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FREE BOUNDARY REGULARITY OF SOME NON-HOMOGENEOUS PROBLEMS

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Introduction

In this work, we deal with the study of a free boundary problem governed by a non-homogeneous equation. We begin this thesis reviewing the paper by Daniela De Silva "Free boundary regularity for a problem with right hand side", see [11].
In particular, we study the free boundary problem governed by an elliptic equation in non-divergence form defined on a bounded connected, possibly regular, subset Ω in $\mathbb{R}^n$.
For the sake of simplicity, here we state the problem in the easier way by considering simply the Laplace operator, namely:

$$\begin{cases} \Delta u = f & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}^{\circ}, \\ |\nabla u| = 1 & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{1}$$

A function u is a solution of the problem (1) if u satisfies the equation $\Delta u = f$ when u is strictly positive and in addition the condition $|\nabla u| = 1$ is fulfilled in a proper unknown subset of Ω, called the *free boundary* of the problem.
In particular, $F(u) = \partial\Omega^+(u) \cap \Omega$ denotes the *free boundary* of the solution u and we point out that the set $F(u)$ is an unknown of the problem. Indeed, we want to discover more information about the properties of the set $F(u)$. For instance, is $F(u)$ a graph? Is $F(u)$ regular? Which type of regularity does $F(u)$ satisfy?
Figure 1 describes a possible geometrical situation associated to free boundary problems.

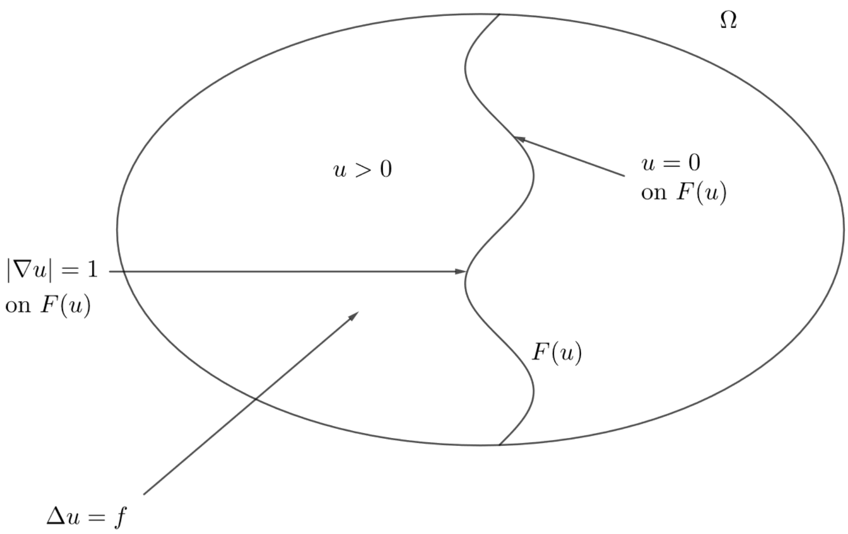

Figure 1: Example of free boundary problem (1).

An important contribution in the comprehension of the problem in the homogeneous case has been obtained by L. Caffarelli in a series of papers, [4], [5], [6], see also [8] for a complete bibliography. Further results about the non-homogeneous problem are collected in [11], [12] and [13].

Before studying the regularity of $F(u)$, it is necessary to spend some words about the correct setting of our problem. In our case, at first we need to introduce the definition of viscosity solution, otherwise some difficulties about the correct notion of solution may arise. For instance, it is known that the regularity up to the boundary of the solution of a Dirichlet problem, in a given set, depends on the regularity of the boundary itself. Consequently, a free boundary problem cannot be reduced to a Dirichlet problem, otherwise the condition $|\nabla u| = 1$ on $F(u)$ could be meaningless in the classical sense (see Lemma A.4 in Appendix A). For example, in case $F(u)$ was not smooth,

which is the right meaning of the condition $|\nabla u| = 1$ on the set $F(u)$? Caffarelli faced the problem in a geometric sense, by applying many ideas coming from the viscosity theory thanks to the flexibility of these notions.
The problem (1) is a particular case of the following family of problems discussed in [11]:

$$\begin{cases} \sum_{i,j} a_{ij}(x) u_{ij} = f & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\ |\nabla u| = g & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{2}$$

Here Ω is as usual a bounded connected set in $\mathbb{R}^n$ and u_{ij} denotes the second derivative of u with respect to x_i, x_j. We also assume the following hypotheses: the coefficients $a_{ij} \in C^{0,\beta}(\Omega)$, $f \in C(\Omega) \cap L^\infty(\Omega)$ and $g \in C^{0,\beta}(\Omega)$, $g \geq 0$. Moreover, the matrix $(a_{ij}(x))_{1 \leq i,j \leq n}$ is positive definite, that is there exists $\lambda > 0$ such that $\forall \xi \in \mathbb{R}^n \setminus \{0\}$, $\forall x \in \Omega$, $A(x)\xi \cdot \xi \geq \lambda |\xi|^2$. Thus, in case $(a_{ij}(x))_{1 \leq i,j \leq n} = (\delta_{ij})_{1 \leq i,j \leq n}$ and $g \equiv 1$, we obtain (1).
We deal with viscosity solutions of problem (2), see Chapter 1 for this definition and the Appendix B for basic definitions about viscosity solution theory.
The main theorem in [11] is the following one:

Theorem 0.1 (**Flatness implies $C^{1,\alpha}$**). *Let u be a viscosity solution to (2) in B_1. Assume that $0 \in F(u)$, $g(0) = 1$ and $a_{ij}(0) = \delta_{ij}$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if the graph of u is $\bar{\varepsilon}$-flat in B_1, i.e.*

$$(x_n - \bar{\varepsilon})^+ \leq u(x) \leq (x_n + \bar{\varepsilon})^+, \quad x \in B_1,$$

and

$$[a_{ij}]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon}, \quad \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \quad [g]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon},$$

then $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

The key idea described in [11] concerns the fact that a flat set to any scale has to be $C^{1,\alpha}$-smooth.
The strategy used in [11] for proving Theorem 0.1 can be summarized as follows:

(i) assuming that

$$\|f\|_{L^\infty(\Omega)} \leq \varepsilon^2, \quad \|g-1\|_{L^\infty(\Omega)} \leq \varepsilon^2, \quad \|a_{ij} - \delta_{ij}\|_{L^\infty(\Omega)} \leq \varepsilon,$$

with $0 < \varepsilon < 1$, then a Harnack type inequality is satisfied by solutions of problem (2).

Roughly saying, with the Harnack inequality we achieve that if the graph of u oscillates εr away from x_n^+ in B_r, then it oscillates $(1-c)\varepsilon r$ in $B_{r/20}$. This property reproduces the effects of the classical Harnack inequality, even if in a different context, on the solutions of problem (2). In this framework, we remark that the Harnack type inequality is rather different from the classical one, see Theorem 2.1, in comparison with the classical Harnack inequality, see Theorem C.7 in Appendix C;

(ii) from previous Harnack type inequality, follows that the graphs of the solutions of problem (2) enjoy an "improvement of flatness" property. In other words, if the graph of a solution oscillates ε away from a hyperplane in B_1, then in B_{r_0} it oscillates $\varepsilon r_0/2$ away from, possibly, a different hyperplane. This fact is introduced in the "improvement of flatness" lemma, see Lemma 3.1;

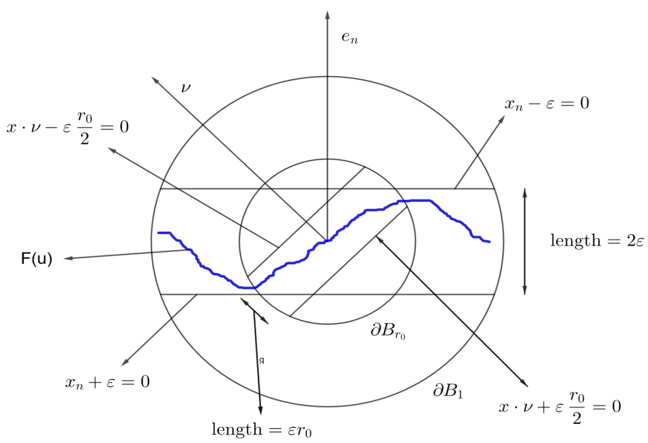

Figure 2: Improvement of flatness

(iii) in conclusion, Theorem 0.1 follows from the "improvement of flatness" lemma via an iterative argument, see Theorem 4.2 and its proof in Chapter 4.

We point out that Theorem 0.1 also follows from the regularity properties of solutions to the following classical Neumann problem for the Laplace operator in a half plane:

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_\rho \cap \{x_n > 0\}\,, \\ \frac{\partial \tilde{u}}{\partial x_n} = 0 & \text{on } B_\rho \cap \{x_n = 0\}\,, \end{cases} \tag{3}$$

where $\frac{\partial \tilde{u}}{\partial x_n}$ denotes $\frac{\partial \tilde{u}}{\partial \nu}$, and ν is the inward pointing unit normal vector respect to $B_\rho \cap \{x_n = 0\}$. In this case, $\nu = e_n$.
In order to clarify this claim, we argue in this way, see for instance [8].
Let u be a solution of (1). We ask for every small $\varepsilon > 0$ that $u_\varepsilon = u + \varepsilon\varphi$ has to be still a solution of (1) for a proper choice of a function φ. As a consequence, since $\Delta u_\varepsilon = f$, we have

$$f = \Delta u_\varepsilon = \Delta(u + \varepsilon\varphi) = \Delta u + \varepsilon\Delta\varphi = f + \varepsilon\Delta\varphi,$$

thus

$$\varepsilon\Delta\varphi = 0$$

and, recalling that $\varepsilon > 0$,

$$\Delta\varphi = 0.$$

Moreover, $|\nabla u_\varepsilon| = 1$ implies

$$|\nabla u_\varepsilon| = 1 \leftrightarrow |\nabla u_\varepsilon|^2 = 1 \leftrightarrow |\nabla u|^2 + 2\varepsilon\nabla u \cdot \nabla\varphi + \varepsilon^2 |\nabla\varphi|^2 = 1.$$

Therefore, seeing as how $|\nabla u|^2 = 1$, we have, inasmuch $\varepsilon > 0$,

$$\varepsilon(2\nabla u \cdot \nabla\varphi + \varepsilon |\nabla\varphi|^2) = 0 \leftrightarrow 2\nabla u \cdot \nabla\varphi + \varepsilon |\nabla\varphi|^2 = 0$$

and for $\varepsilon \to 0$, we obtain

$$2\nabla u \cdot \nabla\varphi = 0,$$

that is

$$\nabla u \cdot \nabla\varphi = 0.$$

Now, on $F(u)$, $|\nabla u| = 1$ and hence $\nabla u \neq 0$. Then, the inward pointing unit normal vector is $\nu = \frac{\nabla u}{|\nabla u|}$, thus from $\nabla u \cdot \nabla \varphi = 0$, we also get, inasmuch as $|\nabla u| > 0$,

$$|\nabla u| \frac{\nabla u}{|\nabla u|} \cdot \nabla \varphi = 0 \leftrightarrow \nu \cdot \nabla \varphi = 0 \leftrightarrow \frac{\partial \varphi}{\partial \nu} = 0,$$

namely $\frac{\partial \varphi}{\partial \nu} = 0$ on $F(u)$, whenever $F(u)$ is sufficiently smooth.
Summarizing, φ satisfies:

$$\begin{cases} \Delta \varphi = 0 & \text{in } \Omega^+(u) \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } F(u). \end{cases}$$

As a consequence, recalling that u_ε is a solution, we can expect that $\varphi = \frac{u_\varepsilon - u}{\varepsilon}$ is indeed a solution to the transmission problem (3). In our case, let be given a solution u of our free boundary problem. We subtract to u the special solution $(x \cdot \nu)^+$ and we divide by $\varepsilon > 0$ in a neighborhood of 0. Here, we have assumed that 0 belongs to $F(u)$ and ν is a constant vector. We expect that $\frac{u_\varepsilon - u}{\varepsilon}$ is a solution to (3) when ε goes to 0. As a byproduct, the function u, in some way, inherits the regularity properties of the solutions of the Neumann problem.
We would like to spend few words about the importance of problem (1). In literature there is a typical model problem arising in classical fluid-dynamics.
We roughly describe this physical situation (see [13]) representing a one-phase problem: a traveling two-dimensional gravity wave of an incompressible, inviscid, heavy fluid moves with constant speed over a horizontal surface. Since the fluid is incompressible, the flow can be described by a stream function u which solves the following free boundary problem (in $2D$):

$$\begin{cases} \Delta u = -\gamma(u) & \text{in } \Omega := \{(x,y) \in \mathbb{R}^2 : 0 < u(x,y) < B\}, \\ 0 \leq u \leq B & \text{in } \overline{\Omega} \\ u = B & \text{on } y = 0, \\ |\nabla u|^2 + 2gy = Q & \text{on } S := \{u = 0\}. \end{cases}$$

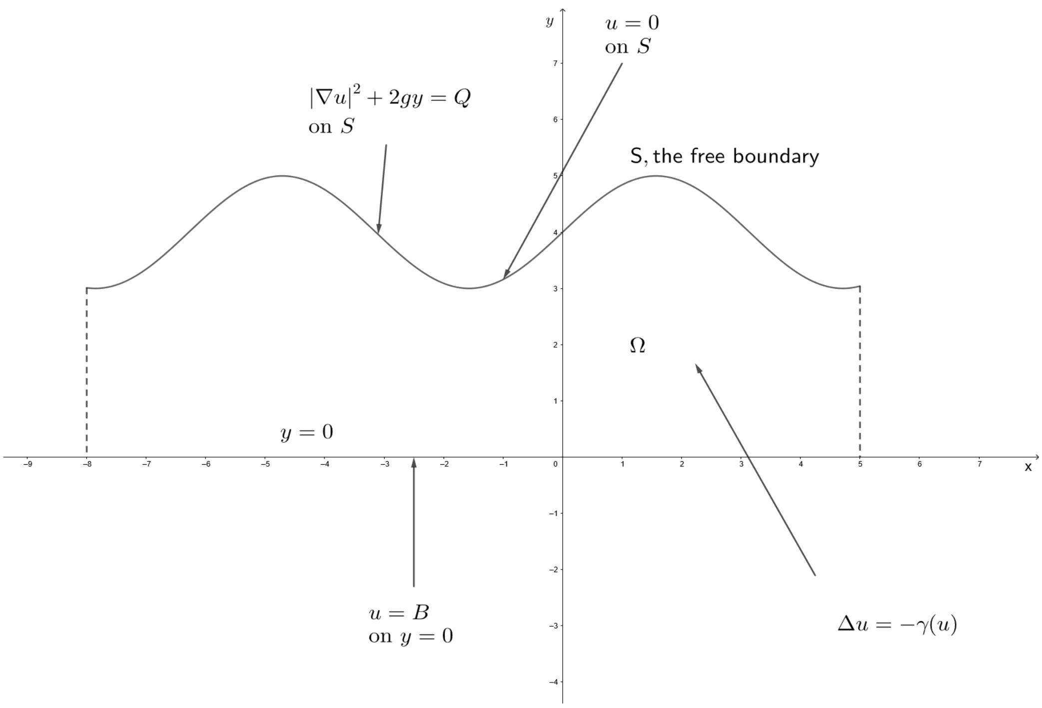

Figure 3: Geometrical representation of the physical example in $\mathbb{R}^2$.

Here Q is a constant, B, g are positive constants, $\gamma : [0, B] \to \mathbb{R}$ is called *vorticity function* and S is the free boundary of the problem, whenever a function u satisfying the above system exists. Given that $u^- \equiv 0$, we have a *one-phase* free boundary problem.

In this thesis, we adapt the proof of Theorem 0.1 to slightly more general operators having an additional term depending on the gradient of the solution. In this way, we study the free boundary regularity for a solution to the following problem:

$$\begin{cases} \sum\limits_{i,j} a_{ij}(x)u_{ij} + \sum\limits_{i} b_i(x) \cdot u_i = f & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\ |\nabla u| = g & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases}$$

with $b_i \in C(\Omega) \cap L^\infty(\Omega)$ and assuming the conditions listed in (2) on Ω, f, g and a_{ij}. Furthermore, u_i denotes the derivative of u with respect to x_i.

In the long run, we also would like to extend our investigation to two-phase problems starting from the results described in: [12], [14], [17],[18], [19], in

order to prove further regularity results, for instance higher regularity of the free boundary for fully non-linear operators, see [15] and [16].
Moreover, we also would like to improve this research, by attacking the non-homogeneous one-phase parabolic problem. Indeed, concerning evolutive problems, there exist few regularity results, see for instance [2] and [3] in the homogeneous framework.
In perspective, further new interesting problems that we would like to consider are associated with degenerate operators like the Kohn-Laplace one in the Heisenberg group.
Specifically, this thesis is organized as follows. In Chapter 1, we introduce notation, definitions and results, which we will use throughout the paper, and we prove a regularity result for viscosity solutions to a Neumann problem which we will use in the proof of Theorem 4.2.
Next, in Chapter 2, we prove our Harnack inequality. In Chapter 3, we prove the main "improvement of flatness" lemma, see Lemma 3.1, from which Theorem 4.2 will follow by an iterative argument. In Chapter 4, we exhibit the proofs of Theorems 4.2 and 4.1. From Chapter 1 to Chapter 4, we strictly follow the organization of the paper [11]. In particular, we review the proofs showing all the details. In Chapter 6, we analyze the same problem in the case of operators with additional term depending on the gradient. For exposure convenience, we conclude the work with an Appendix. This conclusive part is subdivided in some sections collected by homogeneity arguments. Indeed, we list some more or less well-known results in literature by showing in many cases a detailed proof. The main goal of this Appendix, hopefully, is helping the reader in the comprehension of all the steps of this thesis.

Contents

Chapter 1

Prerequisites

We introduce, in this chapter, some tools which will be used throughout the work. We also present an auxiliary result, Lemma 1.8, which will be useful in the proof of our main Theorem 4.2.

Let us start with notation.

$B_\rho(x_0) \subset \mathbb{R}^n$ denotes the open ball of radius ρ centered at x_0 and we write $B_\rho = B_\rho(0)$.

For any continuous function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ we denote

$$\Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \qquad F(u) := \partial\Omega^+(u) \cap \Omega.$$

We refer to the set $F(u)$ as the *free boundary* of u, while $\Omega^+(u)$ is its *positive phase* (or *side*).

We remark that, since u is continuous, then obviously $u = 0$ on $F(u)$.

Indeed, the continuity of u implies that the set $\Omega^+(u)$ is open, thus if $x_0 \in \Omega^+(u)$, we can find a ball $B_r(x)$, such that $B_r(x) \subset \Omega^+(u)$, and hence $B_r(x) \cap \Omega^+(u)^c = \emptyset$.

Analogously, the continuity of u also entails that the set $\{x \in \Omega : u(x) < 0\}$ is open, therefore, if $x \in \{x \in \Omega : u(x) < 0\}$, we can find a ball $B_r(x)$ such that $B_r(x) \subset \{x \in \Omega : u(x) < 0\}$, in other words $B_r(x) \cap \Omega^+(u) = \emptyset$.

Now, if $x \in F(u)$, in particular $x \in \partial\Omega^+(u)$, thus we have $B_r(x) \cap \Omega^+(u) \neq \emptyset$ and $B_r(x) \cap \Omega^+(u)^c \neq \emptyset \; \forall B_r(x)$, so for what we have said above, $x \notin \Omega^+(u)$ and $x \notin \{x \in \Omega : u(x) < 0\}$, that is necessary $u(x) = 0$.

In this thesis, we deal with the one-phase free boundary problem

$$\begin{cases} \sum\limits_{i,j} a_{ij}(x)u_{ij} + \sum\limits_{i} b_i(x)u_i = f & \text{in } \Omega^+(u), \\ |\nabla u| = g & \text{on } F(u). \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ (a domain is a connected open subset), $a_{ij} \in C^{0,\beta}(\Omega)$, $b_i \in C(\Omega) \cap L^\infty(\Omega)$, $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C^{0,\beta}(\Omega)$, $g \geq 0$, the matrix $(a_{ij}(x))$ is positive definite. Formally, u_i denotes the derivative of u with respect to x_i, while u_{ij} the second derivative of u with respect to x_i, x_j.

Specifically, we begin our analysis from the particular case given by $b_i = 0$ for every $i = 1, \dots, n$, i.e. with the one-phase free boundary problem

$$\begin{cases} \sum\limits_{i,j} a_{ij}(x)u_{ij} = f & \text{in } \Omega^+(u), \\ |\nabla u| = g & \text{on } F(u), \end{cases} \tag{1.2}$$

which has been studied by Daniela de Silva in [11].

We state the definition of viscosity solution to (1.2) and for this purpose, we need some basic notions.

Definition 1.1. Given $u, \varphi \in C(\Omega)$, we say that φ *touches* u *from below* (resp. *above*) at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$ and

$$u(x) \geq \varphi(x) \quad (\text{resp. } u(x) \leq \varphi(x)) \quad \text{in a neighborhood } O \text{ of } x_0.$$

If this inequality is strict in $O \setminus \{x_0\}$, we say that φ touches u *strictly* from below (resp. above).

Definition 1.2. Let u be a nonnegative continuos function in Ω. We say that u is a *viscosity solution* to (1.2) in Ω if the following conditions are satisfied:

(i) $\sum\limits_{i,j} a_{ij}(x)u_{ij} = f$ in $\Omega^+(u)$ in the viscosity sense, i.e. if $\varphi \in C^2(\Omega^+(u))$ touches u from below (resp. above) at $x_0 \in \Omega^+(u)$ then

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \leq f(x_0) \qquad \left(\text{resp. } \sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \geq f(x_0)\right).$$

(ii) If $\varphi \in C^2(\Omega)$ and φ^+ touches u from below (resp. above) at $x_0 \in F(u)$ and $|\nabla\varphi|(x_0) \neq 0$ then

$$|\nabla\varphi|(x_0) \leq g(x_0) \quad (\text{resp. } |\nabla\varphi|(x_0) \geq g(x_0)).$$

At this point, we provide the notion of comparison subsolution / supersolution, which is useful to be able to employ comparison techniques.

Definition 1.3. Let $v \in C^2(\Omega)$. We say that v is a *strict (comparison) subsolution* (resp. *supersolution*) to (1.2) in Ω if the following conditions are satisfied:

(i) $\sum\limits_{i,j} a_{ij}(x)v_{ij} > f(x)$ $\left(\text{resp. } \sum\limits_{i,j} a_{ij}(x)v_{ij} < f(x)\right)$ in $\Omega^+(v)$.

(ii) If $x_0 \in F(v)$, then

$$|\nabla v|(x_0) > g(x_0) \quad (\text{resp. } 0 < |\nabla v|(x_0) < g(x_0)).$$

Remark 1.4. We point out that, if v is a strict subsolution / supersolution to (1.2), from (ii) in Definition 1.3, $|\nabla v| > 0$ on $F(v)$, which gives $\nabla v \neq 0$ on $F(v)$. Therefore, recalling that $v \in C^2(\Omega)$, $v = 0$ on $F(v)$ and $\nabla v \neq 0$ on $F(v)$, we can apply the implicit function theorem and we obtain that $F(v)$ is a C^2 hypersurface.

The following lemma is an immediate consequence of the definitions above.

Lemma 1.5. *Let u, v be respectively a solution and a strict subsolution to (1.2) in Ω. If $u \geq v^+$ in Ω then $u > v^+$ in $\Omega^+(v) \cup F(v)$.*

Proof. Assume for contradiction that $\exists\, x_0 \in \Omega^+(v) \cup F(v)$ such that
$u(x_0) = v^+(x_0)$.
We have two different cases.

(i) If $x_0 \in \Omega^+(v)$, i.e. $v(x_0) > 0$, $v^+(x_0) = v(x_0)$.
Therefore, since $u \geq v^+$ in $\Omega \supseteq \Omega^+(v)$, $\forall x \in \Omega^+(v)$

$$u(x) \geq v^+(x) = v(x) > 0,$$

that is $x \in \Omega^+(u)$ and thus $\Omega^+(v) \subseteq \Omega^+(u)$.
In particular, given that $x_0 \in \Omega^+(v)$, $x_0 \in \Omega^+(u)$.
Using that $u(x_0) = v^+(x_0) = v(x_0)$, namely $u(x_0) = v(x_0)$, together with the fact that $u \geq v^+ \geq v$ in Ω, in other words $u \geq v$ in Ω, since Ω is open, we can find an open neighborhood O of x_0 where $u \geq v$ in O and $u(x_0) = v(x_0)$, so we obtain that v touches u from below at $x_0 \in \Omega^+(u)$.
In addition, $v \in C^2(\Omega^+(u))$ because v is a strict subsolution to (1.2) and thus $v \in C^2(\Omega)$, therefore, given that u is a solution to (1.2), we get

$$\sum_{i,j} a_{ij}(x_0)v_{ij}(x_0) \leq f(x_0). \tag{1.3}$$

On the other hand, since v is a strict subsolution to (1.2), we have

$$\sum_{i,j} a_{ij}(x)v_{ij}(x) > f(x) \quad \text{in } \Omega^+(v)$$

and hence, since $x_0 \in \Omega^+(v)$,

$$\sum_{i,j} a_{ij}(x_0)v_{ij}(x_0) > f(x_0),$$

which entails from (1.3)

$$f(x_0) < \sum_{i,j} a_{ij}(x_0)v_{ij}(x_0) \leq f(x_0),$$

namely $f(x_0) < f(x_0)$, which is a contradiction.

(ii) If $x_0 \in F(v)$, $v(x_0) = 0 = v^+(x_0) = u(x_0)$, that is $u(x_0) = 0$ and $v(x_0) = u(x_0)$. Furthermore, $\forall B_r(x_0)$, $B_r(x_0) \cap \Omega^+(v) \neq \emptyset$ and $B_r(x_0) \cap \Omega^+(v)^c \neq \emptyset$.
Since $\Omega^+(v) \subseteq \Omega^+(u)$ from case (i),

$$B_r(x_0) \cap \Omega^+(u) \supseteq B_r(x_0) \cap \Omega^+(v) \neq \emptyset$$

and thus $B_r(x_0) \cap \Omega^+(u) \neq \emptyset$, $\forall B_r(x_0)$.
This fact, together with $u(x_0) = 0$ and so $B_r(x_0) \cap \Omega^+(u)^c \neq \emptyset \; \forall B_r(x_0)$,

implies that $x_0 \in F(u)$.

Now, inasmuch v is a strict subsolution to (1.2) and $x_0 \in F(v)$, we have

$$|\nabla v|(x_0) > g(x_0), \tag{1.4}$$

in other words, seeing as how $g(x_0) \geq 0$,

$$|\nabla v|(x_0) > 0,$$

and hence

$$|\nabla v|(x_0) \neq 0. \tag{1.5}$$

Moreover, $v \in C^2(\Omega)$ since v is a strict subsolution to (1.2), and v^+ touches u from below at $x_0 \in F(u)$, given that $v^+(x_0) = u(x_0)$, $v^+ \leq u$ in Ω, with Ω open and as a consequence, we can find an open neighborhood O of x_0 where $u \geq v^+$.

These two conditions, together with (1.5) and the fact that u is a solution to (1.2), give us

$$|\nabla v|(x_0) \leq g(x_0)$$

that is, from (1.4),

$$g(x_0) < |\nabla v|(x_0) \leq g(x_0),$$

i.e. $g(x_0) < g(x_0)$, which is a contradiction.

Hence, $\nexists\ x_0 \in \Omega^+(v) \cup F(v)$ such that $u(x_0) = v^+(x_0)$, hence, because $u \geq v^+$ in $\Omega \supseteq \Omega^+(v) \cup F(v)$, namely $u \geq v^+$ in $\Omega^+(v) \cup F(v)$, $u > v^+$ in $\Omega^+(v) \cup F(v)$. $\square$

Our main Theorem 4.2 will follow from the regularity properties of solutions to the classical Neumann problem for the Laplace operator. Precisely, we consider the following boundary value problem:

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_\rho \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_\rho \cap \{x_n = 0\}. \end{cases} \tag{1.6}$$

Here $\tilde{u}_n$ is the normal derivative of $\tilde{u}$, which corresponds to $\frac{\partial \tilde{u}}{\partial x_n}$, since the unit normal vector to the surface $B_\rho \cap \{x_n = 0\}$ is e_n.
We use the notion of viscosity solution to (1.6). For completeness (and for helping the reader), we recall standard notions and we prove regularity of viscosity solutions, see also Appendix B.

Definition 1.6. Let $\tilde{u}$ be a continuos function on $B_\rho \cap \{x_n \geq 0\}$. We say that $\tilde{u}$ is a *viscosity solution* to (1.6) if given a quadratic polynomial $P(x)$ touching $\tilde{u}$ from below (resp. above) at $\bar{x} \in B_\rho \cap \{x_n \geq 0\}$,

(i) if $\bar{x} \in B_\rho \cap \{x_n > 0\}$ then $\Delta P \leq 0$ (resp. $\Delta P \geq 0$), i.e. $\tilde{u}$ is harmonic in the viscosity sense;

(ii) if $\bar{x} \in B_\rho \cap \{x_n = 0\}$ then $P_n(\bar{x}) \leq 0$ (resp. $P_n(\bar{x}) \geq 0$).

Remark 1.7. Notice that in the definition above we can choose polynomials P that touch $\tilde{u}$ strictly from below/above.
Indeed, suppose that Definition 1.6 holds for polynomials that touch $\tilde{u}$ strictly from below/above. Let then P be a polynomial touching $\tilde{u}$ from below at $\bar{x} \in B_\rho \cap \{x_n \geq 0\}$, i.e

$$P(\bar{x}) = \tilde{u}(\bar{x})$$

and

$$P(x) \leq \tilde{u}(x) \quad \text{in a neighborhood } O \text{ of } \bar{x}.$$

Let now

$$P_\eta(x) = P(x) - \eta\, |x - \bar{x}|^2 .$$

Notice that, with $\eta > 0$, we have

$$P_\eta(x) = P(x) - \eta\, |x - \bar{x}|^2 < P(x) \leq \tilde{u}(x) \quad \text{in } O \setminus \{\bar{x}\} ,$$

in other words

$$P_\eta(x) < P(x) \quad \text{in } O \setminus \{\bar{x}\} , \tag{1.7}$$

and

$$P_\eta(\bar{x}) = P(\bar{x}) - \eta\, |\bar{x} - \bar{x}|^2 = P(\bar{x}) = \tilde{u}(\bar{x}),$$

namely

$$P(\eta)(\bar{x}) = \tilde{u}(\bar{x}). \tag{1.8}$$

Consequently, from (1.7) and (1.8), we achieve that P_η touches $\tilde{u}$ strictly from below at $\bar{x} \in B_\rho \cap \{x_n \geq 0\}$.

Suppose now that $\bar{x} \in B_\rho \cap \{x_n > 0\}$.

Since P_η touches $\tilde{u}$ strictly from below at $\bar{x}$, from (i) of Definition 1.6, we have

$$\begin{aligned}
\Delta P_\eta &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\left(P(x) - \eta\,|x-\bar{x}|^2\right) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\left(P(x) - \eta \sum_{j=1}^{n}(x_j - \bar{x}_j)^2\right) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial P}{\partial x_i} - 2\eta(x_i - \bar{x}_i)\right) = \sum_{i=1}^{n}\left(\frac{\partial^2 P}{\partial x_i^2} - 2\eta\right) \\
&= \sum_{i=1}^{n} \frac{\partial^2 P}{\partial x_i^2} - \sum_{i=1}^{n} 2\eta = \Delta P - 2n\eta \leq 0,
\end{aligned}$$

that is

$$\Delta P_\eta = \Delta P - 2n\eta \leq 0. \tag{1.9}$$

Now, if we let η go to 0 in (1.9), we obtain

$$\lim_{\eta \to 0} \Delta P_\eta = \Delta P \leq 0,$$

and thus P satisfies (i).

Assume, instead, that $\bar{x} \in B_\rho \cap \{x_n = 0\}$.

Always since P_η touches $\tilde{u}$ strictly from below at $\bar{x}$, from (ii) of Definition 1.6, we have

$$\begin{aligned}
\frac{\partial P_\eta}{\partial x_n}(\bar{x}) &= \frac{\partial}{\partial x_n}\left(P(x) - \eta\,|x-\bar{x}|^2\right)(\bar{x}) = \frac{\partial}{\partial x_n}\left(P(x) - \eta \sum_{j=1}^{n}(x_j - \bar{x}_j)^2\right)(\bar{x}) \\
&= \left(\frac{\partial P}{\partial x_n}(x) - 2\eta(x_n - \bar{x}_n)\right)(\bar{x}) = \frac{\partial P}{\partial x_n}(\bar{x}) \leq 0,
\end{aligned}$$

in other words

$$\frac{\partial P}{\partial x_n}(\bar{x}) \leq 0 \tag{1.10}$$

and hence P satisfies (ii).

At the same time, if P touches $\tilde{u}$ from above at $\bar{x} \in B_\rho \cap \{x_n \geq 0\}$, we use

the same argument, with slightly differences. Specifically, we have opposite inequalities in (1.9) and (1.10) and we take $\eta < 0$ in P_η so that P_η touches $\tilde{u}$ strictly from above at $\bar{x}$.
Also, it suffices to verify that (ii) holds for polynomials $\tilde{P}$ with $\Delta\tilde{P} > 0$. Indeed, let P touching $\tilde{u}$ from below at $\bar{x}$ and thus we have

$$\tilde{u}(\bar{x}) = P(\bar{x})$$

and

$$P(x) \leq \tilde{u}(x) \quad \text{in a neighborhood } O \text{ of } \bar{x}.$$

Then

$$\tilde{P} = P - \eta(x_n - \bar{x}_n) + C(\eta)(x_n - \bar{x}_n)^2$$

touches $\tilde{u}$ from below at $\bar{x}$ for a sufficiently small constant $\eta > 0$ and a large constant $C > 0$ depending on η.
Precisely, $\tilde{P}$ satisfies

$$\tilde{P}(\bar{x}) = P(\bar{x}) - \eta(\bar{x}_n - \bar{x}_n) + C(\eta)(\bar{x}_n - \bar{x}_n)^2 = P(\bar{x}) = \tilde{u}(\bar{x}),$$

i.e.

$$\tilde{P}(\bar{x}) = \tilde{u}(\bar{x}), \tag{1.11}$$

and

$$\tilde{P}(x) \leq P(x) \leq \tilde{u}(x) \quad \text{in } O,$$

in other words

$$\tilde{P}(x) \leq \tilde{u}(x) \quad \text{in } O, \tag{1.12}$$

with $\eta > 0$ and $C(\eta) > 0$ chosen so that $\tilde{P}$ verifies (1.12).
Notice that

$$O \subset B_\rho \cap \{x_n \geq 0\},$$

so since $\bar{x} \in B_\rho \cap \{x_n = 0\}$

$$x_n - \bar{x}_n \geq 0 \quad \text{in O}.$$

Also,

$$\begin{aligned}\Delta\tilde{P} &= \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\Big(P(x)-\eta(x_n-\bar{x}_n)+C(\eta)(x_n-\bar{x}_n)^2\Big)\\ &= \sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}\left(\frac{\partial P}{\partial x_i}\right)+\frac{\partial}{\partial x_n}\left(\frac{\partial P}{\partial x_n}-\eta+2C(\eta)(x_n-\bar{x}_n)\right)\\ &= \Delta P+2C(\eta)>0,\end{aligned}$$

namely

$$\Delta\tilde{P}>0, \tag{1.13}$$

choosing $C(\eta)>-\frac{\Delta P}{2}$, $C(\eta)>0$ and such that η and $C(\eta)$ satisfy (1.12). Furthermore,

$$\begin{aligned}\tilde{P}_n(\bar{x}) &= \frac{\partial}{\partial x_n}\Big(P(x)-\eta(x_n-\bar{x}_n)+C(\eta)(x_n-\bar{x}_n)^2\Big)(\bar{x})\\ &= (P_n(x)-\eta+2C(\eta)(x_n-\bar{x}_n))(\bar{x})=P_n(\bar{x})-\eta,\end{aligned}$$

which gives

$$\tilde{P}_n(\bar{x})=P_n(\bar{x})-\eta. \tag{1.14}$$

Now, from (1.11) and (1.12), we achieve that $\tilde{P}$ touches $\tilde{u}$ from below at $\bar{x}\in B_\rho\cap\{x_n=0\}$.
Therefore, if (ii) holds for strictly subharmonic polynomials, inasmuch $\Delta\tilde{P}>0$ from (1.13), we get from (1.14)

$$\tilde{P}_n(\bar{x})=P_n(\bar{x})-\eta\le 0$$

that is $P_n(\bar{x})\le\eta$, which by letting η go to 0 implies $P_n(\bar{x})\le 0$ and thus P satisfies (ii).
Analogously, if P touches $\tilde{u}$ from above at $\bar{x}$, we have

$$P(\bar{x})=\tilde{u}(\bar{x})$$

and

$$P(x)\ge\tilde{u}(x)\quad\text{in a neighborhood O of }\bar{x}.$$

Then
$$\tilde{P} = P - \eta(x_n - \bar{x}_n) + C(\eta)(x_n - \bar{x}_n)^2$$
touches $\tilde{u}$ from above at $\bar{x}$ with a constant $\eta > 0$ sufficiently small and a large constant $C > 0$ depending on η such that $\tilde{P}(x) \geq P(x) \geq \tilde{u}(x)$ in O. Exactly with the analogous computations used to get (1.13) and (1.14), we obtain
$$\Delta\tilde{P} > 0$$
and
$$\tilde{P}_n(\bar{x}) = P_n(\bar{x}) - \eta.$$
Now, if (ii) holds for strictly subharmonic polynomials, we get
$$\tilde{P}_n(\bar{x}) = P_n(\bar{x}) - \eta \geq 0$$
that is $P_n(\bar{x}) \geq \eta > 0$, which by letting η go to 0 implies $P_n(\bar{x}) \geq 0$ and thus P satisfies (ii).

We show now that viscosity solutions to (1.6) are smooth up to boundary, using a classical argument consisting on an extension by reflection of the function.

Lemma 1.8. *Let $\tilde{u}$ be a viscosity solution to (1.6). Then $\tilde{u}$ is a classical solution to (1.6). In particular, $\tilde{u} \in C^\infty(B_\rho \cap \{x_n \geq 0\})$.*

Proof. Let
$$u^*(x) = \begin{cases} \tilde{u}(x) & \text{if } x \in B_\rho \cap \{x_n \geq 0\} \\ \tilde{u}(x', -x_n) & \text{if } x \in B_\rho \cap \{x_n < 0\} \end{cases}$$
where $x' = (x_1, \dots, x_{n-1})$.

We claim that u^* is harmonic (in the viscosity sense), and hence smooth, in B_ρ.
Precisely, let P be a polynomial touching u^* at $\bar{x} \in B_\rho$ strictly from below (for what we have remarked before, in Definition 1.6 we can choose only polynomials that touch possible viscosity solutions strictly from below/above). We need to show that $\Delta P \leq 0$. Clearly, we only need to consider the case when

$\bar{x} \in \{x_n = 0\}$.
Indeed, if $\bar{x}_n \neq 0$, we can use the fact that $\tilde{u}$ is a viscosity solution in $B_\rho \cap \{x_n \geq 0\}$.
In particular, we have two different cases.

(i) If $\bar{x}_n > 0$, $\bar{x} \in B_\rho \cap \{x_n \geq 0\}$ and we have

$$u^*(\bar{x}) = \tilde{u}(\bar{x}).$$

So, since

$$u^*(x) = \tilde{u}(x) \quad \text{if } x \in B_\rho \cap \{x_n \geq 0\}$$

and P touches u^* strictly from below at $\bar{x}$, P touches $\tilde{u}$ strictly from below at $\bar{x}$, provide that making the neighborhood smaller to remain in $B_\rho \cap \{x_n \geq 0\}$, if necessary.
Hence, because $\tilde{u}$ is a viscosity solution to (1.6) and $\bar{x} \in B_\rho \cap \{x_n > 0\}$, we get $\Delta P \leq 0$.

(ii) If $\bar{x}_n < 0$, $\bar{x} \in B_\rho \cap \{x_n < 0\}$ and we have

$$u^*(\bar{x}) = \tilde{u}(\bar{x}', -\bar{x}_n).$$

Also,

$$u^*(x) = \tilde{u}(x', -x_n) \quad \text{if } x \in B_\rho \cap \{x_n < 0\}$$

and if we define

$$\tilde{P}(x) = P(x', -x_n),$$

$\tilde{P}$ touches $\tilde{u}$ strictly from below at $(\bar{x}', -\bar{x}_n)$, since P touches u^* strictly from below at $\bar{x}$, provide that making the neighborhood smaller to remain in $B_\rho \cap \{x_n < 0\}$, if necessary.
Sure enough,

$$\tilde{P}(\bar{x}', -\bar{x}_n) = P(\bar{x}) = u^*(\bar{x}) = \tilde{u}(\bar{x}', -\bar{x}_n)$$

and

$$
\begin{aligned}
&\tilde{P}(x', -x_n) = P(x) \\
&\leq u^*(x) = \tilde{u}(x', -x_n) \quad \text{in a neighborhood } O \subset B_\rho \cap \{x_n < 0\}\,.
\end{aligned}
$$

Now, $\tilde{u}$ is a viscosity solution to (1.6) and $(\bar{x}', -\bar{x}_n) \in B_\rho \cap \{x_n > 0\}$, since $\|\bar{x}\| = \|(\bar{x}', -\bar{x}_n)\| < \rho$, so we get $\Delta\tilde{P} \leq 0$.

Moreover, inasmuch P is a quadratic polynomial, ΔP is a constant so

$$
\begin{aligned}
\Delta\tilde{P} &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\Big(P(x', -x_n)\Big) \\
&= \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}\left(\frac{\partial P}{\partial x_i}(x', -x_n)\right) - \frac{\partial}{\partial x_n}\left(\frac{\partial P}{\partial x_n}(x', -x_n)\right) \\
&= \sum_{i=1}^{n-1} \left(\frac{\partial^2 P}{\partial x_i^2}\right)(x', -x_n) + \left(\frac{\partial^2 P}{\partial x_n^2}\right)(x', -x_n) \\
&= \Delta P,
\end{aligned}
$$

in other words

$$\Delta\tilde{P} = \Delta P \tag{1.15}$$

and thus $\Delta P \leq 0$ because $\Delta\tilde{P} \leq 0$.

Hence, remain to consider only the case when $\bar{x} \in \{x_n = 0\}$.
Consider the polynomial

$$S(x) = \frac{P(x) + P(x', -x_n)}{2}.$$

Then, from (1.15),

$$\Delta S = \frac{1}{2}\Big(\Delta P + \Delta(P(x', -x_n))\Big) = \frac{1}{2}(2\Delta P) = \Delta P$$

and

$$
\begin{aligned}
S_n(x', 0) &= \frac{1}{2}\left(\frac{\partial}{\partial x_n}\Big(P(x) + P(x', -x_n)\Big)\right)(x', 0) \\
&= \frac{1}{2}\Big(P_n(x) + (-1)(P_n)(x', -x_n)\Big)(x', 0) \\
&= \frac{1}{2}\Big(P_n(x', 0) - P_n(x', 0)\Big) = 0.
\end{aligned}
$$

All in all, we have

$$\Delta S = \Delta P, \quad S_n(x', 0) = 0. \tag{1.16}$$

Also, S still touches u^* strictly from below at $\bar{x}$.

Indeed, we know that P touches u^* at $\bar{x} \in B_\rho$ strictly from below, thus

$$u^*(\bar{x}) = P(\bar{x})$$

and

$$P(x) < u^*(x) \quad \text{in } O \setminus \{\bar{x}\}$$

where O is a neighborhood of $\bar{x}$, $O \subset B_\rho$.

Remark that, since $\bar{x} \in \{x_n = 0\}$,

$$\bar{x} = (\bar{x}', 0).$$

Hence,

$$\begin{aligned} S(\bar{x}) = S(\bar{x}', 0) &= \left(\frac{P(x) + P(x', -x_n)}{2} \right) (\bar{x}', 0) \\ &= \frac{1}{2} \left(P(\bar{x}', 0) + P(\bar{x}', 0) \right) \\ &= \frac{1}{2} (2P(\bar{x}', 0)) = P(\bar{x}', 0) = P(\bar{x}) = u^*(\bar{x}). \end{aligned}$$

Furthermore,

$$S(x) = \frac{P(x) + P(x', -x_n)}{2} < \frac{u^*(x) + u^*(x', -x_n)}{2} \quad \forall x \in O' \setminus \{\bar{x}\},$$

where $O' \subseteq O$ is a neighborhood of $\bar{x}$ symmetric respect to $B_\rho \cap \{x_n = 0\}$, if O is not.

Thus, if we show that

$$\frac{u^*(x) + u^*(x', -x_n)}{2} = u^*(x)$$

we get that S touches u^* strictly from below at $\bar{x}$.

Now, if $x \in B_\rho \cap \{x_n > 0\}$, $(x', -x_n) \in B_\rho \cap \{x_n < 0\}$ and

$$\frac{u^*(x) + u^*(x', -x_n)}{2} = \frac{\tilde{u}(x) + \tilde{u}(x', -(-x_n))}{2} = \frac{1}{2}(2\tilde{u}(x)) = \tilde{u}(x) = u^*(x)$$

and analogously if $x \in B_\rho \cap \{x_n = 0\}$, since $x = (x', 0) = (x', -0)$ and $u^*(x) = u^*(x', 0) = \tilde{u}(x', 0)$.
Instead, if $x \in B_\rho \cap \{x_n < 0\}$, $(x', -x_n) \in B_\rho \cap \{x_n > 0\}$ and

$$\begin{aligned}\frac{u^*(x) + u^*(x', -x_n)}{2} &= \frac{\tilde{u}(x', -x_n) + \tilde{u}(x', -x_n)}{2} \\ &= \frac{1}{2}(2\tilde{u}(x', -x_n)) = \tilde{u}(x', -x_n) = u^*(x).\end{aligned}$$

Hence

$$u^*(x) = \frac{u^*(x) + u^*(x', -x_n)}{2} \quad \forall x \in B_\rho$$

and S touches u^* strictly from below at $\bar{x}$.
Now, consider the family of polynomials

$$S_\varepsilon = S + \varepsilon x_n, \quad \varepsilon > 0.$$

For ε small S_ε will touch u^* from below at some point x_ε, since S touches u^* strictly from below at $\bar{x}$.
Indeed, since $\overline{O} \subseteq B_\rho$ and $u^* \in C(B_\rho)$, $S \in C(B_\rho)$, it suffices to take

$$\varepsilon \leq \frac{\min\limits_{x \in \overline{O}}(u^*(x) - S(x))}{\sup\limits_{x \in \overline{O}} x_n}$$

where O is the neighborhood of $\bar{x}$ where $S < u^*$, and we obtain

$$\begin{aligned}S(x) + \varepsilon x_n &\leq S(x) + \varepsilon \sup_{x \in \overline{O}} x_n \\ &\leq S(x) + \frac{\min\limits_{x \in \overline{O}}(u^*(x) - S(x))}{\sup\limits_{x \in \overline{O}} x_n} \sup_{x \in \overline{O}} x_n \\ &= S(x) + \min_{x \in \overline{O}}(u^*(x) - S(x)) \\ &\leq S(x) + u^*(x) - S(x) = u^*(x) \quad \text{in } O.\end{aligned}$$

Therefore, because O is open, we can find a neighborhood O' of x_ε where $S_\varepsilon \leq u^*$ and $S_\varepsilon(x_\varepsilon) = u^*(x_\varepsilon)$.
Now, if x_ε belongs to $\{x_n = 0\}$, $u^*(x_\varepsilon) = \tilde{u}(x_\varepsilon)$ and thus S_ε touches $\tilde{u}$ from

below at x_ε, in a neighborhood given by the intersection of O' with $B_\rho \cap \{x_n \geq 0\}$.
Hence, since $x_\varepsilon \in \{x_n = 0\}$ and $\tilde{u}$ is a viscosity solution to (1.6), $\tilde{u}_n(x'_\varepsilon, 0) = 0$ in the viscosity sense, so we obtain

$$\begin{aligned}(S_\varepsilon)_n(x'_\varepsilon, 0) &= \frac{\partial}{\partial x_n}\Big(S + \varepsilon x_n\Big)(x'_\varepsilon, 0) \\ &= (S_n + \varepsilon)(x'_\varepsilon, 0) = S_n(x'_\varepsilon, 0) + \varepsilon \leq 0\end{aligned}$$

which implies $S_n(x'_\varepsilon, 0) \leq -\varepsilon < 0$, contradicting (1.16).
Thus $x_\varepsilon \in B_\rho \setminus \{x_n = 0\}$.
Now, since S_ε touches u^* from below at x_ε, repeating the argument used to analyze the cases when $\bar{x} \in \{x_n \neq 0\}$, we get from (1.16)

$$\Delta S_\varepsilon = \Delta S + \Delta(\varepsilon x_n) = \Delta S = \Delta P \leq 0,$$

i.e.

$$\Delta P \leq 0.$$

Analogously, if P touches u^* at $\bar{x} \in B_\rho$ strictly from above, we obtain
$\Delta P \geq 0$.
In conclusion, u^* is harmonic in the viscosity sense in B_ρ.
Now, we want to show that u^* is harmonic in B_ρ in the classical sense.

Remark. Notice that there is an other definition of harmonic function in the viscosity sense.

Definition 1.9. Let $\Omega \subseteq \mathbb{R}^n$ be an open connex set. Let $u \in C(\Omega)$. We say that u is harmonic in the viscosity sense if the following conditions are satisfied:

(i) For every $\varphi \in C^2(\Omega)$ and for every $x_0 \in \Omega$, if $u - \varphi$ realizes a local maximum at x_0, then $\Delta\varphi(x_0) \geq 0$.

(ii) For every $\varphi \in C^2(\Omega)$ and for every $x_0 \in \Omega$, if $u - \varphi$ realizes a local minimum at x_0, then $\Delta\varphi(x_0) \leq 0$.

Recall that $u - \varphi$ realizes a local maximum \ minimum at x_0 if there exists a neighborhood of x_0 where $u - \varphi$ has a maximum \ minimum at x_0.

We need to show that the definitions are equivalent. For exposure convenience, we repeat the definition with polynomials.

Definition 1.10. Let $\Omega \subseteq \mathbb{R}^n$ be an open connex set. Let $u \in C(\Omega)$. We say that u is harmonic in the viscosity sense if the following conditions are satisfied:

(i) If P is a quadratic polynomial touching u from below at $x_0 \in \Omega$, $\Delta P \leq 0$.

(ii) If P is a quadratic polynomial touching u from above at $x_0 \in \Omega$, $\Delta P \geq 0$.

Now, suppose that Definition 1.9 holds. If P is a quadratic polynomial touching u from below at $x_0 \in \Omega$, $P \in C^2(\Omega)$ and $u - P$ realizes a local minimum at x_0, so we get $\Delta P(x_0) = \Delta P \leq 0$. Analogously, if P is a quadratic polynomial touching u from above at $x_0 \in \Omega$, $P \in C^2(\Omega)$ and $u - P$ realizes a local maximum at x_0, so we obtain $\Delta P(x_0) = \Delta P \geq 0$. Hence, Definition 1.9 implies Definition 1.10.
Conversely, suppose that Definition 1.10 holds and we take $\varphi \in C^2(\Omega)$ that $u - \varphi$ realizes a local maximum at $x_0 \in \Omega$, that is

$$u - \varphi \leq (u - \varphi)(x_0) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{1.17}$$

Since $\varphi \in C^2(\Omega)$, we can write the Taylor expansion of φ, that is

$$\begin{aligned}\varphi(x) = \varphi(x_0) + \nabla\varphi(x_0) \cdot (x - x_0) + \frac{1}{2} D^2\varphi(x_0)(x - x_0) \cdot (x - x_0) \\ + o(|x - x_0|^2).\end{aligned}$$

Therefore, from (1.17) we achieve

$$\begin{aligned}
u(x) &\leq \varphi(x) + u(x_0) - \varphi(x_0) \\
&= \varphi(x_0) + \nabla\varphi(x_0)\cdot(x-x_0) + \frac{1}{2}D^2\varphi(x_0)(x-x_0)\cdot(x-x_0) \\
&+ o(|x-x_0|^2) + u(x_0) - \varphi(x_0) \\
&= u(x_0) + \nabla\varphi(x_0)\cdot(x-x_0) + \frac{1}{2}D^2\varphi(x_0)(x-x_0)\cdot(x-x_0) \\
&+ o(|x-x_0|^2) = P_{x_0}(x) + o(|x-x_0|^2) \quad \text{in } O,
\end{aligned}$$

in other words

$$u(x) \leq P_{x_0}(x) + o(|x-x_0|^2) \quad \text{in } O, \tag{1.18}$$

where

$$P_{x_0}(x) := u(x_0) + \nabla\varphi(x_0)\cdot(x-x_0) + \frac{1}{2}D^2\varphi(x_0)(x-x_0)\cdot(x-x_0)$$

is a quadratic polynomial.
Also, if we fix $\varepsilon > 0$,

$$o(|x-x_0|^2) \leq \varepsilon\,|x-x_0|^2\,,$$

thus from (1.18)

$$u(x) \leq P_{x_0}(x) + \varepsilon\,|x-x_0|^2 \quad \forall x \in O. \tag{1.19}$$

Now, we define

$$P_\varepsilon(x) := P_{x_0}(x) + \varepsilon\,|x-x_0|^2\,. \tag{1.20}$$

Notice that P_ε is still a quadratic polynomial, since P_{x_0} is a quadratic polynomial.
We can rewrite P_ε as

$$P_\varepsilon(x) = u(x_0) + \nabla\varphi(x_0)\cdot(x-x_0) + \frac{1}{2}(D^2\varphi(x_0) + 2\varepsilon I)(x-x_0)\cdot(x-x_0).$$

In particular, we have

$$P_\varepsilon(x_0) = u(x_0)$$

and, in view of (1.19) and (1.20),

$$P_\varepsilon(x) \geq u(x) \quad \text{in } O,$$

that is P_ε touches u from above at $x_0 \in \Omega$.

Hence, from Definition 1.10, we obtain

$$\begin{aligned}
\Delta P_\varepsilon &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\bigg(u(x_0) + \nabla\varphi(x_0)\cdot(x-x_0) \\
&+ \frac{1}{2}(D^2\varphi(x_0) + 2\varepsilon I)(x-x_0)\cdot(x-x_0)\bigg) \\
&= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}(u(x_0)) + \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\bigg(\sum_{j=1}^{n} \frac{\partial\varphi}{\partial x_j}(x_0)(x_j - x_{0_j})\bigg) \\
&+ \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\bigg(\sum_{h,j=1}^{n} \frac{\partial^2\varphi}{\partial x_h \partial x_j}(x_0)(x_h - x_{0_h})(x_j - x_{0_j})\bigg) \\
&+ \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\bigg(\sum_{j=1}^{n} \varepsilon(x_j - x_{0_j})^2\bigg) \\
&= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\bigg(\frac{\partial\varphi}{\partial x_i}(x_0)(x_i - x_{0_i})\bigg) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\bigg(\frac{\partial^2\varphi}{\partial x_i^2}(x_0)(x_i - x_{0_i})^2\bigg) \\
&+ \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\bigg(\sum_{\substack{h=1\\ h\neq i}}^{n} 2\frac{\partial^2\varphi}{\partial x_h \partial x_i}(x_0)(x_h - x_{0_h})(x_i - x_{0_i})\bigg) \\
&+ \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\bigg(\varepsilon(x_i - x_{0_i})^2\bigg) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\bigg(\frac{\partial\varphi}{\partial x_i}(x_0)\bigg) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\bigg(2\frac{\partial^2\varphi}{\partial x_i^2}(x_0)(x_i - x_{0_i})\bigg) \\
&+ \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\bigg(\sum_{\substack{h=1\\ h\neq i}}^{n} 2\frac{\partial^2\varphi}{\partial x_h \partial x_i}(x_0)(x_h - x_{0_h})\bigg) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\bigg(2\varepsilon(x_i - x_{0_i})\bigg) \\
&= \frac{1}{2}\sum_{i=1}^{n} 2\frac{\partial^2\varphi}{\partial x_i^2}(x_0) + \sum_{i=1}^{n} 2\varepsilon = \sum_{i=1}^{n} \frac{\partial^2\varphi}{\partial x_i^2}(x_0) + 2\varepsilon\sum_{i=1}^{n} 1 \\
&= \Delta\varphi(x_0) + 2\varepsilon n \geq 0,
\end{aligned}$$

namely

$$\Delta P_\varepsilon = \Delta\varphi(x_0) + 2\varepsilon n \geq 0,$$

and letting ε go to 0,

$$\lim_{\varepsilon\to 0} \Delta P_\varepsilon = \lim_{\varepsilon\to 0}(\Delta\varphi(x_0) + 2\varepsilon n) = \Delta\varphi(x_0) \geq 0,$$

that is $\Delta\varphi(x_0) \geq 0$.
Analogously, repeating the same argument, if $\varphi \in C^2(\Omega)$ and $u - \varphi$ realizes a local minimum at $x_0 \in \Omega$, $\Delta\varphi(x_0) \leq 0$.
To sum it up, Definition 1.10 implies Definition 1.9 and thus Definition 1.9 and Definition 1.10 are equivalent.

Now, u^* satisfies Definition 1.10 in B_ρ, thus u^* also satisfies
Definition 1.9.
We want to show that if u^* satisfies Definition 1.10, u^* is harmonic in the classical sense.
Notice that, since $\tilde{u} \in C(B_\rho \cap \{x_n \geq 0\})$, $u^* \in C(B_\rho)$.
First of all, we prove that for every ball $B_r \subset\subset B_\rho$,

$$\max_{\overline{B_r}} u^* = \max_{\partial B_r} u^* \quad \text{and} \quad \min_{\overline{B_r}} u^* = \min_{\partial B_r} u^*.$$

Fix $B_r \subset\subset B_\rho$ and assume for contradiction that $\max_{\overline{B_r}} u^* \neq \max_{\partial B_r} u^*$.
In particular, because $\partial B_r \subset \overline{B_r}$, it means that $\max_{\overline{B_r}} u^* > \max_{\partial B_r} u^*$, i.e. there exists $x_0 \in B_r$ such that $u^*(x_0) = \max_{\overline{B_r}} u^*$ and $u^*(x_0) > M = \max_{\partial B_r} u^*$.
Let us define now the auxiliary function

$$w(x) = u^* - (M - \varepsilon\,|x - x_0|^2), \quad \varepsilon > 0,$$

in such a way that $w(x) < w(x_0)$ on ∂B_r.
To obtain such a function, it is sufficient to remark that $\forall x \in \partial B_r$, given that $|x - x_0| \leq |x| + |x_0|$, $|x| = r$, $|x_0| < r$ and $u^*(x) \leq M$,

$$\begin{aligned} w(x) &= u^*(x) - M + \varepsilon\,|x - x_0|^2 \\ &\leq u^*(x) - M + \varepsilon(|x| + |x_0|)^2 \\ &< u^*(x) - M + \varepsilon(2r)^2 \\ &= u^*(x) - M + 4\varepsilon r^2 \leq 4\varepsilon r^2 \end{aligned}$$

and require that $4\varepsilon r^2 < w(x_0) = u^*(x_0) - M$ in order to get $w(x) < w(x_0)$ on ∂B_r.

Thus for every $\varepsilon < \frac{u^*(x_0)-M}{4r^2}$ there exists $x_\varepsilon \in B_r$ such that

$$\max_{\overline{B_r}} w = w(x_\varepsilon), \tag{1.21}$$

seeing as how $w|_{\partial B_r} < w(x_0)$ and $x_0 \in B_r$, so $\max\limits_{\overline{B_r}} w$ is reached in an internal point of $\overline{B_r}$.

In this case the function $\varphi_\varepsilon = M - \varepsilon |x - x_0|^2$ is $C^2(B_\rho)$ and $x_\varepsilon \in B_r$.

At this point, in view of (1.21), $\forall x \in B_r$ we have

$$u^*(x) - \varphi_\varepsilon(x) = w(x) \leq w(x_\varepsilon) = u^*(x_\varepsilon) - \varphi_\varepsilon(x_\varepsilon),$$

that is x_ε is a maximum for $u^* - \varphi_\varepsilon$ in B_r.

Also,

$$\begin{aligned}
\Delta\varphi_\varepsilon(x_\varepsilon) &= \Bigg(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \Big(M - \varepsilon |x - x_0|^2 \Big) \Bigg)(x_\varepsilon) \\
&= \Bigg(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \Big(M \Big) + \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \Big(-\varepsilon \sum_{h=1}^n (x_h - x_{0_h})^2 \Big) \Bigg)(x_\varepsilon) \\
&= \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \Big(-\varepsilon (x_i - x_{0_i})^2 \Big)(x_\varepsilon) = \Bigg(\sum_{i=1}^n \frac{\partial}{\partial x_i} \Big(-2\varepsilon (x_i - x_{0_i}) \Big) \Bigg)(x_\varepsilon) \\
&= \Bigg(\sum_{i=1}^n -2\varepsilon \Bigg)(x_\varepsilon) = -2n\varepsilon < 0,
\end{aligned}$$

in other words

$$\Delta\varphi_\varepsilon(x_\varepsilon) = -2n\varepsilon < 0. \tag{1.22}$$

Now, since u^* is harmonic in the viscosity sense, $\varphi_\varepsilon \in C^2(B_\rho)$ and $u^* - \varphi_\varepsilon$ realizes a local maximum at x_ε,

$$\Delta\varphi_\varepsilon(x_\varepsilon) \geq 0,$$

which contradicts (1.22).

Thus, for every ball $B_r \subset\subset B_\rho$,

$$\max_{\overline{B_r}} u^* = \max_{\partial B_r} u^*$$

and analogously, repeating the same argument,

$$\min_{\overline{B_r}} u^* = \min_{\partial B_r} u^*.$$

Let us prove now that $u^* \in C^2(B_\rho)$.
The strategy is the following: let us fix $B_r \subset\subset B_\rho$ and taking h solution of the Dirichlet problem

$$\begin{cases} \Delta h = 0 & \text{in } B_r \\ h = u^* & \text{on } \partial B_r. \end{cases}$$

We want to show that $h = u^*$ for every $x \in B_r$.
We know that, since h is solution of the Dirichlet problem, $u^* - h \in C(B_r)$.
For every $\psi \in C^2(B_\rho)$ such that

$$(u^* - h) - \psi \leq (u^* - h)(x_0) - \psi(x_0),$$

with $x_0 \in B_r$, we get $\Delta\psi(x_0) \geq 0$ because

$$u^* - (h + \psi) = (u^* - h) - \psi \leq (u^* - h)(x_0) - \psi(x_0) = u^*(x_0) - (h + \psi)(x_0),$$

that is $h + \psi \in C^2(B_r)$ (h is solution of the Dirichlet problem) is such that $u^* - (h + \psi)$ realizes a local maximum at x_0.
Therefore, inasmuch u^* is harmonic in the viscosity sense and h is solution of the Dirichlet problem in $\overline{B_r}$,

$$\Delta(h + \psi)(x_0) = \Delta h(x_0) + \Delta\psi(x_0) = \Delta\psi(x_0) \geq 0.$$

As a consequence, $u^* - h$ satisfies Definition 1.9 and, as a byproduct, for what we have seen before, $u^* - h$ satisfies the maximum principle, namely

$$\min_{\partial B_r}(u^* - h) = \min_{\overline{B_r}}(u^* - h), \quad \max_{\partial B_r}(u^* - h) = \max_{\overline{B_r}}(u^* - h).$$

In particular, we have $\forall x \in B_r$

$$0 = \min_{\partial B_r}(u^* - h) = \min_{\overline{B_r}}(u^* - h) \leq u^* - h \leq \max_{\overline{B_r}}(u^* - h) = \max_{\partial B_r}(u^* - h) = 0$$

Hence, $u^* - h = 0$ in B_r, i.e. $u^* = h$ in B_r and since h is solution of the Dirichlet problem in B_r, $u^* \in C^2(B_r)$.

Now, since $\overline{B_\rho}$ is a compact, we can cover B_ρ with a finite number of balls B_r, where u^* is equal to the solution of the Dirichlet problem in B_r and $u^* \in C^2(B_r)$, thus u^* is harmonic in the classical sense in B_ρ and $u^* \in C^2(B_\rho)$. In particular $u^* \in C^\infty(B_\rho)$, hence $\tilde{u} \in C^\infty(B_\rho \cap \{x_n \geq 0\})$ and is harmonic in the classical sense in $B_\rho \cap \{x_n > 0\}$, in other words,

$$\Delta \tilde{u} = 0 \quad \text{in } B_\rho \cap \{x_n > 0\} \quad \text{in the classical sense.} \tag{1.23}$$

Remain to show that $\tilde{u}$ satisfies $\tilde{u}_n = \frac{\partial \tilde{u}}{\partial x_n} = 0$ on $B_\rho \cap \{x_n = 0\}$ in the classical sense.
First of all, notice that $\frac{\partial \tilde{u}}{\partial x_n}$ exists on $B_\rho \cap \{x_n = 0\}$, because
$\tilde{u} \in C^\infty(B_\rho \cap \{x_n \geq 0\})$.
Analogously, $\frac{\partial u^*}{\partial x_n}$ exists on $B_\rho \cap \{x_n = 0\}$, given that $u^* \in C^\infty(B_\rho)$.
In addition, if $\bar{x} \in B_\rho \cap \{x_n = 0\}$,

$$\frac{\partial u^*}{\partial x_n}(\bar{x}) = \lim_{t \to 0^+} \frac{u^*(\bar{x} + te_n) - u^*(\bar{x})}{t} = \lim_{t \to 0^-} \frac{u^*(\bar{x} + te_n) - u^*(\bar{x})}{t}. \tag{1.24}$$

Now, if $\bar{x} \in B_\rho \cap \{x_n = 0\}$, $\bar{x} = (\bar{x}', 0)$ and

$$\begin{aligned} \lim_{t \to 0^+} \frac{u^*(\bar{x} + te_n) - u^*(\bar{x})}{t} &= \lim_{t \to 0^+} \frac{u^*(\bar{x}', t) - u^*(\bar{x}', 0)}{t} \\ = \lim_{t \to 0^+} \frac{\tilde{u}(\bar{x}', t) - \tilde{u}(\bar{x}', 0)}{t} &= \frac{\partial \tilde{u}}{\partial x_n}(\bar{x}', 0), \end{aligned} \tag{1.25}$$

seeing as how $t > 0$, hence $u^*(\bar{x}', t) = \tilde{u}(\bar{x}', t)$, while

$$\begin{aligned} &\lim_{t \to 0^-} \frac{u^*(\bar{x} + te_n) - u^*(\bar{x})}{t} = \lim_{t \to 0^-} \frac{u^*(\bar{x}', t) - u^*(\bar{x}', 0)}{t} \\ &= \lim_{t \to 0^-} \frac{\tilde{u}(\bar{x}', -t) - \tilde{u}(\bar{x}', 0)}{t} = \lim_{t \to 0^-} -\frac{\tilde{u}(\bar{x}', -t) - \tilde{u}(\bar{x}', 0)}{-t} \\ &= -\lim_{t \to 0^-} \frac{\tilde{u}(\bar{x}', -t) - \tilde{u}(\bar{x}', 0)}{-t} \overset{h=-t}{=} -\lim_{h \to 0^+} \frac{\tilde{u}(\bar{x}', h) - \tilde{u}(\bar{x}', 0)}{h} \\ &= -\frac{\partial \tilde{u}}{\partial x_n}(\bar{x}', 0) \end{aligned} \tag{1.26}$$

since $t < 0$, hence $u^*(\bar{x}', t) = \tilde{u}(\bar{x}', -t)$.
Therefore, from (1.24), (1.25) and (1.26), we achieve

$$\frac{\partial \tilde{u}}{\partial x_n}(\bar{x}', 0) = -\frac{\partial \tilde{u}}{x_n}(\bar{x}', 0)$$

and thus

$$\frac{\partial \tilde{u}}{\partial x_n}(\bar{x}', 0) = 0.$$

For the arbitrariness of $\bar{x} \in B_\rho \cap \{x_n = 0\}$, we get

$$\frac{\partial \tilde{u}}{\partial x_n} = 0 \quad \text{on } B_\rho \cap \{x_n = 0\} \quad \text{in the classical sense.} \tag{1.27}$$

In conclusion, from (B.13) and (B.15) we obtain that $\tilde{u}$ is a classical solution of

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_\rho \cap \{x_n > 0\} \\ \tilde{u}_n = \frac{\partial \tilde{u}}{x_n} = 0 & \text{on } B_\rho \cap \{x_n = 0\} \, . \end{cases}$$

$\square$

Chapter 2

A Harnack inequality for a one-phase free boundary problem

In this chapter, we will show that a Harnack type inequality is satisfied by a solution u to our problem

$$\begin{cases} \sum\limits_{i,j} a_{ij}(x)u_{ij} = f & \text{in } \Omega^+(u), \\ |\nabla u| = g & \text{on } F(u), \end{cases} \tag{2.1}$$

under the assumption $(0 < \varepsilon < 1)$

$$\|f\|_{L^\infty(\Omega)} \leq \varepsilon^2, \quad \|g-1\|_{L^\infty(\Omega)} \leq \varepsilon^2, \quad \|a_{ij} - \delta_{ij}\|_{L^\infty(\Omega)} \leq \varepsilon. \tag{2.2}$$

This theorem, although it is called "Harnack inequality", is rather different from the classical Harnack inequality.
Indeed, it roughly says that if the graph of u oscillates εr away from x_n^+ in B_r, then it oscillates $(1-c)\varepsilon r$ in $B_{r/20}$, with $0 < c < 1$.
As regards the proof of this Harnack inequality, it relies on Lemma 2.3, which will be introduced and proved after the statement of the theorem. As a matter of fact, before Lemma 2.3, a remark concerning the Harnack inequality will lead to a corollary, which will be a key tool in the proof of Theorem 4.2.

