Alma Mater Studiorum · Università di Bologna

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea Magistrale in Matematica

KAZHDAN-LUSZTIG CONJECTURE AND MOMENT GRAPHS

Tesi di Laurea in Geometria

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Introduzione

La categoria $\mathcal{O}$ fu introdotta da J. Bernstein, I. Gel'fand e S. Gel'fand negli anni '70 (cfr. [5]) nel tentativo di estendere i risultati classici sulle rappresentazioni di dimensione finita di un'algebra di Lie semisemplice $\mathfrak{g}$, in particolare le formule di Weyl per i caratteri dei moduli irriducibili. Si tratta di una sottocategoria piena di $\mathfrak{g}$-mod i cui oggetti godono di determinate proprietà di finita generabilità e diagonalizzabilità rispetto all'azione di una fissata sottoalgebra di Cartan $\mathfrak{h}$. Benché i suoi oggetti non siano più semisemplici, come invece i moduli di dimensione finita (teorema di Weyl), la categoria $\mathcal{O}$ è abeliana ed artiniana, ed è possibile estendere la nozione di carattere formale. Si pone pertanto in modo naturale il problema di trovare delle formule per i caratteri dei suoi oggetti semplici. La categoria $\mathcal{O}$ possiede una sottocategoria $\mathcal{O}_0$, detta *blocco principale*, alla quale è possibile restringere il problema, ed ammette alcuni oggetti speciali, i *moduli di Verma*, di cui è particolarmente semplice calcolare il carattere formale. Gli oggetti semplici e i moduli di Verma di $\mathcal{O}_0$ sono in numero finito e sono indicizzati dagli elementi del gruppo di Weyl di $\mathfrak{g}$. È possibile scrivere i caratteri cercati in funzione di quelli dei moduli di Verma nella forma seguente

$$\operatorname{ch} L_w = \sum_{y \in W} a_{y,w} \operatorname{ch} M_y$$

ovvero il problema si riconduce a trovare i coefficienti $a_{y,w}$.

In un articolo del 1979 (cfr. [28]) D. Kazhdan e G. Lusztig mostrarono che nello studio dell'algebra di Hecke di un sistema di Coxeter qualunque (W, S) emerge una famiglia di polinomi $P_{y,w}$, chiamati *polinomi di Kazhdan-Lusztig*, associati a (W, S) e indicizzati da coppie di elementi di W. Essi congetturarono che i polinomi di Kazhdan-Lusztig associati al gruppo di Weyl di $\mathfrak{g}$, fornissero i coefficienti cercati, precisamente

$$\operatorname{ch} L_w = \sum_{y \in W} (-1)^{l(w)-(y)} P_{y,w}(1) \operatorname{ch} M_y$$

È questa la cosiddetta *congettura di Kazhdan-Lusztig*: un enunciato dunque, a priori, puramente algebrico.

In caratteristica zero, la congettura fu dimostrata già nei primi anni '80, in modo indipendente da A. Beilinson e J. Bernstein da un lato (cf. [3]) e da J. Brylinski e M. Kashiwara dall'altro (cfr. [12]), usando idee e strumenti provenienti dalla geometria algebrica quali i *D-moduli* e la *corrispondenza di Riemann-Hilbert*.

Il legame con la geometria era già suggerito dalla relazione che gli stessi Kazhdan e Lusztig (cfr. [29]) mostrarono tra questi polinomi e le dimensioni della coomologia di intersezione locale delle varietà di Schubert, un fatto che peraltro mostra, in questo caso, la non negatività dei coefficienti di tali polinomi[1]. Questo punto di incontro tra la teoria delle rappresentazioni e la geometria algebrica diede origine a un intero settore di ricerca noto come *teoria di Kazhdan-Lusztig*.

Negli anni e decenni successivi furono esplorate nuove strade per provare la congettura e fu trovata da Fiebig una seconda dimostrazione, seguendo l'approccio di Soergel (cfr. [35] e [18]). Ancora una volta l'idea fu di trovare un'opportuna traduzione della congettura in un problema di geometria algebrica per poi usare i potenti strumenti che questa disciplina fornisce per risolverlo. Questa seconda prova si basa sulla nozione di *grafi di momento*, introdotti da Braden e MacPherson (cfr. [10]) e usati per dare una descrizione combinatoria della coomologia delle varietà di Schubert: in particolare essi codificano la cosiddetta *coomologia equivariante* di queste varietà (cfr. [6]). Fiebig mostrò invece il loro legame con la categoria $\mathcal{O}$ (o più precisamente con una sua *deformazione*), stabilendo, in caratteristica zero, l'equivalenza tra la cosiddetta *congettura delle molteplicità* sui ranghi delle spighe dei *fasci di Braden-MacPherson*, ovvero

$$\operatorname{rk} \mathscr{B}(w)_y = P_{y,w}(1)$$

e la congettura di Kazhdan-Lusztig. A questo punto mostrò la congettura delle molteplicità attraverso la relazione nota tra i polinomi di Kazhdan-Lusztig e la coomologia di intersezione locale delle varietà di Schubert.

In questa tesi ci proponiamo innanzitutto di enunciare la congettura, dunque di descrivere la categoria $\mathcal{O}$ e porre il problema dei caratteri dei moduli semplici (nei capitoli §1,2 e 3). Successivamente introdurremo i grafi di momento (capitolo §4) e mostreremo l'equivalenza tra la congettura di Kazhdan-Lusztig e quella delle molteplicità dei fasci di Braden-MacPherson in caratteristica zero, seguendo Fiebig (capitolo §5).

[1] Questo problema, nel caso generale, rimase irrisolto fino ai risultati di B. Elias e G. Williamson, cfr. [16], che permettono inoltre una dimostrazione completamente algebrica della congettura di Kazhdan-Lusztig.

Contents

Chapter 1

Kazhdan-Lusztig Polynomials

In this chapter, we shall introduce Coxeter systems and Kazhdan-Lusztig polynomials associated with them. We shall exhibit some examples of computation in small order cases.

1.1 Coxeter Systems

Definition 1.1.1. A *Coxeter system* is a pair (W, S), where W is a finite group with a presentation

$$W = \left\langle s \in S \mid (s_1 s_2)^{m(s_1,s_2)} = e, \forall s_1, s_2 \in S \right\rangle$$

where e is the neutral element of W and

$$\begin{cases} m(s,s) = 1 & \forall s \in S \\ 2 \leq m(s_1, s_2) \leq \infty & \forall s_1 \neq s_2 \end{cases}$$

(the case $m(s_1, s_2) = \infty$ meaning that $s_1 s_2$ has infinite order, i.e. there's no relation of the form $(s_1 s_2)^m = e$)

Elements of S are often called *simple reflections* and conjugates of elements of S are called *reflections*.

We will always consider the case where S is a finite set. In this case one can represent the system (W, S) with an $|S| \times |S|$-matrix with elements in $\{1, 2, \ldots, \infty\}$, defined by $m = (m(s_1, s_2))_{s_1, s_2 \in S}$ which is called the *Coxeter matrix* corresponding (W, S). Similarly one can represent it with its *Coxeter graph* which is defined to have a vertex for each element of S and an edge linking the vertices s_1 and s_2 if $m(s_1, s_2) \geq 3$, labelled with the number $m(s_1, s_2)$ if this exceeds 3.

One can write any element of W in the form $w = s_1 \cdots s_k$, with s_i in S: we call $\underline{s} = (s_1, \dots, s_k)$ an *expression* for w. If, furthermore, k is minimal, one says that $\underline{s}$ is a *reduced expression* for w and the number k, denoted $l(w)$, is called *length* of w .

Example 1.1.2. Symmetric groups form Coxeter systems: if $n \in \mathbb{N}$ and $W = S_{n+1}$, let $S = \{s_1, \dots, s_n\}$, where s_i denotes the transposition $(i, i+1)$ that switches i and $i+1$ and leaves every other element fixed. One can find, for every $i, j = 1, \dots, n$, the relations

$$\begin{cases} s_i s_j = s_i^2 = e & \text{if } |i-j| = 0 \\ (s_i s_j)^3 = e & \text{if } |i-j| = 1 \\ (s_i s_j)^2 = e & \text{if } |i-j| > 1 \end{cases}$$

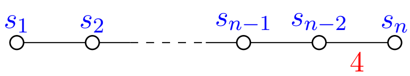
Coxeter matrices and graphs of these systems are of the form:

$$\begin{pmatrix} 1 & 3 & 2 & \dots & 2 \\ 3 & 1 & 3 & \dots & 2 \\ 2 & 3 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & 1 \end{pmatrix}$$

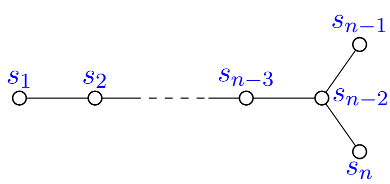
The group W is in this case isomorphic to the symmetry group of the standard n-simplex, and it is said to be of *type* A_n

Example 1.1.3. Other examples of Coxeter systems are given by the following matrices and graphs:

$$\begin{pmatrix} 1 & \dots & 2 & 2 & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & \dots & 1 & 3 & 2 \\ 2 & \dots & 3 & 1 & \mathbf{4} \\ 2 & \dots & 2 & \mathbf{4} & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & \dots & 2 & 2 & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & \dots & 1 & 3 & \mathbf{3} \\ 2 & \dots & 3 & 1 & \mathbf{2} \\ 2 & \dots & \mathbf{3} & \mathbf{2} & 1 \end{pmatrix}$$



In these cases we say that (W, S) is of *type* B_n, resp. D_n.

One can define the following partial order on W, called the *Bruhat order*[1]: let $y \leq w$ if and only if $l(y) \leq l(w)$ and y has a reduced expression that appears as a sub-expression in a reduced expression for w, i.e. there exists a reduced expression $\underline{s} = (s_1, \dots, s_k)$ for w and $1 \leq i_1 < i_2 < \dots < i_h \leq k$ such that $\underline{s}' = (s_{i_1} \dots s_{i_h})$ is a reduced expression for y.

One can show (see [7], prop. 2.3.1) that if W is finite one always has a unique element of maximal length, called the *longest element*, that we shall denote w_0. It follows that it is actually maximal with respect to the Bruhat order.

Example 1.1.4. In figure 1.1 and 1.2, we present, for example, Hasse diagrams associated to the Bruhat order of Coxeter systems of type A_2, B_2 and A_3.

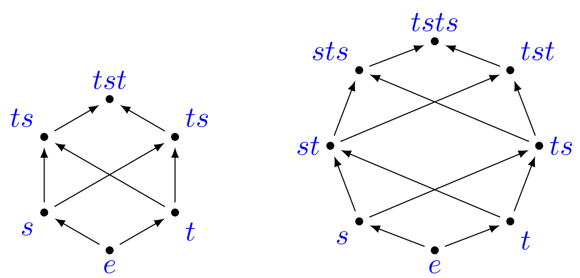


Figure 1.1: Hasse diagrams of the Bruhat orders: type A_2 and B_2

Given a Coxeter system, one can build a simplicial complex that represents it. Let us consider a standard $|S|$-simplex for each element of W, with faces indexed by elements of S, and let us glue together all these simplexes in such a way that the simplex w is adjacent to the simplex ws along the face s. In this way one obtains the so-called *Coxeter complex*.

We can also build a canonical representation of a Coxeter system, called *geometric representation*. Let us take a basis $\{e_s\}$ of $\mathbb{R}^{|S|}$ indexed by the elements of S and let us define the bilinear form $(-,-)$ by assigning its values on elements of the basis: let $(e_s, e_{s'}) = -\cos\left(\frac{\pi}{m(s,s')}\right)$. One can show (see [7] or [25])that

$$s \mapsto \left(v \mapsto v - 2\frac{(v, e_s)}{(e_s, e_s)} e_s\right)$$

[1]The definition that one usually finds in the literture is the following: let us write $w \to w'$ if there exists a reflection t (i.e. the conjugate of an element of S) such that $w' = wt$ and $l(w') > l(w)$ and let us denote $\leq$ the partial order relation generated by $\to$. One can show that this is equivalent to our definition (see [7] prop. 2.2.2)

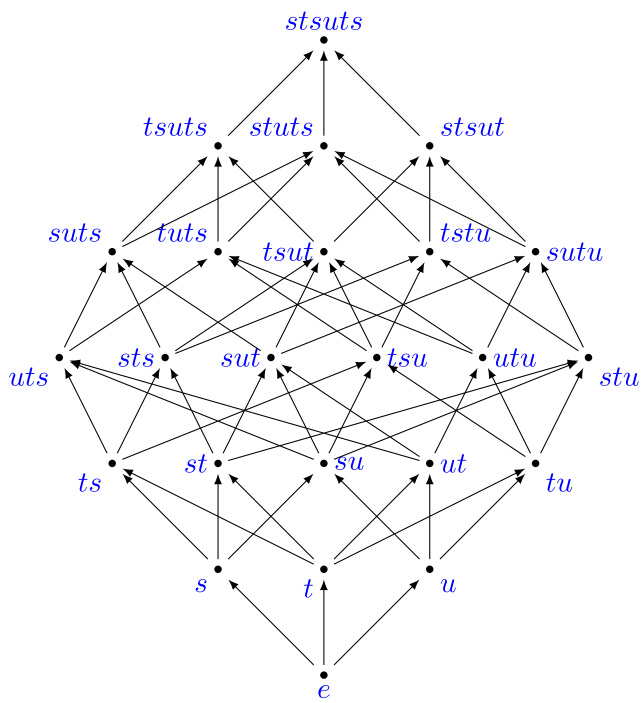
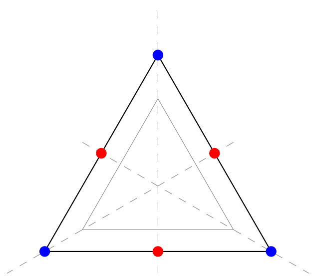
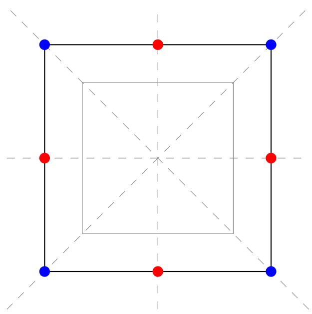


Figure 1.2: Hasse diagram of the Bruhat order: type A_3. Cf. [7] fig. 2.4

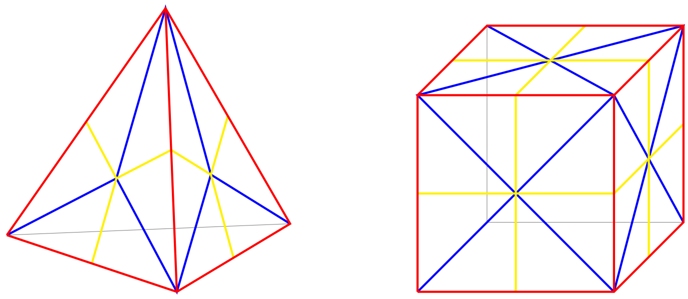
defines a representation of W on $\mathbb{R}^{|S|}$.

Example 1.1.5. The dihedral group of order $2n$, i.e. the group of the symmetries of a regular n-gone, is a Coxeter system with respect to the set $S = \{s, t\}$ of symmetries associated to two consecutive axes, the only non trivial relation being $(st)^n = e$. In the next figure we show the Coxeter complexes associated to some dihedral groups.





Example 1.1.6. Here are some examples of Coxeter complexes of dimension 3 (they represent types A_3 and B_3)



We conclude this section by introducing a fundamental class of Coxeter systems, namely Weyl groups. Recall that, given a euclidean vector space V, with scalar product $(-,-)$, a root system Φ is a finite set of vectors of V such that (see [24], §9.2)

i) The set Φ spans V and $0 \notin \Phi$;

ii) If $\alpha \in \Phi$ then the only multiples of α contained in Φ are $\pm\alpha$;

iii) If $\alpha \in \Phi$ then the reflection s_α, which sends α to $-\alpha$ and fixes the hyperplane $\{(v,\alpha)=0\}$, leaves Φ invariant;

iv) If α and β belong to Φ then $2\dfrac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Recall also that one can chose a set Δ of *simple roots* such that $\Phi \subset \mathbb{Z}\Delta$ (where $\mathbb{Z}\Delta$ denotes the set of $\mathbb{Z}$-linear combinations of elements of Δ). Furthermore, denoting $\Gamma = \mathbb{Z}^+\Delta$ (with similar meaning), where $\mathbb{Z}^+ = \{n \in \mathbb{Z} | n \geq 0\}$, one has a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$ such that $\Phi^+ \subset \Gamma$ and $\Phi^- = -\Phi^+$. The *Weyl group* associated to Φ is the subgroup of $\mathrm{GL}(V)$ that leaves Φ invariant, which is precisely the one generated by the s_α's with $\alpha \in \Phi$: one can actually show that it is generated also by *simple reflections*, i.e. reflections associated to simple roots. The group W has a natural structure of Coxeter system, taking S to be the set of simple reflections. Notice that in this case, geometric representation of W coincides with its inclusion in $\mathrm{GL}(V)$. Observe also that that W respects the *cristallographic condition*, i.e. it fixes a lattice in V: in fact it fixes $\mathbb{Z}\Delta$. One can show that in this case, $m(s_1, s_2) \in \{2,3,4,6\}$, for any s_1 and s_2 in S.

Example 1.1.7. Coxeter system shown in examples 1.1.2 and 1.1.3 are associated to Weyl groups

1.2 Hecke Algebras

In this section we define Kazhdan-Lusztig polynomials associated with a given finite Coxeter system (W, S). For this purpose we need to introduce the *Hecke algebra* of (W,S).

Let us denote $\mathscr{H}_{(W,S)}$ the free $\mathbb{Z}[v,v^{-1}]$-module generated by the elements T_w, one for each $w \in W$, i.e. $\bigoplus_{w\in W} \mathbb{Z}[v,v^{-1}]T_w$, with the associative algebra structure given by the relations[2] [3].

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } l(ww') = l(w) + l(w') \\ T_s^2 = v^{-2} + (v^{-2}-1)T_s & \text{for } s \in S \end{cases} \tag{1.1}$$

Notice that in particular T_e is the unit of this algebra: we shall denote it by 1.

We shall use a slightly different base of $\mathscr{H}_{(W,S)}$ that turns out to be more convenient: for each $w \in W$ let $H_w = v^{l(w)}T_w$. Thus in particular $H_e = T_e = 1$ and $H_s = vT_s$. Relations (1.1) become

$$\begin{cases} H_w H_{w'} = H_{ww'} & \text{if } l(ww') = l(w) + l(w') \\ H_s^2 = 1 + (v^{-1}-v)H_s & \text{for } s \in S \end{cases} \tag{1.2}$$

Notice that by the second equality one finds $H_s^{-1} = H_s - (v^{-1} - v)$.

Let us take $w \in W$ and $s \in S$ we have two possibilities: either $ws > w$ or $ws < w$ (see definition of the Bruhat order in §1.1). Using first line of (1.2) one deduces $H_w H_s = H_{ws}$ in the first case and $H_{ws}H_s = H_w$ in the second case. Hence (multiplying the second equality by H_s on the right) we have the following rules:

$$H_w H_s = \begin{cases} H_{ws} & \text{if } ws > w \\ H_{ws} + (v^{-1}-v)H_w & \text{if } ws < w \end{cases} \tag{1.3}$$

Similarly one gets

$$H_s H_w = \begin{cases} H_{sw} & \text{if } sw > w \\ H_{sw} + (v^{-1}-v)H_w & \text{if } sw < w \end{cases} \tag{1.4}$$

Let us now introduce a duality endomorphism $d : \mathscr{H}_{(W,S)} \to \mathscr{H}_{(W,S)}$ defined by $v \mapsto v^{-1}$ and $H_w \mapsto H_{w^{-1}}^{-1}$: one can check that it behaves properly on products of the forms appearing in (1.2), defining a ring endomorphism. We denote $\overline{H} := d(H)$ for every $H \in \mathscr{H}_{(W,S)}$. In particular one has

$$\overline{H_s} = H_s - (v^{-1} - v) = H_s + (v - v^{-1})$$

for every $s \in S$, because $s^{-1} = s$. Actually we have the following general result

[2]In [28], parameter q corresponds to our v^{-2}. Here we follow [36]. In chapter §2 we shall present an interpretation of this variable v in geometric terms.

[3]One should show that these relations actually define unambiguously an algebra structure (see [9])

Proposition 1.2.1. *Let $w \in W$, then*

$$\overline{H_w} = H_w + \sum_{y<w} (v - v^{-1})^{l(w)-l(y)} H_y$$

Proof. Let us proceed by induction on the Bruhat order, the cases $w = e$ (and also $w \in S$) being already checked. Let $w \in W$ with $w \neq e$, there exists $s \in S$ such that $ws < w$, hence

$$\begin{aligned}
\overline{H_w} &= \overline{H_{ws}H_s} = \overline{H_{ws}}(H_s + (v - v^1)) = \\
&= H_{ws}(H_s + (v - v^{-1})) + \sum_{y<ws} (v - v^{-1})^{l(ws)-l(y)} H_y (H_s + (v - v^{-1})) = \\
&= H_{ws}H_s + (v - v^{-1})H_{ws} + \sum_{y<ws} (v - v^{-1})^{l(ws)-l(y)} H_y H_s + \\
&\quad + \sum_{y<ws} (v - v^{-1})^{l(ws)-l(y)+1} H_y = \\
&= H_w + (v - v^{-1})H_{ws} + \sum_{\substack{y<ws \\ ys>y}} (v - v^{-1})^{l(ws)-l(y)} H_{ys} + \\
&\quad - \sum_{\substack{y<ws \\ ys<y}} (v - v^{-1})^{l(ws)-l(y)+1} H_y + \sum_{\substack{y<ws \\ ys>y}} (v - v^{-1})^{l(ws)-l(y)+1} H_y + \\
&\quad + \sum_{\substack{y<ws \\ ys<y}} (v - v^{-1})^{l(ws)-l(y)+1} H_y = \\
&= H_w + (v - v^{-1})H_{ws} + \sum_{\substack{y<ws \\ ys>y}} (v - v^{-1})^{l(ws)-l(y)} H_{ys} + \\
&\quad + \sum_{\substack{y<ws \\ ys>y}} (v - v^{-1})^{l(ws)-l(y)+1} H_y
\end{aligned}$$

This concludes because any $x < w$ is either ws or of one of the two forms[4]

1. y with $y < ws$;
2. ys with $y < ws$.

□

The following theorem, which is the most important of this section shows us that there is a third basis which is particularly interesting:

[4] This is a property of the Bruhat order on Coxeter systems: see [7] th. 2.2.2.

Theorem 1.2.2 ([28], th. 1.1;[36], th. 2.1). *For every $w \in W$ there exists a unique element*

$$\underline{H}_w \in H_w + \sum_{y<w} v\mathbb{Z}[v]H_y$$

which is self-dual ($\overline{\underline{H}_w} = \underline{H}_w$). In particulr, the $\underline{H}_w$'s form a basis for $\mathscr{H}_{(W,S)}$.

Proof. (cf. [36]) Let us remark that $C_s := H_s + v$ is such that $\overline{C_s} = C_s$. Right multiplication by C_s, thanks to (1.3), obeys to the following rules:

$$\begin{cases} H_x C_s = H_{xs} + vH_x & \text{if } xs > x \\ H_x C_s = H_{xs} + v^{-1}H_x & \text{otherwise} \end{cases} \tag{1.5}$$

In order to show existence, let us procede by induction on the Bruhat order. We already found $\underline{H}_e := H_e = 1$ and $\underline{H}_s := H_s + v$. Let $w \in W$ and let us suppose that there exists a self-dual element $\underline{H}_y$, of the desired form, for every $y < w$. As $w \neq e$, there exists an element $s \in S$ such that $ws < w$: by assumption, exponents of v appearing in the expression of $\underline{H}_{ws}$ are all positive. Let us consider the element $\underline{H}_{ws}C_s$: rules (1.5) imply that exponents of v appearing in it are non-negative, i.e.

$$\underline{H}_{ws}C_s = H_w + \sum_{y<w} h_y(v)H_y$$

for certain $h_y(v) \in \mathbb{Z}[v]$. So let $\underline{H}_w = \underline{H}_{xs}C_s - \sum_{y<w} h_y(0)\underline{H}_y$: it is necessarily self-dual and of the desired form.

We still have to prove uniqueness of elements of this form. It is enough to prove that the only self-dual element of the set

$$\sum_{w\in W} v\mathbb{Z}[v]H_w$$

is 0. For every $w \in W$, by writing H_w in the self-dual basis which we have found, we get $H_w \in \underline{H}_w + \sum_{y<w} \mathbb{Z}[v, v^{-1}]\underline{H}_y$, so, by taking the dual and considering the form of the $\underline{H}_y$'s, one obtains

$$\overline{H_w} \in \underline{H}_w + \sum_{y<w} \mathbb{Z}[v, v^{-1}]\underline{H}_y \subset H_w + \sum_{y<w} \mathbb{Z}[v, v^{-1}]H_y$$

Let us suppose that there exists a non-zero self-dual element

$$H = \sum_{z\in W} h_z(v)H_z$$

with $h_z(v) \in v\mathbb{Z}[v]$. Let us take $z_0 \in W$, maximal with respect to the Bruhat order, such that $h_{z_0}(v) \neq 0$, then, by prop. 1.2.1, the coefficient of H_{z_0} in $\overline{H}$ is $\overline{h_{z_0}(v)} = h_{z_0}(v^{-1})$. But $h_{z_0}(v) = h_{z_0}(v^{-1})$ entails $h_{z_0}(v) = 0$ and one obtains a contradiction. □

Therefore, for every $w \in W$ one has

$$\underline{H}_w = \sum_{y \leq w} h_{y,w}(v) H_y \tag{1.6}$$

with $h_{w,w}(v) = 1$ and $h_{y,w}(v) \in v\mathbb{Z}[v]$ well-determined for $y < w$ in W.

