Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea in Matematica

Equilibrium measures on trees and square tilings

Tesi di Laurea in Analisi Matematica

Relatore:
Chiar.mo Prof.
Nicola Arcozzi

Presentata da:
Davide Cordella

III Sessione
Anno Accademico 2016/2017

A mia sorella Sofia
To my sister Sofia

Introduzione

In questa tesi verranno trattati alcuni argomenti di teoria del potenziale. In particolare ci interesseranno le nozioni di *capacità* di un insieme, *funzione di equilibrio* e *misura di equilibrio*.

Nel primo capitolo l'esposizione verterà sulle definizioni e sui principali risultati della teoria del potenziale assiomatica, in termini di funzioni L^p, così come è presentata in [AH] (Capitolo 2).

Nel secondo capitolo i risultati precedenti verranno tradotti nel contesto discreto degli *alberi*, che sono dei particolari grafi. Verrà definito il *bordo di un albero*, sottolineando le sue proprietà metriche. Vedremo inoltre come le funzioni (e le misure) di equilibrio di sottoalberi si ottengano riscalando quelle dell'albero di partenza. La teoria del potenziale sugli alberi è interessante di per sé e poiché alcuni risultati possono essere usati per dimostrare risultati analoghi nella teoria classica. Si veda per esempio l'articolo di N. Arcozzi, R. Rochberg, E. T. Sawyer and B. D. Wick [ARSW]. Ulteriori esempi e risultati sugli alberi sono esposti in [Be]. Per sviluppi moderni riguardo la teoria del potenziale su alberi e grafi più generali, buoni punti di riferimento sono il libro di P. M. Soardi [So] e il lavoro recente di R. Lyons e Y. Peres [LP].

Nel terzo capitolo verrà data una caratterizzazione delle misure di equilibrio sugli alberi seguendo [AL]: esse sono precisamente le soluzioni di una equazione discreta non lineare. Questo è significativo poiché in generale la nozione di misura di equilibrio è di difficile comprensione.

Inoltre vedremo come ad una misura d'equilibrio si possa associare una *tassellazione con quadrati* (*square tiling*), che è la suddivisione di un rettangolo in piastrelle quadrate non sovrapposte. Tale tipo di costruzione è oggetto di studio sin dal classico articolo del 1940 *'The dissection of rectangles into squares'* di R. L. Brooks, C. A. B. Smith, A. H. Stone e W. T.

Tutte [BSST], dove viene posto il problema di classificare le suddivisioni con quadrati di dimensioni tutte diverse, associando ad esse delle reti elettriche (cioè dei grafi). Alla fine del capitolo verranno esposti alcuni esempi di alberi, misure d'equilibrio e tassellazioni associate.

Nel quarto capitolo si darà una interpretazione delle misure d'equilibrio e delle capacità su alberi in termini probabilistici considerando passeggiate aleatorie. Ciò verrà fatto mostrando un parallelo con un teorema di I. Benjamini e O. Schramm [BS] che vale più in generale per grafi e nel quale viene costruita una tassellazione con 'quadrati' di un cilindro. Il teorema del Capitolo 3 fornisce l' implicazione inversa del teorema di cui sopra nel caso speciale degli alberi.

Infine, nel quinto capitolo verrà esposto un teorema di O. Schramm [Sc], in cui una tassellazione finita viene costruita in termini puramente combinatori a partire da una triangolazione planare, usando la nozione di *metrica estremale* che è analoga a quella di funzione/misura d'equilibrio. Tale costruzione permetterà di fare qualche osservazione anche sul caso degli alberi.

Introduction

In this thesis some topics of potential theory will be studied. In particular, the notions of *capacity* of a set, *equilibrium function* and *equilibrium measure* are significant.

In the first chapter, the exposition will focus on the definitions and main results of the axiomatic potential theory , in terms of L^p functions, as it is presented by D. R. Adams and L. I. Hedberg ([AH], Chapter 2).

In the second chapter, the previous results will be translated in the discrete context of trees, which are particular graphs. We will define the *boundary* of a tree, focusing on its metric properties. We will also see how equilibrium functions (and measures) on subtrees are obtained by rescaling those of the starting tree. Potential theory on trees is interesting in itself and because some results can be used to prove analogous results in the classical theory. See for example N. Arcozzi, R. Rochberg, E. T. Sawyer and B. D. Wick's paper [ARSW]. Further examples and results on trees are shown in [Be]. For some modern developments about potential theory on trees and more general graphs, some good reference are P. M. Soardi's book [So] and the recent work by R. Lyons, Y. Peres [LP].