Notation. A positive constant depending only on the dimension n is called a *universal constant.* We often use c, c_i to denote small universal constants, and C, C_i to denote large universal constants.

Theorem 2.1 (**Harnack inequality**). *There exists a universal constant $\bar{\varepsilon}$ such that if u solves (2.1)-(2.2), and for some point $x_0 \in \Omega^+(u) \cup F(u)$,*

$$(x_n + a_0)^+ \leq u(x) \leq (x_n + b_0)^+ \quad \text{in } B_r(x_0) \subset \Omega \tag{2.3}$$

with

$$b_0 - a_0 \leq \varepsilon r, \quad \varepsilon \leq \bar{\varepsilon},$$

then

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{r/20}(x_0)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1-c)\varepsilon r,$$

and $0 < c < 1$ universal.

Before showing the proof of Theorem 2.1, we observe that if Theorem 2.1 holds, it follows an important corollary which we will use in the proof of our main result.

Corollary 2.2. *Let u be a solution to (2.1)-(2.2) satisfying (2.3) for $r = 1$. Then in $B_1(x_0)$,*

$$\tilde{u}_\varepsilon(x) = \frac{u(x) - x_n}{\varepsilon}$$

has a Hölder modulus of continuity at x_0, outside the ball of radius $\varepsilon/\bar{\varepsilon}$, i.e. for all $x \in (\Omega^+(u) \cup F(u)) \cap B_1(x_0)$ with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$,

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C\, |x - x_0|^\gamma .$$

Proof. Let us begin the proof claiming that if u is a solution to (2.1)-(2.2) satisfying (2.3) with $r = 1$, then we can apply the Harnack inequality repeatedly to obtain

$$(x_n + a_m)^+ \leq u(x) \leq (x_n + b_m)^+ \quad \text{in } B_{20^{-m}}(x_0) \tag{2.4}$$

with

$$b_m - a_m \leq (1-c)^m \varepsilon$$

for all m's such that

$$(1-c)^{m-1} 20^{m-1} \varepsilon \leq \bar{\varepsilon}.$$

This result follows by an induction on m's such that

$$(1-c)^{m-1} 20^{m-1} \varepsilon \leq \bar{\varepsilon}.$$

Precisely, for $m = 1$, applying the Harnack inequality con $r = 1$, we get

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{20^{-1}}(x_0)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1-c)\varepsilon$$

and

$$(1-c)^0 20^0 \varepsilon = \varepsilon \leq \bar{\varepsilon}.$$

Suppose now that the result holds for m and we show that it holds for $m+1$. From the hypothesis of induction, we have

$$(x_n + a_m)^+ \leq u(x) \leq (x_n + b_m)^+ \quad \text{in } B_{20^{-m}}(x_0)$$

with

$$b_m - a_m \leq (1-c)^m \varepsilon$$

and

$$(1-c)^{m-1} 20^{m-1} \varepsilon \leq \bar{\varepsilon}.$$

To apply the Harnack inequality, we must have

$$b_m - a_m \leq \delta\, 20^{-m}$$

with

$$\delta \leq \bar{\varepsilon}.$$

Specifically, we know from the hypothesis of induction that

$$b_m - a_m \leq (1-c)^m \varepsilon = (1-c)^m \varepsilon\, 20^m 20^{-m} = (1-c)^m 20^m \varepsilon\, 20^{-m},$$

hence, if

$$(1-c)^m 20^m \varepsilon \leq \bar{\varepsilon},$$

we can apply the Harnack inequality and we obtain

$$(x_n + a_{m+1})^+ \leq u(x) \leq (x_n + b_{m+1})^+ \quad \text{in } B_{20^{-(m+1)}}(x_0)$$

with

$$b_{m+1} - a_{m+1} \leq (1-c)(1-c)^m 20^m \varepsilon 20^{-m} = (1-c)^{m+1}\varepsilon.$$

Notice that when we apply the Harnack inequality repeatedly, given that u solves (2.1)-(2.2) with ε, u solves (2.1)-(2.2) even with $(1-c)^{m-1}20^{m-1}\varepsilon$, so we can apply the Harnack inequality repeatedly.
This result implies that for all such m's, the oscillation of the function

$$\tilde{u}_\varepsilon(x) = \frac{u(x) - x_n}{\varepsilon}$$

in $(\Omega^+(u) \cup F(u)) \cap B_r(x_0) = (\Omega^+(u) \cap B_r(x_0)) \cup (F(u) \cap B_r(x_0))$, $r = 20^{-m}$, is less than $(1-c)^m = 20^{-\gamma m} = r^\gamma$.
Indeed, $\forall x \in \Omega^+(u) \cap B_r(x_0)$, we have

$$0 < u(x) \leq (x_n + b_m)^+,$$

thus, since $(x_n + b_m)^+ > 0$,

$$(x_n + b_m)^+ = x_n + b_m$$

and from (2.4)

$$x_n + a_m \leq (x_n + a_m)^+ \leq u(x) \leq (x_n + b_m)^+ = x_n + b_m \quad \text{in } \Omega^+(u) \cap B_r(x_0),$$

in other words

$$x_n + a_m \leq u(x) \leq x_n + b_m \quad \text{in } \Omega^+(u) \cap B_r(x_0). \tag{2.5}$$

Furthermore, in view of (2.5), we have

$$a_m \leq u(x) - x_n \leq b_m \leq a_m + (1-c)^m \varepsilon \quad \text{in } \Omega^+(u) \cap B_r(x_0),$$

that is

$$a_m \leq u(x) - x_n \leq a_m + (1-c)^m \varepsilon \quad \text{in } \Omega^+(u) \cap B_r(x_0),$$

which entails

$$\begin{aligned}\operatorname*{osc}_{\Omega^+(u)\cap B_r(x_0)} (u - x_n) &= \sup_{\Omega^+(u)\cap B_r(x_0)} (u - x_n) - \inf_{\Omega^+(u)\cap B_r(x_0)} (u - x_n) \\ &\leq a_m + (1-c)^m \varepsilon - a_m = (1-c)^m \varepsilon,\end{aligned}$$

i.e.

$$\operatorname*{osc}_{\Omega^+(u)\cap B_r(x_0)} (u - x_n) \leq (1-c)^m \varepsilon. \tag{2.6}$$

Consequently, from (2.6), we achieve

$$\begin{aligned}\operatorname*{osc}_{\Omega^+(u)\cap B_r(x_0)} \tilde{u}_\varepsilon &= \operatorname*{osc}_{\Omega^+(u)\cap B_r(x_0)} \left(\frac{u - x_n}{\varepsilon} \right) = \frac{1}{\varepsilon} \operatorname*{osc}_{\Omega^+(u)\cap B_r(x_0)} (u - x_n) \\ &\leq \frac{(1-c)^m \varepsilon}{\varepsilon} = (1-c)^m,\end{aligned}$$

which gives

$$\operatorname*{osc}_{\Omega^+(u)\cap B_r(x_0)} \tilde{u}_\varepsilon \leq (1-c)^m. \tag{2.7}$$

On $F(u) \cap B_r(x_0)$, instead, we have from (2.4)

$$(x_n + a_m)^+ \leq u(x) = 0 \quad \forall x \in F(u) \cap B_r(x_0)$$

and thus, since $0 \leq (x_n + a_m)^+$,

$$(x_n + a_m)^+ = 0 \quad \text{on } F(u) \cap B_r(x_0),$$

which also gives

$$x_n + a_m \leq 0 \quad \text{on } F(u) \cap B_r(x_0)$$

and

$$x_n \leq -a_m \quad \text{on } F(u) \cap B_r(x_0). \tag{2.8}$$

Now, from (2.4), if $(x_n + b_m)^+ = 0$, that is $x_n + b_m \leq 0$ and $x_n \leq -b_m$, we have $u = 0$, inasmuch as u is a solution to (2.1) and as a consequence $u \geq 0$. Also, if we take a point $\bar{x} \in B_r(x_0) \cap \{x_n < -b_m\}$, since $B_r(x_0) \cap \{x_n < -b_m\}$

is open, we can find a ball $B_{\bar{r}}(\bar{x}) \subset B_r(x_0) \cap \{x_n < -b_m\}$, where $u = 0$ and thus $B_{\bar{x}}(\bar{r}) \cap \Omega^+(u) = \emptyset$, in other words $\bar{x} \notin F(u)$.
Therefore, $F(u) \cap (B_r(x_0) \cap \{x_n < -b_m\}) = \emptyset$, namely

$$x_n \geq -b_m \quad \text{in } F(u) \cap B_r(x_0). \tag{2.9}$$

To sum it up, if $x \in F(u) \cap B_r(x_0)$, we have, in view of (2.8) and (2.9),

$$x_n \leq -a_m \quad \text{and} \quad x_n \geq -b_m,$$

hence

$$a_m \leq -x_n \quad \text{and} \quad -x_n \leq b_m,$$

which implies

$$a_m \leq -x_n \leq b_m \leq a_m + (1-c)^m \varepsilon,$$

that is

$$a_m \leq -x_n \leq a_m + (1-c)^m \varepsilon \quad \text{on } F(u) \cap B_r(x_0). \tag{2.10}$$

Notice that, because $u = 0$ on $F(u) \cap B_r(x_0)$,

$$u(x) - x_n = -x_n \quad \text{on } F(u) \cap B_r(x_0),$$

thus, in view of (2.10)

$$a_m \leq u(x) - x_n \leq a_m + (1-c)^m \varepsilon,$$

which implies, repeating the same calculations done to get (2.6) with $F(u) \cap B_r(x_0)$ in place of $\Omega^+(u) \cap B_r(x_0)$,

$$\operatorname*{osc}_{F(u) \cap B_r(x_0)} (u - x_n) \leq (1-c)^m \varepsilon. \tag{2.11}$$

As a consequence, repeating the same computations done to obtain (2.7) with $F(u) \cap B_r(x_0)$ in place of $\Omega^+(u) \cap B_r(x_0)$, we have

$$\operatorname*{osc}_{F(u) \cap B_r(x_0)} \tilde{u}_\varepsilon \leq (1-c)^m. \tag{2.12}$$

Hence, inasmuch

$$(\Omega^+(u) \cup F(u)) \cap B_r(x_0) = (\Omega^+(u) \cap B_r(x_0) \cup (F(u) \cap B_r(x_0)),$$

from (2.7) and (2.12) we achieve that for all m's such that

$$(1-c)^{m-1}20^{m-1}\varepsilon \leq \bar{\varepsilon},$$

we have

$$\operatorname*{osc}_{(\Omega^+(u)\cup F(u))\cap B_{20^{-m}}(x_0)} \tilde{u}_\varepsilon \leq (1-c)^m = 20^{-m\gamma}.$$

Moreover, if $x \in (\Omega^+(u) \cup F(u)) \cap B_{20^{-m}}(x_0)$, seeing as how $x_0 \in (\Omega^+(u) \cup F(u)) \cap B_{20^{-m}}(x_0)$ by the hypothesis of the Harnack inequality,

$$\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0) \leq \operatorname*{osc}_{(\Omega^+(u)\cup F(u))\cap B_{20^{-m}}(x_0)} \tilde{u}_\varepsilon \leq 20^{-m\gamma},$$

$$\tilde{u}_\varepsilon(x_0) - \tilde{u}_\varepsilon(x) \leq \operatorname*{osc}_{(\Omega^+(u)\cup F(u))\cap B_{20^{-m}}(x_0)} \tilde{u}_\varepsilon \leq 20^{-m\gamma},$$

and these two conditions imply

$$\max\left(\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0), \tilde{u}_\varepsilon(x_0) - \tilde{u}_\varepsilon(x)\right) = |\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq 20^{-m\gamma},$$

i.e.

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq 20^{-m\gamma}. \tag{2.13}$$

In particular, we can choose c such that $(1-c)20 > 1$, so there exists $\overline{m}$ that satisfies

$$(1-c)^{\overline{m}}20^{\overline{m}}\varepsilon > \bar{\varepsilon},$$

hence

$$(1-c)^{\overline{m}}\frac{\varepsilon}{\bar{\varepsilon}} > 20^{-\overline{m}}$$

and raising both the terms of the inequality to γ, with $0 < \gamma < 1$, recalling that both the terms are positive or equal to 0,

$$(1-c)^{\overline{m}\gamma}\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^\gamma > 20^{-\overline{m}\gamma}. \tag{2.14}$$

Now, if $x \in (\Omega^+(u) \cup F(u)) \cap B_1(x_0)$, with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$, there exists m such that

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq 20^{-m\gamma}$$

from (2.13).
Furthermore, from (2.14), we have

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq 20^{-m\gamma} = 20^{-m\gamma}\frac{20^{-\overline{m}\gamma}}{20^{-\overline{m}\gamma}} \leq \frac{20^{-m\gamma}}{20^{-\overline{m}\gamma}}(1-c)^{\overline{m}\gamma}\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{\gamma},$$

in other words

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq \frac{20^{-m\gamma}}{20^{-\overline{m}\gamma}}(1-c)^{\overline{m}\gamma}\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{\gamma}. \tag{2.15}$$

As a consequence, because $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$, from (2.15) we get

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq \frac{20^{-m\gamma}}{20^{-\overline{m}\gamma}}(1-c)^{\overline{m}\gamma}\,|x-x_0|^{\gamma} = C\,|x-x_0|^{\gamma},$$

namely

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C\,|x-x_0|^{\gamma}$$

$\forall x \in (\Omega^+(u) \cup F(u)) \cap B_1(x_0)$, $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$.
Thus, $\tilde{u}_\varepsilon$ has a Hölder modulus of continuity at x_0, outside the ball of radius $\varepsilon/\bar{\varepsilon}$. □

The proof of the Harnack inequality relies on the following lemma.

Lemma 2.3. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u is a solution to (2.1)-(2.2) in B_1 with $0 < \varepsilon \leq \bar{\varepsilon}$ and u satisfies*

$$p(x)^+ \leq u(x) \leq (p(x)+\varepsilon)^+, \quad x \in B_1,\ p(x) = x_n + \sigma,\ |\sigma| < 1/10, \tag{2.16}$$

then if at $\bar{x} = \frac{1}{5}e_n$

$$u(\bar{x}) \geq \left(p(\bar{x}) + \frac{\varepsilon}{2}\right)^+, \tag{2.17}$$

then

$$u \geq (p + c\varepsilon)^+ \quad \text{in } \overline{B}_{1/2} \tag{2.18}$$

for some $0 < c < 1$. Analogously, if

$$u(\bar{x}) \leq \left(p(\bar{x}) + \frac{\varepsilon}{2}\right)^+,$$

then

$$u \leq (p + (1-c)\varepsilon)^+ \quad \text{in } \overline{B}_{1/2}.$$

Proof. We prove the first statement.
From (2.16), since $p^+ \geq p$,
$$u \geq p \quad \text{in } B_1. \tag{2.19}$$
Let
$$w = c(|x-\bar{x}|^{-\gamma} - (3/4)^{-\gamma}) \tag{2.20}$$
be defined in the closure of the annulus
$$A := B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x}).$$
The constant c is such that w satisfies the boundary conditions
$$\begin{cases} w = 0 & \text{on } \partial B_{3/4}(\bar{x}), \\ w = 1 & \text{on } \partial B_{1/20}(\bar{x}). \end{cases}$$
In particular, we have
$$w = c((3/4)^{-\gamma} - (3/4)^{-\gamma}) = 0 \quad \text{in } \partial B_{3/4}(\bar{x}),$$
and
$$w = c((1/20)^{-\gamma} - (3/4)^{-\gamma}) = 1 \quad \text{in } \partial B_{1/20}(\bar{x}),$$
thus
$$c = \frac{1}{(1/20)^{-\gamma} - (3/4)^{-\gamma}}$$
and
$$w = \frac{1}{(1/20)^{-\gamma} - (3/4)^{-\gamma}}(|x-\bar{x}|^{-\gamma} - (3/4)^{-\gamma}).$$
Also, because $\|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \leq \varepsilon$, as long as ε is small enough, the matrix (a_{ij}) is uniformly elliptic, (see Lemma A.5 in Appendix A for the proof of this result) and we can choose the constant γ universal so that
$$\sum_{i,j} a_{ij}(x) w_{ij} \geq \delta > 0 \quad \text{in } A$$
with δ universal.
Notice that $w \in C^\infty(A)$, so all the second derivatives of w exist and are

continuous in A.

Let us show that we can choose γ as we have said above.

Precisely, keeping c in the expression of w for the sake of simplicity, we have

$$\begin{aligned}\frac{\partial w}{\partial x_i} &= \frac{\partial}{\partial x_i}\Big(c(|x-\bar{x}|^{-\gamma} - (3/4)^{-\gamma})\Big) = -\gamma c\,|x-\bar{x}|^{-\gamma-1}\frac{\partial}{\partial x_i}\Big(\,|x-\bar{x}|\Big)\\ &= -\gamma c\,|x-\bar{x}|^{-\gamma-1}\frac{x_i-\bar{x}_i}{|x-\bar{x}|} = -\gamma c\,|x-\bar{x}|^{-\gamma-2}(x_i-\bar{x}_i),\end{aligned}$$

in other words

$$\frac{\partial w}{\partial x_i} = -\gamma c\,|x-\bar{x}|^{-\gamma-2}(x_i-\bar{x}_i), \tag{2.21}$$

and from (2.21)

$$\begin{aligned}\frac{\partial^2 w}{\partial x_j \partial x_i} &= \frac{\partial}{\partial x_j}\Big(-\gamma c\,|x-\bar{x}|^{-\gamma-2}(x_i-\bar{x}_i)\Big)\\ &= -\gamma c\frac{\partial}{\partial x_j}\Big(\,|x-\bar{x}|^{-\gamma-2}\Big)(x_i-\bar{x}_i) - \gamma c\,|x-\bar{x}|^{-\gamma-2}\frac{\partial}{\partial x_j}\Big(x_i-\bar{x}_i\Big)\\ &= c\gamma(\gamma+2)\,|x-\bar{x}|^{-\gamma-3}\frac{(x_j-\bar{x}_j)}{|x-\bar{x}|}(x_i-\bar{x}_i) - c\gamma\delta_{ij}\,|x-\bar{x}|^{-\gamma-2}\\ &= c\gamma(\gamma+2)\,|x-\bar{x}|^{-\gamma-4}(x_i-\bar{x}_i)(x_j-\bar{x}_j) - c\gamma\,|x-\bar{x}|^{-\gamma-2}\delta_{ij},\end{aligned}$$

which gives

$$\frac{\partial^2 w}{\partial x_j \partial x_i} = c\gamma(\gamma+2)\,|x-\bar{x}|^{-\gamma-4}(x_i-\bar{x}_i)(x_j-\bar{x}_j) - c\gamma\,|x-\bar{x}|^{-\gamma-2}\delta_{ij}. \tag{2.22}$$

Hence, from (2.21) and (2.22), we obtain, inasmuch (a_{ij}) is uniformly elliptic,

$$\begin{aligned}\sum_{i,j} a_{ij}(x)w_{ij} &= c\gamma(\gamma+2)\,|x-\bar{x}|^{-\gamma-4}\sum_{i,j} a_{ij}(x)(x_i-\bar{x}_i)(x_j-\bar{x}_j)\\ &\quad - c\gamma\,|x-\bar{x}|^{-\gamma-2}\sum_{i,j} a_{ij}\delta_{ij}\\ &\geq \lambda c\gamma(\gamma+2)\,|x-\bar{x}|^{-\gamma-4}\,|x-\bar{x}|^2 - c\gamma\,|x-\bar{x}|^{-\gamma-2}\sum_i a_{ii}\\ &= c\gamma\Big(\lambda(\gamma+2) - Tr(A)\Big)\,|x-\bar{x}|^{-\gamma-2}\\ &\geq c\gamma\,(\lambda(\gamma+2) - n\Lambda)\,|x-\bar{x}|^{-\gamma-2},\end{aligned}$$

i.e.

$$\sum_{i,j} a_{ij}(x) w_{ij} \geq c\gamma \left(\lambda(\gamma+2) - n\Lambda\right) |x - \bar{x}|^{-\gamma-2}. \tag{2.23}$$

Moreover, in A we have $|x - \bar{x}| \leq 3/4$, thus, since $\gamma > 0$,

$$|x - \bar{x}|^{-\gamma-2} \geq (3/4)^{-\gamma-2} \quad \text{in } A. \tag{2.24}$$

Therefore, if we take

$$\lambda(\gamma+2) > n\Lambda,$$

that is

$$\gamma + 2 > n\frac{\Lambda}{\lambda},$$

and

$$\gamma > n\frac{\Lambda}{\lambda} - 2,$$

we get in view of (2.23) and (2.24)

$$\sum_{i,j} a_{ij}(x) w_{ij} \geq c\gamma \Big(\lambda(\gamma+2) - n\Lambda\Big) \left(\frac{3}{4}\right)^{-\gamma-2} = \delta > 0 \quad \text{in } A,$$

in other words

$$\sum_{i,j} a_{ij}(x) w_{ij} \geq \delta \tag{2.25}$$

with δ universal, as desired.

Extend now w to be equal to 1 on $B_{1/20}(\bar{x})$.

Notice that because $|\sigma| < 1/10$, using (2.19), we obtain

$$B_{1/10}(\bar{x}) \subset B_1^+(u). \tag{2.26}$$

In particular, first of all we prove that $B_{1/10}(\bar{x}) \subset B_1$.

Remark that $\bar{x} = \frac{1}{5} e_n$, thus $|\bar{x}| = \frac{1}{5}$.

Now, if $x \in B_{1/10}(\bar{x})$ we have

$$|x| = |x - \bar{x} + \bar{x}| \leq |x - \bar{x}| + |\bar{x}| < \frac{1}{10} + \frac{1}{5} = \frac{3}{10} < 1,$$

that is $|x| < 1$, and hence

$$B_{1/10}(\bar{x}) \subset B_1. \tag{2.27}$$

As consequence, we obtain from (2.19) and (2.27)

$$u \geq p \quad \text{in } B_{1/10}(\bar{x}). \tag{2.28}$$

Also, if $x \in B_{1/10}(\bar{x})$ we have

$$|x - \bar{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \left(x_n - \frac{1}{5}\right)^2} < \frac{1}{10},$$

thus

$$\left|x_n - \frac{1}{5}\right| \leq \sqrt{x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \left(x_n - \frac{1}{5}\right)^2} < \frac{1}{10},$$

i.e.

$$\left|x_n - \frac{1}{5}\right| < \frac{1}{10},$$

which implies

$$-\frac{1}{10} < x_n - \frac{1}{5} < \frac{1}{10}$$

and

$$x_n > \frac{1}{5} - \frac{1}{10} = \frac{1}{10}. \tag{2.29}$$

Now, inasmuch $|\sigma| < \frac{1}{10}$, $\sigma > -\frac{1}{10}$, so, from (2.29), we get

$$p(x) = x_n + \sigma > x_n - \frac{1}{10} > \frac{1}{10} - \frac{1}{10} = 0 \quad \text{in } B_{1/10}(\bar{x}),$$

namely

$$p(x) > 0 \quad \text{in } B_{1/10}(\bar{x}),$$

which entails from (2.28) and (2.27)

$$u > 0 \quad \text{in } B_{1/10}(\bar{x}) \subset B_1,$$

that is

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

In addition to this fact, we have

$$B_{1/2} \subset\subset B_{3/4}(\bar{x}) \subset\subset B_1,$$

in other words,

$$\overline{B}_{1/2} \subset B_{3/4}(\bar{x}) \quad \text{and} \quad \overline{B}_{3/4}(\bar{x}) \subset B_1,$$

inasmuch $\overline{B}_{1/2}$ and $\overline{B}_{3/4}(\bar{x})$ are compacts.
Indeed, if $x \in \overline{B}_{1/2}$,

$$|x - \bar{x}| \leq |x| + |\bar{x}| \leq \frac{1}{2} + \frac{1}{5} = \frac{7}{10} < \frac{3}{4},$$

namely

$$|x - \bar{x}| < \frac{3}{4},$$

which gives

$$\overline{B}_{1/2} \subset B_{3/4}(\bar{x}). \tag{2.30}$$

At the same time, if $x \in \overline{B}_{3/4}(\bar{x})$,

$$|x| = |x - \bar{x} + \bar{x}| \leq |x - \bar{x}| + |\bar{x}| \leq \frac{3}{4} + \frac{1}{5} = \frac{19}{20} < 1,$$

i.e.

$$|x| < 1,$$

which entails

$$\overline{B}_{3/4}(\bar{x}) \subset B_1. \tag{2.31}$$

As a consequence, from (2.30) and (2.31), we achieve

$$\overline{B}_{1/2} \subset B_{3/4}(\bar{x}) \quad \text{and} \quad \overline{B}_{3/4}(\bar{x}) \subset B_1. \tag{2.32}$$

Notice that $u - p$ solves, in the viscosity sense, a uniformly elliptic equation in $B_{1/10}(\bar{x})$ with right-hand side f.
Precisely, let us take $\varphi \in C^2(B_{1/10}(\bar{x}))$ touching $u - p$ from below at $x_0 \in B_{1/10}(\bar{x})$.
Therefore we have

$$\varphi(x_0) = (u - p)(x_0) = u(x_0) - p(x_0),$$

which gives

$$(\varphi + p)(x_0) = \varphi(x_0) + p(x_0) = u(x_0), \tag{2.33}$$

and

$$\varphi(x) \leq (u-p)(x) = u(x) - p(x) \quad \text{in a neighborhood } O \text{ of } x_0,$$

which implies

$$(\varphi + p)(x) = \varphi(x) + p(x) \leq u(x) \quad \text{in } O. \tag{2.34}$$

Hence, in view of (2.33) and (2.34), we obtain that $(\varphi + p)$ touches u from below at x_0, with $\varphi + p \in C^2(B_{1/10}(\bar{x}))$, since $p = x_n + \sigma \in C^\infty(B_1)$ and $B_{1/10}(\bar{x}) \subset B_1$ from (2.27).
To use the fact that u is a viscosity solution in B_1, we have to show that $x_0 \in B_1^+(u)$, but $x_0 \in B_{1/10}(\bar{x})$, thus from (2.26), $x_0 \in B_1^+(u)$.
Therefore, since u is a viscosity solution to (2.1) in B_1 and $(\varphi + p) \in C^2(B_{1/10}(\bar{x}))$ touches u from below at $x_0 \in B_1^+(u)$, we get, from the definition of viscosity solution,

$$\begin{aligned}
\sum_{i,j} a_{ij}(x_0)(\varphi + p)_{ij}(x_0) &= \sum_{i,j} a_{ij}(x_0)(\varphi + x_n + \sigma)_{ij}(x_0) \\
&= \sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + \sum_{i,j} a_{ij}(x_0)(x_n + \sigma)_{ij}(x_0) \\
&= \sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \leq f(x_0)
\end{aligned}$$

hence

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \leq f(x_0). \tag{2.35}$$

We repeat the same argument if $\varphi \in C^2(B_{1/10})$ touches $u - p$ from above at $x_0 \in B_{1/10}(\bar{x})$, but with opposite inequalities, and we achieve from (2.35) that $u - p$ solves, in the viscosity sense, the uniformly elliptic equation

$$\sum_{i,j} a_{ij}(x)(u-p)_{ij} = f \quad \text{in } B_{1/10}(\bar{x}). \tag{2.36}$$

In addition from (2.28) and (2.27), we have $u - p \geq 0$ in $B_{1/10}(\bar{x})$. Consequently, in view of this fact, together with (2.36), we can apply the Harnack inequality to obtain

$$\sup_{\overline{B}_{1/20}(\bar{x})} (u-p) \leq C_1 \left(\inf_{\overline{B}_{1/20}(\bar{x})} (u-p) + C_2 \|f\|_{L^\infty} \right)$$

thus, inasmuch $u(\bar{x}) - p(\bar{x}) \leq \sup_{\overline{B}_{1/20}(\bar{x})} (u-p)$ and $\inf_{\overline{B}_{1/20}(\bar{x})} (u-p) \leq u(x) - p(x)$ $\forall x \in \overline{B}_{1/20}(\bar{x})$,

$$u(\bar{x}) - p(\bar{x}) \leq C_1(u(x) - p(x) + C_2 \|f\|_{L^\infty}) \quad \text{in } \overline{B}_{1/20}(\bar{x}),$$

that is, calling $\frac{1}{C_1} = c$ and $C = C_2$

$$u(x) - p(x) \geq c(u(\bar{x}) - p(\bar{x})) - C \|f\|_{L^\infty} \quad \text{in } \overline{B}_{1/20}(\bar{x}). \tag{2.37}$$

Now, from (2.17), we get,

$$u(\bar{x}) \geq (p(\bar{x}) + \varepsilon/2)^+ \geq p(\bar{x}) + \varepsilon/2,$$

i.e.

$$u(\bar{x}) - p(\bar{x}) \geq \varepsilon/2.$$

In view of this fact, together with the first inequality in (2.2), namely $\|f\|_{L^\infty} \leq \varepsilon^2$, we achieve from (2.37)

$$u - p \geq c\frac{\varepsilon}{2} - C\varepsilon^2 = \varepsilon\left(\frac{c}{2} - C\varepsilon\right) \geq c_0\varepsilon \quad \text{in } \overline{B}_{1/20}(\bar{x}),$$

in other words

$$u - p \geq c_0\varepsilon \quad \text{in } \overline{B}_{1/20}(\bar{x}), \tag{2.38}$$

as long as ε is small enough to satisfy $\frac{c}{2} - C\varepsilon > 0$, i.e. $\varepsilon < \frac{c}{2C}$. Now set

$$v(x) = p(x) + c_0\varepsilon(w(x) - 1), \quad x \in \overline{B}_{3/4}(\bar{x}), \tag{2.39}$$

and for $t \geq 0$,

$$v_t(x) = v(x) + t, \quad x \in \overline{B}_{3/4}(\bar{x}). \tag{2.40}$$

Remark that, from (2.39) and (2.40) we have

$$\begin{aligned}
\sum_{i,j} a_{ij}(x)(v_t)_{ij} &= \sum_{i,j} a_{ij}(x)(v(x) + t)_{ij} \\
&= \sum_{i,j} a_{ij}(x)(p(x) + c_0\varepsilon(w(x) - 1) + t)_{ij} \\
&= \sum_{i,j} a_{ij}(x)(x_n + \sigma + c_0\varepsilon(w(x) - 1) + t)_{ij} \\
&= \sum_{i,j} a_{ij}(x)c_0\varepsilon w_{ij} = c_0\varepsilon \sum_{i,j} a_{ij}(x)w_{ij},
\end{aligned}$$

i.e.

$$\sum_{i,j} a_{ij}(x)(v_t)_{ij} = c_0\varepsilon \sum_{i,j} a_{ij}(x)w_{ij}. \tag{2.41}$$

Thus, in view of (2.25), inasmuch $c_0\varepsilon > 0$, we obtain from (2.41)

$$\sum_{i,j} a_{ij}(x)(v_t)_{ij} \geq c_0\delta\varepsilon > \varepsilon^2 \quad \text{in } A,$$

that is

$$\sum_{i,j} a_{ij}(x)(v_t)_{ij} > \varepsilon^2 \quad \text{in } A, \tag{2.42}$$

if we take ε such that $0 < \varepsilon < c_0\delta$.

Now, according to the definition of v_t in (2.40) we have

$$v_0(x) = v(x) = p(x) + c_0\varepsilon(w(x) - 1) \leq p(x) \leq u(x), \quad x \in \overline{B}_{3/4}(\bar{x}),$$

in other words

$$v_0(x) \leq u(x), \quad x \in \overline{B}_{3/4}(\bar{x}),$$

since $\overline{B}_{3/4}(\bar{x}) \subset B_1$ from (2.32), therefore $p(x) \leq u(x)$ in $\overline{B}_{3/4}(\bar{x})$ from (2.19), and $w \leq 1$ in $\overline{B}_{3/4}(\bar{x})$.

Concerning the last condition, indeed, for definition of w, we have

$$w = 1 \quad \text{in } \overline{B}_{1/20}(\bar{x}) \quad \text{and} \quad w = 0 \quad \text{in } \partial B_{3/4}(\bar{x}). \tag{2.43}$$

Moreover, in $B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})$, since $\gamma > 0$,

$$w = \frac{1}{(1/20)^{-\gamma} - (3/4)^{-\gamma}}(|x - \bar{x}|^{-\gamma} - (3/4)^{-\gamma}) \leq \frac{(1/20)^{-\gamma} - (3/4)^{-\gamma}}{(1/20)^{-\gamma} - (3/4)^{-\gamma}} = 1,$$

i.e.

$$w \leq 1 \quad \text{in } B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x}),$$

which implies, together with (2.43), $w \leq 1$ in $\overline{B}_{3/4}(\bar{x})$.

Let now $\bar{t}$ be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}).$$

Notice that $\bar{t}$ exists, since for $t = 0$, $v_0(x) \leq u(x)$ in $\overline{B}_{3/4}(\bar{x})$.

We want to show that $\bar{t} \geq c_0\varepsilon$. Indeed, if this condition is satisfied, we achieve

the desired result.
Precisely, suppose $\bar{t} \geq c_0\varepsilon$. Then, using the definition (2.39) of $v(x)$ we get

$$
\begin{aligned}
u(x) \geq v_{\bar{t}}(x) = v(x) + \bar{t} &= p(x) + c_0\varepsilon(w(x) - 1) + \bar{t} \\
&= p(x) + c_0\varepsilon w(x) - c_0\varepsilon + \bar{t} \geq p(x) + c_0\varepsilon w(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}),
\end{aligned}
$$

in other words

$$
u(x) \geq p(x) + c_0\varepsilon w(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}). \tag{2.44}
$$

Now, we state that on $\overline{B}_{1/2} \subset B_{3/4}(\bar{x})$ one has $w(x) \geq c_2$ for some universal constant c_2.
Sure enough, for definition, $w \in C(\overline{B}_{3/4}(\bar{x}))$ and $w > 0$ in $B_{3/4}(\bar{x})$, thus, inasmuch as $\overline{B}_{1/2} \subset B_{3/4}(\bar{x})$ from (2.32), $w \in C(\overline{B}_{1/2})$ and $w > 0$ on $\overline{B}_{1/2}$. Therefore, for Weierstrass extreme values theorem, since $\overline{B}_{1/2}$ is a compact

$$
w \geq \min_{\overline{B}_{1/2}} w = c_2 > 0 \quad \text{on } \overline{B}_{1/2},
$$

that is $w \geq c_2$ on $\overline{B}_{1/2}$ for some universal constant c_2.
Consequently, we obtain from (2.44)

$$
u(x) \geq p(x) + c_0\varepsilon w(x) \geq c_0\varepsilon c_2 = p(x) + c\varepsilon \quad \text{on } \overline{B}_{1/2},
$$

which gives

$$
u(x) \geq p(x) + c\varepsilon \quad \text{on } \overline{B}_{1/2}. \tag{2.45}
$$

In particular, we notice that we have found c as $c = c_0c_2$, where $0 < c_2 \leq 1$, recalling that $w \leq 1$ in $\overline{B}_{3/4}(\bar{x})$ and thus also in $\overline{B}_{1/2} \subset B_{3/4}(\bar{x}) \subset \overline{B}_{3/4}(\bar{x})$ from (2.32). In addition, we have taken $c_0 = \frac{c}{2} - C\varepsilon$ in (2.38), which satisfies $0 < c_0 < 1$, if $\frac{c}{2} - C\varepsilon < 1$, which gives $C\varepsilon > \frac{c}{2} - 1$ and $\varepsilon > \frac{c}{2C} - \frac{1}{C}$, which is trivially verified if $\frac{c}{2C} - \frac{1}{C} < 0$. Otherwise, we have already chosen ε so that $\varepsilon < \frac{c}{2C}$, therefore, inasmuch as $\frac{c}{2C} - \frac{1}{C} < \frac{c}{2C}$, we choose ε such that $\frac{c}{2C} - \frac{1}{C} < \varepsilon < \frac{c}{2C}$.
To sum it up, we have $0 < c < 1$.
Also, we know that $u \geq 0$ in $B_1 \supset \overline{B}_{1/2}$, since u is a viscosity solution to

(2.1) in B_1.
Hence, from (2.45), we get

$$u(x) \geq \max(p(x) + c\varepsilon, 0) = (p(x) + c\varepsilon)^+ \quad \text{on } \overline{B}_{1/2},$$

in other words

$$u(x) \geq (p(x) + c\varepsilon)^+ \quad \text{on } \overline{B}_{1/2},$$

with $0 < c < 1$, as desired.

Suppose now $\bar{t} < c_0\varepsilon$. Then at some $\tilde{x} \in \overline{B}_{3/4}(\bar{x})$ we have

$$v_{\bar{t}}(\tilde{x}) = u(\tilde{x}).$$

Indeed, if for contradiction $\tilde{x}$ does not exist, we have $u(x) - v_{\bar{t}}(x) > 0$ $\forall x \in \overline{B}_{3/4}(\bar{x})$, seeing as how $v_{\bar{t}}(x) \leq u(x)$ in $\overline{B}_{3/4}(\bar{x})$.
Moreover, because $u \in C(B_1)$ with $B_1 \supset \overline{B}_{3/4}(\bar{x})$ from (2.32), $p \in C^\infty(B_1)$, thus $p \in C(\overline{B}_{3/4}(\bar{x}))$, and $w \in C(\overline{B}_{3/4}(\bar{x}))$, $u - v_{\bar{t}} \in C(\overline{B}_{3/4}(\bar{x}))$, so for Weierstrass extreme values theorem, given that $\overline{B}_{3/4}(\bar{x})$ is a compact, we can define

$$t_* := \min_{\overline{B}_{3/4}(\bar{x})} (u - v_{\bar{t}}), \tag{2.46}$$

which satisfies $t^* > 0$, recalling that $u(x) - v_{\bar{t}}(x) > 0$ $\forall x \in \overline{B}_{3/4}(\bar{x})$.
Now, for the definition of t_* in (2.46), we have

$$t_* \leq u(x) - v_{\bar{t}}(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}),$$

which gives

$$v_{\bar{t}}(x) + t_* = v_{\bar{t}+t_*}(x) \leq u(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}),$$

namely

$$v_{\bar{t}+t_*} \leq u(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}).$$

Therefore, inasmuch $t_* > 0$, we have found $\bar{t} + t_* > \bar{t}$ that realizes

$$v_{\bar{t}+t_*}(x) \leq u(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}),$$

contradicting the definition of $\bar{t}$.
As a consequence, $\tilde{x}$ exists.
We show that such a touching point can only occur on $\overline{B}_{1/20}(\bar{x})$.
Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar{x})$, from the definition (2.40) of v_t we get

$$v_{\bar{t}}(x) = p(x) + c_0\varepsilon(w(x) - 1) + \bar{t} = p(x) - c_0\varepsilon + \bar{t} \quad \text{on } \partial B_{3/4}(\bar{x}),$$

i.e.

$$v_{\bar{t}}(x) = p(x) - c_0\varepsilon + \bar{t} \quad \text{on } \partial B_{3/4}(\bar{x}). \tag{2.47}$$

Using that $\bar{t} < c_0\varepsilon$ together with the fact that $u \geq p$ in B_1 and thus also on $\partial B_{3/4}(\bar{x})$, because $\partial B_{3/4}(\bar{x}) \subset B_1$ from (2.32), we then obtain from (2.47)

$$v_{\bar{t}}(x) = p(x) - c_0\varepsilon + \bar{t} < p(x) \leq u(x) \quad \text{on } \partial B_{3/4}(\bar{x}),$$

namely

$$v_{\bar{t}} < u \quad \text{on } \partial B_{3/4}(\bar{x})$$

and hence $\tilde{x}$ cannot belong to $\partial B_{3/4}(\bar{x})$.

We now show that $\tilde{x}$ cannot belong to the annulus A.
First of all, in view of (2.42), we have for each $t \geq 0$ and thus also for $\bar{t}$,

$$\sum_{i,j} a_{ij}(x)(v_{\bar{t}})_{ij} > \varepsilon^2 \quad \text{in } A$$

and moreover

$$\begin{aligned} |\nabla v_{\bar{t}}| &= |\nabla(v + \bar{t})| = |\nabla v| \geq |v_n| \\ &= \left| \frac{\partial}{\partial x_n} \Big(p(x) + c_0\varepsilon(w(x) - 1) \Big) \right| \\ &= \left| \frac{\partial}{\partial x_n} \Big(x_n + \sigma + c_0\varepsilon(w(x) - 1) \Big) \right| \\ &= |1 + c_0\varepsilon w_n| \quad \text{in } A, \end{aligned}$$

i.e.

$$|\nabla v_{\bar{t}}| \geq |1 + c_0\varepsilon w_n| \quad \text{in } A. \tag{2.48}$$

At this point, we claim that

$$w_n(x) \geq c_1 \quad \text{on } \{v_{\bar{t}} \leq 0\} \cap A,$$

for a universal constant c_1.
Precisely, since w is radially symmetric, keeping c in the expression of w for the sake of simplicity,

$$\begin{aligned} w_n(x) &= \frac{\partial}{\partial x_n}\Big(c(|x-\bar{x}|^{-\gamma} - (3/4)^{-\gamma})\Big) \\ &= -c\gamma\,|x-\bar{x}|^{-\gamma-1}\,\frac{x_n-\bar{x}_n}{|x-\bar{x}|}, \end{aligned}$$

which gives

$$w_n(x) = -c\gamma\,|x-\bar{x}|^{-\gamma-1}\,\frac{x_n-\bar{x}_n}{|x-\bar{x}|}, \tag{2.49}$$

and furthermore,

$$\begin{aligned} \nabla w(x) &= \Big(-c\gamma\,|x-\bar{x}|^{-\gamma-1}\,\frac{x_1-\bar{x}_1}{|x-\bar{x}|}, \dots, -c\gamma\,|x-\bar{x}|^{-\gamma-1}\,\frac{x_n-\bar{x}_n}{|x-\bar{x}|}\Big) \\ &= -c\gamma\,|x-\bar{x}|^{-\gamma-1}\,\frac{x-\bar{x}}{|x-\bar{x}|}, \end{aligned}$$

namely

$$\nabla w(x) = -c\gamma\,|x-\bar{x}|^{-\gamma-1}\,\frac{x-\bar{x}}{|x-\bar{x}|}. \tag{2.50}$$

As a consequence, from (2.50), we achieve, because $c, \gamma > 0$,

$$|\nabla w(x)| = c\gamma\,|x-\bar{x}|^{-\gamma-1}\left|\frac{x-\bar{x}}{|x-\bar{x}|}\right| = c\gamma\,|x-\bar{x}|^{-\gamma-1},$$

which entails from (2.49) with $x \in A$, recalling that w is defined in $\overline{B}_{3/4}(\bar{x}) \supset A$,

$$w_n(x) = |\nabla w(x)|\,\nu_x \cdot e_n, \quad x \in A, \tag{2.51}$$

where ν_x is the unit direction of $\bar{x} - x$.
Also, from the formula for w in (2.20), we get $|\nabla w| > c$ on A for a constant c.
Indeed, since $|x-\bar{x}| < 3/4$ in A and $\gamma > 0$

$$|\nabla w(x)| = \frac{\gamma\,|x-\bar{x}|^{-\gamma-1}}{(1/20)^{-\gamma} - (3/4)^{-\gamma}} > \frac{\gamma(3/4)^{-\gamma-1}}{(1/20)^{-\gamma} - (3/4)^{-\gamma}} = c \quad \text{on } A,$$

namely

$$|\nabla w(x)| > c \quad \text{on } A. \tag{2.52}$$

In addition, $\nu_x \cdot e_n$ is bounded below in the region $\{v_{\bar{t}} \leq 0\} \cap A$.
Precisely, we declare that for ε small enough,

$$\{v_{\bar{t}} \leq 0\} \cap A \subset \{p \leq c_0\varepsilon\} = \{x_n \leq -\sigma + c_0\varepsilon\} \subset \{x_n < 3/20\}.$$

Indeed, on $\{v_{\bar{t}} \leq 0\} \cap A$, we have

$$v_{\bar{t}} \leq 0 \Leftrightarrow p(x) + c_0\varepsilon(w(x) - 1) + \bar{t} \leq 0 \Leftrightarrow p(x) \leq c_0\varepsilon(1 - w(x)) - \bar{t},$$

as a consequence, seeing as how $\bar{t} \geq 0$, thus $-\bar{t} \leq 0$ and $0 \leq w(x) \leq 1$ in A, so $1 - w(x) \leq 1$, we obtain

$$p(x) \leq c_0\varepsilon \quad \text{on } \{v_{\bar{t}} \leq 0\} \cap A,$$

namely

$$\{v_{\bar{t}} \leq 0\} \cap A \subset \{p \leq c_0\varepsilon\}. \tag{2.53}$$

Now, recalling that $p(x) = x_n + \sigma$

$$\{p \leq c_0\varepsilon\} = \{x_n + \sigma \leq c_0\varepsilon\} = \{x_n \leq -\sigma + c_0\varepsilon\},$$

which gives

$$\{p \leq c_0\varepsilon\} = \{x_n \leq -\sigma + c_0\varepsilon\}. \tag{2.54}$$

Furthermore, given that $|\sigma| < 1/10$, so $\sigma > -1/10$ and $-\sigma < 1/10$, for ε small enough such that $c_0\varepsilon < 1/20$, i.e. $\varepsilon < \frac{1/20}{c_0}$,

$$\{x_n \leq -\sigma + c_0\varepsilon\} \subset \{x_n < 1/10 + 1/20\} = \{x_n < 3/20\},$$

in other words

$$\{x_n \leq -\sigma + c_0\varepsilon\} \subset \{x_n < 3/20\},$$

which implies from (2.53) and (2.54)

$$\{v_{\bar{t}} \leq 0\} \cap A \subset \{x_n < 3/20\}. \tag{2.55}$$

To show that $\nu_x \cdot e_n$ is bounded below in the region $\{v_{\bar{t}} \leq 0\} \cap A$, we remember that $\bar{x} = \frac{1}{5} e_n$, in other words $\bar{x}_n = \frac{1}{5}$, hence, in view of (2.55) and because $|x - \bar{x}| < 3/4$ in A,

$$\nu_x \cdot e_n = \frac{\bar{x}_n - x_n}{|\bar{x} - x|} = \frac{\frac{1}{5} - x_n}{|\bar{x} - x|} > \frac{\frac{1}{5} - \frac{3}{20}}{\frac{3}{4}} = \frac{\frac{1}{20}}{\frac{3}{4}} = \frac{1}{15} \quad \text{on } \{v_{\bar{t}} \leq 0\} \cap A,$$

i.e.

$$\nu_x \cdot e_n \geq \frac{1}{15} \quad \text{on } \{v_{\bar{t}} \leq 0\} \cap A. \tag{2.56}$$

Hence, from (2.51), for (2.52) and (2.56), we achieve

$$w_n(x) \geq \frac{1}{15} c = c_1 > 0 \quad \text{on } \{v_{\bar{t}} \leq 0\} \cap A,$$

namely

$$w_n(x) \geq c_1 > 0 \quad \text{on } \{v_{\bar{t}} \leq 0\} \cap A. \tag{2.57}$$

Consequently, in view of (2.57), we deduce from (2.48) that

$$\begin{aligned} |\nabla v_{\bar{t}}| &\geq |1 + c_0 \varepsilon w_n| = 1 + c_0 \varepsilon w_n \\ &\geq 1 + c_0 \varepsilon c_1 = 1 + c_2 \varepsilon \qquad \text{on } \{v_{\bar{t}} \leq 0\} \cap A, \end{aligned}$$

that is

$$|\nabla v_{\bar{t}}| \geq 1 + c_2 \varepsilon \quad \text{on } \{v_{\bar{t}} \leq 0\} \cap A, \tag{2.58}$$

given that, if $w_n(x) \geq c_1 > 0$ on $\{v_{\bar{t}} \leq 0\} \cap A$, $c_0 \varepsilon w_n > 0$ on $\{v_{\bar{t}} \leq 0\} \cap A$ and thus $|1 + c_0 \varepsilon w_n| = 1 + c_0 \varepsilon w_n$ on $\{v_{\bar{t}} \leq 0\} \cap A$.
In particular, for ε small enough such that $c_2 \varepsilon > \varepsilon^2$, i.e. $\varepsilon < c_2$, we get from (2.58)

$$|\nabla v_{\bar{t}}| (x) > 1 + \varepsilon^2 \geq g(x) \quad \text{for } x \in A \cap \{v_{\bar{t}} \leq 0\},$$

in other words

$$|\nabla v_{\bar{t}}| > g(x) \quad \text{for } x \in A \cap \{v_{\bar{t}} \leq 0\}, \tag{2.59}$$

inasmuch in view of the second inequality in (2.2) $\|g - 1\|_{L^\infty(B_1)} \leq \varepsilon^2$, thus $|g(x) - 1| \leq \varepsilon^2$, $\forall x \in B_1 \supset A$, which gives $|g(x) - 1| \leq \varepsilon^2$ $\forall x \in A$ and $g(x) - 1 \leq \varepsilon^2$ $\forall x \in A$, which also entails $g(x) \leq 1 + \varepsilon^2$ $\forall x \in A$and $g(x) \leq 1 + \varepsilon^2$

$\forall x \in A \cap \{v_{\bar{t}} \leq 0\}$, given that $A \cap \{v_{\bar{t}} \leq 0\} \subset A$.
In addition, from (2.59) we also obtain

$$|\nabla v_{\bar{t}}|\,(x) > g(x) \quad \text{for } x \in A \cap F(v_{\bar{t}}) \tag{2.60}$$

seeing as how $F(v_{\bar{t}}) \cap A \subset \{v_{\bar{t}} = 0\} \cap A \subset \{v_{\bar{t}} \leq 0\} \cap A$.
At this point, we have

$$\sum_{i,j} a_{ij}(x)(v_{\bar{t}})_{ij} > \varepsilon^2 \geq f(x) \quad \text{in } A \supset A^+(v_{\bar{t}}),$$

i.e.

$$\sum_{i,j} a_{ij}(x)(v_{\bar{t}})_{ij} > f(x) \quad \text{in } A^+(v_{\bar{t}}), \tag{2.61}$$

and from (2.60)

$$|\nabla v_{\bar{t}}| > g(x) \quad \text{for } x \in A \cap F(v_{\bar{t}}). \tag{2.62}$$

Furthermore, $v_{\bar{t}} \in C^2(A)$, given that $p \in C^\infty(B_1)$, with $B_1 \supset A$ and $w \in C^\infty(A)$.
Therefore, from (2.61) and (2.62), together with the fact that $v_{\bar{t}} \in C^2(A)$, we get that $v_{\bar{t}}$ is a strict subsolution to (2.1) in A.
Moreover, for the definition of $v_{\bar{t}}$, we have $v_{\bar{t}} \leq u$ in $\overline{B}_{3/4}(\bar{x}) \supset A$, which gives $v_{\bar{t}} \leq u$ in A. In addition, $u \geq 0$ in $B_1 \supset \overline{B}_{3/4}(\bar{x}) \supset A$, so $u \geq 0$ in A, thus $u \geq \max(v_{\bar{t}}, 0) = v_{\bar{t}}^+$ in A, in other words $u \geq v_{\bar{t}}^+$ in A.
To sum it up, we have that $v_{\bar{t}}$ is a strict subsolution to (2.1) in A, u solves (2.1) in A and $u \geq v_{\bar{t}}^+$ in A.
Hence, according to Lemma 1.5, $u > v_{\bar{t}}^+ \geq v_{\bar{t}}$ in $A^+(v_{\bar{t}}) \cup (A \cap F(v_{\bar{t}}))$, that is $u > v_{\bar{t}}$ in $A^+(v_{\bar{t}}) \cup (A \cap F(v_{\bar{t}}))$ and so

$$\tilde{x} \notin A^+(v_{\bar{t}}) \cup (A \cap F(v_{\bar{t}})). \tag{2.63}$$

Consequently, if $\tilde{x} \in A$, it means that $\tilde{x} \in A \setminus (A^+(v_{\bar{t}}) \cup (A \cap F(v_{\bar{t}})))$, which entails $v_{\bar{t}}(\tilde{x}) \leq 0$ and inasmuch $u \geq 0$ in $B_1 \supset A$, the only possibility is that $v_{\bar{t}}(\tilde{x}) = u(\tilde{x}) = 0$, with $\tilde{x} \notin F(v_{\bar{t}})$.
Let us show that also this situation is not possible.
Indeed, for definition,

$$v_{\bar{t}}(x) = p(x) + c_0\varepsilon(w(x) - 1) + \bar{t} = x_n + \sigma + c_0\varepsilon(w(x) - 1) + \bar{t}, \quad x \in \overline{B}_{3/4}(\bar{x}),$$

thus if we fix a value of x_n, $\bar{x}_n$, and we consider $x = (x', \bar{x}_n)$, we have

$$v_{\bar{t}}(x', \bar{x}_n) = \bar{x}_n + \sigma + c_0\varepsilon(w(x', \bar{x}_n) - 1) + \bar{t},$$

and, from the formula of w in (2.20), $v_{\bar{t}}(x', \bar{x}_n)$ can vanish in A for only one value of $|x'|$, which we call ρ.
In addition, w is strictly decreasing and continuous in A, hence also $v_{\bar{t}}(x', \bar{x}_n)$, which thus change its sign in a neighborhood of points $(x', \bar{x}_n)$ with $|x'| = \rho$. As a consequence, for these points, $\forall B_r(x', \bar{x}_n)$, $B_r(x', \bar{x}_n) \cap \{v_{\bar{t}} > 0\} \neq \emptyset$ and $B_r(x', \bar{x}_n) \cap \{v_{\bar{t}} \leq 0\} \neq \emptyset$, also only from $v_{\bar{t}}(x', \bar{x}_n) = 0$.
Therefore, $(x', \bar{x}_n) \in F(v_{\bar{t}})$.
From the arbitrariness of $\bar{x}_n$, we hence achieve that $v_{\bar{t}}$ only vanishes in A in points which also belong to $F(v_{\bar{t}})$, consequently it cannot occur $u(\tilde{x}) = v_{\bar{t}}(\tilde{x}) = 0$ with $\tilde{x} \in A$ and $\tilde{x} \notin F(v_{\bar{t}})$. Thus

$$\tilde{x} \notin A \setminus (A^+(v_{\bar{t}}) \cup (A \cap F(v_{\bar{t}}))). \tag{2.64}$$

Now, putting together (2.63) and (2.64), we get that $\tilde{x}$ cannot belong to A.

As a consequence, $\tilde{x} \in \overline{B}_{3/4}(\bar{x}) \setminus (A \cup \partial B_{3/4}(\bar{x})) = \overline{B}_{1/20}(\bar{x})$ and, given that $w \equiv 1$ in $\overline{B}_{1/20}(\bar{x})$ and we have supposed $\bar{t} < c_0\varepsilon$,

$$u(\tilde{x}) = v_{\bar{t}}(\tilde{x}) = p(\tilde{x}) + c_0\varepsilon(w(\tilde{x}) - 1) + \bar{t} = p(\tilde{x}) + \bar{t} < p(\tilde{x}) + c_0\varepsilon,$$

which implies

$$u(\tilde{x}) - p(\tilde{x}) < c_0\varepsilon,$$

contradicting (2.38), seeing as how $\tilde{x} \in \overline{B}_{1/20}(\bar{x})$. □

We are now ready to give the proof of the Harnack inequality.

Proof of Theorem 2.1. Assume without loss of generality

$$x_0 = 0, \quad r = 1.$$

According to (2.3),

$$p(x)^+ \leq u(x) \leq (p(x) + \varepsilon)^+ \quad \text{in } B_1$$

with $p(x) = x_n + a_0$.
Sure enough, from the statement of Theorem 2.1, we have with $x_0 = 0$ and $r = 1$

$$(x_n + a_0)^+ \le u(x) \le (x_n + b_0)^+ \quad \text{in } B_1, \tag{2.65}$$

together with

$$b_0 - a_0 \le \varepsilon$$

and

$$b_0 \le a_0 + \varepsilon.$$

Hence,

$$x_n + b_0 \le x_n + a_0 + \varepsilon \quad \text{in } B_1,$$

which implies

$$(x_n + b_0)^+ \le (x_n + a_0 + \varepsilon)^+ \quad \text{in } B_1,$$

and according to (2.65)

$$(x_n + a_0)^+ \le u(x) \le (x_n + a_0 + \varepsilon)^+ \quad \text{in } B_1,$$

namely

$$p(x)^+ \le u(x) \le (p(x) + \varepsilon)^+ \quad \text{in } B_1, \tag{2.66}$$

with $p(x) = x_n + a_0$.
Now, if $|a_0| < 1/10$, since u solves (2.1)-(2.2) in $\Omega \supset B_1$ and u satisfies

$$p(x)^+ \le u(x) \le (p(x) + \varepsilon)^+, \quad x \in B_1,\ p(x) = x_n + a_0,\ |a_0| < 1/10,$$

then we can apply Lemma 2.3, and we achieve the desired result.
Indeed, for Lemma 2.3, if in $\bar{x} = \frac{1}{5}e_n$,

$$u(\bar{x}) \ge (p(\bar{x}) + \varepsilon/2)^+,$$

then

$$u \ge (p + c\varepsilon)^+ \quad \text{in } \overline{B}_{1/2} \tag{2.67}$$

for

$$0 < c < 1 \text{ universal.} \tag{2.68}$$

Therefore, given that $\overline{B}_{1/2} \subset B_1$, we have from (2.66) and (2.67)

$$(p(x) + c\varepsilon)^+ \leq u(x) \leq (p(x) + \varepsilon)^+ \quad \text{in } \overline{B}_{1/2} \supset B_{1/20},$$

but according to (2.3), it is also satisfied

$$(p(x) + c\varepsilon)^+ = (x_n + a_0 + c\varepsilon)^+ \leq u(x) \leq (x_n + b_0)^+ \quad \text{in } B_{1/20},$$

with $b_0 - a_0 - c\varepsilon \leq \varepsilon - c\varepsilon = (1 - c)\varepsilon$.
Thus, if there exists b_1, with $a_0 + c\varepsilon < b_1 < b_0$, such that

$$u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{1/20},$$

we can take $a_1 = a_0 + c\varepsilon$, with $a_1 > a_0$, thereby we get

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{1/20}$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0,$$

and

$$b_1 - a_1 = b_1 - a_0 - c\varepsilon \leq b_0 - a_0 - c\varepsilon \leq \varepsilon - c\varepsilon = (1 - c)\varepsilon,$$

with $0 < c < 1$ universal from (2.68), as desired.
Otherwise, we can take $b_1 = b_0$ and $a_1 = a_0 + c\varepsilon$ and we obtain

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{1/20}$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0$$

and

$$b_1 - a_1 = b_0 - a_0 - c\varepsilon \leq \varepsilon - c\varepsilon = (1 - c)\varepsilon,$$

with $0 < c < 1$ universal from (2.68), as desired. Instead, if in $\bar{x} = \frac{1}{5}e_n$,

$$u(\bar{x}) \leq (p(\bar{x}) + \varepsilon/2)^+,$$

then

$$u \leq (p + (1 - c)\varepsilon)^+ \quad \text{in } \overline{B}_{1/2}$$

for

$$0 < c < 1 \text{ universal.} \tag{2.69}$$

Therefore, from (2.66)

$$p(x)^+ \le u(x) \le (p(x) + (1-c)\varepsilon)^+ \quad \text{in } \overline{B}_{1/2} \supset B_{1/20},$$

but according to (2.3), we also have

$$p(x)^+ \le u(x) \le (x_n + b_0)^+ \quad \text{in } B_{1/20}.$$

Now, we have two different situations.

(i) When $b_0 \le a_0 + (1-c)\varepsilon$, if there exists $a_0 \le b_1 < b_0 \le a_0 + (1-c)\varepsilon$ such that

$$(x_n + a_0)^+ \le u(x) \le (x_n + b_1)^+ \quad \text{in } B_{1/20},$$

and furthermore, if there exists a_1, with $a_0 < a_1 \le b_1$ such that

$$(x_n + a_1)^+ \le u(x) \le (x_n + b_1)^+ \quad \text{in } B_{1/20},$$

we get the desired result with

$$a_0 \le a_1 \le b_1 \le b_0$$

and

$$b_1 - a_1 \le b_1 - a_0 \le a_0 + (1-c)\varepsilon - a_0 = (1-c)\varepsilon.$$

Otherwise if such a_1 does not exist, we can take $a_1 = a_0$ and we achieve

$$(x_n + a_1)^+ \le u(x) \le (x_n + b_1)^+ \quad \text{in } B_{1/20},$$

with

$$a_0 \le a_1 \le b_1 \le b_0$$

and

$$b_1 - a_1 \le b_1 - a_0 \le b_0 - a_0 \le a_0 + (1-c)\varepsilon - a_0 = (1-c)\varepsilon,$$

with $0 < c < 1$ universal from (2.69), as desired.

If instead there does not exist b_1 as before, we can take $b_1 = b_0 \leq a_0 + (1-c)\varepsilon$ and, exactly how when b_1 exists, we can also take a_1, with $a_0 \leq a_1 \leq b_0$, such that

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{1/20},$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0$$

and

$$b_1 - a_1 \leq b_1 - a_0 \leq b_0 - a_0 \leq a_0 + (1-c)\varepsilon - a_0 = (1-c)\varepsilon,$$

with $0 < c < 1$ universal from (2.69), as desired.

(ii) When $b_0 > a_0 + (1-c)\varepsilon$, for every b_1, with $a_0 + (1-c)\varepsilon \leq b_1 \leq b_0$, we have

$$u(x) \leq (x_n + a_0 + (1-c)\varepsilon)^+ \leq (x_n + b_1)^+ \leq (x_n + b_0)^+ \quad \text{in } B_{1/20},$$

thus, if there exists a_1, with $b_1 - (1-c)\varepsilon \leq a_1 \leq a_0 + (1-c)\varepsilon \leq b_1$, such that

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{1/20},$$

we get the desired result with

$$a_1 \geq b_1 - (1-c)\varepsilon \geq a_0 + (1-c)\varepsilon - (1-c)\varepsilon = a_0,$$

so

$$a_0 \leq a_1 \leq b_1 \leq b_0$$

and

$$b_1 - a_1 \leq b_1 - b_1 + (1-c)\varepsilon = (1-c)\varepsilon.$$

Otherwise, if such a_1 does not exist, we can take $b_1 = a_0 + (1-c)\varepsilon < b_0$, $a_1 = a_0$ and we obtain

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{1/20},$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0$$

and

$$b_1 - a_1 \leq b_1 - a_0 \leq a_0 + (1-c)\varepsilon - a_0 = (1-c)\varepsilon,$$

with $0 < c < 1$ universal from (2.69), as desired.

Now, suppose instead that $|a_0| \geq 1/10$.
If $a_0 \leq -1/10$, then (for ε small) 0 belongs to the zero phase of $(p(x)+\varepsilon)^+$.
Indeed, $p(0)+\varepsilon = a_0 + \varepsilon$, hence if $0 < \varepsilon < -a_0$, with ε small, we have

$$p(0) + \varepsilon = a_0 + \varepsilon < a_0 - a_0 = 0,$$

that is

$$(p(x)+\varepsilon)^+(0) = 0$$

and furthermore, we can find a ball B_r, with $r < \min(-a_0 - \varepsilon, 1)$ (notice that $-a_0 - \varepsilon > 0$ for the choice of ε), such that if $x \in B_r$,

$$p(x) + \varepsilon = x_n + a_0 + \varepsilon < r + a_0 + \varepsilon < -a_0 - \varepsilon + a_0 + \varepsilon = 0,$$

given that $x_n \leq |x_n| \leq |x| < r$, i.e. $x_n < r$, and $r < \min(-a_0 - \varepsilon, 1)$.
Therefore, $p(x) + \varepsilon < 0$ in B_r, so $(p(x)+\varepsilon)^+ = 0$ in B_r, which implies that 0 belongs to the zero phase of $(p(x)+\varepsilon)^+$.
Also, we have from (2.66)

$$0 \leq p(x)^+ \leq u(x) \leq (p(x)+\varepsilon)^+ = 0 \quad \text{in } B_r \subset B_1,$$

namely $u \equiv 0$ in B_r.
Hence, if $u \equiv 0$ in B_r, seeing as how $0 \in B_r$, $u(0) = 0$, i.e. $0 \notin \Omega^+(u)$.
In addition, if $u \equiv 0$ in B_r, $B_r \cap \Omega^+(u) = \emptyset$, thereby $0 \notin \partial\Omega^+(u) \supset \partial\Omega^+(u) \cap \Omega = F(u)$, that is $0 \notin F(u)$.
Considering these two facts together, we achieve that $0 \notin \Omega^+(u) \cup F(u)$, which contradicts the hypothesis $0 \in \Omega^+(u) \cup F(u)$.
If instead $a_0 \geq 1/10$, then $B_{1/10} \subset B_1^+(u)$.

Precisely, $B_{1/10} \subset B_1$ and moreover if $x \in B_{1/10}$, $|x_n| \leq \|x\| < 1/10$, i.e. $|x_n| < 1/10$ and $x_n > -1/10$, hence

$$p(x) = x_n + a_0 > -1/10 + a_0 \geq -1/10 + 1/10 = 0 \quad \text{in } B_{1/10},$$

that is $p(x) > 0$ in $B_{1/10}$, which entails $p(x)^+ = p(x)$ in $B_{1/10}$ and as a consequence $p(x)^+ > 0$ in $B_{1/10}$.
Therefore, in view of (2.66)

$$0 < p(x)^+ \leq u(x) \quad \text{in } B_{1/10},$$

thus $u > 0$ in $B_{1/10} \subset B_1$, namely

$$B_{1/10} \subset B_1^+(u). \tag{2.70}$$

We now distinguish two cases, if $u(0) - p(0) \geq \varepsilon/2$ or if $u(0) - p(0) < \varepsilon/2$. Let us analyze the two cases separately.