Proposition 1.2.3. *For every $y \leq w$ in W, the leading term of $h_{y,w}(v)$ is $v^{l(w)-l(y)}$ and the other exponents of v have all the same parity.*

Proof. Let $w \in W$, we proceed by decreasing induction, with respect to the Bruhat order, on elements $y \leq w$, the case $y = w$ being trivial. We have, using lemma 1.2.1:

$$\begin{aligned}
\underline{H}_w = \overline{\underline{H}_w} = \overline{H_w} + \sum_{y<w} \overline{h_{y,w}(v) H_y} = H_w + \sum_{y<w} (v - v^{-1})^{l(w)-l(y)} H_y + \\
+ \sum_{y<w} h_{y,w}(v^{-1}) \left(H_y + \sum_{x<y} (v - v^{-1})^{l(y)-l(x)} H_x \right) = \\
= H_w + \sum_{y<w} \left((v - v^{-1})^{l(w)-l(y)} + h_{y,w}(v^{-1}) \right) H_y + \\
+ \sum_{x<w} \sum_{y<x} h_{x,w}(v^{-1}) (v - v^{-1})^{l(x)-l(y)} H_y = \\
= \sum_{y<w} \sum_{y<x<w} h_{x,w}(v^{-1}) (v - v^{-1})^{l(x)-l(y)} H_y
\end{aligned}$$

So for every $y < w$

$$(v - v^{-1})^{l(w)-l(y)} + h_{y,w}(v^{-1}) + \sum_{y<x<w} h_{x,w}(v^{-1})(v - v^{-1})^{l(x)-l(y)} = h_{y,w}(v)$$

Now observe that the statement is true for $h_{y,w}(v)$ if and only if it is true for $h_{y,w}(v) - h_{y,w}(v^{-1})$, i.e. for

$$(v - v^{-1})^{l(w)-l(y)} + \sum_{y<x<w} h_{x,w}(v^{-1})(v - v^{-1})^{l(x)-l(y)}$$

But now one can use induction hypothesis. □

By the proposition one has thaa $v^{l(y)-l(w)} h_{y,w}(v)$ is a polynomial in v^{-2} constant term 1, *i.e* $v^{l(y)-l(w)} h_{y,w} = P_{y,w}(v^{-2})$. We call then $P_{y,w}$ *Kazhdan-Lusztig polynomials.*

One has the following *inversion formula*:

Proposition 1.2.4. *If W is finite and w_0 is the longest element, then for every x and y in W*

$$\sum_{x\leq w\leq y}(-1)^{l(x)+l(w)}h_{x,w}h_{w_0y,w_0w}=\delta_{x,y}$$

or, equivalently,

$$\sum_{x\leq w\leq y}(-1)^{l(x)+l(w)}P_{x,w}P_{w_0y,w_0w}=\delta_{x,y}$$

Proof. (see [36] §3) Let us introduce $\mathscr{H}^*_{(W,S)}:=\mathrm{Hom}_{\mathbb{Z}[v,v^{-1}]}(\mathscr{H}_{(W,S)},\mathbb{Z}[v,v^{-1}])$. A natural basis is given by the elements H^*_w defined by $H^*_w(H_y)=\delta_{w,y}$. We define a duality endomorphism on $\mathscr{H}^*_{(W,S)}$ by $\overline{F}(M):=\overline{F(\overline{M})}$, so, by prop. 1.2.1, we obtain

$$\overline{H^*_x}=\sum_{z>x}(v^{-1}-v)^{l(z)-l(x)}H^*_z$$

Now let $H^w=(-1)^{l(w)}H^*_w$: we obtain a basis as well, and one has

$$\overline{H^x}\in H^x+\sum_{z>x}\mathbb{Z}[v,v^{-1}]H^z$$

If one poses $\underline{H}^x(\underline{H}_y)\in(-1)^{l(x)}\delta_{x,y}$ one obtains self-dual elements (it suffices to compute the dual on the $\underline{H}_y$'s) such that $\underline{H}^x\in H^x+\sum v\mathbb{Z}[v]H^z$ (in fact $H_y\in\underline{H}_y+\sum_{z<y}\underline{H}_z$). Then like in the proof of th. 1.2.2, one shows that they are the only elements with this property. Let us denote $h^{z,x}$ the Laurent polynomials identified by $\underline{H}^x$, i.e. such that

$$\underline{H}^x=\sum_{z\in W}h^{z,x}(v)H^z$$

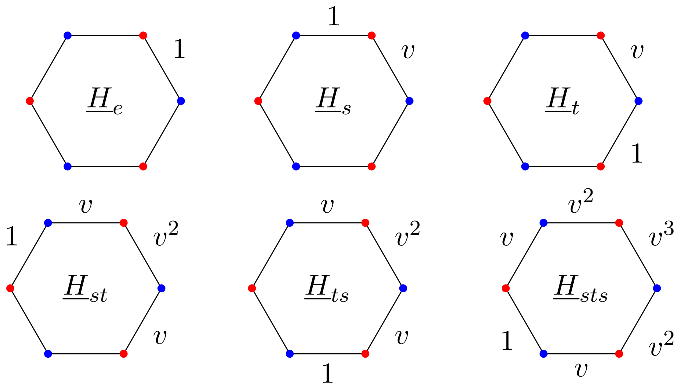
then one gets

$$\sum_z(-1)^{l(x)+l(z)}h^{z,x}h_{z,y}=\delta_{x,y}$$

Finally, considering that the morphism $H_x\mapsto H^{w_0x}$ preserves duality, we obtain that $h^{z,x}=h_{w_0z,w_0x}$. □

1.3 Examples of computation

Example 1.3.1. One can compute explicitly Kazhdan-Lusztig polynomials for dihedral Coxeter systems (as for example those of type A_2 and B_2) and find that they are all trivial. One can represent them on Coxeter complexes in the following way (type A_2: the north-east edge corresponds to identity of W)



Example 1.3.2. The following table gives Kazhdan-Lusztig polynomials for type A_3: notice that we find non-trivial polynomials for elements of length 4 and 5.

	id	(1 2)	(2 3)	(3 4)	(1 2 3)	(1 3 2)	(1 2)(3 4)	(2 3 4)	(2 4 3)	(1 3)	(1 2 3 4)	(1 2 4 3)	(1 3 4 2)	(1 4 3 2)	(2 4)	(1 3 4)	(1 4 3)	(1 3)(2 4)	(1 2 4)	(1 4 2)	(1 3 2 4)	(1 4)	(1 4 2 3)	(1 4)(2 3)
id	1																							
(1 2)	v	1																						
(2 3)	v		1																					
(3 4)	v			1																				
(1 2 3)	v^2	v	v		1																			
(1 3 2)	v^2	v	v			1																		
(1 2)(3 4)	v^2	v		v			1																	
(2 3 4)	v^2		v	v				1																
(2 4 3)	v^2		v	v					1															
(1 3)	v^3	v^2	v^2		v	v				1														
(1 2 3 4)	v^3	v^2	v^2	v^2	v		v	v			1													
(1 2 4 3)	v^3	v^2	v^2	v^2	v		v		v			1												
(1 3 4 2)	v^3	v^2	v^2	v^2		v	v	v					1											
(1 4 3 2)	v^3	v^2	v^2	v^2		v	v		v					1										
(2 4)	v^3		v^2	v^2				v	v						1									
(1 3 4)	v^4	v^3	v^3	v^3	v^2	v^2	v^2	v^2		v	v		v			1								
(1 4 3)	v^4	v^3	v^3	v^3	v^2	v^2	v^2		v^2	v		v		v			1							
(1 3)(2 4)	v^4	v^3	v^3+v	v^3	v^2	v^2	v^2	v^2	v^2	v		v	v		v			1						
(1 2 4)	v^4	v^3	v^3	v^3	v^2		v^2	v^2	v^2		v	v			v				1					
(1 4 2)	v^4	v^3	v^3	v^3		v^2	v^2	v^2	v^2				v	v	v					1				
(1 3 2 4)	v^5	v^4	v^4	v^4	v^3	v^3	v^3	v^3	v^3	v^2	v^2	v^2	v^2		v^2	v		v	v		1			
(1 4)	v^5+v^3	v^4+v^2	v^4	v^4+v^2	v^3	v^3	v^3+v	v^3	v^3	v^2	v^2	v^2	v^2	v^2	v^2	v	v		v	v		1		
(1 4 2 3)	v^5	v^4	v^4	v^4	v^3	v^3	v^3	v^3	v^3		v^2	v^2	v^2	v^2	v^2		v	v		v			1	
(1 4)(2 3)	v^6	v^5	v^5	v^5	v^4	v^4	v^4	v^4	v^4	v^3	v^3	v^3	v^3	v^3	v^3	v^2	v^2	v^2	v^2	v^2	v	v	v	1

The following result is true in general

Proposition 1.3.3 (see [36], prop. 2.9). *Let us suppose W finite and let w_0 the longest element, then*

$$\underline{H}_{w_0} = \sum_{y \in W} v^{l(w_0)-l(y)} H_y \tag{1.7}$$

In other words the Kazhdan-Lusztig polynomials P_{y,w_0} are all 1

This property will translate the fact that flag varieties are smooth: cf. example 2.2.1.

In the next chapter we will see that if our Coxeter system is associated with an algebraic group, then all Kazhdan-Lusztig polynomials have positive coefficients: this was proven already in 1980 by Kazhdan and Lusztig themselves.

Chapter 2

Local Intersection Cohomology of Schubert Varieties

In this chapter we present an important link between Hecke algebras and the geometry of Schubert varieties. For instance, we explain how coefficients of Kazhdan-Lusztig polynomials can give dimensions of their local intersection cohomology. This will show in particular that Kazhdan-Lusztig polynomials have positive coefficients for Coxeter systems associated to algebraic groups.

We shall consider a reductive algebraic group G over $\mathbb{C}$, with a fixed Borel subgroup B which contains a maximal torus H. Recall that the group $X(H) = \mathrm{Hom}(H, GL_1)$ of characters of H is a free abelian group, and we put $L := X(H) \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote $W = W(H,G) = N_G(H)/H$ the Weyl group associated to H (where $N_G(H)$ is the normalizer of H in G): recall that for any other choice of a Borel subgroup and a maximal torus we would obtain a Weyl group isomorphic to this one. Recall also that W is finite and it acts faithfully on $X(H)$ by $[n]\cdot\phi(h) = \phi(nhn^{-1})$, hence it identifies with its image in the group of automorphisms of $X(H)$. Denote $\mathfrak{g}$ the Lie algebra associated with the group G: we have an adjoint representation $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ given by $\mathrm{Ad}(g) = d_e i_g$, where $i_g\colon G \to G$ is the conjugation by g and d_e denotes the differential at the neutral element e of G. As H is a torus, the restriction of the adjoint representation to H is completely reducible and we get

$$\mathfrak{g} = \bigoplus_{\alpha \in X(H)} \mathfrak{g}_\alpha$$

with $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{Ad}(h)x = \alpha(h)x\}$. Setting

$$\Phi(H,G) := \{\alpha \in X(H) \mid \mathfrak{g}_\alpha \neq 0\} \setminus \{0\}$$

we have that $\Phi(H, G)$ is a root system in $L \otimes_{\mathbb{Q}} \mathbb{R}$. Then we have that W coincides with the Weyl group of $\Phi(H, G)$. In particular it is generated by simple reflections and we call $S \subset W$ the subset of these refelctions.

Denoting $N := N_G(H)$, the system (G, B, N, S) satisfy the following properties:

i) G is generated by B and N;

ii) $H = B \cap N$ is a normal subgroup of N;

iii) $W = N/H$ is generated by S whose elements have order 2;

iv) $\dot{s}B\dot{w} \subset B\dot{s}\dot{w}B \cup B\dot{w}B$, for any $\dot{s}$ and $\dot{w}$ (representatives of elements) respectively in S and W;

v) $\dot{s}B\dot{s} \neq B$, for every $s \in S$;

One then says that (G, B, N, S) is a (B, N)*-pair* or a *Tits system*.

Recall that (in general for Tits systems and in particular in our case) we have a Bruhat decomposition $G = \bigsqcup_{w \in W} B\dot{w}B$, where $\dot{w}$ is any representative of w. We call X the quotient G/B (whose elements are denoted by gB), where we consider the action of B on G given by $g \mapsto gb$. The group G acts on X by multiplication on the left $g \cdot hB = ghB$. We have a natural bijection $gB \mapsto gBg^{-1}$ between the variety X and the set $\mathcal{B}$ of Borel subgroups of G, which are all conjugates of B: this bijection gives a structure of G-algebraic variety on $\mathcal{B}$. It is a homogeneous G-variety, so in particular it is smooth[1] (hence reduced). The Bruhat decomposition is preserved and becomes $X = \bigsqcup_{w \in W} B\dot{w}B/B$: the subvarieties $C_w := B\dot{w}B/B$ are called *Schubert cells* and we denote $i_w : C_w \hookrightarrow X$ the inclusions, whereas the closed subvarieties $X_w := \overline{C_w}$ are called *Schubert varieties* and we denote $\underline{i}_w : X_w \hookrightarrow X$ the inclusions.

For more details about these notions see [8].

2.1 Some Geometry of Schubert Varieties

Recall that a reduced complex algebraic variety admits an open covering such that each open set is an affine reduced algebraic variety, which therefore admits an embedding in $\mathbb{C}^n$ for some n. Hence our variety inherits a natural topology from the one of the complex numbers. In the following we shall

[1] Here we are using the fact that we work in characteristic zero: this ensures us that there is at least one non singular point.

consider complex alebraic varieties, so in particular our Schubert varieties, as topological spaces in this way.

We have already said that X is homogeneous as a G-variety, hence it is smooth. The strata of the Bruhat decompositon are also smooth: more precisely we have the following result (for a proof, see [8] 14.4 and 14.12)

Proposition 2.1.1. *For every $w \in W$, the stratum C_w is isomorphic to $\mathbb{A}_{\mathbb{C}}^{l(w)}$, where $l(w)$ is the length of w in (W, S).*

Example 2.1.2. For $G = \mathrm{GL}_n(\mathbb{C})$, taken $B = B_n$ to be the subgroup of upper triangular matrices

$$B_n = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\}$$

and $H = T_n$ to be the subgroup of diagonal matrices, we have that $N_G(H) = N_{\mathrm{GL}_n(\mathbb{C})}(T_n)$ is the subgroup of matrices having exactly one non-zero entry for each row and column. Hence representatives of elements of the Weyl group are permutation matrices. Then we can write explicitly the Bruhat decomposition of GL_n and of GL_n / B_n, as we shall do for $n = 2, 3, 4$.

The flag variety $\mathrm{GL}_n(\mathbb{C})/B$ can be viewed as the set of flags

$$\big\{(V_1, V_2, \ldots, V_n) \mid V_1 \subset V_2 \subset \ldots V_n = \mathbb{C}^n\big\}$$

or as the set of equivalence classes (under right multiplication by B) of matrices in GL_n

$$\big\{gB \mid g \in \mathrm{GL}_2(\mathbb{C})\big\}$$

$n = 2$**:** In this case $W = \{e, s\} \cong S_2$ and the Bruhat decomposition is then the following

$$\mathrm{GL}_2(\mathbb{C}) = B \sqcup B\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)B$$

the Bruhat decomposition of the flag variety is

$$\mathrm{GL}_2(\mathbb{C})/B = B/B \sqcup B\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)B/B$$

where $C_e = B/B$ is a single point, namely $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)B$, which can be viewed as the standard flag $(\mathbb{C}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \mathbb{C}^2)$, whereas C_s is the set of points of the form $\left(\begin{smallmatrix} * & 1 \\ 1 & 0 \end{smallmatrix}\right)B$ and can be viewed as the set of flags $\big(V_1, \mathbb{C}^2\big)$ with $V_1 \neq \mathbb{C}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$: it is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$.

$n = 3$**:** the Bruhat decomposition (directly of the flag variety) is the following

$$\mathrm{GL}_3(\mathbb{C}) = B/B \sqcup B\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)B/B \sqcup B\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)B/B \sqcup$$
$$\sqcup\, B\left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)B/B \sqcup B\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)B/B \sqcup B\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)B/B$$

Schubert cells are then

$$\left\{\left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix}\right)B\right\}$$

$$\left\{\left(\begin{smallmatrix} *&1&0\\1&0&0\\0&0&1 \end{smallmatrix}\right)B\right\} \qquad \left\{\left(\begin{smallmatrix} 1&0&0\\0&*&1\\0&1&0 \end{smallmatrix}\right)B\right\}$$

$$\left\{\left(\begin{smallmatrix} *&*&1\\1&0&0\\0&1&0 \end{smallmatrix}\right)B\right\} \qquad \left\{\left(\begin{smallmatrix} *&1&0\\ *&0&1\\1&0&0 \end{smallmatrix}\right)B\right\}$$

$$\left\{\left(\begin{smallmatrix} *&*&1\\ *&1&0\\1&0&0 \end{smallmatrix}\right)B\right\}$$

and each one is isomorphic to $\mathbb{A}_{\mathbb{C}}^{l(w)}$ with $l(w)$ the corresponding element on the Weyl group, which is the number of free elements in the representatives above. These representatives can also be viewed as bases which define flags of $\mathbb{C}^n$.

Consider now Schubert varieties X_w: each one admits a stratification induced by the Bruhat decomposition, given by $X_w = \bigsqcup_{y \leq w} B\dot{y}B/B = \bigsqcup_{y \leq w} C_y$.

Remark 2.1.3. The Bruhat decomposition, together with proposition 2.1.1, gives us a way to compute Betti numbers for Schubert varieties X_w: the odd ones are zero, whereas for each $i \geq 0$ the $2i$-th one is precisely the number of strata of complex dimension i, that is, the number of element $y \leq w$ such that $l(y) = i$.

We know that Schubert varieties are projective complex varieties, so in particular, as topological spaces, they are compact and they admit a natural orientation, hence if a Schubert variety is smooth it must respect Poincaré duality. By the preceding remark is then easy to find that Schubert varieties are in general not smooth: it is sufficient to find an element $w \in W$ such that the number of $y \leq w$ for which $l(y) = i$ does not always equal the number of $z \leq w$ for which $l(z) = l(w) - i$. For example in tyoe A_3 (with the notation of example 1.1.4) the element $tsut$ has four smaller elements of length 3, five of length 2 and three of length 1, so X_{tsut} is not smooth.

Nevertheless we will show that Schubert varieties admit resolutions of singularities. First of all, for $s \in S$, let P_s be the subgroup of G generated by B and any $\dot{s}$ representing s (in fact as $H \subset B$ this subgroup does not depend on the particular representative chosen): groups of this form are called *minimal parabolic subgroups* of G. We have $P_s = B \sqcup B\dot{s}B$, in fact by property (iv) of Tits systems $\dot{s}B\dot{s} \subset B \cup B\dot{s}B$. Hence $P_s/B \cong X_s \cong \mathbb{P}^1_{\mathbb{C}}$.

We now construct inductively *Bott-Samelson varieties* associated to reduced expressions in (W, S). Let $Z_{(s)} := X_s = P_s/B$ for any $s \in S$ and suppose that we have built a G-variety $Z_{\underline{s'}}$ for any reduced expression $\underline{s'}$ of length lower than a positive integer k. Let then $\underline{s} = (s_1, \ldots, s_k)$ be a reduced expression: we consider $P_{s_1} \times Z_{\underline{s'}}$, where $\underline{s'} = (s_2, \ldots, s_k)$. It admits a free action of B given by $b \cdot (g, z) = (gb^{-1}, b \cdot z)$, and we denote by $P_{s_1} \times^B Z_{\underline{s'}}$ the quotient, which we define as $Z_{\underline{s}}$.

In other words we have

$$Z_{\underline{s}} := P_{s_1} \times^B P_{s_2} \times^B \cdots \times^B P_{s_k}/B$$

We ignored parentheses because we can also see $Z_{\underline{s}}$ directly as the quotient of

$$P_{s_1} \times \cdots \times P_{s_k}/B$$

by the following action of B^{k-1}

$$(b_1, \ldots, b_{k-1}) \cdot (g_1, g_2 \ldots, g_k B) = (g_1 b_1^{-1}, b_1 g_2 b_2^{-1}, \ldots, b_{k-1} g_k B)$$

We denote elements of $Z_{\underline{s}}$ in the form $[g_1 : g_2 : \cdots : g_{k-1} : g_k B]$.

One has the following result

Theorem 2.1.4. *a) Bott-Samelson varieties $Z_{\underline{s}}$ are smooth.*

b) There exists a morphism $\pi_{\underline{s}} \colon Z_{\underline{s}} \to X_w$ proper and birational, which is then a resolution of singularities.

Proof. a) We show that $Z_{\underline{s}}$ is an iterated locally trivial fibration. If $k = 1$, we have $P_{s_1}/B = X_{s_1} \cong \mathbb{P}^1_{\mathbb{C}}$ which is smooth. If $k = 2$ then $Z_{\underline{s}} = P_{s_1} \times^B X_{s_2}$. Consider the morphism $P_{s_1} \times X_{s_2} \to P_{s_1}/B$ given by $(g, hB) \mapsto gB$: it factors through the quotient and gives

$$P_{s_1} \times^B X_{s_2} \to P_{s_1}/B$$

It is a locally trivial fibration with fiber $B \times^B X_{s_2} \cong X_{s_2}$: in fact if we take $C_{s_1} \subset P_{s_1}/B$, its inverse image is $B\dot{s}_1 B \times^B X_{s_2} \cong C_{s_1} \times B \times^B X_{s_2}$, hence it is locally trivial for points in C_{s_1}, but then so is also for $\dot{e}B$ by P_{s_1}-homogeneity. Now for a general k, consider, for $i = 1, \ldots, k-1$ the morphisms

$$\begin{aligned} \phi_i \colon P_{s_1} \times^B P_{s_2} \times^B \cdots \times^B P_{s_{i+1}}/B &\to P_{s_1} \times^B P_{s_2} \times^B \cdots \times^B P_{s_i}/B \\ [g_1 : g_2 : \cdots : g_{i+1}B] &\mapsto [g_1 : g_2 : \cdots : g_i B] \end{aligned}$$

Each one, by the case $k = 2$ is a locally trivial fibration.

b) For any $w \in W$ and $s \in S$, consider the morphism

$$\begin{aligned} \pi_{s,w} \colon P_s \times^B X_w &\to X \\ [g : hB] &\mapsto ghB \end{aligned}$$

It is a proper morphism with image P_sX_w: hence this image is the closure of $B\dot{s}C_w$. Let us notice that, if $sw > w$, then $B\dot{s}C_w = C_{ws}$, hence in this case the closure of $B\dot{s}C_w$, that is the image of $\pi_{s,w}$ is X_{sw}. One has that the restriction

$$B\dot{s}B \times^B C_w \to C_{sw}$$

is an isomorphism, so $\pi_{s,w}$ is a birational morphism.

Now consider the morphism $\pi_{\underline{s}} : Z_{\underline{s}} \to X_w$, given by

$$[g_1 : \cdots : g_{k-1} : g_kB] \mapsto g_1 \cdots g_kB$$

It is the composition of the morphisms

$$P_{s_1} \times^B \cdots \times^B P_{s_{i-1}} \times^B X_{s_i \cdots s_k} \to P_{s_1} \times^B \cdots \times^B P_{s_{i-2}} \times^B X_{s_{i-1} \cdots s_k}$$
$$[g_1 : \cdots : g_{i-1} : hB] \mapsto [g_1 : \cdots : g_{i-2} : g_{i-1}hB]$$

and each of these is induced by the morphism $\pi_{s,s_{i-1}\cdots s_k}$ described above, hence it is proper and birational.

□

Remark 2.1.5. By part (a) of the proof we have that $Z_{\underline{s}}$ has dimension k.

Following [33] we introduce the variety $\mathfrak{X} = X \times X$ which also admits a G-action given componentwise by the one on X. It also admits a stratification induced by the Bruhat decomposition

$$\mathfrak{X} = \bigsqcup_{w \in W} G \cdot (\dot{e}B, \dot{w}B) \tag{2.1}$$

Analogously we call $\mathfrak{C}_w$ the stratum $G \cdot (\dot{e}B, \dot{w}B)$ and $\mathfrak{X}_w$ its closure. We have similarly $\mathfrak{X}_w = \bigsqcup_{y \leq w} \mathfrak{C}_y$. Furthermore one can show that $\mathfrak{C}_w$ is smooth and simply connected and that $\dim \mathfrak{X}_w = \dim X + l(w)$. In fact we have the following isomorphisms

$$\mathfrak{C}_w \cong G \times^B C_w \qquad \mathfrak{X}_w \cong G \times X_w \tag{2.2}$$

Example 2.1.6. Let us now look at some examples of the variety $\mathfrak{X}$ and its decomposition in the case $G = \mathrm{GL}_n(\mathbb{C})$: we can see it as the set of pairs of flags in $\mathbb{C}^n$. Let us examine the cases $n = 1$ and $n = 2$.

$n = 2$**:** The decomposition induced by the Bruhat one on $\mathrm{GL}_2(\mathbb{C})$ is the following

$$\mathrm{GL}_2(\mathbb{C})/B_2 \times \mathrm{GL}_2(\mathbb{C})/B_2 =$$
$$= \underbrace{\left\{(gB_2, gB_2) \mid g \in \mathrm{GL}_2(\mathbb{C})\right\}}_{=\mathfrak{C}_e} \sqcup \underbrace{\left\{(gB_2, g\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)B) \mid g \in \mathrm{GL}_2(\mathbb{C})\right\}}_{=\mathfrak{C}_s}$$

and we can view $\mathfrak{C}_e$ as the set of pairs of equal flags and $\mathfrak{C}_s$ as that of different flags.

$n = 3$**:** our decomposition is now the following

$$\mathrm{GL}_3(\mathbb{C})/B_3 \times \mathrm{GL}_3(\mathbb{C})/B_3 = \underbrace{\Big\{(gB_2, gB_2) \mid g \in \mathrm{GL}_3(\mathbb{C})\Big\}}_{=\mathfrak{C}_e} \sqcup$$

$$\sqcup \underbrace{\Big\{(gB_2, g\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right)B) \mid g \in \mathrm{GL}_3(\mathbb{C})\Big\}}_{=\mathfrak{C}_s} \sqcup \underbrace{\Big\{(gB_2, g\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)B) \mid g \in \mathrm{GL}_3(\mathbb{C})\Big\}}_{=\mathfrak{C}_t} \sqcup$$

$$\sqcup \underbrace{\Big\{(gB_2, g\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}\right)B) \mid g \in \mathrm{GL}_3(\mathbb{C})\Big\}}_{=\mathfrak{C}_{st}} \sqcup \underbrace{\Big\{(gB_2, g\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right)B) \mid g \in \mathrm{GL}_3(\mathbb{C})\Big\}}_{=\mathfrak{C}_{ts}} \sqcup$$

$$\sqcup \underbrace{\Big\{(gB_2, g\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}\right)B) \mid g \in \mathrm{GL}_3(\mathbb{C})\Big\}}_{=\mathfrak{C}_{sts}}$$

We can view these cells as subsets consisting of pairs

$$\big((V_1, V_2, V_3), (W_1, W_2, W_3)\big)$$

of flags (where $V_3 = W_3 = \mathbb{C}^3$) such that:

$\mathfrak{C}_e$:	$V_1 = W_1\,,\ V_2 = W_2$
$\mathfrak{C}_s$:	$V_1 \neq W_1\,,\ V_2 = W_2$
$\mathfrak{C}_t$:	$V_1 = W_1\,,\ V_2 \neq W_2$
$\mathfrak{C}_{st}$:	$V_1 \neq W_1\,,\ V_2 \neq W_2\,,\ V_1 \subset W_2\,,\ W_1 \not\subset V_2$
$\mathfrak{C}_{ts}$:	$V_1 \neq W_1\,,\ V_2 \neq W_2\,,\ V_1 \not\subset W_2\,,\ W_1 \subset V_2$
$\mathfrak{C}_{sts}$:	$V_1 \neq W_1\,,\ V_2 \neq W_2\,,\ V_1 \not\subset W_2\,,\ W_1 \not\subset V_2$

Now we want to give an analogous resolution of singularities of varieties $\mathfrak{X}_w$. For each $s \in S$, we put $Y_s = G/P_s$: it can be interpreted as a partial flag variety. We have the following natural morphisms

$$X \to Y_s \quad gB \mapsto gP_s \tag{2.3}$$

Given a reduced expression $\underline{s}$ for $w \in W$ let us consider the variety

$$\mathfrak{Z}_{\underline{s}} := X \times_{Y_{s_1}} X \times_{Y_{s_2}} \cdots \times_{Y_{s_k}} X$$

where fiber products are taken with respect to morphisms (2.3). It is a smooth variety and taking the projection on the first and last component, we obtain a morphism $\mathfrak{Z}_{\underline{s}} \to \mathfrak{X}$ whose image is $\mathfrak{X}_w$. In fact let $(g_1, \dots, g_{k+1}) \in \mathfrak{Z}_{\underline{s}}$, we have

$$g_1^{-1}g_2 \in P_{s_1} = B \sqcup B\dot{s_1}B \quad \dots \quad g_k^{-1}g_{k+1} \in P_{s_k} = B \sqcup B\dot{s_k}B$$

So $g_1^{-1}g_{k+1} \in \bigcup_{\epsilon_i=0,1} B\dot{s_1}^{\epsilon_1} \dots \dot{s_k}^{\epsilon_k}B$ which is contained in $\mathfrak{X}_w$. Hence we call

$$\varpi_{\underline{s}} : \mathfrak{Z}_{\underline{s}} \to \mathfrak{X}_w$$

the obtained morphism. One can show that it is again a resolution of singularities. In fact we have the isomorphism

$$\begin{aligned} \Psi_{\underline{s}} \colon G \times^B Z_{\underline{s}} &\to \mathfrak{Z}_{\underline{s}} \\ [g : g_1 : \dots : g_k B] &\mapsto (gB, gg_1B, \dots, gg_1 \cdots g_kB) \end{aligned}$$

which makes the following diagram commute

$$\begin{array}{ccc} G \times^B Z_{\underline{s}} & \xrightarrow{\varpi_{\underline{s}}} & G \times^B X \\ \downarrow{\Psi_{\underline{s}}} & & \downarrow \\ \mathfrak{Z}_{\underline{s}} & \xrightarrow{\pi_{\underline{s}}} & X \times X \end{array}$$

where the right vertical arrow is given by $[g : hB] \mapsto (gB, ghB)$ and is again an isomorphism.

