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A fibre bundle approach to U(1) symmetries in physics

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Sommario

In questa tesi si utilizza il formalismo dei fibrati principali per descrivere le proprietà topologiche globali di sistemi fisici classici e quantistici che presentano simmetrie legate all'azione del gruppo $U(1)$. Nel primo capitolo è contenuta una esposizione della teoria matematica dei fibrati, con un particolare riguardo ai fibrati principali ed alle strutture differenziali definibili su di essi (forme differenziali di connessione e curvatura). Nel secondo capitolo si impiega il formalismo precedentemente sviluppato per trattare le proprietà del monopolo magnetico di Dirac e si ottiene una quantizzazione della carica magnetica sulla base di considerazioni di natura topologica. Inoltre, si mostra l'impiego dei fibrati principali $U(1)$ nella costruzione di una descrizione Lagrangiana globale per sistemi quali una particella carica nel campo del monopolo e una particella classica con spin in campo magnetico. Nel terzo capitolo, si descrive teoricamente la comparsa di una fase geometrica (fase di Berry) in sistemi quantistici che evolvono adiabaticamente nel tempo, e si fornisce un'interpretazione geometrica di tale fase come olonomia in un fibrato principale $U(1)$. Il moto di una particella quantistica con spin in campo magnetico quasi-statico e l'effetto Aharonov-Bohm vengono presentati come esempi tipici di manifestazione di una fase geometrica.

Contents

Introduction

The aim of the present work is to give an overview on the possible applications of the theory of fibre bundles in investigating the global properties of some peculiar physical systems, within both a classical and a quantum theoretical framework.
The first Chapter of the thesis will be entirely devoted to an exhaustive exposition of the necessary topological and analytic apparatus, which is that of fibre bundle structures (specifically, principal bundle ones) and connections over them. Roughly speaking, a fibre bundle is a structure which locally resembles a product of spaces: each point on a total bundle space is identified by its coordinates along a certain "fibre"and by the position on a second space -the base manifold- where the former is attached. A projection map allows to project down all the points in the same fibre to a single point in the base space. In a principal bundle structure, there is a structure group (typically a Lie group) which maps a point on a fibre to another one on the same fibre, so that the projection map is left unchanged under the action of the group. Here it is where we make a contact with *gauge theories*: if the Action of a given physical system is a function on the base manifold of a certain principal bundle, it will be invariant with respect to the action of the structure group, which can be therefore regarded as a gauge group. In light of this identification, a gauge fixing is a bijiective association of each point of the base manifold to a point on the total bundle space. This association is realized by means of a connection on the principal bundle, the geometrical counterpart of a gauge potential: if the latter is globally defined then we have a global gauge fixing, and this is eventually possible only if the bundle has a trivial topological structure, i.e. it is globally (and not only locally) a product of spaces. What is fundamental to stress is thus that the symmetries of a certain physical systems are mirrored by the *global* topological features of a bundle structure.

It is commonly thought that the theory of fibre bundles can find some relevant physical applications only when one deals with gauge *field* theories, i.e. continuous systems with an infinite number of degrees of freedom. This is a false impression. Indeed, there are many examples from the physics of point-like particles in which gauge symmetries find an appropriate description within the framework of principal bundles. This will be the object of Chapter 2, in which we will concentrate on two of such systems: the Dirac magnetic monopole and the classical non-relativistic particle with spin. The Dirac

monopole itself highlights the power of the bundle approach, since it furnishes an example of a gauge theory (electromagnetism) in which the topological non-triviality of the principal bundle structure directly leads to a quantization condition. Even more striking results emerge when one studies the motion of an electric charge in the field of the monopole: the charge-monopole system does not admit a global Lagrangian description unless one enlarges the configuration space to a principal bundle whose fibres are element of a $U(1)$ gauge group. In doing so, however, the system acquires a redundant degree of freedom which cannot be eliminated through a global gauge-fixing since the bundle is non trivial. Precisely the same situation is encountered in the study of the classical spinning particle, whose Lagrangian is formally identical to the one of the previous system. There is also a deeper physical reason for the analogy between the spinning particle and the charge-monopole system, which will be explored as well. The conclusion of the Chapter will be dedicated to the exposition of a general technique for the construction of global Lagrangians by means of $U(1)$ principal bundles, recovering the charge-monopole system as an immediate application of the theory.

In Chapter 3 we will turn to the applications of the fibre bundle approach within the context of quantum mechanics. Since the physical observables of a quantum system are defined up to a complex phase, the appropriate geometrical background for the study of the former naturally appears to be an $U(1)$ principal bundle, being the state of the system (i.e. its wavefunction) interpreted as a section over the base manifold. The starting point will be the derivation of the *geometric phase factor* developed by a system which undergoes an adiabatic evolution in time along a closed curve in a suitable parameter space. As we will show, such a phase contains no information about the dynamics of the system, and has to be regarded as a *holonomy* on the bundle over the parameter space. That is, the geometric -or adiabatic- phase measures how much a loop in the parameter space "fails"in remaining closed when lifted up to the total bundle space through an adequate *parallel transport* rule. As an illustrative example we will consider the phase acquired by a spinning particle moving in a slowly rotating magnetic field during a cyclic evolution: this will enable us to clarify once again the relation intervening between this system and that of a magnetic monopole fixed in space. Employing the geometric tools developed so far, in the final part of the Chapter -and of the entire work- we will focus our attention on the theoretical analysis of the so-called "Aharonov-Bohm effect", whose experimental observation led to a critical rethinking of the role played by the electromagnetic potential as a physical observable in quantum mechanics.

Chapter 1

The geometric apparatus: fibre bundles and connections

A fibre bundle is a topological space which is locally similar to a direct product of spaces, i.e. it is homeomorphic to a certain set $\mathbb{X} \times \mathbb{Y}$, as much as a topological manifold is locally homeomorphic to $\mathbb{R}^n$. In a differentiable fibre bundle the homeomorphism is promoted to a diffeomorphism and this allows us to build differential structures on it, such as connections and curvature (which will be the object of the next Section). The importance of these topological structures lies in the role they play as the ideal framework for the geometric formulation of many physical problems. After a long but necessary introduction on the general theory of fibre and vector bundles, we will turn to principal bundle structures, which constitute the real mathematical groundwork for the theory developed in the following Chapters.
Notions and results from differential geometry are widely used within the entire work: the main references for the subject are [1], [2], [3]. For a complete and rigorous exposition of the theory of fibre bundles the reader is instead referred to [4], [5].

1.1 Vector and principal bundles

1.1.1 Basic definitions and examples

An introductory example: the tangent bundle
Let us consider, as a starting point, the **tangent bundle** of a manifold. If M is an n-dimensional manifold, its tangent bundle is defined as the (disjoint) union of all the tangent spaces at each point:

$$\mathrm{T}M \equiv \bigcup_{p \in M} \mathrm{T}_p M \,. \tag{1.1.1.1}$$

The space M is called the **base manifold** of the total bundle space $\mathrm{T}M$. Let $U_\alpha \subset M$ be an open set and $\phi_\alpha(p) = (x^1, ..., x^n)$ a chart over it. Then at each point p in U_α a basis of the tangent space T_pM is given by $\left(\frac{\partial}{\partial x^1}|_p, ..., \frac{\partial}{\partial x^n}|_p\right)$ so that $V_p \in \mathrm{T}_pM$ reads in coordinates[1] as $V_p^i \frac{\partial}{\partial x^i}|_p$. Now since U_α is diffeomorphic to $\mathbb{R}^n$ through the chart ϕ_α, and each T_pM is itself an n-dimensional vector space, then there exists a function, called **local trivialization**, which establishes a diffeomorphism between $\mathrm{T}U_\alpha$ and $\mathbb{R}^n \times \mathbb{R}^n$:

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \quad , \quad V \longmapsto (x^1, ..., x^n, V_p^1, ...V_p^n)\,, \tag{1.1.1.2}$$

where we have also made use of the **canonical projection** $\pi : \mathrm{T}M \longrightarrow M$, a surjective map from the total space onto the manifold, such that if $V \in \mathrm{T}_pM$ then $\pi(V) = p$. Notice that this map is not injective, since no trace is left of the particular vector V projected onto p. In this way, $\pi^{-1}(U_\alpha) = \mathrm{T}U_\alpha$, and we define $\pi^{-1}(p) = \mathrm{T}_pM$ the **fibre** at p.
Consider now a different chart $\Phi_\beta = (y^1, ..., y^n)$ defined on U_β such that $U_\alpha \cap U_\beta \neq \emptyset$. Then if $V_p \in \mathrm{T}_pM$, $p \in U_\alpha \cap U_\beta$, the two coordinate expressions of V_p are related by:

$$V_p = V_p^i \left.\frac{\partial}{\partial x^i}\right|_p = \tilde{V}_p^i \left.\frac{\partial}{\partial y^i}\right|_p \quad \Longrightarrow \quad \tilde{V}_p^i = \left.\frac{\partial y^i}{\partial x^j}\right|_p V_p^j\,. \tag{1.1.1.3}$$

A consistency relation for the charts requires that $\det\left(\frac{\partial y^i}{\partial x^j}\right) \neq 0$, so that the matrix which provides for the coordinate change is an element of $\mathrm{GL}(n, \mathbb{R})$. This is called the **structure group** of the tangent bundle, and its elements are the **transition functions** between charts on the bundle:

$$\tilde{V}_p^i = (t_{\beta\alpha}(p))_j^i\, V_p^j \quad , \quad t_{\beta\alpha}(p) \in \mathrm{GL}(n, \mathbb{R})\,. \tag{1.1.1.4}$$

As we shall see, the transition functions determine the topological properties of the bundle. There is another structure related with the tangent bundle, i.e. a vector field: a map V which smoothly associates each point p in $U \subset M$ to a vector $V_p \in \mathrm{T}_pM \subset \mathrm{T}M$. This map is of course such that $\pi \circ V = \mathrm{id}_U$, and it is called a **local section** of the bundle. If $U = M$ then the section is global: the topological **triviality** of the bundle determines whether or not we are able to build a global section over it.
Let us now give some general definitions.

Differentiable fibre bundle. A differentiable fibre bundle is a structure (E, π, M, F, G) consisting of the following elements:

1. Three differentiable manifolds E , M , F respectively called the total space, the base space and the typical fibre.

[1]Summation over repeated indices is intended unless otherwise specified.

2. A global surjective map $\pi : E \longrightarrow M$. The inverse image at each point p of M, $F_p \equiv \pi^{-1}(p) \subset M$, is called the fibre at p.

3. A structure group G for which a left action on the fibre F is defined:

$$L_g : G \times F \longrightarrow F \quad , \quad (g, f) \longmapsto L_g(f) \equiv gf \, . \tag{1.1.1.5}$$

4. A set of charts $(U_\alpha \, , \Phi_\alpha)$, where $\{U_\alpha\}$ is an open covering of of M and the functions:

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F \quad , \quad \Phi_\alpha(u) = (\pi(u), f) \, , \tag{1.1.1.6}$$

which are called local trivializations of the bundle, are diffeomorphisms such that $\pi \circ \Phi_\alpha^{-1} = \mathrm{pr}_1|_{U_\alpha}$, where $\mathrm{pr}_1 : M \times F \longrightarrow M$ is the projection onto the first factor. In other words, the local trivializations map each open set of the total space into a subset of the product space in such a way that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times F \\ & \searrow^{\pi} & \downarrow{\scriptstyle \mathrm{pr}_1|_{U_\alpha}} \\ & & U_\alpha \end{array}$$

Notice that if we define the map $\Phi_{\alpha,p}^{-1} \equiv \Phi_\alpha^{-1}(p\,, \cdot) : F \longrightarrow F_p$, $p \in U_\alpha$ then its inverse $\Phi_{\alpha,p} : F_p \longrightarrow F$ is also well defined and is a diffeomorphism.

5. For each pair U_i , U_j such that $M \supset U_i \cap U_j \neq \emptyset$ there is a transition function:

$$t_{ij} : U_i \cap U_j \longrightarrow \mathrm{diff}(F) \quad , \quad t_{ij}(p) \equiv \Phi_{i,p} \circ \Phi_{j,p}^{-1} \, . \tag{1.1.1.7}$$

In this way, the two local trivializations $\Phi_i(u) = (p, f_i)$, $\Phi_j(u) = (p, f_j)$, with $p \in U_i \cap U_j$, $u = \pi^{-1}(p)$, are related by a diffeomorphism between the respective coordinates on the fibre:

$$(p, f_i) = (p, \Phi_{i,p} \circ \Phi_{j,p}^{-1} f_j) = (p, t_{ij}(p) f_j) \, . \tag{1.1.1.8}$$

Moreover, although it is not strictly necessary, we require the diffeomorphism $t_{ij}(p)$, for p fixed, to be an element of the group G.
In order to ensure consistency when passing from a chart to another, the transition functions must obey the following **cocycle conditions**:

(i) $t_{ii}(p) = \mathrm{id}_G \quad p \in U_i \, ;$

(ii) $t_{ij}(p) = t_{ji}^{-1}(p) \quad p \in U_i \cap U_j \, ;$

(iii) $t_{ij}(p) \cdot t_{jk}(p) = t_{ik}(p) \quad p \in U_i \cap U_j \cap U_k \, .$

It is also sometimes useful to define the **principal part** of a local trivialization Φ_α as the diffeomorphism between the fibre at a point and the typical fibre:

$$\phi_\alpha \, : \pi^{-1}(U_\alpha) \longrightarrow F \, . \tag{1.1.1.9}$$

Hence, the action of the local trivialization is decomposed as:

$$\Phi_\alpha(u) = (\pi(u), \phi_\alpha(u)) \, . \tag{1.1.1.10}$$

Strictly speaking, as any ordinary manifold, a fibre bundle must be independent of the charts we choose. Therefore we can define two **coordinate bundles**

$$(E, \pi, M, F, G, \{(U_\alpha, \Phi_\alpha)\}) \quad , \quad (E, \pi, M, F, G, \{(U_\beta, \Phi_\beta)\}) \, , \tag{1.1.1.11}$$

and say that they are **equivalent** if $(E, \pi, M, F, G, \{(U_\alpha \cap U_\beta, \Phi_\alpha \cap \Phi_\beta)\})$ is still a coordinate bundle. Then the proper fibre bundle will be the equivalence class of the latter. If there is no need to specify the typical fibre or the structure group we will often adopt the shorthand notation $E \xrightarrow{\pi} M$ when referring to (E, π, M, F, G).

Bundle maps and equivalence, trivial bundles.
Consider two bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$. They are said to be diffeomorphic if the smooth maps $f : M' \to M$ and $\tilde{f} : E' \to E$ are such that each fibre $F'_p \subset E'$ is mapped into $F_{f(p)} \subset E$, so that $f \circ \pi' = \pi \circ \tilde{f}$, that is, the following diagram commutes:

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f} & M \end{array}$$

If $M' = M$ and f is the identity map id_M then the two bundles are equivalent (observe that this definition of bundle equivalence is in accordance with the one previously given). A **trivial bundle** is a bundle which is equivalent to the product bundle $E \equiv M \times F$: this means that the function $\tilde{f} : E' \to M \times F$ is a *global* trivialization. It easily follows then that if the bundle is trivial all the transition functions can be taken to be the identity map (this is so because in the equivalence class there exists an atlas consisting of a single chart).

Remark. The set of transition functions on a fibre bundle is not unique. In fact, let us consider two sets of charts $\{(U_i, \Phi_i)\}$, $\{(U_i, \Psi_i)\}$, with $\{\Phi_i\}$ and $\{\Psi_i\}$ local trivializations giving rise to equivalent bundles. For each $p \in U_i$ there are two different transition functions:

$$t_{ij}(p) = \Phi_{i,p} \circ \Phi_{j,p}^{-1} \in G \quad , \quad s_{ij}(p) = \Psi_{i,p} \circ \Psi_{j,p}^{-1} \in G \, . \tag{1.1.1.12}$$

This naturally defines a map:

$$g_i(p) : F \longrightarrow F \quad , \quad g_i(p) \equiv \Phi_{i,p} \circ \Psi_{i,p}^{-1} , \tag{1.1.1.13}$$

where we make the consistency requirement that $g_i(p) \in G$ and its action on F is at least homeomorphic. In this way the transformation law for the transition functions reads:

$$s_{ij}(p) = g_i(p)^{-1} \circ t_{ij}(p) \circ g_j(p) . \tag{1.1.1.14}$$

Local section. Let (E, π, M, F, G) be a fibre bundle. A local section on $U \subset M$ is a map:

$$\sigma : U \longrightarrow E \quad \text{such that} \quad \pi \circ \sigma = \mathrm{id}_U . \tag{1.1.1.15}$$

Notice that, however, $\sigma \circ \pi$ is not the identity map, since π is not injective. The set of all the smooth local sections on U is denoted by $\Gamma(U, E)$: for instance, on a tangent bundle $\Gamma(M, \mathrm{T}M)$ is the set of all the smooth vector fields on M. It is crucial to stress that a global section on a fibre bundle does not necessarily exist: we will prove that this is the case precisely when the latter is trivial.

Example: the cylinder and the Moebius strip
Consider a fibre bundle $E \xrightarrow{\pi} S^1$ whose typical fibre is a segment on the real line symmetric with respect to the origin, say $F \equiv [-1, 1] \subset \mathbb{R}$. Let $\{U_1, U_2\}$ be an open covering of S^1, with:

$$U_1 \equiv \left\{ (x^1, x^2) \in \mathbb{R}^2 \middle| \, x^1 = \cos\phi \, , x^2 = \sin\phi \, , \phi \in]0, 2\pi[\right\} , \tag{1.1.1.16}$$

$$U_2 \equiv \left\{ (x^1, x^2) \in \mathbb{R}^2 \middle| \, x^1 = \cos\phi \, , x^2 = \sin\phi \, , \phi \in]-\pi, \pi[\right\} . \tag{1.1.1.17}$$

The simplest choice for the respective trivializations is:

$$\Phi_1(u) = (p, t) \, , p = \pi(u) \in U_1 \, , \, t \in F \quad \text{and} \quad \Phi_2(u) = (p, t) \, , p = \pi(u) \in U_2 \, , \, t \in F \, , \tag{1.1.1.18}$$

so that there actually is a global trivialization on $\pi^{-1}(U_1 \cap U_2)$, and the transition function is the identity map $t_{12}(p) : t \to t$. The total space is therefore the trivial bundle $S^1 \times F$ (the cylinder).
However, we can make a different choice of the transition function, in order to "twist" the bundle over one or more points on the base manifold. For instance, let:

$$t_{12}(p) : \begin{cases} t \to t, & \text{if } p \in A \\ t \to -t, & \text{if } p \in B \end{cases} , \tag{1.1.1.19}$$

where

$$A \equiv \left\{ (x^1, x^2) \in \mathbb{R}^2 \middle| \, x^1 = \cos\phi \, , x^2 = \sin\phi \, , \phi \in]0, \pi[\right\} , \tag{1.1.1.20}$$

$$B \equiv \left\{ (x^1, x^2) \in \mathbb{R}^2 \middle| \, x^1 = \cos\phi \, , x^2 = \sin\phi \, , \phi \in]\pi, 2\pi[\right\} . \tag{1.1.1.21}$$

This transition function is obtained when Φ_1 is defined as in Eq. (1.1.1.18) whereas:

$$\Phi_2(u) = (p, t)\,, p \in A \quad , \quad \Phi_2(u) = (p, -t)\,, p \in B\,. \tag{1.1.1.22}$$

The non-trivial bundle represented here is a Moebius strip in which the twists occur over the points in S^1 with $\phi = \pi$, $\phi = 2\pi$. We could of course have chosen different points or even more than two twists by an adequate definition of the transition functions. Notice that in the case of the cylinder the structure group acting on F is $G = \{e\}$, while for the Moebius strip it is the cyclic group $G = \{e, g\} \simeq \mathbb{Z}_2$: in both cases, G is a *finite* group.

Bundle reconstruction The example above suggests us that if we are given a base space M together with an open covering $\{U_i\}$, a typical fibre F and a set of transition functions $t_{ij}(p) \in G$, then we can reconstruct the entire bundle structure (E, π, M, F, G). This is indeed so, and it proceeds as follows. Define:

$$X \equiv \bigcup_i U_i \times F\,, \tag{1.1.1.23}$$

and an equivalence relation on X:

$$\{(p, f) \sim (q, f')\} \Leftrightarrow \{p = q \in U_i \cap U_j \neq \emptyset\,,\ f' = t_{ij}(p)f \text{ for some } t_{ij}(p)\}\,, \tag{1.1.1.24}$$

where $p \in U_i$, $q \in U_j$, $f \in F$. Then the fibre bundle space is *uniquely* defined as the quotient space:

$$E = X/\sim \equiv \{[(p, f)]_\sim\}\,, \tag{1.1.1.25}$$

together with a global projection $\pi : [(p, f)]_\sim \mapsto p$ and the local trivializations:

$$\Phi_i\ : \pi^{-1}(U_i) \longrightarrow U_i \times F \quad , \quad \Phi_i([(p, f)]_\sim) = (p, f)\,. \tag{1.1.1.26}$$

This procedure shows that, once we are in possession of the base manifold and of the typical fibre space, the topological structure of the bundle (e.g. the presence or absence of twists in the example above) is entirely determined by the choice of the transition functions.

1.1.2 Vector bundles

Vector bundle. A k-dimensional vector bundle is a fibre bundle (E, π, M, F, G) whose typical fibre is a k-dimensional vector space, $F \simeq \mathbb{R}^k$. If $\dim(M) = m$ then $\dim(E) = m + k$. Further, the fibre at each point $E_p \equiv \pi^{-1}(p)$ is also a k-dimensional vector space and we require the local trivialization at each point:

$$\Phi_i : E_p \longrightarrow \{p\} \times F \quad , \quad p \in U_i\,, \tag{1.1.2.1}$$

to be a linear spaces isomorphism, so that for each $U_i \subset M$, $\pi^{-1}(U_i) \simeq U_i \times \mathbb{R}^k$.

A few examples.
The tangent bundle $\mathrm{T}M$ described in the previous section is a vector bundle whose fibre at a point T_pM is isomorphic to $\{p\} \times \mathbb{R}^n$, $n = \dim(M)$.
A similar vector bundle structure is inherited by the *cotangent bundle* on a n-dimensional manifold M:

$$\mathrm{T}^*M = \bigcup_{p \in M} \mathrm{T}^*_pM \,. \tag{1.1.2.2}$$

Let $\left\{ \frac{\partial}{\partial x^j} |_p \right\}_{j=1,...,n}$ be the basis of T_pM , $p \in U_i$, induced by the coordinates $(x^1, ..., x^n)$. The dual basis on T^*M is then $\{\mathrm{d}x^j|_p\}$, $\mathrm{d}x^j \left(\frac{\partial}{\partial x^k}\right) = \delta^j_k$. As for the tangent bundle, if $\omega \in \mathrm{T}^*_pM$ then in two local charts we have:

$$\omega = \omega_i \mathrm{d}x^i = \tilde{\omega}_j \mathrm{d}y^j \Longrightarrow \tilde{\omega}_j = \omega_i \left. \frac{\partial x^i}{\partial y^j} \right|_p \equiv (t_{\beta\alpha}(p))^i_j \omega_i \,, \tag{1.1.2.3}$$

where $t_{\beta\alpha}(p) \in \mathrm{GL}(n, \mathbb{R})$ is the transition function between the trivializations:

$$\Phi_\alpha(\omega) = (x^1, ..., x^n, \omega_1, ...\omega_n) \quad , \quad \Phi_\beta(\omega) = (y^1, ..., y^n, \tilde{\omega}_1, ...\tilde{\omega}_n) \,, \tag{1.1.2.4}$$

for $p \in U_\alpha \cap U_\beta$. It is well known that the matrix acting on the components of ω is the inverse transpose of the one acting on the components of $V \in \mathrm{T}_pM$. The sections on M are in this case one-form fields: $\Gamma(M, \mathrm{T}M) = \Omega^1(M)$.
The structure of a cotangent bundle is that of a **dual bundle**: if (E, π, M, F) is a vector bundle then the former is defined as (E^*, π, M, F^*) being E^* (F^*) the set of linear maps from E (F) to $\mathbb{R}$. If $\{e_i(p)\}$ is a basis of F_p , $p \in M$, its dual basis $\{\tilde{e}^j(p)\}$ on F^*_p is simply defined by $\tilde{e}^j(p)(e_i(p)) = \delta^j_i$.

Other vector bundles which we will consider in the following are the **line bundles**, i.e. vector bundles whose characteristic fibres are one-dimensional spaces, $F \simeq \mathbb{R}$ or $F \simeq \mathbb{C}$. The cylinder and the Moebius strip considered earlier are respectively a trivial and a non-trivial real line bundle. As an example, associated with a free non-relativistic quantum particle moving in $\mathbb{R}^3$ there is the trivial complex line bundle isomorphic to $\mathbb{R}^3 \times U(1)$, of which the wave function $\psi(\mathbf{x})$ is just a section. When we are instead considering the motion in the field of a magnetic monopole then the complex line bundle is non trivial and the base space becomes $\mathbb{R}^3 \setminus 0$, which is homotopic to S^2 [6].

Local frames.
The fact that the fibre at each point is a vector space allows us to build reference frames over vector bundles. Suppose that, $\forall p \in U_i$, $\{\tilde{e}_1(p), ..., \tilde{e}_k(p)\}$ is a basis for E_p. Then the sections:

$$\sigma_\mu : U_i \longrightarrow \sigma_\mu(U_i) \subset \pi^{-1}(U_i) \quad , \quad \sigma_\mu(p) = \tilde{e}_\mu(p) \,, \, \mu = 1, ..., k \,, \tag{1.1.2.5}$$

are said to define a *local frame* over U_i. Once we have a local frame, an associated local trivialization is naturally defined. Indeed, let $V \equiv V^\mu \tilde{e}_\mu(p) = V^\mu \sigma_\mu(p) \in E_p$; the associated local trivialization over U_i is:

$$\Phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k \quad , \quad \Phi_i(V^\mu \tilde{e}_\mu(p)) = (p, \{V^1, ..., V^k\}) \,. \tag{1.1.2.6}$$

Conversely, if we consider the canonical basis of $\mathbb{R}^k$, $e_\mu = (0, ..., 1, ..., 0)$ then by the above definition $\Phi_i^{-1}(p, e_\mu) = \tilde{e}_\mu(p)$ so that we are led to define the (local) **trivial section** induced by Φ_i as:

$$\sigma_\mu : p \longmapsto \Phi_i^{-1}(p, e_\mu) \,. \tag{1.1.2.7}$$

Therefore we can always build a local trivialization on a vector bundle when a local frame is given and vice versa.
Let now $\{\tilde{e}_1(p), ..., \tilde{e}_k(p)\}$ and $\{\tilde{e}'_1(p), ..., \tilde{e}'_k(p)\}$ be two local frames on the fibre over $p \in U_i \cap U_j$. Then a local *change of frame* is given by:

$$\tilde{e}_\alpha(p) = G^\beta_\alpha(p)\, \tilde{e}'_\beta(p) \,,\, G(p) \in \mathrm{GL}(k, \mathbb{R}) \,. \tag{1.1.2.8}$$

Thus, $\forall\, V \in E_p$:

$$V = V^\alpha \tilde{e}_\alpha(p) = V'^\beta \tilde{e}'_\beta(p) \Longrightarrow V^\alpha = G^\alpha_\beta(p)^{-1} V'^\beta \,, \tag{1.1.2.9}$$

where $G^\beta_\alpha(p)\, G^\alpha_\gamma(p)^{-1} = \delta^\beta_\gamma$. The transition functions performing a local change of frames are thus matrices with non vanishing determinant:

$$t_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}(k, \mathbb{R}) \quad , \quad t_{ij}(p) = \{G^\alpha_\beta(p)\} \quad \text{s.t.} \tag{1.1.2.10}$$

$$\Phi_i(\boldsymbol{V}) = (p, \{V^\alpha\}) = (p, \{G^\alpha_\beta(p)^{-1} V'^\beta\}) \quad , \quad \Phi_j(\boldsymbol{V}) = (p, \{V'^\beta\}) \,. \tag{1.1.2.11}$$

This result of course generalizes what we have seen in the case of the tangent and cotangent bundle, where the transition functions realize admissible transformations between physical reference frames.

