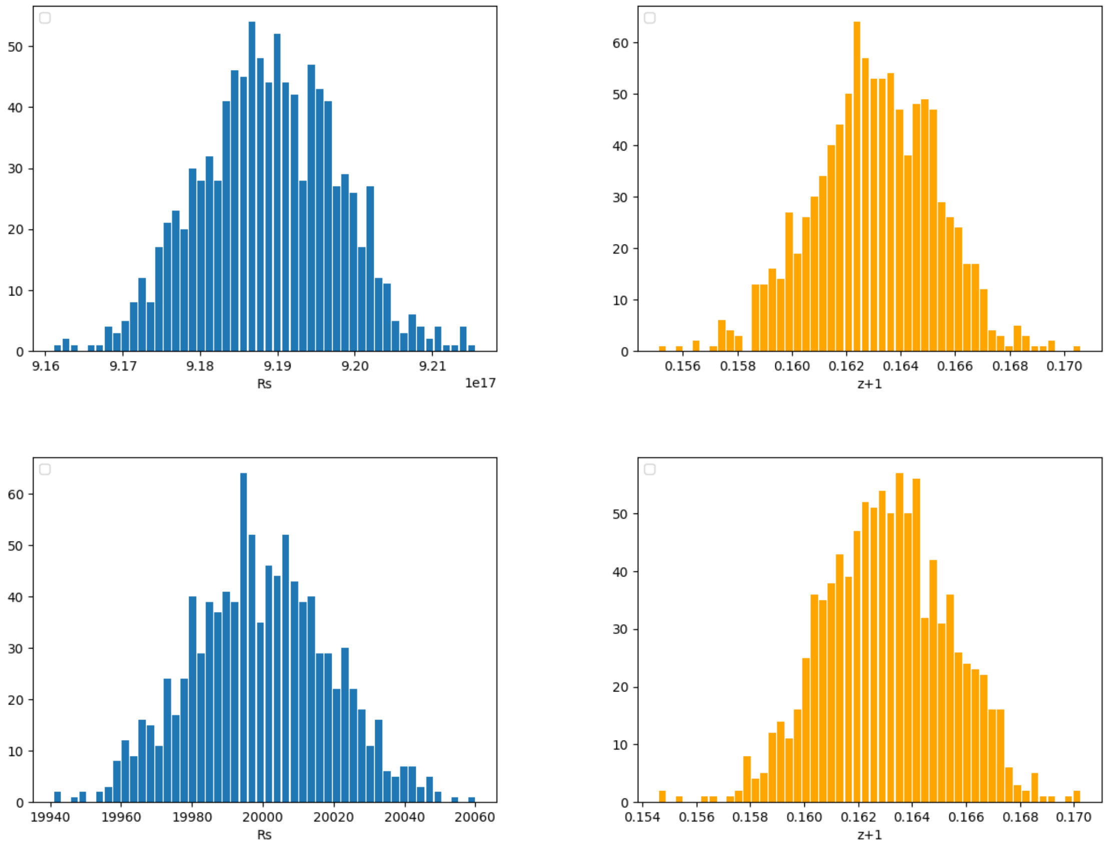

Figure 5.4: Distribution of Schwarzschild radius (left) and redshift (right) for a black hole with mass values M_1(top), M_2 (bottom) for a mode near the event horizon.

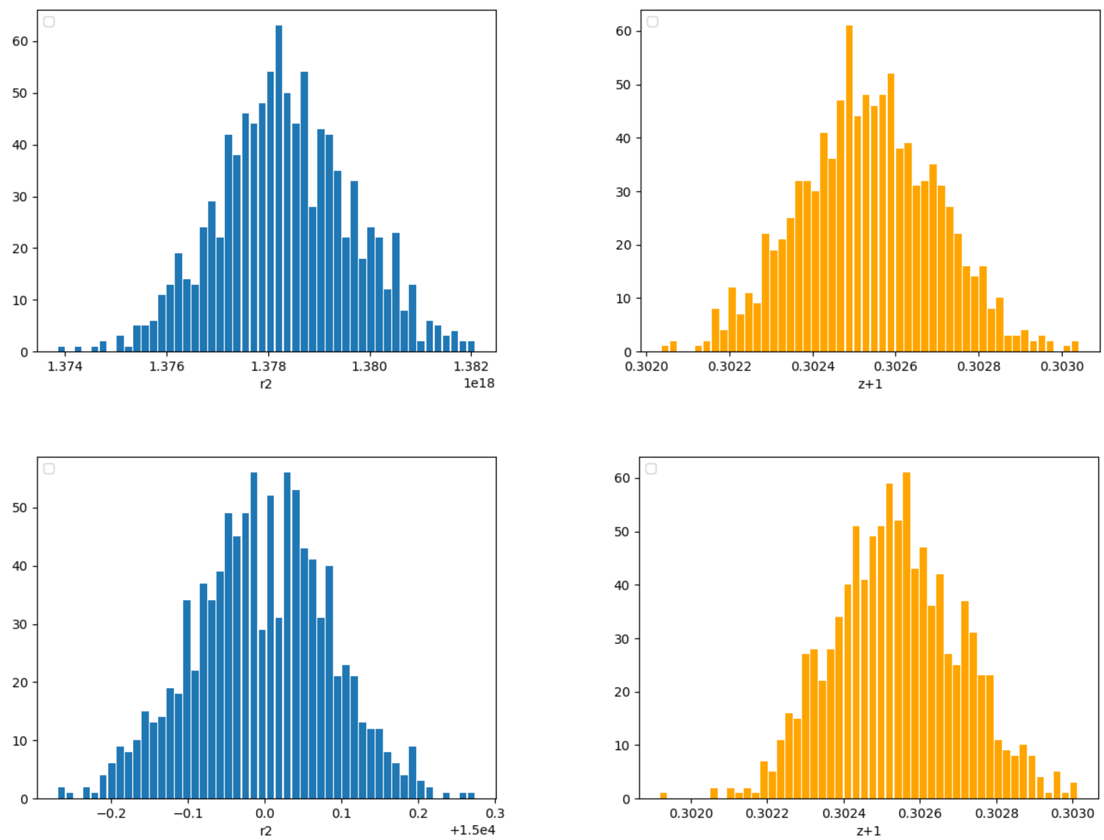

Figure 5.5: Distribution of r_2 (left) and redshift (right) for a black hole with average mass M_1 (top), M_2 (bottom) for a mode near the event horizon.

Einstein field equations describing the effect of gravitational field on the geometry of the spacetime need to be corrected. The semiclassical Einstein field equations are quite different from those written in Eq. (2.8), as they contain additive terms both on left and right side. In particular, the source of the field in a manifold $(\mathcal{M}, g)$ is no more given by the classical stress-energy tensor $T_{\mu\nu}$, but from the expectation value of the stress-energy tensor operator $\left\langle \hat{T}_{\mu\nu}[g] \right\rangle$. For a free quantum field, this theory is self-consistent and fairly well understood. As long as the gravitational field is assumed to be described by a classical metric, the semiclassical Einstein equations seems to be the only plausible dynamical equations for this metric. The stress-energy tensor couples the metric with the matter field and for a given quantum state the only physically observable stress-energy tensor that one can construct is the renormalized expectation value. Nonetheless, the theory is expected to break down when the fluctuations of $\left\langle \hat{T}_{\mu\nu}[g] \right\rangle$ are large. Thus, an extension of the semiclassical theory is needed in order to account for these fluctuations in a self-consistent way. Stochastic gravity is actually carrying on this work [6], through the definition of a physical observable that describes these fluctuations to lowest order, called the noise kernel bi-tensor, which is defined through the two-point correlation of the stress-energy operator. The starting point of the Stochastic gravity is however the well-defined and well-understood theory of semiclassical gravity.
A further, non-trivial request of fluctuating horizons analysis may be to see what happens if the quantum field is acting on a quantum spacetime. In this case, we have to face with the lack of a full consistent quantum field theory of gravity. Gambini et al. [4] chose a full quantum model consisting of a collapsing thin shell in a quantum spacetime, where the quantum spacetime is constructed by a superposition of the classical geometry associated with collapsing shell (Schwarzschild outside the shell, Minkowski inside the shell) with uncertainty in its position and mass. Specifically, the mass and position are supposed to follow a Gaussian distribution. These oscillations carries on many consequences, mostly concerning the Hawking effect described in Section 4.4, showing that, differently from a semiclassical analysis, a full quantum treatment may allow us to retrieve the information about the initial state of the collapsing shell. We will discuss Hawking radiation and black hole information theory in the next part of the thesis.

Part III

Black holes and information

Chapter 6

Entropy and information

In the previous part we have analyzed the motion of a falling radiation towards the black hole. We have found that from the equation of motion, it is possible to obtain an approximate solution. The corresponding wavepacket analysis brought us to the definition of a particular radiation out of the event horizon, called the Hawking radiation. However, the unknown exact form of the scalar field brought us to analyze the redshift. We have noticed that approaching the event horizon, the information detected from an observer at infinity was minimized, until the observer was not able to detect any more information once the radiation reached the event horizon. However, we used a simplified model, where we studied the radial ingoing motion and redshift distribution of a single mode of radiation in a static, classical spacetime. No quantum effect were taken into account. We may build up a more general configuration of fluctuating horizons where quantum fluctuations are non negligible. A full quantum treatment arises the problem of information retrieval in black holes, one of the biggest questions concerning the Hawking effect and quantum black holes theory, that we will expose in this last part of the thesis [17].

6.1 Entanglement, Von Neumann entropy and thermal entropy

Let us consider two non-interacting systems A and B, with $\mathcal{H}_A$ and $\mathcal{H}_B$ respective Hilbert spaces. The composite system's Hilbert space is given by the product $\mathcal{H}_A \otimes \mathcal{H}_B$. Defining a basis $|\psi_i\rangle_A$ for $\mathcal{H}_A$ and a basis $|\psi_j\rangle_B$ for $\mathcal{H}_B$, the most general state in $\mathcal{H}_A \otimes \mathcal{H}_B$ is:

$$|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |\psi_i\rangle_A \otimes |\psi_j\rangle_B \tag{6.1}$$

This state is said to be *separable* if there exists two vectors $[c_i^A], [c_j^B]$ such that $c_{ij} = c_i^A c_j^B$, so that $|\psi\rangle_A = \sum_i c_i^A |\psi_i\rangle_A$ and $|\psi\rangle_B = \sum_i c_j^B |\psi_j\rangle_B$. If for any vectors $[c_i^A], [c_j^B]$ such that for at least one coordinate we have $c_{ij} \neq c_i^A c_j^B$, the state can not be separated and it is called *entangled* state.

If, instead of single states (*pure* states), we have less information about the system, that is called ensemble (*mixed* states) and it is described by a density matrix ρ, a positive semi-definite operator with trace 1. The general form of the matrix is:

$$\rho = \sum_i p_i |\alpha_i\rangle\langle\alpha_i| \tag{6.2}$$

being p_i the probabilities (sum up to 1) and $|\alpha_i\rangle$ the states of the system with proportion p_i.

Extending the definition of separability as for the pure state, a mixed state is separable if it can be written as:

$$\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B \tag{6.3}$$

where p_i are positive probabilities and ρ_i^A, ρ_i^B are mixed states of subsystems A and B respectively. So a state is separable if it represents a probability distribution over uncorrelated states. If A and B are not separable, they are said to be entangled.
If we want the density matrix of the single subsystem A or B, we calculate the so-called reduced density matrix, given by:

$$\rho_A = Tr_B \rho_{AB} \tag{6.4}$$

From the density matrix, we can calculate the Von Neumann Entropy or the Entangled Entropy, defined as:

$$S = -Tr\rho \log \rho = -\sum_i \rho_i \log \rho_i \tag{6.5}$$

For a pure state, where all the eigenvalues except one vanish, $S = 0$. For mixed states, $S > 0$.
Physically, The Von Neumann Entropy measures the number of dominant states in a statistical ensemble, so states with an appreciable probability. It is also called Entangled entropy because it also measures the degree of entanglement between two subsystem A and B.
In case of a total incoherent[1] mixed density matrix, the Von Neumann Entropy gets the maximum value $S_{max} = \log N$.
Consider now a total system Σ divided into many subsystems Σ_i. Assume that Σ is

[1]The system is said to be coherent if there exists a definite phase relation between different states.

described by a pure state with vanishing entropy and that the subsystems Σ_i are thermal and described by a Boltzmann distribution density matrix:

$$\rho_i = \frac{e^{-\beta H_i}}{Z_i} \tag{6.6}$$

being H_i the Hamiltonian and Z_i the partition function of the subsystem Σ_i. This distribution maximizes the entropy. The thermodynamic entropy of the total system will be:

$$S_{therm} = \sum_i S_i \tag{6.7}$$

An important property of the thermodynamic entropy of a subsystem Σ_1 is the fact that it is always larger than its entanglement entropy:

$$S_{therm}\left(\Sigma_1\right) = \sum_i S_i \geq S\left(\Sigma_1\right) = -Tr\rho\log\rho \tag{6.8}$$

This because $S_{therm} \geq 0$ while as $\Sigma_1 \to \Sigma$, the entanglement entropy approaches zero.