Finally the map

$$(g_1B, \dots, g_{k+1}B) \mapsto \big((g_1B, g_2B), \dots (g_kB, g_{k+1}B)\big)$$

gives an isomorphism

$$\mathfrak{Z}_{\underline{s}} \cong \mathfrak{X}_{s_1} \times_X \cdots \times_X \mathfrak{X}_{s_k} \tag{2.4}$$

where fibered products are taken with respect to projection (second projection on the left and first projection on the right).

2.2 Intersection Cohomology and Hecke Algebras

Given any complex algebraic variety Y we denote by $\mathrm{Sh}(Y, \mathrm{Vect}_{\mathbb{Q}})$ the category of sheaves in $\mathbb{Q}$-vector spaces on Y, and with $D(Y)$ its derived category. If we can write $Y = \bigsqcup_{\lambda \in \Lambda} Y_\lambda$, where $\mathcal{L} = \{Y_\lambda \mid \lambda \in \Lambda\}$ is a Whitney stratification, we denote $D^b_{\mathcal{L}}(Y)$ the full subcategory of $D(Y)$ whose objects are bounded complexes (by which we mean cohomologically bounded) with cohomology sheaves that are constructible with respect to $\mathcal{L}$. We denote by

$(-)[1]$ the shift functor of $D(Y)$ (and of $D^b_{\mathcal{L}}(Y)$), and by $(-)[n]$ its n-th iteration, for any integer number n. We recall that $\overline{Y_\lambda}$ is a closed subvariety, union of strata, and if $\underline{i}_\lambda$ denotes the inclusion in Y and $\mathcal{E}$ is an irreducible local system on Y_λ, then the complex[2] $(\underline{i}_\lambda)_* \operatorname{IC}^\bullet(\overline{Y_\lambda}, \mathcal{E})[\dim \overline{Y_\lambda}]$ is a simple object in the abelian category of perverse sheaves with respect to the stratification $\mathcal{L}$ and all simple objects of this category are of this form.

In the following, by abuse of notation, we shall still denote $\underline{\mathbb{Q}}_{\overline{Y_\lambda}}$ the complex that, rigorously, should be denoted $(\underline{i}_\lambda)_* \underline{\mathbb{Q}}_{\overline{Y_\lambda}}[0]$. Similarly we shall sometimes omit the extension by zero from closed subvarieties: for instance we shall still denote $\operatorname{IC}^\bullet(\overline{Y_\lambda}, \mathcal{E})[\dim \overline{Y_\lambda}]$ the complex in $D^b_{\mathcal{L}}(Y)$ that, rigorously, should be denoted $(\underline{i}_\lambda)_* \operatorname{IC}^\bullet(\overline{Y_\lambda}, \mathcal{E})[\dim \overline{Y_\lambda}]$.

We recall also that if Z is another complex algebraic variety endowed with a Whitney stratification $\mathcal{M}$ and $f : Z \to Y$ is a proper morphism compatible with stratifications, then we have the Decomposition Theorem: given any simple object $\operatorname{IC}^\bullet(\overline{Z_\mu}, \mathcal{F})$ in the category of perverse sheaves with respect to $\mathcal{M}$ we have that its derived direct image by f_* admits the following decomposition in $D_{\mathcal{L}}(Y)$:

$$f_*\big(\operatorname{IC}^\bullet(\overline{Z_\mu}, \mathcal{F}[\dim \overline{Z_\mu}])\big) = \bigoplus_{\lambda \in \Lambda} \operatorname{IC}^\bullet(\overline{Y_\lambda}, \mathcal{E}_\lambda[\dim \overline{Y_\lambda}]) \otimes_{\mathbb{Q}} V_\lambda$$

for certain graded vector spaces V_λ. For details about derived categories one can see for example [30], whereas for perverse sheaves and intersection complexes one can see [1] and [2], as well as [4].

In our case, the Bruhat decomposition gives a Whitney stratification of X that we denote by $\mathcal{W}$. We denote by $\operatorname{IC}(X_w)$ the shifted intersection complex $\operatorname{IC}^\bullet(X_w, \underline{\mathbb{Q}}_{C_w})[\dim X] = \operatorname{IC}^\bullet(X_w, \underline{\mathbb{Q}}_{C_w})[l(w)]$. Analogously, (2.1) gives a Whitney stratification of $\mathfrak{X}$, which we denote $\mathfrak{W}$ and we denote $\operatorname{IC}(\mathfrak{X}_w)$ the shifted intersection complex $\operatorname{IC}^\bullet(X_w, \underline{\mathbb{Q}}_{\mathfrak{C}_w})[\dim \mathfrak{X}_w] = \operatorname{IC}^\bullet(X_w, \underline{\mathbb{Q}}_{\mathfrak{C}_w})[l(w) + \dim X]$

We shall now present a function that enables us to describe the structure of complexes in $D^b_{\mathcal{W}}(X)$ via the Hecke algebra $\mathscr{H}_{(W,S)}$ (see §1.2)f. Let us define $h_{\mathcal{W}} : \operatorname{Ob} D^b_{\mathcal{W}}(X) \to \mathscr{H}_{(W,S)}$ in the following way

$$\mathscr{F}^\bullet \mapsto \sum_{w \in W} \sum_{i \in \mathbb{Z}} \dim\big(\mathscr{H}^i_w(\mathscr{F}^\bullet)\big) v^{-i-l(w)} H_w$$

where $\mathscr{H}^i_w(\mathscr{F}^\bullet)$ is the stalk at *any* point of C_w of the cohomology sheaf $\mathscr{H}^i(\mathscr{F}^\bullet)$: in fact we know that for every $\mathscr{F}^\bullet$ in $D^b_{\mathcal{W}}(X)$, the complex $i_w^* \mathscr{F}^\bullet = \mathscr{F}^\bullet|_{C_w}$ has cohomology sheaves which are locally constant, but since $C_w \cong \mathbb{A}^{l(w)}$ is simply connected, they need to be constant.

[2]Note that $(\underline{i}_\lambda)_*$ is the same as $(\underline{i}_\lambda)_!$, because $\overline{Y_\lambda}$ is closed.

The function $h_{\mathcal{W}}$ can also be defined in this way: by what we said $i_w^*\mathscr{F}^\bullet$ can be written uniquely in the form

$$\underline{\mathbb{Q}}_{C_w}[l(w)] \otimes_{\mathbb{Q}} V_w$$

where V_w is a graded $\mathbb{Q}$-vector space. Then

$$h_{\mathcal{W}}(\mathscr{F}^\bullet) = \sum_{w\in W}\sum_{i\in\mathbb{Z}}(\dim V_w^{-i})v^i H_w$$

We have that, by definition, $h_{\mathcal{W}}$ is an additive function in the following sense:

$$h_{\mathcal{W}}(\mathscr{F}_1^\bullet \oplus \mathscr{F}_2^\bullet) = h_{\mathcal{W}}(\mathscr{F}_1^\bullet) + h_{\mathcal{W}}(\mathscr{F}_2^\bullet)$$

Example 2.2.1. Let us consider $\underline{\mathbb{Q}}_X[\dim X] \in D^b_{\mathcal{W}}(X)$. We have:

$$i_w^*\underline{\mathbb{Q}}_X[\dim X] = \underline{\mathbb{Q}}_{C_w}[\dim X] = \underline{\mathbb{Q}}_{C_w}[\dim C_w][\dim X - \dim C_w]$$

and $h_{\mathcal{W}}(\underline{\mathbb{Q}}_X[\dim X]) = \sum_{w\in W} v^{\dim X - l(w)} H_w$.

Let us notice that, still calling w_0 the longest element of W, the preceeding example can be rewritten $h_{\mathcal{W}}\big(\mathrm{IC}(X_{w_0})\big) = \underline{H}_{w_0}$, because X is smooth, hence $\mathrm{IC}(X_{w_0}) = \mathrm{IC}(X) = \underline{\mathbb{Q}}_X[\dim X]$, and because of proposition 1.3.3. Actually this holds in general, giving the desired link between Kazhdan-Lusztig polynomials and local cohomology of Schubert varieties.

Theorem 2.2.2. *For any $w \in W$ we have*

$$h_{\mathcal{W}}\big(\mathrm{IC}(X_w)\big) = \underline{H}_w$$

where the $\underline{H}_w$'s are the elements of the self-dual basis for $\mathscr{H}_{(W,S)}$ found in 1.2.2.

Therefore, by proposition 1.2.3, dimension of $\mathscr{H}_y^i\big(\mathrm{IC}(X_w)\big)$ vanishes if i has not the same parity of $l(w)$ and otherwise it is equal to the coefficient of $h_{y,w}$ of degree $-i - l(y)$ (or the one of $P_{y,w}$ of degree $(i + l(w))/2$). We observe that in the same way as in remark 2.1.3, we could already know this parity vanishing property.

Theorem 2.2.2 answers our question on positivity of coefficients of Kazhdan-Lusztig polynomials (in this case of algebraic groups).

In order to prove it, it turns out to be convenient to work on $\mathfrak{X}$ rather then directly on X. We define similarly a function that we denote by $h_{\mathfrak{W}}$, from $\mathrm{Ob}\, D^b_{\mathfrak{W}}(\mathfrak{X})$ to $\mathscr{H}_{(W,S)}$ in the following way

$$\mathscr{G}^\bullet \mapsto \sum_{w\in W}\sum_{i\in\mathbb{Z}}(\dim \mathscr{H}_w^i(\mathscr{G}^\bullet))v^{-i-l(w)} H_w$$

where, analogously, $\mathscr{H}^i_w(\mathscr{G}^\bullet)$ denotes the stalk of the cohomology sheaf $\mathscr{H}^i(\mathscr{G}^\bullet)$ at *any* point of $\mathfrak{C}_w$: in fact, as we noticed, this is again smooth and simply connected.

It is important to notice that, by (2.2), $\mathscr{H}^i_y \operatorname{IC}(\mathfrak{X}_w) = \mathscr{H}^{i+\dim X}_y \operatorname{IC}(X_w)$. This entails

$$h_{\mathcal{W}}\big(\operatorname{IC}(X_w)\big) = v^{\dim X} h_{\mathfrak{W}}(\operatorname{IC}(\mathfrak{X}_w)) \tag{2.5}$$

More symmetrical structure of $\mathfrak{X}$ allows us to introduce a binary operation $*$ on $\operatorname{Ob} D^b_{\mathfrak{W}}(\mathfrak{X})$ given by

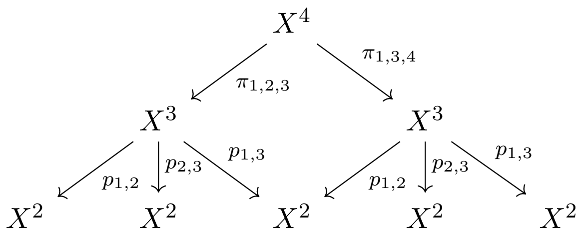
$$\mathscr{G}^\bullet_1 * \mathscr{G}^\bullet_2 := (p_{1,3})_*(p^*_{1,2}\mathscr{G}^\bullet_1 \otimes_{\mathbb{Q}} p^*_{2,3}\mathscr{G}^\bullet_2)$$

where $p_{i,j}$ are the projections $X^3 = X \times X \times X \to \mathfrak{X}$. The complex $\underline{\mathbb{Q}}_{\mathfrak{X}_e}$ is a neutral element of this operation: in fact, call ι the injection

$$\mathfrak{X} \hookrightarrow X^3$$

which sends (gB, hB) to (gB, gB, hB). Then $p_{1,2} \circ \iota = \mathrm{id}_{\mathfrak{X}}$ and, as the stalk of $\underline{\mathbb{Q}}_{\mathfrak{X}_e}$ is $\mathbb{Q}$ at points in the image of ι and zero elsewhere, then $p^*_{1,2}\underline{\mathbb{Q}}_{\mathfrak{X}_e} \otimes p^*_{2,3}\mathscr{G}^\bullet$ is $\iota_!\mathscr{G}^\bullet$, and $(p_{1,3})_*\iota_!\mathscr{G}^\bullet = \mathscr{G}^\bullet$.

We have that $*$ is associative, thanks to base change formula. In fact call $\pi_{i,j,k}\colon X^4 \to X^3$ the projection onto the components i, j and k. Base change formula applied to the upper cartesian square of the following diagram



together with projection formula, allow:

$$\begin{aligned}
&\big(\mathscr{G}^\bullet_1 * \mathscr{G}^\bullet_2\big) * \mathscr{G}^\bullet_3 = \\
&\quad = (p_{1,3})_*\big(p^*_{1,2}(p_{1,3})_*(p^*_{1,2}\mathscr{G}^\bullet_1 \otimes p^*_{2,3}\mathscr{G}^\bullet_2) \otimes p^*_{2,3}\mathscr{G}^\bullet_3\big) = \\
&\quad = (p_{1,3})_*\big((\pi_{1,3,4})_*\pi^*_{1,2,3}(p^*_{1,2}\mathscr{G}^\bullet_1 \otimes p^*_{2,3}\mathscr{G}^\bullet_2) \otimes p^*_{2,3}\mathscr{G}^\bullet_3\big) = \\
&\quad = (p_{1,3})_*\big((\pi_{1,3,4})_*(\pi^*_{1,2,3}p^*_{1,2}\mathscr{G}^\bullet_1 \otimes \pi^*_{1,2,3}p^*_{2,3}\mathscr{G}^\bullet_2) \otimes p^*_{2,3}\mathscr{G}^\bullet_3\big) = \\
&\quad = (p_{1,3})_*(\pi_{1,3,4})_*\big(\pi^*_{1,2}\mathscr{G}^\bullet_1 \otimes \pi^*_{2,3}\mathscr{G}^\bullet_2 \otimes \pi^*_{1,3,4}p^*_{2,3}\mathscr{G}^\bullet_3\big) = \\
&\quad = (\pi_{1,4})_*\big(\pi^*_{1,2}\mathscr{G}^\bullet_1 \otimes \pi^*_{2,3}\mathscr{G}^\bullet_2 \otimes \pi^*_{3,4}\mathscr{G}^\bullet_3\big)
\end{aligned}$$

where $\pi_{i,j}$ denote projections from X^4 to X^2. The same we would obtain from $\mathscr{G}^\bullet_1 * \big(\mathscr{G}^\bullet_2 * \mathscr{G}^\bullet_3\big)$.

The following lemma will be useful in order to prove theorem 2.2.2.

Lemma 2.2.3. *(cf. [33], lemma 3.1.1) Let $\mathscr{G}^\bullet$ be an object of $D^b_{\mathfrak{W}}(\mathfrak{X})$ such that $\mathscr{H}^i(\mathscr{G}^\bullet) = 0$ when i is even (resp. odd). Then $\underline{\mathbb{Q}}_{\mathfrak{X}_s} * \mathscr{G}^\bullet$ has the same property and*

$$h_{\mathfrak{W}}(\underline{\mathbb{Q}}_{\mathfrak{X}_s} * \mathscr{G}^\bullet) = v^{-1}\underline{H}_s h_{\mathfrak{W}}(\mathscr{G}^\bullet)$$

Proof. Let us first compute the right hand side: we use rules (1.5), that still hold even if we change the order of s and w, thanks to (1.4).

$$v^{-1}\underline{H}_s h_{\mathfrak{W}}(\mathscr{G}^\bullet) = \sum_{w\in W}\sum_{i\in\mathbb{Z}}(\dim \mathscr{H}^i_w(\mathscr{G}^\bullet))v^{-i-1-l(w)}\underline{H}_s H_w = \\ = S_1 + S_2 + S_3 + S_4$$

where

$$S_1 = \sum_{\substack{w\in W\\ sw>w}}\sum_{i\in\mathbb{Z}}(\dim \mathscr{H}^i_w(\mathscr{G}^\bullet))v^{-i-1-l(w)}H_{sw}$$
$$S_2 = \sum_{\substack{w\in W\\ sw>w}}\sum_{i\in\mathbb{Z}}(\dim \mathscr{H}^i_w(\mathscr{G}^\bullet))v^{-i-l(w)}H_{w}$$
$$S_3 = \sum_{\substack{w\in W\\ sw<w}}\sum_{i\in\mathbb{Z}}(\dim \mathscr{H}^i_w(\mathscr{G}^\bullet))v^{-i-1-l(w)}H_{sw}$$
$$S_4 = \sum_{\substack{w\in W\\ sw<w}}\sum_{i\in\mathbb{Z}}(\dim \mathscr{H}^i_w(\mathscr{G}^\bullet))v^{-i-2-l(w)}H_{w}$$

Now, noticing that $l(sw) = l(w) \pm 1$ depending on whether $sw > w$ or not, we obtain

$$S_1 + S_3 = \sum_{\substack{w\in W\\ sw>w}}\sum_{i\in\mathbb{Z}}\left(\dim \mathscr{H}^i_w(\mathscr{G}^\bullet) + \dim \mathscr{H}^{i-2}_{sw}(\mathscr{G}^\bullet)\right) v^{-i-l(w)}H_w$$
$$S_2 + S_4 = \sum_{\substack{w\in W\\ sw<w}}\sum_{i\in\mathbb{Z}}\left(\dim \mathscr{H}^i_{sw}(\mathscr{G}^\bullet) + \dim \mathscr{H}^{i-2}_{w}(\mathscr{G}^\bullet)\right) v^{-i-l(w)}H_w$$

Therefore it suffices to prove that

$$\dim \mathscr{H}^i_w(\underline{\mathbb{Q}}_{\mathfrak{X}_s} * \mathscr{G}^\bullet) = \begin{cases} \dim \mathscr{H}^i_w(\mathscr{G}^\bullet) + \dim \mathscr{H}^{i-2}_{sw}(\mathscr{G}^\bullet) & \text{if } sw > w \\ \dim \mathscr{H}^i_{sw}(\mathscr{G}^\bullet) + \dim \mathscr{H}^{i-2}_{w}(\mathscr{G}^\bullet) & \text{if } sw < w \end{cases}$$

Consider the subvariety $\widetilde{X_s}^w$ of X^3 defined by

$$\widetilde{X_s}^w := \{\dot{e}B\} \times X_s \times \{\dot{w}B\} = \{(\dot{e}B, gB, \dot{w}B) \mid g \in P_s\}$$

It is isomorphic to $X_s \cong \mathbb{P}^1_{\mathbb{C}}$. Now denote ι the inclusion of $\widetilde{X_s}^w$ in X^3 and let us consider the restriction

$$\widetilde{\mathscr{G}^\bullet} = \iota^*(p_{1,2}^*\underline{\mathbb{Q}}_{\mathfrak{X}_s} \otimes_{\mathbb{Q}} p_{2,3}^*\mathscr{G}^\bullet) = (p_{1,2}^*\underline{\mathbb{Q}}_{\mathfrak{X}_s} \otimes_{\mathbb{Q}} p_{2,3}^*\mathscr{G}^\bullet)|_{\widetilde{X_s}^w}$$

We have

$$\mathscr{H}_w^i(\underline{\mathbb{Q}}_{\mathfrak{X}_s} * \mathscr{G}^\bullet) = \mathbb{H}^i(\widetilde{X_s}^w, \widetilde{\mathscr{G}^\bullet})$$

where $\mathbb{H}$ denotes hypercohomology. In fact one can apply base change formula to the cartesian square

$$\begin{array}{ccc} \widetilde{X_s}^w & \hookrightarrow & X \times_{Y_s} X \times X \\ \downarrow & & \downarrow p_{1,3} \\ \{\cdot\} & \hookrightarrow & \mathfrak{X} \end{array}$$

Suppose $sw > w$: notice that $p_{1,2}$ sends every point of $\widetilde{X_s}^w$ in $\mathfrak{X}_s$, whereas $p_{2,3}$ sends only the point $(\dot{e}B, \dot{e}B, \dot{w}B)$ in $\mathfrak{C}_w$ and any other point in $\mathfrak{C}_{sw}$. Denote $Z := \{\dot{e}B, \dot{e}B, \dot{w}B)\}$ and U its complement, and denote i_Z and j_U the associated inclusions in $\widetilde{X_s}^w$. Then we have

$$\begin{aligned} i_Z^*\widetilde{\mathscr{G}^\bullet} = i_Z^*\iota^*(p_{1,2}^*\underline{\mathbb{Q}}_{\mathfrak{X}_s} \otimes_{\mathbb{Q}} p_{2,3}^*\mathscr{G}^\bullet) = (i_Z^*\iota^*p_{1,2}^*\underline{\mathbb{Q}}_{\mathfrak{X}_s}) \otimes_{\mathbb{Q}} (i_Z^*\iota^*p_{2,3}^*\mathscr{G}^\bullet) = \\ = (p_{1,2} \circ \iota \circ i_Z)^*\underline{\mathbb{Q}}_{\mathfrak{X}_s} \otimes_{\mathbb{Q}} (p_{2,3} \circ \iota \circ i_Z)^*\mathscr{G}^\bullet \end{aligned}$$

but $p_{1,2} \circ \iota \circ i_Z$ is simply the inclusion of the point $(\dot{e}B, \dot{e}B)$ in $\mathfrak{X}$ and $p_{2,3} \circ \iota \circ i_Z$ is the inclusion of the point $(\dot{e}B, \dot{w}B)$ in $\mathfrak{X}$. Hence $i_Z^*\widetilde{\mathscr{G}^\bullet}$ is the complex $\mathscr{G}^\bullet_{(\dot{e}B,\dot{w}B)}$, which is the same as $\mathscr{H}_w^\bullet(\mathscr{G}^\bullet$. Similarly we have that U identifies with a subset of $\mathfrak{C}_{sw}$ in $\mathfrak{X}$ and one has that $j_U^*\widetilde{\mathscr{G}^\bullet}$ is a constant complex with fiber $\mathscr{G}^\bullet_{(\dot{e}B,\dot{s}\dot{w}B)}$ then we can write $j_U^*\widetilde{\mathscr{G}^\bullet} = \underline{\mathbb{Q}}_U \otimes \mathscr{H}_{sw}^\bullet(\mathscr{G}^\bullet)$.

Now consider the distinguished triangle

$$j_{U!}j_U^*\widetilde{\mathscr{G}^\bullet} \to \widetilde{\mathscr{G}^\bullet} \to i_{Z*}i_Z^*\widetilde{\mathscr{G}^\bullet} \xrightarrow{+1}$$

It produces a long exact sequence in cohomology with compact support

$$\cdots \longrightarrow \mathbb{H}_c^{i-1}(U, j_U^*\widetilde{\mathscr{G}^\bullet}) \longrightarrow$$

$$\longrightarrow \mathbb{H}_c^i(Z, i_Z^*\widetilde{\mathscr{G}^\bullet}) \longrightarrow \mathbb{H}_c^i(\widetilde{X_s}^w, \widetilde{\mathscr{G}^\bullet}) \longrightarrow \mathbb{H}_c^i(U, j_U^*\widetilde{\mathscr{G}^\bullet}) \longrightarrow$$

$$\longrightarrow \mathbb{H}_c^{i+1}(Z, i_Z^*\widetilde{\mathscr{G}^\bullet}) \longrightarrow \cdots$$

Now observing that

1) $\mathbb{H}^i_c(Z, i_Z^* \widetilde{\mathscr{G}}^\bullet) = \mathscr{H}^i_w(\mathscr{G}^\bullet)$, as we pointed out;

2) $\mathbb{H}^i_c(\widetilde{X_s}^w, \widetilde{\mathscr{G}}^\bullet) = \mathbb{H}^i(\widetilde{X_s}^w, \widetilde{\mathscr{G}}^\bullet)$, as $\widetilde{X_s}^w$ is compact;

3) $\mathbb{H}^i_c(U, j_U^* \widetilde{\mathscr{G}}^\bullet) = \mathscr{H}^{i-2}_{sw}(\mathscr{G}^\bullet)$, by what we remarked above and the fact that $\mathbb{H}^i_c(U, \underline{\mathbb{Q}}_U) = \mathbb{Q}$ if $i = 2$ and 0 otherwise;

we can conclude by the vanishing assumption on $\mathscr{G}^\bullet$.

In the same way we can treat the case $sw < w$. □

Now we can prove the following theorem, which is the analogous for $\mathfrak{X}$ of the desired result 2.2.2.

Theorem 2.2.4 (cf. [33], th. 3.2.1). *For any $w \in W$ one has*

$$h_{\mathfrak{W}}(\mathrm{IC}(\mathfrak{X}_w)) = v^{\dim X} \underline{H}_w$$

From this, using (2.5), we can deduce 2.2.2.

