Alma Mater Studiorum · Università di Bologna

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI

Corso di Laurea in Matematica

COMPLEXITY OF SEIFERT MANIFOLDS

Tesi di Laurea Magistrale in Topologia Geometrica

Relatore:
Dott.ssa
Alessia Cattabriga

Presentata da:
Valentina Falcioni

Sessione V
Anno Accademico 2017/2018

The essence of mathematics lies in its freedom.

G. Cantor

Contents

Introduction

The 3-manifolds world is topologically much richer than the surfaces realm, while yet not so crazy as the four-manifolds universe, which cannot be classified in any reasonable sense. The 3-manifolds lie in the middle: we do not have yet a complete satisfactory picture, but we understand them a good deal. The branch of mathematics that studies these topics is called low-dimensional topology and it has recently been undergoing an intense development. On the one hand, the exponential advancement of computer technologies has made it possible to conduct sophisticated computer experiments and to implement algorithmic solutions, which have in turn provided a motivation to search for new and better algorithms. On the other hand, low-dimensional topology has received an additional boost because of the discovery of numerous connections with theoretical physics.

Among all 3-manifolds, a particular class introduced by H. Seifert in 1933, known as Seifert manifolds or Seifert fibre spaces, has been widely studied, well understood, and is having a great impact for understanding 3-manifolds. They suit many nice properties, whom majority were already known since the deep work of Seifert. Seifert fibered spaces constitute a large of class of 3-manifolds and are totally classified by mean of a finite set of invariants. They have widely appeared in the literature for playing a central key-role in the topology of compact 3-manifolds, and nowadays they are very well known and understood. They have allowed the developing of central concepts in the study of 3-manifolds such as the JSJ-decomposition and the Thurston's geometrization conjecture.

It's a result of D. Epstein that the Seifert fibre spaces are characterized as those 3-manifolds which admit a foliation by circles. In fact this definition is a little more general than the original definition of Seifert, in order to correctly englobe the case of non-orientable 3-manifolds; it has now become the modern usual terminology for Seifert fibre spaces.

In order to have an overview over the set of 3-manifolds and to put some order into their chaos, Matveev has introduced the theory of complexity. Indeed, the complexity supplies the set of 3-manifolds with a filtration by finite subsets (of 3-manifolds of a bounded complexity), and this allows to break up the classification problem for all 3-manifolds into an infinite number of classification problems for finite subsets. The complexity is a function that associates a compact 3-manifold to a non-negative integer number and it has the following properties: it is additive under connected sum; for any $k \in \mathbb{Z}$, there are only finitely many closed irreducible manifolds with complexity k; it does not increase when cutting along incompressible surfaces. The problem of calculating the complexity of any given 3-manifold is very difficult. On the contrary, the task of giving an upper bound for the complexity is very easy, but this bound may be not at all sharp. In order to understand what the complexity is and how to estimate it, we'll take as examples of 3-manifolds the Seifert fibre spaces for their easyness.

Since Seifert fibre spaces are a kind of circle bundle over a 2-dimensional orbifold, for talking about them we need some prerequisites that we will introduce in Chapter 1, which are the concepts of 2-dimensional orbifold, fibre bundle and circle bundle. Afterwards, in Chapter 2, we will study Seifert fibre spaces, looking at their properties and their classification up to fibre-preserving homeomorphism and up to homeomorphism, giving a combinatorial description of such manifolds. In Chapter 3, we will introduce the complexity theory, at first in a general way that concerns all compact 3-manifolds and then, from Section 3.3 on, focusing ourselves on the estimation of the complexity of Seifert fibre spaces. In Section 3.4, we will also see some examples in which the estimation ensures us that the complexity of the manifold is zero (i.e. when we get that the complexity muse be less or equal than zero), constructing what is called an *almost simple spine* of such manifold.

Chapter 1

Prerequisites

In this chapter we introduce some concepts that are essential prerequisites for the understanding of Seifert fibre spaces. Indeed, such 3-manifolds can be seen as circle bundles over 2-dimensional orbifolds, therefore we will give some notions about 2-dimensional orbifolds, fibre bundles and circle bundles, taken from [Sco], [Hat], [Mar] and [Fin].

1.1 2-dimensional orbifolds

Definition 1.1. An *n-dimensional orbifold* (without boundary) is a Hausdorff, paracompact space which is locally diffeomorphic to the quotient space of $\mathbb{R}^n$ by a finite group action (eventually trivial).
On the other hand, we can also define *n-dimensional orbifolds with boundary* as spaces locally diffeomorphic to the quotient of $\mathbb{R}^n$ or $\mathbb{R}^n_+$ (where $\mathbb{R}^n_+$ denotes the points of $\mathbb{R}^n$ having non-negative last coordinate) by a finite group action. The *boundary* of the orbifold consists of points locally homemorphic to the quotient of $\mathbb{R}^n_+$ by a finite group action.

From the previous definition it follows that each point x of an orbifold is associated with a group G_x, well defined up to isomorphism, such that a local coordinate system in a neighborhood of x has the form $U \cong \widetilde{U}/G_x$ with $\widetilde{U} = \mathbb{R}^n$, $\mathbb{R}^n_+$.

Definition 1.2. The set $\mathcal{S}$ consisting of the points x of the orbifold such that $G_x \neq 1$ is called the *singular locus* of the orbifold.

Clearly an orbifold is a manifold if the singular locus is empty. We can define the following equivalence relation on the set of the orbifolds.

Definition 1.3. An *isomorphism of orbifolds* is a diffeomorphism which respects the given quotient structures on open subsets of the orbifolds.

Recall that a continuous map between topological spaces $f : X \to Y$ is a covering if any point $y \in Y$ has a neighborhood U such that $f^{-1}(U)$ is the disjoint union of sets V_λ, such that $f_{|V_\lambda} : V_\lambda \to U$ is a homeomorphism.

Definition 1.4. If X and Y are orbifolds and $f : X \to Y$ is an orbifold map, an *orbifold covering* is defined in the same way as a covering except that one allows $f_{|V_\lambda} : V_\lambda \to U$ to be the natural quotient map between two quotients of $\mathbb{R}^n$ by finite groups, one of which is a subgroup of the other.

In the following we will focus on the 2-dimensional case, so we will consider $\mathbb{E}^2$, i.e., $\mathbb{R}^2$ with the euclidean geometry on it, its finite groups of isometries and the corresponding quotient spaces.
A finite group G of isometries of $\mathbb{E}^2$ will be as one of the following three types: a cyclic group of order n generated by a rotation of $2\pi/n$ about a point; a cyclic group of order two generated by a reflexion in a line; a dihedral group of order $2n$ generated by a rotation of order n and a reflexion in a line passing through the rotation center. Each of these three kinds of group will generate a different kind of singularity, as we will see.

Let us consider the first case, that is when G is the cyclic group of order n generated by a rotation α of $2\pi/n$ about a point $P \in \mathbb{E}^2$. Considering the action of G on $\mathbb{E}^2$ and referring to Fig. 1.1, we can notice that the orbit of each point has exactly one point in the region W delimited by the two half lines l_1 and l_2, except that each point of l_1 lies in the same orbit as one point of l_2. It follows that the quotient space $\mathbb{E}^2/G$, which is obtained by identifying each orbit to a single point, is the same as the space obtained from W by gluing l_1 to l_2, that is a cone C with cone angle $2\pi/n$ at a vertex P, as shown in Fig. 1.1. There is a natural metric inherited by the surface

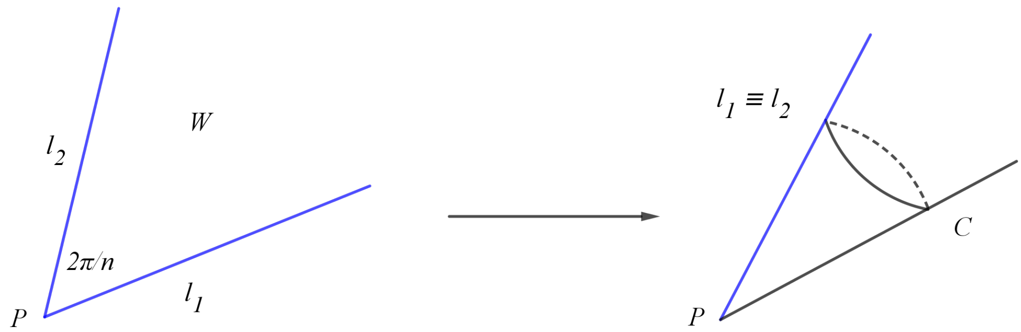

Figure 1.1: The region W and the cone C.

$\mathbb{E}^2/G$ from $\mathbb{E}^2$. This metric has a singularity at the vertex P of the cone C: such singularity is called *cone point*. This means that the metric of C restricted to $C \setminus \{P\}$ is Riemannian, but the metric on C is not Riemannian.

Consider now G as a cyclic group of order two generated by a reflexion in a line l. $\mathbb{E}^2/G$ again inherits a natural metric from $\mathbb{E}^2$ and it is isometric to a half-plane whose boundary line is the image of l, as we can see in Fig. 1.2. Such boundary line consists of singular points and is called *reflector line*.

If G is the dihedral group of order $2n$ generated by a rotation of order n about a point P and the reflexion in a line through P, then $\mathbb{E}^2/G$ again inherits a natural metric and it is isometric to an infinite wedge with angle π/n (see Fig. 1.3). In this case, there are two semi-infinite boundary lines of singular points called *corner reflectors*, which meet in an "even more singular" point P.

Remark 1.1. Notice that generally a n-dimensional manifold doesn't need to be topologically a manifold. For example, if the group $\mathbb{Z}_2$ acts on $\mathbb{E}^3$ by the map $x \mapsto -x$, then the quotient space is diffeomorphic to a cone

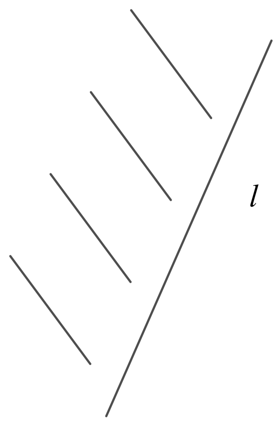

Figure 1.2: The half plane.

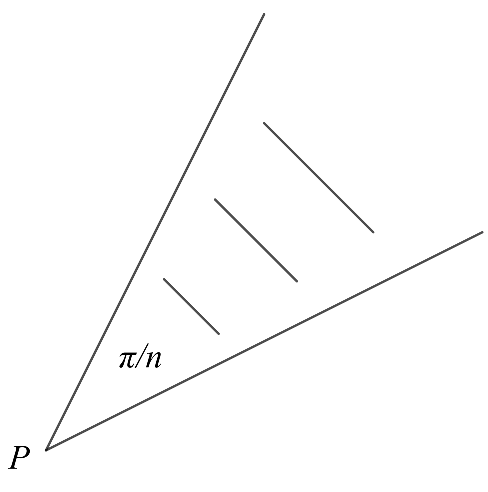

Figure 1.3: The infinite wedge with angle π/n.

on $\mathbb{RP}^2$ and this fails to be a manifold at the cone point. So we see that in general an orbifold is not even diffeomorphic to a manifold. However, in dimension two, any orbifold is diffeomorphic to a manifold, because the only possible types of singular point are: cone points, reflector lines and corner reflectors. It is important to realize that an orbifold with cone points is diffeomorphic to a manifold but isn't isomorphic to one. Therefore one must distinguish carefully between a two-dimensional orbifold and the underlying surface. Note also that if an orbifold has reflexion curves, the points on these curves are not boundary points of the orbifold.

Given a 2-dimensional orbifold X, we can compute its *orbifold fundamental group* as follows.
Let U be a regular neighborhood of the singular set of X and let N be the closure of $X \setminus U$. As N is a surface, we can find its fundamental group without troubles. We need now to focus on the components of U, each of which can be of one of the three types seen above. In order to describe what we mean for fundamental group in this case, we will use the fact that in all the three cases $U \cong \mathbb{R}^2/G$. Firstly, one can have a cone that is the quotient of $\mathbb{R}^2$ by a finite cyclic group of rotations. The orbifold fundamental group of a cone is, of course, finite cyclic. Secondly, one can have an orbifold whose underlying space is $\mathbb{R} \times I$ and which has one reflector line and one boundary line: this orbifold has fundamental group $\mathbb{Z}_2$. Finally, one can have an orbifold whose underlying space is $S^1 \times I$ and which has one reflector circle and one boundary circle: the fundamental group of this orbifold is $\mathbb{Z} \times \mathbb{Z}_2$. Now X is the union of the surface N with the orbifolds which are the components of U and so $\pi_1^{orb}(X)$ can be calculated using van Kampen's Theorem (see page 43 of [Hat]).

We end this section by introducing two particular orbifolds with boundary that will be recalled in the next chapter: $S^2(p)$ and $S^2(p,q)$ (see Fig. 1.4). The first one $S^2(p)$ is called *teardrop orbifold* and is the orbifold having as underlying surface S^2 and one cone point of cone angle $2\pi/p$ where $p \neq 1$.

The second one $S^2(p,q)$ is called *spindle orbifold* and is the orbifold having S^2 as underlying surface and two cone points, respectively of cone angle $2\pi/p$ and $2\pi/q$, where $p \neq q$.

Figure 1.4: On the left, the teardrop orbifold $S^2(p)$ and on the right the spindle orbifold $S^2(p,q)$.

1.2 Fibre bundles

For further details about the contents of this section, see [Hat] and [Mar].

Definition 1.5. A map $p : E \to B$ is said to have the *homotopy lifting property* with respect to a space X if, given a homotopy $g_t : X \to B$ and a map $\tilde{g}_0 : X \to E$ lifting g_0 (i.e. $p \circ \tilde{g}_0 = g_0$), then there exists a homotopy $\tilde{g}_t : X \to E$ lifting g_t.

Definition 1.6. A *fibration* is a map $p : E \to B$ having the homotopy lifting property with respect to all spaces X.

For example, a projection $B \times F \to B$ is a fibration since we can choose lifts of the form $\tilde{g}_t(x) = (g_t(x), h(x))$ where $\tilde{g}_0(x) = (g_0(x), h(x))$.

Definition 1.7. A *fibre bundle structure* on a space E, with fibre F, consists of a projection map $p : E \to B$ such that each point of B has a neighborhood U for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ making the diagram below commute, where the unlabelled map is a projection onto the first factor.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & U & \end{array}$$

Commutativity of the diagram means that h carries each fibre $F_b = p^{-1}(b)$ homeomorphically onto the copy $\{b\} \times F$ of F. Thus the fibres F_b are arranged locally as in the product $B \times F$, though not necessarily globally. An h as above is called a *local trivialization* of the bundle. Since the first coordinate of h is just p, h is determined by its second coordinate, a map $p^{-1}(U) \to F$ which is a homeomorphism on each fibre F_b.
The fibre bundle structure is determined by the projection map $p : E \to B$, but to indicate what the fibre is we sometimes write a fibre bundle as a short exact sequence of spaces $F \to E \to B$. The space B is called the *base space* of the bundle, E is the *total space* of the bundle and F is the *fibre* of the bundle. Besides, a *section* of the bundle is a map $s : B \to E$ such that $p \circ s = Id_B$. A *fibre-preserving homeomorphism* of two fibre bundles $p : E \to B$ and $p' : E' \to B'$ is given by a couple of maps $\psi : E \to E'$ and $\varphi : B \to B'$ such that $\varphi \circ p = p' \circ \psi$. If $B = B'$ and $\varphi = Id_B$ then we say that p and p' are *isomorphic*.

Remark 1.2. A theorem of Huebsch and Hurewicz proved in §2.7 of [Spa] says that fibre bundles over paracompact base spaces are fibrations, having the homotopy lifting property with respect to all spaces.

Example 1.1. Let E be the product $F \times B$ with $p : E \to B$ the projection onto the first factor. Then E is not just locally a product but globally one. Any such fibre bundle is called a *trivial bundle.*

Example 1.2. A fibre bundle with fibre a discrete space is a covering space. Conversely, a covering space whose fibres all have the same cardinality, for example a covering space over a connected base space, is a fibre bundle with discrete fibre.

Example 1.3. General fibre bundles can be thought of as twisted products. Familiar examples are the Moebius band, which is a twisted annulus with line segments as fibres, and the Klein bottle, which is a twisted torus with circles as fibres. In particular, the Moebius band is a bundle over S^1 with fibre an interval: take E to be the quotient of $I \times [-1, 1]$ under the identifications $(0, v) \sim (1, -v)$, with $p : E \to S^1$ induced by the projection $I \times [-1, 1] \to I$, so the fibre is $[-1, 1]$. Gluing two copies of E together by the identity map between their boundary circles produces a Klein bottle, a bundle over S^1 with fibre S^1.

Let us end this section with the following result about properties of homotopy groups of the spaces involved in a fibre bundle.

Theorem 1.2.1. *Let $p : E \to B$ be a fibre bundle. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. Hence if B is path-connected, there is a long exact sequence*

$$... \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to ... \to \pi_0(E, x_0) \to 0. \tag{1.1}$$

For the proof of the above theorem, see page 376 of [Hat].

