Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Dipartimento di Fisica e Astronomia
Corso di Laurea in Fisica

The thermodynamic theory of black holes

Relatore:
Prof. Roberto Balbinot

Presentata da:
Giacomo Giorgi

Anno Accademico 2017/2018

Abstract

Studiando le equazioni di Einstein-Maxwell si ricava la soluzione di Kerr-Newman che descrive il comportamento generale di un buco nero rotante e carico. Questa soluzione dipende da tre parametri m, a e Q che definiscono rispettivamente la massa, il momento angolare e la carica del buco nero. Si ha un buco nero se $m^2 \geq a^2 + Q^2$.

Per $a = Q = 0$ si ottiene la soluzione di Schwarzschild che presenta un orizzonte degli eventi in $r = 2m$.

Per $a = 0$ si ottiene la soluzione di Reissner-Nordstrøm per buchi neri carichi e per $Q = 0$ la soluzione di Kerr per buchi neri rotanti.

La termodinamica dei buchi neri è l'area di studio che cerca di estendere ai buchi neri le leggi e i principali risultati della termodinamica classica e di farli riconciliare con l'esistenza degli orizzonti degli eventi.

Ciò è possibile solo con l'inclusione della meccanica quantistica. Nonostante il collasso gravitazionale conduca, apparentemente, a uno stato di entropia illimitata, l'inclusione di questi effetti quantistici elimina questa divergenza, assegnando a un buco nero una entropia definita. Jacob Bekenstein, nel 1972, congetturò che l'entropia del buco nero fosse proporzionale all'area del suo orizzonte degli eventi A. Hawking, nel 1974, mostrò che i buchi neri emettono radiazione termica corrispondente a una certa temperatura (temperatura di Hawking). Questo permette di di fissare il coefficiente di proporzionalità tra S e A.

Sulla base di questi risultati si dimostra che i buchi neri sono soggetti almeno alle prime due leggi della termodinamica, mentre le condizioni di Nernst per la terza legge della termodinamica non sono soddisfatte completamente: non c'è una chiara ragione termodinamica per cui un buco nero non possa essere raffreddato sotto lo zero assoluto e convertito in una singolarità nuda.

Tra i risultati di questa teoria si ha che tutta l'informazione riguardo allo stato termodinamico del buco nero è contenuta nella relazione di Smarr, che lega M, J e Q. Da questa relazione si trova l'espressione del primo principio della termodinamica per i buchi neri e si ricavano altre grandezze termodinamiche. Si trova, inoltre, che i buchi neri di Kerr-Newman subiscono una transizione di fase, dove la capacità termica cambia segno attraverso una discontinuità infinita.

Abstract

The Kerr-Newman solution that describes the general behaviour for a charged rotating black hole is found studying the Einstein-Maxwell's equations. The solution depends on three parameters m, a e Q defining the mass, the angular momentum and the charge of the black hole. To have a black hole, the condition is $m^2 \geq a^2 + Q^2$.

When $a = Q = 0$, the Schwarzschild solution, which has an event horizon at $r = 2m$, is obtained.

The Reissner-Nordstrøm solution for charged black hole is obtained when $a = 0$ and the Kerr solution for rotating black hole when $Q = 0$.

Black hole thermodynamics is the area of study that seeks to extend to black holes the laws and the main results of classical thermodynamics and to reconcile them with the existence of event horizons.

This is possible with the inclusion of quantum mechanics. Despite the gravitational collapse apparently leads to a state of unbound entropy, the inclusion of quantum effects damps out this divergence and assigns a definite entropy to the black hole. Jacob Bekenstein, in 1972, conjectured that the black hole entropy was proportional to the area of its event horizon A. Hawking, in 1974, showed that black holes emit thermal radiation corresponding to a certain temperature (Hawking temperature). In this way, it is possible to fix the constant of proportionality between S and A.

Due to these results, it is found that black holes are subject at least to the first and second laws of thermodynamics, while the Nernst conditions for the third law of thermodynamics are not fully satisfied: there is no obvious thermodynamic reason why a black hole may not be cooled down below absolute zero and converted into a naked singularity.

Among the results of this theory, there is that all the information about the thermodynamic state of black hole matter is contained in the Smarr relation, which depends on M, J and Q. From this relation, the expression of the first law of thermodynamics for black holes and other thermodynamic quantities are found. Furthermore, it is found that Kerr-Newman black holes undergo a phase transition, where the heat capacity changes sign through an infinite discontinuity.

Contents

Chapter 1

General Relativity

1.1 Principles of general relativity

The general theory of relativity is the geometric theory of gravitation published by Albert Einstein in 1916 including the expression of his field equations. In the general principle of relativity we find the theory's foundations:

General Principle of Relativity. *All observers are equivalent.*

Every observer can discover the same laws of physics. The situation is different in special relativity, where a preferred coordinates system exists: Minkowski coordinates. This suggests that the theory should be invariant under a coordinate transformation. The principle of general covariance follows:

Principle of General Covariance. *The equations of physics should have tensorial form.*

This principle was of fundamental importance to Einstein. It states that it is necessary to change all equations to their covariant form to switch from special to general relativity. Nowadays many doubt the importance of this principle because it is now known that it is possible to formulate any physical theory in tensorial form.

The principle of equivalence is another fundamental principle of general relativity; it can be stated in its strong and weak form:

The Principle of Equivalence (Strong). *The motion of a gravitational test particle in a gravitational field is independent of its mass and composition.*

The Principle of Equivalence (Weak). *The gravitational field is coupled with everything.*

The last formulation of the theorem is also known as the Galilean principle and can be tested by verifying the equivalence of inertial and gravitational mass.

From this, it follows that no body can be shielded from the gravitational field. However, it is possible to remove gravitational effects locally from our theory and thus regain special relativity. We identify an inertial frame in a freely falling reference system, at least locally in space and time. This leads to Einstein's principle of equivalence:

The Principle of Equivalence. *There are no local experiments which can distinguish non-rotating free fall in a gravitational field from uniform motion in the space in the absence of a gravitational field.*

These requests bring forward the idea of a curved space-time in place of a plain one. In Minkowski coordinates, the test particle's equation of motion is given by

$$\frac{d^2x^\alpha}{d\tau^2} = 0,$$

that can be easily extended to a more generic expression through the metric connection $\Gamma^\alpha_{\beta\gamma}$, dependant on the metric tensor $g_{\alpha\beta}$:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0.$$

The affine connection's dependence on the metric is given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\nu}(g_{\nu\gamma,\beta} + g_{\nu\beta,\gamma} - g_{\beta\gamma,\nu}), \tag{1.1}$$

in which we have defined

$$g_{\nu\gamma,\beta} = \partial_\beta g_{\nu\gamma} = \frac{\partial}{\partial x^\beta}g_{\nu\gamma}.$$

Through the covariant derivative this expression can be written as

$$u^\mu \nabla_\mu u^\alpha = (\nabla_{\boldsymbol{u}} \boldsymbol{u})^\mu = 0$$

and $u^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity. This is the geodesic's equation, where a geodesic extends the concept of straight lines to a curved manifold. The affine connection is dependent on the metric tensor, that is not anymore just a flat space's one. A freely falling particle follows the geodesics, determined by the metric, that in a curved manifold are no longer straight lines.

Furthermore, not only are geodesic curves followed by massive particle, but also by photons. From the derivative of the relation $u_\mu u^\mu = 0$, valid for photons in special relativity, we obtain:

$$0 = \frac{d(0)}{d\lambda} = u^\mu \nabla_\mu (0) = u^\mu \nabla_\mu (u_\alpha u^\alpha) = u^\mu u_\alpha \nabla_\mu (u^\alpha) + u^\mu u^\alpha \nabla_\mu (u_\alpha);$$

$$0 = 2u_\alpha (u^\mu \nabla_\mu u^\alpha).$$

We have obtained the geodesics' equation again.

1.2 Field equations

The considerations about the principles expressed led us to conclude that, locally, namely neglecting variations in the gravitational field, we can regain special relativity. However, we require a curved metric which may be thought of as the gravity field's potentials in a non-local situation. Correspondence with Newton's theory then suggests us to search for second-order field equations in such potentials and, from the principle of covariance, these equations have to be in covariant form.

We now introduce the ten complete field equations that we can apply in the presence of other fields beyond the gravitational one through the stress-energy tensor $T^{\alpha\beta}$:

$$G^{\alpha\beta} = kT^{\alpha\beta}. \tag{1.2}$$

k is a constant of proportionality called coupling constant. On the left side, we have the Einstein tensor that depends directly on the Ricci tensor $R^{\alpha\beta}$:

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R, \tag{1.3}$$

where R is the scalar curvature or Ricci scalar, $R = R^{\alpha}_{\alpha}$. The Riemann-Christoffel tensor is defined:

$$R^{\alpha}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}_{\beta\delta} - \partial_{\delta}\Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\epsilon}_{\beta\delta}\Gamma^{\alpha}_{\epsilon\gamma} - \Gamma^{\epsilon}_{\beta\gamma}\Gamma^{\alpha}_{\epsilon\delta}, \tag{1.4}$$

from which Ricci tensor follows as

$$R_{\alpha\beta} = R_{\beta\alpha} = R^{\gamma}_{\gamma\alpha\beta}.$$

The stress-energy tensor has become the field equations' source. This tensor follows a conservation law in special relativity:

$$\partial_{\beta} T^{\alpha\beta} = 0,$$

that can be extended to general relativity by a covariant derivative:

$$\nabla_{\beta} T^{\alpha\beta} = 0.$$

Follows that $\nabla_{\beta} G^{\alpha\beta}$=0. We know that the covariant derivative of the Einstein tensor vanishes through the Bianchi identities:

$$\nabla_{\alpha} R_{\delta\epsilon\beta\gamma} + \nabla_{\gamma} R_{\delta\epsilon\alpha\beta} + \nabla_{\beta} R_{\delta\epsilon\gamma\alpha} = 0,$$

from which the contracted Bianchi identities follow as

$$\nabla_{\beta} G^{\beta}_{\alpha} = 0.$$

The coupling constant k's value can be found considering that Newton equation must hold in the non-relativistic limit. Then, the coupling constant in non-relativist units is:

$$k = 8\pi \frac{G}{c^2},$$

where G is the universal gravitational constant and c is the light velocity in the vacuum. We will, mainly, work in relativist units, in which we can take both $G = 1$ and $c = 1$. Therefore, the coupling constant is:

$$k = 8\pi.$$

The vacuum field equations of general relativity are field equations in the absence of any field beyond the gravitational one. Then, the stress-energy tensor vanishes and our equations take the form:

$$G^{\alpha\beta} = 0. \tag{1.5}$$

Chapter 2

Schwarzschild Metric

2.1 Spherically symmetric solutions

The simplest case to find solutions to the vacuum field equations 1.5 is that of spherical symmetry. To proceed, we need to define what a Killing vector is.
In differential geometry we have a symmetry if, moving along a direction, defined by a vector $\boldsymbol{v}$, our quantity of interest, described by a tensor T, does not change. We can express this idea through the use of Lie derivatives:

$$\pounds_{\boldsymbol{v}} T = 0.$$

If the vector $\boldsymbol{v}$ stand for the axes direction x^i, we have:

$$\pounds_{\boldsymbol{v}} T = \frac{\partial T}{\partial x^i} = 0.$$

Isometries are the symmetries of the metric tensor, then $T = g$:

$$\pounds_{\boldsymbol{v}} g = 0. \tag{2.1}$$

The Killing vectors are the vectors that define the directions along we move.

A time-independent solution is called stationary, but this does not mean that it is not evolutionary, only that time does not enter into it explicitly. Being static is a stronger requirement for a solution because it is a solution that cannot be evolutive and nothing would change if, at any time, we ran the time backwards; then there is a temporal symmetry for any origin of time. We can express a stationary solution through the equation 2.1 mathematically; there is an isometry with respect to the Killing vector that is the time coordinate x^0:

$$\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0.$$

The spherical symmetry can be expressed by Killing vector fields too. A space-time is called spherically symmetric only when it admits three linearly independent spacelike Killing vector fields X^α whose orbits are closed and satisfy

$$[X^1, X^2] = X^3, \quad [X^2, X^3] = X^1, \quad [X^3, X^1] = X^2.$$

It can be proven that in a spherically symmetric space-time, there exists a coordinates system (x^α), called Cartesian, in which the Killing vector fields X^α are

$$X^0 = 0, \quad X^\alpha = \omega^\alpha_\beta x^\beta \quad \text{with} \quad \omega_{\alpha\beta} = -\omega_{\beta\alpha}.$$

The quantity $\omega_{\alpha\beta}$ depends on three parameters that state the three spacelike rotations. These results lead to the line element canonical form, but here we will give a heuristic way to reach it. The spherical symmetry suggests the existence of a favourite point in space, called origin O. The system must be invariant under rotations about this point. Therefore, we can consider the spherical coordinates of centre O with the polar angle ϕ and azimuthal angle θ, as shown in figure 2.1.