Proof. We will procede by induction on the Bruhat order. The case $w = e$ is trivial since $\mathfrak{X}_e$ is isomorphic to the diagonal in $X \times X$ hence it is isomorphic to X which is smooth. So let us suppose $w \neq e$. Recall (2.4): applying repeatedly base change formula one obtains

$$\underline{\mathbb{Q}}_{\mathfrak{X}_{s_1}} * \cdots * \underline{\mathbb{Q}}_{\mathfrak{X}_{s_k}} = (p_{1,k+1})_* \big(p^*_{1,2} \underline{\mathbb{Q}}_{\mathfrak{X}_{s_1}} \otimes \cdots \otimes p^*_{k,k+1} \underline{\mathbb{Q}}_{\mathfrak{X}_{s_k}}\big)$$

where $p_{i,j}$ denotes the projection from X^{k+1} onto components i and j. Now, $p^*_{1,2} \underline{\mathbb{Q}}_{\mathfrak{X}_{s_1}} \otimes \cdots \otimes p^*_{k,k+1} \underline{\mathbb{Q}}_{\mathfrak{X}_{s_k}}$ has stalk $\mathbb{Q}$ at points of $\mathfrak{Z}_{\underline{s}}$ and zero elsewhere, then it is equal to $\iota_! \underline{\mathbb{Q}}_{\mathfrak{Z}_{\underline{s}}}$, and as $p_{1,k+1} \circ \iota = \varpi_{\underline{s}}$ we obtain:

$$(\varpi_{\underline{s}})_* (\underline{\mathbb{Q}}_{\mathfrak{Z}_{\underline{s}}}) = \underline{\mathbb{Q}}_{\mathfrak{X}_{s_1}} * \cdots * \underline{\mathbb{Q}}_{\mathfrak{X}_{s_k}}$$

Hence, by lemma 2.2.3 we get

$$h_{\mathfrak{W}}\big((\varpi_{\underline{s}})_* \underline{\mathbb{Q}}_{\mathfrak{Z}_{\underline{s}}}[k]\big) = \underline{H}_{s_1} \cdots \underline{H}_{s_k} \tag{2.6}$$

We know that $\mathfrak{Z}_{\underline{s}}$ is smooth, so

$$\mathrm{IC}^\bullet(\mathfrak{Z}_{\underline{s}})[\dim \mathfrak{Z}_{\underline{s}}] = \underline{\mathbb{Q}}_{\mathfrak{Z}_{\underline{s}}}[\dim \mathfrak{Z}_{\underline{s}}] = \underline{\mathbb{Q}}_{\mathfrak{Z}_{\underline{s}}}[k + \dim X]$$

We can thus use Decomposition Theorem and obtain

$$(\varpi_{\underline{s}})_* (\underline{\mathbb{Q}}_{\mathfrak{Z}_{\underline{s}}})[k + \dim X] = \bigoplus_{y \leq w} \mathrm{IC}^\bullet(\mathfrak{X}_y) \otimes V_y$$

where the V_y's are graded finite dimensional $\mathbb{Q}$-vector spaces. Recall that $\varpi_{\underline{s}}$ is an isomorphism over $\mathfrak{C}_w$, hence V_w must be $\mathbb{Q}$. Recall also that $\varpi_{\underline{s}}$

is proper so $(\varpi_{\underline{s}})_*(\underline{\mathbb{Q}}_{\mathfrak{Z}_{\underline{s}}})$ is self-dual, with respect to Grothendieck-Verdier duality, hence each V_y is self-dual (because intersection complexes are self-dual). We obtain

$$h_{\mathfrak{W}}\big(\varpi_{\underline{s}*}(\underline{\mathbb{Q}}_{\mathfrak{Z}_{\underline{s}}})[k+\dim X]\big) = h_{\mathfrak{W}}(\mathrm{IC}^\bullet(\mathfrak{X}_w)) + \sum_{y<w} q_y(v) h_{\mathfrak{W}}(\mathrm{IC}^\bullet(\mathfrak{X}_y))$$

For some $q_y(v)$ that by self-duality of V_y must satisfy $q_y(v^{-1}) = q_y(v)$. Now by induction hypothesis we have $h_{\mathfrak{W}}(\mathrm{IC}(\mathfrak{X}_y)) = v^{\dim X}\underline{H}_s$ for any $y < w$, hence the element $v^{-\dim X}h_{\mathfrak{W}}(\mathrm{IC}(\mathfrak{X}_y))$ is stable under duality in $\mathscr{H}_{(W,S)}$. But then also $v^{-\dim X}\sum_{y<w} q_y(v)h_{\mathfrak{W}}(\mathrm{IC}^\bullet(\mathfrak{X}_y))$ is stable, and by (2.6) so is $v^{-\dim X}h_{\mathfrak{W}}(\varpi_{\underline{s}*}(\underline{\mathbb{Q}}_{\mathfrak{Z}_{\underline{s}}})[k+\dim X]$. So we conclude that the same is true for $v^{-\dim X}h_{\mathfrak{W}}(\mathrm{IC}(\mathfrak{X}_w))$ Now by properties of intersection complexes we know that $v^{-\dim X}h_{\mathfrak{W}}(\mathrm{IC}(\mathfrak{X}_w))$ must belong to $H_w + \sum_{y<w} v\mathbb{Z}[v]H_y$, so we can conclude by theorem 1.2.2. □

As we remarked this theorem implies theorem 2.2.2.

As we said at the end of last chapter this is only a partial answer to the general question of whether Kazhdan-Lusztig polynomials of general Coxeter systems have positive coefficients: a complete answer was given by Elias and Williamson (see [16]). On the other hand one has a result in the opposite direction. Given any polynomial with positive coefficient and constant term 1, it is the $P_{y,w}$ for some pair (y, w) of elements of some Coxeter system (W, S) and actually one can always choose as W a symmetric group S_n of appropriate order. This is shown in [31] using geometrical methods as the ones presented in this chapter.

Chapter 3

Category $\mathcal{O}$

In this chapter we shall present the statement of Kazhdan-Lusztig conjecture. We first introduce the category $\mathcal{O}$ of Bernstein-Gel'fand-Gel'fand and describe its basic properties.

In what follows we shall denote $\mathfrak{g}$ a finite dimensional semisimple Lie algebra over an algebraically closed field k of characteristic 0 and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan sub-algebra such that $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ where

$$\mathfrak{n}_- = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha} \qquad \mathfrak{n}_+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha}$$

Here $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}^*\}$ and $\Phi = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$ is the associated root system, where we fixed a set Δ of simple roots and a corresponding set Φ^+ of positive roots. It is a root system in the sense of §1.1 with respect to the euclidean vector space $E = E_0 \otimes_{\mathbb{Q}} \mathbb{R}$, where E_0 is the $\mathbb{Q}$-subvector space of $\mathfrak{h}^*$ generated by Φ and where the non-degenerate bilinear form is given by that of $\mathfrak{h}$, namely the Killing form. We denote by W the Weyl group associated to this root system and S the set of simple reflections (which generates it).

Recall that Δ is a basis of $\mathfrak{h}^*$ and that, fixing a numbering $\alpha_1, \ldots, \alpha_n$ of Φ^+ such that $\{\alpha_1, \ldots, \alpha_l\} = \Delta$, we can consider a standard basis of $\mathfrak{g}$, given by

$$\{y_i, h_j, x_i \mid i = 1, \ldots, n \quad j = 1 \ldots, l\}$$

such that $\mathfrak{g}_{\alpha_i} = kx_i$, $\mathfrak{g}_{-\alpha_i} = ky_i$ and $\mathfrak{h} = kh_1 \oplus \cdots \oplus kh_l$, with conditions $[x_j, y_j] = h_i$ and $\alpha_j(h_j) = 2$, for each $j = 1, \ldots, l$. One also has in general that $\alpha_j(h_i) \in \mathbb{Z}$. Observe that one has an action of W over $\mathfrak{h}^*$ defined as the one over E, namely

$$s_{\alpha_j}(\lambda) := \lambda - \lambda(h_j)\alpha_j$$

for any $\alpha_j \in \Delta$.

We call *integral weights* the elements $\varpi \in \mathfrak{h}^*$ such that, for every i, $\varpi(h_i) \in \mathbb{Z}$ (in particular they belong to $E_0 \subset \mathfrak{h}^*$): among them we have the *fundamental weights* $\varpi_1, \ldots, \varpi_l$ such that $\varpi_i(h_j) = \delta_{ij}$ and the *special weight* $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha = \sum_i \varpi_i$.

We denote $U(\mathfrak{g})$ the envelopping algebra of $\mathfrak{g}$: recall that Poincaré-Birkhoff-Witt theorem tells us that it can be written as $U(\mathfrak{n}_-)U(\mathfrak{h})U(\mathfrak{n}_+)$, i.e. a basis is given by the monomials

$$y_1^{s_1} \cdots y_n^{s_n} h_1^{t_1} \cdots h_l^{t_l} x_1^{r_1} \cdots x_n^{r_n}$$

Let us recall that the datum of a $\mathfrak{g}$-module (i.e. a representation of $\mathfrak{g}$) is equivalent to that of a $U(\mathfrak{g})$-module (one has an equivalence of categories $\mathfrak{g}$-mod $\cong U(\mathfrak{g})$-mod).

For the details of all these notions see [24], §8, §10, §17.

3.1 Definition and first properties

We shall be concerned in representations of $\mathfrak{g}$, i.e. $\mathfrak{g}$-modules. The study of these modules considerably simplifies if one restricts to a particuar subcategory called *category* $\mathcal{O}$. In this and in the following chapter we refer to [23].

Definition 3.1.1. The category $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$- mod, whose objects are the $\mathfrak{g}$-modules M satisfying the following properties:

(*i*) M is $\mathfrak{h}$-semisimple, i.e. $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$, where

$$M_\lambda = \{v \in M \mid h \cdot v = \lambda(h)v, \forall h \in \mathfrak{h}\}$$

(*ii*) M is finitely generated as a $U(\mathfrak{g})$-module: there exists a finiet number of elements $v_1, \ldots, v_n$ such that $M = \sum_i U(\mathfrak{g}) \cdot v_i$

(*iii*) M is locally $\mathfrak{n}_+$-finite, i.e. for every $v \in M$, the subspace $U(\mathfrak{n}_+) \cdot v$ has finite dimension.

Therefore, roughly speaking, we are considering weight modules (i.e. decomposable, as vector spaces, in weight spaces) with certain finiteness conditions. In this category we find in particular the classical finite-dimensional modules (the axioms being readily verified in this case), for which we know classical results such as Weyl character formula. Recall that, given a finite dimensional $\mathfrak{g}$-module M, one defines its *formal character* as

$$\operatorname{ch} M := \sum_\lambda \dim M_\lambda e^\lambda$$

where e^λ is a formal symbol representing λ (the sum being finite by finite dimensionality of M). We can give these symbols also the more precise meaning of functions $e^\lambda : \mathfrak{h}^* \to \mathbb{Z}$ associating 1 to λ and 0 to any other element. In this way also $\operatorname{ch} M$ assumes the meaning of a function $\mathfrak{h}^* \to \mathbb{Z}$. Recall then that we can introduce a product in the set of function $\mathfrak{h}^* \to \mathbb{Z}$ having finite support (i.e. vanishing outside a finite subset of $\mathfrak{h}^*$) in the following way. Let $f, g : \mathfrak{h}^* \to \mathbb{Z}$ with finite support, we put

$$f * g\,(\lambda) = \sum_{\mu+\nu=\lambda} f(\mu)g(\nu) \tag{3.1}$$

Hence in particular $e^\lambda * e^\mu = e^{\lambda+\mu}$, explaining the exponential notation. In the following we shall denote $*$ simply by juxtaposition.

These characters furnish a way to describe the structure of finite dimensional $\mathfrak{g}$-modules. In particular, Weyl formula gives us characters of simple modules. Recall that every simple finite dimensional module is generated by a vector v of weight λ, with λ integral and *dominant*, i.e. $\lambda \in \mathbb{Z}^+\varpi_1 + \ldots \mathbb{Z}^+\varpi_l$, and it is uniquely determined by this λ. Let $L(\lambda)$ be this module, then we have

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w\in W}(-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\sum_{w\in W}(-1)^{l(w)} e^{w(\rho)-\rho}} \tag{3.2}$$

This formula, along with Weyl complete reducibility theorem (see [24], §6.3) gives a complete description of finite dimensional modules.

What we want to describe is an attempt of generalization of this description to modules in $\mathcal{O}$.

We begin by deducing some basic properties from the axioms.

Proposition 3.1.2. *Let M be a module in $\mathcal{O}$:*

a) For every $\lambda \in \mathfrak{h}^*$, the weight space M_λ has finite dimension.

b) The weights of M (i.e. the λ's in $\mathfrak{h}^*$ such that $M_\lambda \neq 0$) belong to a finite union of subsets of the form $\lambda_i - \Gamma$, for certain λ_i's.

c) M is noetherian (hence category $\mathcal{O}$ is noetherian).

d) Every quotient and every submodule of M belongs to $\mathcal{O}$ (hence category $\mathcal{O}$ is abelian).

Proof. *a,b*) Thanks to the axioms one can suppose that the generators of M belong to finitely many weight spaces M_{λ_1}, ..., M_{λ_n} such that the action of $U(\mathfrak{n}^+)$ stabilize $\bigoplus_{i=1}^n M_{\lambda_i}$. The algebra $U(\mathfrak{h})$ stabilizes each of them, whereas each monomial of $U(\mathfrak{n}_-)$ sends a generator $v \in M_{\lambda_i}$ to a vector of weight in $\lambda_i - \Gamma$, and each weight can be reached by the action of only a finite number of monomials.

c) One can use the fact that $U(\mathfrak{g})$ is noetherian and every M in $\mathcal{O}$ is finitely generated.

d) A submodule N of an $\mathfrak{h}$-semisimple module M is still $\mathfrak{h}$-semisimple and in particular its weight spaces are subspaces of the weight spaces of the starting module, namely $N = \bigoplus_\lambda N_\lambda$, where $N_\lambda = M_\lambda \cap N$; furthermore it is finitely generated by (c). The quotient $N' = M/N$ is finitely generated and decomposable as $\bigoplus_\lambda N'_\lambda$ where $N'_\lambda = M_\lambda + N/N$. □

Now we can generalize the notion of formal characters to the category $\mathcal{O}$. We define for every module M in $\mathcal{O}$, its formal character as the function $\operatorname{ch} M : \mathfrak{h}^* \to \mathbb{Z}$, which associates λ to $\dim M_\lambda$. One can then write it as the formal sum

$$\operatorname{ch} M = \sum_{\lambda \in \mathfrak{h}^*} \dim M_\lambda e^\lambda$$

One can also extend the product: instead of considering only function with finite support one can take functions $f, g : \mathfrak{h}^* \to \mathbb{Z}$ whose support is contained in a finite union of sets of the form $\lambda - \Gamma$ and let

$$f * g\,(\lambda) := \sum_{\mu+\nu=\lambda} f(\mu) g(\nu)$$

Formal character of modules in $\mathcal{O}$, by proposition 3.1.2, have the desired condition on supports. Inspired by the finite dimensional case we would like to describe characters of objects in $\mathcal{O}$. Before we have to further investigate the structure of category $\mathcal{O}$, in particular we want to find its simple objects. For this purpose we generalize the classical notion of highest weight modules.

An element v^+ of a $\mathfrak{g}$-module M is called *maximal vector of weight* λ if $v^+ \in M_\lambda$ and $\mathfrak{n}_+ \cdot v^+ = 0$. One says that M is a *highest weight module of weight* λ if it is generated, as a $U(\mathfrak{g})$-module by a maximal vector of weight λ: one can check that M is actually an object of $\mathcal{O}$ (see [23], §1.2). Every module of $\mathcal{O}$ possesses at least one maximal vector (thanks to his property of local $\mathfrak{n}^+$-finiteness), so at least one submodule which is a highest weight module. A fundamental property of highest weight modules is the following.

Proposition 3.1.3. *Every highest weight module admits a unique maximal submodule and a unique simple quotient.*

Proof. It is enough to remark that any proper submodule (which is a direct sum of sub-weight spaces) cannot contain multiples of the generator v^+, so the sum of all proper submodules is still a proper submodule. □

We call *Verma module of weight* λ the module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

where $\mathbb{C}_\lambda$ is $\mathbb{C}$, regarded as a $U(\mathfrak{b})$-module where $\mathfrak{h}$ acts via λ and $\mathfrak{n}_+$ acts trivially (by 0). One can verify that it is a highest weight module of weight λ with maximal vector $1 \otimes 1$. By Poincaré-Birkhoff-Witt theorem $M(\lambda)$ is isomorphic, as $U(\mathfrak{n}_-)$-module, to $U(\mathfrak{n}_-)$ itself.

Verma modules are *universal* among highest waight modules in the sense that, given any highest weight module M of weight λ, there exists, unique up to scalar multiples, a surjection $M(\lambda) \to M$. This holds because M is generated by a highest weight vector of weight λ hence every morphism from $M(\lambda)$ must send $1 \otimes 1$ in a multiple of this vector, and we obtain a surjection if and only if this multiple is not zero. By proposition 3.1.3, $M(\lambda)$ admits s unique simple quotient which we call $L(\lambda)$.

By what we have said so far, we have the following result.

Theorem 3.1.4. *The $L(\lambda)$'s are (up to isomorphism) the only simple modules in $\mathcal{O}$.* □

In particular $L(\lambda)$ must coincide with a simple finite dimensional module if λ is integral and dominant. Then it is natural to suppose that making clear the structure of these $L(\lambda)$'s should be determinant in order to investigate the one of any module in $\mathcal{O}$. To make this precise we have to describe the categorical role of simple objects in category $\mathcal{O}$.

3.2 Composition series, central characters and blocks

We cannot expect any module in $\mathcal{O}$ to be semisimple as in the finite dimensional case (e.g. Verma modules are not), but we have the following result.

Theorem 3.2.1. *The following properties hold*

i) Category $\mathcal{O}$ is artinian.

ii) Every module M of $\mathcal{O}$ admits a composition series

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M$$

such that $M_i/M_{i-1} \cong L(\lambda_i)$ for a certain $\lambda_i \in \mathfrak{h}^*$.

Part (ii) entails, thanks to Jordan-Hölder theorem, that the multiplicity of each $L(\lambda)$ as subquotient of M is well-defined and does not depend on the choice of the series: it is denoted $\left[M : L(\lambda)\right]$. It is therefore natural to ask how to compute these multiplicities. For example what are the multiplicities $\left[M(\lambda) : L(\mu)\right]$ for λ and μ in $\mathfrak{h}^*$?

Remark 3.2.2. We can already have a first result in the direction of theorem 3.2.1. Let M be a module of $\mathcal{O}$: it admits a finite-dimensional subspace V which generates M as a $U(\mathfrak{g})$-module. We take a maximal vector in V which then generates a highest weight submodule M_1 of M. The quotient M/M_1 admits a subspace of generators of dimension strictly lower than $\dim V$, so by induction one obtains that M has a filtration of submodules

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

such that M_{i+1}/M_i is a highest weight module. In particular we have that part (i) of the theorem entails part (ii).

In order to show theorem 3.2.1 we have to introduce some other more sophisticated tools: in particular we shall be concerned about the action of the center $Z(\mathfrak{g})$ of the envelopping algebra $U(\mathfrak{g})$ on modules of $\mathcal{O}$.

Let M be a highest weight module of weight λ generated by a vector v. For every $z \in Z(\mathfrak{g})$, because of its commutation properties, we have that $z \cdot v \in M_\lambda$, which is one-dimensional, so $z \cdot v = \chi_\lambda(z)v$ for a certain scalar $\chi_\lambda(z)$. But v is a $U(\mathfrak{g})$-generator of M, so one has $z \cdot u = \chi_\lambda(z)u$ for every $u \in M$. Furthermore $\chi_\lambda(z)$ does not depend on the particular module chosen (because for example one can use the surjection from a Verma module). One obtains therefore the function

$$\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$$

which, as one can check, turns out to be a $\mathbb{C}$-algebras homomorphism and which is called *central character* associated to λ.

Remark 3.2.3. One can have a glimpse of the importance of central characters, in order to describe the structure of modules in $\mathcal{O}$, by observing that if a highest weight module M of weight λ admits a subquotient which is a highest weight module of weight μ, i.e. $M \supset M' \supset M''$ with M'/M'' a highest weight module of weight μ, then there exists a $w \in (M')_\mu$ such that for any $z \in Z(\mathfrak{g})$ one has $z \cdot w - \chi_\mu(z)w \in M''$. But $z \cdot w = \chi_\lambda(z)w$ forcing $\chi_\lambda(z)w - \chi_\mu(z)w$ to be in M'', hence to be zero, as M'' cannot contain the subspace generated by w. So $\chi_\lambda = \chi_\mu$.

One can now define a *shifted* action of the group W over $\mathfrak{h}^*$ (also called *dot-action*):

$$w \cdot \lambda := w(\lambda + \rho) - \rho$$

If we consider the action on E, this shifted action is obtained by translating the origin on the point $-\rho$. In this way one has that $\mathfrak{h}^*$ is split into W-orbits with respect to the dot-action, called *linkage classes.*

We will use the following fundamental theorem, that we only state here

Theorem 3.2.4 (Harish-Chandra)(see [24], §23.3; [23], §1.10). *The following properties hold*

i) For every λ, μ in $\mathfrak{h}^*$, $\chi_\lambda = \chi_\mu$ if and only if λ and μ lie in the same linkage classe.

ii) Every central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is of the form $\chi = \chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$.

Let us use this result to show theorem 3.2.1.

Proof of theorem 3.2.1. Thanks to remark 3.2.2, it is enough to show part (i).

First we consider the case $M = M(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. Let

$$V = \sum_{w \in W} M_{w \cdot \lambda}$$

Note that it has finite dimension. Let N and N' be submodules of M such that $N \supset N'$ is a proper inclusion. The subquotient N/N' admits at least one maximal vector of weight μ, then, by the remark 3.2.3, we have $\chi_\lambda = \chi_\mu$. Harish-Chandra theorem implies that $\mu = w' \cdot \lambda$ for some $w' \in W$, hence $N \cap V \neq \{0\}$ and $\dim N \cap V > \dim N' \cap V$. Therefore any sequence of proper inclusions of submodules of M must terminate.

Now, let M be a highest weight module of weight λ. Through a surjection $M(\lambda) \to M$, every sequence of proper inclusions in M translates into a sequence of proper inclusions in $M(\lambda)$.

Finally let us take any module M: by the remark 3.2.2, it admits a sequence of sub-modules with subquotients which are highest weight modules, hence we have the following short exact sequences

$$\begin{gathered} 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_2/M_1 \longrightarrow 0 \\ 0 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow M_3/M_2 \longrightarrow 0 \\ \cdots \\ 0 \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_n/M_{n-1} \longrightarrow 0 \end{gathered} \tag{3.3}$$

Hence it is enough to observe that if in a short exact sequence $0 \to M' \to M \to M'' \to 0$ the modules M' and M'' are artinians, then M is also aritinian. $\square$

As we said, the importance of central characters is not confined to what we have seen so far: they furnish a way to simplify the study of category $\mathcal{O}$. In particular they allow a decomposition of the category in blocks.

For every module M in $\mathcal{O}$ we define the subspace

$$M^\chi := \left\{ v \in M \mid \exists\, n > 0 \text{ s.t. } \big(z - \chi(z)\big)^n v = 0 \right\}$$

We define $\mathcal{O}_\chi$ the full subcategory of $\mathcal{O}$ whose objects are modules M such that $M = M^\chi$. For example highest weight modules of weight λ are clearly in $\mathcal{O}_{\chi_\lambda}$.

Theorem 3.2.5. *Category $\mathcal{O}$ decomposes as follows:*

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{O}_{\chi_\lambda}$$

Proof. Let M be a module in $\mathcal{O}$: for every $\lambda \in \mathfrak{h}^*$, the weight space M_λ is a $Z(\mathfrak{g})$-invariant subspace on which $Z(\mathfrak{g})$ gives a family of commuting endomorphisms, so $M_\lambda = \bigoplus_\chi (M_\lambda \cap M^\chi)$. But then M is the direct sum of modules $\bigoplus_{\lambda \in \mathfrak{h}^*} (M_\lambda \cap M^\chi) = M^\chi$. Therefore one has $\mathcal{O} = \bigoplus_\chi \mathcal{O}_\chi$, and one concludes thanks to part (ii) of theorem 3.2.4. $\square$

We can then concentrate on the study of subcategories $\mathcal{O}_{\chi_\lambda}$ with $\lambda \in \mathfrak{h}^*$, each of which contains a finite number of Verma modules, and hence of simple modules: one for every element of the linkage class of λ.

We shall study in particular the block $\mathcal{O}_{\chi_0}$, called *pricipal block*. The orbit of 0 has exactly $|W|$ elements, so we say that 0 is a *regular weight*, which is not true for example for the weight $-\rho$ which is the only element ot its orbit. The study of this principal block turns out to be of central importance because one can extend it, as we shall make more precise, to any block $\mathcal{O}_{\chi_\lambda}$ associated to an integral weight λ.

3.3 Grothendieck group

When dealing with abelian categories it is useful to consider their so-called *Grothendieck group* which encodes some structural properties of the category.

We denote $K(\mathcal{O})$ the Grothendieck group of category $\mathcal{O}$: it is defined as follows. Let us consider the free abelian group $\widetilde{K}$ generated by objects of $\mathcal{O}$, and take the subgroup generated by elements $M - M' - M''$ such that there exists a short exact sequence $0 \to M' \to M \to M'' \to 0$. The Grothendieck group $K(\mathcal{O})$ is defined to be the quotient by this subgroup and we denote $[M]$ the elemet of $K(\mathcal{O})$ corresponding to M. In particular, if $M \cong M'$, then $[M] = [M']$.

If $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ is a composition series for M then we have the short exact sequences given in (3.3). So we have $[M] = \sum_{i=1}^{n} \big[M_i/M_{i-1}\big]$. Hence by theorem 3.2.1, the elements $\big[L(\lambda)\big]$ form a basis of $K(\mathcal{O})$, in particular:

$$[M] = \sum_{\lambda \in \mathfrak{h}^*} \big[M : L(\lambda)\big] \big[L(\lambda)\big] \tag{3.4}$$

Formal characters described in section § 2.1 are additive on short exact sequences, i.e. if $0 \to M' \to M \to M'' \to 0$ is exact then $\operatorname{ch} M = \operatorname{ch} M' + \operatorname{ch} M''$. So the function $\operatorname{ch} : \operatorname{Ob}(\mathcal{O}) \to (\mathbb{Z})^{\mathfrak{h}^*}$ induces a morphism of abelian groups:

$$\begin{aligned} K(\mathcal{O}) &\to (\mathbb{Z})^{\mathfrak{h}^*} \\ [M] &\mapsto \operatorname{ch} M \end{aligned}$$

This morphism is, furthermore, injective by (3.4), hence it identifies $K(\mathcal{O})$ with the subgroup of $(\mathbb{Z})^{\mathfrak{h}^*}$ generated by the $\operatorname{ch} M$'s. Hence, as we expected, formal characters express exhaustively the structure of modules they are associated to.

We have (by (3.4), but also directly by the additive nature of ch):

$$\operatorname{ch} M = \sum_{\lambda \in \mathfrak{h}^*} \big[M : L(\lambda)\big] \operatorname{ch} L(\lambda)$$

Then going back to our problem to determine characters of modules in $\mathcal{O}$ we have found that it is sufficient to determine those of the simpe objects $L(\lambda)$.

3.4 Some known formal characters

Computing the character of Verma modules is particularly easy: one can take advantage of their structure of $U(\mathfrak{n}_-)$-modules in order to determine the character. In particular, if $\beta \in \Gamma$, the weight space $M_{\lambda-\beta}$ has dimension equal to the number of monomials

$$y_1^{s_1} \cdots y_n^{s_n}$$

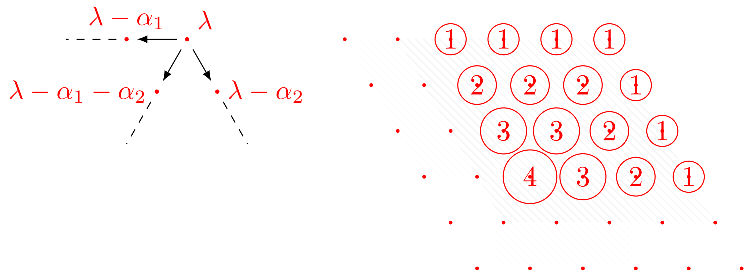
of $U(\mathfrak{n}_-)$ such that $\sum_i s_i \alpha_i = \beta$. This can be expressed in the following way:

$$\operatorname{ch} M(\lambda) = e^{\lambda} \prod_{\alpha} \left(1 + e^{-\alpha} + e^{-2\alpha} + \dots\right) \tag{3.5}$$

Example 3.4.1. Formula (3.5) tells us that in order to compute the dimension of the weight space $M(\lambda)_{\lambda-\beta}$ of the Verma module $M(\lambda)$, we have to count how many ways there are to write β as a sum of positive roots. Consider the type A_2 with $\Delta = \{\alpha_1, \alpha_2\}$: one finds

$$\dim M(\lambda)_{\lambda - n\alpha_1 - m\alpha_2} = \min\{n, m\} + 1$$

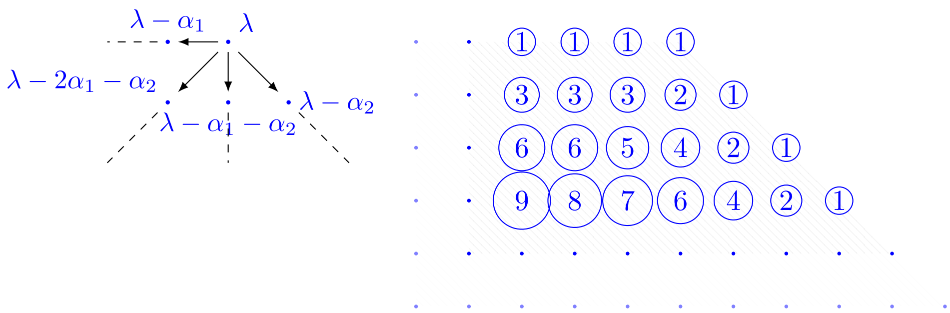
as the following figure represents



For type B_2 we have:

$$\dim M(\lambda)_{\lambda - n\alpha_1 - m\alpha_2} = \begin{cases} \max\{1, 2n, 2m\} & \text{if } n \leq m \\ n + m & \text{if } m < n \leq 2m \\ \max\{1, 3m\} & \text{if } n > 2m \end{cases}$$

And one has the following figure



In general, dimension of weight spaces of Verma modules are given by the *Kostant partition functions* (see [23] §1.16)

One could use characters of Verma modules to prove, in this framework, the classical Weyl formula (see [23], §2.4), which we can restate as follows

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w\in W}(-1)^{l(w)} e^{w\cdot\lambda}}{\sum_{w\in W}(-1)^{l(w)} e^{w\cdot 0}}$$

We shall now deal with the problem of finding characters of simple modules of integral weight, not necessarily dominant. Kazhdan-Lusztig conjecture gives a way to compute these characters in the principal block $\mathcal{O}_0$. This actually allows to compute them for any integral weight, thanks to *translation functors*, introduced by Zuckermann and Jantzen[1], which allow, under certain conditions, to transfer the results about a block to another one. In particular we call *integral antidominant* a weight in $\mathbb{Z}^-\varpi_1 + \dots \mathbb{Z}^-\varpi_l$. One has

Theorem 3.4.2 (see [23], §7.8)**.** *Let λ, $\mu \in \mathfrak{h}^*$ be integral regular antidominant weights. Then there exist functors $T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ and $T_\mu^\lambda : \mathcal{O}_\mu \to \mathcal{O}_\lambda$ that establish an equivalence of categories and are such that:*

i) $T_\lambda^\mu M(w\cdot\lambda) = M(w\cdot\mu)$

ii) $T_\lambda^\mu L(w\cdot\lambda) = L(w\cdot\mu)$

iii) T_λ^μ *et* T_λ^μ *induce mutually inverse isomorphisms between Grothendieck groups* $K(\mathcal{O}_\lambda)$ *and* $K(\mathcal{O}_\mu)$

3.5 Statement of Kazhdan-Lusztig conjecture

Let us consider the principal block $\mathcal{O}_0$: as we noticed, it contains precisely $|W|$ simple modules, one for each element of the linkage class of 0. Let us consider the weight $\lambda = -2\rho$.