(i) First, we suppose $u(0) - p(0) \geq \varepsilon/2$.
Now, from (2.66), since $p \leq p^+$, we have $u - p \geq 0$ in $B_1 \supset B_{1/10}$, i.e. $u - p \geq 0$ in $B_{1/10}$,. Furthermore, u solves, in the viscosity sense, a uniformly elliptic equation in $\Omega^+(u)$, thus also in $B_{1/10}$, recalling that $\Omega \supset B_1$ by hypothesis and $B_1^+(u) \supset B_{1/10}$, hence $\Omega^+(u) \supset B_1^+(u) \supset B_{1/10}$.
Consequently, repeating the same argument used in the proof of Lemma 2.3 to achieve (2.36), $u - p$ solves, in the viscosity sense, a uniformly elliptic equation in $B_{1/10}$ with right hand side f.
Therefore, in view of this fact, together with $u - p \geq 0$ in $B_{1/10}$, we can apply the classical Harnack inequality to obtain

$$\sup_{\overline{B}_{1/20}} (u - p) \leq C_1 \left(\inf_{\overline{B}_{1/20}} (u - p) + C_2 \|f\|_{L^\infty} \right). \tag{2.71}$$

In particular, from (2.71), repeating the same calculations done in the proof of Lemma 2.3 to get (2.37), we achieve

$$u(x) - p(x) \geq c(u(0) - p(0)) - C \|f\|_{L^\infty} \quad \text{in } \overline{B}_{1/20},$$

which implies,

$$u(x) - p(x) \geq c\frac{\varepsilon}{2} - C\varepsilon^2 \quad \text{in } \overline{B}_{1/20}, \tag{2.72}$$

inasmuch $u(0) - p(0) \geq \frac{\varepsilon}{2}$, and in view of the first inequality in (2.2), in other words $\|f\|_{L^\infty} \leq \varepsilon^2$, which also gives $-\|f\|_{L^\infty} \geq -\varepsilon^2$.
In addition, we can rewrite (2.72) as

$$u(x) - p(x) \geq \varepsilon\left(\frac{c}{2} - C\varepsilon\right) = c_0\varepsilon \quad \text{in } \overline{B}_{1/20},$$

i.e.

$$u(x) - p(x) \geq c_0\varepsilon \quad \text{in } \overline{B}_{1/20}, \tag{2.73}$$

where we want to choose c_0 so that $0 < c_0 < 1$, and it is possible if we choose ε such that

$$0 < \frac{c}{2} - C\varepsilon < 1 \leftrightarrow \frac{c}{2} - 1 < C\varepsilon < \frac{c}{2} \leftrightarrow \frac{c}{2C} - \frac{1}{C} < \varepsilon < \frac{c}{2C},$$

namely, seeing as how $\varepsilon > 0$,

$$\max\left(0, \frac{c}{2C} - \frac{1}{C}\right) = \left(\frac{c}{2C} - \frac{1}{C}\right)^+ < \varepsilon < \frac{c}{2C}$$

and hence

$$\left(\frac{c}{2C} - \frac{1}{C}\right)^+ < \varepsilon < \frac{c}{2C}. \tag{2.74}$$

Now, from ((ii)), we have, calling $c = c_0$,

$$u(x) \geq p(x) + c\varepsilon \quad \text{in } \overline{B}_{1/20},$$

with $0 < c < 1$, which entails, recalling that u is a viscosity solution to (2.1) in Ω, and therefore $u \geq 0$ in $\Omega \supset B_1 \supset \overline{B}_{1/20}$, in other words $u \geq 0$ in $\overline{B}_{1/20}$,

$$u(x) \geq \max(p(x) + c\varepsilon, 0) = (p(x) + c\varepsilon)^+ \quad \text{in } \overline{B}_{1/20},$$

i.e.

$$u(x) \geq (p(x) + c\varepsilon)^+ \quad \text{in } \overline{B}_{1/20}$$

and in particular

$$u(x) \geq (p(x) + c\varepsilon)^+ \quad \text{in } B_{1/20},$$

with $0 < c < 1$ universal.

The precise conclusion of Theorem 2.1 follows from the same argument used in case of $|a_0| < 1/10$, after we have applied Lemma 2.3 with the hypothesis $u(\bar{x}) \geq (p(\bar{x}) + \varepsilon/2)^+$ satisfied.

(ii) Suppose instead that $u(0) - p(0) < \varepsilon/2$. In particular, inasmuch as $B_{1/10} \subset B_1^+(u)$ from (2.70), we have from (2.66)

$$0 < u(x) \leq (p(x) + \varepsilon)^+ \quad \text{in } B_{1/10}, \tag{2.75}$$

which gives $(p(x) + \varepsilon)^+ > 0$ in $B_{1/10}$, and thus $(p(x) + \varepsilon)^+ = p(x) + \varepsilon$. As a consequence, from (2.75), we also obtain

$$0 < u(x) \leq p(x) + \varepsilon \quad \text{in } B_{1/10}$$

and

$$p(x) + \varepsilon - u(x) \geq 0 \quad \text{in } B_{1/10}. \tag{2.76}$$

Furthermore, we claim that $p + \varepsilon - u$ solves, in the viscosity sense, a uniformly elliptic equation in $B_{1/10}$.

Indeed, if $\varphi \in C^2(B_{1/10})$ touches $p + \varepsilon - u$ from below at $x_0 \in B_{1/10}$, we have

$$\varphi(x_0) = (p + \varepsilon - u)(x_0) = p(x_0) + \varepsilon - u(x_0) \tag{2.77}$$

and

$$\varphi(x) \leq (p + \varepsilon - u)(x) = p(x) + \varepsilon - u(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{2.78}$$

In particular, (2.77) and (2.78) read

$$u(x_0) = p(x_0) + \varepsilon - \varphi(x_0) \tag{2.79}$$

and

$$u(x) \leq p(x) + \varepsilon - \varphi(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{2.80}$$

Therefore, from (2.79) and (2.80), we get that $p+\varepsilon-\varphi$ touches u from above at $x_0 \in B_{1/10}$, since $(p+\varepsilon-\varphi)(x) = p(x)+\varepsilon-\varphi(x)$.
In addition, given that $p(x) = x_n + a_0 \in C^\infty(B_1)$, with $B_1 \supset B_{1/10}$, $(p+\varepsilon-\varphi) \in C^2(B_{1/10})$.
To sum it up, we have $(p+\varepsilon-\varphi) \in C^2(B_{1/10})$ touching u from above at $x_0 \in B_{1/10}$, with in particular $x_0 \in B_1^+(u) \subset \Omega^+(u)$, recalling that $B_{1/10} \subset B_1^+(u)$ from (2.70) and $B_1 \subset \Omega$.
Consequently, because u is a viscosity solution to (2.1) in Ω, we achieve

$$\begin{aligned}\sum_{i,j} a_{ij}(x_0)(p+\varepsilon-\varphi)_{ij}(x_0) &= \sum_{i,j} a_{ij}(x_0)(x_n+a_0+\varepsilon-\varphi)_{ij}(x_0)\\ &= \sum_{i,j} a_{ij}(x_0)(-\varphi)_{ij}(x_0)\\ &= \sum_{i,j} a_{ij}(x_0)(-\varphi_{ij}(x_0))\\ &= -\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \geq f(x_0),\end{aligned}$$

in other words

$$-\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \geq f(x_0),$$

and

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \leq -f(x_0). \tag{2.81}$$

Repeating the same argument if $\varphi \in C^2(B_{1/10})$ touches $p+\varepsilon-u$ from above at $x_0 \in B_{1/10}$, but with opposite inequalities, we obtain that $p+\varepsilon-u$ solves, in the viscosity sense, the uniformly elliptic equation

$$\sum_{i,j} a_{ij}(x)(p+\varepsilon-u)_{ij} = -f \quad \text{in } B_{1/10}.$$

In view of this fact, together with (2.76), we can apply the Harnack inequality to get

$$\sup_{\overline{B}_{1/20}} (p+\varepsilon-u) \leq C_1\left(\inf_{\overline{B}_{1/20}} (p+\varepsilon-u) + C_2 \left\|-f\right\|_{L^\infty}\right),$$

and repeating the same computations done in the proof of Lemma 2.3 to achieve (2.37), but with $p+\varepsilon-u$ in place of $u-p$, we obtain

$$p(x)+\varepsilon-u(x) \geq c(p(0)+\varepsilon-u(0)) - C\left\|-f\right\|_{L^\infty} \quad \text{in } \overline{B}_{1/20}. \quad (2.82)$$

At this point, we know that $u(0)-p(0) < \frac{\varepsilon}{2}$, which also gives $p(0)-u(0) > -\frac{\varepsilon}{2}$, hence

$$p(0)+\varepsilon-u(0) = p(0)-u(0)+\varepsilon > -\frac{\varepsilon}{2}+\varepsilon = \frac{\varepsilon}{2},$$

namely

$$p(0)+\varepsilon-u(0) > \frac{\varepsilon}{2},$$

which entails, from (2.82),

$$p(x)+\varepsilon-u(x) \geq c\frac{\varepsilon}{2} - C\varepsilon^2 \quad \text{in } \overline{B}_{1/20}, \quad (2.83)$$

inasmuch from the first inequality in (2.2), $\left\|-f\right\|_{L^\infty} = \left\|f\right\|_{L^\infty} \leq \varepsilon^2$, i.e. $\left\|-f\right\|_{L^\infty} \leq \varepsilon^2$ and $-\left\|-f\right\|_{L^\infty} \geq -\varepsilon^2$.
Now, repeating the same argument used in case of $u(0)-p(0) \geq \frac{\varepsilon}{2}$ to achieve (), we obtain from (2.83)

$$p(x)+\varepsilon-u(x) \geq c_0\varepsilon \quad \text{in } \overline{B}_{1/20}, \quad (2.84)$$

with $c_0 = \frac{c}{2} - C\varepsilon$ and ε as in (2.74), in order to have $0 < c_0 < 1$.
In particular, calling $c = c_0$, we can rewrite (2.84) as

$$p(x)+\varepsilon-u(x) \geq c\varepsilon \quad \text{in } \overline{B}_{1/20},$$

with $0 < c < 1$, which implies

$$p(x)+\varepsilon-c\varepsilon \geq u(x) \quad \text{in } \overline{B}_{1/20},$$

with $0 < c < 1$, in other words

$$p(x)+(1-c)\varepsilon \geq u(x) \quad \text{in } \overline{B}_{1/20},$$

with $0 < c < 1$ and in particular

$$p(x)+(1-c)\varepsilon \geq u(x) \quad \text{in } B_{1/20}. \quad (2.85)$$

Moreover, from (2.70) $u > 0$ in $B_{1/10} \supset B_{1/20}$ and thus also $u > 0$ in $B_{1/20}$, which gives from (2.85) $p + (1-c)\varepsilon > 0$ in $B_{1/20}$, that is $(p + (1-c)\varepsilon)^+ = p + (1-c)\varepsilon$ in $B_{1/20}$.
Therefore, in view of (2.85), we get

$$(p(x) + (1-c)\varepsilon)^+ \geq u(x) \quad \text{in } B_{1/20},$$

with $0 < c < 1$ universal.
At this point, the precise conclusion of Theorem 2.1 follows repeating the same argument used in case of $|a_0| < 1/10$ after we have applied Lemma 2.3 with the hypothesis $u(\bar{x}) \leq (p(\bar{x}) + \varepsilon/2)^+$ satisfied.

$\square$

Chapter 3

Free boundary improvement of flatness

In this chapter, we prove the main "improvement of flatness" lemma, see Lemma 3.1. This is the key tool for proving Theorem 4.2, which will follow from Lemma 3.1 via an iterative argument. Roughly saying, the meaning of this lemma may be described as follows. If the graph of a solution u to (2.1)-(2.2) in B_1 oscillates ε away from a hyperplane in B_1, then in B_{r_0} it still remains in a $\varepsilon r_0/2$ -neighborhood of a, possibly different, hyperplane.

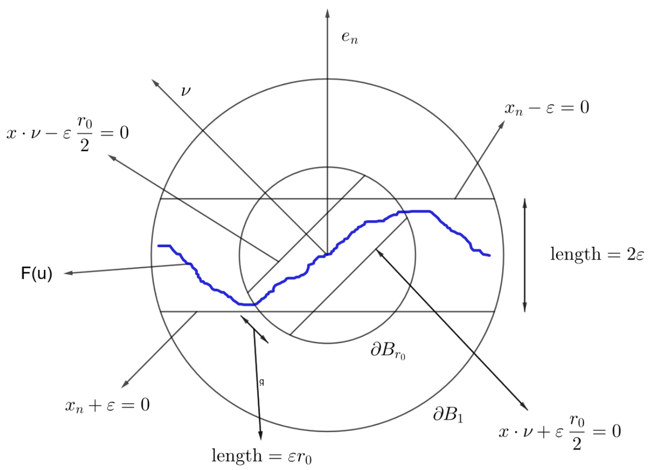

Figure 3.1: Improvement of flatness

We now state and prove the "improvement of flatness " lemma.

Lemma 3.1 (**Improvement of flatness**). *Let u be a solution to (2.1)-(2.2) in B_1 satisfying*

$$(x_n - \varepsilon)^+ \le u(x) \le (x_n + \varepsilon)^+ \quad \textit{for } x \in B_1, \tag{3.1}$$

and with $0 \in F(u)$. If $0 < r \le r_0$ for r_0 a universal constant and $0 < \varepsilon \le \varepsilon_0$ for some ε_0 depending on r, then

$$(x \cdot \nu - r\varepsilon/2)^+ \le u(x) \le (x \cdot \nu + r\varepsilon/2)^+ \quad \textit{for } x \in B_r, \tag{3.2}$$

with $|\nu| = 1$ and $|\nu - e_n| \le C\varepsilon$ for a universal constant C.

Proof. We divide the proof into three steps. We use the following notation:

$$\Omega_\rho(u) := (B_1^+(u) \cup F(u)) \cap B_\rho.$$

Step 1: Compactness. Fix $r \le r_0$ with r_0 universal (the precise r_0 will be given in Step 3). Assume for contradiction that we can find a sequence

$\varepsilon_k \to 0$ and a sequence u_k of solutions to (2.1) in B_1 with coefficients a_{ij}^k, right hand side f_k and free boundary conditions g_k satisfying (2.2), such that u_k satisfies (3.1), i.e.

$$(x_n - \varepsilon_k)^+ \leq u_k(x) \leq (x_n + \varepsilon_k)^+ \quad \text{for } x \in B_1, \ 0 \in F(u_k), \tag{3.3}$$

but it does not satisfy the conclusion (3.2) of the lemma.
Precisely, we are denying for contradiction the statement of Lemma 3.1, that is we suppose that fixed r_0 universal and $0 < r \leq r_0$, $\forall \varepsilon_0 \; \exists \; \bar{\varepsilon}$ such that $0 < \bar{\varepsilon} \leq \varepsilon_0$ and there exists a solution $\bar{u}$ to (2.1)-(2.2) in B_1 such that $\bar{u}$ satisfies (3.1) with $\bar{\varepsilon}$ but not the conclusion (3.2).
Therefore, letting ε_0 go to 0, we can find a sequence $\varepsilon_k \to 0$ such that for every k, ε_k satisfies the same conditions of $\bar{\varepsilon}$. Furthermore, calling u_k the corresponding solution to (2.1)-(2.2) in B_1 that satifies (3.1) with ε_k but not (3.2), we can find the sequence u_k described before.

Set

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k}, \quad x \in \Omega_1(u_k). \tag{3.4}$$

Notice that, since $F(u_k) = \partial B_1^+(u_k) \cap B_1$,

$$\begin{aligned} \Omega_1(u_k) &= (B_1^+(u_k) \cup F(u_k)) \cap B_1 \\ &= (B_1^+(u_k) \cap B_1) \cup (F(u_k) \cap B_1) = B_1^+(u_k) \cup F(u_k), \end{aligned}$$

in other words

$$\Omega_1(u_k) = B_1^+(u_k) \cup F(u_k). \tag{3.5}$$

Then (3.3) gives

$$-1 \leq \tilde{u}_k(x) \leq 1 \quad \text{for } x \in \Omega_1(u_k). \tag{3.6}$$

Indeed, according to (3.3), we have

$$x_n - \varepsilon_k \leq (x_n - \varepsilon_k)^+ \leq u_k(x) \leq (x_n + \varepsilon_k)^+ \quad \text{in } B_1 \supset \Omega_1(u_k),$$

thus

$$u_k(x) \geq x_n - \varepsilon_k \quad \text{in } \Omega_1(u_k),$$

which also gives

$$u_k(x) - x_n \geq -\varepsilon_k \quad \text{in } \Omega_1(u_k)$$

and dividing by $\varepsilon_k > 0$, from (3.4),

$$\frac{u_k(x) - x_n}{\varepsilon_k} = \tilde{u}_k(x) \geq -1 \quad \text{in } \Omega_1(u_k),$$

i.e.

$$\tilde{u}_k(x) \geq -1 \quad \text{in } \Omega_1(u_k). \tag{3.7}$$

Now, we have to show that $\tilde{u}_k \leq 1$ in $\Omega_1(u_k)$, but given that from (3.5), $\Omega_1(u_k) = B_1^+(u_k) \cup F(u_k)$, we can consider at first the case of $B_1^+(u_k)$ and then that of $F(u_k)$.
According to (3.3)

$$0 < u_k(x) \leq (x_n + \varepsilon_k)^+ \quad \text{in } B_1^+(u_k),$$

hence $(x_n + \varepsilon_k)^+ > 0$ in $B_1^+(u_k)$, i.e. $(x_n + \varepsilon_k)^+ = x_n + \varepsilon_k > 0$ in $B_1^+(u_k)$ and

$$u_k(x) \leq x_n + \varepsilon_k \quad \text{in } B_1^+(u_k),$$

which gives

$$u_k(x) - x_n \leq \varepsilon_k \quad \text{in } B_1^+(u_k),$$

and dividing by $\varepsilon_k > 0$ from (3.4)

$$\frac{u_k(x) - x_n}{\varepsilon_k} = \tilde{u}_k(x) \leq 1 \quad \text{in } B_1^+(u_k),$$

namely

$$\tilde{u}_k(x) \leq 1 \quad \text{in } B_1^+(u_k). \tag{3.8}$$

On $F(u_k)$, instead, we know that $u_k \equiv 0$. Also, from (3.3) where $(x_n + \varepsilon_k)^+ = 0$ in B_1, we have

$$0 \leq (x_n - \varepsilon_k)^+ \leq u_k(x) \leq (x_n + \varepsilon_k)^+ = 0,$$

in other words $0 \leq u_k(x) \leq 0$ and thus $u_k(x) = 0$, where $(x_n + \varepsilon_k)^+ = 0$, $x \in B_1$.

Now, $(x_n + \varepsilon_k)^+ = 0$ if $x_n + \varepsilon_k \leq 0$, that is $x_n \leq -\varepsilon_k$.
As a consequence,

$$u_k(x) = 0 \quad \text{with } x_n \leq -\varepsilon_k, \ \ x \in B_1. \tag{3.9}$$

Moreover, if we take $\bar{x} \in \{x \in B_1, \ x_n < -\varepsilon_k\}$, since $\{x \in B_1, \ x_n < -\varepsilon_k\} = B_1 \cap \{x_n < -\varepsilon_k\}$ is an open set, we can find a ball $B_r(\bar{x})$ such that $B_r(\bar{x}) \subset \{x \in B_1, x_n < -\varepsilon_k\}$, and hence from (3.9) $u_k \equiv 0$ in $B_r(\bar{x})$, i.e. $B_r(\bar{x}) \cap B_1^+(u_k) = \emptyset$.
Thus, given that $F(u_k) = \partial B_1^+(u_k) \cap B_1$, $\bar{x} \notin F(u_k)$, which implies that $x_n \geq -\varepsilon_k$ on $F(u_k)$, so $-x_n \leq \varepsilon_k$ on $F(u_k)$.
Consequently, in view of this fact, together with $u_k \equiv 0$ on $F(u_k)$, we achieve from (3.4)

$$\frac{u_k(x) - x_n}{\varepsilon_k} = \tilde{u}_k(x) = -\frac{x_n}{\varepsilon_k} \leq \frac{\varepsilon_k}{\varepsilon_k} = 1 \quad \text{on } F(u_k),$$

i.e.

$$\tilde{u}_k(x) \leq 1 \quad \text{on } F(u_k). \tag{3.10}$$

Therefore, from (3.8) and (3.10), we get in view of (3.5)

$$\tilde{u}_k(x) \leq 1 \quad \text{in } B_1^+(u_k) \cup F(u_k) = \Omega_1(u_k),$$

which together with (3.7) give us (3.6).
From Corollary 2.2, it follows that the function $\tilde{u}_k$ satisfies

$$|\tilde{u}_k(x) - \tilde{u}_k(y)| \leq C\, |x - y|^\gamma \tag{3.11}$$

for C universal and

$$|x - y| \geq \varepsilon_k / \bar{\varepsilon}, \quad x, \ y \in \Omega_{1/2}(u_k).$$

From (3.3) it follows that $F(u_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance, see Definition A.2.
To show this fact, first of all we notice that $F(u_k) \subset \{x \in B_1, \ -\varepsilon_k \leq x_n \leq \varepsilon_k\}$

for every k.
Precisely, as shown before to obtain (3.10), we have

$$x_n \geq -\varepsilon_k \quad \text{on } F(u_k) \subset B_1. \tag{3.12}$$

In addition, from (3.3), where $(x_n-\varepsilon_k)^+ > 0$ in B_1, $u_k > 0$, and $(x_n-\varepsilon_k)^+ > 0$ if $x_n - \varepsilon_k > 0$, i.e. $x_n > \varepsilon_k$. Hence, $u_k > 0$ in $B_1 \cap \{x_n > \varepsilon_k\}$, that is since $u_k \equiv 0$ on $F(u_k)$, $x_n \leq \varepsilon_k$ on $F(u_k)$.
As a consequence, in view of this fact, together with (3.12), we get

$$F(u_k) \subset B_1 \cap \{-\varepsilon_k \leq x_n \leq \varepsilon_k\}. \tag{3.13}$$

Now, we want to show that $d_H(F(u_k), B_1 \cap \{x_n = 0\}) \overset{k\to\infty}{\to} 0$, where d_H denotes the Hausdorff distance.
In particular, if $x \in F(u_k)$, from (3.13), we have $x \in B_1$ and $-\varepsilon_k \leq x_n \leq \varepsilon_k$, namely $|x_n| \leq \varepsilon_k$. Thus, if we write $x = (x', x_n)$, we can take $\bar{y}$ such that $\bar{y} = (x', 0)$. Notice that $\bar{y} \in B_1 \cap \{x_n = 0\}$. Indeed,

$$|\bar{y}| = |(x', 0)| \leq |x| < 1,$$

namely $|\bar{y}| < 1$, and hence $\bar{y} = (x', 0) \in B_1 \cap \{x_n = 0\}$.
Moreover, inasmuch $|x_n| \leq \varepsilon_k$, we have

$$\begin{aligned} |x - \bar{y}| &= \sqrt{(x_1 - x_1)^2 + (x_2 - x_2)^2 + \ldots + (x_{n-1} - x_{n-1})^2 + (x_n - 0)^2} \\ &= |x_n| \leq \varepsilon_k, \end{aligned}$$

in other words,

$$|x - \bar{y}| \leq \varepsilon_k,$$

which implies

$$\inf_{y \in B_1 \cap \{x_n = 0\}} |x - y| = d(x, B_1 \cap \{x_n = 0\}) \leq |x - \bar{y}| \leq \varepsilon_k,$$

i.e.

$$d(x, B_1 \cap \{x_n = 0\}) \leq \varepsilon_k, \quad x \in F(u_k). \tag{3.14}$$

At this point, seeing as how (3.14) holds $\forall x \in F(u_k)$, ε_k is an upper bound of the set $\{d(x, B_1 \cap \{x_n = 0\}),\ x \in F(u_k)\}$ and hence

$$\sup_{x \in F(u_k)} d(x, B_1 \cap \{x_n = 0\}) = e(F(u_k), B_1 \cap \{x_n = 0\}) \le \varepsilon_k,$$

namely

$$e(F(u_k), B_1 \cap \{x_n = 0\}) \le \varepsilon_k. \tag{3.15}$$

In parallel, if $y \in B_1 \cap \{x_n = 0\}$, $y = (y', 0)$. Also, since $u_k \equiv 0$ in $B_1 \cap \{x_n < -\varepsilon_k\}$, $u_k > 0$ in $B_1 \cap \{x_n > \varepsilon_k\}$ and $u_k \in C(B_1)$, $\exists\ \bar{x} = (y', \bar{x}_n) \in B_1$ such that $\forall B_r(\bar{x})$, $B_r(\bar{x}) \cap (\{u_k > 0\} \cap \{x' = y'\}) \neq \emptyset$ and $B_r(\bar{x}) \cap (\{u_k \equiv 0\} \cap \{x' = y'\}) \neq \emptyset$, so $\bar{x} \in F(u_k)$ and thus from (3.13), $|\bar{x}_n| \le \varepsilon_k$.
Furthermore, in view of $|x_n| \le \varepsilon_k$, we also have

$$\begin{aligned} |\bar{x} - y| &= \sqrt{(y_1 - y_1)^2 + (y_2 - y_2)^2 + \ldots + (y_{n-1} - y_{n-1})^2 + (\bar{x}_n - 0)^2} \\ &= |\bar{x}_n| \le \varepsilon_k, \end{aligned}$$

which gives

$$|\bar{x} - y| \le \varepsilon_k,$$

and hence

$$\inf_{x \in F(u_k)} |x - y| = d(F(u_k), y) \le |\bar{x} - y| \le \varepsilon_k,$$

i.e.

$$d(F(u_k), y) \le \varepsilon_k, \quad y \in B_1. \tag{3.16}$$

Now, since (3.16) holds $\forall y \in B_1 \cap \{x_n = 0\}$, ε_k is an upper bound of the set $\{d(F(u_k), y),\ y \in B_1 \cap \{x_n = 0\}\}$ and therefore

$$\sup_{y \in B_1 \cap \{x_n = 0\}} d(F(u_k), y) = e(B_1 \cap \{x_n = 0\}, F(u_k)) \le \varepsilon_k,$$

in other words

$$e(B_1 \cap \{x_n = 0\}, F(u_k)) \le \varepsilon_k. \tag{3.17}$$

Therefore, from (3.15) and (3.17) we obtain

$$\begin{aligned} 0 &\le \max\left(e(F(u_k), B_1 \cap \{x_n = 0\}), e(B_1 \cap \{x_n = 0\}, F(u_k))\right) \\ &= d_H(F(u_k), B_1 \cap \{x_n = 0\}) \le \varepsilon_k, \end{aligned}$$

i.e.

$$d_H(F(u_k), B_1 \cap \{x_n = 0\}) \leq \varepsilon_k \tag{3.18}$$

and letting k go to ∞, since $\varepsilon_k \to 0$, we achieve from (3.18)

$$d_H(F(u_k), B_1 \cap \{x_n = 0\}) \overset{k\to\infty}{\to} 0,$$

that is $F(u_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance.
This fact and (3.11) together with Ascoli-Arzelà give that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ over $\Omega_{1/2}(u_k)$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2} \cap \{x_n \geq 0\}$.
Step 2: Limiting Solution. We now show that $\tilde{u}$ solves

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}, \end{cases} \tag{3.19}$$

in the sense of Definition 1.10.

Let $P(x)$ be a quadratic polynomial touching $\tilde{u}$ at $\bar{x} \in B_{1/2} \cap \{x_n \geq 0\}$ strictly from below (for what we have seen in Chapter 1, it suffices to show that Definition 1.10 is satisfied by polynomials touching strictly from below/above). We need to show that

(i) if $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ then $\Delta P \leq 0$;

(ii) if $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$ then $P_n(\bar{x}) \leq 0$.

Since $\tilde{u}_k \to \tilde{u}$ in the sense specified above, there exist points $x_k \in \Omega_{1/2}(u_k)$, $x_k \to \bar{x}$, and constants $c_k \to 0$ such that

$$P(x_k) + c_k = \tilde{u}_k(x_k) \tag{3.20}$$

and

$$\tilde{u}_k \geq P + c_k \quad \text{in a neighborhood of } x_k. \tag{3.21}$$

In particular, from the definition (3.4) of $\tilde{u}_k$, we have in (3.20)

$$P(x_k) + c_k = \tilde{u}_k(x_k) = \frac{u_k(x_k) - (x_k)_n}{\varepsilon_k},$$

namely

$$\varepsilon_k(P(x_k) + c_k) = u_k(x_k) - (x_k)_n$$

and

$$\varepsilon_k(P(x_k) + c_k) + (x_k)_n = u_k(x_k). \tag{3.22}$$

At the same time, in (3.21) we have, always from the definition (3.4) of $\tilde{u}_k$,

$$\tilde{u}_k = \frac{u_k - x_n}{\varepsilon_k} \geq P + c_k \quad \text{in a neighborhood of } x_k,$$

thus, given that $\varepsilon_k > 0$

$$u_k - x_n \geq \varepsilon_k(P + c_k) \quad \text{in a neighborhood of } x_k$$

and

$$u_k \geq x_n + \varepsilon_k(P + c_k) \quad \text{in a neighborhood of } x_k. \tag{3.23}$$

Hence, (3.22) and (3.23) read

$$u_k(x_k) = Q(x_k) \tag{3.24}$$

and

$$u_k(x) \geq Q(x) \quad \text{in a neighborhood of } x_k \tag{3.25}$$

where

$$Q(x) = \varepsilon_k(P(x) + c_k) + x_n.$$

We now distinguish two cases.

(i) If $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ then $x_k \in B^+_{1/2}(u_k)$ (for k large). In addition, from (3.24) and (3.25), Q touches u_k from below at x_k, with $Q \in C^2(B_{1/2})$, inasmuch $P \in C^\infty(B_{1/2})$ and $x_n \in C^\infty(B_{1/2})$, and hence in particular $Q \in C^2(B^+_{1/2}(u_k))$.
To sum it up, for k large, we have $Q \in C^2(B^+_{1/2}(u_k))$ touching u_k from below at $x_k \in B^+_{1/2}(u_k)$.
As a consequence, since u_k is a solution to (2.1) in B_1, and thus also

in $B_{1/2}$, with coefficients a_{ij}^k, right hand side f_k and free boundary condition g_k satisfying (2.2) with ε_k, we get

$$\begin{aligned}\sum_{i,j} a_{ij}^k(x_k)Q_{ij}(x_k) &= \sum_{i,j} a_{ij}^k(x_k)(\varepsilon_k(P(x)+c_k)+x_n)_{ij}(x_k)\\ &= \sum_{i,j} a_{ij}^k(x_k)\varepsilon_k P_{ij}(x_k) = \varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k)\\ &\le f_k(x_k) \le \varepsilon_k^2,\end{aligned}$$

in other words

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \le \varepsilon_k^2, \tag{3.26}$$

seeing as how $\|f_k\|_{L^\infty(B_1)} \le \varepsilon_k^2$ and $x_k \in B^+_{1/2}(u_k) \subset B_{1/2} \subset B_1$, namely $x_k \in B_1$, so $f_k(x_k) \le |f_k(x_k)| \le \|f_k\|_{L^\infty(B_1)} \le \varepsilon_k^2$, i.e. $f_k(x_k) \le \varepsilon_k^2$.
In particular, from (3.26) we achieve, given that $\varepsilon_k > 0$

$$\sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \le \frac{\varepsilon_k^2}{\varepsilon_k} = \varepsilon_k,$$

i.e.

$$\sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \le \varepsilon_k. \tag{3.27}$$

In addition, from the last inequality in (2.2), that is $\left\|a_{ij}^k - \delta_{ij}\right\|_{L^\infty(B_1)} \le \varepsilon_k$ we have, because $x_k \in B_1$ as said before,

$$\left|a_{ij}^k(x_k) - \delta_{ij}\right| = \left|\delta_{ij} - a_{ij}^k(x_k)\right| \le \left\|a_{ij}^k - \delta_{ij}\right\|_{L^\infty(B_1)} \le \varepsilon_k,$$

which gives

$$\left|\delta_{ij} - a_{ij}^k(x_k)\right| \le \varepsilon_k$$

and

$$-\varepsilon_k \le \delta_{ij} - a_{ij}^k(x_k) \le \varepsilon_k. \tag{3.28}$$

Thus, in view of (3.27) and (3.28), we achieve

$$
\begin{aligned}
\Delta P &= Tr(D^2P) = Tr((D^2P)I) = \sum_i ((D^2P)I)_{ii} \\
&= \sum_{i,j} P_{ij}\delta_{ji} = \sum_{i,j} P_{ij}\delta_{ij} = \sum_{i,j} \delta_{ij}P_{ij} \\
&= \sum_{i,j} (\delta_{ij} - a^k_{ij}(x_k) + a^k_{ij}(x_k))P_{ij} \\
&= \sum_{i,j} (\delta_{ij} - a^k_{ij}(x_k))P_{ij} + \sum_{i,j} a^k_{ij}(x_k)P_{ij} \\
&\leq \sum_{\substack{i,j \\ P_{ij}\geq 0}} \varepsilon_k P_{ij} + \sum_{\substack{i,j \\ P_{ij}<0}} -\varepsilon_k P_{ij} + \varepsilon_k \\
&= \Bigg(\sum_{\substack{i,j \\ P_{ij}\geq 0}} P_{ij} - \sum_{\substack{i,j \\ P_{ij}<0}} P_{ij} + 1 \Bigg)\varepsilon_k = C\varepsilon_k, \qquad (3.29)
\end{aligned}
$$

because $P(x)$ is a quadratic polynomial and therefore P_{ij} is a constant for every i, j, which also entails $P_{ij}(x_k) = P_{ij}$.
Consequently, from (3.29), we obtain

$$\Delta P \leq C\varepsilon_k \qquad (3.30)$$

and because $\varepsilon_k \to 0$ as $k \to \infty$ and C is a constant, we conclude that $\Delta P \leq 0$.

(ii) If $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$, as observed in the Remark 1.7, we can assume that $\Delta P > 0$. We claim that for k large enough, $x_k \in F(u_k)$. Otherwise, we can find a subsequence $k_n \to \infty$ of $k \to \infty$, such that $x_{k_n} \in B_1^+(u_{k_n})$, recalling that $x_{k_n} \in \Omega_{1/2}(u_{k_n})$, but not on $F(u_k)$.
Therefore, as in case (i)

$$\Delta P \leq C\varepsilon_{k_n} \qquad (3.31)$$

and letting $k_n \to \infty$ in (3.31), inasmuch as $\varepsilon_k \to 0$ and ε_{k_n} is a subsequence of ε_k, $\varepsilon_{k_n} \to 0$ and we have that $\Delta P \leq 0$, contradicting the fact that P is strictly subharmonic. Thus $x_k \in F(u_k)$ for k large.

Now notice that

$$\begin{aligned}\nabla Q &= \left(\frac{\partial Q}{\partial x_1}, \frac{\partial Q}{\partial x_2}, \dots, \frac{\partial Q}{\partial x_n}\right)\\ &= \left(\frac{\partial}{\partial x_1}\left(\varepsilon_k(P+c_k)+x_n\right), \frac{\partial}{\partial x_2}\left(\varepsilon_k(P+c_k)+x_n\right), \dots,\right.\\ &\left.\dots, \frac{\partial}{\partial x_n}\left(\varepsilon_k(P+c_k)+x_n\right)\right)\\ &= \left(\varepsilon_k\frac{\partial P}{\partial x_1}, \varepsilon_k\frac{\partial P}{\partial x_2}, \dots, \varepsilon_k\frac{\partial P}{\partial x_n}+1\right) = \varepsilon_k\nabla P + e_n,\end{aligned}$$

in other words

$$|\nabla Q| = \varepsilon_k\nabla P + e_n. \tag{3.32}$$

Consequently, for k large, $|\nabla Q| > 0$.

Precisely, from (3.32), we achieve

$$\begin{aligned}|\nabla Q| &= |\varepsilon_k\nabla P + e_n| \geq |e_n| - |\varepsilon_k\nabla P|\\ &\overset{\varepsilon_k\geq 0}{=} 1 - \varepsilon_k|\nabla P| \geq 1 - \varepsilon_k \sup_{B_{1/2}}|\nabla P|,\end{aligned}$$

which gives

$$|\nabla Q| \geq 1 - \varepsilon_k \sup_{B_{1/2}}|\nabla P|, \tag{3.33}$$

where $\sup\limits_{B_{1/2}}|\nabla P| \leq C$, because $|\nabla P| \leq C$ in $B_{1/2}$, since $P(x)$ is a quadratic polynomial and $B_{1/2}$ is a bounded set.

Hence, if $\sup\limits_{B_{1/2}}|\nabla P| = 0$, that is $|\nabla P| = 0$ in $B_{1/2}$, $|\nabla Q| \geq 1 > 0$ $\forall k$.

Otherwise ($\sup\limits_{B_{1/2}}|\nabla P| > 0$), seeing as how $\varepsilon_k \to 0$, for the definition of limit, $\exists\ \bar{k} \in \mathbb{N}$ such that

$$|\varepsilon_k| < \frac{1}{\sup_{B_{1/2}}|\nabla P|}, \quad \forall k \in \mathbb{N},\ k \geq \bar{k}$$

i.e. since $\varepsilon_k > 0$ and thus $|\varepsilon_k| = \varepsilon_k$

$$\varepsilon_k < \frac{1}{\sup_{B_{1/2}}|\nabla P|}, \quad \forall k \in \mathbb{N},\ k \geq \bar{k}.$$

This fact, together with (3.33), implies that $|\nabla Q| > 0$ for k large.

Now, we have that Q touches u_k from below at $x_k \in F(u_k)$ for k large.

Therefore, given that $u_k \geq 0$ in B_1, recalling that u_k is a viscosity solution to (2.1) in B_1, Q^+ touches u_k from below at x_k.
Indeed, from (3.24), if $u_k(x_k) = 0$, $Q(x_k) = 0 = \max(Q(x_k), 0) = Q^+(x_k)$, namely $Q^+(x_k) = u_k(x_k)$; if instead $u_k(x_k) > 0$, $Q(x_k) > 0$, hence $Q(x_k) = \max(Q(x_k), 0) = Q^+(x_k)$ and $Q^+(x_k) = u_k(x_k)$.
Consequently,

$$u_k(x_k) = Q^+(x_k). \tag{3.34}$$

In addition, inasmuch as $u_k \geq 0$, we obtain from (3.25)

$$u_k(x) \geq \max(0, Q(x)) = Q^+(x) \quad \text{in a neighborhood of } x_k,$$

in other words

$$u_k(x) \geq Q^+(x) \quad \text{in a neighborhood of } x_k. \tag{3.35}$$

Considering (3.34) and (3.35) together, we get that Q^+ touches u_k from below at x_k.
Moreover, $Q \in C^2(B_1)$ because $P \in C^\infty(B_1)$ and $x_n \in C^\infty(B_1)$.
To sum it up, we have $Q \in C^2(B_1)$ such that Q^+ touches u_k from below at x_k, with, for k large, $x_k \in F(u_k)$ and $|\nabla Q| > 0$, which gives $|\nabla Q|(x_k) > 0$.
Thus, for these k's, seeing as how u_k is a solution to (2.1) in B_1 with coefficients a_{ij}^k, right hand side f_k and free boundary condition g_k satisfying (2.2) with ε_k, we get

$$|\nabla Q|(x_k) \leq g_k(x_k) \leq 1 + \varepsilon_k^2,$$

namely

$$|\nabla Q|(x_k) \leq 1 + \varepsilon_2^k, \tag{3.36}$$

since $\|g_k - 1\|_{L^\infty(B_1)} \leq \varepsilon_k^2$ and $x_k \in F(u_k) \subset B_1$, i.e. $x_k \in B_1$, so $g_k(x_k) - 1 \leq |g_k(x_k) - 1| \leq \|g_k - 1\|_{L^\infty(B_1)} \leq \varepsilon_k^2$, which implies $g_k(x_k) - 1 \leq \varepsilon_k^2$ and $g_k(x_k) \leq 1 + \varepsilon_k^2$.

Also, (3.32) and (3.36) give, because $|\nabla Q|(x_k) \geq 0$ and $1+\varepsilon_k^2 \geq 0$

$$\begin{aligned}
|\nabla Q|^2(x_k) &= |\varepsilon_k \nabla P + e_n|^2 (x_k)\\
&= (\varepsilon_k \nabla P(x_k) + e_n)\cdot(\varepsilon_k \nabla P(x_k) + e_n)\\
&= \varepsilon_k^2 \nabla P(x_k)\cdot \nabla P(x_k) + e_n \cdot e_n + 2\varepsilon_k \nabla P(x_k)\cdot e_n\\
&= \varepsilon_k^2 |\nabla P|^2(x_k) + 1 + 2\varepsilon_k P_n(x_k)\\
&\leq (1+\varepsilon_k^2)^2 = 1 + \varepsilon_k^4 + 2\varepsilon_k^2\\
&\leq 1 + \varepsilon_k^2 + 2\varepsilon_k^2 = 1 + 3\varepsilon_k^2 \qquad (3.37)
\end{aligned}$$

given that $0 < \varepsilon_k < 1$ $\forall k \in \mathbb{N}$ for the choice of ε_k. Therefore from (3.37), we achieve

$$\varepsilon_k^2 |\nabla P|^2(x_k) + 1 + 2\varepsilon_k P_n(x_k) \leq 1 + 3\varepsilon_k^2,$$

in other words

$$\varepsilon_k^2 |\nabla P|^2(x_k) - 3\varepsilon_k^2 + 2\varepsilon_k P_n(x_k) \leq 0$$

and thus dividing by $\varepsilon_k > 0$

$$\varepsilon_k |\nabla P|^2(x_k) - 3\varepsilon_k + 2P_n(x_k) \leq 0. \qquad (3.38)$$

Passing to the limit in (3.38) as $k \to \infty$, we obtain $2P_n(\bar{x}) \leq 0$ and hence $P_n(\bar{x}) \leq 0$, seeing as how $\varepsilon_k \to 0$ and $P_n(x_k) \to P_n(\bar{x})$, recalling that $x_k \to \bar{x}$ and $P \in C^\infty(B_1)$.

Let $P(x)$ be instead a quadratic polynomial touching $\tilde{u}$ at $\bar{x} \in B_{1/2} \cap \{x_n \geq 0\}$ strictly from above. This time, we need to show that

(i) if $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ then $\Delta P \geq 0$;

(ii) if $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$ then $P_n(\bar{x}) \geq 0$.

Always since $\tilde{u}_k \to \tilde{u}$ in the sense specified above, there exist points $x_k \in \Omega_{1/2}(u_k)$, $x_k \to \bar{x}$, and constants $c_k \to 0$ such that

$$P(x_k) + c_k = \tilde{u}_k(x_k) \qquad (3.39)$$

and

$$\tilde{u}_k \leq P + c_k \quad \text{in a neighborhood of } x_k. \tag{3.40}$$

As we have shown before, from the definition of $\tilde{u}_k$, (3.39) and (3.40) read

$$u_k(x_k) = Q(x_k) \tag{3.41}$$

and

$$u_k(x) \leq Q(x) \quad \text{in a neighborhood of } x_k \tag{3.42}$$

where

$$Q(x) = \varepsilon_k(P(x) + c_k) + x_n.$$

We distinguish two cases again.

(i) If $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ then $x_k \in B^+_{1/2}(u_k)$ (for k large). Moreover, from (3.41) and (3.42), Q touches u_k from above at $x_k \in B^+_{1/2}(u_k)$, with $Q \in C^2(B_{1/2})$, inasmuch as $P \in C^\infty(B_{1/2})$ and $x_n \in C^\infty(B_{1/2})$, and hence in particular $Q \in C^2(B^+_{1/2}(u_k))$.
To sum it up, for k large, we have $Q \in C^2(B^+_{1/2}(u_k))$ touching u_k from above at $x_k \in B^+_{1/2}(u_k)$.
Consequently, because u_k is a solution to (2.1) in B_1, and thus also in $B_{1/2}$, with coefficients a^k_{ij}, right hand side f_k and free boundary condition g_k satisfying (2.2) with ε_k, we get, thanks to the previous calculation,

$$\sum_{i,j} a^k_{ij}(x_k) Q_{ij}(x_k) = \varepsilon_k \sum_{i,j} a^k_{ij}(x_k) P_{ij} \geq f_k(x_k) \geq -\varepsilon_k^2 \tag{3.43}$$

given that $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2$ and $x_k \in B^+_{1/2}(u_k) \subset B_1$, namely $x_k \in B_1$, thereby $|f_k(x_k)| \leq \|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2$ and thus $|f_k(x_k)| \leq \varepsilon_k^2$, which implies $f_k(x_k) \geq -\varepsilon_k^2$.
In particular, from (3.43) we achieve, seeing as how $\varepsilon_k > 0$

$$\sum_{i,j} a^k_{ij}(x_k) P_{ij} \geq -\varepsilon_k. \tag{3.44}$$

Therefore, in view of (3.44), (3.28) and recalling the case of $P(x)$ touching $\tilde{u}$ from below at $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ to get (3.29),

$$\begin{aligned}\Delta P &= \sum_{i,j}(\delta_{ij} - a^k_{ij}(x_k))P_{ij} + \sum_{i,j} a^k_{ij}(x_k)P_{ij} \\ &\geq \sum_{\substack{i,j \\ P_{ij}\geq 0}} -\varepsilon_k P_{ij} + \sum_{\substack{i,j \\ P_{ij}<0}} \varepsilon_k P_{ij} - \varepsilon_k \\ &= \Bigg(- \sum_{\substack{i,j \\ P_{ij}\geq 0}} P_{ij} + \sum_{\substack{i,j \\ P_{ij}<0}} P_{ij} - 1\Bigg)\varepsilon_k = C\varepsilon_k, \end{aligned} \tag{3.45}$$

because $P(x)$ is a quadratic polynomial. As a consequence, P_{ij} is a constant for every i, j, which entails $P_{ij}(x_k) = P_{ij}$.
Thus, from (3.45), we obtain

$$\Delta P \geq C\varepsilon_k \tag{3.46}$$

and since $\varepsilon_k \to 0$ as $k \to \infty$ and C is a constant, we conclude that $\Delta P \geq 0$.

(ii) If $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$, arguing as in Remark 1.7, we can assume that $\Delta P < 0$. We claim that for k large enough, $x_k \in F(u_k)$. Otherwise, as we have said before, we can find a subsequence $k_n \to \infty$ such that $x_{k_n} \in B^+_1(u_{k_n})$.
Therefore, as in case (i)

$$\Delta P \geq C\varepsilon_{k_n} \tag{3.47}$$

and letting $k_n \to \infty$, $\varepsilon_{k_n} \to 0$ and we have that $\Delta P \geq 0$, contradicting the fact that P is strictly superharmonic. Thus $x_k \in F(u_k)$ for k large.
As shown before,

$$\nabla Q = \varepsilon_k \nabla P + e_n \tag{3.48}$$

and for k large $|\nabla Q| > 0$.
Now, we have that Q touches u_k from above at $x_k \in F(u_k)$ for k large.
Therefore, seeing as how $u_k \geq 0$ in B_1, recalling that u_k is a viscosity

solution to (2.1) in B_1, Q^+ touches u_k from above at x_k.
Indeed, from (3.41), repeating the considerations done above, we get

$$u_k(x_k) = Q^+(x_k). \tag{3.49}$$

Furthermore, since $u_k \geq 0$, from (3.42) we achieve

$$0 \leq u_k(x) \leq Q(x) \quad \text{in a neighborhood of } x_k$$

that is $Q(x) \geq 0$ in this neighborhood and hence $Q(x) = \max(0, Q(x)) = Q^+(x)$, which implies

$$u_k(x) \leq Q^+(x) \quad \text{in a neighborhood of } x_k. \tag{3.50}$$

Considering (3.49) and (3.50) together, we obtain that Q^+ touches u_k from above at x_k.
and repeating the same argument used in case of $P(x)$ touching $\tilde{u}$ from below at $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$, we have $Q \in C^2(B_1)$ and for k large $x_k \in F(u_k)$, with $|\nabla Q|(x_k) > 0$. Consequently, for these k's, recalling that u_k is a solution to (2.1) in B_1 with coefficients a_{ij}^k, right hand side f_k and free boundary condition g_k satisfying (2.2) with ε_k, we get

$$|\nabla Q|(x_k) \geq g_k(x_k) \geq 1 - \varepsilon_k^2 \tag{3.51}$$

given that $\|g_k - 1\|_{L^\infty(B_1)} \leq \varepsilon_k^2$ and $x_k \in F(u_k) \subset B_1$, namely $x_k \in B_1$, so $|g_k(x_k) - 1| \leq \|g_k - 1\|_{L^\infty(B_1)} \leq \varepsilon_k^2$, in other words $|g_k(x_k) - 1| \leq \varepsilon_k^2$, which implies $g_k(x_k) - 1 \geq -\varepsilon_k^2$ and $g_k(x_k) \geq 1 - \varepsilon_k^2$.
In addition, (3.48) and (3.51) give, because of $|\nabla Q|(x_k) \geq 0$ and $1 - \varepsilon_k^2 \geq 0$, inasmuch as $0 < \varepsilon_k < 1$ and thanks to the previous computation

$$\begin{aligned} |\nabla Q|^2(x_k) &= \varepsilon_k^2 |\nabla P|^2(x_k) + 1 + 2\varepsilon_k P_n(x_k) \\ &\geq (1 - \varepsilon_k^2)^2 = 1 + \varepsilon_k^4 - 2\varepsilon_k^2 \overset{\varepsilon_k^4 \geq 0}{\geq} 1 - 2\varepsilon_k^2, \end{aligned}$$

that is

$$\varepsilon_k^2 |\nabla P|^2(x_k) + 1 + 2\varepsilon_k P_n(x_k) \geq 1 - 2\varepsilon_k^2$$

and

$$\varepsilon_k^2 |\nabla P|^2 (x_k) + 2\varepsilon_k P_n(x_k) + 2\varepsilon_k^2 \geq 0.$$

Hence, dividing by $\varepsilon_k > 0$ the last inequality found,

$$\varepsilon_k |\nabla P|^2 (x_k) + 2P_n(x_k) + 2\varepsilon_k \geq 0 \tag{3.52}$$

and passing to the limit as $k \to \infty$ we obtain $2P_n(\bar{x}) \geq 0$, i.e. $P_n(\bar{x}) \geq 0$, seeing as how $\varepsilon_k \to 0$ and $P_n(x_k) \to P_n(\bar{x})$, since $x_k \to \bar{x}$ and $P \in C^\infty(B_1)$.

Step 3: Improvement of flatness. From the previous step, $\tilde{u}$ solves (3.19) and from (3.6),

$$-1 \leq \tilde{u} \leq 1 \quad \text{in } B_{1/2} \cap \{x_n \geq 0\}.$$

Sure enough, fixed $\bar{x} \in B_{1/2} \cap \{x_n \geq 0\}$, because $\tilde{u}_k \to \tilde{u}$ in the sense specified in *Step 1*, we can find a sequence of points $x_k \in \Omega_{1/2}(u_k)$ such that $\tilde{u}_k(x_k) \to \tilde{u}(\bar{x})$.
Moreover, given that $B_{1/2} \subset B_1$ and for the definition of $\Omega_\rho(u_k)$, $\Omega_{1/2}(u_k) \subset \Omega_1(u_k)$, $x_k \in \Omega_1(u_k)$ and from (3.6),

$$-1 \leq \tilde{u}_k(x_k) \leq 1. \tag{3.53}$$

Passing to the limit as $k \to \infty$ in (3.53), since $\tilde{u}_k(x_k) \to \tilde{u}(\bar{x})$, we achieve, for the properties of the sequence limit,

$$-1 \leq \tilde{u}(\bar{x}) \leq 1,$$

and for the arbitrariness of $\bar{x} \in B_{1/2} \cap \{x_n \geq 0\}$,

$$-1 \leq \tilde{u} \leq 1 \quad \text{in } B_{1/2} \cap \{x_n \geq 0\}.$$

Now, from Lemma 1.8 we find that, for the given r,

$$|\tilde{u}(x) - \tilde{u}(0) - \nabla\tilde{u}(0) \cdot x| \leq C_0 r^2 \quad \text{in } B_r \cap \{x_n \geq 0\},$$

for a universal constant C_0.
Precisely, since $\tilde{u}$ solves (3.19), from Lemma 1.8 $\tilde{u} \in C^\infty(B_{1/2} \cap \{x_n \geq 0\})$,

so for the formula of Taylor expansion around 0 up to second degree, we get locally for the given r

$$\tilde{u}(x) = \tilde{u}(0) + \nabla\tilde{u}(0)\cdot x + \frac{1}{2}D^2\tilde{u}(0)x\cdot x + O(|x|^2) \quad \text{in } B_r \cap \{x_n \geq 0\}\,,$$

which gives

$$\tilde{u}(x) - \tilde{u}(0) - \nabla\tilde{u}(0)\cdot x = \frac{1}{2}D^2\tilde{u}(0)x\cdot x + O(|x|^2) \quad \text{in } B_r \cap \{x_n \geq 0\}\,. \tag{3.54}$$

Now, we have

$$\left|O(|x|^2)\right| \leq \overline{C}\,|x|^2 \tag{3.55}$$

with $\overline{C}$ a universal constant, and for the Cauchy-Schwarz inequality

$$\left|D^2\tilde{u}(0)x\cdot x\right| \leq \left|D^2\tilde{u}(0)x\right||x| \leq \left\|D^2\tilde{u}(0)\right\||x|\,|x| = \left\|D^2\tilde{u}(0)\right\||x|^2\,, \quad \forall x \neq 0,$$

in other words

$$\left|D^2\tilde{u}(0)x\cdot x\right| \leq \left\|D^2\tilde{u}(0)\right\||x|^2\,, \quad \forall x \neq 0. \tag{3.56}$$

Therefore, in view of (3.55) and (3.56), we obtain from (3.54)

$$\begin{aligned}
|\tilde{u}(x) - \tilde{u}(0) - \nabla\tilde{u}(0)\cdot x| &= \left|\frac{1}{2}D^2\tilde{u}(0)x\cdot x + O(|x|^2)\right| \\
&\leq \left|\frac{1}{2}D^2\tilde{u}(0)x\cdot x\right| + \left|O(|x|^2)\right| \\
&\leq \frac{1}{2}\left\|D^2\tilde{u}(0)\right\||x|^2 + \overline{C}\,|x|^2 \\
&= \left(\frac{1}{2}\left\|D^2\tilde{u}(0)\right\| + \overline{C}\right)|x|^2 = C_0\,|x|^2 \\
&\leq C_0 r^2 \quad \text{in } B_r \cap \{x_n \geq 0\}\,, \quad x \neq 0
\end{aligned}$$

namely

$$|\tilde{u}(x) - \tilde{u}(0) - \nabla\tilde{u}(0)\cdot x| \leq C_0 r^2 \quad \text{in } B_r \cap \{x_n \geq 0\}\,, \quad x \neq 0, \tag{3.57}$$

for the triangular inequality of $|\cdot|$ applied to $\left|\frac{1}{2}D^2\tilde{u}(0)x\cdot x + O(|x|^2)\right|$ and recalling that $|x| \leq r$ in $B_r \cap \{x_n \geq 0\}$.

Notice that if $x = 0$, $|\tilde{u}(x) - \tilde{u}(0) - \nabla\tilde{u}(0) \cdot x| (0) = 0 \leq C_0 r^2$. As a consequence, from this consideration and (3.57), we achieve

$$|\tilde{u}(x) - \tilde{u}(0) - \nabla\tilde{u}(0) \cdot x| \leq C_0 r^2 \quad \text{in } B_r \cap \{x_n \geq 0\}\,, \tag{3.58}$$

for a universal constant C_0.
At this point, we can rewrite (3.58) as

$$|\tilde{u}(x) - \tilde{u}(0) - \nabla\tilde{u}(0)' \cdot x' - \tilde{u}_n(0)x_n| \leq C_0 r^2 \quad \text{in } B_r \cap \{x_n \geq 0\}\,. \tag{3.59}$$

In particular, because $0 \in F(\tilde{u})$, and thus $\tilde{u}(0) = 0$, and also $\tilde{u}_n(0) = 0$, recalling that $\tilde{u}$ solves (3.19) and $0 \in B_{1/2} \cap \{x_n = 0\}$, we obtain from (3.59)

$$|\tilde{u}(x) - \nabla\tilde{u}(0)' \cdot x'| \leq C_0 r^2 \quad \text{in } B_r \cap \{x_n \geq 0\}\,,$$

which implies

$$-C_0 r^2 \leq \tilde{u}(x) - \nabla\tilde{u}(0)' \cdot x' \leq C_0 r^2 \quad \text{in } B_r \cap \{x_n \geq 0\}\,,$$

and

$$x' \cdot \tilde{\nu} - C_0 r^2 \leq \tilde{u}(x) \leq x' \cdot \tilde{\nu} + C_0 r^2 \quad \text{in } B_r \cap \{x_n \geq 0\}\,, \tag{3.60}$$

where $x' \cdot \tilde{\nu} = \tilde{\nu} \cdot x'$ for the symmetry of the scalar product and $\tilde{\nu}_i = \tilde{u}_i(0)$, $i = 1, \ldots, n-1$, with $|\tilde{\nu}| \leq \tilde{C}$, $\tilde{C}$ a universal constant.
Therefore, for k large enough from (3.60) we get, inasmuch $\tilde{u}_k \to \tilde{u}$ in the sense specified in *Step 1*,

$$x' \cdot \tilde{\nu} - C_1 r^2 \leq \tilde{u}_k(x) \leq x' \cdot \tilde{\nu} + C_1 r^2 \quad \text{in } \Omega_r(u_k). \tag{3.61}$$

From the definition (3.4) of $\tilde{u}_k$ the inequality in (3.61) reads

$$x' \cdot \tilde{\nu} - C_1 r^2 \leq \frac{u_k(x) - x_n}{\varepsilon_k} \leq x' \cdot \tilde{\nu} + C_1 r^2 \quad \text{in } \Omega_r(u_k),$$

in other words, seeing as how $\varepsilon_k > 0$,

$$\varepsilon_k x' \cdot \tilde{\nu} - \varepsilon_k C_1 r^2 \leq u_k(x) - x_n \leq \varepsilon_k x' \cdot \nu + \varepsilon_k C_1 r^2 \quad \text{in } \Omega_r(u_k)$$

and

$$\varepsilon_k x' \cdot \tilde{\nu} + x_n - \varepsilon_k C_1 r^2 \leq u_k \leq \varepsilon_k x' \cdot \tilde{\nu} + x_n + \varepsilon_k C_1 r^2 \quad \text{in } \Omega_r(u_k). \tag{3.62}$$

Let us set now

$$\nu = \frac{(\varepsilon_k \tilde{\nu}, 1)}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}}.$$

Notice that

$$|\nu| = \frac{1}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1} = 1,$$

that is

$$|\nu| = 1 \tag{3.63}$$

and

$$\begin{aligned}
|\nu - e_n| &= \left| \left(\frac{\varepsilon_k \tilde{\nu}}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}}, \frac{1}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} - 1 \right) \right| \\
&= \left| \left(\frac{\varepsilon_k \tilde{\nu}}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}}, \frac{1 - \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \right) \right| \\
&= \frac{1}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \left| \left(\varepsilon_k \tilde{\nu}, \left(1 - \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}\right) \frac{1 + \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}}{1 + \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \right) \right| \\
&= \frac{1}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \left| \left(\varepsilon_k \tilde{\nu}, \frac{1 - \varepsilon_k^2 |\tilde{\nu}|^2 - 1}{1 + \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \right) \right| \\
&= \frac{1}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \left| \left(\varepsilon_k \tilde{\nu}, -\frac{\varepsilon_k^2 |\tilde{\nu}|^2}{1 + \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \right) \right| \\
&\leq \left| \left(\varepsilon_k \tilde{\nu}, -\frac{\varepsilon_k^2 |\tilde{\nu}|^2}{1 + \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \right) \right|,
\end{aligned}$$

in other words

$$|\nu - e_n| \leq \left| \left(\varepsilon_k \tilde{\nu}, -\frac{\varepsilon_k^2 |\tilde{\nu}|^2}{1 + \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \right) \right|, \tag{3.64}$$

inasmuch $\frac{1}{\sqrt{\varepsilon_k^2|\tilde{\nu}|^2+1}} \leq 1$.
In addition, we have

$$\left|\left(\varepsilon_k\tilde{\nu}, -\frac{\varepsilon_k^2\left|\tilde{\nu}\right|^2}{1+\sqrt{\varepsilon_k^2\left|\tilde{\nu}\right|^2+1}}\right)\right| = \sqrt{\varepsilon_k^2\tilde{\nu}_1^2+\ldots+\varepsilon_k^2\tilde{\nu}_{n-1}^2+\left(-\frac{\varepsilon_k^2\left|\tilde{\nu}\right|^2}{1+\sqrt{\varepsilon_k^2\left|\tilde{\nu}\right|^2+1}}\right)^2}$$
$$= \sqrt{\varepsilon_k^2(\tilde{\nu}_1^2+\ldots+\tilde{\nu}_{n-1}^2)+\frac{\varepsilon_k^4\left|\tilde{\nu}\right|^4}{\left(1+\sqrt{\varepsilon_k^2\left|\tilde{\nu}\right|^2+1}\right)^2}}$$
$$\leq \sqrt{\varepsilon_k^2\left|\tilde{\nu}\right|^2+\varepsilon_k^4\left|\tilde{\nu}\right|^4} = \sqrt{\varepsilon_k^2\left|\tilde{\nu}\right|^2(1+\varepsilon_k^2\left|\tilde{\nu}\right|^2)} = \varepsilon_k\left|\tilde{\nu}\right|\sqrt{1+\varepsilon_k^2\left|\tilde{\nu}\right|^2},$$

namely

$$\left|\left(\varepsilon_k\tilde{\nu}, -\frac{\varepsilon_k^2\left|\tilde{\nu}\right|^2}{1+\sqrt{\varepsilon_k^2\left|\tilde{\nu}\right|^2+1}}\right)\right) \leq \varepsilon_k\left|\tilde{\nu}\right|\sqrt{1+\varepsilon_k^2\left|\tilde{\nu}\right|^2}, \tag{3.65}$$

because $\varepsilon_k > 0$ and $\left(1+\sqrt{\varepsilon_k^2\left|\tilde{\nu}\right|^2+1}\right)^2 \geq 1$, which gives

$$\frac{\varepsilon_k^4\left|\tilde{\nu}\right|^4}{\left(1+\sqrt{\varepsilon_k^2\left|\tilde{\nu}\right|^2+1}\right)^2} \leq \varepsilon^4\left|\tilde{\nu}\right|^4.$$

Now, we know that the sequence ε_k is convergent and hence it is bounded, i.e. for every k, $0 < \varepsilon_k \leq \overline{C}$ with $\overline{C}$ a universal constant.
This fact, together with $|\tilde{\nu}| \leq \tilde{C}$, with $\tilde{C}$ a universal constant, implies in view of (3.64) and (3.65)

$$|\nu - e_n| \leq \tilde{C}\varepsilon_k\sqrt{1+\overline{C}^2\tilde{C}^2} = C\varepsilon_k,$$

i.e.

$$|\nu - e_n| \leq C\varepsilon_k \quad \text{for every } k. \tag{3.66}$$

As a consequence, from (3.63) and (3.66) ν satisfies the hypotheses of Lemma 3.1.

