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Adiabatic theory for slowly varying Hamiltonian systems with applications to beam dynamics

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Qual è 'l geomètra che tutto s'affige
per misurar lo cerchio, e non ritrova,
pensando, quel principio ond'elli indige,

tal era io a quella vista nova:
veder voleva come si convenne
l'imago al cerchio e come vi s'indova.

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Abstract

In questo lavoro si presentano due modelli in cui la teoria dell'invarianza adiabatica può essere applicata per ottenere peculiari effetti nel campo della dinamica dei fasci, grazie all'attraversamento di separatrici nello spazio delle fasi causato dal passaggio attraverso determinate risonanze d'un sistema. In particolare, si esporrà un modello bidimensionale con cui è possibile trasferire emittanza tra due direzioni nel piano trasverso: si fornirà una spiegazione del meccanismo per cui tale fenomeno ha luogo e si mostrerà come prevedere i valori finali di emittanza che un sistema raggiunge in tale configurazione. Questi risultati sono confermati da simulazioni numeriche. Simulazioni numeriche e relativi studî parametrici sono anche i risultati che vengono presentati per un altro modello, stavolta unidimensionale: si mostrerà infatti come un eccitatore esterno oscillante dipolare la cui frequenza passi attraverso un multiplo del *tune* della macchina permetta di catturare le particelle d'un fascio in un certo numero d'isole stabili.

Contents

Introduction

The importance of particle accelerators in modern high-energy physics experiments caused the rise of interest in topics belonging to the never-outdated domain of classical mechanics, which, while not addressing fundamental questions abount Universe, had retreated in Mathematics departments and was still one of the main theoretical framework in the field of celestial mechanics, i.e. in the low-energy, long-distance region of the Okun cube.

Although fundamental principles of classical mechanics have been well known for centuries, from Newton and then with the more convenient reformulations by Euler-Lagrange, Hamilton and Jacobi, when non-linear contributions appear in the equations, some highly non-trivial results need to be applied if one wants to illustrate something about a given system.

Non-linear interactions are present both as gravitational perturbative influence of planets, asteroids and comets on other celestial bodies, but also as specific magnets act on charged particles in a particle accelerator. Thus, in the sideral distances of outer space and in the 100 m deep LHC tunnel in the proximity of Geneva, some similar equations holds.

We are going to focus, in this Master thesis, on some of these equations: and one should not be surprised, when noticing in the bibliography books and review articles on celestial dynamics. More specifically, we are going to inspect some effects that we expect to happen when the energy of a system is *slowly* modulated. From the mathematical point of view, this will lead us to talk about *adiabatic invariance*, one of the main tools that "modern classical mechanics" (however this can be interpreted as an oxymoron) offer us.

It turns out in fact that when energies are slowly varied, some quantities remain quasi-constant. We will, of course, discuss what *slowly* and *quasi* exactly mean). It is clear that this property has a great power and is extremely useful when trying to *exploit* our

non-linear effects.

More specifically, we will address two different models which rely on such properties. The first one is a bidimensional one which concerns *emittance exchange*, i.e. the handling of a particles' beam width in phase space. The second one, on which we will present a sort of update of previous studies [7] is related to *beam splitting*. In both cases the same principles are used: we try to exploit the adiabatic invariance break down when a system crosses a separatrix of phase space as a result of a passage through a resonance. In both cases, however, these models are not only theoretically interesting, but also take on technological appeal as they may help to achieve better performance in beam handling or in beam splitting performance in a broader variety of conditions.

Actually, these projects are *spin-off* of a pioneering technique which takes advantage of the same properties and which has already lived its *cursus honorum* from a purely mathematical idea, through numerical validation, experimental proof-of-concept and, finally, has become implemented as a standard in one of the CERN accelerators. We are talking about *Multi-Turn extraction* (MTE) and which we will now briefly present.

Multi-turn extraction

First proposed in 2002 by Cappi and Giovannozzi [8], MTE is a method of splitting beams trapping particles into different phase space stable islands which are obtained by slowly varying the tune of the particle accelerator. It was designed to substitute the old and flawed mechanical system of beam shaving which was responsible of considerable beam losses and posed various maintenance issues. After many years of development phase, in 2015 MTE was finally put in operation [13] at the CERN Proton Synchrotron (PS).

In CERN accelerator chain, PS comes before the Super Proton Synchrotron (SPS) which is 11 times longer: when particles reach 14 GeV energy they are injected into SPS to continue their acceleration process. Now, in order to speed up the extraction-injection process, PS beams are splitted into 5 beamlets which are then injected one at a time into SPS: this allows 10/11 of SPS circumeference to be filled every 2 PS cycles, each lasting 1.2 s.

MTE has the advantage, compared to the precedent method, not to rely on any mechanical actions, but is only dependent on the ef-

fect of magnetic fields. Thus, when a stable resonance, excited by means of magnets generating non-linear fields, is crossed the phase space separatrices modify their topology: a number of stable islands (four if the resonance 1/4 is crossed, as it happens in PS) appears and bunches of particles are trapped as the stable islands pass by: the final effect is the splitting of a single-Gaußian initial beam distribution into several beamlets. Of course, this manipulation can be generalised in several ways, e.g., by changing the resonance order or by time-reversal, thus having Multi-Turn Injection (MTI).

Structure of the work

An interesting thing that one can notice from the history of the MTE project is that, despite more than a decade occurred from the initial to idea to a definitive implementation, is the fact that accelerator physics is one of the few branches of physics where a single person, or a small group can still be held accountable for a theory from its mathematical conception until its hoped experimental verification. In a certain sense, this means applying Galileian method in one of its purest forms. This is what we will humbly attempt in this work: our two models will undergo a theoretical analysis and some numerical validation. A work — which is still in progress at CERN Beam department — which tries to link the models with their possible experimental implementation (experiencing the painful and error-prone work of translation from the mathematical ὑπερουράνιον to the harsh reality.) will me mentioned in the end.

In greater detail, this Master thesis is structured as follows.

There is a **first part** which we called **toolbox** where general topics in classical mechanics and accelerator physics that we will later take advantage of while inspecting our models are briefly presented.

In the **first chapter**, devoted to **Hamiltonian mechanics** we will review the main properties of the Hamiltonian rewriting of classical dynamics: we will present the theory Hamilton equations and canonical transformation as well as Liouville theorem. Then some important sections will be pledged to the theory of action-angle variables and of the adiabatic invariance. We will briefly talk about Birkhoff resonant forms, due to their importance in approximating accelerator dynamics near a resonance, and finally present the break of adiabatic invariance at separatrix crossing, computing the deterministic jump according to adiabatic theory and a stochastic

correction due to the different averaging of oscillations.

In the **second chapter**, on the other hand, we will focus on **circular accelerators** presenting the coordinate system, writing the system Hamiltonian, describing the consequent motion and introducing the concept of *emittance*, which will be central in the following part, proving its main properties.

Then comes the **second part** which is focused on the **emittance exchange model**. In the **third chapter**, after having discussed the resulting Hamiltonian when the system crosses the $(1, 2)$ resonance, we give an argument in terms of adiabatic invariant jump which confirms and extends some preliminary inferences made by [18], about the possibility of exchanging emittance between the horizontal and vertical direction of accelerators' transverse space. Furthermore, a computation of the correction to this jump following what we explained in Chapter 1 is also attempted. Finally, some notes are made about a generalization to different resonances. We shall emphasize the fact that, whereas the idea to study such a system has been first presented in [9, 18], we now achieve a complete theoretical explaination of the phenomenon, which was absent in the cited references.

In the **fourth chapter**, **numerical simulations** on the motion equations derived in the precedent chapter are performed. We integrate equations not only to confirm the claims about emittance exchange but also to show that the mechanism we propose to explain this phenomenon actually holds. We then proceed to make some parametric study on the model in order to evaluate the emittance exchange performance depending on the free parameters of the system.

After this part, a final **fifth chapter** is present, where we talk about a **transverse beam splitting** model which, as usual, relies on adiabatic theory, but that differs from MTE because the accelerator tune is kept constant while a dipolar external excitator is driven through a resonance. There, we review some already-known theory on the topic, before presenting numerical simulations and parametric studies.

Part I

The toolbox

Chapter 1

Elements of Hamiltonian mechanics

Maintenant, voici ce principe, si sage, si digne de l'Être Suprême: lorsqu'il arrive quelque changement dans la Nature, la quantité d'action employée pour ce changement est toujours la plus petite qu'il soit possible

Pierre-Louis Moreau de Maupertuis (1744)

Since the study of beam dynamics in particle accelerators, when collective effects are neglected, reduces to the analysis of the motion of initial conditions with a certain distribution subject to external non-linear forces, the theoretical framework in which this type of study is generally built upon is that of classical mechanics, expecially in its Hamiltonian formulation, which is the best for inspecting effects due to perturbations, bifurcations and resonances effects.

We will first conduct a brief review of fundamental concepts of Hamiltonian mechanics, recalling the equations of motion, the theory of canonical transformations and Liouville theorem. These basic instruments will be needed almost everywere throughout this work. Then, adiabatic invariance framework, which is commonly used in accelerator physics field, will be presented.

The phenomena this work is centered upon, emittance exchange and beam splitting, although grealy different from an experimental point of view, rely their theoretical fundations on the combination of the same two aspects of some more advanced Hamiltonian dy-

namics, that are not commonly taught in undergraduate Analytical mechanics courses: the averaging of perturbation in the vicinity of a resonance and the separatrix crossing of a point and the subsequent loss of adiabatic invariance, according to the *neo-adiabatic* theory

1.1 Hamiltonian mechanics

Given a set of coordinates $(p_i(t),\, q_i(t))$ and a function $\mathcal{H}(p_i,\, q_i,\, t)$, where $1 \leq in$

a system is said to be *Hamiltonian* if the following equations hold:

$$\begin{aligned} \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \\ \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i}\,. \end{aligned} \tag{1.1}$$

These equations are called *Hamilton equation* while $\mathcal{H}$ is the *Hamilton function*, or *Hamiltonian.*

These equations are related to the Least Action Principle via the 1-form (we use Einstein summation convention)

$$\omega = p_i \mathrm{d}q^i - \mathcal{H}\mathrm{d}t\,. \tag{1.2}$$

In fact, the integral of the form on the trajectory $q_0 \to q_1$ which is described by the motion of a particle in the time interval $[t_0,\, t_1]$ yields

$$\int_{q_0=q(t_0)}^{q_1=q(t_1)} \left(p_i \mathrm{d}q^i - \mathcal{H}\mathrm{d}t\right) = \int_{t_0}^{t_1} \left(p_i \frac{\mathrm{d}q^i}{\mathrm{d}t} - \mathcal{H}\right)\mathrm{d}t = \int_{t_0}^{t_1} \mathcal{L}\mathrm{d}t$$

where $\mathcal{L} = p_i \dot{q}^i - \mathcal{H}$ is the Lagrangian, obtained by Legendre trasformation of the Hamiltonian.

Moreover, the Least Action Principle has a geometrical meaning for the Hamiltonian systems and the physical trajectories are the *rotor curves* of the 1-form ω.

1.1.1 Canonical transformations

The power of the Hamiltonian formulation, and the real reason of its adoption as the standard framework in which nonlinear problems

are studied, is that Hamilton equations treat in the same way every coordinate, time included.

Yet, the choice of coordinates is unconstrained: impulses and coordinates are independent (differently from the Lagrangian formulation) and they can exchange their role, or being mixed in complete freedom, while the Hamilton equations maintain their form.

We can therefore find, for an Hamiltonian $\mathcal{H}(p, q, t)$ and a trasformation $(p_i, q_i) \to (P_i, Q_i)$, a function $\mathcal{K}(P, Q, t)$ for which

$$\dot{P}_i = -\frac{\partial \mathcal{K}}{\partial Q_i} \qquad \dot{Q}_i = \frac{\partial \mathcal{K}}{\partial P_i}$$

Both formulations needs to fulfill Least Action Principle, written out as a condition on rotor curves as

$$\mathrm{d}\big(p_i \mathrm{d}q^i - \mathcal{H}\mathrm{d}t\big) = \mathrm{d}\big(P_i \mathrm{d}Q^i\big) - \mathcal{K}\mathrm{d}t$$

This means that the two forms can differ only for a closed form $\mathrm{d}F$ and

$$\mathrm{d}F = p_i \mathrm{d}q^i - P_i \mathrm{d}Q^i + (\mathcal{K} - \mathcal{H})\mathrm{d}t$$

which yields the transformation laws

$$p_i = \frac{\partial F}{\partial q_i} \qquad q_i = -\frac{\partial F}{\partial Q_i} \qquad \mathcal{K} = \mathcal{H} + \frac{\partial F}{\partial t}$$

It should be noted that the form of the Hamiltonian needs to be updated only for time-dependent trasformation.

The function $F(q, Q, t)$ is called *generatrix* or *generating function.*

Another interesting property of Hamiltonian systems, which we will take advantage from, is the possibility of giving the role of time to every coordinate of the system.

Let us express the $k-$th term of the form as $p_k = \tilde{p}$, $q_k = \tilde{q}$. Hence we can write

$$p_i \mathrm{d}q^i - \mathcal{H}\mathrm{d}t = \sum_{i \neq k} p_i \mathrm{d}q^i + (-\mathcal{H})\mathrm{d}t + (-\tilde{p})\mathrm{d}\tilde{q}$$

This form, which is formally equivalent to the previous one, makes it possible to express the motion as a function of $\tilde{q}$, whilst $-\tilde{p}$ plays the role of the Hamiltonian in the equivalent equations of motion

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tilde{q}} = -\frac{\partial \tilde{p}}{\partial q_i} \qquad \frac{\mathrm{d}q_i}{\mathrm{d}\tilde{q}} = \frac{\partial \tilde{p}}{\partial p_i} \qquad (i \neq k)$$
$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\tilde{q}} = -\frac{\partial \tilde{p}}{\partial t} \qquad \frac{\mathrm{d}t}{\mathrm{d}\tilde{q}} = \frac{\partial \tilde{p}}{\partial \mathcal{H}}$$

Finally, one of the most useful properties of the Hamiltonian formulation is that even the time evolution of a system can be seen as a peculiar canonical transformation.

In fact, being p_i' and q_i' the values of the canonical variables after a small time evolution of Δt, thanks to Hamilton equations

$$p_i' = p_i + \Delta t \dot{p}_i = p_i - \Delta t \frac{\partial \mathcal{H}}{\partial q_i}$$

and

$$q_i' = q_i + \Delta t \dot{q}_i = q_i + \Delta t \frac{\partial \mathcal{H}}{\partial p_i}$$

which, keeping only the $\mathcal{O}(\Delta t)$ terms, correspods to

$$q_i = q_i' - \Delta t \dot{q}_i = q_i + \Delta t \frac{\partial \mathcal{H}}{\partial p_i'}$$

which is the canonical trasformation given by the generating function

$$F = p_i' q^i + \mathcal{H} \Delta t$$

This property[1] justifies the construction of *symplectic integrators*, i.e. algorithms of numerical simulation that preserve the Hamiltonian structure of the system, but is also needed to prove the following theorem.

1.2 Liouville theorem

Let us consider the case in which, during a time evolution described by Hamilton equations, every point in the $2n-$dimensional phase-space evolves. Liouville theorem states that every volume element in phase space during such an evolution conserves its volume.

[1] Our simple argument is only valid for small values of Δt. A full proof can be made using Hamilton-Jacobi equation.

Now, the volume element $\mathrm{d}z$ in phase space is given by

$$\mathrm{d}z = \mathrm{d}q_1 \wedge \ldots \mathrm{d}q_n \wedge \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n$$

We know that a time evolution is the result of successive canonical transformation, thus if we prove that phase space volume is invariant under any canonical tranformation, Liouville theorem will be just a corollary of such an assertion.

Therefore, let $(p_i, q_i) \to (P_i, Q_i)$ be a canonical transformation. The volume element will transform according to the Jacobian determinant

$$\mathrm{d}Z = \left|\det \frac{\partial Z_i}{\partial z_j}\right| \mathrm{d}z$$

where z and Z encode the $2n$ canonical variables in the original and in the transformed system.

Volume is preserved if the Jacobian determinant is equal to 1.

Let us also define the ensemble of coordinates

$$\tilde{z} = (q_i, P_i)$$

Our Jacobian will read

$$\det \frac{\partial Z_i / \partial \tilde{z}_k}{\partial z_j / \partial \tilde{z}_k} = \frac{\det \partial Z_i / \partial \tilde{z}_j}{\det \partial z_i / \partial \tilde{z}_j} \tag{1.3}$$

Z and $\tilde{z}$ share the same second part, so all derivatives involving those coordinates will cancel out, and

$$\det \frac{\partial Z_i}{\partial \tilde{z}_j} = \det \frac{\partial Q_i}{\partial q_j} \tag{1.4}$$

and, for the same reason, but applied to the first half,

$$\det \frac{\partial z_i}{\partial \tilde{z}_j} = \det \frac{\partial p_i}{\partial P_j}$$

We now need an expression for $\partial Q_i / \partial q_k$. Being F the generating function of the transformation

$$Q_i = \frac{\partial F}{\partial P_i}$$

and

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial^2 F}{\partial P_i \, \partial q_j}$$

while, in the denominator determinant

$$\frac{\partial p_i}{\partial P_j} = \frac{\partial^2 F}{\partial q_i \, \partial P_j}$$

Thus, the two matrices in Eq. (1.3) are connected by a transposition, and being the determinant independent on trasposition their ratio turns out to be equal to 1, proving our theorem [15].

1.3 Action-angle variables and adiabatic invariance

Let us take into account an Hamiltonian system where the Hamiltonian function depends of a varying parameter λ.

If this variation is sufficiently *slow* (we will discuss what this *slowness* actually means), there are variable which are quasi-invariant. Also this *quasi*-invariance, and its relationship with the parameter variation speed needs further discussion.

Historical note. Although methods related to adiabatic invariance are now standard in the mathematical treatment of classical mechanics, their history does not date back to the golden age of the mathematical reformulations of Newtonian mechanics. It was introduced neither by Lagrange nor by Hamilton or by Jacobi. In fact, the first physicist who mentioned an experiment of *a pendulum whose leg is slowly shortened* was actually Albert Einstein, in the beginning of the quantum adventure.

Proceedings of the 1911 Solvay conference report an interesting debate between Einstein and Lorenz: [16]

> M. LORENTZ se rappelle une conversation qu'il eut avec M. EINSTEIN il y a déjà quelque temps, et dans laquelle il fut question d'un pendule simple qu'on raccourcirait en tenant le fil entre deux doigts, qu'on glisse vers le bas. Supposons qu'au commencement le pendule ait exactement un élément d'énergie tel qu'il correspond à la fréquence de ses oscillations, il semble alors qu'à la

> fin de l'expérience son énergie sera moindre que l'élément qui correspond à la nouvelle fréquence.
>
> M. Einstein. — Si l'on modifie la longueur du pendule de manière infiniment lente, l'énergie de l'oscillation reste égale à $h\nu$, si elle était primitivement égale à $h\nu$; elle varie proportionellement à la fréquence. Il en est de même pour un circuit électrique oscillant, dépourvu de résistance, et aussi pour le rayonnement libre.
>
> M. Lorentz. — Ce résultat est très curieux et fait disparaître la difficulté. En général, l'hypothèse des éléments d'énergie donne lieu à des problèmes intéressants dans tous les cas où l'on peut changer à volonté la fréquence des vibrations.
>
> M. Warburg. — La fréquence d'un pendule filiforme en oscillation peut être augmentée sans échange de travail, si, comme le faisait Galilée, on fait buter un point du fil contre un arrêt, au moment où il passe par sa position d'équilibre, et si l'on fixe ce point pendant que le pendule continue son mouvement dans la même direction.

The contribution of Warburg is also noteworthy. Arnol'd [1] would resolve the dilemma requiring that the experimenter is blind, i.e. cannot use information about the state of the pendulum in order to modulate the shortening of the leg. This request can be mathematically formalized simply asking a sufficient smooth leg-lenght variation.

The discussion topic was the quantization problem. What are the classical properties that in the quantum world turn out to be discretized? Arnold Sommerfeld proposed as an answers *adiabatic invariants*, which were first called this way by Paul Ehrenfest [11] (*Over adiabatische veranderingen van een stelsel in verband met de theorie der quanta*) owing to the fact that the slow modulation resembled the one needed in thermodynamics to perform an adiabatic transformation. The Bohr-Sommerfeld *old quantum theory* was finally overtaken by the later Schrödinger and Dirac axiomatic formualation, and adiabatic invariance theory migrated to departments of Mathematics — mainly in Soviet union — where classical mechanics was still alive. The modern definition, which we will report, of adiabatic invariant dates back to 1928 and is due to Leonid I. Mandel'štam and his students, A. A. Andronov and E. A. Leontovič

[19]. However, in 1963 Vladimir Igorevič Arnol'd would write [2] that adiabatic invariance *has been little studied by mathematicians in spite of its importance and interest* in a paper primarily devoted to celestial mechanics. More than astrophysics, though, common interest in plasma and accelerator physics, due also both to the implications of such researches, being just technological or related to fundamental physics issues, contributed to revive this topic.

We can now give a definition of *adiabatic invariance*:

Adiabatic invariant. A variable $J(p, q, \lambda = \delta t)$ is said to be an *adiabatic invariant* if, for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ so that, for every $\delta < \delta_\epsilon$ and $t < 1/\delta$

$$|J(p(t), q(t), \delta t) - J(p(0), q(0), 0)| < \epsilon$$

In a less formal way, we want that the adiabatic invariant, over a time evolution of $t \sim \epsilon^{-1}$, only undergoes a change which is $\mathcal{O}(\epsilon)$. It should be emphasized that this definition holds independently for every disconnected region of the phase space.

Thus, the speed ϵ of parameter variation is related to the time in which our variable variation is constrained.

A stronger requirement of a variable is to be a *perpetual* adiabatic invariant: the variation of J should be $\mathcal{O}(\epsilon)$ for *every* time. At theorem, proven by Arnol'd, states that, for 1D hamiltonians, if the parameter variation is periodic, adiabatic invariants are also perpetual.[2]

The most known and principal example of adiabatic invariant is the *action* variable.

1.3.1 Action-angle variables

First of all, we need to introduce the action-angle variable.

In a n degree of freedom system, i.e. a $2n-$dimensional symplectic manifold described by the phase variables (p_i, q_i), where n integrals of motion $I_i = k_i$ in involution are known, a theorem due to Liouville states that if the set M_k — we define $k = (k_1, \ldots, k_n)$ — of the points for which the integral have the same values $k_1, \ldots, k_n$ is compact, then it is diffeomorphic to the $n-$torus $\mathbb{T}^n$. We also find that for some functions ω_i which we will call *frequencies* we have

$$\dot{\phi}_i = \omega_i(k_1 \ldots, k_n) \tag{1.5}$$

This assures that, setting as the Hamiltonian one of the integrals, e.g. $\mathcal{H} = I_1$, we have the equations

$$\dot{I}_i = 0 \qquad \dot{\phi}_i = \omega_i(I_1 \ldots, I_n) \tag{1.6}$$

which describe the phase flow relative to the chosen Hamiltonian.

Now we have the coordinates (I_i, ϕ_i) but nothing guarantees that they are symplectic, i.e. that the trasformation $(p_i, q_i) \to (I_i, \phi_i)$ is canonical.

It is possibile, however, to introduce different functions $J_i = J_i(I_1, \ldots, I_n)$ which fulfill

$$\dot{J}_i = 0 \qquad \dot{\phi}_i = \omega_i(J_1 \ldots, J_n) \tag{1.7}$$

and keep the canonical change of variables.

These functions can be explicitly written.

