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On the trace anomaly of a Weyl fermion in a gauge background

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Sommario

In questa tesi studiamo l'anomalia di traccia di un fermione di Weyl in un *background* abeliano. Inizialmente introduciamo il concetto di anomalia in Teoria dei Campi Quantistici e mostriamo esempi di un'anomalia globale e una di *gauge*. Poi, esaminiamo le lagrangiane di un fermione di Weyl e di un fermione di Dirac, i modelli su cui concentreremo i calcoli di anomalie chirali e di traccia. Visto che calcoleremo le anomalie utilizzando una regolarizzazione di Pauli-Villars (PV), presentiamo differenti masse di PV e discutiamo quali simmetrie classiche esse rompano. Identifichiamo gli operatori differenziali che compaiono nei nostri schemi di regolarizzazione e analizziamo il metodo che utilizziamo per calcolare le anomalie: le leggiamo dal *path integral*, *à la* Fujikawa, e poi le valutiamo con le formule dell'*heat kernel*. Infine, calcoliamo l'anomalia chirale e di traccia dei modelli cui siamo interessati. L'anomalia chirale è ampiamente studiata nella letteratura e riproduciamo il risultato standard. L'anomalia di traccia è il nostro risultato originale e, anche se la presenza di un'anomalia chirale implica una rottura dell'invarianza di *gauge*, troviamo che l'anomalia di traccia può essere espressa in una forma *gauge* invariante.

L'argomento è analogo a quello recentemente discusso nella letteratura riguardo a un possibile contributo di un termine dispari sotto parità all'anomalia di traccia di un fermione di Weyl in uno spaziotempo curvo. In un *background* abeliano, questo termine dispari sarebbe una densità di Chern-Pontryagin, che non appare nei nostri risultati.

Abstract

In this thesis we study the trace anomaly of a Weyl fermion in an abelian gauge background. We first introduce the topic of anomaly in Quantum Fields Theory and provide case studies of a global and a gauge anomaly. Then, we review the lagrangians of the Weyl fermion and Dirac fermion, the models that are the focus of our chiral and trace anomaly computations. Since we evaluate the anomalies using Pauli-Villars (PV) regularization, we present different PV masses and discuss the classical symmetries they break. We identify the differential operators that enter our regularization schemes and we review the method that we use to evaluate anomalies: we read them from the path integral *à la* Fujikawa and compute them with heat kernel formulas. Then, we evaluate the chiral and trace anomaly of the models we are interested in. The chiral anomaly is well studied in the literature and we reproduce the standard result. The trace anomaly is our original result and, although the presence of the chiral anomaly implies a breakdown of gauge invariance, we find that the trace anomaly can be cast in a gauge invariant form.

The issue is analogous to the one recently discussed in the literature about a conjectured contribution of an odd-parity term to the trace anomaly of a Weyl fermion in curved backgrounds. With an abelian gauge background, this odd-parity term would be a Chern-Pontryagin density, that does not appear in our final results.

Motivations

In this thesis we study the trace anomaly of a chiral fermion coupled to an abelian gauge field in four dimensions. It is well-known that the model contains an anomaly in the axial gauge symmetry, preventing the quantization of the gauge field in a consistent manner. Nevertheless, it is useful to study the explicit structure of the trace anomaly emerging in the axial $U(1)$ background.

One reason to study the problem is that an analogous situation has recently been addressed for a Weyl fermion coupled to gravity. In particular, the presence of an odd-parity term (the Pontryagin density of the curved background) in the trace anomaly has been reported in [1], and further elaborated upon in [2, 3]. This anomaly was envisaged also in [4], and discussed more recently in [5]. However, there are many indications that such an anomaly cannot be present in the theory of a Weyl fermion. The explicit calculation carried out in [6] confirms this last point of view.

One of the reasons why one does not expect the odd-parity contribution to the trace anomaly is that by CPT in four dimensions a left handed fermion has a right handed antiparticle, expected to contribute oppositely to any chiral imbalance in the coupling to gravity. To see that, one may cast the quantum field theory of a Weyl fermion as the quantum theory of a Majorana fermion. The latter shows no sign of an odd-parity trace anomaly. Indeed, the functional determinant that arises in a path integral quantization can be regulated using Pauli-Villars Majorana fermions with Majorana mass, so to keep the determinant manifestly real, thereby excluding the appearance of a phase that might produce an anomaly (the odd-parity term would carry an imaginary coefficient) [7]. Recently, this has been verified again using Feynman diagrams [8], confirming the results of [6]. An additional piece of evidence comes from studies of the 3-point functions of conserved currents in four dimensional CFT, which exclude odd-parity terms in the correlation function of three stress tensors at non-coinciding points [9, 10], seemingly excluding its presence also in the trace anomaly (see however [11]).

Here we analyze the analogous situation of a chiral fermion coupled to an abelian $U(1)$ gauge background. The theory exhibits a chiral anomaly that implies a breakdown of gauge invariance. Nevertheless, we wish to compute its trace anomaly. Apart from the standard gauge invariant contribution ($\sim F^2$) and possible gauge noninvariant terms, which as we shall show can be canceled by counterterms, one might expect a contribution from the odd-parity Chern-Pontryagin density $F\tilde{F}$. Indeed the latter satisfies the consistency conditions for trace anomalies. In addition, the fermionic functional determinant is complex in euclidean space and thus carries a phase (which is responsible for the $U(1)$ axial anomaly). On the other hand, the structure of the 3-point function of the stress tensor with two $U(1)$ currents in generic CFTs does not allow for odd-parity terms [9, 10] that could signal a corresponding anomaly in the trace of the stress tensor in a $U(1)$ background. Apart from few differences, the case seems analogous to that of the chiral fermion in curved space, and is worth addressing.

To ascertain the situation we compute explicitly the trace anomaly of a Weyl fermion coupled to a $U(1)$ gauge field. Using a Pauli-Villars regularization we find that no odd-parity term emerges in the quantum trace of the stress tensor. We use a Majorana mass for computing the trace anomaly, as this mass term can be covariantized (in curved space) without the need of introducing additional fields of opposite chirality, as would be required by a Dirac mass. The coupling to gravity (needed only at linear order) is used to treat the metric (or vierbein) as an external source for the stress tensor, and to relate the trace of the latter to a Weyl rescaling of the metric (or vierbein). The manifest covariance of the Majorana mass guarantees that the stress tensor can be kept conserved and symmetric also at the quantum level, i.e. without general coordinate (Einstein) and local Lorentz anomalies. We repeat part of our calculations with a Dirac mass as well. In addition, we consider the anomalies of a massless Dirac fermion for comparison and as a test on the scheme adopted. We verify the consistency of the different regularizations, and report the local counterterms that relate them. The results of our findings have also been published as a preprint [12].

We organize the thesis as follows. In chapter 1 we introduce the topic of anomaly in Quantum Field Theory, providing a brief history of anomalies and the example of a global and a gauge anomaly calculation. In particular, the latter is presented with two different approach: Feynman diagram calculation and Fujikawa method. In chapter 2 we review the lagrangians of the Weyl fermion and Dirac fermion, the main models of interest for our anomaly calculation, and identify the relevant differential operators that enter our regularization schemes. In chapter 3 we review the method that we choose for computing the chiral and trace anomalies and in chapter 4 we present our final results. Then we conclude, confining to the appendices notational conventions, heat kernels formulas, Seeley-DeWitt coefficients and covariant counterterms.

Because adventure asks
for adventure:
I have said it several times,
it is easier to begin
than to stop.
Robert Peroni

Chapter 1

Introduction to anomalies

Symmetries play a crucial role in modern physics [13–15]. Actions are built in order to be invariant under the laws of Nature and, guided by symmetries, physicists constructed a successful description of the known interactions. However, physics does not stop at the action: in order to follow the rules of Quantum Mechanics, we have to deal with Feynman path integral. In a nutshell, take an action, exponentiate it and then functionally integrate over the field variables. This seems like a natural procedure from the point of view of the classical limit, since the action is loaded with the symmetries of the physical laws. Nonetheless, the path integral does not distinguish the integrand from the measure and in the construction of a quantum theory we can lose the symmetries of the classical theory [16]. When a symmetry of a classical theory is not a symmetry of the quantum theory, the symmetry is said to be *anomalous*[1]. Another way to understand anomalies is via Feynman diagrams: in order to make sense out of diverging diagrams, we have to regulate our quantum theory. If the regulator does not preserve the classical symmetries, then the quantum theory is anomalous. If one can find a regulator (or a path integral measure) that is invariant under the same symmetries of the classical theory, then no symmetry is broken in the quantization process and the quantum theory is not anomalous.

Of course, since symmetries are extremely important for determining the structure of a theory, anomalies are extremely important as well. One can distinguish two types. When the anomalous symmetry is global, we have a *global anomaly*. There are a lot of global anomalies in the Standard Model, because nothing goes terribly wrong in this case since global anomalies do not affect the renormalizability of a theory. As an example, a global anomaly occurs in QED when computing the decay rate $\pi^0 \to \gamma\gamma$. Anomalies of symmetries associated with gauge bosons, i.e. anomalies associated with gauged symme-

[1] The choice of the word "anomaly" to indicate the quantum breaking of a classical symmetry relies of course on an anthropic point of view, according to which the classical theory is a preferred frame. From Nature point of view, there is no "anomaly": classical and quantum systems simply behave according to different rules.

tries, are called *gauge anomalies*. These anomalies afflict symmetries that are necessary to renormalize the theory and, since they have to be avoided, they become tools to spot ill-defined Quantum Field Theories and this is why anomalies are as important as symmetries. Examples of gauge anomalies are the ones related to gauge chiral symmetries acting on fermions and actually these were the first anomalies to puzzle physicists in the 1940's [17].

1.1 A brief history of anomalies

Already in the 1930's physicists encountered divergences arising from loop diagrams when they tried to compute even the simplest radiative corrections to processes in QED. In the 1940's the efforts were focused on a particular diagram, that of the photon selfenergy with a fermionic loop (see Fig. 1.1). Due to gauge invariance, this diagram needs to be

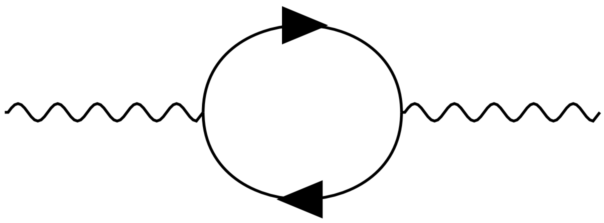

Figure 1.1: Photon selfenergy with a fermionic loop.

transversal and it should vanish on-shell since the photon is massless. However, Tomonaga and collaborators[2], when studying the e^2 corrections to the Klein-Nishina formula for the Compton scattering, found it to be divergent as well as gauge non-invariant. They reported that "there is an infinity containing [the] electromagnetic potential . . . [that] cannot be subtracted by amalgamation as in the case of mass-type and charge-type infinities". In modern terms, the divergence could be identified with a photon mass, but it could not be removed by renormalization (*amalgamated*) because there is no bare photon mass (as required by gauge invariance). Thus, it could not be dealt with as in the case of electron mass and charge infinities, which could be reabsorbed in the electron bare mass and charge.

Oppenheimer suggested that the difficulties were due to an inadequate identification of the photon selfenergies and Fukuda and Miyamoto, two of Tomonaga's collaborators, decided to address this problem in a different way by studying another simple diagram, namely the triangle diagram which was supposed to describe the pion decay into two photons $\pi^0 \to \gamma\gamma$ mediated by fermions (see Fig. 1.2). They considered the cases that Yukawa's U particle, the neutral pion π^0, was a scalar, a pseudoscalar or a pseudovector

[2]The references to the papers mentioned in this section can be found in [17].

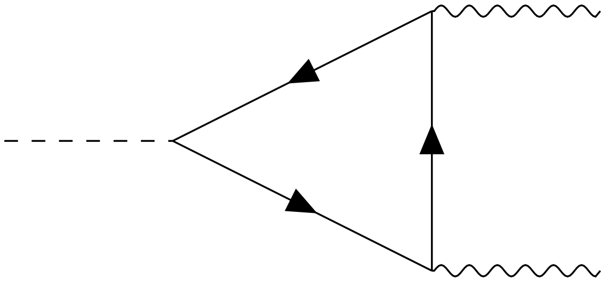

Figure 1.2: Triangle diagrams describing the pion decay into two photons.

with couplings $\lambda U\overline{\psi}\psi$, $\lambda U\overline{\psi}\gamma^5\psi$ and $(\lambda/2m)\overline{\psi}\gamma^5\gamma^a\psi U_a$ respectively, with m the proton mass. They found that the results were not gauge covariant and the pseudovector and pseudoscalar decays were not consistent with each other, even if the models are the same after setting $U_a = \partial_a U$, partial integrating the interaction and using the equation of motion. They ascribed the inconsistency to the mathematical difficulty of handling the Pauli-Jordan distribution and concluded that they did not know how to deal with that ambiguity.

Interested in the work of Fukuda and Miyamoto, Steinberger decided to apply the recently proposed Pauli-Villars regularization scheme to the triangle diagram. Tomonaga followed the same direction, and their studies were a partial success. Indeed, they succeeded in obtaining a finite result out of the diagram which was also in a gauge and Lorentz invariant form. However, the result seemed to depend on how the calculation was performed and the relation between the pseudoscalar and the pseudovector was still not satisfied. They were discovering the chiral anomaly, but they did not understand it at that time. Thus, they concluded that the lifetime of the neutral pion was ambiguous and they hoped in some experimental measure of the decay rate that could help in clarifying the use of the regulator.

In 1951 Schwinger solved the puzzle by introducing a new regularization scheme (Schwinger's proper time) which preserves gauge invariance at all intermediate stages. He calculated anew the photon selfenergy and the triangle diagram and he obtained finite, gauge invariant results that were apparently free from ambiguity. There the calculation rested for two decades, since field theory was almost abandoned in favour of theories like Regge theory, the S-matrix program of Chew and current algebra, which is an infinite-dimensional Lie algebra defined by commutation relations among current operators defined in field theories.

Then, in 1969 two papers were submitted for publication, one by Bell and Jackiw and the other by Adler. Bell and Jackiw noted that the amplitude for the pion decay into two photons could be parametrized as

$$M^{ab}(q_1, q_2) = \epsilon^{abcd} q^1_a q^2_b M(k^2) \tag{1.1}$$

where q_1 and q_2 are the photon momenta and $k = q_1 + q_2$ is the pion momentum.

Their amplitude satisfied gauge invariance (i.e. the Ward identities $q^1_a M^{ab}(q_1, q_2) = q^2_b M^{ab}(q_1, q_2) = 0$ were satisfied) as well as Bose symmetry $M^{ab}(q_1, q_2) = M^{ba}(q_2, q_1)$. They noted also that Steinberger had calculated $M(k^2)$ using the same diagrams that appear in the linear sigma model and had found $M(0) = g4\pi^2/m$. However, Veltman and Sutherland had found that $M(0) = 0$ if one used an off-shell massive pion field that was equal to the divergence of the axial current (PCAC, the partially conserved axial-vector current studied by Gell-Mann and Levy in 1960). Bell and Jackiw decided to tackle the puzzle of the $M(0)$ that seemed to be both vanishing and non-vanishing. They developed a variation which respects PCAC as well as Lorentz and gauge invariance, and their calculation also yielded $M(0) = 0$. They also noted that they were dealing with a linearly divergent integral which picks up a boundary term with a shift of variable. Thus, they saw the hallmark of an anomaly.