At this point, we can rewrite (3.62) as

$$\frac{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \varepsilon_k x' \cdot \tilde{\nu} + \frac{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} x_n - \varepsilon_k C_1 r^2 \leq u_k \leq \frac{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} \varepsilon_k x' \cdot \tilde{\nu}$$
$$+ \frac{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}}{\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}} x_n + \varepsilon_k C_1 r^2 \quad \text{in } \Omega_r(u_k),$$

which gives for the definition of ν,

$$\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}\, x \cdot \nu - \varepsilon_k C_1 r^2 \leq u_k$$
$$\leq \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}\, x \cdot \nu + \varepsilon_k C_1 r^2 \quad \text{in } \Omega_r(u_k). \quad (3.67)$$

Moreover, we remark that $1 \leq \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1} \leq 1 + \varepsilon_k^2 |\tilde{\nu}|^2 / 2$.
Indeed, as regards the first inequality, it suffices to observe that $\varepsilon_k^2 |\tilde{\nu}|^2 \geq 0$ and thus for the monoticity of $\sqrt{\cdot}$, $1 = \sqrt{1} \leq \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}$.
As regards the second inequality, instead,

$$\left(1 + \frac{\varepsilon_k^2 |\tilde{\nu}|^2}{2}\right)^2 = 1 + \frac{\varepsilon_k^4 |\tilde{\nu}|^4}{4} + \varepsilon_k^2 |\tilde{\nu}|^2 \geq \varepsilon_k^2 |\tilde{\nu}|^2 + 1,$$

given that $\varepsilon_k^4 |\tilde{\nu}|^4 / 4 \geq 0$ and raising both the terms of the inequality to $1/2$, recalling that both the terms are positive or equal to 0, we achieve

$$1 + \frac{\varepsilon_k^2 |\tilde{\nu}|^2}{2} \geq \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1},$$

as desired.
Consequently, from (3.67) we have

$$x \cdot \nu - \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} - C_1 r^2 \varepsilon_k \leq u_k \leq x \cdot \nu + \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k).$$

To show this fact, we distinguish two cases.
If $x \cdot \nu \geq 0$, since $\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1} \geq 1$ for what we have said before, $x \cdot \nu \leq \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}\, x \cdot \nu$, so, seeing as how $-\varepsilon_k^2 |\tilde{\nu}|^2 r/2 \leq 0$, we get from (3.67)

$$x \cdot \nu - \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} - C_1 r^2 \varepsilon_k \leq \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}\, x \cdot \nu - \varepsilon_k C_1 r^2 \leq u_k \quad \text{in } \Omega_r(u_k)$$

and hence

$$x \cdot \nu - \varepsilon_k^2 \left|\tilde{\nu}\right|^2 \frac{r}{2} - C_1 r^2 \varepsilon_k \leq u_k \quad \text{in } \Omega_r(u_k). \tag{3.68}$$

In addition, always if $x \cdot \nu \geq 0$, $\sqrt{\varepsilon_k^2 \left|\tilde{\nu}\right|^2 + 1}\, x \cdot \nu \leq (1 + \varepsilon_k^2 \left|\tilde{\nu}\right|^2 /2)\, x \cdot \nu$, seeing as how $\sqrt{\varepsilon_k^2 \left|\tilde{\nu}\right|^2 + 1} \leq 1 + \varepsilon_k^2 \left|\tilde{\nu}\right|^2 /2$ for what we have shown above, and $x \cdot \nu \leq |x \cdot \nu| \leq |x|\, |\nu| \leq r$ in $\Omega_r(u_k)$, i.e. $x \cdot \nu \leq r$, recalling that $|\nu| = 1$ and $|x| \leq r$ in $\Omega_r(u_k) \subset B_r$. As a consequence, inasmuch as $\varepsilon_k^2 \left|\tilde{\nu}\right| r/2 \geq 0$, we get from (3.67)

$$\begin{aligned}
u_k &\leq \sqrt{\varepsilon_k^2 \left|\tilde{\nu}\right|^2 + 1}\, x \cdot \nu + C_1 r^2 \varepsilon_k \\
&\leq \left(1 + \frac{\varepsilon_k^2 \left|\tilde{\nu}\right|^2}{2}\right) x \cdot \nu + C_1 r^2 \varepsilon_k \\
&= x \cdot \nu + \frac{\varepsilon_k^2 \left|\tilde{\nu}\right|^2}{2} x \cdot \nu + C_1 r^2 \varepsilon_k \\
&\leq x \cdot \nu + \varepsilon_k^2 \left|\tilde{\nu}\right|^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k),
\end{aligned}$$

in other words,

$$u_k \leq x \cdot \nu + \varepsilon_k^2 \left|\tilde{\nu}\right|^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k), \tag{3.69}$$

which, together with (3.68), implies

$$x \cdot \nu - \varepsilon_k^2 \left|\tilde{\nu}\right|^2 \frac{r}{2} - C_1 r^2 \varepsilon_k \leq u_k \leq x \cdot \nu + \varepsilon_k^2 \left|\tilde{\nu}\right|^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k). \tag{3.70}$$

If instead $x \cdot \nu < 0$,

$$\sqrt{\varepsilon_k^2 \left|\tilde{\nu}\right|^2 + 1}\, x \cdot \nu \geq (1 + \varepsilon_k^2 \left|\tilde{\nu}\right|^2 /2)\, x \cdot \nu,$$

because $\sqrt{\varepsilon_k^2 \left|\tilde{\nu}\right|^2 + 1} \leq 1 + \varepsilon_k^2 \left|\tilde{\nu}\right|^2 /2$, and for what we have shown before, $|x \cdot \nu| \leq r$ in $\Omega_r(u_k)$ and thus $x \cdot \nu \geq -r$ in $\Omega_r(u_k)$. Consequently, since

$\varepsilon_k^2 |\tilde{\nu}|^2 \geq 0$, we get from (3.67)

$$\begin{aligned} u_k &\geq \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}\, x \cdot \nu - C_1 r^2 \varepsilon_k \\ &\geq \left(1 + \frac{\varepsilon_k^2 |\tilde{\nu}|^2}{2}\right) x \cdot \nu - C_1 r^2 \varepsilon_k \\ &= x \cdot \nu + \frac{\varepsilon_k^2 |\tilde{\nu}|^2}{2} x \cdot \nu - C_1 r^2 \varepsilon_k \\ &\geq x \cdot \nu - \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} - C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k) \end{aligned}$$

i.e.

$$x \cdot \nu - \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} - C_1 r^2 \varepsilon_k \leq u_k \quad \text{in } \Omega_r(u_k). \tag{3.71}$$

In addition, always if $x \cdot \nu < 0$, $\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}\, x \cdot \nu \leq x \cdot \nu$, seeing as how $\sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1} \geq 1$, thereby, given that $\varepsilon_k^2 |\tilde{\nu}|^2 r/2 \geq 0$, we achieve from (3.67)

$$u_k \leq \sqrt{\varepsilon_k^2 |\tilde{\nu}|^2 + 1}\, x \cdot \nu + C_1 r^2 \varepsilon_k \leq x \cdot \nu + \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k),$$

namely

$$u_k \leq x \cdot \nu + \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k), \tag{3.72}$$

which, together with (3.71), gives

$$x \cdot \nu - \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} - C_1 r^2 \varepsilon_k \leq u_k \leq x \cdot \nu + \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k). \tag{3.73}$$

Therefore, considering (3.70) and (3.73) together, we obtain

$$x \cdot \nu - \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} - C_1 r^2 \varepsilon_k \leq u_k \leq x \cdot \nu + \varepsilon_k^2 |\tilde{\nu}|^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k), \tag{3.74}$$

regardless of the sign of $x \cdot \nu$ and hence $\forall x \in \Omega_r(u_k)$.
In particular, if r_0 is such that $C_1 r_0 \leq 1/4$, that is $r_0 \leq \frac{1}{4C_1}$ and moreover k is large enough so that $\varepsilon_k \leq \frac{1}{2|\tilde{\nu}|^2}$, we achieve from (3.74)

$$x \cdot \nu - \varepsilon_k \frac{r}{2} \leq u_k \leq x \cdot \nu + \varepsilon_k \frac{r}{2} \quad \text{in } \Omega_r(u_k).$$

Precisely, if $r_0 \leq \frac{1}{4C_1}$, given that $0 < r \leq r_0$, $0 < r \leq \frac{1}{4C_1}$. Furthermore, inasmuch as $\varepsilon_k \to 0$, for the definition of limit, we can find $\bar{k} \in \mathbb{N}$ such that

$$|\varepsilon_k| \leq \frac{1}{2\,|\tilde{\nu}|^2} \quad \forall k \in \mathbb{N},\ k \geq \bar{k}$$

and thus for these k's $\varepsilon_k \leq |\varepsilon_k| \leq \frac{1}{2|\tilde{\nu}|^2}$, i.e. $\varepsilon_k \leq \frac{1}{2|\tilde{\nu}|^2}$.
To sum it up, $0 < r \leq \frac{1}{4C_1}$ and for k large, $\varepsilon_k \leq \frac{1}{2|\tilde{\nu}|^2}$.
Hence, for these k's

$$\varepsilon_k^2 \left|\tilde{\nu}\right|^2 \frac{r}{2} \leq \varepsilon_k \left|\tilde{\nu}\right|^2 \frac{1}{2\left|\tilde{\nu}\right|^2} \frac{r}{2} = \varepsilon_k \frac{r}{4},$$

in other words

$$\varepsilon_k^2 \left|\tilde{\nu}\right|^2 \frac{r}{2} \leq \varepsilon_k \frac{r}{4} \tag{3.75}$$

and

$$\varepsilon_k C_1 r^2 \leq \varepsilon_k C_1 r \frac{1}{4C_1} = \varepsilon_k \frac{r}{4},$$

which gives

$$\varepsilon_k C_1 r^2 \leq \varepsilon_k \frac{r}{4}. \tag{3.76}$$

As a consequence, in view of (3.75) and (3.76), which also imply $-\varepsilon_k^2 \left|\tilde{\nu}\right|^2 r/2 \geq -\varepsilon_k r/4$ and $-\varepsilon_k C_1 r^2 \geq -\varepsilon_k r/4$, we get from (3.74)

$$x \cdot \nu - \varepsilon_k \frac{r}{4} - \varepsilon_k \frac{r}{4} \leq u_k \leq x \cdot \nu + \varepsilon_k \frac{r}{4} + \varepsilon_k \frac{r}{4} \quad \text{in } \Omega_r(u_k),$$

which gives

$$x \cdot \nu - \varepsilon_k \frac{r}{2} \leq u_k \leq x \cdot \nu + \varepsilon_k \frac{r}{2} \quad \text{in } \Omega_r(u_k). \tag{3.77}$$

Remark that we have assumed $\tilde{\nu} \neq 0$ to write $\varepsilon_k \leq \frac{1}{2|\tilde{\nu}|^2}$.
If instead $\tilde{\nu} = 0$ then $\nu = e_n$. Thus, from previous computation, it follows that

$$x_n - \varepsilon_k C_1 r^2 \leq u_k \leq x_n + \varepsilon_k C_1 r^2 \quad \text{in } \Omega_r(u_k).$$

Moreover, recalling that $\varepsilon_k^2 r/2 \geq 0$,

$$\begin{aligned} x \cdot \nu - \varepsilon_k^2 \frac{r}{2} - C_1 r^2 \varepsilon_k &\leq x \cdot \nu - C_1 r^2 \varepsilon_k \leq u_k \\ &\leq x_n + C_1 r^2 \varepsilon_k \\ &\leq x_n + \varepsilon_k^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k), \end{aligned}$$

i.e.

$$x \cdot \nu - \varepsilon_k^2 \frac{r}{2} - C_1 r^2 \varepsilon_k \leq u_k \leq x \cdot \nu + \varepsilon_k^2 \frac{r}{2} + C_1 r^2 \varepsilon_k \quad \text{in } \Omega_r(u_k). \tag{3.78}$$

Therefore, repeating the above arguments, if $C_1 r_0 \leq 1/4$ and k is large enough so that $\varepsilon_k \leq 1/2$, we achieve from (3.78)

$$x \cdot \nu - \varepsilon_k r/2 \leq u_k \leq x \cdot \nu + \varepsilon_k r/2 \quad \text{in } \Omega_r(u_k).$$

Hence, we get (3.77) one more time. Now, (3.77), together with (3.3), entails that

$$\left(x \cdot \nu - \varepsilon_k \frac{r}{2}\right)^+ \leq u_k \leq \left(x \cdot \nu + \varepsilon_k \frac{r}{2}\right)^+ \quad \text{in } B_r. \tag{3.79}$$

The u_k satisfy the conclusion of Lemma 3.1, obtaining a contradiction, inasmuch we have supposed that the u_k did not satisfy the conclusion (3.2) of Lemma 3.1.

Let us show that (3.79) holds.

From (3.77), since $x \cdot \nu + \varepsilon_k r/2 \leq \max(0, x \cdot \nu + \varepsilon_k r/2) = (x \cdot \nu + \varepsilon_k r/2)^+$, we achieve

$$u_k \leq \left(x \cdot \nu + \varepsilon_k \frac{r}{2}\right)^+ \quad \text{in } \Omega_r(u_k). \tag{3.80}$$

In addition, since $u_k \geq 0$ in $B_1 \supset B_r \supset \Omega_r(u_k)$, namely $u_k \geq 0$ in $\Omega_r(u_k)$, recalling that u_k is a viscosity solution to (2.1) in B_1, we have from (3.77)

$$\max\left(0, x \cdot \nu - \varepsilon_k \frac{r}{2}\right) = \left(x \cdot \nu - \varepsilon_k \frac{r}{2}\right)^+ \leq u_k \quad \text{in } \Omega_r(u_k),$$

in other words

$$\left(x \cdot \nu - \varepsilon_k \frac{r}{2}\right)^+ \leq u_k \quad \text{in } \Omega_r(u_k). \tag{3.81}$$

Recall now that for the definition of $\Omega_r(u_k)$, $\Omega_r(u_k) \supset B_1^+(u_k) \cap B_r$ and $B_1^+(u_k) \cap B_r = B_r^+(u_k)$, hence, seeing as how $u_k \geq 0$ in $B_1 \supset B_r$, for what we have noticed above, i.e. $u_k \geq 0$ in B_r, $u_k = 0$ in $B_r \setminus \Omega_r(u_k)$. Consequently, given that $(x \cdot \nu + \varepsilon_k r/2)^+ \geq 0$ in B_r, we have

$$u_k \leq \left(x \cdot \nu + \varepsilon_k \frac{r}{2}\right)^+ \quad \text{in } B_r \setminus \Omega_r(u_k),$$

which, together with (3.80), implies

$$u_k \leq \left(x \cdot \nu + \varepsilon_k \frac{r}{2}\right)^+ \quad \text{in } B_r. \tag{3.82}$$

At this point, from (3.3), we achieve

$$\left(x \cdot \nu - \varepsilon_k \frac{r}{2}\right)^+ \leq u_k \quad \text{in } B_r \setminus \Omega_r(u_k),$$

which gives from (3.81)

$$\left(x \cdot \nu - \varepsilon_k \frac{r}{2}\right)^+ \leq u_k \quad \text{in } B_r. \tag{3.83}$$

To sum it up, in view of (3.82) and (3.83) we obtain that (3.79) holds. $\square$

Chapter 4

Proofs of the main theorems

We prove our main results, in other words Theorem 0.1 and the following 4.1.

Theorem 4.1 (**Lipschitz implies** $C^{1,\alpha}$**).** *Let u be a viscosity solution to (2.1). Assume $0 \in F(u)$ and $g(0) > 0$. If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\alpha}$ in a (smaller) neighborhood of 0.*

We begin from Theorem 0.1 and for the reader convenience, we recall below its statement given in the introduction.

Theorem 4.2 (**Flatness implies** $C^{1,\alpha}$**).** *Let u be a viscosity solution to (2.1) in B_1. Assume that $0 \in F(u)$, $g(0) = 1$ and $a_{ij}(0) = \delta_{ij}$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if the graph of u is $\bar{\varepsilon}$-flat in B_1, i.e.*

$$(x_n - \bar{\varepsilon})^+ \leq u(x) \leq (x_n + \bar{\varepsilon})^+, \quad x \in B_1, \tag{4.1}$$

and

$$[a_{ij}]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon}, \quad \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \quad [g]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon}, \tag{4.2}$$

then $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

Remark. As observed in [11], the assumptions on the coefficients $a_{ij}(x)$ in Theorem (4.2) can be weakened to a Cordes-Nirenberg type condition

$$\|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \leq \delta(n).$$

Proof of Theorem 4.2. Let u be a viscosity solution to (1.2) in B_1 with $0 \in F(u)$, $g(0) = 1$ and $a_{ij}(0) = \delta_{ij}$. Consider the sequence of rescalings

$$u_k(x) := \frac{u(\rho_k x)}{\rho_k}, \quad x \in B_1,$$

with $\rho_k = \bar{r}^k$, $k = 0, 1, \ldots$, for a fixed $\bar{r}$ such that

$$\bar{r}^\beta \leq \frac{1}{4}, \quad \bar{r} \leq r_0,$$

with r_0 the universal constant of Lemma 3.1.
Notice that if $\bar{r}^\beta \leq 1/4$, raising both the terms of the inequality to $1/\beta$, with $0 < \beta \leq 1$, since both the terms are positive, we get

$$\bar{r} \leq \left(\frac{1}{4}\right)^{1/\beta}$$

and given that $1/\beta > 0$, $(1/4)^{1/\beta} < 1$, thus $\bar{r} < 1$.
As a consequence, $\rho_k = \bar{r}^k$, $k = 0, 1 \ldots$, is such that $\rho_0 = 1$ and $\rho_k < 1$ $\forall k \in \mathbb{N}$ and hence u_k is well-defined $\forall k$.
Indeed, if $x \in B_1$, since $0 < \rho_k \leq 1$, we have

$$|\rho_k x| = \rho_k |x| < \rho_k \leq 1$$

that is

$$\rho_k x \in B_1 \tag{4.3}$$

and so u_k is well-defined, in view of its definition.

Now, we state that each u_k solves (2.1) in B_1 with coefficients $a^k_{ij}(x) := a_{ij}(\rho_k x)$, right hand side $f_k(x) := \rho_k f(\rho_k x)$, and free boundary condition $g_k(x) := g(\rho_k x)$.
Specifically, we need to show that

(i) if $\varphi \in C^2(B_1^+(u_k))$ touches u_k from below (above) at $x_0 \in B_1^+(u_k)$ then

$$\sum_{i,j} a^k_{ij}(x_0)\varphi_{ij}(x_0) \leq f_k(x_0) \quad \left(\text{resp. } \sum_{i,j} a^k_{ij}(x_0)\varphi_{ij}(x_0) \geq f_k(x_0)\right);$$

(ii) if $\varphi \in C^2(B_1)$ and φ^+ touches u_k from below (above) at $x_0 \in F(u_k)$ and $|\nabla\varphi|(x_0) \neq 0$ then

$$|\nabla\varphi|(x_0) \leq g_k(x_0) \quad (\text{resp. } |\nabla\varphi|(x_0) \geq g_k(x_0)).$$

For this purpose, let us take $\varphi \in C^2(B_1^+(u_k))$ that touches u_k from below at $x_0 \in B_1^+(u_k)$ and we have

$$\varphi(x_0) = u_k(x_0) \tag{4.4}$$

and

$$\varphi(x) \leq u_k(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{4.5}$$

In particular, for the definition of u_k, we can rewrite (4.4) as

$$\varphi(x_0) = u_k(x_0) = \frac{u(\rho_k x_0)}{\rho_k},$$

therefore

$$\rho_k\varphi(x_0) = (\rho_k\varphi)(x_0) = u(\rho_k x_0)$$

and in addition

$$(\rho_k\varphi)\left(\frac{\rho_k x_0}{\rho_k}\right) = u(\rho_k x_0). \tag{4.6}$$

Analogously, from (4.5) we have

$$\varphi(x) \leq u_k(x) = \frac{u(\rho_k x)}{\rho_k} \quad \text{in } O,$$

which implies, inasmuch $\rho_k > 0$,

$$\rho_k\varphi(x) = (\rho_k\varphi)(x) \leq u(\rho_k x) \quad \text{in } O$$

and also

$$(\rho_k\varphi)\left(\frac{\rho_k x}{\rho_k}\right) \leq u(\rho_k x) \quad \text{in } O. \tag{4.7}$$

Notice that if $x \in O$, with O neighborhood of x_0, $\rho_k x \in \rho_k O = O'$, with O' neighborhood of $\rho_k x_0$. For instance, if we take O as $B_r(x_0)$ and $x \in O$,

$$|x - x_0| < r,$$

thus, given that $\rho_k > 0$,

$$|\rho_k x - \rho_k x_0| = \rho_k |x - x_0| < \rho_k r,$$

i.e. $\rho_k x \in B_{\rho_k r}(\rho_k x_0) = \rho_k B_r(x_0)$, which is a neighborhood of $\rho_k x_0$.
Consequently, from this remark, together with (4.6) and (4.7), we obtain that $(\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right)$ touches u from below at $\rho_k x_0$.
To use the fact that u is a viscosity solution to (2.1) in B_1, we need to verify that $\rho_k x_0 \in B_1^+(u)$ and $(\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right) \in C^2(B_1^+(u))$, or however in a neighborhood of $\rho_k x_0$.
As regards the first condition, we know that $x_0 \in B_1^+(u_k)$ and hence, for the definition of u_k, we have $\frac{u(\rho_k x_0)}{\rho_k} > 0$, namely $u(\rho_k x_0) > 0$, because $\rho_k > 0$ and so, seeing as how $\rho_k x_0 \in B_1$, as we have shown before, $\rho_k x_0 \in B_1^+(u)$.
As regards the second condition, instead, $(\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right) \in C^2(O')$, recalling that if $x \in O'$, we can write $x = \rho_k y$, with $y \in O$, for what we have said above, and

$$(\rho_k \varphi)\left(\frac{x}{\rho_k}\right) = (\rho_k \varphi)\left(\frac{\rho_k y}{\rho_k}\right) = (\rho_k \varphi)(y) = \rho_k \varphi(y),$$

namely

$$(\rho_k \varphi)\left(\frac{x}{\rho_k}\right) = \rho_k \varphi(y). \tag{4.8}$$

Moreover, provided that making O smaller, inasmuch as $B_1^+(u_k)$ is open and $x_0 \in B_1^+(u_k)$, we can take $O \subset B_1^+(u_k)$, thus, since $\varphi \in C^2(B_1^+(u_k))$, $\varphi \in C^2(O)$ and from (4.8) $(\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right) \in C^2(O')$.
To sum it up, we have $(\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right) \in C^2(O')$ that touches u from below at $\rho_k x_0 \in B_1^+(u)$.
Therefore, given that u is a viscosity solution to (2.1) in B_1, we get

$$\sum_{i,j} a_{ij}(\rho_k x_0) \left((\rho_k \varphi) \left(\frac{\cdot}{\rho_k}\right)\right)_{ij} (\rho_k x_0) \leq f(\rho_k x_0). \tag{4.9}$$

Now,

$$\begin{aligned}\left((\rho_k \varphi) \left(\frac{\cdot}{\rho_k}\right)\right)_{ij} &= \rho_k \left(\varphi \left(\frac{\cdot}{\rho_k}\right)\right)_{ij} = \rho_k \left(\frac{1}{\rho_k} \varphi_j \left(\frac{\cdot}{\rho_k}\right)\right)_{i} \\ &= \frac{\rho_k}{\rho_k} \left(\varphi_j \left(\frac{\cdot}{\rho_k}\right)\right)_{i} = \frac{1}{\rho_k} \varphi_{ij} \left(\frac{\cdot}{\rho_k}\right),\end{aligned}$$

which implies

$$\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)_{ij}=\frac{1}{\rho_k}\varphi_{ij}\left(\frac{\cdot}{\rho_k}\right)$$

and thus

$$\begin{aligned}\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)_{ij}(\rho_k x_0)&=\frac{1}{\rho_k}\varphi_{ij}\left(\frac{\cdot}{\rho_k}\right)(\rho_k x_0)\\&=\frac{1}{\rho_k}\varphi_{ij}\left(\frac{\rho_k x_0}{\rho_k}\right)=\frac{1}{\rho_k}\varphi_{ij}\left(x_0\right),\end{aligned}$$

in other words

$$\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)(\rho_k x_0)=\frac{1}{\rho_k}\varphi_{ij}(x_0). \tag{4.10}$$

As a consequence, we achieve from (4.9) and (4.10)

$$\sum_{i,j}a_{ij}(\rho_k x_0)\frac{1}{\rho_k}\varphi_{ij}(x_0)\leq f(\rho_k x_0),$$

which implies, because $\rho_k>0$

$$\sum_{i,j}a_{ij}(\rho_k x_0)\varphi_{ij}(x_0)\leq \rho_k f(\rho_k x_0)$$

and for the definitions of a_{ij}^k and f_k,

$$\sum_{i,j}a_{ij}^k(x_0)\varphi_{ij}(x_0)\leq f_k(x_0),$$

that is

$$\sum_{i,j}a_{ij}^k(x)(u_k)_{ij}=f_k \quad \text{in } B_1^+(u_k) \text{ in the viscosity sense,} \tag{4.11}$$

repeating an analogous reasoning with opposite inequalities, if $\varphi\in C^2(B_1^+(u_k))$ touches u_k from above at $x_0\in B_1^+(u_k)$.

To show instead that $|\nabla u_k|=g_k$ on $F(u_k)$, let us consider $\varphi\in C^2(B_1)$ such that φ^+ touches u_k from below at $x_0\in F(u_k)$ and $|\nabla\varphi|(x_0)\neq 0$. Seeing as how φ^+ touches u_k from below at $x_0\in F(u_k)$, we have

$$\varphi^+(x_0)=u_k(x_0) \tag{4.12}$$

and

$$\varphi^+ \le u_k(x) \quad \text{in a neighborhood of } x_0. \tag{4.13}$$

From the definition of u_k, (4.12) reads

$$\varphi^+(x_0) = \frac{u(\rho_k x_0)}{\rho_k},$$

hence, given that $\rho_k > 0$,

$$\rho_k \varphi^+(x_0) = (\rho_k \varphi)^+(x_0) = u(\rho_k x_0)$$

and also

$$(\rho_k \varphi)^+ \left(\frac{\rho_k x_0}{\rho_k} \right) = u(\rho_k x_0). \tag{4.14}$$

Likewise, we have from (4.13)

$$\varphi^+(x) \le \frac{u(\rho_k x)}{\rho_k} \quad \text{in } O,$$

which gives, always since $\rho_k > 0$,

$$\rho_k \varphi^+(x) = (\rho_k \varphi)^+(x) \le u(\rho_k x) \quad \text{in } O$$

and moreover

$$(\rho_k \varphi)^+ \left(\frac{\rho_k x}{\rho_k} \right) \le u(\rho_k x) \quad \text{in } O. \tag{4.15}$$

For what we have noticed before, $\rho_k x \in O'$, where O' is a neighborhood of $\rho_k x_0$ and thus from (4.14) and (4.15), we obtain that $(\rho_k \varphi)^+ \left(\frac{\cdot}{\rho_k} \right)$ touches u from below at $\rho_k x_0$.

To use the fact that u is a solution to (2.1) in B_1, this time, we need to prove that $\rho_k x_0 \in F(u)$, $(\rho_k \varphi) \left(\frac{\cdot}{\rho_k} \right) \in C^2(B_1)$, or however in a neighborhood of $\rho_k x_0$, and $\left| \nabla \left((\rho_k \varphi) \left(\frac{\cdot}{\rho_k} \right) \right) \right| (\rho_k x_0) \neq 0$.

With respect to the first condition, we know that $x_0 \in F(u_k)$, i.e. $u_k(x_0) = 0$ and $\forall B_r(x_0)$, $B_r(x_0) \cap B_1^+(u_k) \neq \emptyset$ and $B_r(x_0) \cap B_1^+(u_k)^c \neq \emptyset$. From the definition of u_k, we get $\frac{u(\rho_k x_0)}{\rho_k} = 0$, namely $u(\rho_k x_0) = 0$ and so, $\forall B_r(\rho_k x_0)$, $B_r(\rho_k x_0) \cap B_1^+(u)^c \neq \emptyset$.

Furthermore, if $\forall B_r(x_0)$, $B_r(x_0) \cap B_1^+(u_k) \neq \emptyset$, it means that there exist at

least a point $\bar{x} \in B_r(x_0)$, such that $\bar{x} \in B_1$ and $u_k(\bar{x}) > 0$, thus for the definition of u_k, $u(\rho_k \bar{x}) > 0$, because $\rho_k > 0$.
In addition, for what we have shown above, since $\bar{x} \in B_r(x_0) \cap B_1$, $\rho_k \bar{x} \in B_{\rho_k r}(\rho_k x_0) \cap B_1$ and hence, inasmuch $u(\rho_k \bar{x}) > 0$, $\rho_k \bar{x} \in B_{\rho_k r}(\rho_k x_0) \cap B_1^+(u)$.
In summary, we have that $\forall B_{\rho_k r}(\rho_k x_0)$, $B_{\rho_k r}(\rho_k x_0) \cap B_1^+(u) \neq \emptyset$.
To show that $\rho_k x_0 \in F(u)$, remain to verify that $\forall B_r(\rho_k x_0)$, $B_r(\rho_k x_0) \cap B_1^+(u) \neq \emptyset$, but if we fix a ball $B_{\bar{r}}(\rho_k x_0)$, we can consider $B_{\frac{\bar{r}}{\rho_k}(x_0)}$ and for what we have said before, $B_{\rho_k \frac{\bar{r}}{\rho_k}}(\rho_k x_0) \cap B_1^+(u) \neq \emptyset$, that is $B_{\bar{r}}(\rho_k x_0) \cap B_1^+(u) \neq \emptyset$ and therefore $\forall B_r(\rho_k x_0)$, $B_r(\rho_k x_0) \cap B_1^+(u) \neq \emptyset$, which, together with $u(\rho_k x_0) = 0$ and $B_r(\rho_k x_0) \cap B_1^+(u)^c \neq \emptyset$, $\forall B_r(\rho_k x_0)$, gives $\rho_k x_0 \in F(u)$.
With reference to the second condition, repeating the same reasoning done above, recalling that B_1 is open and $x_0 \in F(u) \subset B_1$, $(\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right) \in C^2(O')$.
Concerning the third condition, instead,

$$\begin{aligned}
&\nabla\left((\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right)\right) = \rho_k \nabla\left(\varphi\left(\frac{\cdot}{\rho_k}\right)\right) \\
&= \rho_k \left(\frac{\partial}{\partial x_1}\left(\varphi\left(\frac{\cdot}{\rho_k}\right)\right), \frac{\partial}{\partial x_2}\left(\varphi\left(\frac{\cdot}{\rho_k}\right)\right), \dots, \frac{\partial}{\partial x_n}\left(\varphi\left(\frac{\cdot}{\rho_k}\right)\right)\right) \\
&= \rho_k \left(\frac{1}{\rho_k}\frac{\partial \varphi}{\partial x_1}\left(\frac{\cdot}{\rho_k}\right), \frac{1}{\rho_k}\frac{\partial \varphi}{\partial x_2}\left(\frac{\cdot}{\rho_k}\right), \dots, \frac{1}{\rho_k}\frac{\partial \varphi}{\partial x_n}\left(\frac{\cdot}{\rho_k}\right)\right) \\
&= \frac{\rho_k}{\rho_k}\nabla\varphi\left(\frac{\cdot}{\rho_k}\right) = \nabla\varphi\left(\frac{\cdot}{\rho_k}\right),
\end{aligned}$$

which gives

$$\nabla\left((\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right)\right) = \nabla\varphi\left(\frac{\cdot}{\rho_k}\right),$$

and thus

$$\left|\nabla\left((\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right)\right)\right|(\rho_k x_0) = \left|\nabla\varphi\left(\frac{\cdot}{\rho_k}\right)\right|(\rho_k x_0) = \left|\nabla\varphi\right|(x_0) \neq 0,$$

i.e.

$$\left|\nabla\left((\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right)\right)(\rho_k x_0) = \nabla\varphi\right|(x_0), \tag{4.16}$$

and

$$\left|\nabla\left((\rho_k \varphi)\left(\frac{\cdot}{\rho_k}\right)\right)\right|(\rho_k x_0) \neq 0.$$

To sum it up, we have $(\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right) \in C^2(O')$ such that $(\rho_k\varphi)^+\left(\frac{\cdot}{\rho_k}\right)$ touches u from below at $\rho_k x_0 \in F(u)$ and $\left|\nabla\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)\right|(\rho_k x_0) \neq 0$. Consequently, inasmuch as u is a solution to (2.1) in B_1,

$$\left|\nabla\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)\right|(\rho_k x_0) \leq g(\rho_k x_0)$$

which gives from (4.16)

$$|\nabla\varphi|(x_0) \leq g(\rho_k x_0)$$

and for the definition of g_k,

$$|\nabla\varphi|(x_0) \leq g_k(x_0),$$

that is

$$|\nabla u_k| = g_k \quad \text{on } F(u_k) \text{ in the viscosity sense,} \tag{4.17}$$

repeating an analogous reasoning with opposite inequalities, if $\varphi \in C^2(B_1)$ is such that φ^+ touches u_k from above at $x_0 \in F(u_k)$ and $|\nabla\varphi|(x_0) \neq 0$.
Therefore, considering together (4.11) and (4.17), we obtain that each u_k solves (2.1) in B_1 with coefficients a_{ij}^k, right hand side f_k and free boundary condition g_k.
Now, for the chosen $\bar{r}$, by taking $\bar{\varepsilon} = \varepsilon_0(\bar{r})^2$ the assumption (2.2) holds for $\varepsilon = \varepsilon_k := 2^{-k}\varepsilon_0(\bar{r})$.
Indeed, in B_1, given that from (4.3), $\rho_k x \in B_1$, if $x \in B_1$ and $\|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}$ in view of the second inequality in (4.2), we have

$$|f_k(x)| = |\rho_k f(\rho_k x)| = \rho_k\,|f(\rho_k x)| \leq \rho_k\,\|f\|_{L^\infty(B_1)} \leq \rho_k\bar{\varepsilon} = \bar{r}^k\bar{\varepsilon}, \tag{4.18}$$

seeing as how $\rho_k > 0$ and $\rho_k = \bar{r}^k$. In addition, from the condition $\bar{r}^\beta \leq 1/4$, since $\bar{r} < 1$, as we have shown before and $0 < \beta \leq 1$, we get $\bar{r} \leq \bar{r}^\beta \leq 1/4$, namely $\bar{r} \leq 1/4 = 2^{-2}$ and thus $\bar{r}^k \leq (1/4)^k = 2^{-2k}$ for $k = 0, 1, \ldots$ As a consequence, from (4.18), we achieve for the definition of $\bar{\varepsilon}$

$$|f_k(x)| \leq \bar{\varepsilon}\bar{r}^k \leq \varepsilon_0(\bar{r})^2 2^{-2k} = \varepsilon_k^2,$$

i.e. $|f_k(x)| \leq \varepsilon_k^2$, with $x \in B_1$, so ε_k^2 is an upper bound of the set $\{|f_k(x)|\,,\ \ x \in B_1\}$ and hence

$$\|f_k\|_{L^\infty(B_1)} = \sup_{x\in B_1} |f_k(x)| \leq \varepsilon_k^2,$$

which gives $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2$, as desired.
Concerning the free boundary condition g_k, instead, always since from (4.3), $\rho_k x \in B_1$, if $x \in B_1$, given that $g(0) = 1$, in view of the third inequality in (4.2), $[g]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon}$, and the definition of $[g]_{C^{0,\beta}(B_1)}$ (see Definition A.1), we have in B_1

$$|g_k(x) - 1| = |g(\rho_k x) - g(0)| \leq [g]_{C^{0,\beta}(B_1)} |\rho_k x|^\beta \leq \bar{\varepsilon}\rho_k^\beta = \bar{\varepsilon}\bar{r}^{k\beta}, \qquad (4.19)$$

because $|x| < 1$, with $x \in B_1$ and thus $|x|^\beta < 1$, recalling that $0 < \beta \leq 1$ and always since $\rho_k > 0$ and $\rho_k = \bar{r}^k$. Furthermore, we know that $\bar{r}^\beta \leq 1/4$, hence, inasmuch as $\bar{r}^\beta \geq 0$ and $1/4 \geq 0$, $\bar{r}^{k\beta} \leq (1/4)^k = 2^{-2k}$, for $k = 0, 1, \ldots$ Therefore, from (4.19), we obtain for the definition of $\bar{\varepsilon}$

$$|g_k(x) - 1| \leq \bar{\varepsilon}\bar{r}^{k\beta} \leq \varepsilon_0(\bar{r})^2 2^{-2k} = \varepsilon_k^2,$$

in other words, $|g_k(x) - 1| \leq \varepsilon_k^2$, with $x \in B_1$, thereby ε_k^2 is an upper bound of the set $\{|g_k(x) - 1|\,,\ \ x \in B_1\}$ and hence

$$\|g_k - 1\|_{L^\infty(B_1)} = \sup_{x\in B_1} |g_k(x) - 1| \leq \varepsilon_k^2,$$

which gives $\|g_k - 1\|_{L^\infty(B_1)} \leq \varepsilon_k^2$, as desired.
Finally, as regards the coefficients a_{ij}^k, always since from (4.3), $\rho_k x \in B_1$ if $x \in B_1$, given that $a_{ij}(0) = \delta_{ij}$, in view of the first inequality in (4.2), $[a_{ij}]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon}$ and the definition of $[a_{ij}]_{C^{0,\beta}(B_1)}$ (see Definition A.1), we have in B_1

$$\left|a_{ij}^k(x) - \delta_{ij}\right| = |a_{ij}(\rho_k x) - a_{ij}(0)| \leq [a_{ij}]_{C^{0,\beta}(B_1)} |\rho_k x|^\beta \leq \bar{\varepsilon}\rho_k^\beta = \bar{\varepsilon}\bar{r}^{k\beta} \quad (4.20)$$

seeing as how $|x|^\beta < 1$ for what we have said before and always since $\rho_k > 0$ and $\rho_k = \bar{r}^k$. Consequently, inasmuch as $\bar{r}^{k\beta} \leq 2^{-2k}$ for $k = 0, 1, \ldots$, as shown above, we get from (4.20), for the definition of $\bar{\varepsilon}$

$$\left|a_{ij}^k(x) - \delta_{ij}\right| \leq \bar{\varepsilon}\bar{r}^{k\beta} \leq \varepsilon_0(\bar{r})^2 2^{-2k} = \varepsilon_k^2 \leq \varepsilon_k, \qquad (4.21)$$

because $0 < \varepsilon_k < 1$, recalling that $0 < \varepsilon_0(\bar{r}) < 1$. Therefore, from (4.21), we achieve $\left|a_{ij}^k(x) - \delta_{ij}\right| \leq \varepsilon_k$, with $x \in B_1$, thus ε_k is an upper bound of the set $\left\{\left|a_{ij}^k(x) - \delta_{ij}\right|, \ x \in B_1\right\}$ and hence

$$\left\|a_{ij}^k - \delta_{ij}\right\|_{L^\infty(B_1)} = \sup_{x \in B_1} \left|a_{ij}^k(x) - \delta_{ij}\right| \leq \varepsilon_k,$$

which gives $\left\|a_{ij}^k - \delta_{ij}\right\|_{L^\infty(B_1)} \leq \varepsilon_k$, as desired.
To sum it up, we have shown that the assumption (2.2) holds for ε_k, for every $k = 0, 1, \dots$ and thus each u_k is a solution to (2.1) in B_1 with coefficients a_{ij}^k, right hand side f_k and free boundary condition g_k, which satisfy (2.2) with ε_k.
Now, the hypothesis (4.1) guarantees that for $k = 0$ also the flatness assumption (3.1) in Lemma 3.1 is satisfied by u_0. Precisely, with $k = 0$, we have $\rho_0 = \bar{r}^0 = 1$, which gives, for the definition of u_k, $u_0 = u$. As a consequence, from (4.1),

$$(x_n - \bar{\varepsilon})^+ \leq u_0(x) \leq (x_n + \bar{\varepsilon})^+, \quad x \in B_1 \tag{4.22}$$

and given that $0 < \varepsilon_0(\bar{r}) < 1$, $\bar{\varepsilon} = \varepsilon_0(\bar{r})^2 \leq \varepsilon_0(\bar{r})$, hence the flatness assumption (3.1) in Lemma 3.1 is satisfied by u_0. In addition, since $\bar{\varepsilon} \leq \varepsilon_0(\bar{r})$ and writing ε_0 for $\varepsilon_0(\bar{r})$, $x_n + \bar{\varepsilon} \leq x_n + \varepsilon_0$, with $x \in B_1$, which implies

$$(x_n+\bar{\varepsilon})^+ = \max(0, x_n+\bar{\varepsilon}) \leq \max(0, x_n+\varepsilon_0) = (x_n+\varepsilon_0)^+, \quad x \in B_1. \tag{4.23}$$

Analogously, because $-\bar{\varepsilon} \geq -\varepsilon_0$, if $\bar{\varepsilon} \leq \varepsilon_0$, $x_n - \bar{\varepsilon} \geq x_n - \varepsilon_0$ with $x \in B_1$, which implies

$$(x_n-\bar{\varepsilon})^+ = \max(0, x_n-\bar{\varepsilon}) \geq \max(0, x_n-\varepsilon_0) = (x_n-\varepsilon_0)^+, \quad x \in B_1. \tag{4.24}$$

Therefore, from (4.22), (4.23) and (4.24), we achieve

$$(x_n - \varepsilon_0)^+ \leq u_0(x) \leq (x_n + \varepsilon_0)^+, \quad x \in B_1. \tag{4.25}$$

In addition, we can write $x_n = x \cdot e_n$ and setting $\nu_0 = e_n$, we get from (4.25)

$$(x \cdot \nu_0 - \varepsilon_0)^+ \leq u_0(x) \leq (x \cdot \nu_0 + \varepsilon_0)^+ \quad x \in B_1. \tag{4.26}$$

Consequently, we state that it follows by an induction on k and Lemma 3.1 that each u_k, with $k \geq 1$, satisfies

$$(x \cdot \nu_k - \varepsilon_k)^+ \leq u_k(x) \leq (x \cdot \nu_k + \varepsilon_k)^+ \quad x \in B_1,$$

with $|\nu_k| = 1$ and $|\nu_k - \nu_{k-1}| \leq C\varepsilon_{k-1}$ for a universal constant C.
Let us analyze the case of $k = 1$.
For what we have shown above, we have that u_0 is a solution to (2.1)-(2.2) in B_1 satisfying (4.25), with $0 \in F(u_0)$, recalling that $0 \in F(u)$ and $u_0 = u$. Hence, because we have chosen $\bar{r}$ such that $\bar{r} \leq r_0$, where r_0 is the universal constant of Lemma 3.1, we can apply Lemma 3.1 with $\bar{r}$ and $\varepsilon_0 = \varepsilon_0(\bar{r})$ to obtain

$$\left(x \cdot \nu_1 - \varepsilon_0 \frac{\bar{r}}{2}\right)^+ \leq u_0(x) \leq \left(x \cdot \nu_1 + \varepsilon_0 \frac{\bar{r}}{2}\right)^+, \quad x \in B_{\bar{r}}, \tag{4.27}$$

with $|\nu_1| = 1$ and $|\nu_1 - e_n| = |\nu_1 - \nu_0| \leq C\varepsilon_0$, i.e. $|\nu_1 - \nu_0| \leq C\varepsilon_0$, for a universal constant C.
Notice that for $k = 1$, $\rho_1 = \bar{r}$, thus we can rewrite (4.27)

$$\left(x \cdot \nu_1 - \varepsilon_0 \frac{\rho_1}{2}\right)^+ \leq u_0(x) \leq \left(x \cdot \nu_1 + \varepsilon_0 \frac{\rho_1}{2}\right)^+, \quad x \in B_{\rho_1}. \tag{4.28}$$

Furthermore, if $x \in B_{\rho_1}$, we can write $x = \rho_1 y$, with $y \in B_1$. Indeed, fixed $\bar{x} \in B_{\rho_1}$, we can take $\bar{y}$ as $\bar{y} = \frac{\bar{x}}{\rho_1}$, seeing as how $\rho_1 = \bar{r} \neq 0$, with $|\bar{y}| = \left|\frac{\bar{x}}{\rho_1}\right| = \frac{1}{\rho_1}|\bar{x}| < \frac{\rho_1}{\rho_1} = 1$, inasmuch as $\rho_1 > 0$, i.e. $|\bar{y}| < 1$, thus $\bar{y} \in B_1$ and moreover $\bar{x} = \rho_1 \bar{y}$ for the definition of $\bar{y}$. Conversely, if $\bar{y} \in B_1$, $\bar{x} = \rho_1 \bar{y}$ is such that $|\bar{x}| = |\rho_1 \bar{y}| = \rho_1 |\bar{y}| < \rho_1$, given that $\rho_1 > 0$, that is $|\bar{x}| < \rho_1$ and $\bar{x} \in B_{\rho_1}$. As a consequence, from (4.28), we get, since $(\rho_1 y) \cdot \nu_1 = \rho_1(y \cdot \nu_1)$,

$$\left(\rho_1(y \cdot \nu_1) - \varepsilon_0 \frac{\rho_1}{2}\right)^+ \leq u_0(\rho_1 y) \leq \left(\rho_1(y \cdot \nu_1) + \varepsilon_0 \frac{\rho_1}{2}\right)^+, \quad y \in B_1$$

and dividing by $\rho_1 > 0$

$$\frac{1}{\rho_1}\left(\rho_1(y \cdot \nu_1) - \varepsilon_0 \frac{\rho_1}{2}\right)^+ \leq \frac{u_0(\rho_1 y)}{\rho_1} \leq \frac{1}{\rho_1}\left(\rho_1(y \cdot \nu_1) + \varepsilon_0 \frac{\rho_1}{2}\right)^+, \; y \in B_1. \tag{4.29}$$

Also, for what we have said before, $u_0 = u$, hence for the definition of u_1, $\frac{u_0(\rho_1 y)}{\rho_1} = u_1(y)$ and from (4.29) we achieve

$$\frac{1}{\rho_1}\left(\rho_1(y\cdot\nu_1) - \varepsilon_0\frac{\rho_1}{2}\right)^+ \leq u_1(y) \leq \frac{1}{\rho_1}\left(\rho_1(y\cdot\nu_1) + \varepsilon_0\frac{\rho_1}{2}\right)^+, \;\; y \in B_1. \tag{4.30}$$

In addition, because $\rho_1 > 0$,

$$\begin{aligned}\frac{1}{\rho_1}\left(\rho_1(y\cdot\nu_1) - \varepsilon_0\frac{\rho_1}{2}\right)^+ &= \left(\frac{1}{\rho_1}\left(\rho_1(y\cdot\nu_1) - \varepsilon_0\frac{\rho_1}{2}\right)\right)^+ \\ &= \left(\frac{\rho_1}{\rho_1}(y\cdot\nu_1) - \frac{\varepsilon_0}{\rho_1}\frac{\rho_1}{2}\right)^+ = \left(y\cdot\nu_1 - \frac{\varepsilon_0}{2}\right)^+\end{aligned}$$

and analogously,

$$\frac{1}{\rho_1}\left(\rho_1(y\cdot\nu_1) + \varepsilon_0\frac{\rho_1}{2}\right)^+ = \left(y\cdot\nu_1 + \frac{\varepsilon_0}{2}\right)^+,$$

therefore from (4.30),

$$\left(y\cdot\nu_1 - \frac{\varepsilon_0}{2}\right)^+ \leq u_1(y) \leq \left(y\cdot\nu_1 + \frac{\varepsilon_0}{2}\right)^+, \quad y \in B_1,$$

that is recalling $y = x$ and given that for the definition of ε_1, $\frac{\varepsilon_0}{2} = \varepsilon_0 2^{-1} = \varepsilon_1$

$$(x\cdot\nu_1 - \varepsilon_1)^+ \leq u_1(x) \leq (x\cdot\nu_1 + \varepsilon_1)^+, \quad x \in B_1, \tag{4.31}$$

with $|\nu_1| = 1$ and $|\nu_1 - \nu_0| \leq C\varepsilon_0$, for a universal constant C, namely the thesis holds for $k = 1$.
Suppose now that the thesis holds for k and show that holds for $k+1$.
We have from the hypothesis of induction that

$$(x\cdot\nu_k - \varepsilon_k)^+ \leq u_k(x) \leq (x\cdot\nu_k + \varepsilon_k)^+, \quad x \in B_1 \tag{4.32}$$

with $|\nu_k| = 1$ and $|\nu_k - \nu_{k-1}| \leq C\varepsilon_{k-1}$.
To apply Lemma 3.1 with ν_k in place of e_n and thus $x\cdot\nu_k$ in place of x_n, we need to show that $0 \in F(u_k)$. In particular, we know from the hypothesis of Theorem 4.2 that $0 \in F(u)$, in other words $\forall B_r$, $B_r \cap B_1^+(u) \neq \emptyset$ and $B_r \cap B_1^+(u)^c \neq \emptyset$, and $u(0) = 0$. Notice that for the definition of u_k, $u_k(0) =$

$\frac{u(\rho_k 0)}{\rho_k} = \frac{u(0)}{\rho_k} = 0$, namely $u_k(0) = 0$, hence $0 \in B_1^+(u_k)^c$ and $\forall B_r$, $B_r \cap B_1^+(u_k)^c \neq \emptyset$. Now, we want to prove that $\forall B_r$, $B_r \cap B_1^+(u_k) \neq \emptyset$. As a consequence, let us fix B_{r_0}. For what we have said above, recalling that $0 \in F(u)$, if we consider $B_{\rho_k r_0}$, $B_{\rho_k r_0} \cap B_1^+(u) \neq \emptyset$, that is there exist points x such that $x \in B_{\rho_k r_0} \cap B_1^+(u)$. Furthermore, we can assume that there exists $\bar{x} \in B_{\rho_k r_0} \cap B_1^+(u)$ such that $|\bar{x}| < \rho_k$. Indeed, if $r_0 \leq 1$, since $\bar{x} \in B_{\rho_k r_0}$, $|\bar{x}| < \rho_k r_0 \leq \rho_k$, that is $|\bar{x}| < \rho_k$, while if $r_0 > 1$, seeing as how also $B_{\rho_k} \cap B_1^+(u) \neq \emptyset$, because $0 \in F(u)$, we can take $\bar{x} \in B_{\rho_k} \cap B_1^+(u)$ and given that $\rho_k < \rho_k r_0$, inasmuch as $r_0 > 1$, $B_{\rho_k} \subset B_{\rho_k r_0}$ and thus $B_{\rho_k} \cap B_1^+(u) \subset B_{\rho_k r_0} \cap B_1^+(u)$, i.e. $\bar{x} \in B_{\rho_k r_0} \cap B_1^+(u)$, as we have supposed. In addition, since $\bar{x} \in B_{\rho_k r_0}$, we can write $\bar{x} = \rho_k \bar{y}$, with $\bar{y} \in B_{r_0}$, repeating the same reasoning done to show that if $x \in B_r$, we can write $x = ry$, with $y \in B_1$. Nevertheless, seeing as how $|\bar{x}| < \rho_k$, $|\rho_k \bar{y}| = |\bar{x}| < \rho_k$, hence $|\rho_k \bar{y}| < \rho_k$ and given that $\rho_k > 0$, $\rho_k |\bar{y}| < \rho_k$, which implies $|\bar{y}| < \frac{\rho_k}{\rho_k} = 1$, that is $|\bar{y}| < 1$ and $\bar{y} \in B_1$. On the other hand, $\bar{x} \in B_1^+(u)$, therefore $u(\bar{x}) > 0$ and recalling that $\bar{x} = \rho_k \bar{y}$, $u(\rho_k \bar{y}) > 0$, which gives, inasmuch as $\rho_k > 0$, $\frac{u(\rho_k \bar{y})}{\rho_k} = u_k(\bar{y}) > 0$, namely $u_k(\bar{y}) > 0$. To sum it up, we have shown that $\bar{y} \in B_1$ and $u_k(\bar{y}) > 0$, in other words $\bar{y} \in B_1^+(u_k)$. Moreover, $\bar{y} \in B_{r_0}$, thus $\bar{y} \in B_{r_0} \cap B_1^+(u_k)$ and $B_{r_0} \cap B_1^+(u_k) \neq \emptyset$. For the arbitrariness of B_{r_0}, we achieve that $B_r \cap B_1^+(u_k) \neq \emptyset \; \forall B_r$ and hence, putting together this fact and $B_r \cap B_1^+(u_k)^c \neq \emptyset \; \forall B_r$, we obtain that $0 \in F(u_k)$, as desired.

Now, because u_k is a solution to (2.1)-(2.2) in B_1 satisfying (4.32), with $0 \in F(u_k)$, we can apply Lemma 3.1 with radius $\bar{r}$, for what we have said in the case of $k = 1$, and with $\varepsilon_k = \varepsilon_0(\bar{r}) 2^{-k} \leq \varepsilon_0(\bar{r})$, i.e. $\varepsilon_k \leq \varepsilon_0(\bar{r})$, and we get

$$\left(x \cdot \nu_{k+1} - \varepsilon_k \frac{\bar{r}}{2}\right)^+ \leq u_k(x) \leq \left(x \cdot \nu_{k+1} + \varepsilon_k \frac{\bar{r}}{2}\right)^+, \quad x \in B_{\bar{r}}, \tag{4.33}$$

with $|\nu_{k+1}| = 1$ and $|\nu_{k+1} - \nu_k| \leq C\varepsilon_k$ for a universal constant C.
In addition, if $x \in B_{\bar{r}}$, $x = \bar{r} y$ with $y \in B_1$, thus we can rewrite (4.33)

$$\left((\bar{r} y) \cdot \nu_{k+1} - \varepsilon_k \frac{\bar{r}}{2}\right)^+ \leq u_k(\bar{r} y) \leq \left((\bar{r} y) \cdot \nu_{k+1} + \varepsilon_k \frac{\bar{r}}{2}\right)^+, \quad y \in B_1, \tag{4.34}$$

and dividing by $\bar{r} > 0$, namely $\bar{r} \neq 0$,

$$\frac{1}{\bar{r}}\left((\bar{r}y)\cdot\nu_{k+1} - \varepsilon_k\frac{\bar{r}}{2}\right)^+ \leq \frac{u_k(\bar{r}y)}{\bar{r}} \leq \frac{1}{\bar{r}}\left((\bar{r}y)\cdot\nu_{k+1} + \varepsilon_k\frac{\bar{r}}{2}\right)^+ \quad y \in B_1,$$

which implies, given that $(\bar{r}y)\cdot\nu_{k+1} = \bar{r}(y\cdot\nu_{k+1})$,

$$\frac{1}{\bar{r}}\left(\bar{r}(y\cdot\nu_{k+1}) - \varepsilon_k\frac{\bar{r}}{2}\right)^+ \leq \frac{u_k(\bar{r}y)}{\bar{r}} \leq \frac{1}{\bar{r}}\left(\bar{r}y(\cdot\nu_{k+1}) + \varepsilon_k\frac{\bar{r}}{2}\right)^+ \quad y \in B_1$$

and, since $\bar{r} > 0$, analogously to the case of $k = 1$ with ρ_1,

$$\left(\frac{\bar{r}}{\bar{r}}(y\cdot\nu_{k+1}) - \frac{\varepsilon_k}{\bar{r}}\frac{\bar{r}}{2}\right)^+ \leq \frac{u_k(\bar{r}y)}{\bar{r}} \leq \left(\frac{\bar{r}}{\bar{r}}(y\cdot\nu_{k+1}) + \frac{\varepsilon_k}{\bar{r}}\frac{\bar{r}}{2}\right)^+ \quad y \in B_1,$$

that is

$$\left(y\cdot\nu_{k+1} - \frac{\varepsilon_k}{2}\right)^+ \leq \frac{u_k(\bar{r}y)}{\bar{r}} \leq \left(y\cdot\nu_{k+1} + \frac{\varepsilon_k}{2}\right)^+ \quad y \in B_1. \tag{4.35}$$

Now, for the definition of u_k

$$\frac{u_k(\bar{r}y)}{\bar{r}} = \frac{u(\rho_k\bar{r}y)}{\rho_k}\frac{1}{\bar{r}} = \frac{u(\rho_k\bar{r}y)}{\rho_k\bar{r}},$$

in other words,

$$\frac{u_k(\bar{r}y)}{\bar{r}} = \frac{u(\rho_k\bar{r}y)}{\rho_k\bar{r}}, \tag{4.36}$$

and because $\rho_k = \bar{r}^k$, $\rho_k\bar{r} = \bar{r}^k\bar{r} = \bar{r}^{k+1} = \rho_{k+1}$, thus in view of (4.36) and for the definition of u_k

$$\frac{u_k(\bar{r}y)}{\bar{r}} = \frac{u(\rho_{k+1}y)}{\rho_{k+1}} = u_{k+1}(y),$$

which gives from (4.35)

$$\left(y\cdot\nu_{k+1} - \frac{\varepsilon_k}{2}\right)^+ \leq u_{k+1}(y) \leq \left(y\cdot\nu_{k+1} + \frac{\varepsilon_k}{2}\right)^+ \quad y \in B_1. \tag{4.37}$$

Furthermore, we have $\varepsilon_k = 2^{-k}\varepsilon_0(\bar{r})$, therefore

$$\frac{\varepsilon_k}{2} = \varepsilon_k 2^{-1} = 2^{-k}\varepsilon_0(\bar{r})2^{-1} = 2^{-k}2^{-1}\varepsilon_0(\bar{r}) = 2^{-(k+1)}\varepsilon_0(\bar{r}) = \varepsilon_{k+1},$$

namely

$$\frac{\varepsilon_k}{2} = \varepsilon_{k+1},$$

which implies from (4.37)

$$(y \cdot \nu_{k+1} - \varepsilon_{k+1})^+ \leq u_{k+1}(y) \leq (y \cdot \nu_{k+1} + \varepsilon_{k+1})^+ \quad y \in B_1.$$

Consequently, setting $y = x$, we have obtained

$$(x \cdot \nu_{k+1} - \varepsilon_{k+1})^+ \leq u_{k+1}(x) \leq (x \cdot \nu_{k+1} + \varepsilon_{k+1})^+ \quad x \in B_1$$

together with $|\nu_{k+1}| = 1$ and $|\nu_{k+1} - \nu_k| \leq C\varepsilon_k$ for a universal constant C.
Summarizing, we have shown by induction on $k \geq 1$ that

$$(x \cdot \nu_k - \varepsilon_k)^+ \leq u_k(x) \leq (x \cdot \nu_k + \varepsilon_k)^+ \quad x \in B_1,$$

with $|\nu_k| = 1$ and $|\nu_k - \nu_{k-1}| \leq C\varepsilon_{k-1}$ for a universal constant C.
Let us show now that there exists a vector ν such that $\nu_k \to \nu$ as $k \to \infty$.
For this purpose, it suffices to verify that the condition $|\nu_k - \nu_{k-1}| \leq C\varepsilon_{k-1}$ implies that the sequence ν_k is a Cauchy sequence and thus convergent.
In particular, we have to prove that $\forall \delta > 0$, there exists $\bar{k} \in \mathbb{N}$ such that

$$|\nu_k - \nu_h| < \delta \quad \forall k, h \in \mathbb{N}, \;\; k, h \geq \bar{k}.$$

To this end, notice that we can assume without loss of generality that $k > h$ and we can write

$$\begin{aligned} |\nu_k - \nu_h| &= |\nu_k - \nu_{k-1} + \nu_{k-1} - \nu_{k-2} + \ldots + \nu_{h+1} - \nu_h| \\ &= |(\nu_k - \nu_{k-1}) + (\nu_{k-1} - \nu_{k-2}) + \ldots + (\nu_{h+1} - \nu_h)|, \end{aligned}$$

which gives for the triangular inequality of $|\cdot|$,

$$|\nu_k - \nu_h| \leq |\nu_k - \nu_{k-1}| + |\nu_{k-1} - \nu_{k-2}| + |\ldots| + |\nu_{h+1} - \nu_h|$$

and hence, using the condition $|\nu_k - \nu_{k-1}| \leq C\varepsilon_{k-1}$, $\forall k$, we obtain

$$|\nu_k - \nu_h| \leq C\varepsilon_{k-1} + C\varepsilon_{k-2} + \ldots + C\varepsilon_h = C\left(\sum_{j=h}^{k-1} \varepsilon_j\right),$$

namely

$$|\nu_k - \nu_h| \leq C\left(\sum_{j=h}^{k-1} \varepsilon_j\right). \tag{4.38}$$

Moreover, we remark that for the definition of ε_k,

$$\varepsilon_j = \varepsilon_0 2^{-j} = \varepsilon_0 2^{-(j-h+h)} = \varepsilon_0 2^{-h} 2^{-(j-h)} = \varepsilon_h 2^{-(j-h)} \quad j = h, \dots, k-1,$$

that is

$$\varepsilon_j = \varepsilon_h 2^{-(j-h)} \quad j = h, \dots, k-1,$$

therefore from (4.38) we get

$$|\nu_k - \nu_h| \leq C \left(\sum_{j=h}^{k-1} \varepsilon_h 2^{-(j-h)} \right) = C\varepsilon_h \left(\sum_{j=h}^{k-1} 2^{-(j-h)} \right),$$

i.e.

$$|\nu_k - \nu_h| \leq C\varepsilon_h \left(\sum_{j=h}^{k-1} 2^{-(j-h)} \right),$$

and calling $l = j - h$, which varies from 0 to $k-1-h$, if j varies from h to $k-1$,

$$|\nu_k - \nu_h| \leq C\varepsilon_h \left(\sum_{l=0}^{k-1-h} 2^{-l} \right). \tag{4.39}$$

In addition, because $2^{-l} \geq 0$,

$$\sum_{l=0}^{k-1-h} 2^{-l} \leq \sum_{l=0}^{\infty} 2^{-l} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2,$$

in other words

$$\sum_{l=0}^{k-1-h} 2^{-l} \leq 2,$$

which implies from (4.39)

$$|\nu_k - \nu_h| \leq 2C\varepsilon_h. \tag{4.40}$$

At this point, if we fix $\delta > 0$ and we want $|\nu_k - \nu_h| < \delta$ with $k,\ h \geq \bar{k}$, we can observe that $\varepsilon_h = \varepsilon_0 2^{-h} \leq \varepsilon_0 2^{-\bar{k}}$, recalling that $h \geq \bar{k}$, as a consequence from (4.40), we achieve

$$|\nu_k - \nu_h| \leq 2C\varepsilon_0 2^{-\bar{k}}, \tag{4.41}$$

hence if we set

$$2C\varepsilon_0 2^{-\bar{k}} < \delta, \tag{4.42}$$

we have from (4.41)

$$|\nu_k - \nu_h| < \delta, \quad \forall k, h \in \mathbb{N}, \ \ k, h \geq \bar{k},$$

given that $h \geq \bar{k}$ and $k > h$ for what we have supposed, and thus ν_k is a Cauchy sequence.

If we want to establish $\bar{k}$ with more precision, we have from (4.42)

$$2^{-\bar{k}} < \frac{\delta}{2C\varepsilon_0},$$

which gives, since $2 > 1$

$$-\bar{k} < \log_2 \frac{\delta}{2C\varepsilon_0}$$

and

$$\bar{k} > -\log_2 \frac{\delta}{2C\varepsilon_0},$$

hence we can take $\bar{k}$ as, for instance, $\bar{k} = \lceil -\log_2 \frac{\delta}{2C\varepsilon_0} \rceil$.

Consequently, seeing as how ν_k is a Cauchy sequence, there exists ν such that $\nu_k \to \nu$ as $k \to \infty$.

Now, we want to show that $u \in C^{1,\alpha}(F(u) \cap B_{1/2})$.

Precisely, we claim that

$$\frac{|u(x) - u(0) - x \cdot \nu|}{|x|} \to 0 \quad |x| \to 0, \ \ x \in (B_1^+(u) \cup F(u)),$$

and therefore $\nu = \nabla u(0)$.

To prove this fact, first of all we notice that $u(0) = 0$, recalling that $0 \in F(u)$, thus

$$|u(x) - u(0) - x \cdot \nu| = |u(x) - x \cdot \nu| \,.$$

Also, if $|x| \to 0$, $x \neq 0$, and we can suppose that $|x| \leq \bar{r} = \rho_1$, namely $x \in B_{\rho_1}$.

So, assume that $x \in B_{\rho_1} \cap (B_1^+(u) \cup F(u))$, $x \neq 0$.

In particular, inasmuch as $x \neq 0$ and $x \in B_{\rho_1}$, there exists an integer k with

$k \geq 0$, such that $\rho_{k+1} \leq |x| \leq \rho_k$, i.e. $x \in B_{\rho_k}$, given that $\rho_k = \bar{r}^k \to 0$ as $k \to \infty$, since $\bar{r} < 1$, as we have already shown. As a consequence, because $x \in B_{\rho_k}$, $x = \rho_k y$, $y \in B_1$, thus for the definition of u_k, $\frac{u(x)}{\rho_k} = u_k(y)$ and from

$$(x \cdot \nu_k - \varepsilon_k)^+ \leq u_k(x) \leq (x \cdot \nu_k + \varepsilon_k)^+, \quad x \in B_1,$$

calling $x = y$, we have

$$(y \cdot \nu_k - \varepsilon_k)^+ \leq u_k(y) \leq (y \cdot \nu_k + \varepsilon_k)^+, \quad y \in B_1,$$

which implies, inasmuch $y = \frac{x}{\rho_k}$ and $\frac{u(x)}{\rho_k} = u_k(y)$,

$$\left(\left(\frac{x}{\rho_k}\right) \cdot \nu_k - \varepsilon_k\right)^+ \leq \frac{u(x)}{\rho_k} \leq \left(\left(\frac{x}{\rho_k}\right) \cdot \nu_k + \varepsilon_k\right)^+ \quad x \in B_{\rho_k},$$

and multiplying by $\rho_k > 0$, seeing as how $\left(\frac{x}{\rho_k}\right) \cdot \nu_k = \frac{1}{\rho_k}(x \cdot \nu_k)$,

$$\rho_k \left(\frac{1}{\rho_k}(x \cdot \nu_k) - \varepsilon_k\right)^+ \leq u(x) \leq \rho_k \left(\frac{1}{\rho_k}(x \cdot \nu_k) + \varepsilon_k\right)^+ \quad x \in B_{\rho_k},$$

which implies, as we have said above, because $\rho_k > 0$

$$\left(\frac{\rho_k}{\rho_k}(x \cdot \nu_k) - \rho_k\varepsilon_k\right)^+ \leq u(x) \leq \left(\frac{\rho_k}{\rho_k}(x \cdot \nu_k) + \rho_k\varepsilon_k\right)^+ \quad x \in B_{\rho_k},$$

that is

$$(x \cdot \nu_k - \rho_k\varepsilon_k)^+ \leq u(x) \leq (x \cdot \nu_k + \rho_k\varepsilon_k)^+ \quad x \in B_{\rho_k}. \tag{4.43}$$

In addition, $x \in B_1^+(u) \cup F(u)$, therefore $x \in B_1^+(u)$ or $x \in F(u)$. Let us analyze the two cases separately.