Let us start with the unidimensional case, where the only integral is the Hamiltonian, and its conserved value h, and the set M_k redusces to M_h.

A canonical transformation is given by the generating function $S(J, q)$ and

$$p = \frac{\partial S}{\partial q} \qquad \phi = \frac{\partial S}{\partial J} \qquad \mathcal{H}\left(p = \frac{\partial S}{\partial q}, q\right) = h(J) \tag{1.8}$$

The invariant torus reduces to the closed curves that are identified just from the value of the energy h, and, being $h = h(J)$, by that of J. The differential $\mathrm{d}S$, for constant J, reads

$$\mathrm{d}S = \frac{\partial S}{\partial q}\,\mathrm{d}q = p\,\mathrm{d}q \tag{1.9}$$

so

$$S = \int p\,\mathrm{d}q \tag{1.10}$$

which is the generating function.

The $1-$form $p\,\mathrm{d}q$ is closed, and therefore locally exact on invariant surfaces. Thus, on a level curve of energy, the whole change of S is

$$\Delta S(J) = \oint_{M_{h(J)}} p\,\mathrm{d}q \tag{1.11}$$

that. thanks to Stokes' theorem, corresponds to the area inside the curve.

We can rewrite, from the second equation of (1.8),

$$\oint_{M_{h(J)}} \mathrm{d}\phi = 2\pi \tag{1.12}$$

Now, the periodicity of ϕ on the torus means that

$$\oint_{M_{h(J)}} \mathrm{d}\left(\frac{\partial S}{\partial J}\right) = \frac{\partial \Delta S(J)}{\partial J} = 2\pi \tag{1.13}$$

from which we finally get

$$J = \frac{1}{2\pi} \oint_{M_h} p \, \mathrm{d}q \tag{1.14}$$

On the other hand, differentiating (1.10) in J we retrieve, from the definition of ϕ

$$\phi = \frac{1}{2\pi} \frac{\partial}{\partial J} \int p \, \mathrm{d}q \tag{1.15}$$

The generalization to systems with more degrees of freedom is straightforward, if we are working in a $\mathbb{R}^{2n}$ phase space. In fact, being $\gamma_1 \dots \gamma_n$ 1D cycles which form a basis on the torus M_{k_i}, which means that the variation of the angular variable ϕ_i on the cycle γ_j is equal to $2\pi\delta_{ij}$, δ_{ij} being the Kroenecker symbol.

The actions J_i are then defined to be

$$J_i(k_1, \dots, k_n) = \frac{1}{2\pi} \int_{\gamma_i} p_j \, \mathrm{d}q^j \tag{1.16}$$

It can be shown that these integrals are independent from the choice of the γ_i.

1.3.2 Adiabatic invariance of the action variable

We can now prove that action variables are actually adiabatic invariants.

Starting from the time-dependent Hamiltonian $\mathcal{H}(p, q, \lambda(t))$, we have, differentiating

$$\frac{\partial \mathcal{H}}{\partial t} = \dot{\lambda}\frac{\partial \mathcal{H}}{\partial \lambda} \tag{1.17}$$

we can take the average of this equality. If $\dot{\lambda}$ is constant,

$$\left\langle \frac{\partial \mathcal{H}}{\partial t} \right\rangle = \dot{\lambda}\left\langle \frac{\partial \mathcal{H}}{\partial \lambda} \right\rangle = \frac{\dot{\lambda}}{T}\int_0^T \mathrm{d}t \frac{\partial \mathcal{H}}{\partial \lambda} \tag{1.18}$$

according to the definition of λ.

It is convenient to rewrite

$$\mathrm{d}t = \frac{\mathrm{d}q}{\partial \mathcal{H}/\partial p} \tag{1.19}$$

thanks to Hamilton equations. Moreover, the period T can be written as $\int_0^T \mathrm{d}t$, so

$$\left\langle \frac{\partial \mathcal{H}}{\partial t} \right\rangle \oint \frac{\mathrm{d}q}{\partial \mathcal{H}/\partial p} = \dot{\lambda} \oint \mathrm{d}q \frac{\partial \mathcal{H}/\partial \lambda}{\partial \mathcal{H}/\partial p} \tag{1.20}$$

where the closed-line integral is extended to one orbit of the system in phase space.

We rewrite this expression in a single integral in the variable q:

$$\oint \mathrm{d}q \left[\left\langle \frac{\partial \mathcal{H}}{\partial t} \right\rangle \frac{\partial p}{\partial \mathcal{H}} - \dot{\lambda}\frac{\partial \mathcal{H}}{\partial \lambda}\left(\frac{\partial \mathcal{H}}{\partial p}\right)^{-1} \right] = 0 \tag{1.21}$$

If λ variation is slow, during one orbit its value can be seen as constant, so the total derivative of $\mathcal{H}$ w.r.t. λ should be zero, i.e.

$$\frac{\partial \mathcal{H}}{\partial \lambda} = -\frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial \lambda} \tag{1.22}$$

thanks to Leibniz's rule.

Substituting this result into the Equation (1.21) we get

$$\oint \mathrm{d}q \left[\left\langle \frac{\partial \mathcal{H}}{\partial t} \right\rangle \frac{\partial p}{\partial \mathcal{H}} + \dot{\lambda}\frac{\partial p}{\partial \lambda} \right] = 0 \tag{1.23}$$

which turns out to be equal to

$$\oint \mathrm{d}q \left[\left\langle \frac{\partial p}{\partial t} \right\rangle + \dot{\lambda}\frac{\partial p}{\partial \lambda} \right] = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \oint p\mathrm{d}q \right\rangle = 0 \tag{1.24}$$

This means that the average variation of the action

$$J = (2\pi)^{-1} \oint p\mathrm{d}q$$

on a orbit is zero, i.e. that the action integral is an adiabatic invariant.

1.3.3 Improved adiabatic invariant

However, something better can be achieved. If we define

$$\tilde{J} = J + \epsilon u \tag{1.25}$$

where (being T the motion period and performing the integral on a unperturbed trajectory)

$$u = \frac{1}{2\pi} \int_0^T \mathrm{d}t \left(\frac{T}{2} - t \right) \frac{\partial \mathcal{H}}{\partial \lambda} \tag{1.26}$$

the *improved* adiabatic invariant $\tilde{J}$ is preserved up to $\mathcal{O}(\epsilon^2)$ [21].

1.4 Fast and slow variables. Averaging.

If we slowly perturb an integrable system, integrals of motion of the original system are expected to slowly change, and the order of magnitude of the time scale over which this change becomes considerable is generally dependent on the inverse of the perturbation parameter.

Now — sticking to the 1D case, the generalization to multiple degrees of freedom being straightforward — if a system is integrable, there exist an integral I and an angle coordinate ϕ so that

$$\dot{I} = 0 \qquad \dot{\phi} = \omega(I) \tag{1.27}$$

where ω is the frequency.

When a perturbation, parametrized by ϵ is introduced, the equations become

$$\dot{I} = \epsilon f(I, \phi, \epsilon) \qquad \dot{\phi} = \omega(I) + \epsilon g(I, \phi, \epsilon) \tag{1.28}$$

where the two functions f and g are periodic in ϕ. I, whose variation is $\mathcal{O}(\epsilon)$, is said to be a *slow variable*, while ϕ, which varies $\mathcal{O}(1)$, is called *fast variable*.

The idea behind *averaging* is to substitute I with a new variable, J, whose equation of motion reads

$$\dot{J} = \epsilon \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi f(J, \phi, 0) \tag{1.29}$$

where, because ϕ evolves faster than I, its contribution on a period to the variation of J can be averaged. This is called *averaging principle*.

Vladimir I. Arnol'd specifies in *Mathematical methods of Classical Mechanics* [1], that

> This principle is not a theorem, but a physical proposition, that is, a vaguely stated, and, strictly speaking, false assertion. Such assertions often happen to be fruitful sources for mathematical theorems.

But how well the averaged system approximates the original one? For single-frequency systems which fulfill some technical assumptions, it can be shown that

$$|I(t) - J(t)| = \mathcal{O}(\epsilon) \qquad \text{for } t < \epsilon^{-1} \tag{1.30}$$

In fact (we follow [1, p. 289]) let

$$P = I + \epsilon k(I, \phi) \tag{1.31}$$

where k is a periodic function. Differentiating, we get

$$\begin{aligned} \dot{P} &= \dot{I} + \epsilon \frac{\partial k}{\partial I} \dot{I} + \epsilon \frac{\partial k}{\partial \phi} \dot{\phi} \\ &= \epsilon f(I, \phi) + \epsilon^2 \frac{\partial k}{\partial I} f(I, \phi) + \epsilon \frac{\partial k}{\partial \phi} (\omega(I) + \epsilon g(I, \phi)) \\ &= \epsilon \left(g(I, \phi) + \frac{\partial k}{\partial \phi} \omega(I) \right) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{1.32}$$

Inverting Eq. (1.32), we can also write

$$I = P + \epsilon h(P, \phi, \epsilon) \tag{1.33}$$

for some function h. We are not actually interested in h, because, in our approximation,

$$\dot{P} = \epsilon \left(g(P, \phi) + \frac{\partial k}{\partial \phi} \omega(P) \right) + \mathcal{O}(\epsilon^2) \tag{1.34}$$

where we need to assume from the beginning that ω, f, g and their derivatives, up to the second order, are bounded.

In order for this to be true, also k and h and their first and second derivatives needs to be bounded. We shall prove that this holds.

We want to connect this result for P with the definition of J. If we tried to set $\dot{P} = \mathcal{O}(\epsilon^2)$ we would solve

$$-\frac{g(P,\phi)}{\omega(P)} = \frac{\partial k}{\partial \phi} \tag{1.35}$$

This is impossible, if we need k to be periodic, because if we take the average of both members of the equation we get

$$\begin{aligned} -\frac{1}{2\pi\omega(P)} \int_0^{2\pi} \mathrm{d}\phi, g(P,\phi) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \, \frac{\partial k}{\partial \phi} \\ = k(P,2\pi) - k(P,0) = 0 \end{aligned} \tag{1.36}$$

which is, in general, not true.

The solution is, being $\overline{g}(P)$ the average of g over a period, to set

$$g(P,\phi) + \frac{\partial k}{\partial \phi}\omega(P) = \overline{g}(P) \tag{1.37}$$

or, equivalently,

$$k(P,\phi) = -\int_0^{\phi} \mathrm{d}\varphi \frac{g(P,\varphi) - \overline{g}(P)}{\omega(P)} \tag{1.38}$$

which is bounded unless the frequency ω is null. Moreover, h is also limited, thanks to the inverse function theorem.

Now,

$$\dot{J} = \epsilon \overline{g}(P) \tag{1.39}$$

from the averaging principle, and

$$\dot{P} = \epsilon \overline{g}(P) + \mathcal{O}(\epsilon^2) \tag{1.40}$$

it follows that

$$|\dot{P} - \dot{J}| = \mathcal{O}(\epsilon^2) \tag{1.41}$$

and, for $t = \mathcal{O}(\epsilon^-1)$, integrating we get

$$|P - J| = \mathcal{O}(\epsilon) \tag{1.42}$$

On the other hand,

$$|P - I| = \epsilon|k| = \mathcal{O}(\epsilon) \tag{1.43}$$

and from the triangular inequality

$$|I - J| < |P - I| + |P - J| = \mathcal{O}(\epsilon) \tag{1.44}$$

This concludes our proof.

1.5 Birkhoff normal forms

Let $\mathcal{H}$ be an n-degree of freedom Hamiltonian with an equilibrium position. The Hamiltonian is composed of a linear oscillator part $\mathcal{H}_0$

$$\mathcal{H}_0 = \sum_{k=1}^{n} \frac{1}{2}\omega_k (p_k + q_k)^2 \tag{1.45}$$

plus some interaction terms, i.e.

$$\mathcal{H} = \mathcal{H}_0 + \sum_{l=1}^{m} \mathcal{H}_l \tag{1.46}$$

where the terms $\mathcal{H}_l$ have degree l in the p_i and q_i variables.

After a change of variables $(p_i, q_i) \to (x_i, y_i)$ and a further one to polar coordinates

$$x_i = \sqrt{2J_i}\cos\phi_i \qquad y_i = \sqrt{2J_i}\cos\phi_i \tag{1.47}$$

we define *resonant normal form* for our Hamiltonian at a given resonance a polynomial function in (x_i, y_i) which, in polar coordinates, depends only on angles in the linear combination selected by the resonance.

The possibility of such a process of variables is assured by a theorem which we will now prove.

Let S be the generating function of the $(p_i, q_i) \to (x_i, y_i)$ transformations.

We have

$$S = x_i q^i + \tilde{S}(x_i, q_j) \tag{1.48}$$

The new Hamiltonian $\tilde{H}$ obeys the equation

$$\mathcal{H}\left(\frac{\partial S}{\partial q_i}, p_j\right) = \tilde{\mathcal{H}}\left(x_i, q_j + \frac{\partial S}{\partial y_i}\right) \tag{1.49}$$

Thus, calling S_k each term of $\tilde{S}$ of degree k, we get, being F_l a generic form,

$$\sum_j \omega_j \left(x_j \frac{\partial S_l}{\partial q_j} - q_j \frac{\partial S_l}{\partial x_j}\right) = \tilde{\mathcal{H}}_l - F_k \tag{1.50}$$

where $\tilde{\mathcal{H}}_l$ corresponds to $\mathcal{H}_l$ under the $(p, q) \to (x, y)$ transformation

Passing to polar coordinates (J_i, ϕ_i) the last equation becomes

$$\omega_i \frac{\partial S_l}{\partial \phi_i} = \tilde{\mathcal{H}}_l - F_l \tag{1.51}$$

Thus, we can choose

$$S_l = \sum_r i \frac{f_{k_r}(J_r)}{k_j \omega^j} \exp\{i k_i \phi^i\} \tag{1.52}$$

where the coefficients f_{k_r} come from the Fourier expansion of F_l and the sum is extended to all the k_r possible combination that are not resonant.

Equation (1.52) therefore means that the resulting Hamiltonian is in the correct form, i.e. only have the required angular dependence.

1.6 Separatrix crossing and area variation

Let us consider a time-dependent Hamiltonian which causes the separatrices to move. In general, this is achieved by introducing a parameter λ in our Hamiltonian, with $\lambda = \epsilon t$.

If ϵ is sufficiently small, we can apply the adiabatic approximation to our system.

Given initial conditions, the trajectory stays in the initial region of phase space, moving, for each λ close to the energy level curves for $\mathcal{H}_\lambda$.

If the separatrices motion causes the region our system is enclosed in to shrink, eventually the trajectory will occupy all the region area. Thus, adiabatic approximation is no more valid, and the trajectory can cross the separatrix and enter a different region, dramatically changing its area value.

In reality, the jump of the adiabatic invariant we observe is composed of two different contributions: one explained by the standard theory of adiabatic invariance, and one which is a result from the *neo-adiabatic theory* and acts as a correction for the classical jump.

1.6.1 The standard theory

When an averaging is done, the action variable of the unperturbed, which was still adiabatically invariant in the original, perturbed system, becomes an integral of motion in the averaged system, and, for each region i of the phase space, the following equation holds

$$\Theta_i := -\oint_{\gamma_i} \frac{\partial \mathcal{H}}{\partial \lambda} \mathrm{d}t = \frac{\partial A_i}{\partial \lambda}$$

where A_i is the area of the region i enclosed by the sepatratrix γ_i.

In fact, being the action J integral of the averaged system, and performing a gauge choice of the Hamiltonian such as the value of the energy on the sepatrix is null, we have, from the definition of action

$$\frac{\partial A_i}{\partial \lambda} = 2\pi \frac{\partial J(h,\lambda)}{\partial \lambda} = -\oint_{\mathcal{H}=h} \mathrm{d}t \, \frac{\partial \mathcal{H}}{\partial \lambda}$$

Before crossing the separatrix we assume the motion to happen with constant action J_0, which is given by the initial conditions, in a region G_0, At a certain time $\lambda = \lambda^*$, when the equation

$$A_0(\lambda^*) = 2\pi J_0$$

the separatrix is crossed. A point can now enter an adjacent region G_j with probability

$$P_j = \frac{\Theta_j(\lambda^*)}{\Theta_0(\lambda^*)}$$

It should be emphasized that entrance into a certain region is not a deterministic event.

This formula for the probability can be explained in terms of Liouville theorem, i.e. from the conservation of phase space volume during the motion.

Let δA_i be the area difference of G_i as λ varies from in $[\lambda, \lambda+\delta\lambda]$. Conversely, ΔA_j is the volume captured into G_j. The probability is

$$P(\lambda^*) = \lim_{\substack{\delta\lambda\to 0 \\ \epsilon\to 0}} \frac{\delta A_j}{\delta A_i} \tag{1.53}$$

In fact, if

$$\delta A_i = A_i(\lambda^* + \delta\lambda) - A_i(\lambda^*)$$

and

$$\delta A_j = \lim_{\epsilon\to 0} A_j(\lambda^* + \delta\lambda) - A_j(\lambda^*)$$

their ratio becomes, in the limit

$$\lim_{\delta\lambda\to 0} \frac{\delta A_j}{\delta A_i} = \frac{\partial A_j/\partial\lambda}{\partial A_i/\partial\lambda}$$

and formula (1.53) follows.

Finally, after the crossing, the point captured in G_j moves with an action

$$J_f = \frac{A_j(\lambda^*)}{2\pi}$$

which is correspondent to the area of the landing region at the crossing time [22].

1.6.2 Improved theory

Nevertheless, the aforementioned standard theory can be corrected, since significant effects can arise not only in the exact instant of the separatrix cross and of the subsequent discontinuous jump, but contributions from both the approach and the subsequent departure to and from the seapratrices have to be taken into account.

In fact, when no separatrix crossings are involved, values of adiabatic invariants undergo natural small oscillations, which are $\mathcal{O}(\epsilon)$. In average, as we have shown, these oscillations cancel each other

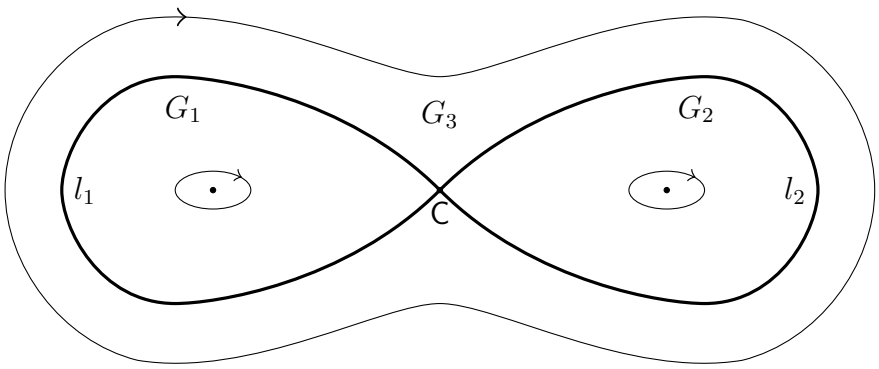


FIGURE 1.1 – Phase diagram of the phase space of the model studied in [21].

and J can be taken as constant. But when a separatrix is crossed and the action dramatically changes value, oscillations before and after the crossing now does not cancel each other anymore, and a residual effect is present and computable.

Of course, we expect this contribution to be at least of order $\mathcal{O}(\epsilon)$, since it need to disappear in the limit $\epsilon \to 0$.

This phenomenon, while first put into light by Tennyson *et al.* [23], has been fully mathematically exploited in the seminal work of Anatolij I. Nejštadt [21], whose general result we will now review, before applying the same method to the specific class of Hamiltonians we are interested in.

Let $\mathcal{H}$ be a 1D Hamiltonian, dependent on the parameter $\lambda = \epsilon t$, whose phase space is described by Fig. 1.1, where C is the hyperbolic point where the separatrices cross. The space is thus divided into three regions, G_1, G_2 and G_3. We will also set, on a separatrix, $\mathcal{H} = 0$, taking advantage of the gauge freedom.

If h, $h \ll 1$ is the value of the energy close to our separatrix. The period of the motion depends logarithmically on h

$$T_i = -a_i \log|h| + b_i + \mathcal{O}(h \log|h|) \tag{1.54}$$

Because, inverting the formula for the frequency

$$T_i = \frac{2\pi}{\omega} = 2\pi \frac{\partial J}{\partial \mathcal{H}} = 2\pi \frac{\partial J}{\partial h} \tag{1.55}$$

integrating (1.54) we get

$$2\pi J = S_i - a_i h \log|h| + (a_i + b_i)h + \mathcal{O}(h^2 \log|h|) \tag{1.56}$$

where the integration constant S_i is the area of the region i, and accounts for the value of J when the energy h is exactly zero.

We can get the expression for $a = a(\lambda)$ via linearization of the motion in the vicinity of the hyperbolic point.

In fact, close to a saddle point, our Hamiltonian is well approximated by an hyperbolic pendulum with frequency ω, i.e., the motion reads

$$x(t) = x_0 \cosh(\omega t) \tag{1.57}$$

which, upon inversion, yields

$$t(x) = \omega^{-1} \log \left[\frac{x}{x_0} \pm \sqrt{\left(\frac{x}{x_0}\right)^2 - 1} \right] \sim \omega^{-1} \log|h| \tag{1.58}$$

Thus, a is given by the hyperbolic frequency of the linearized motion, which is given by

$$a = \omega^{-1} = \left(\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}x^2}\bigg|_{x_{\text{hyp}}} \right)^{-1/2} \tag{1.59}$$

where V_{eff} is the resultant effective potential upon linearization of $\mathcal{H}$ close to the hyperbolic point x_{hyp}.

For an hamiltonian which is not written in diagonal form, this result can be straightforwardly generalized as

$$a = (\det \mathsf{H})^{-1/2}\Big|_{x_{\text{hyp}}} \tag{1.60}$$

where H is the Hessian matrix of the Hamiltonian evaluated on the saddle point.

If we stop at the $\mathcal{O}(h \log|h|)$ order of magnitude, we therefore obtain, differentiating, that, on a trajectory

$$\oint \mathrm{d}t \, \frac{\partial \mathcal{H}}{\partial \lambda} = -2\pi \frac{\partial J}{\partial \lambda} = -\frac{\partial S_i}{\partial \lambda} + \mathcal{O}(h \log|h|) =: -\Theta_i + \mathcal{O}(h \log|h|) \tag{1.61}$$

Then, we proceed calculating the improved adiabatic invariant via Eq. 1.26.

Inside the separatrix, i.e. either in the G_1 or G_2 region, we have, linearizing motion

$$2\pi u = d_i + \mathcal{O}(|h|^{1/2} \log |h|) \tag{1.62}$$

where in the region G_3, if h is small motion is close to the separatrices and the integral reduces to

$$\begin{aligned} 2\pi u &= \frac{1}{2}\Theta_1 T_2 - \frac{1}{2}\Theta_2 T_1 + d_3 + \mathcal{O}(|h|^{1/2} \log |h|) \\ &= \frac{a}{2}(\Theta_2 - \Theta_1) \ln |h| + \frac{b_2}{2}\Theta_1 + \frac{b_1}{2}\Theta_2 + + d_3 + \mathcal{O}(|h|^{1/2} \log |h|) \end{aligned} \tag{1.63}$$

If a point starts in the outer region, it revolves around the separatrix. Let us call J_-, $\tilde{J}_-$ and τ_-, respectively, the initial values of the action, of the improved action and of the slow time. In the same way, we will refer to their values at the separatrix crossing as J_*, $\tilde{J}_*$, τ_*.