6.2 Information conservation principle

Given a subsystem σ_1, the amount of information in the subsystem is defined as the difference between thermal and Von Neumann entropy:

$$I = S_{therm}\left(\sigma_1\right) - S\left(\sigma_1\right) = \sum_i S_i + Tr\rho\log\rho \tag{6.9}$$

If σ_1 is the total system, $S = 0$, thus the information comes from the thermal entropy. While, if the subsystem is very small, $S_{therm} = S \Longrightarrow I = 0$. It can be proved that this is true for all subsystem smaller than one half of the total system. In particular, assume that the total system σ is made of two subsystems, σ_1 and $\sigma - \sigma_1$. The Von Neumann entropies are equal and the amount of information in the two subsystem is:

$$I\left(\sigma_1\right) = S_{therm}\left(\sigma_1\right) - S\left(\sigma_1\right) \qquad I\left(\sigma - \sigma_1\right) = S_{therm}\left(\sigma - \sigma_1\right) - S\left(\sigma - \sigma_1\right) \tag{6.10}$$

We have two cases:

- if $\sigma_1 << \sigma/2$, then $I\left(\sigma_1\right) = S_{therm}\left(\sigma_1\right) - S\left(\sigma_1\right) = 0$
- if $\sigma_1 >> \sigma/2$, then $\sigma - \sigma_1 << \sigma/2$, thus $I\left(\sigma - \sigma_1\right) = S_{therm}\left(\sigma - \sigma_1\right) - S\left(\sigma - \sigma_1\right) = 0$

We shall now give an example to explain the Information Conservation Principle. Suppose that the system σ_1 is a bomb placed in a box B, with reflecting walls and a hole that let the electromagnetic radiation escape. The system $A = \sigma - \sigma_1$ will be the environment. Before the explosion, the systems A and B are in their ground pure states and their entropies are such that:

$$S(A) = S(B) = S_{therm}(A) = S_{therm}(B) = 0 \quad \Longrightarrow I(A) = 0 \tag{6.11}$$

When the bomb explodes, the radiation has not flown out from the box yet. The thermal entropy in the box increases, while all the others are still null:

$$S(A) = S(B) = 0 \qquad S_{therm}(A) = 0, S_{therm}(B) \neq 0 \uparrow \; \Longrightarrow I(A) = 0 \tag{6.12}$$

The initial information is $I(B) = S_{therm}(B)$.
The radiation starts to flow out, therefore the Von Neumann entropies increase, i.e the entanglement between A and B increases. The thermal entropy of the box decreases, while that of the environment increases:

$$S(A) = S(B) \neq 0 \qquad S_{therm}(A) \neq 0 \uparrow, S_{therm}(B) \neq 0 \downarrow \; \Longrightarrow I(A) = 0 \tag{6.13}$$

The information outside the box is still negligible.
At some point, the thermal entropies become equal. When this happens, the entanglement between A and B begins to decrease and so the information starts increasing.
When all photons are out, the box thermal entropy becomes null together with the Von Neumann entropy, being no entanglement inside anymore:

$$S(A) = S(B) = 0 \qquad S_{therm}(A) \neq 0, S_{therm}(B) = 0 \quad \Longrightarrow I(A) = S_{therm}(A) \tag{6.14}$$

From the second law of thermodynamics, the final value of $S^{final}_{therm}(A) > S^{initial}_{therm}(B)$, i.e the information in the outgoing radiation is more than the information initially inside the box.
During the process, $S(A)$, $S(B) < S_{therm}(A)$, $S_{therm}(B)$. In fact, at the beginning the information is mostly given by $S_{therm}(B)$ and $I(A) = 0$, which means that $S = S_{thermal}(A) < S_{thermal}(B)$. In the end we have the reverse situation. In short, we have $S \leq S_{therm}(A)$ or $S_{therm}(B)$. Therefore, information is conserved during the process of explosion. This means that the final state of the radiation is a pure state, although it may seem a thermal state in lower scales.

Chapter 7

Hawking evaporation

In the previous chapter we exposed the information conservation principle and the unitarity of the S matrix. We now briefly describe the Unruh and Hawking effects, showing in particular that the Hawking effect brings to violation of unitarity.

7.1 The Unruh effect

Let us consider the Minkowski metric in two dimensions:

$$ds^2 = -dt^2 + dx^2 \tag{7.1}$$

The Minkowski spacetime is a static globally hyperbolic spacetime. The part of this spacetime with $x > |t|$ is the right Rindler wedge while the part with $x < -|t|$ is the left Rindler wedge (see Appendix A).
Minkowski spacetime is invariant under the following boost:

$$\begin{aligned} t &\longrightarrow t\cosh\beta + x\sinh\beta \\ x &\longrightarrow t\sinh\beta + x\cosh\beta \end{aligned} \tag{7.2}$$

with β boost parameter. Therefore, it is reasonable to choose a new set of coordinates such that:

$$t = \rho\sinh\eta \qquad x = \rho\cosh\eta \tag{7.3}$$

with ρ,η taking any real value. These new coordinates cover only the right and left Rindler wedges. In this new set, the metric takes the form:

$$ds^2 = \rho^2 d\eta^2 - d\rho^2 \tag{7.4}$$

which is independent of η. The worldline with a fixed value of ρ moves with a uniform acceleration ρ^{-1}.

It may be convenient a further coordinate transformation for the right Rindler wedge, i.e:

$$\rho = \frac{1}{a}e^{a\xi} \qquad \eta = a\tau \Longrightarrow t = \frac{1}{a}e^{a\xi}\sinh a\tau \qquad x = \frac{1}{a}e^{a\xi}\cosh a\tau \tag{7.5}$$

with a positive constant. The metric takes the form:

$$ds^2 = e^{2a\xi}\left(d\tau^2 - d\xi^2\right) \tag{7.6}$$

The utility of this new set is that the worldline with $\xi = 0$ moves with a constant acceleration a.
For the left Rindler wedge the coordinates $\left(\bar{\tau}, \bar{\xi}\right)$ are:

$$t = \frac{1}{a}e^{a\bar{\xi}}\sinh a\bar{\tau} \qquad x = \frac{1}{a}e^{a\bar{\xi}}\cosh a\bar{\tau} \tag{7.7}$$

Let us now consider a massless quantum scalar field in two dimentions, $\hat{\Psi}(t, x)$, satisfying the equation:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\hat{\Psi}(t, x) = 0 \tag{7.8}$$

This field can be expanded as:

$$\hat{\Psi}(t, x) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}}\left[\hat{b}_{-k}e^{-ik(t-x)} + \hat{b}_{+k}e^{-ik(t+x)} + \hat{b}^\dagger_{-k}e^{ik(t-x)} + \hat{b}^\dagger_{+k}e^{ik(t+x)}\right] \tag{7.9}$$

The annihilation and creation operators satisfy the known commutation relations expressed in Appendix B. Defining $U = t - x$ and $V = t + x$, the field can be expressed as:

$$\hat{\Psi}(t, x) = \hat{\Psi}_-(U) + \hat{\Psi}_+(V) \tag{7.10}$$

Where $\hat{\Psi}_+(V)$, $\hat{\Psi}_-(U)$ are the left-moving and the right-moving sectors of the field respectively. Specifically:

$$\hat{\Psi}_+(V) = \int_0^\infty dk\left(\hat{b}_{+k}f_k(V) + \hat{b}^\dagger_{+k}f_k^*(V)\right) \qquad f_k = \frac{e^{-ikV}}{\sqrt{4\pi k}} \tag{7.11}$$

similarly for $\hat{\Psi}_-(U)$. The left and right sector of the field do not interact with each other, so we can consider only the left-moving sector field $\hat{\Psi}_+(V)$.
The field equation (7.8) can be expressed in the right Rindler wedge, i.e. with ξ and τ, assuming the same form:

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2}\right)\hat{\Psi}(t, x) = 0 \tag{7.12}$$

The solution can be once again divided into right-moving and left-moving sector, depending on $u = \tau - \xi$ and $v = \tau + \xi$ respectively, related to U and V by:

$$U = t - x = -\frac{1}{a}e^{-au} \qquad V = t + x = -\frac{1}{a}e^{av} \tag{7.13}$$

Thus we have:

$$\hat{\Psi}_+(V) = \int_0^\infty \left(\hat{a}^R_{+\omega} g_\omega(v) + \hat{a}^{R\dagger}_{+\omega} g^*_\omega(v) \right) \qquad g_\omega(v) = \frac{e^{-i\omega v}}{\sqrt{4\pi\omega}} \tag{7.14}$$

where $\hat{a}^R_{+\omega}$,$\hat{a}^{R\dagger}_{+\omega}$ are the creation and annihilation operators in the right Rindler wedge following the usual commutation relations.
The field $\hat{\Psi}_+(V)$ can be expressed in the left Rindler wedge by using the condition $V < 0 < U$, with the left Rindler coordinates (7.7). One defines $\bar{v} = \bar{\tau} - \bar{\xi}$ such that $V = -\frac{1}{a}e^{-a\bar{v}}$ and in the field (7.14) v is replaced by $\bar{v}$ and $\hat{a}^R_{+\omega}$,$\hat{a}^{R\dagger}_{+\omega}$ replaced by $\hat{a}^L_{+\omega}$,$\hat{a}^{L\dagger}_{+\omega}$ with same commutation relations. The two types of annihilation and creation operators are such that $\hat{a}^R_{+\omega}\big|0_R\big\rangle = \hat{a}^L_{+\omega}\big|0_R\big\rangle = 0$, where $\big|0_R\big\rangle$ is the static vacuum state in the left and right Rindler wedges.
From the Quantum Field Theory exposed in Appendix B, we can now calculate the Bogoliubov coefficients $\alpha^{R,L}_{\omega k}$,$\beta^{R,L}_{\omega k}$, where:

$$\Theta(V)g_\omega(V) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}} \left(\alpha^R_{\omega k} e^{-ikV} + \beta^R_{\omega k} e^{ikV} \right) \tag{7.15}$$

$$\Theta(-V)g_\omega(\bar{v}) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}} \left(\alpha^L_{\omega k} e^{-ikV} + \beta^L_{\omega k} e^{ikV} \right) \tag{7.16}$$

being:

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{7.17}$$

the Heaviside function expressing the fact that we may be in the left or in the right Rindler wedge. Multiplying for $e^{ikV}/2\pi$, $k > 0$, and integrating over V, we find the expression for $\alpha^R_{\omega k}$:

$$\alpha^R_{\omega k} = \sqrt{4\pi k} \int_0^\infty \frac{dV}{2\pi} g_\omega(v) e^{ikV} = \sqrt{\frac{k}{\omega}} \int_0^\infty \frac{dV}{2\pi} e^{ikV} (aV)^{-i\frac{\omega}{a}} \tag{7.18}$$

Similarly for $\beta^R_{\omega k}$, replacing e^{ikV} with e^{-ikV} and for $\alpha^L_{\omega k}$ and $\beta^L_{\omega k}$. We find that these coefficients obey the following relations:

$$\beta^L_{\omega k} = -e^{-\frac{\pi\omega}{a}} \alpha^{R*}_{\omega k} \qquad \beta^R_{\omega k} = -e^{-\frac{\pi\omega}{a}} \alpha^{L*}_{\omega k} \tag{7.19}$$

Substituting in (7.15) and (7.16), we find that the following functions are linear combination of positive frequency modes e^{-ikV} in Minkowski spacetime:

$$\begin{aligned} G_\omega(V) &= \Theta(V)g_\omega(v) + \Theta(-V)e^{-\frac{\pi\omega}{a}} g^*_\omega(\bar{v}) \\ \bar{G}_\omega(V) &= \Theta(-V)g_\omega(\bar{v}) + \Theta(V)e^{-\frac{\pi\omega}{a}} g^*_\omega(v) \end{aligned} \tag{7.20}$$

which were demonstrated by Unruh to be purely positive-frequency solutions in Minkowski spacetime.
Now, inverting the above equations such that:

$$\begin{aligned}\Theta(V)g_\omega(v) &\propto G_\omega(V) - e^{-\frac{\pi\omega}{a}}\bar{G}^*_\omega(V)\\ \Theta(-V)g_\omega(\bar{v}) &\propto \bar{G}_\omega(V) - e^{-\frac{\pi\omega}{a}}G^*_\omega(V)\end{aligned} \tag{7.21}$$

and substituting in the left-moving sector field:

$$\hat{\Psi}_+(V) = \int_0^\infty d\omega \left[\Theta(V)\left(\hat{a}^R_{+\omega}g_\omega(v) + \hat{a}^{R\dagger}_{+\omega}g^*_\omega(v)\right) + \Theta(-V)\left(\hat{a}^L_{+\omega}g_\omega(\bar{v}) + \hat{a}^{L\dagger}_{+\omega}g^*_\omega(\bar{v})\right)\right] \tag{7.22}$$

we find that the argument of the integral is proportional to:

$$G_\omega(V)\left[\hat{a}^R_{+\omega} - e^{-\frac{\pi\omega}{a}}\hat{a}^{L\dagger}_{+\omega}\right] + \bar{G}_\omega(V)\left[\hat{a}^L_{+\omega} - e^{-\frac{\pi\omega}{a}}\hat{a}^{R\dagger}_{+\omega}\right] \tag{7.23}$$

The operators $\hat{a}^R_{+\omega} - e^{-\frac{\pi\omega}{a}}\hat{a}^{L\dagger}_{+\omega}$ and $\hat{a}^L_{+\omega} - e^{-\frac{\pi\omega}{a}}\hat{a}^{R\dagger}_{+\omega}$ annihilate the Minkowski vacuum state $|0_M\rangle$[1], being $G_\omega(V)$ and $\bar{G}_\omega(V)$ positive frequency solutions (with respect to the usual time translation) in the Minkowski spacetime[2], i.e.:

$$\left[\hat{a}^R_{+\omega} - e^{-\frac{\pi\omega}{a}}\hat{a}^{L\dagger}_{+\omega}\right]|0_M\rangle = 0 \tag{7.24}$$

$$\left[\hat{a}^L_{+\omega} - e^{-\frac{\pi\omega}{a}}\hat{a}^{R\dagger}_{+\omega}\right]|0_M\rangle = 0 \tag{7.25}$$

We want now to show how to express $|0_M\rangle$ in the Fock space on the Rindler vacuum state $|0_R\rangle$ in order to obtain the Unruh effect. Let us firstly make an approximation where the Rindler energy states ω are discrete. Thus, we use ω_i instead of ω and we write the usual commutation relations:

$$\left[\hat{a}^R_{+\omega_i}, \hat{a}^{R\dagger}_{+\omega_j}\right] = \left[\hat{a}^L_{+\omega_i}, \hat{a}^{L\dagger}_{+\omega_j}\right] = \delta_{ij} \tag{7.26}$$

and all other commutators vanishing. Writing the discrete version of Eq. (7.24) and Eq. (7.25), we obtain:

$$\langle 0_M|\hat{a}^{R\dagger}_{+\omega_i}\hat{a}^R_{+\omega_i}|0_M\rangle = e^{-\frac{2\pi\omega_i}{a}}\langle 0_M|\hat{a}^{L\dagger}_{+\omega_i}\hat{a}^L_{+\omega_i}|0_M\rangle + e^{-\frac{2\pi\omega_i}{a}} \tag{7.27}$$

The same relation is obtained replacing in the above equation $\hat{a}^{R\dagger}_{+\omega_i}$ with $\hat{a}^{L\dagger}_{+\omega_i}$ and $\hat{a}^R_{+\omega_i}$ with $\hat{a}^L_{+\omega_i}$. Solving the two equations simultaneously, we find:

$$\langle 0_M|\hat{a}^{R\dagger}_{+\omega_i}\hat{a}^R_{+\omega_i}|0_M\rangle = \langle 0_M|\hat{a}^{L\dagger}_{+\omega_i}\hat{a}^L_{+\omega_i}|0_M\rangle = \frac{1}{e^{\frac{2\pi\omega_i}{a}} - 1} \tag{7.28}$$

[1] $\hat{b}_{+k}|0_M\rangle = 0$ for all k.