In the third chapter, we will give a characterization of the equilibrium measures on trees following [AL]: they are exactly the solutions to a discrete nonlinear equation. This is significant since in general equilibrium measures are not well understood.

Moreover we will see how an equilibrium measure is associated with a *square tiling*, which is the subdivision of a rectangle into non overlapping squares. This kind of constructions is an object of study since the classic paper of 1940 *'The dissection of rectangles into squares'* by R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte [BSST], where the problem of classifying subdivisions into squares of different sizes is considered, associating them with electrical networks (i.e. graphs). At the end of the chapter some examples of trees, equilibrium measures and associated tilings will be presented.

In the fourth chapter, we will give a probabilistic interpretation of equilibrium measures and capacities on trees by considering random walks. This will be done by showing a parallel with a theorem by I. Benjamini and O. Schramm [BS] that holds for more general graphs and in which a 'square' tiling of a cylinder is constructed. The Theorem in Chapter 3 provides the inverse of the theorem above in the special case of trees.

Finally, in the fifth chapter, we will state a theorem by O. Schramm [Sc], in which a finite square tiling is built, in purely combinatorial terms, starting from a planar triangulation and using the notion of *extremal metric* which is analogous to equilibrium function/measure. This construction will allow us to make some remarks on the case of trees as well.

Contents

Chapter 1

General theory for L^p-capacity

1.1 Preliminaries

1.1.1 Radon measures and weak* convergence

Let X be a locally compact Hausdorff space and let $\mathcal{C}_c(X)$ be the space of all the functions $f : X \longrightarrow \mathbb{C}$ with compact support. By the Riesz-Markov-Kakutani Representation Theorem (see [Ru], Theorem 2.14), there exists a bijection between

- *Positive linear functionals* on $\mathcal{C}_c(X)$, i.e. $\Lambda : \mathcal{C}_c(X) \longrightarrow \mathbb{R}$ such that $\Lambda\varphi \geq 0$ for all $\varphi \in \mathcal{C}_c(X)$ real-valued and nonnegative.

- Positive measures μ, defined on the σ-algebra of the Borel sets $\mathfrak{B}(X)$, for which

  - $\mu(K) < +\infty$ if $K \in \mathfrak{B}(X)$ is compact.
  - (*outer regularity*) For all $A \in \mathfrak{B}(X)$
  $$\mu(A) = \inf\{\mu(O) |\, O \text{ open, } A \subset O \subset X\}.$$
  - (*inner regularity*) If $A \in \mathfrak{B}(X)$ is open or $\mu(A) < +\infty$, then
  $$\mu(A) = \sup\{\mu(K) |\, K \text{ compact, } K \subset A\}.$$

The correspondence is given by the map

$$\mu \longmapsto \Lambda,\ \Lambda(\varphi) = \int_X \varphi(x) d\mu(x).$$

Definition 1.1. We denote by $\mathcal{M}_+(X)$ the space of the measures we mentioned above. The Representation Theorem shows that this space is the positive cone of $\mathcal{M}(X) := \mathcal{C}_c(X)^*$, the so-called space of *Radon measures* on X. If $K \subset X$ is compact, then $\mathcal{M}_+(K)$ denotes the subspace of $\mathcal{M}_+(X)$ consisting of those measures whose supports lie on K; note that they are all finite measures.

$\mathcal{M}_+(X)$ is a subspace of a dual space, so it can be equipped with the weak* topology.

Definition 1.2. We say that a sequence $\{\mu_j\}_{j=1}^{+\infty} \subset \mathcal{M}_+(X)$ *converges weak** to $\mu \in \mathcal{M}(X)$ if for each $\varphi \in \mathcal{C}_c(X)$

$$\lim_{j \longrightarrow +\infty} \int_X \varphi(x) d\mu_j(x) = \int_X \varphi(x) d\mu(x).$$

Obviously $\mu \in \mathcal{M}_+(X)$.

1.1.2 Uniformly convex Banach spaces

Definition 1.3. Let $(E, \|\cdot\|)$ be a Banach space. Then it is *uniformly convex* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every $f,\, g \in E$

$$\|f\| < 1 + \delta,\ \|g\| < 1 + \delta,\ \left\| \frac{f+g}{2} \right\| \geq 1 \Longrightarrow \|f - g\| < \varepsilon.$$

Theorem 1.1. *The spaces L^p are uniformly convex for $1 < p < +\infty$.*

Proof. This is a direct consequence of the Clarkson's inequalities:

1. For every $f,\, g \in L^p$ with $2 \leq p < +\infty$

$$\left\| \frac{f-g}{2} \right\|_p^p + \left\| \frac{f+g}{2} \right\|_p^p \leq \frac{1}{2}(\|f\|_p^p + \|g\|_p^p).$$

2. For every $f,\, g \in L^p$ with $1 < p \leq 2$

$$\left\| \frac{f-g}{2} \right\|_p^{p'} + \left\| \frac{f+g}{2} \right\|_p^{p'} \leq \left(\frac{1}{2}(\|f\|_p^p + \|g\|_p^p) \right)^{p'-1}$$

(where p' is the Hölder-conjugate exponent of p).