Then, we define a map which follows the evolution of the system before the capture where every integer step coincides with a crossing of the y axis every time closer to the saddle point. We label these steps with the integers $-N, -N+1, \ldots, 0$ ($N \gg 1$), where the index 0 corresponds to the capture instant (i.e. $\tau_0 = \tau_*$) and $-N$ needs to be sufficiently close to initial conditions.

We will now express the values of energy, slow time, action and improved action for the n+1-th step as function of the same variables at the n-th iteration.

First of all, for the energy h

$$h_{n+1} = h_n + \epsilon(\Theta_3 + \mathcal{O}(h_{n+1}^{1/2})) \tag{1.64}$$

For the slow time τ, we need to remember that a particle spends most of the time in the vicinity of a saddle point, thus the only relevant contribution to period motion calculation are the logarithmic ones which are given by Eq. 1.53, where the right energy has to be taken into account.

In fact, during a turn around the G_3 region our point passes three times next to the saddle point. The first in at start, with energy h_n. Though, only half of the passage is included in the $n \to n+1$ step:

this explains the 1/2 factor we will multiply to this contribution. Then, a full passage is made at the opposite side of the y axis, when the energy is grown as $h_n + \epsilon\Theta_1$. Finally, another half contribution has to be taken into account when energy has finally reached the value h_{n+1}.

This means that the slow time will evolve as

$$\begin{aligned}\tau_{n+1} = \tau_n + \epsilon \Bigg[& \frac{1}{2}(a \log |h_n| - b_3) + (a \log |h_n + \epsilon\Theta 1| - b_3) + \\ & + \frac{1}{2}(a \log |h_{n+1}| - b_3) + \mathcal{O}(h_{n+1}^{1/2}) \Bigg]\end{aligned} \tag{1.65}$$

According to (1.56), for the action J we have

$$\begin{aligned}2\pi J_n &= S_3(\tau_n) - 2ah_n \log |h_n| + (2a + b_3)h_n + \mathcal{O}(h_{n+1}^2 \ln^2 h_{n+1}) \\ &= S_3(\tau_*) + \Theta_3(\tau_n - \tau_*) - 2ah_n \log |h_n| + (2a + b_3)h_n \\ &\qquad + \mathcal{O}(h_{n+1}^2 \ln^2 h_{n+1})\end{aligned} \tag{1.66}$$

where we rewrote $S_3(\tau_n)$ integrating the relation $\Theta_3 = \mathrm{d}S_3/\mathrm{d}\lambda$.

Finally, for the improved action $\tilde{J} = J + \epsilon u$ we also need a map for u, that, following (1.63) reads

$$2\pi u_{n+1} = \frac{a}{2}(\Theta_2 - \Theta_1) \ln |h| + \frac{b_2}{2}\Theta_1 + \frac{b_1}{2}\Theta_2 + + d_3 + \mathcal{O}(|h|^{1/2} \log |h|) \tag{1.67}$$

In order to obtain the improved adiabatic invariant difference *before* the separatrix crossing we need to calculate the value

$$\Delta\tilde{J} = \tilde{J}_* - \tilde{J}_- = (\tilde{J}_0 - \tilde{J}_N) + (\tilde{J}_N - \tilde{J}_-) \tag{1.68}$$

We want to neglect the second term, in order to get the final value only with the telescopic sum of the values of $\tilde{J}_n = J_n + \epsilon u_n$. If we want this to be true, the difference $(\tilde{J}_0 - \tilde{J}_N)$ needs to be smaller than the remainder of $(\tilde{J}_0 - \tilde{J}_N)$, and this will result in a constraint about how *close* J_N must be to J_-.

We can now work on the main sum. Let us start with the easy part. It is straightforward to see that, for every n,

$$h_n = h_0 + n\epsilon\Theta_3 \tag{1.69}$$

Things get more difficult with τ. We have

$$\begin{aligned}\tau_n &= \tau_0 + \sum_0^{n-1}(\tau_{n+1} - \tau_n) + \\ &= \tau_0 + \epsilon \sum \left[\frac{1}{2}(a\log|h_n| - b_3)\right. \\ &\qquad + (a\log|h_n + \epsilon\Theta 1| - b_3) + \\ &\qquad \left. + \frac{1}{2}(a\log|h_{n+1}| - b_3) + \mathcal{O}(h_{n+1}^{1/2})\right]\end{aligned} \tag{1.70}$$

and, using the last result for h_n

$$\begin{aligned}\tau_n = \tau_0 + \epsilon\Bigg[&\frac{1}{2}\sum_k (a\log(h_0 + k\epsilon\Theta_3) - b_3) \\ &+ \sum_k (a\log(h_0 + \epsilon\Theta_1 + k\epsilon\Theta_3) - b_3) + \\ &+ \frac{1}{2}\sum_k (a\log(h_0 + k\epsilon\Theta_3) - b_3) + \\ &+ \frac{a}{2}\log\frac{h_0 + n\epsilon\Theta_3}{h_0} + \mathcal{O}(h_{n+1}^{1/2})\Bigg]\end{aligned} \tag{1.71}$$

It is useful to remember that

$$\sum_{k=0}^{N_1}\log(x + ky) = \log y^N + \log\left(\frac{x}{y}\right)_N = N\log y + \log\frac{\Gamma(x/y + N)}{\Gamma(x/y)} \tag{1.72}$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol. One should not be surprised that the fact that the sum of the logarithm of a succession values that differ by one integer is something related to the factorial, i.e. to the Gamma function.

Applying this result to our sum for τ we get, defining $\xi = h/(\Theta_3\epsilon)$

$$\tau_n = \tau_0 - \epsilon\Bigg\{ - nb_3 + a\Bigg[2N\ln(\epsilon\Theta_3) + \ln\frac{\Gamma(\xi+n)}{\Gamma(\xi)} + \ln\frac{\Gamma(\xi+\Theta_1/\Theta_3+n)}{\Gamma(+\Theta_1/\Theta_3)} + \frac{1}{2}\ln\frac{\xi+n}{\xi}\Bigg]\Bigg\} \tag{1.73}$$

We can take advantage of the Stirling approximation to get, for large values of n

$$\log\Gamma(z) = \log\sqrt{2\pi} + \left(z-\frac{1}{2}\right)\log z - z + \mathcal{O}(z^{-1}) \tag{1.74}$$

and, substituting,

$$\tau_n = \tau_0 - n\epsilon(-b_3 - 2a + 2a\ln\epsilon\Theta_3) + \epsilon a\Bigg[\ln\frac{2\pi}{\Gamma(\xi)\Gamma(\xi+\Theta_1/\Theta_3)} + 2\left(\xi+n+\frac{\Theta_1}{2\Theta_3}-\frac{1}{4}\right)\log(\xi+n) - 2\xi - \frac{1}{2}\log\xi\Bigg] + \mathcal{O}(\epsilon^{3/2}\log\epsilon) \tag{1.75}$$

Finally, we can substitute into the action J and find

$$2\pi(J_N - J_0) = \Theta_3\Delta\tau - 2ah_N\log h_N + 2ah_0\log h_0 + (2a+b)\Delta h \tag{1.76}$$

where, of course, $\Delta\tau = \tau_N - \tau_0$ and $\Delta h = h_N - h_0 = N\epsilon\Theta_3$ which means

$$2\pi(J_N - J_0) = \Theta_3\Delta\tau - 2a(h_0 + N\epsilon\Theta_3)\log(h_0 + N\epsilon\Theta_3) + 2ah_0\log(h_0) + N\epsilon\Theta_3(2a+b) \tag{1.77}$$

and, for the improved action,

$$2\pi(u_N - u_0) = \frac{a}{2}(\Theta_2 - \Theta_1)\log\frac{h_N}{h_0} = \frac{a}{2}(\Theta_2 - \Theta_1)\log\frac{h_0+\epsilon\Theta_3}{h_0} \tag{1.78}$$

Combining these results, and neglecting $\mathcal{O}(\epsilon^2)$ terms, we get the final estimate for the improved adiabatic invariant variation *before* the crossing as

$$2\pi(\Delta\tilde{J})_- = 2\epsilon a\Theta_3\left[-\frac{1}{2}\ln\frac{2\pi}{\Gamma(\xi)\Gamma(\xi+\frac{\Theta_1}{\Theta_3})}+\xi+\left(\frac{\Theta_2}{2\Theta_3}-\xi\right)\ln\xi\right] + \mathcal{O}(\epsilon^{3/2}\ln\epsilon) \tag{1.79}$$

Nevertheless, this was only the first step. We need now to analyze what happens when the separatrix is crossed.

In this case, we need a better approximation for the crossing time t_*, which we will call $t_*^{(i)}$ where the index i denotes the attainment region G_1 or G_2.

We will use

$$t_*^{(i)} = t_* - \frac{a}{2}\ln h_0 - \frac{a}{2}\ln|h_*^i| + b_i + \mathcal{O}(\epsilon\ln^2\epsilon) \tag{1.80}$$

where

$$h_*^i = h_* - \epsilon\Theta_2 \tag{1.81}$$

Then, denoting as $\tilde{J}_*^{(i)}$ the value for $\tilde{J}_*$ which comes from the integration of Eq. (1.55), we have, defining

$$\xi_i = \frac{|h_*^{(i)}|}{\epsilon\Theta_i} \tag{1.82}$$

and thus using the substitutions

$$h_* = \epsilon\xi\Theta_3 \qquad h_*^i = \epsilon\xi\Theta_i \tag{1.83}$$

the final result becomes (being $\Delta\Theta_i = (-1)^i(\Theta_2-\Theta_1)$)

$$\begin{aligned}2\pi(\Delta\tilde{J})_{*,i} =& \frac{\Theta_i}{\Theta_3}(2\pi J - S_3(\tau_*)) + a\epsilon\Theta_i\left(\xi_i-\frac{1}{2}\right)\ln(\xi_i\Theta_i\epsilon)\\ &-\left(2\frac{\Theta_i}{\Theta_3}\ln(\xi\Theta_3\epsilon)\right)+a\epsilon\frac{\Theta_i}{\Theta_3}(\Delta\Theta_i\xi_i-2\Theta_i)\\ &+\Theta_i\epsilon\left(\frac{1}{2}-\xi_i\right)\left(b_i-\frac{\Theta_i}{\Theta_3}b_3\right)\\ &+\epsilon\left(d_i-\frac{\Theta_i}{\Theta_3}d_3\right)+\mathcal{O}(\epsilon^{3/2}\ln\epsilon)\end{aligned} \tag{1.84}$$

Finally, the point enters either in G_1 or in G_2. As done before, we use a map where every step coincides with the passage on the x axis, either at the left or at the right of the saddle point.

Nevertheless, the calculations and the involved approximation are the same of those we used to obtain (1.79), but this time the calculation of τ_N is simpler, because the particle now passes close to the hyperbolic point only twice, moving for half a separatrix when the energy is $h_n = h_0 - n\epsilon\Theta_i$ and for another half with the increased energy step h_{n+1}.

This means that, having defined $\tilde{J}_+$ as the final value for the improved adiabatic invariant

$$2\pi(\Delta\tilde{J})_{+,i} = \epsilon a\Theta_i\left[-\ln\frac{\sqrt{2\pi}}{\Gamma(\xi_i)} + \xi_i + \left(\frac{1}{2} - \xi_i\right)\log\xi_i\right] + \mathcal{O}(\epsilon^{3/2}\log\epsilon) \tag{1.85}$$

We can now add the three contributions (1.79, 1.84, 1.85) to get the final result in all of its glory:

$$\begin{aligned} 2\pi\Delta\tilde{J}_i = 2\pi(\tilde{J}_+ - \tilde{J}_-) &= \epsilon a\Theta_i\left(\xi_i - \frac{1}{2}\right)\left[\ln(\epsilon\Theta_i) - \frac{2\Theta_i}{\Theta_3}\ln(\epsilon\Theta_3)\right] \\ &- \epsilon a\Theta_i \ln\frac{(2\pi)^{3/2}}{\Gamma(\xi_i)\Gamma\left(\frac{\Theta_i}{\Theta_3}(1-\xi_i)\right)\Gamma\left(1-\frac{\Theta_i}{\Theta_3}\xi_i\right)} \\ &+ \epsilon\Theta_i\left(\frac{1}{2} - \xi_i\right)\left(b_i - \frac{\Theta_i}{\Theta_3}b_3\right) + \epsilon\left(d_i - \frac{\Theta_i}{\Theta_3}d_3\right) + \\ &+ \mathcal{O}(\epsilon^{3/2}(\ln\epsilon - (1-\xi_i)^{-1})) \end{aligned} \tag{1.86}$$

Chapter 2

Fundamentals of beam dynamics

> I sometimes think about the tower at Pisa as the first particle accelerator, a (nearly) vertical linear accelerator that Galileo used in his studies.
>
> Leon M. Lederman

> The first accelerator dates back to prehistoric-historic times, when men built bows and arrows for hunting.
>
> S.Y. Lee [17]

2.1 Particle accelerators

The importance of particle accelerators in the recent history of experimental physics needs not any further explaination. Breakthrough discoveries in the field of high-energy particle physics, from the gauge bosons to the top quark and the Higgs boson have been made by large collaborations of scientists using accelerators, and many Nobel prizes have been awarded for such discoveries.

Anyway, the principle on which complex machines like LHC are built is quite simple: charged particles are accelerated by the action of strong electric fields in order to increase their energy and observe

what happens when they collide with either a fixed target or other accelerated particles.

Two are the main challenges in such experiments: one, which is the main duty of high-energy physicists, is to measure and recognize the collision products, and to verify their relations with theoretically expected values.

The other one, where this work belongs, is the task of the accelerator physicists: to explain and control the dynamics of the ensemble of particles (*beam*) inside the accelerator. [25]

One could think that this task is far simpler than the first, because the physical and matematical tools needed to study these issues — electromagnetism, and thus Maxwell equations since electromagnetic fields are involved, and classical mechanics, mainly in the Hamiltonian formulation, as particle move in certain potentials — have foundations firmly built centuries ago, and are well known and widely taught.

Actually, the solid foundations of these physical tools do not mean that problems are *easy* to solve. As soon as nonlinearities appear, i.e. systems are complex just a bit more than toy models, our knowledge fails and research-level work is needed to overcome these difficulties.

We will now present, mainly following [17], a brief review of some fundamental concepts of beam dynamics that we need to tackle the problems we are focusing this work on. First of all, it should be noted that all that follows is related to *circular* accelerators. Of course, in similar machines, electric fields are used to increase particles' speed while magnetic fields are first of all needed to keep the charges on the circular trajectory, but they also account for effects in the transverse plane.

Thus, we will describe the peculiar (and more convenient) coordinate choice that is standard to describe motion in accelerators, then we will justify an expression for the Hamiltonian we will introduce in our equations which accounts for the magnetic effects. The next step is to derive the basic equations of motion for a beam particle and analyze some of their properties. Finally, we deal with the notion of *emittance* which is central in this work; its basic properties will then be presemted.

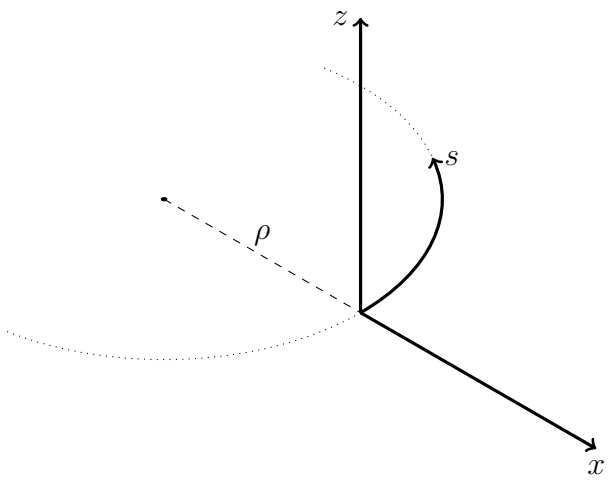


FIGURE 2.1 – The Frenet-Serret coordinate system which we will use throughout this work. The dotted line is the particle trajectory, whose curvature radius is ρ. The s coordinate is measured along the trajectory while x and z are orthogonal to it.

2.2 Co-ordinates and motions

In order to describe the motion of particles inside accelerators a coinvenient choice of co-ordinates needs to be performed, taking advantage of the toroidal symmetry of the system.

First of all, we should distinguish between two types of motion:

- *longitudinal*, along the accelerator circumference (*reference trajectory*)
- *transverse*, in the plane normal to the longitudinal motion

In an analogy with celestial mechanics, longitudinal motion is the planet motion in the orbital plane, while transverse motion corresponds to evolution of planetary positions in the meridional plane.

We can introduce the Frenet-Serret coordinate system. It should be natural to choose one curvilinear coordinate s along the longitudinal motion and two cartesian coordinates x, z for the transverse one (see. Fig. 2.1)

Let ρ be the accelerator radius. The cartesian system (X, Y, Z), centered in the accelerator centre can be mapped into the Frenet-Serret system via the following transformations:

$$X = (x + \rho)\cos(s/\rho) - \rho \qquad Y = z \qquad Z = (x + \rho)\sin(s/\rho)$$

Our goal now is to rewrite the Hamiltonian for a particle in an accelerator in this coordinate system.

Results will generally be derived in their complete form, where a function $\beta(s)$, which accounts for longitudinal coupling to transverse effects will be present. Nevertheless, in the limit of no longitudinal coupling, $\beta(s) \to 1$ and the only role assigned to longitudinal coordinate s will be to act as a timestamp for our system.

It actually turns out that it is convenient to regard s as the time coordinate (due to canonical coupling, the conjugated moment p_s will work as the Hamiltonian function).

The Hamiltonian we will apply to particlse in circular accelerators is given by the Lorentz force, as long as charged particles are accelerated in modulus by the action of an electric field $\mathbf{E}$ (or, better, by a scalar potential Φ), and bent in order to keep their circular orbit by a magnetic field $\mathbf{B}$ which will be more conveniently expressed by the vector potential $\mathbf{A}$ ($\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$).

A linear electromagnetic field keeps particles on a planar circular motion with radius ρ, where ρ is given by the equilibrium between magnetic and centrifugal force.

The quantity $B\rho$ is called *beam rigidity* and corresponds to

$$B\rho = \frac{p}{e}$$

where p is the particle momentum while e its charge.

Now, the Lorentz EM Hamiltonian for a relativistic particle reads

$$\mathcal{H} = e\Phi + \sqrt{m^2c^4 + (c\mathbf{p} - e\mathbf{A})^2}$$

We should express the square norm of $(c\mathbf{p} - e\mathbf{A})$ in the Frenet-Serret coordinates, whose metric tensor reads

$$g_{ij} = \mathrm{diag}(1 + x/\rho,\, 1\, 1)$$

Thus,

$$\mathcal{H} = e\Phi + \sqrt{m^2c^4 + \frac{(cp_s - eA_s)^2}{(1 + x/\rho)^2} + (cp_x - eA_x)^2 + (cp_z - eA_z)^2}$$

We will take advantage of the power of Hamiltonian mechanics to democratically treat every variable, and we will substitute the actual time variable with the longitudinal variable s. In this way, the conjugated variable $-p_s$ will be promoted to the rank of new Hamiltonian $\tilde{\mathcal{H}}$. Solving the latter equation for p_s, we obtain

$$\tilde{\mathcal{H}} = -\left(1 - \frac{x}{\rho}\right)\sqrt{\frac{E^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2} - eA_s$$

where $E = \mathcal{H} - e\Phi$. According to special relativity

$$E^2/c^2 = p^2 + m^2c^2$$

and our Hamiltonian reduces to

$$\tilde{\mathcal{H}} = -\left(1 - \frac{x}{\rho}\right)\sqrt{p^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2} - eA_s$$

where the equation of motion are

$$x_i' = \frac{\partial\tilde{\mathcal{H}}}{\partial p_i} \qquad p_i' = -\frac{\partial\tilde{\mathcal{H}}}{\partial x_i} \qquad (i = x,\, z)$$
$$t' = \frac{\partial\tilde{\mathcal{H}}}{\partial\mathcal{H}} \qquad \mathcal{H}' = -\frac{\partial\tilde{\mathcal{H}}}{\partial t}$$

and the prime refers to differentiation with respect to s.

Because the motion in the longitudinal direction is far faster than in the transverse ones, we have $p \gg p_x$ and $p \gg p_z$ and the Hamiltonian can be expanded, using the fact that $\sqrt{1+x} \sim 1 + x/2$, as

$$\tilde{\mathcal{H}} = \left(1 + \frac{x}{\rho}\right)\left[-p + \frac{1}{2p}\left(p_x^2 + p_z^2\right)\right] - eA_s \tag{2.1}$$

where, as far as we are taking into account only transverse effects, we can assume there is no magnetic field in the longitudinal direction, thus the only contribution to the vector potential is along s, and $A_x = A_z = 0$.

2.3 Betatronic motion

From the Hamiltonian (2.1), we can retrieve the equation of motion of x and z. For the x direction, we have

$$x' = \left(1 + \frac{x}{\rho}\right)\frac{p_x}{p} \qquad p'_x = \frac{p}{\rho}\left(1 + \frac{x}{\rho}\right) + e\frac{\partial A_s}{\partial x}$$

while for z

$$z' = \left(1 + \frac{x}{\rho}\right)\frac{p_x}{p} \qquad p'_z = e\frac{\partial A_s}{\partial z}$$

Now, we can express the partial derivatives of A_s as a function of the x and z components of the magnetic field $\mathbf{B}$.

Expressing the curl in Frenet-Serret coordinates, we have

$$\boldsymbol{\nabla} \times \mathbf{A} = -\frac{\hat{x}}{1 + x/\rho}\frac{\partial A_s}{\partial z} + \frac{\hat{z}}{1 + x/\rho}\frac{\partial A_s}{\partial x} = B_x\hat{x} + B_z\hat{z}$$

which yields

$$\frac{\partial A_s}{\partial x} = -\left(1 + \frac{x}{\rho}\right)B_z \qquad \frac{\partial A_s}{\partial z} = \left(1 + \frac{x}{\rho}\right)B_x$$

The equations of motion now read

$$x' = \left(1 + \frac{x}{\rho}\right)\frac{p_x}{p} \qquad p'_x = \left(1 + \frac{x}{\rho}\right)\left[\frac{p}{\rho} - eB_z\right]$$

and

$$z' = \left(1 + \frac{x}{\rho}\right)\frac{p_x}{p} \qquad p'_z = e\left(1 + \frac{x}{\rho}\right)B_x$$

which can also be written as second-order equation

$$x'' = \frac{1}{\rho} + \frac{x}{\rho^2} + \frac{eB_z}{p}\left(1 + \frac{x}{\rho}\right)^2$$

$$z'' = \frac{eB_x}{p}\left(1 + \frac{x}{\rho}\right)$$

Now, $p = eB\rho$, so we can rewrite the equations as

$$x'' = \frac{1}{\rho} + \frac{x}{\rho^2} + \frac{B_z}{B\rho}\left(1 + \frac{x}{\rho}\right)^2$$

$$z'' = \frac{B_x}{B\rho}\left(1 + \frac{x}{\rho}\right)^2$$

Let now the magnetic field be linear both in x and z. We set $B_z = B\rho K_1 x$ and $B_x = B\rho K_2 z$. If we define $K_x = \frac{1}{\rho^2} - K_1$ and $K_z = K_2$ our equation become

$$x'' + K_x(s)x = 0 \qquad z'' + K_z(s)z = 0$$

which are the well-known *Hill equations.*

Now, let y be either the x or the z coordinate. Hill's equation reads

$$y'' + K(s)y = 0 \tag{2.2}$$

We can perform the *Floquet trasformation*

$$y = aw(s)\exp\{i\psi(s)\}$$

Taking the second derivative, we get

$$ae^{i\psi(s)}\big(w''(s) + 2iw'\psi' + iw\psi'' - w\psi'^2\big) = ae^{i\psi s}K(s)w \tag{2.3}$$

and, from the imaginary part

$$2w'\psi' + w\psi'' = 0$$

which, by means of variable separation reduces to

$$\log\psi' = -2\log w$$

id est

$$\psi' = \frac{1}{w^2}$$

Substituting this result in the real part of equation (2.3 (and taking account of the reality of K) we finally get

$$w'' + Kw - \frac{1}{w^3} = 0 \tag{2.4}$$

Setting $u = w^2$, the equation

$$u''' + 4Ku' + 2K'u = 0$$

corresponds to the first derivative of Eq. (2.4) and is still solved by the functions

$$u_1 = we^{i\phi} \qquad u_2 = we^{-i\psi}$$

The Wronskian

$$u_1u_2' - u_1'u_2 = w(w' - iw\psi') - (w' + iw\psi')w = -2iw^2\psi' = -2i$$

turns out to be constant. Thus, the full solution y of Hill's equation (2.2) is a linear combination of u_1 and u_2. If we set w^2 to be the *betatron function* $\beta(s)$, we can write

$$\psi(s) = \int_0^s \frac{\mathrm{d}s'}{\beta(s')}$$

and

$$y = a\beta^{1/2}\cos(\psi + \xi) \tag{2.5}$$

where ξ is a phase. Letting $\eta = y\beta^{-1/2}$ $\phi = \psi/\nu$, Hill's equation becomes

$$\eta'' + \nu^2\eta = 0$$

which is a linear harmonic oscillator equation, where

$$\nu = \frac{1}{2\pi}\int_0^C \frac{\mathrm{d}s}{\beta(s)}$$

is the change in ψ on one accelerator turn (we set C to be the machine circumference) and is called *tune*.