1.3 Circle bundles

We now study a particular class of fibre bundles: the circle bundles over some compact connected surface S. Particularly we want to classify circle bundles up to fibre-preserving homeomorphism; in order to do so, we will distinguish the case in which the surface has non-empty boundary from the one in which the boundary is empty. For further details about the contents of this section, see [Sco] and [Fin].

The case with non-empty boundary

We start by considering the case where the base surface S has non-empty boundary: in this case every bundle M over S is a 3-manifold with boundary.

Since S is a non-closed surface, then S is homotopy equivalent to a wedge of circles, so that a bundle over S is determined by its restriction to the corresponding loops in S. We now look at those loops (homeomorphic to S^1): there are only two circle bundles over S^1 and their total spaces are the torus and the Klein bottle. Hence a circle bundle η over S determines a homomorphism $\omega : \pi_1(S) \to C_2$, where $C_2 = (\{1, -1\}, \cdot)$, such that if $\lambda : S^1 \to S$ is a loop on S, then the restriction of η to S^1 is the trivial circle bundle over S^1 if and only if $\omega([\lambda]) = 1$. Clearly this gives a bijection between isomorphism classes of circle bundles over S and homomorphisms $\pi_1(S) \to \mathbb{Z}_2$, which in turn correspond to elements of $H^1(S, \mathbb{Z}_2)$. So, this gives a precise classification of circle bundles over S up to isomorphism, which means that two such are considered equivalent only if there is a fibre-preserving homeomorphism between them which covers the identity map of S (see page 7).

Now we wish to classify S^1-bundles over S up to fibre-preserving homeomorphism. In order to do so, let us introduce the following notion: an automorphism of $\pi_1(S)$ is called *geometric* if it is induced by a homeomorphism of S. The following lemma is easily verified.

Lemma 1.3.1. *There exists a fibre-preserving homeomorphism between the S^1-bundles $p_1 : M_1 \to S$ and $p_2 : M_2 \to S$ if and only if there is a geometric automorphism α of $\pi_1(S)$ such that $\omega_{p_2} \circ \alpha = \omega_{p_1}$, where ω_{p_i} is the homomorphism $\pi_1(S) \to C_2$ corresponding to $p_i : M_i \to S$.*

Thus, in order to classify S^1-bundles up to fibre-preserving homemorphism, it suffices to classify homomorphisms $\pi_1(S) \to C_2$ up to the equivalence given by $\omega_1 \sim \omega_2$ if there is a geometric automorphism α of $\pi_1(S)$ such that $\omega_2 \circ \alpha = \omega_1$. To simplify the problem notice that since C_2 is abelian, the homomorphism ω factors through the homomorphism $\pi_1(S)/[\pi_1(S), \pi_1(S)] =$

$H_1(S) \to C_2$, that we still denote with ω.

If S has genus $g \geq 0$ and $n > 0$ boundary components then, referring to Fig. 1.5,

$$H_1(S) = \langle a_i, b_i, s_j \mid s_1 + \cdots + s_n = 0 \rangle_{i=1,\dots,g,\ j=1,\dots,n}$$

if S is orientable, and

$$H_1(S) = \langle v_i, s_j \mid s_1 + \cdots + s_n + 2v_1 + \cdots + 2v_g = 0 \rangle_{i=1,\dots,g,\ j=1,\dots,n} \qquad (g \geq 1)$$

if S is non-orientable. We say that the circle bundle $\eta : M \to S$ is of type:

- o_1 if $\omega(a_i) = \omega(b_i) = 1$ for all $i = 1, \dots, g$;
- o_2 if $\omega(a_i) = \omega(b_i) = -1$ for all $i = 1, \dots, g$ $(g \geq 1)$;
- n_1 if $\omega(v_i) = 1$ for all $i = 1, \dots, g$ $(g \geq 1)$;
- n_2 if $\omega(v_i) = -1$ for all $i = 1, \dots, g$ $(g \geq 1)$;
- n_3 if $\omega(v_1) = 1$ and $\omega(v_i) = -1$ for all $i = 2, \dots, g$ $(g \geq 2)$;
- n_4 if $\omega(v_1) = \omega(v_2) = 1$ and $\omega(v_i) = -1$ for all $i = 3, \dots, g$ $(g \geq 3)$.

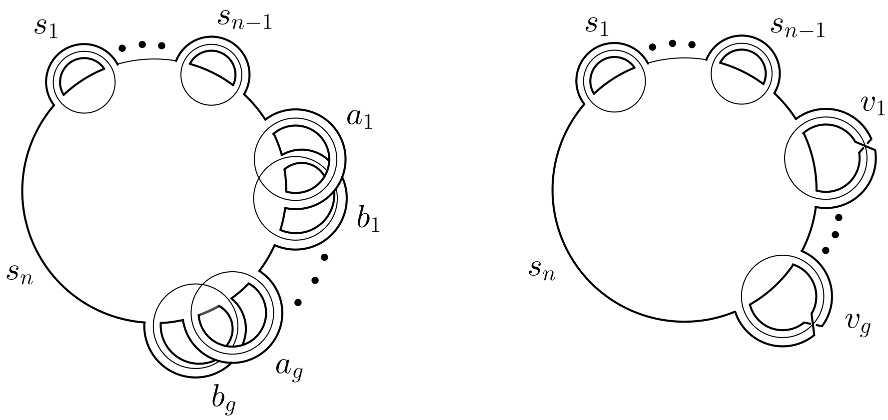

Figure 1.5: Generators of $H_1(B)$.

The following theorem, taken from [Fin], describes the classification of circle bundles over a fixed surface, up to fibre-preserving homeomorphisms.

Theorem 1.3.2. *Let S be a compact connected surface with non-empty boundary. The fibre-preserving homeomorphism classes of circle bundles over S are in 1-1 correspondence with the pairs $(k;\epsilon)$, where k is an even non-negative number which counts the number of s_j such that $\omega(sj) = -1$ and*

(a) $\epsilon = o_1,\ o_2$ when S is orientable and $\epsilon = n_1,\ n_2,\ n_3,\ n_4$ when S is non-orientable, if $k = 0$ or

(b) $\epsilon = o$ with $o := o_1 = o_2$ when S is orientable and $\epsilon = n$ with $n := n_1 = n_2 = n_3 = n_4$ when S is non-orientable, if $k > 0$.

The case with empty boundary

Now we study how to classify circle bundles over a closed surface S, following the approach of [Sco]. Any of such bundles still determines a homomorphism $\omega : \pi_1(S) \to \mathbb{Z}_2$ and any such homomorphism can occur, but in order to determine the bundle η one needs an extra invariant, denoted $b(\eta)$ and called the *Euler number* of η, which is the obstruction to the existence of a section of the bundle η. The invariant b is an integer if the total space of η is orientable and lies in $\mathbb{Z}_2$ otherwise[1]. It can take any value and a circle bundle η over S is determined by the homomorphism $\omega : \pi_1(S) \to \mathbb{Z}_2$ and by $b(\eta)$. The following naturality result explains how b alters under finite covers.

Lemma 1.3.3. *Let η be a circle bundle over a closed surface S with orientable total space M. Let $\widetilde{M}$ be a finite cover of M of degree d, so that $\widetilde{M}$ is the total space of a circle bundle $\tilde{\eta}$ over a surface $\widetilde{S}$. Let the covering $\widetilde{S} \to S$ have degree l and let m denote the degree with which the fibres of $\tilde{\eta}$ cover fibres of η so that $lm = d$.*
Then $b(\tilde{\eta}) = b(\eta) \cdot l/m$.

Note that $b(\tilde{\eta})$ must be an integer, so that the possible values of l and m are somewhat restricted. Moreover, to define b correctly, in the orientable

[1] Note that this terminology is not uniform in literature. For example in [Sco] the Euler number is defined as $-b(\eta)$ when $b(\eta)$ is an integer and 0 otherwise.

case we have to fix an orientation on M and $b(-M) = -b(M)$. In the non-orientable case b is zero if M admits a section and 1 otherwise.

In order to better understand the topological meaning of b, we can use the notion of Dehn filling: indeed a circle bundle over a closed surface can be seen as the result of filling the boundary component of a circle bundle having one toric boundary component. Let us see what this means by introducing some notions (see [Mar]). If a 3-manifold M has a spherical boundary component, we can cap it off with a ball. If M has a toric boundary component, there is no canonical way to cap it off: the simplest object that we can attach to it is a solid torus $S^1 \times D^2$, but the resulting manifold depends on the gluing map. This operation is called a Dehn filling and we now see it in detail.
Let M be a 3-manifold and $T \subset \partial M$ be a boundary torus component.

Definition 1.8. A *Dehn filling* of M along T is the operation of gluing a solid torus $S^1 \times D^2$ to M via a diffeomorphism $\varphi : S^1 \times \partial D^2 \to T$.

The closed curve $\{x\} \times \partial D^2$ is glued to some simple (i.e., without self-intersections) closed curve $\gamma \subset T$, as shown in Fig. 1.6. The result of this operation is a new manifold M^{fill}, which has one boundary component less than M.

Lemma 1.3.4. *The manifold M^{fill} depends only on the isotopy class of the unoriented curve γ.*

Proof. Decompose S^1 into two closed segments $S^1 = I \cup J$ with coinciding endpoints. The attaching of $S^1 \times D^2$ may be seen as the attaching of a 2-handle $I \times D^2$ along $I \times \partial D^2$, followed by the attaching of a 3-handle $J \times D^2$ along its full boundary. If we change γ by an isotopy, the attaching map of the 2-handle changes by an isotopy and hence gives the same manifold. The attaching map of the 3-handle is irrelevant (see Proposition 9.2.1 of [Mar]). □

We say that the Dehn filling *kills* the curve γ, since this is what really happens on the fundamental group: after the filling, γ becomes a representative

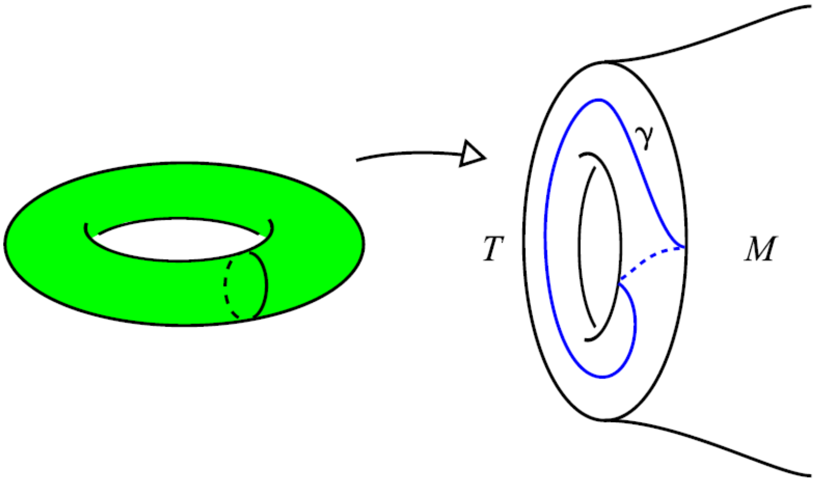

Figure 1.6: Example of Dehn filling (figure taken from [Mar]).

of the trivial element of $\pi_1(M^{fill})$.

Definition 1.9. A *slope* on a torus T is the isotopy class γ of an unoriented homotopically non-trivial simple closed curve.

We indicate the set of slopes on T by $\mathscr{S}$. If we fix a basis $(m;l)$ for $H_1(T;\mathbb{Z}) = \pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$, one can prove that every slope can be written as $\gamma = \pm(pm + ql)$ for some coprime pair $(p;q)$. Therefore we get a 1-1 correspondence $\mathscr{S} \leftrightarrow \mathbb{Q} \cup \{\infty\}$ by sending γ to $\frac{p}{q}$. If T is a boundary component of M, every number $\frac{p}{q}$ determines a Dehn filling of M that kills the corresponding slope γ.

Let $\eta : M \to S$ be circle bundle with one toric boundary component and let c be the corresponding boundary component of S. Since, as we have seen in the previous section, the base space has the homotopy type of a wedge of circles, it is easy to see that η admits a section s. Moreover, it is possile to prove that the slope $d = s(c)$ on ∂M does not depend on the choice of the section (see Corollary 10.2.3 of [Mar]). Moreover, denote with f a fibre contained in ∂M. If M is oriented, its boundary ∂M is given the orientation which, followed by an inward normal, coincides with the orientation of M. We orient d and f such that (d, f) is a positively oriented base of $H_1(\partial M)$.

Then the $\frac{b}{1}$ filling of M is a closed circle bundle with Euler number b.

Chapter 2

Seifert fibre spaces

In this chapter we will introduce the theory of Seifert fibre spaces and their main properties, following [Sco]. We will also give a combinatorial description of such manifolds, according to the approach of [CMMN], which will lead to their classification up to fibre-preserving homeomorphism and up to homeomorphism.

2.1 Definition and properties

A Seifert fibre space is a 3-manifold which can be expressed as a union of disjoint circles, called fibres, in a particular way. The definition that we will give, taken from [Sco], is a little more general than Seifert's original definition, which can be found in [Sei]. We first need some notions, before giving such definition.

Definition 2.1. The *trivial fibred solid torus* is $S^1 \times D^2$ with the trivial product fibration. Thus the fibres of $S^1 \times D^2$ are the circles $S^1 \times \{y\}$, for $y \in D^2$.

Definition 2.2. The *fibred solid torus* $T(p, q)$ of type (p, q) with $p, q \in \mathbb{Z}$, $p > 0$ and $\gcd(p, q) = 1$ is the 3-manifold constructed from a trivial fibred solid torus by cutting it open along $\{x\} \times D^2$ for some $x \in S^1$, rotating one of the discs so obtained through $\frac{q}{p}$ of a full turn and finally gluing the two discs back together. If we identify the disc D^2 with the unit disc in the complex plane, than such identification is made by gluing the two boundary discs of $[0,1]\times D^2$ with the homeomorphism $\varphi_{p,q}$ defined by $\varphi_{p,q} : (0,z) \to (1, ze^{2\pi i \frac{q}{p}})$. The fibre corresponding to $z = 0$ is called *central fibre*.

Thus $T(p,q)$ is a solid torus which is finitely covered by a trivial fibred solid torus, as we will see at page 20.

Remark 2.1. If a fibred solid torus $T(p,q)$ is constructed by a trivial one by cutting open along a 2-disc and glueing with $\frac{q}{p}$ of a full twist, then clearly all the fibres in T a part from the central one represent p times the generator of $\pi_1(T)$ and they also wind q times around the central fibre. Hence if two fibred solid tori $T(p,q)$ and $T(p',q')$ are isomorphic, then $p = p'$ and $q \equiv \pm q'$ (mod p). Note that one can alter q by an integral multiple of p by cutting $T(p,q)$ along a 2-disk and glueing back with a full twist. Hence p is an isomorphism invariant of a fibred solid torus $T(p,q)$ and q will also be an invariant if we normalise q so that $0 \leqslant q \leqslant \frac{1}{2}p$.

Definition 2.3. The invariants (p,q), normalized such that $0 \leqslant q \leqslant \frac{1}{2}p$, are called the *orbit invariants* of the central fibre of $T(p,q)$.

Definition 2.4. A *fibred solid Klein bottle* is constructed from a trivial fibred solid torus by cutting it open along $\{x\} \times D^2$ for some $x \in S^1$ and gluing the two discs back together by a reflexion. If we identify the disc D^2 with the unit disc in the complex plane, than such identification is made by gluing the two boundary discs of $[0,1] \times D^2$ with the orientation reversing homeomorphism φ defined by $\varphi : (0,z) \to (1,\bar{z})$.

In particular, as all reflexions of a disc are conjugate, there is only one fibred solid Klein bottle up to fibre-preserving homeomorphism and this is double covered by a trivial fibred solid torus, as we will see at page 20.

Definition 2.5. A *half solid torus* (resp. *half solid Klein bottle*) is the fibred manifold obtained from $I \times D^2_+$ [1] by gluing $\{0\} \times D^2_+$ with $\{1\} \times D^2_+$ by the restriction of $\varphi_{1,0}$ (resp. φ) to $\{0\} \times D^2_+$.

[1] With D^2_+ we denote the points of the disc having non-negative real part.

Definition 2.6. A *Seifert fibre space* M is a compact connected 3-manifold admitting a decomposition into disjoint circles, called *fibres*, such that each fibre has a neighborhood in M which is a union of fibres and it is fibre-preserving homeomorphic to:

- either a fibred solid torus or Klein bottle, if the fibre is contained in $\text{int}(M)$;
- either a half solid torus or a half solid Klein bottle, if the fibre is contained in ∂M.

We can deduce from the definition that any circle bundle over a surface is a Seifert fibre space.

Seifert's original definition of Seifert fibre space excluded the case of the fibred solid Klein bottle, but there are advantages in allowing such phenomenon, as we can notice from the following theorem. Let's first introduce the notion of foliation.