Remark. Let σ and σ' be sections of a vector bundle $E \xrightarrow{\pi} M$ defined over $U \subset M$. Then $\Gamma(U, E)$ acquires a linear structure if one defines sections sum and multiplication by a scalar:

$$(\sigma + \sigma')(p) \equiv \sigma(p) + \sigma'(p) \quad , \quad (\phi\sigma)(p) \equiv \phi(p)\sigma(p) \,, \tag{1.1.2.12}$$

with $p \in U$, $\phi : U \to \mathbb{R}$. Therefore it is possible to define $\forall p \in M$ a **null section** σ_0 such that:

$$\Phi_i(\sigma_0(p)) = (p, 0) \,, \tag{1.1.2.13}$$

being 0 the null vector of $F \simeq \mathbb{R}^k$: since the latter is unchanged under the action of the transition functions $t_{ij}(p) \in \mathrm{GL}(k, \mathbb{R})$ the null section is independent of any particular choice of the local trivializations. Moreover, being defined at every point of M, σ_0 is also a *global section.*

1.1.3 Principal bundles

Principal bundle. A principal bundle (P, π, M, G) is a bundle whose fibre F is identical[2] to the structure group G. The left action Eq. (1.1.1.5) is then just the usual group multiplication. In addition, there is a *right action* of G on P:

$$R_g : P \times G \longrightarrow P \quad , \quad (u, g) \longmapsto R_g(u) \equiv ug \,, \tag{1.1.3.1}$$

which is *fibre-preserving*. This means that If $u \in \pi^{-1}(p)$, $p \in M$, $g \in G$ then also $ug \in \pi^{-1}(p)$. Equivalently:

$$\pi(ug) = \pi(u) \quad \forall u \in P \,,\, g \in G \,, \tag{1.1.3.2}$$

so that two points which are related by the right action of G stay in the same fibre. Further, consider a local trivialization:

$$\Phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times G \quad , \quad u \longmapsto (\pi(u), \phi(u)) \equiv (p, h_i) \,. \tag{1.1.3.3}$$

The right action on $\pi^{-1}(U_i)$ is defined by:

$$\Phi_i(ug) = (\pi(ug), \phi(ug)) \equiv (\pi(u), \phi(u)g) = (p, h_i g) \tag{1.1.3.4}$$

$$\Longrightarrow \Phi_i^{-1}(p, h_i g) = ug = \Phi_i^{-1}(p, h_i)g \quad \forall u \in \pi^{-1}(U_i) \,. \tag{1.1.3.5}$$

The definition is well-posed, since it is independent of the chosen trivialization; Indeed, if $p \in U_i \cap U_j$:

$$ug = \Phi_i^{-1}(p, h_i g) = \Phi_i^{-1}(p, t_{ji}(p) h_j g) = \Phi_j^{-1}(p, h_j g) \,. \tag{1.1.3.6}$$

Moreover, the action is *transitive* and *free*. This comes from the very definition of the action and from the properties of the right group multiplication $G \times G \to G$ [3]. Namely,

1. G acts transitively on itself and $\pi^{-1}(p) \simeq G$ so the property is inherited:

$$\forall u_1 \,, u_2 \in \pi^{-1}(p) \; \exists! \, g \in G \text{ s.t. } u_1 = u_2 g \,. \tag{1.1.3.7}$$

   In this way we can reconstruct a whole fibre through the action of G:

$$\pi(u) = p \Longrightarrow \pi^{-1}(p) = \{ug \,,\, g \in G\} \,. \tag{1.1.3.8}$$

2. Suppose that $ug = u$ for some $u \in P$. Then in a local trivialization we have:

$$ug = \Phi_i^{-1}(p, h_i g) = \Phi_i^{-1}(p, h_i)g = \Phi_i^{-1}(p, h_i) = u \,, \tag{1.1.3.9}$$

   so that the action is actually free:

$$ug = u \quad , \quad u \in P \Longrightarrow g = e \in G \,. \tag{1.1.3.10}$$

[2] By which we mean globally diffeomorphic.

Canonical local trivialization.
Suppose that we have a local section on a principal bundle (P, π, M, G) , $\sigma_i : p \longmapsto \sigma_i(p) \in \pi^{-1}(p)$, $p \in U_i \subset M$. Since the G-action is free and transitive, for each $u \in \pi^{-1}(p)$ there is a unique $g_u \in G$ such that $u = \sigma_i(p)g_u$. Consequently, we can define a local trivialization as follows:

$$\Phi_i : u \longmapsto (p, g_u) \quad \text{so that} \quad \sigma_i(p) = \Phi_i^{-1}(p, e)\,. \tag{1.1.3.11}$$

This is called the **canonical local trivialization** on U_i. Thus,

$$\Phi_i^{-1}(p, g) = \Phi_i^{-1}(p, e)g = \sigma_i(p)g \quad \forall p \in U_i\,, g \in G\,. \tag{1.1.3.12}$$

Having at our disposal the canonical trivializations we can show how the transition functions relate different local sections on the same fibre. Let Φ_i , Φ_j be the canonical trivializations corresponding to the sections σ_i , σ_j , and $p \in U_i \cap U_j \neq \emptyset$. Then:

$$\sigma_i(p) = \Phi_i^{-1}(p, e) = \Phi_j^{-1}(p, t_{ji}(p)e) = \Phi_j^{-1}(p, t_{ji}(p)) \tag{1.1.3.13}$$

$$= \Phi_j^{-1}(p, e)t_{ji}(p) = \sigma_j(p)t_{ji}(p)\,. \tag{1.1.3.14}$$

In the physical applications, as we shall see, through local sections one builds local *gauge potentials* over the base manifold, and the transition functions represent admissible *gauge transformations* between them.

Associated bundles.
It is always possible to obtain a vector bundle from a principal one. Suppose (P, π, M, G) is a principal bundle and F is a differential manifold such that, togheter with the right action on P, there is a left G-action on F:

$$(u, f) \longmapsto (ug, g^{-1}f) \quad , \quad \forall (u, f) \in P \times F,\, g \in G\,. \tag{1.1.3.15}$$

The associated fibre bundle is then (P_F, π, M, F), where $P_F \equiv (P \times F)/\sim$ is the quotient space defined with respect to the equivalence relation:

$$\{(u, f) \sim (w, h)\} \Leftrightarrow \{g \in G | w = ug\,,\, h = g^{-1}f\}\,. \tag{1.1.3.16}$$

Let us consider the more specific case in which $F \simeq V$ is a k-dimensional vector space. Then if $\rho : G \longrightarrow \mathrm{GL}(k, \mathbb{R})$ denotes the k-dimensional representation of G [7], the total vector space $P_V \equiv (P \times V)/\sim$ is obtained as above by identifying the points (u, v) and $(ug, \rho(g)^{-1}v)$.
Therefore we have a fibre bundle $(P_V, \tilde{\pi}, M, V)$, where the canonical projection is defined as follows:

$$\tilde{\pi} : P_V \longrightarrow M \quad , \quad \tilde{\pi}(u, v) \equiv \pi(u)\,, \tag{1.1.3.17}$$

being $\pi : P \longrightarrow M$ the projection on the principal bundle. The definition is well-posed, meaning that $\tilde{\pi}$ is independent of a particular choice in the equivalence class. Indeed, if $u \in P$, $v \in V$, $g \in G$:

$$\pi(u) = \pi(ug) \Longrightarrow \tilde{\pi}(ug, \rho(g)^{-1}v) = \pi(ug) = \tilde{\pi}(u, v)\,. \tag{1.1.3.18}$$

Further, the local trivializations are naturally defined by:

$$\Phi_i : \tilde{\pi}^{-1}(U_i) \longrightarrow U_i \times V \quad , \quad [(u, v)]_\sim \longmapsto (u, v)\,. \tag{1.1.3.19}$$

It is also possible to build up a principal bundle once a vector bundle is given. In fact, as we have shown, in a vector bundle the transition functions t_{ij} are elements of $\mathrm{GL}(n, \mathbb{R})$, so that the latter can be regarded as the structure group and by adopting the same transition functions the principal bundle is build following the reconstruction procedure exposed in Section 1.1.1.

Remark. The previous construction clarifies the geometry (and the physics) underlying a principal bundle structure (P, π, M, G): the base manifold of the latter is always diffeomorphic to the *space of orbits* P/G, that is, the space obtained by the identification of points of P lying on the same fibre, related by the action of G:

$$P/G = \{[u]_\sim\} \quad , \quad [u]_\sim \equiv \{w \in P \text{ s.t. } w = ug\,, g \in G\}\,. \tag{1.1.3.20}$$

In the following Chapter we will see how does this identification physically lead to the addiction of redundant gauge degrees of freedom.

1.1.4 The Hopf Map

As an important example of a principal bundle we will now consider in detail the Hopf fibration of the three-sphere and its generalizations.
In 1931 H. Hopf showed that the hypersphere S^3 can be mapped onto the ordinary sphere S^2 in such a way that each distinct circle of S^3 corresponds to a point in S^2 [9]. Stated in other terms, since $S^1 \simeq U(1)$, there is a principal bundle structure $(S^3, \pi, S^2, U(1))$, often denoted by $S^1 \to S^3 \to S^2$. A graphical representation of the fibration is given in Figure 1.1. The bundle is obviously non trivial since[3] $S^3 \neq S^2 \times U(1)$, and the fibration proceeds as follows.
Let us consider an embedding of the three-sphere in $\mathbb{R}^4$ and of the two-sphere in $\mathbb{R}^3$:

$$S^3 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 | (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\}\,, \tag{1.1.4.1}$$
$$S^2 = \{(y^1, y^2, y^3) \in \mathbb{R}^3 | (y^1)^2 + (y^2)^2 + (y^3)^2 = 1\}\,. \tag{1.1.4.2}$$

[3]This can be inferred by a connectedness argument. In fact, $S^3 = S^2 \times U(1)$ would imply $\pi_1 S^3 = \pi_1(S^2 \times U(1)) \simeq \pi_1(S^2) \oplus \pi_1(U(1))$, where we have used a property of the fundamental group of product spaces [6]: $\pi_1(\mathbb{X} \times \mathbb{Y}) \simeq \pi_1(\mathbb{X}) \times \pi_1(\mathbb{Y}) \simeq \pi_1(\mathbb{X}) \oplus \pi_1(\mathbb{Y})$. However, the left-hand side $\pi_1(S^2)$ is the trivial group, while the right-hand side $\pi_1(U(1))$is isomorphic to $\mathbb{Z}$, so that we have an absurd.

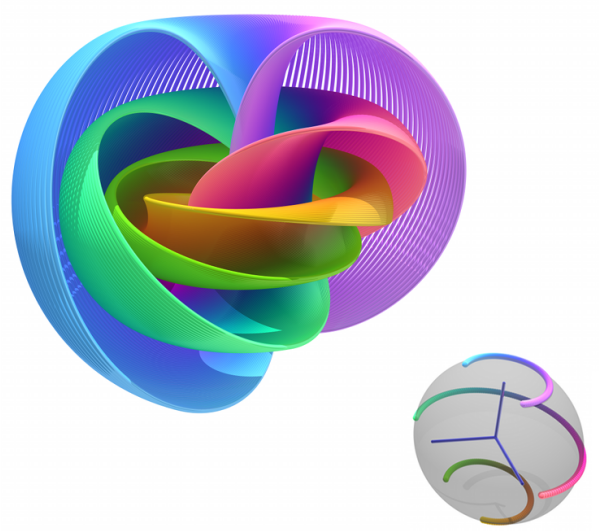

Figure 1.1: Hopf fibration of the three sphere. On the left, a pictorial representation of S^3, whose different fibres are in different colors: each fibre corresponds to a point of the same color on S^2, on the right. The image is taken from [8]

The Hopf fibration is defined by the canonical projection

$$\pi : S^3 \to S^2 \quad , \quad (x^1, x^2, x^3, x^4) \mapsto (y^1, y^2, y^3)\,, \tag{1.1.4.3}$$

with:

$$\begin{cases} y^1 = & 2(x^1x^3 + x^2x^4) \\ y^2 = & 2(x^2x^3 - x^1x^4) \\ y^3 = & (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 \end{cases} \tag{1.1.4.4}$$

and it is easily checked that $(y^1)^2 + (y^2)^2 + (y^3)^2 = 1$. We have now to define the local trivializations and the corresponding transition functions; a useful way to do that in view of further generalizations is by a shift to complex variables. Define $z^0 \equiv x^1 + ix^2$, $z^1 \equiv x^3 + ix^4$ so that the embedding condition for S^3 becomes:

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1 \Longrightarrow \left|z^0\right|^2 + \left|z^1\right|^2 = 1\,. \tag{1.1.4.5}$$

Let also $S_N\,, S_U \subset S^2$ be two open charts covering the northern and the southern hemisphere respectively, so that $S_N \cap S_S$ is an (arbitrarily) thin strip containing the equator. Let then (X, Y) and (U, V) be the stereographic coordinates of a point in the *southern* hemisphere projected from the *North* Pole and of a point in the *northern*

hemisphere projected from the *South* Pole respectively, i.e.:[4]

$$(X,Y)=\left(\frac{y^1}{1-y^3},\frac{y^2}{1-y^3}\right),\boldsymbol{y}\in S_S\quad,\quad(U,V)=\left(\frac{y^1}{1+y^3},-\frac{y^2}{1+y^3}\right),\boldsymbol{y}\in S_N\,. \tag{1.1.4.7}$$

Then if we define $Z=X+iY$ and $W=U+iV$ we easily find:

$$Z=\frac{y^1+iy^2}{1-y^3}=\frac{x^1+ix^2}{x^3+ix^4}=\frac{z^0}{z^1}\quad,\boldsymbol{y}\in S_S\,, \tag{1.1.4.8}$$

$$W=\frac{y^1-iy^2}{1+y^3}=\frac{x^3+ix^4}{x^1+ix^2}=\frac{z^1}{z^0}\quad,\boldsymbol{y}\in S_N\,. \tag{1.1.4.9}$$

$$\tag{1.1.4.10}$$

In this way the coordinates Z, W are well defined on all U_S, U_N respectively and are contained within the unitary circle in the complex plane so that the latter is put in a bijective map with S^2. Therefore, using the complex coordinates z^0, z^1, Z, W we can define the local trivializations:

$$\Phi_S:\,S^3\supset\pi^{-1}(S_S)\mapsto S_S\times U(1)\quad,\quad\Phi_S(z^0,z^1)\equiv\left(Z,\frac{z^1}{|z^1|}\right), \tag{1.1.4.11}$$

$$\Phi_N:\,S^3\supset\pi^{-1}(S_S)\mapsto S_S\times U(1)\quad,\quad\Phi_N(z^0,z^1)\equiv\left(W,\frac{z^0}{|z^0|}\right). \tag{1.1.4.12}$$

Notice that $z^0\neq0$ on S_N and $z^1\neq0$ on S_S so the maps are well defined. Since on the equator $S_{eq}\subset S_S\cap S_N$, $y^3=0$ we have $|z^0|=|z^1|=1/\sqrt{2}$ therein. Therefore:

$$\Phi_S|_{S_{eq}}(z^0,z^1)\equiv(Z,\sqrt{2}z^1)\quad,\quad\Phi_N|_{S_{eq}}(z^0,z^1)\equiv(W,\sqrt{2}z^0)\,. \tag{1.1.4.13}$$

Then on the equator the change of coordinates is performed by an $U(1)$-valued transition function:

$$t_{NS}:\,S_{eq}\simeq S^1\longrightarrow U(1)\,, \tag{1.1.4.14}$$

$$t_{NS}(\boldsymbol{y})\equiv\frac{z^0}{z^1}=y^1+iy^2=\exp\{i\alpha\}\quad\text{for some }\alpha\in\mathbb{R}\,. \tag{1.1.4.15}$$

[4]The stereographic coordinates on $S^n\equiv\{(x^1,...,x^{n+1})\in\mathbb{R}^{n+1}|\sum_{i=1}^{n+1}(x^i)^2=1\}$ are obtained by projecting the points of the sphere on an equatorial plane from an arbitrary pole; e.g. if we choose the North Pole:

$$(x^1,...,x^{n+1})\mapsto\left(\frac{x^1}{1-x^{n+1}},...,\frac{x^n}{1-x^{n+1}}\right), \tag{1.1.4.6}$$

and the map only fails at the projection pole. If we take the South Pole to be the latter we only have to change $x^{n+1}\mapsto-x^{n+1}$ in the denominators. Notice that with the stereographic coordinates over S^2 it is customary to introduce a change of sign when the projection pole is $(0,0,-1)$, in order to maintain the right-handedness of the axis in the projective plane.

This transition function has to be single-valued all along the equator so that $\alpha = 2\pi n$, $n \in \mathbb{Z}$. Thus, through the transition function t_{NS}, the Hopf fibration of the-three sphere represents a homotopy class [6] of the first $U(1)$ fundamental group, $\pi_1(U(1)) \simeq \mathbb{Z}$. This is a feature shared by all the $U(1)$ principal bundles that will show up in the description of Dirac's magnetic monopole.

The above construction can be generalized to other spaces in the following way. First consider the natural isomorphisms (indeed, diffeomorphisms) between the real three-sphere and the complex one-sphere:

$$S^3 \simeq S^1_{\mathbb{C}} \equiv \{(z^0, z^1) \in \mathbb{C}^2 | \left|z^0\right|^2 + \left|z^1\right|^2 = 1\}\,, \tag{1.1.4.16}$$

and between the real two-sphere and the complex projective plane:

$$S^2 \simeq \mathbb{C}P^1 = \{[(z^0, z^1)]\} \quad , \quad [(z^0, z^1)] \equiv \{\lambda(z^0, z^1) | \lambda \in \mathbb{C} \setminus 0\}\,. \tag{1.1.4.17}$$

The Hopf map Eq. (1.1.4.3) reads then:

$$\pi : S^1_{\mathbb{C}} \longrightarrow \mathbb{C}P^1 \quad , \quad (z^0, z^1) \mapsto [(z^0, z^1)]\,. \tag{1.1.4.18}$$

The fact that this is actually the map defined above can be inferred by observing that if (z^0, z^1) is in $S^1_{\mathbb{C}}$ then this is true also for $\lambda(z^0, z^1)$, $|\lambda| = 1$, and all such points are mapped to a single one in $\mathbb{C}P^1$ so there is still an $U(1)$ "gauge freedom".
Then if we consider the *quaternion algebra*:

$$\mathbb{H} \equiv \{q = t + \boldsymbol{i}x + \boldsymbol{i}y + \boldsymbol{i}z + \boldsymbol{i}z | (t, x, y, z) \in \mathbb{R}^4\}\,, \tag{1.1.4.19}$$

with:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1 \tag{1.1.4.20}$$

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}; \tag{1.1.4.21}$$

the following isomorphisms immediately follow:

$$S^3 \simeq \{q \in \mathbb{H} | \, |q|^2 = 1\}, \quad S^7 \simeq S^1_{\mathbb{H}} \equiv \{(q^0, q^1) \in \mathbb{H}^2 | \, \left|q^0\right|^2 + \left|q^1\right|^2 = 1\}\,. \tag{1.1.4.22}$$

The Hopf map still has the same form:

$$\pi : S^1_{\mathbb{H}} \longrightarrow \mathbb{H}P^1 \quad , \quad (q^0, q^1) \mapsto [(q^0, q^1)]\,, \tag{1.1.4.23}$$

where:

$$\mathbb{H}P^1 = \{[(q^0, q^1)]\} \quad , \quad [(q^0, q^1)] \equiv \{\lambda(q^0, q^1) | \lambda \in \mathbb{H} \setminus 0\} \simeq S^4\,. \tag{1.1.4.24}$$

Now the points (q^0, q^1) and $\eta(q^0, q^1)$, $|\eta| = 1$ are both in S^7 and get projected down to a single point, so that the fibration is $S^3 \simeq SU(2) \to S^7 \to S^4$. Curiously, this generalization still has a physical meaning, being used in the description of the so-called "instantons", particular solutions of the Yang-Mills free equations [3]. Using the *octonions* it is still possible to define an Hopf map $S^7 \to S^{15} \to S^8$, even though strictly speaking this is not a principal bundle since S^7 is not a group (multiplication rule is not associative). The peculiar feature of the Hopf Map in all those mentioned case is however that all three of the bundle space, base manifold and typical fibre are hyperspheres.

1.1.5 Triviality of bundles

We shall conclude the Section by illustrating some results on triviality of bundles. Let us first give a few definitions.

Pull-back bundle. Let $E \xrightarrow{\pi} M$ be a fibre bundle with typical fibre F, N a differentiable manifold and $f : N \to M$. the pullback space of E by the function f is:

$$f^*E \equiv \{(p, u) \in N \times E | f(p) = \pi(u)\} . \tag{1.1.5.1}$$

Then if we define the maps $\tilde{f} : f^*E \to E$ and $\tilde{\pi} : f^*E \to N$ by $\tilde{f}(p, u) = u$, $\tilde{\pi}(p, u) = p$ the following holds by construction:

$$\pi \circ \tilde{f} = f \circ \tilde{\pi} , \tag{1.1.5.2}$$

so that $f^*E \xrightarrow{\tilde{\pi}} N$ inherits the structure of a pullback bundle. If $N \equiv M$ and $f = \mathrm{id}_M$ the two bundles are obviously equivalent.

Homotopic maps. Let M and M' be differentiable manifolds and $f , g : M' \longrightarrow M$. f and g are homotopic maps if there exists a smooth map $F : M' \times [0, 1] \longrightarrow M$ s.t.:

$$F(p, 0) = f(p) , F(p, 1) = g(p) \quad \forall p \in M' . \tag{1.1.5.3}$$

We need now a technical lemma, the proof of which can be found in [4]:

Lemma. Let $E \xrightarrow{\pi} M$ be a fibre bundle whose fibre is F, and $f , g : N \longrightarrow M$ be homotopic maps. Then the pullback bundles over N, f^*E and g^*E, are equivalent.

A first result on triviality of a generic fibre bundle follows from the previous lemma. Consider a manifold M which is contractible to a point, i.e. there exists a homotopy $F : M \times [0, 1] \longrightarrow M$ such that:

$$\forall p \in M \quad F(p, 0) = p \quad , \quad F(p, 1) = p_0 , \tag{1.1.5.4}$$

being $p_0 \in M$ fixed. Define a function $h_t : M \ni p \mapsto F(p, t) \in M$, $t \in [0, 1]$, so that $h_0(p) \equiv \mathrm{id}_M$ and $h_1(p) = p_0 \, \forall p \in M$. Consider now the two pullback bundles (over M) h_0^*E and h_1^*E. Since h_0 is the identity map, $h_0^*E \equiv E$, while h_1^*E is the pullback of the trivial bundle $\{p_0\} \times F$ and hence it is itself trivial: $h_1^*E \simeq M \times F$. But from the previous lemma h_0^*E and h_1^*E are equivalent bundle, so that E is a trivial bundle. In synthesis, we have obtained the following:

Theorem. Let $E \to M$ be a fibre bundle. If M is contractible to a point, then the bundle is trivial.

The result is rather strong: it means that, *whichever choice we make of the transition functions*, if the topological structure of M is "good enough", we can only build trivial bundles over it. Note however that the converse is in general not true.
We shall now give a necessary and sufficient condition under which a *principal* bundle is trivial.

Theorem. A principal bundle is trivial if and only if it admits a global section.

Proof. The sketch of the proof strictly follows from the natural association of a trivialization to each section and vice versa. Consider a fibre bundle (P, π, M, G) and a global section $\sigma \in \Gamma(M, P)$, $\sigma : p \mapsto \sigma(p) \in \pi^{-1}(p)$. Let $g \in G$; by the definition of principal bundle if $\sigma(p) \in \pi^{-1}(p)$ then also $\sigma(p)g \in \pi^{-1}(p)$, and since the action is transitive and free any $u \in P$ is uniquely written as $u = \sigma(p)g_u$, $g_u \in G$. Moreover, the section is global, so that when p varies in M we can uniquely obtain each point in P. Therefore there is a global trivialization:

$$\Phi : P \longrightarrow M \times G \quad , \quad \sigma(p)g \mapsto (p, g)\,, \tag{1.1.5.5}$$

which is (at least) a global homeomorphism, so that $P \simeq M \times G$ and hence trivial.
Conversely, if $P \simeq M \times G$ and $\Phi : P \longrightarrow M \times G$ is a global trivialization, then for each $g \in G$ there is a global section:

$$\sigma_g : M \longrightarrow P \quad , \quad \sigma_g(p) \equiv \Phi^{-1}(p, g)\,, \tag{1.1.5.6}$$

and this concludes the proof. □

Remark. As we previously remarked, every vector bundle admits a global null section and therefore the existence of a global section does not ensure the triviality of the bundle in this case. However, we can always build a principal bundle from a vector one, and they will have the same transition functions. Consequently, we easily find that a vector bundle is trivial if and only if its associated principal bundle is trivial.

1.2 Connections and curvature

A connection on a manifold is a *rule* which specifies how to compare vectors in tangent spaces at different points, that is, it defines a *parallel transport* on that space; by inspecting how a vector is parallel transported one can tell whether the manifold is flat or there is a non vanishing *curvature*. If the manifold is endowed with a metric tensor field, i.e. there is a notion of length of curves and angles between vectors, a connection can be uniquely derived from a compatibility condition between the metric and the parallel transport rule: this gives rise to the covariant calculus on Riemannian manifold, a fundamental geometric tool for the development of general relativity. The aim of this section is to introduce the covariant calculus on principal bundles. We will follow the line of [3], [4] and [1].