See [Br] (proof of Theorem 4.10 and Problem 20). □

Lemma 1.2 ([AH] Corollary 1.3.3)**.** *Let* $(E, \|\cdot\|)$ *be a uniformly convex Banach space. If* $\{f_j\}_{j=1}^{+\infty}$ *is a sequence in* E *such that*

$$\lim_{j\longrightarrow+\infty} \|f_j\| = 1, \qquad \liminf_{j,k\longrightarrow+\infty} \left\| \frac{f_j + f_k}{2} \right\| \geq 1,$$

then $\{f_j\}$ *is a Cauchy sequence, so it converges in* E.

Proof. Let $\varepsilon > 0$. For every $0 < \eta < 1$ there exists j_η such that $\|f_j\| < 1+\eta$ and $\|(f_j + f_k)/2\| > 1-\eta$ when j, $k \geq j_\eta$. We define $\widehat{f_j} = f_j/(1-\eta)$, in order to have $\|(\widehat{f_j} + \widehat{f_k})/2\| > 1$ and $\|\widehat{f_j}\| < 1+\delta$, where $\delta = 2\eta/(1-\eta)$ decreases to 0 as η goes to 0. Thus, if we fix η small enough, by uniform convexity $\|f_j - f_k\| < (1-\eta)\varepsilon$ for all j, $k \geq j_\eta$. □

Proposition 1.3 ([AH] Corollary 1.3.4)**.** *Let* Ω *be a convex subset of a uniformly convex Banach space* E*. Then there exists a unique element in the closure* $\overline{\Omega}$ *with least norm.*

Sketch of the proof. If $0 \in \overline{\Omega}$, it is obviously the element of least norm. Otherwise, if $\{f_j\} \subset \Omega$ minimizes the norm, we suppose that $\lim_{j\longrightarrow+\infty} \|f_j\| = 1$ by possibly rescaling the sequence. By convexity and Lemma 1.2 we have the convergence to an element of least norm in the closure. Uniqueness follows by uniform convexity of E. □

1.2 Kernel, potentials and capacity

Potential theory (in terms of L^p functions) can be develop in a quite abstract context. For that we summarize the main definitions and results from the book of D. R. Adams and L. I. Hedberg ([AH], Chapter 2).

Definition 1.4. Let (X, ρ) be a locally compact metric space and let (M, ν) equipped with a σ-finite ν. measure. A *kernel* is a function $g : X \times M \longrightarrow \mathbb{R}_+$ such that:

- For each $y \in M$ the map $X \longrightarrow \mathbb{R}_+$, $x \longmapsto g(x, y)$ is a lower semi-continuous (LSC) function.

- For each $x \in X$, the map $M \longrightarrow \mathbb{R}_+$, $y \longmapsto g(x,y)$ is a ν-measurable function.

Let $f : M \longrightarrow [0,+\infty]$ be a ν-measurable function and $\mu \in \mathcal{M}_+(X)$.

i) The *potential* of f is the function

$$\mathcal{G}f : X \longrightarrow [0,+\infty] \qquad \mathcal{G}f(x) := \int_M g(x,y)f(y)d\nu(y).$$

ii) The *(co)-potential* of μ is the function

$$\check{\mathcal{G}}\mu : M \longrightarrow [0,+\infty] \qquad \check{\mathcal{G}}\mu(y) := \int_X g(x,y)d\mu(x).$$

iii) The *mutual energy* of μ and f is the quantity

$$\mathcal{E}(\mu,f) \in [0,+\infty] \quad \mathcal{E}(\mu,f) := \int_X \mathcal{G}f(x)d\mu(x) = \int_M \check{\mathcal{G}}\mu(y)f(y)d\nu(y)$$

where the last inequality follows from Fubini-Tonelli Theorem.