Definition 2.7. A *foliation by circles* of a 3-dimensional compact manifold M is a decomposition of M into a union of disjoint embedded circles, called the *leaves* of the foliation, with the following property: every point in M has a neighborhood U and a system of local coordinates $x = (x_1, x_2, x_3) : U \to \mathbb{R}^3$ such that for each leaf S^1 the components of $U \cap S^1$ are described by the equations $x_2 =$ constant, $x_3 =$ constant.

Theorem 2.1.1. [Eps] *Having a compact 3-manifold M, M is a Seifert space if and only if M is foliated by circles.*

This simple statement would be false if one kept Seifert's original definition of a Seifert fibre space.
Let us introduce some notions about the fibres of a Seifert manifold M.

Definition 2.8. A fibre of a Seifert manifold M is called *regular* if it has a neighborhood isomorphic to a trivial fibred solid torus or to a half solid torus and *critical* otherwise.

Since in the definition of Seifert fibre space we say that each fibre has a neighborhood in M which is isomorphic to a fibred solid torus or Klein bottle or half of them, saying that a fibre is critical means that it has a neighborhood in M which is isomorphic to a fibred solid torus which is *not* trivial or to a fibred solid Klein bottle (or half of it).

Remark 2.2. It follows that in a Seifert fibre space critical fibres are either isolated, corresponding to the axis of $T(p,q)$ with $p>1$, or form properly embedded compact surfaces, corresponding to the points $(I\times\{z\})/\sim_\varphi$ in a solid Klein bottle with $\mathrm{Im}(z)=0$. Indeed, a fibred solid torus has at most one critical fibre, namely the central one (while all the other fibres are regular). Moreover, each connected critical surface is either a properly embedded annulus or it is a closed surface obtained by gluing together the two boundaries of an annulus, so it is either a torus or a Klein bottle. It follows that the union of all the critical fibres in a Seifert fibre space M consists of isolated fibres together with annuli, tori or Klein bottles. We denote by $E(M)$ (resp. $SE(M)$) the union of all isolated (resp. non-isolated) critical fibres of M and call *E-fibre* (resp. *SE-fibre*) any fibre contained in $E(M)$ (resp. $SE(M)$). Finally, we set $SE(M)=CE(M)\cup AE(M)$, where $CE(M)$ contains the closed components of $SE(M)$, while $AE(M)$ contains the non-closed ones. Note that if M is orientable then $SE(M)=\emptyset$.

The components of ∂M are either tori or Klein bottles: the toric components are regularly fibred, while a Klein bottle component is either regularly fibred (see the left part of Fig. 2.1) or it contains two critical fibres of $AE(M)$ (see the right part of Fig. 2.1).

Example 2.1. Let us see an example of Seifert fibre space: the lens spaces. There are two equivalent ways of defining lens spaces: one as Dehn fillings and one as quotients of the 3-sphere.

- Let $M=S^1\times D^2$ be the solid torus where the oriented meridian $m=\{y\}\times S^1$ and longitude $l=\{x\}\times S^1$ form a basis for $H_1(\partial M,\mathbb{Z})$. The lens space $L(p,q)$ is the result of a Dehn filling (see Definition 1.8)

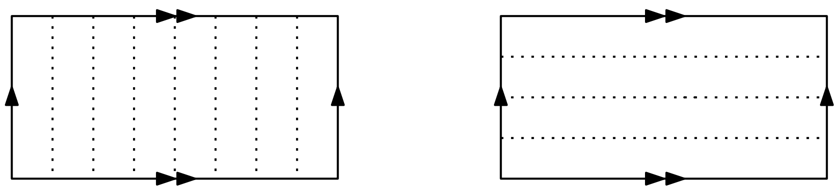

Figure 2.1: The two different fibre structures of the Klein bottle boundary components of a Seifert fibre space.

of M that kills the slope $qm+pl$, which means that $L(p,q)$ is the (q,p)-Dehn filling of the solid torus. Therefore, $L(p,q)$ is a three-manifold which decomposes into two solid tori, individuated by the slope $qm+pl$ in ∂M.

- Consider the 3-sphere S^3 as the submanifold of unit vectors in $\mathbb{C}^2$, let $f : S^3 \to S^3$ be the homemorphism defined by $f(z,w) = (e^{\frac{2\pi i}{p}}, e^{\frac{2q\pi i}{p}})$ and $\Gamma = \langle f \rangle$. We define the lens space $L(p,q)$ as the quotient S^3/Γ. One can show (as it is done in [Mar] at page 302) that the manifold S^3/Γ is the (q,p)-Dehn filling of the solid torus, therefore the two definitions are equivalent.

If we consider just the orientable ones, we can see Seifert manifolds as Dehn fillings of trivial bundles over surfaces with boundary. Lens spaces are an example of such case. Indeed, the lens space $L(p,q)$ is a Seifert fibre space with an exceptional fibre of type (q,p) and with base space an orbifold with a cone point of type p and undelying manifold S^2.

2.1.1 Base spaces

The reason for the terminology "fibre" is that one can think of a Seifert fibre space M as a generalized kind of bundle in which the circles of the foliation of M are the fibres. We say that this is a generalized concept because the presence of the critical fibres makes a Seifert fibre space a "singular" circle bundle. Indeed, if we consider M excluding a regular neighborhood (see Definition 3.5) of each critical fibre, we get a circle bundle (otherwise, M itself is not a circle bundle).

Definition 2.9. The *base space* X of a Seifert fibre space M is the quotient space of M obtained by identifying each circle to a point.

In order to understand how the base space of a Seifert fibre space is, we will look at the elementary pieces that compose such Seifert fibre space.

- If M is a trivial fibred solid torus $S^1 \times D^2$, then the base space X is clearly a 2-disc and the projection $M \to X$ is a bundle map.

- If M is a fibred solid torus $T(p,q)$, then M is p-fold covered by a trivial fibred solid torus $S^1 \times D^2$. The corresponding action of $\mathbb{Z}_p$ on $S^1 \times D^2$ is generated by a homeomorphism which is simply the product of a rotation through $\frac{2\pi}{p}$ on the S^1-factor with a rotation through $\frac{2\pi q}{p}$ on the D^2-factor. Notice that this action of $\mathbb{Z}_p$ on $S^1 \times D^2$ induces an action of $\mathbb{Z}_p$ on the base space D^2, which is generated by a rotation through $\frac{2\pi q}{p}$. It follows that the base space X obtained from M by identifying each fibre to a point can be naturally identified with the quotient of D^2 by this action of $\mathbb{Z}_p$, that is, a cone orbifold (i.e. a 2-dimensional orbifold with a cone point) with cone angle $\frac{2\pi}{p}$.
  Notice that the projection $M \to X$ is not a bundle map in the usual sense (because of the singularity at the cone point), but we will think of it as a bundle map in a generalized sense.

- If M is a fibred solid Klein bottle, then M is double covered by a trivial fibred solid torus $S^1 \times D^2$. The corresponding action of $\mathbb{Z}_2$ on $S^1 \times D^2$ is generated by a homeomorphism which is simply the product of a rotation through π on the S^1-factor with a reflexion of the D^2-factor. Clearly the space X obtained from M by identifying fibres to a point can be naturally identified with the quotient of D^2 by this action of $\mathbb{Z}_2$, which is an orbifold with a reflector line.

Since locally any Seifert fibre space M is made by fibred solid tori and fibred solid Klein bottles, the quotient space X of M obtained by identifying each fibre to a point is locally as one of the types just analysed. Therefore X is, topologically speaking, a surface and naturally has an orbifold structure (see Definition 1.1) in which cone points correspond to orientation preserving critical fibres (i.e. the central fibre of non-trivial solid tori) and points on reflexion curves correspond to orientation reversing critical fibres (i.e. the critical fibres of solid Klein bottles). From the previous analysis, it also follows that the orbifold X will have no corner reflectors. Besides, any 2-dimensional orbifold without corner reflectors is the base space of at least one Seifert fibre space. It is natural to construct a Seifert bundle over such an orbifold X by starting with a circle bundle over a surface (which is obtained from X excluding the interior of a regular neighborhoods of each of its singular points) and then gluing on pieces corresponding to the components of the singular set of X. We will see such construction in Section 2.2.1.

Remark 2.3. If M is a manifold without boundary, then the base space X is an orbifold without boundary, as reflector curves do not form part of the boundary of an orbifold. In general, ∂X is the image of ∂M under the projection $M \to X$. Note that, since M is compact by definition, ∂M is a union of tori and Klein bottles.

2.1.2 Universal covering

As we have seen in Remark 2.2, the union of all the regular fibres in any connected fibre space M is connected and forms a bundle (in the usual sense). In particular, all the regular fibres of M are freely homotopic to each others and any critical fibre has a power which is freely homotopic to a regular fibre. As a consequence, in a covering $\widetilde{M}$ of M, the foliation of M by circles gives rise to a foliation of $\widetilde{M}$ by circles (which must again be a Seifert fibration) or to a foliation of $\widetilde{M}$ by lines. In either case, the base space of $\widetilde{M}$, defined by identifying each leaf of the foliation to a point, is an orbifold covering of the base space of M. This is obvious if M is a fibred solid torus or Klein bottle,

and then it follows for any Seifert fibre space. Conversely, if M is a Seifert fibre space over an orbifold X and if $\widetilde{X}$ is an orbifold covering of X, there is a natural covering space $\widetilde{M}$ of M with an orbit space $\widetilde{X}$ which should be thought as the bundle over $\widetilde{X}$ induced by the projection $\widetilde{X} \to X$ (i.e. the foliation is maintained). Again this statement is clear if M is a fibred solid torus or Klein bottle and then follows for any Seifert fibre space.
We are now in a position to show a significant fact about Seifert fibre spaces.

Lemma 2.1.2. *Let M be a Seifert fibre space without boundary. Then the universal covering $\widetilde{M}$ of M is homeomorphic to one of S^3, $\mathbb{R}^3$ or $S^2 \times \mathbb{R}$. Further, the induced foliation of $\widetilde{M}$ by circles or lines gives $\widetilde{M}$ one of the following structures:*

(a) a Seifert bundle over one of the orbifolds S^2, $S^2(p)$, $S^2(p,q)$ where p and q are coprime (see page 6);

(b) a product line bundle over $\mathbb{R}^2$;

(c) a product line bundle over S^2.

Proof. First suppose that the natural foliation of $\widetilde{M}$ is by circles, so that $\widetilde{M}$ is a Seifert fibre space. As $\widetilde{M}$ is simply connected, it has no proper coverings. Hence the base space of $\widetilde{M}$ is an orbifold $\widetilde{X}$ with no proper coverings. The only such orbifolds are S^2, $S^2(p)$, $S^2(p,q)$, where p and q are coprime, and $\mathbb{R}^2$. The case $\widetilde{X} = \mathbb{R}^2$ cannot occurr, for then $\widetilde{M}$ would be $S^1 \times \mathbb{R}^2$ and so not be simply connected. In the other cases, we write $\widetilde{X} = D_1 \cup D_2$, where D_1 and D_2 are 2-discs with a possible interior cone point. Thus $\widetilde{M}$ is the union of two fibred solid tori T_1 and T_2 where T_i has base space D_i for $i = 1, 2$; and hence is a lens space (see Example 2.1). As $\widetilde{M}$ is simply connected, it must be S^3.
If $\widetilde{M}$ is foliated by lines, then $\widetilde{M}$ is a line bundle over its base space $\widetilde{X}$ and $\widetilde{X}$ is an orbifold without singularities. Again $\widetilde{X}$ is also simply connected so that $\widetilde{X}$ must be S^2 or $\mathbb{R}^2$. This gives the cases *(b)* and *(c)* of the lemma. □

2.1.3 Fundamental group

We can now see the rather special structure of $\pi_1(M)$, when M is a Seifert fibre space.

Lemma 2.1.3. *Let M be a Seifert space with base orbifold X. There is an exact sequence*

$$1 \longrightarrow J \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(X) \longrightarrow 1$$

where J denotes the cyclic subgroup of $\pi_1(M)$ generated by a regular fibre and $\pi_1^{orb}(X)$ denotes the orbifold fundamental group of X. The group J is infinite except in the cases where M is covered by S^3.

Proof. For $\pi_1(M)$ acts on $\widetilde{M}$ preserving the natural foliation, there is an induced action of $\pi_1(M)$ on $\widetilde{X}$. This gives a natural homomorphism $\pi_1(M) \to \pi_1^{orb}(X)$. The kernel J of this map consists of covering translations of $\widetilde{M}$ which project to the identity map on $\widetilde{X}$. As J acts freely on any one of the fibres of $\widetilde{M}$, we see that J is infinite cyclic if $\widetilde{M}$ is not compact and it is finite cyclic when M is S^3. As J is normal in $\pi_1(M)$, there are no base point problems with the statement that J is generated by a regular fibre. □

This exact sequence gives another reason for regarding M as a kind of bundle over X. Indeed for a circle bundle $\eta : M \to S$ over a surface different from the sphere or the projective space, the exact sequence of Theorem 1.2.1 reduces to

$$1 \to \pi_1(S^1) \to \pi_1(M) \to \pi_1(S) \to 1$$

since the universal covering of S is contractible being $\mathbb{R}^2$, and so $\pi_2(S) = 1$ (see page 342 of [Hat]).

2.2 Classification

Our aim now is to classify Seifert fibre spaces: firstly up to fibre-preserving homeomorphism and secondly up to homeomorphism. We will also see the difference between the two classifications.

2.2.1 Classification up to fibre-preserving homeomorphism

In Subsection 2.1.1 we have seen Seifert fibre spaces as fibre bundles over orbifolds and now we want to classify them up to fibre-preserving homeomorphism, which is a generalization of bundle isomorphism for bundles over orbifolds. The first invariant of a Seifert fibre space M is the base orbifold X. Let X' denote $X \setminus \mathrm{int}(N)$, where N is a regular neighborhood of the singular locus. Then we have a circle bundle over X', which is a surface, and we have already discussed the classification of such objects in Section 1.3. There are no extra invariants to attach to the reflector lines and circles of X because a circle bundle over X' determines uniquely a Seifert fibre space over the union of X' with the components of N which contain reflector lines or circles. Our given Seifert fibre space M over X is completed by adding the fibred solid tori corresponding to the cone points of X. As we have already seen at page 16, each fibred solid torus determines a pair of coprime integers (p, q), called orbit invariants, which are invariants for the total Seifert fibre space too. One more invariant is needed to complete the classification of Seifert fibre spaces in the case when the base space is a closed orbifold. This is a generalization of the invariant b defined earlier for circle bundles over closed surfaces at page 11. As in that case, b is an integer if the Seifert fibre space is orientable and lies in $\mathbb{Z}_2$ otherwise. Note that if X has reflector curves, then b should be defined to be zero. On the other hand, if X has no reflector curves, Seifert showed that b could take any value except that if some pair of orbit invariants is $(2, 1)$, then b must be zero.

To sum up, a Seifert fibre space is determined by:

- the base orbifold X;
- the circle bundle over X' obtained restricting the circle bundle over the orbifold X defined by M;
- the orbit invariants (p, q) of the critical fibres corresponding to cone points;

- the value of the invariant b.

A combinatorial description of Seifert fibre spaces is given in [Fin] for the closed case and in [CMMN] has been extended to the boundary case. Let us first introduce some notation so that we can construct Seifert fibre spaces in a combinatorial way and then get their classification. We start with the combinatorial description for Seifert fibre spaces, recovering the notation seen at pages 10 and 18. Let

- g, t, k, m_+, m_-, r be non-negative integers such that $k+m_-$ is even and $k \leq t$;
- ϵ be a symbol belonging to the set $\mathcal{E} = \{o, o_1, o_2, n, n_1, n_2, n_3, n_4\}$ such that (i) $\epsilon = o, n$ if and only if $k + m_- > 0$, (ii) if $\epsilon = n_4$ then $g \geq 3$, (iii) if $\epsilon = n_3$ then $g \geq 2$ and (iv) if $\epsilon = o_2, n, n_1, n_2$ then $g \geq 1$;
- $h_1, \ldots, h_{m_+}$ and $k_1, \ldots, k_{m_-}$ be non-negative integers such that $h_1 \leq \cdots \leq h_{m_+}$ and $k_1 \leq \cdots \leq k_{m_-}$;
- (p_j, q_j) be lexicographically ordered pairs of coprime integers such that $0 < q_j < p_j$ if $\epsilon = o_1, n_2$ and $0 < q_j \leq p_j/2$ otherwise, for $j = 1, \ldots, r$;
- b be an arbitrary integer if $t = m_+ = m_- = 0$ and $\epsilon = o_1, n_2$; $b = 0$ or 1 if $t = m_+ = m_- = 0$ and $\epsilon = o_2, n_1, n_3, n_4$ and no $p_j = 2$; $b = 0$ otherwise.

The previous parameters with the given conditions are called *normalized*, and we denote by

$$\left\{b; (\epsilon, g, (t, k)); \left(h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}\right); ((p_1, q_1), \ldots, (p_r, q_r))\right\}$$

the Seifert fibre space constructed as follows.