We can also substitute $w = \beta^{1/2}$ in Eq. 2.2, and obtain

$$\frac{\beta''}{2} + K\beta - \frac{1}{\beta}(1 + \alpha^2) = 0$$

where

$$\alpha = -\beta'/2$$

or

$$\alpha' = K\beta - \frac{1}{\beta}(1 + \alpha^2)$$

Finally, if we define the Hamiltonian

$$\mathcal{H}(y, y', s) = \frac{1}{2}y'^2 + \frac{1}{2}K(s)y^2 \tag{2.6}$$

we can get back Hill's equation from the related equations of motion. We now want to find the action-angle variable change, thus we need a generating function F so that $y' = \partial F/\partial y$.

From the solution (2.5) , we get, upon differentiation

$$y' = \frac{a}{\beta^{1/2}}(\beta' \cos\psi - \sin\psi) = -\frac{y}{\beta}(\tan\psi + \alpha)$$

and, integrating,

$$F = \int \mathrm{d}y y' = -\frac{y^2}{2\beta}(\tan\psi + \alpha)$$

Now, the action J reads

$$J = -\frac{\partial F}{\partial \psi} = \frac{y^2}{2\beta\cos^2\psi}$$

while the new Hamiltonian, differentiating F w.r.t. s becomes simply

$$\tilde{\mathcal{H}} = \frac{J}{\beta}$$

From the expression for the action variable, we get

$$y = \sqrt{2\beta J}\cos\psi \qquad y' = -\sqrt{2J/\beta}(\sin\psi + \alpha\cos\psi)$$

which can be also cast in the form

$$y = \sqrt{2\beta_y J_y}\cos\psi_y \qquad p_y = -\sqrt{2\beta_y J_y}\sin\psi_y$$

A final trasformation can be made in order to use the orbital angle $\theta = s/R$ and not the longitudinal coordinate s as the independent variable. Using

$$\tilde{\psi} = \psi - \int_0^s \frac{\mathrm{d}s}{\beta} + \omega\theta$$

and keeping the same J, the new Hamiltonian, being the generating function $F = \tilde{\psi}J$ becomes

$$\mathcal{H} = \omega J$$

and the phase-space coordinates finally read

$$y = \sqrt{2\beta J}\cos\left(\tilde{\psi} + \int_0^s \frac{\mathrm{d}s'}{\beta} - \omega\theta\right)$$
$$p_y = -\sqrt{2\beta J}\sin\left(\tilde{\psi} + \int_0^s \frac{\mathrm{d}s}{\beta} - \omega\theta\right)$$

2.4 Emittance

Let y be one of the transverse coordinates. Betatron motion is described by Hill equation

$$y'' + K(s)y = 0 \tag{2.7}$$

It is possible to write a general solution of equation (2.7) as

$$y(s) = a\sqrt{\beta_y(s)}\cos\left(\psi_y(s) + \xi_s\right)$$

where a and ξ are fixed by the initial conditions, while

$$\psi_y(s) = \int_0^s \frac{\mathrm{d}s'}{\beta_y(s')}$$

Now we perform the transformation into action-angle variables

$$y = \sqrt{2\beta_y J_y}\cos\psi_y \qquad p_y = -\sqrt{2\beta_y J_y}\sin\psi_y$$

It can also be shown that the following relation holds true:

$$p_y = \beta_y\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{1}{2}\frac{\mathrm{d}\beta_y}{\mathrm{d}s}y$$

Substituting the solution of $y(s)$ in the latter expression, we have

$$p_y = \beta_y\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{1}{2}\frac{\mathrm{d}\beta_y}{\mathrm{d}s}y = -\frac{a}{\sqrt{\beta(s)}}\sin(\psi_s)$$

and we can define the Courant-Snyder ellipse

$$C(y, y') = \frac{y^2 + p_y^2}{\beta_y(s)} = \gamma y^2 + 2\alpha y y' + \beta y'^2$$

Substituting the solutions for y and p_y we obtain

$$C(y, y') = \frac{1}{\beta}\left(a^2\beta\cos^2\psi + a^2\beta\sin^2\psi\right) = a^2 =: \varepsilon$$

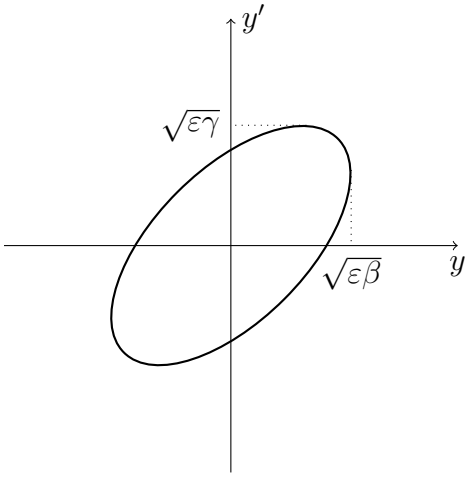


FIGURE 2.2 – The Courant-Snyder ellipse $\gamma y'^2 + 2\alpha y y' + \beta y'^2$. The area encoled by the ellipse is equal to $\pi\varepsilon$.

The area enclosed by such an ellipse, that we show in Fig. 2.2, will be equal to $\pi\varepsilon$.

Being also $a = \sqrt{\varepsilon}$ we can write

$$y(s) = \sqrt{\beta_y \varepsilon_y}\cos\psi_y$$

It is clear that $\varepsilon_y = 2J_y$, i.e. twice the value of the action variable of our particle.

This ε is called *Courant-Snyder invariant* and is a property relative to a single particle. It should not be confused with *emittance*, which is relative to a beam, i.e. a distribution of particles in phase space. However, these two quantities are deeply related, as we show.

Let $\rho(y, y')$ be such a (normalized) distribution. Average and standard deviations read

$$\langle y\rangle = \int \mathrm{d}y\mathrm{d}y'\, y\rho(y,y') \qquad \langle y'\rangle = \int \mathrm{d}y\mathrm{d}y'\, y'\rho(y,y')$$

$$\sigma_y^2 = \int \mathrm{d}y\mathrm{d}y'\, (y-\langle y\rangle)^2\rho(y,y') \qquad \sigma_{y'}^2 = \int \mathrm{d}y\mathrm{d}y'\, (y'-\langle y'\rangle)^2\rho(y,y')$$

$$\sigma_{yy'} = \int \mathrm{d}y\mathrm{d}y'\, (y-\langle y\rangle)(y'-\langle y'\rangle)\rho(y,y')$$

Thus, we can define the *rms emittance* as

$$\varepsilon_{\text{rms}} = \sqrt{\sigma_y^2 \sigma_{y'}^2 - \sigma_{yy'}^2} \tag{2.8}$$

We should relate the single-particle, betatron emittance with this statistically inferred value.

It is straightforward to show that, if the average of y and y' is zero, substituting in $C(y, y')$ the rms values for y and y' we obtain

$$C(y, y') = \int \mathrm{d}y\mathrm{d}y' \rho(y, y') \left(\gamma y^2 + 2\alpha y y' + \beta y'^2\right) = \varepsilon$$

thanks to the normalization of ρ.

Thus, the beam emittance is the Courant-Snyder invariant of the root mean square particle in the beam.

It must also be noted that in some literature a factor 4 is introduced in this definition, so that $\varepsilon := 4\varepsilon_{\text{rms}}$. Anyway, throughout this work, we will stick to the definition $\varepsilon = \varepsilon_{\text{rms}}$.

If we differentiate the expression of ε^2 from Eq. (2.8) w.r.t. s we get

$$\frac{\mathrm{d}\varepsilon^2}{\mathrm{d}s} = \sigma_{y'}^2 \frac{\mathrm{d}\sigma_y^2}{\mathrm{d}s} + \sigma_y^2 \frac{\mathrm{d}\sigma_{y'}^2}{\mathrm{d}s} - 2\sigma_{yy'} \frac{\mathrm{d}\sigma_{yy'}}{\mathrm{d}s}$$

Now, remembering that $\mathrm{d}y/\mathrm{d}s = y'$,

$$\begin{aligned} \frac{\mathrm{d}\sigma_y^2}{\mathrm{d}s} &= \int \mathrm{d}y\mathrm{d}y' \rho(y, y') \frac{\mathrm{d}}{\mathrm{d}s} (y - \langle y \rangle)^2 \\ &= 2 \int \mathrm{d}y\mathrm{d}y' \rho(y, y') (y y' - y' \langle y \rangle - y \langle y' \rangle + \langle y \rangle \langle y' \rangle) \end{aligned}$$

We can also use the definition of $\langle y \rangle$ and $\langle y' \rangle$ and, after having taken averaged values out of the integration, by normalization of ρ we obtain

$$\frac{\mathrm{d}\sigma_y^2}{\mathrm{d}s} = 2(\langle y y' \rangle - \langle y \rangle \langle y' \rangle) = 2\sigma_{yy'}$$

where we have used the definition of average for yy'

$$\langle y y' \rangle = \int \mathrm{d}y\mathrm{d}y' \rho(y, y') y y'$$

For the second term, we write, analogously,

$$\begin{aligned}\frac{\mathrm{d}\sigma_{y'}^2}{\mathrm{d}s} &= \int \mathrm{d}y\mathrm{d}y'\rho(y,y')\frac{\mathrm{d}}{\mathrm{d}s}(y'-\langle y'\rangle)^2 \\ &= 2\int \mathrm{d}y\mathrm{d}y'\rho(y,y')(y'y''-y''\langle y'\rangle - y'\langle y''\rangle + \langle y'\rangle\langle y''\rangle) \\ &= 2(\langle y'y''\rangle - \langle y'\rangle\langle y''\rangle) =: 2\sigma_{y'y''}\end{aligned}$$

Finally, the covariance needs differentiation

$$\frac{\mathrm{d}\sigma_{yy'}}{\mathrm{d}s} = \int \mathrm{d}y\mathrm{d}y'\rho(y,y')\frac{\mathrm{d}}{\mathrm{d}s}(yy' - y\langle y'\rangle - \langle y\rangle y' + \langle y\rangle\langle y'\rangle)$$

which, performing the same operations reduces to

$$\frac{\mathrm{d}\sigma_{yy'}}{\mathrm{d}s} = \langle y'^2\rangle - \langle y'\rangle^2 - \langle yy''\rangle - \langle y\rangle\langle y'\rangle = \sigma_{y'}^2 + \sigma_{yy''}$$

being $\sigma_{yy''}$ the correlation between y and y''. Merging these results, we finally get

$$\frac{\mathrm{d}\varepsilon^2}{\mathrm{d}s} = 2\sigma_{y'}^2\sigma_{yy'} + 2\sigma_y^2\sigma_{y'y''} - 2\sigma_{yy'}(\sigma_{y'}^2 + \sigma_{yy''}) = 2\sigma_y^2\sigma_{y'y''} - 2\sigma_{yy'}\sigma_{yy''}$$

Now, if the Hamiltonian is linear, i.e. only dipole and quadrupole magnets act on the beam, $y'' = -Ky$ and, following the definition,

$$\sigma_{yy''} = -K\sigma_y^2 \qquad \sigma_{y'y''} = -K\sigma_{yy'}$$

so that

$$\frac{\mathrm{d}\varepsilon^2}{\mathrm{d}s} = 2K\left(\sigma_y^2\sigma_{yy'} - \sigma_y^2\sigma_{yy'}\right) = 0$$

and conservation of emittance follows.

When we add nonlinear terms to the Hamiltonian (e.g., a sestupole term $k_3y^3/3$, which, means, upon differentiation, $y'' = Kx - k_3x^2$) we get

$$\frac{\mathrm{d}\varepsilon^2}{\mathrm{d}s} = 2k_3\left[\sigma_y^2\left(\langle y'y^2\rangle - \langle y'\rangle\langle y^2\rangle\right) - \sigma_{yy'}\left(\langle y^3\rangle - \langle y\rangle\langle y^2\rangle\right)\right]$$

i.e., the emittance evolution is equal to the strenght of the nonlinearity multiplied by a number which is a property only of the distribution.

More generally, we would have, for a nonlinear $k_n x^n/n$ hamiltonian term

$$\frac{\mathrm{d}\varepsilon_n^2}{\mathrm{d}s} = 2k_n\left[\sigma_y^2\left(\left\langle y' y^{n-1}\right\rangle - \left\langle y'\right\rangle \left\langle y^{n-1}\right\rangle\right) - \sigma_{yy'}\left(\left\langle y^n\right\rangle - \left\langle y\right\rangle \left\langle y^{n-1}\right\rangle\right)\right]$$

2.5 Advanced techniques

In the previous section we saw that, as long as only linear motion is taken into account, emittance is an invariant. Thus, if we want to manipulate beams' transverse shape we need to break down the invariance. This can be done thanks to the action of nonlinearities and taking advantage of the time modulation of the strength of magnets which act on the accelerator, which can be used to excite some resonances.

Here lies the connection between beam dynamics and adiabatic invariance theory. We described in Chapter 1 how adiabatic invariance can be broken when crossing a separatrix: applying that theory to particle accelerators we could obtain effects which would be impossible in the realm of linear motion.

Part II

Emittance exchange

Chapter 3

Theoretical results

Manipulating transverse distribution is one of the tasks that are the responsibility of modern accelerator physicists. Certain distribution might need to be attained, diverse applications may require beams to be widened or squeezed, sometimes an useful geometry alteration should be achieved only either in the horizontal or in the vertical plane.

An example of this is given by the current MTE configuration at PS. The beamlets which result from the splitting process have an high vertical width, and do not perfectly match with SPS vertical mechanical aperture. Thus, a system which could redistribute emittances between the two transverse planes would be then warmly welcomed.

A work by Lee *et al.* [18] suggests that crossing a specific 2D resonance a system undergoes an emittance exchange in the transverse plane. As the goal of the authors of [18] is to *avoid* this exchange during the resonance crossing, they only take care, to identify a configuration which minimizes the emittance exchange. Conversely, our aim is exactly opposite: we want to fully exploit the mechanism, and, as we present a novel argument to explain how the exchange works, we will be able to predict the final emittances of a system that undergoes the crossing process.

3.1 The model

For the study of the emittance exchange at the crossing of a Walkinshaw resonance, we start from ([18]), whose idea, as suggested by [3, p. 409] is to reduce the Hamiltonian of the particle in

the accelerator to a two-degrees of freedom resonant normal form, then to reduce the system to only one degree of freedom, using one of the action variables, which turns out to be invariant throughout the whole motion, thus also during the separatrix crossing, as a parameter.

Let J_x and J_z be, respectively, the transverse horizontal and vertical actions of a particle in a beam.

A Gaußian two-dimensional distribution in x, p_x, z, p_z reads, if transformed into J_x, J_z, ϕ_x and ϕ_z coordinates

$$\rho(J_x, J_y) = \frac{1}{\varepsilon_x \varepsilon_z} \exp\left(-\frac{J_x}{\varepsilon_x} - \frac{J_z}{\varepsilon_z}\right) \tag{3.1}$$

i.e., it becomes a negative-exponential distribution in terms of the actions.

The mean values of J_x and J_z, which would correspond to the variance of the x and z coordinate in the Gaußian distribution, are the emittances

$$\langle J_x \rangle = \varepsilon_x \qquad \langle J_z \rangle = \varepsilon_z$$

Now, we can write an Hamiltonian in the x and z variables for a particle in an accelerator which takes account for the linear motion plus the sextupole and octupole magnets contributions. We will have Hill's hamiltonian (2.6) for the x and z coordinates, and two potentials V_3 and V_4 which account for the nonlinear terms.

$$\mathcal{H} = \frac{1}{2}(x'^2 + K_x x^2 + z'^2 + K_z z^2) + V_3(x, z, s) + V_4(x, z, s)$$

From the multipolar expansion of the magnetic field we also get

$$V_3 = -\frac{1}{6} k_3(s)(x^3 - 3xz^2)$$

where we have set

$$k_3 = \left.\frac{\partial^2 B_z}{\partial x^2}\right|_{x=z=0}$$

We can now trasform the Hamiltonian into action-angle variables $(J_x, J_z, \psi_x, \psi_z)$. For the harmonic part,

$$\mathcal{H}(J_x, J_z, \psi_x, \psi_z) = \omega_x J_z + \omega_z J_z$$

where $\omega_{x,z}$ are the machine *frequncies*, which differ from the *tunes* as they include a 2π factor.

Putting the tranformation $x = \sqrt{2\beta_x J_x}\cos\psi_x$, and the analogous one for z in the expressions for V_3 we get

$$\begin{aligned} V_3 = &-\frac{2^{3/2}}{6}k_3 J_x^{3/2}\beta_x^{3/2}\cos^3\psi_x \\ &+\sqrt{2}k_3 J_x^{1/2}J_z\beta_x^{1/2}\beta_z\cos\psi_x\cos^2\psi_z \end{aligned}$$

From the exponential representation of the cosine function, it is straightforward to show that

$$\cos^3\psi_x = \frac{1}{4}\cos 3\psi_x + \frac{3}{8}\cos\psi_x$$

and

$$\cos^2\psi_z\cos\psi_x = \frac{1}{4}[\cos(2\psi_z - \psi_x) + \cos(2\psi_z + \psi_x) + 2\cos\psi_x]$$

and perform the averaging in the vicinity of the $(1, 2)$ resonance, where the (m, n) symbol identifies the resonance given by $m\omega_x - n\omega_z = \ell$. This means, according to Birkhoff normal forms theory, that the only surviving angular-dependent term is the $J_x^{1/2}J_z$ one.

We therefore obtain

$$\begin{aligned} \mathcal{H}(J_x,\, J_z,\, \phi_x,\, \phi_z) = \omega_x J_x + \omega_z J_z + \frac{1}{2}\alpha_{xx}J_x^2 + \alpha_{xz}J_xJ_z + \frac{1}{2}\alpha_{zz}J_z^2 \\ + GJ_x^{1/2}J_z\cos(\phi_x - 2\phi_z - \ell\theta + \xi) \end{aligned} \tag{3.2}$$

where we have also introduced the *detuning parameters* which account for the first-order correction to the x and z tunes. They read

$$\begin{aligned} \overline{\omega}_x &= \omega_x + \alpha_{xx}J_x + \alpha_{xz}J_z \\ \overline{\omega}_z &= \omega_z + \alpha_{xz}J_x + \alpha_{zz}J_z \end{aligned}$$

As long as the motion undergoes separatrix crossing, J_x and J_z, although being adiabatic invariants, are not going to have their value conserved during the evolution of the system.

However, this Hamiltonian admits a integral, which is a linear combination of J_x and J_z. In fact, performing the canonical transformation

$$\begin{aligned} J_1 &= J_x \qquad J_2 = 2J_x + J_z \\ \phi_1 &= \phi_x - 2\phi_z - \ell\theta + \xi \qquad \phi_2 = \phi_z \end{aligned}$$

we get the new Hamiltonian

$$\begin{aligned} \mathcal{H} = \delta J_1 + \frac{1}{2}\alpha_{11}J_1^2 + \alpha_{12}J_1J_2 + GJ_1^{1/2}(J_2 - 2J_1)\cos\phi_1 \\ + \omega_z J_2 + \alpha_{22}J_2^2 \end{aligned} \tag{3.3}$$

where δ is the resonance distance parameter given by

$$\delta = \omega_x - 2\omega_z - \ell$$

and the detuning parameters have conveniently been redefined as

$$\alpha_{11} = \alpha_{xx} - 4\alpha_{zz} \qquad \alpha_{12} = \alpha_{xz} - 2\alpha_{zz} \qquad \alpha_{22} = \alpha_{zz}$$

Now, the Hamiltonian (3.3) is not dependent on the angle ϕ_2, and according to Hamilton equations,

$$\dot{J}_2 = \frac{\partial \mathcal{H}}{\partial \phi_2} = 0$$

which means that J_2 remains constant during the resonance passage (i.e., the invariant does not undergo destruction at separatrix crossing).

Moreover, the transformed Hamiltonian can be cast in the form $\mathcal{H}_1 + \mathcal{H}_2$ where $\mathcal{H}_2$ recollects the terms which are only dependent on J_2. As far as we consider J_2 to be constant, $\mathcal{H}_2$ is constant too, and can safely be neglected. Since J_2 is constant, from the definition

$$J_2 = 2J_x + J_z$$

we immediately get that, over the whole evolution

$$2\Delta J_x = -\Delta J_z$$

and, passing to averages and thus to emittances

$$2\Delta\varepsilon_x = -\Delta\varepsilon_z$$

We succeeded to couple the x and z directions and reduce the 2D problem to a unidimensional one.

Now, equations of motions for J_1 and ϕ_1 read

$$\begin{aligned}\dot{J}_1 &= GJ_1^{1/2}(J_2 - 2J_1)\sin\phi_1 \\ \dot{\phi}_1 &= \delta + \alpha_{12}J_2 + \alpha_{11}J_1 + G\frac{J_2 - 6J_1}{2J_1^{1/2}}\cos\phi_1\end{aligned}$$

3.2 Distributions

Lee *et al.* [18] show the plausibility of a certain emittance exchange from an analysis of the evolution of beam distribution given the transformation $(J_x, J_z) \to (J_1, J_2)$.

We now report their argument, which we will compare with our more general result, which will predict the exact value of the final emittances.

Tranforming the Gaußian distribution (3.1) we get

$$\rho(J_1, J_2) = \frac{1}{\varepsilon_x\varepsilon_z}\exp\left(-\frac{J_1}{\varepsilon_x} - \frac{J_2 - 2J_1}{\varepsilon_z}\right)$$

Using the variable $u = J_1/J_2$ ($0 \le u \le 1/2$, if we want J_z to remain positive) the distribution becomes

$$\rho(u, J_2) = \frac{J_2}{\varepsilon_x\varepsilon_z}\exp\left\{\left(-\frac{u}{\varepsilon_x} - \frac{1-2u}{\varepsilon_z}\right)J_2\right\}$$

and integrating over J_2 we finally get

$$\rho(u) = \frac{\varepsilon_x/\varepsilon_z}{[\varepsilon_x/\varepsilon_z + (1 - 2\varepsilon_x/\varepsilon_z)u]^2}$$

When the initial emittances $\varepsilon_{x,0}$, $\varepsilon_{z,0}$ are in the ratio

$$\frac{\varepsilon_{x,0}}{\varepsilon_{z,0}} = \frac{1}{2}$$

the distribution $\rho(u)$ is uniform and we do not expect any emittance exchange.