1.2.1 Connections and connection one-forms on principal bundles

Horizontal and vertical subspaces
Let (P, π, M, G) be a principal bundle and $G_p \equiv \pi^{-1}(p)$, $p \in M$. The **vertical subspace** $V_u P$ at $u \in G_p$ is a subset of $\mathrm{T}_u P$ consisting of all the vectors tangent to the fibre at that point, i.e.

$$V_u P \equiv \left\{ A_p^{\#} \text{ s.t. } A_p^{\#}(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f(u \exp\{tA\}))\Big|_{t=0},\ A \in \mathfrak{g} \right\} , \tag{1.2.1.1}$$

where $f \in \mathcal{C}^\infty(P, \mathbb{R})$, t is real and $\mathfrak{g}$ is the Lie Algebra of the (connected) structure group G, so that each element $g \in G$ can be expressed as $g = \exp\{tA\}$ for some $A \in \mathfrak{g}$. To see that the definition is well-posed observe that since P is a principal bundle if $\pi(u) = p$ then also $\pi(ug) = p \, \forall g \in G$ so that the integral curve of $A \in \mathfrak{g}$ starting at u:

$$\phi_p^A : \mathbb{R} \longrightarrow P \quad , \quad \phi_p^A(t) \equiv u \exp\{tA\} = ug \, , \tag{1.2.1.2}$$

lies entirely in G_p. The vector field $A^{\#} : P \to V_u P$ is then naturally defined and it is sometimes referred to as the *fundamental vector field* generated by A. From the properties of the exponential function it can be easily shown [3] that the map $\# : A \mapsto A_u^{\#}$ establishes a natural algebras homomorphism between $\mathfrak{g}$ and $V_u P$, that is, the two linear spaces are isomorphic and the Lie parenthesis are preserved:

$$[A, B]_u^{\#} = [A_u^{\#}, B_u^{\#}] \quad \forall A \, , B \, \in \mathfrak{g} \, . \tag{1.2.1.3}$$

Notice also that if we consider the pushforward [1] of the canonical projection map:

$$\pi_* : \mathrm{T}_u P \longrightarrow \mathrm{T}_{\pi(u)} M \quad , \quad (\pi_* V_u)(f) \equiv V_u(f \circ \pi) \, , \tag{1.2.1.4}$$

for all the smooth functions $f : M \to \mathbb{R}$, then the vertical subspace $V_u P$ might be equivalently defined as the kernel of π_*, since $\forall \, A^{\#} \in V_u P$ we have:

$$(\pi_* A^{\#})(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f(\pi(u \exp\{tA\})))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(f(\pi(u)))\Big|_{t=0} = 0 \, . \tag{1.2.1.5}$$

The **horizontal subspace** $H_u P$ at $u \in P$ is a complement set of $V_u P$ in $\mathrm{T}_u P$; the former is in general not uniquely specified unless a *connection* is given, i.e. a rule which allows us to write at each point $p \in P$ the tangent space $\mathrm{T}_u P$ as a direct sum of the vertical and horizontal subspaces. Formally:

Connection. Let (P, π, M, G) be a principal bundle. A connection on it is a choice of horizontal spaces $\{H_u P\}_{u \in P} \subset \mathrm{T}_u P$ such that:

(i) $\forall\, u \in P \quad \mathrm{T}_u P = V_u P \oplus H_u P\,.$

(ii) Each smooth vector field $X \in \Gamma(P, \mathrm{T}P)$ is separated into a horizontal and a vertical component:

$$X = X^h + X^v \quad , \quad X^v \in \Gamma(P, \mathrm{V}_u P) \quad , \quad X^v \in \Gamma(P, \mathrm{H}_u P)\,. \tag{1.2.1.6}$$

(iii) For any $u \in P$, $g \in G$:

$$R_{g*}(\mathrm{H}_u P) = \mathrm{H}_{ug} P\,, \tag{1.2.1.7}$$

where $R_{g*} : \mathrm{T}_u P \to \mathrm{T}_{ug} P$ is the pushforward of the right action, $(R_{g*} V_u)(f) \equiv V_u(f \circ R_g)\, \forall f \in \mathcal{C}^\infty(P, \mathbb{R})$.

Condition iii) in the definition above means that the right action on the fibre G_p induces a map between horizontal spaces at different points in the same fibre, so that all the spaces $\{\mathrm{H}_u P\}$, $u \in \pi^{-1}(p)$, are defined once one of them is given. As we shall see, this also ensures that if a point $u \in \pi^{-1}(p)$ is *parallel transported* so are the points ug, $g \in G$.
This definition of a connection does not give us a hint on how to practically associate a horizontal subspace to each point. The latter is yielded by a **connection one-form** on the bundle:

Connection one-form. A connection one-form on a principal bundle (P, π, M, G) (also known as an **Ehresmann connection**) is a Lie-algebra-valued one form[5] $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ such that:

(i)

$$\forall\, A \in \mathfrak{g}\,, \forall\, u \in P \quad \omega(A^\#_u) \equiv A\,, \tag{1.2.1.10}$$

being $A^\#$ the fundamental vector field generated by A, $A^\#(u) \equiv A^\#_u \in \mathrm{V}_u P \simeq \mathfrak{g}$.

(ii)

$$\forall\, g \in G \quad , \quad R^*_g \omega \equiv \mathrm{Ad}_{g^{-1}} \omega\,, \tag{1.2.1.11}$$

where $\mathrm{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ is an adjoint map on the algebra [1] and $R^*_g \omega$ is the pullback of ω induced by the right action.

[5] This is an example of a *vector-valued one form*. A general V-valued k-form on a manifold P is a multilinear map :

$$\phi : \mathrm{T}P \wedge ... \wedge \mathrm{T}P \longrightarrow V\,, \tag{1.2.1.8}$$

being V a vector space. The set of all such forms is then denoted as $\Omega^k(P) \otimes V$, and if $\{e_i\}_{i=1,...,n}$ is a basis of V, $\dim(V) = n$ we can write the most general form $\phi \in \Omega^k(P) \otimes V$ as:

$$\phi = \sum_{i=1}^{i=n} \phi^i \otimes e_i \quad , \quad \phi^i \in \Omega^k(P)\,. \tag{1.2.1.9}$$

If we consider a finite dimensional representation of the structure group, so that we can identify $G \simeq \mathrm{GL}(n,\mathbb{R})$, then equality ii) assumes a more manageable form:

$$(R_g^*\omega_{ug})(X_u) \equiv \omega_{ug}(R_{g*}X_u) = g^{-1}\omega_u(X_u)g\,, \tag{1.2.1.12}$$

with $X_u \in \mathrm{T}_uP$, $\omega_u \equiv \omega(u) \in \mathrm{T}_u^*P$. A connection one-form ω_u is just a projection of T_uP into the vertical component $\mathrm{V}_uP \simeq \mathfrak{g}$, and the horizontal subspace H_uP at a point $u \in P$ can therefore be defined as the the kernel of the former:

$$\mathrm{H}_uP \equiv \{X_u \in \mathrm{T}_uP | \omega_u(X_u) = 0\}\,. \tag{1.2.1.13}$$

It is now quite straightforward to verify that this definition is consistent with the one of a connection given above. Indeed, since by construction $\forall\, u \in P\ \mathrm{Im}(\omega_u) = \mathrm{V}_uP$ and by definition $\mathrm{Ker}(\omega_u) = \mathrm{H}_uP$ it immediately follows that $\mathrm{T}_uP = \mathrm{H}_uP \oplus \mathrm{V}_uP$. Moreover, take $X_u \in \mathrm{H}_uP$ and consider its pushforward induced by R_g, $R_{g*}X_u \in \mathrm{T}_{ug}P$. Then, being $\omega_u(X_u) = 0$:

$$\omega_{ug}(R_{g*}X_u) \equiv (R_g^*\omega_{ug})(X_u) = g^{-1}\omega_u(X_u)g = 0 \Longrightarrow R_{g*}X_u \in \mathrm{H}_{ug}P\,. \tag{1.2.1.14}$$

But R_{g*} is an invertible map so that each $Y \in \mathrm{H}_{ug}P$ is written as $Y = R_{g*}X_u$ for some $X_u \in \mathrm{H}_uP$; this eventually proves that $R_{g*}(\mathrm{H}_uP) = H_{ug}P$. A connection and a connection one-form are then completely equivalent, but the approach we have adopted has the advantage to distinguish between the (purely geometric) separation of T_uP in its two subspaces and the practical way to obtain such a separation.

The local connection form. Let us observe that a connection one-form is defined on the entire bundle P. Therefore, if $\{U_i\}$ is an open covering of M and $\{\sigma_i\}$, $\sigma_i : U_i \longrightarrow \pi^{-1}(U_i)$ are the corresponding local section, it is possible to define the **local connection** one forms $\mathcal{A}_i$ as the pullbacks of ω induced by the sections:

$$\mathcal{A}_i \equiv \sigma_i^*(\omega) \in \Omega^1(U_i) \otimes \mathfrak{g}\,, \tag{1.2.1.15}$$

so that by definition:

$$\forall\, p \in U_i\,,\ \forall\, X_p \in \mathrm{T}_pU_i \quad \mathcal{A}_i(X_p) \equiv \omega_{\sigma_i(p)}(\sigma_{i*}X_p)\,. \tag{1.2.1.16}$$

However, of far more importance is the converse result, that is, the fact that is always possible to reconstruct an Ehresmann connection once that a set of Lie-algebra-valued one-forms $\mathcal{A}_i \in \Omega^1(U_i) \otimes \mathfrak{g}$ is given; further, the connection can be made *unique* if some consistency conditions are fulfilled.

Theorem. Let (P,π,M,G) be a principal bundle, $U_i \subset M$, $\sigma_i : U_i \longrightarrow \pi^{-1}(U_i)$, $\mathcal{A}_i \in \Omega^1(U_i) \otimes \mathfrak{g}$. Then there exists a connection one-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ such that $\mathcal{A}_i = \sigma_i^*(\omega)$.

Proof. Let $p \in U_i$ and $g_i \in G$ be the only element[6] such that, for any $u \in \pi^{-1}(U_i)$, $u = \sigma_i(p)g_i$ if $\Phi_i(u) = (p, g_i)$, being Φ_i the canonical local trivialization on $\pi^{-1}(U_i)$. Then let $\omega_i \in \Omega^1(\pi^{-1}(U_i)) \otimes \mathfrak{g}$ be defined by:

$$\omega_i \equiv g_i^{-1}\pi^*\mathcal{A}_i\, g_i + g_i^{-1}\mathrm{d}g_i\,, \tag{1.2.1.17}$$

where d denotes the exterior derivative[7] on P. Now let $X \in \mathrm{T}_pM$, so that we have $\sigma_{i*}X \in \mathrm{T}_{\sigma_i(p)}$ and $g_i = e$ at $u = \sigma_i(p)$. Therefore:

$$\begin{aligned}(\sigma_i^*\omega_i)(X) \equiv \omega_i(\sigma_{i*}X) &= (\pi^*\mathcal{A}_i)(\sigma_i^*X) + \mathrm{d}g_i(\sigma_i^*X) \\ &= \mathcal{A}_i(\pi_*\sigma_{i*}X) + \sigma_i^*X(e) = \mathcal{A}_i(X)\,,\end{aligned} \tag{1.2.1.19}$$

the last equality following from the identity $\pi_*\sigma_{i*} = \mathrm{id}_{\mathrm{T}_pM}$. Since the above equation holds for any $X \in \mathrm{T}_pM$ we have:

$$\sigma_i^*\omega_i = \mathcal{A}_i\,. \tag{1.2.1.20}$$

A direct calculation [3] shows that properties i) and ii) of the definition of a connection-one form are satisfied by ω_i; accordingly, an Ehresmann connection on P is given by:

$$\omega \in \Omega^1(P) \otimes \mathfrak{g} \quad \text{such that} \quad \omega\big|_{\pi^{-1}(U_i)} \equiv \omega_i \,\forall\, U_i \,\subset M\,. \tag{1.2.1.21}$$

□

Let us now examine the problem of uniqueness. Clearly the connection one-form obtained above is uniquely defined if, for each pair of open sets $U_i\,,\, U_j \subset M$ with a non-empty intersection, we have $\omega_i = \omega_j$ on $U_i \cap U_j$. This leads to a peculiar transformation law which must be fulfilled by the A_i's. To derive the latter we need the following intermediate result:

Lemma. Let (P, π, M, G) be a principal bundle, $p \in U_i \cap U_j$, σ_i and σ_j local sections defined respectively over $U_i\ ,\ U_j$; let also $t_{ij} : U_i \cap U_j \to G$ be the transition function. Then:

$$\sigma_{j*}X = R_{t_{ij}*}(\sigma_{i*}X) + (t_{ij}^{-1}\mathrm{d}t_{ij}(X))^{\#} \quad \forall\, X \,\in \mathrm{T}_pM\,. \tag{1.2.1.22}$$

[6]This is a little abuse of notation, since we are denoting $g_i \equiv g_i(u)$.

[7]By the footnote above, $g_i : \pi^{-1}(U_i) \to G$, so that $\mathrm{d}g_i$ is actually a $\mathfrak{g}$-valued one-form on $\pi^{-1}(U_i)$. In particular, if $u \in \pi^{-1}(U_i)$, $X_u \equiv \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}$, $g_{i,u} = \exp\{tA\}$, $A \in \mathfrak{g}$ we have:

$$\mathrm{d}g_{i,u}(X_u) \equiv X_u(g_{i,u}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp\{tA\} = A\,. \tag{1.2.1.18}$$

By applying the one form ω defined by Eq. (1.2.1.17) and Eq. (1.2.1.21) to both sides of Eq. 1.2.1.22 we get:

$$\begin{aligned}(\sigma_j^*\omega)(X) &= (R^*_{t_{ij}}\omega)(\sigma_{i*}X) + \omega(t_{ij}^{-1}\mathrm{d}t_{ij}(X))^{\#} \\ &= t_{ij}^{-1}\omega(\sigma_{i*}X)t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}(X)\,, \end{aligned} \tag{1.2.1.23}$$

where in the last equality we have made use of both the defining properties i) and ii). Since Eq. (1.2.1.23) holds for any X, we finally get the transformation law for the local connections:

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}\,. \tag{1.2.1.24}$$

U(1) bundles and the electromagnetic potentials. Let us now make a brief contact with gauge theories, considering electromagnetism as a simple example. First, we recall that a principal bundle (P, π, M, G) is trivial if and only if it admits a global section $\sigma : M \to P$. By what we have seen, if such a section does exist then it is also possible to define a global connection one-form on M, $\mathcal{A} \equiv \sigma^*\omega$. As we shall see, in a physical theory which admits some gauge degrees of freedom, we can globally "fix the gauge"if (and only if) there is a global section on the manifold M in which our Lagrangian (or our Action) takes values: this suggests us that the "gauge potential(s)"of the theory must be identified with the local connection(s) on M. This is indeed so, as is shown by a plain calculation in the case of electromagnetism.

Let $(P, \pi, M, U(1))$ be a principal bundle, and $\mathcal{A}_i$, $\mathcal{A}_j$ two local connection forms over the overlapping regions U_i , U_j. The transition function $t_{ij} : U_i \cap U_j \to U(1)$ is given by:

$$t_{ij}(p) \equiv \exp\{i\Lambda(p)\} \quad , \quad \Lambda(p) \in \mathbb{R}\,, \tag{1.2.1.25}$$

so that Eq. (1.2.1.24) becomes:

$$\begin{aligned}\mathcal{A}_j(p) &= t_{ij}^{-1}(p)\mathcal{A}_i t_{ij}(p) + t_{ij}^{-1}(p)\mathrm{d}t_{ij}(p) \\ &= \mathcal{A}_i(p) + i\mathrm{d}\Lambda(p)\,. \end{aligned} \tag{1.2.1.26}$$

If $\{x^\mu\}$ is a set of coordinates on $U_i \cap U_j$ then Eq. (1.2.1.26) reads in components:

$$(\mathcal{A}_j)_\mu = (\mathcal{A}_i)_\mu + i\partial_\mu\Lambda\,, \tag{1.2.1.27}$$

which is indeed a very familiar expression, since if we define the four-potential $A_\mu \equiv (-A_0, \mathbf{A}) \equiv -i\mathcal{A}_\mu$ the above equation simply represents a gauge transformation of the latter, namely[8]:

$$(A_j)_\mu = (A_i)_\mu + \partial_\mu\Lambda\,. \tag{1.2.1.28}$$

[8]We are implicitly referring to a four-dimensional manifold M endowed with the Minkowski metric $g_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$.

This entitles our hypothesis on the physical meaning of the local connections. Finally, also recall the result of Section 1.1.5 according to which a bundle over a contractible manifold is necessarily trivial. This means that electromagnetism in $\mathbb{R}^4$ (with the Minkowski metric) is described by a trivial bundle $(P \simeq \mathbb{R}^4 \times U(1), \pi, \mathbb{R}^4, U(1))$: this is confirmed by the fact that we *can* globally fix a gauge in that situation (e.g. the transverse gauge or the Lorenz gauge). Things become more involved when we deal with topologically non trivial base manifolds, as we shall see in the case of the the magnetic monopole and the spinning particle.

1.2.2 Parallel transport and holonomy

As we have already anticipated, a connection on a principal bundle allows us to define a unique notion of parallel transport throughout the fibres in the total space. Roughly speaking, we would say that a point is parallel transported along a curve $\tilde{\gamma}$ in the total bundle space P with respect to a given curve γ in the base manifold M if γ is the projection of $\tilde{\gamma}$, i.e. $\pi(\tilde{\gamma}) = \gamma$. However, this definition is not accurate enough to ensure the uniqueness of $\tilde{\gamma}$, since the tangent vector of the latter at a point $u \in P$ can span the entire space $\mathrm{T}_u P$ and thus γ and $\tilde{\gamma}$ might be not parallel at all. What we need is that $\tilde{\gamma}$ be the *horizontal lift* of γ:

Horizontal lift of a curve. Let (P, π, M, G) be a principal bundle and $\gamma : [0,1] \to M$ a curve in M. $\tilde{\gamma} : [0,1] \to P$ is a horizontal lift of γ if:

(i) $\pi(\tilde{\gamma}(t)) = \gamma(t) \quad \forall\, t \in [0,1]$;

(ii) if $\tilde{X}$ is the tangent vector field to $\tilde{\gamma}$, then $\tilde{X} \in \mathrm{H}_{\tilde{\gamma}(t)} P \ \forall\, t \in [0,1]$; equivalently, $\pi_*(\tilde{X}) = X$, where $X \equiv \frac{\mathrm{d}}{\mathrm{d}t}$ is tangent to γ.

If we have a connection one-form ω on P, then by Eq. (1.2.1.13) $\omega(\tilde{X}) = 0$ so that existence and uniqueness of the horizontal lift simply follow from the fundamental theorems on ordinary differential equations:

Theorem. Let $\gamma : [0,1] \to M$ be a curve in M starting at $p_0 = \gamma(0)$. Then there is a unique horizontal lift $\tilde{\gamma} \in \mathcal{C}^1([0,1], P)$ such that $\tilde{\gamma}(0) = u_0$, $u_0 \in \pi^{-1}(p_0)$.

Proof. Let U_i be a chart on M containing γ, $\gamma([0,1]) \subset U_i \subset M$, and let $\sigma_i : U_i \to \pi^{-1}(U_i)$ be a section on it. Define then a curve on P by $\tilde{\gamma}(t) \equiv \sigma_i(\gamma(t)) g_i(t)$ for some $g_i(t) \equiv g_i(\gamma(t)) \in G$: it is not restrictive to suppose that $\sigma_i(\gamma(0)) = \tilde{\gamma}(0)$, so that $g_i(0) = e$. By construction, condition i) of the definition of horizontal lift is verified. Now let $X \equiv \frac{\mathrm{d}}{\mathrm{d}t}$ be tangent to $\gamma(t)$ at $p = \gamma(0)$, so that $\tilde{X} \equiv \tilde{\gamma}_*$ is tangent to $\tilde{\gamma}$ at $u = \tilde{\gamma}(0)$. by adapting Eq. (1.2.1.22) to the present context we have:

$$\tilde{X} = g_i(t)^{-1} (\sigma_{i*} X) g_i(t) + [g_i(t)^{-1} \mathrm{d} g_i(X)]^{\#} . \tag{1.2.2.1}$$

We further impose $\omega(\tilde{X}) = 0$, which ensures condition ii) to hold. By the previous Eq. (1.2.2.1) we get:

$$0 = \omega(\tilde{X}) = 0 = g_i(t)^{-1}\omega((\sigma_{i*}X))g_i(t) + g_i(t)^{-1}\frac{\mathrm{d}g_i(t)}{\mathrm{t}}$$
$$\Longrightarrow \frac{\mathrm{d}g_i(t)}{\mathrm{d}t} = -\omega((\sigma_{i*}X))g_i(t) = -\mathcal{A}_i(X)g_i(t)\,, \tag{1.2.2.2}$$

since by definition $\omega(\sigma_{i*}X) = (\sigma_i^*\omega)(X) \equiv \mathcal{A}_i(X)$ and $\mathrm{d}g_i(\mathrm{d}/\mathrm{d}t) = \mathrm{d}g_i(t)/\mathrm{d}t$. Eq. (1.2.2.2) is then a firs-order ordinary differential equation for $g_i(t)$ with initial condition $g_i(0) = e$, whose formal solution is:

$$g_i(\gamma(t)) = P\exp\left\{-\int_0^t \mathcal{A}_{i\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\,\mathrm{d}t\right\} \tag{1.2.2.3}$$
$$= P\exp\left\{-\int_{\gamma(0)}^{\gamma(t)} \mathcal{A}_{i\mu}(\gamma(t))\,\mathrm{d}x^\mu\right\}, \tag{1.2.2.4}$$

being $\{x^\mu\}$ a set of local coordinates on U_i and P a *time-ordering operator*[9] (see e.g. [10]), whose action is required since in general $\mathfrak{g}$ is not commutative, so that the exponential might not be well-defined. The horizontal lift is then given by $\tilde{\gamma(t)} = \sigma_i(\gamma(t))g_i(t)$. □

Since our choice of the initial value $g_i(0) = e$ is arbitrary, we might also choose a different horizontal lift $\tilde{\gamma}'$. However, it is easy to see as an immediate corollary of the previous theorem that if $\tilde{\gamma}'$ is such that $\tilde{\gamma}'(0) = \gamma(0)g$, $g \in G$, then also $\tilde{\gamma}'(t) = \gamma(t)g$ at any time $t \in [0,1]$.
We are now ready to give the definition of parallel transport on a fibre bundle along a given curve and to introduce the notion of holonomy.

Parallel transport. Let (P, π, M, G) be a principal bundle, $\gamma : [0,1] \to M$ a curve, $u_0 \in \pi^{-1}(\gamma(0))$, and $\tilde{\gamma}$ the unique horizontal lift of γ through u_0. The parallel transported of the point u_0 along γ is the point $u_1 \equiv \tilde{\gamma}(1) \in \pi^{-1}(\gamma(1))$.

Parallel transport therefore defines a map between fibres at different points, namely:

$$\Gamma(\tilde{\gamma}) : G_0 \equiv \pi^{-1}(\gamma(0)) \to G_1 \equiv \pi^{-1}(\gamma(1)) \quad , \quad u_0 \mapsto u_1\,. \tag{1.2.2.6}$$

From the uniqueness of horizontal lift it also follows that each point $u_0 g \in G_0$ is mapped into a unique $u_1 g = \Gamma(\tilde{\gamma})(u_0 g) \in G_1$; further, if R_g is the right action then:

$$(R_g\Gamma(\tilde{\gamma}))(u_0) = u_1 g \quad , \quad (\Gamma(\tilde{\gamma})R_g)(u_0) = \Gamma(\tilde{\gamma})(u_0 g)\,. \tag{1.2.2.7}$$

[9] If $A_1(t)\,, A_2(t)\,, ..., A_n(t)$ are time-dependent matrices P acts as follows:

$$P[A_1(t_1)\,A_2(t_2)\,...A_n(t_n)] \equiv A_{i_1}(t_{i_1})\,A_{i_2}(t_{i_2})\,...A_{i_n}(t_{i_n})\,, \tag{1.2.2.5}$$

being $t_{i_1} \leq t_{i_2} \leq ...t_{i_n}$.

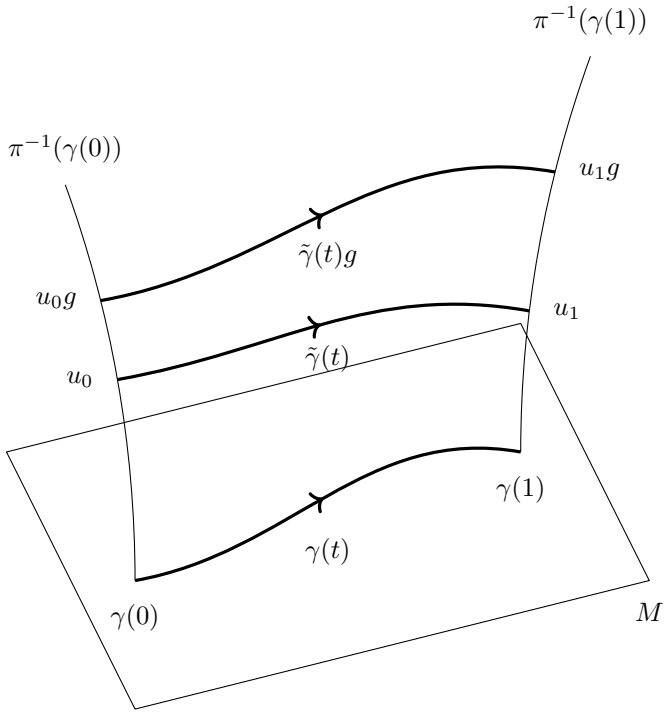

Figure 1.2: Parallel transport of the points u_0, $u_0 g$ along a curve γ.

Thus we find that the right action and the parallel transport *commute*:

$$R_g \circ \Gamma(\tilde{\gamma}) = \Gamma(\tilde{\gamma}) \circ R_g \,. \tag{1.2.2.8}$$

Let us now consider how parallel transport affects loops. Take two curves $\gamma_1 \,, \gamma_2 : [0,1] \to M$ such that $\gamma_1(0) = \gamma_2(0) = p_0$, $\gamma_1(1) = \gamma_2(1) = p_1$ and construct their horizontal lifts such that $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = u_0$: nothing ensures that is also $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$; as a consequence, if we consider a single loop at a point p, i.e. $\gamma : [0,1] \to M$, $\gamma(0) = \gamma(1) = p$, then in general $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$. Hence, a loop γ at $p \in M$, together with a connection one-form ω which assures uniqueness of the horizontal lift, defines an automorphism on the fibre at p:

$$h_\gamma : \pi^{-1}(p) \to \pi^{-1}(p) \quad , \quad h_\gamma(\tilde{\gamma}(0)) = \tilde{\gamma}(1) \,, \tag{1.2.2.9}$$

which, in view of Eq. (1.2.2.8), is compatible with the right action on the fibre, that is $h_\gamma(ug) = h_\gamma(u)g$.
Now, for any given loop γ, $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ are in general different points but still on the same fibre, and must therefore be related by the right action of some $g \in G$. The set of such elements is a proper subgroup of G, and is known as the *holonomy group* (at a point):

Holonomy group. Let $u \in P$, $\pi(u) = p \in M$, and

$$\mathcal{L}_p(M) \equiv \{\gamma : [0,1] \to M | \gamma(0) = \gamma(1)\}\,, \tag{1.2.2.10}$$

be the set of loop based at p. The holonomy group at u is:

$$\mathcal{H}_u \equiv \{g \in G | h_\gamma(u) = ug\,,\, \gamma \in \mathcal{L}_p(M)\}\,. \tag{1.2.2.11}$$

With an appropriate definition of the inverse of a loop and of loops product one easily checks that $\mathcal{H}_u$ is indeed a group [3].

Remark. Let $U_i \subset M$ be a chart with coordinates $\{x^\mu\}$, $\mathcal{A}_i = \mathcal{A}_{i\mu}\mathrm{d}x^\mu$ a local connection over U_i and γ a loop in U_i. Let also $u \in P$, $g_\gamma \in \mathcal{H}_u$ such that $h_\gamma(u) = ug_\gamma$. Then from Eq. (1.2.2.3) one immediately sees that the coordinate expression of the holonomy group element g_γ is:

$$g_\gamma = P \exp\left\{ - \oint_\gamma \mathcal{A}_{i\mu}\mathrm{d}x^\mu \right\}. \tag{1.2.2.12}$$

The physical meaning of this expression will become evident later on, when we will deal with the Berry phase in quantum mechanics.