Remark 1.1. $\mathcal{G}$ and $\check{\mathcal{G}}$ act as linear operators:

$$\mathcal{G}(\alpha f + \beta \widetilde{f}) = \alpha(\mathcal{G}f) + \beta(\mathcal{G}\widetilde{f})$$

$$\check{\mathcal{G}}(\alpha\mu + \beta\widetilde{\mu}) = \alpha(\check{\mathcal{G}}\mu) + \beta(\check{\mathcal{G}}\widetilde{\mu})$$

for all f, $\widetilde{f} : M \longrightarrow [0,+\infty]$ ν-measurable function, μ, $\widetilde{\mu} \in \mathcal{M}_+(X)$, $\alpha, \beta \geq 0$ scalar factors. The mutual energy $\mathcal{E}$ is linear in both entries.

Example 1.1. If we consider $X = M = \mathbb{R}^3$ (with the standard Lebesgue measure) and the kernel

$$g(x,y) = \frac{1}{|x-y|}$$

(which is the *Newtonian kernel* on dimension 3), then for every positive charge distribution μ on a compact set $K \subset \mathbb{R}^3$ (i.e. $\mu \in \mathcal{M}_+(K)$) the potential is

$$\check{\mathcal{G}}\mu(y) = \int_K \frac{d\mu(x)}{|x-y|},$$

which is the potential (*Newtonian potential*) of classical electrostatics.

First, we establish some regularity properties.

Proposition 1.4 ([AH] Prop. 2.3.2)**.** *Let g be a fixed kernel, f a nonnegative ν-measurable function on M and $y \in M$.*

1. *The map $x \longmapsto \mathcal{G}f(x)$ is LSC on X.*
2. *The map $\mu \longmapsto \check{\mathcal{G}}\mu(y)$ is LSC on $\mathcal{M}_+(X)$, with the weak* topology.*
3. *The map $\mu \longmapsto \mathcal{E}(\mu, f)$ is LSC on $\mathcal{M}_+(X)$, with the weak* topology.*

Proof of 1. Let $x_0 \in X$ and $\{x_j\}_{j=1}^{+\infty} \subset X$ such that

$$x_j \overset{j\longrightarrow+\infty}{\longrightarrow} x_0 \qquad \lim_{j\longrightarrow+\infty} \mathcal{G}f(x_j) = \liminf_{x\longrightarrow x_0} \mathcal{G}f(x). \tag{1.1}$$

Thus $g(x_0, y) \leq \liminf\limits_{j\longrightarrow+\infty} g(x_j, y)$, as g is LSC. By using Fatou's Lemma

$$\mathcal{G}f(x_0) = \int_M g(x_0, y)f(y)d\nu(y) \leq \int_M \liminf_{j\longrightarrow+\infty} g(x_j, y)f(y)d\nu(y) \leq$$

$$\overset{\text{(Fatou)}}{\leq} \liminf_{j\longrightarrow+\infty} \int_M g(x_j, y)f(y)d\nu(y) \overset{(1.1)}{=} \liminf_{x\longrightarrow x_0} \mathcal{G}f(x)$$

so the lower semi-continuity is proven. □

Proof of 2. A nonnegative LSC function on X is approximated by an increasing sequence of function in $\mathcal{C}_c(X)$, converging pointwise [1]. So we pick $\{h_k\}_{k=1}^{+\infty} \subset \mathcal{C}_c(X)$ such that $h_k(x) \uparrow g(x, y)$. Now if $\mu \in \mathcal{M}_+(X)$ and $\{\mu_j\}_{j=1}^{+\infty} \subset \mathcal{M}_+(X)$ is such that $\mu_j \longrightarrow \mu$ weak*, then for all k

$$\int_X h_k(x)d\mu(x) = \lim_{j\longrightarrow+\infty} \int_X h_k(x)d\mu_j(x) \leq \liminf_{j\longrightarrow+\infty} \underbrace{\int_X g(x, y)d\mu_j(x)}_{=\check{\mathcal{G}}\mu_j(y)},$$

so by Monotone Convergence Theorem

$$\check{\mathcal{G}}\mu(y) = \int_X g(x, y)d\mu(x) \overset{\text{(MCT)}}{=} \lim_{k\longrightarrow+\infty} \int_X h_k(x)d\mu(x) \leq \liminf_{j\longrightarrow+\infty} \check{\mathcal{G}}\mu_j(y). \quad \square$$

[1] This can be done by suitably modifying the continuous functions that appear in [Ru], Exercise 2.22.