Adler studied the axial-vector-vector triangle graph (i.e. the triangle graph with one axial and two vector vertices) in spinor QED and proved the uniqueness of the triangle diagrams by imposing vector gauge invariance. He discussed the possible connection with the $\pi^0 \to \gamma\gamma$ decay and found that the PCAC had to be modified, completely altering its prediction for the decay process. In 1963, Rosenberg had already considered the triangle graph of a vector-axial theory and imposing vector gauge invariance he had already expressed the divergent factors in terms of convergent ones and in fact his results were used by Adler. However, Adler, as well as Bell and Jackiw, studied in addition the behaviour of the axial vector Ward identity, which was not considered by Rosenberg. Thus, the axial anomaly clearly appeared in the works by Adler and Bell and Jackiw.

When in 1971 't Hooft proved that nonabelian pure gauge theories are renormalizable, it was clear that gauge anomalies were a problem, because they spoil unitarity and renormalizability. Thus, physicists had to make sure that anomalies in gauge transformations of chiral quarks and leptons would cancel, without ruining the theory. Phrased it differently, if we consider the Standard Model with the $SU(3)_{QCD} \times SU(2)_{weak} \times U(1)_Y$ gauge symmetries, we have to check that the currents associated with these symmetries are non-anomalous. Indeed, since left-handed Weyl fermions and right-handed Weyl fermions have opposite contributions to the anomaly and the sum of the electric charges of all quarks and leptons in a given family is zero, for the $U(1)_Y$ symmetry the anomaly vanishes. Things would not change much if one considered the $U(1)_{EM}$ as in QED, since all fermions in this case are Dirac fermions which produce a vanishing gauge anomaly. Similarly, since QCD is non-chiral there is no anomaly from the $SU(3)_{QCD}$ symmetry, while $SU(2)_{weak}$ has no anomaly because all of its representations are pseudoreal and they cannot develops gauge anomalies. Thus, the Standard Model proved to be anomaly free.

It was later realized by Kimura, Delbourgo and Salam, Eguchi and Freund, that anomalies were not an issue of non-gravitational theories alone, but they affected gravity as well. In particular, coupling a fermion to gravity and considering triangle diagrams in four dimensions with Dirac fermions in the loop, one vertex given by the axial current and the other two vertices given by $h^{\mu\nu}T_{\mu\nu}$, $T_{\mu\nu}$ being the stress tensor for fermions, they found anomalies proportional to $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma\alpha\beta}$.

Then the question was: are there anomalies in the conservation of the stress tensor of chiral fermions in a gravitational background? They would be the counterpart of the gauge anomalies present in chiral gauge theories. Since the conservation of the stress tensor is ensured by the local Lorentz invariance, an anomaly in the Lorentz symmetry could lead to the stress tensor non-conservation. However, it was then found that there is no potential problem for the Standard Model, because gravitational contributions to chiral anomalies cancel, while local Lorentz anomalies can only occur in $4k+2$ dimensions. Thus, it was realized that gravity fit well as an external field in the Standard Model. In addition to anomalies in chiral models, trace anomalies can occur when (global or local) scale invariance of the classical action is broken in the quantum theory. In 1971 Coleman and Jackiw analysed the quantum breaking of global scale transformation, while in 1974 Capper and Duff studied the trace anomaly of classical Weyl invariant massless vectors and spinors coupled to gravity in 4 dimensions. In the spinors case, the trace anomaly was found to be

$$\begin{aligned}\langle T^{\mu}{}_{\mu}\rangle =&aR^2 + bR^2_{\mu\nu} + cR^2_{\mu\nu\rho\sigma} + d\Box R\\ =&\alpha\left(C^2_{\mu\nu\rho\sigma} + \frac{2}{3}\Box R\right) + \beta E_4\end{aligned} \tag{1.2}$$

with $\Box R$ removable by a local counterterm, $C^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ is the square of the Weyl tensor, $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ yields the Euler invariant and the constants α, β parametrize the one-loop divergences.

In 2014, Bonora and collaborators [1] computed the trace anomaly of a Weyl fermion coupled to a gravitational background and found an additional Pontryagin term in it

$$\langle T^{\mu}{}_{\mu}\rangle = \frac{1}{360(4\pi)^2}\left(9C^2_{\mu\nu\rho\sigma} - \frac{11}{2}E_4 - i\frac{15}{2}P\right) \tag{1.3}$$

with the Pontryagin density given by $P = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}$. The imaginary unit brings back unitarity problems and a couple of years later Bastianelli and Martelli [6] computed the trace anomaly of the same system considered by Bonora. They found

$$\langle T^{\mu}{}_{\mu}\rangle = \frac{1}{360(4\pi)^2}\left(18C^2_{\mu\nu\rho\sigma} - 11E_4\right) \tag{1.4}$$

with no contribution from the Pontryagin density. Up to now, the controversy remain unsolved and in this thesis we will indirectly address it by considering the trace anomaly of a Weyl fermion coupled to a $U(1)$ gauge field instead of the gravitational background.

Before proceeding with the models and the anomalies of main interest for the present thesis, we now provide the details of the computation of one global and one gauge anomaly (respectively, the one that occur when computing the pion decay in two photons and the chiral anomaly) via Feynman diagrams calculations. In the case of the chiral

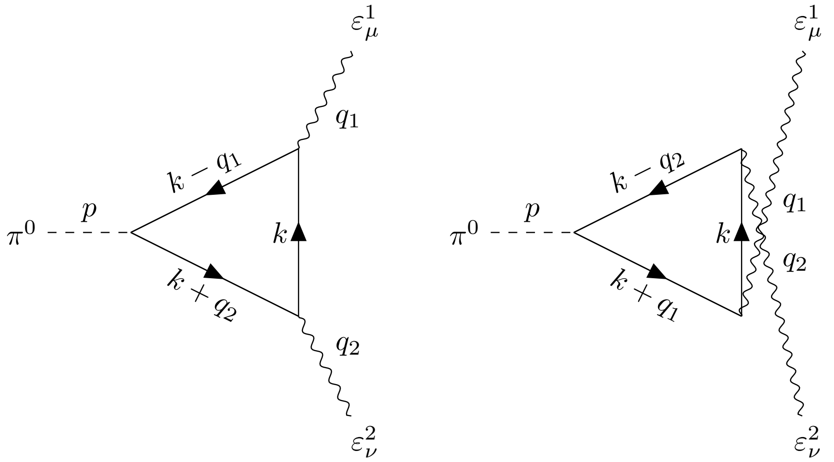

Figure 1.3: The two 1-loop diagrams that contributes to $\pi^0 \to \gamma\gamma$.

anomaly, we will also describe the Fujikawa method that relies on the non-invariance of the path integral and that will be useful to introduce our approach in chapter 3.

1.2 Global anomaly: $\pi^0 \to \gamma\gamma$ decay rate

Consider the QED lagrangian with a Yukawa coupling between a fermion ψ and a pseudoscalar π:

$$\mathcal{L}_{YUK} = -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}\pi(-\Box + m_\pi^2)\pi - \overline{\psi}(\gamma^a\partial_a - ie\gamma^a A_a + m)\psi + i\lambda\pi\overline{\psi}\gamma^5\psi \tag{1.5}$$

where one can think of π as the neutral pion, ψ as the proton with charge e, the Yukawa coupling as $\lambda = m/m_\pi$ and of course $F_{ab} = \partial_a A_b - \partial_b A_a$. There are two 1-loop diagrams that contributes to the process, which are shown in Fig. 1.3. Their sum is

$$\mathcal{M} = (-1)(-\lambda)(-e)^2\varepsilon^1_a\varepsilon^2_b M^{ab}(q_1, q_2) \tag{1.6}$$

where ε^1_a and ε^2_b are the polarization vectors of the two outgoing photons and

$$\begin{aligned} M^{ab}(q_1, q_2) = \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\Bigg[&\gamma^a\frac{-i\not{k} + m}{k^2 + m^2}\gamma^b\frac{-i(\not{k} + \not{q}_2) + m}{(k + q_2)^2 + m^2}\gamma^5\frac{-i(\not{k} - \not{q}_1) + m}{(k - q_1)^2 + m^2} \\ &+ \gamma^b\frac{-i\not{k} + m}{k^2 + m^2}\gamma^a\frac{-i(\not{k} + \not{q}_1) + m}{(k + q_1)^2 + m^2}\gamma^5\frac{-i(\not{k} - \not{q}_2) + m}{(k - q_2)^2 + m^2}\Bigg] . \end{aligned} \tag{1.7}$$

It is easy to see that M^{ab} must be UV finite (although $M^{ab} \sim \int \frac{d^4k}{k^3}$ looks divergent). By Lorentz invariance and symmetry under exchanging $1 \leftrightarrow 2$, $a \leftrightarrow b$ we can only have

$M^{ab} \sim q_2^a q_1^b$ or $M^{ab} \sim \epsilon^{abcd} q_c^1 q_d^2$. Thus, the integral has to scale at worst as $M^{ab} \sim q^2 \int \frac{d^4k}{k^5}$ which is UV finite.

First we trace over the spinor indices (see appendix A for our conventions)

$$\mathrm{tr}\left\{\gamma^a[-i\not{k}+m]\gamma^b[-i(\not{k}+\not{q}_2)+m]\gamma^5[-i(\not{k}-\not{q}_1)+m]\right\} = 4im\epsilon^{abcd}q_c^1 q_d^2 \tag{1.8}$$

so that

$$M^{ab}(q_1,q_2) = 8im\epsilon^{abcd}q_c^1 q_d^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2+m^2][(k+q_2)^2+m^2][(k-q_1)^2+m^2]}\ . \tag{1.9}$$

Now to evaluate the integral we use Feynman parameters[3] to recast it as

$$M^{ab}(q_1,q_2) = 8im\epsilon^{abcd}q_c^1 q_d^2 \frac{-i}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, \frac{1}{-m^2+x(1+x)q_1^2-y(1-y)q_2^2-xy(s^2-q_1^2-q_2^2)} \tag{1.10}$$

where the Mandelstam variable $s=(q_1+q_2)^2$ is understood. We can set the on-shell conditions $q_1^2=q_2^2=0$ and $s=m_\pi^2$ and disregard the pion mass with respect to the proton mass $m_\pi \ll m_p = m$. Then the double integral gives $-1/2m^2$ and

$$\mathcal{M} = -\lambda\frac{e^2}{4\pi^2 m}\epsilon^{abcd}\varepsilon_a^1\varepsilon_b^2 q_c^1 q_d^2\ . \tag{1.11}$$

From the 1-loop correction to the scattering process, we can evaluate the decay rate that we only report for completeness:

$$\Gamma(\pi^0\to\gamma\gamma) = \frac{\alpha_e^2}{64\pi^3}\lambda^2\frac{m_\pi^3}{m^2} \tag{1.12}$$

where $\alpha_e = e^2/4\pi$ is the fine structure constant.

So far we have dealt with a scattering process. Now the question is: what has this to do with global anomalies? If we consider the QED lagrangian

$$\begin{aligned}\mathcal{L}_{QED} &= -\,\overline{\psi}(\gamma^a\partial_a - ie\gamma^a A_a + m)\psi \\ &= -\,\overline{\psi}_L\gamma^a(\partial_a - ieA_a)\psi_L - \overline{\psi}_R\gamma^a(\partial_a - ieA_a)\psi_R - m(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L)\end{aligned} \tag{1.13}$$

we see that in the $m\to 0$ limit it is invariant under two global symmetries, a vector symmetry

$$\psi(x)\to\psi'(x) = e^{i\alpha}\psi(x) \tag{1.14}$$

and a chiral symmetry

$$\psi(x)\to\psi'(x) = e^{i\beta\gamma^5}\psi(x) \tag{1.15}$$

[3]We recall the formula $(ABC)^{-1} = 2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)[xA+yB+zC]^{-3}$ to be used with $A=(k-q_1)^2+m^2$, $B=(k+q_2)^2+m^2$, $C=k^2+m^2$.

to which we associate two Noether currents, the vector and axial current respectively

$$\begin{aligned} J^a &= i\overline{\psi}\gamma^a\psi \\ J_5^a &= i\overline{\psi}\gamma^a\gamma^5\psi \ . \end{aligned} \tag{1.16}$$

From the equations of motion we have

$$\begin{aligned} \partial_a J^a &= 0 \\ \partial_a J_5^a &= 2im\overline{\psi}\gamma^5\psi \ . \end{aligned} \tag{1.17}$$

i.e. the vector symmetry is classically conserved while the chiral symmetry is only conserved in the massless limit. Now we recall the result of our loop calculation in (1.11). We can interpret this result by saying that the operator $i\overline{\psi}\gamma^5\psi$ to which the pion couples has a non-zero value in the presence of a vector background field, that is[4]

$$\langle A|i\overline{\psi}\gamma^5\psi|A\rangle = \frac{e^2}{32\pi^2}\frac{1}{m}\epsilon^{abcd}F_{ab}F_{cd} \ . \tag{1.18}$$

However, eq. (1.18) is consistent with eq. (1.17) only if the axial current is not conserved

$$\langle A|\partial_a J_5^a|A\rangle = \frac{e^2}{16\pi^2}\epsilon^{abcd}F_{ab}F_{cd} \ . \tag{1.19}$$

It is important to remark that even if $m = 0$, i.e. even if the axial current is classically conserved, eq. (1.19) holds. There is no inconsistency in this reasoning only if the symmetry violation arises due to quantum effects, that is to say only if the chiral symmetry is anomalous. In order to better understand the situation, we will show now what happens if we take $m = 0$ from the beginning.

1.3 Gauge anomaly: massless fermion

It is not hard to see that the massless limit of the 1-loop calculation is problematic. Indeed, the integral in (1.9) seems to vanish, since it is proportional to m, while the final result in (1.11) blows up because it goes with $1/m$. Thus, we have to proceed carefully and it is reasonable to recast the calculation as matrix elements of currents by considering the correlation function $\langle J_5^a(x)J^b(y)J^c(z)\rangle$. In particular, since we want to see whether the classical conservation laws $\partial_a J^a = \partial_a J_5^a = 0$ hold also in the quantum theory, we need to study the quantity $\partial_b\langle J_5^a(x)J^b(y)J^c(z)\rangle$. We are going to address this issue using two different method, i.e. by calculating the relevant Feynman diagrams in perturbation theory and then with the Fujikawa method.

[4] To see the link between (1.11) and (1.18) we should digress into the Euler-Heisenberg lagrangian and Schwinger proper time. We avoid this here, but refer the interested reader to [15].