If $b = 0$, denote by B^* a compact connected genus g surface having $s = r+t+m_++m_-$ boundary components and being orientable if $\epsilon = o, o_1, o_2$ and non-orientable otherwise. By Theorem 1.3.2 there is a unique S^1-bundle over B^* associated to the pair $(k+m_-, \epsilon)$, up to fibre-preserving homeomorphism:

call it M^*. Note that M^* has $k + m_-$ boundary components which are Klein bottles and the remaining $r + t - k + m_+$ ones are tori. Denote by $c_1, \ldots, c_s$ the boundary components of B^*, numbering them such that the last $k + m_-$ correspond to Klein bottles in M^*. Let $\partial_1 B^* = c_1 \cup \ldots \cup c_{r+t-k+m_+}$ and $\partial_2 B^* = \partial B^* \setminus \partial_1 B^*$. Finally, denote by $s^* : B^* \to M^*$ a section of $f^* : M^* \to B^*$.

(a) For $j = 1, \ldots, r$ fill the toric boundary component $(f^*)^{-1}(c_j)$ of M^* with a solid torus by sending the boundary of a meridian disk of the solid torus into the curve $p_j d_j + q_j f_j$, where f_j is a fibre and $d_j = s^*(c_j)$;

(b) for $i = 1, \ldots, m_+$ (resp. $j = 1, \ldots, m_-$) consider h_i (resp. k_j) disjoint closed arcs inside the boundary component c_{i+r} of $\partial_1 B^*$ (resp. $c_{j+r+t-k+m_+}$ of $\partial_2 B^*$) and, for each arc and each point x of the arc, attach a Möbius strip along the boundary to the fibre $(f^*)^{-1}(x)$, where the Möbius strip is foliated by circles. On the whole, we attach h_i (resp. k_j) disjoint copies of $N \times I$ to the boundary of M^* corresponding to the counter-image of c_{i+r} (resp. $c_{j+r+t-k+m_+}$). So the boundary component remains unchanged if $h_i = 0$ (resp. $k_j = 0$) and it is partially filled otherwise. In the latter case instead of the initial boundary component we have h_i (resp. k_j) Klein bottle boundary components;

(c) for $i = 1, \ldots, t - k$ (resp. $j = 1, \ldots, k$) glue a copy of $N \times S^1$ (resp. $N \widetilde{\times} S^1$) to each toric (resp. Klein bottle) boundary component of M^* along the boundary via a homeomorphism which is fibre-preserving with respect to the fibration of the components to glue. Namely, as in the previous step, for each point $x \in c_{i+r+m_+}$ (resp. $x \in c_{j+r+t-k+m_++m_-}$) we attach a Möbius strip along the boundary to the fibre $(f^*)^{-1}(x)$.

If $b \neq 0$ (and therefore $t = m_+ = m_- = 0$) we modify the above construction as follows: we take a surface B^* with $r + 1$ boundary components and fill the first r-ones boundary components of M^* as described in (a) and the last one by sending the boundary of a meridian disk of the solid torus into $d_{r+1} + bf_{r+1}$ (i.e., with $(p_{r+1}, q_{r+1}) = (1, b)$).

The resulting manifold is the Seifert fibre space

$$M = \left\{b; (\epsilon, g, (t, k)) ; \left(h_1, \dots, h_{m_+} \mid k_1, \dots, k_{m_-}\right) ; ((p_1, q_1), \dots, (p_r, q_r))\right\}.$$

Note that when $t = m_+ = m_- = 0$, the above construction gives the classical closed Seifert fibre space $(b; \epsilon, g; (p_1, q_1), \dots, (p_r, q_r))$ of [Sei].

From the above construction it follows that the critical set of M is composed by: (a) an E-fibre of type[2] (p_j, q_j) for $j = 1, \dots, r$, (b) t closed critical surfaces, k of which are Klein bottles while the remaining $t - k$ are tori and (c) $t' = h_1 + \dots + h_{m_+} + k_1 + \dots + k_{m_-}$ critical surfaces homeomorphic to annuli. Moreover, the boundary of M has t' components which are Klein bottles with two critical fibres (contained in $AE(M)$) and, for each $h_i = 0$ (resp. $k_j = 0$), a toric (resp. Klein bottle) boundary component without critical fibres.

The singular locus of the base orbifold B consists of: (a) r cone points of cone angles $2\pi/p_1, \dots, 2\pi/p_r$ (in figures each cone point will be decorated with the corresponding pair (p_j, q_j)), (b) t reflector circles and (c) t' reflector arcs. The underlying surface of the orbifold has genus g and it is orientable if and only if $\epsilon = o, o_1, o_2$. Moreover, it has $m_+ + m_- + t$ boundary components: one boundary component containing h_i (resp. k_j) disjoint reflector arcs for each $i = 1, \dots, m_+$ (resp. $j = 1, \dots, m_-$), and one boundary components for each reflector circle. We decorate by the symbol "$-$" each boundary component of the underlying surface having a Klein bottle as counterimage in M.

Remark 2.4. Conditions on the invariants ensuring the orientability and the closeness of a Seifert fibre space are the following:

(i) M is orientable if and only if $t = m_- = 0$, $h_i = 0$ for all $i = 1, \dots, m_+$ and $\epsilon = o_1, n_2$;

[2]Note that a fibred tubular neighborhood of an E-fibre of type (p_j, q_j) is fibre-preserving equivalent to $T(p_j, r_j)$ with $r_j q_j \equiv 1 \mod p_j$.

(ii) M is closed if and only if $m_+ = m_- = 0$.

Example 2.2. The Seifert fibre space $\{0; (o, 4, (1,1)); (1 \mid 0); ((3,1), (5,2))\}$ is depicted in Fig. 2.2. The thick lines and points represent the singular locus of the base orbifold B. The Seifert fibre space has two E-fibres of type $(3,1)$ and $(5,2)$, one Klein bottle critical surface and one annulus critical surface. The boundary consists of two Klein bottles, one with two critical fibres and another without critical fibres.

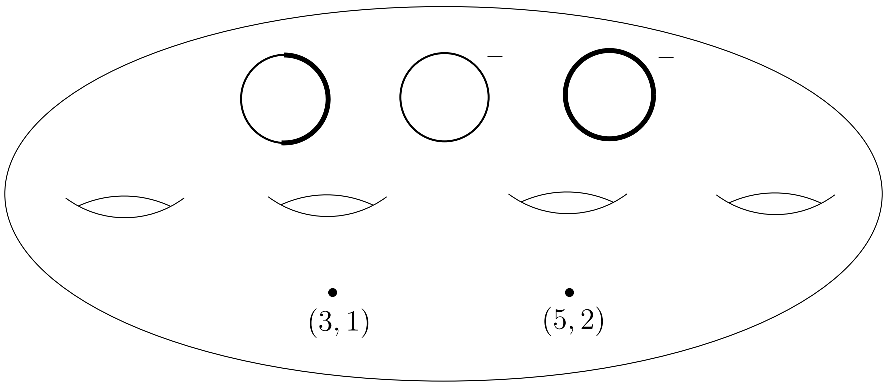

Figure 2.2: The Seifert fibre space $\{0; (o, 4, (1,1)); (1 \mid 0); ((3,1), (5,2))\}$.

Theorem 2.2.1. *Every Seifert fibre space is uniquely determined, up to fibre-preserving homeomorphism, by the normalized set of parameters*

$$\left\{b; (\epsilon, g, (t,k)); \left(h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}\right); ((p_1, q_1), \ldots, (p_r, q_r))\right\},$$

and, when $M \setminus SE(M)$ is orientable (i.e., $\epsilon \in \{o_1, n_2\}$), by the following additional conditions: (i) if M is closed and orientable, then $b \geq -r/2$ and, if $b = -r/2$, $0 < q_l < p_l/2$, (ii) if M is non-closed or non-orientable then $0 < q_l < p_l/2$; where l is the minimum j, if any, such that $p_j > 2$.

For the proof of the above theorem, see page 15 of [CMMN].

2.2.2 Classification up to homeomorphism

The classification of Seifert fibre spaces up to fibre-preserving homeomorphism and the one up to homeomorphism don't coincide in general, but they do in most of the cases, as the following result shows.

Theorem 2.2.2. *Let M be a compact 3-manifold homeomorphic to two Seifert fibre spaces which are not fibre-preserving homeomorphic. Then one of the following cases occurs:*

(a) M is covered by S^3 or $S^2 \times \mathbb{R}$;

(b) M is covered by $S^1 \times S^1 \times S^1$;

(c) M is covered by $S^1 \times D^2$ or a I-bundle over the torus or Klein bottle.

For the proof of the above theorem, see page 439 of [Sco].

The conclusion to be drawn from the previous result is that, while the fibre-preserving homeomorphism of Seifert bundle structures always implies the existence of a homeomorphism between the Seifert spaces, the vice versa is not always true but it holds in most of the cases.
In fact, a still stronger result holds: for most manifolds which admit a Seifert fibration, this fibration is unique up to homotopy, and not just up to fibre-preserving homeomorphism as we have just seen. The precise statement is as follows.

Theorem 2.2.3. *Let M be a compact Seifert fibre space and let $f : M \to N$ be a homeomorphism. Then f is homotopic to a fibre preserving homeomorphism (and hence an isomorphism of Seifert bundles), unless one of the following occurs.*

(a) M is covered by S^3 or $S^2 \times \mathbb{R}$;

(b) M is covered by $S^1 \times S^1 \times S^1$;

(c) M is covered by $S^1 \times D^2$ or a I-bundle over the torus or Klein bottle.

For the proof of the above theorem, see page 440 of [Sco].

We conclude this chapter with some results that allow us to characterise 3-manifolds having a Seifert fibre space structure. We first see the cases in which the fundamental group of the manifold M is infinite, in particular when M is $\mathbb{P}^2$-irreducible (Theorem 2.2.4) and when M is non-irreducible (Theorem 2.2.5), and finally a result about the case in which the fundamental group is finite (Theorem 2.2.6). For a deeper analysis of the topic, see [Pré]. Before stating the results, we give the following definitions.

Definition 2.10. A 3-manifold M is called *irreducible* if every 2-sphere in M bounds a 3-ball. Otherwise, M is called *reducible.*

Definition 2.11. A 3-manifold M is called $\mathbb{P}^2$*-irreducible* when it is irreducible and it contains no 2-sided $\mathbb{RP}^2$.

Remark 2.5. An orientable manifold is $\mathbb{P}^2$-irreducible if and only if it is irreducible.

Remark 2.6. If M is reducible, then either it can be decomposed into a non-trivial connected sum (see Definition 3.21), or is one of the manifolds $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, the non-orientable S^2-bundle over S^1.

Theorem 2.2.4. *Let M be a $\mathbb{P}^2$-irreducible 3-manifold whom π_1 is infinite and contains a non-trivial cyclic normal subgroup. Then M is a Seifert bundle.*

Theorem 2.2.5. *Let M be an orientable non-irreducible 3-manifold whose π_1 contains an infinite cyclic normal subgroup. Then the manifold obtained by filling all spheres in ∂M with balls is a Seifert fiber space.*

Theorem 2.2.6. *A 3-manifold with finite π_1 containing no sphere in its boundary is a closed Seifert fibered space when orientable and $\mathbb{P}^2 \times I$ when non-orientable.*

Chapter 3

Complexity

In this chapter we will introduce the complexity theory: at first, we will see the topic in a general way that concerns all compact 3-manifolds, following the approach of [Mat]. Then, from Section 3.3 on, we will focus on the estimation of the complexity of Seifert fibre spaces showing some results from [CMMN] and in Section 3.4 we will see some examples in which we construct what is called an *almost simple spine* of manifolds having complexity zero.

3.1 Spines of 3-manifolds

We want to study the geometry and topology of 3-manifolds using their complexity as a tool. For such purpose, we will need the central notion of spine of a 3-manifold.

3.1.1 Collapsing

In order to discuss spines, we need to introduce the definition of collapsing. We start with the definition of an elementary simplicial collapse.
Let K be a simplicial complex, and let σ^n, $\delta^{n-1} \in K$ be two open simplices such that σ is principal, i.e. σ is not a proper face of any simplex in K, and δ is a free face of it, i.e. δ is not a proper face of any simplex in K other than σ.

Definition 3.1. The transition from K to $K \setminus (\sigma \cup \delta)$ is called an *elementary simplicial collapse*, see Fig. 3.1.
A *simplicial collapse* of a simplicial complex K onto its subcomplex L is a sequence of elementary simplicial collapses transforming K into L.

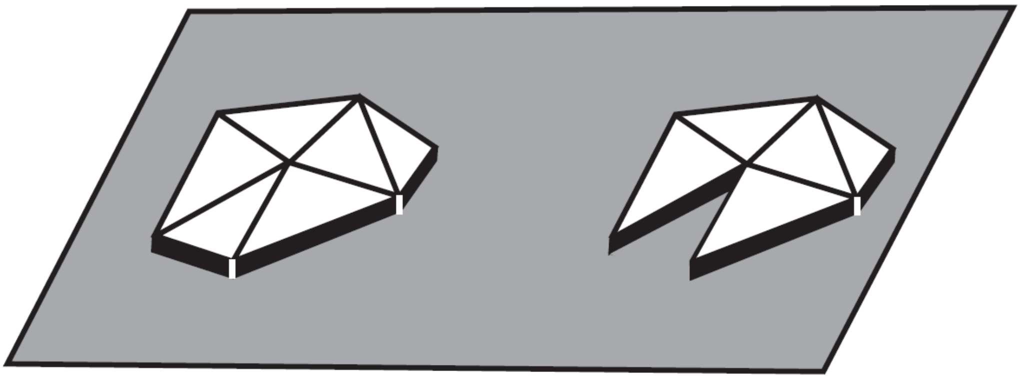

Figure 3.1: Elementary simplicial collapse (figure taken from [Mat]).

Definition 3.2. A polyhedron P *collapses* to a subpolyhedron Q (notation: $P \searrow Q$) if for some triangulation (K, L) of the pair (P, Q) the complex K collapses onto L by a sequence of elementary simplicial collapses.

In general, there is no need to triangulate the polyhedron P to construct a collapse $P \searrow Q$; for this purpose one can use larger blocks instead of simplexes. It is clear that any n-dimensional cell B^n collapses to any $(n-1)$-dimensional face $B^{n-1} \subset \partial B^n$. It follows that the collapse of P to Q can be performed at once by removing cells as we see in the following definition.

Definition 3.3. Let P a polyhedron and Q a subpolyhedron of P, such that $P = Q \cup B^n$ and $Q \cap B^n = B^{n-1}$, where B^n is an n-cell and B^{n-1} is an $(n-1)$-dimensional face of B^n. The transition from P to Q is called an *elementary polyhedral collapse*, see Fig. 3.2.

It is easy to see that an elementary simplicial collapse is a special case of an elementary polyhedral collapse. Likewise, it is possible to choose a

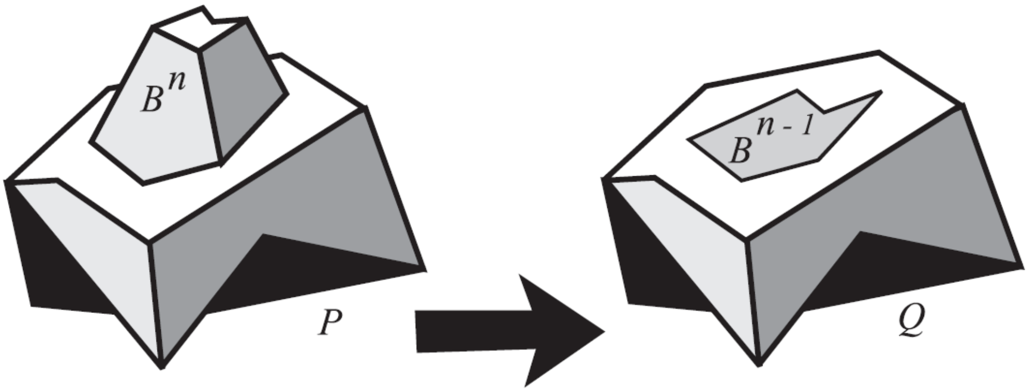

Figure 3.2: Elementary polyhedral collapse (figure taken from [Mat]).

triangulation of the ball B^n such that the collapse of B^n onto its face B^{n-1} can be expressed as a sequence of elementary simplicial collapses. It follows that the same is true for any elementary polyhedral collapse.
We now extend the notion of polyhedral collapse.

Definition 3.4. A *polyhedral collapse* of a polyhedron P onto its subpolyhedron Q is a sequence of elementary polyhedral collapses.

The notion of collapse allows us to give the definition of regular neighborhood, taken from [Hud].

Definition 3.5. Let P, Q be compact polyhedra in an n-manifold M. We say that Q is a *regular neighborhood* of P in M if

(a) Q is an n-manifold,

(b) Q is a topological neighborhood of P in M,

(c) $Q \searrow P$.

The above definition can be extended from polyhedra to compact manifolds, up to homeomorphism, therefore it makes sense to talk about regular neighborhoods of manifolds.

3.1.2 Spines

Definition 3.6. Let M be a compact connected 3-dimensional manifold with boundary. A subpolyhedron $P \subset M$ is called a *spine* of M if $M \searrow P$, that is, M collapses to P.