[2] As for QFT in Minkowski spacetime, the scalar field is expanded in terms of the energy momentum eigenfunctions, and the vacuum state is defined as the state annihilated by all annihilation operators, i.e. the coefficient operators of the positive-frequency eigenfunctions.

We have found that the expectation value of the Rindler-particle number is that of a Bose-Einstein particle in a thermal bath with temperature $T = \frac{a}{2\pi}$. Thus, the Minkowski state can be expressed as a thermal state in the Rindler wedge and the boost generator is the Hamiltonian. We can now express (7.28) without discretization, defining:

$$\hat{a}^R_{+f} \equiv \int_0^\infty d\omega f(\omega) \hat{a}^R_{+\omega} \qquad \int_0^\infty d\omega \left| f(\omega) \right|^2 = 1 \tag{7.29}$$

Such that:

$$\left\langle 0_M \middle| \hat{a}^{R\dagger}_{+f} \hat{a}^R_{+f} \middle| 0_M \right\rangle = \int_0^\infty d\omega \frac{\left| f(\omega) \right|^2}{e^{\frac{2\pi\omega}{a}} - 1} \tag{7.30}$$

The same for the left Rindler number operator.
Nonetheless, in order to conclude that the Minkowski vacuum state restricted to the right or left Rindler wedge is a thermal state, it is necessary to show that the probability of each right/left Rindler energy eigenstate corresponds to the grand canonical ensemble if the other Rindler wedge is disregarded.
Discretizing (7.24) and (7.25), one obtains:

$$\left(\hat{a}^{R\dagger}_{+\omega_i} \hat{a}^R_{+\omega_i} - \hat{a}^{L\dagger}_{+\omega_i} \hat{a}^L_{+\omega_i} \right) \left| 0_M \right\rangle = 0 \tag{7.31}$$

Therefore, the number of the left Rindler particles is the same as that of the right Rindler particles for each ω_i. So we can write:

$$\left| 0_M \right\rangle \propto \prod_i \sum_{n_i=0}^{\infty} \frac{C_{n_i}}{n_i!} \left(\hat{a}^{R\dagger}_{+\omega_i} \hat{a}^{L\dagger}_{+\omega_i} \right)^{n_i} \left| 0_R \right\rangle \tag{7.32}$$

The discretization of (7.24) and (7.25) gives us the recursion formula for K_{n_i}:

$$K_{n_i+1} = e^{-\frac{\pi\omega_i}{a}} K_{n_i} \Longrightarrow K_{n_i} = e^{-\frac{\pi\omega_i}{a}} K_0 \tag{7.33}$$

for which we can write:

$$\left| 0_M \right\rangle = \prod_i \left(\sqrt{1 - e^{-\frac{2\pi\omega_i}{a}}} \sum_{n_i=0}^{\infty} e^{-\frac{\pi\omega_i}{a}} \left| n_i, R \right\rangle \otimes \left| n_i, L \right\rangle \right) \tag{7.34}$$

Where $\left| n_i, R \right\rangle \otimes \left| n_i, L \right\rangle$ denotes the state with n_i left-moving particles with Rindler energy ω_i in each of the left and right Rindler wedges, thus:

$$\prod_i \left| n_i, R \right\rangle \otimes \left| n_i, L \right\rangle \equiv \left[\prod_i \frac{1}{n_i!} \left(\hat{a}^{R\dagger}_{+\omega_i} \hat{a}^{L\dagger}_{+\omega_i} \right)^{n_i} \right] \left| 0_R \right\rangle \tag{7.35}$$

Examining only the right Rindler wedge, the Minkowski vacuum density matrix is described by tracing out the left Rindler states, i.e.:

$$\hat{\rho_R} = \prod_i \left[\left(1 - e^{-\frac{2\pi\omega_i}{a}}\right) \sum_{n_i=0}^{\infty} e^{-\frac{2\pi n_i \omega_i}{a}} \left|n_i, R\right\rangle\left\langle n_i, R\right| \right] \tag{7.36}$$

This is the density matrix for a system of free bosons with temperature $T = \frac{a}{2\pi}$. Therefore, the Minkowski vacuum state for the left moving particle and restricted to the left Rindler wedge is a thermal state with temperature $T = \frac{a}{2\pi}$ and as Hamiltonian the boost generator normalized on $x^2 - t^2 = \frac{1}{a^2}$. In other words, an uniformly accelerated observer in the Minkowski vacuum observes a thermal spectrum near the horizon with temperature $T = \frac{a}{2\pi}$. This is the Unruh effect [17][2]. It is clear that the same result would be obtained for the right Rindler wedge.

7.2 Hawking radiation

Let us now consider the Schwarzschild metric neglecting the angular terms:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{7.37}$$

We now define new coordinates in region I:

$$U = -\sqrt{r_s\left(r - r_s\right)} e^{\frac{r-t}{2r_s}} \qquad V = \sqrt{r_s\left(r - r_s\right)} e^{\frac{r+t}{2r_s}} \tag{7.38}$$

called the Kruskal ingoing and outgoing null coordinates respectively. The metric then becomes:

$$ds^2 = -\frac{4r_s}{r} e^{-\frac{r}{r_s}} dU dV \tag{7.39}$$

which is a form valid not only in region I, but also in all the spacetime.
We consider now an inertial observer falling towards the horizon $r_s = 2M$. The radius r of the freely falling object is related to the Schwarzschild observer time t, near the horizon, by the formula:

$$r - r_s = e^{-\frac{t}{r_s}} \tag{7.40}$$

A distant initial observer at a fixed radial distance $r = r_\infty$ will observe a proper time, related to the Schwarzschild time t, given by the equation:

$$\tau_\infty = \sqrt{1 - \frac{r_s}{r_\infty}} t \tag{7.41}$$

Therefore, a distant observer measures $\tau \to \infty$ as $r \to r_s$, i.e. she will never see the falling object actually crossing the horizon.

Let us now define the outgoing and ingoing null coordinates u and v, defined only in quadrant I of the Rindler spacetime and given by the tortoise coordinate $r_* = r - r_s + r_s \log\left(\frac{r}{r_s} - 1\right)$, i.e.:

$$u = t - r_* = -2r_s \log\left(-\frac{U}{r_s}\right) + r_s \qquad v = t + r_* = 2r_s \log\left(\frac{V}{r_s}\right) - r_s \tag{7.42}$$

Thus, the metric in this new system will be:

$$ds^2 = -\frac{r_s}{r} e^{\frac{v-u}{2r_s}} e^{\frac{r-r_s}{r_s}} du dv \tag{7.43}$$

We have now configured the system and we are ready to start the quantum field theory analysis. First of all, let us make the following consideration: in the Schwarzschild geometry, the solutions of the equation of motion are spherically symmetric, thus they can be expressed in terms of the spherical harmonics Y_{lm}, i.e.

$$\psi = \sum_{lm} Y_{lm} \psi_{lm} \tag{7.44}$$

where ψ_{lm} solves the Schrdinger equation:

$$\left(\partial_t^2 - \partial_{r_*}^2 + V(r_*)\right) \psi_{lm} = 0 \tag{7.45}$$

With potential of the form:

$$V(r_*) = \frac{r - r_s}{r} \left[\frac{r_s}{r} + \frac{l(l+1)}{r^2}\right] \tag{7.46}$$

In the limit $r_* \to \infty$, so $r \to \infty$ (asymptotically flat spacetime), the potential near zero is $V \cong \frac{l(l+1)}{r^2}$. Thus, the particle is free in this limit. In the near horizon, $r \to r_s$, the tortoise coordinate behaves as $r_* \to -\infty$ and the potential goes to zero as $V \cong \frac{r-rs}{r}$. The particle is free in this regime too.
Therefore, near infinity and near horizon, the solutions are plane wave of the form e^{iku} and e^{ikv}.

The equation of motion of the scalar field is:

$$\partial_u \partial_v \phi = \partial_U \partial_V \phi = 0 \tag{7.47}$$

with solution:

$$\phi = \phi_L(u) + \phi_R(v) = \phi_L(U) + \phi_R(V) \tag{7.48}$$

We will consider only the right moving part.
Let us now replace u and v in region I with two new coordinates $\eta = t = \frac{u+v}{2}$ (time)

and $\xi = r_* = -\frac{u-v}{2}$ (ray). These new coordinates define the Rindler quadrant with acceleration $a = \frac{1}{2r_s}$ and the metric becomes:

$$ds^2 = e^{2a\xi}\left(-d\eta^2 + d\xi^2\right) \tag{7.49}$$

The positive frequency normalized modes in region I take the form:

$$g_k^{(1)} = \frac{1}{\sqrt{4\pi|k|}} e^{-i|k|\eta + ik\xi} \tag{7.50}$$

The right moving part corresponds to $k > 0$, therefore:

$$g_k^{(1)} = \frac{1}{\sqrt{4\pi|k|}} e^{-i|k|u} \tag{7.51}$$

The right moving part with negative frequency corresponds to $g_k^{(1)*}$. Then, the right moving field operator will be expanded as:

$$\hat{\phi}_R(u) = \int_0^\infty dk \left(\frac{\hat{b}_k}{\sqrt{4\pi|k|}} e^{-i|k|u} + \frac{\hat{b}_k^\dagger}{\sqrt{4\pi|k|}} e^{i|k|u} \right) \tag{7.52}$$

We make a change of variable $\omega = k$ and $\hat{b}_k = \frac{\hat{b}_\omega}{\sqrt{2\pi}}$, so we get:

$$\hat{\phi}_R(u) = \int_0^\infty \frac{d\omega}{2\pi} \left(\frac{\hat{b}_\omega}{\sqrt{2\omega}} e^{-i\omega u} + \frac{\hat{b}_\omega^\dagger}{\sqrt{2\omega}} e^{i\omega u} \right) \tag{7.53}$$

with $\hat{b}_\omega$, $\hat{b}_\omega^\dagger$ satisfying all the known commutation relations, being g_k^1 normalized such that $(g_k^{(1)}, g_{k'}^{(1)}) = \delta(k - k')$.
The field operator in the Schwarzschild tortoise coordinates (t, r_*) is given by:

$$\hat{\phi}(t, r_*) = \int_{-\infty}^\infty \frac{dk}{2\pi} \left(\frac{\hat{b}_k}{\sqrt{2|k|}} e^{-i|k|t + ikr_*} + \frac{\hat{b}_k^\dagger}{\sqrt{2|k|}} e^{i|k|t - ikr_*} \right) \tag{7.54}$$

with frequency $\omega = |k|$ and t the proper time at infinity where Schwarzschild becomes Minkowski. The vacuum state with respect to the inertial asymptotic tortoise Schwarzschild observer, also known as Boulware vacuum, is:

$$\hat{b}_k |0_T\rangle = 0 \tag{7.55}$$

for all k. The mode expansion in the Kruskal coordinates (U, V), with proper time $T = \frac{U+V}{2}$, space like coordinate $X = -\frac{U-V}{2}$ and frequency $|k| = \nu$ is:

$$\phi(T, X) = \int_{-\infty}^\infty \frac{dk}{2\pi} \left(\frac{\hat{a}_k}{\sqrt{4\pi\nu}} e^{-i\nu T + ikX} + \frac{\hat{a}_k^\dagger}{\sqrt{4\pi\nu}} e^{i\nu T - ikX} \right) \tag{7.56}$$

U is the equivalent of proper time τ of the infalling observer. The Kruskal vacuum is defined as $a_k|0_K\rangle = 0$ for all k.
The right moving field corresponds to $k > 0$, while the left moving to $k < 0$. We then write the field as:

$$\phi(T, X) = \int_0^\infty \frac{d\nu}{2\pi} \left(\frac{\hat{a}_\nu}{\sqrt{2\nu}} e^{-i\nu U} + \frac{\hat{a}^\dagger_{-\nu}}{\sqrt{2\nu}} e^{-i\nu V} + h.c. \right) \tag{7.57}$$

or in the tortoise Schwarzschild coordinates:

$$\phi(t, r_*) = \int_0^\infty \frac{d\omega}{2\pi} \left(\frac{\hat{b}_\omega}{\sqrt{2\omega}} e^{-i\omega u} + \frac{\hat{b}^\dagger_{-\omega}}{\sqrt{2\omega}} e^{-i\omega v} + h.c. \right) \tag{7.58}$$