1.3.1 Feynman diagrams calculation

In momentum space, we want to compute

$$\begin{aligned}
&iM_5^{abc}(p,q_1,q_2)(2\pi)^4\delta^4(p-q_1-q_2) = \\
&= i\int d^4x\int d^4y\int d^4z\, e^{-ipx}e^{iq_1y}e^{iq_2z}\langle J_5^a(x)J^b(y)J^c(z)\rangle \\
&= \int d^4x\int d^4y\int d^4z\, e^{-ipx}e^{iq_1y}e^{iq_2z}\langle[\overline{\psi}(x)\gamma^a\gamma^5\psi(x)][\overline{\psi}(y)\gamma^b\psi(y)][\overline{\psi}(z)\gamma^c\psi(z)]\rangle
\end{aligned} \tag{1.20}$$

where brackets denote contracted spinor indices. Thus, the leading diagrams are precisely the one in Fig. 1.3 without the external lines and the coupling constants and the 1-loop correlation function in momentum space reads

$$\begin{aligned}
iM_5^{abc}(p,q_1,q_2) = \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\Big[&\gamma^b\frac{-i\not{k}}{k^2}\gamma^c\frac{-i(\not{k}+\not{q}_2)}{(k+q_2)^2}\gamma^a\gamma^5\frac{-i(\not{k}-\not{q}_1)}{(k-q_1)^2} \\
&+\gamma^c\frac{-i\not{k}}{k^2}\gamma^b\frac{-i(\not{k}+\not{q}_1)}{(k+q_1)^2}\gamma^a\gamma^5\frac{-i(\not{k}-\not{q}_2)}{(k-q_2)^2}\Big] .
\end{aligned} \tag{1.21}$$

Instead of evaluating the integral and then contracting the result with the momenta, it is easier to proceed the other way around. Contracting the axial current with p^a gives

$$p_aM_5^{abc}(p,q_1,q_2) = \int\frac{d^4k}{(2\pi)^4}\left[\frac{\mathrm{tr}\,[\gamma^b\not{k}\gamma^c(\not{k}+\not{q}_2)\not{p}\gamma^5(\not{k}-\not{q}_1)]}{k^2(k+q_2)^2(k-q_1)^2}+\begin{pmatrix} b\leftrightarrow c\\ 1\leftrightarrow 2\end{pmatrix}\right] . \tag{1.22}$$

Using the property $\{\gamma^5,\gamma^a\}=0$ and the relation $p^a=q_1^a+q_2^a$ we write

$$\not{p}\gamma^5 = (\not{q}_1+\not{q}_2)\gamma^5 = \gamma^5(\not{k}-\not{q}_1)+(\not{k}+\not{q}_2)\gamma^5 \tag{1.23}$$

to simplify the integral as

$$\begin{aligned}
p_aM_5^{abc}(p,q_1,q_2) &= \int\frac{d^4k}{(2\pi)^4}\left[\frac{\mathrm{tr}\,[\gamma^b\not{k}\gamma^c(\not{k}+\not{q}_2)\gamma^5]}{k^2(k+q_2)^2}+\frac{\mathrm{tr}\,[\gamma^b\not{k}\gamma^c(\not{k}-\not{q}_1)\gamma^5]}{k^2(k-q_1)^2}+\begin{pmatrix} b\leftrightarrow c\\ 1\leftrightarrow 2\end{pmatrix}\right] \\
&= -4i\epsilon^{bcde}\int\frac{d^4k}{(2\pi)^4}\left[\frac{k^dq_2^e}{k^2(k+q_2)^2}+\frac{k^dq_1^e}{k^2(k-q_1)^2}\right]+\begin{pmatrix} b\leftrightarrow c\\ 1\leftrightarrow 2\end{pmatrix} .
\end{aligned} \tag{1.24}$$

Every term in brackets has only q_1 or q_2, so that by Lorentz invariance we expect the integral to give either $q_1^dq_1^e$ or $q_2^dq_2^e$ for every term, that vanish when contracted with ϵ^{bcde}. Thus, in contrast to our expectations, $p_aM_5^{abc}$ seems to vanish.

However, let us have a look at $q_b^1M_5^{abc}$. In this case we should get zero if we want the Ward identity to be satisfied. The contraction leads to

$$q_b^1M_5^{abc}(p,q_1,q_2) = \int\frac{d^4k}{(2\pi)^4}\left[\frac{\mathrm{tr}\,[\not{q}_1\not{k}\gamma^c(\not{k}+\not{q}_2)\gamma^a\gamma^5(\not{k}-\not{q}_1)]}{k^2(k+q_2)^2(k-q_1)^2}+\frac{\mathrm{tr}\,[\gamma^c\not{k}\not{q}_1(\not{k}+\not{q}_1)\gamma^a\gamma^5(\not{k}-\not{q}_2)]}{k^2(k+q_1)^2(k-q_2)^2}\right] . \tag{1.25}$$

If we replace $\not{q}_1 = \not{k} - (\not{k} - \not{q}_1)$ in the first term and $\not{q}_1 = (\not{k} + \not{q}_1) - \not{k}$ in the second term, then the integral becomes

$$\begin{aligned}
q_b^1 M_5^{abc}(p, q_1, q_2) = \int \frac{d^4k}{(2\pi)^4} \Bigg[& \frac{\mathrm{tr}\,[\gamma^c(\not{k} + \not{q}_2)\gamma^a\gamma^5(\not{k} - \not{q}_1)]}{(k - q_1)^2(k + q_2)^2} - \frac{\mathrm{tr}\,[\not{k}\gamma^c(\not{k} + \not{q}_2)\gamma^a\gamma^5]}{k^2(k + q_2)^2} \\
& + \frac{\mathrm{tr}\,[\gamma^c\not{k}\gamma^a\gamma^5(\not{k} - \not{q}_2)]}{k^2(k - q_2)^2} - \frac{\mathrm{tr}\,[\gamma^c(\not{k} + \not{q}_1)\gamma^a\gamma^5(\not{k} - \not{q}_2)]}{(k + q_1)^2(k - q_2)^2} \Bigg] \\
= 4i\epsilon^{acde} \int \frac{d^4k}{(2\pi)^4} & \left[\frac{(k - q_1)^d(k + q_2)^e}{(k - q_1)^2(k + q_2)^2} - \frac{(k - q_2)^d(k + q_1)^e}{(k - q_2)^2(k + q_1)^2} \right]
\end{aligned} \tag{1.26}$$

Now we could shift $k \to k+q_1$ in the first integrand and $k \to k+q_2$ in the second integrand to get zero identically. However, this would be incorrect because we are dealing with a linearly divergent integral and this shift would be illegal. Making sense out of this integral is in fact the main difficulty that physicist faced in the 1940's when they were about to discover anomalies. To solve it, we first take a little digression into linearly divergent integrals to see how we can compute them without introducing a regulator in the theory.

Consider the one-dimensional integral

$$I(\xi) = \int_{-\infty}^{+\infty} dx\,[f(x + \xi) - f(x)] \tag{1.27}$$

where the function $f(x)$ takes a constant value at $x = -\infty$ and a different constant value at $x = +\infty$. Each integrand is then linearly divergent and we want to see whether $I(\xi)$ is finite or not. If we Taylor expand around $\xi = 0$ we get

$$I(\xi) = \int_{-\infty}^{+\infty} dx \left[\xi f'(x) + \frac{\xi^2}{2} f''(x) + \dots\right] = \xi[f(+\infty) - f(-\infty)] \tag{1.28}$$

because the higher derivative terms do not contribute at infinity. Thus the difference between a linearly divergent integral and its shifted value depends linearly on the shift. We can proceed analogously in four dimensions. If we consider the integral

$$I^a(\xi^b) = \int \frac{d^4k}{(2\pi)^4}(F^a[k + \xi] - F^a[k]) \tag{1.29}$$

and Wick rotate it (with $k_0 \to ik_0$)

$$I^a(\xi^b) = i \int \frac{d^4k_E}{(2\pi)^4}(F^a[k_E + \xi] - F^a[k_E]) \tag{1.30}$$

we can Taylor expand it around $\xi = 0$

$$I^a(\xi^b) = i \int \frac{d^4 k_E}{(2\pi)^4} \left[\xi^b \frac{\partial}{\partial k_E^b} F^a[k_E] + \frac{1}{2} \xi^b \xi^c \frac{\partial}{\partial k_E^b} \frac{\partial}{\partial k_E^c} F^a[k_E] + \dots \right] \tag{1.31}$$

Since the integral is taken to be linearly divergent, we can suppose that (dropping the euclidean subscript for simplicity)

$$\lim_{|k| \to \infty} F^a(k) = C \frac{k^a}{k^4} \tag{1.32}$$

and only the term with one derivative contributes to the integral. Thus, we cast it as a surface integral

$$I^a(\xi^b) = i\xi^b \int \frac{d^4 k}{(2\pi)^4} \frac{\partial}{\partial k^b} F^a[k] = i\xi^b \int \frac{d^3 S_b}{(2\pi)^4} F^a[k] \tag{1.33}$$

with $d^3 S_b = k^2 k_b \, d\Omega_4$ and write

$$I^a(\xi^b) = i\xi^b \lim_{|k| \to \infty} \int \frac{d\Omega_4}{(2\pi)^4} C \frac{k_b k^a}{k^2} \; . \tag{1.34}$$

Now we use $k^a k^b = \frac{1}{4} k^2 \delta^{ab}$ and $\Omega_4 = 2\pi^2$ to get

$$I^a(\xi^b) = \frac{i}{32\pi^2} C \xi^a \tag{1.35}$$

This is the general result that we need to evaluate a linearly divergent integral that would vanish if we could shift the integration variable.

Now we are able to solve the integral (1.26) with the general result in (1.35). In fact, we note that part of (1.26) is quadratically divergent, but since $\epsilon^{acde} k_d k_e = 0$ we are left with a linear divergence. Then, if we introduce $\xi^e = q_2^e - q_1^e$ we see that the integral is of the same form as (1.30). From the divergence in the second term we identify

$$F^d(k) = 4i\epsilon^{acde} \frac{(q_1 + q_2)^d k^e}{(k + q_1)^2 (k - q_2)^2} \xrightarrow{|k| \to \infty} 4i\epsilon^{acde} (q_1 + q_2)^d \frac{k^e}{k^4} \tag{1.36}$$

so that

$$q_b^1 M_5^{abc}(p, q_1, q_2) = -\frac{1}{4\pi^2} \epsilon^{acde} q_d^1 q_e^2 \neq 0 \tag{1.37}$$

Thus, in contrast to our expectations, it seems that the Ward identity is violated for the vector current but it is satisfied for the axial current.

The problem is that, although the integral we calculated is finite, its value depends on the shift between the two integrands in (1.26). However, the choice of k is arbitrary because it is the loop variable and the only constraint is that once we have choose it we need to be consistent with our choice. Thus, let us be as general as possible: consider again the diagrams in Fig. 1.3 without the external legs and let us change the momentum k^a in the first graph with

$$k^a \to k^a + \xi_1 q_1^a + \xi_2 q_2^a \, . \tag{1.38}$$

To preserve Bose symmetry, we have to take $k^a \to k^a + \xi_2 q_1^a + \xi_1 q_2^a$ in the second graph. Then, the result of the loop calculation will be

$$\begin{aligned} q_b^1 M_5^{abc}(p, q_1, q_2) &= -\frac{1}{8\pi^2}\epsilon^{acde}(q_d^1 + q_d^2)(1 - \xi_1 + \xi_2)(q_e^2 - q_e^1) \\ &= -\frac{1}{4\pi^2}\epsilon^{acde} q_d^1 q_e^2 (1 - \xi_1 + \xi_2) \end{aligned} \tag{1.39}$$

and similarly

$$p_a M_5^{abc}(p, q_1, q_2) = -\frac{1}{4\pi^2}\epsilon^{bcde} q_d^1 q_e^2 (\xi_1 - \xi_2) \tag{1.40}$$

Now we see that if we take $\xi_1 = \xi_2$ then

$$\begin{aligned} q_b^1 M_5^{abc}(p, q_1, q_2) &= -\frac{1}{4\pi^2}\epsilon^{acde} q_d^1 q_e^2 \\ p_a M_5^{abc}(p, q_1, q_2) &= 0 \end{aligned} \tag{1.41}$$

the axial current is conserved but the vector current is not. On the other hand, if we take $\xi_1 - \xi_2 = 1$ then

$$\begin{aligned} q_b^1 M_5^{abc}(p, q_1, q_2) &= 0 \\ p_a M_5^{abc}(p, q_1, q_2) &= -\frac{1}{4\pi^2}\epsilon^{bcde} q_d^1 q_e^2 \end{aligned} \tag{1.42}$$

the vector current is conserved while the axial current is not. With this choice we reproduce the result obtained in the massive case. In that case there was no ambiguity because the mass term breaks the chiral symmetry already in the classical theory and then only the vector symmetry can be conserved also in the quantum theory. Here we have more freedom since the classical theory is that of a massless fermion.

Thus, we have seen that if one insists in preserving gauge invariance, we can have $\partial_b \langle J_5^a J^b J^c \rangle = \partial_c \langle J_5^a J^b J^c \rangle = 0$, so that the Ward identity is satisfied, at the price of violate the conservation of the axial current, i.e. $\partial_a \langle J_5^a J^b J^c \rangle \neq 0$. This behaviour is entirely due to quantum effects and it is independent of the method we use to check the quantum current conservation. We now show how we can come to the same conclusion by using a different approach.

1.3.2 Fujikawa method

Fujikawa's intuitive idea is that anomalies arise when there are symmetries of the action that are not symmetries of the functional measure of the path integral [16, 18, 19]. In this framework, the connection between the anomaly and the violation of the classical symmetry is more explicit, because it is not mediated by the choice of a loop momentum.

We start with a quick review of the conservation law from the path integral point of view. Given a gauge invariant operator $O(x_1, \ldots, x_n)$, e.g. $O = J^a(x)J^b(y)$, we define

$$\langle O(x_1, \ldots, x_n)\rangle = \int D\overline{\psi} D\psi \, \exp\left[-i\int d^4x \, \overline{\psi}\partial\!\!\!/\psi\right] O(x_1, \ldots, x_n) \; . \tag{1.43}$$

The action is invariant under the global symmetry (1.14). If we now take $\alpha = \alpha(x)$ in (1.14), we have that

$$\overline{\psi}\partial\!\!\!/\psi \to \overline{\psi}\partial\!\!\!/\psi + i\overline{\psi}\gamma^a\psi\partial_a\alpha(x) \; . \tag{1.44}$$

However, the path integral is invariant under any field redefinition, because it integrates over all field configurations. Thus, the additional term proportional to α must vanish and, expanding to first order in α and integrating by parts, we get

$$0 = \int d^4z \, \alpha(z) \int D\overline{\psi} D\psi \, \exp\left[-i\int d^4x \, \overline{\psi}\partial\!\!\!/\psi\right] \frac{i\partial}{\partial z^a}\left[i\overline{\psi}(z)\gamma^a\psi(z)\right] O(x_1, \ldots, x_n) \tag{1.45}$$

that must hold for all $\alpha(z)$. Thus

$$\partial_a\langle\left[i\overline{\psi}(x)\gamma^a\psi(x)\right] O(x_1, \ldots, x_n)\rangle = \partial_a\langle J^a(x) O(x_1, \ldots, x_n)\rangle = 0 \; . \tag{1.46}$$

This reasoning is correct, but it is only valid if the path integral measure does not change under the field transformations. Let us see how the field transformation affects the measure.

Consider a general linear transformation

$$\psi(x) \to \Delta(x)\psi(x) \tag{1.47}$$

which generates a Jacobian factor that for fermionic fields has the form

$$D\overline{\psi} D\psi \to |\mathcal{J}|^{-2} D\overline{\psi} D\psi \tag{1.48}$$

with

$$\mathcal{J} = \det\Delta = \exp \mathrm{tr} \, \ln\Delta \tag{1.49}$$

where the trace sums over the eigenvalues of $\ln\Delta$. Thus, if we consider the vector transformation $\Delta(x) = e^{i\alpha(x)}$, we can write

$$\mathrm{tr} \, \ln\Delta = i\int d^4x \, \alpha(x) \tag{1.50}$$

and the Jacobian is

$$\mathcal{J} = \exp\left(i\int d^4x\,\alpha(x)\right) \tag{1.51}$$

with $|\mathcal{J}| = 1$. Thus, the result in (1.46) is indeed correct because the path integral measure is invariant under a vector transformation.