1.2.3 Curvature

Strictly related to the notion of parallel transport there is that of covariant derivation. Through the covariant derivative of a connection, we can define the curvature of a principal bundle and show how the latter determines the appearence of holonomy: that is, loops are not preserved by the horizontal lift if the bundle is not "flat".
Covariant derivation of differential forms generalizes the notion of exterior derivation [2], [1]. The latter is a map which takes a k-form on a manifold M into a $(k+1)$-form, and this is immediately extended to vector-valued differential forms:

$$\mathrm{d} : \Omega^k(M) \otimes V \longrightarrow \Omega^{k+1}(M) \otimes V \quad , \quad \mathrm{d}\phi \equiv (\mathrm{d}\phi^\alpha) \otimes e_\alpha\,, \tag{1.2.3.1}$$

where $\alpha = 1, ..., \dim(V)$, $\phi^\alpha \in \Omega^k(M)$ and $\{e_\alpha\} \subset V$ is a basis. Then we can give the the following definitions:

Covariant derivative on a principal bundle. Let (P, π, M, G) be a principal bundle endowed with an Ehresmann connection $\omega \in \Omega^1(P) \otimes \mathfrak{g}$; let V be a vector space, $\phi \in \Omega^k(P) \otimes V$, $u \in P$ and $X_1, ..., X_{k+1} \in \mathrm{T}_u P$. The covariant derivative of ϕ is the map:

$$\begin{aligned} &\mathrm{D} : \Omega^k(P) \otimes V \longrightarrow \Omega^{k+1}(P) \otimes V\,, \\ &\mathrm{D}\phi(X_1, ..., X_{k+1}) \equiv \mathrm{d}\phi(X_1^h, ..., X_{k+1}^h)\,, \end{aligned} \tag{1.2.3.2}$$

where $X_i^h \in \mathrm{H}_u P$ is the horizontal component of the vector, $\omega_u(X_i^h) = 0$.

Curvature. the curvature two-form Ω on a principal bundle P is the covariant derivative of the Ehresmann connection:

$$\Omega \equiv \mathrm{D}\omega \in \Omega^2(P) \otimes \mathfrak{g}\,. \tag{1.2.3.3}$$

Take $g \in G$. Recalling that if $G \subset \mathrm{GL}(n, \mathbb{R})$ then $R_g^* \omega = g^{-1}\omega g$, and that the right action preserves the horizontal subspaces, $(R_{g*}X)^h = R_{g*}(X^h)$, we easily find that the right action acts on the curvature as:

$$R_g^* \Omega = g^{-1}\Omega g\,. \tag{1.2.3.4}$$

Moreover, there is an useful equation which relates the actions of a connection one-form and that of a curvature on vectors, namely:

$$\Omega(X, Y) = \mathrm{d}\omega(X, Y) + [\omega(X), \omega(Y)] \qquad u \in P\,,\ X\,, Y\ \in \mathrm{T}_u P\,, \tag{1.2.3.5}$$

so that we can write:

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega\,. \tag{1.2.3.6}$$

Eq. (1.2.3.6) is known as the **Cartan's structure equation**, for a derivation of which the reader is referred to [1]: as we shall momentarily see, it embodies the geometric meaning of the curvature.

Consider indeed two horizontal vectors $X, Y \in \mathrm{H}_u P$, so that $\omega(X) = \omega(X^h) = 0$ and $\omega(Y) = \omega(Y^h) = 0$. Then:

$$\mathrm{d}\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) = -\omega([X, Y])\,, \tag{1.2.3.7}$$

where the first equation is a general property of the exterior derivative on one-forms. By Eq. (1.2.3.6) it follows that:

$$\Omega(X, Y) = \mathrm{d}\omega(X, Y) + [\omega(X), \omega(Y)] = -\omega([X, Y])\,. \tag{1.2.3.8}$$

Now let $(U_i, \{x^\mu\})$ be a chart on M and $\gamma \in U_i$ an infinitesimal parallelogram whose corners are $Q = \{0, 0, ..., 0\}$, $R = \{\epsilon, 0, ..., 0\}$, $S = \{\epsilon, \delta, ..., 0\}$, $T = \{0, \delta, ..., 0\}$. If we define the coordinate vectors $V \equiv \frac{\partial}{\partial x^1}$, $W \equiv \frac{\partial}{\partial x^2}$ then ϵV and δW approximately agree to the lengths of the edges QR and QT respectively. Further, let $\tilde{\gamma}$ be the horizontal lift of γ induced by ω and X, Y its tangent vectors in $\mathrm{H}P$ such that $\pi_* X = \epsilon V$, $\pi_* Y = \delta W$. It follows then:

$$\pi_*([X, Y]^h) = \pi_*([X^h, Y^h]) = \epsilon\delta[V, W] = \epsilon\delta \left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right] = 0\,, \tag{1.2.3.9}$$

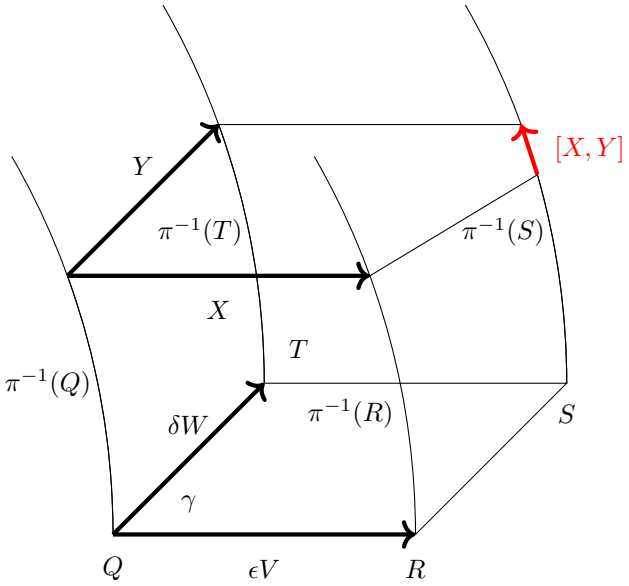

Figure 1.3: Geometrical meaning of the curvature on a principal bundle.

which means that $[X^h, Y^h] = [X, Y]$ is a *vertical* vector, $[X, Y] \in \mathrm{V}P$. Since geometrically $[X, Y]$ measures the distance of the two ending points obtained by horizontally lifting the curves $QR \cap RS$ and $QT \cap TS$ (see Figure 1.3), this means that γ is *not a closed curve*, because the lifted ending points of the aforementioned paths lie in the same fibre ($\pi^{-1}(S)$) but are not the same points. Their vertical distance is precisely measured by the curvature. In fact, if $[X, Y] \in \mathrm{V}P$ then $\exists A \in \mathfrak{g}$ such that $[X, Y] = A^{\#}$, and therefore:

$$\Omega(X, Y) = -\omega([X, Y]) = -A\,. \tag{1.2.3.10}$$

However, since the discrepancy between the ending points of the horizontal lift of a closed loop is given by an element of the holonomy group, there must be a relation between the latter and the curvature. This is the object of the following theorem, a proof of which is found in [1]:

Theorem. (Ambrose-Singer). Let (P, π, M, G) be a principal bundle, with M a connected manifold, and let $\mathfrak{h}_u$ be the Lie algebra of the holonomy group $\mathcal{H}_u$ at $u \in$. Then $\mathfrak{h}_u$ is a subalgebra of $\mathfrak{g}$, given by:

$$\mathfrak{h}_u = \{A = \Omega_u(X, Y) \,|\, X, Y \in \mathrm{H}_u P\} \subset \mathfrak{g} \simeq \mathrm{V}_u P\,. \tag{1.2.3.11}$$

The local expression of curvature. Let $\sigma : M \supset U \longrightarrow P$ be a local section of a principal bundle P and ω an Ehresmann connection over it. As much as it is possible to define a local connection $\mathcal{A} = \sigma^*\omega$, which is a one-form over U, it is also possible to

define the local form of the curvature Ω of P, which will be a $\mathfrak{g}$-valued two-form on U:

$$\mathcal{F} \equiv \sigma^*\Omega \in \Omega^2(U) \otimes \mathfrak{g}\,. \tag{1.2.3.12}$$

By Cartan's structure equation (1.2.3.6) and by observing that the pullback of a one-form commutes with the exterior derivative and with the wedge product [2], [1], i.e. for any three one-forms ω, ρ, η we have $\sigma^*\mathrm{d}\omega = \mathrm{d}\sigma^*\omega$ and $\sigma^*(\rho \wedge \eta) = \sigma^*\rho \wedge \sigma^*\eta$, we find an expression of $\mathcal{F}$ as a function of the local connection $\mathcal{A}$:

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}\,, \tag{1.2.3.13}$$

so that for each $X\,,\, Y \in \mathrm{T}U$:

$$\mathcal{F}(X,Y) = \mathrm{d}\mathcal{A}(X,Y) + [\mathcal{A}(X), \mathcal{A}(Y)]\,. \tag{1.2.3.14}$$

If we now introduce a set of coordinates $\{x^\mu\}$ over U, so that we can write $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu$ and $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ then Eq. (1.2.3.13) reads in components:

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]\,. \tag{1.2.3.15}$$

Recall now the relation established in Section 1.2.1 between local connection one-forms and gauge potentials, namely in the case of $U(1)$ principal bundles, where $A_\mu \equiv i\mathcal{A}_\mu$ can be regarded as an electromagnetic four-potential. Since $U(1)$ is commutative, Eq. (1.2.3.15) simplifies into $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$; therefore if we set $F_{\mu\nu} \equiv i\mathcal{F}_{\mu\nu}$:

$$F_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu\,, \tag{1.2.3.16}$$

which shows that $F_{\mu\nu}$ is nothing but the *electromagnetic field tensor*. An immediate generalization then leads us to the conclusion that the local expression of the curvature two-form on a principal bundle must be identified with the strength field tensor associated to the gauge potential.

Finally, consider the overlapping charts $U_i\,,\, U_j \subset M$ and the respective sections σ_i, σ_j. Let also t_{ij} be the transition function on $U_i \cap U_j$. From the transformation law Eq. (1.2.1.24) and the identity $\mathrm{d}t^{-1} = -t^{-1}\mathrm{d}t\,t^{-1}$ we obtain the compatibility condition between $\mathcal{F}_i$ and $\mathcal{F}_j$, that is, the transformation law for the local form of the curvature:

$$\mathcal{F}_j = \mathrm{Ad}_{t_{ij}^{-1}}\mathcal{F}_i = t_{ij}^{-1}\mathcal{F}_i t_{ij}\,. \tag{1.2.3.17}$$

Once again, if we examine the specific case of $U(1)$, the previous equation simply reduces to $\mathcal{F}_j = \mathcal{F}_i$, which expresses the well known gauge-invariance of the electromagnetic field under the transformation of the four-potential.

Chapter 2

U(1) bundles and gauge theories in classical physics

In the present Chapter, we will apply the mathematical tools previously introduced to the study of two physical systems: the magnetic monopole and the classical non-relativistic spinning particle. As we shall see, the construction of an $U(1)$ principal bundle over the configuration space becomes essential if one wants to provide a global Lagrangian description for such systems. Starting from these particular models, we will prove a general result on the construction of global Lagrangians through $U(1)$ principal bundles. Within this context, we will also show how some purely topological considerations yield a quantization prescription for the magnetic monopole and related systems. The main references for this Chapter are [11], [12].

2.1 Magnetic monopole

In 1931 P.A.M. Dirac proposed an adjustment to the Maxwell's equations in order to allow for the existence of an hypothetical *magnetic monopole*, i.e. a point-like magnetic charge [11]. If such a particle did exist[1], Maxwell's equation would assume the more symmetrical form (in the standard Gaussian cgs units):

$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi\rho_e \\ \nabla \cdot \mathbf{B} = 4\pi\rho_m \end{cases} \quad \text{and} \quad \begin{cases} c\,\nabla \times \mathbf{E} + \partial_t \mathbf{B} = -4\pi \mathbf{j}_m \\ c\,\nabla \times \mathbf{B} - \partial_t \mathbf{E} = +4\pi \mathbf{j}_e \end{cases} , \tag{2.1.0.1}$$

where the new terms ρ_m, $\mathbf{j}_m$ respectively represent the magnetic charge density and magnetic current density. In spite of the evident symmetry, these equations give now rise to some problems when one tries to define the electromagnetic potentials. Leaving aside the interacting equations on the right-hand side, we shall now turn to the analysis

[1] There is currently no reliable empirical evidence of the existence of the latter in nature.

of the magnetic field generated by a Dirac's monopole. In the following, unless otherwise specified, we will employ Gaussian cgs units.

2.1.1 Physical and geometric setting of the monopole

Suppose that a monopole of charge g is kept in a fixed position at $\mathbf{r} = 0$, so that Gauss's Law reads:

$$\nabla \cdot \mathbf{B} = 4\pi g \delta^{(3)}(\mathbf{r})\,. \tag{2.1.1.1}$$

Now since $\nabla(1/r) = -\mathbf{r}/r^3$ and the fundamental solution of the 3-dimensional Laplace operator is given by $\nabla^2(1/r) = -4\pi\delta^{(3)}(\mathbf{r})$, the above equation is solved by a Coulomb-like field:

$$\mathbf{B} = \frac{g\mathbf{r}}{r^3} = \frac{g\hat{r}}{r^2}\,. \tag{2.1.1.2}$$

Eq. (2.1.1.1) thus implies that $\mathbf{B}$ is *no more solenoidal*, i.e. that does not exist a *regular* potential $\mathbf{A}$ such that

$$\mathbf{B} = \nabla \times \mathbf{A}\,, \tag{2.1.1.3}$$

holds everywhere. In fact if it existed we would have, by Stokes'theorem:

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot (\nabla \times \mathbf{A})\, \mathrm{d}^3 r = 0\,, \tag{2.1.1.4}$$

for every closed 2-surface S such that $S = \partial V$. This is of course absurd since if we take S to be the sphere of unitary radius centered in the origin then the integration of Eq. (2.1.1.1) immediately gives:

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{B}\, \mathrm{d}^3 r = 4\pi g\,. \tag{2.1.1.5}$$

However it is clear that Eq. (2.1.1.3) must hold almost everywhere, since $\nabla \cdot \mathbf{B} = 0$ for any $\mathbf{r} \neq 0$. In his original work Dirac explicitly found a vector potential such that Eq. (2.1.1.3) is fulfilled everywhere except for a line, which he chose to be the positive axis $z > 0$:

$$\mathbf{A}^S(\mathbf{r}) = \left(\frac{gy}{r(r-z)}, -\frac{gx}{r(r-z)}, 0 \right) = -\frac{g(1+\cos\theta)}{r\sin\theta}\mathbf{e}_\phi = A^S_\phi \mathbf{e}_\phi\,, \tag{2.1.1.6}$$

where, adopting spherical coordinates, $\mathbf{e}_\phi = -\sin\phi\mathbf{e}_x + \cos\phi\mathbf{e}_y$. Indeed, the right-hand side of the previous equation diverges only at $\theta = 0$, while is regular at $\theta = \pi$ and a direct calculation furnishes:

$$\nabla \times \mathbf{A}^S = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A^S_\phi \sin\theta) = \frac{g}{r^2}\hat{r} \quad \theta \neq 0\,. \tag{2.1.1.7}$$

The solution of Eq. (2.1.1.5) is therefore written as

$$\mathbf{B} = \nabla \times \mathbf{A}^S + 4\pi g \delta(x)\delta(y)\Theta(z)\mathbf{e}_z \,, \tag{2.1.1.8}$$

being $\Theta(z)$ the Heaviside distribution. The above potential is defined and smooth everywhere except for a line emanating from the monopole (this is a so-called *Dirac string singularity*) and in a completely analogous way we might have chosen the singularity to lie in the negative semiaxis, by defining:

$$\mathbf{A}^N(\mathbf{r}) = \left(\frac{-gy}{r(r+z)}, \frac{gx}{r(r+z)}, 0 \right) = \frac{g(1-\cos\theta)}{r\sin\theta}\mathbf{e}_\phi = A^S_\phi \mathbf{e}_\phi \,, \tag{2.1.1.9}$$

hence in this case:

$$\mathbf{B} = \nabla \times \mathbf{A}^N - 4\pi g \delta(x)\delta(y)\Theta(-z)\mathbf{e}_z \,. \tag{2.1.1.10}$$

Nonetheless, there is a way to avoid the string singularity. Indeed, one could use *more* than a single potential, e.g. by adopting $\mathbf{A}^S$ in the southern hemisphere and $\mathbf{A}^N$ in the northern and "pasting"them together through an appropriate gauge transformation along the equator. The idea was brought up by Wu and Yang [13], who also gave a description of the monopole gauge potentials in terms of connection one-forms on a U(1) principal bundle.
First, let us notice that at the equator ($\theta = \pi/2\,,\ \phi \in [0, 2\pi]$) the potentials are related by a transformation of the form $\mathbf{A}^N = \mathbf{A}^S + \nabla\Lambda$. In fact, from Eq. (2.1.1.6) and Eq. (2.1.1.9):

$$\mathbf{A}^N - \mathbf{A}^S = \frac{2g}{r\sin\theta}\,\mathbf{e}_\phi = \nabla(2g\phi) \,, \quad \Lambda \equiv 2g\phi \,. \tag{2.1.1.11}$$

This is obviously an admissible gauge transformation since $\mathbf{B}$ is left unchanged; moreover, $\nabla(2g\phi)$ is singular only at $\theta = 0\ ,\ \pi$; and Eq. (2.1.1.11) is therefore well defined on the equatorial strip. The total magnetic flux through a closed 2-surface $S = S^S \cup S^N$ containing the monopole is then given by:

$$\begin{aligned} \mathbf{\Phi} = \oint_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} &= \int_{S^N} (\nabla \times \mathbf{A}^N)\cdot \mathrm{d}\mathbf{S} + \int_{S^S} (\nabla \times \mathbf{A}^S)\cdot \mathrm{d}\mathbf{S} \\ &= \oint_{\partial S^N} \mathbf{A}^N \cdot \mathrm{d}\mathbf{r} + \oint_{\partial S^S} \mathbf{A}^S \cdot \mathrm{d}\mathbf{r} = \oint_{\partial S} (\mathbf{A}^N - \mathbf{A}^S)\cdot \mathrm{d}\mathbf{r} \\ &= \oint_{\partial S} \nabla(2g\phi)\cdot \mathrm{d}\mathbf{r} = 4\pi g \,, \end{aligned} \tag{2.1.1.12}$$

where we have used again Stokes' theorem and assumed the orientation of the equatorial boundary ∂S to be that of ∂S^N. As expected, the result is in agreement with Eq. (2.1.1.5).

We shall now turn to the topological description of the system [3]. The (static) field of the monopole is defined in $\mathbb{R}^3 \setminus \{0\}$ but since the relevant homotopy properties of

this space are the same of the 2-sphere [6] we can limit ourselves to the study of an $U(1)$ principal bundle over S^2, $(P, \pi, S^2, U(1))$. Let $\{S_N, S_S\}$ be an open covering of the sphere, with the two hemispheres intersecting along the equator:

$$S_N \equiv \{(\theta, \phi) | \theta \in [0, \pi/2 + \epsilon] \, , \, \phi \in [0, 2\pi[\} \, , \tag{2.1.1.13}$$

$$S_S \equiv \{(\theta, \phi) | \theta \in [\pi/2 - \epsilon, \pi] \, , \, \phi \in [0, 2\pi[\} \, . \tag{2.1.1.14}$$

Let then Φ_N and Φ_S be the local trivializations over $\pi^{-1}(S_N)$, $\pi^{-1}(S_S)$, whose actions are given by:

$$\Phi_N(u) = (p \, , \, e^{i\alpha_N}) \quad , \quad \Phi_S(u) = (p \, , \, e^{i\alpha_S}) \quad , \quad \alpha_N \, , \, \alpha_S \, \in [0, 2\pi] \tag{2.1.1.15}$$

$\forall u \in \pi^{-1}(S_N \cap S_S)$, $p = \pi(u)$. Then at each point $p = p(\phi)$ on the equator we have a transition function $g_{NS}(p) : U(1) \longrightarrow U(1)$, $g_{NS}(p) = e^{in\phi}$ with n an integer so that the function is single-valued. The transition function precisely characterizes the topological structure of the bundle since n, being integer, specifies the homotopy class of the latter in the first homotopy group $\pi_1(U(1)) \simeq \mathbb{Z}$. The fibre coordinates are related on the equator by:

$$e^{i\alpha_N} = e^{in\phi} e^{i\alpha_S} \, , \tag{2.1.1.16}$$

and when $n = 0$ the bundle is trivial, i.e. $P \simeq S^2 \times S^1$, while if $n \neq 0$ the fibre bundle is "twisted"on the equator. Furthermore, since $U(1)$ is abelian, the right and left action of an element $g = e^{i\Lambda} \in U(1)$ over P are equivalent and the structure group's action on a fibre is represented by:

$$\Phi_N(ug) = (p \, , \, e^{i(\alpha_N + \Lambda)}) \quad , \quad \Phi_S(ug) = (p \, , \, e^{i(\alpha_S + \Lambda)}) \, . \tag{2.1.1.17}$$

Supposed $n \neq 0$, the gauge properties of the magnetic monopole system are described by a connection on a non trivial $U(1)$ bundle. Indeed, on the two charts S_S and S_N previously defined we have two local sections: $\sigma_N : \, S_N \longrightarrow P$, $\sigma_S : \, S_S \longrightarrow P$ through which we can construct two local connection one-forms, i.e. local gauge potentials:

$$\mathcal{A}_N = \sigma_N^* \omega \quad , \quad \mathcal{A}_S = \sigma_S^* \omega \, , \tag{2.1.1.18}$$

being ω a given Ehresmann connection on P. $\mathcal{A}_N$ and $\mathcal{A}_S$ can be taken of the Wu-Yang form:

$$\mathcal{A}_N = \frac{ig(1 - \cos\theta)}{r \sin\theta} \mathrm{d}\phi \quad , \quad \mathcal{A}_S = -\frac{ig(1 + \cos\theta)}{r \sin\theta} \mathrm{d}\phi \, , \tag{2.1.1.19}$$

since if we consider a generic transition function on the equator:

$$g_{NS} : S_S \cap S_N \supset S^1 \longrightarrow U(1) \quad , \quad g_{NS}(\phi) = e^{i\alpha(\phi)} \, , \tag{2.1.1.20}$$

with $\alpha : S^1 \longrightarrow \mathbb{R}$, then we immediately find that the transformation law for the local connections on S^1 is:

$$\begin{aligned} &\mathcal{A}_N = g_{NS}^{-1}\mathcal{A}_S g_{NS} + g_{NS}^{-1}\mathrm{d}g_{NS} = \mathcal{A}_S + i\mathrm{d}\alpha \\ &\Longrightarrow \mathrm{d}\alpha = i(\mathcal{A}_S - \mathcal{A}_N) = \frac{2g}{r\sin\theta}\mathrm{d}\phi = 2g\mathrm{d}\phi\,; \end{aligned} \tag{2.1.1.21}$$

where we have used the fact that $r = 1$ and $\theta = \pi/2$. Eq.(2.1.1.11) is therefore reproduced, with $\mathbf{A}^N \equiv -i\mathcal{A}_N$, $\mathbf{A}^S \equiv -i\mathcal{A}_S$.

Finally, when ϕ runs from 0 to 2π , $\alpha(\phi)$ undergoes the variation:

$$\Delta\alpha = \int_0^{2\pi} 2g\,\mathrm{d}\phi = 4\pi g\,, \tag{2.1.1.22}$$

and, in accordance to what previously said, the above quantity must be an integer multiple of 2π for g_{NS} to be uniquely defined on the equator:

$$\Delta\alpha = 2\pi n \Longrightarrow 2g = n \in \mathbb{Z}\,. \tag{2.1.1.23}$$

We have thus obtained a *quantization condition* for the monopole in a purely topological way. Eq. (2.1.1.23) shows that it is the (dimensionless) "strength"of the monopole which eventually yields the homotopy class of the bundle: the latter is trivial only if the former is absent.

2.1.2 Electric charge quantization

Let us now derive the electric charge quantization condition in the presence of a monopole. The Hamiltonian of a point-like particle with electric charge e and mass m moving in an electromagnetic field is [14]:

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\phi\,, \tag{2.1.2.1}$$

being $\mathbf{A}$ and ϕ the vector and scalar potentials. The Lagrangian in turn reads:

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{r}} - e\phi\,. \tag{2.1.2.2}$$

Here we make the important observation that the canonical momentum $\mathbf{p}$ and velocity $\dot{\mathbf{r}}$ are related by:

$$\mathbf{p} \equiv \frac{\partial L}{\partial\dot{\mathbf{r}}} = m\dot{\mathbf{r}} + \frac{e}{c}\mathbf{A}\,, \tag{2.1.2.3}$$

i.e. $\mathbf{p}$ is not the kinetic momentum: this is so because the potential contains a linear term in $\dot{\mathbf{r}}$. If we limit ourselves to the field generated by a Dirac monopole, so that $\phi = 0$, the (time independent) Schroedinger equation for the particle is:

$$\hat{H}\psi(\mathbf{r}) \equiv \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}}\right)^2 \psi(\mathbf{r}) = E\,\psi(\mathbf{r})\,. \tag{2.1.2.4}$$

When the vector potential undergoes the local transformation $\mathbf{A} \mapsto \mathbf{A} + \nabla\Lambda(\mathbf{r})$ the Hamiltonian changes as:

$$\hat{H} \mapsto \hat{U}^\dagger \hat{H} \hat{U} \quad , \quad \hat{U} \equiv \exp\left(-\frac{ie\Lambda}{\hbar c}\right) , \tag{2.1.2.5}$$

as can be seen by recalling that in the position space $\hat{\mathbf{p}}\psi(\mathbf{r}) \equiv -i\hbar\nabla\psi(\mathbf{r})$ and then "completing the square". The Schroedinger equation however must be gauge-invariant, and this is achieved only if under the same transformation the wave function *acquires a phase factor*:

$$\psi(\mathbf{r}) \mapsto \exp\left(\frac{ie\Lambda}{\hbar c}\right) \psi(\mathbf{r}) . \tag{2.1.2.6}$$

If we consider the two Wu-Yang potentials of a magnetic monopole related according to Eq. (2.1.1.11) then:

$$\psi^S(\mathbf{r}) \mapsto \exp\left(-\frac{i2eg\phi}{\hbar c}\right) \psi^N(\mathbf{r}) . \tag{2.1.2.7}$$

Let now ϕ vary from 0 to 2π along the equator; the wave function is required to be single-valued and this is possible only if

$$\frac{2eg}{\hbar c} = n , \quad n \in \mathbb{Z} . \tag{2.1.2.8}$$

The above relation is the famous Dirac quantization condition [11], [15], which reduces to Eq. (2.1.1.23) if one puts $\hbar = e = c = 1$. Eq. (2.1.2.8) however leads to a very remarkable conclusion: the fact that, if one hypothesizes the existence of a single magnetic monopole, then *all the electric charges must be quantized*.