To sum up, the asymptotic tortoise Schwarzschild observer, defined for $r > r_s$, is the analogue of the Rindler uniformly accelerated observer with acceleration $\frac{1}{2r_s}$, while the freely falling Kruskal observer is the analogue of the inertial Minkowski observer defined in all the spacetime manifold.
The right moving field of the Schwarzschild observer at $r > r_s$ is:

$$\hat{\phi}_R(u) = \int_0^\infty d\omega \left(v_\omega \hat{b}_\omega + v^*_\omega \hat{b}^\dagger_\omega \right) \qquad v_\omega = \frac{e^{-i\omega u}}{\sqrt{4\pi\omega}} \tag{7.59}$$

with $(v_\omega, v_{\omega'}) = \delta(\omega - \omega')$ and the known commutation relations for $\hat{b}_\omega$, $\hat{b}^\dagger_\omega$.
The Kruskal observer right moving field is:

$$\hat{\phi}_R(U) = \int_0^\infty d\nu \left(u_\nu \hat{a}_\nu + u^*_\nu \hat{a}^\dagger_\nu \right) \qquad u_\nu = \frac{e^{-i\nu U}}{\sqrt{4\pi\nu}} \tag{7.60}$$

with $(u_\nu, u_{\nu'}) = \delta(\nu - \nu')$ and the known commutation relations for $\hat{a}_\nu$, $\hat{a}^\dagger_\nu$.
These two observers see the following vacuum states respectively:

$$\begin{aligned} \hat{b}_\omega|0_T\rangle &= 0 \\ \hat{a}_\nu|0_K\rangle &= 0 \end{aligned} \tag{7.61}$$

We can now write the annihilation and creation operators and the the modes in terms of the Bogoliubov coefficients:

$$\begin{aligned} v_\omega &= \int_0^\infty d\nu \left(\alpha_{\omega\nu} u_\nu + \beta_{\omega\nu} u^*_\nu \right) & u_\nu &= \int_0^\infty d\omega \left(\alpha^*_{\omega\nu} v_\omega - \beta_{\omega\nu} v^*_\omega \right) \\ \hat{a}_\nu &= \int_0^\infty d\omega \left(\alpha_{\omega\nu} \hat{b}_\omega + \beta^*_{\omega\nu} \hat{b}^\dagger_\omega \right) & \hat{b}_\omega &= \int_0^\infty d\nu \left(\alpha^*_{\omega\nu} \hat{a}_\nu - \beta^*_{\omega\nu} \hat{a}^\dagger_\nu \right) \end{aligned} \tag{7.62}$$

The Bogoliubov coefficients are, explicitly:

$$\begin{aligned}\alpha_{\omega\nu} = (v_\omega, u_\nu) = -\int_{-\infty}^{+\infty}\frac{du}{2\pi}\sqrt{\frac{\omega}{\nu}}e^{-i\omega u}e^{i\nu U} \Longrightarrow \alpha^*_{\omega\nu} = -\sqrt{\frac{\omega}{\nu}}F(\omega,\nu)\\ \beta_{\omega\nu} = -(v_\omega, u^*_\nu) = \int_{-\infty}^{+\infty}\frac{du}{2\pi}\sqrt{\frac{\omega}{\nu}}e^{-i\omega u}e^{-i\nu U} \Longrightarrow \beta^*_{\omega\nu} = \sqrt{\frac{\omega}{\nu}}F(\omega,-\nu)\end{aligned} \tag{7.63}$$

where the function:

$$F(\omega,\nu) = \int_{-\infty}^{+\infty}\frac{du}{2\pi}e^{i\omega u}e^{-i\nu U} = \int_{-\infty}^{+\infty}\frac{du}{2\pi}e^{i\omega u - i\nu U_0 e^{-au}} \tag{7.64}$$

can be written in terms of the Euler gamma function:

$$F(\omega,\nu) = \frac{1}{2\pi a}e^{\frac{i\omega}{a}\log(i\nu U_0)}\Gamma\left(-\frac{i\omega}{a}\right) \tag{7.65}$$

Now, we would like to calculate the number of particles with frequency ω as seen by the Schwarzschild observer. It is the expectation value of the number operator $N_\omega = \hat{b}^\dagger_\omega b_\omega$. For the tortoise vacuum, this expectation value is of course zero, but the actual vacuum state, i.e the state with lowest energy of quantum scalar field, in the presence of a semiclassical black hole is the Kruskal vacuum $|0_K\rangle$. The reason is that the Schwarzschild observer is the analogue of the Rindler observer, while the Kruskal is the analogue of Minkowski. Thus, the vacuum state $|0_K\rangle$ actually contains the number of particles as seen from the Schwarzschild observer:

$$\langle 0_K|N_\omega|0_K\rangle = \langle 0_K|\hat{b}^\dagger_\omega\hat{b}_\omega|0_K\rangle = \int_0^\infty d\nu|\beta_{\omega\nu}|^2 \tag{7.66}$$

In order to compute this integral, firstly we need to calculate the function $F(\omega,\nu)$, i.e:

$$F(\omega,\nu) = e^{\frac{\omega\pi}{a}}F(\omega,-\nu) \tag{7.67}$$

Then, from the normalization condition:

$$\begin{aligned}\delta(\omega-\omega') = (v_\omega, v_{\omega'}) &= \int_0^\infty d\nu\,(\alpha_{\omega\nu}\alpha^*_{\omega'\nu} - \beta_{\omega\nu}\beta^*_{\omega'\nu})\\ &= \left(e^{\frac{\pi(\omega+\omega')}{a}} - 1\right)\int_0^\infty d\nu\frac{\sqrt{\omega\omega'}}{\nu}F^*(\omega,-\nu)F(\omega',-\nu)\end{aligned} \tag{7.68}$$

We rewrite this equation as:

$$\int_0^\infty d\nu\frac{\sqrt{\omega\omega'}}{\nu}F^*(\omega,-\nu)F(\omega',-\nu) = \frac{\delta(\omega-\omega')}{e^{\frac{\pi(\omega+\omega')}{a}} - 1} \tag{7.69}$$

For $\omega = \omega'$ we get:

$$\int_0^\infty d\nu \frac{\omega}{\nu} |F(\omega, -\nu)|^2 = \frac{\delta(0)}{e^{\frac{2\pi\omega}{a}} - 1} \tag{7.70}$$

Therefore, the expectation value of the particle operator number, i.e the density of particle in the black hole vacuum state $\left|0_K\right\rangle$ is given by:

$$n_\omega = \left\langle 0_K \middle| N_\omega \middle| 0_K \right\rangle = \left\langle 0_K \middle| \hat{b}^\dagger_\omega \hat{b}_\omega \middle| 0_K \right\rangle = \int_0^\infty d\nu |\beta_{\omega\nu}|^2 = \frac{1}{2\pi} \frac{1}{e^{\frac{2\pi\omega}{a}} - 1} \tag{7.71}$$

We recognize the Black Body Planck spectrum with temperature:

$$T_H = \frac{a}{2\pi} = \frac{1}{4\pi r_s} = \frac{1}{8\pi M} \tag{7.72}$$

which in SI units becomes:

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} \tag{7.73}$$

This is the Hawking temperature as described in [17]. Thus, the black hole as seen from a distant observer is radiating energy, so its masses decrease, consequently its temperature increases and the black hole evaporates.

In the previous part of this thesis, we have found that in the event horizon neighborhood, it is possible to define s-modes, which we considered to be the main constituents of the Hawking radiation. In order to understand the reason of this statement, we will now quantitatively describe the Hawking radiation following [16]. Consider the Schwarzschild metric in tortoise coordinates:

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr_*^2\right) + r^2 d\Omega^2 \tag{7.74}$$

Computing the action of a massless scalar field in this background, and expanding the field in spherical coordinates such that:

$$\psi = \sum_{lm} \psi_{lm} Y_{lm} \tag{7.75}$$

The equation of motion of the field is:

$$-\partial_t^2 \psi_{lm} + \partial_{r_*}^2 \psi_{lm} = V(r_*) \psi_{lm} \tag{7.76}$$

A stationary solution of this equation may be $\psi_{lm} = e^{i\nu t} \bar{\psi}_{lm}$, where the field $\bar{\psi}_{lm}$ obeys:

$$-\partial_{r_*}^2 \bar{\psi}_{lm} + V(r_*) \bar{\psi}_{lm} = \nu^2 \bar{\psi}_{lm} \tag{7.77}$$

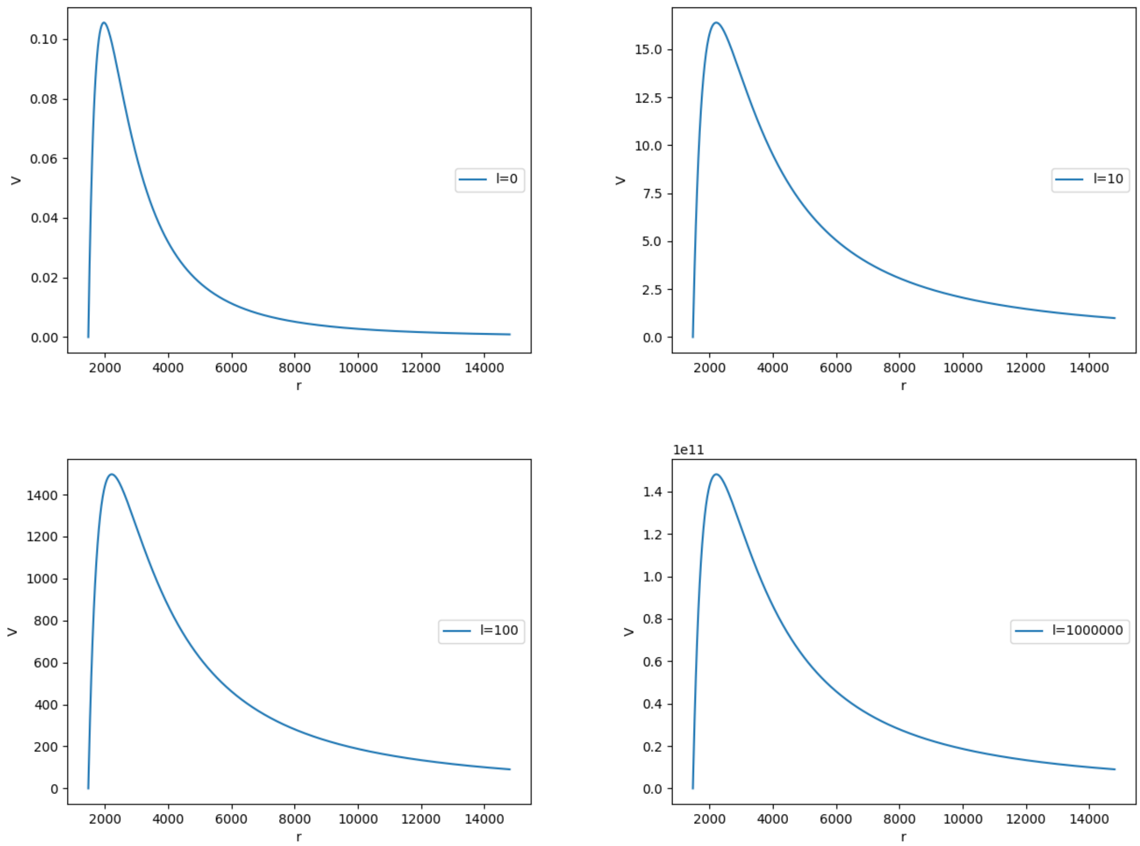

Figure 7.1: Potential for different values of the angular momentum component. The radial distance and the potential have been rescaled such that they result dimensionless.