However, if we consider a chiral transformation with

$$\Delta(x) = e^{i\beta(x)\gamma^5} \tag{1.52}$$

then the Jacobian becomes

$$\mathcal{J} = \exp\left(i\int d^4x\,\beta(x)\mathrm{tr}\,\gamma^5\right) \tag{1.53}$$

that appears to vanish, making the measure singular. Thus, we need to proceed carefully and let us take into consideration the path integral

$$\int D\overline{\psi}D\psi\, e^{iS_{QED}(m=0)} \tag{1.54}$$

where

$$S_{QED}(m=0) = \int d^4x\mathcal{L}_{QED}(m=0) = -\int d^4x\,\overline{\psi}\gamma^a(\partial_a - ieA_a)\psi \tag{1.55}$$

To regulate the divergence of the measure, we introduce a one-particle Hilbert space $\{|x\rangle\}$ so that $\Delta(x) = \langle x|\Delta(\hat{x})|x\rangle$ and

$$\mathcal{J} = \exp\left(i\int d^4x\,\mathrm{tr}\left[\langle x|\beta(\hat{x})\gamma^5|x\rangle\right]\right). \tag{1.56}$$

We introduce also an exponential regulator

$$\exp\left(\frac{\hat{\not{\Pi}}^2}{\Lambda^2}\right) \quad , \quad \hat{\not{\Pi}} = \hat{\not{p}} - e\not{A}(\hat{x}) \tag{1.57}$$

where Λ plays the role of a UV cutoff and $\hat{p}$ is the operator conjugate to $\hat{x}$. Using the Dirac algebra, we write

$$\hat{\not{\Pi}}^2 = (\eta^{ab} + \gamma^{ab})\hat{\Pi}_a\hat{\Pi}_b = \hat{\Pi}^2 - \frac{ie}{2}\gamma^{ab}F_{ab} \tag{1.58}$$

with $\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$. Then

$$\begin{aligned}\mathrm{tr}\left[\langle x|\beta(\hat{x})\gamma^5|x\rangle\right] &= \lim_{\Lambda\to\infty}\mathrm{tr}\left[\langle x|\beta(\hat{x})\gamma^5 e^{\hat{\not{\Pi}}^2/\Lambda^2}|x\rangle\right] \\ &= \lim_{\Lambda\to\infty}\beta(x)\mathrm{tr}\left[\langle x|\gamma^5\exp\left(\frac{(\hat{p} - eA(\hat{x}))^2}{\Lambda^2}\right)\exp\left(\frac{-\frac{ie}{2}\gamma^{ab}F_{ab}}{\Lambda^2}\right)|x\rangle\right]\end{aligned} \tag{1.59}$$

If we Taylor expand the second exponential, the first term which gives a non-vanishing trace is the term of order $1/\Lambda^4$. Thus

$$\mathrm{tr}\left[\langle x|\beta(\hat{x})\gamma^5|x\rangle\right] = -\beta(x)\frac{ie^2}{2}\epsilon^{abcd}F_{ab}F_{cd}\lim_{\Lambda\to\infty}\left[\frac{1}{\Lambda^4}\langle x|e^{(p-eA)^2/\Lambda^2}|x\rangle + o(\Lambda^{-5})\right] \quad (1.60)$$

Now, to extract the leading contribution in e, we set $A=0$ in the exponential and we insert $\mathbb{1} = \int d^4k\,|k\rangle\langle k|$ with $\hat{p}|k\rangle = k|k\rangle$ in order to have

$$\frac{1}{\Lambda^4}\langle x|e^{\hat{p}^2/\Lambda^2}|x\rangle = \frac{1}{\Lambda^4}\int\frac{d^4k}{(2\pi)^4}e^{k^2/\Lambda^2} = \frac{i}{\Lambda^4}\int\frac{d^4k_E}{(2\pi)^4}e^{-k_E^2/\Lambda^2} = \frac{i}{16\pi^2} \quad (1.61)$$

Since $\Lambda\to\infty$ we find a finite result, that is

$$\mathcal{J} = \exp\left[i\int d^4x\,\beta(x)\frac{e^2}{32\pi^2}\epsilon^{abcd}F_{ab}F_{cd}\right] \quad (1.62)$$

Thus, under an axial transformation the path integral changes as

$$\begin{aligned}&\int D\overline{\psi}D\psi\,e^{i\int d^4x\,\mathcal{L}_{QED}(m=0)}\\ &\to\int D\overline{\psi}D\psi\,\exp\left[i\int d^4x\left(\mathcal{L}_{QED}(m=0)+\beta(x)\partial_a J_5^a(x)-\beta(x)\frac{e^2}{32\pi^2}\epsilon^{abcd}F_{ab}F_{cd}\right)\right]\end{aligned} \quad (1.63)$$

Then, repeating the steps that led us to eq. (1.46) using this time the axial transformation, we get

$$\partial_a\langle J_5^a(x)O(x_1,\dots,x_n)\rangle = \frac{e^2}{16\pi^2}\langle\epsilon^{abcd}F_{ab}F_{cd}O(x_1,\dots,x_n)\rangle \quad (1.64)$$

that is usually written as

$$\partial_a\langle J_5^a\rangle = \frac{e^2}{16\pi^2}\epsilon^{abcd}F_{ab}F_{cd} \quad (1.65)$$

which agrees with eq. (1.19).

Chapter 2

Actions and symmetries

In this chapter we present the classical models that are the focus of the present thesis. The model of main interest is a massless Weyl fermion coupled to an abelian gauge field. We describe its symmetries and the mass terms to be used in a Pauli-Villars regularization. For comparison, we consider also a massless Dirac fermion coupled to vector and axial abelian gauge fields, a set-up used by Bardeen to compute systematically the anomalies in vector and axial currents [20]. Our notation is commented upon and recapitulated in appendix A.

2.1 The Weyl fermion

The lagrangian of a left handed Weyl spinor λ coupled to a $U(1)$ gauge field is[1]

$$\mathcal{L}_W = -\overline{\lambda}\gamma^a(\partial_a - iA_a)\lambda = -\overline{\lambda}\gamma^a D_a(A)\lambda = -\overline{\lambda}\not{D}(A)\lambda \tag{2.1}$$

where the chirality of the spinor is defined by the constraint $\gamma^5\lambda = \lambda$, or equivalently $\lambda = \frac{1+\gamma^5}{2}\lambda$. It is classically gauge invariant and conformally invariant. Both symmetries become anomalous at the quantum level.

In the following we find it convenient to use the charge conjugated spinor λ_c, which has the opposite chirality of λ

$$\lambda_c = C^{-1}\overline{\lambda}^T \ , \qquad \gamma^5\lambda_c = -\lambda_c \ . \tag{2.2}$$

The lagrangian can be cast in equivalent forms using λ_c rather then $\overline{\lambda}$

$$\mathcal{L}_W = \lambda_c^T C\not{D}(A)\lambda = \lambda^T C\not{D}(-A)\lambda_c = \frac{1}{2}\left(\lambda_c^T C\not{D}(A)\lambda + \lambda^T C\not{D}(-A)\lambda_c\right) \tag{2.3}$$

with the last two forms valid up to boundary terms (we perform partial integrations in the action and drop boundary terms). We will use the last form in our calculations.

[1] From now on we set the spinor charge $e = 1$.

The gauge transformations can be written as

$$\begin{cases} \lambda(x) & \rightarrow & \lambda'(x) = e^{i\alpha(x)}\lambda(x) \\ \overline{\lambda}(x) & \rightarrow & \overline{\lambda}'(x) = e^{-i\alpha(x)}\overline{\lambda}(x) \\ \lambda_c(x) & \rightarrow & \lambda_c'(x) = e^{-i\alpha(x)}\lambda_c(x) \\ A_a(x) & \rightarrow & A_a'(x) = A_a(x) + \partial_a\alpha(x) \end{cases} \tag{2.4}$$

and the action $S_W = \int d^4x\, \mathcal{L}_W$ is gauge invariant. Recall also that A_a can be used as an external source for the current

$$J^a = i\overline{\lambda}\gamma^a\lambda\ . \tag{2.5}$$

Varying only A_a in the action under a gauge transformation with infinitesimal parameter $\alpha(x)$ produces

$$\delta_\alpha^{(A)} S_W = -\int d^4x\, \alpha(x)\partial_a J^a(x) \tag{2.6}$$

and the full gauge symmetry ($\delta_\alpha S_W = 0$) guarantees that the $U(1)$ current is conserved on-shell (i.e. using the fermion equations of motion)

$$\partial_a J^a(x) = 0\ . \tag{2.7}$$

Similarly, one may check that the action is classically conformal invariant and that the stress tensor has a vanishing trace. To see this one couples the model to gravity by introducing the vierbein $e_\mu{}^a$ (and related spin connection $\omega_\mu{}^{ab}$), and realizes that the action is invariant under general coordinate, local Lorentz, and Weyl transformations. The energy momentum tensor, or stress tensor, is defined as usual by

$$T^{\mu a}(x) = \frac{1}{e}\frac{\delta S_W}{\delta e_{\mu a}(x)} \tag{2.8}$$

where e is the determinant of the vierbein, and is covariantly conserved, symmetric, and traceless on-shell, as consequence of diffeomorphisms, local Lorentz invariance, and Weyl symmetry, respectively

$$\nabla_\mu T^{\mu a} = 0\ , \qquad T_{ab} = T_{ba}\ , \qquad T^a{}_a = 0 \tag{2.9}$$

(indices are made "curved" or "flat" by using the vierbein and its inverse). The vierbein can be used as an external source for the stress tensor, and an infinitesimal Weyl transformation on the vierbein acts as a source for the trace $T^a{}_a$ of the stress tensor. In the following we only need a linearized coupling to gravity to produce a single insertion of the stress tensor in correlation functions. Otherwise we are only interested in flat space results. In any case, the full coupling to gravity reads

$$\mathcal{L}_W = -e\,\overline{\lambda}\gamma^\mu\nabla_\mu\lambda \tag{2.10}$$

where $\gamma^\mu = e^\mu{}_a\gamma^a$ are the gamma matrices with curved indices, $e^\mu{}_a$ is the inverse vierbein, and ∇_μ is the covariant derivative containing both the $U(1)$ gauge field A_μ and spin connection $\omega_{\mu ab}$

$$\nabla_\mu = \partial_\mu - iA_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b \ . \tag{2.11}$$

The local Weyl symmetry is given by

$$\begin{cases} \lambda(x) & \to \quad \lambda'(x) = e^{-\frac{3}{2}\sigma(x)}\lambda(x) \\ \overline{\lambda}(x) & \to \quad \overline{\lambda}'(x) = e^{-\frac{3}{2}\sigma(x)}\overline{\lambda}(x) \\ A_a(x) & \to \quad A'_a(x) = A_a(x) \\ e^a{}_\mu(x) & \to \quad e'^a{}_\mu(x) = e^{\sigma(x)}e^a{}_\mu(x) \end{cases} \tag{2.12}$$

where $\sigma(x)$ is an arbitrary function. Varying in the action only the vierbein with an infinitesimal Weyl transformation produces the trace of the stress tensor

$$\delta_\sigma^{(e)} S_W = -\int d^4x e\, \sigma(x) T^a{}_a(x) \tag{2.13}$$

and the full Weyl symmetry of the action ($\delta_\sigma S_W = 0$) guarantees that it is traceless on-shell

$$T^a{}_a(x) = 0 \ . \tag{2.14}$$

For completeness, we record the form of the stress tensor in flat space emerging form the previous considerations and simplified by using the equations of motion

$$T_{ab} = \frac{1}{4}\overline{\lambda}\left(\gamma_a \overleftrightarrow{D}_b + \gamma_b \overleftrightarrow{D}_a\right)\lambda \tag{2.15}$$

where $\overleftrightarrow{D}_a = D_a - \overleftarrow{D}_a$ (in terms of the gauge covariant derivative). Obviously, it is traceless on-shell.

2.1.1 Mass terms

To compute the anomalies in the quantum theory we regularize the latter using massive Pauli-Villars (PV) fields, with the anomalies eventually coming from the noninvariance of the mass term. For the massless Weyl fermion, one can take as PV field a Weyl fermion of the same chirality with a Majorana mass added. The mass term is Lorentz invariant, but breaks the gauge and conformal symmetries. It takes many equivalent forms

$$\begin{aligned} \Delta_{\scriptscriptstyle M}\mathcal{L}_W &= \frac{M}{2}\left(\lambda^T C\lambda + \text{h.c.}\right) = \frac{M}{2}\left(\lambda^T C\lambda - \overline{\lambda}C^{-1}\overline{\lambda}^T\right) \\ &= \frac{M}{2}\left(\lambda^T C\lambda + \lambda_c^T C\lambda_c\right) \end{aligned} \tag{2.16}$$

where h.c. denotes the hermitian conjugate and M is a real mass parameter. Since the charge conjugation matrix C is antisymmetric this term is nonvanishing for an anticommuting spinor[2].

Casting the full massive PV action $\mathcal{L}_{PV} = \mathcal{L}_W + \Delta_M \mathcal{L}_W$ in the following compact form

$$\mathcal{L}_{PV} = \frac{1}{2}\phi^T T\mathcal{O}\phi + \frac{1}{2}M\phi^T T\phi \,, \tag{2.18}$$

where ϕ is a column vector containing both λ and λ_c (ϕ is thus 8 dimensional)

$$\phi = \begin{pmatrix} \lambda \\ \lambda_c \end{pmatrix} \,, \tag{2.19}$$

permits the identification of the operators

$$T\mathcal{O} = \begin{pmatrix} 0 & C\not{D}(-A)P_R \\ C\not{D}(A)P_L & 0 \end{pmatrix} \,, \qquad T = \begin{pmatrix} CP_L & 0 \\ 0 & CP_R \end{pmatrix} \tag{2.20}$$

and

$$\mathcal{O} = \begin{pmatrix} 0 & \not{D}(-A)P_R \\ \not{D}(A)P_L & 0 \end{pmatrix} \,, \qquad \mathcal{O}^2 = \begin{pmatrix} \not{D}(-A)\not{D}(A)P_L & 0 \\ 0 & \not{D}(A)\not{D}(-A)P_R \end{pmatrix} \,. \tag{2.21}$$

The latter will be used in our anomaly calculations. The chiral projectors P_L and P_R

$$P_L = \frac{\mathbb{1} + \gamma^5}{2} \,, \qquad P_R = \frac{\mathbb{1} - \gamma^5}{2} \tag{2.22}$$

have been introduced to stress that the matrix T is not invertible in the full 8 dimensional space on which ϕ lives. An advantage of the Majorana mass term is that it can be constructed without the need of introducing extra degrees of freedom (as required by a Dirac mass term). Moreover, it can be covariantized under Einstein (general coordinate) and local Lorentz symmetries. The covariantization is achieved by multiplying it with the determinant of the vierbein e

$$\Delta_M \mathcal{L}_W = \frac{eM}{2}\left(\lambda^T C\lambda + \lambda_c^T C\lambda_c\right) \,. \tag{2.23}$$

An alternative mass term is the Dirac mass. To use it one must introduce in addition an uncoupled right handed PV fermion ρ (satisfying $\rho = P_R\rho$), so that the full massive PV lagrangian reads

$$\tilde{\mathcal{L}}_{PV} = -\overline{\lambda}\not{D}(A)\lambda - \overline{\rho}\not{\partial}\rho - M(\overline{\lambda}\rho + \overline{\rho}\lambda) \tag{2.24}$$