Moreover, we can expect that if the beam reaches in its evolution the state where $\varepsilon_x/\varepsilon_z = 1/2$, no more emittance will be exchanged and this ratio will be kept till the end. This advocates the final emittance values

$$\varepsilon_{z,f} = 2\varepsilon_{x,0} \tag{3.4}$$

and, according to the constancy of J_2

$$\varepsilon_{x,f} = \frac{\varepsilon_{z,0}}{2} \tag{3.5}$$

are plausible. However, we will prove that this actually occurs according to separatrix crossing theory.

3.3 Phase space

From the Hamiltonian $\mathcal{H}_1$ we can describe the evolution of the phase space topology following the variation of δ, i.e. the passing through the Walkinshaw resonance. First of all, the condition $J_z > 0$, which translates to $J_1 < J_2/2$ limits the motion inside the circle with radius $\sqrt{J_2/2}$, which we will call *outer circle*.

Now, it is convenient to perform the transformation to the Cartesian $(x,\, y)$ coordinates defined as

$$x = \sqrt{2J_1}\cos\phi \qquad y = \sqrt{2J_1}\sin\phi$$

whence

$$\mathcal{H}(x,y) = \frac{\delta+\alpha_{12}J_2}{2}(x^2+y^2)+\frac{\alpha_{11}}{2}\left(\frac{x^2+y^2}{2}\right)^2+\frac{G}{\sqrt{2}}x\big(J_2-x^2-y^2\big) \tag{3.6}$$

It is then possibile to identify ellyptic and hyperbolic points by inspecting the first derivative of the Hamiltonian. We have

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial x} &= \big[\delta+\alpha_{12}+\alpha_{11}(x^2+y^2)\big]x+\sqrt{2}G(J_2-3x^2-y^2)\\ \frac{\partial\mathcal{H}}{\partial y} &= \Big[\delta+\alpha_{12}+\frac{\alpha_{11}}{2}(x^2+y^2)\Big]y-\sqrt{2}Gxy\end{aligned} \tag{3.7}$$

We can write explicitly our separatrices

$$x^2 + y^2 = J_2 \qquad \frac{\alpha_{11}}{4}(x^2 + y^2) - \sqrt{2}Gx + \delta + \alpha_{12}J_2 + \frac{1}{4}\alpha_{11}J_2 = 0$$

which are the *outer circle* (fixed) and a *coupling arc*, i.e. an arc of a circumference with centre

$$\mathsf{C} = \left(\frac{2\sqrt{2}G}{\alpha_{11}}, 0\right)$$

and radius

$$r = \frac{1}{|\alpha_{11}|}\sqrt{8G^2 - \alpha_{11}J_2(\alpha_{11} + 4\alpha_{12}) - 4\alpha_{11}\delta}$$

which keeps expanding while δ decreases if $\alpha_1 1$ is positive (if it were negative, we would have to cross the resonance in the opposite way, *increasing* δ from a negative value to a positive one). We can find some equilibria inside the circle, for a range of δ sufficiently close to 0, i.e.

- for $y = 0$ two elliptic points[1]
- two hyperbolic points on the outer circle, at the intersection with the coupling arc

Conversely, when $|\delta|$ is bigger than a critic value, there is only a close-to-origin elliptic point, and the motion, while constrained in the outer circle, remains quasi-periodic.

The $\alpha_{12}J_2$ term only contributes in translating the point of resonance passage, while the α_{11} term, which controls the quadratic dependence in J_1 causes the coupling arc to increase its curvature. In the notable case $\alpha_{11} = 0$ the circumference centre and the radius go to infinity, and the coupling arc becomes just the straight line

$$x = x_0 = \frac{\delta + \alpha_{12}J_2}{\sqrt{2}G}$$

In this case, area calculations are notably simpler and the equilibria are

[1]Substituting $y = 0$ in Eq. (3.7) we would find a cubic equation in x which has 3 real solutions. Nevertheless, one of the roots is always extern to the outer circle, and does not have any effects on the motion.

- Elliptic: at $y = 0$ and

$$x = \sqrt{2}\frac{\delta + a_{12}J_2 \pm \sqrt{a_{12}^2 J_2^2 + 2J_2(\delta a_{12} + 3G^2) + \delta^2}}{6G}$$

- Hyperbolic: at $x = x_0$ and $y = \pm\sqrt{J_2 - x_0^2}$

Simulations performed by Lee *et al.* actually show that emittance exchange is actually independent from the value of α_{11}. Anyway, we will give an argument accounting for the extension of our results to systems with $\alpha_{11} \neq 0$.

3.4 Separatrix crossing: classic theory

According to Nejštadt theory of separatrix crossing, crossing probability between two phase space regions is a function of the relative area variation velocity of those regions.

In our case, we have only two regions, all enclosed in a fixed circle. Thus, the area gain of a region is equal to the area loss of the other one, so the crossing probability is equal to 1.

Now, according to the trapping mechanism, the system preserves area until it fills the available region, then jumps with probability 1 into the other region, as it is physically impossible to go outside of the circle. We have

$$A_\mathrm{f} = A_\mathrm{o.c.} - A_\mathrm{i}$$

where A_f is the final area, A_i the initial one and $A_\mathrm{o.c.}$ the outer circle area, because, thanks to area adiabatic invariance, final area is the area of the region the particle is trapped into when δ is greater than the critical value δ_*.

Following these prescriptions, we immediately have

$$A_\mathrm{i} = 2\pi J_{1,0}$$

and

$$A_\mathrm{o.c.} = \pi J_2$$

since its radius is $\sqrt{J_2}$. Thus,

$$A_\mathrm{f} = \pi(J_2 - 2J_{1,0}) \tag{3.8}$$

This means that final J_1 ($J_{1,f}$) reads

$$J_{1,f} = \frac{A_\mathrm{f}}{2\pi} = \frac{J_2}{2} - J_{1,0}$$

We can go back to the J_x, J_z coordinates and we find

$$J_{x,f} = J_{1,f} = \frac{1}{2}(2J_{x,0} + J_{z,0}) - J_{x,0} = \frac{J_{z,0}}{2}$$

and, from the definition of J_2, thanks to its invariance,

$$J_{z,f} = J_2 - 2J_{x,f} = J_2 - J_{z,0} = 2J_{x,0}$$

For a Gaußian distribution of the coordinates, i.e. an exponential one for the actions, emittances ε_x and ε_z are just mean values of J_x and J_z, so their change follows J_x and J_z, and we have:

$$\varepsilon_{x,f} = \frac{\varepsilon_{z,i}}{2} \tag{3.9}$$

$$\varepsilon_{z,f} = 2\varepsilon_{x,i} \tag{3.10}$$

This confirms the simulated data shown by [18], and supports the possibility of performing an emittance exchange between the x and z directions in the transverse plane, as the final x emittance depends on the z initial one, and vice-versa.

Another useful distribution whose evolution is interesting to inspect is the Kapčinskij-Vladimirskij (KV) one [25], which is defined, in the action-angle variables as

$$\rho_{KV}(J_x, J_z) = \frac{1}{4\varepsilon_x\varepsilon_z}\,\delta\!\left(\frac{J_x}{2\varepsilon_x} + \frac{J_z}{2\varepsilon_z} - 1\right)$$

where $\delta(x)$ is the Dirac delta distribution. This means that the distribution is uniform on the line where

$$\frac{J_x}{2\varepsilon_x} + \frac{J_z}{2\varepsilon_z} = 1$$

Now, performing the transformations (3.9, 3.10), since the emittances transform coherently, we immediately get to know that the KV distribution is kept after the emittance exchange, of course with new emittances.

In general, this result is guaranteed for any distribution in which actions and emittances appear only in the ratios J_x/ε_x and J_z/ε_z.

It should be noted that also the normalization, if dependent from the product $\varepsilon_x\varepsilon_z$, remains invariant.

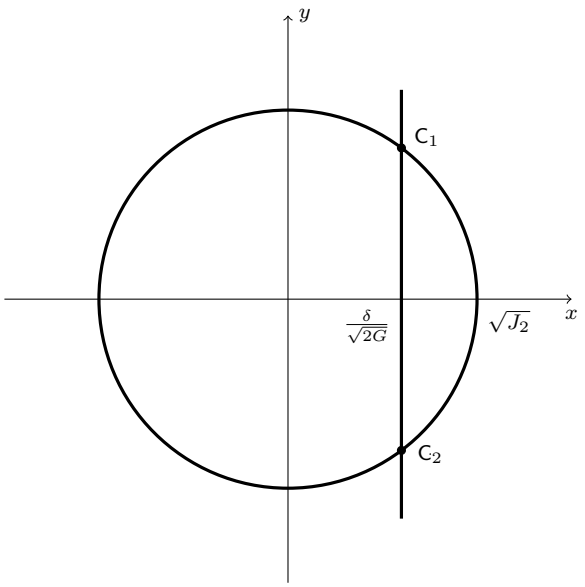


FIGURE 3.1 – Phase space of the Hamiltonian system (3.6) with $\alpha_{11} = \alpha_{12} = 0$.

3.5 Separatrix crossing: improved theory

Let us now apply the improved adiabatic approach to our Hamiltonian.

Differently from the calculations shown in §1.6.2, we are interested in a separatrix crossing only between two phase space regions. In particular, we are analyzing a phase space whose topology is shown in Fig. 3.1, where an outer fixed circle constraints the motion, and a single separatrix curve moves throughout the circle.

We have two saddle point, C_1 and C_2 which share the same motion in their vicinity, as the Hamiltonian, and thus the phase space, are symmetric along the horizontal axis.

Therefore, using the same notation as before, we can label with $-N, \ldots, -1, 0, 1, \ldots, n, n+1, \ldots$ the closest passages near one of the two symmetrical hyperbolic points; the 0 label corresponding to the separatrix crossing.

We have, like before

$$h_{n+1} = h - \epsilon\Theta \tag{3.11}$$

and

$$\tau_{n+1} = \tau_n - \epsilon a \left[\frac{1}{2} \log h_n + \log\left(h_n - \epsilon \frac{\Theta}{2} \right) + \frac{1}{2} \log h_{n+1} \right] + \epsilon b \quad (3.12)$$

because there are three hyperbolic (and thus, dominant) contributions to the period of the motion: the initial one close to the starting hyperbolic point, which counts for a half and with a h_n energy; a full passage near the opposite saddle point, where energy is increasing between h_n and h_{n+1}: for sufficiently large N, it can be safely estimated as the middle point between the two energy steps; and, finally, the return to the starting saddle with an half-passage at the next energy step $h_{n+1} = h_n - \epsilon\Theta$

Everything follows just as before. The energy at the $n-$th step will be given by $h_n - n\epsilon\Theta$ and the slow time τ by

$$\tau_n = \tau_0 - n\epsilon b + \epsilon a \left[n \log \epsilon\Theta + \log \frac{\Gamma(\xi + n)}{\Gamma(\xi)} + n \log \frac{\xi\Theta}{2} + \right.$$
$$\left. + \log \frac{\Gamma(\xi/2 + n)}{\xi/2} + \frac{1}{2} \log(\xi + n) - \frac{1}{2} \log \xi \right]$$

and, following [14], we expect that the $\tilde{J}$ variation before crossing reads

$$2\pi(\tilde{J}_{-N} - \tilde{J}_*) = a\epsilon\Theta \left[\log \frac{2\pi}{\Gamma(\xi)\Gamma(\xi/2)} + \xi \log \xi - \frac{3}{2}\xi + \right.$$
$$\left. + \frac{\xi}{2} \log \frac{\xi}{2} - \frac{1}{2} \log \xi \right]$$

After crossing, the particle lands in a region whose area evolution, being the motion confined in a fixed circle, is exactly opposite of the starting region. Then, we can simply reuse Equation (3.13), but having care to replace $\Theta' = -\Theta$ and $\xi' = 1 - \xi$.

We get

$$2\pi(\tilde{J}_0 - \tilde{J}_N) = a\epsilon\Theta\Bigg[\log\frac{2\pi}{\Gamma(1-\xi)\Gamma(1-\xi/2)} - (1-\xi)\log(1-\xi) + \frac{3}{2}(1-\xi) - \left(1-\frac{\xi}{2}\right)\log\left(1-\frac{\xi}{2}\right) - \frac{1}{2}\log(1-\xi)\Bigg] \tag{3.13}$$

Then, we need a formula for the change of $\tilde{J}$ at the exact resonance crossing. However, following [14] we expect this contribution to cancel all terms in the previous expressions except the one with the Gamma functions, so that the final result reads

$$2\pi\Delta\tilde{J} = a\epsilon\Theta\log\frac{4\pi^2}{\Gamma(\xi)\Gamma(1-\xi)\,\Gamma(\xi/2)\Gamma(1-\xi/2)} \tag{3.14}$$

and thanks to the fact that

$$\Gamma(x)\Gamma(1-x) = -\frac{\pi}{\sin\pi x} \tag{3.15}$$

we finally get

$$2\pi\Delta\tilde{J} = a\epsilon\Theta\log\left(4\sin(\pi\xi)\sin\frac{\pi\xi}{2}\right) \tag{3.16}$$

Because $\xi \sim 1/\epsilon$ changes dramatically if the initial conditions differ only by $\mathcal{O}(\epsilon)$, it can be regarded as a random variable uniformly distributed in the $(0,1)$ interval.

Now, the function

$$\log\left(4\sin(\pi\xi)\sin\frac{\pi\xi}{2}\right) \tag{3.17}$$

is at least $\mathcal{L}^2([0,1])$, so for the jump ΔJ we can easily get the average

$$\left\langle 2\pi\Delta\tilde{J}\right\rangle = 0 \tag{3.18}$$

and the dispersion

$$\sigma^2 = 2.47\,a^2\epsilon^2\Theta^2 \tag{3.19}$$

where the constant has been obtained evaluating numerically the integral.

For actual models the quantity $a\Theta$ can be retrieved just knowing the dynamics close to the saddle points and the area variation.

We can now apply these results to our Hamiltonian (3.3), in the absence of detuning case ($\alpha_{11} = \alpha_{12} = 0$). Detuning can actually be introduced without any bigger conceptual effort, yet algebraic calculations become lengthy and cumbersome.

Now, our Hamiltonian reads

$$\mathcal{H} = \delta J_1 + G J_1^{1/2}(J_2 - 2J_1)\cos\phi_1 \tag{3.20}$$

and admits two hyperbolic points at

$$J_1 = \frac{J_2}{2} \qquad \cos\phi_1 = \frac{\delta}{G\sqrt{2J_2}} \tag{3.21}$$

The determinant of the Hessian matrix of the Hamiltonian at the saddle points is

$$\det H = \delta^2 - 2J_2G^2 \tag{3.22}$$

and a, as a function of crossing δ_* is given by

$$a = \frac{1}{\sqrt{\delta_*^2 - 2J_2G^2}} \tag{3.23}$$

Being the separatrix a straight line, it is also easy to calculate the two regions' area and thus Θ. We have

$$S(\delta) = \frac{1}{2}\left(\frac{J_2}{2}\right)^2 (2\phi_C - \sin 2\phi_C) \tag{3.24}$$

where ϕ_C is the angle coordinate of the saddle point at δ_*. Performing all the substitutions and differentiating we get

$$\Theta(\delta_*) = -\frac{J_2\sqrt{\delta_*^2 - 2J_2G^2}}{4G^2} \tag{3.25}$$

and we find that

$$a\Theta = -\frac{J_2}{4G^2} \tag{3.26}$$

does not depend on δ_*.

3.6 Generalization to higher resonances

Let us now take into account a class of Hamiltonians similar to (3.2) that are the result of averaging near a generic resonance, i.e. when the x and z tunes are in the relation

$$m\omega_x - n\omega_z = \ell$$

Starting from the initial Hamiltonian (3.2), the same approach of averaging which was presented in the previous section would lead us to the Hamiltonian

$$\begin{aligned}\mathcal{H}_{m,n}(J_x, \phi_x, J_z, \phi_z) = \omega_x J_x + \omega_z J_z + \alpha_{xx} J_x^2 + \alpha_{zz} J_z^2 \\ + G J_x^{\frac{|m|}{2}} J_z^{\frac{|n|}{2}} \cos(m\phi_x - n\phi_z - \ell\theta + \xi)\end{aligned}$$

Now, given an integer order $r = |m| + |n|$, a resonance of that order can be excited at least by a $2r-$pole magnetic element.

In fact, the potential V_r term for a $2r-$pole magnet is

$$V_r \sim \Re(x + iz)^r = \sum_{|a|+|b|=r} J_x^{|a|/2} J_z^{|b|/2} \cos(a\psi_x + b\psi_z)$$

and, for a (m, n) resonance we only keep

$$V_r \sim J_x^{|m|/2} J_z^{|n|/2} \cos(m\psi_x - n\psi_z)$$

which is present in the latter Hamiltonian.

It should be emphasized that this approach actually corresponds to finding the resonant Birkhoff normal form for the given Hamiltonian.

Now, applying the same procedure, we would like to find a canonical transformation into two new actions, J_1 and J_2, where one, being (adiabatically) invariant also during the separatrix crossing, can be treated as a parameter, while the other one undergoes time evolution.

Such a transformation can be generated by the function [3, p. 410]

$$S = (m\phi_x + n\phi_z)(l_2 J_1) J_1 + (l_1\phi_x + l_2\phi_z) J_2$$

where

$$J_1 = l_1 J_1 - l_2 J_2$$

and

$$J_2 = -nJ_1 + mJ_2$$

The values l_1 and l_2 needs to be chosen by keeping the unitarity of the transformation, i.e., computing the determinant

$$ml_1 + nl_2 = 1$$

If we want $l_2 = 0$, we need $l_1 = 1/m$ and the transformation reads

$$J_x = mJ_1 \qquad \phi_1 = m\,\phi_x - n\,\phi_z - l\theta + \mu \tag{3.27}$$

$$J_z = J_2 - nJ_1 \qquad \phi_2 = \phi_z \tag{3.28}$$

and in the rotating frame the Hamiltonian becomes

$$\begin{aligned}\mathcal{H}(J_1, \phi_1, J_2) = \delta J_1 + \alpha_{11} J_1^2 + \alpha_{12} J_1 J_2 \\ + G(mJ_1)^{\frac{|m|}{2}} (J_2 - nJ_1)^{\frac{|n|}{2}} \cos\phi_1 \\ + \left[\omega_z J_2 + \alpha_{22} J_2^2\right]\end{aligned} \tag{3.29}$$

where $\delta = m\omega_x - n\omega_z$ is the resonance distance parameter and the new constant α_{11}, α_{12} and α_{22} are still function of α_{xx}, α_{xz} and α_{zz}.

As usual, we neglect the part of Hamiltonian only dependent on J_2 and we get the equations of motion:

$$\begin{aligned}\dot{\phi}_1 = \frac{\partial \mathcal{H}}{\partial J_1} = \delta + 2\alpha_{11} J_1 + \alpha_{12} J_2 \\ + \frac{m}{2} G (mJ_1)^{\frac{m}{2}-1} (J_2 - nJ_1)^{\frac{n}{2}-1} \left[mJ_2 - n(m+n)J_1\right] \cos\phi_1 \\ \dot{J}_1 = -\frac{\partial \mathcal{H}}{\partial \phi_1} = G\,(mJ_1)^{\frac{m}{2}} (J_2 - nJ_1)^{\frac{n}{2}} \sin\phi_1\end{aligned}$$

Being J_2 constant, the evolution of J_x and J_z should follow the rule

$$\Delta J_z = -\frac{n}{m} \Delta J_x \tag{3.30}$$

Now, the phase space which these equations describe is far more complex that the old $(1, 2)$ one. However, some familiar elements

are still present. First of all, we still have an outer circle where all particles are needed to live in. The condition $J_z > 0$ is now

$$J_1 < \frac{J_2}{n}$$

which, in the (x, y) coordinates, describes a circle with radius $\sqrt{2J_2/n}$. Then, a coupling arc which acts as a separatrix is still present, because two symmetric hyperbolic points on the circle can still be found. In fact, inserting the constraint $J_1 = J_2/n$ in the equation $\dot{\phi}_1 = 0$ we get

$$\cos\phi_1 = \frac{2(J_2(2\alpha_{11} + n\alpha_{12}) + n\delta)(m/nJ_2)^{-m/2}}{Gn^3}$$

The coupling arc can now have a different geometry: it is not any longer a circumference arc or a straight line, even in the absence of the detuning parameters. However, our argument for the final values of the emittances, as long as the improved calculation of adiabatic jump is not involved, is just topological, and this should not be a problem.

We could think, then, that the problem is solved, and that we can easily get the emittance exchange for any resonance. Sadly, this is not the end of the story.

For higher values of m and n (i.e., for $|m|+|n| \geq 3$), the degree of the polynomial equation of motion, cast into the (x, y) coordinates, increases, and thus also the number of the equilibrium points.

In fact, setting $\phi = 0$ or $\phi = \pi$ in the equations, as the maximum power that J_1 can reach is $(m + n)/2$, we can get multiple (more than two!) solutions which will become, alternatively, elliptic or hyperbolic points.

This means that lots of new separatrices can grow and shrink and move in the phase space, and, which is the worst thing, can be crossed by particles. An example is given by the phase space portraits presented in Fig. 3.2, which represent what happens, at different values of δ, for a $(3, 1)$ resonance.

If they were absent, or at least negligible, we would be able to repeat the argument and get the final emittances.