If $x \in B_1^+(u)$, $u(x) > 0$, hence from (4.43) we obtain $(x \cdot \nu_k + \rho_k\varepsilon_k)^+ > 0$, i.e. $(x \cdot \nu_k + \rho_k\varepsilon_k)^+ = x \cdot \nu_k + \rho_k\varepsilon_k$ and given that $x \cdot \nu_k - \rho_k\varepsilon_k \leq (x \cdot \nu_k - \rho_k\varepsilon_k)^+$, we achieve from (4.43)

$$x \cdot \nu_k - \rho_k\varepsilon_k \leq u(x) \leq x \cdot \nu_k + \rho_k\varepsilon_k \quad x \in B_{\rho_k} \cap B_1^+(u),$$

which gives

$$-\rho_k\varepsilon_k \leq u(x) - x \cdot \nu_k \leq \rho_k\varepsilon_k \quad x \in B_{\rho_k} \cap B_1^+(u),$$

namely

$$|u(x) - x \cdot \nu_k| \le \rho_k \varepsilon_k \quad x \in B_{\rho_k} \cap B_1^+(u). \tag{4.44}$$

If instead $x \in F(u)$, repeating the reasoning done in the proof of Lemma 3.1, we have from (4.43)

$$-\rho_k \varepsilon_k \le x \cdot \nu_k \le \rho_k \varepsilon_k \quad x \in B_{\rho_k} \cap F(u),$$

i.e.

$$|x \cdot \nu_k| \le \rho_k \varepsilon_k \quad x \in B_{\rho_k} \cap F(u),$$

which implies, recalling that $u(x) = 0$ with $x \in B_{\rho_k} \cap F(u)$, and $|x \cdot \nu_k| = |-x \cdot \nu_k|$,

$$|u(x) - x \cdot \nu_k| \le \rho_k \varepsilon_k \quad x \in B_{\rho_k} \cap F(u). \tag{4.45}$$

Consequently, putting together (4.44) and (4.45), seeing as how $B_{\rho_k} \cap (B_1^+(u) \cup F(u)) = (B_{\rho_k} \cap B_1^+(u)) \cup (B_{\rho_k} \cup F(u))$, we get

$$|u(x) - x \cdot \nu_k| \le \rho_k \varepsilon_k \quad x \in B_{\rho_k} \cap (B_1^+(u) \cup F(u)). \tag{4.46}$$

In addition, since $|x| \ge \rho_{k+1}$ for what we have said above, and for the definition of ρ_k, $\rho_{k+1} = \bar{r}\rho_k$, we obtain from (4.46), because $\bar{r} \neq 0$

$$|u(x) - x \cdot \nu_k| \le \frac{\bar{r}}{\bar{r}} \rho_k \varepsilon_k = \rho_{k+1} \frac{\varepsilon_k}{\bar{r}} \le |x| \frac{\varepsilon_k}{\bar{r}} \quad x \in B_{\rho_k} \cap (B_1^+(u) \cup F(u)),$$

in other words

$$|u(x) - x \cdot \nu_k| \le \frac{\varepsilon_k}{\bar{r}} |x| \quad x \in B_{\rho_k} \cap (B_1^+(u) \cup F(u)). \tag{4.47}$$

Let us consider now $|u(x) - x \cdot \nu|$ with $x \in B_{\rho_k} \cap (B_1^+(u) \cup F(u))$ and we can write

$$\begin{aligned} |u(x) - x \cdot \nu| &= |u(x) - x \cdot \nu_k + x \cdot \nu_k - x \cdot \nu| \\ &= |(u(x) - x \cdot \nu_k) + (x \cdot \nu_k - x \cdot \nu)|, \end{aligned}$$

for k chosen before, which gives, for the triangular inequality of $|\cdot|$,

$$|u(x) - x \cdot \nu| \le |u(x) - x \cdot \nu_k| + |x \cdot \nu_k - x \cdot \nu|$$

and from (4.47)

$$|u(x) - x \cdot \nu| \leq \frac{\varepsilon_k}{\bar{r}} |x| + |x \cdot \nu_k - x \cdot \nu| \quad x \in B_{\rho_k} \cap (B_1^+(u) \cup F(u)). \quad (4.48)$$

Furthermore,

$$|x \cdot \nu_k - x \cdot \nu| = |x \cdot (\nu_k - \nu)|,$$

and for the Cauchy-Schwarz inequality

$$|x \cdot \nu_k - x \cdot \nu| \leq |x| |\nu_k - \nu|,$$

where, for the considerations done above, inasmuch as $\nu = \lim_{k\to\infty} \nu_k$, with $k \in \mathbb{N}$,

$$|\nu_k - \nu| \leq 2C\varepsilon_k,$$

therefore

$$|x \cdot \nu_k - x \cdot \nu| \leq 2C\varepsilon_k |x|,$$

and from (4.48), we achieve

$$\begin{aligned} |u(x) - x \cdot \nu| &\leq \frac{\varepsilon_k}{\bar{r}} |x| + 2C\varepsilon_k |x| \\ &= \left(\frac{1}{\bar{r}} + 2C\right) \varepsilon_k |x| = \tilde{C}\varepsilon_k |x| \quad x \in B_{\rho_k} \cap (B_1^+(u) \cup F(u)), \end{aligned}$$

that is, given that $x \neq 0$,

$$\frac{|u(x) - x \cdot \nu|}{|x|} \leq \tilde{C}\varepsilon_k \quad x \in B_{\rho_k} \cap (B_1^+(u) \cup F(u)). \quad (4.49)$$

At this point, if we let $|x|$ go to 0, it is possible to choose the integer k such that $k \to \infty$, recalling that $\rho_k \to 0$ as $k \to \infty$, and with this choice, $\varepsilon_k = 2^{-k}\varepsilon_0 \to 0$, thus from (4.49), since $\frac{|u(x)-x\cdot\nu|}{|x|} \geq 0$, we obtain that

$$\frac{|u(x) - x \cdot \nu|}{|x|} \to 0, \quad |x| \to 0, \ x \in B_1^+(u) \cup F(u),$$

i.e.

$$u(x) - x \cdot \nu = o(|x|), \quad x \in (B_1^+(u) \cup F(u))$$

and seeing as how $u(x) - x \cdot \nu = u(x) - u(0) - x \cdot \nu$,

$$u(x) - u(0) - x \cdot \nu = 0(|x|),$$

which means that $\nabla u(0) = \nu$, with $0 \in F(u)$, and we recall $\nu = \nu(0)$, in order to distinguish this ν from ν's which we get if we repeat the same argument $\forall x_0 \in F(u)$.
As a consequence, we achieve that $\forall x_0 \in F(u)$, $\nabla u(x_0) = \nu(x_0)$.
So, we can consider the function $\nu(x)$ with $x \in F(u)$, which represents $\nabla u(x)$, with $x \in F(u)$ and we want to show that $|\nu(x) - \nu(y)| \leq C\,|x-y|^{\alpha}$, with x, $y \in F(u) \cap B_{1/2}$, which gives $u \in C^{1,\alpha}(F(u) \cap B_{1/2})$.
To prove this fact, we notice, first of all, that if x, $y \in B_{1/2}$, $|x-y| \leq |x| + |y| \leq 1/2 + 1/2 = 1$, namely $|x-y| \leq 1$, hence, given that $\rho_0 = \bar{r}^0 = 1$, there exists an integer k, with $k \geq 0$, such that $\rho_{k+1} \leq |x-y| \leq \rho_k$. In correspondence with this k, we consider $|\nu_k(x) - \nu_k(y)|$ and we can write

$$|\nu_k(x) - \nu_k(y)| = |\nu_k(x) - e_n + e_n - \nu_k(y)| = |(\nu_k(x) - e_n) + (e_n - \nu_k(y))|\,,$$

which gives for the triangular inequality of $|\cdot|$,

$$|\nu_k(x) - \nu_k(y)| \leq |\nu_k(x) - e_n| + |e_n - \nu_k(y)|\,,$$

and inasmuch as $|\nu_k(\bar{x}) - e_n| \leq 2C\varepsilon_k$, with $\varepsilon_k = 2^{-k}\varepsilon_0$, independently from $\bar{x} \in F(u)$, we have

$$|\nu_k(x) - \nu_k(y)| \leq 2C\varepsilon_k + 2C\varepsilon_k = 4C\varepsilon_k,$$

i.e.

$$|\nu_k(x) - \nu_k(y)| \leq 4C\varepsilon_k. \tag{4.50}$$

In particular, because $\varepsilon_k = 2^{-k}\varepsilon_0$, we can rewrite (4.50) as

$$|\nu_k(x) - \nu_k(y)| \leq 4C\varepsilon_k = 4C2^{-k}\varepsilon_0 = 4C(\bar{r}^{\log_{\bar{r}} 2^{-1}})^k\varepsilon_0 = 4C(\bar{r}^{\alpha})^k\varepsilon_0,$$

that is

$$|\nu_k(x) - \nu_k(y)| \leq 4C(\bar{r}^k)^{\alpha}\varepsilon_0, \tag{4.51}$$

where $\alpha = \log_{\bar{r}} 2^{-1} = \log_{\bar{r}} \frac{1}{2}$, and seeing as how $\bar{r} \leq (1/4)$ for what we have shown before, raising both the terms of the inequality to $1/2$, recalling that both are positive, $\bar{r}^{1/2} \leq 1/2$, which gives, since $0 < \bar{r} < 1$, $1/2 \geq \log_{\bar{r}} \frac{1}{2} = \alpha$,

in other words $\alpha \leq 1/2$. Also, because $\bar{r} < 1$, $\alpha = \log_{\bar{r}} \frac{1}{2} > \log_{\bar{r}} 1 = 0$, therefore we have $0 < \alpha \leq 1/2$.
In addition, from (4.51), we obtain

$$|\nu_k(x) - \nu_k(y)| \leq 4C(\bar{r}^k)^\alpha \varepsilon_0 = 4C\frac{(\bar{r}^{k+1})^\alpha}{\bar{r}^\alpha}\varepsilon_0$$

and thus, given that $\rho_{k+1} \leq |x-y|$, $\rho_{k+1} = \bar{r}^{k+1}$ and $\alpha > 0$, we achieve

$$|\nu_k(x) - \nu_k(y)| \leq \frac{4C}{\bar{r}^\alpha} |x-y|^\alpha \varepsilon_0 = C |x-y|^\alpha ,$$

calling $C = \frac{4C}{\bar{r}^\alpha}\varepsilon_0$, namely

$$|\nu_k(x) - \nu_k(y)| \leq C |x-y|^\alpha \quad x, y \in F(u) \cap B_{1/2}. \tag{4.52}$$

Now, passing to the limit in (4.52) as $k \to \infty$ we achieve, recalling that $\nu_k(x) \to \nu(x)$, $\nu_k(y) \to \nu(y)$, and hence $\nu_k(x) - \nu_k(y) \to \nu(x) - \nu(y)$, which also gives $|\nu_k(x) - \nu_k(y)| \to |\nu(x) - \nu(y)|$,

$$|\nu(x) - \nu(y)| \leq C |x-y|^\alpha , \quad x, y \in F(u) \cap B_{1/2},$$

as desired.
Consequently, we have shown that $u \in C^{1,\alpha}(F(u) \cap B_{1/2})$.
Furthermore, we have $\nabla u(x_0) = \nu(x_0)$ $\forall x_0 \in F(u) \cap B_{1/2}$, with $|\nu(x_0)| = 1$ and thus $\nu(x_0) \neq 0$, which gives $\nabla u(x_0) \neq 0$. Therefore, given that $u = 0$ on $F(u) \cap B_{1/2}$ and supposing that, provided that changing the order of the variables, $\frac{\partial u}{\partial x_n}(x_0) \neq 0$, with $x_0 \in F(u) \cap B_{1/2}$, we can apply the implicit function theorem and $\forall x_0 \in F(u) \cap B_{1/2}$ there exists an open neighborhood of x_0', $V_{x_0'}$, an open neighborhood of x_{0_n}, $V_{x_{0_n}}$, and a unique function $\varphi_{x_0} : V_{x_0'} \to V_{x_{0_n}}$ such that $\varphi_{x_0}(x_0') = x_{0_n}$ and

$$(F(u) \cap B_{1/2}) \cap (V_{x_0'} \times V_{x_{0_n}}) = \{(x', x_n), \quad x_n = \varphi_{x_0}(x')\} ,$$

with $\varphi_{x_0} \in C^{1,\alpha}(V_{x_0'})$.
In particular, provided that enlarging $V_{x_0'} \times V_{x_{0_n}}$, if necessary, the set $\left\{V_{x_0'} \times V_{x_{0_n}}, \quad x_0 \in F(u) \cap B_{1/2}\right\}$ cover $\overline{F(u) \cap B_{1/2}}$, which is a compact, since

it is a closed set and bounded, seeing as how subset of $\overline{B}_{1/2}$. As a consequence, we can find a finite number m of $V_{x_0'} \times V_{x_{0n}}$ such that $\bigcup_{i=1}^{m} (V_{x_0'} \times V_{x_{0n}})_i \supset \overline{F(u) \cap B_{1/2}} \supset F(u) \cap B_{1/2}$, and thus $\bigcup_{i=1}^{m} (V_{x_0'} \times V_{x_{0n}}) \supset F(u) \cap B_{1/2}$. Hence, putting together the corresponding functions φ_{x_0}, which coincide in the intersection of $V_{x_0'} \times V_{x_{0n}}$ for the uniqueness of φ_{x_0}, we can find a function $\varphi : (F(u) \cap B_{1/2})' \to \mathbb{R}$ such that

$$F(u) \cap B_{1/2} = \{(x', x_n), \quad x_n = \varphi(x')\},$$

with $\varphi \in C^{1,\alpha}((F(u) \cap B_{1/2})')$, that is $F(u) \cap B_{1/2} \in C^{1,\alpha}$, in other words $F(u) \in C^{1,\alpha}$ in $B_{1/2}$. $\square$

Before starting the proof of Theorem 4.1, we remark that in Theorem 4.1, the size of the neighborhood where $F(u)$ is $C^{1,\alpha}$ depends on the radius ρ of the ball B_ρ where $F(u)$ is Lipschitz, on the Lipschitz norm of $F(u)$, on $[a_{ij}]_{C^{0,\beta}(B_\rho)}$, $\|g\|_{C^{0,\beta}(B_\rho)}$, and $\|f\|_{L^\infty(B_\rho)}$.

Proof of Theorem 4.1. Let u be a viscosity solution to (2.1) in Ω with $0 \in F(u)$ and $g(0) > 0$. Without loss of generality, assume $\Omega = B_1$ and $g(0) = 1$. Indeed, concerning the assumption $g(0) = 1$, if $g(0) \neq 1$, because $g(0) > 0$ and thus $g(0) \neq 0$, we can divide g by $g(0)$ to get $\tilde{g} := \frac{g}{g(0)}$, and if we set $\tilde{u} := \frac{u}{g(0)}$, we claim that $\tilde{u}$ is a viscosity solution to (2.1) in Ω with coefficients a_{ij}, free boundary condition $\tilde{g}$ and right hand side $\tilde{f} := \frac{f}{g(0)}$.
Precisely, if $\varphi \in C^2(B_1^+(\tilde{u}))$ touches $\tilde{u}$ from below at $x_0 \in B_1^+(\tilde{u})$, we have

$$\varphi(x_0) = \tilde{u}(x_0) \tag{4.53}$$

and

$$\varphi(x) \leq \tilde{u}(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{4.54}$$

In particular, for the definition of $\tilde{u}$, (4.53) reads

$$\varphi(x_0) = \frac{u(x_0)}{g(0)},$$

i.e.

$$g(0)\varphi(x_0) = u(x_0), \tag{4.55}$$

and analogously (4.54) reads

$$g(0)\varphi(x) \leq u(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{4.56}$$

Consequently, from (4.55) and (4.56), seeing as how $g(0)\varphi(x) = (g(0)\varphi)(x)$, we obtain that $g(0)\varphi$ touches u from below at x_0. Notice that, inasmuch as $\tilde{u}(x) = \frac{u(x)}{g(0)}$, $\tilde{u}(x) > 0$ if and only if $u(x) > 0$, hence $B_1^+(\tilde{u}) = B_1^+(u)$, which implies that $x_0 \in B_1^+(u)$ and $g(0)\varphi \in C^2(B_1^+(u))$. Therefore, we have that $g(0)\varphi \in C^2(B_1^+(u))$ touches u from below at $x_0 \in B_1^+(u)$ and hence, recalling that u is a viscosity solution to (2.1) in B_1, we achieve

$$\begin{aligned}\sum_{i,j} a_{ij}(x_0)(g(0)\varphi)_{ij}(x_0) &= \sum_{i,j} a_{ij}(x_0)g(0)\varphi_{ij}(x_0)\\ &= g(0)\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \leq f(x_0),\end{aligned}$$

namely

$$g(0)\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \leq f(x_0),$$

which gives, because $g(0) > 0$,

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \leq \frac{f(x_0)}{g(0)},$$

that is for the definition of $\tilde{f}$,

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) \leq \tilde{f}(x_0).$$

As a consequence, repeating the same argument if $\varphi \in C^2(B_1^+(\tilde{u}))$ touches $\tilde{u}$ from above at $x_0 \in B_1^+(\tilde{u})$, but with opposite inequalities, we obtain that

$$\sum_{i,j} a_{ij}(x)\tilde{u}_{ij} = \tilde{f} \quad \text{in } B_1^+(\tilde{u}) \text{ in the viscosity sense.} \tag{4.57}$$

In parallel, if $\varphi \in C^2(B_1)$ is such that φ^+ touches $\tilde{u}$ from below at $x_0 \in F(\tilde{u})$, with $|\nabla\varphi|(x_0) \neq 0$, repeating the considerations done above, we get

$$g(0)\varphi^+(x_0) = u(x_0)$$

and

$$g(0)\varphi(x)^+ \leq u(x) \quad \text{in a neighborhood of } x_0,$$

which imply, inasmuch as $g(0) > 0$, that $g(0)\varphi^+ = (g(0)\varphi)^+$ touches u from below at x_0.
Now, $x_0 \in F(\tilde{u})$, thus $\forall B_r(x_0)$, $B_r(x_0) \cap B_1^+(\tilde{u}) \neq \emptyset$ and $B_r(x_0) \cap B_1^+(\tilde{u})^c \neq \emptyset$, hence, because $B_1^+(\tilde{u}) = B_1^+(u)$ for what we have said before, $B_r(x_0) \cap B_1^+(u) \neq \emptyset$ and $B_r(x_0) \cap B_1^+(u)^c \neq \emptyset$, $\forall B_r(x_0)$, i.e. $x_0 \in F(u)$.
In addition, since $g(0) > 0$ and $|\nabla\varphi|(x_0) \neq 0$, $|\nabla(g(0)\varphi)|(x_0) = g(0)|\nabla\varphi|(x_0) \neq 0$, namely $|\nabla(g(0)\varphi)|(x_0) \neq 0$.
To sum it up, we have $g(0)\varphi \in C^2(B_1)$ such that $(g(0)\varphi)^+$ touches u from below at $x_0 \in F(u)$ and $|\nabla(g(0)\varphi)|(x_0) \neq 0$, therefore, since u is a viscosity solution to (2.1) in B_1,

$$|\nabla(g(0)\varphi)|(x_0) = g(0)|\nabla\varphi|(x_0) \leq g(x_0),$$

which gives

$$|\nabla\varphi|(x_0) \leq \frac{g(x_0)}{g(0)} = \tilde{g}(x_0),$$

that is

$$|\nabla\varphi|(x_0) \leq \tilde{g}(x_0).$$

Consequently, repeating the same reasoning in the case of $\varphi \in C^2(B_1)$ such that φ^+ touches $\tilde{u}$ from above at $x_0 \in F(\tilde{u})$, with $|\nabla\varphi|(x_0) \neq 0$, but with opposite inequalities, we achieve

$$|\nabla\tilde{u}| = \tilde{g} \quad \text{on } F(\tilde{u}) \text{ in the viscosity sense.} \tag{4.58}$$

Putting together (4.57) and (4.58), we obtain that $\tilde{u}$ is a viscosity solution to (2.1) in B_1 with coefficients a_{ij}, right hand side $\tilde{f}$ and free boundary condition $\tilde{g}$.
Also, for simplicity we take $a_{ij}(0) = \delta_{ij}$.

At this point, consider the blow-up sequence

$$u_k := u_{\delta_k}(x) = \frac{u(\delta_k x)}{\delta_k},$$

with $\delta_k \to 0$ as $k \to \infty$.

In particular, repeating the same argument used in the proof of Theorem 4.2, each u_k solves (2.1) with coefficients $a^k_{ij}(x) := a_{ij}(\delta_k x)$, right hand side $f_k(x) := \delta_k f(\delta_k x)$, and free boundary condition $g_k(x) := g(\delta_k x)$. Furthermore, for k large, the assumption (4.2) is satisfied for the universal constant $\bar{\varepsilon}$ of Theorem 4.2. In fact, in B_1, we have, given that $\delta_k > 0$,

$$|f_k(x)| = |\delta_k f(\delta_k x)| = \delta_k \, |f(\delta_k x)| \, , \tag{4.59}$$

and, seeing as how $\delta_k \to 0$ as $k \to \infty$, there exists $\bar{k} \in \mathbb{N}$ such that $|\delta_k| < 1$ with $k \geq \bar{k}$, $k \in \mathbb{N}$, namely, because $\delta_k > 0$, $\delta_k < 1$ for k large. Thus, for these k's, $|\delta_k x| = \delta_k \, |x| < |x| < 1$, with $x \in B_1$, which gives from (4.59)

$$|f_k(x)| \leq \delta_k \, \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon},$$

in other words

$$|f_k(x)| \leq \bar{\varepsilon}, \tag{4.60}$$

always since $\delta_k \to 0$ as $k \to \infty$, and hence there exists $\bar{k} \in \mathbb{N}$ such that $\delta_k \leq \frac{\bar{\varepsilon}}{\|f\|_{L^\infty(B_1)}}$, with $k \geq \bar{k}$, that is for k large enough. As a consequence, from (4.60), we get

$$\sup_{x \in B_1} |f_k(x)| = \|f_k\|_{L^\infty(B_1)} \leq \bar{\varepsilon},$$

i.e.

$$\|f_k\|_{L^\infty(B_1)} \leq \bar{\varepsilon} \tag{4.61}$$

because $\bar{\varepsilon}$ is an upper bound of the set $\{|f_k(x)| \, , \quad x \in B_1\}$.

Moreover, always in B_1, seeing as how $g_k(0) = g(0) = 1$ and in view of the definition of $[g]_{C^{0,\beta}}$, (see Definition A.1)

$$|g_k(x) - 1| = |g_k(x) - g_k(0)| \leq |x|^\beta \, [g_k]_{C^{0,\beta}(B_1)} \leq [g_k]_{C^{0,\beta}(B_1)}, \tag{4.62}$$

inasmuch as $|x|^\beta \leq 1$, given that $x \in B_1$ and $\beta > 0$.

Notice now that $[g_k]_{C^{0,\beta}(B_1)} = \delta_k^\beta [g]_{C^{0,\beta}(B_{\delta_k})}$. Indeed,

$$
\begin{aligned}
[g_k]_{C^{0,\beta}(B_1)} &= \sup_{\substack{x,y\,\in\, B_1\\ x\neq y}} \frac{|g_k(x)-g_k(y)|}{|x-y|^\beta} = \sup_{\substack{x,y\,\in\, B_1\\ x\neq y}} \frac{|g(\delta_k x)-g(\delta_k y)|}{|x-y|^\beta}\\
&= \frac{\delta_k^\beta}{\delta_k^\beta} \sup_{\substack{x,y\in B_1\\ x\neq y}} \frac{|g(\delta_k x)-g(\delta_k y)|}{|x-y|^\beta} = \delta_k^\beta \sup_{\substack{x,y\,\in\, B_1\\ x\neq y}} \frac{|g(\delta_k x)-g(\delta_k y)|}{\delta_k^\beta\,|x-y|^\beta}\\
&= \delta_k^\beta \sup_{\substack{x,y\,\in\, B_1\\ x\neq y}} \frac{|g(\delta_k x)-g(\delta_k y)|}{|\delta_k(x-y)|^\beta} = \delta_k^\beta \sup_{\substack{x,y\,\in\, B_1\\ x\neq y}} \frac{|g(\delta_k x)-g(\delta_k y)|}{|\delta_k x-\delta_k y|^\beta},
\end{aligned}
$$

namely

$$
[g_k]_{C^{0,\beta}(B_1)} = \delta_k^\beta \sup_{\substack{x,y\,\in\, B_1\\ x\neq y}} \frac{|g(\delta_k x)-g(\delta_k y)|}{|\delta_k x-\delta_k y|^\beta},
$$

and since $\delta_k x$, $\delta_k y$ vary in B_{δ_k} if x, y vary in B_1,

$$
[g_k]_{C^{0,\beta}(B_1)} = \delta_k^\beta \sup_{\substack{\delta_k x,\delta_k y\,\in\, B_{\delta_k}\\ \delta_k x\neq \delta_k y}} \frac{|g(\delta_k x)-g(\delta_k y)|}{|\delta_k x-\delta_k y|^\beta} = \delta_k^\beta [g]_{C^{0,\beta}(B_{\delta_k})},
$$

that is

$$
[g_k]_{C^{0,\beta}(B_1)} = \delta_k^\beta [g]_{C^{0,\beta}(B_{\delta_k})}.
$$

Therefore, from (4.62), we obtain

$$
|g_k(x)-1| \leq [g_k]_{C^{0,\beta}(B_1)} = \delta_k^\beta [g]_{C^{0,\beta}(B_{\delta_k})} \leq \delta_k^\beta [g]_{C^{0,\beta}(B_1)},
$$

in other words

$$
|g_k(x)-1| \leq [g_k]_{C^{0,\beta}(B_1)} \leq \delta_k^\beta [g]_{C^{0,\beta}(B_1)}, \tag{4.63}
$$

inasmuch as for k large $\delta_k < 1$, for what we have said before, and thus $B_{\delta_k} \subset B_1$, which implies $[g]_{C^{0,\beta}(B_{\delta_k})} \leq [g]_{C^{0,\beta}(B_1)}$.
In addition, since $\delta_k \to 0$ as $k \to \infty$, with $0 < \beta \leq 1$, $\delta_k^\beta \to 0$ as $k \to \infty$, hence there exists $\bar{k} \in \mathbb{N}$ such that $\delta_k^\beta \leq \frac{\bar{\varepsilon}}{[g]_{C^{0,\beta}(B_1)}}$, with $k \geq \bar{k}$, $k \in \mathbb{N}$, i.e. for k large $\delta_k^\beta \leq \frac{\bar{\varepsilon}}{[g]_{C^{0,\beta}(B_1)}}$, as a consequence from (4.63), we achieve for k large

$$
|g_k(x)-1| \leq [g_k]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon}
$$

which gives at the same time

$$[g_k]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon} \tag{4.64}$$

and

$$\sup_{x \in B_1} |g_k(x) - 1| = \|g_k - 1\|_{L^\infty(B_1)} \leq \bar{\varepsilon},$$

namely

$$\|g_k - 1\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \tag{4.65}$$

given that $\bar{\varepsilon}$ is an upper bound of the set $\{|g_k(x) - 1|, \quad x \in B_1\}$.
As regards the inequalities which concern coefficients a_{ij}^k, we consider always in B_1, $\left|a_{ij}^k(x) - \delta_{ij}\right|$, and because $a_{ij}^k(0) = a_{ij}(0) = \delta_{ij}$, we have

$$\left|a_{ij}^k(x) - \delta_{ij}\right| = \left|a_{ij}^k(x) - a_{ij}^k(0)\right|,$$

which entails for the definition of $[a_{ij}^k]_{C^{0,\beta}(B_1)}$, (see Definition A.1)

$$\left|a_{ij}^k(x) - \delta_{ij}\right| \leq |x|^\beta \, [a_{ij}^k]_{C^{0,\beta}(B_1)} \leq [a_{ij}^k]_{C^{0,\beta}(B_1)} \tag{4.66}$$

recalling that $|x|^\beta \leq 1$ in view of what we have said above.
Repeating the considerations done above, we also get from (4.66),

$$\left|a_{ij}^k(x) - \delta_{ij}\right| \leq [a_{ij}^k]_{C^{0,\beta}(B_1)} \leq \delta_k^\beta [a_{ij}]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon},$$

which gives

$$[a_{ij}^k]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon} \tag{4.67}$$

and

$$\left\|a_{ij}^k - \delta_{ij}\right\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \tag{4.68}$$

seeing as how $\bar{\varepsilon}$ is an upper bound of the set $\left\{\left|a_{ij}^k(x) - \delta_{ij}\right|, \quad x \in B_1\right\}$.
To sum it up, for k large from (4.61), (4.64) and (4.67), f_k, g_k and a_{ij}^k satisfy the assumption (4.2) in B_1 with $\bar{\varepsilon}$, while from (4.61), (4.65) and (4.68), f_k, g_k and a_{ij}^k satisfy (2.2) in B_1 with $\bar{\varepsilon}$.
Therefore, using nondegeneracy and uniform Lipschitz continuity of the u_k's (see Lemma 5.1), standard arguments (see for instance [1]) imply that (up to extracting a subsequence):

(i) $u_k \to u_0$,

(ii) $\partial \{u_k > 0\} \to \partial \{u_0 > 0\}$ locally in the Hausdorff distance,

for a globally defined function $u_0 : \mathbb{R}^n \to \mathbb{R}$.
Let us show that (i) is verified.
Precisely, we have, for what we have said above, that each u_k solves (2.1) in B_1 with coefficients a_{ij}^k, right hand side f_k and free boundary condition g_k and in addition for k large f_k, g_k and a_{ij}^k satisfy (2.2) in B_1 with $\bar{\varepsilon}$. We want to show that $F(u_k)$ is a Lipschitz graph in a neighborhood of 0 and $F(u_k) \cap B_1 \neq \emptyset$.
In particular, as we have shown in the proof of Theorem 4.2, we have $0 \in F(u_k)$ $\forall k$, thus $F(u_k) \cap B_1 \neq \emptyset$ $\forall k$. In addition, we know that $F(u)$ is a Lipschitz graph in a neighborhood O of 0, that is

$$F(u) \cap O = \{(x', \psi(x'))\},$$

with ψ a Lipschitz function in $(F(u) \cap O)'$. Always for what we have shown in the proof of the Theorem 4.2, $x_0 \in F(u)$ if and only if $\frac{x_0}{\delta_k} \in F(u_k)$, as a consequence $F(u_k) = \frac{1}{\delta_k} F(u)$ and we can define $\psi_k(y') := \frac{\psi(\delta_k y')}{\delta_k}$, which satisfies

$$\psi_k \left(\frac{x'}{\delta_k} \right) = \frac{\psi\left(\delta_k \frac{x'}{\delta_k}\right)}{\delta_k} = \frac{\psi(x')}{\delta_k},$$

in other words

$$\psi_k \left(\frac{x'}{\delta_k} \right) = \frac{\psi(x')}{\delta_k},$$

and

$$\left| \psi_k \left(\frac{x'}{\delta_k} \right) - \psi_k \left(\frac{y'}{\delta_k} \right) \right| = \left| \frac{\psi(x')}{\delta_k} - \frac{\psi(y')}{\delta_k} \right|,$$

which gives with x', $y' \in (F(u) \cap O)'$, recalling that ψ is a Lipschitz function in $(F(u) \cap O)'$ with Lipschitz constant that we call C_ψ,

$$\left| \psi_k \left(\frac{x'}{\delta_k} \right) - \psi_k \left(\frac{y'}{\delta_k} \right) \right| \leq C_\psi \left| \frac{x'}{\delta_k} - \frac{y'}{\delta_k} \right|,$$

hence ψ_k is a Lipschitz function in $\frac{1}{\delta_k}(F(u) \cap O)'$.
Now, if $y' \in \frac{1}{\delta_k}(F(u) \cap O)'$, $y' = \frac{x'}{\delta_k}$ with $x' \in (F(u) \cap O)'$, therefore $x' \in F(u)'$

and $x' \in O'$, thereby $y' \in \frac{1}{\delta_k}F(u)' = F(u_k)'$ and $y' \in \frac{1}{\delta_k}O' = V'$ where $V = \frac{1}{\delta_k}O$ is a neighborhood of $\frac{0}{\delta_k} = 0$, thus $y' \in (F(u_k) \cap V)'$. Consequently, ψ_k is a Lipschitz function in $(F(u_k) \cap V)'$ and we can write

$$\frac{1}{\delta_k}(F(u) \cap O)' = (F(u_k) \cap V)' = \left\{ \left(\frac{1}{\delta_k}x', \frac{\psi(\frac{x'}{\delta_k})}{\delta_k} \right) \right\} = \left\{ (y', \psi_k(y')) \right\},$$

which implies that $F(u_k)$ is a Lipschitz graph in a neighborhood of 0.
To sum it up, we have, for k large, that u_k is a solution to (2.1)-(2.2) with $\varepsilon_k \leq \bar{\varepsilon}$, $F(u_k) \cap B_1 \neq \emptyset$ and $F(u_k)$ is a Lipschitz graph in a neighborhood of 0, so we can apply Lemma 5.1 and hence for these k's u_k is Lipschitz, in other words

$$|u_k(x) - u_k(y)| \leq C_k \left| x - y \right|, \quad \forall x, y. \tag{4.69}$$

In particular fix one of these k's and we call it $\bar{k}$. As a consequence, from (4.69) we have

$$|u_{\bar{k}}(x) - u_{\bar{k}}(y)| \leq C_{\bar{k}} \left| x - y \right|, \quad \forall x, y. \tag{4.70}$$

At this point, notice that for every k with $k \geq \bar{k}$ we have from the definition of u_k

$$u_k(x) = \frac{u(\delta_k x)}{\delta_k} = \frac{u(\delta_{\bar{k}}\frac{\delta_k}{\delta_{\bar{k}}}x)}{\delta_{\bar{k}}\frac{\delta_k}{\delta_{\bar{k}}}} = \frac{1}{\frac{\delta_k}{\delta_{\bar{k}}}}u_{\bar{k}}\left(\frac{\delta_k}{\delta_{\bar{k}}}\right) = \frac{\delta_{\bar{k}}}{\delta_k}u_{\bar{k}}\left(\frac{\delta_k}{\delta_{\bar{k}}}x\right),$$

i.e.

$$u_k(x) = \frac{\delta_{\bar{k}}}{\delta_k}u_{\bar{k}}\left(\frac{\delta_k}{\delta_{\bar{k}}}x\right). \tag{4.71}$$

Therefore, in view of (4.70) and (4.71), we obtain

$$\begin{aligned}
|u_k(x) - u_k(y)| &= \left| \frac{\delta_{\bar{k}}}{\delta_k}u_{\bar{k}}\left(\frac{\delta_k}{\delta_{\bar{k}}}x\right) - \frac{\delta_{\bar{k}}}{\delta_k}u_{\bar{k}}\left(\frac{\delta_k}{\delta_{\bar{k}}}y\right) \right| \\
&= \left| \frac{\delta_{\bar{k}}}{\delta_k} \right| \left| u_{\bar{k}}\left(\frac{\delta_k}{\delta_{\bar{k}}}x\right) - u_{\bar{k}}\left(\frac{\delta_k}{\delta_{\bar{k}}}y\right) \right| \leq \left| \frac{\delta_{\bar{k}}}{\delta_k} \right| C_{\bar{k}} \left| \frac{\delta_k}{\delta_{\bar{k}}}x - \frac{\delta_k}{\delta_{\bar{k}}}y \right| \\
&= \left| \frac{\delta_{\bar{k}}}{\delta_k} \right| C_{\bar{k}} \left| \frac{\delta_k}{\delta_{\bar{k}}}(x-y) \right| = \left| \frac{\delta_{\bar{k}}}{\delta_k} \right| C_{\bar{k}} \left| \frac{\delta_k}{\delta_{\bar{k}}} \right| |x-y| \\
&= \frac{|\delta_{\bar{k}}|}{|\delta_k|} C_{\bar{k}} \frac{|\delta_k|}{|\delta_{\bar{k}}|} |x-y| = C_{\bar{k}} \left| x - y \right|,
\end{aligned}$$

namely

$$|u_k(x) - u_k(y)| \leq C_{\bar{k}}\,|x-y| \quad \forall x, y \quad \forall k,\ \ k \geq \bar{k}. \tag{4.72}$$

Consequently, in view of (4.72), for k large u_k is uniformly Lipschitz continuous and hence equicontinuous. Indeed, in B_1, if we fix $\varepsilon > 0$, calling $C = C_{\bar{k}}$ in (4.72), we can take $\eta > 0$, $\eta = \frac{\varepsilon}{C}$ such that if x, $y \in B_1$, $|x-y| < \eta$ we get from (4.72)

$$|u_k(x) - u_k(y)| \leq C\,|x-y| < C\frac{\varepsilon}{C} = \varepsilon,$$

namely there exists $\eta > 0$ such that

$$|u_k(x) - u_k(y)| < \varepsilon,$$

if x, $y \in B_1$, $|x-y| < \eta$ and for k large, i.e. u_k is equicontinuous.
Now, from Lemma 5.1 which we have applied for these k's, we also obtain

$$c_0 d(z) \leq u_k(z) \leq C_0 d(z), \quad \text{for all } z \in B_1^+(u_k), \tag{4.73}$$

with $d(z) = \text{dist}(z, F(u_k))$, and c_0, C_0 universal constants independent from k.
In particular, seeing as how $0 \in F(u_k)\ \forall k$,

$$d(z) = \inf_{y \in F(u_k)} |z-y| \leq |z-0| = |z| < 1,$$

because $z \in B_1^+(u_k)$, which entails $|z| < 1$, that is $d(z) < 1$, which gives from (4.73)

$$u_k(z) \leq C_0 \quad z \in B_1^+(u_k). \tag{4.74}$$

Moreover, given that $u_k \geq 0$ in B_1, $u_k = 0$ in $B_1 \setminus B_1^+(u_k)$ and thus, inasmuch as $C_0 \geq 0$, $u_k(z) \leq C_0$ with $z \in B_1 \setminus B_1^+(u_k)$, as a consequence we achieve from (4.74)

$$u_k(z) \leq C_0, \quad z \in B_1,$$

i.e. since $u_k \geq 0$ and hence $|u_k| = u_k$,

$$|u_k| \leq C_0 \quad \text{in } B_1.$$

Therefore, we have shown that the sequence u_k is uniformly bounded in B_1 and because u_k is also equicontinuous in B_1 with k large, we can apply the Ascoli-Arzelà theorem (see Theorem A.3) and we get that there exists a subsequence which we still call u_k such that u_k converges uniformly to u_0 in K with K a compact subset of B_1.
In addition, we notice that u_k is well-defined also in $B_{\frac{1}{\delta_k}}$, recalling that if $x \in B_{\frac{1}{\delta_k}}$ $\delta_k x \in B_1$, where u is well-defined and hence for the definition of u_k, u_k is well-defined in $B_{\frac{1}{\delta_k}}$. As a consequence, seeing as how $\delta_k \to 0$, $\frac{1}{\delta_k} \to \infty$, so for every compact K, given that there exists a ball $B_{\bar{r}}$, with $\overline{B_{\bar{r}}} \supset K$, we can find $\bar{k} \in \mathbb{N}$, such that $\frac{1}{\delta_k} > \bar{r}$, for $k \in \mathbb{N}$, $k \geq \bar{k}$, thereby we can repeat the same reasoning done before to obtain that u_k converges uniformly to u_0 in K. Thanks to this fact, we can consider u_0 as a globally defined function. Now, using a similar argument to that used in Lemma 3.1 to show that $\tilde{u}$ solves (3.19), we get that the blow-up limit u_0 is a global solution to the free boundary problem

$$\begin{cases} \Delta u_0 = 0 & \text{in } \{u_0 > 0\}, \\ |\nabla u_0| = 1 & \text{on } F(u_0). \end{cases} \tag{4.75}$$

and since $F(u)$ is a Lipschitz graph in a neighborhood of 0, we also see from (i)-(ii) that $F(u_0)$ is Lipschitz continuous. Thus, it follows from [4] that u_0 is a so-called one-plane solution, i.e. (up to rotations) $u_0 = x_n^+$.
Combining the facts above, one concludes that for all k large enough, u_k is $\bar{\varepsilon}$-flat say in B_1, in other words

$$(x_n - \bar{\varepsilon})^+ \leq u_k(x) \leq (x_n + \bar{\varepsilon})^+, \quad x \in B_1.$$

Precisely, since $u_0 = x_n^+$ and $u_k \to u_0$ uniformly, we have for k large enough, for instance $k \geq k_{\bar{\varepsilon}}$,

$$\left| u_k(x) - x_n^+ \right| \leq \bar{\varepsilon}, \quad x \in B_1, \tag{4.76}$$

which gives

$$-\bar{\varepsilon} \leq u_k(x) - x_n^+, \quad x \in B_1$$

and

$$x_n^+ - \bar{\varepsilon} \leq u_k(x), \quad x \in B_1. \tag{4.77}$$

Therefore, from (4.77), seeing as how $x_n \leq x_n^+$, we achieve

$$x_n - \bar{\varepsilon} \leq u_k(x), \quad x \in B_1,$$

which implies, given that $u_k \geq 0$ in B_1,

$$\max(x_n - \bar{\varepsilon}, 0) = (x_n - \bar{\varepsilon})^+ \leq u_k(x) \quad x \in B_1,$$

i.e. for k large enough

$$(x_n - \bar{\varepsilon})^+ \leq u_k(x), \quad x \in B_1. \tag{4.78}$$

Furthermore, from (4.76), we also have

$$u_k(x) - x_n^+ \leq \bar{\varepsilon}, \quad x \in B_1,$$

and

$$u_k(x) \leq x_n^+ + \bar{\varepsilon}, \quad x \in B_1,$$

which entails, where $x_n \geq 0$ in B_1, in other words in $B_1 \cap \{x_n \geq 0\}$,

$$u_k(x) \leq x_n + \bar{\varepsilon}, \quad x \in B_1 \cap \{x_n \geq 0\}, \tag{4.79}$$

recalling that if $x_n \geq 0$, $x_n^+ = x_n$.
In addition, if $x_n \geq 0$, $x_n + \bar{\varepsilon} \geq \bar{\varepsilon} > 0$, which gives $x_n + \bar{\varepsilon} > 0$ and hence $x_n + \bar{\varepsilon} = (x_n + \bar{\varepsilon})^+$, as a consequence from (4.79), we get

$$u_k(x) \leq (x_n + \bar{\varepsilon})^+, \quad x \in B_1 \cap \{x_n \geq 0\},$$

which also gives from (4.78)

$$(x_n - \bar{\varepsilon})^+ \leq u_k(x) \leq (x_n + \bar{\varepsilon})^+ \quad x \in B_1 \cap \{x_n \geq 0\},$$

and using the fact that $\partial \{u_k > 0\} \to \partial \{u_0 > 0\}$ locally in the Hausdorff distance,

$$(x_n - \bar{\varepsilon})^+ \leq u_k(x) \leq (x_n + \bar{\varepsilon})^+ \quad x \in B_1.$$

Consequently, u_k satisfies the assumptions of Theorem 4.2, and our conclusion follows. $\square$

Chapter 5

Nondegeneracy property of the solutions

In this chapter, we state and prove the nondegeneracy of a solution u to (2.1)-(2.2). This property has been used in the proof of Theorem 4.1.

Lemma 5.1. *Let u be a solution to (2.1)-(2.2) with $\varepsilon \leq \tilde{\varepsilon}$ a universal constant. If $F(u) \cap B_1 \neq \emptyset$, $F(u)$ is a Lipschitz graph in B_2, then u is Lipschitz and nondegenerate in $B_1^+(u)$, i.e.*

$$c_0 d(z) \leq u(z) \leq C_0 d(z) \quad \text{for all } z \in B_1^+(u),$$

with $d(z) = \operatorname{dist}(z, F(u))$ and c_0, C_0 universal constants.

Proof. Assume without loss of generality that $0 \in B_1^+(u)$ and set $d := d(0)$. Consider the rescaled function

$$\tilde{u}(x) = \frac{u(dx)}{d}, \quad x \in B_1. \tag{5.1}$$

Repeating the reasoning done in Theorem 4.2, we obtain that $\tilde{u}$ satisfies (2.1) in B_1 with coefficients $\tilde{a}_{ij}(x) := a_{ij}(dx)$, right hand side $\tilde{f}(x) := df(dx)$ and free boundary condition $\tilde{g}(x) = g(dx)$.
Now, we notice that $d \leq 1$. Indeed, given that $F(u) \cap B_1 \neq \emptyset$, there exists a point $\bar{x} \in F(u) \cap B_1$ and which satisfies thus $|\bar{x}| \leq 1$. As a consequence, we

have, seeing as how $\bar{x} \in F(u)$ if $\bar{x} \in F(u) \cap B_1$,

$$d = \mathrm{dist}(0, F(u)) = \inf_{x \in F(u)} |x| \leq |\bar{x}| \leq 1,$$

i.e.

$$d \leq 1.$$

In particular, since $d \leq 1$, the assumption (2.2) holds.
Precisely, fixed $x \in B_1$, we have, because $d \geq 0$, recalling that d is a distance, and $d \leq 1$,

$$\left|\tilde{f}(x)\right| = |df(dx)| = d\,|f(dx)| \leq |f(dx)| \leq \|f\|_{L^\infty},$$

namely

$$\left|\tilde{f}(x)\right| \leq \|f\|_{L^\infty}. \tag{5.2}$$

Furthermore, inasmuch $\|f\|_{L^\infty} \leq \varepsilon^2$, recalling that u is a solution to (2.1)-(2.2), we get from (5.2),

$$\left|\tilde{f}(x)\right| \leq \varepsilon^2. \tag{5.3}$$

As a consequence, from (5.3), we achieve that ε^2 is an upper bound of the set $\left\{\left|\tilde{f}(x)\right|, \quad x \in B_1\right\}$, and thus

$$\left\|\tilde{f}\right\|_{L^\infty(B_1)} = \sup_{x \in B_1} \left|\tilde{f}(x)\right| \leq \varepsilon^2,$$

which gives

$$\left\|\tilde{f}\right\|_{L^\infty(B_1)} \leq \varepsilon^2. \tag{5.4}$$

As regards the second inequality in (2.2), instead, we fix $x \in B_1$, and we have

$$|\tilde{g}(x) - 1| = |g(dx) - 1| \leq \|g - 1\|_{L^\infty},$$

in other words,

$$|\tilde{g}(x) - 1| \leq \|g - 1\|_{L^\infty}. \tag{5.5}$$

Moreover, inasmuch as u is a solution to (2.1)-(2.2), $\|g - 1\|_{L^\infty} \leq \varepsilon^2$, hence from (5.5) we obtain

$$|\tilde{g}(x) - 1| \leq \varepsilon^2,$$

which entails that ε^2 is an upper bound of the set $\{|\tilde{g}(x) - 1|\,, \quad x \in B_1\}$, and therefore, we get

$$\|\tilde{g}(x) - 1\|_{L^\infty(B_1)} \leq \varepsilon^2. \tag{5.6}$$

Concerning the third inequality in 2.2, we fix $x \in B_1$ and we have

$$|\tilde{a}_{ij}(x) - \delta_{ij}| = |a_{ij}(dx) - \delta_{ij}| \leq \|a_{ij} - \delta_{ij}\|_{L^\infty}\,,$$

that is

$$|\tilde{a}_{ij}(x) - \delta_{ij}| \leq \|a_{ij} - \delta_{ij}\|_{L^\infty}\,. \tag{5.7}$$

In addition, u is a solution to (2.1)-(2.2) and thus $\|a_{ij} - \delta_{ij}\|_{L^\infty} \leq \varepsilon$, as a consequence from (5.7) we achieve

$$|\tilde{a}_{ij}(x) - \delta_{ij}| \leq \varepsilon,$$

which implies that ε is an upper bound of the set $\{|\tilde{a}_{ij}(x) - \delta_{ij}|\,, \quad x \in B_1\}$, and hence we obtain

$$\|\tilde{a}_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \leq \varepsilon. \tag{5.8}$$

Considering together (5.4), (5.6) and (5.8), we get that the assumption (2.2) holds for $\tilde{f}$, $\tilde{g}$ and $\tilde{a}_{ij}$.
At this point, we wish to show that

$$c_0 \leq \tilde{u}_0 \leq C_0.$$

For this purpose, assume for contradiction that $\tilde{u}(0) > C_0$, with C_0 to be made precise later.

Now, let

$$G(x) = C(|x|^{-\gamma} - 1) \tag{5.9}$$

be defined on the closure of the annulus $B_1 \setminus \overline{B}_{1/2}$.
In particular, in view of the uniform ellipticity of the coefficients (see Lemma A.5 in Appendix A), repeating the same computation described for proving Lemma 2.3, we can choose γ large universal so that (for ε small)

$$\sum_{i,,j} \tilde{a}_{ij} G_{ij} > \varepsilon^2 \quad \text{on } B_1 \setminus \overline{B}_{1/2}. \tag{5.10}$$

In addition we can choose C so that

$$G = 1 \quad \text{on } \partial B_{1/2}.$$

Indeed, since $|x| = 1/2$ on $\partial B_{1/2}$, if we take

$$C = \frac{1}{(1/2)^{-\gamma} - 1}, \tag{5.11}$$

we achieve

$$G(x) = \frac{1}{(1/2)^{-\gamma} - 1}((1/2)^{-\gamma} - 1) = 1 \quad x \in \partial B_{1/2},$$

i.e. $G = 1$ on $\partial B_{1/2}$.

Notice now that $\tilde{u} > 0$ in B_1. Indeed, if $x \in B_1$, inasmuch $d \geq 0$, $|dx| = d\,|x| < d$, which gives $dx \in B_d$, where $u > 0$, and as a consequence $\tilde{u} > 0$ in B_1. To show that $u > 0$ in B_d, we recall that $d = \operatorname{dist}(0, F(u))$ and thus $B_d \cap F(u) = \emptyset$, otherwise there would exist $\bar{x} \in B_d \cap F(u)$, which satisfies $|\bar{x}| < d$, $\bar{x} \in F(u)$, therefore we would have $d = \inf\limits_{x \in F(u)} |x| \leq |\bar{x}| < d$, that is $d < d$, which is a contradiction. Moreover, seeing as how u is continuous in B_d, it can not exist $\bar{x} \in B_d$ such that $u(\bar{x}) = 0$ and $\bar{x} \notin F(u)$, otherwise, given that $u(0) > 0$, there would exist x^*, for instance in the line which connects 0 and $\bar{x}$, so that $x^* \in F(u) \cap B_d$ and as before, we reach a contradiction. To sum it up, we have shown that $u > 0$ in B_d and hence $\tilde{u} > 0$ in B_1.

Consequently, inasmuch $\tilde{u} > 0$ in B_1 and solves, in the viscosity sense, a uniformly elliptic equation in B_1 with right hand side $\tilde{f}$, we can apply the Harnack inequality to obtain

$$\sup_{\overline{B}_{1/2}} \tilde{u} \leq C_1 \left(\inf_{\overline{B}_{1/2}} \tilde{u} + C_2 \left\| \tilde{f} \right\|_{L^\infty(B_1)} \right),$$

which implies,

$$\begin{aligned} \tilde{u}(0) \leq \sup_{\overline{B}_{1/2}} \tilde{u} &\leq C_1 \left(\inf_{\overline{B}_{1/2}} \tilde{u} + C_2 \left\| \tilde{f} \right\|_{L^\infty(B_1)} \right) \\ &\leq C_1 \left(\tilde{u} + C_2 \left\| \tilde{f} \right\|_{L^\infty(B_1)} \right) \quad \text{on } \overline{B}_{1/2} \end{aligned}$$

namely

$$\tilde{u}(0) \leq C_1 \left(\tilde{u} + C_2 \left\| \tilde{f} \right\|_{L^\infty(B_1)} \right) \quad \text{on } \overline{B}_{1/2}. \tag{5.12}$$

At this point, from (5.12) we get

$$\frac{1}{C_1} \tilde{u}(0) \leq \tilde{u} + C_2 \left\| \tilde{f} \right\|_{L^\infty(B_1)} \quad \text{on } \overline{B}_{1/2},$$

which also gives

$$\frac{1}{C_1} \tilde{u}(0) - C_2 \left\| \tilde{f} \right\|_{L^\infty(B_1)} \leq \tilde{u} \quad \text{on } \overline{B}_{1/2}. \tag{5.13}$$

In particular, because $\left\| \tilde{f} \right\|_{L^\infty(B_1)} \leq \varepsilon^2$, and thus $- \left\| \tilde{f} \right\|_{L^\infty(B_1)} \geq -\varepsilon^2$, we achieve from (5.13)

$$\tilde{u} \geq \frac{1}{C_1} \tilde{u}(0) - C_2 \varepsilon^2 \quad \text{on } \overline{B}_{1/2}. \tag{5.14}$$

In addition, using the contradiction hypothesis, i.e. $\tilde{u}(0) > C_0 > 0$, which means that $\tilde{u}(0)$ is large enough, we can choose $\varepsilon > 0$ such that $\varepsilon < \tilde{u}(0)$, therefore from (5.14) we obtain

$$\tilde{u} \geq \frac{1}{C_1} \tilde{u}(0) - C_2 \varepsilon \tilde{u}(0) = \left(\frac{1}{C_1} - C_2 \varepsilon \right) \tilde{u}(0) \quad \text{on } \overline{B}_{1/2}, \tag{5.15}$$

and taking ε small enough so that $\frac{1}{C_1} - C_2 \varepsilon > 0$, in other words $\varepsilon < \frac{1}{C_1 C_2}$, calling $c = \frac{1}{C_1} - C_2 \varepsilon$, we get from (5.15)

$$\tilde{u} \geq c \tilde{u}(0) \quad \text{on } \overline{B}_{1/2}, \tag{5.16}$$

with $\varepsilon < \min(\tilde{u}(0), \frac{1}{C_1 C_2})$.
Let us call now $v(x) := c\tilde{u}(0) G(x)$ and we claim that $\tilde{u} - v$ satisfies

$$\sum_{i,j} \tilde{a}_{ij} (\tilde{u} - v)_{ij} \leq 0 \quad \text{in } B_1 \setminus \overline{B}_{1/2}$$

in the viscosity sense, i.e. $\tilde{u} - v$ is a viscosity supersolution of $\sum_{i,j} \tilde{a}_{ij} (\tilde{u} - v)_{ij} = 0$ in $B_1 \setminus \overline{B}_{1/2}$, see Definition B.4 in Appendix B.

Precisely, if $\varphi \in C^2(B_1 \setminus \overline{B}_{1/2})$ touches $\tilde{u} - v$ from below at $x_0 \in B_1 \setminus \overline{B}_{1/2}$, we have

$$\varphi(x_0) = (\tilde{u} - v)(x_0) = \tilde{u}(x_0) - v(x_0) \tag{5.17}$$

and

$$\varphi(x) \leq (\tilde{u}(x) - v(x)) = \tilde{u}(x) - v(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{5.18}$$

As a consequence, (5.17) and (5.18) read

$$\varphi(x_0) + v(x_0) = (\varphi + v)(x_0) = \tilde{u}(x_0) \tag{5.19}$$

and

$$\varphi(x) + v(x) = (\varphi + v)(x) \leq \tilde{u}(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{5.20}$$

In particular, let us remark that $G \in C^\infty(B_1 \setminus \overline{B}_{1/2})$, thus also $G \in C^2(B_1 \setminus \overline{B}_{1/2})$, which implies $v \in C^2(B_1 \setminus \overline{B}_{1/2})$, because $v = c\tilde{u}(0)G$, with $c\tilde{u}(0)$ constant.

This fact, together with (5.19) and (5.20), gives that $(\varphi + v) \in C^2(B_1 \setminus \overline{B}_{1/2})$ touches $\tilde{u}$ from below at x_0.

Furthermore, we have $\tilde{u}(x_0) > 0$, inasmuch, as observed above, $\tilde{u} > 0$ in B_1 and hence in $B_1 \setminus \overline{B}_{1/2}$.

Therefore, since $\tilde{u}$ is a solution to (2.1) in B_1 and thus also in $B_1 \setminus \overline{B}_{1/2}$ and $(\varphi + v) \in C^2(B_1 \setminus \overline{B}_{1/2})$ touches $\tilde{u}$ from below at $x_0 \in (B_1 \setminus \overline{B}_{1/2})^+(\tilde{u})$, we get

$$\begin{aligned}
&\sum_{i,j} \tilde{a}_{ij}(x_0)(\varphi + v)_{ij}(x_0) = \sum_{i,j} \tilde{a}_{ij}(x_0)(\varphi + c\tilde{u}(0)G)_{ij}(x_0) \\
&= \sum_{i,j} \tilde{a}_{ij}(x_0)(\varphi_{ij}(x_0) + c\tilde{u}(0)G_{ij}(x_0)) \\
&= \sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + \sum_{i,j} \tilde{a}_{ij}(x_0)c\tilde{u}(0)G_{ij}(x_0) \\
&= \sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + c\tilde{u}(0)\sum_{i,j} \tilde{a}_{ij}(x_0)G_{ij}(x_0) \leq \tilde{f}(x_0)
\end{aligned}$$

in other words

$$\sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + c\tilde{u}(0)\sum_{i,j} \tilde{a}_{ij}(x_0)G_{ij}(x_0) \leq \tilde{f}(x_0),$$

which entails

$$\sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) \leq \tilde{f}(x_0) - c\tilde{u}(0)\sum_{i,j} \tilde{a}_{ij}(x_0)G_{ij}(x_0). \tag{5.21}$$

Now, in view of (5.10), given that $x_0 \in (B_1 \setminus \overline{B}_{1/2})$, we achieve from (5.22) taking $\varepsilon^2 = c\tilde{u}(0)\varepsilon^2$,

$$\sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) \leq \tilde{f}(x_0) - \varepsilon^2 \leq \tilde{f}(x_0) - \tilde{f}(x_0) = 0, \tag{5.22}$$

seeing as how from the first inequality in (2.2) we have $\left\|\tilde{f}\right\|_{L^\infty(B_1)} \leq \varepsilon^2$, which also gives $\left|\tilde{f}(x)\right| \leq \varepsilon^2$, $\forall x \in B_1$. Thus, inasmuch $x_0 \in (B_1 \setminus \overline{B}_{1/2}) \subset B_1$, namely $x_0 \in B_1$, we have $\left|\tilde{f}\right|(x_0) \leq \varepsilon^2$ and hence $\tilde{f}(x_0) \leq \varepsilon^2$.
To sum it up, from (5.22), we have obtained

$$\sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) \leq 0,$$

which implies that $\tilde{u} - v$ is a viscosity supersolution to $\sum_{i,j} \tilde{a}_{ij}(\tilde{u} - v)_{ij} = 0$ in $B_1 \setminus \overline{B}_{1/2}$.
Consequently, we can apply the maximum principle and we get

$$\inf_{B_1 \setminus \overline{B}_{1/2}} (\tilde{u} - v) = \inf_{\partial(B_1 \setminus \overline{B}_{1/2})} (\tilde{u} - v) = \inf_{\partial B_1 \cup \partial B_{1/2}} (\tilde{u} - v). \tag{5.23}$$

In addition, we have $G = 1$ on $\partial B_{1/2}$, hence from (5.16), we achieve

$$\tilde{u} \geq c\tilde{u}(0)G \quad \text{on } \partial B_{1/2}. \tag{5.24}$$

At the same time, we have $G = 0$ on ∂B_1, therefore, because $\tilde{u} \geq 0$ on ∂B_1 we achieve

$$\tilde{u} \geq c\tilde{u}(0)G \quad \text{on } \partial B_1 \tag{5.25}$$

Thus, from (5.24) and (5.25) we obtain for definition of v, $\tilde{u} \geq v$ on $\partial B_1 \cup \partial B_{1/2}$, that is $\tilde{u} - v \geq 0$ on $\partial B_1 \cup \partial B_{1/2}$. As a consequence, 0 is a lower bound of the set

$$\left\{ \tilde{u}(x) - v(x), \quad x \in \partial B_1 \cup \partial B_{1/2} \right\},$$

which entails

$$\inf_{\partial B_1 \cup \partial B_{1/2}} (\tilde{u} - v) \geq 0. \tag{5.26}$$

Therefore, from (5.23) and (5.26), we also get, since $\tilde{u} - v(x) \geq \inf_{B_1 \setminus \overline{B}_{1/2}} (\tilde{u} - v)$ $\forall x \in B_1 \setminus \overline{B}_{1/2}$,

$$0 \leq \tilde{u}(x) - v(x) \quad \forall x \in B_1 \setminus \overline{B}_{1/2},$$

which gives, together with $\tilde{u}(x) - v(x) \geq 0$ $\forall x \in \partial B_1 \cup \partial B_{1/2}$,

$$\tilde{u}(x) \geq v(x) \quad \text{on } \overline{B}_1 \setminus B_{1/2}. \tag{5.27}$$

At this point, we notice that $d > 0$, recalling that $B_1^+(u)$ is an open set, inasmuch as $u \in C(\Omega)$ and B_1 is an open set, and thus we can find a ball $B_{\bar{r}}$ such that $B_{\bar{r}} \subset B_1^+(u)$, that is $u > 0$ in $B_{\bar{r}}$. This fact, specifically, entails $B_{\bar{r}} \cap F(u) = \emptyset$, because $u = 0$ on $F(u)$, and hence $d = \inf_{x \in F(u)} |x| \geq \bar{r} > 0$, in other words, $d > 0$.

In particular, if we call $r^* = \sup \left\{ r \mid \quad B_r \subset B_1^+(u) \right\}$, we have $r^* < 1$, inasmuch $F(u) \cap B_1 \neq \emptyset$, and we claim that there exists $z \in \partial B_{r^*}$ so that $z \in F(u)$. Indeed, if for contradiction such z does not exist, we have $\overline{B}_{r^*} \subset B_1^+(u)$ or there exists $x_0 \in F(u)$, with $|x_0| < r^*$. With respect to this second possibility, however, we would have that $|x_0|$ would be an upper bound of the set $\left\{ B_r \mid \quad B_r \subset B_1^+(u) \right\}$, and as a consequence, for definition of sup, we would have $r^* \leq |x_0| < r^*$, namely $r^* < r^*$, which is a contradiction. Therefore, we have $\overline{B}_{r^*} \subset B_1^+(u)$, but, given that $B_1^+(u)$ is an open set, $\operatorname{dist}(\overline{B}_{r^*}, \partial B_1^+(u)) > 0$, thus if we call $\delta := \operatorname{dist}(\overline{B}_{r^*}, \partial B_1^+(u))$, and we take $r^* + \frac{\delta}{2}$, $B_{r^* + \frac{\delta}{2}} \subset B_1^+(u)$, which implies, for definition of sup, $r^* + \frac{\delta}{2} \leq r^*$, which is a contradiction, recalling that $\frac{\delta}{2} > 0$.

To sum it up, we have proved that there exists $z \in \partial B_{r^*}$, with $z \in F(u)$.

We show now that z is the point where d is achieved, that is $d = |z|$. Precisely, if for contradiction $d \neq |z|$, seeing as how $z \in F(u)$, we have $d < |z|$. Furthermore, $|z| = r^*$, inasmuch $z \in \partial B_{r^*}$, hence $d < r^*$. Consequently, $\frac{r^*-d}{2} > 0$ and if we set $\varepsilon = \frac{r^*-d}{2} > 0$, since $d = \inf\limits_{x \in F(u)} |x|$, there exists $\bar{x} \in F(u)$ such that $d \leq |\bar{x}| \leq d + \varepsilon$, which entails that $r^* \leq d + \varepsilon$, but for the choice of ε,

$$d + \varepsilon = d + \frac{r^* - d}{2} < d + r^* - d = r^*,$$

i.e. $d + \varepsilon < r^*$, which contradicts $r^* \leq d + \varepsilon$.
Thus, z is the point where d is achieved and $|z| = d$.
Moreover, $z \in F(u)$, hence $u(z) = 0$ and for definition of $\tilde{u}$, $\tilde{u}(\frac{z}{d}) = \frac{u(d\frac{z}{d})}{d} = \frac{u(z)}{d} = 0$. As a consequence $\forall B_r\left(\frac{z}{d}\right)$ $B_r\left(\frac{z}{d}\right) \cap (\overline{B}_1 \setminus B_{1/2})^+(\tilde{u})^c \neq \emptyset$, and seeing as how $\tilde{u} > 0$ in B_1, for what we have said above, also $B_r\left(\frac{z}{d}\right) \cap (\overline{B}_1 \setminus B_{1/2})^+(\tilde{u}) \neq \emptyset$ $\forall B_r\left(\frac{z}{d}\right)$, recalling that $\frac{z}{d} \in \partial B_1$, and hence $B_r\left(\frac{z}{d}\right) \cap B_1 \neq \emptyset$ $\forall B_r\left(\frac{z}{d}\right)$. Therefore, $\frac{z}{d} \in \partial(\overline{B}_1 \setminus B_{1/2})^+(\tilde{u}) \cap (\overline{B}_1 \setminus B_{1/2})$.
Nevertheless, given that $\tilde{u}\left(\frac{z}{d}\right) = 0$, we also have that $B_r\left(\frac{z}{d}\right) \cap \overline{B}_1^+(\tilde{u})^c \neq \emptyset$, $\forall B_r\left(\frac{z}{d}\right)$, and if $B_r\left(\frac{z}{d}\right) \cap (\overline{B}_1 \setminus B_{1/2})^+(\tilde{u})$, $\forall B_r\left(\frac{z}{d}\right)$, since $\overline{B}_1 \setminus B_{1/2} \subset \overline{B}_1$, $B_r\left(\frac{z}{d}\right) \cap \overline{B}_1^+(\tilde{u}) \cap \emptyset$, $\forall B_r\left(\frac{z}{d}\right)$ as well.
To sum it up, $\frac{z}{d} \in \partial\overline{B}_1^+(\tilde{u}) \cap \overline{B}_1$.
Now, from (5.27), inasmuch as $\tilde{u}\left(\frac{z}{d}\right) = 0$ and $v \geq 0$, recalling that $\tilde{u}(0) > 0$ and $G \geq 0$, for definition, in $\overline{B}_1 \setminus B_{1/2}$, we obtain $v(\frac{z}{d}) = 0$, which implies from (5.27), that v touches $\tilde{u}$ at $\frac{z}{d} \in \partial\overline{B}_1^+(\tilde{u}) \cap \overline{B}_1$, with $v \in C^2(\overline{B}_1 \setminus B_{1/2})$.
Consequently, because $\tilde{u}$ is a solution to (2.1)-(2.2) in B_1 with free boundary condition $\tilde{g}$, and repeating the same argument, also in $\overline{B}_1$, we get, inasmuch $\frac{z}{d} \in \partial\overline{B}_1^+(\tilde{u}) \cap \overline{B}_1$, which is the free boundary in $\overline{B}_1$,

$$|\nabla v|\left(\frac{v}{d}\right) \leq \tilde{g}\left(\frac{z}{d}\right) = g\left(d\frac{z}{d}\right) = g(z),$$

that is

$$|v|\left(\frac{z}{d}\right) \leq g(z). \tag{5.28}$$

In particular, seeing as how $C, c, \tilde{u}(0), \gamma > 0$, we can rewrite the first term in

(5.28) as

$$\begin{aligned}|\nabla v|\left(\frac{z}{d}\right) &= \left|\nabla(c\tilde{u}(0)C(|x|^{-\gamma}-1))\right|\left(\frac{v}{d}\right)\\ &= \left|c\tilde{u}(0) - \gamma C\,|x|^{-\gamma-1}\,\frac{x}{|x|}\right|\left(\frac{v}{d}\right)\\ &= (c\tilde{u}(0)C\gamma\,|x|^{-\gamma-1})\left(\frac{z}{d}\right) = c\tilde{u}(0)C\gamma\left|\frac{z}{d}\right|^{-\gamma-1},\end{aligned}$$

i.e.

$$|\nabla v|\left(\frac{z}{d}\right) \leq c\tilde{u}(0)C\gamma\left|\frac{z}{d}\right|^{-\gamma-1},$$

which gives, because $\left|\frac{z}{d}\right| = 1$,

$$|\nabla v|\left(\frac{z}{d}\right) = c\tilde{u}(0)C\gamma. \tag{5.29}$$

Consequently, from (5.28) and (5.29), we achieve

$$c\tilde{u}(0)C\gamma \leq g(z) \leq 1+\varepsilon^2 \leq 2,$$

namely

$$c\tilde{u}(0)C\gamma \leq 2, \tag{5.30}$$

given that $\|g-1\| \leq \varepsilon^2$, thus $g(z)-1 \leq \varepsilon^2$, and $g(z) \leq 1+\varepsilon^2$, and inasmuch as $\varepsilon^2 \leq 1$.
Now, from (5.30) we obtain

$$\tilde{u}(0) \leq \frac{2}{cC\gamma}, \tag{5.31}$$

but we have supposed for contradiction $\tilde{u}(0) > C_0$, thus if we take $C_0 > \frac{2}{cC\gamma}$, we get from (5.31),

$$\frac{2}{cC\gamma} < \frac{2}{cC\gamma},$$

which is a contradiction.
To sum it up, we have shown that

$$\tilde{u}(0) \leq C_0, \tag{5.32}$$

with $C_0 > \frac{2}{cC\gamma}$.
To prove the lower bound, instead, let

$$\tilde{G}(x) = \eta(1 - G(x)) = \eta(1 - C(|x|^{-\gamma} - 1)), \tag{5.33}$$

with η (depending on γ) such that

$$\left|\nabla\tilde{G}\right| < 1 - \varepsilon^2 \quad \text{on } \partial B_{1/2}. \tag{5.34}$$

Specifically, we have, seeing as how $C, \gamma, \eta > 0$

$$\begin{aligned}\left|\nabla\tilde{G}\right| &= \left|\nabla(\eta(1 - C(|x|^{-\gamma} - 1)))\right| \\ &= \eta C\gamma\,|x|^{-\gamma-1}\left|\frac{x}{|x|}\right| = \eta C\gamma\,|x|^{-\gamma-1},\end{aligned}$$

in other words,

$$\left|\nabla\tilde{G}\right| = \eta C\gamma\,|x|^{-\gamma-1},$$

which entails, since $|x| = \frac{1}{2}$ on $\partial B_{1/2}$,

$$\left|\nabla\tilde{G}\right| = \eta C\gamma\left(\frac{1}{2}\right)^{-\gamma-1} \quad \text{on } \partial B_{1/2}. \tag{5.35}$$

Therefore, if we impose that $\left|\nabla\tilde{G}\right| < 1 - \varepsilon^2$ on $\partial B_{1/2}$, we obtain from (5.35)

$$\eta C\gamma\left(\frac{1}{2}\right)^{-\gamma-1} < 1 - \varepsilon^2,$$

which gives

$$\eta < \frac{1 - \varepsilon^2}{C\gamma\left(\frac{1}{2}\right)^{-\gamma-1}},$$

and hence we choose $\eta > 0$ so that this condition on η is satisfied.
Now, assume without loss of generality that $F(u)$ is a Lipschitz graph in the x_n direction, otherwise we can apply a rotation to the coordinates to achieve this fact. In addition, we suppose that the Lipschitz constant is equal to 1. At this point, we translate the graph of $\tilde{G}$ by $-te_n$, with $t \in \mathbb{R}$, $t > 0$ i.e. if we denote with

$$\Gamma_{\tilde{G}} := \left\{(x, \tilde{G}(x)), \quad x \in \overline{B}_1 \setminus B_{1/2}\right\},$$

the graph of $\tilde{G}$, we can write the translation as

$$\begin{aligned}\Gamma_{\tilde{G}} - te_n &= \Big\{(x - 4e_n, \tilde{G}(x)), \quad x \in \overline{B}_1 \setminus B_{1/2}\Big\} \\ &= \Big\{(x_1, \dots, x_n - t, \tilde{G}(x)), \quad x \in \overline{B}_1 \setminus B_{1/2}\Big\}\end{aligned}$$

where $x = (x_1, \dots, x_n)$.
In particular, we can take t large enough so that $\tilde{u} \equiv 0$ in $B_1(-te_n)$.
Furthermore, we remark that from (5.9) and (5.11),

$$0 \leq G = \frac{1}{(1/2)^{-\gamma} - 1}(|x|^{-\gamma} - 1) \leq \frac{(1/2)^{-\gamma} - 1}{(1/2)^{-\gamma} - 1} = 1, \quad \text{on } \overline{B}_1 \setminus B_{1/2},$$

namely

$$0 \leq G \leq 1 \quad \text{on } \overline{B}_1 \setminus B_{1/2}.$$

As a consequence, we have from (5.33) that $0 \leq \tilde{G} \leq \eta$ and thus $\Gamma_{\tilde{G}} - te_n$ is above the graph of $\tilde{u}$, since $\tilde{u} \equiv 0$ in $B_1(-te_n)$, for t large enough. We slide then the graph of $\tilde{G}$ in the e_n direction till we touch the graph of $\tilde{u}$, in a point which we call $\tilde{z}$. Moreover, we call $\tilde{t}$ the value of t for which this contact is verified.
At this point, we define

$$\tilde{G}_{\tilde{t}}(x) = \tilde{G}(x + \tilde{t}e_n), \tag{5.36}$$

and we notice that $\tilde{G}_{\tilde{t}}$ is defined on $\overline{B}_1(-\tilde{t}e_n) \setminus B_{1/2}(-\tilde{t}e_n)$, given that $\tilde{G}$ is defined on $\overline{B}_1 \setminus B_{1/2}$. Indeed, from definition of $\tilde{G}_{\tilde{t}}$, since $\tilde{G}$ is defined on $\overline{B}_1 \setminus B_{1/2}$, we must impose

$$\frac{1}{2} \leq \big|x + \tilde{t}e_n\big| = \big|x - (-\tilde{t}e_n)\big| \leq 1,$$

that is $\tilde{G}_{\tilde{t}}$ is defined on $\overline{B}_1(-\tilde{t}e_n) \setminus B_{1/2}(-\tilde{t}e_n)$.
In addition, we claim that $\tilde{G}_{\tilde{t}}$ is a strict supersolution to our free boundary problem on $\overline{B}_1(-\tilde{t}e_n) \setminus B_{1/2}(-\tilde{t}e_n)$.
Precisely, from (5.10), we get

$$-\sum_{i,j} \tilde{a}_{ij} G_{ij} < -\varepsilon^2 \quad \text{on } B_1 \setminus \overline{B}_{1/2},$$

which gives,

$$\begin{aligned}\sum_{i,j} \tilde{a}_{ij}\tilde{G}_{\tilde{t}_{ij}} &= \sum_{i,j} \tilde{a}_{ij}(\tilde{G}(x+\tilde{t}e_n))_{ij} = \sum_{i,j} \tilde{a}_{ij}\tilde{G}_{ij}(x+\tilde{t}e_n) \\ &= \sum_{i,j} \tilde{a}_{ij}(\eta(1-G))_{ij}(x+\tilde{t}e_n) = \sum_{i,j} \tilde{a}_{ij}(-\eta G_{ij})(x+\tilde{t}e_n) \\ &= -\eta \sum_{i,j} \tilde{a}_{ij}G_{ij}(x+\tilde{t}e_n) < -\eta\varepsilon^2 \quad \text{on } B_1(-\tilde{t}e_n) \setminus \overline{B}_{1/2}(-\tilde{t}e_n),\end{aligned}$$

in other words,

$$\sum_{i,j} \tilde{a}_{ij}\tilde{G}_{\tilde{t}_{ij}} < -\varepsilon^2 \quad \text{on } B_1(-\tilde{t}e_n) \setminus \overline{B}_{1/2}(-\tilde{t}e_n), \tag{5.37}$$

calling $-\eta\varepsilon^2 = -\varepsilon^2$.
Moreover, seeing as how $\left\|\tilde{f}\right\|_{L^\infty} \leq \varepsilon^2$, we have $\left|\tilde{f}\right|(x) \leq \varepsilon^2 \ \forall x$ which entails $\tilde{f}(x) \geq -\varepsilon^2 \ \forall x$.
Therefore, from (5.37), we obtain

$$\sum_{i,j} \tilde{a}_{ij}\tilde{G}_{\tilde{t_{ij}}} < \tilde{f} \quad \text{on } B_1(-\tilde{t}e_n) \setminus \overline{B}_{1/2}(-\tilde{t}e_n). \tag{5.38}$$

On the other hand, we also have $\|\tilde{g}-1\|_{L^\infty} \leq \varepsilon^2$, which implies, repeating the reasoning done above, $\tilde{g}(x) - 1 \geq -\varepsilon^2$, $\forall x$, namely $\tilde{g}(x) \geq 1-\varepsilon^2$, $\forall x$. Consequently, inasmuch $x+\tilde{t}e_n \in \partial B_{1/2}$ if $x \in \partial B_{1/2}(-\tilde{t}e_n)$, we achieve from (5.34)

$$\begin{aligned}\left|\nabla \tilde{G}_{\tilde{t}}\right| = \left|\nabla(\tilde{G}(x+\tilde{t}e_n))\right| &= \left|\nabla \tilde{G}(x+\tilde{t}e_n)\right| \\ &< 1-\varepsilon^2 \leq \tilde{g} \quad \text{on } \partial B_{1/2}(-\tilde{t}e_n),\end{aligned}$$

i.e.

$$\left|\nabla \tilde{G}_{\tilde{t}}\right| < \tilde{g} \quad \text{on } \partial B_{1/2}(-\tilde{t}e_n). \tag{5.39}$$

To sum it up, from (5.38) and (5.39) we have that $\tilde{G}_{\tilde{t}}$ is a strict supersolution to our free boundary problem on $\overline{B}_1(-\tilde{t}e_n) \setminus B_{1/2}(-\tilde{t}e_n)$.
Now, we remark that if we define $\tilde{G}_t$ as we have done for $\tilde{G}_{\tilde{t}}$, we have $\tilde{G}_t \equiv 0$ on $\partial B_{1/2}(-\tilde{t}e_n)$ $\forall t$. As a consequence, from the choice of $\tilde{t}$, the touching point

$\tilde{z}$ can occur only on $F(\tilde{u})$ or where $\tilde{u}$ is positive.