This is the Schroedinger equation with potential V and energy ν^2. We obtain then the potential, given by:

$$V(r_*) = \frac{r-2M}{r}\left[\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right] \tag{7.78}$$

The potential vanishes at $r = 2M$ and for $r \to \infty$. Thus, it must have a maximum which we can calculate from the first derivative, obtaining:

$$r_{\pm} = 3M\left[\frac{1}{2} - \frac{1}{2l(l+1)} \pm \frac{1}{2}\sqrt{1 + \frac{7l^2+7l+4}{4l^2(l+1)^2}}\right] \tag{7.79}$$

and the physical solution is given by:

$$r_{+} \equiv r_{max} = 3M\left[\frac{1}{2} - \frac{1}{2l(l+1)} + \frac{1}{2}\sqrt{1 + \frac{7l^2+7l+4}{4l^2(l+1)^2}}\right] \tag{7.80}$$

We notice that for very large angular momentum, the maximum of the potential is $3M$. For $r \ll 3M$, the potential is repulsive and goes as $\frac{l(l+1)}{r^2}$, similar to a centrifugal potential, while for $r < 3M$ the potential becomes attractive. This means that every particle in this region, with zero initial speed, will fall spirally into the horizon. For high angular momentum, the maximum of the potential is very large and thus for such fields it is more difficult to escape or penetrate the potential barrier. The trend of the potential is shown in Fig. 7.1. Consider now a near horizon geometry, with $u = \log \rho$ as the tortoise coordinate:

$$ds^2 = e^{2u}\left(-d\omega^2 + du^2\right) + dY^2 + dZ^2 \tag{7.81}$$

As before, we write the action of the scalar field and expand the field into transverse wave plane as:

$$\phi = \int \frac{dk_2}{2\pi}\frac{dk_3}{2\pi} e^{i(k_2 Y + k_3 Z)}\psi(k_2, k_3, \omega, u) \tag{7.82}$$

we obtain the Rindler potential:

$$V = e^{2u}k^2 = \rho^2 k^2 \propto l^2 \qquad l = 2M|k| \tag{7.83}$$

This potential confines particles to the region near the horizon, as well as in the case of the Schwarzschild black hole. However, the potential in the Schwarzschild black hole becomes repulsive at $r > 3M$, which is equivalent to $\rho > M$, therefore the potential barrier for the Schwarzschild black hole is cut off for $\rho > M$, while the Rindler potential barrier keeps increasing as ρ^2 without any bound. Now, as a Schwarzschild black hole near the horizon will appear as Rindler and as the Rindler observer sees the Minkowski vacuum as a thermal ensemble with temperature $T = \frac{1}{2\pi}$, we expect that an identical thermal effect is observed near the Schwarzschild black hole horizon, with the crucial difference that while in the Rindler case the Rindler potential fully confines the thermal atmosphere, in the Schwarzschild case the thermal atmosphere is not fully confined by the potential (7.78). Therefore, in the Schwarzschild case particles can leak out of the thermal atmosphere, thus black hole evaporates. The temperature measured by Schwarzschild observer can be calculated from the relation between the Rindler time ω and the Schwarzschild time t:

$$\omega = \frac{t}{4M} \tag{7.84}$$

Therefore, the frequency measured by the Schwarzschild observer is red-shifted with respect to that measured by the Rindler observer, thus the temperature as measured by the Schwarzschild observer will result reddened:

$$\nu = \frac{\nu_R}{4M} \Longrightarrow T = \frac{T_R}{4M} = \frac{1}{8\pi M} \tag{7.85}$$

which is the Hawking temperature in natural units.
As we have seen, the potential (7.78) has a barrier around $3M$. As for modes with $l = 0$ (s-waves), the height of the barrier is given by:

$$V_{max}(l = 0) = \frac{27}{1024M^2} \qquad r_{max}(l = 0) = \frac{8M}{3} \tag{7.86}$$

As the energy in the potential is $E = \nu^2$, the modes $l = 0$ will escape the barrier coming from the event horizon if $E \geq V_{max}(l = 0)$, i.e.:

$$\nu \geq \frac{3\pi\sqrt{3}}{4}T \tag{7.87}$$

However, these modes can quite easily escape the potential barrier, because they are in a thermal state with Hawking temperature, thus their energies are of the order of T. While, for modes with higher l, the potential barrier goes as $\frac{l^2}{M^2}$, so it is very high compared to the thermal scale of Hawking radiation, hence these modes do not escape easily from the horizon as the s-waves. This is a peculiarity of Hawking radiation.

7.3 Information loss in black hole evaporation

Despite the Hawking analysis about the evaporation process of black holes seems to be theoretically correct, the study of black hole evaporation has shown a transition from a pure entangled state near the horizon to a mixed state outside the horizon, showing unitarity violations [17]. Consider a black hole formed during gravitational collapse in some pure quantum state $|\psi\rangle$[3]. Out of the horizon, we can define a state which can be seen as an outgoing wave packet P centered around some positive frequency ω with support only at large r at late times $t \to \infty$. This wave packet is a solution of the Klein Gordon equation which, at infinity, behaves as $e^{-i\omega t}$, so near the event horizon depends on the outgoing (or right moving) coordinates $u = t - r_*$, therefore $P \propto e^{-i\omega u}$. P can be expressed in terms of an annihilation operator $\hat{a}(P)$ and in terms of a field operator ϕ, solution of Klein Gordon equation (see Appendix B), i.e.:

$$a(P) = (\phi, P) \tag{7.88}$$

Let us consider this wave packet going backwards in time, so falling towards the black hole. It will be divided into a reflected part R, scattered away from the black hole to large radii and a transmitted part with support only immediately outside the event horizon, i.e:

$$P = R + T \tag{7.89}$$

[3]The gravitational collapse is not treated in this thesis. We just remark the fact that as for this theory, the formed black hole can be described by a pure quantum state $|\psi\rangle$.

Being the black hole metric stationary, R and T will have the same positive frequency with respect to the asymptotic Schwarzschild observer as P. But the infalling observer intersecting the trajectory of T at the event horizon will see both positive and negative frequencies in T. The reason is the following. Consider a freely falling observer with geodesic $x^\mu(\tau)$, being τ her proper time, crossing the horizon at $\tau = 0$. For this observer, the wavepacket T will be τ dependent, i.e. $T = T(x^\mu(\tau))$ vanishing at $\tau > 0$ as the transmitted part has no support behind the horizon. The function $T = T(x^\mu(\tau))$ cannot have only positive frequencies. In fact, from mathematical analysis, we know that a function that vanishes on a continuous arc in a domain of analyticity, then vanishes everywhere in that domain. Therefore, any positive frequency of the form:

$$h(\tau) = \int_0^\infty d\omega e^{-i\omega\tau}\tilde{h}(\omega) \tag{7.90}$$

is analytic in the lower half τ plane, since the addition of a negative imaginary part to τ leaves the integral convergent. Now, the positive real τ axis is the limit of an arc in the lower half plane, thus if $h(\tau) = 0$ for $\tau > 0$, it must vanish for $\tau < 0$ too. So, the function $T = T(x^\mu(\tau))$ must have a negative frequency component too as seen from the falling observer. A similar motivation is given for the asymptotic Schwarzschild observer. In fact, a function that is analytic on the lower half τ plane and that does not diverge for $|\tau| \to \infty$, must contain only positive frequencies as $e^{-i\omega\tau}$ diverges for $\tau \to \infty$ if ω is negative.
The annihilation operator decomposes as:

$$\hat{a}(P) = \hat{a}(R) + \hat{a}(T) \tag{7.91}$$

The action of this operator on the state $|\psi\rangle$ gives contributions only for the transmitted part, as $\hat{a}(R)$ annihilate $|\psi\rangle$ because the reflected wave packet has only support very far away outside from the black hole horizon. While T, containing both positive frequencies as well as negative with respect to the freely falling observer proper time, can be decomposed as:

$$T = T^+ + T^- \implies \hat{a}(T) = \hat{a}(T^+) + \hat{a}(T^-) \tag{7.92}$$

If T would have annihilated $|\psi\rangle$ too, $|\psi\rangle \equiv |0_T\rangle$. From the Klein Gordon inner product $(\phi_1, \phi_2)^* = -(\phi_1^*, \phi_2^*)$, we obtain $\hat{a}^\dagger(\bar{T}^-) = -\hat{a}(T^-)$.
Having T support only near the event horizon, $T \sim e^{-i\omega u}$. From metric (7.43):

$$\frac{du}{d\tau} \sim e^{\frac{u}{2r_s}} \implies \tau \cong -\tau_0 e^{-\frac{u}{2r_s}} \tag{7.93}$$

where τ_0 is a constant depending on the speed of the freely fall observer. Thus, T behaves as follows:

$$T \quad \begin{cases} \sim e^{2ir_s\omega \log(-\tau)} & \tau < 0 \\ = 0 & \tau > 0 \end{cases} \tag{7.94}$$

T is a positive frequency mode as for the Schwarzschild observer, thus it contains positive and negative frequencies as for the freely falling observer (analogously $g_k^{(1)}$ was a positive frequency solution only for the Rindler observer in quadrant I, but as for Minkowski observer it contained both positive and negative frequencies). We want now to extend Eq. (7.94) inside the horizon, i.e. with $\tau > 0$ and so obtain the positive and negative frequency extensions T^+ and T^-. The extension of T from $\tau < 0$ to $\tau > 0$ can be obtained by making an analytic continuation of the term $\log(-\tau)$ from $\tau < 0$ to $\tau > 0$ in the lower half complex τ plane, given by $\log(\tau + i\pi)$ with $\tau > 0$. The wave packet solution inside the horizon will be:

$$\begin{aligned} \tilde{T} &= T(-\tau) = e^{2ir_s\omega\log(\tau)} && \tau > 0 \\ \tilde{T} &= 0 && \tau < 0 \end{aligned} \tag{7.95}$$

The total wave packet T^+, positive frequency extension of T, will be of the form:

$$T^+ = c_+ \left(T + \tilde{T}e^{-\frac{\pi\omega}{\kappa}}\right) \tag{7.96}$$

Where we have substitute $\kappa = \frac{1}{2r_s}$, being κ the surface gravity of the black hole. The negative frequency extension of T is obtained by the analytic continuation in the upper half complex τ plane, given by $\log(\tau - i\pi)$ with $\tau > 0$, i.e.:

$$T^- = c_- \left(T + \tilde{T}e^{\frac{\pi\omega}{\kappa}}\right) \tag{7.97}$$

From the boundary conditions:

$$\begin{aligned} T^+ + T^- &= T && \tau < 0 \Longrightarrow c_+ + c_- = 1 \\ T^+ + T^- &= 0 && \tau > 0 \Longrightarrow c_+e^{-\frac{\pi\omega}{\kappa}} + c_-e^{\frac{\pi\omega}{\kappa}} = 0 \end{aligned} \tag{7.98}$$

we get:

$$c_+ = \frac{1}{1 - e^{-\frac{2\pi\omega}{\kappa}}} \qquad c_- = \frac{1}{1 - e^{\frac{2\pi\omega}{\kappa}}} \tag{7.99}$$

We can now define the annihilation and creation operators:

$$\hat{a}(T) = (\phi, T) \qquad \hat{a}^\dagger(T) = -(\phi, \bar{T}) \tag{7.100}$$

$$\hat{a}(\tilde{T}) = (\phi, \tilde{T}) \qquad \hat{a}^\dagger(\tilde{T}) = -(\phi, \bar{\tilde{T}}) \tag{7.101}$$

$$\hat{a}(T^+) = (\phi, T^+) = c_+\hat{a}(T) + c_+e^{-\frac{\pi\omega}{\kappa}}\hat{a}(\tilde{T}) \tag{7.102}$$

$$\hat{a}(T^-) = (\phi, \bar{T}^-) = -c_-\hat{a}^\dagger(T) - c_-e^{\frac{\pi\omega}{\kappa}}\hat{a}^\dagger(\tilde{T}) \tag{7.103}$$

As $\tilde{T}$ has negative norm, $\hat{a}(\tilde{T}) = -\hat{a}^\dagger(\bar{\tilde{T}})$ and $\hat{a}^\dagger(\tilde{T}) = -\hat{a}(\bar{\tilde{T}})$. Thus, the vacuum conditions on the state $|\psi\rangle$, i.e. $\hat{a}(T^+)|\psi\rangle = 0$ and $\hat{a}(\bar{T}^-)|\psi\rangle = 0$, become:

$$\begin{aligned} \left[\hat{a}(T) - e^{-\frac{\pi\omega}{\kappa}}\hat{a}^\dagger(\bar{\tilde{T}})\right]|\psi\rangle = 0 \\ \left[-\hat{a}^\dagger(T) + e^{\frac{\pi\omega}{\kappa}}\hat{a}(\bar{\tilde{T}})\right]|\psi\rangle = 0 \end{aligned} \tag{7.104}$$

Now, assuming that the wave packets T and $\bar{\tilde{T}}$ are normalized, so $\left[\hat{a}(T), \hat{a}^{\dagger}(T)\right] = 1$ and $\left[\hat{a}(\bar{\tilde{T}}), \hat{a}^{\dagger}(\bar{\tilde{T}})\right] = 1$, the annihilation and creation operators can be written as $\hat{a}(T) = \frac{\partial}{\partial \hat{a}^{\dagger}(T)}$ and $\hat{a}(\bar{\tilde{T}}) = \frac{\partial}{\partial \hat{a}^{\dagger}(\bar{\tilde{T}})}$. Therefore, (7.104) can be written as:

$$\begin{aligned} \frac{\partial}{\partial \hat{a}^{\dagger}(T)}\big|U\big\rangle &= e^{-\frac{\pi\omega}{\kappa}}\hat{a}^{\dagger}(\bar{\tilde{T}})\big|U\big\rangle \\ \frac{\partial}{\partial \hat{a}^{\dagger}(\bar{\tilde{T}})}\big|U\big\rangle &= e^{-\frac{\pi\omega}{\kappa}}\hat{a}^{\dagger}(T)\big|U\big\rangle \end{aligned} \tag{7.105}$$

with solution:

$$\big|U\big\rangle = \mathcal{N} e^{-\frac{-\pi\omega}{\kappa}}\hat{a}^{\dagger}(T)\hat{a}^{\dagger}(\bar{\tilde{T}})\big|B\big\rangle \tag{7.106}$$

where $\mathcal{N}$ is the normalization constant and $\big|B\big\rangle$ is the Boulware vacuum defined by (7.55). Eq. (7.104) defines the Unruh vacuum $\big|U\big\rangle$, which represents the vacuum state of the black hole. Indeed, the vacuum state of a black hole can not be the Boulware vacuum which, from its definition, is annihilated by the transmitted part, so $\hat{a}(T) = \hat{a}(\bar{\tilde{T}})\big|B\big\rangle = 0$. This state is different from the black hole initial state $\big|\psi\big\rangle$. The Unruh vacuum state can be interpreted as the in state as well as $\big|\psi\big\rangle$ can be interpreted as the out state of the black hole. It is also called the squeezed state, and it is a 2-mode entangled pure state, with the two modes corresponding to T outside the horizon and $\tilde{T}$ inside the horizon. It can also be rewritten as:

$$\big|U\big\rangle = \mathcal{N}\sum_n \frac{1}{n!}e^{-\frac{\pi\omega}{\kappa}}\left(\hat{a}^{\dagger}(T)\right)^n\left(\hat{a}^{\dagger}(\bar{\tilde{T}})\right)^n\big|B\big\rangle \tag{7.107}$$

which as first approximation can be represented as:

$$\big|U\big\rangle \cong \mathcal{N}\sum_n e^{-\frac{\pi\omega}{\kappa}}\big|n_R\big\rangle\big|n_L\big\rangle \tag{7.108}$$

where $\big|n_R\big\rangle \cong \frac{1}{\sqrt{n!}}\left(\hat{a}^{\dagger}(T)\right)^n\big|B_R\big\rangle$ and $\big|n_L\big\rangle \cong \frac{1}{\sqrt{n!}}\left(\hat{a}^{\dagger}(\bar{\tilde{T}})\right)^n\big|B_L\big\rangle$ are the states at n-excitations level of exterior modes T and interior modes $\bar{\tilde{T}}$, respectively.