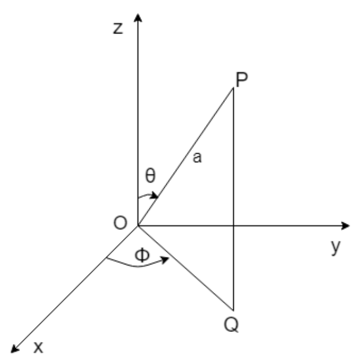

Figure 2.1: The standard spherical coordinates.

At a fixed moment in time, we consider a point P at a distance a from the origin O. Through the space-rotations the point P describes a 2-sphere centred on O. Q is the projection of P on the xy-plane. ϕ and θ, to represent all points on the 2-sphere, take values in the ranges:

$$-\pi < \phi \leq \pi, \qquad 0 \leq \theta \leq \pi.$$

Moreover, the line element of the 2-sphere is

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2}$$

In four dimensions we can consider adding an arbitrary timelike coordinate t and a radial parameter r to ϕ and θ, so the line element is reduced to the expression 2.2 on a 2-sphere $t =$ constant, $r =$ constant. Spherical symmetry requires that the line element does not change when angles ϕ and θ are varied, so ϕ and θ appear in the line element only in the form $(d\theta^2 + \sin^2\theta d\phi^2)$. Furthermore, there must be no crossed terms in $d\phi$ or $d\theta$ because the metric has to be separately invariant under the reflections:

$$\theta \to \theta' = \pi - \theta \quad \text{and} \quad \phi \to \phi' = -\phi.$$

We start, then, from the hypothesis that exists a reference frame

$$(x^\alpha) = (x^0, x^1, x^2, x^3) = (t, x, y, z)$$

in which the line element takes the form

$$ds^2 = Adt^2 - 2Bdtdr - Cdr^2 - D(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.3}$$

where A, B, C and D are functions of t and r

$$A = A(t, r), \quad B = B(t, r), \quad C = C(t, r), \quad D = D(t, r).$$

We can introduce a new radial coordinate by the transformation:

$$r \rightarrow r' = D^{\frac{1}{2}},$$

whereby 2.3 becomes:

$$ds^2 = A'(t,r')dt^2 - 2B'(t,r')dtdr' - C'(t,r')dr'^2 - r'^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Considering the differential

$$A'(t,r')dt - B'(t,r')dr',$$

the theory of ordinal differential equations claims that we can always multiply it by an integrating factor, $I = I(t,r')$, which makes it a perfect differential. We use this result to define a new time coordinate t' by requiring:

$$dt' = I(t,r')[A'(t,r')dt - B'(t,r')dr'].$$

Squaring:

$$dt'^2 = I^2(A'^2dt^2 - 2A'B'dtdr' + B'^2dr'^2);$$

$$A'dt^2 - 2B'dtdr' = A'^{-1}I^{-2}dt'^2 - A'^{-1}B'^2dr'^2,$$

the line element becomes

$$A'^{-1}I^{-2}dt'^2 - (C' - A'^{-1}B'^2)dr'^2 - r'^2(d\theta^2 + \sin^2\theta d\phi^2).$$

We can define two new functions ν and λ by

$$A'^{-1}I^{-2} = e^{\nu} \tag{2.4}$$

$$\text{and} \qquad C' + A'^{-1}B'^2 = e^{\lambda}, \tag{2.5}$$

$$\text{where} \quad \nu = \nu(t',r'), \quad \lambda = \lambda(t',r').$$

Dropping the primes, we obtain the line element:

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.6}$$

The definitions of ν and λ in 2.4 and 2.5 are given in exponential terms, which, being always positive, ensure that the signature of the metric is always -2.

2.2 The Schwarzschild solution

We use the vacuum field equations 1.5 to determinate the unknown functions ν and λ in the line element 2.6. The covariant metric is

$$g_{\alpha\beta} = \mathrm{diag}(e^\nu, -e^\lambda, -r^2, -r^2\sin^2\theta).$$

We know that

$$g_{\alpha\beta}g^{\beta\gamma} = \delta^\alpha_\gamma,$$

where δ^α_γ is the Kronecker tensor; the contravariant metric must be diagonal as well and its elements must be the multiplicative inverse of those of the covariant one:

$$g^{\alpha\beta} = \mathrm{diag}(e^{-\nu}, -e^{-\lambda}, -r^{-2}, -r^{-2}\sin^{-2}\theta). \tag{2.7}$$

We use the expressions 2.2 and 2.7 to find the non-vanishing elements of the Einstein tensor through its definition 1.3, having found the affine connection 1.1 and, then, the Riemann tensor 1.4:

$$G^0_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}; \tag{2.8}$$

$$G^1_0 = -e^{-\lambda}r^{-1}\dot{\lambda} = -e^{\lambda-\nu}G^0_1; \tag{2.9}$$

$$G^1_1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}; \tag{2.10}$$

$$G^2_2 = G^3_3 = \frac{1}{2}e^{-\lambda}\left(\frac{\nu'\lambda'}{2} + \frac{\lambda'}{r} - \frac{\nu'}{r} - \frac{\nu'^2}{2} - \nu''\right) + \frac{1}{2}e^{-\nu}\left(\ddot{\lambda} + \frac{\ddot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right). \tag{2.11}$$

In these expressions we have written λ' and ν' for the derivatives in relation to R and $\dot{\lambda}$ and $\dot{\nu}$ for the derivatives in relation to t. The contracted Bianchi identities 1.2 reveals that equation 2.11 vanishes if the equations 2.8, 2.9 and 2.10 all vanish as well. Thus, it remains only three linearly independent equations to solve:

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0; \tag{2.12}$$

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}=0; \tag{2.13}$$

$$\dot{\lambda}=0. \tag{2.14}$$

Adding equation 2.12 to 2.13, we obtain:

$$\lambda'+\nu'=0;$$

which, integrated, returns

$$\lambda+\nu=h(t),$$

where h is an arbitrary function of integration. λ is only function of r by equation 2.14, thus equation 2.8 is an ordinary differential equation which can be written as

$$e^{-\lambda}-re^{-\lambda}\lambda'=(re^{-\lambda})'=1.$$

Integrating, we obtain

$$re^{-\lambda}=r+\text{constant}.$$

Choosing the constant to be $-2m$, we obtain:

$$e^{\lambda}=(1-2m/r)^{-1},$$

which reduces our metric 2.2 to:

$$g_{\alpha\beta}=\text{diag}(e^{h(t)}(1-2m/r),-(1-2m/r)^{-1},-r^2,-r^2\sin^2\theta).$$

It remains the dependence on h that can be removed by a transformation to a new temporal coordinate t', that is defined by the relation

$$t' = \int_c^t \mathrm{e}^{\frac{1}{2}h(u)}\,\mathrm{d}u,$$

where c is an arbitrary constant. Clearly, only the component of the metric g_{00} changes under this transformation:

$$t' = \int_c^t \mathrm{e}^{\frac{1}{2}h(u)}\,\mathrm{d}u, \quad dt' = e^{\frac{1}{2}h(t)}dt \implies dt^2 = e^{-h(t)}dt'^2;$$

$$g'_{00} = \left(1 - \frac{2m}{r}\right).$$

Dropping the primes, there is always a coordinate system where the most generic spherically symmetric solution to the vacuum field equations is

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.15}$$

This is the Schwarzschild line element.

The solution 2.15 is both time-symmetric, since it is invariant under temporal inversion $t \to t' = -t$, and time translation invariant, since it is invariant under temporal translation $t \to t' = t + \text{constant}$. Hence, this solution is static.

In this way, we have proven Birkhoff's theorem.

Birkhoff's Theorem. *A spherically symmetric vacuum solution in the exterior region is necessary static.*

This theorem implies that, if a spherically symmetrical source, like a star, changes its shape maintaining spherical symmetry, it cannot spread any radiation in the space around. If a spherically symmetrical source is restricted to a limited region $r \leq a$, for a certain $a > 2m$, the solution for $r > a$ is the Schwarzschild solution, called Schwarzschild exterior solution. However, it is not true the opposite: a source does not inherit the

symmetries of its field, then the source is not necessarily spherically symmetrical even if its field is the Schwarzschild exterior solution.
Considering the limit of 2.15 as $r \to \infty$, we regain the flat metric of special relativity in spherical polar coordinates (that was our purpose), that is:

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Then, a vacuum spherically symmetrical solution is necessary asymptotically flat. Knowing this and that we have to find the Newton potential $\phi = -\frac{GM}{r}$ in the weak-field limit, with a mass M situated at the origin O, we find that m value is:

$$m = \frac{GM}{c^2}.$$

m has the dimensions of length and is sometimes called geometric mass. In relativistic units, m is the mass of the object at the origin O.

2.3 Coordinates for the Schwarzschild solution

Now we have to consider the Schwarzschild solution in the form expressed in the equation 2.15. The components of the metric are:

$$g^{00} = \left(1 - \frac{2m}{r}\right)^{-1}, \quad g^{11} = -\left(1 - \frac{2m}{r}\right), \quad g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2\sin^2\theta}.$$

In this coordinates system, t is timelike and r, θ and ϕ are spacelike. The matrix is independent of t and there are not crossed terms in dt, hence the solution is static. r is a radial parameter which has the property that the 2-sphere $t =$ constant, $r =$ constant has the standard line element

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

from which it follows that the surface area of the 2-sphere is $4\pi r^2$. θ and ϕ are the usual polar coordinates on the 2-sphere and are defined invariantly by the spherical symmetry.

The Schwarzschild coordinates (t, r, θ, ϕ) are canonical coordinates, defined invariantly by the symmetries.

Nevertheless, the Schwarzschild coordinates have some singularities. There are coordinates singularities because these coordinates do not cover the axes $\theta = 0, \pi$, where the line element becomes degenerate; this degeneration can be removed using Cartesian coordinates (x, y, z):

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

This kind of singularities reflects deficits of the reference frame and can be easily removed changing the frame itself. There are other two coordinates values that return singularities: $r = 0$ and the Schwarzschild radius $r = 2m$. The latter is a removable coordinate singularity too, as indicated by the Riemann tensor scalar invariant

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 48\frac{m^2}{r^6},$$

which is finite for $r = 2m$. Its value is the same in every and each reference frame being it a scalar. Instead, in the origin $r = 0$, this invariant diverges: in this case the singularity is real and irremovable, therefore is variously called intrinsic, curvature, physic, essential or real singularity. $r = 2m$ is a null hypersurface that divides the manifold into two separated regions $2m < r < \infty$ and $0 < r < 2m$. In the second region the coordinates r and t reverse their character, t becomes spacelike and r timelike. The Schwarzschild solution is an exterior vacuum solution for every spherically symmetrical body of radius $a > 2m$. A different metric would describe the body itself for $r < a$ but then it would correspond to a matter distribution, that is a not vanishing stress-energy tensor.