[2]In terms of the 2-component left handed Weyl spinor l_α this mass terms reads as

$$\Delta_M \mathcal{L}_W = \frac{M}{2}\left(l_\alpha(-i\sigma^2)^{\alpha\beta}l_\beta + l^*_{\dot\alpha}(i\sigma^2)^{\dot\alpha\dot\beta}l^*_{\dot\beta}\right) \tag{2.17}$$

and it does not contain any other spinor apart from l_α and its complex conjugate $l^*_{\dot\alpha}$. In the chiral representation of the gamma matrices the 2-component spinor l_α sits inside λ as in eq. (A.12).

or, equivalently,

$$\begin{aligned}\tilde{\mathcal{L}}_{PV} =& \frac{1}{2}\left(\lambda_c^T C \not{D}(A)\lambda + \lambda^T C \not{D}(-A)\lambda_c\right) + \frac{1}{2}\left(\rho_c^T C \not{\partial}\rho + \rho^T C \not{\partial}\rho_c\right) \\ &+ \frac{M}{2}(\lambda_c^T C\rho + \rho^T C\lambda_c + \rho_c^T C\lambda + \lambda^T C\rho_c)\ . \end{aligned} \tag{2.25}$$

Casting this PV lagrangian in the general form in (2.18) with

$$\phi = \begin{pmatrix} \lambda \\ \lambda_c \\ \rho \\ \rho_c \end{pmatrix} \tag{2.26}$$

where each entry is a 4 dimensional Dirac spinor (with chiral projectors), one finds

$$T\mathcal{O} = \begin{pmatrix} 0 & C\not{D}(-A)P_R & 0 & 0 \\ C\not{D}(A)P_L & 0 & 0 & 0 \\ 0 & 0 & 0 & C\not{\partial}P_L \\ 0 & 0 & C\not{\partial}P_R & 0 \end{pmatrix} \tag{2.27}$$

$$T = \begin{pmatrix} 0 & 0 & 0 & CP_L \\ 0 & 0 & CP_R & 0 \\ 0 & CP_R & 0 & 0 \\ CP_L & 0 & 0 & 0 \end{pmatrix} \tag{2.28}$$

and

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & \not{\partial}P_R & 0 \\ 0 & 0 & 0 & \not{\partial}P_L \\ \not{D}(A)P_L & 0 & 0 & 0 \\ 0 & \not{D}(-A)P_R & 0 & 0 \end{pmatrix} \tag{2.29}$$

$$\mathcal{O}^2 = \begin{pmatrix} \not{\partial}\not{D}(A)P_L & 0 & 0 & 0 \\ 0 & \not{\partial}\not{D}(-A)P_R & 0 & 0 \\ 0 & 0 & \not{D}(A)\not{\partial}P_R & 0 \\ 0 & 0 & 0 & \not{D}(-A)\not{\partial}P_L \end{pmatrix} . \tag{2.30}$$

These differential operators appear also in [21], where definitions for the determinant of a chiral Dirac operator were studied with the purpose of addressing chiral anomalies.

A drawback of the Dirac mass term, as regulator of the Weyl theory, is that one cannot covariantize it while keeping the auxiliary right handed spinor ρ free in the kinetic term (it cannot be coupled to gravity, otherwise it would not regulate properly the original chiral theory). One can still use the regularization keeping ρ free in the kinetic term, but as the mass term breaks the Einstein and local Lorentz symmetries explicitly, one would get anomalies in the conservation $\partial_a T^{ab}$ and antisymmetric part $T^{[ab]}$ of the stress tensor. Then, one is forced to study the counterterms that reinstate conservation and symmetry of the stress tensor (this can always be done in 4 dimensions [7, 22]), and check which trace anomaly one is left with. As this is rather laborious, we do not use this mass term to calculate the trace anomaly in the Weyl theory[3].

2.2 The Dirac fermion

We consider also the more general model of a massless Dirac fermion coupled to vector and axial $U(1)$ gauge fields A_a and B_a. The lagrangian is

$$\begin{aligned}\mathcal{L}_D &= -\overline{\psi}\gamma^a(\partial_a - iA_a - iB_a\gamma^5)\psi = -\overline{\psi}\not{D}(A,B)\psi \\ &= \frac{1}{2}\psi_c^T C\not{D}(A,B)\psi + \frac{1}{2}\psi^T C\not{D}(-A,B)\psi_c \end{aligned} \tag{2.31}$$

where the last form is valid up to boundary terms. A chiral projector emerges when $A_a = \pm B_a$, and we use this model to address again the issue of the chiral fermion in flat space (the limit $A_a = B_a \to \frac{A_a}{2}$ reproduces the massless part of (2.24)).

The lagrangian is invariant under the local $U(1)_V$ vector transformations

$$\left\{\begin{array}{rcl} \psi(x) & \to & \psi'(x) = e^{i\alpha(x)}\psi(x) \\ \overline{\psi}(x) & \to & \overline{\psi}'(x) = e^{-i\alpha(x)}\overline{\psi}(x) \\ \psi_c(x) & \to & \psi_c'(x) = e^{-i\alpha(x)}\psi_c(x) \\ A_a(x) & \to & A_a'(x) = A_a(x) + \partial_a\alpha(x) \\ B_a(x) & \to & B_a'(x) = B_a(x) \end{array}\right. \tag{2.32}$$

and local $U(1)_A$ axial transformations

$$\left\{\begin{array}{rcl} \psi(x) & \to & \psi'(x) = e^{i\beta(x)\gamma^5}\psi(x) \\ \overline{\psi}(x) & \to & \overline{\psi}'(x) = \overline{\psi}(x)e^{i\beta(x)\gamma^5} \\ \psi_c(x) & \to & \psi_c'(x) = e^{i\beta(x)\gamma^5}\psi_c(x) \\ A_a(x) & \to & A_a'(x) = A_a(x) \\ B_a(x) & \to & B_a'(x) = B_a(x) + \partial_a\beta(x)\ . \end{array}\right. \tag{2.33}$$

Again one can use A_a and B_a as sources for $J^a = i\overline{\psi}\gamma^a\psi$ and $J_5^a = i\overline{\psi}\gamma^a\gamma^5\psi$, respectively.

[3] A possibility to simplify the calculation would be to use the axial metric background introduced in [2, 3], but here we will not follow this direction either.

Under infinitesimal variation of these external sources one finds

$$\begin{aligned} \delta^{(A)}_{\alpha} S_{D} &= -\int d^4x\, \alpha(x)\partial_a J^a(x) \\ \delta^{(B)}_{\beta} S_{D} &= -\int d^4x\, \beta(x)\partial_a J^a_5(x) \end{aligned} \tag{2.34}$$

and the classical gauge symmetries imply that J^a and J^a_5 are conserved on-shell

$$\begin{aligned} \partial_a J^a(x) &= 0 \\ \partial_a J^a_5(x) &= 0 \ . \end{aligned} \tag{2.35}$$

A coupling to gravity shows that the stress tensor is traceless because of the Weyl symmetry. The Weyl transformations rules have the same form as in (2.12), extended to B_a by leaving it invariant. An infinitesimal Weyl variation on the vierbein produces the trace of the stress tensor

$$\delta^{(e)}_{\sigma} S_{D} = -\int d^4x e\, \sigma(x) T^a{}_a(x) \ . \tag{2.36}$$

and the Weyl symmetry implies that it vanishes on-shell

$$T^a{}_a(x) = 0 \ . \tag{2.37}$$

2.2.1 Mass terms

To regulate the one-loop graphs we introduces massive PV fields. The standard Dirac mass term

$$\Delta_M \mathcal{L}_D = -M\overline{\psi}\psi = \frac{M}{2}(\psi_c^T C\psi + \psi^T C\psi_c) \tag{2.38}$$

preserves vector gauge invariance, and casting the PV lagrangian

$$\mathcal{L}_{PV} = \mathcal{L}_D + \Delta_M \mathcal{L}_D \tag{2.39}$$

in the form (2.18), now with $\phi = \begin{pmatrix} \psi \\ \psi_c \end{pmatrix}$, allows to recognize the operators

$$T\mathcal{O} = \begin{pmatrix} 0 & C\not{D}(-A,B) \\ C\not{D}(A,B) & 0 \end{pmatrix} , \qquad T = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \tag{2.40}$$

and

$$\mathcal{O} = \begin{pmatrix} \not{D}(A,B) & 0 \\ 0 & \not{D}(-A,B) \end{pmatrix} , \qquad \mathcal{O}^2 = \begin{pmatrix} \not{D}(A,B)^2 & 0 \\ 0 & \not{D}(-A,B)^2 \end{pmatrix} . \tag{2.41}$$

This mass terms mixes the two chiral parts λ and ρ of the Dirac fermion $\psi = \lambda + \rho$, see eqs. (2.24) or (2.25) that makes it immediately visible. After covariantization to gravity the decoupling of the two chiralities is not easily achievable, and relations between the trace anomaly of a Dirac fermion and the trace anomaly of a Weyl fermion cannot be studied directly by using the Dirac mass in the PV regularization.

Thus, it is useful to consider a Majorana mass as well. It breaks both vector and axial symmetries

$$\tilde{\Delta}_M \mathcal{L}_D = \frac{M}{2}(\psi^T C \psi + \text{h.c.}) = \frac{M}{2}(\psi^T C \psi + \psi_c^T C \psi_c) \tag{2.42}$$

and one finds from the alternative PV lagrangian

$$\tilde{\mathcal{L}}_{PV} = \mathcal{L}_D + \tilde{\Delta}_M \mathcal{L}_D \tag{2.43}$$

the operators

$$T\mathcal{O} = \begin{pmatrix} 0 & C\not{D}(-A,B) \\ C\not{D}(A,B) & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \tag{2.44}$$

and

$$\mathcal{O} = \begin{pmatrix} 0 & \not{D}(-A,B) \\ \not{D}(A,B) & 0 \end{pmatrix}, \quad \mathcal{O}^2 = \begin{pmatrix} \not{D}(-A,B)\not{D}(A,B) & 0 \\ 0 & \not{D}(A,B)\not{D}(-A,B) \end{pmatrix}. \tag{2.45}$$

Covariantization to gravity does not mix the chiral parts of the Dirac fermion, and a decoupling limit to the chiral theory of a Weyl fermion λ is now attainable.

Chapter 3

Regulators and consistent anomalies

To compute the anomalies we employ a Pauli-Villars regularization [23]. Following the scheme of refs. [24, 25] we cast the calculation in the same form as the one obtained by Fujikawa in analyzing the measure of the path integral [16, 18, 19]. This makes it easier to use heat kernel formulas [26, 27] to evaluate the anomalies explicitly. At the same time, the method guarantees that one obtains consistent anomalies, i.e. anomalies that satisfy the consistency conditions [28, 29].

Let us review the scheme of ref. [24]. One considers a lagrangian for a field φ

$$\mathcal{L} = \frac{1}{2}\varphi^T T \mathcal{O} \varphi \tag{3.1}$$

which is invariant under a linear symmetry

$$\delta\varphi = K\varphi \tag{3.2}$$

that generically acts also on the operator $T\mathcal{O}$, which may depend on background fields. The one-loop effective action can be regulated by subtracting a loop of a massive PV field ϕ with action

$$\mathcal{L}_{PV} = \frac{1}{2}\phi^T T \mathcal{O}\phi + \frac{1}{2}M\phi^T T\phi \tag{3.3}$$

where M is a real parameter[1]. The mass term identifies the operator T that in turn allows to find the operator $\mathcal{O}$. As we shall see, in fermionic theories with a first order differential operator $\mathcal{O}$ in the kinetic term, the operator $\mathcal{O}^2$ acts as a regulator in the final formula for the anomaly. The invariance of the original action extends to an invariance of the massless part of the PV action by defining

$$\delta\phi = K\phi \tag{3.4}$$

[1]To be precise, one should employ a set of PV fields with mass M_i and relative weight c_i in the loop to be able to regulate and cancel all possible one-loop divergences [23]. For simplicity, we consider only one PV field with relative weight $c = -1$, as this is enough for our purposes. The weight $c = -1$ means that we are subtracting a massive PV loop from the original one.

so that only the mass term may break the symmetry

$$\delta\mathcal{L}_{PV} = \frac{1}{2}M\phi^T(TK + K^TT + \delta T)\phi = M\phi^T(TK + \frac{1}{2}\delta T)\phi \ . \tag{3.5}$$

The path integral Z and the one-loop effective action Γ are regulated by the PV field

$$Z = e^{i\Gamma} = \int D\varphi\; e^{iS} \qquad \to \qquad Z = e^{i\Gamma} = \int D\varphi D\phi\; e^{i(S+S_{PV})} \tag{3.6}$$

where it is understood that one should take the $M \to \infty$ limit, with all divergences canceled as explained in footnote 1. The anomalous response of the path integral under a symmetry is due to the PV mass term only, as one can define the measure of the PV field so to make the whole path integral measure invariant [24]. In a hypercondensed notation, where a term like $\phi^T\phi$ includes in the sum of the indices a spacetime integration as well, a lagrangian like the one in (3.3) is equivalent to the action, and one may compute the symmetry variation of the regulated path integral to obtain

$$\begin{aligned} i\delta\Gamma = i\langle\delta S\rangle &= \lim_{M\to\infty}\; iM\langle\phi^T(TK + \frac{1}{2}\delta T)\phi\rangle \\ &= -\lim_{M\to\infty} \mathrm{Tr}\left[\left(K + \frac{1}{2}T^{-1}\delta T\right)\left(1 + \frac{\mathcal{O}}{M}\right)^{-1}\right] \end{aligned} \tag{3.7}$$

where brackets $\langle \ldots \rangle$ denote normalized correlation functions. For our purposes, it is convenient to cast it in an equivalent form [25]

$$i\delta\Gamma = i\langle\delta S\rangle = -\lim_{M\to\infty} \mathrm{Tr}\left[\left(K + \frac{1}{2}T^{-1}\delta T + \frac{1}{2}\frac{\delta\mathcal{O}}{M}\right)\left(1 - \frac{\mathcal{O}^2}{M^2}\right)^{-1}\right] \tag{3.8}$$

which is obtained by using the identity $1 = (1 - \frac{\mathcal{O}}{M})(1 - \frac{\mathcal{O}}{M})^{-1}$ and the invariance of the massless action

$$\delta\mathcal{L} = \varphi^T\left(T\mathcal{O}K + \frac{1}{2}\delta T\mathcal{O} + \frac{1}{2}T\delta\mathcal{O}\right)\varphi = 0 \ . \tag{3.9}$$

In deriving these expressions, we have considered a fermionic theory, used the PV propagator

$$\langle\phi\phi^T\rangle = \frac{i}{T\mathcal{O} + TM} \ , \tag{3.10}$$

taken into account the opposite sign for the PV field in the loop, and considered an invertible mass matrix T. In the limit $M \to \infty$ the regulating term $(1 - \frac{\mathcal{O}^2}{M^2})^{-1}$ inside (3.8) can be replaced by $e^{\frac{\mathcal{O}^2}{M^2}}$. This is allowed as for extracting the limit these regulators cut off the ultraviolet frequencies in an equivalent way (we assume that $\mathcal{O}^2$ is negative definite after a Wick rotation to euclidean space). Clearly, if one finds a symmetrical mass term, then the symmetry would remain automatically anomaly free.