Proof of 3. This is analogous to point 2. The map $x \longmapsto \mathcal{G}f(x)$ is LSC, so there exists an increasing sequence $\{h_k\}_{k=1}^{+\infty} \subset \mathcal{C}_c(X)$ such that $h_k(x) \uparrow \mathcal{G}f(x)$ for $k \longrightarrow +\infty$. Proceeding as before, if $\mu \in \mathcal{M}_+(X)$ and $\mu_j \longrightarrow \mu$ weak*, then for all $k \geq 1$

$$\int_X h_k(x)d\mu(x) = \lim_{j\longrightarrow+\infty} \int_X h_k(x)d\mu_j(x) \leq \liminf_{j\longrightarrow+\infty} \underbrace{\int_X \mathcal{G}f(x)d\mu_j(x)}_{=\mathcal{E}(\mu_j,f)},$$

hence $\mu \longmapsto \mathcal{E}(\mu, f)$ is LSC because

$$\mathcal{E}(\mu, f) = \int_X \mathcal{G}f(x)d\mu(x) \overset{\text{(MCT)}}{=} \lim_{k\longrightarrow+\infty} \int_X h_k(x)d\mu(x) \leq \liminf_{j\longrightarrow+\infty} \mathcal{E}(\mu_j, f). \qquad \square$$

Definition 1.5 (L^p-Capacity)**.** Let $1 \leq p < +\infty$, and let g be a fixed kernel. For $A \subset X$ we consider the set

$$\Omega_A := \{f \in L^p_+(\nu)|\, \mathcal{G}f(x) \geq 1 \text{ for all } x \in A\}.$$

Then the L^p-capacity of A is

$$Cap_p(A) := \inf_{f\in\Omega_A} \|f\|^p_{L^p(\nu)}.$$

By convention $\inf \emptyset = +\infty$. Note that the definitions above depend on the choice of a kernel g.

Cap_p is a function from the subsets of X to $[0, +\infty]$, so we may think it is a kind of 'measure'. Like measures, $Cap_p(\emptyset) = 0$ because $\Omega_\emptyset$ degenerates to $L^p_+(\nu)$ and then the constant function $0 \in \Omega_\emptyset$. The next three Propositions assert that L^p-capacity has respectively the properties of *monotonicity*, *outer regularity* and *σ-sub-additivity.*

Proposition 1.5 ([AH] Proposition 2.3.4)**.** *If $A \subset \widetilde{A} \subset X$, then $Cap_p(A) \leq Cap_p(\widetilde{A})$.*

Proof. If $\mathcal{G}f \geq 1$ on $\widetilde{A}$, in particular $\mathcal{G}f \geq 1$ on A, so $\Omega_A \supset \Omega_{\widetilde{A}}$ and

$$Cap_p(A) = \inf_{f\in\Omega_A} \|f\|^p_{L^p(\nu)} \leq \inf_{f\in\Omega_{\widetilde{A}}} \|f\|^p_{L^p(\nu)} = Cap_p(\widetilde{A}). \qquad \square$$

Proposition 1.6 ([AH] Proposition 2.3.5)**.** *For every $A \subset X$*

$$Cap_p(A) = \inf\{Cap_p(O)|\, O \text{ open}, A \subset O \subset X\}.$$

Proof. $Cap_p(O) \geq Cap_p(A)$ for all O open subset including A, so it is sufficient to find a sequence of open sets whose capacities converge to $Cap_p(A)$. By definition of capacity, for every $n \in \mathbb{N}$ there exists $f_n \in \Omega_A$ such that $\|f_n\|^p_{L^p(\nu)} < Cap_p(A) + 1/n$ and we define the set

$$O_n := \left\{ x \in X \,|\, \mathcal{G}f_n(x) > 1 - \frac{1}{n} \right\}.$$

They are open sets because $x \longmapsto \mathcal{G}f_n(x)$ is LSC (Proposition 1.4(1)) and they contain A since $f_n \in \Omega_A$. Since $(1-1/n)^{-1} f_n \in \Omega_{O_n}$,

$$Cap_p(A) \leq Cap_p(O_n) \leq \frac{\|f_n\|^p_{L^p(\nu)}}{(1-1/n)^p} < \frac{Cap_p(A) + 1/n}{(1-1/n)^p} \longrightarrow Cap_p(A)$$

as $n \longrightarrow +\infty$. □

Proposition 1.7 ([AH] Proposition 2.3.6)**.** *Let* $A = \bigcup_{j=1}^{+\infty} A_j \subset X$*. Then*

$$Cap_p(A) \leq \sum_{j=1}^{+\infty} Cap_p(A_j). \tag{1.2}$$

Proof. If the right hand side of (1.2) is $+\infty$ there is nothing to prove. Otherwise, let $\varepsilon > 0$. For each $j \in \mathbb{N}$ there exists a function $f_j \in \Omega_{A_j}$ such that