2.4 A radially infalling particle

Considering a particle of mass μ infalling radially toward the origin, we can notice that it moves on a timelike geodesic. Our line element is 2.15; we can use the variational principle to find the trajectory's equation

$$S[x^\mu(\tau)] = \mu \int_A^B dS = \mu \int_A^B \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\tau, \qquad (2.16)$$

where τ is the proper time of the particle and $g_{\mu\nu}, x^\mu, \dot{x}^\mu = \frac{dx^\mu}{d\tau}$ are respectively elements of the metric, the 4-vector and the 4-velocity in tensorial notation. We can define $2T = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ whose value can be considered 1. Then the equation for the variations 2.16 becomes:

$$\delta S = \mu \int \frac{\delta T}{\sqrt{2T}} d\tau = \mu \int \delta T d\tau.$$

T is the Lagrangian of the system and, having required the condition $2T = 1$, we obtain:

$$2T = \left(1 - \frac{2m}{r}\right) \dot{t}^2(\tau) - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2(\tau) - r^2(\dot{\theta}^2(\tau) + \sin^2\theta \dot{\phi}^2(\tau)) = 1. \qquad (2.17)$$

Considering $\mu = 1$ for convenience, the Euler-Lagrange equations are:

$$\frac{d}{d\tau}\left[\left(1 - \frac{2m}{r}\right)\dot{t}\right] = 0;$$

$$\frac{d}{d\tau}(r^2\dot{\theta}) - r^2 \sin\theta \cos\theta \dot{\phi}^2 = 0;$$

$$\frac{d}{d\tau}\left(r^2 \sin^2\theta \dot{\phi}\right) = 0.$$

It will not be necessary to resolve these equations because there are already three first integrals of motion, namely T and the generalized momenta of cyclic coordinates t and ϕ. Then, from equation 2.17, we define the conserved quantities:

$$E = \left(1 - \frac{2m}{r}\right)\dot{t}, \qquad (2.18)$$

$$J = r^2 \sin^2\theta \dot{\phi}. \qquad (2.19)$$

We find that, if the motion takes place on the equatorial plane, it will remain there like in classical mechanics; hence, thanks to the spherical symmetries of the gravitational field, we can always consider $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$ and $\dot{\phi} = 0$. Therefore, equation 2.19 becomes:

$$J = r^2 \dot{\phi} = 0.$$

Since $2T = 1$, we obtain the equation

$$\left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 = 1. \tag{2.20}$$

E can be considered the energy of the particle and its value corresponds to different initial conditions. Let's consider the case, in relativistic coordinates $G = 1$, $C = 1$, of a particle with initial zero velocity, thus $E = 1$. Therefore, for $r \longrightarrow \infty$, it is $\dot{t} \simeq 1$, that is, asymptotically $\dot{t} \simeq 1$. From 2.18 and 2.20, it is obtained

$$\left(\frac{d\tau}{dr}\right)^2 = \frac{r}{2m}.$$

We consider the negative root of this equation because the particle is getting closer to the origin. Integrating:

$$\tau - \tau_0 = \frac{2}{3(2m)^{\frac{1}{2}}} (r_0^{\frac{3}{2}} - r^{\frac{3}{2}}), \tag{2.21}$$

where the particle is in the position r_0 at the proper time τ_0. At the Schwarzschild radius, there is no singular behaviour and the particle falls continuously to $r = 0$ in a finite proper time.

Things change if the motion of the particle is described in terms of the Schwarzschild coordinate time t:

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = -\left(\frac{r}{2m}\right)^{\frac{1}{2}} \left(1 - \frac{2m}{r}\right)^{-1}.$$

Integrated, it returns

$$t - t_0 = -\frac{2}{3(2m)^{\frac{1}{2}}} (r^{\frac{3}{2}} - r_0^{\frac{3}{2}} + 6mr^{\frac{1}{2}} - 6mr_0^{\frac{1}{2}}) + 2m \ln \frac{[r^{\frac{1}{2}} + (2m)^{\frac{1}{2}}][r_0^{\frac{1}{2}} - (2m)^{\frac{1}{2}}]}{[r_0^{\frac{1}{2}} + (2m)^{\frac{1}{2}}][r^{\frac{1}{2}} - (2m)^{\frac{1}{2}}]}. \tag{2.22}$$

For situations where r and r_0 are much larger than $2m$, the results of the equations 2.21

and 2.22 are approximately the same. Instead, if r values is very close to $2m$, we find

$$r - 2m = (r_0 - 2m)e^{-(t-t_0)/2m}.$$

It clearly follows:

$$t \longrightarrow \infty \implies r - 2m \longrightarrow 0.$$

So we see that $2m$ is approached without ever being reached. t corresponds to the proper time measured by a observer at rest far away from the origin; from this observer, the particle takes an infinite amount of time to reach $2m$.

The singularity at $r = 2m$ follows from the coordinates system used and can be removed using a more adapt system, namely Eddington-Finkelstein coordinates. We apply the coordinate change

$$t \longrightarrow \bar{t} = t + 2m \ln{(r - 2m)}.$$

Applying this change to the line element 2.15, we obtain the Eddington-Finkelstein form

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\bar{t}^2 - \frac{4m}{r} d\bar{t}dr - \left(1 + \frac{2m}{r}\right) dr^2 - r^2(d\theta + \sin^2\theta d\phi^2). \tag{2.23}$$

We can write the solution 2.23 in a simpler form introducing the null coordinate

$$v = \bar{t} + r,$$

which is called an advanced time parameter. The resulting line element is

$$ds^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2dvdr - r^2(d\theta + \sin^2\theta d\phi^2). \tag{2.24}$$

Now the solution is regular at $r = 2m$, the surface there lets only the radially outgoing photons remain where they are while all the rest is pulled toward the centre. At $r < 2m$ all photons are drawn to the centre singularity too. The surface $r = 2m$ acts as a one-way membrane, letting future-directed timelike and null curves cross only from the outside to the inside. This surface is called event horizon because it represents the limit of all events visible by an external observer. The Schwarzschild event horizon is absolute because it seals off all internal events from every external observer.

2.5 Non-rotating black holes

The theory of stellar evolution tells us that stars with a mass of the order of the sun can reach a final equilibrium state as white dwarfs or neutron stars. For much larger masses, this equilibrium is not possible; in these cases, the star will contract until the gravitation effects overcome the internal pressure and stresses which will no longer be able to halt the further collapse. The theory of general relativity foretells that a spherically symmetrical star will necessarily contract until all its matter arrives at the singularity in the centre.

Now the collapse of a non-rotating spherically symmetrical star that continues until the star's surface approaches the Schwarzschild radius has to be considered. The Schwarzschild vacuum solution remains the star exterior field, but a signal at $r = 2m$ will never escape the surface $r = 2m$ and, for $r < 2m$, all signals will necessarily fall toward the singularity at the centre. A distant external observer will always be able to see only the star's surface as it was before reaching the Schwarzschild radius. In practice, however, the observer will never be able to see the surface of the star as it was before it plunged through the Schwarzschild radius and the star would quickly fade from sight leaving behind a black hole in space.

It would seem that the request for a spherically symmetrical solution, which does not even consider charge or rotation, is too tight. However, although there are different details in different cases, the main characteristics of black holes (absolute event horizons and singularities) persist.

The idea of the black hole as an object with such a gravitational field from which not even light can escape is a consequence of the Newton corpuscular theory of light. Considering a particle with mass m radially moving away from a matter distribution with radius R

and mass M, if the particle has a velocity v at a distance r from the centre, its energy is

$$E = \textit{kinetic energy} + \textit{potential energy} = \\ = \frac{1}{2}mv^2 - \frac{GMm}{r}.$$

The escape velocity v_0 is defined as the velocity at the surface of the distribution of matter which allows the particle to escape to infinity with zero velocity. So $v_0 = 0$ as $r \longrightarrow \infty$, that leads to $E = 0$. Hence, the escape velocity is

$$v_0^2 = 2GM/r.$$

The velocity of light in the vacuum is c and it escapes toward infinity only when it is related to R and M through the relation

$$c^2 = 2GM/R.$$

The distribution of matter is, in this way, bound by the radius limit condition

$$R = \frac{2GM}{c^2},$$

which corresponds to our definition of the Schwarzschild radius in non-relativistic coordinates.

Chapter 3

Reissner-Nordstrøm Metric

3.1 Maxwell's equations

Working in Heavyside-Lorentz units with $c = 1$, we can rewrite Maxwell's equations for special relativity; thus we find that Maxwell's equations in vacuo for the electromagnetic field split up into two pairs of equations: the source equations

$$\mathrm{div}\boldsymbol{E} = \rho \tag{3.1}$$

$$\mathrm{curl}\boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j}, \tag{3.2}$$

and the internal equations

$$\mathrm{div}\boldsymbol{B} = 0 \tag{3.3}$$

$$\mathrm{curl}\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}, \tag{3.4}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ the magnetic induction, ρ the charge density and $\boldsymbol{j}$ the current density. ρ and $\boldsymbol{j}$ cannot be assumed independent, but they must follow a continuity equation:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\boldsymbol{j} = 0. \tag{3.5}$$

To write these equations in a tensorial form, we define an anty-symmetric tensor $F^{\alpha\beta}$, called the electromagnetic field tensor or Maxwell tensor, by

$$F^{\alpha\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \tag{3.6}$$

and the current density or source 4-vector j^α by

$$j^\alpha = (\rho, \boldsymbol{j}).$$

The continuity equation 3.5 becomes

$$\partial_\alpha j^\alpha = 0.$$

Hence, the source equations, 3.1-3.2, and internal equations, 3.3-3.4, can be written in the form

$$\partial_\beta F^{\alpha\beta} = j^\alpha, \tag{3.7}$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0. \tag{3.8}$$

The anti-symmetry of $F_{\alpha\beta}$ means that the equations 3.8 can be written simply as

$$\partial_{[\alpha} F_{\beta\gamma]} = 0.$$

However, rather than working with the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, it is more convenient working with the potentials: the scalar potential ϕ and the vector potential $\boldsymbol{A}$ defined by

$$\boldsymbol{E} = -\text{grad}\phi - \frac{\partial \boldsymbol{A}}{\partial t},$$

$$\boldsymbol{B} = \text{curl}\boldsymbol{A}.$$

We define the 4-potential by

$$\phi^\alpha = (\phi, \boldsymbol{A}),$$

from which we regain the Maxwell tensor 3.6 as

$$F_{\alpha\beta} = \partial_\beta \phi_\alpha - \partial_\alpha \phi_\beta. \tag{3.9}$$

The 4-potential is not defined uniquely by this equation because we may perform a gauge transformation

$$\phi_\alpha = \bar{\phi}_\alpha = \phi_\alpha + \partial_\alpha \psi, \tag{3.10}$$

where ψ is an arbitrary scalar field. The gauge transformation does not alters $F_{\alpha\beta}$ and, therefore, $\boldsymbol{E}$ and $\boldsymbol{B}$. In solving particular problems, it is often convenient to reduce the gauge freedom by imposing a constraint on ϕ_α, called a gauge condition. An important gauge condition to discuss electromagnetic radiation is given by the Lorentz gauge

$$\eta^{\alpha\beta}\phi_{\alpha,\beta} = 0, \tag{3.11}$$

where $\eta^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. Imposing this constraint on 3.10, we find a scalar field that is no longer arbitrary but it must be a solution of the wave equation

$$\Box\psi \equiv \eta^{\alpha\beta}\psi_{\alpha,\beta} = 0,$$

where $\Box$ is the d'Alambertian operator

$$\Box \equiv \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2.$$

The internal equations 3.8 are automatically satisfied by the definition 3.9, and the source equations 3.7, in terms of the 4-potential, become, raising the indexes with the Minkowski metric:

$$\partial_\beta[\eta^{\alpha\gamma}\eta^{\beta\delta}(\partial_\delta\phi_\gamma - \partial_\gamma\phi_\delta)] = j^\alpha,$$

which, with the Lorentz gauge 3.11, becomes

$$\Box\phi^{\alpha} = j^{\alpha}.$$

In a source-free region, the equation simplifies to

$$\Box\phi^{\alpha} = 0.$$

Then, ϕ^{α} and $F^{\alpha\beta}$, and thus $\boldsymbol{E}$ and $\boldsymbol{B}$, are solutions to the wave equation.