We can extend the notion of spine also to 3-manifolds which are not compact and connected as follows.

Definition 3.7. A *spine* of a closed connected 3-manifold M is a spine of $M \setminus \text{int}(B^3)$ where B^3 is a 3-ball in M.
A *spine* of a disconnected 3-manifold is the union of spines of its connected components.

Remark 3.1. A simple argument shows that any compact triangulated 3-manifold M always has a spine of dimension ≤ 2. Indeed, let M collapse to a subcomplex K. If K contains a 3-simplex, then K contains a 3-simplex with a free face, so the collapsing can be continued.

It is often convenient to view 3-manifolds as mapping cylinders over their spines and as regular neighborhoods of the spines. The following theorem justifies these points of view. We first recall the definition of a mapping cylinder.

Definition 3.8. Let $f : X \to Y$ be a map between topological spaces. The *mapping cylinder* C_f is defined as $Y \cup (X \times [0,1])/\sim$, where the equivalence relation is generated by identifications $(x,1) = f(x)$ for all $x \in X$. If Y is a point, then C_f is called the *cone* over X. See Fig. 3.3.

Theorem 3.1.1. *The following conditions on a compact subpolyhedron* $P \subset \text{int}(M)$ *of a compact 3-manifold* M *with boundary are equivalent:*

(a) P *is a spine of* M*;*

(b) M *is homeomorphic to a regular neighborhood of* P *in* M*;*

(c) M *is homeomorphic to the mapping cylinder of a map* $f : \partial M \to P$*;*

Figure 3.3: The mapping cylinder and the cone (figure taken from [Mat]).

(d) The manifold $M \setminus P$ is homeomorphic to $\partial M \times [0,1)$.

Proof. $(a) \Rightarrow (b)$. This implication is valid in view of the following property of a regular neighborhood of P in M: it is a submanifold of M that can be collapsed onto P, as we have seen in Definition 3.5.
$(b) \Rightarrow (c)$ Let a pair (K, L) of simplicial complexes triangulate the pair (M, P). Denote by $\mathrm{St}(L, K'')$ the star of L in the second barycentric subdivision K'' of K. According to the theorem on regular neighborhoods of [RS], M can be identified with the underlying space $N = |\mathrm{St}(L, K'')|$ of the star. The possibility of representing the manifold N in the form of the cylinder of a map $f : \partial N \to P$ is one of the properties of the star.
$(c) \Rightarrow (d)$. This implication is obvious.
$(d) \Rightarrow (a)$. Suppose the manifold $M \setminus P$ is homeomorphic to $\partial M \times [0,1)$. Denote by N a small regular neighborhood of P in M. Since we have proved the implications $(b) \Rightarrow (c) \Rightarrow (d)$, we can apply them to N. Therefore the manifold $N \setminus P$ is homeomorphic to $\partial N \times [0,1)$. Note that the embedding of $N \setminus P$ into $\partial M \times [0,1)$ is proper in the following sense: the intersection of any compact set $C \subset \partial M \times [0,1)$ with $N \setminus P$ is compact. In this case the manifold $\mathrm{Cl}(M \setminus N)$, i.e., the closure of $M \setminus N$, is homeomorphic to $\partial N \times I$. Since $\partial N \times I \searrow \partial N \times \{0\}$ and $N \searrow P$, it follows that $M \searrow P$. □

3.1.3 Simple and special spines

A spine of a 3-manifold M carries much information about M. In particular, if $\partial M \neq \emptyset$ then any spine P of M is homotopy equivalent to M and hence determines the homotopy type of M. Nevertheless, it is possible for two non-homeomorphic manifolds to have homeomorphic spines. In order to eliminate this difficulty, we will restrict our class of spines to those called special spines. We will give a precise definition shortly afterwards. First we must define the notion of simple polyhedron.

Definition 3.9. A compact polyhedron P is called *simple* if the link of each point $x \in P$ is homeomorphic to one of the following 1-dimensional polyhedra:

(a) a circle (such a point x is called *non-singular*);

(b) a circle with a diameter (such a point x is a *triple point* and a line made of triple points is said *triple line*);

(c) a circle with three radii (such a point x is a *true vertex*).

Typical neighborhoods of points of a simple polyhedron are shown in Fig. 3.4. The polyhedron used here to illustrate the true vertex singularity will

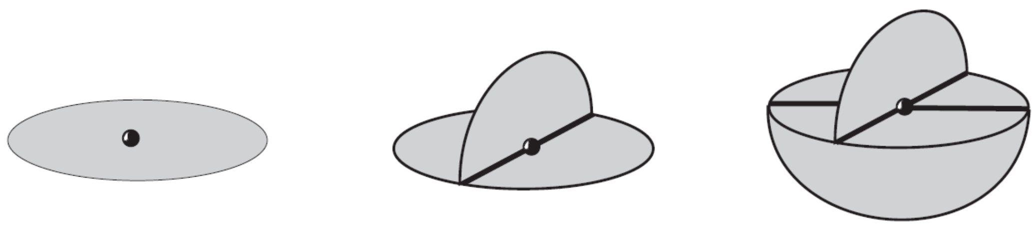

Figure 3.4: Allowable neighborhoods in a simple polyhedron (respectively, nonsingular point, triple point, true vertex) (figure taken from [Mat]).

be denoted by E and we will call it a *butterfly* (see the right part of Fig. 3.4). Its body consists of four segments having a common endpoint, and it has six *wings*. Each wing spans two segments, and each pair of the segments is spanned by exactly one wing.

Definition 3.10. The set of singular points of a simple polyhedron (that is, the union of its true vertices and triple lines) is called its *singular graph* and is denoted by SP.

In general, SP is not a graph whose vertices are the true vertices of P, since it can contain closed triple lines without true vertices. If there are no closed triple lines, then SP is a regular graph of degree 4, i.e. every true vertex of SP is incident to exactly four edges.
Let us consider the structure of simple polyhedra in detail. Each simple polyhedron is naturally stratified. In this stratification each stratum of dimension 2, which is called a *2-component*, is a connected component of the set of non-singular points. Strata of dimension 1, called *1-strata*, consist of open or closed triple lines, and dimension 0 strata are true vertices.

Definition 3.11. A simple polyhedron P is called *special* if:

(a) each 1-stratum of P is an open 1-cell;

(b) each 2-component of P is an open 2-cell.

Definition 3.12. A spine of a 3-manifold is called *simple* or *special* if it is a simple or special polyhedron, respectively.

Example 3.1. An example of special spine of the 3-ball is shown in Fig. 3.5: Bing's House with two rooms, which is a cube B decomposed by the middle section into two rooms. Each room has a vertical tube entrance joined to the walls by a quadrilateral membrane.
Let us describe a collapse of the 3-ball onto Bing's House. First we collapse the 3-ball onto a cube B which is contained in it. Next we penetrate through the upper tube into the lower room and exhaust the interior of the room keeping the quadrilateral membrane fixed. Finally, we do the same with the upper room.

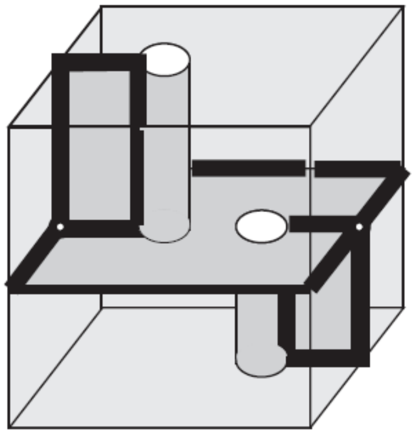

Figure 3.5: Bing's House with two rooms (figure taken from [Mat]).

Before stating the following theorem, we need the notion of k-handles attached to 3-manifolds, taken from the definition of [Mar] of k-handles attached to n-manifolds.

Definition 3.13. Let M be a (possibly empty or disconnected) 3-manifold with boundary and $0 \leq k \leq 3$. A *k-handle* is the 3-dimensional ball decomposed as $D^k \times D^{3-k}$, where D^i denotes the i-th ball. By attaching a k-handle to M we mean gluing $D^k \times D^{3-k}$ to M using a diffeomorphism $\phi : \partial D^k \times D^{3-k} \to Y \subset \partial M$. Clearly, by attaching a k-handle to M we create a new manifold that, generally, won't be diffeomorphic to M.

Theorem 3.1.2. *Any compact 3-manifold possesses a special spine.*

Proof. Let M be a 3-manifold with boundary and let T be a triangulation of M. Consider the handle decomposition generated by T. This means the following: we replace each vertex with a ball B_i (a handle of index 0), each edge with a *beam* C_j (a handle of index 1), and each triangle with a *plate* P_k (a handle of index 2), see Fig. 3.6. The rest of M consists of index 3 handles. Let P be the union of the boundaries of all handles: $P = \bigcup_{i,j,k} \partial B_i \cup \partial C_j \cup \partial P_k$ (the boundaries of index 3 handles do not contribute to the union). Then P is a special polyhedron and is indeed a special spine of M with an open ball removed from each handle. Alternatively, one can construct a special spine of multipunctured M by taking the union of ∂M and the 2-dimensional skeleton of the cell decomposition dual to T.

It remains to show that if M with $m > 1$ balls removed has a special spine, then M with $m - 1$ balls removed also has a special spine. We do that in two steps. First, we show that as long as the number of removed balls is greater than one, there exist two distinct balls separated by a 2-component of P. This can be achieved by considering a general position arc connecting two distinct balls and observing that it must pass transversally through at least one separating 2-component.

The second step consists in puncturing the spine to fuse these two balls into one so that the remaining spine is also special. If we just made a hole to cut our way through the 2-component, the boundary of the hole would contain points of forbidden types. One can try to collapse the punctured spine as long as possible with hoping to get a special polyhedron, but sometimes we would end up with a polyhedron which is not even simple. So we must find a way to avoid this. The *arch construction* illustrated in Fig. 3.7 gives us a solution.

The arch connects two different balls separated by a 2-cell C in such a way as to form a special polyhedron. To see this, consider how we get such an arch: first add a “blister” to the spine as illustrated in Fig. 3.7. This is done by considering a neighborhood of the spine and then collapsing most of it (except the blister) back down to the spine. Squeeze in the blister until what remains is a filled tube attached by a membrane F to the spine. From each end of the tube, push in its contents until all that remains is a disk in the middle of the tube. Now remove this disk.

The claim is that we get a special spine for M with the number of removed balls decreased by one. The crux of the matter is that each of the 2-components of the new spine is a 2-cell. Actually the only “suspicious” 2-component is D, that appeared after joining 2-components A and B by the arch. Clearly, D is a 2-cell provided $A \neq B$ (if $A = B$, we get either an annulus or a Moebius band). To see that the proviso always holds, one should use the fact that we have started with two distinct balls separated by the 2-component C: A differs from B, since they separate different pairs of balls.

After a few such steps we get a special spine P' of once punctured M. If M is closed, then we are done. If not, we slightly push P' into the interior of M and use the arch construction again to unite the ball and a component of $M \setminus P'$ homeomorphic to $\partial M \times [0, 1)$. □

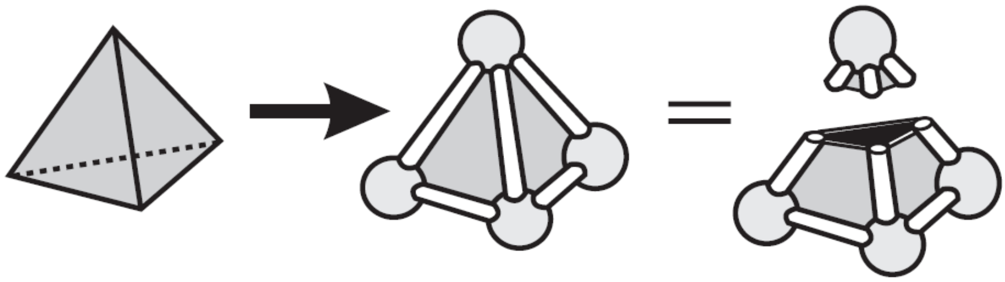

Figure 3.6: Going from a triangulation to a handle decomposition (figure taken from [Mat]).

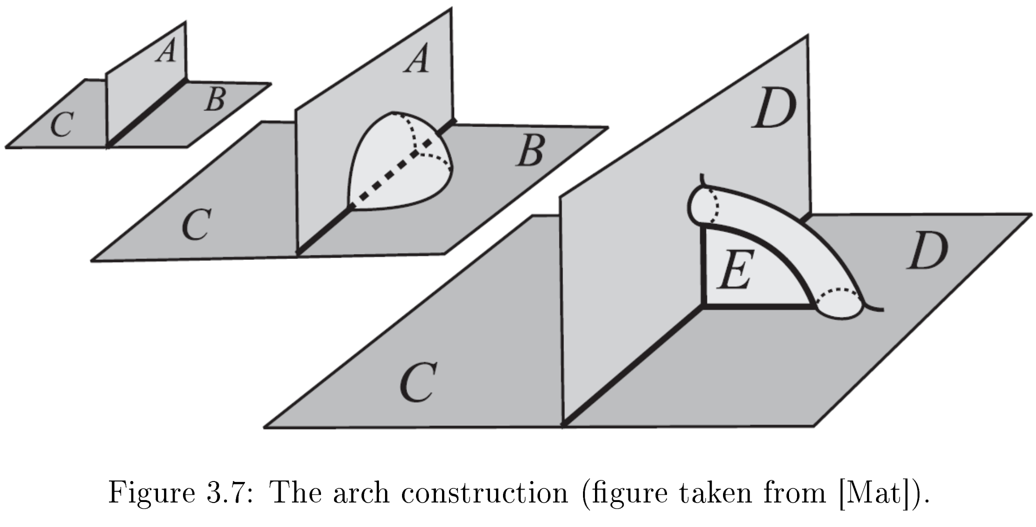

Figure 3.7: The arch construction (figure taken from [Mat]).

Theorem 3.1.3. *If two compact connected 3-manifolds have homeomorphic special spines and either both are closed or both have non-empty boundaries, then these 3-manifolds are homeomorphic.*

The proof of the above theorem can be found at page 8 of [Mat]. Its meaning is that any special spine of a manifold determines it uniquely. It follows that special spines may be viewed as presentations of 3-manifolds.
One should point out that, in contrast to group theory where every presentation determines a group, not every special polyhedron presents a 3-manifold. It is because there exist *unthickenable* special polyhedra which cannot be embedded into 3-manifolds.

Example 3.2. We attach the disc D^2 by its boundary to the projective plane $\mathbb{RP}^2$ along the projective line $\mathbb{RP}^1$. All the 2-components of the 2-polyhedron P obtained in this way are 2-cells. However, P cannot be embedded into a 3-manifold M. Indeed, if this were possible, the restriction to $\mathbb{RP}^1$ of the trivial normal bundle of D^2 in M would be isomorphic to the non-trivial normal bundle of $\mathbb{RP}^1$ in $\mathbb{RP}^2$.
Since P has no true vertices, it is not special. Nevertheless, it is easy to attach to P additional 2-cells (bubbles) to get an unthickenable special polyhedron.

It turns out that the "normal bundle obstruction" described above is the only thing that can make a special polyhedron unthickenable.
Moreover, a 3-manifold M can be reconstructed from a regular neighborhood $N(SP)$ in P of the singular graph SP of P: starting from $N(SP)$, one can easily reconstruct P by attaching 2-cells to all the circles in $\partial N(SP)$, and then reconstruct M. If M is orientable, then $N(SP)$ can be embedded into $\mathbb{R}^3$. This gives us a very convenient way for presenting 3-manifolds: we simply draw a picture, see Fig. 3.8 for the representation of the Bing's House with two rooms, a special spine of the 3-ball .

Theorem 3.1.4. *For any integer k there exists only a finite number of special spines with k true vertices. All of them can be constructed algorithmically.*

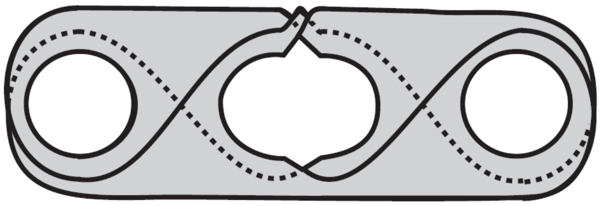

Figure 3.8: Bing's House with two rooms presented as regular neighborhood of its singular graph (figure taken from [Mat]).