Suppose, hence, that $\tilde{z} \in F(\tilde{u})$. We notice that, because $\tilde{u}(\tilde{z}) = 0$, $\tilde{G}_{\tilde{t}}(\tilde{z}) = 0$, thus $\tilde{z} \in \partial B_{1/2}(-\tilde{t}e_n)$ necessary. Then, from (5.35) and in view of the calculation for achieving (5.39), we obtain

$$\left|\nabla \tilde{G}_{\tilde{t}}\right| = \left|\nabla \tilde{G}(x + \tilde{t}e_n)\right| = \eta C\gamma \left(\frac{1}{2}\right)^{-\gamma-1} \quad \text{on } \partial B_{1/2},$$

which entails $\left|\nabla \tilde{G}_{\tilde{t}}\right| \neq 0$ on $\partial B_{1/2}(-\tilde{t}e_n)$ and in particular $\left|\nabla \tilde{G}\right|(\tilde{z}) \neq 0$.

At this point, for the choice of $\tilde{t}$, we can find a neighborhood $O \subset \overline{B}_1(-\tilde{t}e_n) \setminus B_{1/2}(-\tilde{t}e_n)$ of $\tilde{z}$ such that $\tilde{G}_{\tilde{t}}$ touches $\tilde{u}$ from above at $\tilde{z}$ and furthermore, inasmuch as $\tilde{G}_{\tilde{t}} \geq 0$, $\tilde{G}^+_{\tilde{t}} = \tilde{G}_{\tilde{t}}$, that is we also have that $\tilde{G}^+_{\tilde{t}}$ touches $\tilde{u}$ from above at $\tilde{z}$.

Therefore, summarizing, we have $\tilde{G}^+_{\tilde{t}}$ touching $\tilde{u}$ from above at $\tilde{z} \in F(\tilde{u})$, with $\left|\nabla \tilde{G}_{\tilde{t}}\right|(\tilde{z}) \neq 0$ and $\tilde{G}_{\tilde{t}} \in C^\infty(\overline{B}_1(-\tilde{t}e_n) \setminus B_{1/2}(-\tilde{t}e_n)$, and hence $\tilde{G}_{\tilde{t}} \in C^2(\overline{B}_1(-\tilde{t}e_n) \setminus B_{1/2}(-\tilde{t}e_n)$. So, seeing as how $\tilde{u}$ is a solution to (2.1)-(2.2), we get

$$\left|\nabla \tilde{G}_{\tilde{t}}\right|(\tilde{z}) \geq \tilde{g}(\tilde{z}),$$

which gives from (5.39), since $\tilde{z} \in \partial B_{1/2}(-\tilde{t}e_n)$,

$$\tilde{g}(\tilde{z}) \leq \left|\nabla \tilde{G}\right|(\tilde{z}) < \tilde{g}(\tilde{z}),$$

in other words $\tilde{g}(\tilde{z}) < \tilde{g}(\tilde{z})$, which is a contradiction.

Consequently, $\tilde{z} \in \{x, \quad \tilde{u}(x) > 0\}$ and also $\tilde{G}_{\tilde{t}}(\tilde{z}) > 0$, which implies that $\tilde{z} \in \overline{B}_1(-\tilde{t}e_n) \setminus \overline{B}_{1/2}(-\tilde{t}e_n)$. In particular, we claim that $\tilde{z} \in \partial B_1(-\tilde{t}e_n)$, where $\tilde{G}_{\tilde{t}} \equiv \eta$, for definition of $\tilde{G}$, i.e. $\tilde{z}$ occur on the η level set.

Precisely, for what we have said before, we have that $\tilde{G}_{\tilde{t}}$ touches $\tilde{u}$ from above at $\tilde{z} \in \{x, \quad \tilde{u}(x) > 0\}$, with $\tilde{G}_{\tilde{t}} \in C^2(\overline{B}_1(-\tilde{t}e_n) \setminus B_{1/2}(-\tilde{t}e_n))$, as observed above, and thus, given that $\tilde{u}$ is a solution to (2.1)-(2.2), we achieve

$$\sum_{i,j} \tilde{a}_{ij}(\tilde{z})\tilde{G}_{\tilde{t}_{ij}}(\tilde{z}) \geq \tilde{f}(\tilde{z}),$$

which entails from (5.39), if $\tilde{z} \in B_1(-\tilde{t}e_n) \setminus \overline{B}_{1/2}(-\tilde{t}e_n)$,

$$\tilde{f}(\tilde{z}) \leq \sum_{i,j} \tilde{a}_{ij}(\tilde{z})\tilde{G}_{\tilde{t}}(\tilde{z}) < \tilde{f}(\tilde{z}),$$

namely $\tilde{f}(\tilde{z}) < \tilde{f}(\tilde{z})$, which is a contradiction.
Therefore, we have obtained that $\tilde{z} \in \partial B_1(-\tilde{t}e_n)$ and hence $\tilde{z}$ occurs on the η level set.
Furthermore, if we denote $\tilde{d} := \text{dist}(\tilde{z}, F(\tilde{u}))$, $\tilde{d} \leq 1$.
Indeed, because $\tilde{G}_{\tilde{t}}$ is above $\tilde{u}$, and $\tilde{G} \equiv 0$ on $\partial B_{1/2}(-\tilde{t}e_n)$, we have $\tilde{u} \equiv 0$ on $\partial B_{1/2}(-\tilde{t}e_n)$. As a consequence, inasmuch $\tilde{u}$ is continuous and $\tilde{u}(\tilde{z}) > 0$, with $\tilde{z} \in \partial B_1(-\tilde{t}e_n)$, there exists a point $\tilde{x} \in F(\tilde{u})$ so that

$$\text{dist}(\tilde{x}, \tilde{z}) = |\tilde{x} - \tilde{z}| \leq \text{dist}(\tilde{z}, \partial B_{1/2}(-\tilde{t}e_n)) = \frac{1}{2} \leq 1,$$

in other words

$$|\tilde{x} - \tilde{z}| \leq 1,$$

which implies, seeing as how $\tilde{x} \in F(\tilde{u})$,

$$\tilde{d} = \inf_{x \in F(\tilde{u})} |x - \tilde{z}| \leq |\tilde{x} - \tilde{z}| \leq 1,$$

that is $\tilde{d} \leq 1$.
Now, from the first part, $\tilde{u}$ is Lipschitz continuous, namely

$$|\tilde{u}(x) - \tilde{u}(y)| \leq L\,|x - y|\,, \tag{5.40}$$

calling L its Lipschitz constant.
In particular, the Lipschitz continuity of $\tilde{u}$ implies that also u is Lipschitz continuous.
Indeed, from (5.40) we have

$$|\tilde{u}(x) - \tilde{u}(y)| = \left|\frac{u(dx)}{d} - \frac{u(dy)}{d}\right| \qquad = \frac{1}{d}\,|u(dx) - u(dy)| \leq L\,|x - y|$$

i.e.

$$\frac{1}{d}\,|u(dx) - u(dy)| \leq L\,|x - y|\,,$$

which implies

$$|u(dx) - u(dy)| \leq dL\,|x - y| = L\,|d(x - y)| = L\,|dx - dy|\,,$$

in other words
$$|u(dx) - u(dy)| \leq L\,|dx - dy|\,,$$
which gives the Lipschitz continuity of u. Consequently, if we take $x \in F(\tilde{u})$, $\tilde{u}(x) = 0$, hence from (5.40) we get
$$|\tilde{u}(\tilde{z})| = |\tilde{u}(\tilde{z}) - \tilde{u}(x)| \leq L\,|\tilde{z} - x|\,,$$
in other words, inasmuch as $\tilde{u} \geq 0$ and thus $|\tilde{u}(\tilde{z})| = \tilde{u}(\tilde{z})$,
$$\tilde{u}(\tilde{z}) \leq L\,|\tilde{z} - x|\,,$$
and
$$\frac{\tilde{u}(\tilde{z})}{L} \leq |\tilde{z} - x|\,. \tag{5.41}$$
In particular, from the arbitrariness of $x \in F(\tilde{u})$, we achieve that $\frac{\tilde{u}(\tilde{z})}{L}$ is a lower bound of the set $\{|\tilde{z} - x|\,,\ x \in F(\tilde{u})\}$, therefore
$$\frac{\tilde{u}(\tilde{z})}{L} \leq \inf_{x \in F(\tilde{u})} |\tilde{z} - x| = \tilde{d},$$
which implies
$$\frac{\tilde{u}(\tilde{z})}{L} \leq \tilde{d}$$
and
$$\tilde{u}(\tilde{z}) \leq L\tilde{d}. \tag{5.42}$$
In addition, we know that $\tilde{u}(\tilde{z}) = \eta$, as a consequence, from (5.42) we also have
$$\eta \leq L\tilde{d},$$
which gives
$$L^{-1}\eta \leq \tilde{d} \leq 1,$$
that is, $\tilde{d}$ is comparable to 1.

At this point, we notice that $F(\tilde{u})$ is Lipschitz.

Precisely, since $F(u)$ is Lipschitz and we have supposed that $F(u)$ is a Lipschitz graph in the x_n direction with Lipschitz constant equal to 1, we have
$$F(u) = \{(x', \psi(x'))\}\,, \tag{5.43}$$

with

$$|\psi(x') - \psi(y')| \leq |x' - y'|, \quad (x', \psi(x')), (y', \psi(y')) \in F(u). \tag{5.44}$$

Now, if $x_0 \in F(u)$, we have $u(x_0) = 0$, and $\forall B_r(x_0)$, $B_r(x_0) \cap \{x, \ u(x) > 0\} \neq \emptyset$. Therefore, for definition of $\tilde{u}$, $\tilde{u}\left(\frac{x_0}{d}\right) = \frac{u\left(d\frac{x_0}{d}\right)}{d} = \frac{u(x_0)}{d} = 0$, i.e. $\tilde{u}\left(\frac{x}{d}\right) = 0$ and thus in particular $B_r\left(\frac{x}{d}\right) \cap \{x, \ \tilde{u}(x) > 0\}^c$.

Moreover, if we fix $B_{\bar{r}}\left(\frac{x_0}{d}\right)$ and we consider $B_{d\bar{r}}(x_0)$, $B_{d\bar{r}}(x_0) \cap \{x, \ u(x) > 0\} \neq \emptyset$, namely there exist a point $z_0 \in B_{d\bar{r}}(x_0) \cap \{x, \ u(x) > 0\}$, which satisfies $u(z_0) > 0$ and it can be written $z_0 = d\bar{z}$, with $\bar{z} \in B_{\bar{r}}\left(\frac{x_0}{d}\right)$, see proof of Theorem 4.2. Hence, for definition of $\tilde{u}$, $\tilde{u}(\bar{z}) = \frac{u(d\bar{z})}{d} > 0$, in other words $\tilde{u}(\bar{z}) > 0$ and, given that $\bar{z} \in B_{\bar{r}}\left(\frac{x_0}{d}\right)$, $\bar{z} \in B_{\bar{r}}\left(\frac{x_0}{d}\right) \cap \{x, \ \tilde{u}(x) > 0\}$, in other words $B_{\bar{r}}\left(\frac{x}{d}\right) \cap \{x, \ \tilde{u}(x) > 0\} \neq \emptyset$.

As a consequence, for the arbitrariness of $B_{\bar{r}}\left(\frac{x_0}{d}\right)$, we obtain $B_r\left(\frac{x_0}{d}\right) \cap \{x, \ \tilde{u}(x) > 0\} \neq \emptyset$, $\forall B_r\left(\frac{x_0}{d}\right)$. To sum it up, we have $\tilde{u}\left(\frac{x_0}{d}\right) = 0$ and $B_r\left(\frac{x_0}{d}\right) \cap \{x, \ \tilde{u}(x) > 0\} \neq \emptyset$ and $B_r\left(\frac{x_0}{d}\right) \cap \{x, \ \tilde{u}(x) > 0\}^c \neq \emptyset$, $\forall B_r\left(\frac{x_0}{d}\right)$, which implies $\frac{x_0}{d} \in F(\tilde{u})$.

Consequently, for the arbitrariness of $x_0 \in F(u)$ and repeating the same argument used to show that $d\bar{x} \in F(u)$ if we have $\bar{x} \in F(\tilde{u})$, we get from (5.43)

$$F(\tilde{u}) = \frac{1}{d}F(u) = \frac{1}{d}\{(x', \psi(x'))\} = \left\{\frac{x'}{d}, \frac{\psi(x')}{d}\right\},$$

namely

$$F(\tilde{u}) = \left\{\frac{x'}{d}, \frac{\psi(x')}{d}\right\}. \tag{5.45}$$

At this point, if we define

$$\tilde{\psi}(x') := \frac{\psi(dx')}{d},$$

we achieve

$$\tilde{\psi}\left(\frac{x'}{d}\right) = \frac{\psi\left(d\frac{x'}{d}\right)}{d} = \frac{\psi(x')}{d},$$

that is

$$\tilde{\psi}\left(\frac{x'}{d}\right) = \frac{\psi(x')}{d}. \tag{5.46}$$

Therefore, from (5.46) and (5.47), we obtain

$$F(\tilde{u}) = \left\{ \frac{x'}{d}, \tilde{\psi}\left(\frac{x'}{d}\right) \right\} \tag{5.47}$$

with, in view of (5.44) and (5.46)

$$\begin{aligned}
\left| \tilde{\psi}\left(\frac{x'}{d}\right) - \tilde{\psi}\left(\frac{y'}{d}\right) \right| &= \left| \frac{\psi(x')}{d} - \frac{\psi(y')}{d} \right| \\
&= \frac{1}{d} \left| \psi(x') - \psi(y') \right| \\
&\leq \frac{1}{d} \left| x' - y' \right| = \left| \frac{x'}{d} - \frac{y'}{d} \right|,
\end{aligned}$$

in other words

$$\left| \tilde{\psi}\left(\frac{x'}{d}\right) - \tilde{\psi}\left(\frac{y'}{d}\right) \right| \leq \left| \frac{x'}{d} - \frac{y'}{d} \right|. \tag{5.48}$$

As a consequence, from (5.47) and (5.48), we get that $F(\tilde{u})$ is Lipschitz.
Now, because $F(\tilde{u})$ is Lipschitz, we can connect 0 and $\tilde{z}$ with a chain of intersecting balls included in the positive side of $\tilde{u}$ with radii comparable to 1.
Specifically, let us call this chain

$$\left\{ B_{r_i}(x_i), \ i = 0, \dots, N \right\},$$

with x_i in the positive side of $\tilde{u}$, r_i comparable to 1 and which satisfies $0 \in B_{r_0}(x_0)$ and $\tilde{z} \in B_{r_N}(x_N)$. Furthermore, the number N of these balls is bounded by a universal constant. In particular, we want to apply the Harnack inequality repeatedly to compare $\tilde{u}(0)$ with $\tilde{u}(\tilde{z})$, thus we suppose that also $B_{2r_i}(x_i)$ is in the positive side of $\tilde{u}$.
At this point, we are ready to apply the Harnack inequality in each ball.
Let us begin from the first ball and repeating the reasoning done to achieve (5.16), we get

$$\tilde{u}(0) \geq c_1 \tilde{u}(\tilde{x}_1) \geq c_1 c_2 \tilde{u}(\tilde{x}_2) \geq c_1 c_2 \dots c_{N+1} \tilde{u}(\tilde{z}) = c \tilde{u}(\tilde{z}), \tag{5.49}$$

with $\tilde{x}_i \in B_{r_{i-1}}(x_{i-1}) \cap B_{r_i}(x_i)$ and where we take ε small enough, with $\left\| \tilde{f} \right\|_{L^\infty} \leq \varepsilon$, such that we can obtain a result analogous to (5.16) in each ball.

Let us remark, moreover, that we can find this ε since the number of balls is bounded by a universal constant.
In particular, from (5.49) we have

$$\tilde{u}(0) \geq c\tilde{u}(\tilde{z}) = c_0,$$

i.e.

$$\tilde{u}(0) \geq c_0,$$

which entails from (5.32)

$$c_0 \leq \tilde{u}(0) \leq C_0,$$

and from definition of $\tilde{u}$, see (5.1),

$$c_0 d(0) \leq u(0) \leq C_0 d(0), \tag{5.50}$$

inasmuch $d = d(0) = \mathrm{dist}(0, F(u))$.
Now, if in place of 0, we have $x_0 \in B_1^+(u)$, $x_0 \neq 0$, we can repeat exactly the same argument with

$$\tilde{u}(x) = \frac{u(x_0 + d(x_0)x)}{d(x_0)},$$

where $d(x_0) = \mathrm{dist}(x_0, F(u))$, and we achieve

$$c_0 d(x_0) \leq u(x_0) \leq C_0 d(x_0),$$

which gives, together with (5.50)

$$c_0 d(z) \leq u(z) \leq C_0 d(z), \quad \text{for all } z \in B_1^+(u),$$

as desired. $\square$

Chapter 6

The one-phase problem for equations with first order term

We return now to the more general problem (1.1) introduced in Chapter 1.
For exposure convenience, we rewrite here the problem, that is:

$$\begin{cases} \sum_{i,j} a_{ij}(x)u_{ij} + \sum_{i} b_i(x) \cdot u_i = f & \text{in } \Omega^+(u) \\ |\nabla u| = g & \text{on } F(u) \end{cases} \tag{6.1}$$

with $b_i \in C(\Omega) \cap L^\infty(\Omega)$ and the same conditions listed in Chapter 1 for Ω, f, g and a_{ij}. Moreover, u_i denotes the first derivative of u respect to x_i and u_{ij} the second derivative of u with respect to x_i and x_j.

6.1 Definition and properties of viscosity solutions

The definition of viscosity solution to (6.1) can be easily deduced. However, for the reader convenience, we introduce in this framework the explicit statements. See also Appendix B for a basic introduction to viscosity solutions.

Definition 6.1. Let u be a nonnegative continuous function in Ω. We say that u is a *viscosity solution* to (6.1) in Ω if the following conditions are satisfied:

(i) $\sum\limits_{i,j} a_{ij}(x)u_{ij} + \sum\limits_{i} b_i(x)u_i = f$ in $\Omega^+(u)$ in the viscosity sense, i.e. if $\varphi \in C^2(\Omega^+(u))$ touches u from below (resp. above) at $x_0 \in \Omega^+(u)$ then

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i b_i(x_0)\varphi_i(x_0) \le f(x_0) \quad \Bigg(\text{resp. } \sum_{i,j} a_{ij}\varphi_{ij}(x_0) + \sum_i b_i(x_0)\varphi_i(x_0) \ge f(x_0)\Bigg).$$

(ii) If $\varphi \in C^2(\Omega)$ and φ^+ touches u from below (resp. above) at $x_0 \in F(u)$ and $|\nabla\varphi|(x_0) \neq 0$ then

$$|\nabla\varphi|(x_0) \le g(x_0) \quad (\text{resp. } |\nabla\varphi|(x_0) \ge g(x_0)).$$

We present, at this point, the notion of comparison subsolution / supersolution, which will be used in the same way as we have used it in case of problem (1.2).

Definition 6.2. Let $v \in C^2(\Omega)$. We say that v is a strict (comparison) subsolution (resp. supersolution) to (6.1) in Ω if the following conditions are satisfied:

(i) $\sum\limits_{i,j} a_{ij}(x)v_{ij} + \sum\limits_{i} b_i(x)v_i > f(x)$ (resp. $\sum\limits_{i,j} a_{ij}(x)v_{ij} + \sum\limits_{i} b_i(x)v_i < f(x)$) in $\Omega^+(v)$.

(ii) If $x_0 \in F(v)$, then

$$|\nabla v|(x_0) > g(x_0) \quad (\text{resp. } 0 < |\nabla v|(x_0) < g(x_0)).$$

Remark. Repeating the same argument used in the Remark 1.4, if v is a strict subsolution / supersolution to (6.1) then $F(v)$ is a C^2 hypersurface.

It is possible to give the same lemma valid in case of system (1.2).

Lemma 6.3. *Let u, v be respectively a solution and a strict subsolution to (6.1) in Ω. If $u \geq v^+$ in Ω then $u > v^+$ in $\Omega^+(v) \cup F(v)$.*

Proof. Suppose for contradiction that there exists $x_0 \in \Omega^+(v) \cup F(v)$ such that $u(x_0) = v^+(x_0)$.
In particular, we distinguish two different cases.

(i) If $x_0 \in \Omega^+(v)$, we have, repeating the same reasoning done in the proof of Lemma 1.5 in the case (i), that v touches u from below at $x_0 \in \Omega^+(u)$, with $\varphi \in C^2(\Omega^+(u))$, consequently, inasmuch u is a solution to (6.1) in Ω,
$$\sum_{i,j} a_{ij}(x_0)v_{ij}(x_0) + \sum_i b_i(x_0)v(x_0) \leq f(x_0). \tag{6.2}$$
On the other hand, since v is a strict subsolution to (6.1) in Ω, we achieve
$$\sum_{i,j} a_{ij}(x)v_{ij} + \sum_i b_i(x)v_i > f(x) \text{ in } \Omega^+(v),$$
hence, given that $x_0 \in \Omega^+(v)$,
$$\sum_{i,j} a_{ij}(x_0)v_{ij}(x_0) + \sum_i b_i(x_0)v_i(x_0) > f(x_0),$$
which implies from (6.2)
$$f(x_0) < \sum_{i,j} a_{ij}(x_0)v_{ij}(x_0) + \sum_i b_i(x_0)v_i(x_0) \leq f(x_0),$$
i.e. $f(x_0) < f(x_0)$, which is a contradiction.

(ii) If $x_0 \in F(v)$ we can repeate the whole reasoning done in the proof of Lemma 1.5 in the case (ii) and we reach a contradiction.

Therefore, $\nexists\ x_0 \in \Omega^+(v) \cup F(v)$ such that $u(x_0) = v^+(x_0)$, in other words, seeing as how $u \geq v^+$ in $\Omega \supset (\Omega^+(v) \cup F(v))$, i.e. $u \geq v^+$ in $\Omega^+(v) \cup F(v)$, $u > v^+$ in $\Omega^+(v) \cup F(v)$. $\square$

6.2 Harnack inequality

Arguing in parallel with the case of problem (1.2), we show that, provided giving a further condition on the coefficient b, a solution to (6.1) satisfies the same Harnack type inequality expressed by Theorem 2.1. In particular, for exposure convenience, we recall here the same assumption done in (2.2), in other words

$$\|f\|_{L^\infty(\Omega)} \leq \varepsilon^2, \quad \|g-1\|_{L^\infty(\Omega)} \leq \varepsilon^2, \quad \|a_{ij}-\delta_{ij}\|_{L^\infty(\Omega)} \leq \varepsilon, \tag{6.3}$$

with $0 < \varepsilon < 1$.

Theorem 6.4 (**Harnack inequality**). *There exists a universal constant $\bar{\varepsilon}$ such that if u solves (6.1)-(6.3) under the assumption*

$$\|b\|_{L^\infty(\Omega)} \leq \varepsilon^2. \tag{6.4}$$

Suppose also that for some point $x_0 \in \Omega^+(u) \cup F(u)$

$$(x_n + a_0)^+ \leq u(x) \leq (x_n + b_0)^+ \quad \text{in } B_r(x_0) \subset \Omega \tag{6.5}$$

with

$$b_0 - a_0 \leq \varepsilon r, \quad \varepsilon \leq \bar{\varepsilon},$$

then

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{r/20}(x_0)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1-c)\varepsilon r,$$

and $0 < c < 1$ universal.

For completeness, because it will be used in the proof of "improvement of flatness " lemma, we state now the same corollary, introduced in Chapter 2 after Theorem 2.1.

Corollary 6.5. *Let u be a solution to (6.1)-(6.3)-(6.4) satisfying (6.5) for $r = 1$. Then in $B_1(x_0)$,*

$$\tilde{u}_\varepsilon(x) = \frac{u(x) - x_n}{\varepsilon}$$

has a Hölder modulus of continuity at x_0, outside the ball of radius $\varepsilon/\bar{\varepsilon}$, i.e. for all $x \in (\Omega^+(u) \cup F(u)) \cap B_1(x_0)$ with $|x - x_0| \leq \varepsilon/\bar{\varepsilon}$,

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C\,|x - x_0|^\gamma\,.$$

Proof. The proof is the same provided in Chapter 2 for Corollary 2.2. □

As in Chapter 2, Harnack inequality is a consequence of the following lemma.

Lemma 6.6. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u is a solution to (6.1)-(6.3)-(6.4) in B_1 with $0 < \varepsilon \leq \bar{\varepsilon}$ and u satisfies*

$$p(x)^+ \leq u(x) \leq (p(x) + \varepsilon)^+, \quad x \in B_1,\ \ p(x) = x_n + \sigma,\ \ |\sigma| < 1/10, \tag{6.6}$$

then if at $\bar{x} = \frac{1}{5}e_n$,

$$u(\bar{x}) \geq (p(\bar{x}) + \varepsilon/2)^+, \tag{6.7}$$

then

$$u \geq (p + c\varepsilon)^+ \quad \text{in } \overline{B}_{1/2} \tag{6.8}$$

for some $0 < c < 1$. Analogously, if

$$u(\bar{x}) \leq (p(\bar{x}) + \varepsilon)^+,$$

then

$$u \leq (p + (1-c)\varepsilon)^+ \quad \text{in } \overline{B}_{1/2}.$$

Proof. We argue as in the proof of Lemma 2.3, explaining only the main differences and referring to the proof of Lemma 2.3 for all details.
As in the proof of Lemma 2.3, we prove the first statement.
First of all, from (6.6), we obtain

$$u \geq p \quad \text{in } B_1. \tag{6.9}$$

Let

$$w(x) = c(|x - \bar{x}|^{-\gamma} - (3/4)^{-\gamma}),$$

be defined on the closure of the annulus

$$A := B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x}).$$

The constant c is chosen so that w satisfies the boundary conditions

$$\begin{cases} w = 0 & \text{on } \partial B_{3/4}(\bar{x}), \\ w = 1 & \text{on } \partial B_{1/20}(\bar{x}). \end{cases}$$

Repeating the calculation done in the proof of Lemma 2.3, we achieve

$$c = \frac{1}{(1/2)^{-\gamma} - (3/4)^{-\gamma}}.$$

Now, the condition $\|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \leq \varepsilon$ implies that the matrix (a_{ij}) is uniformly elliptic, as long as ε is small enough, see Lemma A.5 in Appendix A.

Consequently, we can choose the constant γ universal so that

$$\sum_{i,j} a_{ij}(x)w_{ij} + \sum_i b_i(x)w_i \geq \delta > 0 \quad \text{in } A,$$

with δ universal. Precisely, from (2.21) and (2.22) in the proof of Lemma 2.3, we have, keeping c in the expression of w,

$$\frac{\partial w}{\partial x_i} = -\gamma c \, |x - \bar{x}|^{-\gamma-2} (x_i - \bar{x}_i) \tag{6.10}$$

and

$$\frac{\partial^2 w}{\partial x_j \partial x_i} = c\gamma(\gamma+2) \, |x - \bar{x}|^{-\gamma-4} (x_i - \bar{x}_i)(x_j - \bar{x}_j) - c\gamma \, |x - \bar{x}|^{-\gamma-2} \delta_{ij}. \tag{6.11}$$

Therefore, since (a_{ij}) is uniformly elliptic, from (6.10) and (6.11), repeating the same arguments described in (2.23), we obtain

$$\begin{aligned} \sum_{i,j} a_{ij}(x)w_{ij} + \sum_i b_i(x)w_i \geq{}& c\gamma \left(\lambda(\gamma + 2) - n\Lambda\right) |x - \bar{x}|^{-\gamma-2} \\ & + \sum_i b_i(x)(-c\gamma \, |x - \bar{x}|^{-\gamma-2} (x_i - \bar{x}_i)) \\ & = c\gamma \left(\lambda(\gamma + 2) - n\Lambda\right) |x - \bar{x}|^{-\gamma-2} \\ & - c\gamma \, |x - \bar{x}|^{-\gamma-2} \, b(x) \cdot (x - \bar{x}) \\ & = c\gamma \left(\lambda(\gamma + 2) - n\Lambda - b(x) \cdot (x - \bar{x})\right) |x - \bar{x}|^{-\gamma-2} \end{aligned}$$

which implies,

$$\sum_{i,j} a_{ij}(x)w_{ij} + \sum_i b_i(x)w_i \geq c\gamma\left(\lambda(\gamma+2) - n\Lambda - |b(x)|\,|x-\bar{x}|\right)|x-\bar{x}|^{-\gamma-2} \tag{6.12}$$

given that for the Cauchy-Schwarz inequality $|b(x)\cdot(x-\bar{x})| \leq |b(x)|\,|x-\bar{x}|$, thus $b(x)\cdot(x-\bar{x}) \leq |b(x)|\,|x-\bar{x}|$ and $-b(x)\cdot(x-\bar{x}) \geq -|b(x)|\,|x-\bar{x}|$.
At this point, we know from (6.4) that

$$\|b\|_{L^\infty(B_1)} = \max_{i=1,\ldots,n} \|b_i\|_{L^\infty(B_1)} \leq \varepsilon^2, \tag{6.13}$$

which entails $\|b_i\|_{L^\infty(B_1)} \leq \varepsilon^2$, $\forall i = 1,\ldots,n$, and thus $|b_i|(x) \leq \varepsilon^2$, $\forall i = 1,\ldots,n$ and for all $x \in B_1$.
As a consequence, inasmuch $|b_i(x)|$ and ε^2 are positive or equal to 0, $|b_i(x)|^2 \leq \varepsilon^4$, i.e. $b_i(x)^2 \leq \varepsilon^4$ and hence

$$|b(x)| = \sqrt{b_1(x)^2 + \ldots + b_n(x)^2} \leq \sqrt{\varepsilon^4 + \ldots + \varepsilon^4} = \sqrt{n\varepsilon^4} = \sqrt{n}\varepsilon^2,$$

namely

$$|b(x)| \leq \sqrt{n}\varepsilon^2. \tag{6.14}$$

Now, from (6.12) and (6.14), which also gives $-|b(x)| \geq -\sqrt{n}\varepsilon^2$, we achieve

$$\sum_{i,j} a_{ij}(x)w_{ij} + \sum_i b_i(x)w_i \geq c\gamma\left(\lambda(\gamma+2) - n\Lambda - \sqrt{n}\varepsilon^2\,|x-\bar{x}|\right)|x-\bar{x}|^{-\gamma-2},$$

which implies,

$$\sum_{i,j} a_{ij}(x)w_{ij} + \sum_i b_i(x)w_i \geq c\gamma\left(\lambda(\gamma+2) - n\Lambda - \sqrt{n}\varepsilon^2\frac{3}{4}\right)\left(\frac{3}{4}\right)^{-\gamma-2} \text{ in } A, \tag{6.15}$$

since in A $|x-\bar{x}| \geq 3/4$, which gives $-|x-\bar{x}| \geq -3/4$ and $|x-\bar{x}|^{-\gamma-2} \geq (3/4)^{-\gamma-2}$, recalling that $\gamma > 0$.
In particular, if we take

$$\lambda(\gamma+2) - n\Lambda - \sqrt{n}\varepsilon^2\frac{3}{4} > 0,$$

in other words

$$\gamma + 2 > n\frac{\Lambda}{\lambda} + \sqrt{n}\frac{3\varepsilon^2}{4\lambda},$$

and

$$\gamma > n\frac{\Lambda}{\lambda} + \sqrt{n}\frac{3\varepsilon^2}{4\lambda} - 2,$$

we get from (6.15)

$$\sum_{i,j} a_{ij}(x)w_{ij} + \sum_i b_i(x)w_i \geq C\gamma\left(\lambda(\gamma+2) - n\Lambda - \sqrt{n}\varepsilon^2\frac{3}{4}\right)\left(\frac{3}{4}\right)^{-\gamma-2}$$
$$= \delta > 0 \quad \text{in } A,$$

namely

$$\sum_{i,j} a_{ij}(x)w_{ij} + \sum_i b_i(x)w_i \geq \delta \quad \text{in } A, \tag{6.16}$$

with δ universal, as desired.

Extend now w to be equal to 1 on $B_{1/20}(\bar{x})$.

Repeating the considerations done in the proof of Lemma 2.3, we obtain from (6.9), inasmuch $|\sigma| < 1/10$,

$$B_{1/10}(\bar{x}) \subset B_1^+(u). \tag{6.17}$$

Moreover, in the same way of the proof of Lemma 2.3, we achieve

$$B_{1/2} \subset\subset B_{3/4}(\bar{x}) \subset\subset B_1,$$

which can be rewrite

$$\overline{B}_{1/2} \subset B_{3/4}(\bar{x}) \quad \text{and} \quad \overline{B}_{3/4}(\bar{x}) \subset B_1. \tag{6.18}$$

Notice at this point that $u-p$ solves, in the viscosity sense, a uniformly elliptic equation in $B_{1/10}(\bar{x})$ as in the proof of Lemma 2.3, but with a different right hand side.

Indeed, if we take $\varphi \in C^2(B_{1/10}(\bar{x}))$ touching $u-p$ from below at $x_0 \in B_{1/10}(\bar{x})$, we have

$$\varphi(x_0) = (u-p)(x_0) = u(x_0) - p(x_0) \tag{6.19}$$

and

$$\varphi(x) \leq (u-p)(x) = u(x) - p(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{6.20}$$

In particular, (6.19) and (6.20) read

$$\varphi(x_0) + p(x_0) = u(x_0) \tag{6.21}$$

and

$$\varphi(x) + p(x) \leq u(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{6.22}$$

In addition, since $B_{1/10}(\bar{x})$ is open and $x_0 \in B_{1/10}(\bar{x})$, we can suppose $O \subset B_{1/10}(\bar{x})$, and we have $(\varphi + p) \in C^2(O)$, recalling that $\varphi \in C^2(B_{1/10}(\bar{x}))$ and $p \in C^\infty(B_1)$, with $B_1 \supset B_{1/10}(\bar{x}) \supset O$ from (6.17), because $B_1^+(u) \subset B_1$. Therefore, from this fact, together with (6.21) and (6.22), we get that $(\varphi+p)$ touches u from below at $x_0 \in B_{1/10}$, seeing as how $(\varphi + p)(x) = \varphi(x) + p(x)$. In particular, from (6.17), we have $x_0 \in B_1^+(u)$.
As a consequence, we have that $(\varphi+p)$ touches u from below at $x_0 \in B_1^+(u)$, hence, since u is a viscosity solution to (6.1) in B_1, we obtain

$$\begin{aligned}
&\sum_{i,j} a_{ij}(x_0)(\varphi + p)_{ij}(x_0) + \sum_i b_i(x_0)(\varphi + p)_i(x_0) \\
&= \sum_{i,j} a_{ij}(x_0)(\varphi + x_n + \sigma)_{ij}(x_0) + \sum_i b_i(x_0)(\varphi + x_n + \sigma)_i(x_0) \\
&= \sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + \sum_{i,j} a_{ij}(x_0)(x_n + \sigma)_{ij}(x_0) \\
&+ \sum_i b_i(x_0)\varphi_i(x_0) + \sum_i b_i(x_0)(x_n + \sigma)_i(x_0) \\
&= \sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i b_i(x_0)\varphi_i(x_0) + b_n(x_0) \leq f(x_0),
\end{aligned}$$

which gives

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i b_i(x_0)\varphi_i(x_0) + b_n(x_0) \leq f(x_0),$$

which also entails

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i b_i(x_0)\varphi_i(x_0) \leq f(x_0) - b_n(x_0). \tag{6.23}$$

Repeating the same argument if $\varphi \in C^2(B_{1/10})$ touches u from above at $x_0 \in B_{1/10}(\bar{x})$, but with opposite inequalities, we achieve from (6.23) that

$u-p$ solves, in the viscosity sense, the uniformly elliptic equation

$$\sum_{i,j} a_{ij}(x)(u-p)_{ij} + \sum_i b_i(x)(u-p)_i = f - b_n \quad \text{in } B_{1/10}(\bar{x}).$$

Furthermore, we have $u-p \geq 0$ in $B_{1/10}(\bar{x})$, given that $u-p \geq 0$ in B_1 from (6.9) and $B_{1/10}(\bar{x}) \subset B_1$ for what we have said before. As a consequence, because $u-p \geq 0$ in $B_{1/10}(\bar{x})$ and $u-p$ solves (6.23) in the viscosity sense, we can apply the Harnack inequality to obtain

$$\sup_{\overline{B}_{1/20}(\bar{x})} (u-p) \leq C_1 \left(\inf_{\overline{B}_{1/20}(\bar{x})} (u-p) + C_2 \left\| f - b_n \right\|_{L^\infty} \right),$$

which implies, in view of the same steps done in the proof of Lemma 2.3,

$$u(x) - p(x) \geq c(u(\bar{x}) - p(\bar{x})) - C \left\| f - b_n \right\|_{L^\infty} \quad \text{in } \overline{B}_{1/20}(\bar{x}). \tag{6.24}$$

Now, we have for definition of $\|b\|_{L^\infty}$, see (6.13),

$$|f(x) - b_n(x)| \leq |f(x)| + |b_n(x)| \leq \|f\|_{L^\infty} + \|b_n\|_{L^\infty} \leq \|f\|_{L^\infty} + \|b\|_{L^\infty},$$

in other words

$$|f(x) - b_n(x)| \leq \|f\|_{L^\infty} + \|b\|_{L^\infty},$$

which gives

$$\sup_x |f(x) - b_n(x)| = \|f - b_n\|_{L^\infty} \leq \|f\|_{L^\infty} + \|b\|_{L^\infty},$$

namely

$$\|f - b_n\|_{L^\infty} \leq \|f\|_{L^\infty} + \|b\|_{L^\infty}. \tag{6.25}$$

In addition, we know, from (6.3)-(6.4), that $|f\|_{L^\infty} \leq \varepsilon^2$ and $\|b\|_{L^\infty} \leq \varepsilon^2$, thus from (6.25) we achieve

$$\|f - b_n\|_{L^\infty} \leq 2\varepsilon^2,$$

and hence

$$-\|f - b_n\|_{L^\infty} \geq -2\varepsilon^2,$$

which entails from (6.24)

$$u(x) - p(x) \geq c(u(\bar{x}) - p(\bar{x})) - 2C\varepsilon^2 \quad \text{in } \overline{B}_{1/20}(\bar{x}). \tag{6.26}$$

In particular, repeating the same computations done in the proof of Lemma 2.3, we get from (6.7)

$$u(\bar{x}) - p(\bar{x}) \geq \frac{\varepsilon}{2},$$

which implies, in view of (6.26),

$$u(x) - p(x) \geq c\frac{\varepsilon}{2} - 2C\varepsilon^2 = \varepsilon\left(\frac{c}{2} - 2C\varepsilon\right) = c_0\varepsilon \quad \text{in } \overline{B}_{1/20}(\bar{x}),$$

that is

$$u - p \geq c_0\varepsilon \quad \text{in } \overline{B}_{1/20}(\bar{x}), \tag{6.27}$$

provided that taking ε small enough so that $\frac{c}{2} - 2C\varepsilon > 0$, in other words $\varepsilon < \frac{c}{4C}$.
At this point, analogously to the proof of Lemma 2.3, we set

$$v(x) = p(x) + c_0\varepsilon(w(x) - 1), \quad x \in \overline{B}_{3/4}(\bar{x}), \tag{6.28}$$

and for $t \geq 0$,

$$v_t(x) = v(x) + t, \quad x \in \overline{B}_{3/4}(\bar{x}). \tag{6.29}$$

Notice that, from (6.28) and (6.29), we obtain

$$
\begin{aligned}
&\sum_{i,j} a_{ij}(x)(v_t)_{ij} + \sum_i b_i(x)(v_t)_i = \sum_{i,j} a_{ij}(x)(v(x)+t)_{ij} + \sum_i b_i(x)(v(x)+t)_i \\
&= \sum_{i,j} a_{ij}(x)(p(x) + c_0\varepsilon(w(x)-1)+t)_{ij} \\
&+ \sum_i b_i(x)(p(x) + c_0\varepsilon(w(x)-1)+t)_i \\
&= \sum_{i,j} a_{ij}(x)(x_n + \sigma + c_0\varepsilon(w(x)-1)+t)_{ij} \\
&+ \sum_i b_i(x)(x_n + \sigma + c_0\varepsilon(w(x)-1)+t)_i \\
&= \sum_{i,j} a_{ij}(x) c_0\varepsilon w_{ij} + \sum_{\substack{i \\ i\neq n}} b_i(x) c_0\varepsilon w_i + b_n(x)(1 + c_0\varepsilon w_n) \\
&= c_0\varepsilon \sum_{i,j} a_{ij}(x) w_{ij} + c_0\varepsilon \sum_{\substack{i \\ i\neq n}} b_i(x) w_i + b_n(x) + c_0\varepsilon b_n(x) w_n \\
&= c_0\varepsilon \sum_{i,j} a_{ij}(x) w_{ij} + c_0\varepsilon \sum_i b_i(x) w_i + b_n(x) \\
&= c_0\varepsilon \left(\sum_{i,j} a_{ij}(x) w_{ij} + \sum_i b_i(x) w_i \right) + b_n(x),
\end{aligned}
$$

therefore, in view of (6.16), inasmuch as $c_0\varepsilon > 0$,

$$
\sum_{i,j} a_{ij}(x)(v_t)_{ij} + \sum_i b_i(x)(v_t)_i = c_0\varepsilon\delta + b_n(x) \quad \text{in } A. \tag{6.30}
$$

Moreover, for what we have shown above, we have $|b_n|(x) \leq \varepsilon^2$, $\forall x \in B_1$, which gives $b_n(x) \geq -\varepsilon^2$, $\forall x \in B_1$ and thus also $\forall x \in A$, recalling that $A \subset B_1$. Consequently, from (6.30), we get

$$
\sum_{i,j} a_{ij}(x)(v_t)_{ij} + \sum_i b_i(x)(v_t)_i \geq c_0\varepsilon\delta - \varepsilon^2 > \varepsilon^2 \quad \text{in } A,
$$

namely

$$
\sum_{i,j} a_{ij}(x)(v_t)_{ij} + \sum_i b_i(x)(v_t)_i > \varepsilon^2 \quad \text{in } A, \tag{6.31}
$$

if we take ε such that

$$c_0\varepsilon\delta - \varepsilon^2 > \varepsilon^2 \leftrightarrow c_0\delta\varepsilon - 2\varepsilon^2 > 0 \leftrightarrow \varepsilon(c_0\delta - 2\varepsilon) > 0 \leftrightarrow 0 < \varepsilon < \frac{c_0\delta}{2},$$

in other words if ε satisfies $0 < \varepsilon < \frac{c_0\delta}{2}$.
Now, from the definition of $v_{\bar{t}}$ in (6.29) we have

$$v_0(x) = v(x) = p(x) + c_0\varepsilon(w(x) - 1) \leq p(x) \leq u(x), \quad x \in \overline{B}_{3/4}(\bar{x}),$$

i.e

$$v_0(x) \leq u(x), \quad x \in \overline{B}_{3/4}(\bar{x}),$$

recalling that $\overline{B}_{3/4}(\bar{x}) \subset B_1$ from (6.18) and hence from (6.9), $p(x) \leq u(x)$, with $x \in \overline{B}_{3/4}(\bar{x})$, and $w \leq 1$ in $\overline{B}_{3/4}(\bar{x})$, from the proof of Lemma 2.3.
Let then $\bar{t}$ be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}).$$

Remark that $\bar{t}$ exists, given that for $t = 0$ we have $v_0(x) \leq u(x)$.
We want to show that $\bar{t} \geq c_0\varepsilon$. Indeed, if this condition is satisfied, exactly how in the proof of Lemma 2.3, we obtain

$$u(x) \geq (p(x) + c\varepsilon)^+ \quad \text{on } \overline{B}_{1/2},$$

with $0 < c < 1$ universal, as desired.
The continuance of the proof is the same of the proof of Lemma 2.3, observing that (6.31) is satisfied for every $t \geq 0$ and hence also for $\bar{t}$. □

As in case of problem (1.2), we can provide, at this point, the proof of Harnack inequality.

Proof of Theorem 6.4. As in the proof of Theorem 2.1, we assume without loss of generality

$$x_0 = 0, \quad r = 1.$$

According to (6.5) and repeating the same argument used in the proof of Theorem 2.1, we achieve

$$p(x)^+ \leq u(x) \leq (p(x) + \varepsilon)^+ \quad \text{in } B_1 \tag{6.32}$$

with $p(x) = x_n + a_0$.
The proofs of the cases in which $|a_0| < 1/10$ and $a_0 \leq -1/10$ are analogous to those given in the proof of Theorem 2.1.
Consequently, remain to show the result if $a_0 \geq 1/10$.
Repeating the same argument used in the proof of Theorem 2.1, we achieve that $B_{1/10} \subset B_1^+(u)$ and as in that proof, we distinguish two cases, if $u(0) - p(0) \geq \varepsilon/2$ or $u(0) - p(0) < \varepsilon/2$.

(i) First, we suppose $u(0) - p(0) \geq \varepsilon/2$.
At this point, from (6.32) we get, recalling that $p \leq p^+$, $u \geq p$ in $B_1 \supset B_1^+(u) \supset B_{1/10}$, which entails $u \geq p$ in $B_{1/10}$ and $u - p \geq 0$ in $B_{1/10}$.
In addition, u solves, in the viscosity sense, a uniformly elliptic equation in $\Omega^+(u) \supset B_1^+(u)$, seeing as how $\Omega \supset B_1$ from the hypothesis of Theorem 6.4, and hence we can repeat the same argument used in the proof of Lemma 6.6 to obtain that $u - p$ solves, in the viscosity sense, the uniformly elliptic equation

$$\sum_{i,j} a_{ij}(x)(u-p)_{ij} + \sum_i b_i(x)(u-p)_i = f - b_n \quad \text{in } B_{1/10}.$$

In view of this fact, together with $u - p \geq 0$ in $B_{1/10}$, we can apply the Harnack inequality to achieve

$$\sup_{\overline{B}_{1/20}} (u-p) \leq C_1 \left(\inf_{\overline{B}_{1/20}} (u-p) + C_2 \left\| f - b_n \right\|_{L^\infty} \right),$$

which implies, repeating the same calculations done in the proof of Lemma 6.6,

$$u(x) - p(x) \geq c_0 \varepsilon \quad \text{in } \overline{B}_{1/20}, \tag{6.33}$$

with $c_0 = \frac{c}{2} - 2C\varepsilon$ and ε such that $0 < c_0 < 1$, in other words

$$0 < \frac{c}{2} - 2C\varepsilon < 1 \leftrightarrow \frac{c}{2} - 1 < 2C\varepsilon < \frac{c}{2} \leftrightarrow \frac{c}{4C} - \frac{1}{2C} < \varepsilon < \frac{c}{4C},$$

i.e.

$$\frac{c}{4C} - \frac{1}{2C} < \varepsilon < \frac{1}{2C},$$

which also gives, because $\varepsilon > 0$

$$\max\left(0, \frac{c}{4C} - \frac{1}{2C}\right) = \left(\frac{c}{4C} - \frac{1}{2C}\right)^+ < \varepsilon < \frac{c}{4C},$$

namely

$$\left(\frac{c}{4C} - \frac{1}{2C}\right)^+ < \varepsilon < \frac{c}{4C}.$$

In particular we get from (6.33), calling $c = c_0$ and given that $B_{1/20} \subset \overline{B}_{1/20}$,

$$u(x) - p(x) \geq c\varepsilon \quad \text{in } B_{1/20},$$

which also entails

$$u(x) \geq p(x) + c\varepsilon \quad \text{in } B_{1/20}, \tag{6.34}$$

with $0 < c < 1$ universal.

Now, we know that $u \geq 0$ in $\Omega \supset B_1 \supset B_{1/20}$, that is $u \geq 0$ in $B_{1/20}$, since u is a viscosity solution to (6.1) in Ω. As a consequence, from (6.34) we obtain

$$u(x) \geq \max(p(x) + c\varepsilon, 0) = (p(x) + c\varepsilon)^+ \quad \text{in } B_{1/20},$$

in other words

$$u(x) \geq (p(x) + c\varepsilon)^+ \quad \text{in } B_{1/20},$$

with $0 < c < 1$ universal.

The precise conclusion of Theorem 6.4 follows from case (i) in the proof of Theorem 2.1 when $a_0 \geq 1/10$.

(ii) Suppose now that $u(0) - p(0) < \varepsilon/2$. Repeating the same argument used in case (ii) in the proof of Theorem 2.1 when $a_0 \geq 1/10$, we achieve

$$p(x) + \varepsilon - u(x) \geq 0 \quad \text{in } B_{1/10}. \tag{6.35}$$

At this point, we state that $p + \varepsilon - u$ solves, in the viscosity sense, a uniformly elliptic equation in $B_{1/10}$.

Precisely, if $\varphi \in C^2(B_{1/10})$ touches $p+\varepsilon-u$ from below at $x_0 \in B_{1/10}$, we have

$$\varphi(x_0) = (p+\varepsilon-u)(x_0) = p(x_0)+\varepsilon-u(x_0) \tag{6.36}$$

and

$$\varphi(x) \leq (p+\varepsilon-u)(x) = p(x)+\varepsilon-u(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{6.37}$$

In particular, (6.36) and (6.37) read

$$u(x_0) = p(x_0)+\varepsilon-\varphi(x_0) \tag{6.38}$$

and

$$u(x) \leq p(x)+\varepsilon-\varphi(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{6.39}$$

Therefore, from (6.38) and (6.39), we get that $p+\varepsilon-\varphi$ touches u from above at $x_0 \in B_{1/10}$, inasmuch $(p+\varepsilon-\varphi)(x) = p(x)+\varepsilon-\varphi(x)$.
Furthermore, since $B_{1/10}$ is open and $x_0 \in B_{1/10}$, we can take $O \subset B_{1/10}$ and we have $(p+\varepsilon-\varphi) \in C^2(O)$ inasmuch as $p(x) = x_n + a_0 \in C^\infty(B_1)$ and $B_1 \supset B_{1/10} \supset O$.
To sum it up, we have $(p+\varepsilon-\varphi) \in C^2(O)$ touching u from above at $x_0 \in B_{1/10}$, with in particular $x_0 \in \Omega^+(u)$, given that $B_{1/10} \subset B_1^+(u) \subset \Omega^+(u)$, inasmuch as $B_1 \subset \Omega$ from the hypothesis of Theorem 6.4. Consequently, seeing as how u is a solution to (6.1) in Ω, we obtain

$$\begin{aligned}
&\sum_{i,j} a_{ij}(x_0)(p+\varepsilon-\varphi)_{ij}(x_0) + \sum_i b_i(x_0)(p+\varepsilon-\varphi)_i(x_0) \\
&= \sum_{i,j} a_{ij}(x_0)(x_n+a_0+\varepsilon-\varphi) + \sum_i b_i(x_0)(x_n+a_0+\varepsilon-\varphi)_i(x_0) \\
&= \sum_{i,j} a_{ij}(x_0)(-\varphi)_{ij}(x_0) + \sum_{\substack{i \\ i\neq n}} b_i(x_0)(-\varphi)_i(x_0) + b_n(x_0)(x_n-\varphi)_n(x_0) \\
&= \sum_{i,j} a_{ij}(x_0)(-\varphi_{ij}(x_0)) + \sum_{\substack{i \\ i\neq n}} b_i(x_0)(-\varphi_i(x_0)) + b_n(x_0) - b_n(x_0)\varphi_n(x_0) \\
&= -\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) - \sum_i b_i(x_0)\varphi_i(x_0) + b_n(x_0) \geq f(x_0),
\end{aligned}$$

namely

$$-\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) - \sum_i b_i(x_0)\varphi_i(x_0) + b_n(x_0) \geq f(x_0),$$

which implies

$$-\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) - \sum_i b_i(x_0)\varphi_i(x_0) \geq f(x_0) - b_n(x_0)$$

and

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i b_i(x_0)\varphi_i(x_0) \leq b_n(x_0) - f(x_0). \tag{6.40}$$

Repeating the same argument if $\varphi \in C^2(B_{1/10})$ touches $p + \varepsilon - u$ from above at $x_0 \in B_{1/10}$, but with opposite inequalities, we achieve that $p + \varepsilon - u$ solves, in the viscosity sense, the uniformly elliptic equation

$$\sum_{i,j} a_{ij}(p + \varepsilon - u)_{ij} + \sum_i b_i(x)(p + \varepsilon - u)_i = b_n - f \quad \text{in } B_{1/10}.$$

In view of this fact, together with (6.35), we can apply the Harnack inequality to get

$$\sup_{\overline{B}_{1/20}} (p + \varepsilon - u) \leq C\left(\inf_{\overline{B}_{1/20}} (p + \varepsilon - u) + C_2 \|b_n - f\|_{L^\infty} \right),$$

which entails, repeating the same calculations used for instance in the proof of Lemma 2.3 to obtain (2.37),

$$p(x) + \varepsilon - u(x) \geq c(p(0) + \varepsilon - u(0)) - C\|b_n - f\|_{L^\infty} \quad \text{in } \overline{B}_{1/20}. \tag{6.41}$$

Now, $\|b_n - f\|_{L^\infty} = \|f - b_n\|_{L^\infty}$, hence, repeating the same computations used in the proof of Lemma 6.6, we have $\|b_n - f\|_{L^\infty} \leq 2\varepsilon^2$, which also gives $-\|b_n - f\|_{L^\infty} \geq -2\varepsilon^2$. As a consequence, we get from (6.41)

$$p(x) + \varepsilon - u(x) \geq c(p(0) + \varepsilon - u(0)) - 2C\varepsilon^2 \quad \text{in } \overline{B}_{1/20}. \tag{6.42}$$

Moreover, we have supposed $u(0) - p(0) < \varepsilon/2$, which also gives $p(0) - u(0) > -\varepsilon/2$, thus

$$p(0) + \varepsilon - u(0) = p(0) - u(0) + \varepsilon > -\frac{\varepsilon}{2} + \varepsilon = \frac{\varepsilon}{2},$$

i.e.

$$p(0) + \varepsilon - u(0) > \frac{\varepsilon}{2},$$

which implies from (6.42)

$$p(x) + \varepsilon - u(x) \geq c\frac{\varepsilon}{2} - 2C\varepsilon^2 \quad \text{in } \overline{B}_{1/20}. \tag{6.43}$$

At this point, repeating the same argument used in (i), we achieve from (6.43)

$$p(x) + \varepsilon - u(x) \geq c_0\varepsilon \quad \text{in } \overline{B}_{1/20},$$

with $0 < c_0 < 1$ universal, namely calling $c = c_0$

$$p(x) + \varepsilon - u(x) \geq c\varepsilon \quad \text{in } \overline{B}_{1/20},$$

which also gives

$$p(x) + \varepsilon - c\varepsilon = p(x) + (1-c)\varepsilon \geq u(x) \quad \text{in } \overline{B}_{1/20},$$

in other words, since $B_{1/20} \subset \overline{B}_{1/20}$,

$$p(x) + (1-c)\varepsilon \geq u(x) \quad \text{in } B_{1/20}. \tag{6.44}$$

In addition, for what we have said above, $u > 0$ in $B_{1/10} \supset B_{1/20}$, that is $u > 0$ in $B_{1/20}$. Consequently, from (6.44) we get that $p+(1-c)\varepsilon > 0$ in $B_{1/20}$, which entails $(p+(1-c)\varepsilon)^+ = p+(1-c)\varepsilon$ in $B_{1/20}$ and therefore from (6.44)

$$(p(x) + (1-c)\varepsilon)^+ \geq u(x) \quad \text{in } B_{1/20}.$$

Now, the precise conclusion of Theorem 6.4 follows from case (ii) in the proof of Theorem 2.1 when $a_0 \geq 1/10$.

$\square$

6.3 Improvement of flatness

We introduce here the "improvement of flatness" property also for the graph of a solution to (6.1)-(6.3)-(6.4).

Lemma 6.7 (Improvement of flatness). *Let u be a solution to (6.1)-(6.3)-(6.4) in B_1 satisfying*

$$(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \quad \textit{for } x \in B_1, \tag{6.45}$$

and with $0 \in F(u)$. If $0 < r \leq r_0$ for r_0 a universal constant and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r, then

$$(x \cdot \nu - r\varepsilon/2)^+ \leq u(x) \leq (x \cdot \nu + r\varepsilon/2)^+ \quad \textit{for } x \in B_r, \tag{6.46}$$

with $|\nu| = 1$ and $|\nu - e_n| \leq C\varepsilon$ for a universal constant C.

Proof. We proceed as in the proof of Lemma 3.1, explaining only the main differences and referring to the proof of Lemma 3.1 for all the details.
As in the proof of Lemma 3.1, we divide the proof into three steps and we introduce the following notation:

$$\Omega_\rho(u) := (B_1^+(u) \cup F(u)) \cap B_\rho.$$

Step 1: Compactness. Fix $r \leq r_0$ with r_0 universal (the precise r_0 is given in Step 3 of the proof of Lemma 3.1). Assume for contradiction that there exist a sequence $\varepsilon_k \to 0$ and a sequence u_k of solutions to (6.1) in B_1 with coefficients a_{ij}^k and b_i^k, right hand side f_k and free boundary condition g_k satisfying (6.3)-(6.4), such that u_k satisfies (6.45), namely

$$(x_n - \varepsilon_k)^+ \leq u_k(x) \leq (x_n + \varepsilon_k)^+ \quad \text{for } x \in B_1,\ \ 0 \in F(u_k), \tag{6.47}$$

but it does not satisfy the conclusion (6.46) of the lemma.
The explanation of how we can take these sequences is the same provided in the proof of Lemma 3.1.
As in the proof of Lemma 3.1, we set

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k}, \quad x \in \Omega_1(u_k),$$

where, for what we have noticed in the proof of Lemma 3.1, $\Omega_1(u_k) = B_1^+(u_k) \cup F(u_k)$.

Repeating the same computations used in the proof of Lemma 3.1, we obtain from (6.47) that

$$-1 \le \tilde{u}_k(x) \le 1 \quad \text{for } x \in \Omega_1(u_k).$$

and from Corollary 6.5 we achieve that the function $\tilde{u}_k$ satisfies

$$|\tilde{u}_k(x) - \tilde{u}_k(y)| \le C\,|x-y|^\gamma \tag{6.48}$$

for C universal and

$$|x-y| \ge \varepsilon_k/\bar{\varepsilon}, \quad x, y \in \Omega_{1/2}(u_k).$$

Repeating the same argument used in the proof of Lemma 3.1, we get that $F(u_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance and using this fact and (6.48) together with Ascoli-Arzelà, we obtain that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ over $\Omega_{1/2}(u_k)$ converge (up to subsequence) in the Hausdorff distance to the graph of a of a Hölder continuous function $\tilde{u}$ over $B_{1/2} \cap \{x_n \ge 0\}$.
Step 2: Limiting solution. We prove, at this point, that, as in case of the proof of Lemma 3.1, $\tilde{u}$ solves

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}, \end{cases}$$

in the sense of Definition 1.6.
As observed in the Remark following 1.6, we can verify that Definition 1.6 is satisfied only by polynomials touching strictly from below/above.

Let thus $P(x)$ be a quadratic polynomial touching $\tilde{u}$ at $\bar{x} \in B_{1/2} \cap \{x_n \ge 0\}$ strictly from below. Specifically, we need to show that

(i) if $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ then $\Delta P(\bar{x}) \le 0$;

(ii) if $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$ then $P_n(\bar{x}) \le 0$.

Now, given that $\tilde{u}_k \to \tilde{u}$ in the sense specified above, we can find points $x_k \in \Omega_{1/2}(u_k)$, $x_k \to \bar{x}$, and constants $c_k \to 0$ so that

$$P(x_k) + c_k = \tilde{u}_k(x_k) \tag{6.49}$$

and

$$\tilde{u}_k \geq P + c_k \quad \text{in a neighborhood of } x_k. \tag{6.50}$$

In particular, from the definition of $\tilde{u}_k$ and repeating the same calculations done in the proof of Lemma 3.1, (6.49) and (6.50) read

$$u_k(x_k) = Q(x_k) \tag{6.51}$$

and

$$u_k(x) \geq Q(x) \quad \text{in a neighborhood of } x_k \tag{6.52}$$

where

$$Q(x) = \varepsilon_k(P(x) + c_k) + x_n.$$

As in the proof of Lemma 3.1, we now distinguish two cases.

(i) If $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ then, as in the proof of Lemma 3.1, we get that $x_k \in B^+_{1/2}(u_k)$ for k large. In addition, from (6.51) and (6.52) we have that Q touches u_k from below at x_k, where $Q \in C^2(B_{1/2})$, inasmuch $P \in C^\infty(B_{1/2})$ and $x_n \in C^\infty(B_{1/2})$, hence in particular $Q \in C^2(B^+_{1/2}(u_k))$.
To sum it up, for k large, we have $Q \in C^2(B^+_{1/2}(u_k))$ touching u_k from below at $x_k \in B^+_{1/2}(u_k)$.
Therefore, inasmuch as u_k is a solution to (6.1) in B_1, and thus also in $B_{1/2}$, with coefficients a^k_{ij} and b^k_i, right hand side f_k and free boundary condition g_k satisfying (6.3)-(6.4) with ε_k, we obtain

$$\begin{aligned}
&\sum_{i,j} a^k_{ij}(x_k)Q_{ij}(x_k) + \sum_i b^k_i(x_k)Q_i(x_k) \\
&= \sum_{i,j} a^k_{ij}(x_k)(\varepsilon_k(P(x) + c_k) + x_n)_{ij}(x_k) \\
&+ \sum_i b^k_i(x_k)(\varepsilon_k(P(x) + c_k) + x_n)_i \\
&= \sum_{i,j} a^k_{ij}(x_k)\varepsilon_k P_{ij}(x_k) + \sum_{\substack{i \\ i\neq n}} b^k_i(x_k)\varepsilon_k P_i(x_k) + b^k_n(x_k)(\varepsilon_k P_n(x_k) + 1) \\
&= \varepsilon_k \sum_{i,j} a^k_{ij}(x_k)P_{ij}(x_k) + \varepsilon_k \sum_i b^k_i(x_k)P_i(x_k) + b^k_n(x_k) \leq f_k(x_k),
\end{aligned}$$

i.e.

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) + \varepsilon_k \sum_i b_i^k(x_k)P_i(x_k) + b_n^k(x_k) \le f_k(x_k),$$

which implies

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \le f_k(x_k) - \varepsilon_k \sum_i b_i^k(x_k)P_i(x_k) - b_n^k(x_k). \tag{6.53}$$

Now, from the first inequality in (6.3), namely $\|f_k\|_{L^\infty(B_1)} \le \varepsilon_k^2$, we achieve, seeing as how $x_k \in B_{1/2}^+(u_k) \subset B_1$, that is $x_k \in B_1$,

$$f_k(x_k) \le |f_k(x_k)| \le \varepsilon_k^2,$$

in other words $f_k(x_k) \le \varepsilon_k^2$, which gives from (6.53)

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \le \varepsilon_k^2 - \varepsilon_k \sum_i b_i^k(x_k)P_i(x_k) - b_n^k(x_k). \tag{6.54}$$

In addition, we know from (6.4) that $\|b_k\|_{L^\infty(B_1)} \le \varepsilon_k^2$, i.e.

$$\|b_k\|_{L^\infty(B_1)} = \max_{i=1,\dots,n} \left\|b_i^k\right\|_{L^\infty(B_1)} \le \varepsilon_k^2,$$

which entails $\left\|b_n^k\right\|_{L^\infty(B_1)} \le \varepsilon_k^2$, and thus, given that $x_k \in B_1$ for what we have said above,

$$-b_n^k(x_k) \le \left|b_n^k(x_k)\right| \le \left\|b_n^k\right\|_{L^\infty(B_1)} \le \varepsilon_k^2,$$

i.e.

$$-b_n^k(x_k) \le \varepsilon_k^2, \tag{6.55}$$

which gives from (6.54)

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \le 2\varepsilon_k^2 - \varepsilon_k \sum_i b_i^k(x_k)P_i(x_k). \tag{6.56}$$

As regards $-\sum_i b_i^k(x_k)P_i(x_k)$, we can rewrite it as $-b^k(x_k) \cdot \nabla P(x_k)$, and from the Cauchy-Schwarz inequality, we get $-b^k(x_k) \cdot \nabla P(x_k) \le$

$\left|b^k(x_k)\right| |\nabla P(x_k)|$.