Until now, we have restricted our attention only to special relativity in Minkowski coordinates. From the general covariance principle, we know that, to obtain the covariant formulation, it is simply necessary to exchange all ordinary derivatives with covariant derivatives. Nevertheless, 3.8 does not change because

$$\nabla_{[\alpha}F_{\beta\gamma]} = \partial_{[\alpha}F_{\beta\gamma]} \quad \text{and} \quad \nabla_{[\alpha}\phi_{\beta\gamma]} = \partial_{[\alpha}\phi_{\beta\gamma]}.$$

The covariant formulation of Maxwell's equations in the vacuo is

$$\nabla F^{\alpha\beta} = j^{\alpha} \tag{3.12}$$

$$\partial_{[\alpha}\phi_{\beta\gamma]} = 0, \tag{3.13}$$

with the restrain

$$\nabla_{\alpha}j^{\alpha} = 0.$$

3.2 The Maxwell energy-momentum tensor

Our goal is to build the energy-momentum tensor for the electromagnetic field and, to do so, we will use the variational principle. To simplify the problem, we will work in vacuo in special relativity in Minkowski coordinates and restrict our attention to a source-free region, that is a region where j^{α} vanishes. Hence, let's consider the Lagrangian for the electromagnetic field

$$\mathscr{L}_E(\phi_\alpha, F_{\alpha\beta}) = \frac{1}{4\pi}\left[-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + (\phi_{\alpha,\beta} - \phi_{\beta,\alpha})F^{\alpha\beta}\right].$$

The field equations corresponding to a variation with respect to ϕ_α are

$$\frac{\delta\mathscr{L}_E}{\delta\phi_\alpha} = \frac{\partial\mathscr{L}_E}{\partial\phi_\alpha} - \left(\frac{\partial\mathscr{L}_E}{\partial\phi_{\alpha,\beta}}\right)_{,\beta} = 0 - \frac{1}{4\pi}(F^{\alpha\beta} - F^{\beta\alpha})_{,\beta};$$

$$(F^{\alpha\beta} - F^{\beta\alpha})_{,\beta} = 0. \tag{3.14}$$

In the same way, the field equations corresponding to a variation with respect to $F_{\alpha\beta}$ are

$$\begin{aligned}\frac{\delta\mathscr{L}_E}{\delta_{\alpha\beta}} = \frac{\partial\mathscr{L}_E}{\partial F_{\alpha\beta}} &= \frac{\partial}{\partial F_{\alpha\beta}}\frac{1}{4\pi}\left[-\frac{1}{2}\eta^{\gamma\epsilon}\eta^{\delta\nu}F_{\gamma\delta}F_{\epsilon\nu} + \eta^{\gamma\epsilon}\eta^{\delta\nu}(\phi_{\gamma,\delta} - \phi_{\delta,\gamma})F_{\epsilon\nu}\right]\\ &= \frac{1}{4\pi}\left[-\frac{1}{2}\eta^{\alpha\epsilon}\eta^{\beta\nu}F_{\epsilon\nu} - \frac{1}{2}\eta^{\gamma\alpha}\eta^{\delta\beta}F_{\gamma\delta} + \eta^{\gamma\alpha}\eta^{\delta\beta}(\phi_{\gamma,\delta} - \phi_{\delta,\gamma})\right]\\ &= \frac{\eta^{\alpha\gamma}\eta^{\beta\delta}}{4\pi}[-F_{\gamma\delta} + (\phi_{\gamma,\delta} - \phi_{\delta,\gamma})];\end{aligned}$$

$$F_{\alpha\beta} = \phi_{\alpha,\beta} - \phi_{\beta,\alpha}. \tag{3.15}$$

This last equation defines $F_{\alpha\beta}$ in terms of the 4-potential, as seen in 3.9, and it shows us that $F_{\alpha\beta}$ is anti-symmetric. This equation also implies that the internal equations 3.8 are automatically fulfilled and equations 3.14 become

$$F^{\alpha\beta}{}_{,\beta} = 0,$$

that are the source equations in a source-free region. As a result of equations 3.15, we can rewrite the Lagrangian in the form

$$\mathscr{L}_E = \frac{1}{8\pi}\eta^{\alpha\gamma}\eta^{\delta\beta}F_{\alpha\delta}F_{\gamma\delta}. \tag{3.16}$$

Let's assume that the Lagrangian 3.16, in the transition to the complete theory, becomes

$$\mathscr{L}_E = \frac{(-g)^{\frac{1}{2}}}{8\pi} g^{\alpha\gamma} g^{\delta\beta} F_{\alpha\beta} F_{\gamma\delta}.$$

The factor $(-g)^{\frac{1}{2}}$ is included to ensure that $\mathscr{L}_E$ is a scalar density, in Minkowski's coordinates it becomes 1. g is the determinant of the metric and its differential is

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\alpha\beta} \delta g^{\alpha\beta}.$$

Hence, we find

$$\begin{aligned}\frac{\partial \mathscr{L}_E}{\partial g^{\alpha\beta}} &= \frac{\partial}{\partial g^{\alpha\beta}} \frac{(-g)^{\frac{1}{2}}}{8\pi} g^{\epsilon\gamma} g^{\delta\nu} F_{\epsilon\nu} F_{\gamma\delta} \\ &= \frac{(-g)^{\frac{1}{2}}}{8\pi} g^{\delta\nu} F_{\alpha\nu} F_{\beta\delta} + \frac{(-g)^{\frac{1}{2}}}{8\pi} g^{\epsilon\gamma} F_{\epsilon\beta} F_{\gamma\alpha} + \\ &- \frac{1}{2} \frac{(-g)^{\frac{1}{2}}}{8\pi} g_{\beta\alpha} g^{\epsilon\gamma} g^{\delta\nu} F_{\epsilon\nu} F_{\gamma\delta}.\end{aligned}$$

Remembering that $g^{\alpha\beta}$ is symmetric and $F^{\alpha\beta}$ antisymmetric, the relation reduces to

$$\frac{\partial \mathscr{L}_E}{\partial g^{\alpha\beta}} = -\frac{(-g)^{-\frac{1}{2}}}{4\pi} \left(-g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} + \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \right) = -(-g)^{\frac{1}{2}} T_{\alpha\beta}.$$

In the last line, we have introduced the stress-energy tensor, in such a way we define the Maxwell energy-momentum tensor in a source-free region by

$$T_{\alpha\beta} = \frac{1}{4\pi} \left(-g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} + \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \right). \tag{3.17}$$

The trace of this tensor vanishes for the properties of the metric tensor $g^{\alpha\beta}$ and of the Maxwell tensor $F^{\alpha\beta}$:

$$T^{\alpha}_{\alpha} = g^{\alpha\beta} T_{\alpha\beta} = 0.$$

3.3 The Reissner-Nordstrøm solution

We are searching for the Reissner-Nordstrøm solution for a charged mass point. Therefore, we will look for a static, asymptotically flat, spherically symmetric solution of the Maxwell-Einstein field equations. We have seen that the Maxwell-Einstein field equations 1.2 in relativist coordinates are

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{3.18}$$

where $T_{\alpha\beta}$ is the stress-energy tensor which vanishes in a source-free region, as we have seen in § 1.2, returning as a result the vacuum field equations 1.5.

In this particular case, the stress-energy tensor we will use is the Maxwell energy-momentum tensor 3.17 the trace of which vanishes, as we have said. Knowing the expression for Einstein tensor 1.3, we see that the trace of the Ricci tensor, the curvature scalar R, must vanish as well. Hence, we can also work with equations equivalent to 3.18, namely

$$R_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{3.19}$$

Furthermore, the Maxwell's tensor $F_{\alpha\beta}$ 3.6 must satisfy Maxwell's equations 3.12 and 3.13 in a source-free region:

$$\nabla F^{\alpha\beta} = 0, \tag{3.20}$$

$$\partial_{[\alpha}\phi_{\beta\gamma]} = 0. \tag{3.21}$$

The assumption of spherical symmetry means we can introduce a new set of coordinates (t, r, θ, ϕ) in which the line element reduces itself to the canonical form 2.6:

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where ν and λ are functions of t and r but, if we impose to the solution to be static, they become only functions of r:

$$\nu = \nu(r), \qquad \lambda = \lambda(r).$$

To work in polar coordinates we need the Maxwell tensor 3.6 expressed in this coordinates, namely a tensor that describes the electromagnetic field of a charged mass point; thus, we will find that the magnetic field vanishes because the charge is still and that the electric field is a function only of r for the spherical symmetry of the charge. In the end, it remains

$$F_{\alpha\beta} = E(r) \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.22}$$

We find that the Maxwell's equations 3.21 are immediately satisfied by the tensor 3.22. Now we apply this tensor and this line element to the equations 3.20. Considering $g_{\alpha\beta} = \mathrm{diag}(e^{\nu}, -e^{\lambda}, -r^2, -r^2 \sin^2\theta)$, we obtain

$$\begin{aligned} \nabla_\beta F^{\alpha\beta} &= \partial_\beta F^{\alpha\beta} + \Gamma^{\alpha}_{\gamma\beta} F^{\gamma\beta} \\ &= \partial_\beta F^{\alpha\beta} + \frac{1}{2} g^{\alpha\delta}(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\gamma\beta}) F^{\gamma\beta} = 0; \end{aligned}$$

$$\frac{d}{dr}(e^{\frac{1}{2}(\nu+\lambda)} r^2 E) = 0.$$

Integrating, it returns

$$E(r) = e^{\frac{1}{2}(\nu+\lambda)} \frac{Q}{r^2},$$

where Q is a constant of integration. The solution must be asymptotically flat, hence $\nu, \lambda \longrightarrow 0$ as $r \longrightarrow \infty$. Thus, asymptotically, it must be $E \sim Q/r^2$. This result is the same we obtain classically for a mass point of charge Q in the origin. Therefore, we can consider Q as the charge of the mass point.

Now, we can find the Maxwell's energy-momentum tensor by its definition 3.17. Plugging this tensor in the field equations 3.19, we find that the 00 and 11 equations lead to

$$\frac{d\lambda}{dr} + \frac{d\nu}{dr} = 0$$

which result in $\lambda = -\nu$ for the asymptotic conditions. The 22 equation is the one to remain independent and it leads to

$$\frac{d}{dr}(re^{\nu}) = 1 - \frac{Q^2}{r^2},$$

the integration of which returns

$$e^{\nu} = 1 - \frac{2m}{r} + \frac{Q^2}{r^2},$$

where m is the constant of integration. We have finally arrived at the Reissner-Nordstrøm solution:

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (3.23)$$

When $Q = 0$, we regain the Schwarzschild line element 2.15 and thus we identify m as the geometric mass.

As for Schwarzschild solution, the assumptions of a static and asymptotically flat solution were not necessary. We can find, hence, an analogue to Birkhoff theorem (§ 2.2).

Theorem. *A spherically symmetrical exterior solution of the Einstein-Maxwell field equations is necessary static.*

3.4 Coordinates for the R-N solution

Considering the coefficients

$$g_{00} = -(g_{11})^{-1} = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} = \frac{G}{r^2},$$

where

$$G = r^2 - 2mr + Q^2.$$

The discriminant of the quadratic G is

$$\Delta = m^2 - Q^2.$$

If this is negative, that is $Q^2 > m^2$, the quadratic has no real roots and it is positive for all values of r except at the origin $r = 0$. Hence, it follows that the line element 3.23 is non-singular for all values of r except at the origin $r = 0$. However, the solution has an intrinsic singularity at $r = 0$, as can be seen calculating the Riemann invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$. This fact does not come as a surprise because in the origin the point charge producing the field is located.

The more interesting case occurs when $Q^2 \leq m^2$, then the metric has singularities when Q vanishes, namely at $r = r_+$ and $r = r_-$, where

$$r_\pm = m \pm (m^2 - Q^2)^{\frac{1}{2}}.$$

The line element 3.23 is regular in the regions:

I. $r_+ < r < \infty$,

II. $r_- < r < r_+$,

III. $0 < r < r_-$.

If $Q^2 = m^2$, only the regions I and III exist. The regions are separated by null hypersurfaces $r = r_+$ and $r = r_-$. The situation at $r = r_+$ is rather similar to the Schwarzschild case, $g_{00} = 1 - \frac{2m}{r}$, at $r = 2m$. The coordinates t and r are timelike and spacelike, respectively, in the regions I and III, but interchange their character in region II. Thus, regions I and III are static, but region II is not. As in the case of the Schwarzschild solution, it seems as that the regions I, II and III are totally disconnected. Therefore, as we did for the Schwarzschild solution, we will not pursue the structure of the solution in these coordinates but we will look for the analogue of the Eddington-Finkelstein coordinates.

We consider the case $Q^2 < m^2$ and we apply the coordinate change for $r > r_+$

$$\bar{t} = t + \frac{r_+^2}{r_+ - r_-}\ln(r - r_+) - \frac{r_-^2}{r_+ - r_-}\ln(r - r_-).$$

The line element 3.23 takes the form

$$ds^2 = (1 - f)d\bar{t}^2 - 2fd\bar{t}dr - (1 + f)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where we have defined

$$f = 1 - g_{00} = \frac{2m}{r} - \frac{Q^2}{r^2}.$$

Now, we can introduce the advanced time parameter, as we did for the Schwarzschild solution,

$$v = \bar{t} + r,$$

hence, we can write the Reissner-Nordstrøm solution in advanced Eddington-Finkelstein coordinates:

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dv^2 - 2fdvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (3.24)$$

We find that there is an absolute event horizon at $r = r_+$, hence no signal can escape from region II to region I. Any particle entering region II will move necessary toward the centre until it crosses $r = r_-$ or reaches it asymptotically. However, in the region III, particles do not need to fall into the singularity at the centre and, in the case of neutral particles, they cannot reach the singularity.