Now, reducing this state to the outside of the black hole event horizon, we obtain:

$$\rho_R = Tr_L\big|U\big\rangle\big\langle U\big| = \mathcal{N}^2\sum_n e^{-\frac{2n\pi\omega}{\kappa}}\big|n_R\big\rangle\big\langle n_R\big| \tag{7.109}$$

which is a mixed state representing a thermal canonical ensemble. Thus, this is one of the expressions of the information loss problems in black holes: an entangled pure state near the horizon has transformed into a thermal mixed state outside the horizon.

Moreover, we notice that near the event horizon T consists of rapid oscillations, with T^{+}, $\bar{T}^{-}$ positive high frequency modes, which are not initially contained in the black hole state $|\psi\rangle$. So, evolving backward in time means that the frequencies are blueshifted (so increase) and therefore a freely falling observer will measure an increased frequency until it becomes infinite near the event horizon, so characterized by a Trans-Planckian wavelength. In other words, the backward modes seem to arise deep in the UV region, called the Trans-Planckian reservoir. The appearance of quantities below Planck scale may represent an issue as for the validity of Hawking theory, as we are still not able to describe the physics at these scales. However, it has been demonstrated by Unruh [15] that this problem actually does not really occur, as he showed that Hawking radiation is unaffected by a truncation of free field theory at the Planck scale. He used an hydrodynamical model to demonstrate this, basing his analysis on the analogy between the hydrodynamic equations of motion of sound waves in a non static fluid and those that governs s-waves emission of a massless scalar field from an evaporating Schwarzschild black hole in free field theory. So, if the properties of the Hawking effect do not depend on the existence of such a trans-Planckian reservoir neither from a reservoir of modes with frequency arbitrarily far beyond the Planck frequency, where does these outgoing modes that carry the Hawking radiation really emerge from? A possible answer may be that the Hawking particles are the result of the tidal effects induced by a non-static curved metric during the process of collapse in the neighborhood of the surface of the collapsing matter, such as a star. In fact, particle creation is a generic feature of curved spacetime quantum field theory (see Appendix B). If the non static gravitational field is due to the collapse of a star, vacuum pairs of particles antiparticles are ripped apart from the tidal forces, such that positive energy particles escape and travel to future infinity becoming part of Hawking radiation, while the negative energy particles fall inward in the collapsing matter. Thus, Hawking radiation is produced before the black hole has formed, otherwise the surface gravity, so the Hawking temperature, would increase with time, leading to a non thermal radiation [10].

Going back to the information loss, considering the Hawking radiation, the transition from a pure to a mixed state can be studied starting from the Schroedinger equation:

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \tag{7.110}$$

The evolution of this state can be written as:

$$|\psi_{final}\rangle = S|\psi_{initial}\rangle \Longrightarrow \psi^n_{final} = S^{mn}\psi^m_{initial} \tag{7.111}$$

S is the scattering matrix, used in the Schroedinger equation solution to express the evolution from pure quantum states to pure quantum states. But, as we have seen above, in black holes transition from pure to mixed states occurs, i.e:

$$\rho_{initial} = |\psi_{initial}\rangle\langle\psi_{initial}| \tag{7.112}$$

$$\rho_{final} = \sum_i p_i|\psi_{final}\rangle\langle\psi_{final}| \tag{7.113}$$

which can be expressed in terms of a new matrix, the so-called dollar matrix or the "non-S matrix" [5][12][8]:

$$\rho_{final}^{mm'} = \$_{mm',nn'}\rho_{initial}^{nn'} \tag{7.114}$$

which in case of the Schroedinger equation gets the form $\$_{mm',nn'} = S_{mn}S^*_{n'm'}$. So, as for the Schroedinger equation solution, we have the scattering matrix expressing the transition from pure states to pure states, while in case of black holes we have the dollar matrix taking pure states to mixed states. From the black hole evaporation process perspective, this transition from pure to mixed states translates into transition from gravitational collapse and black hole formation to Hawking radiation and black hole evaporation. Since the outgoing Hawking radiation is independent of the initial state, different initial states result in the same final mixed state, for which black hole evaporation does not conserve information, violating the information conservation principle in quantum mechanics, thus the unitarity of the S matrix.

7.4 Page analysis

To conclude, we give a brief description of the time evolution of a black hole quantum state and its radiation [17][11]. A rigorous way is by making a split of the Hawking radiation into late and early radiation. This may be interpreted as the fact that photons in late region R are already out in the radiation while photons in early region BH have still to be emitted from the black hole. Thus, the Hawking radiation can be decomposed into a bipartite system:

$$\mathcal{H}_{out} = \mathcal{H}_R \otimes \mathcal{H}_{BH} \tag{7.115}$$

We can now compute the entanglement entropy S_R of system R as a function of time, obtaining the so-called Page curve. We expect that at the beginning, the black hole is in a pure state, so the radiation field is such that $S_R = 0$. As the black hole begins to radiate, S_R starts increasing but, at some point it must come back to zero as all the radiation is out so that the black hole must be in a pure state. However, there is not a unique Page curve. Many [17] argue that computing the Page curve means to have solved the black hole information problem.
Consider now a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_A$ with $dimA < dimB$. A system is said to be maximally entangled if the state ρ_{AB} is pure but the reduced state $\rho_A = Tr_B\rho_{AB}$ is proportional to the identity operator on $\mathcal{H}_A$, i.e. $\rho_A \propto \mathbb{1}_A$. Then, Page's theorem states that if $dimA \ll dimB$, a randomly chosen pure state ρ_{AB} in $\mathcal{H}_{AB}$ is likely to be very close to a maximally entangled pure state.
We can now apply this theorem to the entanglement entropy of the black hole. We have seen that Hawking radiation is mostly made of s-waves, which can be described as 1+1 dimensional free scalar field with an Hawking temperature $T_H = 1/8\pi M = 1/4\pi r_s$.

Each escaping particle is carrying an energy given by Hawking temperature, $\nu \sim T_H$. We assume that a single quanta (l=0) per unit Rindler time $\omega = t/2r_s$ will escape the barrier, thus $1/2r_s$ quanta per unit Schwarzschild time will escape the barrier. Then, the total energy carried out of the black hole will be given by:

$$\frac{dE_R}{dt} = \frac{C}{G^2M^2} \tag{7.116}$$

being C a constant of proportionality. The energy rate, by energy conservation principle, is given by negative the mass rate, such that:

$$\frac{dM}{dt} = -\frac{C}{G^2M^2} \Longrightarrow Cdt = -G^2M^2dM \tag{7.117}$$

In order to apply Page's theorem, we first assume that the pure state of the early Hawking radiation R and the black hole is random. At early times, $dimR \ll dimBH$, so the entanglement entropy is given by the theorem:

$$S_E = S_R \sim \log(dimR) \tag{7.118}$$

Integrating (7.117) in a small time interval in which M is assumed to remain constant, we obtain the energy carried out by the radiation:

$$E_R = \frac{C}{G^2M^2}t \tag{7.119}$$

From the Thermodynamic of black holes, we know that $E_R/S_R \sim 1/r_s$, thus $S_R = tT$. This is valid only for times such that $S_R << S_{BH} \sim M^2$, i.e. for $t << M^3$. Therefore, at early times we expect the following linear behaviour of the entanglement entropy:

$$S_E \sim tT \qquad t << M^3 \tag{7.120}$$

After Page time, for which $\log(dimR) \sim \log(dimBH)$, we can assume $dimBH \ll dimR$, thus apply the Page theorem in the opposite direction:

$$S_E = S_{BH} \sim \log(dimBH) \tag{7.121}$$

Integrating (7.117) between t and $t_{evaporation}$ where $t_{page} \leq t \leq t_{evaporation}$ and from a value of the mass M till 0 (the black hole has evaporated), we obtain:

$$(t_{evaporation} - t)^{2/3} \sim M^2 \tag{7.122}$$

From the first law of thermodynamic $dE = TdS$, we can write the entropy of the black hole, knowing that the energy of the black hole E as seen from a distant observer goes as M. Thus:

$$dM = \frac{dS}{8\pi MG} \Longrightarrow S = 4\pi M^2G = \frac{\pi r_s^2}{G} \tag{7.123}$$

The area of the Schwarzschild black hole event horizon is $4\pi r_s^2$, so we obtain:

$$S_{BH} = \frac{Area}{4G} \tag{7.124}$$

known as the Bekenstein-Hawking entropy. Since the black hole entropy is proportional to the area, so to its mass squared, we can write:

$$S_{BH} \sim (t_{evaporation} - t)^{2/3} \Rightarrow S_E \sim (t_{evaporation} - t)^{2/3}, \quad t_{page} \leq t \leq t_{evaporation} \tag{7.125}$$

which is the Entanglement entropy of the black hole in relation with the Schwarzschild time.

Conclusions

The presence of black holes in the Universe can be proven by studying their Gravitational Field effects on the environment, such as interaction with object and fields. In this thesis we have studied the behavior of a massless scalar field radially falling towards the Schwarzschild black hole event horizon. After an introduction in Part I in which the main features of black holes were exposed, in Part II we have focused on the calculation of equation of motion of the freely falling Scalar Field. The introduction of a new coordinate system, namely Painlevé-Gullstrand metric, has been necessary since the Schwarzschild metric is singular in the Schwarzschild radius $r_s = 2M$. Moreover, we have built up a lattice configuration in these new coordinates, expressing the two-dimensional system in (y, t), being y a radial coordinate constant along the geodesic and t is the Painlevé-Gullstrand time coordinate. Applying the classical Scalar Field Theory in this background, we obtained an extremely complex equation of motion, for which no analytical solution could be found but an approximation through the so-called WKB approximation. Nonetheless, this approximation did not give an accurate form of the Scalar Field, thus we could not use it for the further analysis of the Redshift. Indeed, no exact frequency was computed, therefore we chose a configuration in which an asymptotic observer measures the frequency of a single radially falling mode, studying the problem kinematically. The final formula showed the expected behavior, and particularly it confirmed the peculiarity of the black hole event horizon: once the mode reaches the event horizon at $r_s = 2M$, the measured frequency at infinity vanishes, in accordance with the property of the black hole horizon. However, as in physics we do not deal with precise values of quantities, as any measurement implies a standard deviation around the expected value, we have studied the fluctuation of the redshift related to any fluctuation of the mass. Specifically the redshift depends on the Schwarzschild radius, thus on the black hole mass, so any fluctuation of the mass propagates to the redshift. In particular, we considered a gaussian distribution of the Schwarzschild radius, and successfully verified that a similar trend occurred for the redshift value. However the analysis of the redshift was simplified: first of all, we did not consider any field theory. Second of all, we considered a static spacetime. Then, no quantum fluctuations of the system were taken into account. For a field theory in a non static spacetime, which is what actually occurs in reality for a freely falling observer, we would have to work in semiclassical gravity, based on the semiclassical Einstein equations with sources given by the expectation value of the stress-energy tensor of quantum fields and we would have had to correct our theory starting from the correction of Einstein equations, obtaining the so-called Einstein-Langevin equations, which brings to a additive term to the stress-energy tensor, called noise Kernel. The calculation of noise-kernel is carried on by the Stochastic Semiclassical Gravity [6]. The scope and limits of semiclassical gravity are still unknown, because we still lack a fully well-understood consistent quantum theory of gravity. Therefore, from the semiclassical Einstein equations it seems clear that the semiclassical theory should break down when the quantum fluctuations of the stress tensor are large.

Any fluctuation of the mass treated in semiclassical gravity may be due to Hawking Radiation. This raised many issues and implied an accurate analysis of the Information Problem in black holes, exposed in Part III. Stephen Hawking showed that treating black holes physics as semiclassical leads to thermodynamical properties and in particular, he obtained that black holes event horizon as seen from a distant observer emits thermal radiation with temperature $T_H = \hbar c^3/8\pi GMk_B$, i.e. black holes evaporate. This radiation is mainly made of s-waves, which have enough energy to take over the potential barrier near the event horizon. The discovery of this radiation, despite being analytically correct, brought up several issues, such as the violation of the information conservation principle and the appearance of quantities below the Planck length, where no physics is yet known. If the Trans-Planckian problem is resolved showing that it does not actually represent an issue for the Hawking radiation, the information problem still occurs. Black holes form from the Gravitational Collapse of matter, which can be described by a pure state. Then, due to the radiation, they evaporate until they vanish, a process that can be described as a mixed state. The mixed nature of the thermal final state reflects on phase information loss as for any infalling particle. Since the outgoing radiation is completely independent from the initial state, different initial states end up in the same final state, violating the information conservation principle. Thus, as the basis framework of quantum mechanics is changed, one would expect this effect to appear elsewhere, maybe only at the Planck scale. The absence of a Theory of quantum gravity does not shed any light on what actually happens in micro-physics world, so Hawking carried on his theory arguing that intuitively micro-physical laws should also accommodate violations of quantum mechanics [3]. A possible solution for the Information conservation principle is considering this process of evaporation only in a coarse-grained sense. This means that the final state of the radiation becomes purified and information is carried out through Hawking radiation in subtle quantum correlations between late and early particles. The final pure state of the radiation is so complicated that Hawking thermal analysis is a good approximation. This solution, in which unitarity is maintained and information is conserved, implies a breakdown of the semi-classical description of gravitational fields, as we have been studying quantum gravity as an effective field theory, and this has led us to information loss [12]. The third interpretation has to do with the so-called remnant.