2.1.3 Motion of an electric charge in the field of the monopole

We will now discuss the classical Hamiltonian and Lagrangian formalism for the motion of a non-relativistic charged particle in the field of a Dirac monopole [12], [16]. As we shall see, the peculiar features of the system are once again elegantly described within the topological framework of $U(1)$ principal bundles: specifically, this is realized by employing the *Hopf fibration* of the three-sphere (see Section 1.1.4). For the sake of simplicity, we shall only consider *separated* magnetic and electric charge: the generalization to the case of Dyons (i.e. particles with both electric and magnetic charge) can be found in [17].

Equations of motion.
Let $\mathbf{r} = (r_1, r_2, r_3)$ denote the position of the particle, whose electric charge is[2] $-e$, and

[2]The minus signs is chosen in order to present the Lagrangian of the system in a convenient fashion.

let m be the reduced mass of the monopole-charge system. The magnetic field, as previously shown, is $\mathbf{B} = \frac{g\mathbf{r}}{r^3}$; the motion is governed by the Lorentz force equation, which -in the absence of any other source an with $c = 1$- reads:

$$\mathbf{F} = -e\dot{\mathbf{r}} \times \mathbf{B} = \frac{eg}{r^3}\mathbf{r} \times \dot{\mathbf{r}}\,. \tag{2.1.3.1}$$

Define now $n \equiv eg$. Then Eq. (2.1.3.1) can be written in components:

$$m\ddot{r_i} = \frac{n\epsilon_{ijk}r_j\dot{r_k}}{r^3} \quad , \quad i = 1, 2, 3\,. \tag{2.1.3.2}$$

Remark. Since we are dealing with the classical case, the quantity n in the above equation could in principle assume any real value. However, if one considers the quantization condition Eq. (2.1.2.8) and puts $\hbar = c = 1$ then it follows that n must be *half-integer*: this result establishes a first contact with the spinning particle system which we will treat later on.

It is quite instructive to decompose the motion in its radial and angular components, by defining:

$$r_i \equiv r\hat{r_i} \quad , \quad \sum_{i=1}^{3} \hat{r_i}^2 = 1\,. \tag{2.1.3.3}$$

Then Eq. 2.1.3.2 splits into:

$$\ddot{r} = r\sum_i \dot{\hat{r}_i}^2 \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}[m\epsilon_{ijk}r_j\dot{r_k} + n\hat{r_i}] = 0 \quad , \quad i = 1, 2, 3\,. \tag{2.1.3.4}$$

The radial equation is that of a free particle, as we expected from the very structure of Eq. (2.1.3.1). The angular equation however shows that, if we identify the bracketed expression with the conserved total angular momentum, the latter must be written as:

$$\mathbf{J} \equiv m\mathbf{r} \times \dot{\mathbf{r}} + n\hat{\mathbf{r}}\,. \tag{2.1.3.5}$$

That is, the presence of the magnetic charge determines an additional piece $n\hat{\mathbf{r}}$ which can be interpreted as a *helicity* along the line joining the charge and the monopole, and proportional to the "intensity"of the latter:

$$\mathbf{J} \cdot \hat{\mathbf{r}} = n\,. \tag{2.1.3.6}$$

We will come back to this fact in the following. Let us now turn to the Hamiltonian and Lagrangian description of the system.

The Hamiltonian and Lagrangian formalism.
We shall make the assumption that the moving particle is forbidden to occupy the same

position of the monopole, so that the configuration space of the system is $\mathbb{R}^3 \setminus 0$. One would then naturally define the Hamiltonian of the charged particle as a function on the cotangent bundle of the configuration space, whose canonical coordinates are $\{(\mathbf{r}, \mathbf{p})\}$. In this way, given the Hamiltonian Eq. (2.1.2.1) (with $\phi = 0$ and $e \mapsto -e$), the equations of motion would immediately follow from the canonical Poisson Brackets:

$$\{r_i, r_j\} = \{p_i, p_j\} = 0 \quad , \quad \{r_i, p_j\} = \delta_{ij} \, . \tag{2.1.3.7}$$

However, it will turn out more useful to adopt a sightly different approach, that is, to define the Hamiltonian on the *tangent* bundle of the configuration space, as a function of coordinates and velocities:

$$H(\mathbf{r}, \mathbf{v}) \equiv \frac{1}{2} m \mathbf{v}^2 \, . \tag{2.1.3.8}$$

The reasons of this choice will become less obscure in the following Section. In order to obtain the equations of motion we now have to regard $\mathbf{r}$ and $\mathbf{v}$ as independent variables and appropriately define the Poisson Brackets:

$$\{r_i, r_j\} = 0 \, , \tag{2.1.3.9}$$

$$\{r_i, v_j\} = \frac{\delta_{ij}}{m} \, , \tag{2.1.3.10}$$

$$\{v_i, v_j\} = - \frac{n \epsilon_{ijk} r_k}{m^2 r^3} \, ; \tag{2.1.3.11}$$

so that we have:

$$\dot{r_i} = \{r_i, H\} = v_i \quad , \quad \dot{v_i} = \{v_i, H\} = \frac{n \epsilon_{ijk} r_j \dot{r_k}}{m r^3} \, . \tag{2.1.3.12}$$

The correct equations are thus reproduced.
The Lagrangian L, which in virtue of our choice is defined in the same space of the Hamiltonian, is given by:

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2} m \mathbf{v}^2 + \frac{e}{c} \mathbf{A}(\mathbf{r}) \cdot \mathbf{v} \, , \tag{2.1.3.13}$$

and is easily obtained by temporarily replacing $\mathbf{v} = \frac{1}{m}(\mathbf{p} - \frac{e}{c}\mathbf{A})$ in the Hamiltonian and then taking its Legendre transformation. However, if $\mathbf{A}$ is the Dirac vector potential previously considered, then there is a string singularity along which the Lagrangian is not defined and thus the configuration space is not entirely accessible; if instead we choose to use two different potentials in two different domains of $\mathbb{R}^3 \setminus 0$ to avoid the singularity then the respective Lagrangians also differ. In both cases, while the Hamiltonian Eq. (2.1.3.8) does not explicitly contain the potential and is thus *globally* defined, the Lagrangian Eq. (2.1.3.13) can be only *local*. In order to provide a global Lagrangian description, we need to enlarge the configuration space to an $U(1)$ principal bundle.

Global Lagrangian.
Notice that the decomposition Eq. (2.1.3.3) in radial and angular motion establishes a diffeomorphism:

$$\mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^+ \times S^2 \quad , \quad \mathbf{r} \mapsto (r, \hat{r}) \,, \tag{2.1.3.14}$$

and therefore we can regard $M \equiv \mathbb{R}^+ \times S^2$ as the actual configuration space of the particle.
We can now define a bundle structure $(P, \pi, M, U(1))$ as follows. Let P be the product space:

$$P = \mathbb{R}^+ \times SU(2) = \{(r, s) | r > 0 \,,\, s^\dagger s = \mathbb{1} \,, \det(s) = 1\} \,. \tag{2.1.3.15}$$

The canonical projection from P to M is the map

$$\pi \,:\, SU(2) \simeq S^3 \longrightarrow S^2 \quad , \quad \pi(r, s) \equiv (r, \hat{r}) \,, \tag{2.1.3.16}$$

where $\hat{r} = (\hat{r}_1, \hat{r}_2, \hat{r}_3)$ is such that the following holds:

$$\hat{R} \equiv \sigma_i \hat{r}_i = s \sigma_3 s^{-1} \,. \tag{2.1.3.17}$$

Here σ_1, σ_2, σ_3 denote the three Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,,\, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,,\, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{2.1.3.18}$$

It is not difficult to see that the map defined by Eq. (2.1.3.17) is nothing but the Hopf fibration of S^3, although it is presented in a compact and coordinate-free fashion. Indeed, if we adopt the defining two-dimensional matrix representation of $SU(2)$ we can write $s \in SU(2)$ as:

$$s = \begin{pmatrix} z^0 & , & z^1 \\ -\bar{z^1} & , & \bar{z^0} \end{pmatrix} \quad , \quad z^0 \,, z^1 \in \mathbb{C} \quad \text{s.t.} \ \left|z^0\right|^2 + \left|z^1\right|^2 = 1 \,, \tag{2.1.3.19}$$

where z^0 , z^1 must be regarded as time-dependent. Then Eq. (2.1.3.17) reads:

$$\begin{pmatrix} \hat{r}_3 & , & \hat{r}_1 - i\hat{r}_2 \\ \hat{r}_1 + i\hat{r}_2 & , & -\hat{r}_3 \end{pmatrix} = \begin{pmatrix} |z^0|^2 - |z^1|^2 & , & 2z^0\bar{z^1} \\ 2\bar{z^0}z^1 & , & |z^1|^2 - |z^0|^2 \end{pmatrix} \,. \tag{2.1.3.20}$$

If we define $(x^1, x^2, x^3, x^4) \in S^3$ so that $z^0 = x^1 + ix^2$, $z^1 = x^3 + ix^4$ then[3] the previous equation simply reproduces Eq. (1.1.4.4) with the substitution $\mathbf{y} \mapsto \hat{r}$.

[3]Notice that this correspondence establishes the natural isomorphism between S^3 and the matrix representation of $SU(2)$.

We will now show that the global Lagrangian (defined in P) which yields the required equations of motion is:

$$\begin{aligned} L(r,s) =& \frac{1}{2} m \dot{\mathbf{r}}^2 + in \mathrm{Tr}(\sigma_3 s^{-1} \dot{s}) \\ =& \frac{1}{2} m \dot{r}^2 + \frac{1}{4} m r^2 \mathrm{Tr}\left(\dot{\hat{R}}^2\right) + in \mathrm{Tr}(\sigma_3 s^{-1} \dot{s}) \,, \end{aligned} \tag{2.1.3.21}$$

where we used the identity $\mathrm{Tr}(\hat{R}\dot{\hat{R}}) \equiv \mathrm{Tr}(\sigma_i \hat{r}_i \sigma_j \dot{\hat{r}}_j) = 0$ to write the second expression so that the dependence on $r(t)$, $s(t)$ is made explicit.
By observing that $\mathrm{Tr}\dot{\hat{R}}^2 = 2\sum_i \dot{\hat{r}}_i^2$ the Euler-Lagrangian equation for r immediately reduces to the radial equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \Longrightarrow \ddot{r} = r \sum_i \dot{\hat{r}}_i^{\,2} = 0 \,. \tag{2.1.3.22}$$

Concerning the evolution of the $SU(2)$ coordinate (i.e. the angular motion), a faster and more elegant approach consists in performing a direct variation of $s \in SU(2)$ without any regard to a particular parametrization of the group (which, we recall, is topologically a 3-dimensional manifold and we would therefore need to write down three equations). Namely, since a generic element $s \in SU(2)$ can be written in terms of the Lie-Algebra generators:

$$s = \exp\{(i\lambda_i \sigma_i)\}, \quad \text{with} \quad i\lambda \cdot \sigma \equiv i\lambda_i \sigma_i \in \mathfrak{su}(2) \,,\ \lambda_i = \lambda_i(t) \,,\ i = 1,2,3 \,; \tag{2.1.3.23}$$

we have the first order variation:

$$\delta s = \exp\{(i(\lambda + \epsilon) \cdot \sigma)\} - \exp\{(i\lambda \cdot \sigma)\} = i\epsilon \cdot \sigma \, \exp\{(i\lambda \cdot \sigma)\} = i\epsilon \cdot \sigma \, s \equiv i\epsilon_i \sigma_i s \,, \tag{2.1.3.24}$$

up to terms of order ϵ^2. Because $s^{-1} = s^\dagger$ we also have $\delta s^{-1} = -is^{-1}\epsilon \cdot \sigma$, and hence we get the variations:

$$\delta \hat{R} = i[\epsilon \cdot \sigma, \hat{R}] \quad , \quad \delta \mathrm{Tr}\left(\sigma_3 s^{-1} \dot{s}\right) = i \mathrm{Tr}\left(\dot{\epsilon} \cdot \sigma \hat{R}\right) \,. \tag{2.1.3.25}$$

Using these equations and the cyclic identity for the trace operator:

$$\mathrm{Tr}(A[B,C]) = \mathrm{Tr}(B[C,A]) \quad \forall \, A, B, C \in \mathrm{GL}(n) \,, \tag{2.1.3.26}$$

we arrive at:

$$\delta L = 0 \Longrightarrow \mathrm{Tr}\left\{\epsilon \cdot \sigma \frac{\mathrm{d}}{\mathrm{d}t}\left\{-\frac{1}{2}[\hat{R}, mr^2 \dot{\hat{R}}] + n\hat{R}\right\}\right\} \tag{2.1.3.27}$$

$$= \mathrm{Tr}\left\{\epsilon_i \sigma_i \frac{\mathrm{d}}{\mathrm{d}t}\left\{-\frac{1}{2} m r^2 [\hat{r}_j \sigma_j \,,\, \dot{\hat{r}}_k \sigma_k] + n \hat{r}_l \sigma_l\right\}\right\} \tag{2.1.3.28}$$

$$= 0 \,. \tag{2.1.3.29}$$

Recalling that $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ and $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$, with a little algebra Eq. (2.1.3.27) leads to:

$$\frac{d}{dt}\left\{-\frac{i}{2}[\hat{R}, mr\dot{\hat{R}}] + n\hat{R}\right\} = 0\,, \tag{2.1.3.30}$$

which in turn one easily shows equivalent to $\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0$ so that both the radial and the angular equations of motion are fully reproduced.

Gauge properties and bundle construction.
As we have seen, the fibre bundle structure we are dealing with is the Hopf fibration of the three sphere $U(1) \to S^3 \to S^2$, and the global Lagrangian is defined in the total space.[4] It is convenient to represent the associated $U(1)$ gauge group as the one-dimensional subgroup of $SU(2)$ generating rotations around the z-axis:

$$U(1) \simeq \left\{ g \equiv \exp\left(\frac{i\sigma_3\theta}{2}\right) = \mathrm{diag}\left\{e^{i\theta/2}\,, e^{-i\theta/2}\right\} \middle|\, \theta = \theta(t) \right\}\,. \tag{2.1.3.31}$$

The group acts on the enlarged space $S^3 \simeq SU(2)$ by the right multiplication:

$$R_g :\, SU(2)\times U(1) \longrightarrow SU(2) \quad , \quad R_g(s) \equiv sg = s\exp\left(\frac{i\sigma_3\theta}{2}\right)\,, \tag{2.1.3.32}$$

and the action -as expected- is free, transitive and fibre preserving, since Eq. (2.1.3.17) is left unchanged and this means that $\pi(sg) = \pi(s)$, for every $s \in SU(2)$ and $g \in U(1)$. As we have already anticipated, this implies that the system admits an $U(1)$ gauge degree of freedom. Indeed, the equations of motion Eq. (2.1.3.4), being written in terms of the variables $(r, \hat{r})$ in the *base manifold*, are unaffected by the transformation Eq. (2.1.3.32) and are thus gauge-invariant; on the other hand the Lagrangian Eq. (2.1.3.21) depends on s and is therefore only *weakly invariant*. By this we mean that under the action Eq. (2.1.3.32) L acquires an additional time-derivative, as one can easily check[5]

$$L(r,s) \longmapsto L'(r,s) \equiv L(r,sg) = L(s,g) - n\dot{\theta}\,. \tag{2.1.3.34}$$

This is indeed a gauge freedom since the Action then only changes by a constant, leading to the same equations of motion.

Through the Hopf-fibration we have therefore obtained a global Lagrangian description, but at the price of being now left with an *unavoidable* $U(1)$ gauge degree of freedom.

[4]For the sake of accuracy, the fibration is actually $U(1) \to \mathbb{R}^+ \times S^3 \to \mathbb{R}^+ \times S^2$. However, since π trivially maps $r \in \mathbb{R}^+$ into itself, we can safely take $P \equiv S^3$, $M \equiv S^2$.

[5]It is sufficient to the see that the argument of the Trace operator changes as

$$\sigma_3 s^{-1}\dot{s} \mapsto \sigma_3 \exp\{-i\sigma_3\theta/2\}s^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[s\exp\{i\sigma_3\theta/2\}\right] = \sigma_3 s^{-1}\dot{s} + i\frac{\theta}{2}\mathbb{1}\,. \tag{2.1.3.33}$$

:

In fact, a global gauge fixing is not possible, since the bundle is not trivial. Namely, fixing a global gauge means that we are able to find a map $\tilde{\sigma} : S^2 \longrightarrow SU(2)$ which is a global section on the bundle, so that is smooth $\forall \hat{r} \in S^2$ and:

$$\sigma_i \hat{r_i} = s\left(\hat{r}\right) \sigma_3 s\left(\hat{r}\right)^{-1} \quad , \quad s\left(\hat{r}\right) \equiv \tilde{\sigma}(\hat{r}) \, ; \tag{2.1.3.35}$$

that is, $\pi \circ \tilde{\sigma} = \mathrm{id}_{S^2}$. Such a connection of course cannot exist since $S^3 \neq S^2 \times S^1$. However, we can still find a section which is discontinuous only at a point; if we choose the latter to be the South Pole $\hat{r} = (0, 0, -1)$, such a function is:

$$s\left(\hat{r}\right) = \frac{1}{2} \left\{ \alpha \mathbb{1} - \frac{1}{\alpha} [\sigma_3 \, , \, \sigma_i \hat{r_i}] \right\} \quad , \quad \alpha \equiv [2(1 + \hat{r_3})]^{1/2} \, . \tag{2.1.3.36}$$

A direct calculation shows that $s\left(\hat{r}\right)$ is indeed an element of $SU(2)$. Through this gauge fixing we can rewrite the interacting term in the Lagrangian Eq. (2.1.3.21):

$$in \mathrm{Tr} \left\{ \sigma_3 s(\hat{r})^{-1} \dot{s}(\hat{r}) \right\} = \frac{n \epsilon_{3ij} \hat{r_i} \dot{\hat{r}}_j}{1 + \hat{r}_3} \, . \tag{2.1.3.37}$$

The result is remarkable: when we (locally) fix the redundant gauge degree of freedom the Dirac string singularity shows up, whereas is absent in the Lagrangian 2.1.3.21 which is only weakly gauge-invariant, so that we gain one and lose one.

We conclude the section by recovering also the Wu and Yang singularity-free formulation. Let us cover S^2 with two charts which are as usual the northern and southern hemispheres S_N , S_S: this yields the two local sections $\tilde{\sigma}_N$, $\tilde{\sigma}_S$ and the respective functions $s_N(\hat{r})$, $s_S(\hat{r})$, where s_N is defined as in Eq. (2.1.3.37) and s_S only differs by a change of sign in the definition of α, i.e. $\alpha_S \equiv [2(1 - \hat{r_3})]^{1/2}$. Along the equator we therefore have:

$$\begin{aligned} \sigma_i \hat{r_i} =& s_S\left(\hat{r}\right) \sigma_3 s_S\left(\hat{r}\right)^{-1} = s_N\left(\hat{r}\right) \sigma_3 s_N\left(\hat{r}\right)^{-1} \\ \Longrightarrow & t_{NS} \sigma_3 t_{NS}^{-1} = \sigma_3 \quad , \quad t_{NS} \equiv \left[s_N(\hat{r})^{-1} s_S(\hat{r}) \right] \, . \end{aligned} \tag{2.1.3.38}$$

Eq. (2.1.3.38) shows that s_S and s_N belong to the same fibre and are related by a change of coordinates in the overlapping region:

$$s_N(\hat{r}) = s_S(\hat{r}) \exp\left\{ \left(\frac{i \sigma_3 \theta}{2} \right) \right\} \quad , \quad \text{for some } \theta = \theta(t) \, . \tag{2.1.3.39}$$

That is, the transition function is an element of the gauge structure group. It immediately follows that in $S_S \cap S_N$ the two local Lagrangians differ again by a total time derivative:

$$L_N = L_S - n \dot{\theta} \, . \tag{2.1.3.40}$$

2.2 Non-relativistic spinning particle

The classical treatment of a particle with spin [18] is much similar to the one of an electric charge in the field of a magnetic monopole. Also in this case we can readily get a global Hamiltonian description, while the Lagrangian one cannot be achieved through a canonical Legendre transformation in the phase space but requires the use of a non trivial fibre bundle.

2.2.1 Hamiltonian and Lagrangian description

Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{p} = (p_1, p_2, p_3)$ denote the position and momentum of the particle. Define also its spin $\mathbf{S} = (S_1, S_2, S_3)$ to be a vector with fixed magnitude:

$$\mathbf{S}^2 = \sum_i S_i^2 \equiv \lambda^2 \,. \tag{2.2.1.1}$$

If λ does not vary then the configuration space is $M \equiv \mathbb{R}^3 \times S^2$. Since the particle has spin, it is also endowed with a magnetic moment $\boldsymbol{\mu} \equiv \mu\mathbf{S}$, $\mu \equiv -\mu_B g_s$, so that in the presence of an external magnetic field $\mathbf{B} = (B_1, B_2, B_3)$ the interacting (global) Hamiltonian reads:

$$H = H_0 + \mu\mathbf{B}\cdot\mathbf{S}\,, \tag{2.2.1.2}$$

where $H_0 = \frac{\mathbf{p}^2}{2m}$ is the Hamiltonian of the free particle.
Since the energy is given and the equations of motion for $\mathbf{x}$, $\mathbf{p}$, $\mathbf{S}$ are known (see e.g. [14]), we can easily reproduce them within the Hamiltonian description by means of the canonical Poisson Brackets[6]:

$$\{x_i, x_j\} = \{p_i, p_j\} = 0\,, \tag{2.2.1.3}$$
$$\{x_i, p_j\} = \delta_{ij}\,, \tag{2.2.1.4}$$
$$\{S_i, S_j\} = \epsilon_{ijk} S_k\,. \tag{2.2.1.5}$$

Notice incidentally that the last bracket is defined in perfect analogy with the canonical one for the orbital angular momentum. Now, with regard to the Hamiltonian Eq. (2.2.1.2), we have:

$$\dot{x}_i = \{x_i, H\} = \frac{p_i}{m}\,, \tag{2.2.1.6}$$
$$\dot{p}_i = \{p_i, H\} = -\mu S_j \frac{\partial B_j}{\partial x_i}\,, \tag{2.2.1.7}$$
$$\dot{S}_i = \{S_i, H\} = \mu\epsilon_{ijk} B_j S_k\,. \tag{2.2.1.8}$$

[6]One might directly obtain the evolution of $\mathbf{x}$ and $\mathbf{p}$ through the Hamilton's equations of motion. However, as we shall see, this is more problematic in the case of $\mathbf{S}$.

The last two equations reduce to $\dot{p}_i = 0\,,\ \dot{S}_i = 0$ in the free case ($\mathbf{B} = \mathbf{0}$).

Once the Hamiltonian of a system is given, we can often get its Lagrangian by performing a Legendre transformation: this happens when our coordinates and momenta are *canonical*. This means that if the configuration space is an n-dimensional manifold M whose global coordinates are $(Q_1, ..., Q_n)$, then there is a global chart for the phase space $\mathrm{T}^*M = \{(Q_1, ..., Q_n, P_1, ...P_n)\}$ and the latter is endowed with a globally defined symplectic form [19]:

$$\omega = \mathrm{d}Q_i \wedge \mathrm{d}P_i \in \Omega^2(\mathrm{T}^*M)\,, \tag{2.2.1.9}$$

which induces the canonical Poisson brackets:

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}\,. \tag{2.2.1.10}$$

Under this circumstances, if the Hamiltonian H is a regular function on T^*M with a non-vanishing Hessian determinant then the Legendre transformation simply maps:

$$\mathrm{T}^*M \ni H \mapsto L \equiv \dot{Q}_i P_i - H \in \mathrm{T}M \quad , \quad \dot{Q}_i \equiv \frac{\partial H}{\partial P_i}\,; \tag{2.2.1.11}$$

and the procedure of course also works in the reverse direction. In the present case, however, our system of coordinates is *not* canonical. In fact, we can set $Q_i = x_i$, $P_i = p_i$ for $i = 1, 2, 3$, but then Q_4 , P_4 should be functions of the spin vector $\mathbf{S}$ which in turn spans a 2-dimensional sphere isomorphic to S^2. Hence, since there does not exist a global chart on the latter, any choice of $(Q_4(\mathbf{S}), P_4(\mathbf{S}))$ is singular for at least one[7] $\mathbf{S}$.

Therefore, as it was the case for the monopole (although the reason is slightly different), it is not possible to build up a global Lagrangian over $\mathbb{R}^3 \times S^2$. Nevertheless, we can still derive a global Lagrangian description defined on an enlarged configuration space $\mathbb{R}^3 \times SU(2)$ by employing the very same procedure worked out for the monopole. We can relate a matrix $s \in SU(2)$ to a spin vector through the Hopf map $\pi : SU(2) \to S^2$:

$$\pi(s) = \mathbf{S} \quad \text{s.t.} \quad \mathbf{S} \cdot \sigma \equiv S_i \sigma_i = \lambda s \sigma_3 s^{-1}\,, \tag{2.2.1.12}$$

so that the constraint Eq. (2.2.1.1) is automatically fulfilled.

We will now show that the Lagrangians which yield the correct equations of motion for $\mathbf{x}$ and $\mathbf{S}$ in the free and interacting case are respectively:

$$L_0 = \frac{1}{2} m \dot{\mathbf{x}}^2 + i\lambda \mathrm{Tr}\left(\sigma_3 s^{-1} \dot{s}\right)\,, \tag{2.2.1.13}$$

and

$$L_1 = L_0 - \mu \mathbf{B} \cdot \mathbf{S} = L_0 - \frac{\mu}{2} \mathrm{Tr}(\mathcal{S}\mathcal{B})\,, \tag{2.2.1.14}$$

[7]Incidentally we may observe that if one employs standard spherical coordinates then the singularity is on the polar axis, as it was the case for the Dirac string in the monopole system.

where $\mathcal{S} \equiv S_i\sigma_i = \lambda s\sigma_3 s^{-1}$, $\mathcal{B} \equiv B_i\sigma_i$. Observe that the formal structure of the free Lagrangian is identical to that of the charge-monopole system, with the simple substitution $n \mapsto \lambda$. We will come back on this analogy.
For both the free and the interacting Lagrangian the motion of the position coordinates is readily given by the Euler-Lagrange equations:

$$\frac{\partial L_A}{\partial x_i} - \frac{d}{dt}\frac{\partial L_A}{\partial \dot{x}_i} = 0 \quad A = 0, 1\,, \tag{2.2.1.15}$$

which yield:

$$m\ddot{x}_i = 0 \quad , \quad m\ddot{x}_i = -\mu S_j \frac{\partial B_j}{\partial x_i} \quad , \quad i = 1, 2, 3\,. \tag{2.2.1.16}$$

As we did before, the evolution of the $SU(2)$ parameter is obtained by a direct first-order variation:

$$\delta s \equiv i\epsilon \cdot \sigma\, s \Longrightarrow \delta s^{-1} = -is^{-1}\epsilon \cdot \sigma\,. \tag{2.2.1.17}$$

Hence we get, for the Lagrangian L_0:

$$\delta L_0 = -\lambda \mathrm{Tr}\left(\sigma_3 s^{-1}\dot{\epsilon} \cdot \sigma s\right) = -2S_i\dot{\epsilon}_i = 0 \longrightarrow \dot{S}_i = 0\,, \tag{2.2.1.18}$$

where the last equality follows from an integration by parts. The calculation for L_1 is just a bit more involved; from Eq. (2.2.1.17) and using again the cyclic identity Eq. (2.1.3.26) we have:

$$\delta L_1 = \delta L_0 - i\frac{\mu}{2}\mathrm{Tr}([\mathcal{S}, \mathcal{B}]\epsilon \cdot \sigma) = 0\,, \tag{2.2.1.19}$$

which again, after an integration by parts, yields $\dot{S}_i = \mu\epsilon_{ijk}B_jS_k$. The equations of motion for both $\mathbf{x}$ and $\mathbf{S}$ are thus fully recovered.