Heat kernel formulas may now be directly applied. Denoting

$$J = K + \frac{1}{2}T^{-1}\delta T + \frac{1}{2}\frac{\delta \mathcal{O}}{M} , \qquad R = -\mathcal{O}^2 \tag{3.11}$$

the anomaly is related to the trace of the heat kernel of the regulator R with an insertion of J

$$i\delta\Gamma = i\langle \delta S\rangle = - \lim_{M\to\infty} \mathrm{Tr}[Je^{-\frac{R}{M^2}}] . \tag{3.12}$$

This has the same form that appears in Fujikawa's method for computing anomalies [18, 19], where J is the infinitesimal part of the fermionic jacobian arising from a change of the path integral variables under a symmetry transformation, and R is the regulator. The limit extracts only the mass independent term (negative powers of the mass vanish in the limit, while positive (diverging) powers are made to cancel by using additional PV fields). The PV method guarantees that the regulator R together with J produces consistent anomalies, which follows from the fact that we are computing directly the variation of the effective action.

The heat kernel formulas that we need in the anomaly calculation are well-known, and we report them in appendix B using a minkowskian time. In particular, in four dimensions we just need the so-called Seeley-DeWitt coefficients $a_2(R)$ corresponding to the regulators R associated to the fields assembled into ϕ. These are the only coefficients that survive in the limit $M \to \infty$ (as said, diverging pieces are removed by the PV renomalization). Running through the various cases of the previous section, we can extract the "jacobians" J and regulators R to find the structure of the anomalies. For the Weyl model we find

$$\begin{aligned} \partial_a\langle J^a\rangle &= \frac{i}{(4\pi)^2}\Big\{\mathrm{tr}\,[P_L a_2(R_\lambda)] - \mathrm{tr}\,[P_R a_2(R_{\lambda_c})]\Big\} \\ \langle T^a{}_a\rangle &= -\frac{1}{2(4\pi)^2}\Big\{\mathrm{tr}\,[P_L a_2(R_\lambda)] + \mathrm{tr}\,[P_R a_2(R_{\lambda_c})]\Big\} . \end{aligned} \tag{3.13}$$

These formulas are obtained by considering that for the $U(1)$ symmetry the jacobian J in (3.11) is extracted from the symmetry transformations of λ and λ_c in (2.4)

$$J = \begin{pmatrix} i\alpha P_L & 0 \\ 0 & -i\alpha P_R \end{pmatrix} . \tag{3.14}$$

Only K contributes, as δT vanishes while we have neglected the contribution from $\delta\mathcal{O}$ (it vanishes after taking the traces in (3.13), as checked in the next chapter). The infinitesimal parameter α is eventually factorized away from (3.12) to obtain the local form in (3.13). In computing J from (3.11), it is enough to check that the mass matrix T is invertible on the relevant chiral spaces (extracted by the projectors P_L and P_R). For the Weyl symmetry one uses instead the transformation laws in (2.12) to find

$$J = \begin{pmatrix} \frac{1}{2}\sigma P_L & 0 \\ 0 & \frac{1}{2}\sigma P_R \end{pmatrix} , \tag{3.15}$$

where now it is crucial to consider that the covariant (under gravity) extension of the mass terms contains a factor of e, see eq. (2.23), which brings in a contribution from $\frac{1}{2}T^{-1}\delta T$ to J ($\delta\mathcal{O}$ is neglected again for the same reason as before). This contribution is necessary to guarantee that general coordinate invariance is kept anomaly free in the regularization. The infinitesimal Weyl parameter σ is then factorized away from (3.12) to obtain the second equation in (3.13).

Proceeding in a similar way, we find for the Dirac model

$$\begin{aligned}
\partial_a\langle J^a\rangle &= \frac{i}{(4\pi)^2}\Big\{\mathrm{tr}\, a_2(R_\psi) - \mathrm{tr}\, a_2(R_{\psi_c})\Big\} \\
\partial_a\langle J^a_5\rangle &= \frac{i}{(4\pi)^2}\Big\{\mathrm{tr}\,[\gamma^5 a_2(R_\psi)] + \mathrm{tr}\,\gamma^5[a_2(R_{\psi_c})]\Big\} \\
\langle T^a{}_a\rangle &= -\frac{1}{2(4\pi)^2}\Big\{\mathrm{tr}\, a_2(R_\psi) + \mathrm{tr}\, a_2(R_{\psi_c})\Big\}\ .
\end{aligned} \tag{3.16}$$

All remaining traces are traces on the gamma matrices taken in the standard four dimensional Dirac spinor space.

Chapter 4

Chiral and trace anomalies

In this chapter we compute systematically the chiral and trace anomalies for the Weyl and Dirac models described in chapter 2. We use, when applicable, two different versions of the Pauli-Villars regularization with different mass terms. We verify that the final results are consistent with each other, and coincide after taking into account the variation of local counterterms.

4.1 Chiral and trace anomalies of a Weyl fermion

We consider first the case of a Weyl fermion.

4.1.1 PV regularization with Majorana mass

The regularization of the Weyl fermion coupled to an abelian gauge field is achieved in the most minimal way by using a PV fermion of the same chirality with the Majorana mass term given in eq. (2.16) added. This set-up was already used in [6] to address the case of a Weyl fermion in a gravitational background, without the abelian gauge coupling. The mass term is Lorentz invariant and does not introduce additional chiralities, but breaks the gauge and conformal (and Weyl) symmetries. Therefore, one expects chiral and trace anomalies.

To obtain the anomalies we have to compute the expressions in (3.13) with the regulators contained inside the $\mathcal{O}^2$ given in eq. (2.21). They read

$$\begin{aligned} R_\lambda &= -\not{D}(-A)\not{D}(A)P_L \\ R_{\lambda_c} &= -\not{D}(A)\not{D}(-A)P_R \,. \end{aligned} \tag{4.1}$$

Using the Seeley-DeWitt coefficients a_2 of these regulators[1], we find for the chiral anomaly

$$\partial_a \langle J^a \rangle = \frac{1}{(4\pi)^2} \left(\frac{1}{6} \epsilon^{abcd} F_{ab} F_{cd} - \frac{8}{3} \partial_a (A^a A^2) + \frac{2}{3} \Box (\partial A) \right) \tag{4.2}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$. It contains normal-parity terms that can be canceled by the gauge variation of the local counterterm

$$\Gamma_1 = \int \frac{d^4x}{(4\pi)^2} \left(\frac{2}{3} A^4 - \frac{1}{3} A^a \Box A_a \right) , \tag{4.3}$$

so that the chiral gauge anomaly takes the form

$$\partial_a \langle J^a \rangle = \frac{1}{96\pi^2} \epsilon^{abcd} F_{ab} F_{cd} \tag{4.4}$$

which is the standard result.

Similarly, we compute the trace anomaly which is given by

$$\langle T^a{}_a \rangle = -\frac{1}{(4\pi)^2} \left(\frac{2}{3} (\partial_a A_b)(\partial^a A^b) - \frac{2}{3} (\partial A)^2 - \frac{2}{3} \Box A^2 \right) . \tag{4.5}$$

It does not contain any odd-parity contribution. Gauge invariance is broken by the chiral anomaly, still the trace anomaly can be cast in a gauge invariant form by varying a local counterterm with a Weyl transformation and then restricting to flat space[2]. The (gravity covariant and gauge noninvariant) counterterm is given by

$$\Gamma_2 = \int \frac{d^4x \sqrt{g}}{(4\pi)^2} \left(\frac{1}{3} (\nabla^\mu A^\nu)(\nabla_\mu A_\nu) + \frac{1}{6} R A^2 \right) \tag{4.6}$$

and the trace anomaly takes the form

$$\langle T^a{}_a \rangle = -\frac{1}{48\pi^2} F_{ab} F^{ab} . \tag{4.7}$$

The counterterms Γ_1 and Γ_2 are consistent with each other, and merge into the unique counterterm (needed only at linear order in the metric)

$$\Gamma_3 = \int \frac{d^4x \sqrt{g}}{(4\pi)^2} \left(\frac{2}{3} A^4 + \frac{1}{3} (\nabla^\mu A^\nu)(\nabla_\mu A_\nu) + \frac{1}{6} R A^2 \right) \tag{4.8}$$

where, of course, $A^2 = g^{\mu\nu} A_\mu A_\nu$ and $A^4 = (A^2)^2$.

Thus, we have seen that the trace anomaly of a Weyl fermion does not contain any contribution from the topological density $F\tilde{F}$ (which on the other hand enters the chiral anomaly in (4.4), as well-known). Also, it can be presented in a gauge invariant form by the variation of a local counterterm, and equals half the standard trace anomaly of a Dirac fermion.

[1] In appendix C we describe how we computes the Seeley-DeWitt coefficients a_2.

[2] We list in appendix D the covariant counterterms with a non-trivial transformation under a Weyl symmetry, their variations and their flat spacetime limits.

4.1.2 PV regularization with Dirac mass

In order to use a Dirac mass we have to include also a right handed free fermion in the PV lagrangian. The lagrangian is given in (2.24), and from eq. (2.30) one finds the regulators

$$\begin{aligned} R_\lambda &= -\partial\!\!\!/ D\!\!\!\!/(A) P_L \\ R_{\lambda_c} &= -\partial\!\!\!/ D\!\!\!\!/(-A) P_R \,. \end{aligned} \tag{4.9}$$

Then, from the corresponding heat kernel coefficients a_2 we find the chiral anomaly

$$\partial_a \langle J^a \rangle = \frac{1}{(4\pi)^2} \left(\frac{1}{6} \epsilon^{abcd} F_{ab} F_{cd} - \frac{1}{3} \partial_a (A^a A^2) + \frac{1}{3} \Box (\partial A) \right) . \tag{4.10}$$

It contains noncovariant normal-parity terms, that are canceled by the variation of the local counterterm

$$\Gamma_4 = \int \frac{d^4x}{(4\pi)^2} \left(\frac{1}{12} A^4 - \frac{1}{6} A^a \Box A_a \right) \tag{4.11}$$

so that the anomaly takes the standard form

$$\partial_a \langle J^a \rangle = \frac{1}{96\pi^2} \epsilon^{abcd} F_{ab} F_{cd} \tag{4.12}$$

as in the previous section.

Unfortunately, we cannot proceed to compute in a simple way the trace anomaly using this regularization, as the mass term breaks the Einstein and local Lorentz symmetries as well. The ensuing anomalies should then be computed and canceled by local counterterms, to find eventually the (expected) agreement of the remaining trace anomaly with the one found in the previous section.

4.2 Chiral and trace anomalies of a Dirac fermion

We now consider the case of the massless Dirac spinor coupled to vector and axial gauge fields with lagrangian given in eq. (2.31). The most natural regularization is obtained by employing a Dirac mass for the PV fields, but we also consider a Majorana mass. The latter allows to take a chiral limit in a simple way, which we use to rederive the previous results on the Weyl fermion.

4.2.1 PV regularization with Dirac mass

The relevant regulators are obtained from (2.41) and read

$$\begin{aligned} R_\psi &= -D\!\!\!\!/(A,B)^2 \\ R_{\psi_c} &= -D\!\!\!\!/(-A,B)^2 \,. \end{aligned} \tag{4.13}$$

The vector symmetry is guaranteed to remain anomaly free by the invariance of the mass term, while the chiral anomaly from (3.16) becomes

$$\partial_a \langle J_5^a \rangle = \frac{1}{(4\pi)^2} \left(\epsilon^{abcd} F_{ab}(A) F_{cd}(A) + \frac{1}{3} \epsilon^{abcd} F_{ab}(B) F_{cd}(B) - \frac{16}{3} \partial_a (B^a B^2) + \frac{4}{3} \Box (\partial B) \right) . \tag{4.14}$$

It contains normal-parity terms in the B field. They are canceled by the variation of a local counterterm

$$\Gamma_5 = \int \frac{d^4x}{(4\pi)^2} \left(\frac{4}{3} B^4 - \frac{2}{3} B^a \Box B_a \right) \tag{4.15}$$

so that one ends up with

$$\partial_a \langle J^a \rangle = 0 \tag{4.16}$$

$$\partial_a \langle J_5^a \rangle = \frac{1}{(4\pi)^2} \left(\epsilon^{abcd} F_{ab}(A) F_{cd}(A) + \frac{1}{3} \epsilon^{abcd} F_{ab}(B) F_{cd}(B) \right) . \tag{4.17}$$

As for the trace anomaly, we find from (3.16)

$$\langle T^a{}_a \rangle = -\frac{1}{(4\pi)^2} \left(\frac{2}{3} F_{ab}(A) F^{ab}(A) + \frac{4}{3} (\partial_a B_b)(\partial^a B^b) - \frac{4}{3} (\partial B)^2 - \frac{4}{3} \Box B^2 \right) \tag{4.18}$$

and the counterterm

$$\Gamma_6 = \int \frac{d^4x \sqrt{g}}{(4\pi)^2} \left(\frac{2}{3} (\nabla^\mu B^\nu)(\nabla_\mu B_\nu) + \frac{1}{3} R B^2 \right) \tag{4.19}$$

brings it into the gauge invariant form

$$\langle T^a{}_a \rangle = -\frac{1}{24\pi^2} \Big(F_{ab}(A) F^{ab}(A) + F_{ab}(B) F^{ab}(B) \Big) . \tag{4.20}$$

All these counterterms merge naturally into the complete counterterm

$$\Gamma_7 = \int \frac{d^4x \sqrt{g}}{(4\pi)^2} \left(\frac{4}{3} B^4 + \frac{2}{3} (\nabla^\mu B^\nu)(\nabla_\mu B_\nu) + \frac{1}{3} R B^2 \right) . \tag{4.21}$$

4.2.2 PV regularization with Majorana mass

Finally, we consider the regularization with a Majorana mass. As both vector and chiral symmetries are broken by the mass term, we expect anomalies in both $U(1)$ currents. From eq. (2.45) we find the regulators

$$\begin{aligned} R_\psi &= -\not{D}(-A, B) \not{D}(A, B) \\ R_{\psi_c} &= -\not{D}(A, B) \not{D}(-A, B) . \end{aligned} \tag{4.22}$$

Thus, we compute from (3.16)

$$\partial_a\langle J^a\rangle = \frac{1}{(4\pi)^2}\left(\frac{2}{3}\epsilon^{abcd}F_{ab}(A)F_{cd}(B)+\frac{4}{3}\Box(\partial A)-\frac{16}{3}\partial_a[A^a(A^2+B^2)]-\frac{32}{3}\partial_a(B^aA_bB^b)\right) \tag{4.23}$$

and

$$\begin{aligned}\partial_a\langle J^a_5\rangle = \frac{1}{(4\pi)^2}\Bigg(&\frac{1}{3}\epsilon^{abcd}F_{ab}(A)F_{cd}(A)+\frac{1}{3}\epsilon^{abcd}F_{ab}(B)F_{cd}(B)+\frac{4}{3}\Box(\partial B)\\ &-\frac{16}{3}\partial_a[B^a(A^2+B^2)]-\frac{32}{3}\partial_a(A^aA_bB^b)\Bigg)\ .\end{aligned} \tag{4.24}$$