Chapter 4

Kerr Metric

4.1 Null tetrads

The Kerr solution describes rotating black holes. We will not resolve directly the vacuum field equations for the Kerr solution, but we will, instead, use the Newman and Janis "trick" to obtain the Kerr solution from the Schwarzschild solution. This same trick can be applied on the Reissner-Nordstrøm solution to obtain the most generic solution for charged rotating black holes: the Kerr-Newman solution. In order to use this approach, we introduce the idea of a null tetrad.

The free test particle travels on timelike geodesics. We can consider a 2-surface S ruled by a congruence of timelike geodesics, that is a family of geodesics such that exactly one of the curves goes through every point of S. The parametric equation of S is given by

$$x^{\alpha} = x^{\alpha}(\tau, \nu),$$

where τ is the proper time and ν labels distinct geodesics. We define the vector field on S by

$$v^{\alpha} = \frac{dx^{\alpha}}{d\tau},$$

where v^{α} is the tangent vector to the timelike geodesics at each point.

Now, we introduce an orthogonal frame of three unit spacelike vectors at any point P of the curve C, on which the particle moves:

$$e_{\boldsymbol{a}}^{\alpha} = (e_1^{\alpha}, e_2^{\alpha}, e_3^{\alpha}),$$

which are all orthogonal to v^{α} and where $\boldsymbol{a}$ is a bold label running from 1 to 3. Therefore, we can complete the set of four orthonormal coordinates defining

$$e_0^{\alpha} = v^{\alpha},$$

hence, we have the following orthonormality relations

$$\begin{cases} e_0^{\alpha} e_{0\alpha} = -e_{\boldsymbol{a}}^{\alpha} e_{\boldsymbol{a}\alpha} = 1 \\ e_0^{\alpha} e_{\boldsymbol{a}\alpha} = e_{\boldsymbol{a}}^{\alpha} e_{\boldsymbol{b}\alpha} = 0 \end{cases} \quad \text{with} \quad a, b = 1, 2, 3; \quad a \neq b.$$

The four vectors $e_{\boldsymbol{i}}^{\alpha}$ $(i = 0, 1, 2, 3)$ are said to form a frame or tetrad at P. The orthonormality relations can be summarized by

$$e_{\boldsymbol{i}}^{\alpha} e_{\boldsymbol{j}\alpha} = \eta_{\boldsymbol{ij}},$$

where $\eta_{\boldsymbol{ij}}$ is the Minkowski metric, that is $\mathrm{diag}(1, -1, -1, -1)$. We name the vectors of our frame

$$v^{\alpha} = e_0^{\alpha}, \qquad i^{\alpha} = e_1^{\alpha}, \qquad j^{\alpha} = e_2^{\alpha}, \qquad k^{\alpha} = e_3^{\alpha}.$$

The most important case is when the tetrad vectors are taken to be null vectors. To study a null tetrad, working at a point, we define a matrix of scalar $g_{\boldsymbol{ij}}$, called the frame metric, by

$$g_{\boldsymbol{ij}} = g_{\alpha\beta} e_{\boldsymbol{i}}^{\alpha} e_{\boldsymbol{j}}^{\beta}. \tag{4.1}$$

Since $e_{\boldsymbol{i}}^{\alpha}$ are linearly independent and $g_{\alpha\beta}$, now a more generic metric, is non-singular, then the matrix $g_{\boldsymbol{ij}}$ is non-singular as well and, hence, it is invertible; its inverse is defined by

$$g_{\boldsymbol{ij}} g^{\boldsymbol{jk}} = \delta_{\boldsymbol{i}}^{\boldsymbol{k}}.$$

We can use the frame metric to raise and lower tensor indices like we would use a metric tensor. We find that the inverse relationship to 4.1 is

$$g_{\alpha\beta} = g_{\boldsymbol{ij}} e_{\alpha}^{\boldsymbol{i}} e_{\beta}^{\boldsymbol{j}}.$$

We now take

$$e_0^{\alpha} = l^{\alpha} = \frac{1}{\sqrt{2}}(v^{\alpha} + i\alpha),$$

$$e_1^{\alpha} = n^{\alpha} = \frac{1}{\sqrt{2}}(v^{\alpha} - i\alpha).$$

These are null vectors, that is

$$l^{\alpha} l_{\alpha} = n^{\alpha} n_{\alpha} = 0,$$

and they satisfy the normalization condition

$$l^{\alpha} n_{\alpha} = 1.$$

It is also advantageous to introduce a complex null vector and its complex conjugate

$$m^{\alpha} = \frac{1}{\sqrt{2}}(j^{\alpha} + ik^{\alpha}),$$

$$\bar{m}^{\alpha} = \frac{1}{\sqrt{2}}(j^{\alpha} - ik^{\alpha}).$$

These are null vectors as well

$$m^{\alpha} m_{\alpha} = \bar{m}^{\alpha} \bar{m}_{\alpha} = 0,$$

and they satisfy the normalizing condition

$$m^{\alpha} \bar{m}_{\alpha} = -1.$$

If we choose $(e_0^\alpha, e_1^\alpha, e_2^\alpha, e_3^\alpha) = (l^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$, we then define a null tetrad with frame metric

$$g_{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Thus, we have decomposed the metric tensor in terms of the vectors of the null tetrads as

$$g_{\alpha\beta} = l_\alpha n_\beta + l_\beta n_\alpha - m_\alpha \bar{m}_\beta - m_\beta \bar{m}_\alpha;$$

the contravariant form of which is

$$g^{\alpha\beta} = l^\alpha n^\beta + l^\beta n^\alpha - m^\alpha \bar{m}^\beta - m^\beta \bar{m}^\alpha. \tag{4.2}$$

4.2 The Kerr solution

The Schwarzschild solution in advanced Eddington-Finkelstein coordinates 2.24 returns the contravariant metric in the form

$$g^{01} = -1, \quad g^{11} = -\left(1 - \frac{2m}{r}\right), \quad g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2 \sin^2\theta}.$$

The contravariant metric may be written, using the contravariant frame metric 4.2, in terms of the following null tetrad

$$\begin{cases} l^\alpha = \partial_r, \\ n^\alpha = -\partial_v - \frac{1}{2}\left(1 - \frac{2m}{r}\right)\partial_r, \\ m^\alpha = \frac{1}{\sqrt{2}r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right). \end{cases}$$

The Newman-Janis "trick" starts by extending the coordinate r to take on complex values. In addition to this, certain terms involving r are complex conjugated, while others are left alone. This ambiguous step results in the following tetrad

$$\begin{cases} l^\alpha = \partial_r, \\ n^\alpha = -\partial_v - \frac{1}{2}\left[1 - m(r^{-1} + \bar{r}^{-1})\right]\partial_r, \\ m^\alpha = 1/\sqrt{2}\bar{r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right). \end{cases}$$

The ambiguity in the previous step is reflected by the fact that, if the complex conjugation on r was done in a different way, the desired result at the end of the procedure will not be derived. After this, we let the coordinate v take on complex values and perform the complex coordinate transformation

$$v \longrightarrow v' = v + ia\cos\theta, \quad r \longrightarrow r' = r + ia\cos\theta, \quad \theta \longrightarrow \theta' \quad \phi \longrightarrow \phi'.$$

where a is a constant. The basis vectors transform as follow

$$\begin{aligned} \partial_v &= \partial_{v'}, \\ \partial_r &= \partial_{r'}, \\ \partial_\theta &= \partial_{\theta'} - ia\sin\theta(\partial_{v'} + \partial_{r'}), \\ \partial_\phi &= \partial_{\phi'}. \end{aligned}$$

Applying this complex coordinate transformation to our null tetrad, requiring v', r', θ' and ϕ' to be real and dropping the primes, we obtain

$$\begin{cases} l^\alpha = \partial_r, \\ n^\alpha = -\partial_v - \frac{1}{2}\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right)\partial_r, \\ m^\alpha = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(-ia\sin\theta(\partial_{v'} + \partial_{r'}) + \partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right). \end{cases}$$

Knowing the contravariant frame metric 4.2, we reach the line element

$$ds^2 = \left(1 - \frac{2mr}{\rho^2}\right) dv^2 - 2dvdr + \frac{2mr}{\rho^2}(2a\sin^2\theta)dvd\bar{\phi} + 2a\sin^2\theta drd\bar{\phi} + \\ - \rho^2 d\theta^2 - \left((r^2 + a^2)\sin^2\theta + \frac{2mr}{\rho^2}a^2\sin^4\theta\right) d\bar{\phi}^2,$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta.$$

For convenience, we have replaced ϕ with $\bar{\phi}$. This is the advanced Eddington-Finkelstein form of Kerr's solution. To obtain the analogue of the Schwarzschild solution, we apply the transformation to new coordinates (t, r, θ, ϕ) by

$$dv = dt + \frac{2mr + \Delta}{\Delta}dr, \quad d\bar{\phi} = d\phi + \frac{a}{\Delta}dr,$$

where

$$\Delta = r^2 - 2mr + a^2.$$

The Boyer-Lindquist form of Kerr's solution follows:

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - adt]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2. \tag{4.3}$$

However, Kerr originally discovered the solution in Cartesian-type coordinates $(\bar{t}, x, y, z)$. The Kerr form of the solution is

$$ds^2 = d\bar{t}^2 - dx^2 - dy^2 - dz^2 + \\ - \frac{2mr^3}{r^4 + a^2z^2}\left(d\bar{t} + \frac{r}{a^2 + r^2}(xdx + ydy) + \frac{a}{a^2 + r^2}(ydx - xdy) + \frac{z}{r}dz\right)^2, \tag{4.4}$$

where

$$\begin{aligned} \bar{t} &= v - r, \\ x &= r\sin\theta\cos\phi + a\sin\theta\sin\phi, \\ y &= r\sin\theta\sin\phi - a\sin\theta\cos\phi, \\ z &= r\cos\theta. \end{aligned} \tag{4.5}$$

4.3 Properties of the Kerr solution

To study the properties of the Kerr solution, we will make use of all its three forms. Let's start with the Boyer-Lindquist form 4.3. The solution depends on two parameters, a and m; setting $a = 0$, we regain Schwarzschild solution in Schwarzschild coordinates, hence, we find that m is the geometric mass. The metric coefficients in 4.3 are independent of the coordinates t and ϕ, therefore we find that the Kerr solution is static and axially symmetric and this means that ∂_t and ∂_ϕ are both Killing vector fields. These are the only continuous symmetries. As for the discrete symmetries, the solution is not symmetric separately under time or ϕ reflection, but it is symmetrical under the simultaneous inversion of t and ϕ, that is

$$t \longrightarrow t' = -t, \qquad \phi \longrightarrow \phi' = -\phi.$$

The line element is also invariant under the transformation

$$t \longrightarrow t' = -t, \qquad a \longrightarrow a' = -a.$$

These invariances suggest that the Kerr field may arise from a spinning source, since running time backwards with a negative spin direction is the same as running the time forward with a positive spin direction. Hence, a specifies a spin direction. There are also different arguments that suggest a correlation between the angular velocity and a, and that the angular momentum, as measured at infinity, is ma.

r is not the usual spherical polar radial coordinate except asymptotically. If we consider Kerr solution in Kerr form 4.4, with (x, y, z) the usual Cartesian coordinates, we find that the spherical polar radial coordinate R is defined by

$$R = x^2 + y^2 + z^2,$$

that, considering 4.5, becomes

$$R = r^2 + a^2 \sin^2\theta.$$

For $r >> a$, we can expand R with the Taylor series:

$$R = r + \frac{a^2 \sin^2\theta}{2r} + \dots,$$

which shows that R and r coincide, asymptotically. Nevertheless, they also correspond as $a \longrightarrow 0$ in the Schwarzschild limit. Furthermore, it follows from Kerr solution 4.4 that

$$g_{\alpha\beta} \longrightarrow \eta_{\alpha\beta} \quad \text{as} \quad R \longrightarrow \infty,$$

so that the Kerr solution is asymptotically flat.