2.2.2 Gauge properties; relation between the massless spinning particle and the Charge-Monopole System

The classical spinning particle share the same gauge properties of the charge-monopole system previously described, and therefore also the same description in terms of fibre bundles. Namely, if we neglect the translational degrees of freedom (i.e. $\mathbf{x} \in \mathbb{R}^3$) then we have a principal bundle structure ($SU(2) \simeq S^3, S^2, \pi, U(1)$) which is the Hopf fibration of the sphere with the map defined by Eq. (2.2.1.12). If we employ the representation of $U(1)$ defined by Eq. (2.1.3.31) then the right action on the total bundle space $SU(2)$ is given also in this case by:

$$R_g :\, SU(2) \times U(1) \longrightarrow SU(2) \quad , \quad R_g(s) \equiv sg = s\exp\left(\frac{i\sigma_3\theta}{2}\right)\,. \tag{2.2.2.1}$$

Since Eqs. (2.2.1.6) - (2.2.1.8) only involve variables in the base manifold $\mathbb{R}^3 \times S^2$ they are gauge-invariant, i.e. unaffected by Eq. (2.2.2.1). This is of course so because

$\pi(sg) = \pi(s)\ \forall s \in SU(2)\,,\ g \in U(1)$. The Lagrangians L_A in turn are weakly invariant since they are function on the total bundle space[8] and therefore they change as:

$$L(r,s) \longmapsto L'(r,s) \equiv L(r,sg) = L(s,g) + \lambda\dot{\theta}\,. \tag{2.2.2.2}$$

The above equation is completely analogous in form to Eq. (2.1.3.34), so that also in this case the Action is unaffected and the gauge freedom cannot globally fixed since the bundle is not trivial. Namely, if we define a section:

$$\tilde{\sigma}:\ S^2 \longrightarrow SU(2) \quad , \quad \tilde{\sigma}(\mathbf{S}) \equiv s(\mathbf{S}) \text{ s.t. } S_i\sigma_i = \lambda s(\mathbf{S})\sigma_3 s(\mathbf{S})^{-1}\,, \tag{2.2.2.3}$$

then the latter cannot be continuous for every $\mathbf{S} \in S^2$ and is therefore not a global section, because $S^3 \neq S^2 \times S^1$.

We shall conclude the Section highlighting some features which establish a relation between the charge-monopole system and the spinning particle from a more physical point of view. As we have already pointed out, the Lagrangians of the systems are basically the same, if one identifies the "classical"spin of the particle with the quantity $n = eg$. A more precise correspondence will be formulated in Section 3.3 within the framework of quantum mechanics. However, the physical reasons underlying this identification remain quite obscure unless one turns to the *relativistic* description of the spinning particle. Without entering into details (for which the reader is referred to [12]), we shall limit ourselves to a few considerations on the dynamics of a relativistic *massless* spinning particle.

Let us start by observing that, in view of the Poisson Brackets Eqs. (2.1.3.9) , (2.1.3.11), when one quantizes the charge-monopole system through the standard Dirac prescription [20], the result is a system which is localized is the position space but is not so in the velocity space, since $[v_i, v_j] \neq 0$. On the other hand, if one considers the Hamiltonian of a massless spinning particle ($c = 1$):

$$H = |\mathbf{p}|\,, \tag{2.2.2.4}$$

endowed with the canonical Poisson Brackets (or equivalently, commutation relations):

$$\{p_i, p_j\} = 0\,, \tag{2.2.2.5}$$

$$\{r_i, p_j\} = \delta_{ij}\,, \tag{2.2.2.6}$$

$$\{r_i, r_j\} = -\lambda\epsilon_{ijk}\frac{p_k}{p^3}\,; \tag{2.2.2.7}$$

then the dynamical relation between $\mathbf{r}$ and $\mathbf{p}$ is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\epsilon_{ijk}r_j p_k + \lambda\frac{p_i}{p}\right] = 0\,, \tag{2.2.2.8}$$

[8]For the sake of precision, this is true only for L_0, whose domain is $\mathbf{R}^3 \times SU(2)$, while the interacting Lagrangian L_1, in view of the additional term, is a function on $\mathbf{R}^3 \times SU(2) \times S^2$; however, this is of no influence since the interaction energy $\mu\mathbf{B}\cdot\mathbf{S}$ is obviously gauge-invariant.

while Eqs. (2.2.2.5) , (2.2.2.7) are constraints which show that the particle is localized in the momentum space and not in the position one (as it expected for a massless spinning particle in a relativistic quantum description). In Eq. (2.2.2.6), λ is the *helicity* of the particle, which is related to an Irreducible Representation of the proper ortochronus Poincarè group[9] (e.g. we have a photon for $\lambda = 1$ and a massless neutrino for $\lambda = 1/2$). Moreover, one shows that the J_i's, $J_i \equiv \left[\epsilon_{ijk} r_j p_k + \lambda \frac{p_i}{p}\right]$, $i = 1, 2, 3$, generate the rotation of the system. Therefore, if one compares Eq. (2.2.2.8) with the radial equation Eq. (2.1.3.4) it is clear that the product $n = eg$ actually represents a helicity *provided the charge-monopole system is not localized in the velocity space*. For the sake of completeness, we also observe that the Lagrangian Eq. (2.1.3.21) is the analogous of the non-relativistic limit for the photon Lagrangian:

$$L = p\hat{p}_k \dot{r}_k - i\lambda \mathrm{Tr}[\sigma_3 s^{-1} \dot{s}]\,, \tag{2.2.2.9}$$

where $s \in SU(2)$, $p_k \equiv p\hat{p}_k$ and $\sigma_k p_k = s\sigma_3 s^{-1}$.

2.3 U(1) bundles and global Lagrangians

The two systems we have studied so far (i.e. the monopole-charge system and the spinning particle) are characterized by the following peculiar feature: there exists a phase space endowed with a globally defined energy function, and a symplectic form (equivalently, Poisson Brackets) is also globally defined therein; however, those systems *do not admit global canonical coordinates* in that space and therefore a global Lagrangian cannot be found by means of a Legendre transformation. The latter can nonetheless be performed locally, so that local Lagrangians are obtained in appropriate domains of the manifold. A global Lagrangian for the above systems was built by enlarging the configuration space to a non-trivial $U(1)$ bundle. We shall now present a general procedure for finding a global Lagrangian whenever local ones are given togheter with a globally defined Hamiltonian (and the respective symplectic form). The most salient results can be synthesized in the following points:

(i) A "quantization"condition must be fulfilled in order for the bundle construction to work. This condition emerges classically and has a pure geometric origin. Friedman and Sorkin [17] found that this "light"quantization condition for a system of several interacting electric and magnetic point-like particles is the following:

$$\frac{e_i g_j}{e_k g_l} = q \in \mathbb{Q}\,, \tag{2.3.0.1}$$

[9]It is interesting to note that in the most general relativistic case (i.e. when we allow $m \neq 0$) there also emerges a non trivial bundle fibration $U(1) \to \mathcal{P}_+^\uparrow \to \mathcal{P}_+^\uparrow / U(1)$ within the Lagrangian description.

where the e_i's and g_i's are the electric and the magnetic charges respectively. This condition implies that the two types of charged are separately quantized[10], but it is still a less stringent result than the quantization Eq. (2.1.2.8) found by Dirac using the Schroedinger equation.

(ii) If the above requirement is fulfilled than the global Lagrangian is found by introducing additional $U(1)$ degrees of freedom, i.e. a fibre bundle structure: it is remarkable that nothing more involved of an $U(1)$ principal bundle is required, *independently of the specific functional form of the Lagrangian.*

2.3.1 A technical result

The procedure we will adopt to work out the global Lagrangian is based on the following result, due to A. Weil [21].

Theorem. Let M be a paracompact manifold[11] and $\Omega \in \Omega^2(M)$ a closed two-form on M, $\mathrm{d}\Omega = 0$. Let also:

$$\oint_\Sigma \Omega = 2\pi\nu\lambda \quad , \quad \nu \in \mathbb{Z}\,, \tag{2.3.1.1}$$

for every closed two-surface $\Sigma \subset M$, where $\lambda \in \mathbb{R}$ is constant and ν is characteristic of Σ.
Then there exists a principal bundle $(P, M, \pi, U(1))$ and an *exact* two-form $\tilde{\Omega} \in \Omega^2(P)$ which is the pullback of Ω induced by the canonical projection $\pi : P \longrightarrow M$, i.e.:

1. $\tilde{\Omega} = \mathrm{d}\Lambda$, for some $\Lambda \in \Omega^1(P)$;
2. $\tilde{\Omega} = \pi^*\Omega$.

Before proceeding with the proof, we make a few observations. The exact form $\tilde{\Omega}$, being *globally* defined on P, is a "gauge-invariant"two-form: as we know, this means that if $\sigma_i : U_i \to P$, $\sigma_j : U_j \to P$ are two local sections, $U_i \cap U_j \neq \emptyset$, then $\sigma_i^*\tilde{\Omega} = \sigma_j^*\tilde{\Omega}$. Thus, we may regard $\tilde{\Omega}$ as a curvature two-form and Ω as its local expression. In conventional situations in classical mechanics, there exists a global symplectic form $\mathrm{d}p_i \wedge \mathrm{d}q^i$ (in global canonical coordinates) which is exact, since:

$$\mathrm{d}p_i \wedge \mathrm{d}q^i = \mathrm{d}(p_i \mathrm{d}q)\,. \tag{2.3.1.2}$$

[10]To convince oneself of this it is sufficient to take $e_i = e_k$ and $g_j = g_l$, which respectively show the quantization of the magnetic and of the electric charge.

[11]A paracompact space is a topological space in which any open covering admits a *locally* finite refinement, i.e. for each point in the space there exists a neighbourhood which intersects only a finite number of open sets of the covering.

Then this theorem states under which conditions a symplectic form which is closed but not exact can be turned into an exact one in a total bundle space.
As we will see, Eq. (2.3.1.1) is precisely what gives rise to the quantization condition.

Proof. if $\lambda = 0$ then Ω is exact; henceforth we shall assume that $\lambda \neq 0$. Since M is paracompact, it admits an open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, $U_\alpha \subset M$ which is contractible, i.e. for any $\tilde{\mathcal{A}} \subset \mathcal{A}$ the set $\bigcap_{\alpha\in\tilde{\mathcal{A}}}$ is either empty or can be smoothly contracted to a point. In each of these sets the converse Poincarè lemma is therefore valid. So consider the overlapping open sets U_α , U_β , $\in M$: there exist some locally defined one forms Θ_α , Θ_β such that:

$$\Omega|_{U_\alpha} = \mathrm{d}\Theta_\alpha \quad , \quad \Omega|_{U_\beta} = \mathrm{d}\Theta_\beta \,. \tag{2.3.1.3}$$

Hence,

$$\mathrm{d}(\Theta_\alpha - \Theta_\beta) = 0 \text{ on } U_\alpha \cap U_\beta \quad \Longrightarrow \Theta_\alpha - \Theta_\beta \equiv \mathrm{d}f_{\alpha\beta}\,,\ f_{\alpha\beta} \in \mathcal{C}^1(U_\alpha \cap U_\beta, \mathbb{R})\,. \tag{2.3.1.4}$$

Consider now a third subset U_γ such that $U_\gamma \cap U_\alpha \cap U_\beta \neq \emptyset$ and then define two functions $f_{\alpha\gamma}$, $f_{\beta\gamma}$ as in Eq. 2.3.1.4. We have:

$$\mathrm{d}(f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}) = 0 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma\,, \tag{2.3.1.5}$$

so that $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}$ is constant on the intersection and we can take that constant to be:

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 2\pi\nu_{\alpha\beta\gamma}\lambda \quad , \quad \nu_{\alpha\beta\gamma} \in \mathbb{Z}\,,\ \lambda \in \mathbb{R}\,. \tag{2.3.1.6}$$

One can prove that the above relation is equivalent to the condition Eq. (2.3.1.1) in the statement of the theorem (see [21]). Eq. (2.3.1.6) allows us to define the map:

$$g_{\alpha\beta} \,:\, U_\alpha \cap U_\beta \longrightarrow U(1) \quad , \quad p \longmapsto g_{\alpha\beta}(p) \equiv \exp\left(\frac{i f_{\alpha\beta}}{\lambda}\right)\,; \tag{2.3.1.7}$$

and analogously for $g_{\alpha\gamma}$, $g_{\beta\gamma}$. By construction (notice that $f_{\alpha\beta} = -f_{\beta\alpha}$) these maps fulfill the cocycle conditions:

$$\begin{cases} g_{\alpha\alpha}(p) = \mathrm{id} \quad , \quad p \in U_\alpha \\ g_{\alpha\beta}(p) = g_{\beta\alpha}(p)^{-1} \quad , \quad p \in U_\alpha \cap U_\beta \\ g_{\alpha\beta}(p) \cdot g_{\beta\gamma}(p) \cdot g_{\gamma\alpha}(p) = \mathrm{id} \quad , \quad p \in U_\alpha \cap U_\beta \cap U_\gamma \end{cases}$$

so that they are actually transition functions which make P an $U(1)$ bundle over M. Indeed, if π is the canonical projection map $\pi : P \to M$, the local trivializations:

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times U(1) \quad , \quad \Phi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times U(1)\,, \tag{2.3.1.8}$$

are related on the fibre over $U_\alpha \cap U_\beta \ni p$ by:

$$\Phi_\alpha^{-1}(p, h^{(\alpha)}) = \Phi_\beta^{-1}(p, h^{(\beta)}) = \Phi_\beta^{-1}(p, h^{(\alpha)} g_{\alpha\beta}(p)) \quad , \quad h^{(\alpha)}\,,\, h^{(\alpha)} \in U(1)\,. \tag{2.3.1.9}$$

Now define a one-form over U_α, which in a local chart is written as:

$$m_\alpha \equiv -i\lambda(h^{(\alpha)})^{-1}\mathrm{d}h^{(\alpha)} \in \Omega^1(U_\alpha)\,, \tag{2.3.1.10}$$

so that:

$$\begin{aligned} m_\alpha - m_\beta &= -i\lambda(h^{(\alpha)})^{-1}\mathrm{d}h^{(\alpha)} + i\lambda(h^{(\beta)})^{-1}\mathrm{d}h^{(\beta)} \\ &= -i\lambda(h^{(\alpha)})^{-1}\mathrm{d}h^{(\alpha)} + i\lambda(h^{(\alpha)}g_{\alpha\beta})^{-1}\mathrm{d}h^{(\alpha)}g_{\alpha\beta} \\ &= -i\lambda(h^{(\alpha)})^{-1}\mathrm{d}h^{(\alpha)} + i\lambda(h^{(\alpha)})^{-1}g_{\alpha\beta}^{-1}g_{\alpha\beta}\mathrm{d}h^{(\alpha)} + i\lambda(h^{(\alpha)})^{-1}g_{\alpha\beta}^{-1}h^{(\alpha)}\mathrm{d}g_{\alpha\beta} \\ &= i\lambda g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} = -\mathrm{d}f_{\alpha\beta}\,, \end{aligned} \tag{2.3.1.11}$$

which is a one-form on $U_\alpha \cap U_\beta$. From Eq. (2.3.1.4) and Eq. (2.3.1.11) we then get:

$$\Theta_\alpha + m_\alpha = \Theta_\beta + m_\beta\,, \tag{2.3.1.12}$$

so that the one-form

$$\tilde{\Theta} \equiv \tilde{\Theta}_\alpha + \tilde{m}_\alpha\,, \tag{2.3.1.13}$$

with $\tilde{\Theta}_\alpha \equiv \pi^*\Theta_\alpha$, $\tilde{m}_\alpha \equiv \pi^* m_\alpha$, is a globally defined one form on P. Finally, since $\mathrm{d}\tilde{m}_\alpha = \pi^*(\mathrm{d}m_\alpha) = 0$, we can write $\tilde{\Omega} \equiv \mathrm{d}\tilde{\Theta}$; the proof is thus completed. □

2.3.2 Global Lagrangians through fibre bundles

We shall now apply the previous result to our physical context. In doing that, we shall regard the Hamiltonian as a function of *coordinates and velocities*, rather than of coordinates and momenta, so that it will be defined in the tangent bundle $\mathrm{T}M$ of the configuration space, as well as all the local Lagrangians. This is also the main reason for the choice we made in defining the Hamiltonian of the charged particle in Section 2.1.3.

Construction of the Lagrangian.
Let M be a $2n$-dimensional paracompact manifold which represents the configuration space of the system and $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open covering of the latter. Further, let $(x^1, ..., x^n, \dot{x}^1, ..., \dot{x}^n)$ be a local set of coordinates[12] for the tangent bundle $\mathrm{T}U_\alpha$ over U_α. Suppose that:

(i) The system at stake admits local Lagrangians $L^{(\alpha)}$ defined on the $\mathrm{T}U_\alpha$'s.

(ii) The Hamiltonian is globally defined on $\mathrm{T}M = \bigcup_\alpha \mathrm{T}U_\alpha$, so that in local coordinates:

$$H^{(\alpha)} \equiv \frac{\partial L^{(\alpha)}}{\partial \dot{x}^i}\dot{x}^i - L^{(\alpha)} = \frac{\partial L^{(\beta)}}{\partial \dot{x}^i}\dot{x}^i - L^{(\beta)} \equiv H^{(\beta)}\,, \tag{2.3.2.1}$$

on each non-empty intersection $\mathrm{T}U_\alpha \cap \mathrm{T}U_\beta$.

[12] We shall omit Greek indices referring to the set when there is no risk of ambiguity.

(iii) There is a globally defined two-form $\omega \in \Omega^2(M)$. In local coordinates:

$$\omega^{(\alpha)} \equiv \mathrm{d}\left(\frac{\partial L^{(\alpha)}}{\partial \dot{x}^i}\mathrm{d}x^i\right) = \mathrm{d}\left(\frac{\partial L^{(\beta)}}{\partial \dot{x}^i}\mathrm{d}x^i\right) = \omega^{(\beta)}\,. \tag{2.3.2.2}$$

This is of course nothing but the canonical symplectic form, as one sees when turning to the conjugate coordinates and momenta: $\mathrm{d}p_i \wedge \mathrm{d}x^i = \mathrm{d}(p_i \wedge \mathrm{d}x^i)$.

(iv) The symplectic form Eq. (2.3.2.2) fulfills the quantization condition Eq. (2.3.1.1), which we recall:

$$\oint_\Sigma \omega = 2\pi\nu\lambda \quad , \quad \nu = 0, \pm 1, \pm 2, ... \tag{2.3.2.3}$$

for any closed two-dimensional surface $\Sigma \in M$. Again, λ is a real constant and ν characterizes Σ.

The important result we are now able to prove is that if conditions i) to iv) are satisfied then there exists an $U(1)$ principal bundle P over M and a global Lagrangian defined on $\mathrm{T}M$.

To work out the global Lagrangian we only need adapt to the present context the procedure exposed in the previous section. So let us define a one-form $\psi_\alpha \equiv \frac{\partial L^{(\alpha)}}{\partial \dot{x}^i}\mathrm{d}x^i$ on $U_\alpha \subset M$ and analogously for all the other open subsets of the covering. Then Eq. (2.3.2.2) implies:

$$\mathrm{d}(\psi_\alpha - \psi_\beta) = \mathrm{d}\left[\frac{\partial}{\partial \dot{x}^i}\left(L^{(\alpha)} - L^{(\beta)}\right)dx^i\right] = 0 \quad \text{on } U_\alpha \cap U_\beta\,, \tag{2.3.2.4}$$

so that $\psi_\alpha - \psi_\beta$ is a closed one-form on $U_\alpha \cap U_\beta$, and since the latter is a contractible set the form is locally exact, i.e. $\exists f_{\alpha\beta} \in \mathcal{C}^1(U_\alpha \cap U_\beta)$ such that $(\psi_\alpha - \psi_\beta)|_{U_\alpha \cap U_\beta} = \mathrm{d}f_{\alpha\beta}$. Proceeding exactly as above, we ask $f_{\alpha\beta}$ to satisfy Eq. (2.3.1.6) (which is equivalent to Eq. (2.3.2.3)) so that we can build an $U(1)$ bundle P with fibres over M and the one-forms m so that Eq.(2.3.1.10) holds in each set U_α. Then the form:

$$\xi \equiv \psi_\alpha + m_\alpha\,, \tag{2.3.2.5}$$

is globally defined on M, and therefore

$$\mathrm{d}(\pi^*\xi) = \pi^*(\mathrm{d}\xi) = \pi^*\omega \in \Omega^2(P)\,, \tag{2.3.2.6}$$

being π the canonical projection.

We can now proceed to find the Lagrangian. The energy condition Eq. (2.3.2.1) can be rewritten as:

$$L^{(\alpha)} - L^{(\beta)} = \frac{\partial f_{\alpha\beta}}{\partial x^i}\dot{x}^i\,, \tag{2.3.2.7}$$

and therefore the Lagrangian

$$\tilde{L} = \pi^* \left(L^{(\alpha)} - i\lambda (h^{(\alpha)})^{-1} \frac{\mathrm{d}h^{(\alpha)}}{\mathrm{d}t} \right) , \tag{2.3.2.8}$$

with $h^{(\alpha)} \in U(1)$, is globally defined on $\mathrm{T}P$: this follows from $-i\lambda (h^{(\alpha)})^{-1} \mathrm{d}h^{(\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = m_\alpha \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \in \mathrm{T}U_\alpha$. Moreover, the last term on the right-hand side of the above equation is the total time-derivative of a function, so that $L^{(\alpha)}$, $L^{(\beta)}$ give the same equation of motion. This shouldn't be surprising, since the Eq. (2.3.2.8) represents an $U(1)$ gauge transformation on the total bundle space, under which the global Lagrangian is known to be only weakly invariant.

The charge-monopole system

To see how (and when) does this result practically work let us turn once again to the charge-monopole system. Suppose that $(\xi^1, ..., \xi^{2n})$ is a coordinate set for the phase space $\mathrm{T}M$ and the globally-defined symplectic form ω is expressed using these coordinates as:

$$\omega = \frac{1}{2!} \omega_{ij}(\xi) \mathrm{d}\xi^i \wedge \mathrm{d}x^j \quad , \quad \omega_{ij}(\xi) \equiv \{\xi_i \, ; \, \xi_j\} \, , \tag{2.3.2.9}$$

where $\{ \, ; \, \}$ are the Poisson Brackets defined in Section 2.1.3 by Eqs. (2.1.3.9) - (2.1.3.11) and we lower and raise indices with the euclidean metric so that $\omega_{ij}\omega^{jk} = \delta_i^k$ and there is no distinction between covariant and contravariant indices. Taking $(r_1, r_2, r_3, v_1, v_2, v_3)$ as coordinates then Eq. (2.3.2.9) reads:

$$\omega = 2m \mathrm{d}v_i \wedge \mathrm{d}r_i + \frac{F_{ij}}{2} \mathrm{d}r_i \wedge \mathrm{d}r_j \quad , \quad F_{ij} \equiv n \frac{\epsilon_{ijk} r_k}{r^3} \, , \tag{2.3.2.10}$$

for i , j , k $\in \{1, 2, 3\}$. If we consider a closed two-surface $\Sigma = \partial V$ in $\mathrm{T}M$ (which is a six-dimensional space), then if the latter does not include the monopole (i.e. the origin of the **r**-space) Stokes' theorem yields:

$$\oint_\Sigma \omega = \int_V \mathrm{d}\omega = 0 \, , \tag{2.3.2.11}$$

being the symplectic form locally exact. However, generalizing what we have seen in Section 2.1.1, if one takes the two sphere S^2 in the **r**-space (with outward orientation) then a direct calculation furnishes $\int_{S^2} \omega = -4\pi n$. We may now observe that the integration over the most generic closed two-surface Σ in $\mathrm{T}M$ can be obtained by multiple integrations over S^2 with different orientations, and therefore we can write:

$$\int_\Sigma \omega = 4\pi n \nu_n = 2\pi\lambda\nu \quad , \quad \lambda \equiv 2n \, , \, \nu \equiv \nu_n \in \mathbb{Z} \, . \tag{2.3.2.12}$$

We have thus shown that this simple system satisfies the condition Eq. (2.3.2.3). It is fundamental to stress that Eq. (2.3.2.12) *is not a constraint* on the possible values of n which, following the convention of Section 2.1.3, is given by $n = eg$ and is therefore real-valued. There is no physical quantization yet and the above equation is only used to define λ and ν, only the latter being explicitly defined as an integer.

It is also possible to derive the more general relation Eq. (2.3.0.1) in a simple way. Indeed, suppose that an additional monopole is introduced in the system. For the sake of simplicity we shall assume that the latter can interact only with the electric charge, i.e. we neglect any monopole-monopole interaction. Taking as a coordinate the relative distance between this monopole and the electric charge, $\mathbf{r'} = (r'_1, r'_2, r'_3)$, the symplectic form Eq. (2.3.2.10) gains the term:

$$\frac{1}{2} n' \frac{\epsilon_{ijk} r'_k}{r'3} \mathrm{d}r'_i \wedge \mathrm{d}r'_j \,, \tag{2.3.2.13}$$

being $n' \equiv eg'$. Eq. (2.3.2.12) therefore becomes:

$$\int_\Sigma \omega = 4\pi n \nu_n + 4\pi n' \nu_{n'} \quad , \quad \nu_n \,,\, \nu_{n'} \in \mathbb{Z}. \tag{2.3.2.14}$$

Comparing the above result with Eq. (2.3.2.3) one then gets:

$$\lambda \nu = 2n\nu_n + 2n'\nu_{n'} \,. \tag{2.3.2.15}$$

Since this holds for any two-surface Σ, we may choose the latter such that $\nu_n = 0$ (we recall that ν is characteristic of the surface): this implies that $\lambda = nk$, $k = \frac{2\nu_n}{\nu} \in \mathbb{Q}$; by choosing instead Σ such that $\nu'_n = 0$ one gets $\lambda = n'k$, $k \in \mathbb{Q}$ and therefore the ratio $\frac{n}{n'}$ is rational. By adding also another electric charge to the system one arrives at the quantization condition found by Friedman and Sorkin, Eq. (2.3.0.1).

Chapter 3

Bundle connections and curvature in quantum mechanics: the geometric phase

The main purpose of this Chapter is to elucidate the geometrical meaning of the adiabatic phase acquired by a quantum system during a slow evolution in time. Once again, we will work within the mathematical framework of principal bundles. Namely, the invariance of the observables of a quantum system under a phase transformation of the wavefunction is precisely mirrored by the choice of $U(1)$ as the typical fibre. After a general exposition of the theory, we will turn to two illustrative examples, which are on their own of a discrete physical interest: the spinning particle in a slowly varying magnetic field and the Aharonov-Bohm effect. We will follow the original works on the topic [22], [23], [24], while the notation employed is that of [3].