In particular, we have

$$|\nabla P| \le C, \quad \text{in } B_{1/2}, \tag{6.57}$$

given that $P(x)$ is a quadratic polynomial and $B_{1/2}$ is a bounded set, as a consequence

$$-b^k(x_k) \cdot \nabla P(x_k) \le C \left|b^k(x_k)\right|. \tag{6.58}$$

Furthermore, for what we have shown before, $\left|\left|b_i^k(x_k)\right| \le \varepsilon_k^2, \forall i = 1, \ldots, n\right.$, in other words $b^k(x_k) \le \varepsilon_k^2$, therefore

$$\left|b^k(x_k)\right| = \sqrt{b_1^k(x_k)^2 + b_2^k(x_k)^2 + \ldots + b_n^k(x_k)^2} \le \sqrt{n\varepsilon_k^4} = \sqrt{n}\varepsilon_k^2,$$

namely

$$\left|b^k(x_k)\right| \le \sqrt{n}\varepsilon_k^2, \tag{6.59}$$

which implies from (6.58) and (6.56), since $\varepsilon_k > 0$,

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k) P_{ij}(x_k) \le 2\varepsilon_k^2 + \varepsilon_k C\sqrt{n}\varepsilon_k^2 = \varepsilon_k^2(2 + C\sqrt{n}\varepsilon_k),$$

i.e.

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k) P_{ij}(x_k) \le \varepsilon_k^2(2 + C\sqrt{n}\varepsilon_k),$$

and dividing by $\varepsilon_k > 0$,

$$\sum_{i,j} a_{ij}^k(x_k) P_{ij}(x_k) \le \varepsilon_k(1 + c\sqrt{n}\varepsilon_k). \tag{6.60}$$

At this point, from the last inequality in (6.3), that is $\left\|a_{ij}^k - \delta_{ij}\right\|_{L^\infty(B_1)} \le \varepsilon_k$, we achieve, because $x_k \in B_1$ for what we have said before,

$$\left|a_{ij}^k(x_k) - \delta_{ij}\right| = \left|\delta_{ij} - a_{ij}^k(x_k)\right| \le \left\|a_{ij}^k - \delta_{ij}\right\|_{L^\infty(B_1)} \le \varepsilon_k,$$

which gives

$$-\varepsilon_k \le \delta_{ij} - a_{ij}^k(x_k) \le \varepsilon_k. \tag{6.61}$$

Therefore, in view of this fact and (6.60), repeating the same calculations done in the proof of Lemma 3.1 to get (3.29), we obtain

$$\begin{aligned}
\Delta P &= \sum_{i,j}(\delta_{ij} - a^k_{ij}(x_k))P_{ij} + \sum_{i,j} a^k_{ij}(x_k)P_{ij} \\
&\leq \sum_{\substack{i,j \\ P_{ij}\geq 0}} \varepsilon_k P_{ij} + \sum_{\substack{i,j \\ P_{ij}<0}} -\varepsilon_k P_{ij} + \varepsilon_k(2 + C\sqrt{n}\varepsilon_k) \\
&= \Bigg(\sum_{\substack{ij \\ P_{ij}\geq 0}} P_{ij} - \sum_{\substack{i,j \\ P_{ij}<0}} P_{ij} + 2 + C\sqrt{n}\varepsilon_k \Bigg)\varepsilon_k \\
&= (C_1 + C\sqrt{n}\varepsilon_k)\varepsilon_k,
\end{aligned}$$

namely

$$\Delta P \leq (C_1 + C\sqrt{n}\varepsilon_k)\varepsilon_k, \tag{6.62}$$

since $P(x)$ is a quadratic polynomial and thus P_{ij} is a constant $\forall i, j$ which also entails $P_{ij} = P_{ij}(x_k)$.
Consequently, passing to the limit in (6.62) as $k \to \infty$, we achieve that $\Delta P \leq 0$, as desired, inasmuch $\varepsilon_k \to 0$ and $(C_1 + C\sqrt{n}\varepsilon k) \to C_1$, which is a constant.

(ii) If instead $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$, we argue exactly in the same way of the proof of Lemma 3.1 and we get $P_n(\bar{x}) \leq 0$ as desired.

As in the proof of Lemma 3.1, we also consider the case of a quadratic polynomial $P(x)$ touching $\tilde{u}$ at $\bar{x} \in B_{1/2} \cap \{x_n \geq 0\}$ strictly from above.
In particular, we need to prove that

(i) if $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ then $\Delta P \geq 0$;

(ii) if $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$ then $P_n(\bar{x}) \geq 0$.

Always since $\tilde{u}_k \to \tilde{u}$ in the sense specified above, there exist points $x_k \in \Omega_{1/2}(u_k)$ and constants $c_k \to 0$ such that

$$P(x_k) + c_k = \tilde{u}_k(x_k) \tag{6.63}$$

and

$$\tilde{u}_k \le P + c_k \quad \text{in a neighborhood of } x_k. \tag{6.64}$$

Repeating the same argument used in the proof of Lemma 3.1, from the definition of $\tilde{u}_k$, (6.63) and (6.64) read

$$u_k(x_k) = Q(x_k) \tag{6.65}$$

and

$$u_k(x) \le Q(x) \quad \text{in a neighborhood of } x_k \tag{6.66}$$

where

$$Q(x) = \varepsilon_k(P(x) + c_k) + x_n.$$

We distinguish two cases again.

(i) If $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ then, repeating the argument used in the proof of Lemma 3.1, we achieve that $x_k \in B^+_{1/2}(u_k)$ for k large. Moreover, from (6.65) and (6.66) we have that Q touches u_k from above at x_k, where $Q \in C^2(B_{1/2})$, inasmuch $P \in C^\infty(B_{1/2})$ and $x_n \in C^\infty(B_{1/2})$ and hence in particular, $Q \in C^2(B^+_{1/2}(u_k))$.
To sum it up, for k large, we have $Q \in C^2(B^+_{1/2}(u_k))$ touching u_k from above at $x_k \in B^+_{1/2}(u_k)$.
Therefore, inasmuch u_k is a solution to (6.1) in B_1, and thus also in $B_{1/2}$, with coefficients a^k_{ij} and b^k_i, right hand side f_k and free boundary condition g_k satisfying (6.3)-(6.4) with ε_k, we get

$$\begin{aligned}
&\sum_{i,j} a^k_{ij}(x_k)Q_{ij}(x_k) + \sum_i b^k_i(x_k)Q_i(x_k) \\
&= \sum_{i,j} a^k_{ij}(x_k)(\varepsilon_k(P(x) + c_k) + x_n)_{ij}(x_k) \\
&+ \sum_i b^k_i(x_k)(\varepsilon_k(P(x) + c_k) + x_n)_i(x_k) \\
&= \sum_{i,j} a^k_{ij}(x_k)\varepsilon_k P_{ij}(x_k) + \sum_{\substack{i \\ i\neq n}} b^k_i(x_k)\varepsilon_k P_i(x_k) + b_n(x_k)(\varepsilon_k P_n(x_k) + 1)
\end{aligned}$$

$$\varepsilon_k \sum_{i,j} a^k_{ij}(x_k)P_{ij}(x_k) + \varepsilon_k \sum_i b^k_i(x_k)P_i(x_k) + b^k_n(x_k) \ge f_k(x_k),$$

in other words

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) + \varepsilon_k \sum_i b_i^k(x_k)P_i(x_k) + b_n^k(x_k) \geq f_k(x_k),$$

which implies

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \geq f_k(x_k) - \varepsilon_k \sum_i b_i^k(x_k)P_i(x_k) - b_n^k(x_k). \quad (6.67)$$

Now, from the first inequality of (6.3), i.e. $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2$, we obtain $|f_k(x)| \leq \varepsilon_k^2$, with $x \in B_1$, hence, since $x_k \in B^+_{1/2}(u_k) \subset B_1$, namely $x_k \in B_1$, we have $|f_k(x_k)| \leq \varepsilon_k^2$, which also gives $f_k(x_k) \geq \varepsilon_k^2$. As a consequence, from (6.67) we get

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \geq -\varepsilon_k^2 - \varepsilon_k \sum_i b_i^k(x_k)P_i(x_k) - b_n^k(x_k). \quad (6.68)$$

In addition, repeating the same argument by which we have obtained (6.55) with $b_n^k(x_k)$ in place of $-b_n^k(x_k)$, we also have $b_n^k(x_k) \leq \varepsilon_k^2$, and thus $-b_n^k(x_k) \geq -\varepsilon_k^2$, which entails from (6.68)

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k) \geq -2\varepsilon_k^2 - \varepsilon_k \sum_i b_i^k(x_k)P_i(x_k). \quad (6.69)$$

Concerning $-\sum_i b_i^k(x_k)P_i(x_k)$, as in case of $\bar{x} \in B_{1/2} \cap \{x_n > 0\}$ for P touching $\tilde{u}$ from below at $\bar{x}$, we can rewrite it as $-b^k(x_k) \cdot \nabla P(x_k)$ and this time, for the Cauchy-Schwarz inequality, we get $-b^k(x_k) \cdot \nabla P(x_k) \geq -\left|b^k(x_k)\right| |\nabla P(x_k)|$, which gives from (6.57) and (6.59)

$$-b^k(x_k) \cdot \nabla P(x_k) \geq -C\sqrt{n}\varepsilon_k^2.$$

Consequently, in view of this fact, from (6.69) we have, because $\varepsilon_k > 0$,

$$\varepsilon_k \sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \geq -2\varepsilon_k^2 - \varepsilon_k C\sqrt{n}\varepsilon_k^2,$$

and dividing by $\varepsilon_k > 0$,

$$\sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \geq -2\varepsilon_k - \varepsilon_k C\sqrt{n}\varepsilon_k = (-2 - C\sqrt{n}\varepsilon_k)\varepsilon_k,$$

that is

$$\sum_{i,j} a_{ij}^k(x_k)P_{ij}(x_k) \geq (-2 - C\sqrt{n}\varepsilon_k)\varepsilon_k. \tag{6.70}$$

Therefore, from (6.61) and (6.70), we obtain, repeating the same computations done in the proof of Lemma 3.1 to get (3.45),

$$\begin{aligned}
\Delta P &= \sum_{i,j}(\delta_{ij} - a_{ij}^k(x_k))P_{ij} + \sum_{i,j} a_{ij}^k(x_k)P_{ij} \\
&\geq \sum_{\substack{i,j \\ P_{ij}\geq 0}} -\varepsilon_k P_{ij} + \sum_{\substack{i,j \\ P_{ij}<0}} \varepsilon_k P_{ij} + (-2 - C\sqrt{n}\varepsilon_k)\varepsilon_k \\
&= \Bigg(- \sum_{\substack{i,j \\ P_{ij}\geq 0}} P_{ij} + \sum_{\substack{i,j \\ P_{ij}<0}} P_{ij} - 2 - C\sqrt{n}\varepsilon_k \Bigg)\varepsilon_k \\
&= (C_1 - C\sqrt{n}\varepsilon_k)\varepsilon_k,
\end{aligned}$$

in other words

$$\Delta P \geq (C_1 - C\sqrt{n}\varepsilon_k)\varepsilon_k, \tag{6.71}$$

inasmuch $P(x)$ is a quadratic polynomial and hence P_{ij} is a constant $\forall i, j$, which also implies $P_{ij}(x_k) = P_{ij}$. As a consequence, passing to the limit in (6.71) as $k \to \infty$, we get $\Delta P \geq 0$, as desired, because $\varepsilon_k \to 0$ and $(C_1 - C\sqrt{n}\varepsilon_k) \to C_1$, which is a constant.

(ii) If $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$, we argue exactly in the same way of the proof of Lemma 3.1 and we have $P_n(\bar{x}) \geq 0$ as desired.

Step 3: Improvement of flatness. In this step, we argue exactly in the same way of the final step of Lemma 3.1. □

6.4 Theorems

We introduce here the results for the problem (6.1), corresponding to Theorem 4.2 and 4.1.

Theorem 6.8 (Flatness implies $C^{1,\alpha}$). *Let u be a viscosity solution to (6.1) in B_1. Assume that $0 \in F(u)$, $g_0 = 1$ and $a_{ij}(0) = \delta_{ij}$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if the graph of u is $\bar{\varepsilon}$-flat in B_1, i.e.*

$$(x_n - \bar{\varepsilon})^+ \leq u(x) \leq (x_n + \bar{\varepsilon})^+, \quad x \in B_1, \tag{6.72}$$

and

$$[a_{ij}]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon}, \ \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \ \|b|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \ [g]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon}, \tag{6.73}$$

then $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

Remark. The Remark following the statement of Theorem 4.2 holds also for Theorem 6.8.

Proof. We proceed in the same way of the proof of Theorem 4.2, explaining only the main differences and referring to the proof of Theorem 4.2 for all the details.

Let u be a viscosity solution to (6.1) in B_1 with $0 \in F(u)$, $g(0) = 1$ and $a_{ij}(0) = \delta_{ij}$. Consider the sequence of rescalings

$$u_k(x) := \frac{u(\rho_k x)}{\rho_k}, \quad x \in B_1,$$

with $\rho_k = \bar{r}^k$, $k = 0, 1, \ldots$, for a fixed $\bar{r}$ such that

$$\bar{r}^\beta \leq \frac{1}{4}, \quad \bar{r} \leq r_0,$$

with r_0 the universal constant of Lemma 6.7.

Repeating the same argument used in the proof of Theorem 4.2, we remark that u_k is well-defined.

In parallel to the proof of Theorem 4.2, we claim that each u_k solves a problem of the type satisfied by u.

In particular, we state that each u_k solves (6.1) in B_1 with coefficients $a^k_{ij}(x) := a_{ij}(\rho_k x)$ and $b^k_i(x) := \rho_k b_i(\rho_k x)$, right hand side $f_k(x) := \rho_k f(\rho_k x)$ and free boundary condition $g_k(x) := g(\rho_k x)$.

Specifically, we need to prove that

(i) if $\varphi \in C^2(B_1^+(u_k))$ touches u_k from below (above) at $x_0 \in B_1^+(u_k)$ then

$$\sum_{i,j} a_{ij}^k(x_0)\varphi_{ij}(x_0) + \sum_i b_i^k(x_0)\varphi_i(x_0) \le f_k(x_0) \quad \Bigg(\text{resp. } \sum_{i,j} a_{ij}^k(x_0)\varphi_{ij}(x_0) + \sum_i b_i^k(x_0)\varphi_i(x_0) \ge f_k(x_0)\Bigg);$$

(ii) if $\varphi \in C^2(B_1)$ and φ^+ touches u_k from below (above) at $x_0 \in F(u_k)$ and $|\nabla\varphi|(x_0) \neq 0$ then

$$|\nabla\varphi|(x_0) \le g_k(x_0) \quad (\text{resp. } |\nabla\varphi|(x_0) \ge g_k(x_0)).$$

Let us start showing that (i) is verified. For this purpose, we take $\varphi \in C^2(B_1^+(u_k))$ touching u_k from below at $x_0 \in B_1^+(u_k)$, and we have

$$\varphi(x_k) = u_k(x_k) \tag{6.74}$$

and

$$\varphi(x) \le u_k(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{6.75}$$

In particular, repeating the same argument used in the proof of Theorem 4.2, (6.74) and (6.75) read

$$(\rho_k\varphi)\left(\frac{\rho_k x_0}{\rho_k}\right) = u(\rho_k x_0) \tag{6.76}$$

and

$$(\rho_k\varphi)\left(\frac{\rho_k x}{\rho_k}\right) \le u(\rho_k x) \quad \text{in } O. \tag{6.77}$$

At this point, calling $O' = \rho_k O$, we have, from the proof of Theorem 4.2, that O' is a neighborhood of $\rho_k x_0$ and repeating the same argument used in the proof of Theorem 4.2, we obtain from (6.76) and (6.77) that $(\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right) \in C^2(O')$ touches u_k from below at $\rho_k x_0 \in B_1^+(u)$.

Consequently, since u is a viscosity solution to (6.1) in B_1, we get

$$\sum_{i,j} a_{ij}(\rho_k x_0)\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)_{ij}(\rho_k x_0) + \sum_i b_i(\rho_k x_0)\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)_i(\rho_k x_0) \le f(\rho_k x_0). \tag{6.78}$$

Now, from (4.10), we have

$$\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)_{ij}(\rho_k x_0) = \frac{1}{\rho_k}\varphi_{ij}(x_0). \tag{6.79}$$

In addition,

$$\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)_i = \rho_k\left(\varphi\left(\frac{\cdot}{\rho_k}\right)\right)_i = \rho_k\frac{1}{\rho_k}\varphi_i\left(\frac{\cdot}{\rho_k}\right) = \varphi_i\left(\frac{\cdot}{\rho_k}\right),$$

in other words

$$\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)_i = \varphi_i\left(\frac{\cdot}{\rho_k}\right),$$

which implies

$$\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)_i(\rho_k x_0) = \varphi_i\left(\frac{\rho_k x_0}{\rho_k}\right) = \varphi_i(x_0),$$

namely

$$\left((\rho_k\varphi)\left(\frac{\cdot}{\rho_k}\right)\right)_i(\rho_k x_0) = \varphi_i(x_0). \tag{6.80}$$

Therefore, in view of (6.78), together with (6.79) and (6.80), we obtain

$$\sum_{i,j} a_{ij}(\rho_k x_0)\frac{1}{\rho_k}\varphi_{ij}(x_0) + \sum_i b_i(\rho_k x_0)\varphi_i(x_0) \le f(\rho_k x_0),$$

which also gives, inasmuch $\rho_k > 0$,

$$\sum_{i,j} a_{ij}(\rho_k x_0)\varphi_{ij}(x_0) + \rho_k\sum_i b_i(\rho_k x_0)\varphi_i(x_0) \le \rho_k f(\rho_k x_0),$$

i.e.

$$\sum_{ij} a_{ij}(\rho_k x_0)\varphi_{ij}(x_0) + \sum_i \rho_k b_i(\rho_k x_0)\varphi_i(x_0) \le \rho_k f(\rho_k x_0),$$

and for the definitions of a_{ij}^k, b_i^k and f_k,

$$\sum_{i,j} a_{ij}^k(x_0)\varphi_{ij}(x_0) + \sum_i b_i^k(x_0)\varphi_i(x_0) \le f_k(x_0).$$

Repeating an analogous argument, but with opposite inequalities, if $\varphi \in C^2(B_1^+(u_k))$ touches u_k from above at $x_0 \in B_1^+(u_k)$, we get

$$\sum_{i,j} a_{ij}^k(x)(u_k)_{ij} + \sum_i b_i^k(x)(u_k)_i = f_k \quad \text{in } B_1^+(u_k) \text{ in the viscosity sense.} \tag{6.81}$$

As regards the condition $|\nabla u_k| = g_k$ on $F(u_k)$ in the viscosity sense, we can repeat exactly the argument used in the proof of Theorem 4.2 and we obtain

$$|\nabla u_k| = g_k \quad \text{on } F(u_k) \text{ in the viscosity sense.} \tag{6.82}$$

At this point, putting together (6.81) and (6.82), we have that each u_k is a solution to (6.1) in B_1 with coefficients a_{ij}^k and b_i^k, right hand side f_k and free boundary condition g_k.
Moreover, repeating the same argument used in the proof of Theorem 4.2, we can show that for the chosen $\bar{r}$, a_{ij}^k, f_k and g_k satisfy the assumption (6.3) in B_1, with $\varepsilon_k = 2^{-k}\varepsilon_0(\bar{r})$. In particular, as in the proof of Theorem 4.2, we have $\bar{\varepsilon} = \varepsilon_0(\bar{r})^2$.
We now show that also b^k verifies (6.4) in B_1 with ε_k.
Indeed, if we fix $x \in B_1$, and we consider $b_i^k(x)$ with $i \in \{1, \ldots, n\}$, we have, since $\rho_k > 0$

$$\left|b_i^k(x)\right| = |\rho_k b_i(\rho_k x)| = \rho_k \left|b_i(\rho_k x)\right|,$$

that is

$$\left|b_i^k(x)\right| = \rho_k \left|b_i(\rho_k x)\right|. \tag{6.83}$$

In particular, since from (4.3), $\rho_k x \in B_1$, if $x \in B_1$, we obtain from (6.83)

$$\left|b_i^k(x)\right| \leq \rho_k \left\|b_i\right\|_{L^\infty(B_1)}. \tag{6.84}$$

Furthermore, we know from the definition of $\|b|_{L^\infty(B_1)}$ that

$$\|b_i\|_{L^\infty(B_1)} \leq \max_{i=1,\ldots,n} \|b_i\|_{L^\infty(B_1)} = \|b\|_{L^\infty(B_1)},$$

in other words

$$\|b_i\|_{L^\infty(B_1)} \leq \|b\|_{L^\infty(B_1)},$$

which gives from (6.84)

$$\left|b_i^k(x)\right| \leq \rho_k \|b\|_{L^\infty(B_1)}. \tag{6.85}$$

In addition, we have from (6.73) that $\|b\|_{L^\infty(B_1)} \leq \bar{\varepsilon}$, as a consequence, we get from (6.85)

$$\left|b_i^k(x)\right| \leq \rho_k \bar{\varepsilon},$$

which entails, because $\rho_k = \bar{r}^k$ and $\bar{\varepsilon} = \varepsilon_0(\bar{r})^2$,

$$\left|b_i^k(x)\right| \leq \bar{r}^k \varepsilon_0(\bar{r})^2. \tag{6.86}$$

At this point, repeating the same argument used in the proof of Theorem 4.2 to obtain $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2$, we have $\bar{r} \leq 1/4 = 2^{-2}$ and hence $\bar{r}^k \leq 2^{-2k}$ for $k = 0, 1, \ldots$

Therefore, from (6.86) we get, inasmuch $\varepsilon_k = 2^{-k}\varepsilon_0(\bar{r})$,

$$\left|b_i^k(x)\right| \leq 2^{-2k}\varepsilon_0(\bar{r})^2 = \varepsilon_k^2,$$

i.e.

$$\left|b_i^k(x)\right| \leq \varepsilon_k^2. \tag{6.87}$$

Consequently, for the arbitrariness of $x \in B_1$, we have that ε_k^2 is an upper bound of the set $\left\{ \left|b_i^k(x)\right|, \quad x \in B_1 \right\}$, and thus

$$\left\|b_i^k\right\|_{L^\infty(B_1)} = \sup_{x \in B_1} \left|b_i^k(x)\right| \leq \varepsilon_k^2,$$

namely

$$\left\|b_i^k\right\|_{L^\infty(B_1)} \leq \varepsilon_k^2, \quad i \in \{i, \ldots, n\}. \tag{6.88}$$

In addition, for the definition of $\left\|b^k\right\|_{L^\infty(B_1)}$, (6.88) implies

$$\left\|b^k\right\|_{L^\infty(B_1)} = \max_{i=1,\ldots,n} \left\|b_i^k\right\|_{L^\infty(B_1)} \leq \varepsilon_k^2,$$

that is

$$\left\|b^k\right\|_{L^\infty(B_1)} \leq \varepsilon_k^2,$$

as desired. To sum it up, each u_k solves (6.1) in B_1, with coefficients a_{ij}^k and b_i^k, right hand side f_k and free boundary condition g_k, satisfying (6.3)-(6.4) with ε_k.

This fact allows us to apply Lemma 6.7 with u_k and the continuance of the proof is the same of that of Theorem 4.2. $\square$

Theorem 6.9 (Lipschitz implies $C^{1,\alpha}$). *Let u be a viscosity solution to (6.1). Assume that $0 \in F(u)$ and $g(0) > 0$. If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\alpha}$ in a (smaller) neighborhood of 0.*

Remark. As in Theorem 4.1, the size of the neighborhood where $F(u)$ is $C^{1,\alpha}$ depends on the radius ρ of the ball B_ρ where $F(u)$ is Lipschitz, on the Lipschitz norm of $F(u)$, on $[a_{ij}]_{C^{0,\beta}(B_\rho)}$, $\|g\|_{C^{0,\beta}(B_\rho)}$, $\|f\|_{L^\infty(B_\rho)}$ and $\|b\|_{L^\infty(B_\rho)}$.

Proof. The proof follows the scheme of the proof of Theorem 4.1 and there are only small differences with the proof of Theorem 4.1, which we will explain, while for all the details see the proof of Theorem 4.1.
Let u be a viscosity solution to (6.1) in Ω with $0 \in F(u)$ and $g(0) > 0$. As in the proof of Theorem 4.1, we can assume without loss of generality that $\Omega = B_1$ and $g(0) = 1$.
Indeed, concerning $g(0) = 1$, arguing as in the proof of Theorem 4.1, if $g(0) \neq 1$, since $g(0) > 0$ and hence $g(0) \neq 0$, we can divide g by $g(0)$ to get $\tilde{g} := \frac{g}{g(0)}$, and if we set $\tilde{u} = \frac{u}{g(0)}$, we state that $\tilde{u}$ is a viscosity solution to (6.1) in Ω with coefficients a_{ij} and b_i, free boundary condition $\tilde{g}$ and right hand side $\tilde{f} := \frac{f}{g(0)}$.
Precisely, if $\varphi \in C^2(B_1^+(\tilde{u}))$ touches $\tilde{u}$ from below at $x_0 \in B_1^+(\tilde{u})$, we have

$$\varphi(x_0) = \tilde{u}(x_0) \tag{6.89}$$

and

$$\varphi(x) \leq \tilde{u}(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{6.90}$$

In particular, from the definition of $\tilde{u}$, repeating the same calculations done in the proof of Theorem 4.1, (6.89) and (6.90) read

$$g(0)\varphi(x_0) = u(x_0) \tag{6.91}$$

and

$$g(0)\varphi(x) \leq u(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{6.92}$$

Consequently, repeating the same argument used in the proof of Theorem 4.1, we get from (6.91) and (6.92) that $g(0)\varphi \in C^2(B_1^+(u))$ touches u from below at $x_0 \in B_1^+(u)$.

Therefore, inasmuch u is a viscosity solution to (6.1) in B_1, we have

$$\begin{aligned}
&\sum_{i,j} a_{ij}(x_0)(g(0)\varphi)_{ij}(x_0) + \sum_i b_i(x_0)(g(0)\varphi)_i(x_0)\\
&= \sum_{i,j} a_{ij}(x_0)g(0)\varphi_{ij}(x_0) + \sum_i b_i(x_0)g(0)\varphi_i(x_0)\\
&= g(0)\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + g(0)\sum_i b_i(x_0)\varphi_i(x_0) \le f(x_0),
\end{aligned}$$

namely

$$g(0)\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + g(0)\sum_i b_i(x_0)\varphi_i(x_0) \le f(x_0),$$

which entails, since $g(0) > 0$,

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i b_i(x_0)\varphi_i(x_0) \le \frac{f(x_0)}{g(0)},$$

in other words, for the definition of $\tilde{f}$,

$$\sum_{i,j} a_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i b_i(x_0)\varphi_i(x_0) \le \tilde{f}(x_0).$$

As a consequence, repeating the same argument if $\varphi \in C^2(B_1^+(\tilde{u}))$ touches $\tilde{u}$ from above at $x_0 \in B_1^+(\tilde{u})$, but with opposite inequalities, we obtain

$$\sum_{i,j} a_{ij}(x)\tilde{u}_{ij} + \sum_i b_i(x)\tilde{u}_i = \tilde{f} \quad \text{in } B_1^+(\tilde{u}) \text{ in the viscosity sense.} \tag{6.93}$$

As regards the condition $|\nabla \tilde{u}| = \tilde{g}$ on $F(\tilde{u})$, we can repeat exactly the same argument used in the proof of Theorem 4.1 and we get

$$|\nabla \tilde{u}| = \tilde{g} \quad \text{on } F(\tilde{u}) \text{ in the viscosity sense.} \tag{6.94}$$

Hence, putting together (6.93) and (6.94), we have that $\tilde{u}$ is a viscosity solution to (6.1) in B_1 with coefficients a_{ij} and b_i, right hand side $\tilde{f}$ and free boundary condition $\tilde{g}$.
Moreover, for simplicity we take $a_{ij}(0) = \delta_{ij}$.

Now, as in the proof of Theorem 4.1, we consider the blow-up sequence

$$u_k := u_{\delta_k}(x) = \frac{u(\delta_k x)}{\delta_k},$$

with $\delta_k \to 0$ as $k \to \infty$.
In particular, repeating the same argument used in the proof of Theorem 6.8, each u_k solves (6.1) with coefficients $a_{ij}^k := a_{ij}(\delta_k x)$ and $b_i^k(x) = \delta_k b_i(\delta_k x)$, right hand side $f_k(x) := \delta_k f(\delta_k x)$ and free boundary condition $g_k(x) := g(\delta_k x)$.
Furthermore, repeating the same argument used in the proof of Theorem 4.1, we also have that, for k large, f_k, g_k and a_{ij}^k satisfy (4.2) in B_1 with $\bar{\varepsilon}$ and (6.3) in B_1 with $\bar{\varepsilon}$.
At this point, we prove that b^k satisfies (6.4) in B_1 with $\bar{\varepsilon}$, which is the same condition in (6.73).
Specifically, we fix $x \in B_1$ and we consider $b_i^k(x)$, with $i \in \{1, \ldots, n\}$.
From the definition of b_i^k, we have, because $\delta_k > 0$,

$$\left|b_i^k(x)\right| = |\delta_k b_i(\delta_k x)| = \delta_k \left|b_i(\delta_k x)\right|,$$

i.e.

$$\left|b_i^k(x)\right| = \delta_k \left|b_i(\delta_k x)\right|. \tag{6.95}$$

In particular, since $\delta_k \to 0$ as $k \to \infty$ and $\delta_k > 0$, we have that there exists $\bar{k} \in \mathbb{N}$ such that

$$\delta_k < 1, \quad \forall k \in \mathbb{N}, k \geq \bar{k},$$

in other words for k large $\delta_k < 1$.
Thus, for these k's, if $x \in B_1$, inasmuch $\delta_k > 0$,

$$|\delta_k x| = \delta_k |x| < |x| < 1,$$

that is $\delta_k x \in B_1$. Therefore, consider k large enough so that $\delta_k < 1$, which also gives $\delta_k x \in B_1$ if $x \in B_1$ and as a consequence from (6.95) we obtain

$$\left|b_i^k(x)\right| \leq \delta_k \left\|b_i\right\|_{L^\infty(B_1)}. \tag{6.96}$$

Now, always because $\delta_k \to 0$ as $k \to \infty$, there also exists $\bar{k} \in \mathbb{N}$ such that

$$\delta_k < \frac{\bar{\varepsilon}}{\|b_i\|_{L^\infty(B_1)}}, \quad k \in \mathbb{N}, k \geq \bar{k},$$

i.e. for k large $\delta_k < \frac{\bar{\varepsilon}}{\|b_i\|_{L^\infty(B_1)}}$.
Consequently, if we take k large so that this condition is satisfied, we have from (6.96)

$$\left|b_i^k(x)\right| \leq \delta_k \|b_i\|_{L^\infty(B_1)} \leq \frac{\bar{\varepsilon}}{\|b_i\|_{L^\infty(B_1)}} \|b_i\|_{L^\infty(B_1)} = \bar{\varepsilon},$$

namely

$$\left|b_i^k(x)\right| \leq \bar{\varepsilon}. \tag{6.97}$$

Hence, from the arbitrariness of $x \in B_1$, we get from (6.97) that $\bar{\varepsilon}$ is an upper bound of the set $\left\{\left|b_i^k(x)\right|, \quad x \in B_1\right\}$, and thus

$$\left\|b_i^k\right\|_{L^\infty(B_1)} = \sup_{x \in B_1} \left|b_i^k(x)\right| \leq \bar{\varepsilon},$$

i.e.

$$\left\|b_i^k\right\|_{L^\infty(B_1)} \leq \bar{\varepsilon}. \tag{6.98}$$

As a consequence, from the definition of $\left\|b^k\right\|_{L^\infty(B_1)}$, we obtain

$$\left\|b^k\right\|_{L^\infty(B_1)} = \max_{i=1,\dots,n} \left\|b_i^k\right\|_{L^\infty(B_1)} \leq \bar{\varepsilon},$$

which gives

$$\left\|b^k\right\|_{L^\infty(B_1)} \leq \bar{\varepsilon}.$$

Therefore, b_k satisfies (6.4) with $\bar{\varepsilon}$ for k large so that

$$\delta_k < \min\left(1, \frac{\bar{\varepsilon}}{\max\limits_{i=1,\dots,n} \|b_i\|_{L^\infty(B_1)}}\right).$$

To sum it up, we have for k large that f_k, g_k, a_{ij}^k and b_i^k satisfy (6.73) in B_1 with $\bar{\varepsilon}$ and in parallel f_k, g_k, a_{ij}^k and b_i^k satisfy (6.3)-(6.4) in B_1 with $\bar{\varepsilon}$.
The remaining part of the proof is the same of the proof of Theorem 4.1, remarking these two facts.

(i) Also a solution to (6.1)-(6.3)-(6.4) is Lipschitz continuous and satisfy a nondegeneracy property like that expressed by Lemma 5.1.

(ii) The blow-up limit u_0 is always a global solution to the free boundary problem

$$\begin{cases} \Delta u_0 = 0 & \text{in } \{u_0 > 0\} \\ |\nabla u_0| = 1 & \text{on } F(u_0). \end{cases}$$

□

6.5 Nondegeneracy property

In this section, we provide the nondegeneracy property also for a solution to (6.1)-(6.3)-(6.4).

Lemma 6.10. *Let u be a solution to (6.1)-(6.3)-(6.4) with $\varepsilon \leq \tilde{\varepsilon}$ a universal constant. If $F(u) \cap B_1 \neq \emptyset$ and $F(u)$ is a Lipschitz graph in B_2, then u is Lipschitz and nondgenerate in $B_1^+(u)$, i.e.*

$$c_0 d(z) \leq u(z) \leq C_0 d(z) \quad \textit{for all } z \in B_1^+(u),$$

with $d(z) = \operatorname{dist}(z, F(u))$ and c_0, C_0 universal constants.

Proof. The proof follows exactly the scheme of the proof of Lemma 5.1 and we explain only the main differences, referring to the proof of Lemma 5.1 for all the details.
As in the proof of Lemma 5.1, assume without loss of generality that $0 \in B_1^+(u)$ and set $d := d(0)$.
Consider always the rescaled function

$$\tilde{u}(x) = \frac{u(dx)}{d}, \quad x \in B_1.$$

Repeating the same argument used in the proof of Theorem 6.8, we get that $\tilde{u}$ satisfies (6.1) in B_1 with coefficients $\tilde{a}_{ij}(x) := a_{ij}(dx)$ and $\tilde{b}_i(x) = db_i(dx)$, right hand side $\tilde{f}(x) := df(dx)$ and free boundary condition $\tilde{g}(x) := g(dx)$.

In addition, repeating the same computations done in the proof of Lemma 5.1, we achieve $d \leq 1$ and the assumption (6.3) holds in B_1 for $\tilde{a}_{ij}$, $\tilde{f}$ and $\tilde{g}$. At this point, we claim that $\tilde{b}$ satisfies (6.4) in B_1.
Precisely, if we fix $x \in B_1$, and we consider $\tilde{b}_i(x)$, we have, because $0 \leq d \leq 1$,

$$\left|\tilde{b}_i(x)\right| = |d b_i(dx)| = d\,|b_i(dx)| \leq |b(dx)| \leq \|b_i\|_{L^\infty},$$

namely

$$\left|\tilde{b}_i(x)\right| \leq \|b_i\|_{L^\infty}. \tag{6.99}$$

Moreover, we know from hypothesis that b satisfies (6.4), as a consequence we have

$$\|b_i\|_{L^\infty} \leq \|b\|_{L^\infty} \leq \varepsilon^2,$$

in other words

$$\|b_i\|_{L^\infty} \leq \varepsilon^2,$$

which implies from (6.99)

$$\left|\tilde{b}_i(x)\right| \leq \varepsilon^2. \tag{6.100}$$

Therefore, for the arbitrariness of $x \in B_1$, we obtain from (6.100) that ε^2 is an upper bound of the set $\left\{\left|\tilde{b}_i(x)\right|, \quad x \in B_1\right\}$, and thus

$$\left\|\tilde{b}_i\right\|_{L^\infty(B_1)} = \sup_{x \in B_1} \left|\tilde{b}_i(x)\right| \leq \varepsilon^2,$$

i.e.

$$\left\|\tilde{b}_i\right\|_{L^\infty(B_1)} \leq \varepsilon^2. \tag{6.101}$$

Consequently, from the definition of $\left\|\tilde{b}\right\|_{L^\infty(B_1)}$, we get

$$\left\|\tilde{b}\right\|_{L^\infty(B_1)} = \max_{i=1,\dots,n} \left\|\tilde{b}_i\right\|_{L^\infty(B_1)} \leq \varepsilon^2,$$

that is

$$\left\|\tilde{b}\right\|_{L^\infty(B_1)} \leq \varepsilon^2$$

and hence $\tilde{b}$ satisfies (6.4) in B_1.
At this point, as in the proof of Lemma 5.1, we wish to show that

$$c_0 \leq \tilde{u}(0) \leq C_0.$$

Specifically, we assume for contradiction that $\tilde{u}(0) > C_0$, with C_0 to be made precise later.
As in the proof of Lemma 5.1, let

$$G(x) = C(|x|^{-\gamma} - 1)$$

be defined on the closure of the annulus $B_1 \setminus \overline{B}_{1/2}$.
In particular, in view of the uniform ellipticity of the coefficients $\tilde{a}_{ij}$(see Lemma A.5 in Appendix A), repeating the same calculations done in the proof of Lemma 6.6, we can choose γ large universal so that (for ε small)

$$\sum_{i,j} \tilde{a}_{ij} G_{ij} + \sum_i \tilde{b}_i G_i > \varepsilon^2 \quad \text{on } B_1 \setminus \overline{B}_{1/2}. \tag{6.102}$$

Furthermore, we can choose the constant C so that

$$G = 1 \quad \text{on } \partial B_{1/2},$$

and from the proof of Lemma 5.1, we achieve

$$C = \frac{1}{(1/2)^{-\gamma} - 1}.$$

In addition, repeating the same argument used in the proof of Lemma 5.1, we get $\tilde{u} > 0$ in B_1.
Consequently, in view of this fact and inasmuch $\tilde{u}$ solves, in the viscosity sense, a uniformly elliptic equation in B_1 with right hand side $\tilde{f}$, we can apply the the Harnack inequality to obtain

$$\sup_{\overline{B}_{1/2}} \tilde{u} \le C_1 \left(\inf_{\overline{B}_{1/2}} \tilde{u} + C_2 \left\| \tilde{f} \right\|_{L^\infty(B_1)} \right),$$

which gives, repeating the same computations done in the proof of Lemma 5.1,

$$\tilde{u} \ge c\tilde{u}(0) \quad \text{on } \overline{B}_{1/2}. \tag{6.103}$$

At this point, as in the proof of Lemma 5.1, we define $v(x) := c\tilde{u}(0)G(x)$ and we state that $\tilde{u} - v$ satisfies

$$\sum_{i,j} \tilde{a}_{ij} (\tilde{u} - v)_{ij} + \sum_i \tilde{b}_i (\tilde{u} - v)_i \le 0 \quad \text{in } B_1 \setminus \overline{B}_{1/2}$$

in the viscosity sense, that is $\tilde{u} - v$ is a viscosity supersolution of $\sum\limits_{i,j} \tilde{a}_{ij}(\tilde{u} - v)_{ij} + \sum\limits_{i} \tilde{b}_i(\tilde{u} - v)_i = 0$ in $B_1 \setminus \overline{B}_{1/2}$, see Definition B.4 in Appendix B. Precisely, if $\varphi \in C^2(B_1 \setminus \overline{B}_{1/2})$ touches $\tilde{u} - v$ from below at $x_0 \in (B_1 \setminus \overline{B}_{1/2})$, we have

$$\varphi(x_0) = (\tilde{u} - v)(x_0) = \tilde{u}(x_0) - v(x_0) \tag{6.104}$$

and

$$\varphi(x) \leq (\tilde{u} - v)(x) = \tilde{u}(x) - v(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{6.105}$$

In particular, (6.104) and (6.105) read

$$\varphi(x_0) + v(x_0) = (\varphi + v)(x_0) = \tilde{u}(x_0) \tag{6.106}$$

and

$$\varphi(x) + v(x) = (\varphi + v)(x) \leq \tilde{u}(x) \quad \text{in a neighborhood } O \text{ of } x_0. \tag{6.107}$$

Consequently, from (6.106) and (6.107), repeating the same argument used in the proof of Lemma 5.1, we achieve that $(\varphi + v) \in C^2(B_1 \setminus \overline{B}_{1/2})$ touches $\tilde{u}$ from below at $x_0 \in (B_1 \setminus \overline{B}_{1/2})^+(\tilde{u})$.

Hence, since $\tilde{u}$ is a solution to (6.1) in B_1 and also in $B_1 \setminus \overline{B}_{1/2}$, we get

$$\begin{aligned}
&\sum_{i,j} \tilde{a}_{ij}(x_0)(\varphi+v)_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)(\varphi+v)_i(x_0) \\
&= \sum_{i,j} \tilde{a}_{ij}(x_0)(\varphi + c\tilde{u}(0)G)_{ij}(x_0) + \sum_i b_i(x_0)(\varphi + c\tilde{u}(0)G)_i(x_0) \\
&= \sum_{i,j} \tilde{a}_{ij}(x_0)(\varphi_{ij}(x_0) + c\tilde{u}(0)G_{ij}(x_0)) + \sum_i \tilde{b}_i(x_0)(\varphi_i(x_0) + c\tilde{u}(0)G_i(x_0)) \\
&= \sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + \sum_{i,j} \tilde{a}_{ij}(x_0)c\tilde{u}(0)G_{ij}(x_0) \\
&+ \sum_i \tilde{b}_i(x_0)\varphi_i(x_0) + \sum_i \tilde{b}_i(x_0)c\tilde{u}(0)G_i(x_0) \\
&= \sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)\varphi_i(x_0) \\
&+ \sum_{i,j} \tilde{a}_{ij}(x_0)c\tilde{u}(0)G_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)G_i(x_0) \\
&= \sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)\varphi_i(x_0) \\
&+ c\tilde{u}(0)\sum_{i,j} \tilde{a}_{ij}(x_0)G_{ij}(x_0) + c\tilde{u}(0)\sum_i \tilde{b}_i(x_0)G_i(x_0) \\
&= \sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)\varphi_i(x_0) \\
&+ c\tilde{u}(0)\left(\sum_{i,j} \tilde{a}_{ij}(x_0)G_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)G_i(x_0)\right) \leq \tilde{f}(x_0)
\end{aligned}$$

in other words

$$\begin{aligned}
&\sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)\varphi_i(x_0) \\
&+ c\tilde{u}(0)\left(\sum_{i,j} \tilde{a}_{ij}(x_0)G_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)G_i(x_0)\right) \leq \tilde{f}(x_0),
\end{aligned}$$

which entails

$$\begin{aligned}
&\sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)\varphi_i(x_0) \leq \tilde{f}(x_0) \\
&- c\tilde{u}(0)\left(\sum_{i,j} \tilde{a}_{ij}c\tilde{u}(0)G_{ij}(x_0) - \sum_i \tilde{b}_i(x_0)G_i(x_0)\right). \qquad (6.108)
\end{aligned}$$

In addition, in view of (6.102), inasmuch $x_0 \in (B_1 \setminus \overline{B}_{1/2})$, we obtain from (6.108) taking $\varepsilon^2 = c\tilde{u}(0)\varepsilon^2$,

$$\sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)\varphi_i(x_0) \leq \tilde{f}(x_0) - \varepsilon^2. \tag{6.109}$$

Now, from the first inequality in (6.3), i.e. $\left\|\tilde{f}\right\|_{L^\infty(B_1)} \leq \varepsilon^2$, we also have $\left|\tilde{f}(x)\right| \leq \varepsilon^2$, $\forall x \in B_1$ and thus, because $x_0 \in (B_1 \setminus \overline{B}_{1/2}) \subset B_1$, that is $x_0 \in B_1$, $\left|\tilde{f}(x_0)\right| \leq \varepsilon^2$, which also gives $\tilde{f}(x_0) \leq \varepsilon^2$.
Therefore, in view of this fact, we achieve from (6.109)

$$\sum_{i,j} \tilde{a}_{ij}(x_0)\varphi_{ij}(x_0) + \sum_i \tilde{b}_i(x_0)\varphi_i(x_0) \leq 0,$$

which implies that $\tilde{u} - v$ is a viscosity supersolution to $\sum_{i,j} \tilde{a}_{ij}(\tilde{u} - v)_{ij} + \sum_i \tilde{b}_i(\tilde{u} - v)_i = 0$ in $B_1 \setminus \overline{B}_{1/2}$.
At this point, the remainder of the proof is the same of the proof of Lemma 5.1, with the only difference that $\tilde{G}_{\tilde{t}}$ is a strict supersolution to (6.1), in place to (2.1), but with the same computations to see it. $\square$

Appendix A

Some definitions and auxiliary theorems

We introduce here general tools used in the work.

Definition A.1. Let Ω be an open subset of $\mathbb{R}^n$ and $BC(\Omega)$ be the bounded continuous functions on Ω. For $u \in BC(\Omega)$ and $0 < \beta \leq 1$ let

$$\|u\|_{C(\Omega)} := \sup_{x \in \Omega} |u(x)|$$

and

$$[u]_{C^{0,\beta}(\Omega)} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\beta}.$$

If $[u]_{C^{0,\beta}(\Omega)} < \infty$, then u is Hölder continuos with holder exponent β. The collection of β-Hölder continuos function in Ω will be denoted by

$$C^{0,\beta}(\Omega) := \left\{u \in BC(\Omega) : [u]_{C^{0,\beta}(\Omega)} < \infty\right\}$$

and for $u \in C^{0,\beta}(\Omega)$ let

$$\|u\|_{C^{0,\beta}(\Omega)} := \|u\|_{C(\Omega)} + [u]_{C^{0,\beta}(\Omega)} .$$

Definition A.2. Let (X, d) a metric space and A, $B \subset X$ two non-empty subsets. We define their Hausdorff distance $d_H(A, B)$ by

$$d_H(A, B) := \max\left\{e(A, B), e(B, A)\right\},$$

where

$$e(A,B) := \sup_{x \in A} d(x,B)$$

and

$$d(x,B) := \inf_{y \in B} d(x,y).$$

Theorem A.3 (**Ascoli-Arzelà Theorem**). *Let $K \subset \mathbb{R}^n$ be a compact set. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C(K,\mathbb{R})$ such that*

(i) *$(f_n)_{n\in\mathbb{N}}$ is uniformly bounded, that is $\exists\ M > 0$ such that*

$$|f_n(x)| \leq M \quad \forall x \in K,\ \forall f_n;$$

(ii) *$(f_n)_{n\in\mathbb{N}}$ is equicontinuos, i.e $\forall \varepsilon > 0$, $\exists\ \delta > 0$ such that $\forall x, y \in K$, $d(x,k) < \delta$*

$$|f_n(x) - f_n(y)| < \varepsilon \quad \forall f_n.$$

Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges uniformly.

We provide here two general results.

Lemma A.4. *Let $\Gamma(\theta_0, e_2) = \{\tau :\ \alpha(\tau, e_2) < \theta_0\}$ be the open cone of axis e_2 and aperture θ_0 in $\mathbb{R}^2$, where $e_2 = (0,1)$, $0 < \theta_0$ and $\alpha(\tau, e_2)$ is the angle between the vectors τ and e_2, and let u be a solution to*

$$\begin{cases} \Delta u = f & \text{in } \Gamma(\theta_0, e_2) \\ u = 0 & \text{on } \partial\Gamma(\theta_0, e_2). \end{cases} \tag{A.1}$$

Then u is not necessary Lipschitz.

Proof. First of all, let us do a change of variables and we write

$$\begin{cases} x = \rho\cos(\theta) \\ y = \rho\sin(\theta), \end{cases}$$

with ρ, θ the polar coordinates in $\mathbb{R}^2$.
In particular, after this change of variables, if u is a solution to (A.1), we obtain

$$\begin{cases} \Delta u(\rho\cos(\theta), \rho\sin(\theta)) = f(\rho\cos(\theta), \rho\sin(\theta)) & \text{in } \Gamma(\theta_0, e_2) \\ u(\rho\cos(\theta), \rho\sin(\theta)) = 0 & \text{on } \partial\Gamma(\theta_0, e_2). \end{cases} \tag{A.2}$$

Let us set then $v(\rho,\theta) = u(\rho\cos(\theta), \rho\sin(\theta))$ and let us see what it means that $u(\rho\cos(\theta), \rho\sin(\theta))$ satifies (A.2).
Let us start with calculating

$$\begin{aligned} \frac{\partial v(\rho,\theta)}{\partial\rho} &= \frac{\partial}{\partial\rho}\big(u(\rho\cos(\theta), \rho\sin(\theta))\big) \\ &= \frac{\partial u}{\partial x}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\frac{\partial}{\partial\rho}\big(\rho\cos(\theta)\big) + \frac{\partial u}{\partial y}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\frac{\partial}{\partial\rho}\big(\rho\sin(\theta)\big) \\ &= \frac{\partial u}{\partial x}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\cos(\theta) + \frac{\partial u}{\partial y}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\sin(\theta), \end{aligned}$$

which gives

$$\frac{\partial v}{\partial\rho}(\rho,\theta) = \frac{\partial u}{\partial x}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\cos(\theta) + \frac{\partial u}{\partial y}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\sin(\theta). \tag{A.3}$$

Analogously,

$$\begin{aligned} \frac{\partial v(\rho,\theta)}{\partial\theta} &= \frac{\partial}{\partial\theta}\big(u(\rho\cos(\theta), \rho\sin(\theta))\big) \\ &= \frac{\partial u}{\partial x}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\frac{\partial}{\partial\theta}(\rho\cos(\theta)) + \frac{\partial u}{\partial y}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\frac{\partial}{\partial\theta}\big(\rho\sin(\theta)\big) \\ &= \frac{\partial u}{\partial x}\big(\rho\cos(\theta), \rho\sin(\theta)\big)(-\rho\sin(\theta)) + \frac{\partial u}{\partial y}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\rho\cos(\theta), \end{aligned}$$

namely

$$\begin{aligned} \frac{\partial v}{\partial\theta}(\rho,\theta) = &-\frac{\partial u}{\partial x}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\rho\sin(\theta) \\ &+ \frac{\partial u}{\partial y}\big(\rho\cos(\theta), \rho\sin(\theta)\big)\rho\cos(\theta). \end{aligned} \tag{A.4}$$

At this point, we also calculate the second derivative of $v(\rho,\theta)$ respect to ρ and the second derivative of $v(\rho,\theta)$ respect to θ, in order to find an expression for $\Delta u(\rho\cos(\theta),\rho\sin(\theta))$.

Specifically, from (A.3), we have

$$
\begin{aligned}
&\frac{\partial^2 v(\rho,\theta)}{\partial\rho^2} = \frac{\partial}{\partial\rho}\left(\frac{\partial v(\rho,\theta)}{\partial\rho}\right)\\
&= \frac{\partial}{\partial\rho}\left(\frac{\partial u}{\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\cos(\theta) + \frac{\partial u}{\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\sin(\theta)\right)\\
&= \left(\frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\cos(\theta) + \frac{\partial^2 u}{\partial y\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\sin(\theta)\right)\cos(\theta)\\
&+ \frac{\partial u}{\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\frac{\partial}{\partial\rho}\big(cos(\theta)\big)\\
&+ \left(\frac{\partial^2 u}{\partial x\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\cos(\theta) + \frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\sin(\theta)\right)\sin(\theta)\\
&+ \frac{\partial u}{\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\frac{\partial}{\rho}\big(\sin(\theta)\big)\\
&= \frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\cos^2(\theta) + \frac{\partial^2 u}{\partial y\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\sin(\theta)\cos(\theta)\\
&+ \frac{\partial^2 u}{\partial x\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\cos(\theta)\sin(\theta) + \frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\sin^2(\theta),
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\frac{\partial^2 v}{\partial\rho^2}(\rho,\theta) = \frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\cos^2(\theta)\\
&+ \frac{\partial^2 u}{\partial y\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\sin(\theta)\cos(\theta) + \frac{\partial^2 u}{\partial x\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\cos(\theta)\sin(\theta)\\
&+ \frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\sin^2(\theta).
\end{aligned}
\tag{A.5}
$$

Analogously, from (A.4), we achieve

$$
\begin{aligned}
&\frac{\partial^2 v(\rho,\theta)}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(\frac{\partial v(\rho,\theta)}{\partial\theta}\right)\\
&= \frac{\partial}{\partial\theta}\left(-\frac{\partial u}{\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\sin(\theta) + \frac{\partial u}{\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\cos(\theta)\right)\\
&= \left(-\frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)(-\rho\sin(\theta)) - \frac{\partial^2 u}{\partial y\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)(\rho\cos(\theta))\right)\\
&\times \rho\sin(\theta) - \frac{\partial u}{\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\frac{\partial}{\partial\theta}\big(\rho\sin(\theta)\big)\\
&+ \left(\frac{\partial^2 u}{\partial x\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)(-\rho\sin(\theta)) + \frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\cos(\theta)\right)\\
&\times \rho\cos(\theta) + \frac{\partial u}{\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\frac{\partial}{\partial\theta}\big(\rho\cos(\theta)\big)\\
&= \frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\sin^2(\theta) - \frac{\partial^2 u}{\partial y\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\cos(\theta)\sin(\theta)\\
&- \frac{\partial u}{\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\cos(\theta)\\
&- \frac{\partial^2 u}{\partial x\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\sin(\theta)\cos(\theta) + \frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\cos^2(\theta)\\
&- \frac{\partial u}{\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\sin(\theta),
\end{aligned}
$$

in other words,

$$
\begin{aligned}
&\frac{\partial^2 v}{\partial\theta^2}(\rho,\theta) = \frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\sin^2(\theta)\\
&- \frac{\partial^2 u}{\partial y\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\cos(\theta)\sin(\theta) - \frac{\partial u}{\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\cos(\theta)\\
&- \frac{\partial^2 u}{\partial x\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\sin(\theta)\cos(\theta) + \frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\cos^2(\theta)\\
&- \frac{\partial u}{\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\sin(\theta).
\end{aligned}
\tag{A.6}
$$

In particular, from (A.5) and (A.6), we get

$$
\begin{aligned}
&\rho^2\frac{\partial^2 v}{\partial\rho^2}(\rho,\theta)+\frac{\partial^2 v}{\partial\theta^2}(\rho,\theta)=\frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\cos^2(\theta)\\
&+\frac{\partial^2 u}{\partial y\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\sin(\theta)\cos(\theta)+\frac{\partial^2 u}{\partial x\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\cos(\theta)\\
&\times\sin(\theta)+\frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\sin^2(\theta)+\frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\sin^2(\theta)\\
&-\frac{\partial^2 u}{\partial y\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\cos(\theta)\sin(\theta)-\frac{\partial u}{\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\cos(\theta)\\
&-\frac{\partial^2 u}{\partial x\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\sin(\theta)\cos(\theta)+\frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2\cos^2(\theta)\\
&-\frac{\partial u}{\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\sin(\theta)=\frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2(\cos^2(\theta)+\sin^2(\theta))\\
&+\frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho^2(\sin^2(\theta)+\cos^2(\theta))-\frac{\partial u}{\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\cos(\theta)\\
&-\frac{\partial u}{\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\rho\sin(\theta)=\rho^2\Bigg(\frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)+\frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\\
&\rho\sin(\theta)\big)\Bigg)-\rho\left(\frac{\partial u}{\partial x}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\cos(\theta)+\frac{\partial u}{\partial y}\big(\rho\cos(\theta),\rho\sin(\theta)\big)\sin(\theta)\right),
\end{aligned}
$$

and thus, in view of (A.3) and inasmuch

$$\frac{\partial^2 u}{\partial x^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)+\frac{\partial^2 u}{\partial y^2}\big(\rho\cos(\theta),\rho\sin(\theta)\big)=\Delta u\big(\rho\cos(\theta),\rho\sin(\theta)\big),$$

we obtain

$$\rho^2\frac{\partial^2 v}{\partial\rho^2}(\rho,\theta)+\frac{\partial^2 v}{\theta^2}(\rho,\theta)=\rho^2\Delta u\big(\rho\cos(\theta),\rho\sin(\theta)\big)-\rho\frac{\partial v}{\partial\rho}(\rho,\theta),$$

which implies

$$\rho^2\Delta u\big(\rho\cos(\theta),\rho\sin(\theta)\big)=\rho^2\frac{\partial^2 v}{\partial\rho^2}(\rho,\theta)+\frac{\partial^2 v}{\partial\theta^2}(\rho,\theta)+\rho\frac{\partial v}{\partial\rho}(\rho,\theta),$$

and dividing by ρ^2, which is strictly positive in $\Gamma(\theta_0,e_2)$, given that $\Gamma(\theta_0,e_2)$ is an open,

$$\Delta u(\rho\cos(\theta),\rho\sin(\theta))=\frac{\partial^2 v}{\partial\rho^2}(\rho,\theta)+\frac{1}{\rho^2}\frac{\partial^2 v}{\partial\theta^2}(\rho,\theta)+\frac{1}{\rho}\frac{\partial v}{\partial\rho}(\rho,\theta). \tag{A.7}$$

Consequently, if u solves $\Delta u = f$ in $\Gamma(\theta_0, e_2)$, in polar coordinates we have from (A.7)

$$\frac{\partial^2 v}{\partial \rho^2}(\rho,\theta)+\frac{1}{\rho^2}\frac{\partial^2 v}{\partial \theta^2}(\rho,\theta)+\frac{1}{\rho}\frac{\partial v}{\partial \rho}(\rho,\theta) = f(\rho\cos(\theta),\sin(\theta)) \quad \text{in } \Gamma(\theta_0, e_2). \tag{A.8}$$

Let us consider now the particular case when $f = 0$ in $\Gamma(\theta_0, e_2)$ and we achieve in view of (A.8)

$$\frac{\partial^2 v}{\partial \rho^2}(\rho,\theta)+\frac{1}{\rho^2}\frac{\partial^2 v}{\partial \theta^2}(\rho,\theta)+\frac{1}{\rho}\frac{\partial v}{\partial \rho}(\rho,\theta) = 0 \quad \text{in } \Gamma(\theta_0, e_2). \tag{A.9}$$

This equation lead us to look for the function $v(\rho,\theta)$ in the form $v(\rho,\theta) = \varphi(\rho)\psi(\theta)$ and we obtain from (A.9)

$$\varphi''(\rho)\psi(\theta) + \frac{1}{\rho^2}\varphi(\rho)\psi''(\theta) + \frac{1}{\rho}\varphi'(\rho)\psi(\theta) = 0$$

and dividing by $\varphi(\rho)\psi(\theta)$, which we suppose different from 0 for every (ρ,θ), we get

$$\frac{\varphi''(\rho)}{\varphi(\rho)} + \frac{1}{\rho^2}\frac{\psi''(\theta)}{\psi(\theta)} + \frac{1}{\rho}\varphi'(\rho)\varphi(\rho) = 0,$$

which entails

$$\frac{1}{\rho^2}\frac{\psi''(\theta)}{\psi(\theta)} = -\left(\frac{\varphi''(\rho)}{\varphi(\rho)} + \frac{1}{\rho}\frac{\varphi'(\rho)}{\varphi(\rho)}\right) = -\frac{\rho\varphi''(\rho)+\varphi'(\rho)}{\rho\varphi(\rho)},$$

and multiplying by ρ^2

$$\frac{\psi''(\theta)}{\psi(\theta)} = -\rho^2\frac{\rho\varphi''(\rho)+\varphi'(\rho)}{\rho\varphi(\rho)} = -\rho\frac{\rho\varphi''(\rho)+\varphi'(\rho)}{\varphi(\rho)} = -\rho^2\frac{\varphi''(\rho)}{\varphi(\rho)} - \rho\frac{\varphi'(\rho)}{\varphi(\rho)},$$

namely

$$\frac{\psi''(\theta)}{\psi(\theta)} = -\rho^2\frac{\varphi''(\rho)}{\varphi(\rho)} - \rho\frac{\varphi'(\rho)}{\varphi(\rho)}. \tag{A.10}$$

Notice, at this point, that in (A.10) we have a function $\frac{\psi''(\theta)}{\psi(\theta)}$, which depends only on θ, equal to a function $-\rho^2\frac{\varphi''(\rho)}{\varphi(\rho)} - \rho\frac{\varphi'(\rho)}{\varphi(\rho)}$ which depends only on ρ, for every ρ and for every θ, and this fact implies that the only possibility is that both the functions are constant and seeing as how they are equal, the constant is the same, in other words there exists a constant k such that

$$\frac{\psi''(\theta)}{\psi(\theta)} = k \tag{A.11}$$

and

$$-\rho^2 \frac{\varphi''(\rho)}{\varphi(\rho)} - \rho \frac{\varphi'(\rho)}{\varphi(\rho)} = k. \tag{A.12}$$

We treatise the two equations separately.
As regards the first equation, we can rewrite (A.11) as

$$\psi''(\theta) = k\psi(\theta). \tag{A.13}$$

Let us recall now that $v(\rho, \theta) = u(\rho\cos(\theta), \rho\sin(\theta))$, where $u(\rho\cos(\theta), \rho\sin(\theta))$ satisfies (A.2). As a consequence, $v(\rho, \theta)$ fulfills $v(\rho, \theta) = 0$ on $\partial\Gamma(\theta_0, e_2)$. Specifically, the values of θ which correspond to $\partial\Gamma(\theta_0, e_2)$ are $\dfrac{\pi}{2} - \theta_0$ and $\dfrac{\pi}{2} + \theta_0$, therefore we want to solve the following problem:

$$\begin{cases} \psi''(\theta) = k\psi(\theta) & \text{in } \Gamma(\theta_0, e_2) \\ \psi(\frac{\pi}{2} - \theta_0) = 0 & \text{on } \partial\Gamma(\theta_0, e_2) \\ \psi(\frac{\pi}{2} + \theta_0) = 0 & \text{on } \partial\Gamma(\theta_0, e_2), \end{cases} \tag{A.14}$$

where $\psi''(\theta) = k\psi(\theta)$ is fulfilled in $\Gamma(\theta_0, e_2)$, recalling that this equation derives from (A.9).
We distinguish three cases depending on k.

(i) If $k > 0$, the general integral of (A.13) is

$$\psi(\theta) = C_1 e^{\sqrt{k}\theta} + C_2 e^{-\sqrt{k}\theta},$$

and if we impose the conditions in (A.14), we obtain the system

$$\begin{cases} \psi(\frac{\pi}{2} - \theta_0) = C_1 e^{\sqrt{k}\left(\frac{\pi}{2} - \theta_0\right)} + C_2 e^{-\sqrt{k}\left(\frac{\pi}{2} - \theta_0\right)} = 0 \\ \psi(\frac{\pi}{2} + \theta_0) = C_1 e^{\sqrt{k}\left(\frac{\pi}{2} + \theta_0\right)} + C_2 e^{-\sqrt{k}\left(\frac{\pi}{2} + \theta_0\right)} = 0. \end{cases}$$

Consequently, if we call A the matrix

$$A := \begin{pmatrix} e^{\sqrt{k}\left(\frac{\pi}{2}-\theta_0\right)} & e^{-\sqrt{k}\left(\frac{\pi}{2}-\theta_0\right)} \\ e^{\sqrt{k}\left(\frac{\pi}{2}+\theta_0\right)} & e^{-\sqrt{k}\left(\frac{\pi}{2}+\theta_0\right)}, \end{pmatrix}$$

we have to solve

$$A \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0, \tag{A.15}$$

which admits a solution different from the trivial one only if $\det A = 0$, in other words if

$$e^{\sqrt{k}\left(\frac{\pi}{2}-\theta_0\right)} e^{-\sqrt{k}\left(\frac{\pi}{2}+\theta_0\right)} - e^{-\sqrt{k}\left(\frac{\pi}{2}-\theta_0\right)} e^{\sqrt{k}\left(\frac{\pi}{2}+\theta_0\right)} = 0. \tag{A.16}$$

In particular, we can rewrite the left term in (A.16) as

$$\begin{aligned} &e^{\sqrt{k}\left(\frac{\pi}{2}-\theta_0\right)} e^{-\sqrt{k}\left(\frac{\pi}{2}+\theta_0\right)} - e^{-\sqrt{k}\left(\frac{\pi}{2}-\theta_0\right)} e^{\sqrt{k}\left(\frac{\pi}{2}+\theta_0\right)} = e^{\sqrt{k}\frac{\pi}{2}-\sqrt{k}\theta_0} e^{-\sqrt{k}\frac{\pi}{2}-\sqrt{k}\theta_0} \\ &- e^{-\sqrt{k}\frac{\pi}{2}+\sqrt{k}\theta_0} e^{\sqrt{k}\frac{\pi}{2}+\sqrt{k}\theta_0} = e^{-2\sqrt{k}\theta_0} - e^{2\sqrt{k}\theta_0}, \end{aligned}$$

thus from (A.16), we achieve

$$e^{-2\sqrt{k}\theta_0} - e^{2\sqrt{k}\theta_0} = 0,$$

which implies

$$e^{-2\sqrt{k}\theta_0} = e^{2\sqrt{k}\theta_0},$$

that is

$$\frac{1}{e^{2\sqrt{k}\theta_0}} = e^{2\sqrt{k}\theta_0},$$

and

$$e^{4\sqrt{k}\theta_0} = 1. \tag{A.17}$$

At this point, the only possibility that (A.17) will have a solution is that $k = 0$, but we are in case of $k > 0$, hence (A.17) give a contradiction. As a consequence, the only solution of (A.15) is the trivial one, namely $C_1 = 0$ and $C_2 = 0$, which gives $\psi(\theta) = 0\ \forall\theta$, that contradicts the hypothesis we have done, i.e. $\psi(\theta) \neq 0\ \forall\ \theta$.

(ii) Suppose now that $k = 0$. In this case, the general integral of (A.13) is

$$\psi(\theta) = C_1 + C_2\theta,$$

and imposing the conditions in (A.14), we get

$$\begin{cases} \psi(\frac{\pi}{2} - \theta_0) = C_1 + C_2\left(\dfrac{\pi}{2} - \theta_0\right) = C_1 + C_2\dfrac{\pi}{2} - C_2\theta_0 = 0 \\ \psi(\frac{\pi}{2} + \theta_0) = C_1 + C_2\left(\dfrac{\pi}{2} + \theta_0\right) = C_1 + C_2\dfrac{\pi}{2} + C_2\theta_0 = 0, \end{cases} \tag{A.18}$$

where, subtracting the two equations, we obtain

$$2C_2\theta_0 = 0,$$

and thus, because $\theta_0 \neq 0$, $C_2 = 0$, which gives, from the equations in (A.18), also $C_1 = 0$ and hence we achieve that $\psi(\theta) = 0\ \forall\theta$, contradicting again the hypothesis $\psi(\theta) \neq 0\ \forall\theta$.