Proof. We will construct a finite set of special polyhedra that a fortiori contains all special spines with k true vertices. First, one should enumerate all regular graphs of degree 4 with k true vertices. Clearly, there is only a finite number of them. Given a regular graph, we replace each true vertex v by a copy of the butterfly E that presents a typical neighborhood of a true vertex in a simple polyhedron, see Definition 3.9. Neighborhoods in ∂E of triple points of ∂E (we will call them *triodes*) correspond to edges having an endpoint at v. In Fig. 3.9 the triodes are shown by fat lines. For each edge e, we glue together the triodes that correspond to endpoints of e via a homeomorphism between them. It can be done in six different ways (up to isotopy). We get a simple polyhedron P with boundary. Attaching 2-discs to the circles in ∂P, we get a special polyhedron. Since at each step we have had only a finite number of choices, this method produces a finite set of special polyhedra. Not all of them are thickenable. Nevertheless, the set contains all special spines with k true vertices. □

3.2 Almost simple spines and definition of complexity

It would be a natural idea to measure how complex is a 3-manifold by the number of true vertices of its special spine. This characteristic is convenient

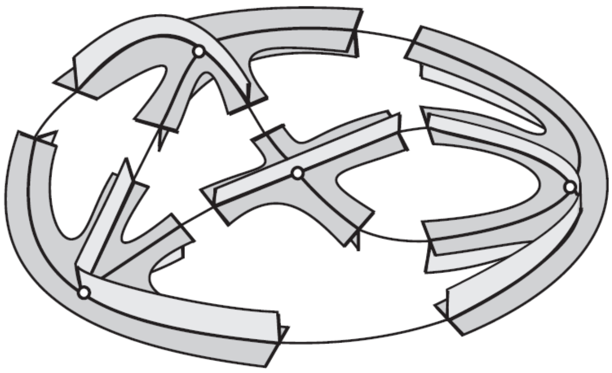

Figure 3.9: A decomposition of $N(SP)$ into copies of E (figure taken from [Mat]).

in that there exists only a finite number of 3-manifolds having special spines with a given number of vertices. But it has two shortcomings. First, it is not additive with respect to connected sums (see Definition 3.21). Second, restricting ourselves to special spines, we lose the possibility to consider very natural spines such as a point for the ball (and S^3), a circle for the solid torus, and a projective plane for the projective space $\mathbb{RP}^3$. Also, working only with special spines, we are sometimes compelled to make artificial tricks to preserve the special polyhedra structure. For example, in the proof of Theorem 3.1.2 we used a delicate arch construction instead of simply making a hole in a 2-cell.

All these shortcomings have the same root: the property of being special is not hereditary. In other words, a subpolyhedron of a special polyhedron may not be special, even if it cannot be collapsed onto a smaller subpolyhedron. This is why we shall widen the class of special polyhedra by considering a class of what we call almost simple polyhedra. Roughly speaking, the class of almost simple polyhedra is the minimal class which contains special polyhedra and is closed with respect to the passage to subpolyhedra.

Definition 3.14. A compact polyhedron P is said to be *almost simple* if the link of any of its points can be embedded into Γ_4, a complete graph with four vertices (see Fig. 3.10).

A spine P of a 3-manifold M is *almost simple*, if it is an almost simple polyhedron.

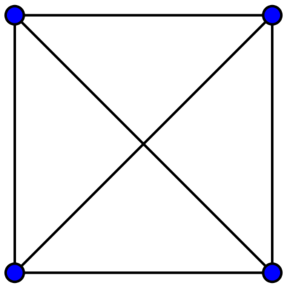

Figure 3.10: The complete graph with four vertices Γ_4.

It is convenient to present Γ_4 as a circle with three radii or as the boundary of the standard butterfly.

Remark 3.2. One usually considers only almost simple polyhedra that cannot be collapsed onto smaller subpolyhedra. One can notice that any proper subpolyhedron of the circle with three radii can be collapsed onto a polyhedron L having one of the following types:

(a) L is either empty or a finite set of $n \geq 2$ points;

(b) L is the union of a finite (possibly empty) set and a circle;

(c) L is the union of a finite (possibly empty) set and a circle with a diameter;

(d) L is a Γ_4.

An almost simple polyhedron P cannot be collapsed onto a smaller subpolyhedron if and only if the link L of any point of P is contained in the above list.

The notions of a true vertex, singular graph, 2-component of an almost simple polyhedron are introduced in the same way as for simple polyhedra, see Section 3.1.3.
Almost simple spines are easier to work with than special spines, since we may puncture cells and stay within the realm of almost simple spines.

Definition 3.15. The *complexity* $c(P)$ of an almost simple polyhedron P is equal to the number of its true vertices.

Definition 3.16. The *complexity* $c(M)$ of a compact 3-manifold M is equal to k if M possesses an almost simple spine with k true vertices and has no almost simple spines with a smaller number of true vertices.
In other words, $c(M) = \min_P c(P)$, where the minimum is taken over all almost simple spines of M.

Let us give some examples. The complexity of S^3, of the projective space $\mathbb{RP}^3$, of the lens space $L(3,1)$ (see Example 2.1), and the manifold $S^2 \times S^1$ is equal to zero, since they possess almost simple spines without true vertices: the point, the projective plane, the triple hat[1], and the wedge of S^2 with S^1, respectively. Among compact manifolds with boundary, zero complexity is possessed by all handlebodies[2] and I-bundles over surfaces. Indeed, any handlebody collapses to a graph that (being considered as an almost simple polyhedron) has no true vertices, while the I-bundles collapse to surfaces.

It is convenient to observe that removing an open ball does not affect the complexity.

Proposition 3.2.1. *Suppose that B is a 3-ball in a 3-manifold M. Then $c(M) = c(M \setminus \mathrm{int}(B))$.*

[1] Recall that by the triple hat we mean the quotient space of D^2 by a free action of the group $\mathbb{Z}_3$ on ∂D^2.

[2] The genus n handlebody H_n is the compact orientable 3-manifold bounded by a compact orientable surface of genus n embedded in $\mathbb{R}^3$.

Proof. If M is closed, then $c(M) = c(M \setminus \mathrm{Int}B)$ since M and $M \setminus \mathrm{Int}B$ have the same spines by definition of the spine of a closed manifold. Let $\partial M \neq \emptyset$, and let P be an almost simple spine of $M \setminus \mathrm{Int}B$ possessing $c(M \setminus \mathrm{Int}B)$ true vertices. Denote by C the connected component of the space $M \setminus P$ containing B. Since M is not closed, there exists a 2-component α of P that separates C from another component of $M \setminus P$. Removing an open 2-disc from α and collapsing yields an almost simple spine $P_1 \subset P$ of M. The number of true vertices of P_1 is no greater than that of P, since puncturing α and collapsing results in no new true vertices. Therefore, $c(M) \leq c(M \setminus \mathrm{Int}B)$.
To prove the converse inequality, consider an almost simple spine P_1 of M with $c(M)$ true vertices. Let us take a 2-sphere S^2 in M such that $S^2 \cap P_1 = \emptyset$. Join S^2 to P^1 by an arc l that has no common points with $P_1 \cup S^2$ except the endpoints. Clearly, $P = P_1 \cup S^2 \cup l$ is an almost simple spine of $M \setminus \mathrm{Int}B$. New true vertices do not arise. It follows that $c(M) \geq c(M \setminus \mathrm{Int}B)$. □

In general, the problem of calculating the complexity $c(M)$ of a 3-manifold M is very difficult. Let us start with a simpler problem of *estimating* $c(M)$. To do that it suffices to construct an almost simple spine P of M. The number of true vertices of P will serve as an upper bound for the complexity. Since an almost simple spine can be easily constructed from practically any presentation of the manifold, the estimation problem does not give rise to any difficulties. Let us describe several estimates of the complexity based on different presentations of 3-manifolds.

Proposition 3.2.2. *Suppose a 3-manifold M is obtained by pasting together n tetrahedra by affine identifications of their faces. Then $c(M) \leq n$.*

Proof. Recall that any tetrahedron Δ contains a canonical copy $P_\Delta = \cup|\mathrm{lk}_i(v_i, \Delta')|$ of the standard butterfly E, where v_i, $0 \leq i \leq 3$, are the vertices of Δ. When pasting together the tetrahedra, these copies are glued together into a simple polyhedron $P \subset M$ that may have a boundary if M is not closed. The polyhedron P has n true vertices and is a spine of M with several balls removed from it. These balls are the neighborhoods of the points which are obtained by gluing the vertices of the tetrahedra and lie in the interior of M. It follows from Proposition 3.2.1 that $c(M) \leq n$. □

Remark 3.3. Indeed for closed irreducible and $\mathbb{P}^2$-irreducible manifolds (see Definition 2.10 and 2.11), the complexity coincides with the minimum number of tetrahedra needed to construct a manifold, with the only exceptions of S^3, $\mathbb{RP}^3$ and $L(3,1)$, all having complexity zero.

Proposition 3.2.3. *Suppose $\widetilde{M}$ is a k-fold covering space of a 3-manifold M. Then $c(\widetilde{M}) \leq kc(M)$.*

Proof. Let P be an almost simple spine of M having $c(M)$ true vertices. Consider the almost simple polyhedron $\widetilde{P} = p^{-1}(P)$, where $p : \widetilde{M} \to M$ is the covering map. Since the degree of the covering is k, the polyhedron $\widetilde{P}$ has $kc(M)$ true vertices. If $\partial M \neq \emptyset$, then $\widetilde{P}$ is an almost simple spine of $\widetilde{M}$, since the collapse of M onto P can be lifted to a collapse of $\widetilde{M}$ onto $\widetilde{P}$. Therefore, $c(\widetilde{M}) \leq kc(M)$.
If M is closed, $\widetilde{P}$ is a spine of the manifold $\widetilde{M} \setminus \pi^{-1}(V)$, where V is an open 3-ball in M. The inverse image $p^{-1}(V)$ consists of k open 3-balls, hence, by Proposition 3.2.1, we have $c(\widetilde{M}) = c(\widetilde{M} \setminus p^{-1}(V)) \leq kc(M)$. □

In [Mat, p.77] an upper bound for the complexity of lens spaces (see Example 2.1) is given. In order to describe it we need the following definition.

Definition 3.17. For two coprime integers p, q with $0 < q < p$ denote by $S(p,q)$ the sum of the coefficients of the expansion of p/q as a continued fraction:

$$\text{if} \quad \frac{p}{q} = a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}, \qquad \text{then } S(p,q) = a_1 + \cdots + a_k.$$

The upper bound for the complexity of lens spaces that we can find in [Mat] is the following:

$$c(L(p,q)) \leq \max\{S(p,q) - 3, 0\}, \tag{3.1}$$

which has been proved to be sharp in many cases (see [JRT, JRT2]).

3.2.1 Converting almost simple spines into special ones

We have already stated the advantages of using almost simple spines, yet there are important downsides too. In general, almost simple spines determine 3-manifolds in a nonunique way, and cannot be represented by regular neighborhoods of their singular graphs alone. Since special spines, as has been mentioned before, are free from such liability, we would like to go from almost simple polyhedra to special ones whenever possible. We will now see when it is possible.
Let P be an almost simple spine of a 3-manifold M that is not a special one. Then P either possesses a 1-dimensional part or has 2-components not homeomorphic to a disc. Our aim is to transform P into a special spine of M without increasing the number of true vertices. In general this is not possible, but there are cases in which it is. To give an exact formulation, we need to recall a few notions of 3-manifold topology.
Recall that a compact surface F in a 3-manifold M is called *proper* if $F \cap \partial M = \partial F$.

Definition 3.18. A 3-manifold M is *boundary irreducible* if for every proper disc $D \subset M$ the curve ∂D bounds a disc in ∂M. Otherwise, M is called *boundary reducible.*

Definition 3.19. Let M be an irreducible and boundary irreducible 3-manifold. A proper annulus $A \subset M$ is called *inessential* if either it is ∂-parallel to an annulus in ∂M, or the core circle of A is contractible in M (in the second case A can be viewed as a tube possessing a meridional disc). Otherwise A is called *essential.*

Definition 3.20. Let P be a simple polyhedron in a 3-manifold M. An open ball $V \subset M \setminus P$ is called *proper* (with respect to P), if $\mathrm{Cl}(V) \setminus V \subset P$.

Theorem 3.2.4. *Suppose M is a compact irreducible boundary irreducible 3-manifold such that $M \neq D^3$, S^3, $\mathbb{RP}^3$, $L(3,1)$ and all proper annuli in M are inessential. Then for any almost simple spine P of M there exists a special spine P_1 of M having the same or a fewer number of true vertices.*

Proof. Identify M (or M with a 3-ball removed, if M is closed) with a regular neighborhood of P. We will assume that P cannot be collapsed to a smaller subpolyhedron. We convert P into P_1 by a sequence of transformations (moves) of three types. To control the number of steps, we assign to any almost simple polyhedron P the following three numbers:

- $c_2(P)$, the number of 2-components of P;
- $-\chi_2(P) = -\sum_\alpha \chi(\alpha)$, where the sum is taken over all 2-components α of P and $\chi(\alpha)$ is the Euler characteristic.
- $c_1(P) = \min e(X_P)$, where the 1-dimensional part X_P of P (i.e., the union of points having 0-dimensional links) is presented as a graph with $e(X_P)$ edges and the minimum is taken over all such presentations.

The triples $(c_2(P), -\chi_2(P), c_1(P))$ will be considered in the lexicographic order.

MOVE 1. Suppose that the 1-dimensional part X_P of P is nonempty. Consider an arc $\gamma \subset X_P$ and a proper disc $D \subset M$ which intersects γ transversally at one point. Since M is irreducible and boundary irreducible, D cuts a 3-ball B out of M. Removing $B \cap P$ from P and collapsing the rest of P as long as possible, we get a new almost simple spine $P' \subset M$. If $B \cap P$ contains at least one 2-component of P, then $c_2(P') < c_2(P)$. If $B \cap P$ is 1-dimensional, then the 2-dimensional parts of P, P' coincide and thus $c_2(P') = c_2(P)$, $-\chi_2(P') = -\chi_2(P)$. Of course, $c_1(P') < c_1(P)$.

Assume that a 2-component α of P contains a nontrivial simple closed curve l so that the restriction to l of the normal bundle ν of α is trivial. If α is not D^2, S^2 or $\mathbb{RP}^2$, then l always exists. It follows that one can find a proper annulus $A \subset M$ that intersects P transversally along l. Since all annuli are inessential, either A is parallel to the boundary or its core circle is contractible.

Move 2. Suppose that A is parallel to the boundary. Then it cuts off a solid torus V from M so that the remaining part of M is homeomorphic to M. Removing $V \cap P$ from P, we obtain (after collapsing) a new almost simple spine $P' \subset M$. This move annihilates α, so $c_2(P') < c_2(P)$.

Move 3. Suppose that the core circle of A is contractible. Then both circles of ∂A are also contractible. Choose one of them. By Dehn's Lemma (see [Pap]), it bounds a disc in M and, since M is boundary irreducible, a disc D in ∂M. It follows that there is a disc $D \subset \text{Int}M$ such that $D \cap P = \partial D = l$. Since $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$, D cuts a proper open 3-ball B out of $M \setminus P$, see Definition 3.20. If we puncture D, collapse B and then collapse the rest of D, we return to P. However, if we get inside the ball B through another 2-component of the free boundary of B (see Fig. 3.11), we get after collapsing a new almost simple spine $P' \subset M$.

Let us analyze what happens to α under this move. If l does not separate α, then the collapse eliminates α completely together with D. In this case we have $c_2(P') < c_2(P)$.

Suppose that l separates α into two parts, α' and α'' (the notation is chosen so that the hole is in α''). Then the collapse destroys α'', and we are left with $\alpha' \cup D$. In this case either $c_2(P') < c_2(P)$ (if the collapse destroys some other 2-components of P), or $c_2(P') = c_2(P)$ and $-\chi_2(P') < \chi_2(P)$ since $-\chi(\alpha' \cup D) < -\chi(\alpha)$.

Now let us perform Steps 1, 2, 3 as long as possible. The procedure is finite, since each step strictly decreases the triple $(c_2(P), -\chi_2(P), c_1(P))$ and hence any monotonically decreasing sequence of triples is finite. Let P_1 be the resulting almost simple spine of M. By construction, P_1 has no 1-dimensional part and no 2-components different from D^2, S^2, and $\mathbb{RP}^2$. The following cases are possible:

- P_1 has no 2-components at all. Since it also has no 1-dimensional part, P_1 is a point and thus $M = S^3$ or $M = D^3$;

- P_1 contains a 2-component which is not homeomorphic to the disc. In this case P_1 is either $\mathbb{RP}^2$ or S^2. Suppose that $P_1 = \mathbb{RP}^2$. Then $M = \mathbb{RP}^2 \times I$ or $\mathbb{RP}^3$. We cannot have $M = \mathbb{RP}^2 \tilde{\times} I$, i.e., the twisted I-bundle over $\mathbb{RP}^2$, since this manifold is a punctured projective space and hence is reducible. For the same reason we cannot have $P_1 = S^2$: the manifold $S^2 \times I$ is reducible;

- All the 2-components of P_1 are discs and P_1 has no true vertices but contains triple points. Denote by k the number of 2-components of P_1. We cannot have $k = 3$, since the union of three discs with common boundary is a spine of S^3 with three punctures, which is a reducible manifold. The simple polyhedron obtained by attaching two discs to a circle is unthickenable, see Example 3.2. We may conclude that P_1 has only one 2-component, which is homeomorphic to the disc. In this case M is homeomorphic to $L(3, 1)$.

- There remains only one possibility: P_1 has true vertices and all its 2-components are discs. In this case P_1 is special.