$$\|f_j\|^p_{L^p(\nu)} < Cap_p(A_j) + \varepsilon 2^{-j}.$$

We pick the supremum $f := \sup_{j \geq 1} f_j$; $\mathcal{G}f \geq 1$ holds in A_j for every index j, so it is true on the union of the sets A_j, which is A. Furthermore, as the f_j are in $L^p_+(\nu)$,

$$\|f\|^p_{L^p(\nu)} = \int_M f(y)^p d\nu(y) \leq \int_M \sum_{j=1}^{+\infty} f_j(y)^p d\nu(y) \overset{\text{(MCT)}}{=} \sum_{j=1}^{+\infty} \|f_j\|^p_{L^p(\nu)} <$$

$$< \sum_{j=1}^{+\infty} \left(Cap_p(A_j) + \frac{\varepsilon}{2^j} \right) = \sum_{j=1}^{+\infty} Cap_p(A_j) + \varepsilon.$$

Then $f \in L^p_+(\nu) \Longrightarrow f \in \Omega_A$ and

$$Cap_p(A) \leq \|f\|^p_{L^p(\nu)} < \sum_{i=1}^{+\infty} Cap_p(A_j) + \varepsilon.$$

(1.2) follows since ε is arbitrary. □

Remark 1.2. In general, it is not true that Cap_p is *σ-additive.* It may happen that

$$Cap_p\left(\bigcup_{j=1}^{+\infty} A_j\right) < \sum_{j=1}^{+\infty} Cap_p(A_j)$$

even if the sets A_j are pairwise disjoint. This will be clear in the case of trees (which is studied in the next two Chapters). Hence Cap_p is not a measure.

Definition 1.6. A property holds *Cap_p-almost everywhere*[2] (*Cap_p-a.e.*) if it is satisfied except for a set B with null capacity: $Cap_p(B) = 0$.

There is a characterization of sets whose capacity is zero.

Proposition 1.8 ([AH] Proposition 2.3.7)**.** *If $A \subset X$ is a nonempty set, then the following are equivalent:*

i) The set A has null L^p-capacity: $Cap_p(A) = 0$.

ii) There exists $h \in L^p_+(\nu)$ such that $\mathcal{G}h(x) = +\infty$ for all $x \in A$.

Proof. We first prove $ii) \Longrightarrow i)$.
For every $\lambda > 0$, $A \subset A_\lambda := \{x \in X | \, \mathcal{G}h(x) \geq \lambda\}$ and $\lambda^{-1}h \in \Omega_{A_\lambda}$. Thus

$$Cap_p(A_\lambda) \leq \frac{\|h\|^p_{L^p(\nu)}}{\lambda^p}, \qquad 0 \leq Cap_p(A) \leq \inf_{\lambda>0} Cap_p(A_\lambda) \leq 0$$

and $i)$ holds.

To prove $i) \Longrightarrow ii)$, as A has null capacity, we choose $h_n \in \Omega_A$ (for each $n \in \mathbb{N}$) so that $\|h_n\|^p_{L^p(\nu)} < 2^{-np}$. Then $h := \sum_{n=1}^{+\infty} h_n$ is a function of $L^p_+(\nu)$ because

$$\|h\|_{L^p(\nu)} \leq \sum_{n=1}^{+\infty} \|h_n\|_{L^p(\nu)} < 1.$$

In addition, $\mathcal{G}h(x) = \sum_{n=1}^{+\infty} \mathcal{G}h_n(x)$ by simply applying Monotone Convergence Theorem and $\mathcal{G}h \equiv +\infty$ on A, as each summand of the last series is greater than 1 when $x \in A$. $\square$

[2] In [AH] the expression (g,p)-*quasi everywhere* is used, in order to distinguish capacities from measures.

Remark 1.3. Every function $f : M \longrightarrow [-\infty, +\infty]$ can be decomposed in its *positive part* f^+ and its *negative part* f^-:

$$f^+ = \max\{f, 0\}, \quad f^- = \max\{-f, 0\}, \quad f = f^+ - f^-.$$

So we define $\mathcal{G}f^+$ and $\mathcal{G}f^+$. Whenever one of the two is finite, we consider

$$\mathcal{G}f := \mathcal{G}f^+ - \mathcal{G}f^-.$$

Proposition 1.8 implies that $\mathcal{G}f$ is defined Cap_p-almost everywhere, whenever $f \in L^p(\nu)$.