The counterterm $\Gamma_8+\Gamma_9$

$$\begin{aligned}\Gamma_8 &= \int\frac{d^4x}{(4\pi)^2}\left(\frac{4}{3}(A^2+B^2)^2+\frac{16}{3}(A^aB_a)^2-\frac{2}{3}A^a\Box A_a-\frac{2}{3}B^a\Box B_a\right)\\ \Gamma_9 &= \int\frac{d^4x}{(4\pi)^2}\left(\frac{8}{3}\epsilon^{abcd}B_aA_b(\partial_cA_d)\right)\end{aligned} \tag{4.25}$$

allows to recover vector gauge invariance, and the anomalies take the form

$$\partial_a\langle J^a\rangle = 0 \tag{4.26}$$

$$\partial_a\langle J^a_5\rangle = \frac{1}{(4\pi)^2}\left(\epsilon^{abcd}F_{ab}(A)F_{cd}(A)+\frac{1}{3}\epsilon^{abcd}F_{ab}(B)F_{cd}(B)\right)\ . \tag{4.27}$$

As for the trace anomaly, we find

$$\langle T^a{}_a\rangle = -\frac{1}{(4\pi)^2}\left(\frac{4}{3}(\partial_aA_b)(\partial^aA^b)-\frac{4}{3}(\partial A)^2-\frac{4}{3}\Box A^2+\frac{4}{3}(\partial_aB_b)(\partial^aB^b)-\frac{4}{3}(\partial B)^2-\frac{4}{3}\Box B^2\right) \tag{4.28}$$

and using the counterterm

$$\Gamma_{10} = \int\frac{d^4x\sqrt{g}}{(4\pi)^2}\left(\frac{2}{3}(\nabla^\mu A^\nu)(\nabla_\mu A_\nu)+\frac{2}{3}(\nabla^\mu B^\nu)(\nabla_\mu B_\nu)+\frac{1}{3}R(A^2+B^2)\right) \tag{4.29}$$

we get the final gauge invariant form

$$\langle T^a{}_a\rangle = -\frac{1}{24\pi^2}\Big(F_{ab}(A)F^{ab}(A)+F_{ab}(B)F^{ab}(B)\Big)\ . \tag{4.30}$$

The counterterms employed in this section are consistent with each other, and combine into a unique final counterterm, which we report for completeness

$$\begin{aligned}\Gamma_{11} = \int\frac{d^4x\sqrt{g}}{(4\pi)^2}\Bigg(&\frac{2}{3}(\nabla^\mu A^\nu)(\nabla_\mu A_\nu)+\frac{2}{3}(\nabla^\mu B^\nu)(\nabla_\mu B_\nu)+\frac{1}{3}R(A^2+B^2)\\ &+\frac{4}{3}(A^2+B^2)^2+\frac{16}{3}(A^\mu B_\mu)^2+\frac{4}{3}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}B_\mu A_\nu F_{\rho\sigma}(A)\Bigg)\ .\end{aligned} \tag{4.31}$$

Evidently, the anomalies computed with the Majorana mass coincide with those obtained with the Dirac mass, after using local counterterms.

The results of this section can be projected consistently to recover the chiral and trace anomalies of the Weyl fermion. Indeed, one can consider the limit $A_a = B_a \to \frac{1}{2}A_a$. In this limit, a chiral projector $P_L = \frac{1+\gamma^5}{2}$ emerges inside the Dirac lagrangian (2.31) to reproduce the Weyl lagrangian (2.1). In addition, in the coupling to gravity, the right handed component of the Dirac field can be kept free both in the kinetic and in the PV mass term, while preserving the covariance of the mass term for the left handed part of the PV Dirac fermion. Thus, the right handed part can be ignored altogether. Indeed, one may verify that the anomalies in subsection 4.1.1 are reproduced by those computed here, including the counterterms, by setting $A_a = B_a \to \frac{1}{2}A_a$ (note that the current J^a in 4.1.1 corresponds to half the sum of J^a and J_5^a of this section).

Finally, we have checked that terms proportional to $\delta\mathcal{O}$ in (3.11) never contribute to the anomalies computed so far, as the extra terms vanish under the Dirac trace.

Conclusions

We introduced the topic of anomalies in Quantum Field Theory and reviewed the computation of one global anomaly and one gauge anomaly. With the latter example we discussed the Fujikawa method to evaluate anomalies which has been useful to introduce our method, that makes use of the path integral and heat kernel formulas.

Since we employed a Pauli-Villars regularization, we discussed Majorana and Dirac mass terms that we used to regularize the models of our interest in order to check the internal consistency of our approach. We considered a Weyl fermion coupled to a $U(1)$ gauge field A_a and a Dirac massless fermion coupled to $U(1)$ vector A_a and axial B_a gauge fields. The Majorana mass term is more naturally used to regularize the Weyl fermion, since this mass allows to treat separately the fermion left and right chiralities, while the Dirac mass term mixes the two chiralities and thus it is more suitable to regularize the Dirac fermion.

Nonetheless, we employed both regularizations in the study of chiral anomalies in our models and we checked that our results are regularization-independent. The chiral anomaly that we obtained for the Weyl fermion agrees with the standard one known in the literature. The Dirac fermion may develops an anomaly both in the vector and in the axial symmetry, but eventually the anomalous behaviour can be shifted entirely in the chiral current with the help of countertems in the effective action. Thus, since the Dirac fermion is equivalent to the Weyl model in the limit $A_a = B_a \to \frac{1}{2}A_a$, we checked that the Dirac chiral anomaly correctly reproduces the Weyl one.

The trace anomaly could not be computed easily in every regularization. Indeed, the Weyl model regulated with a PV field with Dirac mass exhibits Einstein and local Lorentz anomalies as well and we did not pursue this calculation. However, we computed the trace anomaly of the Weyl fermion regulated with a PV field with Majorana mass, which is the main result of our thesis. Even though the chiral anomaly implies a breakdown of gauge invariance, the trace anomaly can be put in a gauge invariant form with the help of local counterterms and we found

$$\langle T^a{}_a \rangle = -\frac{1}{48\pi^2} F_{ab}F^{ab} \; . \tag{4.32}$$

This result is also reproduced in the Dirac model regulated with PV field with Majorana mass in the limit $A_a = B_a \to \frac{1}{2}A_a$.

CONCLUSIONS

We remark that our results show no presence of any odd-parity contribution in the trace anomaly of the Weyl fermion. In particular, there is no Chern-Pontryagin term, even if this term satisfies the consistency conditions for Weyl anomalies. Regarding the controversy about its presence in the trace anomaly of a Weyl fermion in curved background, our result have no direct implications for the curved background case; however, it strengthens the findings of ref. [6].

Appendix A

Conventions

We use a mostly plus Minkowski metric η_{ab}. The Dirac matrices γ^a satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{A.1}$$

and the conjugate Dirac spinor $\overline{\psi}$ is defined using $\beta = i\gamma^0$ by

$$\overline{\psi} = \psi^\dagger \beta \; . \tag{A.2}$$

The hermitian chiral matrix γ^5 is given by

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.3}$$

and used to define the chiral projectors

$$P_L = \frac{\mathbb{1} + \gamma_5}{2} \; , \qquad P_R = \frac{\mathbb{1} - \gamma_5}{2} \tag{A.4}$$

that split a Dirac spinor ψ into its left and right Weyl components

$$\psi = \lambda + \rho \; , \qquad \lambda = P_L\psi \; , \quad \rho = P_R\psi \; . \tag{A.5}$$

The charge conjugation matrix C satisfies

$$C\gamma^a C^{-1} = -\gamma^{aT} \; , \tag{A.6}$$

it is antisymmetric and used to define the charge conjugation of the spinor ψ by

$$\psi_c = C^{-1}\overline{\psi}^T \tag{A.7}$$

for which the roles of particle and antiparticle get interchanged. Note that a chiral spinor λ has its charge conjugated field λ_c of opposite chirality. A Majorana spinor μ is a spinor that equals its charged conjugated spinor

$$\mu = \mu_c \; . \tag{A.8}$$

This constraint is incompatible with the chiral constraint, and Majorana-Weyl spinors do not exist in 4 dimensions.

We find it convenient, as a check on our formulas, to use the chiral representation of the gamma matrices. In terms of 2×2 blocks they are given by

$$\gamma^0 = -i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \qquad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{A.9}$$

where σ^i are the Pauli matrices, so that

$$\gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} , \qquad \beta = i\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{A.10}$$

The chiral representation makes evident that the Lorentz generators in the spinor space $M^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] = \frac{1}{2}\gamma^{ab}$ take a block diagonal form

$$M^{0i} = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} , \qquad M^{ij} = \frac{i}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{A.11}$$

and do not mix the chiral components of a Dirac spinor (as γ^5 is also block diagonal). The usual two-dimensional Weyl spinors appear inside a four-dimensional Dirac spinor as follows

$$\psi = \begin{pmatrix} l \\ r \end{pmatrix} , \qquad \lambda = \begin{pmatrix} l \\ 0 \end{pmatrix} , \qquad \rho = \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{A.12}$$

where l and r indicate two-dimensional independent spinors of opposite chirality. In the chiral representation one may take the charge conjugation matrix C to be given by

$$C = \gamma^2\beta = -i \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \tag{A.13}$$

and satisfies

$$C = -C^T = -C^{-1} = -C^\dagger = C^* \tag{A.14}$$

(some of these relations are representation dependent). In the chiral representation the Majorana constraint (A.8) takes the form

$$\mu = \mu_c \qquad \to \qquad \begin{pmatrix} l \\ r \end{pmatrix} = \begin{pmatrix} i\sigma^2 r^* \\ -i\sigma^2 l^* \end{pmatrix} \tag{A.15}$$

which shows that the two-dimensional spinors l and r cannot be independent. The Majorana condition can be solved in terms of the single two-dimensional left-handed spinor l as

$$\mu = \begin{pmatrix} l \\ -i\sigma^2 l^* \end{pmatrix} \tag{A.16}$$

which, evidently, contains the four-dimensional chiral spinors λ and λ_c defined by

$$\lambda = \begin{pmatrix} l \\ 0 \end{pmatrix} , \qquad \lambda_c = \begin{pmatrix} 0 \\ -i\sigma^2 l^* \end{pmatrix} . \tag{A.17}$$

In a four-dimensional spinors notation one can write

$$\mu = \lambda + \lambda_c . \tag{A.18}$$

Alternatively, the Majorana condition can be solved in terms of the two-dimensional right-handed spinor r as

$$\mu = \begin{pmatrix} i\sigma^2 r^* \\ r \end{pmatrix} \tag{A.19}$$

which contains the four-dimensional chiral spinors ρ and ρ_c

$$\rho = \begin{pmatrix} 0 \\ r \end{pmatrix} , \qquad \rho_c = \begin{pmatrix} i\sigma^2 r^* \\ 0 \end{pmatrix} \tag{A.20}$$

and $\mu = \rho + \rho_c$. This solution is of course the same as the previous one, as one may identify $\lambda = \rho_c$.

The explicit dictionary between Weyl and Majorana spinors shows clearly that the field theory of a Weyl spinor is equivalent to that of a Majorana spinor, as Lorentz symmetry fixes uniquely their actions, which are bound to be identical.

Finally, we normalize our ϵ symbols by $\epsilon_{0123} = -1$ and $\epsilon^{0123} = 1$, so that

$$\frac{1}{4}\mathrm{tr}\,(\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d) = i\epsilon^{abcd} . \tag{A.21}$$

Appendix B

The heat kernel

We consider an operator in flat D dimensional spacetime of the form

$$H = -\nabla^2 + V \tag{B.1}$$

with V a matrix potential and $\nabla^2 = \nabla^a \nabla_a$ constructed with a gauge covariant derivative $\nabla_a = \partial_a + W_a$ that satisfies

$$[\nabla_a, \nabla_b] = \partial_a W_b - \partial_b W_a + [W_a, W_b] = \mathcal{F}_{ab} \,. \tag{B.2}$$

The trace of the corresponding heat kernel is perturbatively given by

$$\begin{aligned} \mathrm{Tr}\left[Je^{-isH}\right] &= \int d^D x \, \mathrm{tr}\, \left[J(x)\langle x|e^{-isH}|x\rangle\right] \\ &= \int d^D x \, \frac{i}{(4\pi i s)^{\frac{D}{2}}} \sum_{n=0}^{\infty} \mathrm{tr}\,[J(x) a_n(x,H)](is)^n \\ &= \int d^D x \, \frac{i}{(4\pi i s)^{\frac{D}{2}}} \mathrm{tr}\,[J(x)(a_0(x,H) + a_1(x,H) is + a_2(x,H)(is)^2 + ...)] \end{aligned} \tag{B.3}$$

where the symbol “tr” is the trace on the remaining discrete matrix indices, $J(x)$ is an arbitrary matrix function, and $a_n(x,H)$ are the so-called Seeley-DeWitt coefficients, or heat kernel coefficients. They are matrix valued, and the first few ones are given by

$$\begin{aligned} a_0(x,H) &= \mathbb{1} \\ a_1(x,H) &= -V \\ a_2(x,H) &= \frac{1}{2}V^2 - \frac{1}{6}\nabla^2 V + \frac{1}{12}\mathcal{F}_{ab}^2 \,. \end{aligned} \tag{B.4}$$

As V is allowed to be a matrix, then $\nabla_a V = \partial_a V + [W_a, V]$.

In the main text, the role of the hamiltonian H is played by the various regulators R, and $is \sim \frac{1}{M^2}$, see eq. (3.12). In $D = 4$ the s independent term is precisely the one with $a_2(x, H)$, which is the coefficient producing the anomalies in 4 dimensions (we use a minkowskian set-up, but justify the heat kernel formulas by Wick rotating to an euclidean time and back, when necessary).

More details on the heat kernel expansion are found in [26, 27], where the coefficients appear with the additional coupling to a background metric. They have been recomputed with quantum mechanical path integrals in [30], a useful report is [31], while in [32] one may find the explicit expression for $a_3(x, H)$, originally calculated by Gilkey [33], which is relevant for calculations of anomalies in 6 dimensions.

Appendix C

Seeley-DeWitt coefficients

Here we show how to compute the Seeley-DeWitt coefficients a_2 for the operators R_λ and R_{λ_c} considered in the main text and how to reproduce the results of section 4.

To start with, if we consider a second order operator H of the Laplace type, defined on a vector bundle B over a Riemannian manifold M, we can locally represent this operator as

$$H = -\left(g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + b\right) \tag{C.1}$$

where a^μ and b are matrix valued functions on M. There is a unique connection on V and a unique endomorphism V of B so that

$$H = -g^{\mu\nu}\nabla_\mu\nabla_\nu + V = -\nabla^2 + V \tag{C.2}$$

which is as in eq. (B.1) and, for the flat space cases of our interests, we will have a gauge covariant derivative $\nabla_a = \partial_a + W_a$. Expanding the covariant derivatives one finds

$$H = -\nabla^2 + V = -\partial^a\partial_a - 2W^a\partial_a - (\partial_a W^a) - W^a W_a + V \ . \tag{C.3}$$

Then, since the Seeley-DeWitt coefficients are given as in (B.4), we only need to find W_a, V and $\mathcal{F}_{ab}$ for the regulators presented in chapter 2.

We structure this appendix as chapter 4 for clarity.

C.1 Weyl fermion

First we consider the regulators for the Weyl fermion.