Calculations of the Riemann invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ reveals that the Kerr metric has an intrinsic singularity at $\rho = 0$. Since

$$\rho^2 = r^2 + a^2 \cos^2\theta = 0,$$

it follows that $r = \cos\theta = 0$, and, for the solution in Kerr form 4.4, this occurs when

$$x^2 + y^2 = a^2, \quad z = 0. \tag{4.6}$$

This singularity is a ring of radius a lying in the equatorial plane $z = 0$. We can also find the surfaces of infinite red-shift searching the values at which g_{00} vanishes. Considering the Boyer-Lindquist form of the solution 4.3, we find

$$g_{00} = (r^2 - 2mr + a^2 \cos^2\theta)/\rho^2 = 0,$$

from which

$$r = r_{s\pm} = m \pm (m^2 - a^2 \cos^2\theta)^{\frac{1}{2}}. \tag{4.7}$$

In the Schwarzschild limit $a \longrightarrow 0$, the surface S_+ reduces to $r = 2m$ and the surface S_- to $r = 0$. Let's consider the case $a^2 < m^2$, namely, the spin is small compared with the mass. The surfaces are axially symmetric, with S_+ possessing a radius $2m$ at the equator and a radius $m + (m^2 - a^2)^{\frac{1}{2}}$ at the poles, and the surface S_- being completely contained inside S_+.

The Killing vector field

$$X^\alpha = (1, 0, 0, 0)$$

has magnitude

$$X^2 = X^\alpha X_\alpha = g_{\alpha\alpha} X^\alpha X^\alpha = g_{00}$$

It follows from 4.6 and 4.7 that X^α is timelike outside S_+ and inside S_-, null on S_+ and S_-, and spacelike between S_+ and S_-. In analogy with the Schwarzschild solution, we search for the event horizon by looking for the hypersurfaces where $r =$ constant becomes null, that is where g^{11} vanishes. From the Boyer-Lindquist form of the solution 4.3, we find

$$g^{11} = -\frac{\Delta}{\rho^2} = -\frac{r^2 - 2mr + a^2}{r^2 + a^2 \cos^2\theta},$$

which results in two null event horizons (assuming $a^2 < m^2$)

$$r = r_\pm = m \pm (m^2 - a^2)^{\frac{1}{2}}.$$

As for Reissner-Nordstrøm solution 3.23, the Kerr solution is regular in the three regions:

I. $r_+ < r < \infty$,

II. $r_- < r < r_+$,

III. $0 < r < r_-$.

In the Schwarzschild limit $a \longrightarrow 0$, the two event horizons reduce to $r = 2m$ and $r = 0$, from which it follows that in the Schwarzschild solution the surfaces of infinite red-shift and the event horizons coincide. The event horizon $r = r_+$ lies entirely within S_+, generating to a region between the two known as the ergosphere.

4.4 The Kerr-Newman solution

We obtain the most general black hole solution applying the Janis-Newman "trick" to the Reissner-Nordstrøm solution in advanced Eddington-Finkelstein coordinates 3.24. We find the solution

$$ds^2 = \left(1 - \frac{2mr}{\rho^2} + \frac{Q^2}{\rho^2}\right) dv^2 - 2dvdr + \frac{2a}{\rho^2}(2mr - Q^2)\sin^2\theta dv d\bar{\phi} + 2a\sin^2\theta dr d\bar{\phi} + \\ - \rho^2 d\theta^2 - [(r^2 + a^2)^2 - (r^2 - 2mr + a^2 + Q^2)a^2\sin^2\theta]\frac{\sin^2\theta}{\rho^2} d\bar{\phi}^2,$$

which is the Kerr-Newman solution in advanced Eddington-Finkelstein coordinates. The solution depends on three parameters m, a and Q, defining the mass, spin and charge. The solution is static and axisymmetric and possess a stationary limit surface

$$r = m + (m^2 - Q^2 - a^2\cos^2\theta)^{\frac{1}{2}}$$

and, provided that $a^2 + Q^2 \leq m^2$, an outer event horizon

$$r = m + (m^2 - Q^2 - a^2)^{\frac{1}{2}}.$$

Chapter 5

The Thermodynamic Theory of Black Holes

5.1 Classical thermodynamics

Classical thermodynamics fails to cope properly with self-gravitating systems: gravitational collapse, apparently, leads to a state of unbound entropy and space curvature, but it has long been established that such a collapse, in the presence of an event horizon, leads to an equilibrium state, the black hole. However, the inclusion of quantum effects by Hawking (1975) damps out this divergence and assigns a definite entropy to the black holes. Furthermore, this entropy has a geometric definition through the general theory of relativity. With the entropy is associated an all range of thermodynamic quantities and the black hole system results thermodynamically completely defined.

Hence, before discussing the thermodynamic theory of black holes, we will recap the most important results of classical thermodynamics that will be needed afterwards.

To define the thermal equilibrium between two systems, we introduce the temperature, T: the two systems are in thermal equilibrium if they have the same temperature and, if a system is in equilibrium, then its temperature is constant. The transitive relation can be applied to this property, leading to the zero law of thermodynamics.

Zero law. *If two systems are both in thermal equilibrium with a third system, then they are in thermal equilibrium with each other.*

The thermodynamic system is described through equations of state that are thermodynamic equations relating state variables which describe the state of matter under a given set of physical conditions. An example is found with the first law of thermodynamics that is, essentially, the law of conservation of energy in the context of thermodynamic systems.

First law. *The increase in internal energy of a closed system is equal to the total of the energy added to the system:*

$$\Delta U = W + \mathcal{Q},$$

where W is the work and $\mathcal{Q}$ the heat.

The second law of thermodynamics shows the irreversibility of natural processes, and their tendency to lead towards spatial homogeneity of matter, of energy and, especially, of temperature. It implies the existence of a quantity called the entropy, S, of a thermodynamic system. The entropy depends on the heat and temperature through the relation

$$\delta \mathcal{Q} = T dS.$$

Furthermore, Boltzmann demonstrated the existence of the equation

$$S = k \ln w, \tag{5.1}$$

where k is the Boltzmann constant and w the statistical weight of the macrostate with entropy S, that is the number of possible microstates consistent with it.

The third law of thermodynamics, also known as Nernst theorem, states that:

Third law. *The entropy differences that can be connected by an isothermal process must vanish as $T \longrightarrow 0$.*

This is sometimes interpreted as implying the attainability of the absolute zero of temperature.

An useful quantity we can define is the heat capacity, or thermal capacity, that is a measurable physical quantity equal to the ratio of the heat added to (or removed from) an object to the resulting temperature change. The thermal capacity of a body, maintaining the quantity x constant, is

$$C_x = \left(\frac{\partial Q}{\partial T}\right)_x = T\left(\frac{\partial S}{\partial T}\right)_x . \tag{5.2}$$

Between thermal capacities, it exists the relation

$$C_p - C_V = TV\frac{\alpha^2}{K},$$

where p is the pressure, α is the coefficient of thermal expansion, and K is the isothermal compressibility. The expression of these coefficients is

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p , \quad K = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T .$$

The Maxwell relations, a set of thermodynamic equations which are derivable from the symmetry of second derivatives and from the definitions of the thermodynamic potentials, are also of great importance. A useful example is given by

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V .$$

The thermodynamic potential is a scalar quantity used to represent the thermodynamic state of a system. A couple of examples are the internal energy and the Gibbs free energy, that is a thermodynamic potential that can be used to calculate the maximum of reversible work that may be performed by a thermodynamic system at a constant temperature and pressure. It is defined by

$$G = U - TS + pV,$$

where p is the pressure and V the volume.

Another thermodynamic potential we will need afterwards is the Helmholtz free energy that measures the useful work obtainable from a closed thermodynamic system at a constant temperature and volume. It is defined by

$$F = U - TS.$$

5.2 The Smarr relation

In order to discuss the black hole thermodynamics, we will give for granted the knowledge of two important results by S. Hawking: the black holes temperature and the evaporation of black holes. The latter is a black hole's emission of black-body radiation, the Hawking radiation, that reduces its mass, and therefore its energy, due to quantum effects near the event horizon. Without another way to replenish the mass, the black hole will shrink and, ultimately, vanish.

The relation between the geometrically defined thermodynamic quantities and their classical counterparts is not yet clear; we can apply, nevertheless, traditional thermodynamics techniques to the Kerr-Newman black holes. The first and second law of thermodynamics may be retrieved through the identification:

$$\begin{cases} S \propto & \text{area of the event horizon} \quad A. \\ T \propto & \text{surface gravity} \quad \kappa, \end{cases}$$

that is

$$\kappa = \frac{\sqrt{M^2 - Q^2 - J^2/M^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - J^2/M^2}},$$

where M is the mass, J the angular momentum and Q the electric charge. The fact that a quantity κ, which is constant over the event horizon, may be defined at all, is equivalent to the zero law of thermodynamics, which asserts that the temperature of a system at thermodynamic equilibrium is constant.

Furthermore, L. Smarr (1973) found, originally for classical black holes systems, the fundamental thermodynamic relation which contains all the information about the thermodynamic state of black hole matter

$$M^2 = \frac{1}{4}\left(\frac{A}{4\pi}\right) + \left(\frac{4\pi}{A}\right)\left(J^2 + \frac{1}{4}Q^4\right) + \frac{1}{2}Q^2. \tag{5.3}$$

It will be $J = 0$ for the Reissner-Nordstrøm solution, $Q = 0$ for the Kerr one and $J = Q = 0$ for the Schwarzschild one. This relation 5.3 can be rewritten identifying A with a real entropy:

$$S = \frac{1}{4}kA.$$

We may regard the parameter S, J and Q as a set of global quantities for black holes matter, $M = M(S, J, Q)$. Total energy is, usually, a homogeneous first order function of the global parameter, but 5.3 is not, that is because black holes matter cannot be divided up into subsystems with a separate identity. Nonetheless, we will consider the global quantities extensive as they are in classical thermodynamics. Then, choosing units to make the value of k to be $1/8\pi$ for convenience, the fundamental equation becomes

$$M = M(S, J, Q) = \sqrt{2S + \frac{1}{8S}\left(J^2 + \frac{1}{4}Q^4\right) + \frac{1}{2}Q^2}. \tag{5.4}$$

It is useful to invert this equation, considering the root with the plus sign because $S > 0$, to obtain

$$S = \frac{1}{4}M^2 - \frac{1}{8}Q^2 + \frac{1}{4}M^2\left[1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}\right]^{\frac{1}{2}}, \tag{5.5}$$

As we know, the first law of thermodynamics states that, in any thermodynamic change, the total energy is conserved; therefore, if M changes by an infinitesimal quantity:

$$dM = TdS + \Omega dJ + \phi dQ, \tag{5.6}$$

where the quantities T, Ω and ϕ are defined by

$$T = \frac{\partial M}{\partial S} = \frac{1}{M}\left[1 - \frac{J^2 + \frac{1}{4}Q^4}{16S^2}\right], \tag{5.7}$$

$$\Omega = \frac{\partial M}{\partial J} = \frac{J}{8MS}, \tag{5.8}$$

$$\phi = \frac{\partial M}{\partial Q} = \frac{Q(Q^2 + 8S)}{16MS}. \tag{5.9}$$

These are the corresponding intensive parameters which are constant on the horizon. Ω is the angular velocity of the event horizon associated with the angular momentum J, ϕ is the electric potential associated with the electric charge Q, and T is the black hole temperature as was found by Hawking (1975) from quantum theory. The three equations 5.7-5.9 are three equations of state.

The relation 5.4 can be made homogeneous of degree 1/2 considering M as a function of Q^2 instead of Q:

$$\frac{1}{2}M = TS + \Omega J + \Theta Q^2 = TS + \Omega J + \frac{1}{2}\phi Q,$$

where

$$\Theta = \frac{\phi}{2Q} = \frac{Q^2 + 8S}{32MS}.$$

This relation, firstly noted by Smarr, is the black hole equivalent of Gibbs-Duhem relation of thermodynamics that shows how the intensive thermodynamic quantities are not independent but related.

5.3 The phase transition

From the Smarr relation 5.3, we can regain the fundamental results of thermodynamics; this is the case for thermal capacity 5.2. Let's suppose to hold a rotating black hole in equilibrium, at some temperature T, in a temperature bath. If the external bath temperature is increased slightly, the black holes will absorb the energy reversibly. This absorption will be isotropic and it will not change, on average, the angular momentum

J and the charge Q. Therefore, we will compute the heat capacity $C_{J,Q}$, where J and Q are held constant, eliminating M through 5.5 and 5.4; hence, the result is

$$C_{J,Q} = T\left(\frac{\partial S}{\partial T}\right)_{J,Q} = \frac{8MS^3T}{J^2 + \frac{1}{4}Q^4 - 8T^2S^3}. \tag{5.10}$$

An interesting example is the Schwarzschild case, $J = Q = 0$; in this case, the thermal capacity C reduces to $C \equiv -M/T \equiv -1/T^2 \equiv -M^2$. This quantity is negative definite, then the Schwarzschild black hole gets hotter as it radiates energy.