(iii) Suppose finally that $k < 0$ and the general integral in this case is

$$\psi(\theta) = C_1\cos(\sqrt{|k|}\theta) + C_2\sin(\sqrt{|k|}\theta). \tag{A.19}$$

Imposing the conditions in (A.14), we get this time

$$\begin{cases} \psi(\frac{\pi}{2} - \theta_0) = C_1\cos(\sqrt{|k|}\left(\frac{\pi}{2} - \theta_0\right)) + C_2\sin(\sqrt{|k|}(\frac{\pi}{2} - \theta_0)) = 0 \\ \psi(\frac{\pi}{2} + \theta_0) = C_1\cos(\sqrt{|k|}\left(\frac{\pi}{2} + \theta_0\right)) + C_2\sin(\sqrt{|k|}(\frac{\pi}{2} + \theta_0)) = 0. \end{cases} \tag{A.20}$$

In particular, using the addition and subtraction formulas for cosine and sine, we can rewrite the first equation in (A.20) as

$$\begin{aligned} &C_1\cos\left(\sqrt{|k|}\left(\frac{\pi}{2} - \theta_0\right)\right) + C_2\sin\left(\sqrt{|k|}\left(\frac{\pi}{2} - \theta_0\right)\right) \\ &= C_1\cos\left(\sqrt{|k|}\frac{\pi}{2} - \sqrt{|k|}\theta_0\right) + C_2\sin\left(\sqrt{|k|}\frac{\pi}{2} - \sqrt{|k|}\theta_0\right) \\ &= C_1\left(\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) + \sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\right) \\ &+ C_2\left(\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) - \cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\right) \\ &= C_1\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) + C_1\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0) \\ &+ C_2\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) - C_2\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0), \end{aligned}$$

and analogously, we can rewrite the second equation in (A.20) as

$$\begin{aligned}
&C_1 \cos\left(\sqrt{|k|}\left(\frac{\pi}{2}+\theta_0\right)\right) + C_2 \sin\left(\sqrt{|k|}\left(\frac{\pi}{2}+\theta_0\right)\right)\\
&= C_1 \cos\left(\sqrt{|k|}\frac{\pi}{2}+\sqrt{|k|}\theta_0\right) + C_2 \sin\left(\sqrt{|k|}\frac{\pi}{2}+\sqrt{|k|}\theta_0\right)\\
&= C_1 \left(\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) - \sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\right)\\
&+ C_2 \left(\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) + \cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\right)\\
&= C_1 \cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) - C_1 \sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\\
&+ C_2 \sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) + C_2 \cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0).
\end{aligned}$$

Consequently, from (A.20), we obtain

$$\begin{cases}
C_1 \cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) + C_1 \sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\\
+C_2 \sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) - C_2 \cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0) = 0\\
C_1 \cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) - C_1 \sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\\
+C_2 \sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0) + C_2 \cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0) = 0
\end{cases} \tag{A.21}$$

and if we call A the matrix

$$A := \begin{pmatrix}
\begin{array}{c}\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0)\\ +\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\end{array} & \begin{array}{c}\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0)\\ -\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\end{array}\\[2em]
\begin{array}{c}\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0)\\ -\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\end{array} & \begin{array}{c}\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0)\\ +\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0),\end{array}
\end{pmatrix}$$

we achieve from (A.21)

$$A\begin{pmatrix} C_1\\ C_2\end{pmatrix} = 0, \tag{A.22}$$

which admits a solution different from the trivial one only if $\det A = 0$, in other words if

$$\begin{aligned}&\left(\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0)+\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\right)\left(\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\right.\\&\left.\times\cos(\sqrt{|k|}\theta_0)+\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\right)-\left(\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\right.\\&\left.\times\cos(\sqrt{|k|}\theta_0)-\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\right)\left(\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0)\right.\\&\left.-\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\right)=0.\end{aligned} \tag{A.23}$$

Developing the left term in (A.23), we have

$$\begin{aligned}&\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos^2(\sqrt{|k|}\theta_0)+\cos^2\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0)\\&\times\sin(\sqrt{|k|}\theta_0)+\sin^2\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\cos(\sqrt{|k|}\theta_0)+\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\\&\times\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin^2(\sqrt{|k|}\theta_0)-\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos^2(\sqrt{|k|}\theta_0)\\&+\cos^2\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\cos(\sqrt{|k|}\theta_0)+\sin^2\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0)\\&\times\sin(\sqrt{|k|}\theta_0)-\cos\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin^2(\sqrt{|k|}\theta_0)=0,\end{aligned}$$

which gives

$$\begin{aligned}&2\cos^2\left(\sqrt{|k|}\frac{\pi}{2}\right)\cos(\sqrt{|k|}\theta_0)\sin(\sqrt{|k|}\theta_0)+2\sin^2\left(\sqrt{|k|}\frac{\pi}{2}\right)\sin(\sqrt{|k|}\theta_0)\\&\times\cos(\sqrt{|k|}\theta_0)=0,\end{aligned}$$

that is

$$\begin{aligned}&2\left(\sin^2\left(\sqrt{|k|}\frac{\pi}{2}\right)+\cos^2\left(\sqrt{|k|}\frac{\pi}{2}\right)\right)\sin(\sqrt{|k|}\theta_0)\cos(\sqrt{|k|}\theta_0)\\&\quad=2\sin(\sqrt{|k|}\theta_0)\cos(\sqrt{|k|}\theta_0)=\sin(2\sqrt{k}\theta_0)=0,\end{aligned}$$

and thus to sum it up, we have $\det A=0$ if $\sin(2\sqrt{|k|}\theta_0)=0$.
Now, $\sin(2\sqrt{|k|}\theta_0)=0$ if and only if

$$2\sqrt{|k|}\theta_0=m\pi\quad\text{with } m\in\mathbb{N}\cup 0, \tag{A.24}$$

where we take $m \in \mathbb{N} \cup 0$ and not in $\mathbb{Z}$, recalling that for hypothesis $0 < \theta_0$ and hence $2\sqrt{|k|}\theta_0$ is positive or equal to 0.
Also, from (A.24), we have

$$\sqrt{|k|} = \frac{m\pi}{2\theta_0}, \quad \text{with } m \in \mathbb{N} \cup 0, \tag{A.25}$$

which entails,

$$|k| = \frac{m^2\pi^2}{4\theta_0^2} \quad \text{with } m \in \mathbb{N} \cup 0, \tag{A.26}$$

raising to 2 both the terms of the inequality in (A.24), which is possible recalling that they are both positive or equal to 0 for what we have said above.
In addition, we recall that in this case $k < 0$, therefore $|k| = -k$, and as a consequence we get in view of (A.26)

$$-k = \frac{m^2\pi^2}{4\theta_0^2}, \quad \text{with } m \in \mathbb{N} \cup 0,$$

which gives

$$k = -\frac{m^2\pi^2}{4\theta_0^2}, \quad \text{with } m \in \mathbb{N} \cup 0,$$

where in particular, given that $k \neq 0$, we have to suppose $m \neq 0$. Consequently, (A.22) admits a solution different from the trivial one if and only if

$$k = -\frac{m^2\pi^2}{4\theta_0^2} \text{with } m \in \mathbb{N}.$$

At this point, we want to look for C_1 and C_2 for these k's.
Specifically, seeing as how $\det A = 0$, it suffices to consider only an equation in (A.21) and we choose the first one, where we substitute $\sqrt{|k|}$ found in (A.25) and we achieve

$$\begin{aligned} &C_1 \cos\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\cos\left(\frac{m\pi}{2\theta_0}\theta_0\right) + C_1 \sin\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2\theta_0}\theta_0\right) \\ &+ C_2 \sin\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\cos\left(\frac{m\pi}{2\theta_0}\theta_0\right) - C_2 \cos\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2\theta_0}\theta_0\right) = 0, \end{aligned}$$

namely

$$\begin{aligned} &C_1 \cos\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\cos\left(\frac{m\pi}{2}\right) + C_1 \sin\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right) \\ &+ C_2 \sin\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\cos\left(\frac{m\pi}{2}\right) - C_2 \cos\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right) = 0. \end{aligned} \tag{A.27}$$

In particular, we know that $\sin\left(\frac{m\pi}{2}\right) = 0$ and $\cos\left(\frac{m\pi}{2}\right) = \pm 1$ if m is even, while $\cos\left(\frac{m\pi}{2}\right) = 0$ and $\sin\left(\frac{m\pi}{2}\right) = \pm 1$ if m is odd, therefore we distinguish two cases in (A.27).

(a) If m is even, inasmuch as $\sin\left(\frac{m\pi}{2}\right) = 0$, from (A.27) we obtain

$$C_1 \cos\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\cos\left(\frac{m\pi}{2}\right) + C_2 \sin\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\cos\left(\frac{m\pi}{2}\right) = 0,$$

and inasmuch $\cos\left(\frac{m\pi}{2}\right) = \pm 1$, we have

$$C_1 \cos\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right) + C_2 \sin\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right) = 0,$$

which gives

$$C_2 = -\text{cotan}\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right) C_1,$$

and substituting into (A.19), we get

$$\psi(\theta) = C_1 \left(\cos\left(\frac{m\pi}{2\theta_0}\theta\right) - \text{cotan}\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2\theta_0}\theta\right)\right),$$

with $\theta \in \left(\frac{\pi}{2} - \theta_0, \frac{\pi}{2} + \theta_0\right)$.

(b) If m is odd, instead, $\cos\left(\frac{m\pi}{2}\right) = 0$, thus from (A.27) we achieve

$$C_1 \sin\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right) - C_2 \cos\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right) = 0,$$

which implies, given that $\sin\left(\frac{m\pi}{2}\right) = \pm 1$,

$$C_1 \sin\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right) - C_2 \cos\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right) = 0,$$

in other words

$$C_2 = \tan\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right) C_1,$$

and substituting into (A.19), we obtain

$$\psi(\theta) = C_1\left(\cos\left(\frac{m\pi}{2\theta_0}\theta\right) + \tan\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2\theta_0}\theta\right)\right),$$

with $\theta \in \left(\frac{\pi}{2} - \theta_0, \frac{\pi}{2} + \theta_0\right)$.

To sum it up, we have found that the solution of (A.14) is

$$\psi(\theta) = \begin{cases} C_1\left(\cos\left(\frac{m\pi}{2\theta_0}\theta\right) - \operatorname{cotan}\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2\theta_0}\theta\right)\right), & m \in \mathbb{N}, m \text{ even}, \\ C_1\left(\cos\left(\frac{m\pi}{2\theta_0}\theta\right) + \tan\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2\theta_0}\theta\right)\right) & m \in \mathbb{N}, m \text{ odd}, \end{cases} \tag{A.28}$$

with $\theta \in \left[\frac{\pi}{2} - \theta_0, \frac{\pi}{2} + \theta_0\right]$.
Considering now (A.12), and we can rewrite this equation as

$$-\rho^2\varphi''(\rho) - \rho\varphi'(\rho) = k\varphi(\rho),$$

i.e.

$$\rho^2\varphi''(\rho) + \rho\varphi'(\rho) + k\varphi(\rho) = 0, \tag{A.29}$$

which is an Euler type differential equation.
Set hence $\rho = e^t$ and we define $\varphi(\rho) = \varphi(e^t) := w(t)$, which satisfies

$$w'(t) = \varphi'(e^t)\frac{d}{dt}\left(e^t\right) = \varphi'(e^t)e^t,$$

that is

$$w'(t) = \varphi'(e^t)e^t, \tag{A.30}$$

and

$$w''(t) = \frac{d}{dt}\left(\varphi'(e^t)e^t\right) = \varphi''(e^t)e^te^t + \varphi'(e^t)e^t = \varphi''(e^t)e^{2t} + \varphi'(e^t)e^t,$$

namely

$$w''(t) = \varphi''(e^t)e^{2t} + \varphi'(e^t)e^t. \tag{A.31}$$

In addition, since $\rho = e^t$, from (A.31), we also get

$$w''(t) = \varphi''(\rho)\rho^2 + \varphi'(\rho)\rho,$$

which implies from (A.29)

$$w''(t) = -k\varphi(\rho) = -kw(t),$$

i.e.

$$w''(t) = -kw(t). \tag{A.32}$$

Now, for what we have achieved before establishing $\psi(\theta)$, we can accept only $k < 0$ and in particular we have found that

$$k = -\frac{m^2\pi^2}{4\theta_0^2},$$

as a consequence from (A.32), we have

$$w''(t) = \frac{m^2\pi^2}{4\theta_0^2}w(t)$$

and the general integral of this equation is

$$w(t) = C_1 e^{\left|\frac{m\pi}{2\theta_0}\right|t} + C_2 e^{-\left|\frac{m\pi}{2\theta_0}\right|t}.$$

Moreover, using the fact that $\rho = e^t$, and $w(t) = \varphi(\rho)$, we can rewrite the general integral as

$$\varphi(\rho) = C_1\rho^{\frac{m\pi}{2\theta_0}} + C_2\rho^{-\frac{m\pi}{2\theta_0}}. \tag{A.33}$$

At this point, let us recall that $v(\rho,\theta) = 0$ on $\partial\Gamma(\theta_0, e_2)$ and this condition implies, as regards the radius ρ in polar coordinates, that $v(0,\theta) = 0$, which give also $\varphi(0) = 0$ for how we have written $v(\rho,\theta)$.
Consequently, if we impose the condition $\varphi(0) = 0$ in (A.33), seeing as how $-\left|\frac{m\pi}{2\theta_0}\right| \leq 0$, we have to set $C_2 = 0$, therefore we get from (A.33)

$$\varphi(\rho) = C_1\rho^{\left|\frac{m\pi}{2\theta_0}\right|}. \tag{A.34}$$

Now, putting together (A.28) and (A.34), where we call C_{1_ψ} the constant C_1 in (A.28) and C_{1_φ} the constant C_1 in (A.34), we obtain, because $v(\rho,\theta) = \varphi(\rho)\psi(\theta)$,

$$v(\rho,\theta) = \begin{cases} C_{1_\varphi}\rho^{\frac{m\pi}{2\theta_0}} C_{1_\psi}\left(\cos\left(\frac{m\pi}{2\theta_0}\theta\right) - \text{cotan}\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2\theta_0}\theta\right)\right) \\ \text{with } m \in \mathbb{N}, m \text{ even} \\ C_{1_\varphi}\rho^{\frac{m\pi}{2\theta_0}} C_{1_\psi}\left(\cos\left(\frac{m\pi}{2\theta_0}\theta\right) + \tan\left(\frac{m\pi}{2\theta_0}\frac{\pi}{2}\right)\sin\left(\frac{m\pi}{2\theta_0}\theta\right)\right) \\ \text{with } m \in \mathbb{N}, m \text{ odd,} \end{cases} \tag{A.35}$$

with $\theta \in \left[\frac{\pi}{2} - \theta_0, \frac{\pi}{2} + \theta_0\right]$ and where we have written $\left|\frac{m\pi}{2\theta_0}\right| = \frac{m\pi}{2\theta_0}$, recalling that $\frac{m\pi}{2\theta_0} > 0$ for what we have said above.
At this point, notice that, always since $\frac{m\pi}{2\theta_0} > 0$, we have

$$\frac{m\pi}{2\theta_0} < 1 \leftrightarrow 2\theta_0 > m\pi \leftrightarrow \theta_0 > \frac{m\pi}{2}.$$

Let us consider then the particular case with $m = 1$, and the condition $\theta_0 > \frac{m\pi}{2}$ becomes $\theta_0 > \frac{\pi}{2}$. Let us take thus $\theta_0 = \frac{3}{4}\pi$, and

$$\frac{m\pi}{2\theta_0} = \frac{\pi}{2\theta_0} = \frac{\pi}{2\frac{3}{4}\pi} = \frac{1}{\frac{3}{2}} = \frac{2}{3},$$

i.e.

$$\frac{m\pi}{2\theta_0} = \frac{2}{3}.$$

This fact, together with $m = 1$, give us from (A.35)

$$v(\rho,\theta) = C_{1_\varphi}\rho^{\frac{2}{3}} C_{1_\psi}\left(\cos\left(\frac{2}{3}\theta\right) + \tan\left(\frac{2}{3}\frac{\pi}{2}\right)\sin\left(\frac{2}{3}\theta\right)\right),$$

namely calling $C = C_{1_\varphi}C_{1_\psi}$ and inasmuch $\tan\left(\frac{2}{3}\frac{\pi}{2}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, that is $\tan\left(\frac{2}{3}\frac{\pi}{2}\right) = \sqrt{3}$,

$$v(\rho,\theta) = C\rho^{\frac{2}{3}}\left(\cos\left(\frac{2}{3}\theta\right) + \sqrt{3}\sin\left(\frac{2}{3}\theta\right)\right). \tag{A.36}$$

Suppose now that $v(\rho,\theta)$ found is Lipschitz, therefore $v(\rho,\theta)$ satisfies

$$|v(\rho_1,\theta_1) - v(\rho_2,\theta_2)| \leq L\,|(\rho_1,\theta_1) - (\rho_2,\theta_2)|, \quad \forall(\rho_1,\theta_1),(\rho_2,\theta_2) \in \Gamma(\theta_0,e_2) \cup \partial\Gamma(\theta_0,e_2). \tag{A.37}$$

In particular, if we take $\theta_1 = \theta_2 = 0$, $\rho_2 = 0$ and $\rho_1 = t$, with $t > 0$, we have $(t, 0)$ and $(0, 0) \in \Gamma(\theta_0, e_2) \cup \partial\Gamma(\theta_0, e_2)$, recalling that $\theta_0 = \frac{3}{4}\pi$ and hence, we achieve from (A.37)

$$|v(t,0) - v(0,0)| \le L\,|(t,0) - (0,0)| = L\,|t|\,,$$

in other words, because $t > 0$, and thus $|t| = t$,

$$|v(t,0) - v(0,0)| \le Lt. \tag{A.38}$$

Let us analyze $|v(t,0) - v(0,0)|$ and we remark that for (A.36), $v(0,0) = 0$ and

$$v(t,0) = Ct^{\frac{2}{3}},$$

as a consequence, always since $t > 0$,

$$|v(t,0) - v(0,0)| = \left|Ct^{\frac{2}{3}}\right| = |C|\,t^{\frac{2}{3}},$$

which entails from (A.38)

$$|C|\,t^{\frac{2}{3}} \le Lt, \quad \forall t > 0,$$

and

$$t^{\frac{2}{3}} \le \frac{L}{|C|}t \quad \forall t > 0, \tag{A.39}$$

where we can divide by $|C|$, inasmuch as $v(\rho,\theta) \neq 0$ in $\Gamma(\theta_0, e_2)$ and hence $C \neq 0$.

At this point, dividing by $t > 0$ in (A.39), we get

$$t^{\frac{2}{3}-1} = t^{-\frac{1}{3}} \le \frac{L}{|C|}, \quad \forall t > 0$$

i.e.

$$t^{-\frac{1}{3}} \le \frac{L}{|C|}, \quad \forall t > 0, \tag{A.40}$$

and letting t go to 0, $t^{-\frac{1}{3}} \to \infty$, therefore, seeing as how $\frac{L}{|C|}$ is a positive constant, we can find $\bar{t} > 0$ such that $\bar{t}^{-\frac{1}{3}} > \frac{L}{|C|}$, which gives from (A.40)

$$\frac{L}{|C|} < \bar{t}^{-\frac{1}{3}} \le \frac{L}{|C|},$$

that is

$$\frac{L}{|C|} < \frac{L}{|C|},$$

which is a contradiction, and the contradiction derives from the fact that we have supposed $v(\rho,\theta)$ Lipschitz.
As a result, $v(\rho,\theta)$ is not Lipschitz.
Now, $v(\rho,\theta) = u(\rho\cos(\theta), \rho\sin(\theta))$, and with $\theta_1 = \theta_2 = 0$, $\rho_2 = 0$, $\rho_1 = t$, with $t > 0$, $(\rho_1\cos(\theta_1), \rho_1\sin(\theta_1)) = (t, 0)$ and $(\rho_2\cos(\theta_2), \rho_2\sin(\theta_2)) = (0,0)$, thus repeating the reasoning done to show that $v(\rho,\theta)$ is not Lipschitz, we obtain that $u(\rho\cos(\theta), \rho\sin(\theta))$ is not Lipschitz and returning to the coordinates (x,y) $u(x,y)$ is not Lipschitz.
To sum it up, we have proved that if u is a solution to

$$\begin{cases} \Delta u = f & \text{in } \Gamma(\theta_0, e_2) \\ u = 0 & \text{on } \partial\Gamma(\theta_0, e_2), \end{cases}$$

then u is not necessary Lipschitz, as desired. □

Lemma A.5. *Let $A : \Omega \to \mathbb{S}^n$, where $\mathbb{S}^n$ is the real symmetric $n \times n$ matrix space and Ω is an open set in $\mathbb{R}^n$. Assume that $a_{ij} \in C^{0,\beta}(\Omega)$, $\forall i$, $j = 1, \ldots, n$ and also that $A(x)$ is positive definite $\forall x \in \Omega$, in other words $A(x)\xi \cdot \xi > 0$ $\forall \xi \in \mathbb{R}^n \setminus \{0\}$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if $\forall i$, $j = 1, \ldots, n$ $\|a_{ij} - \delta_{ij}\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} |a_{ij}(x) - \delta_{ij}| \le \varepsilon$, with $0 < \varepsilon \le \bar{\varepsilon}$, then A is uniformly elliptic, that is there exist $0 < \lambda \le \Lambda$ such that*

$$\lambda |\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda |\xi|^2, \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n.$$

Proof. Let us fix $x \in \Omega$ and we write

$$A(x)\xi \cdot \xi = \sum_{i=1}^n (A(x)\xi)_i \xi_i = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x)\xi_j \right) \xi_i = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j,$$

namely

$$A(x)\xi \cdot \xi = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j. \tag{A.41}$$

Let us start thus from $\sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$ and we have

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j &= \sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij}+\delta_{ij})\xi_i\xi_j\\ &= \sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})\xi_i\xi_j + \sum_{i,j=1}^{n}\delta_{ij}\xi_i\xi_j\\ &= \sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})\xi_i\xi_j + \sum_{i=1}^{n}\xi_i^2, \end{aligned} \tag{A.42}$$

inasmuch $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Therefore, given that $\sum\limits_{i=1}^{n}\xi_i^2 = |\xi|^2$, we achieve from (A.42)

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j = \sum_{i,j}(a_{ij}(x)-\delta_{ij})\xi_i\xi_j + |\xi|^2. \tag{A.43}$$

Now, we have by hypothesis $\|a_{ij}-\delta_{ij}\|_{L^\infty(\Omega)} \leq \varepsilon$, for every $i, j = 1, \ldots, n$, hence for the point $x \in \Omega$ fixed, $|a_{ij}(x)-\delta_{ij}| \leq \|a_{ij}-\delta_{ij}\|_{L^\infty(\Omega)} \leq \varepsilon$, in other words $|a_{ij}(x)-\delta_{ij}| \leq \varepsilon$, for every $i, j = 1, \ldots, n$, which gives $-\varepsilon \leq a_{ij}(x)-\delta_{ij} \leq \varepsilon$ for every $i, j = 1, \ldots, n$.
Consequently, if $\xi_i\xi_j \geq 0$, $(a_{ij}(x)-\delta_{ij})\xi_i\xi_j \leq \varepsilon\xi_i\xi_j$ and $(a_{ij}(x)-\delta_{ij})\xi_i\xi_j \geq -\varepsilon\xi_i\xi_j$, whereas if $\xi_i\xi_j < 0$, $(a_{ij}(x)-\delta_{ij})\xi_i\xi_j \geq \varepsilon\xi_i\xi_j$ and $(a_{ij}(x)-\delta_{ij})\xi_i\xi_j \leq -\varepsilon\xi_i\xi_j$.
Thus, using these facts, we get

$$\sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})\xi_i\xi_j \leq \sum_{\substack{i,j\\ \xi_i\xi_j\geq 0}}\varepsilon\xi_i\xi_j + \sum_{\substack{i,j\\ \xi_i\xi_j< 0}}-\varepsilon\xi_i\xi_j = \varepsilon\sum_{\substack{i,j\\ \xi_i\xi_j\geq 0}}\xi_i\xi_j - \varepsilon\sum_{\substack{i,j\\ \xi_i\xi_j< 0}}\xi_i\xi_j,$$

that is

$$\sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})\xi_i\xi_j \leq \varepsilon\sum_{\substack{i,j\\ \xi_i\xi_j\geq 0}}\xi_i\xi_j - \varepsilon\sum_{\substack{i,j\\ \xi_i\xi_j< 0}}\xi_i\xi_j, \tag{A.44}$$

and

$$\sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})\xi_i\xi_j \geq \sum_{\substack{i,j\\ \xi_i\xi_j\geq 0}}-\varepsilon\xi_i\xi_j + \sum_{\substack{i,j\\ \xi_i\xi_j< 0}}\varepsilon\xi_i\xi_j = -\varepsilon\sum_{\substack{i,j\\ \xi_i\xi_j\geq 0}}\xi_i\xi_j + \varepsilon\sum_{\substack{i,j\\ \xi_i\xi_j< 0}}\xi_i\xi_j,$$

i.e.

$$\sum_{i,j=1}^{n} (a_{ij}(x) - \delta_{ij})\xi_i\xi_j \geq -\varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} \xi_i\xi_j + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} \xi_i\xi_j. \tag{A.45}$$

As a consequence, from (A.43) and (A.44), we obtain

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} \xi_i\xi_j - \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} \xi_i\xi_j + |\xi|^2, \tag{A.46}$$

while, from (A.43) and (A.45), we achieve

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq -\varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} \xi_i\xi_j + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} \xi_i\xi_j + |\xi|^2. \tag{A.47}$$

Now, for Cauchy inequality applied to ξ_i, ξ_j, with $i,\, j \in \{1, \ldots, n\}$, we have

$$\xi_i\xi_j \leq \frac{1}{2}\big(\xi_i^2 + \xi_j^2\big), \tag{A.48}$$

and multiplying by -1 this inequality,

$$-\xi_i\xi_j \geq -\frac{1}{2}\big(\xi_i^2 + \xi_j^2\big). \tag{A.49}$$

Furthermore, seeing as how Cauchy inequality holds for every couple of real numbers, we can apply it also to $-\xi_i$ and ξ_j, and we get, since $(-\xi_i)^2 = \xi_i^2$,

$$-\xi_i\xi_j \leq \frac{1}{2}\big(\xi_i^2 + \xi_j^2\big), \tag{A.50}$$

which entails also, multiplying by -1 this inequality,

$$\xi_i\xi_j \geq -\frac{1}{2}\big(\xi_i^2 + \xi_j^2\big). \tag{A.51}$$

Thus, from (A.46), in view of (A.48) and (A.50), we obtain

$$\begin{aligned}
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j &\leq \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} \xi_i\xi_j - \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} \xi_i\xi_j + |\xi|^2 \\
&\leq \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} \frac{1}{2}\big(\xi_i^2 + \xi_j^2\big) + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} -\xi_i\xi_j + |\xi|^2 \\
&\leq \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} \frac{1}{2}\big(\xi_i^2 + \xi_j^2\big) + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} \frac{1}{2}\big(\xi_i^2 + \xi_j^2\big) + |\xi|^2,
\end{aligned}$$

namely

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} \frac{1}{2}\left(\xi_i^2 + \xi_j^2\right) + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} \frac{1}{2}\left(\xi_i^2 + \xi_j^2\right) + |\xi|^2 . \qquad \text{(A.52)}$$

In addition, given that $\varepsilon > 0$, $\frac{1}{2}\left(\xi_i^2 + \xi_j^2\right) \geq 0$, we can increase the two sums in the right term in (A.52) with $\sum_{i,,j=1}^{n} \frac{1}{2}\left(\xi_i^2 + \xi_j^2\right)$, recalling that the couples of indexes i, j are couples of indexes in $\{i, \ldots, n\}$, and hence the number of these couples is smaller than the number of all the couples of indexes i, j in $\{1, \ldots, n\}$. Therefore, from (A.52), we achieve

$$\begin{aligned}
&\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \varepsilon \sum_{i,j=1}^{n} \frac{1}{2}\left(\xi_i^2 + \xi_j^2\right) + \varepsilon \sum_{i,j=1}^{n} \frac{1}{2}\left(\xi_i^2 + \xi_j^2\right) + |\xi|^2 \\
&= 2\varepsilon \frac{1}{2} \sum_{i,j=1}^{n} \left(\xi_i^2 + \xi_j^2\right) + |\xi|^2 = \varepsilon \sum_{i,j=1}^{n} \left(\xi_i^2 + \xi_j^2\right) + |\xi|^2 \\
&= \varepsilon \left(\sum_{i,j=1}^{n} \xi_i^2 + \sum_{i,j=1}^{n} \xi_j^2 \right) + |\xi|^2 = \varepsilon \sum_{i,j=1}^{n} \xi_i^2 + \varepsilon \sum_{i,j=1}^{n} \xi_j^2 + |\xi|^2 \\
&= \varepsilon \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \xi_i^2 \right) + \varepsilon \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \xi_j^2 \right) + |\xi|^2
\end{aligned}$$

that is

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \varepsilon \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \xi_i^2 \right) + \varepsilon \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \xi_j^2 \right) + |\xi|^2 . \qquad \text{(A.53)}$$

Now, we have $\sum_{i=1}^{n} \xi_i^2 = |\xi|^2$ and $\sum_{j=1}^{n} \xi_j^2 = |\xi|^2$, as a consequence, inasmuch $|\xi|^2$ is a constant with respect to i and to j, we get from (A.53)

$$\begin{aligned}
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j &\leq \varepsilon \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \xi_i^2 \right) + \varepsilon \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \xi_j^2 \right) + |\xi|^2 \\
&= \varepsilon \sum_{j=1}^{n} |\xi|^2 + \varepsilon \sum_{i=1}^{n} |\xi|^2 + |\xi|^2 \\
&= \varepsilon n |\xi|^2 + \varepsilon n |\xi|^2 + |\xi|^2 \\
&= (n\varepsilon + n\varepsilon + 1) |\xi|^2 = (2n\varepsilon + 1) |\xi|^2 = \Lambda |\xi|^2 ,
\end{aligned}$$

setting $\Lambda = 2n\varepsilon + 1$, which implies

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda\left|\xi\right|^2. \tag{A.54}$$

Notice that Λ chosen as above satisfies $\Lambda > 0$, inasmuch as $\varepsilon > 0$.
In parallel, in view of (A.49) and (A.51), (A.47) gives

$$\begin{aligned}
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j &\geq -\varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} \xi_i\xi_j + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} \xi_i\xi_j + \left|\xi\right|^2 \\
&\geq \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} -\xi_i\xi_j + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} -\frac{1}{2}\big(\xi_i^2 + \xi_j^2\big) + \left|\xi\right|^2 \\
&\geq \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} -\frac{1}{2}\big(\xi_i^2 + \xi_j^2\big) + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} -\frac{1}{2}\big(\xi_i^2 + \xi_j^2\big) + \left|\xi\right|^2,
\end{aligned}$$

i.e.

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j \geq 0}} -\frac{1}{2}\big(\xi_i^2 + \xi_j^2\big) + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j < 0}} -\frac{1}{2}\big(\xi_i^2 + \xi_j^2\big) + \left|\xi\right|^2 \tag{A.55}$$

and repeating the considerations done to find (A.54), we obtain from (A.55)

$$
\begin{aligned}
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j &\geq \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j\geq 0}} -\frac{1}{2}\big(\xi_i^2+\xi_j^2\big) + \varepsilon \sum_{\substack{i,j \\ \xi_i\xi_j< 0}} -\frac{1}{2}\big(\xi_i^2+\xi_j^2\big) + |\xi|^2 \\
&\geq \varepsilon \sum_{i,j=1}^{n} -\frac{1}{2}\big(\xi_i^2+\xi_j^2\big) + \varepsilon \sum_{i,j=1}^{n} -\frac{1}{2}\big(\xi_i^2+\xi_j^2\big) + |\xi|^2 \\
&= -2\varepsilon\frac{1}{2}\sum_{i,j=1}^{n}\big(\xi_i^2+\xi_j^2\big) + |\xi|^2 = -\varepsilon\sum_{i,j=1}^{n}\big(\xi_i^2+\xi_j^2\big) + |\xi|^2 \\
&= -\varepsilon\left(\sum_{i,j=1}^{n}\xi_i^2 + \sum_{ij=1}^{n}\xi_j^2\right) + |\xi|^2 = -\varepsilon\sum_{i,j=1}^{n}\xi_i^2 - \varepsilon\sum_{i,j=1}^{n}\xi_j^2 + |\xi|^2 \\
&= -\varepsilon\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\xi_i^2\right) - \varepsilon\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\xi_j^2\right) + |\xi|^2 \\
&= -\varepsilon\sum_{j=1}^{n}|\xi|^2 - \varepsilon\sum_{i=1}^{n}|\xi|^2 + |\xi|^2 \\
&= -\varepsilon n\,|\xi|^2 - \varepsilon n\,|\xi|^2 + |\xi|^2 = (1-2n\varepsilon)\,|\xi|^2 = \lambda\,|\xi|^2\,,
\end{aligned}
$$

setting $\lambda = 1-2n\varepsilon$, which entails

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda\,|\xi|^2\,. \tag{A.56}$$

Notice that λ, established as above, satisfies $\lambda > 0$ if and only if $1-2n\varepsilon > 0$, that is $\varepsilon < \frac{1}{2n}$ and hence we can choice the universal constant $\bar{\varepsilon}$ as, for instance, $\bar{\varepsilon} = \frac{1}{4n}$. So, if we take $0 < \varepsilon \leq \bar{\varepsilon}$, $\lambda > 0$, recalling that $\varepsilon \leq \frac{1}{4n} < \frac{1}{n}$, namely $\varepsilon < \frac{1}{2n}$. In addition, we have also $\Lambda > 0$ and $\Lambda = 1+2n\varepsilon > 1-2n\varepsilon = \lambda$, therefore from (A.54), (A.56) and (A.41), we obtain

$$\lambda\,|\xi|^2 \leq A(x)\xi\cdot\xi \leq \Lambda\,|\xi|^2\,, \quad \forall x\in\Omega, \quad \forall \xi\in\mathbb{R}^n,$$

with $0 < \lambda \leq \Lambda$, i.e. A is uniformly elliptic, as desired. $\square$

Appendix B

Viscosity solutions: a basic introduction

We recall the basic definition of viscosity solution for elliptic partial differential equations. An exhaustive source for this subject it can be found in the following classical papers: [9] and [10]. We refer to them for further details.

Definition B.1. Let Ω be an open set in $\mathbb{R}^n$. We define:

(i) $\mathrm{usc}(\Omega) := \big\{\varphi : \Omega \to \mathbb{R} \mid \varphi$ is upper semicontinuous in Ω and φ is upper bounded$\big\}$, where φ is upper semicontinuous in Ω if

$$\lim_{r\to 0}\left(\sup_{y\,\in\, B_r(x)\,\setminus\,\{x\}} u(y)\right) \leq u(x), \quad \forall x \in \Omega;$$

(ii) $\mathrm{lsc}(\Omega) := \big\{\varphi : \Omega \to \mathbb{R} \mid \varphi$ is lower semicontinuous in Ω and φ is lower bounded$\big\}$, where φ is lower semicontinuous in Ω if

$$\lim_{r\to 0}\left(\inf_{y\,\in\, B_r(x)\,\setminus\,\{x\}} u(y)\right) \geq u(x), \quad \forall x \in \Omega.$$

We now introduce the operators, for which we will provide the definition of viscosity solution.

Definition B.2. Let $F : \mathbb{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a continuous function, where $\mathbb{S}^n$ is the real symmetric $n \times n$ matrix space and Ω is an open set in $\mathbb{R}^n$ and suppose that F satisfies:

(i) decreasing monotonicity in s, that is $\forall r,\ s \in \mathbb{R}$, $\forall M \in \mathbb{S}^n$, $\forall p \in \mathbb{R}^n$, $\forall x \in \Omega$, if $s \le r$, then $F(M, p, r, x) \le F(M, p, s, x)$;

(ii) elliptic degeneracy (monotonicity in M), i.e. $\forall M,\ N \in \mathbb{S}^n$, $\forall p \in \mathbb{R}^n$, $\forall r \in \mathbb{R}$, $\forall x \in \Omega$, if $M \le N$, then $F(M, p, r, x) \le F(N, p, r, x)$. Recall that $M \le N$ if $N - M \ge 0$, in other words $(N - M)\xi \cdot \xi \ge 0\ \forall \xi \in \mathbb{R}^n$.

Definition B.3 (Viscosity subsolution). Let F be as in Definition B.2 and $u \in \mathrm{usc}(\Omega)$. We say that u is a *viscosity subsolution* of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, if $\forall x_0 \in \Omega$, $\forall \varphi \in C^2(\Omega)$, if $u - \varphi$ realizes a local maximum at x_0, then

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \ge 0.$$

Recall that $u - \varphi$ realizes a local maximum at x_0 if there exists a neighborhood of x_0 where $u - \varphi$ has a maximum at x_0.

Definition B.4 (Viscosity supersolution). Let F be as in Definition B.2 and let $u \in \mathrm{lsc}(\Omega)$. We say that u is a *viscosity supersolution* of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, if $\forall x_0 \in \Omega$, $\forall \varphi \in C^2(\Omega)$, if $u - \varphi$ realizes a local minimum at x_0, then

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \le 0.$$

Recall that $u - \varphi$ realizes a local minimum at x_0 if there exists a neighborhood of x_0 where $u - \varphi$ has a minimum at x_0.

Definition B.5 (Viscosity solution). Let $u \in C(\Omega)$ and let F be as in Definition B.2. We say that u is a *viscosity solution* of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, if u is both a viscosity subsolution and a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω.

We provide now other definitions of viscosity subsolution and supersolution for this kind of equations and we prove the equivalence of these definitions and those given before.

Definition B.6 (**Definition of superjet of second order of u in Ω**)**.**
Let $(p, X) \in \mathbb{R}^n \times \mathbb{S}^n$ such that

$$u(y) \leq u(x) + p \cdot (y - x) + \frac{1}{2} X(y - x) \cdot (y - x) + o(|y - x|^2).$$

In this case, we say that (p, X) belongs to the superjet of second order of u in Ω, which is denoted as $J_{\Omega}^{2,+}u(x)$ at point x.

Definition B.7 (**Definition of subjet of second order of u in Ω**)**.**
We define the subjet of second order of u in Ω as

$$\begin{aligned} J_{\Omega}^{2,-}u(x) := \Big\{ &(p, X) \in \mathbb{R}^n \times \mathbb{S}^n \ | \ u(y) \geq u(x) + p \cdot (y - x) \\ &+ \frac{1}{2} X(y - x) \cdot (y - x) + o(|y - x|^2) \Big\}. \end{aligned}$$

Definition B.8 (**Viscosity subsolution using superjet $J_{\Omega}^{2,+}u(x)$**)**.** Let $u \in \mathrm{usc}(\Omega)$ and F as in Definition B.2. If $\forall x \in \Omega$, $\forall (p, X) \in J_{\Omega}^{2,+}u(x)$, it is satisfied

$$F(X, p, u(x), x) \geq 0,$$

then we call u *viscosity subsolution* of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω.

Definition B.9 (**Viscosity supersolution using subjet $J_{\Omega}^{2,-}u(x)$**)**.** Let $u \in \mathrm{lsc}(\Omega)$ and F as in Definition B.2. If $\forall x \in \Omega$, $\forall (p, X) \in J_{\Omega}^{2,-}u(x)$, it is satisfied

$$F(X, p, u(x), x) \leq 0,$$

then we define u *viscosity supersolution* of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω.

Theorem B.10.

(i) *Let* $u \in \mathrm{usc}(\Omega)$ *and* F *as in Definition B.2. Then,* u *is a viscosity subsolution of* $F(D^2u(x), \nabla u(x), u(x), x) = 0$ *in* Ω *if and only if* u *is a viscosity subsolution of* $F(D^2u(x), \nabla u(x), u(x), x) = 0$ *in* Ω *in the sense of the superjet* $J^{2,+}_\Omega u(x)$

(ii) *Let* $u \in \mathrm{lsc}(\Omega)$*. Then,* u *is a viscosity supersolution of* $F(D^2u(x), \nabla u(x), u(x), x) = 0$ *in* Ω *if and only if* u *is a viscosity supersolution of* $F(D^2u(x), \nabla u(x), u(x), x) = 0$ *in* Ω *in the sense of the subjet* $J^{2,-}_\Omega u(x)$.

Proof. Suppose that u is a viscosity subsolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω in the sense of the superjet $J^{2,+}_\Omega u(x)$. Assume also that $u - \varphi$ realizes a local maximum at $x_0 \in \Omega$ with $\varphi \in C^2(\Omega)$. Then, there exists a neighborhood O of x_0 such that

$$u(x) - \varphi(x) \leq u(x_0) - \varphi(x_0) \quad \text{in } O,$$

which implies

$$u(x) \leq u(x_0) - \varphi(x_0) + \varphi(x) \quad \text{in } O. \tag{B.1}$$

In addition, we can write φ with the Taylor expansion around x_0 in O and we obtain from (B.1)

$$\begin{aligned} u(x) &\leq u(x_0) - \varphi(x_0) + \varphi(x_0) + \nabla\varphi(x_0) \cdot (x - x_0) \\ &\quad + \frac{1}{2} D^2\varphi(x_0)(x - x_0) \cdot (x - x_0) + o(|x - x_0|^2) \\ &= u(x_0) + \nabla\varphi(x_0) \cdot (x - x_0) + \frac{1}{2} D^2\varphi(x_0)(x - x_0) \cdot (x - x_0) \\ &\quad + o(|x - x_0|^2) \quad \text{in } O, \end{aligned}$$

namely

$$u(x) \leq u(x_0) + \nabla\varphi(x_0) \cdot (x - x_0) + \frac{1}{2} D^2\varphi(x_0)(x - x_0) \cdot (x - x_0) + o(|x - x_0|^2) \quad \text{in } O.$$

Consequently, for the definition of $J^{2,+}_\Omega u(x_0)$ and inasmuch as $D^2\varphi(x_0)$ is a symmetric matrix, recalling that $u \in C^2(\Omega)$, we have that $(\nabla\varphi(x_0), D^2\varphi(x_0))$

belongs to $J^{2,+}_{\Omega}u(x_0)$, and thus, since u is a viscosity subsolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω in the sense of $J^{2,+}_{\Omega}u(x)$ and $x_0 \in \Omega$, we get

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \geq 0.$$

For the arbitrariness of $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ realizes a local maximum at x_0, we achieve that u is a viscosity subsolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω. Conversely, suppose that u is a viscosity subsolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω. Let us fix $x_0 \in \Omega$ and we take $(p, X) \in J^{2,+}_{\Omega}u(x_0)$. Then, we have

$$u(x) \leq u(x_0)+p\cdot(x-x_0)+\frac{1}{2}X(x-x_0)\cdot(x-x_0)+o(|x-x_0|^2), \quad \text{with } x \in \Omega. \tag{B.2}$$

Now, for definition of $o(|x-x_0|^2)$, $\forall\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\left|o(|x-x_0|^2)\right| \leq \varepsilon \left|x-x_0\right|^2, \quad \forall x \in \Omega \cap B_{\delta_\varepsilon}(x_0),$$

as a consequence, from (B.2), recalling that $o(|x-x_0|^2) \leq \left|o(|x-x_0|^2)\right|$, we obtain that $\forall\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$u(x) \leq u(x_0)+p\cdot(x-x_0)+\frac{1}{2}X(x-x_0)\cdot(x-x_0)+\varepsilon\left|x-x_0\right|^2, \ \forall x \in \Omega\cap B_{\delta_\varepsilon}(x_0). \tag{B.3}$$

Therefore, if we call

$$\varphi_\varepsilon(x) := u(x_0) + p\cdot(x-x_0) + \frac{1}{2}X(x-x_0)(x-x_0) + \varepsilon\left|x-x_0\right|^2,$$

we get from (B.3)

$$u(x) - \varphi_\varepsilon(x) \leq 0, \quad x \in \Omega \cap B_{\delta_\varepsilon}(x_0). \tag{B.4}$$

Moreover,

$$\varphi_\varepsilon(x_0) = u(x_0),$$

which entails from (B.4)

$$u(x_0) - \varphi_\varepsilon(x_0) = 0 \geq u(x) - \varphi_\varepsilon(x), \quad x \in \Omega \cap B_{\delta_\varepsilon}(x_0),$$

that is $u - \varphi_\varepsilon$ realizes a local maximum at $x_0 \in \Omega$.
Furthermore, notice that $\varphi_\varepsilon \in C^2(\Omega)$, hence, given that u is a viscosity subsolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, we achieve

$$F(D^2\varphi_\varepsilon(x_0), \nabla\varphi_\varepsilon(x_0), u(x_0), x_0) \geq 0. \tag{B.5}$$

Let us calculate, at this point, $\nabla\varphi_\varepsilon(x_0)$ and $D^2\varphi_\varepsilon(x_0)$.
In particular, we have

$$\begin{aligned}
\frac{\partial\varphi_\varepsilon}{\partial x_i}(x) &= \frac{\partial}{\partial x_i}\left(u(x_0) + p\cdot(x-x_0) + \frac{1}{2}X(x-x_0)\cdot(x-x_0) + \varepsilon\,|x-x_0|^2\right) \\
&= \frac{\partial}{\partial x_i}\left(u(x_0) + \sum_{i=1}^{n} p_i(x_i - x_{0_i}) + \frac{1}{2}\sum_{i,j=1}^{n} X_{ij}(x_i - x_{0_i})(x_j - x_{0_j}) \right.\\
&\left. + \varepsilon\sum_{i=1}^{n}(x_i - x_{0_i})^2\right) \\
&= p_i + \frac{1}{2}2X_{ii}(x_i - x_{0_i}) + \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n}(X_{ij} + X_{ji})(x_j - x_{0_j}) + \varepsilon 2(x_i - x_{0_i})
\end{aligned}$$

which entails evaluating this equality in x_0,

$$\frac{\partial\varphi_\varepsilon}{\partial x_i}(x_0) = p_i,$$

in other words,

$$\nabla\varphi_\varepsilon(x_0) = p. \tag{B.6}$$

From the calculus to find (B.6), we have obtained

$$\frac{\partial\varphi_\varepsilon}{\partial x_i}(x) = p_i + X_{ii}(x_i - x_{0_i}) + \sum_{\substack{j=1 \\ j\neq i}}^{n} X_{ji}(x_j - x_{0_j}) + 2\varepsilon(x_i - x_{0_i}), \quad \forall i = 1, \ldots, n, \tag{B.7}$$

seeing as how the matrix X is symmetric and thus $X_{ji} + X_{ij} = 2X_{ij}$, for every j.

Consequently, from (B.7), we get

$$\begin{aligned}\frac{\partial^2\varphi_\varepsilon}{\partial x_j\partial x_i}(x_0) &= \frac{\partial}{\partial x_j}\left(p_i + X_{ii}(x_i - x_{0_i}) + \sum_{\substack{h=1\\h\neq i}}^{n} X_{hi}(x_h - x_{0_h}) + 2\varepsilon(x_i - x_{0_i})\right)(x_0)\\ &= \big(X_{ii}\delta_{ij} + X_{ji}(1-\delta_{ij}) + 2\varepsilon\delta_{ij}\big)(x_0)\\ &= X_{ii}\delta_{ij} + X_{ji}(1-\delta_{ij}) + 2\varepsilon\delta_{ij},\end{aligned}$$

which gives

$$\frac{\partial^2\varphi_\varepsilon}{\partial x_j\partial x_i}(x_0) = X_{ii}\delta_{ij} + X_{ji}(1-\delta_{ij}) + 2\varepsilon\delta_{ij}, \tag{B.8}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j\\ 0 & \text{if } i \neq j.\end{cases}$$

Therefore, from (B.8), we achieve

$$D^2\varphi_\varepsilon(x_0) = X + 2\varepsilon I. \tag{B.9}$$

As a consequence, substituting (B.6) and (B.9) in (B.5), we achieve

$$F(X + 2\varepsilon I, p, u(x_0), x_0) \geq 0,$$

and inasmuch F is continuous for hypotheses, letting ε go to 0, we obtain

$$F(X, p, u(x_0), x_0) \geq 0,$$

hence for arbitrariness of $x_0 \in \Omega$ and $(X,p) \in J^{2,+}_\Omega u(x_0)$, we get that u is a viscosity solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω in the sense of the superjet $J^{2,+}_\Omega u(x)$.

Suppose now, instead, that u is a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω in the sense of the subjet $J^{2,-}_\Omega u(x)$. Let us fix thus $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ realizes a local minimum at x_0, then there exists a neighborhood O of x_0, where $u(x) - \varphi(x) \geq u(x_0) - \varphi(x_0)$, in O, and using the Taylor expansion of φ around x_0, we obtain with the same steps done in case of u viscosity subsolution, but with opposite inequalities,

$$u(x) \geq u(x_0) + \nabla\varphi(x_0)\cdot(x - x_0) + \frac{1}{2}D^2\varphi(x_0)(x - x_0)\cdot(x - x_0) + o(|x - x_0|^2) \quad \text{in } O,$$

therefore for the definition of $J_\Omega^{2,-}u(x_0)$ and since $D^2\varphi(x_0)$ is a symmetric matrix, recalling that $\varphi \in C^2(\Omega)$, we have that $(\nabla\varphi(x_0), D^2\varphi(x_0)$ belongs to $J_\Omega^{2,-}u(x_0)$ and hence, given that u is a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω in the sense of the subjet $J_\Omega^{2,-}u(x)$ and $x_0 \in \Omega$, we achieve

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \leq 0.$$

For the arbitrariness of $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ realizes a local minimum at x_0, we get that u is a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω.

Conversely, suppose that u is a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω. Let us fix $x_0 \in \Omega$ and $(p, X) \in J_\Omega^{2,-}u(x_0)$, then we have

$$u(x) \geq u(x_0) + p \cdot (x - x_0) + \frac{1}{2}X(x - x_0) \cdot (x - x_0) + o(|x - x_0|^2), \quad x \in \Omega,$$

and repeating the considerations done in case of u viscosity subsolution, we obtain, seeing as how if $|o(|x - x_0|)^2| \leq \varepsilon |x - x_0|^2$, $o(|x - x_0|^2 \geq -\varepsilon |x - x_0|^2$ and thus $-o(|x - x_0|^2 \leq \varepsilon |x - x_0|^2$,

$$u(x) - \varphi_\varepsilon(x) \geq 0, \quad \forall x \in \Omega \cap B_{\delta_\varepsilon}(x_0), \tag{B.10}$$

where

$$\varphi_\varepsilon(x) := u(x_0) + p \cdot (x - x_0) + \frac{1}{2}X(x - x_0)(x - x_0) - \varepsilon |x - x_0|^2.$$

Furthermore,

$$\varphi_\varepsilon(x_0) = u(x_0),$$

therefore from (B.10), we achieve

$$u(x) - \varphi_\varepsilon(x) \leq u(x_0) - \varphi_\varepsilon(x_0) = 0, \quad \forall x \in \Omega \cap B_{\delta_\varepsilon}(x_0),$$

i.e. $u - \varphi_\varepsilon$ realizes a local minimum at $x_0 \in \Omega$.

In addition, we remark that $\varphi_\varepsilon \in C^2(\Omega)$, hence, inasmuch u is a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, we get

$$F(D^2\varphi_\varepsilon(x_0), \nabla\varphi_\varepsilon(x_0), u(x_0), x_0) \leq 0,$$

and repeating the calculus done in case of u viscosity subsolution, with $-\varepsilon$ in place of ε,

$$F(X - 2\varepsilon I, p, u(x_0), x_0) \leq 0. \tag{B.11}$$

Now, letting ε go to 0 in (B.11), we obtain

$$F(x, p, u(x_0), x_0) \leq 0,$$

which implies, for the arbitrariness of $x_0 \in \Omega$ and $(p, X) \in J_\Omega^{2,-}u(x_0)$, that u is a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω in the sense of the subjet $J_\Omega^{2,-}u(x)$. $\square$

We show, at this point, the equivalence of classical solution of $F = 0$ and viscosity solution of $F = 0$ under certain conditions, where F is as in Definition B.3.

Lemma B.11. *Let F be as in Definition B.2 and let $u \in C^2(\Omega)$. u is a classical solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, if and only if u is a viscosity solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω.*

Proof. Suppose that u is a classical solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, then $F(D^2u(x), \nabla u(x), u(x), x) = 0$ $\forall x \in \Omega$. To prove that u is also a viscosity solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, we need to show that u is both a viscosity subsolution and a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω. For this purpose, let now $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$, such that $u - \varphi$ realizes a local maximum at x_0. Then, given that x_0 is a local maximum for $u - \varphi$, we have:

(i) $D^2(u - \varphi)(x_0) \leq 0$;

(ii) $\nabla(u - \varphi)(x_0) = 0$.

In addition, we know that, in view of the linearity of the partial derivative, $D^2(u - \varphi)(x_0) = D^2u(x_0) - D^2\varphi(x_0)$, as a consequence from (i), we achieve $D^2u(x_0) \leq D^2\varphi(x_0)$. Analogously, $\nabla(u - \varphi)(x_0) = \nabla u(x_0) - \nabla\varphi(x_0)$, hence from (ii), we get $\nabla u(x_0) = \nabla\varphi(x_0)$. To sum it up, we have $D^2u(x_0) \leq$

$D^2\varphi(x_0)$ and $\nabla u(x_0) = \nabla\varphi(x_0)$. Now, seeing as how $D^2u(x_0)$ and $D^2\varphi(x_0)$ $\in \mathbb{S}^n$ for Schwarz's theorem, recalling that $u \in C^2(\Omega)$ and $\varphi \in C^2(\Omega)$, we can use the elliptic degeneracy of F, and

$$F(D^2u(x_0), \nabla\varphi(x_0), u(x_0), x_0) \leq F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0). \quad \text{(B.12)}$$

Furthermore, since $x_0 \in \Omega$ and u is a classical solution of
$F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω,

$$F(D^2u(x_0), \nabla u(x_0), u(x_0), x_0) = 0, \quad \text{(B.13)}$$

thus inasmuch $\nabla\varphi(x_0) = \nabla u(x_0)$, we have from (B.12) and (B.13)

$$0 = F(D^2u(x_0), \nabla u(x_0), u(x_0), x_0) \leq F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0),$$

namely

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \geq 0.$$

Consequently, for the arbitrariness of $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$, such that $u-\varphi$ realizes a local maximum at x_0, and inasmuch as if $u \in C^2(\Omega)$, $u \in \text{usc}(\Omega)$, we obtain that u is a viscosity subsolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω.

Consider now always $x_0 \in \Omega$, and we take $\varphi \in C^2(\Omega)$ such that $u-\varphi$ realizes a local minimum at x_0. In this case, given that x_0 is a local minimum for $u - \varphi$, we have:

(i) $D^2(u - \varphi)(x_0) \geq 0$;

(ii) $\nabla(u - \varphi)(x_0) = 0$.

Repeating the reasoning done for the case when $u - \varphi$ realizes a local maximum at x_0, we achieve that $D^2u(x_0) \geq D^2\varphi(x_0)$ and $\nabla u(x_0) = \nabla\varphi(x_0)$. Therefore, using the elliptic degeneracy of F and the considerations done to show that u is a viscosity subsolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, we get

$$F(D^2u(x_0), \nabla\varphi(x_0), u(x_0), x_0) \geq F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0). \quad \text{(B.14)}$$

Moreover, also in this case, $x_0 \in \Omega$ and u is a classical solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, hence

$$F(D^2u(x_0), \nabla u(x_0), u(x_0), x_0) = 0, \tag{B.15}$$

and as a consequence, seeing as how $\nabla u(x_0) = \nabla\varphi(x_0)$, from (B.14) and (B.15) we obtain

$$0 = F(D^2u(x_0), \nabla u(x_0), u(x_0), x_0) \geq F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0),$$

i.e.

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \leq 0.$$

Consequently, for the arbitrariness of $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$, such that $u-\varphi$ realizes a local minimum at x_0, and inasmuch if $u \in C^2(\Omega)$, $u \in \mathrm{lsc}(\Omega)$, we achieve that u is a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω.

To sum it up, we have proved that u is both a viscosity subsolution and a viscosity supersolution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, so, because $u \in C(\Omega)$, if $u \in C^2(\Omega)$, we get that u is a viscosity solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω.

Conversely, suppose that u is a viscosity solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω and we want to prove that u is also a classical solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω.

For this purpose, let us fix $x_0 \in \Omega$, and inasmuch as $u \in C^2(\Omega)$, we can write the Taylor expansion of u around x_0 in a neighborhood O of x_0, $O \subset \Omega$, and we obtain

$$\begin{aligned} u(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + \frac{1}{2} D^2u(x_0)(x - x_0) \cdot (x - x_0) \\ + o(|x - x_0|^2), \quad x \in O. \end{aligned} \tag{B.16}$$

In particular, for the definition of $J^{2,+}_{\Omega}u(x_0)$ and given that $D^2u(x_0)$ is a symmetric matrix, recalling that $u \in C^2(\Omega)$, we achieve from (B.16) that $(\nabla u(x_0), D^2u(x_0)) \in J^{2,+}_{\Omega}u(x_0)$.

Therefore, recalling that u is a viscosity solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, and thus also a viscosity subsolution, we get, from the equivalence of two definitions of viscosity subsolution shown in Theorem B.10,

$$F(D^2u(x_0), \nabla u(x_0), u(x_0), x_0) \geq 0. \tag{B.17}$$

On the other hand, from (B.16), we also obtain, for the definition of $J^{2,-}_{\Omega}u(x_0)$ and always since $D^2u(x_0)$ is a symmetric matrix, that $(\nabla u(x_0), D^2u(x_0)) \in J^{2,-}_{\Omega}u(x_0)$.
Consequently, because u is a viscosity solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, and hence in particular a viscosity supersolution, we achieve, from the equivalence of two definitions of viscosity supersolution shown in Theorem B.10,

$$F(D^2u(x_0), \nabla u(x_0), u(x_0), x_0) \leq 0. \tag{B.18}$$

Putting together (B.17) and (B.18), we have

$$0 \leq F(D^2u(x_0), \nabla u(x_0), u(x_0), x_0) \leq 0,$$

which entails

$$F(D^2u(x_0), \nabla u(x_0), u(x_0), x_0) = 0,$$

and, from the arbitrariness of $x_0 \in \Omega$, we conclude that u is a classical solution of $F(D^2u(x), \nabla u(x), u(x), x) = 0$ in Ω, as desired. $\square$

Appendix C

The Harnack inequality for elliptic operators

We recall here the classical Harnack inequality for uniformly elliptic operators in non-divergence form. We also cite two other theorems, from which the classical Harnack inequality follows as a corollary. For proofs and further details, see [20].
First of all, we introduce the operators for which we state the classical Harnack inequality.
Specifically, we deal with operators in the general form:

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} + \sum_{i=1}^{n} b_i(x)u_i + c(x)u, \tag{C.1}$$

with coefficients a_{ij}, b_i, c, where i, $j = 1, \ldots, n$ defined on an open connected set Ω in $\mathbb{R}^n$.
In particular, we assume that $(a_{ij}(x))_{i,j}$ is a symmetric matrix $\forall x \in \Omega$, and if we call A the matrix-valued function such that $A(x) = (a_{ij}(x))_{i,j}$, we suppose in our case that A is uniformly elliptic, i.e. there exist $0 < \lambda \leq \Lambda$ such that

$$\lambda \left|\xi\right|^2 \leq A(x)\xi \cdot \xi \leq \Lambda \left|\xi\right|^2, \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n.$$

In addition, we also suppose that b_i and c are bounded in Ω, and accordingly, we fix a constant ν such that

$$\left(\frac{|b|}{\lambda}\right)^2, \frac{|c|}{\lambda} \leq \nu.$$

At this point, we also need to introduce the notion of solution, for which the classical Harnack inequality is satisfied.

Definition C.1 (Weak derivative). Let Ω be an open connected set in $\mathbb{R}^n$, $u \in L^1_{loc}(\Omega)$ and α any multi-index. Then a locally integrable function v is called the α^{th} *weak derivative* of u if it satisfies

$$\int_\Omega \varphi v \, dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi \, dx, \quad \text{for all } \varphi \in C_0^{|\alpha|}(\Omega).$$

We write $v = D^\alpha u$ and we notice that $D^\alpha u$ is uniquely determined up to sets of measure zero.

Remark. Let us recall that we say α is a multi-index if

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n),$$

where $\alpha_i \in \mathbb{N} \cup 0$, $\forall i = 1, \ldots, n$, and we denote $|\alpha|$ with

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

Moreover, with $D^\alpha \varphi$ we refer to

$$D^\alpha \varphi = \frac{1}{|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \varphi.$$

Definition C.2 (Weakly differentiable). Let $u \in L^1_{loc}(\Omega)$, with Ω as in Definition C.1. We say that u is *weakly differentiable* if all its weak derivatives of first order exist and k *times weakly differentiable* if all its weak derivatives exist for orders up to and including k.

Definition C.3 (Sobolev spaces).
Let Ω be as in Definition C.1. Let us denote by $W^k(\Omega)$ the linear space of k times weakly differentiable functions in Ω.
In addition, for $p \geq 1$ and k non-negative integer, we define

$$W^{k,p}(\Omega) := \left\{ u \in W^k(\Omega);\; D^\alpha u \in L^p(\Omega) \;\forall \alpha, \text{ with } |\alpha| \leq k \right\}.$$

Definition C.4 (**Strong solution**).
Let be given an equation of the form

$$Lu = f \quad \text{in } \Omega, \tag{C.2}$$

where Ω is as in Definition C.1, L is an operator of the type introduced in (C.1) and f is a function on Ω.
We say that $u \in W^2(\Omega)$ is a *strong solution* of (C.2) if u satisfies C.2 almost everywhere in Ω.

Remark. Notice that also when we will write the inequality $Lu \geq f$ and $Lu \leq f$ in the following theorems, they will be considered satisfied almost everywhere.

Theorem C.5. *Let Ω as in Definition C.1 and $u \in W^{2,n}(\Omega)$. Suppose also that $Lu \geq f$ in Ω, where $f \in L^n(\Omega)$ and L is an operator of the type introduced in (C.1). Then for any ball $B = B_{2R}(y) \subset \Omega$ and $p > 0$, we have*

$$\sup_{B_R(y)} u \leq C \left\{ \left(\frac{1}{|B|} \int_B (u^+)^p \right)^{1/p} + \frac{R}{\lambda} \|f\|_{L^n(B)} \right\},$$

with $C = C\left(n, \frac{\Lambda}{\lambda}, \nu R^2, p\right)$.

Theorem C.6. *Let Ω as in Definition C.1 and $u \in W^{2,n}(\Omega)$. Suppose that u satisfies $Lu \leq f$ in Ω, where $f \in L^n(\Omega)$ and L is an operator of the type introduced in (C.1). Suppose also that u is non-negative in a ball $B = B_{2R}(y) \subset \Omega$. Then*

$$\left(\frac{1}{|B_R(y)|} \int_{B_R(y)} u^p \right)^{1/p} \leq C \left(\inf_{B_R(y)} u + \frac{R}{\lambda} \|f\|_{L^n(B)} \right),$$

where p and C are positive constants depending only on n, $\frac{\Lambda}{\lambda}$ and νR^2.

Consequently, from (C.5) and (C.6), we can obtain the classical Harnack inequality.

Theorem C.7 (**Classical Harnack inequality**).
Let Ω as in Definition C.1 and $u \in W^{2,n}(\Omega)$. Suppose that u satisfies $Lu = f$ in Ω, where $f \in L^n(\Omega)$ and L is an operator of the type introduced in (C.1). Suppose also that $u \geq 0$ in Ω. Then for any ball $B_{2r}(y) \subset \Omega$, we have

$$\sup_{B_R(y)} u \leq C_1 \left(\inf_{B_R(y)} u + C_2 \left\| f \right\|_{L^n(\Omega)} \right),$$

where C_1 and C_2 are positive constants depending only on n, $\frac{\Lambda}{\lambda}$ and νR^2.

Harnack inequality also holds for fully nonlinear operators, see [7]. For the sake of simplicity, we restrict ourselves to the particular case of uniformly elliptic operators. Specifically, we consider operators of the type:

$$F : \mathbb{S}^n \times \Omega \to \mathbb{R}, \tag{C.3}$$

where Ω is a bounded open connected set in $\mathbb{R}^n$ and $\mathbb{S}^n$ is the space of real $n \times n$ symmetric matrices. In addition, we assume that F is a uniformly elliptic operator, that is,

Definition C.8. F is uniformly elliptic if there are two positive constants $\lambda \leq \Lambda$ (called ellipticity constants) such that $\forall M \in S^n$ and $\forall x \in \Omega$

$$\lambda \left\| N \right\| \leq F(M+N, x) - F(M, x) \leq \Lambda \left\| N \right\| \quad \forall N \geq 0,$$

where we write $N \geq 0$ whenever N is a non-negative definite symmetric matrix. $\|M\|$ denotes the (L^2, L^2)-norm of M (i.e., $\|M\| = \sup_{|x|=1} |Mx|$); therefore $\|N\|$ is equal to the maximum eigenvalue of N whenever $N \geq 0$.

At this point, we need to introduce *Pucci's extremal operators* to state the Harnack inequality.

Definition C.9 (**Pucci's extremal operators.**).
Let $0 < \lambda \leq \Lambda$. For $M \in \mathbb{S}^n$, we define

$$\mathcal{M}^-(M, \lambda, \Lambda) = \mathcal{M}^-(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$$

$$\mathcal{M}^+(M, \lambda, \Lambda) = \mathcal{M}^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $e_i = e_i(M)$ are the eigenvalues of M.

Remark. In particular, let now A be a symmetric matrix whose eigenvalues belong to $[\lambda, \Lambda]$, namely $\lambda |\xi|^2 \le A\xi \cdot \xi \le \Lambda |\xi|^2$ for any $\xi \in \mathbb{R}^n$. We will write in this case that $A \in \mathcal{A}_{\lambda,\Lambda}$.
Define a linear functional L_A on $\mathbb{S}^n$ by

$$L_A M = \operatorname{tr}(AM) = \sum_{i,j=1}^{n} A_{ij} M_{ji} = \sum_{i,j=1}^{n} A_{ij} M_{ij}, \quad M \in \mathbb{S}^n.$$

Since $M \in \mathbb{S}^n$, we have $M = ODO^t$ where $D_{ij} = e_i \delta_{ij}$ (e_i are the eigenvalues of M) and O is an orthogonal matrix, and it proves that

$$\mathcal{M}^-(M, \lambda, \Lambda) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} L_A M$$
$$\mathcal{M}^+(M, \lambda, \Lambda) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} L_A M.$$

We now define the class of functions for which the Harnack inequality holds.

Definition C.10. Let f be a continuous function in Ω, with Ω as in Definition C.3, and let $\lambda \le \Lambda$ be two positive constants. We denote by $\underline{S}(\lambda, \Lambda, f)$ the space of continuous functions u in Ω such that $\mathcal{M}^+(D^2u, \lambda, \Lambda) \ge f(x)$ in the viscosity sense in Ω, in other words if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ and $u - \varphi$ realizes a local maximum at x_0 then

$$\mathcal{M}^+(D^2\varphi(x_0), \lambda, \Lambda) \ge f(x_0).$$

Definition C.11. Let f be a continuous function in Ω, with Ω as in Definition C.3, and let $\lambda \le \Lambda$ be two positive constants. We denote by $\overline{S}(\lambda, \Lambda, f)$ the space of continuous functions u in Ω such that $\mathcal{M}^-(D^2u, \lambda, \Lambda) \le f(x)$ in the viscosity sense in Ω, that is if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ and $u - \varphi$ realizes a local minimum at x_0 then

$$\mathcal{M}^-(D^2\varphi(x_0), \lambda, \Lambda) \le f(x_0).$$

We also define, in the same hypotheses of Definition C.10,

$$S(\lambda, \Lambda, f) := \underline{S}(\lambda, \Lambda, f) \cap \overline{S}(\lambda, \Lambda, f),$$

and

$$S^*(\lambda, \Lambda, f) := \underline{S}(\lambda, \Lambda, -|f|) \cap \overline{S}(\lambda, \Lambda, |f|).$$

In particular, we will call the functions in $\underline{S}(\lambda, \Lambda, f)$, $\overline{S}(\lambda, \Lambda, f)$ $S(\lambda, \Lambda, f)$ subsolutions, supersolutions and solutions, respectively.
We now state a theorem which is the Harnack inequality for viscosity solutions.

Theorem C.12. *Let $u \in S^*(\lambda, \Lambda, f)$ in Q_1, where*

$$Q_1 := \left(-\frac{1}{2}, \frac{1}{2}\right) \times \ldots \times \left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$$

and f is continuous and bounded in Q_1. Suppose also that $u \geq 0$ in Q_1. Then

$$\sup_{Q_{1/2}} u \leq C\left(\inf_{Q_{1/2}} u + \|f\|_{L^n(Q_1)}\right),$$

where C is a universal constant.

Remark. We notice that for the definition of $\underline{S}$, we have $\underline{S}(\lambda, \Lambda, f) \subset \underline{S}(\lambda, \Lambda, -|f|)$, given that $f \geq -|f|$, and analogously, for the definition of $\overline{S}$, $\overline{S}(\lambda, \Lambda, f) \subset \overline{S}(\lambda, \Lambda, |f|)$, inasmuch $f \leq |f|$. Consequently,

$$S(\lambda, \Lambda, f) = \underline{S}(\lambda, \Lambda, f) \cap \overline{S}(\lambda, \Lambda, f) \subset \underline{S}(\lambda, \Lambda, -|f|) \cap \overline{S}(\lambda, \Lambda, |f|) = S^*(\lambda, \Lambda, f),$$

i.e. $S(\lambda, \Lambda, f) \subset S^*(\lambda, \Lambda, F)$ and hence the functions $u \in S(\lambda, \Lambda, f)$, namely the viscosity solutions, satisfy Theorem C.12.

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