In this case, evaporation stops when the decreasing black hole size becomes Planckian and the remaining Planck-sized objects are called a remnant. It is characterized by an extremely large entanglement entropy in order for the total state to remain pure. Therefore, even if this is an object with a finite energy, it is effectively characterized by an infinite number of states and thus the connection between Bekenstein-Hawking entropy and number of states is lost, thus loss of information [17][3].
To conclude, information problem in black holes has not been solved yet, despite many alternatives which still present violation of fundamental principles or theory break-downs. Our work focuses on a simplification of this issue, showing that if the gravitational field is treated classically, a theory can be built up with appropriate simplifications. Despite any simplification, there is still the problem of how to write the scalar field near the event horizon, even if the spacetime is considered as static. As we have not been able to give a consistent answer, so we could not find any boundary condition in order to make a numerical analysis that may have shown the trend of an approximate field, we will leave this question to further studies.

Appendix A

Rindler spacetime

Rindler spacetime is a non-inertial (uniformly accelerated) reference frame with respect to the Minkowski spacetime. This means that there is an artificial gravitational field which can be removed by an appropriate coordinate transformation [17][7].
Physically, the Rindler frame can be visualized as an elevator in the empty space, with no external gravitational field, uniformly accelerated upward. The observer inside the elevator will experience a pressure from the floor, like if he was near a planet, so he will feel an artificial, uniform gravitational field. How does this observer see the outside (Minkowskian) spacetime?
Let us call $\xi^\mu = (\xi^0, \xi^1)$ the coordinate system inside the elevator, which we supposed to be accelerated upward in the x directions with an uniform acceleration a. The empty space is described by the Minkoskian coordinate system $x^\mu = (t, x)$. The motion of the elevator is described by a relation $x^\mu = x^\mu(\xi)$. At the observer proper time $\tau = 0$, the two systems coincide. Let us take the origin as the middle floor of the elevator.
In an infinitesimal time interval $d\tau$, the motion of the elevator can be considered to be approximately inertial, so the speed of the elevator $v = ad\tau$ to be constant, therefore the motion of the elevator is be given by Lorenz transformations:

$$\begin{aligned} \xi^0 &= \gamma\left(x^0 - \frac{v}{c}x^1\right) \Rightarrow d\tau \cong x^0 - ad\tau x^1 \\ \xi^1 &= \gamma\left(x^1 - \frac{v}{c}x^0\right) \Rightarrow \xi^1 \cong x^1 - ad\tau x^0 \end{aligned} \tag{A.1}$$

without considering transversal motion. Using the matrix notation and neglecting $o(d\tau^2)$, we obtain the relation between the elevator coordinates $(d\tau, \xi)$ and the minkowskian one (t, x):

$$\begin{pmatrix} x^0 - d\tau \\ x^1 \end{pmatrix} = \begin{pmatrix} 1 & ad\tau \\ ad\tau & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi^1 \end{pmatrix} \tag{A.2}$$

We can rewrite:

$$\begin{pmatrix} d\tau \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & ad\tau \\ ad\tau & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{a} \end{pmatrix} \tag{A.3}$$

So that:

$$\begin{pmatrix} x^0 \\ x^1 + \frac{1}{a} \end{pmatrix} = \begin{pmatrix} 1 & ad\tau \\ ad\tau & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi^1 + \frac{1}{a} \end{pmatrix} \tag{A.4}$$

and the final Lorentz transformations will be:

$$\begin{pmatrix} x^0 \\ \vec{x} + \frac{\vec{a}}{a^2} \end{pmatrix} = (1 + \delta L) \begin{pmatrix} 0 \\ \vec{\xi} + \frac{\vec{a}}{a^2} \end{pmatrix}, \delta L = \begin{pmatrix} 0 & ad\tau \\ ad\tau & 0 \end{pmatrix} \tag{A.5}$$

Repeating it N times, so at a time $\tau = Nd\tau$, the elevator coordinates are related to the Minkowskian ones by:

$$\begin{pmatrix} x^0 \\ \vec{x} + \frac{\vec{a}}{a^2} \end{pmatrix} = L(\tau) \begin{pmatrix} 0 \\ \vec{\xi} + \frac{\vec{a}}{a^2} \end{pmatrix}, L(\tau) = (1 + \delta L)^N \tag{A.6}$$

We can find $L(\tau)$ from the solutions of the equation:

$$L(\tau + d\tau) = (1 + \delta L) L(\tau) \tag{A.7}$$

for with the solution will be of the form $L(\tau) = \begin{pmatrix} A(\tau) & B(\tau) \\ B(\tau) & A(\tau) \end{pmatrix}$.
Eq. (A.7) can be rewritten as:

$$L(\tau + d\tau) - L(\tau) = \delta L L(\tau) = d\tau \begin{pmatrix} \frac{dA}{d\tau} & \frac{dB}{d\tau} \\ \frac{dB}{d\tau} & \frac{dA}{d\tau} \end{pmatrix} \tag{A.8}$$

Equivalently:

$$\frac{dA}{d\tau} = aB \qquad \frac{dB}{d\tau} = aA \tag{A.9}$$

with solutions:

$$A = \cosh a\tau \qquad B = \sinh a\tau \tag{A.10}$$

We then get:

$$x^0 = \sinh a\tau \left(\xi^1 + \frac{1}{a} \right) x^1 = \cosh a\tau \left(\xi^1 + \frac{1}{a} \right) - \frac{1}{a} \tag{A.11}$$

We can now compute the metric in Rindler spacetime:

$$ds^2 = \eta_{\mu\nu} x^\mu x^\nu = -dt^2 + dx^2 = -a^2 \left(\xi^1 + \frac{1}{a} \right)^2 d\tau^2 + \left(d\xi^1 \right)^2 \tag{A.12}$$

Let us consider now a Schwarzschild black hole and rewrite the Schwarzschild metric in terms of a new coordinate R, which represent a proper distance from the black hole horizon:

$$R = \int_{r_s}^{r} \sqrt{g_{rr}(r')} dr' = \int_{r_s}^{r} \frac{dr'}{\sqrt{1 - \frac{r_s}{r}}} = \sqrt{r (r - r_s)} + r_s \sinh \sqrt{\frac{r}{r_s} - 1} \tag{A.13}$$

In terms of R, the Schwarzschild metric is:

$$ds^2 = -\left(1 - \frac{r_s}{r(R)}\right) dt^2 + dR^2 + r^2(R)d\Omega^2 \tag{A.14}$$

In the horizon neighborhood, so $r = r_s + \delta$, so $R = 2\sqrt{r_s\delta}$, the metric is:

$$ds^2 = -R^2 d\omega^2 + dR^2 + r_s^2 d\Omega^2 \tag{A.15}$$

where $\omega = \frac{t}{2r_s}$ and together with R they can be considered as respectively hyperbolic and radial angular variables of the Minkowski spacetime. Indeed, defining:

$$X = R\cosh\omega \qquad T = R\sinh\omega \qquad Y = r_s \sin\theta\cos\phi \qquad Z = r_s\sin\theta\sin\phi \tag{A.16}$$

we obtain the Minkowski metric:

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2 \tag{A.17}$$

Comparing (A.15) with (A.12), we recognize (A.15) to be the Rindler metric such that $a\tau \rightarrow \omega$, $\xi^2 + 1/a \rightarrow R$. The time ω is called Rindler time and the Minkowskian spacetime approximation near the black hole horizon is called Rindler approximation. It shows that the black hole event horizon is locally non-singular, as the geometry is not distinguishable from the Minkowskian flat one.
The relation between the Minkowskian X and T and the Rindler coordinates R and ω can also be written as:

$$R^2 = X^2 - T^2 \qquad \frac{X}{T} = \tanh\omega \tag{A.18}$$

For $X > |T|$, we have the Rindler spacetime (or Rindler right wedge) and physically the region outside the horizon. It can be visualized as the first of four quadrants defined by the R=constant hyperboles bisectors. The lines of constant ω are straight lines through the origin, while the horizon lies at $R = 0$ or $T = X = 0$ and it is a two dimensional surface at $r = r_s$. Since $g_{00} = 0$ on the surface, it has no time extension.
For $X < -|T|$, we have the Rindler left wedge, visualized as the fourth of the four quadrants. The graphic is shown in Fig. A.1.

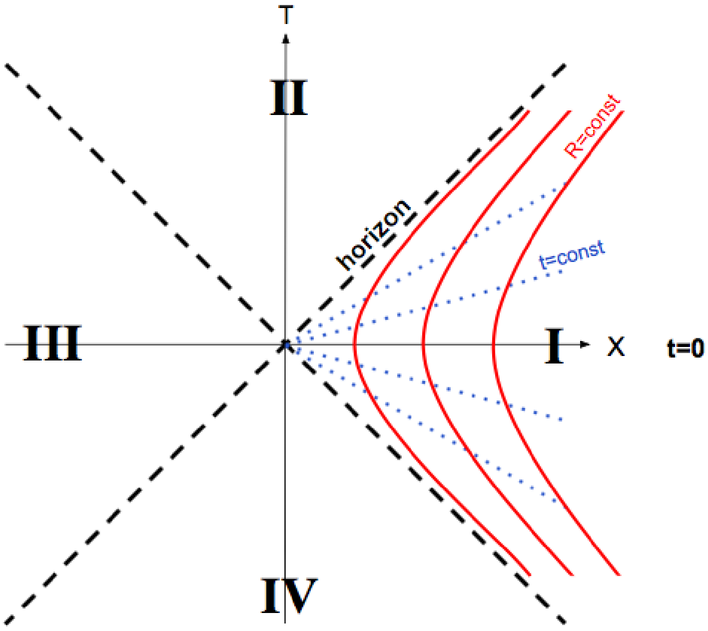

Figure A.1: Rindler spacetime.

Appendix B

Quantum field theory in a curved background

We study other properties of free fields in curved space time [7]. To do this, we will firstly show the particular case of an harmonic oscillator in one dimension, as free fields in curved spacetime are similar to collections of harmonic oscillators with time dependent frequencies. The Lagrangian, so the equation of motion of a particle of mass m moving in a time dependent potential $V(x,t)$ is:

$$L = \frac{m\dot{x}^2}{2} - V(x,t) \Longrightarrow m\ddot{x} + \partial_x V(x,t) \tag{B.1}$$

In order to apply canonical quantization, we firstly define the momentum conjugate to x, $p = \partial L \partial \dot{x} = m\dot{x}$. Then, we replace x and p by the operators $\hat{x}$ and $\hat{p} = -i\hbar\partial_x$ in the position representation, imposing the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. The operators are represented as hermitian linear operators on a Hilbert space such that their spectrum is real. In the Schroedinger picture, the state is time dependent while the operator is time independent. In the Heisenberg picture, the state is time independent while the operator is time dependent. The commutation relation in terms of the position and speed in the Heisenberg picture becomes:

$$[x(t), \dot{x}(t)] = i\hbar/m \tag{B.2}$$

From now on, we use the notation in which we drop the hat that usually distinguishes the number from an operator.
Consider now the specific case of an harmonic oscillator. Thus the potential is $V(x,t) = \frac{1}{2}m\omega^2(t)x^2$, such that the equation of motion takes the form:

$$\ddot{x} + \omega^2(t)x = 0 \tag{B.3}$$

As for the Cauchy theorem, we can find this solution only having the initial condition on the operators $x(0)$ and $\dot{x}(0)$ and the solution will be a linear combination of these operators, being the equation of motion linear. However, it is more convenient to write the solution in terms of two operators, a and $a^\dagger$, known as annihilation and creation operators:

$$x(t) = f(t)a + \bar{f}(t)a^\dagger \tag{B.4}$$

where $f(t)$ is a complex function which satisfies the equation of motion:

$$\ddot{f} + \omega^2(t)f = 0 \tag{B.5}$$

$\bar{f}$ is the complex conjugate of f and $a^\dagger$ is the hermitian conjugate of a. We underline that the operator x written as in (B.4) is non hermitian. The commutation relation (B.2) takes the form:

$$\langle f, f\rangle \left[a, a^\dagger\right] = 1 \qquad \langle f, g\rangle = \left(\frac{im}{\hbar}\right)\left[\hbar f \partial_t g - (\partial_t f)\, g\right] \tag{B.6}$$

If f and g are solution of (B.5), then the bracket $\langle f, g\rangle$ is independent of the time at which it is evaluated, consistently with the time independence of a. Assume now that the solution f is chosen such that $\langle f, f\rangle > 0$ and rescaled such that $\langle f, f\rangle = 1$. In this case the commutation relation of a and $a^\dagger$ are:

$$\left[a, a^\dagger\right] = 1 \tag{B.7}$$

We can now write a and $a^\dagger$ in terms of the bracket with f and x:

$$a = \langle f, x\rangle \qquad a^\dagger = -\langle \bar{f}, x\rangle \tag{B.8}$$

which are time independent as both f and x satisfy the equation of motion. Moving now to the Hilbert representation, we introduce the vacuum state, also called the ground state, $|0\rangle$, defined such that $a|0\rangle = 0$. A state n is such that $|n\rangle = \left(1/\sqrt{n!}\right)\left(a^\dagger\right)^n|0\rangle$ and it is an eigenstate of the number operator $N = a^\dagger a$, with eigenvalue n. These state define the excitation of the ground state $|0\rangle$. For a general solution f, the eigenvalues of the hamiltonian are:

$$H|0\rangle = \frac{m}{2}\left[\left(\dot{f}^2 + \omega^2 f^2\right)^* a^\dagger a^\dagger|0\rangle + \left(|\dot{f}|^2 + \omega^2|f|^2\right)|0\rangle\right] \tag{B.9}$$

If $|0\rangle$ is an eigenstate of H, then the first term vanish for which we have:

$$\dot{f} = \pm i\omega f \tag{B.10}$$

The norm of this f will be $\langle f, f\rangle = \mp(2m\omega/\hbar)|f|^2$. As we required the norm of f to be positive, we choose the plus sign of this norm, so the minus sign for (B.10), obtaining a solution for f:

$$f(t) = \sqrt{\frac{\hbar}{2m\omega}}e^{-i\omega t} \tag{B.11}$$

Thus, the hamiltonian takes the form:

$$H = \hbar\omega \left(N + \frac{1}{2} \right) \tag{B.12}$$

We notice that the ground state has energy $E = \frac{\hbar\omega}{2}$.