We can now state Kazhdan-Lusztig conjecture, which, as we said, soon became a theorem

Theorem 3.5.1. *Given any $w \in W$ we have*

$$\operatorname{ch} L(w\cdot\lambda) = \sum_{y\leq w}(-1)^{l(w)-l(y)} P_{y,w}(1) \operatorname{ch} M(y\cdot\lambda) \tag{3.6}$$

where $P_{y,w}$ are Kazhdan-Lusztig polynomials associated to Coxeter system (W, S)

[1]They introduced them independently respectively in 1977 and 1979

Therefore one obtains the corresponding formula in the Grothendieck group:

$$\big[L(w\cdot\lambda)\big] = \sum_{y\leq w}(-1)^{l(y)-l(w)}P_{y,w}(1)\big[M(y\cdot\lambda)\big]$$

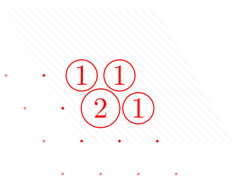
By using inversion formula 1.2.4 one obtains the equivalent statement:

$$\big[M(w\cdot\lambda)\big] = \sum_{y\leq w}P_{w_0w,w_0y}(1)\big[L(y\cdot\lambda)\big] \tag{3.7}$$

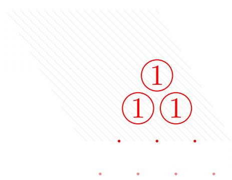
which, by (3.4), is the same as $\big[M(w\cdot\lambda) : L(y\cdot\lambda)\big] = P_{w_0w,w_0y}(1)$.

In the next chapters we shall describe the ideas of the proof of this statement which was completed by Fiebig, using moment graphs.

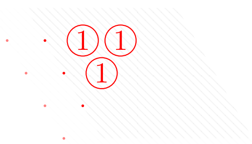
Example 3.5.2. In figure 3.1 we present the characters of the $L(w\cdot\lambda)$ in the principal block for the type A_2



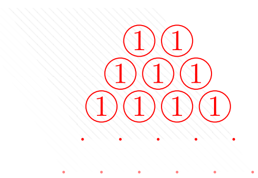
(a) Character of $L(\lambda) = M(\lambda)$



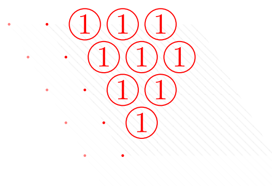
(b) Character of $L(s_{\alpha_1}\cdot\lambda)$



(c) Character of $L(s_{\alpha_2}\cdot\lambda)$



(d) Character of $L(s_{\alpha_2}s_{\alpha_1}\cdot\lambda)$



(e) Character of $L(s_{\alpha_1}s_{\alpha_2}\cdot\lambda)$

(f) Character of $L(s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}\cdot\lambda) = L(0)$

Figure 3.1: Characters of simple modules in the principal block for type A_2

Chapter 4

Moment graphs

This and the following chapter will be devoted to introduce, respectively, the notions of sheaves on moment graphs and that of deformed category $\mathcal{O}$.

In this chapter we will see that one can associate a moment graph to a given root system. Our first aim for this chapter is to define the Braden-MacPherson sheaves, study their categorical properties and state the multiplicity conjecture. Secondly we will describe the localization of $\mathcal{Z}$-modules. We follow [19], [18] and [27].

4.1 Definition

Definition 4.1.1. A *moment graph* $\mathcal{G}$ on a finite dimensional vector space V is the datum of:

i) a set of vertices $\mathcal{V}$;

ii) a set of edges $\mathcal{E}$ linking different vertices (and at most one for any pair of vertices), which we represent $E\colon x - y$, meaning that the edge E links vertices x and y in $\mathcal{V}$. We can think E as the set $\{x, y\}$, hence we will write $x \in E$ whenever x is a vertex of the edge E;

iii) a function $l\colon \mathcal{E} \to \mathbb{P}(V)$ which associates to each edge a one-dimensional subvector space of V (it would be the same to give a function $\alpha\colon \mathcal{E} \to V \setminus \{0\}$).

We shall always consider *finite* and *oriented* moment graphs, i.e. moment graphs such that the set $\mathcal{V}$ is finite and where every edge in $\mathcal{E}$ is oriented (and we then shall represent $x \to y$ the edge starting at x and ending at y) in such a way that there exists a partial order $\leq$ on $\mathcal{V}$ generated by the relation $\to$ (equivalently there are no oriented loops).

The partial order provides us a topology on the set of vertices of any (finite) oriented moment graph, namely the *Alexandrov topology*: we say that a subset $\mathcal{I} \subset \mathcal{V}$ is open if and only if it is an *upper set*, that is, such that for any $x \in \mathcal{I}$ and $y \geq x$, also $y \in \mathcal{I}$. Hence the sets $\{> x\} := \{y \in \mathcal{V} \mid y > x\}$ and $\{\geq x\} := \{y \in \mathcal{V} \mid y \geq x\}$ are open. The latter also form a basis for the topology.

Example 4.1.2. 1) The *generic moment graph* is the graph $\mathcal{G}$ where $\mathcal{V}$ consists of a single element, hence $\mathcal{E}$ is empty.

2) The *subgeneric moment graph* is the graph $\mathcal{G}$ where $\mathcal{V}$ consists of two elements and $\mathcal{E}$ of a single edge E, labelled with $l(E) \in \mathbb{P}(V)$, for a certain vector space V.

In the following we will still say *moment graph* meaning finite and oriented ones.

We can naturally associate moment graphs to root systems. Take a root system Φ in a euclidean vector space V and consider its Weyl group W: we put $\mathcal{V} = W$ and we link w and w' with an edge if and only if there exists a reflection s_α (with $\alpha \in \Phi$) such that $w' = wt$ and $w' > w$ in the Bruhat order. Finally we label this edge by the subspace generated by α.

4.2 Sheaves on Moment Graphs

We now want to define sheaves on moment graphs.

Let $S = S(V)$ denote the symmetric algebra of a k-vector space V, recall that it is a graded algebra. In the following, given a graded S-module M, we shall denote $M[1]$ the shifted module given by $M[1]_i = M_{i+1}$.

Definition 4.2.1. Given a moment graph $\mathcal{G}$ on V, we call a *sheaf on* $\mathcal{G}$ the datum $(\{\mathscr{F}_x\}_{x\in\mathcal{V}}, \{\mathscr{F}_E\}_{E\in\mathcal{E}}, \{\rho_{x,E}\}_{E\in\mathcal{E}, x\in E})$, where:

i) for any $x \in \mathcal{V}$, $\mathscr{F}_x$ is a graded S-module;

ii) for any $E \in \mathcal{E}$, $\mathscr{F}_E$ is a graded S-module such that $l(E)\mathscr{F}_E = 0$, where we view $l(E)$ inside $S(V)$;

iii) for any $E \in \mathcal{E}$ with $x \in E$, $\rho_{x,E}\colon \mathscr{F}_x \to \mathscr{F}_E$ is a morphism of graded S-modules.

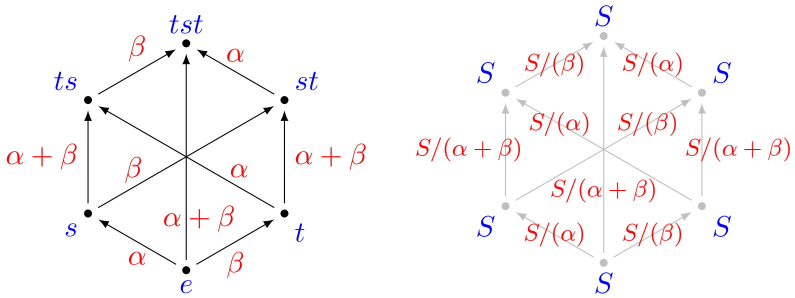
In a similar way, given a sheaf $\mathscr{F}$ on $\mathcal{G}$ we denote $\mathscr{F}[1]$ the sheaf obtained by shifting every S-module of $\mathscr{F}$. We have a natural notion of morphism of sheaves on moment graphs: given two sheaves $\mathscr{F}$ and $\mathscr{G}$, a morphism

$f\colon \mathscr{F} \to \mathscr{G}$ consists of two families $\{f_x\colon \mathscr{F}_x \to \mathscr{G}_x\}$ and $\{f_E\colon \mathscr{F}_E \to \mathscr{G}_E\}$ of morphisms of graded S-modules, such that for any $E \in \mathcal{E}$ and $x \in E$, the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}_x & \xrightarrow{f_x} & \mathscr{G}_x \\ \downarrow{\scriptstyle \rho^{\mathscr{F}}_{x,E}} & & \downarrow{\scriptstyle \rho^{\mathscr{G}}_{x,E}} \\ \mathscr{F}_E & \xrightarrow{f_E} & \mathscr{G}_E \end{array} \tag{4.1}$$

This gives a category of sheaves on the moment graph $\mathcal{G}$, that we denote $\mathrm{Sh}(\mathcal{G})$. We shall actually consider only those sheaves $\mathscr{F}$ for which $\mathscr{F}_x$ is a finitely generated and torsion-free S-module, and we shall denote the full subcategory so-defined by $\mathrm{Sh}^{\mathrm{f}}(\mathcal{G})$.

Example 4.2.2. The most natural example of sheaf on the moment graph $\mathcal{G}$ is the *structure sheaf* $\mathscr{A}$: it is defined via $\mathscr{A}_x = S$ and $\mathscr{A}_E = S/l(E)S$ for any $x \in \mathcal{V}$ and $E \in \mathcal{E}$ with $\rho_{x,E}$'s being the natural quotient maps. The following are the moment graph and the structure sheaf associated with the root system of type A_2



We now want to define sections on these sheaves: for any $\mathcal{I} \subset \mathcal{V}$ (not necessarily open in the Alexandrov topology) we put

$$\Gamma(\mathcal{I}, \mathscr{F}) := \Big\{ (f_x)_{x\in\mathcal{I}} \in \bigoplus_{x\in\mathcal{I}} \mathscr{F}_x \mid \forall E\colon x \to y,\ \rho_{x,E}(f_x) = \rho_{y,E}(f_y) \Big\}$$

It is a graded S-module: actually it is the kernel of the following morphism between graded S-modules:

$$\begin{aligned} \bigoplus_{x\in\mathcal{I}} \mathscr{F}_x &\longrightarrow \bigoplus_{E\in\mathcal{E}_{\mathcal{I}}} \mathscr{F}_E \\ (f_x)_{x\in\mathcal{I}} &\longmapsto \big(\rho_{x,E}(f_x) - \rho_{y,E}(f_y)\big)_{E\in\mathcal{E}_{\mathcal{I}}} \end{aligned}$$

In particular we denote $\Gamma(\mathscr{F})$ the module of *global sections* of $\mathscr{F}$, i.e. $\Gamma(\mathcal{V}, \mathscr{F})$. For any $\mathcal{I} \subset \mathcal{J} \subset \mathcal{V}$ we have the restriction morphism

$$\Gamma(\mathcal{J}, \mathscr{F}) \to \Gamma(\mathcal{I}, \mathscr{F})$$

which is given by projection along the direct sum.

The graded S-module $\mathcal{Z} := \Gamma(\mathscr{A})$ has a structure of S-algebra given by compoentwise multiplication: we call it the *structure algebra* of the moment graph $\mathcal{G}$.

Notice that, given a sheaf $\mathscr{F}$ on $\mathcal{G}$, for any $\mathcal{I} \subset \mathcal{V}$ the S-module $\mathcal{Z}$ acts componentwise on $\Gamma(\mathcal{I}, \mathscr{F})$ which then has a natural structure of $\mathcal{Z}$-module: in particular this holds for $\Gamma(\mathscr{F})$. Hence if we denote $\mathcal{Z}$-mod the category of $\mathcal{Z}$-modules, then we have a functor

$$\Gamma\colon \operatorname{Sh}(\mathcal{G}) \to \mathcal{Z}\text{-mod}$$

In fact, commuting diagram (4.1) allows to define naturally the behaviour of Γ on morphisms.

Now let us denote $\mathcal{Z}\text{-mod}^{\mathrm{f}}$ the full subcategory whose objects are $\mathcal{Z}$-modules which are finitely generated over S and torsion free over S. Then the previous functor naturally induces

$$\Gamma\colon \operatorname{Sh}^{\mathrm{f}}(\mathcal{G}) \to \mathcal{Z}\text{-mod}^{\mathrm{f}}$$

Consider now a prime ideal $\mathfrak{p} \in S$ and the localization $S_{\mathfrak{p}}$: we put

$$\mathcal{Z}_{\mathfrak{p}} := \left\{ (s_x)_{x\in\mathcal{V}} \in \bigoplus_{x\in\mathcal{V}} S_{\mathfrak{p}} \mid \forall E\colon x \to y, s_x - s_y \in l(E)S_{\mathfrak{p}} \right\}$$

In particular one can take $\mathfrak{p} = \mathfrak{h}S$ and obtain

$$\mathcal{Z}_R := \mathcal{Z}_{\mathfrak{h}S} = \left\{ (s_x)_{x\in\mathcal{V}} \in \bigoplus_{x\in\mathcal{V}} R \mid \forall E\colon x \to y, s_x - s_y \in l(E)S_{\mathfrak{p}} \right\}$$

where we denote R the localisation of S at $\mathfrak{h}S$. Notice also that unless $l(E) \subset \mathfrak{p}$, the ideal $l(E)S_{\mathfrak{p}}$ is the whole $S_{\mathfrak{p}}$, and the condition $s_x - s_y \in l(E)S_{\mathfrak{p}}$ is always verified. Hence we can consider the subgraph $\mathcal{G}_{\mathfrak{p}}$ having the same set of vertices as $\mathcal{G}$ but where we remove every edge E with $l(E)$ not contained in $\mathfrak{p}$. This graph is denoted $\mathcal{G}_{\mathfrak{p}}$ and it is called the $\mathfrak{p}$-*reduction* of $\mathcal{G}$.

Definition 4.2.3. We say that a moment graph $\mathcal{G}$ satisfy the *Goresky-Kottwitz-MacPherson-assumption* (or *GKM-assumption*) if for any vertex $x \in \mathcal{V}$ and $E_1, E_2 \in \mathcal{E}^x$ (where $\mathcal{E}^x := \{E \in \mathcal{E} \mid x \in E\}$) we have $l(E_1) \neq l(E_2)$ in $\mathbb{P}(V)$

Remark 4.2.4. Notice that in our characteristic zero case, the GKM-assumption is always verified for moment graphs associated to root systems.

Remark 4.2.5. Notice that the GKM-assumption is equivalent to requiring that for any prime ideal $\mathfrak{p}$ of height one, the $\mathfrak{p}$-reduction of the graph be the union of disjoint generic or subgeneric graphs.

4.3 Flabby Sheaves and Braden-MacPherson Sheaves

Definition 4.3.1. A sheaf $\mathscr{F}$ on a moment graph $\mathcal{G}$ is said to be *flabby* if restriction morphisms are surjective, i.e. for any inclusion of open subsets $\mathcal{I} \subset \mathcal{J} \subset \mathcal{V}$ the morphism $\Gamma(\mathcal{J}, \mathscr{F}) \to \Gamma(\mathcal{I}, \mathscr{F})$ is surjective.

Now let $\mathcal{E}^{\delta x}$ be the subset of $\mathcal{E}$ consisting of edges that start at the point x. We denote by $\mathscr{F}_{\delta x}$ the image of the following composition

$$\Gamma(\{> x\}, \mathscr{F}) \hookrightarrow \bigoplus_{y > x} \mathscr{F}_y \to \bigoplus_{x \to y} \mathscr{F}_y \to \bigoplus_{E \in \mathcal{E}^{\delta x}} \mathscr{F}_E$$

where the first one is simply inclusion, the second one is the projection along the direct sum and the third is induced by the maps $\rho_{y,E}$.

We can characterize flabby sheaves as follows:

Proposition 4.3.2. *Given a sheaf $\mathscr{F}$ on a moment graph $\mathcal{G}$, the following are equivalent:*

i) $\mathscr{F}$ is flabby.

ii) For any $\mathcal{I} \subset \mathcal{V}$, the morphism $\Gamma(\mathscr{F}) \to \Gamma(\mathcal{I}, \mathscr{F})$ is surjective.

iii) For any $x \in \mathcal{V}$ the restriction morphism $\Gamma(\{\geq x\}, \mathscr{F}) \to \Gamma(\{> x\}, \mathscr{F})$ is surjective.

iv) For any $x \in \mathcal{V}$, the morphism $\mathscr{F}_x \xrightarrow{\oplus \rho_{x,E}} \bigoplus_{E \in \mathcal{E}_{\delta x}} \mathscr{F}_E$ contains $\mathscr{F}_{\delta x}$ in its image.

Proof. We show $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ and $(iii) \Leftrightarrow (iv)$.

$(i) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (ii)$. Given any open proper subset $\mathcal{I}$ of $\mathcal{V}$, by finiteness of $\mathcal{V}$ we can find a vertex $x \notin \mathcal{I}$ such that $\{> x\} \subset \mathcal{I}$. Now take any section of $\Gamma(\mathcal{I}, \mathscr{F})$: by (iii) its restriction to $\Gamma(\{> x\}, \mathscr{F})$ can be extended to a section of $\Gamma(\{> x\}, \mathscr{F})$. Now these two sections coincide on the intersection, hence they give a section on $\mathcal{I} \cup \{x\}$. Now by induction we can obtain a global section.

$(ii) \Rightarrow (i)$. Given any inclusion $\mathcal{I} \subset \mathcal{J}$ of open subsets of $\mathcal{V}$ we have the following diagram of restriction morphisms:

$$\begin{array}{ccc} \Gamma(\mathscr{F}) & & \\ \downarrow & \searrow & \\ & & \Gamma(\mathcal{J}, \mathscr{F}) \\ \downarrow & \swarrow & \\ \Gamma(\mathcal{I}, \mathscr{F}) & & \end{array}$$

Hence as the composition is surjective by (*ii*) then also the second map has to be surjective.

(*iii*)⇔(*iv*). Suppose (*iii*) and take an element of $\mathscr{F}_{\delta x}$: by definition it is the image of a section of $\Gamma(\{> x\}, \mathscr{F})$. This by (*iii*) can be extended to a section of $\Gamma(\{\geq x\}, \mathscr{F})$ which in particular gives an element of $\mathscr{F}_x$ whose image via $\oplus \rho_{x,E}$ is, by definition of section, the given element of $\mathscr{F}_{\delta x}$. Conversely suppose (*iv*): any section on $\Gamma(\{> x\}, \mathscr{F})$ gives by definition an element of $\mathscr{F}_{\delta x}$ which by (*iv*) is the image of an element of $\mathscr{F}_x$ via $\oplus \rho_{x,E}$. Adding this element we obtain a section of $\Gamma(\{\geq x\}, \mathscr{F})$ as wanted. □

Now we can introduce the following fundamental notion:

Definition 4.3.3. We say that a sheaf $\mathscr{B}$ is *F-projective*[1] if

i) $\mathscr{B}_x$ is a graded free S-module of finite rank;

ii) for $E\colon x \to y$ the morphism $\mathscr{B}_x \to \mathscr{B}_E$ is surjective with kernel $l(E)\mathscr{B}_x$;

iii) $\mathscr{B}$ is flabby;

iv) The projection $\Gamma(\mathscr{B}) \to \mathscr{B}_x$ is surjective.

We shall now construct the fundamental objects among F-projective sheaves, i.e. the *Braden-MacPherson sheaves* which, by what we will show in next section turn out to be the unique indecomposable F-projective sheaves. The following procedure to construct them is called the *Braden-MacPherson algorithm.*

Let us fix a vertex v of $\mathcal{V}$, we build a moment graph $\mathscr{B}(v)$ associated to v. Let $\mathscr{B}(v)_x = 0$ for every $x \not\leq v$ and put $\mathscr{B}(v)_v = S$. Now take a vertex $x < v$ and suppose we have built $\mathscr{B}(v)_y$ for every $y > x$ and $\mathscr{B}(v)_E$ for every edge of the full subgraph $\{> x\}$. For any edge $E\colon x \to y$, put $\mathscr{B}(v)_E$ to be the quotient $\mathscr{B}(v)_y / l(E)\mathscr{B}(v)_y$ and $\rho_{y,E}$ to be the natural projection map (so that property (*ii*) is satisfied).

Now take $\mathscr{B}(v)_{\delta x}$ to be the image of the composition

$$\Gamma(\{> x\}, \mathscr{B}(v)) \hookrightarrow \bigoplus_{y>x} \mathscr{B}(v)_y \to \bigoplus_{x \to y} \mathscr{B}(v)_y \to \bigoplus_{E \in \mathcal{E}_{\delta x}} \mathscr{B}(v)_E$$

where[2]

$$\Gamma(\{> x\}, \mathscr{B}(v)) = \{(b_y)_{y>x} \in \bigoplus_{y>x} \mathscr{B}(v)_y \mid \rho_{y_1,E}(b_{y_1}) = \rho_{y_2,E}(b_{y_2}), \forall E\colon y_1 \to y_2\}$$

[1]We will use this terminology also for more general situation of A-sheaves (see §4.4): they are in fact two different definitions: this one uses the grading of S.

[2]Notice that, to avoid confusion we are using the same notation as if the sheaf $\mathscr{B}(v)$ had already been built.

Now let us take a projective cover of $\mathscr{B}(v)_{\delta x}$ and let us call it $\mathscr{B}(v)_x$. Notice that $\mathscr{B}(v)_x$ has to be free and of finite rank over S (satisfying condition (i)). Finally for $E'\colon x \to y$, put $\rho_{x,E'}$ to be the composition

$$\mathscr{B}(v)_x \twoheadrightarrow \mathscr{B}(v)_{\delta x} \to \bigoplus_{E \in \mathcal{E}_{\delta x}} \mathscr{B}(v)_E \to \mathscr{B}(v)_{E'}$$

In this way we inductively build a sheaf $\mathscr{B}(v)$ satisfying properties (i) and (ii). By construction the map $\mathscr{B}(v)_x \xrightarrow{\oplus \rho_{x,E}} \bigoplus_{E \in \mathcal{E}_{\delta x}} \mathscr{B}(v)_E$ contains $\mathscr{B}(v)_{\delta x}$ in its image, hence by proposition 4.3.2 $\mathscr{B}(v)$ is flabby. Finally we also have property (iv) because, given an element of $\mathscr{B}(v)^x$, by construction, we automatically get a section in $\Gamma(\{\geq x\}, \mathscr{B}(v))$, which in turn, by flabbyness, comes from a global section in $\Gamma(\mathscr{B}(v))$.

As we announced, one has the following result.

Theorem 4.3.4 (see [27] prop. 3.12)**.** *i) For any $v \in \mathcal{V}$, the sheaf $\mathscr{B}(v)$ is the unique indecomposable sheaf in $\mathrm{Sh}^{\mathrm{f}}(\mathcal{G})$ such that $\mathscr{B}(v)_v = S$ and $\mathscr{B}(v)_x = 0$ when $x \not\leq v$.*

ii) Any Braden-MacPherson sheaf $\mathscr{B}$ is isomorphic to a direct sum

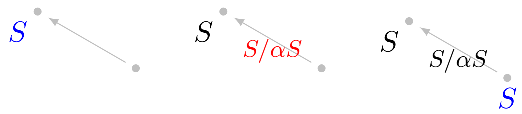
$$\mathscr{B}(v_1)[l_1] \oplus \cdots \oplus \mathscr{B}(v_n)[l_n]$$

for some $v_1, \ldots, v_n \in \mathcal{V}$ and $l_1, \ldots, l_n \in \mathbb{Z}$ uniquely determined.

Example 4.3.5 (Type A_2)**.** We construct explicitly the sheaves $\mathscr{B}(w)_x$ with $w, x \in W$ for a Coxeter system (W, S) of type A_2, whose associated moment graph is show in example 4.2.2

$\mathscr{B}(e)$: this one is trivial, $\mathscr{B}(e)_x = 0$ for every vertex exept e for which it is S, and $\mathscr{B}(e)_E = 0$ for every edge E.