Through 5.7, we can rewrite 5.10 as

$$C_{J,Q} == \frac{MST}{2 - T(2M + ST)}. \tag{5.11}$$

As we approach the extreme case $T \longrightarrow 0$ for the general solution, namely the Kerr-Newman one, we see that $S \longrightarrow \frac{1}{4}M^2 - \frac{1}{8}Q^2 > 0$, so $C_{J,Q} \longrightarrow 0^+$; hence, $C_{J,Q}$ changes sign before the limit.

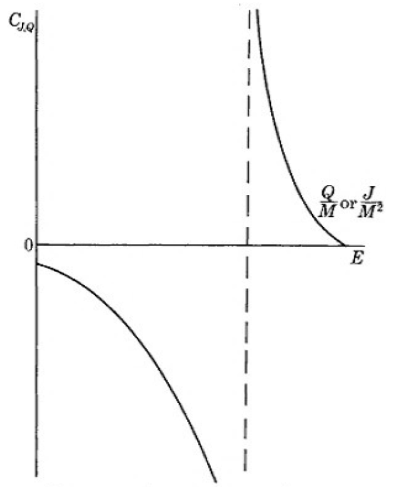

Figure 5.1: General behaviour of heat capacity at constant J and Q. The broken line indicates the position of the phase transition, at which $C_{J,Q}$ suffers an infinite discontinuity. Therefore, for low values of J/M^2 and Q/M, it is negative, but beyond the line it is positive, falling to zero at the extreme value E.

We reach another significant result if we express $C_{J,Q}$ as a function of J, Q and M by eliminating S and T from 5.10, using 5.4 and 5.5. A significant result is reached: there are some values for J and Q where the specific heat presents an infinite discontinuity, and changes from positive to negative. The general heat capacity behaviour is reported in figure 5.1. To explore the meaning behind this discontinuity, we notice that at the critical point S, T, Ω and ϕ are all continuous and finite. In particular, the Gibbs free energy

$$G = M - TS - \Omega J - \phi Q$$

is continuous and so are its gradients $(\partial G/\partial T)_{\Omega,\phi}$, $(\partial G/\partial \Omega)_{T,\phi}$ and $(\partial G/\partial \phi)_{\Omega,T}$. A transition characterized by the continuity of G and its first derivatives and by a discontinuity in the second derivatives, e.g. the heat capacities, can be classified as a second order phase transition.

This discontinuity occurs at $J^2 = \alpha M^4$, $Q^2 = \beta M^2$ for some values of $\alpha, \beta > 0$. From 5.10 follows the restriction $J^2 + \frac{1}{4}Q^4 = 8T^2S^3$. Eliminating T and S with 5.7 and 5.5 in the restriction and substituting J and Q, we obtain the equation

$$\alpha^2 + 6\alpha + 4\beta = 3. \tag{5.12}$$

We have also to impose another restriction: that $\alpha + \beta > 1$. This condition follows from the fact that $T > 0$. Hence, for a given mass M, from 5.7 we obtain

$$J^2 + \frac{1}{4}Q^4 < 16S^2.$$

We substitute 5.5 in this expression to obtain

$$\frac{J^2}{M^4} + \frac{Q^2}{M^2} < 1 \implies \alpha + \beta < 1.$$

In the case of an uncharged black hole, a Kerr one, we have $\beta = 0$, whence

$$\alpha = 2\sqrt{3} - 3 \simeq 0.464.$$

The ratio J/M is usually called a, and so, at the critical point,

$$a \simeq 0.68M.$$

In the case of Reissner-Nordstrøm black holes, we have $J = 0 \implies \alpha = 0$; then, at the

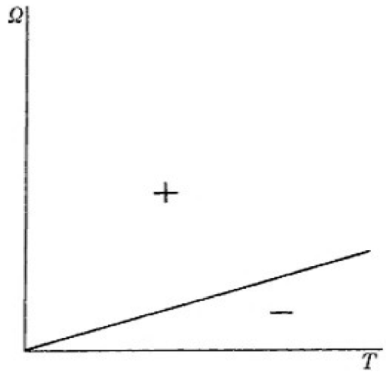

Figure 5.2: Phase diagram for Kerr black hole. The phase ± correspond to positive negative values of the heat capacity at constant J and the two regions are divided by the phase line $\Omega = 0.23T$.

critical point,

$$\beta = \frac{3}{4},$$

$$\text{or} \quad Q = \frac{1}{2}\sqrt{3}M \simeq 0.86M.$$

Knowing the value of α, and hence J, at the critical point for Kerr metric, we can eliminate M and S between 5.6, 5.7 and 5.8 to obtain the constant ratio

$$\frac{\Omega}{T} = \frac{J}{8MS}\frac{16S^2M}{16S^2 - J^2} \simeq 0.23;$$

so that the phase line in figure 5.2 is straight.

For Reissner-Nordstrøm metric we know the value of β, and therefore Q, finding, through the expressions 5.5 and 5.9, that the singularly occurs at

$$\phi = \frac{1}{\sqrt{3}} \simeq 0.58,$$

so that the phase transition takes place always at the same value of the electric potential, irrespective of the mass of the black hole.

Restricting ourself to Kerr and Reissner-Nordstrøm solutions, we will be able to obtain other thermal capacities by computing S as a function of T and Ω for the Kerr case or as a function of T and ϕ for the Reissner-Nordstrøm case:

$$S = \frac{(1 - 16\Omega^2 S)^2}{2T^2(1 - 8\Omega^2 S)} \quad \text{(Kerr)},$$

$$S = \frac{(1 - \phi^2)^2}{2T^2} \quad \text{(R-N)}.$$

We compute, then, the new heat capacities

$$C_\Omega \equiv T\left(\frac{\partial S}{\partial T}\right)_\Omega = -\frac{2TS}{M(16\Omega^2+T^2)}, \tag{5.13}$$

$$C_\phi \equiv T\left(\frac{\partial S}{\partial T}\right)_\phi = -2S. \tag{5.14}$$

Moreover, we can make use of the Maxwell relations between partial derivatives, as seen in § 5.1, to obtain

$$\left(\frac{\partial J}{\partial T}\right)_\Omega = \left(\frac{\partial S}{\partial \Omega}\right)_T = -\frac{8M^3\Omega S}{4M^3\Omega^2+TS}, \tag{5.15}$$

$$\left(\frac{\partial Q}{\partial T}\right)_\phi = \left(\frac{\partial S}{\partial \phi}\right)_T = \frac{2(-2\phi)(1-\phi^2)}{2T^2} = -\frac{Q}{T}. \tag{5.16}$$

Continuing our analogy with the classical thermodynamics and the heat capacities, we can define something similar to the coefficients of thermal expansion and isothermal compressibility:

$$\alpha = \begin{cases} -\frac{1}{J}\left(\frac{\partial J}{\partial T}\right)_\Omega, \\ -\frac{1}{Q}\left(\frac{\partial Q}{\partial T}\right)_\phi; \end{cases} \qquad K = \begin{cases} -\frac{1}{J}\left(\frac{\partial J}{\partial \Omega}\right)_T, \\ -\frac{1}{Q}\left(\frac{\partial Q}{\partial \phi}\right)_T. \end{cases}$$

Here, we can envisage α as a coefficient of thermal rotation or electric charging and K as the ease with which the black hole may be spun, or charged up, at a constant temperature.

Among these four thermal capacities we have defined, as we have seen in the classical case, we can establish the relations

$$C_\Omega - C_J = TJ\frac{\alpha^2}{K}, \quad C_\phi - C_Q = TQ\frac{\alpha^2}{K}. \tag{5.17}$$

From the expressions 5.13 and 5.14, we see that C_Ω and C_ϕ are both continuous across the phase line $\Omega \simeq 0.23T$ or $\phi \simeq 0.58$ as it is α, J and Q. Therefore, it follows from 5.17 that, for C_J and C_Q to have a discontinuity across the phase line, K must vanish at the critical point. That can be easily shown in the Reissner-Nordstrøm case through the relation

$$Q = 2\phi(1-\phi^2)/T$$

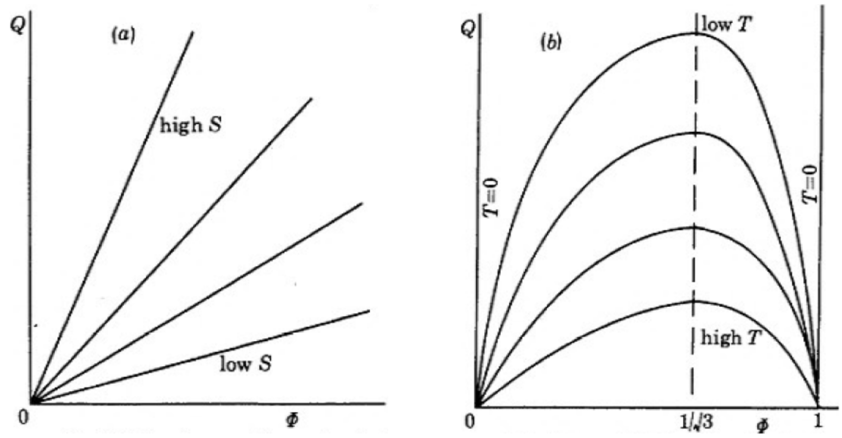

Figure 5.3: (a) Isentropes and (b) isotherms for Reissner-Nordstrøm black hole. The broken line indicates the phase transition at $\phi = 1/\sqrt{3}$. Each Q, T generally corresponds to two values of ϕ. The point $\phi = 1$ is the thermodynamic limit, at which $T = 0$.

that returns

$$K = -2(3\phi^2 - 1)/TQ,$$

which vanishes at $\phi = 1/\sqrt{3}$.

In figures 5.3a and 5.3b the general behaviour of the isotherms (T is a constant) and isentropes (S is a constant) is represented on a ϕ,Q diagram for a Reissner-Nordstrøm black hole. The same is represented in figure 5.4 for the Kerr black hole on an Ω,J diagram. For a given value of Q or J, there are in general two possible values of ϕ or Ω that give the same temperature: one value corresponds to a small mass black hole with high ϕ or Ω, the other to a large mass object with small ϕ or Ω. At the phase transition point, these two values coincide. The same cannot be said for the entropy.

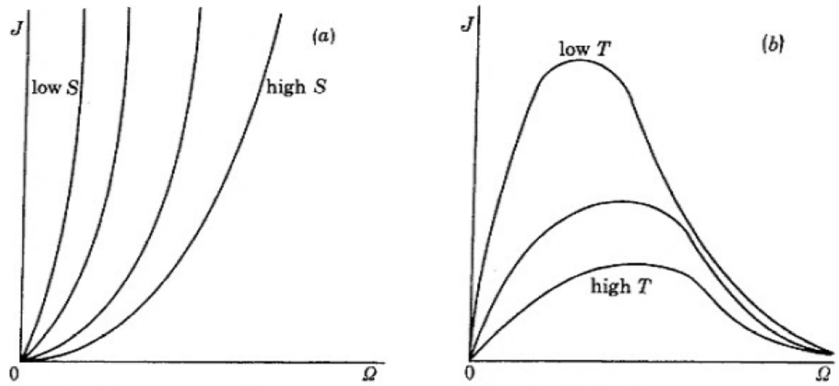

Figure 5.4: (a) Isentropes and (b) isotherms for Kerr black hole. Each J, T generally corresponds to two values of Ω.

As an example of this theory applications, we will consider two Schwarzschild black holes, each in equilibrium at constant temperature inside a small adiabatic enclosure, with masses M, m and temperatures T, t respectively. We will ignore the mass of the radiation, and we will consider the interaction between the two black holes to be small in order to assign definite separate values at the thermodynamic parameter of each black hole. The second condition is never really met, but the interaction can be considered only a negligible disturbance from the equilibrium when the separation between the black holes greatly exceeds their Schwarzschild radii. This two systems can be brought into equilibrium in various ways.

The first is to join the enclosures together and remove the dividing wall, allowing the two black holes to coalesce adiabatically. The final mass is the sum of the initial masses, $M+m$, and the final temperature is given by 5.7, that is $(M+m)^{-1} = Tt/(T+t)$, and it is less than either T or t. The process is highly irreversible and the entropy jumps by Mm from $\frac{1}{2}(M^2+m^2)$ to $\frac{1}{2}(M+m)^2$. The entropy increases even if $T = t$, in contrast to a conventional thermodynamic system.