C.1.1 PV regularization with Majorana mass

The regulators that are used in the model of section 4.1.1 are

$$\begin{aligned} R_\lambda &= -\not{D}(-A)\not{D}(A)P_L \\ R_{\lambda_c} &= -\not{D}(A)\not{D}(-A)P_R \ . \end{aligned} \tag{C.4}$$

Now, let us focus on R_λ. Neglecting the projector, that can be reinstated later, and expanding the derivative one finds

$$\begin{aligned} R_\lambda = -\not{D}(-A)\not{D}(A) &= -D^a(-A)D_a(A) - 2i\gamma^{ab}A_a\partial_b + \frac{i}{2}F_{ab}\gamma^{ab} \\ &= -\partial^a\partial_a + 2i\gamma^{ab}A_b\partial_a + i(\partial^a A_a) - A^a A_a + \frac{i}{2}F_{ab}\gamma^{ab} \end{aligned} \tag{C.5}$$

where $\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$. Comparing (C.3) and (C.5) one fixes

$$W^a = -i\gamma^{ab}A_b \tag{C.6}$$

$$W^a W_a = -\gamma^{ab}A_b\gamma_{ac}A^c = 3A^a A_a \tag{C.7}$$

so that V is given by

$$V = 2A^a A_a + i(\partial^a A_a)\ . \tag{C.8}$$

At this stage one proceeds to evaluate the field strength $\mathcal{F}_{ab}$ in eq. (B.2) associated to this particular W^a, that turns out to be

$$\mathcal{F}_{ab} = i\gamma_{ac}(\partial_b A^c) - i\gamma_{bc}(\partial_a A^c) - 2\gamma_{ac}A_b A^c - 2\gamma_{cb}A_a A^c + 2\gamma_{ab}A^2\ . \tag{C.9}$$

To resume,

$$R_\lambda \Rightarrow \begin{cases} W^a = -i\gamma^{ab}A_b \\ V = 2A^a A_a + i(\partial^a A_a) \\ \mathcal{F}_{ab} = i\gamma_{ac}(\partial_b A^c) - i\gamma_{bc}(\partial_a A^c) - 2\gamma_{ac}A_b A^c - 2\gamma_{cb}A_a A^c + 2\gamma_{ab}A^c A_c \end{cases} \tag{C.10}$$

are the quantities that one needs to evaluate the coefficient $a_2(R_\lambda)$ from $a_2(H)$ of eq. (B.4) (remembering to reinsert the projector). In particular, evaluating the trace[1] one finds

$$\begin{aligned} \mathrm{tr}\,[P_L a_2(R_\lambda)] = &\frac{2}{3}(\partial_a A_b)(\partial^a A^b) - \frac{2}{3}(\partial A)^2 - \frac{2}{3}\Box A^2 \\ &+ i\left(\frac{4}{3}\partial_a(A^a A^2) - \frac{1}{3}\Box(\partial A) - \frac{1}{12}\epsilon^{abcd}F_{ab}F_{cd}\right)\ . \end{aligned} \tag{C.11}$$

Note that this particular coefficient contains an odd-parity term proportional to the topological density $F\tilde{F}$.

Similarly, one proceeds with R_{λ_c}. Neglecting the projector and expanding it in the expression

$$\begin{aligned} R_{\lambda_c} = -\not{D}(A)\not{D}(-A) &= -D^a(A)D_a(-A) + 2i\gamma^{ab}A_a\partial_b - \frac{i}{2}F_{ab}\gamma^{ab} \\ &= -\partial^a\partial_a - 2i\gamma^{ab}A_b\partial_a - i(\partial^a A_a) - A^a A_a - \frac{i}{2}F_{ab}\gamma^{ab} \end{aligned} \tag{C.12}$$

[1] We have checked our trace calculations on the gamma matrices also by computer, employing a notebook developed in [34] using the xAct and xTensor packages [35, 36].

one can compare it with the operator (C.3) to find the relevant quantities

$$R_{\lambda_c} \Rightarrow \begin{cases} W^a = i\gamma^{ab}A_b \\ V = 2A^aA_a - i(\partial^a A_a) \\ \mathcal{F}_{ab} = -i\gamma_{ac}(\partial_b A^c) + i\gamma_{bc}(\partial_a A^c) - 2\gamma_{ac}A_bA^c - 2\gamma_{cb}A_aA^c + 2\gamma_{ab}A^cA_c \end{cases} \tag{C.13}$$

to be used in the coefficients $a_2(R_{\lambda_c})$ and proceeds to evaluate (3.13).

C.1.2 PV regularization with Dirac mass

In section 4.1.2 one deals with the following regulators:

$$\begin{aligned} R_\lambda &= -\partial\!\!\!/ D\!\!\!\!/(A)P_L \\ R_{\lambda_c} &= -\partial\!\!\!/ D\!\!\!\!/(-A)P_R \;. \end{aligned} \tag{C.14}$$

Disregarding the projector, one expands them to find

$$R_\lambda = -\partial^a\partial_a + i(\partial^a A_a) + iA^a\partial_a + i\gamma^{ab}A_b\partial_a + \frac{i}{2}\gamma^{ab}F_{ab} \tag{C.15}$$

and similarly

$$R_{\lambda_c} = -\partial^a\partial_a - i(\partial^a A_a) - iA^a\partial_a - i\gamma^{ab}A_b\partial_a - \frac{i}{2}\gamma^{ab}F_{ab} \;. \tag{C.16}$$

Comparing them with eq. (C.3) one obtains

$$R_\lambda \Rightarrow \begin{cases} W^a = -\frac{1}{2}i(A^a + \gamma^{ab}A_b) \\ V = \frac{1}{2}i(\partial_a A^a) + \frac{1}{4}i\gamma^{ab}F_{ab} + \frac{1}{2}A_aA^a \\ \mathcal{F}_{ab} = -\frac{1}{2}i\left(F_{ab} + \gamma_{bc}(\partial_a A^c) - \gamma_{ad}(\partial_b A^d)\right)) - \frac{1}{2}A^c(\gamma_{ac}A_b - \gamma_{bc}A_a) + \frac{1}{2}\gamma_{ab}A^cA_c \end{cases} \tag{C.17}$$

$$R_{\lambda_c} \Rightarrow \begin{cases} W^a = \frac{1}{2}i(A^a + \gamma^{ab}A_b) \\ V = -\frac{1}{2}i(\partial_a A^a) - \frac{1}{4}i\gamma^{ab}F_{ab} + \frac{1}{2}A_aA^a \\ \mathcal{F}_{ab} = \frac{1}{2}i\left(F_{ab} + \gamma_{bc}(\partial_a A^c) - \gamma_{ad}(\partial_b A^d)\right) - \frac{1}{2}A^c(\gamma_{ac}A_b - \gamma_{bc}A_a) + \frac{1}{2}\gamma_{ab}A^cA_c \end{cases} \tag{C.18}$$

and proceeds to compute a_2 from (B.4). Keeping in mind to reinsert the projectors, one then evaluates (3.13).

C.2 Dirac fermion

We move now to the Seeley-DeWitt coefficients needed in section 4.2.

C.2.1 PV regularization with Dirac mass

The relevant regulators in section (4.2.1) are

$$\begin{aligned} R_\psi &= -\not{D}(A,B)^2 \\ R_{\psi_c} &= -\not{D}(-A,B)^2 \ . \end{aligned} \tag{C.19}$$

Expanding R_ψ one finds

$$\begin{aligned} R_\psi =& -\partial^a\partial_a + i(\partial^a A_a) + 2iA^a\partial_a + A^aA_a + i\gamma^5(\partial^a B_a) - B^aB_a \\ &+ \gamma^{ab}\Big(\frac{i}{2}F_{ab}(A) + \frac{i}{2}\gamma^5 F_{ab}(B) + 2\gamma^5 A_aB_b - 2i\gamma^5 B_a\partial_b\Big) \end{aligned} \tag{C.20}$$

and similarly for R_{ψ_c}. Comparing with (C.3), one finds the quantities to construct the Seeley-DeWitt coefficients from (B.4), namely

$$R_\psi \Rightarrow \begin{cases} W^a = -iA^a + iB_b\gamma^{ba}\gamma^5 \\ V = i\gamma^5(\partial^a B_a) + \dfrac{i}{2}\gamma^{ab}F_{ab}(A) + 2B^aB_a \\ \mathcal{F}_{ab} = -iF_{ab}(A) + i\gamma^5\left(\gamma_{ac}(\partial_b B^c) - \gamma_{bc}(\partial_a B^c)\right) + 2B^cB_c\gamma_{ab} + 2B_c\left(\gamma^c{}_aB_b - \gamma^c{}_bB_a\right) \end{cases} \tag{C.21}$$

$$R_{\psi_c} \Rightarrow \begin{cases} W^a = iA^a + iB_b\gamma^{ba}\gamma^5 \\ V = i\gamma^5(\partial^a B_a) - \dfrac{i}{2}\gamma^{ab}F_{ab}(A) + 2B^aB_a \\ \mathcal{F}_{ab} = iF_{ab}(A) + i\gamma^5\left(\gamma_{ac}(\partial_b B^c) - \gamma_{bc}(\partial_a B^c)\right) + 2B^cB_c\gamma_{ab} + 2B_c\left(\gamma^c{}_aB_b - \gamma^c{}_bB_a\right) \ . \end{cases} \tag{C.22}$$

One proceeds evaluating (3.16).

C.2.2 PV regularization with Majorana mass

The regulators considered in section 4.2.2 are

$$\begin{aligned} R_\psi &= -\not{D}(-A,B)\not{D}(A,B) \\ R_{\psi_c} &= -\not{D}(A,B)\not{D}(-A,B) \ . \end{aligned} \tag{C.23}$$

Focusing on R_ψ, one expands it in the form

$$\begin{aligned} R_\psi =& -\partial^a\partial_a + i(\partial^a A_a) + i\gamma^5(\partial^a B_a) - A^aA_a - 2\gamma^5 A^aB_b - B^aB_a \\ &+ \gamma^{ab}\left(\frac{i}{2}F_{ab}(A) + 2iA_b\partial_a + \frac{i}{2}\gamma^5F_{ab}(B) + 2i\gamma^5 B_b\partial_a\right) \end{aligned} \tag{C.24}$$

and proceeds similarly for R_{ψ_c}. Now it is convenient to introduce the vectors

$$\begin{aligned} C_a &= A_a + \gamma^5 B_a \\ C'_a &= -A_a + \gamma^5 B_a \end{aligned} \tag{C.25}$$

in terms of which one finds, comparing with (C.3),

$$R_\psi \Rightarrow \begin{cases} W^a = -i\gamma^{ab} C_b \\ V = 2C^a C_a + i(\partial^a C_a) \\ \mathcal{F}_{ab} = -i\gamma_{bc}(\partial_a C^c) + i\gamma_{ac}(\partial_b C^c) + 2\gamma_{ab} C^c C_c + 2\gamma_{ac} C_b C^c - 2\gamma_{cb} C_a C^c \end{cases} \tag{C.26}$$

$$R_{\psi_c} \Rightarrow \begin{cases} W^a = -i\gamma^{ab} C'_b \\ V = 2C'^a C'_a + i(\partial^a C'_a) \\ \mathcal{F}_{ab} = -i\gamma_{bc}(\partial_a C'^c) + i\gamma_{ac}(\partial_b C'^c) + 2\gamma_{ab} C'^c C'_c + 2\gamma_{ac} C'_b C'^c - 2\gamma_{cb} C'_a C'^c \,. \end{cases} \tag{C.27}$$

The Seeley-DeWitt coefficients are then computed from (B.4) and one proceeds by evaluating the traces in (3.16).

Appendix D

Covariant counterterms

When computing the trace anomalies of chapter 4 we introduce covariant counterterms in the effective action with a non-trivial transformations with respect to the Weyl symmetry (2.12) in order to put the anomalies in a gauge invariant form. In this appendix we list the possible covariant counterterms and show their variation under Weyl transformation; eventually, we consider the flat spacetime limit that is of interest in the text.

The counterterms we can insert in the effective action of a Weyl or Dirac fermion coupled to a vector field A_μ are:

$$\begin{aligned}
&g^{\mu\rho} g^{\nu\sigma} (\nabla_\mu A_\nu)(\nabla_\rho A_\sigma) \\
&g^{\mu\nu} g^{\rho\sigma} (\nabla_\mu A_\nu)(\nabla_\rho A_\sigma) \\
&R^{\mu\nu} A_\mu A_\nu \\
&R A^\mu A_\mu \ .
\end{aligned} \tag{D.1}$$

where $R^{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar and $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b$ is the covariant derivative containing the spin connection $\omega_{\mu ab}$. These are all the counterterms with a non trivial Weyl transformation that we can build. Counterterms like $g^{\mu\nu} g^{\rho\sigma} A_\mu A_\nu A_\rho A_\sigma$, for example, are Weyl invariant because $\delta_\sigma^{(e)} A_\mu = 0$ and the transformations of the two metrics are cancelled by the measure $d^4x\,\sqrt{g}$ of the curved action.

Varying the counterterms (D.1) with respect to the Weyl transformation (2.12) we have (neglecting the variations which are cancelled by the measure of the action)

$$\begin{aligned}
&\delta_\sigma^{(e)}\Big(g^{\mu\rho} g^{\nu\sigma} (\nabla_\mu A_\nu)(\nabla_\rho A_\sigma)\Big) = -2A^\mu(\nabla_\nu A_\mu)(\nabla^\nu\sigma) - 2A^\mu(\nabla_\mu A_\nu)(\nabla^\nu\sigma) + 2A_\mu(\nabla^\mu\sigma)(\nabla_\nu A^\nu) \\
&\delta_\sigma^{(e)}\Big(g^{\mu\nu} g^{\rho\sigma} (\nabla_\mu A_\nu)(\nabla_\rho A_\sigma)\Big) = 4A_\mu(\nabla^\mu\sigma)(\nabla_\nu A^\nu) \\
&\delta_\sigma^{(e)}\Big(R^{\mu\nu} A_\mu A_\nu\Big) = -A_\mu A_\nu(2\sigma R^{\mu\nu} + 3\nabla^\mu\nabla^\nu\sigma - \nabla^\nu\nabla^\mu\sigma) - A_\mu(2\sigma A_\nu R^{\mu\nu} + A^\mu\nabla_\nu\nabla^\nu\sigma) \\
&\delta_\sigma^{(e)}\Big(R A^\mu A_\mu\Big) = -2\sigma R A^\mu A_\mu - 2A^\mu A_\mu(\sigma R + 3\nabla_\mu\nabla^\mu\sigma) \ .
\end{aligned} \tag{D.2}$$

Now, if we take the flat spacetime limit (with $g_{\mu\nu}$ replaced by η_{ab}) and integrate by part disregarding boundary terms, we get:

$$\begin{aligned}
&\delta_\sigma^{(e)}\Big(g^{\mu\rho}g^{\nu\sigma}(\nabla_\mu A_\nu)(\nabla_\rho A_\sigma)\Big) \to 2\sigma\Big((\partial_a A_b)(\partial^b A^a) + \frac{1}{2}\partial^a\partial_a(A^b A_b) - (\partial^a A_a)^2\Big)\\
&\delta_\sigma^{(e)}\Big(g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu A_\nu)(\nabla_\rho A_\sigma)\Big) \to -4\sigma\Big(A_a\partial^a(\partial^b A_b) + (\partial^a A_a)^2\Big)\\
&\delta_\sigma^{(e)}\Big(R^{\mu\nu}A_\mu A_\nu\Big) \to -\sigma\Big((\partial_a A_b)(\partial^b A^a) + 2A_a\partial^a(\partial^b A_b) - \partial^a\partial_a(A^b A_b)\Big)\\
&\delta_\sigma^{(e)}\Big(RA^\mu A_\mu\Big) \to -6\sigma\partial^a\partial_a(A^b A_b)\ .
\end{aligned} \tag{D.3}$$

In light of these transformations, in chapter 4 we just have to choose the appropriate counterterms in order to have effective actions with gauge invariant trace anomalies.

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