We can now introduce the theory for a scalar field. The action of a scalar field ϕ coupled with the metric [17][7] is:

$$S = \int d^4x \sqrt{-detg} \left(-\frac{1}{2} g_{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) \tag{B.13}$$

We may be interested in an action which is quadratic in the scalar field, therefore we must choose $V(\phi) = \frac{m^2\phi^2}{2}$, being m the mass of the field. In a curved background, there is another term which is quadratic in ϕ and depends on the Ricci tensor R. In an arbitrary dimension, the action is:

$$S = \int d^nx \sqrt{-detg} \left(-\frac{1}{2} g_{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{m^2\phi^2}{2} - \frac{\zeta R \phi^2}{2} \right) \tag{B.14}$$

where $\zeta = \frac{n-2}{4(n-1)}$ is called conformal coupling.
The equation of motion derived from Euler-Lagrange equations is:

$$\left(\nabla_\mu \nabla^\mu - m^2 - \zeta R \right) \phi \tag{B.15}$$

known as the Klein-Gordon equation.
Let ϕ_1 and ϕ_2 be the two solution of the equation of motion. Their inner product is defined as:

$$(\phi_1, \phi_2) = -i \int_\Sigma \left(\phi_1 \partial_\mu \phi_2^* - \partial_\mu \phi_1 \phi_2^* \right) d\Sigma n^\mu \tag{B.16}$$

where $d\Sigma$ is the volume element in the spacelike hypersurface Σ, n^μ the normal timelike unit vector. This inner product is independent from Σ.
We can always find a complete set of solutions u_i and u_i^* of Eq. (B.15) orthonormal in their inner product, i.e:

$$(u_i, u_j) = \delta_{ij} \qquad \left(u_i^*, u_j^* \right) = -\delta_{ij} \qquad \left(u_i, u_j^* \right) = 0 \tag{B.17}$$

The field can be expanded as:

$$\phi = \sum_i \left(a_i u^i + a_i^* u^{i*} \right) \tag{B.18}$$

Let this system now be canonically quantized. We decompose the spacatime in spacelike hypersurfaces. Let Σ be one of the hypersurfaces, with n^μ as unit normal vector, corresponding to a fixed value of $x^0 = t$. The action can be rewritten as $S = \int dx^0 L$, where

$L = \int d^{n-1}x\sqrt{-detg}\mathcal{L}$ and the canonical momentum is $\pi = -\sqrt{-detg}g^{\mu 0}\partial_\mu\phi$.
In order to quantized, ϕ and π are transformed into hermitian operators and imposed to follow the canonical commutation relations:

$$\left[\phi(x^0, x^i), \pi(x^0, y^i)\right] = i\delta^{n-1}(x_i - y_i) \tag{B.19}$$

with delta function satisfying:

$$\int \delta^{n-1}\left(x^i - y^i\right) d^{n-1}y = 1 \tag{B.20}$$

The coefficients in (B.18) become annihilation and creation operators a_i, $a_i^\dagger$ respectively, with commutation relations:

$$\left[a_i, a_j^\dagger\right] = \delta_{ij} \qquad [a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0 \tag{B.21}$$

The state $\left|0\right\rangle_u$ define the vacuum state:

$$a_i\left|0\right\rangle_u = 0 \tag{B.22}$$

The Fock basis can be constructed by repeating the application of a and $a^\dagger$ on the vacuum state.
As the vacuum state is not unique, because u_i and u_i^* are not unique, let us define another set of solutions (v_i, v_i^*) of the equation of motion (B.15), orthonormal in their inner product (B.16). The field is then expanded as:

$$\phi = \sum_i \left(b_i v^i + b_i^* v^{i*}\right) \tag{B.23}$$

and through a canonical quantization, the coefficients become annihilation and creation operators b_i,$b_i^\dagger$ with the same commutation relations of a_i,$a_i^\dagger$ and the vacuum state defined as:

$$b_i\left|0\right\rangle_v = 0 \tag{B.24}$$

The relation between the set of solutions u_i, u_i^* and the set of solutions v_i, v_i^* (two sets of modes seen by two different observers) and the two sets of annihilation and creation operators $a_i, a_i^\dagger$ and $b_i, b_i^\dagger$ is given by the so-called Bogoliubov transformations:

$$\begin{aligned} v_i &= \sum_j \left(\alpha_{ij}u_j + \beta_{ij}u_j^*\right) \qquad & u_i &= \sum_j \left(\alpha_{ji}^* v_j - \beta_{ji}v_j^*\right) \\ a_k &= \sum_i \left(\alpha_{ik}b_i + \beta_{ik}^* b_i^\dagger\right) \qquad & b_k &= \sum_i \left(\alpha_{ki}^* a_i - \beta_{ki}^* a_i^\dagger\right) \end{aligned} \tag{B.25}$$

The Bogoliubov coefficients can be found from:

$$\alpha_{ij} = (v_i, u_j) \qquad \beta_{ij} = -\left(v_i, u_j^*\right) \tag{B.26}$$

and they satisfy normalization conditions:

$$\sum_k (\alpha_{ik}\alpha_{jk} - \beta_{ik}\beta_{jk}) = \delta_{ij} \qquad \sum_k \left(\alpha_{ik}\beta_{jk}^* - \beta_{ik}\alpha_{jk}^*\right) = 0 \tag{B.27}$$

Let us now define the number operator $N_u = \sum_k a_k^\dagger a_k$ with respect to the u-observer. We clearly find that:

$$\left\langle 0_u \middle| N_u \middle| 0_u \right\rangle = 0 \tag{B.28}$$

As for the v-observer, we compute:

$$\left\langle 0_v \middle| a_k^\dagger a_k \middle| 0_v \right\rangle = \sum_i \beta_{ik}\beta_{ik}^* \Rightarrow \left\langle 0_v \middle| N_u \middle| 0_v \right\rangle = Tr\beta\beta^\dagger \tag{B.29}$$

This means that with respect to the v-observer, the vacuum state $\left|0_u\right\rangle$ is not empty but filled with particles. This shows the possibility of particle creation by a gravitational field, a crucial point in the Hawking effect analysis.

Now, analogously with the case of the harmonic oscillator, one can define a bracket from two complex solution f and g of the Klein-Gordon equation, such that:

$$\left\langle f, g \right\rangle = \int_\Sigma d\Sigma_\mu J^\mu \qquad J^\mu(f,g) = \left(\frac{i}{\hbar}\right)\sqrt{|g|}g^{\mu\nu}\left[\bar{f}\partial_\nu g - \left(\partial_\nu \bar{f}\right) g\right] \tag{B.30}$$

The bracket is called the Klein-Gordon inner product and $\left\langle f, f\right\rangle$ the Klein-Gordon norm of f. J^μ is the current density and it is divergenceless if f and g satisfy the Klein-Gordon equation, i.e. $\partial_\mu J^\mu = 0$. In this case, (B.30) does not depend on the surface Σ, provided the function vanishes at spatial infinity. The Klein-Gordon inner product is such that:

$$\overline{\left\langle f, g\right\rangle} = -\left\langle \bar{f}, \bar{g}\right\rangle = \left\langle g, f\right\rangle \qquad \left\langle f, \bar{f}\right\rangle = 0 \tag{B.31}$$

In analogy with the harmonic oscillator case, we can define the annihilation and creation operators in terms of brackets of f with another hermitian field operator ψ satisfying the Klein-Gordon equation:

$$a(f) = \left\langle f, \psi\right\rangle = -a\left(\bar{f}\right) \tag{B.32}$$

The canonical commutation relations can be rewritten as:

$$\left[a(f), a^\dagger(g)\right] = \left\langle f, g\right\rangle \qquad [a(f), a(g)] = -\left\langle f, \bar{g}\right\rangle \qquad \left[a^\dagger(f), a^\dagger(g)\right] = -\left\langle \bar{f}, g\right\rangle \tag{B.33}$$

If f is a positive norm solution with unit norm $\langle f, f \rangle = 1$, then $a(f)$ and $a^\dagger(f)$ satisfy the usual commutation relation.
Suppose now that a state $|\Psi\rangle$ is a normalized quantum state which satisfies $a(f)|\Psi\rangle = 0$. This condition specify one aspect of the state, i.e. for each n, the state $|n, \Psi\rangle = \left(\frac{1}{\sqrt{n!}}\right)\left(a^\dagger(f)\right)^n |\Psi\rangle$ is a normalized eigenstate of the number operator $N(f) = a^\dagger(f)a(f)$ with eigenvalue n. The set of all these states defines a Fock space made of f-wavepacket n-particle excitation on the state $|\Psi\rangle$. The total Hilbert space can be built up from a direct sum of positive norm subspace S_P made of complex solutions to the wave equation of Klein Gordon and its conjugate $\bar{S}_P$, namely:

$$S = S_P \oplus \bar{S}_P \tag{B.34}$$

In these subspaces brackets will be defined as:

$$\langle f, f \rangle > 0, \forall f \in S_P \qquad \langle f, g \rangle = 0, \forall f, g \in \bar{S}_P \tag{B.35}$$

This implies that the following commutation relations among the creation and annihilation operators for f and g must hold:

$$[a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0 \tag{B.36}$$

In such Hilbert space, we can then define a vacuum state $|0\rangle$, called the Fock vacuum, for which $a(f)|0\rangle = 0$, $\forall f \in S_P$ and all the spaces $a^\dagger(f_1)...a^\dagger(f_n)|0\rangle$ for all $f_1, ..., f_n$ in S_P. The Fock vacuum depends on the decomposition we choose for S and it is not necessarily the ground state. Moreover, if the spacetime is not static (e.g. curved spacetime such as the Schwarzschild one), the vacuum is not even well define (as we have seen in the Hawking effect analysis, where we have define the Boulware vacuum and the Unruh vacuum as possible vacuum states near the event horizon).
In the specific case of a flat spacetime, the decomposition of the space of solutions regards the positive and negative frequency with respect to a Minkowski time translation and the Fock vacuum is the ground state. In a flat spacetime, the Klein-Gordon equation of motion is simply:

$$\left(\Box + m^2\right)\psi = 0 \tag{B.37}$$

whose solutions are plane waves. Usually as the plane waves are not normalizable, it is helpful to introduce periodic boundary conditions, with space becoming a large three-dimensional torus with circumferences L and volume $V = L^3$. Allowed wave vectors $\vec{k} = \frac{2\pi}{L}\vec{n}$, being $\vec{n}$ the unitary vector with integer as components. Mode solutions are written as:

$$f_{\vec{k}}(t, \vec{x}) = \sqrt{\frac{\hbar}{2\omega(\vec{k})V}} e^{-i\omega(\vec{k})t} e^{i\vec{k}\vec{x}} \tag{B.38}$$

and $\omega(\vec{k})$ is given by the dispersion relation $\omega(\vec{k}) = \sqrt{|k|^2 + m^2}$, together with the solution obtained for negative frequencies $-\omega(\vec{k})$. The corresponding brackets are:

$$\begin{aligned} \langle f_{\vec{k}}, f_{\vec{h}} \rangle &= \delta_{\vec{k}\vec{h}} \\ \langle \bar{f}_{\vec{k}}, \bar{f}_{\vec{h}} \rangle &= -\delta_{\vec{k}\vec{h}} \\ \langle f_{\vec{k}}, \bar{f}_{\vec{h}} \rangle &= 0 \end{aligned} \tag{B.39}$$

providing an orthogonal decomposition of the solution space S into a positive norm solutions subspace S_P spanned with positive frequency modes as (B.38). This provides a Fock representation of the flat spacetime, as we can define annihilation operator associated with $f_{\vec{k}}$, namely:

$$a_{\vec{k}} = \langle \psi, f_{\vec{k}} \rangle \tag{B.40}$$

together with the corresponding annihilation operator $a^{\dagger}_{\vec{k}}$, such that the field will be expanded in modes:

$$\psi = \sum_{\vec{k}} \left(f_{\vec{k}} a_{\vec{k}} + \bar{f}_{\vec{k}} a^{\dagger}_{\vec{k}} \right) \tag{B.41}$$

and $f_{\vec{k}}$ positive frequency modes. Since the Hamiltonian is a sum over the contributions from each $\vec{k}$ value, the vacuum state is then defined as:

$$a_{\vec{k}} |0\rangle = 0 \tag{B.42}$$

which for all $\vec{k}$ is the ground state of the Hamiltonian. A state of the form $a^{\dagger}_{\vec{k}}|0\rangle$ has energy $\hbar\omega(\vec{k})$ and it is a single particle state. State as $a^{\dagger}_{\vec{k}_1} ... a^{\dagger}_{\vec{k}_n} |0\rangle$ is a n-particle state.

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