3.1 Adiabatic evolution of a quantum system and the Berry phase

Suppose that we have a quantum system which is described by a Hamiltonian $H(\mathbf{R}(t))$, where $\mathbf{R}(t) \equiv (R_1(t), ..., R_n(t))$ are some time-varying parameters. The evolution of the system is governed by the Schroedinger equation:

$$H(\mathbf{R}(t))\,|\psi(t)\rangle = i\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle\ , \tag{3.1.0.1}$$

where we have put $\hbar \equiv 1$. Suppose further that at a fixed time, say e.g. $t = 0$, the energy spectrum is discrete with non-degenerate orthonormal eigenvalues $|n, \mathbf{R}(0)\rangle$, $n \in \mathbb{N}$. The **adiabatic theorem** states that under this assumption, if the time evolution

of the parameters $\mathbf{R}(t)$ is slow enough and the system is initially found in a given energy eigenstate $|\psi(0)\rangle = |n, \mathbf{R}(0)\rangle$, with:

$$H(\mathbf{R}(0))\,|n, \mathbf{R}(0)\rangle = E_n(\mathbf{R}(0))\,|n, \mathbf{R}(0)\rangle\ , \tag{3.1.0.2}$$

then at a later time $t > 0$ the system is always in the same eigenstate, which evolves in time altogether with the Hamiltonian:

$$H(\mathbf{R}(t))\,|n, \mathbf{R}(t)\rangle = E_n(\mathbf{R}(t))\,|n, \mathbf{R}(t)\rangle\ . \tag{3.1.0.3}$$

Stated in other words, if the evolution is adiabatic then the eigenstates change their functional form in such a way that no level crossing takes place. A derivation of the conditions for an adiabatic evolution is found in Appendix A.

A naive guess for the solution of Eq. (3.1.0.1) with the initial condition Eq. (3.1.0.2) is then of the form:

$$|\psi(t)\rangle = \exp\left[-i\int_0^t E_n(\mathbf{R}(s))\,\mathrm{d}s\right]|n, \mathbf{R}(t)\rangle \equiv \exp[i\eta_d(t)]\,|n, \mathbf{R}(t)\rangle\ , \tag{3.1.0.4}$$

where the phase factor keeps track of the evolution of the system and is therefore a *dynamical* phase. As a consequence of the above equation, if at a certain time t the system is back in its original configuration, i.e. $\mathbf{R}(t) = \mathbf{R}(0)$, then it is also found in the initial eigenstate, up to the dynamical phase factor: $|\psi(t)\rangle = \exp[i\eta_d(t)]\,|\psi(0)\rangle$. However a direct substitution of Eq. (3.1.0.4) in Eq. (3.1.0.1) shows that this is *not* a solution. As it was first observed by M.V.Berry in 1983 [23], there is a further contribution to the phase factor, which has a purely geometric character and leads to observable consequences. We shall now derive this *geometric phase*.
Let:

$$|\psi(t)\rangle = \exp\left[i\eta(t)\right]|n, \mathbf{R}(t)\rangle \quad , \quad |\psi(0)\rangle = |n, \mathbf{R}(0)\rangle\ . \tag{3.1.0.5}$$

The total phase[1] $\eta(t)$ is found by substitution of Eq. (3.1.0.5) in Eq. (3.1.0.1). On account of the adiabatic assumption Eq. (3.1.0.3) this gives:

$$i\frac{\mathrm{d}\,|\psi(t)\rangle}{\mathrm{d}t} = i\left(i\frac{\mathrm{d}\eta}{\mathrm{d}t}\,|n, \mathbf{R}(t)\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\,|n, \mathbf{R}(t)\rangle\right)\exp[i\eta(t)] \tag{3.1.0.6}$$

for the right-hand side and:

$$H(\mathbf{R}(t))\,|\psi(t)\rangle = E_n(\mathbf{R}(t))\exp[i\eta(t)]\,|n, \mathbf{R}(t)\rangle \tag{3.1.0.7}$$

for the left-hand side. Equating both sides and left-multiplying by $\langle n, \mathbf{R}(t)|$ we get a differential equation for η:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -E_n(\mathbf{R}(t)) + i\,\langle n, \mathbf{R}(t)|\,\frac{\mathrm{d}}{\mathrm{d}t}\,|n, \mathbf{R}(t)\rangle\ , \tag{3.1.0.8}$$

[1]Notice that the phase must be a function of the eigenstate and should be then more accurately denoted by $\eta(n, t)$.

so that an immediate integration yields:

$$\eta(t) = -\int_0^t E_n(\mathbf{R}(s))\,\mathrm{d}s + i\int_0^t \langle n,\mathbf{R}(s)|\,\frac{\mathrm{d}}{\mathrm{d}s}\,|n,\mathbf{R}(s)\rangle\;\mathrm{d}s \equiv \eta_d(t) + \eta_g(t)\,. \tag{3.1.0.9}$$

Together with the expected dynamical phase η_d a new term η_g has appeared, and one must argue its dependence only on the *geometric path* followed by the system during the evolution because the energy is "confined"in the first term:

$$\eta_g(t) = i\int_0^t \langle n,\mathbf{R}(s)|\,\frac{\mathrm{d}}{\mathrm{d}s}\,|n,\mathbf{R}(s)\rangle\;\mathrm{d}s = i\int_\gamma \langle n,\mathbf{R}|\,\nabla_{\mathbf{R}}\,|n,\mathbf{R}\rangle\;\mathrm{d}\mathbf{R}\,, \tag{3.1.0.10}$$

being $\gamma = \{\mathbf{R}(t)\,,\,0\le s\le t\}$ a path in the parameter space. We can therefore write:

$$|\psi(t)\rangle = \exp\left[i\eta_d(t)\right]\exp\left[i\eta_g(t)\right]|n,\mathbf{R}(t)\rangle\;. \tag{3.1.0.11}$$

Remark. Being $E_n(\mathbf{R}(t))$ real for all t, $i\eta_d(t)$ is obviously purely imaginary. We can observe that, in view of the orthonormality condition $\langle n,\mathbf{R}(t)|m,\mathbf{R}(t)\rangle = \delta_{n,m}$, this is so also for $i\eta_g$. Indeed:

$$\begin{aligned}2\,\mathrm{Re}\left\{\langle n,\mathbf{R}(s)|\,\frac{\mathrm{d}}{\mathrm{d}s}\,|n,\mathbf{R}(s)\rangle\right\} &= \langle n,\mathbf{R}(s)|\,\frac{\mathrm{d}}{\mathrm{d}s}\,|n,\mathbf{R}(s)\rangle + \left(\frac{\mathrm{d}}{\mathrm{d}s}\,\langle n,\mathbf{R}(s)|\right)|n,\mathbf{R}(s)\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\langle n,\mathbf{R}(s)|\,|n,\mathbf{R}(s)\rangle\right) = 0\,,\end{aligned}$$

so that:

$$\eta_g(t) = -\,\mathrm{Im}\left\{\int_0^t \langle n,\mathbf{R}(s)|\,\frac{\mathrm{d}}{\mathrm{d}s}\,|n,\mathbf{R}(s)\rangle\;\mathrm{d}s\right\}. \tag{3.1.0.12}$$

Now suppose that the system executes a closed loop Γ in the parameter space, that is $\mathbf{R}(T) = \mathbf{R}(0)$ for some $T > 0$. Then we have:

$$\eta_g(t) = -\,\mathrm{Im}\left\{\oint_\Gamma \langle n,\mathbf{R}|\,\nabla_{\mathbf{R}}\,|n,\mathbf{R}\rangle\;\mathrm{d}\mathbf{R}\right\}. \tag{3.1.0.13}$$

Since the integrand is not necessarily an exact one-form (i.e. $\langle n,\mathbf{R}(s)|\,\frac{\mathrm{d}}{\mathrm{d}s}\,|n,\mathbf{R}(s)\rangle$ is not a total time derivative) the *geometric phase might not vanish along a closed loop*. Berry showed that this is indeed the case.[2]
Moreover, the geometric phase which is developed in such a situation is *gauge invariant*, that is, it can't be made to vanish by performing a gauge transformation on the instantaneous eigenkets:

$$|n,\mathbf{R}(t)\rangle \mapsto |n,\mathbf{R}(t)\rangle' \equiv \exp[i\alpha(\mathbf{R}(t))]\,|n,\mathbf{R}(t)\rangle\;. \tag{3.1.0.14}$$

[2]Notice that there is however an important situation in which the geometric phase is absent; that is when the energy eigenstates $|n,\mathbf{R}(t)\rangle$ can be chosen real (as it is evident by inspecting Eq. (3.1.0.12)).

In fact we can easily check that under this transformation the integrand changes as:

$$\begin{aligned}\langle n,\mathbf{R}|\,\nabla_{\mathbf{R}}\,|n,\mathbf{R}\rangle &\mapsto \langle n,\mathbf{R}|\exp[-i\alpha(\mathbf{R})]\nabla_{\mathbf{R}}\left(\exp[-i\alpha(\mathbf{R})]\,|n,\mathbf{R}\rangle\right)\\ &= \langle n,\mathbf{R}|\,\nabla_{\mathbf{R}}\,|n,\mathbf{R}\rangle + i\nabla_{\mathbf{R}}(\alpha(\mathbf{R}))\,,\end{aligned} \tag{3.1.0.15}$$

and by an integration of the above transformation rule we get:

$$\eta_g(t)\mapsto\eta_g(t)'\equiv\eta_g(t)-\alpha(\mathbf{R}(t))+\alpha(\mathbf{R}(0))\Longrightarrow\eta_g(T)=\eta_g(T)'\,. \tag{3.1.0.16}$$

The above equation eventually shows the gauge invariance of the Berry phase developed along a closed loop. However, there is another reason which makes Eq. (3.1.0.15) particularly suggestive, that is, the fact that the quantity:

$$\nabla\times\langle n,\mathbf{R}|\,\nabla_{\mathbf{R}}\,|n,\mathbf{R}\rangle \tag{3.1.0.17}$$

is gauge invariant independently of the particular transformation the system undergoes. One is then lead to make a contact with electromagnetism, by regarding $\langle n,\mathbf{R}|\,\nabla_{\mathbf{R}}\,|n,\mathbf{R}\rangle$ as the analogous of a vector potential and its curl as the respective magnetic field. In this way, the geometric phase Eq. (3.1.0.13) would naturally appear as a magnetic flux throughout the surface enclosed by Γ. We will come back on this analogy in the next Sections.

3.2 The geometric phase as a holonomy on a $U(1)$ bundle

We shall now give a description of the geometric phase within the framework we have developed in the first Chapter, that is, principal bundles and connections over them.
Consider the parameter space manifold M, whose local coordinates are $\mathbf{R}=(R_1,...,R_n)$. At each point $p\in M$ we have a Hamiltonian $H(\mathbf{R})$ whose normalized n-th eigenstate is $|n,\mathbf{R}\rangle$: this will be henceforth denoted simply by $|\mathbf{R}\rangle$, since we are assuming that the evolution is adiabatic and there is no state transition. The relevant quantum states are defined up to a phase, so that we can introduce an equivalence relation:

$$|\mathbf{R}\rangle\sim|\mathbf{R}\rangle'\Leftrightarrow|\mathbf{R}\rangle=g\,|\mathbf{R}\rangle\quad,\quad g\in U(1)\,, \tag{3.2.0.1}$$

and define the total space over U as $P\equiv\{[|\mathbf{R}\rangle]_\sim\,;\,\mathbf{R}\in M\}$. If we further define a canonical projection:

$$\pi:P\longrightarrow M\quad,\quad\pi[|\mathbf{R}\rangle]\equiv|\mathbf{R}\rangle\ , \tag{3.2.0.2}$$

then we have specified a principal $U(1)$ bundle P over M, and through local sections we can fix the phase of the quantum state $|\mathbf{R}\rangle$ at each point $\mathbf{R}$ on M. Therefore, as we had

already anticipated in Section 1.1.2, local sections of $U(1)$ bundles are in this context the geometric representations of wavefunctions. Let:

$$\sigma : M \supset U \to P \quad , \quad \sigma(\mathbf{R}) \equiv |\mathbf{R}\rangle = e\,|\mathbf{R}\rangle \ , \tag{3.2.0.3}$$

be the trivial local section over U. The corresponding local trivialization $\Phi : \pi^{-1}(U) \to U \times U(1)$ is then given by:

$$\Phi(|\mathbf{R}\rangle) = (\mathbf{R}, e)\,, \tag{3.2.0.4}$$

and the $U(1)$ right action on P (equivalent to the left one) is:

$$|\mathbf{R}\rangle\, g = \sigma(\mathbf{R})g = \Phi^{-1}((\mathbf{R}, e)g) = \Phi^{-1}(\mathbf{R}, g)\,. \tag{3.2.0.5}$$

We can now provide the principal bundle with a connection. We define the **Berry's connection** over $U \subset M$ as the local connection one-form[3] $\mathcal{A} \in \Omega^1(U)$ which in local coordinates $\mathbf{R} = (R_1, ..., R_n)$ is given by:

$$\mathcal{A} \equiv \mathcal{A}_\mu \mathrm{d}R^\mu = \langle \mathbf{R}|\,\mathrm{d}\,|\mathbf{R}\rangle = \langle \mathbf{R}|\,\frac{\partial}{\partial R^\mu}\,|\mathbf{R}\rangle\,\mathrm{d}R^\mu\,. \tag{3.2.0.6}$$

This is an *anti-Hermitian* connection, since:

$$0 = \frac{\partial}{\partial R^\mu}(\langle \mathbf{R}|\mathbf{R}\rangle) = \left(\frac{\partial\,\langle \mathbf{R}|}{\partial R^\mu}\right)|\mathbf{R}\rangle + \langle \mathbf{R}|\left(\frac{\partial\,|\mathbf{R}\rangle}{\partial R^\mu}\right) \equiv \left(\mathcal{A}^\dagger\right)^\mu + \mathcal{A}^\mu\,. \tag{3.2.0.7}$$

Now, if U_i and U_j are two overlapping charts over M whose respective sections are $\sigma_i(\mathbf{R}) = |\mathbf{R}\rangle_i$ and $\sigma_j(\mathbf{R}) = |\mathbf{R}\rangle_j$ then the latter are related by a transition function $t_{ij} : U_i \cap U_j \to U(1)$ as $|\mathbf{R}\rangle_j = |\mathbf{R}\rangle_i\, t_{ij} = t_{ij}\,|\mathbf{R}\rangle_i$. Consequently:

$$\begin{aligned}\mathcal{A}_j &=_j \langle \mathbf{R}|\,\mathrm{d}\,|\mathbf{R}\rangle_j = t_{ij}(\mathbf{R})^*_i\,\langle \mathbf{R}|\,[(\mathrm{d}\,|\mathbf{R}\rangle_i)t_{ij}(\mathbf{R}) + |\mathbf{R}\rangle_i\,(\mathrm{d}t_{ij}(\mathbf{R}))] \\ &= \mathcal{A}_i + t_{ij}(\mathbf{R})^{-1}\mathrm{d}t_{ij}(\mathbf{R})\,,\end{aligned} \tag{3.2.0.8}$$

where we have used the normalization of the eigenstate and the unitarity of the structure group, i.e. $t_{ij}(\mathbf{R})^* = t_{ij}(\mathbf{R})^{-1}$. This is nothing but the consistency law Eq. (1.2.1.24) in the case of an abelian structure group. Therefore a set of local Berry's connections $\{A_i\}$ over $U_i \subset M$ define through their pullbacks an unique Ehresmann connection on $(P, \pi, M, U(1))$. Further, if one sets $t_{ij}(\mathbf{R}) \equiv \exp\{i\Lambda(\mathbf{R})\}$ the electromagnetic gauge transformation Eq. (1.2.1.27) is recovered. This supports our previous suggestion on the mathematical affinity between the vector potential and the Berry phase, being both just $U(1)$ connections on a principal bundle. The same relationship is of course shared by the electromagnetic field tensor and the local **Berry's curvature**, where the latter is given by:

$$\mathcal{F} = \mathrm{d}\mathcal{A} = (\mathrm{d}\,\langle \mathbf{R}|) \wedge (\mathrm{d}\,|\mathbf{R}\rangle) = \left(\frac{\partial\,\langle \mathbf{R}|}{\partial R^\mu}\right)\left(\frac{\partial\,|\mathbf{R}\rangle}{\partial R^\nu}\right)\mathrm{d}R^\mu \wedge \mathrm{d}R^\nu\,. \tag{3.2.0.9}$$

[3]Observe that the Lie Algebra $\mathfrak{u}(1)$ is one-dimensional so that $\Omega^1(U, \mathfrak{u}(1)) \simeq \Omega^1(U, \mathbb{R})$.

We can now proceed and unveil the role played by the adiabatic phase within this geometric framework. First, let us define -for each energy level n and instant of time t- a "rescaled"Hamiltonian:

$$\tilde{H}(\mathbf{R}(t)) \equiv H(\mathbf{R}(t)) - E_n(\mathbf{R}(t))\,. \tag{3.2.0.10}$$

The eigenvalue relation fulfilled by the instantaneous normalized n-th eigenvector $|\mathbf{R}(t)\rangle$, Eq. (3.1.0.3), reads then:

$$\tilde{H}(\mathbf{R}(t))\,|\mathbf{R}(t)\rangle = 0\,, \tag{3.2.0.11}$$

while the modified Schroedinger equation (i.e. Eq. (3.1.0.1) with H replaced by $\tilde{H}$) is solved by:

$$|\psi(t)\rangle = \exp[i\eta_g(t)]\,|\mathbf{R}(t)\rangle\ , \tag{3.2.0.12}$$

being $\eta_g(t)$ the geometric phase defined by Eq. (3.1.0.12), as one could have also (clumsily) inferred by taking $E_n \equiv 0$ in Eq. (3.1.0.9). The dynamical phase has thus been discarded and will make no appearance in our further considerations. Consider then a loop in the parameter space:

$$\gamma:\,[0,1] \longrightarrow U \subset M \quad , \quad \gamma(t) \equiv \mathbf{R}(t)\,. \tag{3.2.0.13}$$

If $\sigma(\mathbf{R}(t)) = |\mathbf{R}(t)\rangle$ is a local section over U and the respective local connection form $\mathcal{A}$ is given by Eq. (3.2.0.6) then we can follow the procedure adopted in Section 1.2.2 and build the horizontal lift of γ with respect to $\mathcal{A}$ in P:

$$\tilde{\mathbf{R}}(t) \equiv \tilde{\gamma}(t) = \sigma(\mathbf{R}(t))g(\mathbf{R}(t))\,, \tag{3.2.0.14}$$

where the usual choice $g(\mathbf{R}(0)) = e\ \in U(1)$ has been made. The variation of the group element $g(\mathbf{R}(t))$ with respect to t is given by:

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = -\mathcal{A}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) g(t)\,, \tag{3.2.0.15}$$

that is:

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} g(t)^{-1} = -\mathcal{A}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = -\,\langle \mathbf{R}(t)|\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\mathbf{R}(t)\rangle\ . \tag{3.2.0.16}$$

The previous equation defines a rule for parallel transporting a point along the lifted curve. If we now set $g(t) \equiv \exp[i\eta(t)]$ then $\eta(t)$ is easily recognized as the geometric phase, since Eq. (3.2.0.16) yields:

$$i\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\,\langle \mathbf{R}(t)|\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\mathbf{R}(t)\rangle = -\,\langle \mathbf{R}(t)|\,\nabla_{\mathbf{R}}\,|\mathbf{R}(t)\rangle\,\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\,, \tag{3.2.0.17}$$

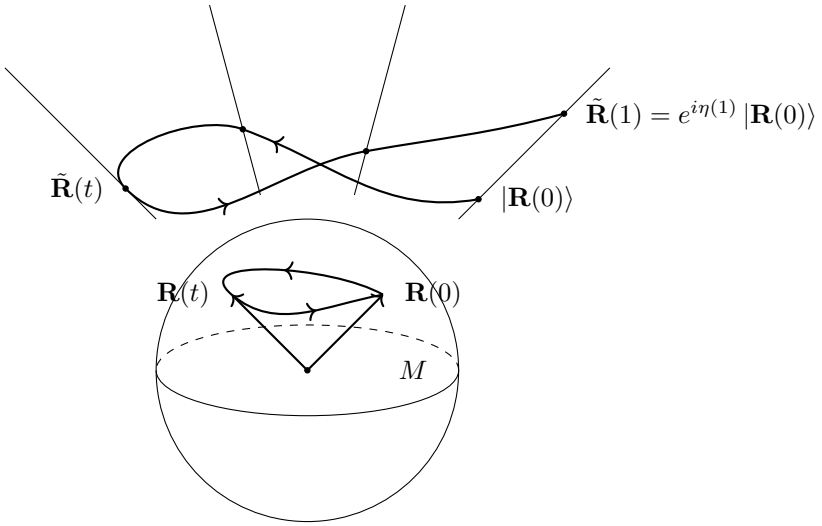

Figure 3.1: Adiabatic phase as a holonomy on a principal bundle.

which is nothing but Eq. (3.1.0.8) once that one has neglected the dynamical term. After an immediate integration we have:

$$\eta(t) = i\int_0^t \langle \mathbf{R}(s)|\,\frac{\mathrm{d}}{\mathrm{d}s}\,|\mathbf{R}(s)\rangle\,\mathrm{d}s \Longrightarrow \eta(1) = i\oint_\gamma \langle \mathbf{R}|\,\mathrm{d}\,|\mathbf{R}\rangle = i\oint_\gamma \mathcal{A}_\mu \mathrm{d}R^\mu\,. \qquad (3.2.0.18)$$

Finally, the parallel transported of the point $\mathbf{R}(0) = \mathbf{R}(1)$ as a function of the Berry's connection is given by:

$$\tilde{\mathbf{R}}(1) = \exp[i\eta(1)]\,|\mathbf{R}(0)\rangle = \exp\left[-\oint_\gamma \mathcal{A}_\mu \mathrm{d}R^\mu\right]|\mathbf{R}(0)\rangle\ , \qquad (3.2.0.19)$$

where we have used the fact that $|\mathbf{R}(0)\rangle = |\mathbf{R}(1)\rangle$. The previous equation elucidates the meaning of the geometric phase acquired by a particle which slowly moves along a loop as a *holonomy* on the $U(1)$ bundle built over the parameter space: $\exp[i\eta(1)]$ measures the distance between the horizontal lift of the starting and the ending point of the loop (see Figure 3.1). The interpretation of the adiabatic phase as a holonomy was firstly given by B. Simon [24].

It is of course also possible to express the previous result in terms of the curvature $\mathcal{F} = \mathrm{d}\mathcal{A}$; if S is a closed two surface in M whose boundary is $\partial S = \gamma$ then Eq. (3.2.0.19) implies:

$$\tilde{\mathbf{R}}(1) = \exp\left[-\int_S \mathcal{F}_{\mu\nu}\mathrm{d}R^\mu \wedge \mathrm{d}R^\nu\right]|\mathbf{R}(0)\rangle\ . \qquad (3.2.0.20)$$

3.3 Spinning particle in a slowly varying magnetic field

As a simple application of the theory previously exposed we shall now derive the geometric phase developed by a particle with a spin magnetic moment in the presence of an adiabatically changing magnetic field. As we shall see, also in the quantum case a surprising similarity between this system and that of a magnetic monopole sitting at a fixed point in space shows up.

Suppose that a spin-1/2 particle is in the presence of a time varying magnetic field $\mathbf{B}(t)$ whose modulus B is fixed. The spin operator for the particle is $\hat{\mathbf{s}} = \frac{\hbar\hat{\sigma}}{2}$ so that the spin magnetic moment is $\hat{\mu}_s = -\frac{\mu_B g_e \hat{\sigma}}{2} = -\mu_B \hat{\sigma}$, where we have assumed the spin g-factor to be $g_e = 2$. Then, neglecting the translational degrees of freedom and taking $\mathbf{R}(\mathbf{t}) \equiv \mu_B \mathbf{B}(t)$ as coordinates on the parameter manifold, the Hamiltonian of the system is an immediate generalization of Eq. (2.2.1.2):

$$H(\mathbf{R}(t)) = -\hat{\boldsymbol{\mu}}_B \cdot \mathbf{B}(t) = \mathbf{R}(t) \cdot \hat{\boldsymbol{\sigma}} = \begin{pmatrix} R_3 & , & R_1 - iR_2 \\ R_1 + iR_2 & , & -R_3 \end{pmatrix} . \tag{3.3.0.1}$$

The time varying parameter is then the direction of the magnetic field, with $\mathbf{R}(t) = (R_1(t), R_2(t), R_3(t))$ spanning a two-sphere. Therefore, if the magnetic field has a fixed magnitude, we can take with no loss of generality $M \equiv S^2$ as the parameter base manifold. Notice that this situation is dual to the one described in Section 2.2, in which the spin vector $\mathbf{S}$ was regarded as varying on a sphere.
Eq. (3.3.0.1) describes a two-state system, for which it is known that, at a fixed time t, the energy eigenvalues are $\pm|\mathbf{R}|$ and the Schroedinger equation Eq. (3.1.0.1) reads:

$$(\mathbf{R} \cdot \hat{\boldsymbol{\sigma}}) \, |\mathbf{R}\rangle_{\pm} = \pm|\mathbf{R}| \, |\mathbf{R}\rangle_{\pm} \, . \tag{3.3.0.2}$$

Let us now adopt spherical coordinates in the $\mathbf{R}$ space, so that:

$$R_1 = R \sin\theta \cos\phi \quad , \quad R_2 = R \sin\theta \sin\phi \quad , \quad R_3 = R \cos\theta \, ; \tag{3.3.0.3}$$

with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$. It is well known from the Pauli theory of two-state systems that if we consider the two eigenvectors of $\hat{\sigma}_3$ (that is, the spin eigenstates when the magnetic field is along the z-axis, $\mathbf{B} = B\mathbf{e}_3$):

$$|+1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |-1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad \hat{\sigma}_3 \, |\pm 1\rangle = \pm \, |\pm 1\rangle \, , \tag{3.3.0.4}$$

then the eigenvectors $|\mathbf{R}\rangle_{\pm}$, i.e. spin eigenstates when the angular coordinates of the field are given by Eq. (3.3.0.3), are obtained by a "rotation"of $|+1\rangle$ and $|-1\rangle$; namely:

$$|\mathbf{R}\rangle_{+} = \cos(\theta/2) \, |+1\rangle + \exp(i\phi) \sin(\theta/2) \, , \tag{3.3.0.5}$$

$$|\mathbf{R}\rangle_{-} = -\exp(-i\phi) \sin(\theta/2) \, |+1\rangle + \cos(\theta/2) \, |-1\rangle \, . \tag{3.3.0.6}$$

Let us concentrate on the positive eigenvalue. According to what we did in the previous section, we can introduce a Hamiltonian:

$$\tilde{H}(\mathbf{R}) \equiv H(\mathbf{R}) - \mathbb{1}|\mathbf{R}|\,, \tag{3.3.0.7}$$

so that the zero-energy eigenstate is precisely given by Eq. (3.3.0.5):

$$|\mathbf{R}\rangle_N \equiv |\mathbf{R}\rangle_+ = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} = [2R(R+R_3)]^{-1/2} \begin{pmatrix} R+R_3 \\ R_1+iR_2 \end{pmatrix}. \tag{3.3.0.8}$$

Here the subscript "N"denotes the fact that this eigenvector is singular when $R = -R_3$, i.e. $\theta = \pi$, and this parametrization is valid only on the Northern hemisphere $S_N \subset S^2$. One immediately verifies that ${}_N\langle\mathbf{R}|\mathbf{R}\rangle_N = 1$. On the southern hemisphere $S_S \subset S^2$ we must choose a different eigenvector; a good choice consists in simply rotating $|\mathbf{R}\rangle_N$ around the z-axis, that is:

$$\begin{aligned} |\mathbf{R}\rangle_S &\equiv \exp(-i\phi)\,|\mathbf{R}\rangle_N \\ &= \begin{pmatrix} e^{-i\phi}\cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} = [2R(R-R_3)]^{-1/2} \begin{pmatrix} R_1-iR_2 \\ R-R_3 \end{pmatrix}. \end{aligned} \tag{3.3.0.9}$$

One can immediately check that $|\mathbf{R}\rangle_S$ is normalized and satisfies $\tilde{H}(\mathbf{R})\,|\mathbf{R}\rangle_S = 0$ (and must therefore not be confused with the *negative* eigenstate defined in Eq. (3.3.0.6)). Moreover, $|\mathbf{R}\rangle_S$ is singular when $R = +R_3$ ($\theta = 0$) and is thus well defined on S_S.