Proposition 1.9 ([AH] Proposition 2.3.8)**.** *Let $\{f_j\}_{j=1}^{+\infty}$ be a Cauchy sequence in $L^p(\nu)$ which converges strongly to f. Then there exists a subsequence $\{f_{j_k}\}_{k=1}^{+\infty}$ such that $\lim\limits_{k \longrightarrow +\infty} \mathcal{G}f_{j_k}(x) = \mathcal{G}f(x)$ Cap_p-almost everywhere in X and the convergence is uniform outside an open set which capacity is arbitrarily small.*

Proof. By Remark 1.3 and σ-sub-additivity (Proposition 1.7) it follows that the potentials $\mathcal{G}f_j$ and $\mathcal{G}f$ are finite outside a set F of null capacity. Since $\{f_j\}$ is a Cauchy sequence, there exists a subsequence indexed by $k \in \mathbb{N}$ such that

$$\|f_{j_k} - f\|_{L^p(\nu)} < 2^{-2k}.$$

For all $k, n \in \mathbb{N}$, we define

$$G_k := \{x \in X | \, \mathcal{G}|f_{j_k} - f|(x) > 2^{-k}\}, \quad H_n := \bigcup_{k=n}^{+\infty} G_k, \quad H := \bigcap_{n=1}^{+\infty} H_n.$$

Thus

$$Cap_p(G_k) \leq 2^{kp} \|f_{j_k} - f\|^p_{L^p(\nu)} \leq 2^{-kp}$$

$$\Longrightarrow Cap_p(F \cup H_n) \leq \sum_{k=n}^{+\infty} \frac{1}{2^{kp}} \overset{n \longrightarrow +\infty}{\longrightarrow} 0, \qquad Cap_p(F \cup H) = 0.$$

Note that G_k and H_n are open sets. By outer regularity (Proposition 1.6) $F \cup H_n$ is included in an open set W_n of arbitrarily small capacity if n is large enough. If $x \notin F \cup H_n$, we have that

$$\text{for all } k \geq n \qquad |\mathcal{G}f_{j_k}(x) - \mathcal{G}f(x)| \leq \mathcal{G}|f_{j_k} - f|(x) \leq 2^{-k} \leq 2^{-n}, \tag{1.3}$$

which proves the uniform convergence in W_n. Furthermore, if $x \notin F \cup H$ there exists $n(x) \in \mathbb{N}$ so that $x \notin F \cup H_{n(x)}$ and again for (1.3) we get the pointwise convergence outside a set of null capacity. $\square$

It would be better if the infimum in the definition of capacity were actually a minimum. The problem is that Ω_A is not a closed subset of $L^p(\nu)$.

Lemma 1.10 ([AH] Proposition 2.3.9)**.** *Let $1 \leq p < +\infty$ and $A \subset X$. Then*

$$\overline{\Omega}_A = \{f \in L^p_+(\nu) | \mathcal{G}f(x) \geq 1 \ Cap_p\text{-a.e. on } A\}. \tag{1.4}$$

Proof. Let $\mathcal{A}$ be the set on the right hand side of (1.4). We first prove that $\mathcal{A}$ is closed in $L^p(\nu)$. From this we could deduce that $\overline{\Omega}_A \subset \mathcal{A}$. Let $\{f_j\}_{j=1}^{+\infty} \subset \mathcal{A}$ be a sequence converging to f strongly. From Proposition 1.9, we suppose that $\lim\limits_{j \longrightarrow +\infty} \mathcal{G}f_j(x) = \mathcal{G}f(x)$ (by possibly extracting a subsequence). Each $\mathcal{G}f_j$ is greater than 1 on $A \setminus N_j$ with N_j of null capacity, so $\mathcal{G}f(x) \geq 1$ if $x \in A \setminus \left(\bigcup\limits_{j=1}^{+\infty} N_j \right)$. The union of N_j has null capacity (for σ-sub-additivity), hence $\mathcal{G}f(x) \geq 1$ Cap_p-a.e. Obviously f is positive, so we have just proven that $f \in \mathcal{A}$.

Now it is enough to show the inclusion $\mathcal{A} \subset \overline{\Omega}_A$. Let $f \in \mathcal{A}$: we are looking for a sequence in Ω_A which converges to f in $L^p(\nu)$-norm. We denote by N the subset of A with $Cap_p(N) = 0$ where $\mathcal{G}f < 1$. Proposition 1.8 says that there exists $h \in L^p_+(\nu)$ such that $\mathcal{G}h \equiv +\infty$ on N. Let $0 < \tau \leq \|h\|_{L^p(\nu)} \in]0, +\infty[$; for all $j \in \mathbb{N}$ we consider the functions $\widetilde{f}_j := f + (\tau j)^{-1}h$. Then $\|\widetilde{f}_j - f\|_{L^p(\nu)} \leq j^{-1}$ for each j and

$$\mathcal{G}\widetilde{f}_j(x) = \begin{cases} \text{if } x \in A \setminus N & \underbrace{\mathcal{G}f(x)}_{\geq 1} + (\tau j)^{-1} \underbrace{\mathcal{G}h(x)}_{\geq 0} \geq 1 + 0 = 1 \\ \text{if } x \in N & \underbrace{\mathcal{G}f(x)}_{\geq 0} + (\tau j)^{-1} \underbrace{\mathcal{G}h(x)}_{=+\infty} = +\infty \geq 1 \end{cases}.$$