□

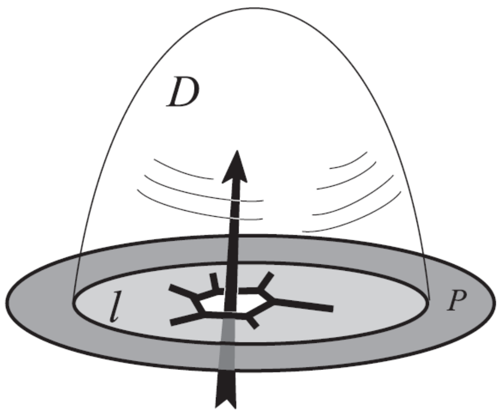

Figure 3.11: Attaching D^2 along l and puncturing another 2-component produces a simpler spine (figure taken from [Mat]).

3.2.2 The finiteness property

Theorem 3.2.5. *For any integer k, there exists only a finite number of distinct compact irreducible boundary irreducible 3-manifolds that contain no essential annuli and have complexity k.*

Proof. Follows immediately from Theorem 3.2.4 and Theorem 3.1.4. □

Restricting ourselves to the most interesting case of closed irreducible 3-manifolds, we immediately get the following corollary.

Corollary 3.2.6. *For any integer k, there exists only a finite number of distinct closed irreducible 3-manifolds of complexity k.*

Remark 3.4. To show that the assumptions of Theorem 3.2.5 are essential, let us describe three infinite sets of distinct 3-manifolds of complexity 0. The sets consist of manifolds that are either reducible (a), or boundary reducible (b), or contain essential annuli (c).

(a) For any integer n, the connected sum (see Definition 3.21) M_n of n copies of the projective space $\mathbb{RP}^3$ is a closed manifold of complexity 0. To construct an almost simple spine of M_n without true vertices, one may take n exemplars of the projective plane $\mathbb{RP}^2$ and join them by arcs.

(b) The genus n handlebody H_n (see footnote at page 45) is irreducible, but boundary reducible. Since it can be collapsed onto a 1-dimensional spine, $c(H_n) = 0$.

(c) Manifolds $\partial H_n \times I$ are irreducible and boundary irreducible, but contain essential annuli. They have complexity 0 since can be collapsed onto the corresponding surfaces.

The property seen in Corollary 3.2.6 has been used in order to construct a census of closed irreducible 3-manifolds according to complexity: exact values of it are listed for the orientable case at `http://matlas.math.csu.ru/?page=search` (up to complexity 12) and for the non-orientable case at `https://regina-normal.github.io` (up to complexity 11).

3.2.3 The additivity property

Let us first recall the notions of connected sum and boundary connected sum of two compact 3-manifolds M_1, M_2.

Definition 3.21. The *connected sum* $M_1 \# M_2$ of two compact 3-manifolds M_1, M_2 is defined as the manifold $(M_1 \setminus \mathrm{int}(B_1)) \cup_h (M_2 \setminus \mathrm{int}(B_2))$, where $B_1 \subset \mathrm{int}(M_1)$, $B_2 \subset \mathrm{int}(M_2)$ are 3-balls, and h is a homeomorphism between their boundaries. If the manifolds are orientable, their connected sum depends on whether h is orientation reversing or preserving. In this case $M_1 \# M_2$ will denote any of the two possible connected sums. Alternatively, one can use signs and write $M_1 \#(\pm M_2)$.

Definition 3.22. To define the *boundary connected sum* $M_1 \amalg M_2$ of M_1, M_2, consider two discs $D_1 \subset \partial M_1$, $D_2 \subset \partial M_2$ in the boundaries of the two 3-manifolds. Glue M_1 and M_2 together by identifying the discs along a homeomorphism $h : D_1 \to D_2$. Equivalently, one can attach an index 1 handle $I \times D^2$ to $M_1 \cup M_2$ such that the base of the handle $\{0, 1\} \times D^2$ coincides with $D_1 \cup D_2$. The manifold M thus obtained is $M_1 \amalg M_2$. Of course, M depends on the choice of the discs (if at least one of the manifolds has disconnected boundary), and on the choice of h (the resulting manifold changes depending on whether h preserves or reverses orientation). Thus the notation $M_1 \amalg M_2$ is slightly ambiguous, like the notation for the connected sum. When shall use it to mean that $M_1 \amalg M_2$ is one of the manifolds that can be obtained by the above gluing.

Theorem 3.2.7. *For any 3-manifolds M_1, M_2 we have:*

(a) $c(M_1 \# M_2) = c(M_1) + c(M_2)$

(b) $c(M_1 \amalg M_2) = c(M_1) + c(M_2)$

Proof. We begin by noticing that the first conclusion of the theorem follows from the second one. To see that, we choose 3-balls $V_1 \subset \mathrm{Int} M_1$, $V_2 \subset \mathrm{Int} M_2$ and $V_3 \subset \mathrm{Int}(M_1 \# M_2)$. It is easy to see that $(M_1 \setminus \mathrm{Int} V_1) \amalg (M_2 \setminus \mathrm{Int} V_2)$

and $(M_1 \# M_2) \setminus V_3$ are homeomorphic, where the index 1 handle realizing the boundary connected sum is chosen so that it joins ∂V_1 and ∂V_2. Assuming *(b)* and using Proposition 3.2.1, we have: $c(M_1 \# M_2) = c((M_1 \# M_2) \setminus V_3) = c(M_1 \setminus \text{Int}V_1) + c(M_2 \setminus \text{Int}V_2) = c(M_1) + c(M_2)$.
Let us prove the second conclusion. The inequality $c(M_1 \amalg M_2) \leq c(M_1) + c(M_2)$ is obvious, since if we join minimal almost simple spines of M_1,M_2 by an arc, we get an almost simple spine of $M_1 \amalg M_2$ having $c(M_1) + c(M_2)$ true vertices.
The proof of the inverse inequality is based on Haken's theory of normal surfaces (see Chapter 3 of [Mat]). So we restrict ourselves to a reference to Corollary 4.2.10 of [Mat], which states that attaching an index 1 handle preserves complexity. □

3.3 Estimation of the complexity of Seifert manifolds

We now show some results about how to estimate the complexity of Seifert fibre spaces. We give a result about how to find an upper bound for their complexity in terms of the invariants of the combinatorial description seen in Section 2.2. All the results are taken from [CMMN], where it is possible to find also the relative proofs.

Theorem 3.3.1. *Let* $M = \{b; (\epsilon, g, (t, k)); (h_1, \dots, h_{m_+} \mid k_1, \dots, k_{m_-}); ((p_1, q_1), \dots, (p_r, q_r))\}$ *be a Seifert fibre space such that* $\partial M \neq \emptyset$ *(i.e.,* $m_+ + m_- > 0$*). Then*

$$c(M) \leq t + \sum_{j=1}^{r} \max\{S(p_j, q_j) - 3, 0\}, \tag{3.2}$$

where $S(p_j, q_j)$ *denotes the sum of the coefficients of the expansion of* p_j/q_j *as a continued fraction.*

Moreover, if $M = N \times S^1, N \widetilde{\times} S^1, D^2 \times S^1, SK$ *then* $c(M) = 0$.

Next corollary characterizes a wide class of Seifert fibre spaces having complexity zero.

Corollary 3.3.2. *Let M be a Seifert fibre space with $\partial M \neq \emptyset$, and such that*

i) $SE(M) = AE(M)$ *(i.e., $t = 0$),*

ii) *the E-fibres of M, if any, are of type $(2,1)$, $(3,1)$ and $(3,2)$,*

then $c(M) = 0$.

Proof. From the above conditions we have $S(p_j, q_j) \leq 3$. So the statement follows directly from (3.2). □

Theorem 3.3.3. *Let $M = \{b; (\epsilon, g, (t,k))\,; (\;|\;)\,; ((p_1, q_1), \ldots, (p_r, q_r))\}$ be a closed Seifert fibre space with $b \geq -r/2$, and let $\chi = 2 - 2g$ if $\epsilon = o, o_1, o_2$ and $\chi = 2 - g$ if $\epsilon = n, n_1, n_2, n_3, n_4$.*

(a) If $\chi = 2$ and $r = t = 0$, then $c(M) \leq \max\{b - 3, 0\}$;

(b) if $\chi = 2$, $t = 0$, $r = 1$ and $b > 0$, then $c(M) \leq \max\{b + S(p_1, q_1) - 3, 0\}$;

(c) if $\chi = 2$, $t = 0$, $r = 1$ and $b = 0$, then $c(M) \leq \max\{S(p_1, q_1) - 3 - \lfloor p_1/q_1 \rfloor, 0\}$, where $\lfloor \cdot \rfloor$ denotes the integer part function;

(d) if $\chi = 1$, $\epsilon = n_1$, $r = t = 0$ and $b = 0$, then $c(M) \leq 1$;

(e) if $\chi = 1$, $\epsilon = n_1$, $r = t = 0$ and $b = 1$, then $c(M) = 0$;

(f) in all other cases:

$$c(M) \leq \max\{b - 1 + \chi, 0\} + 6(1 - \chi) + \sum_{j=1}^{r} \left(S(p_j, q_j) + 1\right), \qquad (3.3)$$

if M is orientable;

$$c(M) \leq 6(1 - \chi) + 6t + \sum_{j=1}^{r} \left(S(p_j, q_j) + 1\right), \qquad (3.4)$$

if M is non-orientable.

Remark 3.5. Both the proofs of the Theorems 3.3.1 and 3.3.3 follow the idea of constructing a particular almost special spine for the Seifert manifold M and counting its true vertices: by definition, the number so obtained gives an upper bound for the complexity of the manifold. Such spine is built as a union of skeletons, each of which is a skeleton of one of the blocks in which M is divided. Indeed, we consider separately the *critical block*, obtained from M by removing neighborhoods of the critical fibres of $CE(M)$ and $E(M)$ (for the meaning of the notation, see page 18), and the *main block*, which is what is left in M from such removal. The critical block gets divided into sub-blocks, depending on the number and the kind of critical fibres that it contains. For each of these sub-blocks there is a particular construction of the corresponding skeleton.
The details of such constructions can be found in [CMMN]. In the next section we describe explicitely some examples of spines for Seifert fibre spaces of complexity zero (i.e., those of Corollary 3.3.2).

Remark 3.6. The complexity estimation given by the formula (3.4) is sharp in most of the cases listed in the catalogues of Seifert manifolds that can be found in [AM] (up to complexity 7), in [Bur] (up to complexity 10) and at the web page `https://regina-normal.github.io` (up to complexity 11). By saying that the complexity estimation is sharp, we mean that it actually coincides with the real value of the complexity. There are some cases of Seifert manifolds for which the estimation (3.4) is not sharp: they are listed in [CMMN].

3.4 Examples of spines of Seifert manifolds

Complexity of closed Seifert manifolds has been studied quite deeply, indeed their catalogues until complexity 11 have been listed and are available, as we said in Remark 3.6. Therefore, we will focus on the bordered case, in order to provide some basic examples of spines of Seifert fibre spaces in this less studied case. In particular, we'll consider the case in which the hypothesis of Corollary 3.3.2 are verified and therefore the complexity of the manifold is zero. So following the combinatorial description of the Seifert fibre spaces

$$M = \{b; (\epsilon, g, (t,k)); (h_1, \ldots, h_{m_+} \mid k_1, \ldots, k_{m_-}); ((p_1, q_1), \ldots, (p_r, q_r))\}$$

that we introduced at page 25, we will consider Seifert manifolds such that: $b = t = k = 0$; $m_+ + m_- \neq 0$; (p_i, q_i), if there are any, are either $= (2,1)$, $(3,1)$ or $(3,2)$.

In all of these examples, we'll start by looking at the base orbifold B of the Seifert fibre space M considered, indicating with $f : M \to B$ the projection of M on its base B. In order to make our analysis easier to understand, we'll consider in separate examples the different kinds of singularities that can appear in the base orbifold B (even though we could find them all in the same one). We'll start with examples in which there are no singular points in B, then we'll consider the cases in which the base contains cone points and finally reflector arcs. These are the only kinds of singularities that we consider on B for being sure that the complexity of M is zero, following the hypothesis of Corollary 3.3.2.

In this last section, following the idea of the proof of Theorem 3.3.1 described in Remark 3.5, we will explain how to obtain a spine for a general Seifert manifold gluing together the skeletons constructed for the sub-blocks composing the manifold.

3.4.1 Base without singular points

Example 3.3. The trivial case.

Let $B = D^2$ and M be the trivial circle bundle over B, therefore M is the trivial solid torus $S^1 \times D^2$, whose combinatorial description is $\{0; (o_1, 0, (0,0)); (0|); \}$. Since the base B is a disk, we can contract it to a point O. Let $P = f^{-1}(O)$: it is a circle in M which is an almost simple spine of M, as we can see in Figure 3.12.

Example 3.4. Let B be the Moebius strip N and M be the trivial circle bundle over B, i.e., $M = S^1 \times N$ with the trivial fibration on it, whose

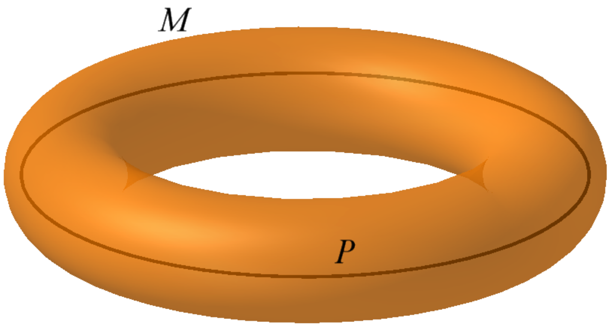

Figure 3.12: The trivial solid torus M and its almost simple spine P.

combinatorial description is $\{0;(n_1,1,(0,0));(0|);\}$. Since the base B is a Moebius strip, we can contract it to its core circle Γ (the thick line in Figure 3.13). Defining $P=f^{-1}(\Gamma)$ we obtain a torus (because $S^1\times\Gamma$ is actually $S^1\times S^1$) which is an almost simple spine of M.

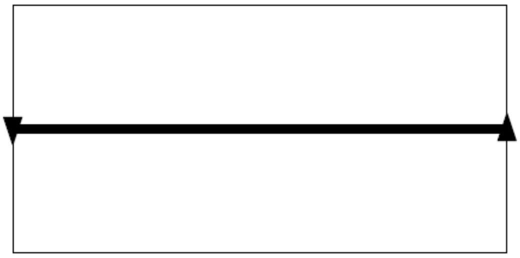

Figure 3.13: The Moebius strip with its core circle depicted with a thick line.

Adding boundary components to M (which corresponds to removing solid tori or Klein bottles from M and removing disks from B) causes an addition of tori or Klein bottles to the spine of M, according to the sign of the function ω that we introduced while characterizing Seifert fibre spaces (see Section 1.3). We just have to glue appropriately these skeletons, so that during the gluing no true vertex appears in the global spine and so its complexity is still zero. Such gluing is made by attaching annuli or Moebius strips to the "partial" skeletons. Let us see a simple example.

Example 3.5. As base B we take a Moebius strip in which a disk has been removed as depicted in Figure 3.14. In this example we don't consider M as the trivial circle bundle over B: the combinatorial description of M is $\{0; (n_1, 1, (0, 0)); (|0, 0); \}$. Indeed, indicating by s_1 and s_2 the two boundary components of B and by v the generator of $H_1(B)$, we have $\omega(s_1) = \omega(s_2) = -1$, $\omega(v) = 1$. Notice that as a conseguence of Theorem 1.3.2, we could not have an odd number of boundary components s_i such that $\omega(s_i) = -1$.
The base B can be contracted to the graph Γ depicted with colours in Figure 3.14. Therefore as a spine of the manifold M we take $P = f^{-1}(\Gamma)$: it is actually an almost simple spine of M. P is composed by a Klein bottle (over the orange circle) and a torus (over the green circle), linked by an annulus (over the blue segment) in such a way that one of the boundary circles of the annulus is glued to a fibre of the torus and its other boundary circle is glued to a fibre of the Klein bottle. P cannot be properly represented in $\mathbb{R}^3$, therefore we represent it in the plane, identifying edges of quadrilaterals in Figure 3.15.

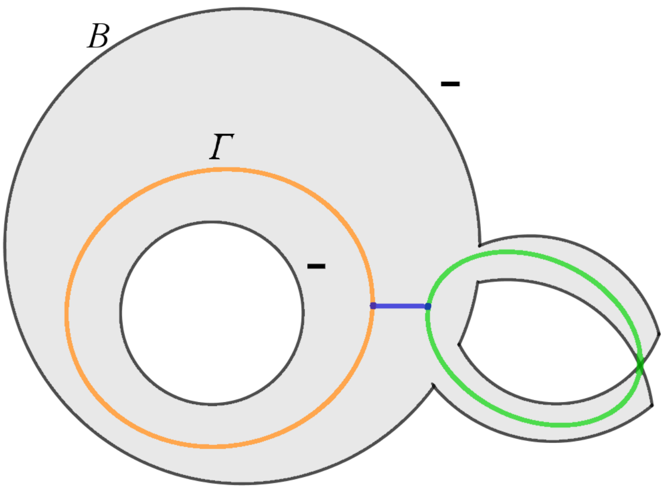

Figure 3.14: The base B in grey and the graph Γ in orange, blue and green.