An $U(1)$ principal bundle structure is once again naturally defined by the previous construction. The base manifold is S^2, and for $n = +1$, $n = -1$ there is a fibre attached to each point corresponding to the eigenstate $|n, \mathbf{R}\rangle$: the fibre at $\mathbf{R} \in M$ is given by $[|\mathbf{R}\rangle]_\sim$, being $\sim$ the equivalence relation defined in Eq. (3.2.0.1). If the adiabatic assumption holds and there is no state change during the evolution[4] then the total bundle space is

$$P \equiv \{[|\mathbf{R}\rangle]_\sim\,;\, \mathbf{R} \in S^2\}\,, \tag{3.3.0.10}$$

and the canonical projection $\pi : P \longrightarrow S^2$ maps each quantum state into the parameter on which it is defined:

$$\pi : \exp(i\alpha)\,|\mathbf{R}\rangle \longmapsto \mathbf{R} \in S^2\,. \tag{3.3.0.11}$$

The bundle is not trivial since at least two charts (and two local sections accordingly) are required to parametrize the states at each point. Notice also that when $\mathbf{R} = 0$ (that is, $\mathbf{B} = 0$) then the separation between the two eigenstate $|\mathbf{R}\rangle_+$ and $|\mathbf{R}\rangle_-$ ceases to exist

[4]In this context the adiabatic assumption simply states that when the magnetic fields rotates, the particle keeps its magnetic moment always aligned to the former. This is approximately so when $\Omega \ll \omega_L$, being Ω a characteristic rotational frequency of the field and $\omega_L \equiv 2\mu_B B/\hbar$ the Larmor frequency of the spin precession. Since the system has only two possible states, when the adiabatic condition breaks then the fields rotates fast enough to make the particle "reverse"its spin.

and there is only a doubly degenerate zero-energy state.
We can now work out the local connections on the two hemispheres. Recalling that in local cartesian coordinates the external derivative simply reads:

$$\mathrm{d}\,|\mathbf{R}\rangle_{N,S} = \frac{\partial|\mathbf{R}\rangle_{N,S}}{\partial R^\mu}\mathrm{d}R^\mu \quad , \quad \mu = 1,2,3\,; \tag{3.3.0.12}$$

a long but plain calculation [3] yields for $\mathcal{A}_N \in \Omega^1(S_N)$:

$$\mathcal{A}_N \equiv_N \langle\mathbf{R}|\,\mathrm{d}\,|\mathbf{R}\rangle_N = -i\frac{R^2\mathrm{d}R^1 - R^1\mathrm{d}R^2}{2R(R+R^3)}\,. \tag{3.3.0.13}$$

An analogous result holds for $\mathcal{A}_S \in \Omega^1(S_S)$:

$$\mathcal{A}_S \equiv_S \langle\mathbf{R}|\,\mathrm{d}\,|\mathbf{R}\rangle_S = +i\frac{R^2\mathrm{d}R^1 - R^1\mathrm{d}R^2}{2R(R-R^3)}\,. \tag{3.3.0.14}$$

The gauge potentials $\mathcal{A}_N$ and $\mathcal{A}_S$ can be expressed in a more suggestive form if one turns to polar coordinates. Indeed, by Eq. (3.3.0.8), Eq. (3.3.0.9) and an elementary application of the chain rule $\mathrm{d}R^\mu = \frac{\partial R^\mu}{\partial\theta}\mathrm{d}\theta + \frac{\partial R^\mu}{\partial\phi}\mathrm{d}\phi$ one gets:

$$\mathcal{A}_N = \frac{1}{2}i(1-\cos\theta)\mathrm{d}\phi \quad , \quad \theta \neq \pi\,, \tag{3.3.0.15}$$

$$\mathcal{A}_S = -\frac{1}{2}i(1+\cos\theta)\mathrm{d}\phi \quad , \quad \theta \neq 0\,, \tag{3.3.0.16}$$

and the transformation law Eq. (3.2.0.8) reads:

$$\mathcal{A}_S = A_N - i\mathrm{d}\phi = \mathcal{A}_N + \exp(i\phi)\mathrm{d}\exp(-i\phi)\,. \tag{3.3.0.17}$$

Therefore one can identify the transition function t_{NS} on the equatorial line as:

$$t_{NS} : S_N \cap S_S \longrightarrow U(1) \quad , \quad t_{NS}(\pi/2,\phi) \equiv \exp(-i\phi)\,. \tag{3.3.0.18}$$

With this choice of coordinates one can also immediately obtain a simple expression for the curvature two-form; namely, from Eq. (3.3.0.15) or Eq. (3.3.0.16):

$$\mathcal{F} \equiv \mathrm{d}\mathcal{A} = i\frac{1}{2}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi, \tag{3.3.0.19}$$

where the subscript has been suppressed since the curvature is gauge invariant.[5]
Now the analogy between this physical system and the magnetic monopole is completely evident: in both cases we have a non-trivial $P(S^2,U(1))$ principal bundle, and by comparing Eq. (3.3.0.15) and (3.3.0.16) with Eq. (2.1.1.19) (with $r=1$, $\theta=\pi/2$) one can

[5]Specifically, minus signs "cancel out"when one takes the exterior derivative of both $\mathcal{A}_N$ and $\mathcal{A}_S$.

identify the local connections for the spinning-particle system with the local vector potentials of a Wu-Yang monopole of strength $g = -1/2$. The $U(1)$ gauge transformation Eq. (2.1.1.21) and Eq. (3.3.0.17) therefore do also coincide.

We now turn to the holonomy and the adiabatic phase. The general procedure to derive the geometric phase is the one followed in the previous section. We choose a loop $\gamma(t)$ on S^2, build its horizontal lift $\tilde{\gamma}(t)$ through a section $\sigma : \mathbf{R}(t) \mapsto |\mathbf{R}(t)\rangle$ and parallel transport $|\mathbf{R}(0)\rangle$ along the loop, obtaining a map:

$$h_{\mathbf{R}} : \pi^{-1}(\mathbf{R}) \longrightarrow \pi^{-1}(\mathbf{R}) \quad , \quad |\mathbf{R}(0)\rangle \mapsto \tilde{\mathbf{R}}(1) = \exp[i\eta(1)]\,|\mathbf{R}(0)\rangle \ . \tag{3.3.0.20}$$

The holonomy group element $\exp[i\eta(1)]$ is precisely the Berry phase, which in the current situation is given by:

$$\eta(1) = i \oint_{\gamma} \mathcal{A} = i \int_{\Sigma} \mathcal{F} = -\frac{1}{2} \int_{\Sigma} \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi = -\frac{1}{2}\Omega(\gamma) \, , \tag{3.3.0.21}$$

where $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi$ denotes an infinitesimal solid angle and therefore $\Omega(\gamma)$ is the total solid angle swept by the magnetic field $\mathbf{B}(t)$ while it rotates on S^2. We come then to the conclusion that the geometric phase developed by a spinning particle when the magnetic field (adiabatically) completes a loop is nothing but the total magnetic flux throughout the corresponding solid angle produced by a magnetic monopole fixed at the origin of S^2.

Finally, we observe that this result is general, i.e. is valid also for different values of the spin. Without entering into details (see [23]), the eigenvalue equation for a generic spin operator $\hat{\mathbf{S}}$ becomes

$$-\hat{\boldsymbol{\mu}}_s \cdot \mathbf{B}\,|m, \mathbf{R}\rangle \equiv \alpha \mathbf{R} \cdot \hat{\mathbf{S}}\,|m, \mathbf{R}\rangle = \alpha m \hbar\,|m, \mathbf{R}\rangle \ , \tag{3.3.0.22}$$

being $\alpha \equiv -\frac{\mu_B B g_s}{\hbar}$ and m the integer which enumerates the eigenvalues of the spin projection along the direction of $\mathbf{R}$. The spin 1/2 case obviously corresponds to $m = +1/2$, $m = -1/2$, and the previous result is recovered when one consider the $m = 1/2$ eigenvalue. Starting from Eq. (3.3.0.22) one obtains a Berry phase:

$$\eta(1) = -m\Omega\,(\gamma) \ , \tag{3.3.0.23}$$

thus showing that the general correspondence between the spin projection m and the monopole strength g is:

$$m = -g \, . \tag{3.3.0.24}$$

3.4 The Aharonov-Bohm effect

As shown in Section 1.2.3, an important feature which establishes a primary distinction between local connection one-forms (gauge potentials) and curvature (strength field

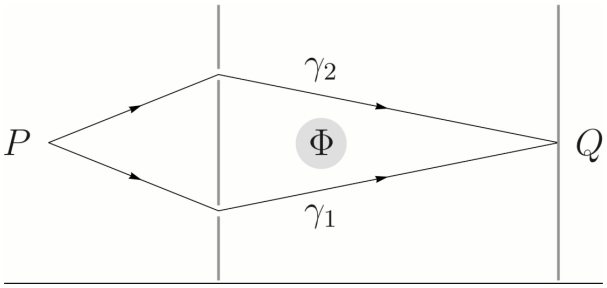

Figure 3.2: Schematical representation of the experimental setup for the Aharonov-Bohm effect.

tensors) is that only the latter are actually gauge invariant and can therefore represent physical observable quantities. This is why classically is the electromagnetic field $F_{\mu\nu}$ to be of principal interest, whereas the four-potential A_μ is only of secondary importance. However, as it was firstly predicted by W. Ehrenberg and R. E. Siday in 1949 [25] and subsequently by Y. Aharonov and D. Bohm in a famous article published ten years later [22], there are certain situations in quantum mechanics in which the role played by the potentials become essential, and give rise to measurable effects. The most famous example is provided by the *Aharonov-Bohm effect*, which we will now briefly expose and describe in terms of quantum geometric phases.

An Aharonov-Bohm experimental apparatus can be schematically represented as in Figure 3.2. Suppose that a beam of electrons with charge $-e$ is incoming from the far left, and that it splits into two different beams at a point distant from the measuring apparatus. Suppose further that there is a solenoid with a finite radius R and that the electrons are prevented from penetrating inside the latter (this is realized e.g. by placing a shield with only two separated slits before the solenoid), so that the two beams can only rejoin at a point on a screen far on the right from the solenoid, where the intensity of the resultant superposition field of the electrons can be measured. For the sake of simplicity, we take the solenoid to be perpendicular to the plane where the motion occurs, so that the accessible configuration space is two dimensional.
If the solenoid is placed along the z-axis then the magnetic field $\mathbf{B} = B\mathbf{e}_3$ is approximately uniform inside the solenoid and is absent everywhere out of it:

$$\mathbf{B}(\mathbf{r}) = 0 \text{ if } r > R \quad , \quad \mathbf{r} = (x, y, 0)\,. \tag{3.4.0.1}$$

Therefore, if Σ is any surface in the x-y plane which encloses the solenoid then the magnetic flux Φ through Σ is fixed and given by:

$$\Phi \equiv \int_\Sigma \mathbf{B}\cdot \mathrm{d}\mathbf{S} = \int_\Sigma \nabla\times\mathbf{A}\cdot \mathrm{d}\mathbf{S} = \oint_{\partial\Sigma} \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \pi B R^2\,, \tag{3.4.0.2}$$

where the vector potential $\mathbf{A}(\mathbf{r})$ is such that $\nabla\times\mathbf{A} = \mathbf{B}$ for each $r < R$ and $\nabla\times\mathbf{A} = 0$ when $r > R$. However, the vanishing of the curl outside of the solenoid does not imply

that $\mathbf{A}$ itself is zero therein. Indeed, if one takes:

$$\mathbf{A}(\mathbf{r}) \equiv \left(-\frac{y\Phi}{2\pi r^2}, \frac{x\Phi}{2\pi r^2}, 0\right) = \frac{\Phi}{2\pi r} \quad , \quad r \geq R \tag{3.4.0.3}$$

$$\mathbf{A}(\mathbf{r}) \equiv \left(-\frac{y\Phi}{2\pi R^2}, \frac{x\Phi}{2\pi R^2}, 0\right) = \frac{\Phi}{2\pi R} \quad , \quad r \leq R\,; \tag{3.4.0.4}$$

then it is immediate to check that Eq. (3.4.0.2) is satisfied, even though $\mathbf{A} \neq 0$ outside of the solenoid.

Remark. One might try to gauge-transform the potential defined by Eqs. (3.4.0.3) - (3.4.0.4) so that $\mathbf{A}' \equiv \mathbf{A} + \nabla\Lambda$ is identically vanishing when $r > R$. Adopting spherical coordinates, that would be possible if there existed a scalar function $\Lambda(r, \theta, \phi)$ such that:

$$\nabla\Lambda(r, \theta, \phi) + \frac{\Phi}{2\pi r}\mathbf{e}_\theta = 0 \Longrightarrow \frac{1}{r}\frac{\partial\Lambda}{\partial\theta} + \frac{\Phi}{2\pi r} = 0\,, \tag{3.4.0.5}$$

where $\mathbf{e}_\theta \equiv -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y$. Up to an integration constant, the above equation admits the solution:

$$\Lambda(r, \theta, \phi) = -\frac{\Phi}{2\pi}\,. \tag{3.4.0.6}$$

Eq. (3.4.0.6) does not define an acceptable gauge transformation since Λ is not single-valued and therefore neither is the transformed wavefunction. This means that we can't make $\mathbf{A}$ vanish everywhere through a global gauge-fixing unless we allow the latter to be defined by a non single-valued scalar function.

Nonetheless, *classically*, the presence of a non-vanishing vector potential is expected to be of no influence on the motion of the electrons, since the latter is governed by the Lorentz force $-e(\mathbf{v} \times \mathbf{B})$ in which only the field makes an appearance. This in contrast with the experimental evidence of an *interference* registered on the screen when the two beams reunite, which indicates that the respective wavefunctions have developed a phase shift along their separated paths. This can be explained on account of the quantum behaviour of the particles.

Let us denote by γ_1 and γ_2 the two paths followed by the particles and by $\psi_i(\mathbf{r}, t)$, $i = 1, 2$, the respective wavefunctions when $\mathbf{A} = 0$ outside of the solenoid. When there is instead a non vanishing vector potential the time-dependent Schroedinger equation for the electrons reads:

$$\hat{H}\psi_i \equiv \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}}\right)^2 \psi(\mathbf{r}, t) = i\hbar\frac{\partial\psi_i}{\partial t} \quad , \quad i = 1, 2\,; \tag{3.4.0.7}$$

where $\hbar$ and c have been reintroduced. As shown in Section 2.1.2 the wavefunctions $\psi_i^{\mathbf{A}}(\mathbf{r}, t)$ in this situation are obtained by gauge-transforming $\psi_i(\mathbf{r}, t)$; namely if the field configuration is stationary ($\mathbf{A}$ is constant in time):

$$\psi_i^{\mathbf{A}}(\mathbf{r}, t) = \exp[i\eta(\gamma_i)]\psi_i(\mathbf{r}, t) \equiv \exp\left[\frac{ie}{\hbar c}\int_{\gamma_i}\mathbf{A}(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\right]\psi_i(\mathbf{r}, t) \quad , \quad i = 1, 2\,; \tag{3.4.0.8}$$

being γ_i two different paths which both start at a very distant point P and have the same ending point Q on the screen. Let $\mathbf{r}_0$, $\mathbf{r}$ denote the position of P and Q respectively. Notice that the integrals appearing in the phases η_i in Eq. (3.4.0.8) would be equal if the region where the motion takes place were simply connected: indeed, since $\nabla \times \mathbf{A} = 0$ outside of the solenoid, the integrand is a closed one-form and therein locally exact, so that the the integrals would depend only on the ending points. However, the region is *not* simply connected because of the solenoid, so that there is an explicit dependence on the path.

We can now compute the resulting wavefunction at $\mathbf{r}$. Since the field is constant we can neglect the explicit time dependence of $\psi_i^{\mathbf{A}}$ (which is simply a phase factor) so that:

$$\begin{aligned}\psi_1^{\mathbf{A}}(\mathbf{r}) + \psi_2^{\mathbf{A}}(\mathbf{r}) =& \exp\left[\frac{ie}{\hbar c}\int_{\gamma_1}\mathbf{A}(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\right]\psi_1(\mathbf{r}) \\ &+ \exp\left[\frac{ie}{\hbar c}\int_{\gamma_2}\mathbf{A}(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\right]\psi_2(\mathbf{r}) \\ =& \exp\left[\frac{ie}{\hbar c}\int_{\gamma_2}\mathbf{A}(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\right]\left\{\exp\left[\frac{ie}{\hbar c}\oint_{\gamma}\mathbf{A}(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\right]\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})\right\},\end{aligned} \tag{3.4.0.9}$$

where $\gamma \equiv \gamma_1 \cup (-\gamma_2)$ is a loop enclosing the solenoid. Recalling Eq. (3.4.0.2) we find that there is an *observable phase shift* between the two wavefunctions, which is given by:

$$\eta(\gamma) \equiv \frac{ie}{\hbar c}\oint_{\gamma}\mathbf{A}(\mathbf{r}')\,\mathrm{d}\mathbf{r}' = \frac{ie\Phi}{\hbar c}\,. \tag{3.4.0.10}$$

The resulting expression is independent of γ as long as the latter actually encloses the solenoid; otherwise, if the beams were to pass-by the solenoid remaining "on the same side"of the latter there would be no interference.
Notice also that two different values of Φ, say Φ_1 and Φ_2, would cause the same shift if:

$$\Delta\Phi = \Phi_1 - \Phi_2 = \frac{2\pi\hbar n}{e} \quad , \quad n \in \mathbb{Z}\,. \tag{3.4.0.11}$$

Curiously, up to an irrelevant geometric factor, this quantization relation for the flux is the same obtained by Dirac for the magnetic monopole, Eq. (2.1.2), with the substitution[6] $\Phi \leftrightarrow g$. It is therefore suggestive to think of the solenoid as a sort of Dirac string singularity caused by a monopole placed at the origin.

We can now turn to the geometry underlying the Aharonov-Bohm effect and explain the geometric phase shift in terms of holonomy. For simplicity, we take the radius of the solenoid to be infinitesimally small (keeping the flux fixed), so that the the base manifold

[6]This is dimensionally consistent since a flux and a magnetic charge share the same units of measurement.

is $M \equiv \mathbb{R}^2 \setminus 0$. The principal bundle structure is $(P, \pi, M, U(1))$, being the total space defined by the equivalence relation of Section 3.2; the associated vector bundle is instead $E \equiv P_{\mathbb{C}} \times \mathbb{C}$, with the sections $\sigma : M \to E$ once again representing wavefunctions. Let now $A_\mu \equiv (0, \mathbf{A})$, with $\mathbf{A}$ given by Eq. (3.4.0.3), and define a local connection one-form:

$$\mathcal{A} \in \Omega^1(M) \quad , \quad \mathcal{A} \equiv i\mathcal{A}_\mu \mathrm{d}x^\mu \, . \tag{3.4.0.12}$$

Such a connection, as we know, uniquely separates (when lifted up to P through a section) $\mathrm{T}P$ into a horizontal and a vertical component at each point, so that a covariant derivative over P (Eq. (1.2.3.2)) is well defined and given by $\mathrm{D} = \mathrm{d} + \mathcal{A}$. Since the origin has been excluded from M we also have $\mathrm{d}\mathcal{A} = \mathcal{F} = 0$ and the bundle is locally flat.
Now restrict the connection to the unit circle enclosing the solenoid, $S^1 = \{e^{i\theta}\, 0 \leq \theta \leq 2\pi\}$, so that the connection Eq. (3.4.0.12) reads therein ($r = 1\,,\, \{x^\mu\} = \theta$):

$$\mathcal{A} = i\frac{\Phi}{2\pi}\mathrm{d}\theta \, . \tag{3.4.0.13}$$

Take $\psi : S^1 \longrightarrow E$ to be a local section. By imposing that ψ is parallel transported along S^1, that is $\mathrm{D}\psi = 0$, we obtain an explicit expression for the section and, so, for the horizontal lift of the circle:

$$\left(\mathrm{d} + i\frac{\Phi}{2\pi}\mathrm{d}\theta\right)\psi(\theta) = \frac{\mathrm{d}\psi}{\mathrm{d}\theta}\mathrm{d}\theta + i\frac{\Phi}{2\pi}\psi\mathrm{d}\theta = 0 \Longrightarrow \psi(\theta) = \exp\left[-\frac{i\Phi\theta}{2\pi}\right] \, ; \tag{3.4.0.14}$$

where we have normalized to 1 the amplitude coefficient. Up to numerical coefficients (that one adjusts by a slight modification in the definition of D), the last term in the equation above is precisely the Aharonov-Bohm phase factor $\exp[\eta(S^1)]$ which we ultimately recognize as a geometric phase, i.e. a holonomy:

$$\Gamma : \pi^{-1}(\theta = 0) \to \pi^{-1}(\theta = 2\pi) \quad , \quad \Gamma[\psi(0)] \equiv e^{-i\Phi}\psi(0) \, . \tag{3.4.0.15}$$

Appendix A

The adiabatic theorem

In Section 3.1 it was shown that if a quantum system evolves according to:[1]

$$\hat{H}(t)\,|\psi(t)\rangle = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle \quad , \quad t \in [t_0, t_1]\,, \tag{A.1}$$

and the discrete, orthonormal and non-degenerate eigenstates $|n,t\rangle$ of $\hat{H}(t)$ are such that the adiabatic assumption holds:

$$\hat{H}(t)\,|n,t\rangle = E_n(t)\,|n,t\rangle\ , \tag{A.2}$$

then if $|\psi(t_0)\rangle = |n,t_0\rangle$, the state at a time $t > t_0$ is given by:

$$|\psi(t)\rangle = \exp[i\eta_d(t)]\exp[i\eta_g(t)]\,|n,t\rangle\ , \tag{A.3}$$

being η_d, η_g the dynamical and geometric phase respectively.
Following [26] we shall now derive a quantitative condition which establishes when the evolution can be considered "slow enough"for Eq. (A.2) to hold. Let $\tau \equiv t_1 - t_0$ be the period of time in which the evolution occurs. If the (generic) state at t_0 is $|\psi(t_0)\rangle$ then its evolution in time is given by:

$$|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle \quad , \quad t \in [t_0, t_1]\,, \tag{A.4}$$

where $\hat{U}$ is the time-evolution operator of the system, which is given by:

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}(t)\hat{U}(t,t_0) \quad , \quad \hat{U}(t_0,t_0) = \mathbb{1}\,. \tag{A.5}$$

[1] Here we tacitly assume that the time-dependence of the Hamiltonian is contained only in the parameters $\mathbf{R}(t)$ so that we can use the shorthand notation $H(\mathbf{R}(t)) \equiv H(t)$. Notice that $\hbar$ has been reintroduced.

Observe that the above equation togheter with the initial condition imply the unitarity of the evolution operator: $\hat{U}(t,t_0)^\dagger \hat{U}(t,t_0) = \mathbb{1}$. Eq. (A.5) is easily rewritten in the integral form:

$$\hat{U}(t,t_0) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(t')\hat{U}(t',t_0)\,\mathrm{d}t'\,, \tag{A.6}$$

whose formal solution is given by the series expansion:

$$\hat{U}(t,t_0) = \mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t} \hat{H}(t)\mathrm{d}t + \frac{1}{(i\hbar)^2}\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t''\hat{H}(t')\hat{H}(t'') + O(\hbar^{-3})\,. \tag{A.7}$$

The adiabatic approximation is valid if at the time t_1 the probability to find the system in a state other than that in which it started at t_0, $|\psi(t_0)\rangle \equiv |0\rangle$, is very low, i.e.:

$$\begin{aligned}\zeta &\equiv \langle 0|0\rangle - |\langle\psi(t_1)|0\rangle|^2 \\ &= \langle\psi(t_1)|\,\hat{U}(t_1,t_0)^\dagger\hat{U}(t_1,t_0)\,|0\rangle - \langle 0|\,\hat{U}^\dagger(t_1,t_0)\,|0\rangle\,\langle 0|\,\hat{U}(t_1,t_0)\,|0\rangle \ll 1\,.\end{aligned} \tag{A.8}$$

We shall now assume to work in the perburbative limit so that we can consider only the first-order term in the series Eq. (A.7):

$$\hat{U}(t_1,t_0) \simeq \mathbb{1} + \frac{\tau}{i\hbar}\hat{H}_\tau \quad , \quad \hat{H}_\tau \equiv \frac{1}{\tau}\int_{t_0}^{t_1}\mathrm{d}t\hat{H}(t)\,. \tag{A.9}$$

Notice that $\hat{H}_\tau$ is a time-independent *operator* and must not be confused with the energy mean value $\langle H(t)\rangle \equiv \langle\psi(t)|\,\hat{H}(t)\,|\psi(t)\rangle$. Substitution of Eq. (A.9) in Eq. (A.8) and a little algebra yield:

$$\begin{aligned}\zeta &= \langle 0|\,(\mathbb{1} + i\hbar^{-1}\tau\hat{H}_\tau)(\mathbb{1} - i\hbar^{-1}\tau\hat{H})\,|0\rangle - \langle 0|\,(\mathbb{1} + i\hbar^{-1}\tau\hat{H}_\tau)\,|0\rangle\,\langle 0|\left(\mathbb{1} - i\hbar^{-1}\tau\hat{H}_\tau\right)|0\rangle \\ &= \frac{\tau^2}{\hbar^2}(\langle 0|\,\hat{H}_\tau^2\,|0\rangle - \langle 0|\,\hat{H}_\tau\,|0\rangle\,\langle 0|\,\hat{H}_\tau\,|0\rangle) = \frac{\tau^2}{\hbar^2}\left(\langle\hat{H}_\tau^2\rangle - \langle\hat{H}_\tau\rangle^2\right) = \frac{\tau^2\sigma^2(\hat{H}_\tau)}{\hbar^2}\,,\end{aligned} \tag{A.10}$$

being $\sigma^2(\hat{H}_\tau)$ the squared deviation of the time-averaged Hamiltonian. The validity condition for the adiabatic approximation reads therefore:

$$\zeta = \frac{\tau^2\sigma^2(\hat{H}_\tau)}{\hbar^2} \ll 1 \Rightarrow \tau \ll \frac{\hbar}{\langle\hat{H}_\tau\rangle}\,. \tag{A.11}$$

Notice that the above condition is consistent with the time-energy indetermination relation. For a more rigorous statement (and proof) of the adiabatic theorem the reader is referred to [27].

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