$\mathscr{B}(s)$, $\mathscr{B}(t)$: we construct $\mathscr{B}(s)$, then symmetrically one obtains $\mathscr{B}(t)$. We have $\mathscr{B}(s)_s = S$ and $\mathscr{B}(s)_x = 0$ for every other vertex exept $\mathscr{B}(e)$. We also have $\mathscr{B}(s)^{e \to s} = S/(\alpha)$ and $\mathscr{B}(s)^E = 0$ for any other edge E. We need only to detrmine $\mathscr{B}(s)_e$ and $\rho_{e,e\to s}$: following the notation used in the description of the Braden-MacPherson algorithm, we have $\Gamma(\{> e\} = S$ and $\mathscr{B}(e)_{\delta e} = S/(\alpha)$. The quotient $S \twoheadrightarrow S/(\alpha)$ is a projective cover, hence we obtain $\mathscr{B}(s)_e = S$ and that $\rho_{e,e\to s}$ is the quotient map



$\mathscr{B}(ts)$, $\mathscr{B}(st)$**:** we have $\mathscr{B}(ts)_{ts} = S$ and by the same reasoning of the preceeding case we obtain $\mathscr{B}(ts)_s = \mathscr{B}(ts)_t = S$ as well as $\mathscr{B}(ts)_{s\to ts} = S/(\alpha+\beta)$ and $\mathscr{B}(ts)_{t\to ts} = S/(\alpha)$. Now we put $\mathscr{B}(ts)_{e\to s} = S/(\alpha)$ and $\mathscr{B}(ts)_{e\to t} = S/(\beta)$. We need only to find $\mathscr{B}(ts)_e$ and the two morphisms $\rho_{e,e\to s}$ and $\rho_{e,e\to t}$. We denote elements of

$$\mathscr{B}(ts)_{ts} \oplus \mathscr{B}(ts)_s \oplus \mathscr{B}(ts)_t = S \oplus S \oplus S$$

in the form $\left(\begin{smallmatrix} & z_{ts} & \\ z_s & & z_t \end{smallmatrix}\right)$, where $z_{ts} \in \mathscr{B}(ts)_{ts}$, $z_s \in \mathscr{B}(ts)_s$ and $z_t \in \mathscr{B}(ts)_t$ (in order to recall positions of modules in the graph). The submodule $\Gamma(\{> e\}, \mathscr{B}(ts))$ is generated by $\left(\begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & 0 & \\ \alpha+\beta & & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} & 0 & \\ 0 & & \alpha \end{smallmatrix}\right)$. The projection

$$\mathscr{B}(ts)_{ts} \oplus \mathscr{B}(ts)_t \oplus \mathscr{B}(ts)_s \to \mathscr{B}(ts)_t \oplus \mathscr{B}(ts)_s$$

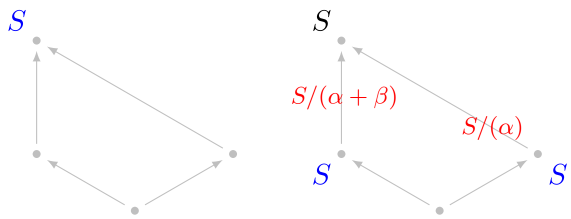
sends them to $(1,1)$, $(\alpha+\beta, 0)$ and $(0,\alpha)$. Finally, applying the map $\rho_{s,e\to s} \oplus \rho_{t,e\to t}$, we obtain that

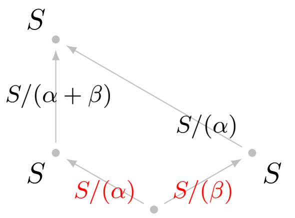
$$([1]_\alpha, [1]_\beta),\ ([\alpha+\beta]_\alpha, [0]_\beta) = ([\beta]_\alpha, [0]_\beta),\ ([0]_\alpha, [\alpha]_\beta)$$

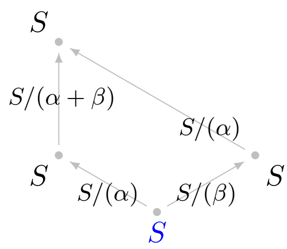
generate $\mathscr{B}(ts)_{\delta e} \subset \mathscr{B}(ts)_{e\to s} \oplus \mathscr{B}(ts)_{e\to t} = S/(\alpha) \oplus S/(\beta)$. Now we observe that $\alpha([1]_\alpha, [1]_\beta) = ([0]_\alpha, [\alpha]_\beta)$ and $\beta([1]_\alpha, [1]_\beta) = ([\beta]_\alpha, [0]_\beta)$, hence the map

$$\begin{aligned} S &\longrightarrow \mathscr{B}(ts)_{\delta e} \\ 1 &\mapsto ([1]_\alpha, [1]_\beta) \end{aligned}$$

is surjective and its kernel is $(\alpha\beta)$ which is a superfluous submodule (i.e. ideal) of S. We conclude that $\mathscr{B}(ts)_e = S$ and $\rho_{e,e\to s}$, $\rho_{e,e\to t}$ are quotient maps.







$\mathscr{B}(sts)$: we start with $\mathscr{B}(sts)_{sts} = S$ and, applying twice the same reasoning of the preceeding case we obtain

$$\mathscr{B}(sts)_{ts} = \mathscr{B}(sts)_{st} = \mathscr{B}(sts)_s = \mathscr{B}(sts)_t = S$$

and $\mathscr{B}(sts)_E = S/l(E)S$ for any edge in the full subgraph $\{> e\}$. We put now $\mathscr{B}(sts)_{e\to s}$, $\mathscr{B}(sts)_{e\to t}$ and $\mathscr{B}(sts)_{e\to sts}$ to be respetcively $S/(\alpha)$, $S/(\beta)$ and $S/(\alpha+\beta)$. With analogous notation we have that $\Gamma(\{> e\}, \mathscr{B}(sts))$ is generated by

$$\begin{pmatrix} & 1 & \\ 1 & & 1 \\ 1 & & 1 \end{pmatrix}, \quad \begin{pmatrix} & 0 & \\ \beta & & 0 \\ \beta & & \alpha+\beta \end{pmatrix}, \quad \begin{pmatrix} & 0 & \\ 0 & & \alpha \\ \alpha+\beta & & \alpha \end{pmatrix}, \quad \begin{pmatrix} & 0 & \\ 0 & & 0 \\ \beta(\alpha+\beta) & & 0 \end{pmatrix}, \quad \begin{pmatrix} & 0 & \\ 0 & & 0 \\ 0 & & \alpha(\alpha+\beta) \end{pmatrix}$$

which, after projection on $\mathscr{B}(sts)_s \oplus \mathscr{B}(sts)_t \oplus \mathscr{B}(sts)_{sts}$, give

$$\begin{pmatrix} & 1 & \\ 1 & & 1 \end{pmatrix}, \quad \begin{pmatrix} & 0 & \\ \beta & & \alpha+\beta \end{pmatrix}, \quad \begin{pmatrix} & 0 & \\ \alpha+\beta & & \alpha \end{pmatrix}, \quad \begin{pmatrix} & 0 & \\ \beta(\alpha+\beta) & & 0 \end{pmatrix}, \quad \begin{pmatrix} & 0 & \\ 0 & & \alpha(\alpha+\beta) \end{pmatrix}$$

Hence, passing to quotients, we obtain that

$$\mathscr{B}(sts)_{\delta e} \subset S/(\alpha) \oplus S/(\alpha+\beta) \oplus S/(\beta)$$

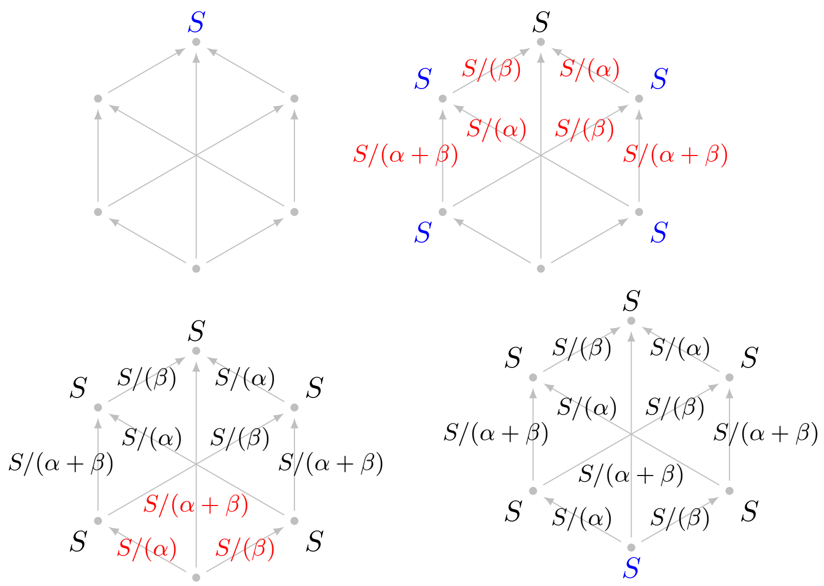
is generated by

$$\begin{aligned} ([1]_\alpha, [1]_{\alpha+\beta}, [1]_\beta) &=: \underline{\mathbf{1}}, \\ ([\beta]_\alpha, [0]_{\alpha+\beta}, [\alpha]_\beta) &= (\alpha+\beta)\underline{\mathbf{1}}, \\ ([\beta^2]_\alpha, [0]_{\alpha+\beta}, [0]_\beta) &= \beta(\alpha+\beta)\underline{\mathbf{1}}, \\ ([0]_\alpha, [0]_{\alpha+\beta}, [\alpha^2]_\beta) &= \alpha(\alpha+\beta)\underline{\mathbf{1}} \end{aligned}$$

Therefore we have that

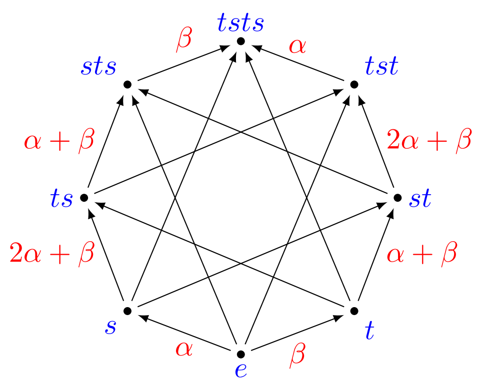
$$\begin{aligned} S &\longrightarrow \mathscr{B}(ts)_{\delta e} \\ 1 &\longmapsto \underline{\mathbf{1}} \end{aligned}$$

is a surjection, and, as its kernel is $(\alpha\beta(\alpha+\beta))$, it is a projective cover. So $\mathscr{B}(sts)_e = S$ and the $\rho_{e,E}$'s are quotient maps.



Observe that in this case we find that the $\mathscr{B}(w)$'s are all restriction of the structure sheaf $\mathscr{A}$ of the moment graph.

Example 4.3.6 (Type B_2)**.** In the same way we can compute these sheaves for a moment graph associated with root system of type B_2, which is the following:



where we only labelled external edges: parallel edges have the same labels.

We compute now $\mathscr{B}(sts)$ and $\mathscr{B}(stst)$, lower cases being obtained automatically from those of type A_2.

$\mathscr{B}(sts)$,$\mathscr{B}(tst)$: we start with $\mathscr{B}(sts)_{sts} = S$. It is easy to determine $\mathscr{B}(sts)_{ts}$, $\mathscr{B}(sts)_{st}$, $\mathscr{B}(sts)_t$ and $\mathscr{B}(sts)_s$ (as well as all the $\mathscr{B}(sts)_E$'s between them): the three first are computed using the same reasoning of type A_2, the fourth is slightly different because we do not have two parallel edges but the result is the same. Then as usual we put $\mathscr{B}(sts)_{e\to x} = S/l(e \to x)S$ for $x = s, t, sts$. Hence we are left to find $\mathscr{B}(sts)_e$ and the maps $\rho_{e,e\to x}$. The submodule $\Gamma(\{> e\}, \mathscr{B}(sts))$ is generated by:

$$\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1\ 1 & \end{pmatrix},$$
$$\begin{pmatrix} & 0 & \\ \alpha+\beta & & 0 \\ & \beta/2\ \ \alpha+\beta & \end{pmatrix}, \begin{pmatrix} & 0 & \\ 0 & & \alpha \\ & \alpha+\beta/2\ \ \alpha & \end{pmatrix},$$
$$\begin{pmatrix} & 0 & \\ 0 & & 0 \\ & \beta(2\alpha+\beta)\ \ 0 & \end{pmatrix}, \begin{pmatrix} & 0 & \\ 0 & & 0 \\ & 0\ \ \alpha(\alpha+\beta) & \end{pmatrix}$$

After projecting and applying the maps ρ's one obtains that

$$\mathscr{B}(sts)_{\delta e} \subset S/(\alpha) \oplus S/(2\alpha+\beta) \oplus S/(\beta)$$

is generated by

$$\begin{aligned} &([1]_\alpha, [1]_{2\alpha+\beta}, [1]_\beta) =: \underline{\mathbf{1}} \\ &([\beta/2]_\alpha, [0]_{2\alpha+\beta}, [\alpha]_\beta) = (\alpha+\beta/2)\underline{\mathbf{1}} \\ &([\beta^2]_\alpha, [0]_{2\alpha+\beta}, [0]_\beta) = \beta(2\alpha+\beta)\underline{\mathbf{1}} \\ &([0]_\alpha, [0]_{2\alpha+\beta}, [\alpha^2]_\beta)\alpha = (\alpha+\beta/2)\underline{\mathbf{1}} \end{aligned}$$

So

$$\begin{aligned} S &\to \mathscr{B}(sts)_{\delta e} \\ 1 &\mapsto \underline{\mathbf{1}} \end{aligned}$$

is a projective cover.

$\mathscr{B}(stst)$**:** applying twice the preceeding case we obtain that $\mathscr{B}(tsts)_x = S$ for every $x > e$ (and that on edges we simply have quotients). The submodule $\Gamma(\{> e\}, \mathscr{B}(tsts))$ is generated by

$$\begin{pmatrix} & & 1 & & \\ & 1 & & 1 & \\ 1 & & & & 1 \\ & 1 & & 1 & \end{pmatrix},$$
$$\begin{pmatrix} & & 0 & & \\ & \beta & & 0 & \\ \beta & & & & 2\alpha+\beta \\ & 2\alpha+2\beta & & 2\alpha+\beta & \end{pmatrix}, \begin{pmatrix} & & 0 & & \\ & 0 & & \alpha & \\ \alpha+\beta & & & & \alpha \\ & \alpha+\beta & & 2\alpha+\beta & \end{pmatrix},$$
$$\begin{pmatrix} & & 0 & & \\ & 0 & & 0 & \\ \beta(\alpha+\beta) & & & & 0 \\ & \beta(\alpha+\beta) & & (\alpha+\beta)(2\alpha+\beta) & \end{pmatrix}, \begin{pmatrix} & & 0 & & \\ & 0 & & 0 & \\ 0 & & & & \alpha(2\alpha+\beta) \\ & (\alpha+\beta)(2\alpha+\beta) & & \alpha(2\alpha+\beta) & \end{pmatrix},$$
$$\begin{pmatrix} & & 0 & & \\ & 0 & & 0 & \\ 0 & & & & 0 \\ & \beta(\alpha+\beta)(2\alpha+\beta) & & 0 & \end{pmatrix}, \begin{pmatrix} & & 0 & & \\ & 0 & & 0 & \\ 0 & & & & 0 \\ & 0 & & \alpha(\alpha+\beta)(2\alpha+\beta) & \end{pmatrix}$$

From which we obtain that

$$\mathscr{B}(tsts)_{\delta e} \subset S/(\alpha) \oplus S/(2\alpha+\beta) \oplus S/(\alpha+\beta) \oplus S/(\beta)$$

is generated by

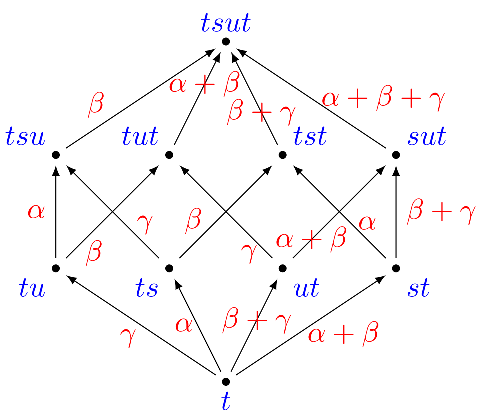
$$
\begin{aligned}
&([1]_\alpha, [1]_{2\alpha+\beta}, [1]_{\alpha+\beta}, [1]_\beta) =: \underline{\mathbf{1}}\\
&([2\beta]_\alpha, [\beta]_{2\alpha+\beta}, [0]_{\alpha+\beta}, [2\alpha]_\beta) = (2\alpha+2\beta)\underline{\mathbf{1}}\\
&([\beta]_\alpha, [0]_{2\alpha+\beta}, [\alpha]_{\alpha+\beta}, [2\alpha]_\beta) = (2\alpha+\beta)\underline{\mathbf{1}}\\
&([\beta^2]_\alpha, [0]_{2\alpha+\beta}, [0]_{\alpha+\beta}, [2\alpha^2]_\beta) = (\alpha+\beta)(2\alpha+\beta)\underline{\mathbf{1}}\\
&([\beta^3]_\alpha, [0]_{2\alpha+\beta}, [0]_{\alpha+\beta}, [0]_\beta) = \beta(\alpha+\beta)(2\alpha+\beta)\underline{\mathbf{1}}\\
&([0]_\alpha, [0]_{2\alpha+\beta}, [0]_{\alpha+\beta}, [4\alpha^3]_\beta) = \alpha(\alpha+\beta)(2\alpha+\beta)\underline{\mathbf{1}}
\end{aligned}
$$

This shows that S is a projective cover of $\mathscr{B}(tsts)_{\delta e}$.

Again we find simply the structure sheaf or its restrictions.

In the following example, instead, we will find a sheaf $\mathscr{B}(w)$ with not all the stalks of rank one.

Example 4.3.7 (Type A_3)**.** Let us consider a root system of type A_3 and let us denote s, t, u the simple reflections of its Weyl group, where $sts = tst$, $tut = utu$ and $su = us$. We consider the sheaf $\mathscr{B}(tsut)$: we want to compute in particular $\mathscr{B}(tsut)_t$. We first consider the associated moment graph: the following is the full subgraph $\{t \leq x \leq tsut\}$, which is the only part we need for our computation.



We have $\mathscr{B}(tsut)_{tsut} = S$, and repeating four times the reasoning made for the type A_2 (to compute $\mathscr{B}(ts)_e$) we obtain that every $\mathscr{B}(tsut)_x$ is S and every $\mathscr{B}(tsut)_E$ is $S/l(E)S$, for $x > t$ and edges E linking vertices bigger than t. Now we put, as usual, $\mathscr{B}(tsut)_{e\to x} = S/l(e \to x)$ for $x = tu, ts, ut, st$. We are left to find $\mathscr{B}(tsut)_t$ and its maps $\rho_{e,e\to x}$. The submodule $\Gamma(\{> t\})$

is generated by the elements (same notation as before):

$$\begin{pmatrix} & 1 & \\ \begin{smallmatrix}1&1\\1&1\end{smallmatrix} & & \begin{smallmatrix}1&1\\1&1\end{smallmatrix}\end{pmatrix},$$

$$\begin{pmatrix} & 0 & \\ \begin{smallmatrix}\beta&0\\\beta&\beta\end{smallmatrix} & & \begin{smallmatrix}0&0\\0&0\end{smallmatrix}\end{pmatrix},\quad \begin{pmatrix} & 0 & \\ \begin{smallmatrix}0&\alpha+\beta\\\alpha&0\end{smallmatrix} & & \begin{smallmatrix}0&0\\\alpha&0\end{smallmatrix}\end{pmatrix},\quad \begin{pmatrix} & 0 & \\ \begin{smallmatrix}0&0\\0&\gamma\end{smallmatrix} & & \begin{smallmatrix}\beta+\gamma&0\\0&\beta+\gamma\end{smallmatrix}\end{pmatrix},\quad \begin{pmatrix} & 0 & \\ \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & & \begin{smallmatrix}0&\alpha+\beta+\gamma\\\gamma&\alpha\end{smallmatrix}\end{pmatrix},$$

$$\begin{pmatrix} & 0 & \\ \begin{smallmatrix}0&0\\\alpha\beta&0\end{smallmatrix} & & \begin{smallmatrix}0&0\\0&0\end{smallmatrix}\end{pmatrix},\quad \begin{pmatrix} & 0 & \\ \begin{smallmatrix}0&0\\0&\beta\gamma\end{smallmatrix} & & \begin{smallmatrix}0&0\\0&0\end{smallmatrix}\end{pmatrix},\quad \begin{pmatrix} & 0 & \\ \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & & \begin{smallmatrix}0&0\\(\alpha+\beta)\gamma&0\end{smallmatrix}\end{pmatrix},\quad \begin{pmatrix} & 0 & \\ \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & & \begin{smallmatrix}0&0\\0&\alpha(\beta+\gamma)\end{smallmatrix}\end{pmatrix}$$

Hence after projecting and applying the ρ's one obtain that $\mathscr{B}(tsut)_{\delta t}$ is generated by

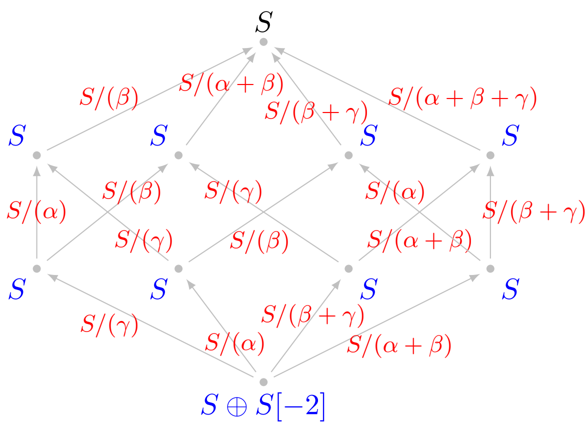
$$\begin{aligned}
&([1]_\gamma, [1]_\alpha, [1]_{\beta+\gamma}, [1]_{\alpha+\beta}) =: \underline{\mathbf{1}},\\
&([\beta]_\gamma, [\beta]_\alpha, [0]_{\beta+\gamma}, [0]_{\alpha+\beta}) =: \underline{\beta},\\
&([\alpha]_\gamma, [0]_\alpha, [\alpha+\beta]_{\beta+\gamma}, [0]_{\alpha+\beta}) = (\alpha+\beta)\underline{\mathbf{1}} - \underline{\beta}\\
&([0]_\gamma, [\gamma]_\alpha, [0]_{\beta+\gamma}, [\beta+\gamma]_{\alpha+\beta}) = (\beta+\gamma)\underline{\mathbf{1}} - \underline{\beta}\\
&([0]_\gamma, [0]_\alpha, [\gamma]_{\beta+\gamma}, [\alpha]_{\alpha+\beta}) = \underline{\beta} - \beta\underline{\mathbf{1}}\\
&([\alpha\beta]_\gamma, [0]_\alpha, [0]_{\beta+\gamma}, [0]_{\alpha+\beta}) = \alpha\underline{\beta}\\
&([0]_\gamma, [\beta\gamma]_\alpha, [0]_{\beta+\gamma}, [0]_{\alpha+\beta}) = \gamma\underline{\beta}\\
&([0]_\gamma, [0]_\alpha, [(\alpha+\beta)\gamma]_{\beta+\gamma}, [0]_{\alpha+\beta}) = (\alpha+\beta)\gamma\underline{\mathbf{1}} - \gamma\underline{\beta}\\
&([0]_\gamma, [0]_\alpha, [0]_{\beta+\gamma}, [\alpha(\beta+\gamma)]_{\alpha+\beta}) = \alpha(\beta+\gamma)\underline{\mathbf{1}} - \alpha\underline{\beta}
\end{aligned}$$

Therefore the morphism

$$\begin{aligned}
S \oplus S[-2] &\longrightarrow \mathscr{B}(tsut)^{\delta t}\\
(1,0) &\longmapsto \underline{\mathbf{1}}\\
(0,1) &\longmapsto \underline{\beta}
\end{aligned}$$

is surjective and its kernel is the superfluous submodule $(\alpha\gamma(\alpha+\beta)(\beta+\gamma)) \oplus (\alpha\gamma)$. Hence we conclude that $\mathscr{B}(tsut)_t = S \oplus S[-2]$ and that the maps ρ are given by projections on the first component, and by the following morphism on the second one

$$\begin{aligned}
S[-2] &\to S/(\gamma) & \qquad S[-2] &\to S/(\alpha)\\
1 &\mapsto [\beta]_\gamma & 1 &\mapsto [\beta]_\alpha\\
S[-2] &\to S/(\beta+\gamma) & S[-2] &\to S/(\alpha+\beta)\\
1 &\mapsto [0]_{\beta+\gamma} & 1 &\mapsto [0]_{\alpha+\beta}
\end{aligned}$$



This last example suggests us the link between the ranks of the stalks of the Braden-MacPherson sheaves and Kazhdan-Lusztig polynomials: in fact we found the first non trivial $\mathscr{B}(w)_y$ for the same pair for which we found (see example 1.3.2) the first non trivial Kazdhan-Lusztig polynomial. We can state the following conjecture (which is a theorem in characteristic 0 case):

Conjecture 4.3.8 (Multiplicity conjecture)**.** *Given a moment graph associated to a root system, satisfying the GKM-assumption, for any pair of elements w, y in the Weyl group W, we have*

$$\mathscr{B}(w)^y = P_{y,w}(1)$$

4.4 Localization of $\mathcal{Z}$-modules

We can generalize our notion of sheaves on a given moment graph $\mathcal{G}$ at the cost of losing grading.

Definition 4.4.1. Given a moment graph $\mathcal{G}$ on V and a unital S-algebra A, we call an *A-sheaf on $\mathcal{G}$* the datum $(\{\mathscr{F}_x\}_{x\in\mathcal{V}}, \{\mathscr{F}_E\}_{E\in\mathcal{E}}, \{\rho_{x,E}\}_{E\in\mathcal{E}, x\in E})$, where:

i) for any $x \in \mathcal{V}$, $\mathscr{F}_x$ is an A-module;

ii) for any $E \in \mathcal{E}$, $\mathscr{F}_E$ is an A-module such that $l(E)\mathscr{F}_E = 0$, where we consider the action of S on $\mathscr{F}_E$.

iii) for any $E \in \mathcal{E}$ with $x \in E$, $\rho_{x,E} \colon \mathscr{F}_x \to \mathscr{F}_E$ is a morphism of A-modules.

Analogously one defines morphisms of A-sheaves, as well as sections $\Gamma(\mathcal{I}, -)$ and the notion of flabby A-sheaves.

In this way we get categories $\mathrm{Sh}_A(\mathcal{G})$ and $\mathcal{Z}_A$-mod where $\mathcal{Z}_A$ is obtained as the global section module of the structure sheaf $\mathscr{A}_A$ defined by:

1. $(\mathscr{A}_A)_x := A$;
2. $(\mathscr{A}_A)_E := A/l(E)A$;
3. $\rho_{x,A}$ is the quotient morphism.

One has the global section functor:

$$\Gamma\colon \mathrm{Sh}_A(\mathcal{G}) \to \mathcal{Z}_A\text{-mod}$$

and in the same way one has the subcategories $\mathrm{Sh}^{\mathrm{f}}_A(\mathcal{G})$ and $\mathcal{Z}_A\text{-mod}^{\mathrm{f}}$ to which the functor Γ restricts properly.

We can define a *base change*: given a ring morphism $A \to A'$ (or, equivalently, given an A algebra A'), and am A-sheaf $\mathscr{F}$ one can define $\mathscr{F} \otimes_A A'$ by

1. $(\mathscr{F} \otimes_A A')_x := \mathscr{F}_x \otimes_A A'$;
2. $(\mathscr{F} \otimes_A A')_E := \mathscr{F}_E \otimes_A A'$;
3. $\rho^{\mathscr{F} \otimes_A A'}_{x,E} := \rho^{\mathscr{F}}_{x,E} \otimes \mathrm{id}_{A'}$.

Finally we have a non-graded general version of the notion of F-projectivity: in definition 4.3.3 it is sufficient to require the stalks to be *finitely generated projective A-modules* instead of *graded free S-modules of finite rank*.

We shall in particular be interested in the case where $A = S_{\mathfrak{p}}$ with $\mathfrak{p}$ is a prime ideal of S, and specifically in the case $\mathfrak{p} = \mathfrak{h}S$: we denote R the localization in this case.

Remark 4.4.2. Notice that in the case $A = S_{\mathfrak{p}}$ we have a natural equivalence $\mathrm{Sh}_A(\mathcal{G}) \cong \mathrm{Sh}_A(\mathcal{G}_{\mathfrak{p}})$ (recall that $\mathcal{G}_{\mathfrak{p}}$ is the $\mathfrak{p}$-reduction of $\mathcal{G}$). Observe also that this is trivial in the case $A = R$ because $\mathcal{G}_{\mathfrak{h}S} = \mathcal{G}$.