In fact, as before, the starting action is $J_{1,0}$ and $A_\mathrm{i} = 2\pi J_{1,0}$. Being $2\pi J_2/n$ the area of the outer circle, we have, at crossing of the coupling arc,

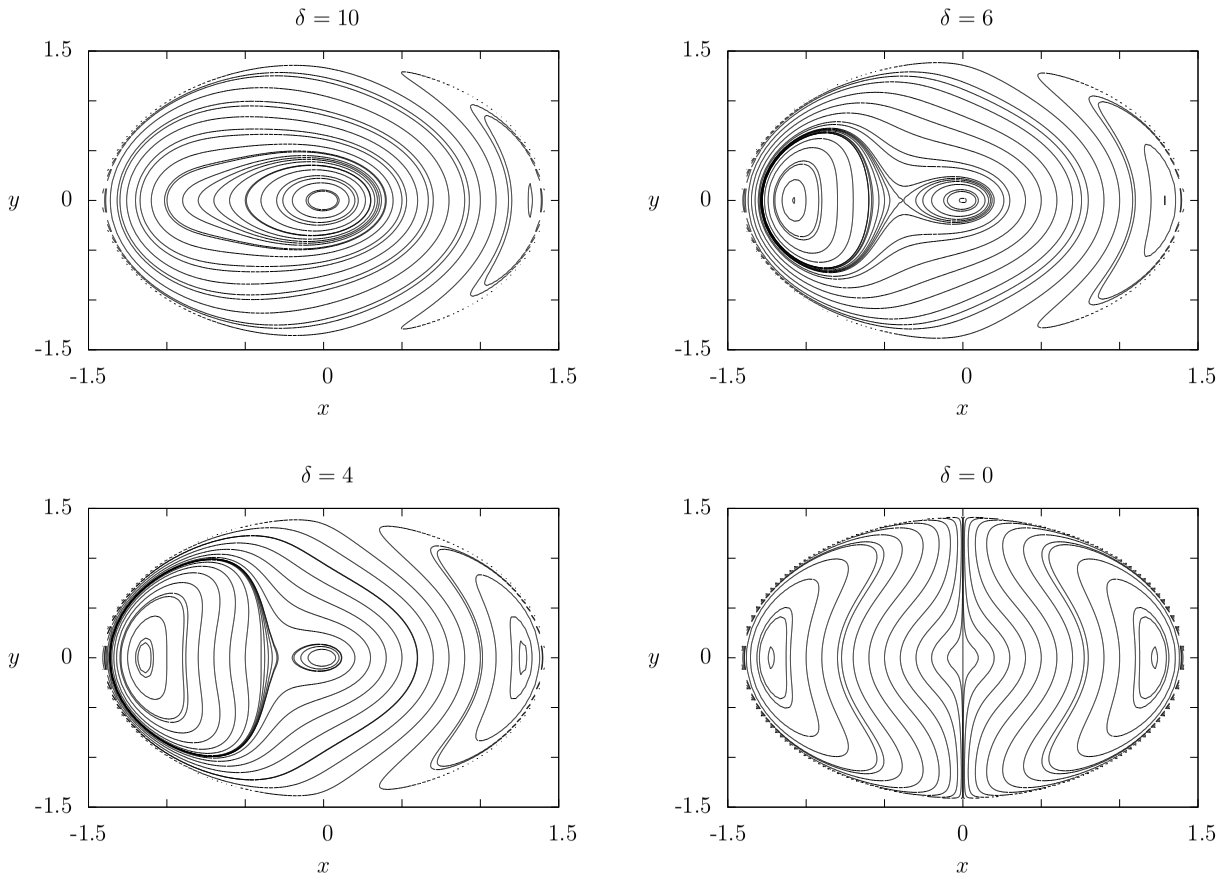


FIGURE 3.2 – Phase space portraits of (3.29) with $m = 3$, $n = 1$, $J_2 = 1$, $\alpha_{11} = \alpha_{12} = 0$ at different values of δ. Although a separatrix still crosses the outer circle other fixed point, either elliptic or hyperbolic, are present.

$$2\pi J_{1,f} = A^* = 2\pi\left(\frac{J_2}{n} - J_{1,0}\right)$$

and, going back to the x and z actions

$$J_{x,f} = mJ_{1,f} = m\left(\frac{J_{z,0} + n/mJ_{x,0}}{n} - \frac{J_{x,i}}{m}\right) = \frac{m}{n}J_{z,0}$$

and, following (3.30)

$$J_{z,f} = \frac{n}{m}J_{x,0}$$

where an emittance exchange, though with different ratios, still happens.

This result could also be expected from the distribution analysis, as if we start with a Gaußian distribution, and we transform into J_1 and J_2 for the (m, n) resonance, we get the distribution

$$\rho_{mn}(J_1,\, J_2) = \frac{m}{\varepsilon_x\varepsilon_y}\exp\left[-J_1\left(\frac{m}{\varepsilon_x} - \frac{n}{\varepsilon_y}\right) - \frac{J_2}{\varepsilon_y}\right]$$

and, passing to the $u = J_1/J_2$ coordinate (this time, $0 \leq u \leq 1/n$)

$$\rho(u, J_2) = \frac{mJ_2}{\varepsilon_x\varepsilon_y}\exp\left[-\left(\frac{m}{\varepsilon_x}u + \frac{1-nu}{\varepsilon_y}\right)J_2\right] \tag{3.31}$$

and, integrating out J_2

$$\rho(u) = \frac{m\,\varepsilon_x/\varepsilon_y}{\left[\varepsilon_x/\varepsilon_y + (m - n\varepsilon_x/\varepsilon_y)\,u\right]^2} \tag{3.32}$$

which is uniform for if $\varepsilon_x/\varepsilon_z = m/n$. [12]

Nevertheless, one could think about choosing detuning parameters in order to lower the polynomial grade of the equation $\partial\mathcal{H}/\partial x = 0$ for $y = 0$ and consequently reduce the number of zeroes and the subsequent separatrices. Of course, the choice needs to be independent of J_2. A dependence on δ could be accepted, if we open to the possibility of coherently vary the strength of the quadrupole magnets which cause the detuning.

Unfortunately, if we inspect what happens at resonances of order 4, we see that such a choice is not possible.

For resonances $(1,3)$ and $(3,1)$ the coefficient of the term J_1^3 of the equation reads

$$16(\alpha_{11}^2 + 27G^2)$$

and can never be set to zero, while the J_1^2 one is dependent on J_2.

Thus, in order to exploit higher-order resonance emittance exchange further inspection is needed in the parameters' space, in order to find appropriate values that could minimize the impact of the other separatrices.

Chapter 4

Simulations

In order to validate our theoretical analysis of the system in the $(1, 2)$ *Walkinshaw* resonance, we performed a number of numerical simulations via symplectic algorithms.

Different kinds of simulations have been carried out:

- At fixed value of J_2 with initial conditions uniformly generated in the circle $J_2 - 2J_1 > 0$. For each particle, initial versus final area is plotted to validate the linear relation (3.8).
- For single trajectories, with area measured every few thousands of evolution steps, with the aim to verify conservation of the adiabatic invariant and to see the behaviour of the adiabatic invariant (i.e., the jump) at separatrix crossing.
- Finally, with bunches of particles at initially random generated J_1 and J_2 according to the distributions (3.1) in order to evaluate the performance of emittance exchange.

4.1 Algorithms

Integration of equation of motions As usual for Haniltonian dynamics, simulation of the particle motion has been mainly performed using a *symplectic integrator*, i.e. the 4-th order Candy-Rozmus algorithm. Symplectic integrators are made in order to guarantee that every discrete integration step is effectively a canonical trasformation, just like, as we saw in Chapter 1, the continuous time evolution of an Hamiltonian system is a canonical transformation itself. They thus preserve the symplectic invariants of the system, like — and this is what we are looking exactly for — area.

The idea is that, if we have a *separable* Hamiltonian

$$\mathcal{H} = \mathcal{H}_1(p) + \mathcal{H}_2(x)$$

and we can exactly solve each the equation in which only one part of the Hamiltonian appears, being Δt the time interval we have that the exact solution for $\mathcal{H}_1$ reads $\exp\{\Delta t D_1\}\mathbf{x}$ and, for $\mathcal{H}_2$, $\exp\{\Delta t D_2\}\mathbf{x}$.

Now, for the phase flow of $\mathcal{H}$ we have

$$\exp\{\Delta t D_{\mathcal{H}}\} - \exp\{\Delta t D_1\}\exp\{\Delta t D_2\} = \frac{(\Delta t)^2}{2} D_{[\mathcal{H}_1,\mathcal{H}_2]} + \mathcal{O}((\Delta t)^3)$$

from the BCH formula [24, p. 424].

This is a 1-st order integrator. We can do better and use a 4-th order one, where the error is just $\mathcal{O}((\Delta t)^4)$. In general, a symplectic integrator acts on $(x,\, y)$ in n steps which are evaluated as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \Delta t\, c_i\, \partial\mathcal{H}/\partial y \;\big|_x \\ y - \Delta t\, d_i\, \partial\mathcal{H}/\partial x \;\big|_y \end{pmatrix}$$

where the c_i and d_i constants are given by the peculiar scheme used. For the Candy-Rozmus algorithm [6] we have

$$c_1 = c_4 = \frac{1}{2(2-2^{1/3})} \qquad c_2 = c_3 = \frac{1-2^{1/3}}{2(2-2^{1/3})}$$
$$d_1 = d_3 = \frac{1}{2-2^{1/3}} \qquad d_2 = -\frac{2^{1/3}}{2-2^{1/3}} \quad d_4 = 0$$

The Hamiltonian (3.3) has two problems: first of all it is singular in J_1. If the values of the J_1 gets under zero the program will halt trying to perform an impossible operation. This can be overcome passing to the $(x,\, y)$ coordinates as done in (3.6).

The second fact is that the Hamiltonian is *not* separable. We could in fact build a real symplectic integrator, which would exactly conserve area (far from separatrices, of course) separating the Hamiltonian into different terms and solving each one exactly. However, this has not been necessary, because, as we show in the subsequent figures, the performance of the basic Candy-Rozmus algorithm with the chosen Δt have been sufficiently satisfactory.

On the other hand, for the $(3, 1)$ resonance, the Candy integrator did not show a sufficient conservation of areas, being the Hamiltonian impossible to solve piecewise, we had to fall back to a usual Runge-Kutta 4th order integrator.

Measuring area In order to evaluate the phase space area of our trajectories, a formula attributed to Gauß and sometimes known as *shoelace*, was used. Given an ordered set of cartesian coordinates $\mathbf{x}_k = (x_k,\, y_k)$ for $1 \leq k \leq N$, the area of the polygon whose vertices are the elements of this set is given by

$$A = \frac{1}{2}\left|\sum_{k=1}^{N}(x_k y_{k+1} - x_{k+1}y_k)\right|$$

where $x_{N+1} \equiv x_1$ and $y_{N+1} \equiv y_1$.

A proof of this formula can be produced in terms of differential manifolds. In fact, we can define the area of a set Ω as

$$A = \int_\Omega \mathrm{d}x \wedge \mathrm{d}y$$

and if

$$\omega = \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x)$$

we have, differentiating,

$$\mathrm{d}\omega = \mathrm{d}x\mathrm{d}y - \mathrm{d}y\mathrm{d}x = \mathrm{d}x \wedge \mathrm{d}y$$

Thus,

$$A = \int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega$$

thanks to Stokes theorem.

Now, the boundary of Ω is defined by N segments S_j, where S_j connects $\mathbf{x}_j$ and $\mathbf{x}_{j+1}$

$$A = \int_{\partial\Omega} \omega = \frac{1}{2}\sum_{j=1}^{N}\int_{S_j} x\mathrm{d}y - y\mathrm{d}x$$

We can write, being A_j segments parametrized by $t \in [0, 1]$

$$x = x_j + (x_{j+1} - x_j)t \qquad \mathrm{d}x = (x_{j+1} - x_j)\mathrm{d}t$$

and the same for y. Substituing, and performing the elementary integration, we get

$$\frac{1}{4}\sum_{j=1}^{N}[(x_j + x_{j+1})(y_{j+1} - y_j) - x \leftrightarrow y]$$

whence the result immediately follows.

From the data of the orbit, thus, area was measured identifying the points which completed a closed orbit and therefore applying the shoelace algorithm. The identification of a closed orbit has been implemented checking the angular position of the particle and halting the search when for the second time the same value, up to a threshold, of the arctangent of that angle had been attained (in order to have the algorithm work also if the orbit did not enclose the origin). Now, the chosen threshold sometimes fails to stop the algotithm at the first orbit but let it continue for a second one. These spurious area values are however easily identified in the resulting data since they account for an area value which is exactly twice than expected and can be filtered out without any difficulty.

4.2 (1,2) resonance

4.2.1 Single trajectories and adiabatic jumps

First of all, in order to show how the separatrix-crossing mechanics works, we present in Fig. 4.1 a trajectory of a particle in the phase space (J_1, ϕ_1) here represented in the Cartesian coordinates (x, y) (colours encode the elapsed time). We see that, at a certain time, the particle suddendly changes regime and lands in a different region. In Fig. 4.2 we show, for another simulated particle, the area of the revolutions as function of the time (datas have been filtered to eliminate spurious values due to imperfections of the algorithm or to the impossibility to measure area in the immediate vicinity of the jump). We see that area remains quasi-constant far from separatrix (with some oscillation that are $\mathcal{O}(\epsilon)$, being ϵ the adiabaticity parameter. When the particle is approaching the separatrix, oscillations amplitude increases. Then the area suddendly jumps to its final value with oscillations decreasing while drifting apart from the separatrix.

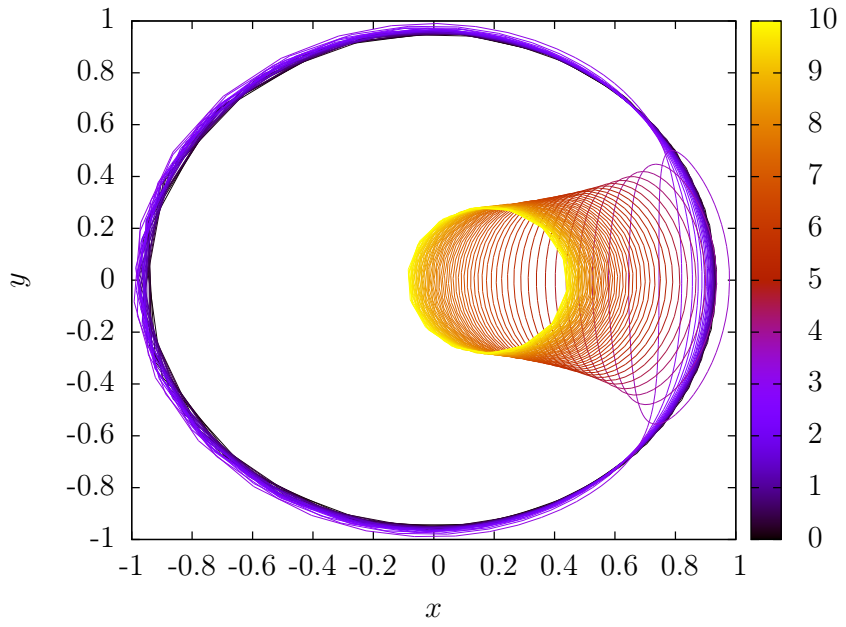


FIGURE 4.1 – Phase trajectory of a particle with initial position ($x = 0.8$, $y = 0.5$) undergoing emittance exchange. Parameters are set as $G = 3$, $\alpha_{11} = \alpha_{12} = 0$. Colours encode time steps in units of 10^6.

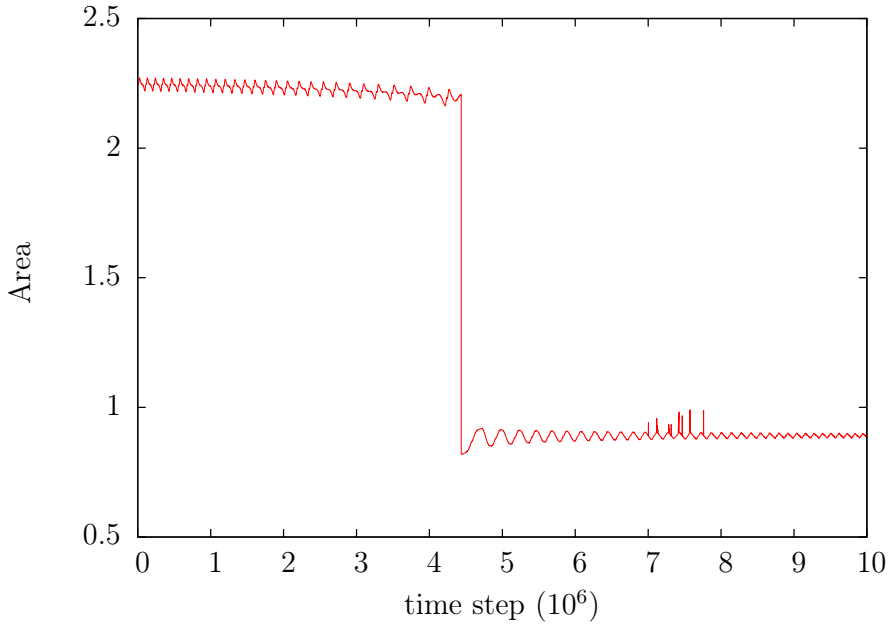


FIGURE 4.2 – Plot of orbit area versus time step in a simulation.

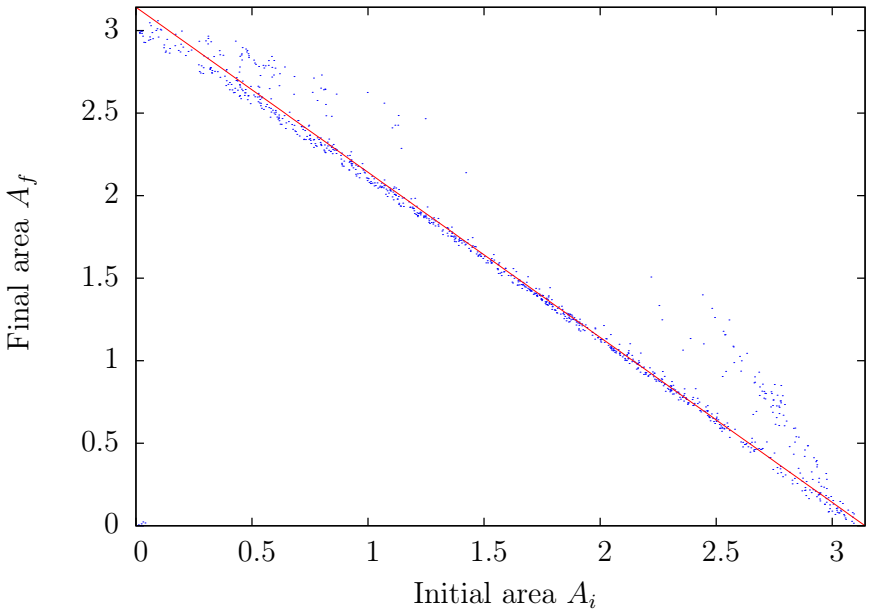


FIGURE 4.3 – Distribution of initial and final orbit area during an emittance exchange process for 1000 uniformly distributed particles at fixed $J_2 = 1$. The continuous red line shows the theoretical line of $A_f = \pi - A_i$.

4.2.2 Initial vs final areas

The next step, as stated in our schedule, is to run simulations of the dynamics of a bunch of particles keeping constants J_2. We performed such a simulation with 1000 particles uniformly distributed inside the *outer circle* $x^2 + y^2 \leq J_2$ (with no detuning and $G = 3$) and we plotted in Fig. 4.3 the first orbit area versus the last one for each particle. The relation

$$A_f = \pi - A_i$$

which we derived in Chapter 3 from adiabatic theory is thus confirmed; as warned before, some data result actually *double* than expected: this is explained by the aforementioned imperfections of the area measuring algorithm. These spurious values are concentrated at the extrema of the range, where either A_i or A_f is very small, and it is easier that the algorithm fails to individuate the end of an orbit.

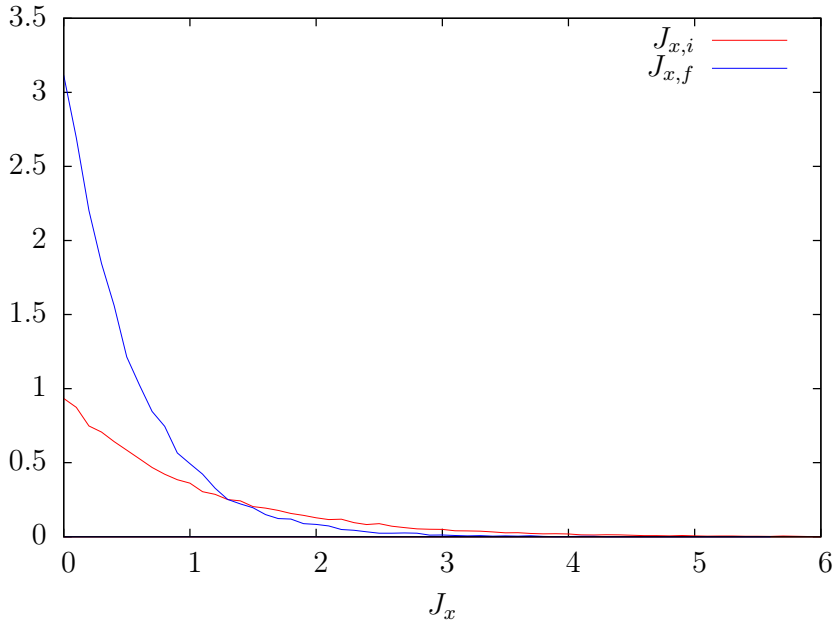


FIGURE 4.4 – Distribution of initial and final J_x after an emittance exchange process.

4.2.3 Emittances exchange

We then proceed to simulate the evolution of a bunch of 30000 particles with $G = 3$ and no detuning and a starting Gaußian distribution with $\varepsilon_x = \varepsilon_z = 1$ in order to verify the proper execution of the emittance exchange.

We plot in Fig. 4.4 and Fig. 4.5 the initial and final distribuition of J_x and J_z for such a process (in which we varied δ from $+20$ to -20 in 10^7 time steps).

The resulting value of emittance ($\varepsilon_x = 0.53$, $\varepsilon_z = 1.99$) are in good corresponance with the theoretical ones ($\varepsilon_x = 0.5, \varepsilon_z = 2$) and the distribution, as better shown in Figg. 4.6 and 4.7, where a logarithmic scale has been used, remain exponential.

4.2.4 Dependence on parameters

Finally, several runs of the simulation have been made to verify the dependence of the emittance exchange on the parameters of the model, i.e. the sextupole resonance strenght G and the detuning parameters α_{11} and α_{12}. The study has been made taking into

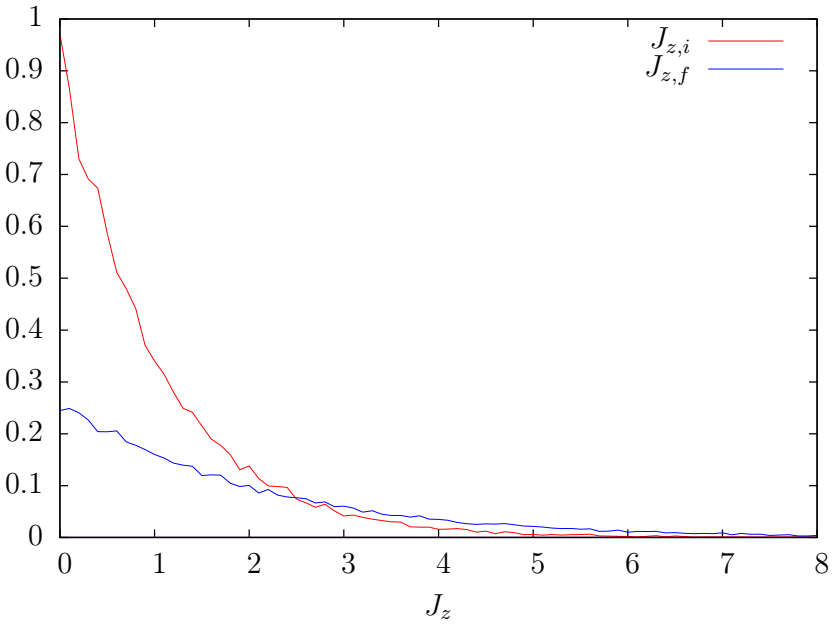


FIGURE 4.5 – Distribution of initial and final J_z after an emittance exchange process.

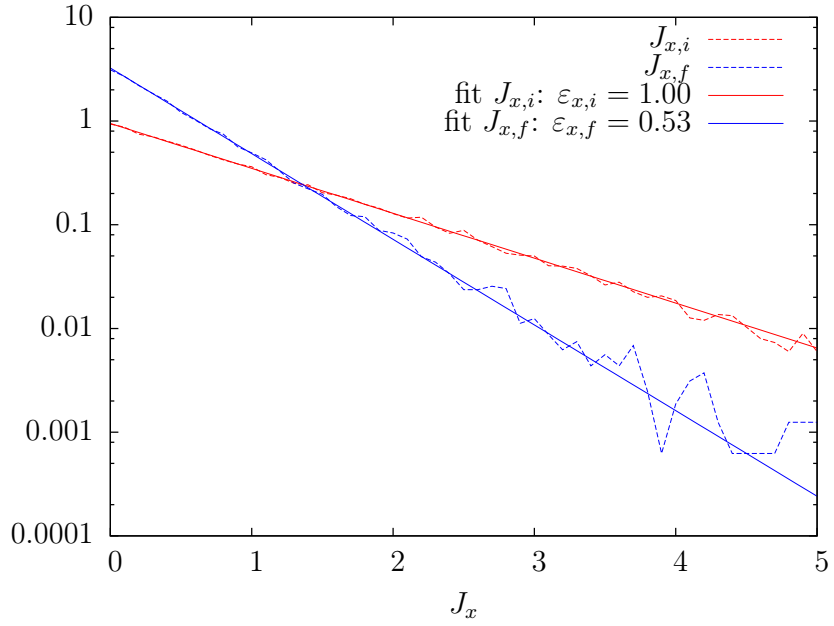


FIGURE 4.6 – Distributions of Fig. 4.4 plotted in log scale. An exponential fit has been performed.