The second method is to join the enclosures and allow a free flow of heat between them, without letting the black holes come into contact. The final entropy is $2 \times \frac{1}{2}((M+m)/2)^2$, that is always less than the initial one $\frac{1}{2}(M^2+m^2)$ if $M \neq m$; the process is therefore forbidden by the second law. The final masses, and hence the temperatures, must be equal for equilibrium to occur at a minimum of the entropy, that is a point of instability. Then, a slight perturbation causes one black hole to feed on the other, drawing energy from it and thereby increasing the temperature difference, accelerating until one black hole is evaporated completely away.

The last method is to transfer energy reversibly from one enclosure to the other. The second method reduces to this one after the equilibrium is broken. This method enables the greatest amount of thermal energy to be extracted from the system in the form of work. A reversible process occurs with no change of total entropy, hence, considering M, m, T and t variables now, follows from the first law 5.6

$$dS = dM/T + dm/t = 0 \implies MdM + mdm = 0,$$

that integrated yields

$$M_f = \left[\frac{1}{2}(M^2 + m^2)\right]^{\frac{1}{2}} \implies T_f = \frac{\sqrt{2}Tt}{(T^2 + t^2)^{\frac{1}{2}}}.$$

These are the final mass and temperature of each black hole. In particular, the temperature is the maximum equilibrium temperature that can be achieved. The energy extracted during the process is

$$M + m - 2\left[\frac{1}{2}(M^2 + m^2)\right]^{\frac{1}{2}}.$$

Numerous other different cases can be studied with this method if $J, Q \neq 0$.

5.4 Stability and equilibrium conditions

In the previous section we have considered a black hole in thermodynamic equilibrium, and in this one, we will search the conditions for stable equilibrium by using the second law of thermodynamics.

The entropy equation 5.5, we have found from the Smarr relation, reveals that, for a given black hole mass M, the entropy is maximized by choosing $J = Q = 0$. From this, it follows that a black hole will spin down and discharge. A rotating black hole obeys the phenomenon of superradiance that predicts that boson quanta of axial angular momentum m and frequency $\omega < m\Omega$ cause stimulated emission of rotational energy. For a charged black hole an analogous process occurs for modes with $\omega < e\phi$, where e is the charge on the emitted quantum. In the former effect, the spin-down rate is very slow while the electric discharge process is very much more efficient. For supermassive black holes, the electric discharge will cease completely because the available thermal and electromagnetic energy is insufficient, and as such unable to cause the emission of charged particles with a non-zero rest mass.

Now, in the context of a black hole immersed in a bath of thermal radiation at the temperature T, it is possible to neglect the superradiance because non-superradiant modes come into equilibrium; at that point J and Q can be considered as constants. The equilibrium will be found at a turning point of Helmholtz free energy F, and it will be stable if the turning point is a minimum. F is defined by

$$F = M - TS.$$

The condition $dF = 0$ reproduces the relation between M, T and S which follows from 5.7-5.9. Imposing $d^2F > 0$

$$2(1-\alpha-\beta)^{\frac{3}{2}} < 6\alpha - \alpha\beta - 3\beta - 2 \implies (4\alpha+\beta^2)(a^2+6\alpha+4\beta-3) > 0,$$

from which it follows the equation $(a^2 + 6\alpha + 4\beta - 3) > 0$, that is the condition for the black holes to be in the higher J, Q phase (see 5.12). In this phase we know that the heat capacity is positive; for values of α and β below the critical transition values the equilibrium is unstable. For a negative heat capacity, the black hole will indefinitely continue to feed on the external heat bath, while for a positive one, as the temperature increases, the black hole will radiate some net energy causing the temperature to increase further and further until the black hole will disappear in an explosion.

A similar technique can be also be applied to study the adiabatic equilibrium. Let's consider a Schwarzschild black hole inside a box with perfectly reflecting walls, then we must take into account the accumulation of thermal evaporation radiation inside the box. The equilibrium condition is that the total entropy of the box contents is an extremum and the equilibrium is stable if that is a maximum. The total entropy is the sum of the ones of the black hole and the radiation:

$$S = \frac{1}{2}M^2 + \frac{4}{3}(aVM_r^3)^{\frac{1}{4}},$$

where M_r denotes the mass of the radiation, V is the volume of the box and a is the radiation constant. The adiabatic condition for equilibrium is that $dS = 0$ with $dM + dM_r$.

The result is

$$V = \frac{M_r M^4}{a}.$$

We are searching for a maximum, and hence, from $d^2S < 0$, it follows

$$M > 4M_r, \qquad V < \frac{1}{4}\frac{M^5}{a}. \tag{5.18}$$

Hawking was the one to point out that Schwarzschild black holes with masses less than four fifths of the total cannot be in stable adiabatic equilibrium with thermal radiation and, therefore, they will either evaporate away or will grow in mass until conditions 5.18 are met.

For $J, Q \neq 0$, in order to maintain the stability, the volume of the box increases until, at the critical values given by 5.12, V is infinite. If we require isothermal or adiabatic equilibrium with Ω and ϕ held constant, it is necessary to deal instead with the Gibbs function G or the enthalpy, respectively.

5.5 The third law

We have seen how the third law of thermodynamics, or Nernst theorem, can be read as the unattainability of the absolute zero of temperature, and from this, it follows, as we have seen, that, for a given mass M, for $T \longrightarrow 0$ we obtain the relation

$$\alpha + \beta < 1.$$

The Kerr-Newman family of solutions must follow this condition to describe black holes and not naked singularities, where a naked singularity is a gravitational singularity without an event horizon. Hence the unattainability of the absolute zero is equivalent to the cosmic censorship hypothesis (R. Penrose 1969) which posit that no naked singularities, other than the Big Bang singularity, exist in the universe.

Therefore, we will need to examine how well the third law holds up for black holes. As $T \longrightarrow 0$, equations 5.15 and 5.16 reveal that

$$\left(\frac{\partial S}{\partial \Omega}\right)_T \longrightarrow -\frac{2S}{\Omega} = -M^3,$$

$$\left(\frac{\partial S}{\partial \phi}\right)_T \longrightarrow \infty.$$

Hence, the isothermal quantity α does not vanish at $T = 0$, in apparent contradiction to the Nernst postulate.

We can, in fact, accept the third law in a stronger form, due to Planck, which states that the entropy of a quantum thermodynamic system vanishes as $T \longrightarrow 0$. The statistical foundation of this law is that there exists only one, unique, microstate associated with the zero temperature state. There is a consistent statistical base for the black hole entropy which is based on an examination of the microscopic particle states, which make up a black hole with given M, J and Q. Nonetheless, neither this formulation of the law is satisfied.

Inspection of 5.11 shows that

$$C_{J,Q} \longrightarrow 0 \quad \text{as} \quad T \longrightarrow 0,$$

so that the entropy change in a constant J, Q process remains finite as $T = 0$ is approached:

$$\Delta S = \int_0^T \frac{C_{J,Q}}{T}\, \mathrm{d}T < \infty.$$

This is also true for a constant Ω process because from 5.13 follows

$$C_\Omega \longrightarrow 0 \quad \text{as} \quad T \longrightarrow 0.$$

Nevertheless, we have seen from 5.5 that, as $T \longrightarrow 0$, $S \longrightarrow \frac{1}{4}M^2 - \frac{1}{8}Q^2 > 0$ and it is not possible to remove this zero point entropy with an additive constant because M and Q are extensive parameters.

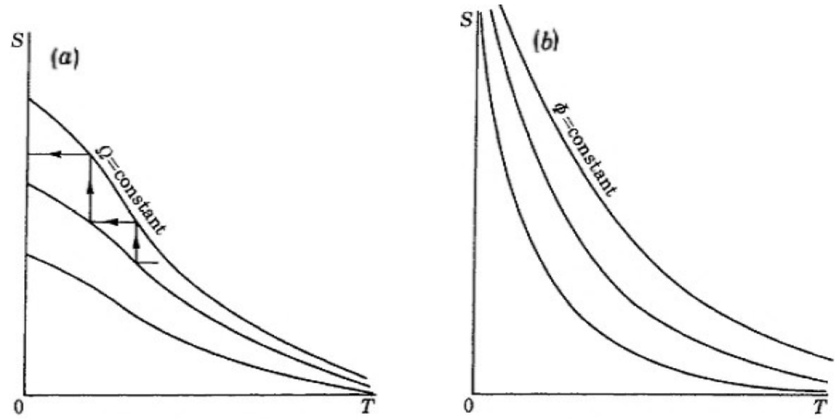

Figure 5.5: (a) The entropy of a Kerr black hole does not vanish at $T = 0$, so lines of constant Ω do not intersect there. A thermodynamically permitted path which reaches absolute zero in a finite sequence of transition is shown.
(b) Lines of constant $\phi < 1$ do not intersect $T = 0$.

For the existence of an M-dependent zero point entropy, lines at constant Ω (with $Q = 0$) do not intersect $T = 0$ on an $S - T$ diagram, see 5.5 (a). This is seen as a violation of the third law because the system can be cooled down to $T = 0$ through a reversible process in a finite number of steps, as represented in the figure through arrows, although this is not enough to conclude that there exist thermodynamic ways to convert a black hole into a naked singularity. There may be reasons, even non-thermodynamic ones, for the process not to occur. For example, from 5.14 we see that C_ϕ remains finite at $T = 0$ in apparent violation of the third law: it is not possible to cool down a black hole along a line of constant ϕ. The reason for this may be found eliminating S from 5.9

$$\frac{2\phi}{1+\phi^2} = \frac{Q}{M} = \beta^{\frac{1}{2}}$$

from which it follows that for a given ϕ, β is constant and thus, it is impossible to approach $T = 0$, $\beta = 1$ with fixed ϕ, as seen in figure 5.5 (b).

It is interesting to imagine what would happen if a black hole was cooled into a naked singularity. For a non-rotating supermassive object carrying an enormous electric charge, the disturbance due to superradiance disappears, and the black hole may be in complete and stable equilibrium with its environment keeping a constant charge. If $Q/M > \sqrt{3}/2$ ($\beta > \sqrt{3}/2$), C_Q is positive, so the black hole may be cooled down in the traditional way

by manipulating the environment and allowing the black hole to radiate energy. Q/M may or may not be decreased to 1 ($T = 0$) depending on whether the third law turns out to apply to black holes.
What happens then can be determined with the use of the techniques developed by P.C.W Davies, S.A.Fulling and W.G.Unruh for studying the energy-momentum tensor of a quantum field in the proximity of a black hole and that may be extended to deal with the naked singularity case. Without giving any detail of the calculation, it is found that, for the Reissner-Nordstrøm case, the calculations are perfectly consistent with the thermodynamic picture given for $Q < M$, but for $Q > M$ the model predicts that "naked" collapse also produces radiation, with such intensity that the collapsing matter is entirely evaporated away before a naked singularity can form.

5.6 Conclusions

The developments on the quantum properties of black holes make it likely that they are subject at least to the first and second laws of thermodynamics. Nevertheless, in classical thermodynamics, the entropy is defined through the Boltzmann's equation 5.1, but a similar definition for a black hole would be only formal because there is no way, even in principle, we could observe the constituent microstates from outside the black hole. Moreover, it is not possible to consider the black hole entropy as being located with a certain density in a certain region of space-time. Black hole entropy is a truly global property.
From all this, it seems that the entropy is more a property of the gravitational field than of the microscopic matter content of the black hole. It could, after all, be made entirely of gravitons.
Furthermore, the temperature, and hence the entropy, is certainly affected by the distribution of matter around it.
If the black hole entropy is really gravitational in origin, we might expect to encounter entropy even in the absence of matter. Empty universes with gravitational fields can occur if Einstein's cosmological constant is non zero, changing the general form of the Einstein equations.

Bibliography

[1] d'Inverno R., *"Intoduzione alla Relatività di Einstein"* , Italian translation by Carlo Bertoni and Fabio Finelli, Cooperativa Libraria Universitaria Editrice Bologna, Bologna 2001 (orig. ed. *"Introducing Einstein's Relativity"* , Oxford University Press, Oxford 1992).

[2] Schutz B., *"Geometrical methods of mathematical physics"* , Cambridge University Press, Cambridge 1999 [1980].

[3] Focardi S., Mazza I., Uguzzoni A. & Villa M., *"Fisica Generale. Meccanica e Termodinamica"* , Casa Editrice Ambrosiana, Milano 2014 [1999].

[4] Davies, P. C. W., *"The Thermodynamic Theory of Black Holes"* , Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, vol. 353, no. 1675, 1977, pp. 499–521.