We already have the functor Γ of global section which goes from sheaves to modules, now we want to describe a *localization functor*[3] $\mathscr{L}_R\colon \mathrm{Sh}^{\mathrm{f}}_R(\mathcal{G}) \to \mathcal{Z}_R\text{-mod}^{\mathrm{f}}$

[3] We will describe this construction for $A = R$, but one could replace R with $S_{\mathfrak{p}}$ in what follows: actually one could work with a general domain A (see [27])

Take Q to be the quotient field of S and notice that we have a natural morphism

$$\mathcal{Z}_R \otimes_R Q \to \Big(\bigoplus_{x\in\mathcal{V}} R\Big) \otimes_R Q = \bigoplus_{x\in\mathcal{V}} Q \tag{4.2}$$

This morphism is an isomorphism: in fact it is injective because Q is a flat R-module and it is surjective because, denoting r_y the element $(\delta_{y,x})_x$ of $\bigoplus_{x\in\mathcal{V}} R$, we obtain $r_y \otimes 1$ as image of

$$\Big(\prod_{E\ni y} \alpha(E)\Big) r_y \otimes \frac{1}{\prod_{E\ni y}\alpha(E)}$$

Notice that the elements $e_y = r_y \otimes 1$ are orthogonal idempotents and $1 = 1\otimes 1 = \sum_y e_y$. So for each $\mathcal{Z} \otimes_R Q$-module N we have the decomposition $N = \bigoplus_{x\in\mathcal{V}} e_x \cdot N$ where projections are given by $\pi_x(v) = e_x \cdot v$. In particular if M is a $\mathcal{Z}$-module in $\mathcal{Z}_R$-mod$^{\mathrm{f}}$, the module $M \otimes_R Q$ decomposes as

$$M \otimes_R Q = \bigoplus_{x\in\mathcal{V}} e_x \cdot (M \otimes_R Q) \tag{4.3}$$

As M is supposed torsion-free over R we have a natural inclusion $M \hookrightarrow M \otimes_R Q$ (given by $m \mapsto m \otimes 1$), hence we can consider it as a sub-R-module of $M \otimes_R Q$.

We want to define a sheaf $\mathscr{L}_R(M)$ on the moment graph.

1. Let $\mathscr{L}_R(M)_x := e_x \cdot M$ for every $x \in \mathcal{V}$

2. Given an edge $E : x \to y$ in $\mathcal{E}$, consider $M(E) := (e_x + e_y)\cdot M + \alpha_E e_x \cdot M = (e_x + e_y)\cdot M + \alpha_E e_y \cdot M$. We define $\mathscr{L}_R(M)_E$ and the maps $\rho_{x,E}$, $\rho_{y,E}$ by the pushout:

$$\begin{array}{ccc} M(E) & \xrightarrow{\pi_x} & \mathscr{L}_R(M)_x \\ \big\downarrow{\scriptstyle \pi_y} & & \big\downarrow{\scriptstyle \rho_{x,E}} \\ \mathscr{L}_R(M)_y & \xrightarrow{\rho_{y,E}} & \mathscr{L}_R(M)_E \end{array} \tag{4.4}$$

Notice that π_x and π_y are surjective, hence so are $\rho_{x,E}$ and $\rho_{y,E}$. We have, by general properties of pushouts that $\ker \rho_{x,E} = \pi_x(\ker \pi_y)$, which implies that $\alpha_E \mathscr{L}_R(M)_E = 0$, because $\alpha_E \mathscr{L}_R(M)_x = \alpha_E e_x \cdot M$ over which π_x is the identity and π_y is zero. This shows that $\mathscr{L}_R(M)$ is a sheaf on $\mathcal{G}$ which, because $\mathscr{L}_R(M)_x$ is finitely generated and torsion-free over R, is an object of $\mathrm{Sh}^{\mathrm{f}}_R(\mathcal{G})$. Each step is functorial, hence one gets the definition of $\mathscr{L}_R$ on morphisms: the following diagram is used for edges.

$$\begin{array}{ccccc} M(E) & \longrightarrow & \mathscr{L}_R(M)_x & \longrightarrow & \mathscr{L}_R(N)_x \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{L}_R(M)_y & \longrightarrow & \mathscr{L}_R(M)_E & \searrow & \\ \downarrow & & & & \downarrow \\ \mathscr{L}_R(N)_y & & \longrightarrow & & \mathscr{L}_R(N)_E \end{array}$$

The maps $g_M \colon M \to \Gamma\big(\mathscr{L}_R(M)\big)$ defined by $v \mapsto (e_x v)_{x \in \mathcal{V}}$ (this is in fact a global section) define a natural transformation g between the functors $\mathrm{id}_{\mathcal{Z}_R\text{-mod}^{\mathrm{f}}}$ and $\Gamma \circ \mathscr{L}_R$.

We can define a functor $f \colon \mathscr{L}_R \circ \Gamma \to \mathrm{id}_{\mathrm{Sh}^{\mathrm{f}}_R(\mathcal{G})}$ in the following way: for $x \in \mathcal{V}$ we have the inlcusion $\mathscr{L}_R(\Gamma(\mathscr{F}))_x = e_x\Gamma(\mathscr{F}) \subset \mathscr{F}_x$, because multiplication by e_x gives in this case the projection on the factor $\mathscr{F}_x \otimes_R Q$ and $\Gamma(\mathscr{F}) \subset \bigoplus_{y \in \mathcal{V}} \mathscr{F}_y$. This defines $f_{\mathscr{F}}$ on vertices. Using pushout universal property one defines also $(f_{\mathscr{F}})_E$ on edges (see [27] §2.13). One can then show

Proposition 4.4.3 (see [27] §2.13)**.** *For $\mathscr{F} \in \mathrm{Sh}^{\mathrm{f}}_R(\mathcal{G})$, the morphism $g_{\Gamma(\mathscr{F})}$ is an isomorphism with inverse $\Gamma(f_{\mathscr{F}})$.*

and

Theorem 4.4.4 (see [27] §2.13)**.** *The functor $\mathscr{L}_R \colon \mathcal{Z}_R\text{-mod}^f \to \mathrm{Sh}^f_R(\mathcal{G})$ is left adjoint to the functor $\Gamma \colon \mathrm{Sh}^f_R(\mathcal{G}) \to \mathcal{Z}_R\text{-mod}^f$.*

Let us then define $\underline{\mathrm{Sh}}_R(\mathcal{G})$ the full subcategory of $\mathrm{Sh}^{\mathrm{f}}_R(\mathcal{G})$ whose objects are those sheaves $\mathscr{F}$ which are *generated by global sections*, i.e. such that $f_{\mathscr{F}}$ is an isomorphism. Similarly define $\mathcal{Z}_R\text{-}\underline{\mathrm{mod}}$ the full subcategory of $\mathcal{Z}_R$-mod whose objects are those modules M that are *determined by local relations*, i.e. for which g_M is an isomorphism. Proposition 4.4.3 says that Γ and $\mathscr{L}_R$ induce mutually inverse equivalence of categories between $\underline{\mathrm{Sh}}(\mathcal{G})$ and $\mathcal{Z}_R\text{-}\underline{\mathrm{mod}}$.

Recall that an R-module M is said to be reflexive if the natural morphism $M \to M^{**}$ (where * denotes the dual $\mathrm{Hom}(-, R)$) is an isomorphism. This is equivalent to say that it is the intersection of its localisations at prime ideals of height one. One can show the following property:

Proposition 4.4.5 (see [27] §2.16)**.** *Suppose that $\mathcal{G}$ satisfies the GKM-assumption, then a module in $\mathcal{Z}_R\text{-mod}^{\mathrm{f}}$ which is reflexive as an R-module lies in $\mathcal{Z}_R\text{-}\underline{\mathrm{mod}}$.*

We conclude this chapter by stating the following generalization of theorem 4.3.4 in the case of A being a localisation of S.

Theorem 4.4.6 (see [27], prop. 3.17)**.** *Let $\mathscr{P}$ be a F-projective R-sheaf, then there exist $v_1, \dots, v_n$ such that*

$$\mathscr{P} \cong \mathscr{B}(v_1) \otimes_S R \oplus \cdots \oplus \mathscr{B}(v_n) \otimes_S R$$

Chapter 5

Deformation of Category $\mathcal{O}$

In this chapter we want to describe a deformed version of category $\mathcal{O}$ which will allow us to make a link with moment graph introduced in last chapter. First of all we have to see some other results concerning non-deformed category $\mathcal{O}$, in particular we have to talk about projective objects, Verma flags and the BGG-reciprocity. Then we shall see how to extend this to the deformed case. We will skip many proofs and refer to the literature.

We will use the same notation introduced in chapter 3.

5.1 The BGG-reciprocity

In this section we refer to [23] (we will give detailed reference every time). We want to prove the following theorem

Theorem 5.1.1. *Category $\mathcal{O}$ has enough projective objects.*

As a first step we shall find *some* projective objects, for instance:

Proposition 5.1.2 ([23], §3.8)**.** *Let $\lambda \in \mathfrak{h}^*$ be dominant then $M(\lambda)$ is projective.*

Proof. Let $M \xrightarrow{\pi} N \to 0$ be exact and let $M(\lambda) \xrightarrow{\phi} N$ be a morphism, that we can suppose non-zero (otherwise we are done): we want to find a lift $M(\lambda) \to M$. By surjectivity of π, if we call v^+ the maximal vector of $M(\lambda)$, there exists a non-zero vector $u \in M$ such that $\pi(u) = \phi(v^+)$, hence u is also of weight λ. Let us consider the $\mathfrak{n}_+$-submodule generated by u in M: it must contain at least one maximal vector u^+, then u belongs to the highest weight submodule of M generated by u^+. If $\mu \geq \lambda$ is the weight of u^+, for any $z \in Z(\mathfrak{g})$, we have $z \cdot u = \chi_\mu(z)u$, hence $z \cdot \pi(u) = \chi_\mu(z)\pi(u)$. But we also have $z \cdot \pi(u) = z \cdot \phi(v^+) = \chi_\lambda(z)\phi(v^+) = \chi_\lambda(z)\pi(u)$. We deduce

$\chi_\lambda = \chi_\mu$, so by theorem 3.2.4, μ is linked to λ. But λ is dominant, hence maximal in its linkage class, so we must have $\lambda = \mu$, which means that u is already a maximal vector ($\mathfrak{n}_+ \cdot u = 0$, hence $u^+ \in ku$). Then $v^+ \mapsto u$ defines a morphism $M(\lambda) \to M$ which lifts π. □

Recall now that if M and N are $\mathfrak{g}$-modules we have a canonical structure of $\mathfrak{g}$-module on $M \otimes N$, given by $x \cdot (m \otimes n) := (x \cdot m) \otimes n + m \otimes (x \cdot n)$. But even if M and N are objects of $\mathcal{O}$ it is not necessarily true that also $M \otimes N$ is in $\mathcal{O}$. Nevertheless one can show the following

Proposition 5.1.3 (see [23], §1.1). *If N is a finite dimensional module of $\mathcal{O}$ then:*

i) for any M in $\mathcal{O}$, $M \otimes N$ is an object of $\mathcal{O}$;

ii) $- \otimes N$ is an exact functor $\mathcal{O} \to \mathcal{O}$.

Recall also that the dual M^* of a $\mathfrak{g}$-module M has a natural structure of $\mathfrak{g}$-module, given by $(x \cdot f)(v) := -f(x \cdot v)$ and that for any other $\mathfrak{g}$-module N, the space $\mathrm{Hom}(M, N)$ has a natural structure of $\mathfrak{g}$-module, isomorphic to $M^* \otimes N$. Again, even if M and N are in $\mathcal{O}$ the same need not hold for these modules, but if M is finite dimensional then so is M^* hence it belongs to $\mathcal{O}$ and by 5.1.3 so does $\mathrm{Hom}(M, N)$. This gives us the adjunction:

$$M \otimes - \quad \dashv \quad \mathrm{Hom}(M, -)$$

which gives us a tool to build other projective objects, as the following proposition illustrates.

Proposition 5.1.4. *Let P be a projective object in $\mathcal{O}$ and L a finite dimensional $\mathfrak{g}$-module. Then $P \otimes L$ is projective.*

Proof. We have the following

$$\mathrm{Hom}_{\mathcal{O}}(P \otimes L, -) \cong \mathrm{Hom}_{\mathcal{O}}(P, \mathrm{Hom}(L, -)) \cong \mathrm{Hom}_{\mathcal{O}}(P, L^* \otimes -)$$

Hence $\mathrm{Hom}_{\mathcal{O}}(P \otimes L, -)$ is the composition of a right exact and an exact functor, hence it is right exact. □

We saw in chapter 3 that any module admits a composition series: there is another kind of filtration that not all modules admit but which is particularly interesting when dealing with projective ones.

Definition 5.1.5. We say that a module M in $\mathcal{O}$ *admits a Verma flag* if there exists a filtration of M with subquotients isomorphic to Verma modules.

By computing formal characters one can find that all Verma flags of a module M (which admits some) have the same length and the same multiplicities which we denote by $(M : M(\lambda))$ We obviously have that any Verma module admits a Verma flag. But we also have the following

Proposition 5.1.6 (see [23] §3.6)**.** *For any $\lambda \in \mathfrak{h}^*$ and M a finite dimensional module, $M(\lambda) \otimes M$ admits a Verma flag with $M(\lambda + \mu)$ occurring as a subquotient exactly* $\dim M_\mu$ *times. Furthermore we can say that it has a submodule isomorphic to $M(\lambda+\nu)$ and a quotient isomorphic to $M(\lambda+w_0\nu)$ where ν is the maximal weight of M.*

We can now prove that $\mathcal{O}$ has enough projectives.

Proof of theorem 5.1.1. We first prove that for any $\lambda \in \mathfrak{h}^*$ the simple module $L(\lambda)$ has a projective cover. For sufficiently large n the weight $\lambda + n\rho$ is dominant hence $M(\lambda + n\rho) \otimes L(n\rho)$ is projective by propositions 5.1.2 and 5.1.4. But by proposition 5.1.6, it admits $M(\lambda = M(\lambda + n\rho - n\rho)) = M(\lambda + n\rho + nw_0\rho)$ (because ρ is stable by $-w_0$) as a quotient, hence also $L(\lambda)$.

Now take a general M in $\mathcal{O}$. We can proceed by induction on Jordan-Hölder length of M, being the first step of the proof the base step. Let

$$0 \to L(\lambda) \to M \to N \to 0$$

be an exact sequence: N has a length lower than M hence it admits a projective cover $P \twoheadrightarrow N$. By projectivity it induces $P \to M$ such that the following diagram commutes

$$\begin{array}{ccccccccc} & & & & & \swarrow & P & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & L(\lambda) & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Now either $P \to M$ is surjective, or it induces a splitting $M \cong L(\lambda) \oplus N$. In the second case a pair of projective objects having surjections towards $L(\lambda)$ and N provides a surjection of their direct sum towards M. $\square$

We saw that $\mathcal{O}$ is an artinian category, hence having enough projectives implies that any object has a projective cover. In partcular, fix a projective cover $P(\lambda) \xrightarrow{\pi_\lambda} L(\lambda)$, then $\ker \pi_\lambda$ is the only maximal submodule of $P(\lambda)$ because the restriction to any submodule $N \subset P(\lambda)$ not contained in $\ker \pi_\lambda$ would be surjective by simplicity of $L(\lambda)$ contradicting essentiality of $P(\lambda)$. In particular we found that $P(\lambda)$ is indecomposable. One can show that the converse is also true

Proposition 5.1.7 (see [23], §3.9). *i) Any indecomposable projective object in $\mathcal{O}$ is isomorphic to some $P(\lambda)$;*

ii) For any projective object P which can be written in the form

$$P = \bigoplus_{\lambda \in \mathfrak{h}^*} P(\lambda)^{\oplus a_\lambda}$$

we have $a_\lambda = \dim \mathrm{Hom}_{\mathcal{O}}(P, L(\lambda))$;

iii) For any M in $\mathcal{O}$, we have $\dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)]$

From this proposition one can deduce the following

Proposition 5.1.8 (see [23] §3.10). *Any projective object in $\mathcal{O}$ admits a Verma flag. The Verma flag of $P(\lambda)$ satisfies: $(P(\lambda) : M(\mu)) \neq 0$ only if $\mu \in \lambda + \Gamma$ and $(P(\lambda) : M(\lambda)) = 1$.*

We can now state the following theorem, known as *BGG-reciprocity*:

Theorem 5.1.9 (see [23], §3.11). *For any $\lambda, \mu \in \mathfrak{h}^*$ we have*

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$$

5.2 Definition of deformed category $\mathcal{O}$

Let us consider the symmetric algebra $S = S(\mathfrak{h}) = U(\mathfrak{h})$ associated with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let A be a commutative, unital, noetherian S-algebra and let $\tau \colon S \to A$ be its structure morphism. The algebra A is called *deformation algebra* and will be the base for our deformed category $\mathcal{O}$.

Instead of $U(\mathfrak{g})$-modules, we now consider $(U(\mathfrak{g}) \otimes_k A)$-modules and as in the non-deformed case we shall restrict to a subcategory. First, for any $(U(\mathfrak{g}) \otimes_k A)$-module M and $\lambda \in \mathfrak{h}^*$ we define

$$M_\lambda := \{m \in M \mid h \cdot m = (h \otimes 1) \cdot m = 1 \otimes \big(\lambda(h) 1_A + \tau(h)\big) m\}$$

where, computing $\tau(h)$ we consider the restriction of τ to $\mathfrak{h} \subset S$.

Definition 5.2.1. The category $\mathcal{O}_A$ is the full subcategory of $(U(\mathfrak{g}) \otimes_k A)$-mod whose objects M satisfy the following properties:

i) M is finitely generated as $(U(\mathfrak{g}) \otimes_k A)$-module.

ii) $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$;

iii) for any $m \in M$ the sub-A-module $U(\mathfrak{n}_+) \cdot m$ is finitely generated (as a A-module).

Again we call *weights* of M the λ's in $\mathfrak{h}$ such that $M_\lambda \neq 0$.

We still have Verma modules, in fact let us denote A_λ, for any $\lambda \in \mathfrak{h}^*$, the $(U(\mathfrak{b}) \otimes_k A)$-module A where A acts by multiplication, $U(\mathfrak{h}) = S(\mathfrak{h})$ via the morphism τ and $U(\mathfrak{n}_+)$ trivially. Now consider

$$M_A(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} A_\lambda$$

We call it the *deformed Verma module* assocuated to weight λ. As an $(U(\mathfrak{n}_-) \otimes_k A)$-module it is isomorphic to $U(\mathfrak{n}_-)$ itself and one can check that it actually belongs to $\mathcal{O}_A$.

We can still define also *highest weight* vectors and modules analogously to the non-deformed case and we can prove, in the same way as its homologues, the following results

Proposition 5.2.2. *i) Objects in $\mathcal{O}_A$ have weights in a finite union of sets of the form $\lambda - \Gamma$;*

ii) $\mathcal{O}_A$ is closed under direct sums, submodules and quotients, hence it is an abelian category;

iii) Any M in $\mathcal{O}_A$ admits a sequence of submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

such that M_{i+1}/M_i is a highest weight (deformed) module. □

We define similarly also the notion of *deformed Verma flag*, as a filtration whose subquotients are deformed Verma modules. If a module admits a deformed Verma flag then multiplicities are well-defined.

Example 5.2.3. We consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and we take $A = S$ (hence $\tau = \mathrm{id}$). We describe $M_S(\lambda)$ for $\lambda \in \mathfrak{h}^*$.

Let $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$, with $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. We have $S = \mathbb{C}[h]$ and also $U(\mathfrak{n}_-) = \mathbb{C}[f]$ and $U(\mathfrak{n}_+) = \mathbb{C}[e]$. The $U(\mathfrak{b})$-module $S_\lambda = \mathbb{C}[h]$ have the action

$$\begin{aligned} e \cdot p(h) &= 0 \\ h \cdot p(h) &= (\lambda(h) + h)p(h) \end{aligned}$$

The module $M_S(\lambda)$ is generated, as a vector space by elements $f^n \otimes h^m$ with $n, m \in \mathbb{N}$ and where $\otimes$ denotes tensor product in $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} S_\lambda$. In order to describe the action of $U(\mathfrak{g}) \otimes_{\mathbb{C}} S$ it is sufficient to describe that of the elements $e \otimes_{\mathbb{C}} 1$, $h \otimes_{\mathbb{C}} 1$ and $f \otimes_{\mathbb{C}} 1$ (here instead we always subscribe $\mathbb{C}$ to avoid confusion). When we write $p(h) \cdot x$ for some $p(h) \in S(\mathfrak{h})$ and $x \in M_S(\lambda)$ we always mean $1 \otimes_{\mathbb{C}} p(h) \cdot x$. We will use the following formulas of the universal envelopping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$:

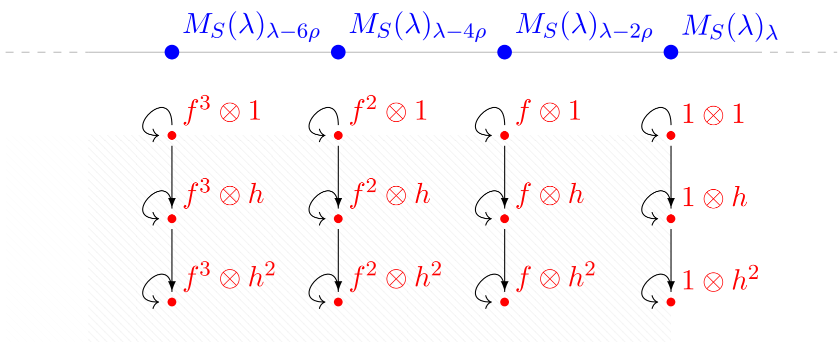
$$\begin{aligned} hf &= -2f + fh \\ hf^2 &= -4f^2 + f^2 h \\ &\dots \\ hf^n &= -2nf^n + f^n h \end{aligned}$$

$$\begin{aligned} ef &= h + fe \\ ef^2 &= -2f + 2fh + f^2 e \\ &\dots \\ ef^n &= -n(n-1)f^{n-1} + nf^{n-1}h + f^n e \end{aligned}$$

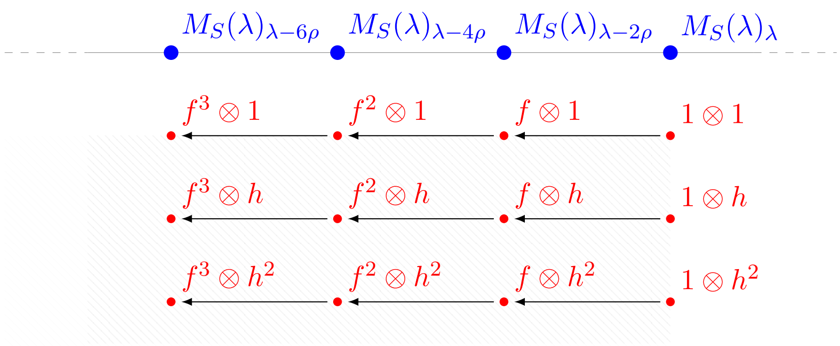
We start by the action of $h \otimes_{\mathbb{C}} 1$ and we notice that fixing $n \in \mathbb{N}$, the vectors $f^n \otimes h^m$ form a basis for $M_S(\lambda)_{\lambda - 2n\rho}$ in fact

$$\begin{aligned} (h \otimes_{\mathbb{C}} 1) \cdot (f^n \otimes h^m) &= hf^n \otimes h^m = -2nf^n \otimes h^m + f^n h \otimes h^m = \\ &= -2nf^n \otimes h^m + f^n \otimes (\lambda(h) + h)h^m = \\ &= (\lambda(h) - 2n + h) \cdot f^n \otimes h^m = \\ &= (\lambda(h) - 2n) \cdot f^n \otimes h^m + f^n \otimes h^{m+1} \end{aligned}$$

We can represent it as follows:



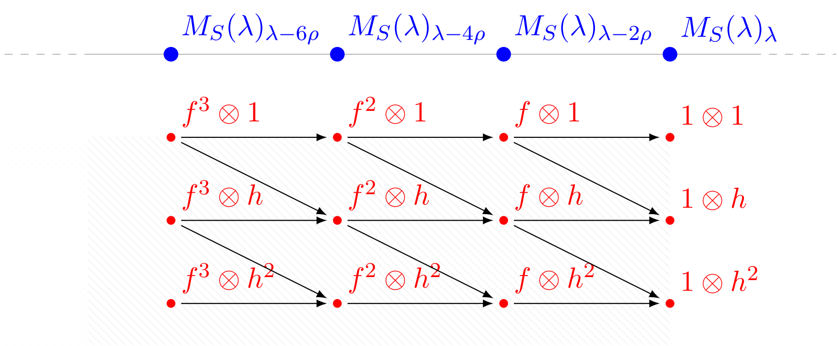
The action of $f \otimes_{\mathbb{C}} 1$ is the simplest: it sends $f^n \otimes h^m$ to $f^{n+1} \otimes h^m$. Hence we can see it in this way:



Finally, the action of $e \otimes_{\mathbb{C}} 1$ is given by

$$\begin{aligned}(e \otimes_{\mathbb{C}} 1) \cdot (f^n \otimes h^m) &= ef^n \otimes h^m = \\ &= -n(n-1)f^n \otimes h^m + nf^n h \otimes h^m + f^n e \otimes h^m = \\ &= (-n(n-1) + n(\lambda(h) + h)) \cdot f^n \otimes h^m = \\ &= n((1-n) + \lambda(h) + h) \cdot f^n \otimes h^m = \\ &= n((1-n) + \lambda(h)) \cdot f^n \otimes h^m + nf^n \otimes h^{m+1}\end{aligned}$$

Hence the picture is the following:



We notice that, in particular, there are *no* highest weight vectors exept those of $M_S(\lambda)_\lambda$, even if $\lambda = 2\rho$, differently from the non-deformed case.

Given a morpshism $A \to A'$ of deformation algebras, by which we mean a morphism of S-algebras (hence commuting with structure morhisms τ and τ'), we have a *base change* functor given by $- \otimes_A A'$. To an object M of $\mathcal{O}_A$ we associate $M \otimes_A A'$ which has a natural structure of module over $(U(\mathfrak{g}) \otimes_k A) \otimes_A A' = U(\mathfrak{g}) \otimes_k A'$, and to a morphism $f \colon M \to N$ we associate $f \otimes \mathrm{id}_{A'}$. One can show that $M \otimes_A A'$ actually belongs to $\mathrm{Ob}\, \mathcal{O}_{A'}$ defining the desired functor.

We now consider the case where A is a local algebra, i.e. it has an only maximal ideal $\mathfrak{m}$: in this case we say that A is a *local deformation algebra*. Let us call K the residue field associated with A, that is, $K = A/\mathfrak{m}$: the quotient map is a morphism of deformation algebras, and we call τ' the structure morphism of K (which is the composition of τ with the quotient map). Hence we consider the base change functor $- \otimes_A K \colon \mathcal{O}_A \to \mathcal{O}_K$.

Notice that $U(\mathfrak{g}) \otimes_k K = U(\mathfrak{g} \otimes_k K)$, and looking at the definition one can see that $\mathcal{O}_K$ is a subcategory of the category $\mathcal{O}$ associated with $\mathfrak{g} \otimes_k K$: in particular its objects are those modules whose weights belong to $\tau' + \mathfrak{h}^* \hookrightarrow \mathrm{Hom}(\mathfrak{h} \otimes_k K, K)$. Hence simple objects of $\mathcal{O}_K$ are parametrized by weights in $\tau' + \mathfrak{h}^*$.

In this case where A is a local deformation algebra we obtain a description of simple objects in $\mathcal{O}_A$, namely

Theorem 5.2.4. *Base change functor $- \otimes_A K$ induces a bijection between isomorphism classes of simple objects in $\mathcal{O}_A$ and isomorphism classes of simple objects in $\mathcal{O}_K$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A and let L be a simple object in $\mathcal{O}_A$. By Nakayama's Lemma we have $\mathfrak{m}L = 0$, then $L \otimes_A K$ is isomorphic, as $(U(\mathfrak{g}) \otimes_k A)$-module to L itself, via

$$\begin{aligned} L &\longrightarrow L \otimes_A K \\ l &\longmapsto l \otimes 1 \end{aligned}$$

Hence any $(U(\mathfrak{g}) \otimes_k K)$-submodule, which is also an $(U(\mathfrak{g}) \otimes_k A)$-submodule must be either 0 or $L \otimes_A K$ itself.