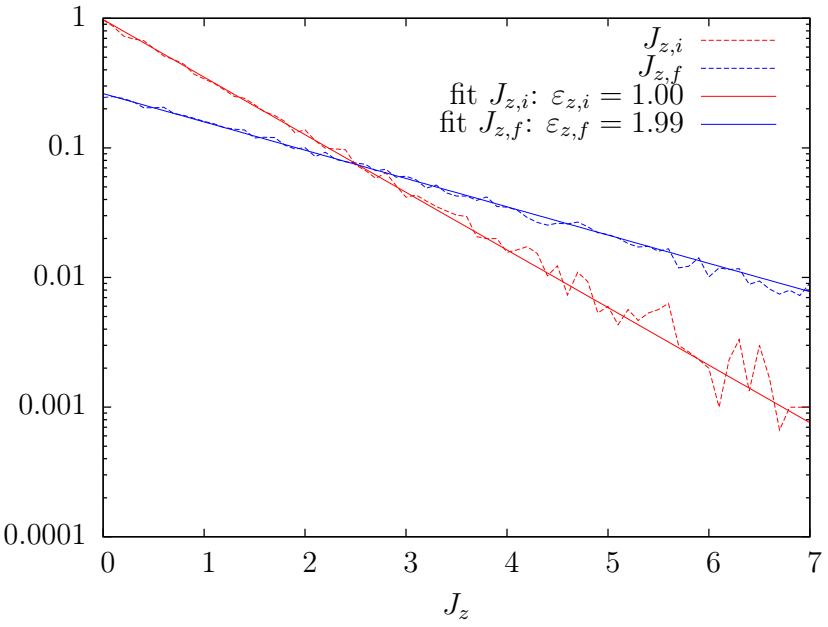


FIGURE 4.7 – Distributions of Fig. 4.5 plotted in log scale. An exponential fit has been performed.

account a Gaußian initial distribution of 3000 particles with $\varepsilon_x = \varepsilon_z = 1$ and performing the emittance exchange modulating δ from $+20$ to -20 in 10^7 time steps.

For each bunch of particles, the final x emittance and the standard deviation of the action J_x, in order to check the preservation of the exponential distribution, have been computed.

First of all, let us take into account the running over the parameter G, where simulations have been performed keeping $\alpha_{11} = \alpha_{12} = 0$. Fig. 4.8 shows that, as expected, if $G = 0$ the equations of motion is exactly correspondant to an harmonic oscillator, thus no exchange is expected. While G increases, but still remains small, the system is not sufficiently adiabatic to perform the emittance exchange, and the final emittance value decreases until an optimal point is reached (around $G = 1$). Then, for higher values of G the performance gets worse, because, as the resonance passage becomes slower, not all particles in the phase space are affected.

Fig. 4.9 shows data regarding the final x emittance for more values of G (a logarithmic scale is used): for extremely high values of G, the emittance exchange performance looks good, but actually the

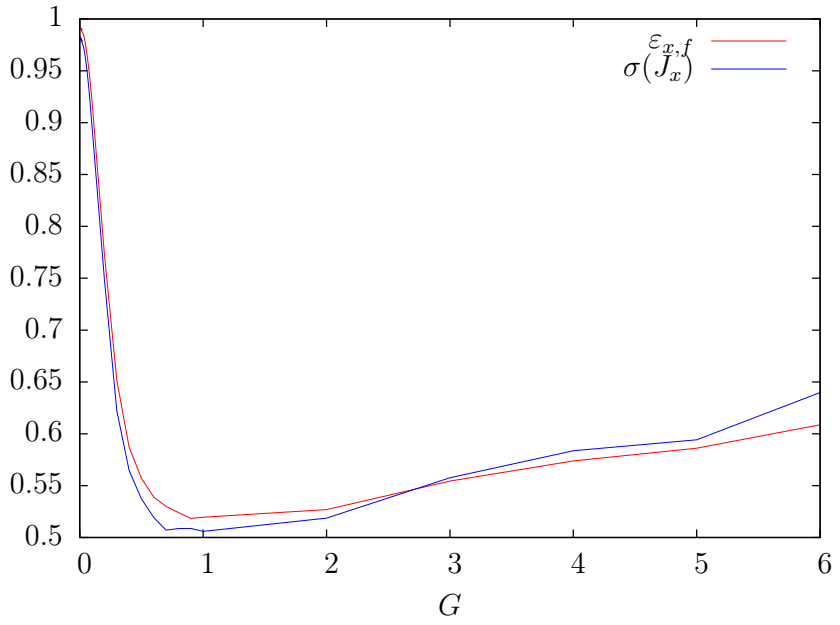


FIGURE 4.8 – Emittance exchange performance as function of the parameter G (with $\alpha_{11} = \alpha_{12} = 0$).

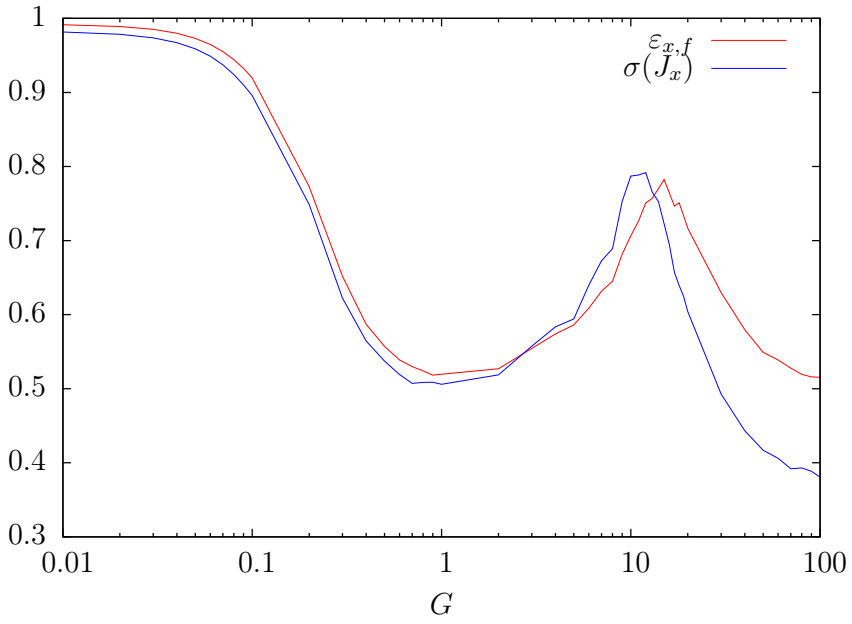


FIGURE 4.9 – Emittance exchange performance as function of the parameter G (log-scale).

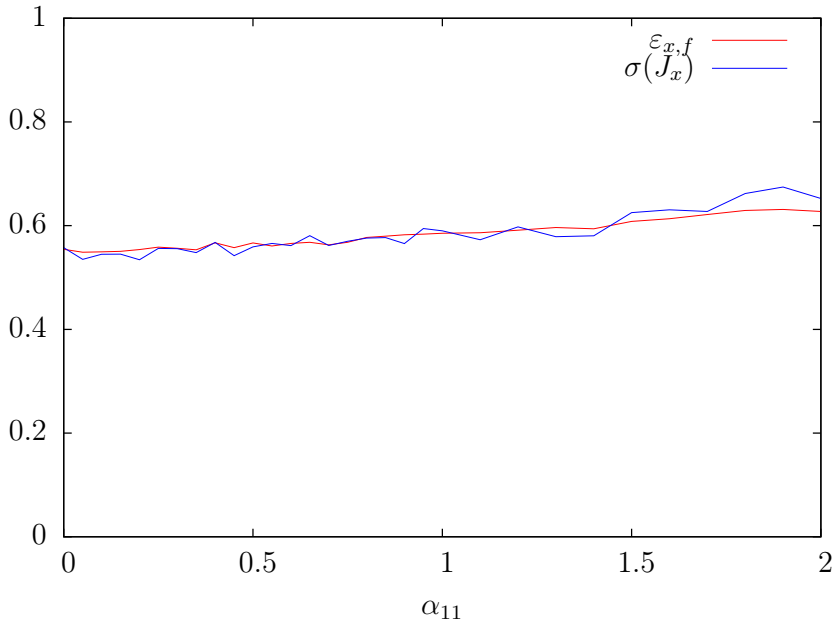


FIGURE 4.10 – Emittance exchange performance with different values of α_{11} ($G = 3$, $\alpha_{12} = 0$)

standard deviation values for such parameters show that the resulting distribution is not any more exponential. We believe that this kind of exchange is due to a different mechanism, which is related to what happens to the equations of motion where δ is negligible compared to G.

For what concerns the detuning parameters, let us start with α_{11}. For a reasonable range of values (Fig. 4.10) the final emittance remains close to the optimal value 0.5, and the standard deviation values advocate for the conservation of the exponential distribution. The same (Fig. 4.11) can be said for α_{12}. The data we present have been obtained with $G = 3$ and keeping one of the detuning parameters switched off while the other undergoes variation.

One could be puzzled seeing that Lee *et al.* in [18] show that the emittance exchange is preserved for values of α_{11} up to $2000\,\pi\mathrm{m}^{-1}$ while we present only data for $\alpha_{11} < 2$. The value difference is explained by the fact that in [18] initial emittances are taken in the order of the micrometer, while in our dimensionless model we are keeping the emittances with an unitary value. Thus, as long as the detuning parameter appears in the Hamiltonian coupled with the J_1^2

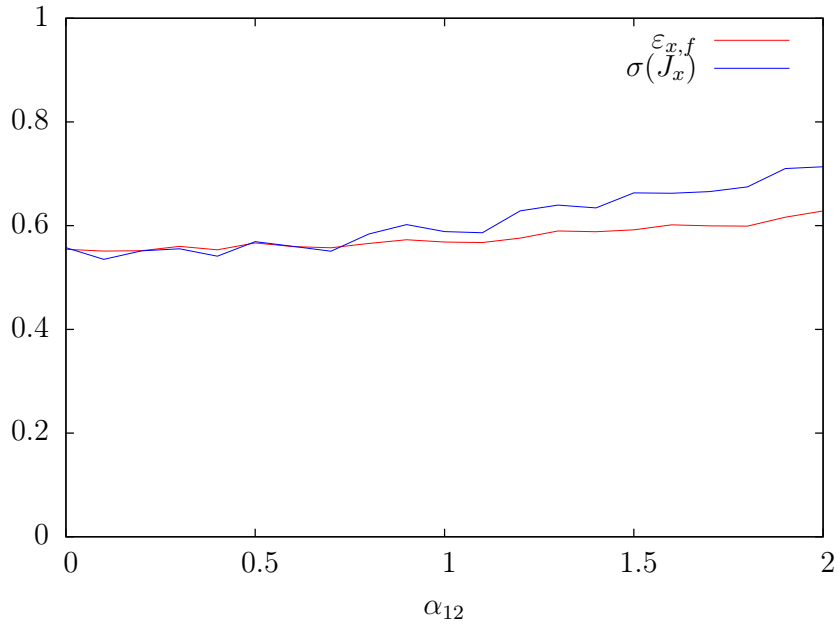


FIGURE 4.11 – Emittance exchange performance with different values of α_{12} ($G = 3$, $\alpha_{11} = 0$)

term, we see that our data also cover the range in which emittance exchange is shown in [18].

4.3 (3,1) resonance

Generalization to higher resonances also in the realm of computer simulation has then be attempted. However, two main obstacles interpose between us and the understanding of the dynamics in such a situation: not only the presence of other separatrices in the phase space which can trap particles inside, but also the fact that in the equations expressed in the (x, y) the quantity

$$\sqrt{2J_2 - x^2 - y^2}$$

appears at denominator, thus creating a singularity close to the the outer circle where integration errors get inflated and a computed trajectory could finish out of the circle, this resulting in an error thrown by the program as it attempts either to compute a division by zero or the square root of a negative number. Thus, we were able

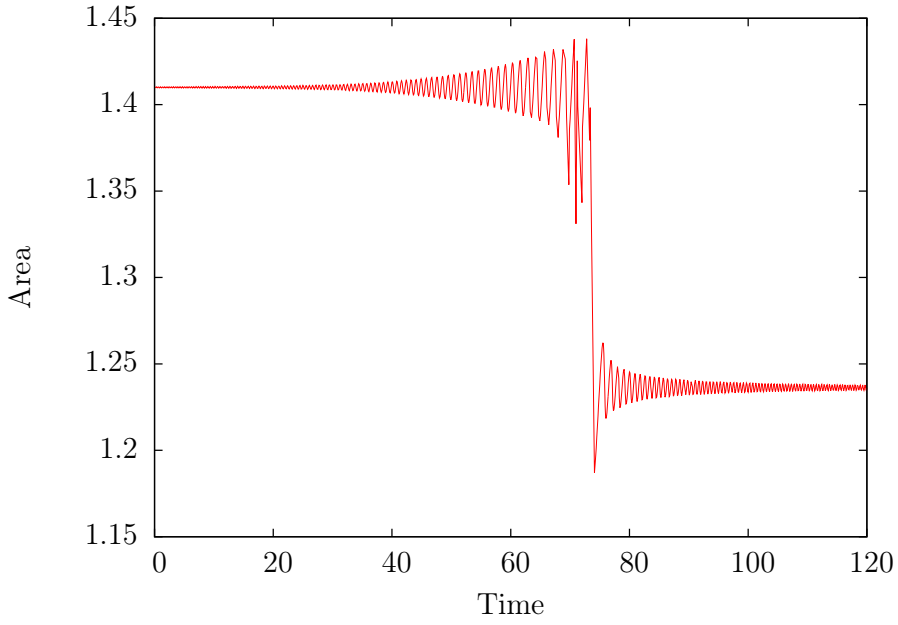


FIGURE 4.12 – Area variation as function of the time for a $(3, 1)$ resonance. Data have been filtered to exclude spurious values.

to track particles only in a limited number of cases and to collect only semi-empirical data about the relationship between initial and final areas this configuration.

For example, in Fig. 4.12 we see the variation of area in a $(3, 1)$ resonance crossing, with the same structure we witnessed at the Walkishaw passage, with the increasing of oscillations close to the separatrix, the jump and the subsequent decreasing of oscillation. However, we do not have enough data to say whether the simple relationship obtained in Chapter 3 for higher-order resonance are confirmed by simulations, as we do not know whether particles are going to be trapped in the other separatrices.

Chapter 5

New approach to beam splitting

The MTE scheme, as it is performed applying an adiabatic modulation to the machine tune, might be limited, when exported to other machines, by possible constraints in the choice of the working point, i.e. the tunes (Q_x, Q_z). One possible solution can be found in the pioneering work of Nejštadt (see, e.g., [20, 22]) who has shown that trapping into stable islands can be achieved also by means of a different approach: instead of exciting a resonance intrinsic to the system under study, one adds an external, time-dependent force, whose frequency is in resonance with that of the original system. The interesting feature of this approach is that the initial system does not need to feature a frequency close to a given resonance, as this condition is created by the external force.

5.1 Theory review

Starting from the multipolar expansion of the magnetic field, and taking into account only transverse motion in the horizontal plane, the Hamiltonian of a particle moving under the influence of magnetic fields in a circular accelerator can be written up to octupole terms [17]

$$\mathcal{H}_0 = \frac{p^2}{2} + \omega_0 \frac{q^2}{2} + k_3 \frac{q^3}{3} + k_4 \frac{q^4}{4} \tag{5.1}$$

We add to this Hamiltonian a new term, called *dipolar kick*, controlled by the amplitude ε. We have,

$$\mathcal{H} = \frac{p^2}{2} + \omega_0 \frac{q^2}{2} + k_3 \frac{q^3}{3} + k_4 \frac{q^4}{4} + \varepsilon q \cos \omega t \tag{5.2}$$

Adiabatic variations of the kick frequency ω, modifying the separatrices, cause the particles to be randomly captured in islands of stability different from the central one. Crossing a r-order resonance (*i.e.* varying ω around $r\,\omega_0$) causes the creation of r stable islands.

The dipolar Hamiltonian has been studied in different works by Nejštadt [20, 22] using an averaging approach. Nejštadt computes a formula for the probability of a single particle to be captured into $1:1$ resonance as a function of the frequency $\tilde{\omega}$ when the separatrix is crossed. This probability reads

$$\mathcal{P} = \frac{\pi - \Theta}{\pi - \Theta/2} \tag{5.3}$$

having defined

$$\Theta = \arccos\left(\frac{\lambda}{2x_{\mathrm{C}}^2} - 2\right) \tag{5.4}$$

and

$$x_{\mathrm{C}} = \frac{\sqrt{6\lambda}}{3} \cos\left[\frac{\pi}{6} + \frac{1}{3} \arcsin\left(\frac{3\sqrt{6}}{4} \frac{\mu}{\lambda^{3/2}}\right)\right] \tag{5.5}$$

where $\lambda \sim \tilde{\omega}/k_4$ and $\mu \sim \varepsilon/k_4$.

Averaging applied to different resonances causes only the appearance of new constants in the expression of μ, thus not changing the scaling laws. Indeed, we expect this probability law to be valid, at least in a scaling sense, for the actual trapping fraction of particles as a function of ε and k_4.

For small values of ε, we have, from (5.3)

$$\mathcal{P} \sim \alpha \varepsilon^{1/2} + \beta \varepsilon + \mathcal{O}(\varepsilon^{3/2}) \tag{5.6}$$

and, for k_4,

$$\mathcal{P} \sim \alpha k_4^{1/4} + \beta k_4^{1/2} + \mathcal{O}(k_4^{3/4}) \tag{5.7}$$

while, for large values of ε, we expect a linear behaviour as $\mathcal{P}$ approaches 1.

Note that, following the same approach, the probability has been as well computed in [7] for a $2:1$ resonance in the *quadrupolar*

case (where the Hamiltonian perturbative term reads $\varepsilon q^2/2\cos(\omega_1 t)$, obtaining

$$\mathcal{P} = \frac{8\arcsin\sqrt{\mu/\lambda}}{2\pi + 4\arcsin\sqrt{\mu/\lambda}} \tag{5.8}$$

where, just as before, $\lambda \sim \tilde{\omega}/k_4$ and $\mu \sim \varepsilon/k_4$.

Thus, as λ/μ does not depend on k_4, we expect k_4 to have little effect in the quadrupolar trapping.

Indeed, for small values of ε the probability (5.8) scales as

$$\mathcal{P} \sim \alpha\varepsilon^{1/2} + \beta\varepsilon + \mathcal{O}(\varepsilon^{3/2}) \tag{5.9}$$

just like the dipolar case.

In the rest of this work, our simulations will be focused on the dipolar excitation, only,

5.2 Simulation results

Numerical simulations have been performed using the 4th-order Candy symplectic integrator. Starting from an initial distribution (either Gaußian or uniform) we computed the final coordinates in phase space of the particles in the beam, identifying, when the trapping succeeded, the different islands appearing. Then, counting the number of particles trapped in the external region, we computed the trapping efficiency as

$$\tau = \frac{N_{\text{trapped}}}{N_{\text{total}}}\,. \tag{5.10}$$

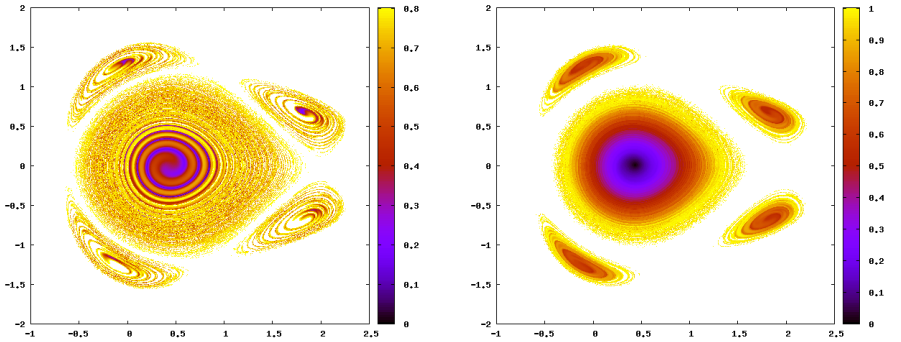
The number of trapped particles can be counted by a mere inspection of the resulting configuration, selecting particles which have moved to different beamlets, or more efficiently, computing the tune of their orbit via the frequency analysis algorithm we present, to identify whether a particle has been trapped.

The initial distributions read, for the Gaußian case

$$f_{\mu_q,\sigma}(p,q) = \frac{1}{2\pi\sigma}\exp\left\{-\frac{p^2 + (q-\mu_q)^2}{2\sigma^2}\right\} \tag{5.11}$$

and, for the uniform one,

$$f_{r_{\max}}(p,q) = \frac{1}{\pi(p^2+q^2)} \qquad p^2 + q^2 \leq r_{\max}\,. \tag{5.12}$$



(a) $\mu_q = 0$, $r_{\max} = 0.8$, $\varepsilon = 30$ (b) $\mu_q = 0.35$, $r_{\max} = 1.0$, $\varepsilon = 30$

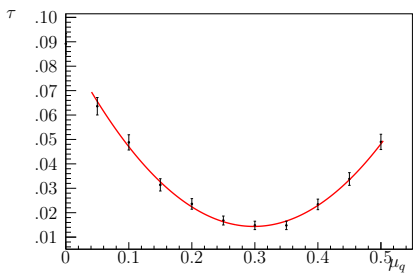
FIGURE 5.1 – Final state in the phase space of a uniform-distributed beam with different centres for dipolar kick. Colour encodes starting distance from the distribution centre.

Extensive scans over the different parameters affecting the trapping efficiency (ε, k_4, σ and μ_q of the initial distribution, number of time iterations) have been performed. In every plot, the error $\delta\tau$ is estimated by the square root rule

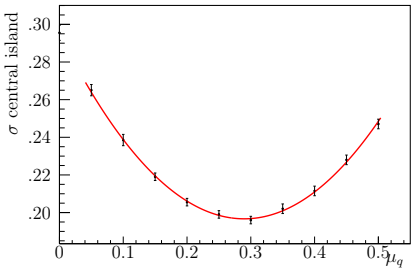
$$\delta\tau = \frac{\sqrt{N_{\text{trapped}}}}{N_{\text{total}}} = \sqrt{\frac{\tau}{N_{\text{total}}}}\,. \tag{5.13}$$

When not differently stated, for the dipolar kick we computed the trapping linearly varying ω near the 4-order resonance, from $\omega_{\text{i}} = 4.00$ to $\omega_{\text{f}} = 3.60$. The frequency ω_0 and the sextupole constant k_3 are merely scaling parameters, and were set $\omega_0 = 1$, $k_3 = -2.41$.

Uniform distribution The final results of a trapping process for the dipolar kick starting from an uniform distribution centred in $(0, 0)$ and from one centred in the fixed point $(0.35, 0)$ are plotted in Fig. 5.1, where the different colours encode the starting radius of the particle. From the plot it is possible to see that there exist a minimum radius $r_{\min}$ so that, for $r < r_{\min}$ no trapping is possible. Moreover, when the initial distribution is centred on the fixed point, the initial amplitude ordering is respected in the final phase space, where particle at small amplitude are those with small initial amplitude. This property is lost when the distribution is centred in zero.