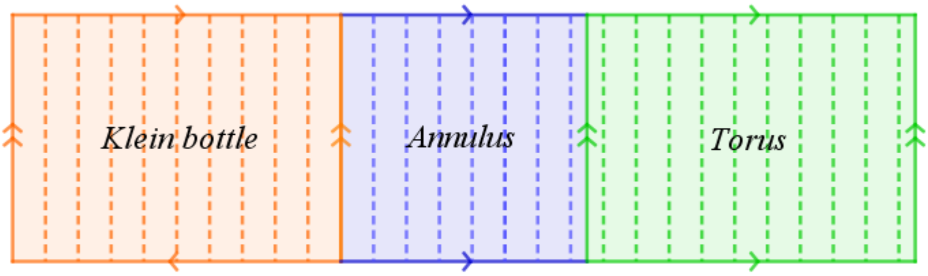

Figure 3.15: A representation of the spine P. The identifications are made between edges of the same colour, with the same arrow depicted on them and in the direction indicated. The dashed lines represent the fibration.

3.4.2 Base with cone points

Now we consider an example in which in the base space appear two types of cone points: $(2,1)$ and $(3,1)$. This choice is due to the fact that these two kinds of singularities give rise to different kinds of blocks, and we want to show all of them. A cone point of kind $(3,2)$ would give the same type of blocks as the cone point $(3,1)$ gives (see [FW] for further explanations).

Example 3.6. Let B be a disk with two cone points of type $(2,1)$ and $(3,1)$ as in Figure 3.16 and let M be the Seifert fibre space with base B such that $M = \{0; (o_1, 0, (0,0)); (0|); (2,1), (3,1)\}$. The presence of the cone point of type $(2,1)$ produces a torus block, called *second torus block* in [FW], having as skeleton a properly embedded Moebius strip as showed in Figure 3.17. On the other hand, the cone point of type $(3,1)$ produces two other blocks: a torus block, called *first torus block* in [FW], having as skeleton a properly embedded Moebius strip with half a disk as a "wing" as represented in Figure 3.18 and another torus block, called *transitional block* in [FW], having as skeleton the polyhedron P_t represented in Figure 3.19. All these blocks belong to the critical block of the manifold, while the main block of M is the trivial circle bundle over the space resulting from the removal of neighborhoods of the cone points from B (in Fig. 3.16, we remove from B the disks delimited by the dashed circles). As a skeleton for the main block, we consider the annulus $f^{-1}(\Gamma)$ where Γ is a graph composed of just one edge (depicted in grey in Fig. 3.16). For getting the global spine of M, we finally need to glue all these pieces together. In particular:

- we glue the border of the wing attached to the Moebius strip (called r in Figure 3.18) of the first torus block to the edge in P_t that we named s in Figure 3.19;
- we glue one of the two boundary components of the annulus $f^{-1}(\Gamma)$ to the boundary of the Moebius strip of the first torus block and the other boundary component of $f^{-1}(\Gamma)$ to the boundary of the Moebius strip of the second torus block.

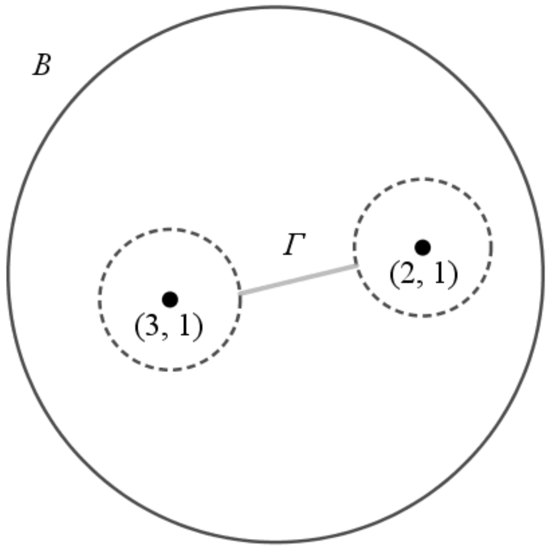

Figure 3.16: The base B with the cone points of type $(2,1)$ and $(3,1)$ and the graph Γ. The dashed circles delimit neighbourhoods of the cone points.

3.4.3 Base with reflector arcs

We now consider the case in which the base space B has reflector arcs as singular points. We will see that a reflector arc in the base space corresponds

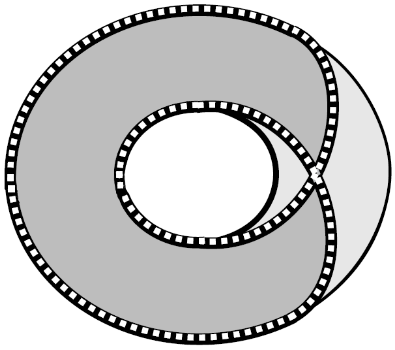

Figure 3.17: The second torus block and its skeleton in darker grey (figure taken from [FW]).

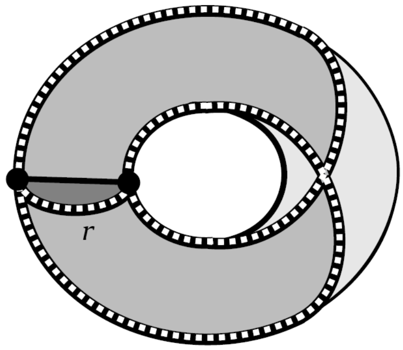

Figure 3.18: The first torus block and its skeleton in darker grey (figure taken from [FW]).

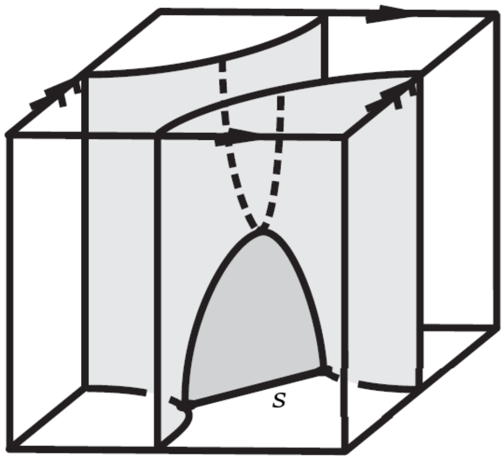

Figure 3.19: The transitional block and its skeleton in grey (figure taken from [FW]).

to a Moebius strip in the construction of the spine of the manifold M. In the following example, we'll see the case in which we have just one reflector arc, but we could actually have several of them, on the same boundary component or on different ones: for each reflector arc we just add a Moebius strip to the spine of M, properly attached to the rest of the spine.

Example 3.7. Let B be an annulus with a reflector arc, as showed in Figure 3.20, and let M be the Seifert fibre space having as base B and as combinatorial description $\{0; (o_1, 0, (0, 0)); (0, 1|); \}$. The base B can be contracted to the graph Γ depicted in orange and green in Figure 3.20. Therefore as a spine for M we take $P = f^{-1}(\Gamma)$: over the orange circle we get a torus and over the green segment a Moebius strip. Such spine is an almost simple spine for M and is a torus with a Moebius strip attached to it along a fibre.

3.4.4 The general case

We end this section with a generalization of the examples that we have just seen. Indeed, we can construct spines for Seifert fibre spaces with more complex base spaces than the ones we saw, still having complexity zero:

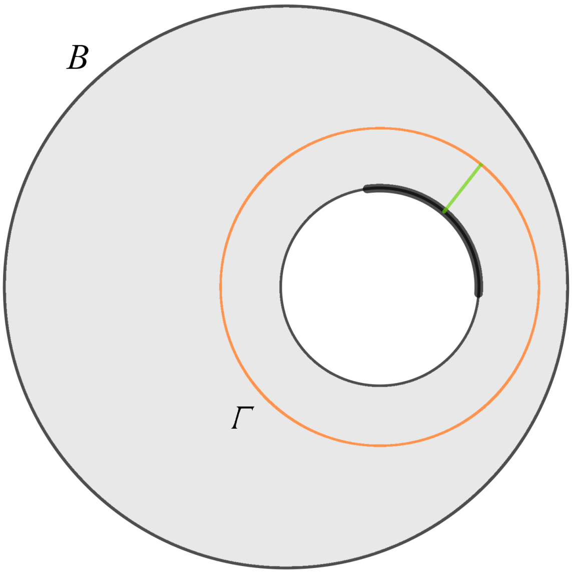

Figure 3.20: The base B in grey and the graph Γ in orange and green.

we can have a different underlying space, several cone points of type $(2,1)$, $(3,1)$ and $(3,2)$, several boundary components and several reflector arcs. The spine will be composed by gluing together the pieces that we have seen in the previous examples, connected in an adequate way. Consider for example the base space B of Figure 3.21 and let M be the Seifert fibre space having base B and combinatorial description $\{0;(n_1,3,(0,0));(0,0,1,2|);\}$. We take as main block the bundle over the base space depicted in Figure 3.21, which is B with the removal of neighborhoods of the cone points. On the other hand, we take as critical block the bundle over the neighborhoods of the cone points. As skeleton of the main block we take $f^{-1}(\Gamma)$, where Γ is the grey graph in Figure 3.22 and as skeleton of the critical block we construct skeletons for the neighborhoods of the cone points as we did in Example 3.6. Then, gluing together such skeletons, we get a spine for M, which will actually be an almost simple spine of M and will have no true vertices.

In a similar way we can obtain spines for all the Seifert fibre spaces of type

$$M = \{0; (\epsilon, g, (0,0)); (h_1, \dots, h_{m_+} \mid k_1, \dots, k_{m_-}); ((p_1, q_1), \dots, (p_r, q_r))\}$$

such that $m_+ + m_- \neq 0$ and (p_i, q_i), if there are any, are either $= (2,1)$, $(3,1)$ or $(3,2)$.

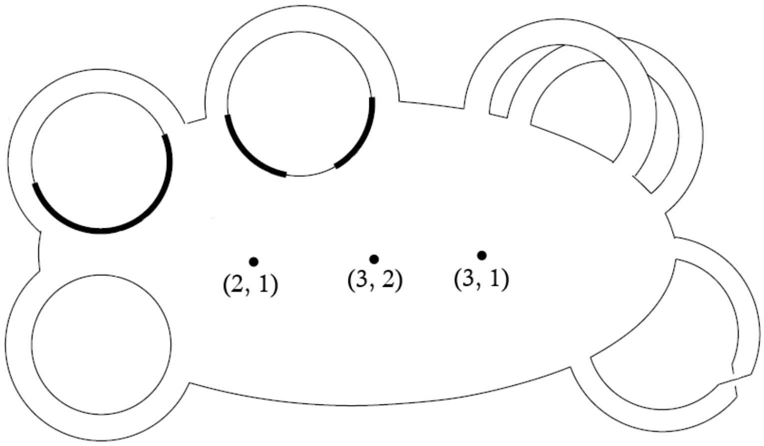

Figure 3.21: The base space B of the Seifert fibre space $M = \{0; (n_1, 3, (0,0)); (0,0,1,2|); \}$.

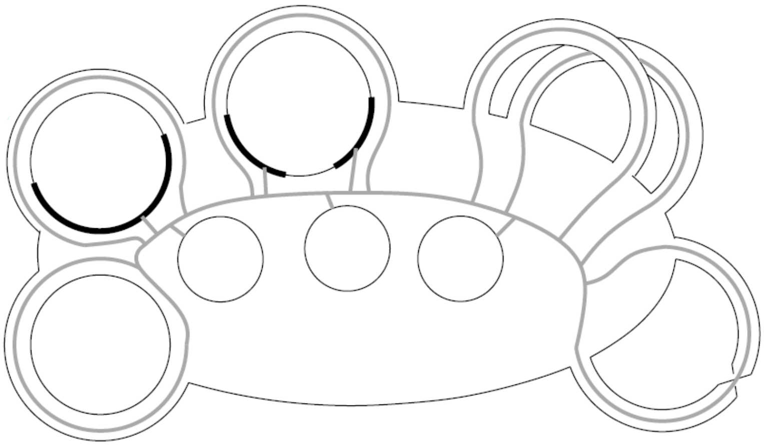

Figure 3.22: The base space of the main block and the graph Γ in grey.

Bibliography

[AM] G. Amendola, B. Martelli, Non-orientable manifolds of complexity up to 7, *Topology Appl.* **150**, 2005, 179–195.

[Bur] B. A. Burton, Enumeration of Non-Orientable 3-Manifolds Using Face-Pairing Graphs and Union-Find, *Discrete Comput. Geom.* **38**, 2007, 527–571.

[CMMN] A. Cattabriga, S. Matveev, M. Mulazzani, T. Nasybullov, On the complexity of non-orientable fibre spaces, 2018, accepted for publication on *Indiana Univ. Math. J.*, 2018, avalaible at http://www.iumj.indiana.edu/IUMJ/forthcoming.php.

[Eps] D. B. A. Epstein, Periodic flows on 3-manifolds, *Ann. of Math.* **95**, 1972, 66–82.

[Fin] R. Fintushel, Local S^1-action on 3-manifolds, *Pac. J. Math.* **66**, 1976, 111–118.

[FW] E. Fominykh, B. Wiest, Upper bounds for the complexity of the torus knot complements, *J. Knot Theory Ramifications* **22**, 2013, 1–19.

[Hat] A. Hatcher, Algebraic Topology, 2001, available at `https://www.math.cornell.edu/~hatcher/AT/AT.pdf`.

[Hud] J.F.P. Hudson, E.C. Zeeman, On regular neighborhoods, *Proc. London Math. Soc.* **14**, 1964, 719–745.

[JRT] W. Jaco, H. Rubinstein, S. Tillmann, Minimal triangulations for an infinite family of lens spaces, *J. Topol.* **2**, 2009, 157–180.

[JRT2] W. Jaco, H. Rubinstein, S. Tillmann, Coverings and Minimal Triangulations of 3-manifolds, *Algebr. Geom. Topol.* **11**, 2011, 1257–1265.

[Mar] B. Martelli, An Introduction to Geometric Topology, 2016, available at `http://people.dm.unipi.it/martelli/Geometric_topology.pdf`.

[Mat] S. Matveev, Algorithmic topology and classification of 3-manifolds, *ACM-Monographs* **9**, *Spinger-Verlag*, *Berlin-Heidelberg-New York*, 2003.

[Pap] Papakyriakopoulos, C. D., On Dehn's lemma and the asphericity of knots, *Ann. of Math. (2)* **66**, 1957, 1–26

[Pré] J. Préaux, A Survey on Seifert Fibre Space Theorem, *G. Martin and J. Porti*, *Aix-Marseille Université*, 2014.

[RS] C. P. Rourke, B. J. Sanderson, Introduction to piecewise-linear topology, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **69**, *Springer-Verlag*, *New York-Heidelberg*, 1972.

[Sco] P. Scott, The geometries of 3-manifolds, *Bull. Lond. Math. Soc.* **15**, 1983, 401–487.

[Sei] H. Seifert, Topologie dreidimensionaler gefaserter Raüme, *Acta Math.* **60**, 1933, 147–238.

[Spa] E. H. Spanier, Algebraic Topology, *McGraw-Hill*, *New York*, 1966.

Acknowledgements

Alla fine di questa tesi, vorrei esprimere la mia gratitudine verso chi mi ha accompagnata lungo questo mio percorso universitario, ognuno in modi e contesti diversi, apportando un contributo essenziale.
Innanzitutto ringrazio la professoressa Alessia Cattabriga, che seguendomi con energia e pazienza in questo lavoro di tesi mi ha dato la possibilità di apprendere nuovi concetti in un contesto di confronto costruttivo.
Grazie agli amici che hanno vissuto con me la quotidianità di questi cinque anni universitari a Bologna, all'interno e all'esterno delle mura del Dipartimento di Matematica, che hanno reso questa città una seconda casa per me.
Grazie agli amici di sempre, su cui so di poter far affidamento in ogni circostanza e, nonostante le distanze che ci separano, continuano ad essere una presenza costante e fondamentale per me.
Grazie al meraviglioso gruppo clown FANEP dell'ospedale Sant'Orsola di cui ho l'onore di far parte da quattro anni, ormai una seconda famiglia per me, che mi ha dato la possibilità di arricchirmi come persona e che ogni settimana mi ricorda quanto sia bello sapersi donare, mettendosi in gioco con tutte le proprie energie e con tutto il proprio cuore.
Grazie agli amici sparsi per l'Europa con cui ho avuto la fortuna di condividere il mio ultimo anno universitario all'estero: è stato l'anno più arricchente della mia vita e questo lo devo anche al confronto che ho potuto instaurare con loro ed ai bellissimi legami che si sono creati.
Infine il ringraziamento più grande va ai miei genitori, perché se sono arrivata al termine di questo percorso così serenamente è grazie a loro, che con

lucidità hanno creduto insieme a me nei miei desideri e nelle mie capacità, e con fiducia mi hanno lasciata libera di fare e di essere.