Conversely, let L be a simple object in $\mathcal{O}_K$, then it is simple also as object of $\mathcal{O}_A$.

By the above and by the fact that $L \otimes_A K \cong L$ for any object in $\mathcal{O}_K$, we established the desired bijection. □

5.3 Projective objects

We shall now concentrate on projective objects in $\mathcal{O}_A$, our aim being to generalize BGG-reciprocity. We introduce the partial order relation $\leq$ on $\mathfrak{h}^*$ defined by:

$$\lambda \leq \mu \Leftrightarrow \mu - \lambda \in \Gamma$$

Let us consider the full subcategory $\mathcal{O}_A^{\leq\lambda}$ whose objects are modules with weights in $\lambda - \Gamma$, i.e. weights μ such that $\mu \leq \lambda$. We have the following

Proposition 5.3.1. *For any $\lambda, \mu \in \mathfrak{h}^*$ there exists an object $Q_A^\lambda(\mu)$ in $\mathcal{O}_A^{\leq\lambda}$ which represents the functor*

$$\begin{aligned} \mathcal{O}_A^{\leq\lambda} &\to A\text{-mod} \\ M &\mapsto M_\mu \end{aligned}$$

and which admits a deformed Verma flag.

Proof. Let us consider the ideal $B = \bigoplus_{\gamma \not\leq \lambda-\mu} U(\mathfrak{b})_\gamma$ of $U(\mathfrak{b})$ and the quotient $U(\mathfrak{b})^{\lambda-\mu} := U(\mathfrak{b})/B$. We define

$$Q_A^\lambda(\mu) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U(\mathfrak{b})^{\lambda-\mu} \otimes_S A_\mu)$$

This is a module in $\mathcal{O}_A^{\leq\lambda}$, in fact consider the monomials $x_1^{r_1} \cdots x_n^{r_n}$ with weight in $\lambda - \mu - \Gamma$, with respect to the adjoint action of $\mathfrak{h}$: there is a finite number of them (which is 0 if $\lambda \not\geq \mu$). Then the module $Q_A^\lambda(\mu)$ is generated, as a $(U(\mathfrak{g}) \otimes_k A)$-module, by these monomials (i.e. by the elements $1 \otimes [x_1^{r_1} \cdots x_n^{r_n}] \otimes 1$), and as a vector space by the elements

$$y_1^{s_1} \cdots y_n^{s_n} \otimes [x_1^{r_1} \cdots x_n^{r_n}] \otimes t$$

that have weights in $\lambda - \Gamma$ and give the decomposition of $Q_A^\lambda(\mu)$ in weight spaces.

Suppose $\lambda - \mu \in \Gamma$. Consider a monomial $x_1^{r_1} \cdots x_n^{r_n}$ of maximal weight among those which generates $Q_A^\lambda(\mu)$: the submodule generated by $1 \otimes [x_1^{r_1} \cdots x_n^{r_n}] \otimes 1$ is isomorphic to the Verma module $M_A(\lambda)$. The quotient will be generated by (representative of) all other monomials: so repeating the same procedure we obtain a submodule which is isomorphic to a Verma module. In this way we obtain a Verma flag for $Q_A^\lambda(\mu)$.

Now let M be in $\mathcal{O}_A^{\leq\lambda}$. We have that $\mathrm{Hom}(Q_A^\lambda(\mu), M)$ is isomorphic to M_μ, as A-modules, via the evaluation on $1 \otimes 1 \otimes 1$. □

Remark 5.3.2. *i)* The objects $Q_A^\lambda(\mu)$'s are projective in $\mathcal{O}_A^\lambda$, because the functor $M \mapsto M_\mu$ is exact;

ii) If a module M is generated by a weight vector $m \in M_\mu$ then we have a surjection $Q_A^\lambda(\mu) \twoheadrightarrow M$. Hence the category $\mathcal{O}_A^\lambda$ has enough projectives;

iii) Every module in $\mathcal{O}_A^\lambda$ is generated by a finite number of weight vectors, then, by the preceeding remark, it is a quotient of a direct sum $\bigoplus Q_A^\lambda(\mu)$. This implies that any projective module is a direct summand of such a sum.

Proposition 5.3.3. *Let $A \to A'$ be a morphism of deformation rings and let P be projective in $\mathcal{O}_A^\lambda$, then $P \otimes_A A'$ is projective in $\mathcal{O}_{A'}^\lambda$*

Proof. By remark 5.3.2 (*iii*), P is a direct summand of a module of the form $\bigoplus Q_A^\lambda(\mu)$, then $P \otimes_A A'$ is a direct summand of $\left(\bigoplus Q_A^\lambda(\mu)\right) \otimes_A A' = \bigoplus Q_{A'}^\lambda(\mu)$, then it is projective. □

One has the following result

Proposition 5.3.4 (see [17], lemma 2.5)**.** *Direct summands of modules with a Verma flag in $\mathcal{O}_A$ admit a Verma flag*

Hence by the preceeding remark one obtains

Corollary 5.3.5. *Projective objects in $\mathcal{O}_A^{\leq\lambda}$ admit a Verma flag*

As well as with simple objects one also obtains the following result for projective objects:

Proposition 5.3.6 (see [17], prop. 2.6)**.** *Let A be a local deformation domain, then base change functor $-\otimes_A K$ induces a bijection between isomorphism classes of projective objects in $\mathcal{O}_A^{\leq\lambda}$ and isomorphism classes of projective objects in $\mathcal{O}_K^{\leq\lambda}$.*

Hence one obtains the following two theorems that generalize our results on non-deformed category $\mathcal{O}$, in particular the BGG-reciprocity (see [17], th. 2.7):

Theorem 5.3.7. *i) Every simple object $L_A(\lambda)$ in $\mathcal{O}_A^{\leq\nu}$ admits a projective cover $P_A^\nu(\lambda)$. The $P_A^\nu(\lambda)$'s are indecomposable and they are the only indecomposable projective objects in $\mathcal{O}_A^{\leq\nu}$;*

ii) **(generalized BGG reciprocity)** *The projective cover $P_A^\nu(\lambda)$ in $\mathcal{O}_A^\nu$ of the simple object $L_A(\lambda)$ has a Verma flag whose multiplicities satisfy:*

$$(P_A^\nu(\lambda) : M_A(\mu)) = [M_K(\mu) : L_K(\lambda)]$$

Remark 5.3.8. In particular $(P_A^\nu(\lambda) : M_A(\mu)) = 0$ unless $\mu - \lambda \in \Gamma$ and $(P_A^\nu(\lambda) : M_A(\lambda)) = 1$.

Now take R to be the localization of S at the ideal $\mathfrak{h}S$: it is a local deformation algebra and its residue filed coincides with k. Then the BGG reciprocity assumes the following form:

$$(P_R^\nu(\lambda) : M_R(\mu)) = [M(\mu) : L(\lambda)]$$

5.4 Block decomposition $\mathcal{O}_A$

Let A be always a local deformation domain. We introduce the equivalence relation $\sim_A$ on $\mathfrak{h}^*$ generated by:

$$\lambda \sim_A \mu \Leftrightarrow \exists \nu \in \mathfrak{h}^* \text{ such that } (P_A^\nu(\lambda) : M_A(\mu)) \neq 0 \Leftrightarrow [M_K(\mu) : L_K(\lambda)] \neq 0$$

By definition we have $\sim_A = \sim_K$.

Let Λ be an equivalence class with respect to $\sim_A$. Define $\mathcal{O}_{A,\Lambda}$ to be the full subcategory of $\mathcal{O}_A$ generated by the $P_A^\nu(\lambda)$ with $\nu \in \mathfrak{h}^*$ and $\lambda \in \Lambda$.

One has the following result

Proposition 5.4.1 (see [17], prop. 2.8)**.** *The functor*

$$\prod_{\Lambda \in \mathfrak{h}^*/\sim_A} \mathcal{O}_{A,\Lambda} \to \mathcal{O}$$

$$(M_\Lambda)_\Lambda \mapsto \bigoplus_\Lambda M_\Lambda$$

establishes an equivalence of categories, i.e. we have a block decomposition of category $\mathcal{O}_A$

Remark 5.4.2. Observe that each block is contained in a suitable $\mathcal{O}_A^{\leq \nu}$ hence our category $\mathcal{O}_A$ has enough projective objects and we could get rid of the superscripts $\leq \nu$ (see [27] §4.7)[1]

The following proposition describes the behaviour of equivalence classes under base change.

Proposition 5.4.3 (see [17], lemma 2.9, cor. 2.10)**.** *Given a morphism of local deformation domains* $A \to A'$*, we have that* $\lambda \sim_{A'} \mu$ *implies* $\lambda \sim_A \mu$*. Hence base change functor induces* $\mathcal{O}_{A,\Lambda} \to \mathcal{O}_{A',\Lambda}$*, for any equivalence class* Λ*.*

Denote Z_A the center of category $\mathcal{O}_A$ and $Z_{A,\Lambda}$ the one of $\mathcal{O}_{A,\Lambda}$, then one can show that we have

Proposition 5.4.4 (see [17], prop. 3.1)**.** *A morphism of local deformation domains* $A \to A'$ *induces a morphism* $Z_A \to Z_{A'}$ *and, for each equivalence class* $\Lambda \in \mathfrak{h}^*$ *a morphism* $Z_{A,\Lambda} \to Z_{A',\Lambda}$*.*

[1] Actually this approach with the subcategories $\mathcal{O}_A^{\leq \lambda}$ can be generalized to Kac-Moody algebras, where we would no more have enough projective objects in general.

We can naturally extend the bilinear form on $\mathfrak{h}^*$ to $\mathfrak{h}^*_A = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, A) = \mathfrak{h}^* \otimes_{\mathbb{C}} A$ by letting $(\lambda \otimes t, \mu \otimes s)_A := (\lambda, \mu)ts$, obtaining an A-bilinear form $(-,-)_A \colon \mathfrak{h}^*_A \times \mathfrak{h}^*_A \to A$. Note that $(\tau, \alpha)_A = (\alpha, \alpha)\tau(\alpha^\vee)$ (where we denote $\alpha^\vee$ the element in $\mathfrak{h}$ which is the coroot of α, such that $\alpha(\alpha^\vee) = 2$).

Analogously we have a K-bilinear form $(-,-)_K$. Let $\Lambda \in \mathfrak{h} * / \sim_A$, we denote $\Delta_A(\Lambda)$ the set

$$\Delta_A(\Lambda) = \{\alpha \in \Phi \mid 2(\lambda + \rho + \tau, \alpha)_K \in \mathbb{Z}(\alpha, \alpha)_K\}$$

Take now, as before, R to be the localisation of S at the ideal $\mathfrak{h}S$. For any $\alpha \in \Phi$ denote R_α the localisation of R at the ideal $(\tau, \alpha)_R R$ and K_α the residue field. Note that for an equivalence class $\Lambda \in \mathfrak{h}^* / \sim_{R_\beta}$:

$$\begin{aligned} \Delta_{R_\beta}(\Lambda) = \{\alpha \in \Phi \mid 2(\lambda + \rho, \alpha) \in \mathbb{Z} \text{ for some } \lambda \in \Lambda \text{ and } \alpha = \pm\beta\} = \\ = \Delta_{\mathbb{C}}(\Lambda) \cap \{\pm\beta\} \end{aligned} \tag{5.1}$$

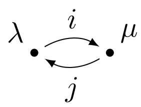
in fact $(\lambda + \rho, \alpha)_{K_\alpha} = (\lambda + \rho, \alpha) 1_{K_\alpha}$ and $(\tau, \alpha)_{K_\alpha} = (\alpha, \alpha)\tau(\alpha^\vee)$ but then we must have $\tau(\alpha^\vee) = 0$ in K_β which is equivalent to $\alpha = \pm\beta$.

One can then show that we obtain that Λ is either $\{\lambda\}$ or $\{\lambda, s_\beta \cdot \lambda\}$ (generic and subgeneric case). We have

Proposition 5.4.5 (see [17], prop. 3.4)**.** *In the subgeneric case with $\Lambda = \{\lambda, \mu\}$, with $\mu = s_\beta \cdot \lambda$ and $\lambda - \mu \in \Gamma$, we have $P_{R_\beta}(\lambda) \cong M_{R_\beta}(\lambda)$ and a shoert exact sequence*

$$0 \to M_{R_\beta}(\lambda) \to P_{R_\beta}(\mu) \to M_{R_\alpha}(\mu) \to 0$$

The block $\mathcal{O}_{R_\beta, \Lambda}$ is equivalent to the category of right representations of the R_β-algebra generated by the path on the quiver



with relations $j \circ i = (\tau, \beta)_{K_\beta} e_\lambda$, where e_λ is the trivial path at λ.

This has the following consequence.

Corollary 5.4.6 (see [17], cor. 3.5)**.** *The evaluation of $Z_{R_\beta, \{\lambda, \mu\}}$ at Verma modules:*

$$\begin{aligned} Z_{R_\beta, \{\lambda, \mu\}} &\to \mathrm{End}(M_{R_\beta}(\lambda)) \oplus \mathrm{End}(M_{R_\beta}(\mu)) = R_\beta \oplus R_\beta \\ z &\mapsto (z(M_{R_\beta}(\lambda)), z(M_{R_\beta}(\mu))) \end{aligned}$$

is injective and induce an isomorphism

$$Z_{R_\beta, \{\lambda, \mu\}} \cong \{(t_\lambda, t_\mu) \in R_\beta \oplus R_\beta \mid t_\lambda - t_\mu \in \frac{(\tau, \beta)_R}{1} R_\alpha\}$$

This result about subgeneric case allows us to describe the center in the situation $A = R$, in the following way:

Theorem 5.4.7 (see [17], th. 3.6)**.** *If $\Lambda \in \mathfrak{h}^*/\sim_R$, evaluation at Verma modules gives an isomorphism*

$$Z_{R,\Lambda} \cong \{(t_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda} R \mid t_\lambda - t_{s_\beta\cdot\lambda} \in (\tau,\beta)_R R\}$$

We conclude this section by explicitely observing the following

Remark 5.4.8. The evaluation

$$\begin{aligned} Z_{R,\Lambda} &\to \mathrm{End}(P_R(\lambda)) \\ \zeta &\mapsto \zeta(P_R(\lambda)) \end{aligned}$$

is an isomorphism, because $P_R(\lambda)$ is a projective generator of $\mathcal{O}_{R,\Lambda}$.

5.5 The structure functor

In this section we present the so-called combinatorial version of Kazhdan-Lusztig conjecture, i.e. we show its equivalence with the statement about the ranks of the stalks of Braden-MacPherson sheaves that we presented in chapter 4. We will only give some of the proofs and we shall just state some of the results which we need: one can find the proofs in [27].

Let us consider as deformation algebra A the localisation R of S at $\mathfrak{h}S$, and let us take the equivalence class $\Lambda \in \mathfrak{h}^*/\sim_R$, containing the antidominant weight $\lambda = -2\rho$. We consider the moment graph $\mathcal{G}$ associated to the Weyl group W: by 5.4.7 we have $\mathcal{Z}_R \cong Z_{R,\Lambda}$.

We define the following functor, called the *structure functor*[2]

$$\begin{aligned} \mathbb{V} = \mathbb{V}_{R,\Lambda} \colon \mathcal{O}_R &\to \text{mod-}\mathrm{End}(P_R(\lambda)) = \mathrm{End}(P_R(\lambda))^{\mathrm{op}}\text{-mod} \\ M &\mapsto \mathrm{Hom}_{\mathcal{O}_R}(P_R(\lambda), M) \end{aligned}$$

By rem. 5.4.8, $\mathrm{End}(P_R(\lambda)) \cong Z_{R,\Lambda} \cong \mathcal{Z}_R$. The functor $\mathbb{V}$ so-defined is an exact functor because $P_R(\lambda)$ is projective. We have the following:

Proposition 5.5.1 (see [27] §4.12)**.** *Let M be a module in $\mathcal{O}_R$ which admits a Verma flag. Then $\mathbb{V}M$ is free of finite rank over R. In particular $\mathbb{V}M$ belongs to $\mathcal{Z}_R\text{-mod}^f$.*

[2] One can actually define it for general deformation algebras.

Proof. We use prop. **??**: by hypothesis we have the exact sequences

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_2/M_1 \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow M_3/M_2 \longrightarrow 0$$

$$\dots$$

$$0 \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_n/M_{n-1} \longrightarrow 0$$

Where each subquotient is a deformed Verma module. Now applying the exact functor $\mathbb{V}$ gives that $\mathbb{V}M$ is free and of finite rank. □

We can then apply functor $\mathscr{L}_R$ and the R-sheaf $\mathscr{L}_R(\mathbb{V}M)$ belongs to $\mathrm{Sh}^{\mathrm{f}}(\mathcal{G})$. One can prove the following statements, for which we refer to [27].

Proposition 5.5.2 (see [27] §4.13). *i) Let $w \in W$ and let $M_R(w \cdot \lambda)$ be the deformed Verma module associated with the weight $w \cdot \lambda$, then $\mathscr{L}_R\big(\mathbb{V}M_R(w \cdot \lambda)\big)$ is the skyscraper sheaf $\mathscr{M}_R(w)$;*

ii) Let M be a module in $\mathcal{O}_R$ admitting a Verma flag, then $\mathbb{V}M$ belongs to $\mathcal{Z}_R$-$\underline{\mathrm{mod}}$.[3]

Let us denote $\mathcal{V}_{\mathcal{O}_R}$ the subcategory of $\mathcal{O}_R$ consisting of object admitting a Verma flag.

Proposition 5.5.3 ([27], §4.18-20). *The following hold:*

i) Let P be a projective module in $\mathcal{O}_R$ then $\mathscr{L}_R(\mathbb{V}P)_w$ is free and of finite rank. This rank is precisely $(P : M_R(w_0 w \cdot \lambda))$;

ii) If P is a projective object in $\mathcal{V}_{\mathcal{O}_R}$, then $\mathscr{L}_R(\mathbb{V}P)$ is F-projective.

And one gets the following theorem:

Theorem 5.5.4 (see [27], §4.21). *We have natural isomorphisms*

$$\mathrm{Hom}(M, N) \cong \mathrm{Hom}(\mathbb{V}M, \mathbb{V}N) \cong \mathrm{Hom}(\mathscr{L}_R\mathbb{V}M, \mathscr{L}_R\mathbb{V}N)$$

These result allow us to finally make our announced link with Braden-MacPherson sheaves:

[3] For this part one uses prop. 4.4.5

Theorem 5.5.5. *We have*

$$\mathscr{L}_R\big(\mathbb{V}P_R(w\cdot\lambda)\big) \cong \mathscr{B}(w_0 w)\otimes_S R$$

Proof. By prop. 5.5.3, part (ii), $\mathscr{L}_R\big(\mathbb{V}P_R(w\cdot\lambda)\big)$ is F-projective and by prop. 4.4.6, it is isomorphic to a direct sum of sheaves of the form $\mathscr{B}(z)\otimes_S R$. Now, $P(w\cdot\lambda)$ is indecomposable, hence the only idempotent elements of End $\big(P(w\cdot\lambda)\big)$ are 0 and 1. But by th. 5.5.4, the same holds for End $\Big(\mathscr{L}_R\big(\mathbb{V}P_R(w\cdot\lambda)\big)\Big)$, then $\mathscr{L}_R\big(\mathbb{V}P_R(w\cdot\lambda)\big)$ is indecomposable. This implies that there exists a $z\in W$ such that $\mathscr{L}_R\big(\mathbb{V}P_R(w\cdot\lambda)\big)\cong\mathscr{B}(z)\otimes_S R$. The element z is characterized as the maximal among the elements x for which $\big(\mathscr{B}(z)\otimes_S R\big)_x\neq 0$, i.e. for which $0\neq\mathscr{L}_R\big(\mathbb{V}P_R(w\cdot\lambda)\big)_x=\big(P_R(w\cdot\lambda):M_R(w_0x\cdot\lambda)\big)$, by prop. 5.5.3, part (i). This, by remark 5.3.8, implies $w_0 z = w$, i.e. $z = w_0 w$. □

Theorem 5.5.5 allows us to finally prove the desired equivalence between the two conjectures

Theorem 5.5.6. *Kazhdan-Lusztig conjecture 3.5.1 is equivalent to multiplicity conjecture 4.3.8 on Braden-MacPherson sheaves in characteristic 0.*

Proof. We have, by generalized BGG reciprocity (th. 5.3.7, (ii)):

$$[M(w\cdot\lambda):L(y\cdot\lambda)] = \big(P_R(y\cdot\lambda):M_R(w\cdot\lambda)\big)$$

By prop. 5.5.3,

$$\big(P_R(y\cdot\lambda):M_R(w\cdot\lambda)\big) = \operatorname{rk}\mathscr{L}_R(\mathbb{V}P(y\cdot\lambda))_{w_0 w}$$

Finally by prop. 5.5.5

$$\operatorname{rk}\mathscr{L}_R(\mathbb{V}P(y\cdot\lambda))_{w_0 w} = \operatorname{rk}\big(\mathscr{B}(w_0 y)\otimes_S R\big)_{w_0 w} = \operatorname{rk}\mathscr{B}(w_0 y)_{w_0 w}$$

□

Bibliography

[1] Alberto Arabia. *Faisceaux pervers sur les variétés algébriques complexes - Correspondance de Springer*. 2014. URL: https://webusers.imj-prg.fr/~alberto.arabia.

[2] Alberto Arabia. *Introduction à la cohomologie d'intersecition*. 2002. URL: https://webusers.imj-prg.fr/~alberto.arabia.

[3] Alexander A. Beilinson and Joseph Bernstein. "Localisation de $\mathfrak{g}$-modules". In: *C.R. Math. Acad. Sci. Paris* 292 (1981), pp. 15–18.

[4] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne. "Faisceaux pervers". In: *Analyse et topologie sur les espaces singuliers, I (Luminy, 1981)*. Ed. by Soc. Math. France. Astérisque 100. 1982, pp. 5–171.

[5] Joseph Bernstein, Israel Gel'fand, and Sergei Gel'fand. "On a category of $\mathfrak{g}$-modules". In: *Funktsional. Anal. i Prilozhen* 10.2 (1976).

[6] Joseph Bernstein and Valery Lunts. *Equivariant Sheaves and Functors*. Lecture Notes in Mathematics 1578. Springer-Verlag, 1994.

[7] Anders Bjorner and Francesco Brenti. *Combinatorics of Coxeter Groups*. Graduate Texts in Mathematics 231. New York: Springer, 2005.

[8] Armand Borel. *Linear Algebraic Groups*. 2nd ed. Graduate Texts in Mathematics 126. New York: Springer, 1991.

[9] Nicolas Bourbaki. *Groupes et algèbres de Lie*. Springer, 2007. Chap. 4,5,6.

[10] Tom Braden and Robert MacPherson. "From Moment Graphs to Intersection Cohomology". In: *Mathematische Annalen* 321 (2001), pp. 533–511.

[11] Michel Brion. *Lectures on the Geometry of Flag Varieties*. 2004. URL: https://arxiv.org/abs/math/0410240v1.

[12] J. L. Brylinski and M. Kashiwara. "Kazhdan-Lusztig Conjecture and Holonomic Systems". In: *Inventiones Mathematicae*. 64. Springer-Verlag, 1981, pp. 387–410.

[13] Mark Andrea A. De Cataldo and Luca Migliorini. “The Decomposition Theorem, Perverse Sheaves and the Topology of Algebraic Maps”. In: *Bulletin of the American Mathematical Society* 46.4 (2009), pp. 535–633.

[14] Michel Demazure. “Désingularisation des variétés de Schubert généralisées”. In: *Annales Scientifiques de l’E.N.S.* 4th ser. 7.1 (1974), pp. 53–88.

[15] David S. Dummit and Richard M. Foote. *Abstract Algebra.* 3rd ed. John Wiley & Sons, Inc., 2004.

[16] Ben Elias and Geordie Williamson. *The Hodge Theory of Soergel Bimodules.* 2012. URL: `https://arxiv.org/pdf/1212.0791.pdf`.

[17] Peter Fiebig. “Centers and Translation Functors for the Category $\mathcal{O}$ over Kac-Moody algebras”. In: *Mathematische Zeitschrift* 243 (2003), pp. 689–717.

[18] Peter Fiebig. *Moment Graphs in Representatio Theory and Geometry.* 2013. URL: `https://arxiv.org/abs/1308.2873v2`.

[19] Peter Fiebig. *Sheaves on Moment Graphs and a Localisation of Verma Flags.* 2005. URL: `https://arxiv.org/abs/math/0505108v2`.

[20] Peter Fiebig. “The Combinatorics of category $\mathcal{O}$ over symmetrizable Kac-Moody algebras”. In: *Transformation Groups* 11.1 (2006), pp. 29–49.

[21] Mark Goresky, Robert Kottwitz, and MacPherson Robert. “Equivariant Cohomology, Koszul Duality, and the Localisation Theorem”. In: *Inventiones Mathematicae.* Vol. 131. Springer-Verlag, 1998, pp. 25–83.

[22] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory.* Vol. 236. Progress in Mathematics. Boston: Birkhäuser, 2008.

[23] James Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$.* Springer, 2008.

[24] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* New York: Springer-Verlag, 1972.

[25] James E. Humphreys. *Reflection Groups and Coxeter Groups.* Cambridge University Press, 1990.

[26] Jens C. Jantzen. *Moduln mit einem höchsten Gewicht.* Lecture Notes in Mathematics. Springer, 1979.

[27] Jens C. Jantzen. *Moment Graphs and Representations*. 2010. URL: https://www-fourier.ujf-grenoble.fr/sites/ifmaquette.ujf-grenoble.fr/files/jantzen_rev.pdf.

[28] David Kazhdan and George Lusztig. "Representations of Coxeter Groups and Hecke Algebras". In: *Inventiones Mathematicae*. Vol. 53. Springer-Verlag, 1979, pp. 165–184.

[29] David Kazhdan and George Lusztig. "Schubert Varieties and Poincaré Duality". In: *Proc. Sympos. Pure Math., American Mathematical Society* XXXVI (1980), pp. 185–203.

[30] Anmon Neeman. *Triangulated Categories*. Vol. 143. Princeton University Press, 2001.

[31] Patrick Polo. "Construction of arbitrary Kazhdan-Lusztig polynomials in Symmetric Groups". In: *An Electronic Journal of the American Mathematical Society*. Representation Theory 3 (1999), pp. 90–104.

[32] Simon Riche. *D-modules, faisceaux pervers et conjecture de Kazhdan-Lusztig (d'après Beilinson-Bernstein)*. 2010. URL: http://math.univ-bpclermont.fr/~riche/exposeKL-v2.pdf.

[33] Simon Riche. *Perverse Sheaves on Flag Manifolds and Kazhdan-Lusztig Polynomials (after Kazhdan-Lusztig, MacPherson, Springer ...)* 2010. URL: http://math.univ-bpclermont.fr/~riche/exposeKL2.pdf.

[34] Alvany Rocha-Caridi and Nolan R. Wallach. "Projective Modules over Graded Lie Algebras. I". In: *Mathematische Zeitschrift*. Vol. 180. Springer-Verlag, 1982.

[35] Wolfgang Soergel. "Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe". In: *Jurnal of the American Mathematical Society* 3 (1990), pp. 421–445.

[36] Wolfgang Soergel. "Kazhdan-Lusztig Polynomials and a Combinatoric for Tilting Modules". In: *An Electronic Journal of the American Mathematical Society* 1 (1997), pp. 83–114.