(a) Trapping probability τ as function of μ_q.



(b) Standard deviation of distance of points in the central island from the centre of distribution as function of μ_q.

FIGURE 5.2 – Behaviour of τ and σ of the central island as function of centre of distribution $(\mu_q, 0)$. Gaußian distributions were used with $\sigma = 0.2$.

Centering of the distribution. Study of σ In Fig. 5.2a we plot the trapping efficiency as a function of the centre $(\mu_q, 0)$ of Gaußian initial distributions with $\sigma = 0.2$. It should be noted that, when the distribution is centred in the fixed point of the frozen phase space, the trapping efficiency reaches a minimum. This could be explained by the existence of a minimal radius (around the fixed point) below which the trapping is not possible. Moving the distribution away from the fixed point gets better trapping because less particles start in the no-trapping region. The same behaviour (and the same quadratic fit) appears in the plot in Fig. 5.2b where the standard deviation of the final central island is computed for different μ_q.

Finally, in Fig. 5.3 we plot the evolution of trapping probability for origin-centred Gaußian distribution of particles with different standard deviation σ.

Study of ε The behaviour of the trapping efficiency versus the dipole strength ϵ is plotted in Fig. 5.4. Gaußian initial distributions centred in the origin have been used, with $\sigma = 0.2$. A comparison with the scaling law (5.6) for small ε has also been made. Trapping probabilities for higher values of ε have not been computed due to lack of sufficient separation between the central and the external islands.

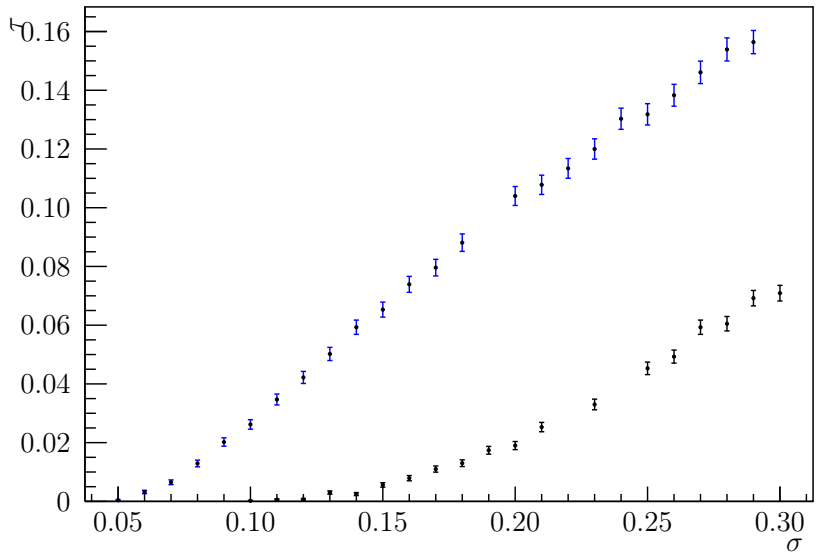


FIGURE 5.3 – Evolution of trapping probability τ for Gaußian distribution with $\mu_q = 0$ and variable σ. The black line shows simulations for $\varepsilon = 20$, while the blue one is for $\varepsilon = 30$.

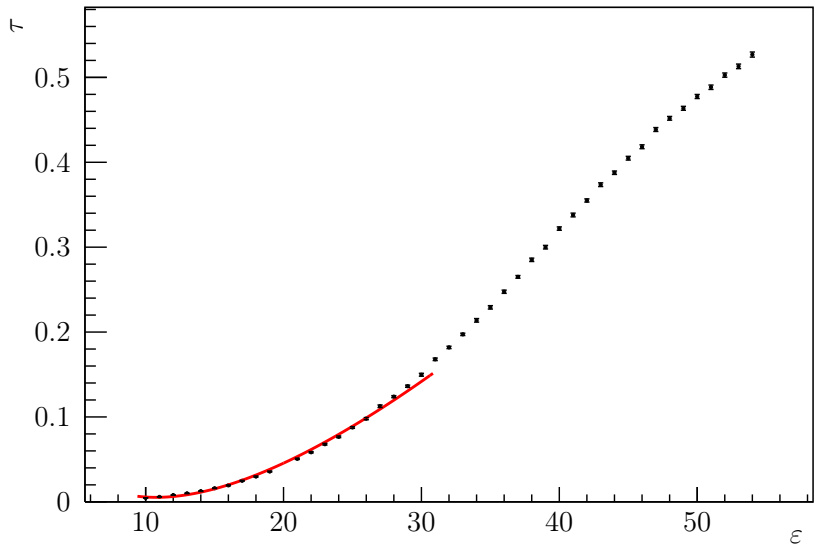


FIGURE 5.4 – Evolution of trapping probability as function of ε. We set $k_4 = 0.24$. The red line shows the fit for $\tau = \text{const.} + \alpha\varepsilon^{1/2} + \beta\varepsilon$.

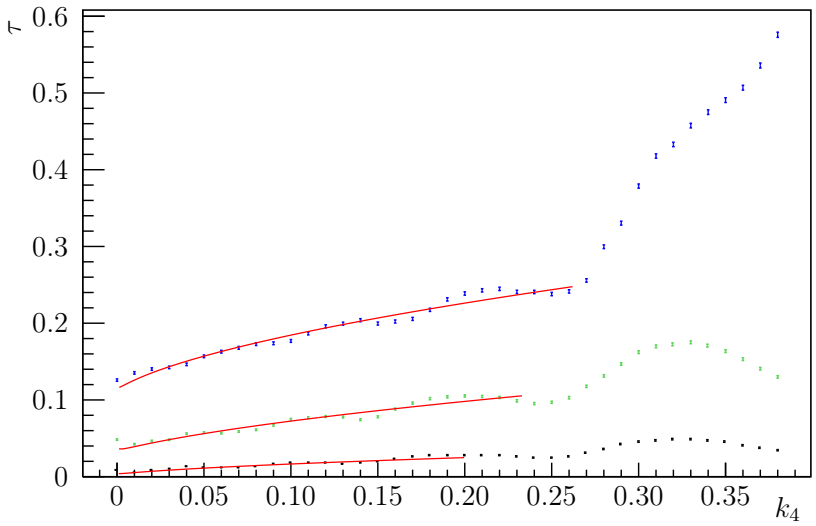


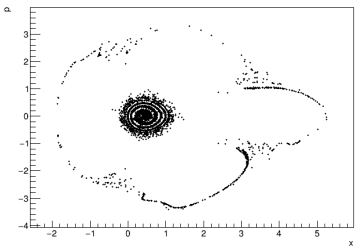
FIGURE 5.5 – Trapping probability in the dipolar case over k_4 for different values of ε (black $\varepsilon = 30$, green $\varepsilon = 35$, blue $\varepsilon = 40$). A fit is performed for the scaling law $\tau = \text{const.} + \alpha k_4^{1/4} + \beta k_4^{1/2}$

Study of k_4 The behaviour of the trapping efficiency versus the octupole constant k_4 is plotted in Fig. 5.5 for different values of ε. A fit for small values of k_4 has been performed to evaluate the agreement with the scaling law given by (5.7).

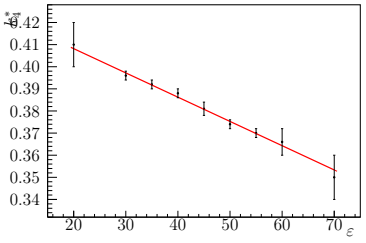
Indeed, we observed a remarkable phenomenon during the scanning of k_4 parameter. For every ε, there exist a maximum value k_4^* for the trapping to be performed. For $k_4 > k_4^*$, the external islands merge, and a single annular region appears outside of the separatrix (Fig. 5.6a). The value of k_4^* versus ε is plotted in Fig. 5.6b, and is consistent with a linear law.

Time iterations The last parameter we focused on in our study is the number of time iterations. For a slower variation of the kick frequency, thus a better approximation of the adiabatic regime, we achieve an increased trapping. We confirm (Fig. 5.7) the scaling law $\tau \sim n_{\text{iter}}^{-0.5}$ stated in [5], at least for $n_{\text{iter}} > 1 \times 10^5$.

Different resonances Finally, a study has been conducted regarding the behaviour of the dipolar trapping for resonances 1, 2,



(a) Final phase space distribution of particles for $\varepsilon = 30$, $k_4 = 0.40 > k_4^*$.



(b) Plot of maximum value of k_4 for the trapping to be performed as function of ε.

FIGURE 5.6 – Study of maximum value of k_4.

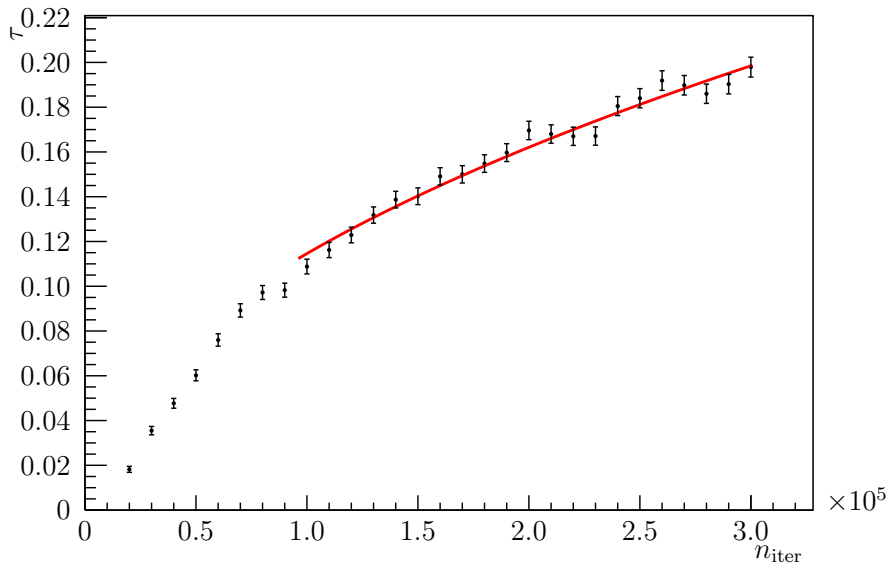
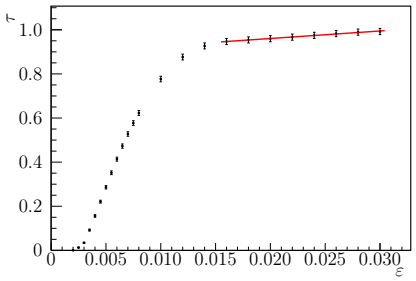
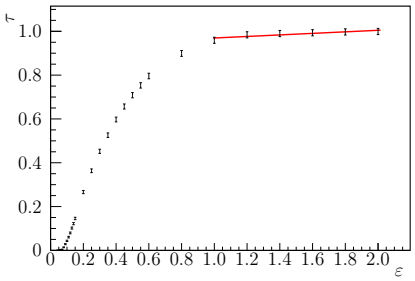


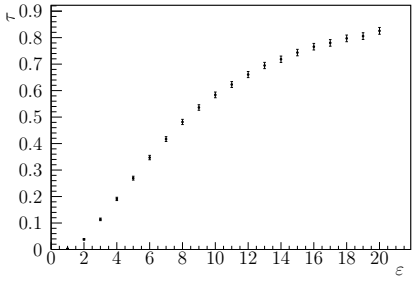
FIGURE 5.7 – Evolution of trapping probability τ in the dipolar case for different numbers of time iterations. A fit is performed for $\tau = \text{const.} + \alpha n_{\text{iter}}^{-1/2}$.



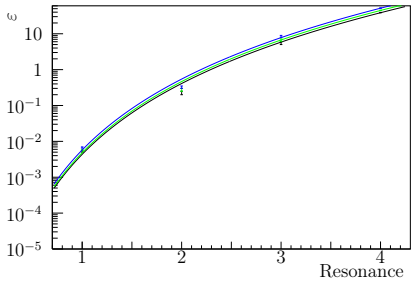
(a) Plot of τ over ε for resonance 1



(b) Plot of τ over ε for resonance 2



(c) Plot of τ over ε for resonance 3



(d) Comparison between ε (log-scale) needed to achieve the same τ for different resonances. Legend: $\tau = 0.3$ black, $\tau = 0.4$ green, $\tau = 0.5$ blue. Power-law fits have been performed.

FIGURE 5.8 – Analysis of different resonances.

and 3.

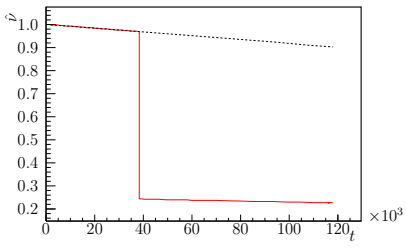
The results are shown in Fig. 5.8. It is clear that, for resonances of lower order a trapping efficiency up to 100% can be achieved. Furthermore, it was possible to verify the linear law we expected as the trapping probability tends to 1. The plots share the same behaviour of the 4-resonance case (Fig. 5.4). In Fig. 5.8d we show a comparison between the ε needed, at different resonances, for achieving the same trapping efficiency. A power-law fit $\varepsilon = \alpha k^{\beta}$, being k the resonance order, gives compatible values of β for the three cases analyzed ($\beta_{\tau=0.3} = 6.56(4)$, $\beta_{\tau=0.4} = 6.59(4)$, $\beta_{\tau=0.5} = 6.56(4)$) suggesting a ~ 6.5-power scaling law for ε as function of resonance order.

5.2.1 Frequency analysis

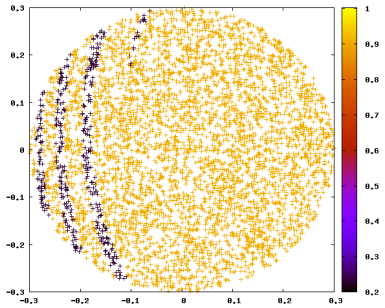
In order to understand what is the relationship between trapping into resonances and frequency variation, a Fast-Fourier-transform analysis has been performed (following [4]) to determine the evolution of betatronic tune of trapped and non-trapped particles.

The algorithm we used performed an improved FFT on the phase space orbit data of a particle for a number ΔT of time iterations, returning the main frequency $\hat{\nu}$.

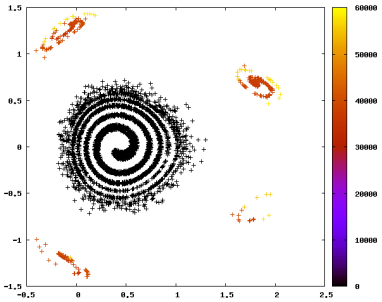
In Fig. 5.9a we show the evolution of $\hat{\nu}$ for a trapped and a non-trapped particle under the dipolar excitation. It is possible to see that particles undergo a sudden tune jump when trapped into resonance, shifting their frequency from ω_0 to 1/4. We also plot, in Fig. 5.9b, the final frequency of particles in a uniform distribution with constant initial tune (the final distribution can be seen in Fig. 5.9d). A peculiar structure, with strips of phase space in which trapping is achieved alternated with ones with no trapping, emerges. Furthermore, in Fig. 5.9c, we show the time of the frequency jump for the same distribution. Knowing the linear relationship between time and kick frequency ω, we can retrieve the separatrix-crossing $\tilde{\omega}$ used by Nejštadt to compute the trapping probability in [20].



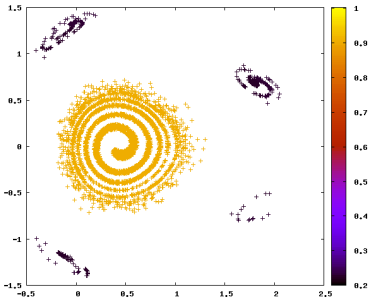
(a) Tune evolution of two different particles: the red line shows a trapped particle, the black one a non-trapped one.



(b) Initial distribution in phase space of particles undergoing the trapping process. Colours encode the *final* tune: black particles are going to be trapped.



(c) Final state in phase space of the uniform distribution of Fig. 5.9b. Colours encode tune-jumping time (black particles do not undergo the frequency change). Total number of time iterations is 120×10^3.



(d) Distribution in phase space of particles from Fig. 5.9b after the trapping process. Colours encode the final tune.

FIGURE 5.9 – Dipole kick: frequency analysis

Conclusions

In this Master thesis, we presented two models which use adiabatic invariance theory to address some specific problems in accelerator physics.

We can summarize our main results:

- We explained the mechanism which is responsible for the exchange of emittances between transverse directions when a $(1, 2)$ resonance is crossed, computing, given an initial Gaußian distribution, what are the expected final emittances. Numerical simulations have then been performed to confirm these results.

- We propose a novel alternative to MTE to achieve beam splitting by means of an external excitator, reviewing previous theoretical results and presenting numerical studies which explore the parameters' space to achieve the best splitting efficiency.

This is the conclusion of the thesis, but not of the research work related to this topic. We need to move these models from the theoretical framework to actual proposals that can be implemented in particle accelerators.

While we write these final notes, an intense work with this exact aim is undergoing at CERN Beams department. In fact, we are trying to adapt our models to the real configuration of the PS accelerator, performing the translation between parameters of the model and currents and magnet strength in the real machine. We use dedicated software like `MAD-X` [10] to perform realistic beam simulations in such configurations and verify whether the desired effects are actually achievable.

Moreover, further theoretical study can be made to complete generalizations of the models, or to analyze the systems in terms of Hénon-like maps.

These topics will be addressed, however, in a Doctoral project which is soon starting: thus the best *explicit* of this work would probably be

TO BE CONTINUED...

Bologna/Meyrin, June–October 2018.

Bibliography

[1] V. I. Arnol'd. *Metodi matematici della meccanica classica*. Mosca/Roma: MIR/Editori Riuniti, 1979. ISBN: 978-88-6473-204-6.

[2] V. I. Arnol'd. "Small denominators and problems of stability of motion in classical and celestial mechanics". *Uspechi Mat. Nauk* 18.6 (1963), pp. 91–192.

[3] V. I. Arnol'd, V. V. Kozlov, and A. I. Neištadt. *Mathematical Aspects of Classical and Celestial Mechanics*. Encyclopaedia of Mathematical Sciences. Berlin: Springer, 2007. ISBN: 9783540489269.

[4] R. Bartolini *et al.* "Algorithms for a precise determination of the betatron tune". *EPAC 96*. Proceedings of the 5th European Particle Accelerator Conference (Barcelona, June 10–14, 1996). Ed. by S. Myers. Bristol: IOP, 1996, pp. 1329–1334. ISBN: 9780750303873.

[5] A. Bazzani *et al.* "Analysis of adiabatic trapping for quasi-integrable area-preserving maps". *Phys. Rev. E* 89 (4 Apr. 2014), p. 042915. DOI: 10.1103/PhysRevE.89.042915.

[6] J. Candy and W. Rozmus. "A symplectic integration algorithm for separable Hamiltonian functions". *Journal of Computational Physics* 92.1 (1991), pp. 230–256. DOI: 10.1016/0021-9991(91)90299-Z.

[7] F. Capoani. "Adiabatic extraction models for beam dynamics". Bachelor thesis. Alma Mater Studiorum — Università di Bologna, Oct. 2016. URL: http://amslaurea.unibo.it/12056/.

[8] R. CAPPI and M. GIOVANNOZZI. "Novel method for multiturn extraction: Trapping charged particles in islands of phase space". *Physical Review Letters* 88 (10 Feb. 2002), p. 104801. DOI: 10.1103/PhysRevLett.88.104801.

[9] H.-C. CHAO. "Emittance evolution in crossing Walkinshaw resonance and envelope dynamics simulations". PhD thesis. Indiana University, 2015.

[10] L. DENIAU, ed. *The MAD-X Program (Methodical Accelerator Design)*. Geneva: CERN, 2018. URL: http://mad.web.cern.ch/mad/releases/5.04.00/madxuguide.pdf.

[11] P. EHRENFEST. "Over adiabatische veranderingen van een stelsel in verband met e theorie de quanta". *Verslagen Kon. Akad. Amsterdam* 25 (1916), pp. 412–433.

[12] M. GIOVANNOZZI and B. PERCIVAL. *Hamiltonian dynamic analysis near a generic resonance*. Private communication. 2018.

[13] A. HUSCHAUER *et al.* "Transverse beam splitting made operational: Key features of the multiturn extraction at the CERN Proton Synchrotron". *Phys. Rev. Accel. Beams* 20 (6 June 2017), p. 061001. DOI: 10.1103/PhysRevAccelBeams.20.061001.

[14] A. P. ITIN *et al.* "Change in the adiabatic invariant in a nonlinear two-mode model of Feshbach resonance passage". *Physica D: Nonlinear Phenomena* 232.2 (2007), pp. 108–115. ISSN: 0167-2789. DOI: 10.1016/j.physd.2007.06.001.

[15] L. D. LANDAU and E. M. LIFŠIC. *Corso di Fisica teorica. 1. Meccanica*. Mosca/Roma: MIR/Editori Riuniti, 1975.

[16] P. LANGEVIN and L. DE BROGLIE, eds. *La théorie du rayonnement et les quanta*. Solvay conference (Bruxelles, 1911). Paris: Gauthier-Villars, 1912.

[17] S. Y. LEE. *Accelerator Physics*. 3rd. World Scientific, 2011. DOI: 10.1142/8335.

[18] S. Y. LEE *et al.* "Evolution of Beam Distribution in Crossing a Walkinshaw Resonance". *Phys. Rev. Lett.* 110 (9 2013), p. 094801. DOI: 10.1103/PhysRevLett.110.094801.

[19] L. I. MANDEL'ŠTAM. *Collected works*. Moskva: Akad. Nauk SSSR, 1948, pp. 297–304.

[20] A. I. Nejštadt, A. A. Vasil'ev, and A. V. Artem'ev. "Capture into resonance and escape from it in a forced nonlinear pendulum". *Regular and Chaotic Dynamics* 18.6 (Nov. 2013), pp. 686–696. ISSN: 1468-4845.

[21] A. I. Nejštadt. "Change in the adiabatic invariant at a separatrix". *Sov. J. Plasma Phys.* 12 (1986), p. 568.

[22] A. I. Nejštadt. "Passage through a separatrix in a resonance problem with a slowly-varying parameter". *Journal of Applied Mathematics and Mechanics* 39 (4 1975), pp. 594–605.

[23] J. L. Tennyson, J. R. Cary, and D. F. Escande. "Change of the Adiabatic Invariant due to Separatrix Crossing". *Phys. Rev. Lett.* 56 (20 1986), pp. 2117–2120.

[24] G. Turchetti. *Dinamica classica dei sistemi fisici*. Bologna: Zanichelli, 1998. ISBN: 9783540489269.

[25] A. Wolski. *Beam Dynamics in High Energy Particle Accelerators*. Imperial College press, 2014. DOI: 10.1142/p899.

Editorial note. Throughout the whole work, bibliography included, Russian names and journal titles have been romanized according to the so-called *scientific*, ISO approved, transliteration, and not following the anglosaxon tradition which is more common in scientific publishing.

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labirinto degli effetti e delle cause
per la diversità delle creature
che compongono questo singolare universo,
per la ragione, che non cesserà di sognare
un qualche disegno del labirinto,
per l'algebra, palazzo dai precisi cristalli,
per il geometrico e bizzarro gioco degli scacchi,
per la tartaruga di Zenone e la mappa di Royce,
per gli intimi doni che non elenco,
per la musica, misteriosa forma del tempo.

Jorge Luis Borges

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