Thus the $\widetilde{f}_j$ are in Ω_A. $\square$

Theorem 1.11 ([AH] Theorem 2.3.10)**.** *Let $1 < p < +\infty$ and $A \subset X$ with $Cap_p(A) < +\infty$. Then there exists a unique* [3] *function $f^A \in L^p_+(\nu)$ such that*

[3] We identify functions in $L^p(\nu)$ if they differ on a set of null ν-measure, as usual.

$\mathcal{G}f(x) \geq 1$ Cap_p*-a.e. on* A *and*

$$Cap_p(A) = \|f^A\|^p_{L^p(\nu)}.$$

Proof. Since $1 < p < +\infty$, $L^p(\nu)$ is uniformly convex and Ω_A is a nonempty convex subset (it is not empty as the capacity is finite, while convexity holds because of linearity of $\mathcal{G}$). By Proposition 1.3, there exists a unique element f^A in $\overline{\Omega}_A$ such that

$$\|f^A\|^p_{L^p(\nu)} = \inf_{f \in \Omega_A} \|f\|^p_{L^p(\nu)} = Cap_p(A). \qquad \square$$

The function f^A is called the *equilibrium* (or *capacitary*) *function* associated with the subset A of X.

Remark 1.4. To define capacity, we have considered only nonnegative functions, but this is not restrictive. Indeed, if $f \in L^p(\nu)$ with $\mathcal{G}f \geq 1$ on a subset $A \subset X$, then $f^+ \in L^p_+(\nu)$, $\mathcal{G}f^+ \geq 1$ on A and $\|f^+\|_{L^p(\nu)} \leq \|f\|_{L^p(\nu)}$.

1.3 Dual definition of capacity

Capacities of compact sets may be defined by means of positive measures on X, instead of functions on the metric space (M, ν).

Theorem 1.12 ([AH] Theorem 2.5.1)**.** *Let* $K \subset X$ *be compact,* $1 < p < +\infty$ *and* p' *its Hölder-conjugate exponent. Then*

$$Cap_p(K)^{1/p} = \sup\{\mu(K) | \, \mu \in \mathcal{M}_+(K), \|\check{\mathcal{G}}\mu\|_{L^{p'}(\nu)} \leq 1\}. \tag{1.5}$$

Proof. Let $A \subset K$ and $f \in \Omega_A$. Then for all $\mu \in \mathcal{M}_+(K)$

$$\mu(A) \leq \int_A \mathcal{G}f(x) d\mu(x) \leq \int_X \mathcal{G}f(x) d\mu(x) \overset{\text{(Fubini)}}{=} \int_M \check{\mathcal{G}}\mu(y) f(y) d\nu(y) \leq$$

$$\overset{\text{(Hölder)}}{\leq} \|\check{\mathcal{G}}\mu\|_{L^{p'}} \|f\|_{L^p},$$

which leads to the inequality

$$\mu(A) \leq \|\check{\mathcal{G}}\mu\|_{L^{p'}} Cap_p(A)^{1/p}, \tag{1.6}$$

which is interesting in itself. By considering $A = K$ we get the inequality

$$Cap_p(K)^{1/p} \geq \sup\{\mu(K) \,|\, \mu \in \mathcal{M}_+(K), \|\check{\mathcal{G}}\mu\|_{L^{p'}} \leq 1\}.$$

Our real aim is to prove the equality. In order to do that, we define the convex spaces $\mathcal{X} = \{\mu \in \mathcal{M}_+(K) \,|\, \mu(K) = 1\}$ and $\mathcal{Y} = \{f \in L^p_+(\nu) \,|\, \|f\|_{L^p} \leq 1\}$. Considering the weak* topology, $\mathcal{X}$ is compact[4] and the map $\mathcal{E}$

$$\mathcal{X} \times \mathcal{Y} \longrightarrow [0, +\infty], \quad (\mu, f) \longmapsto \mathcal{E}(\mu, f)$$

is linear in each entry and LSC in the first one (by Proposition 1.4(3)). Then all the hypothesis of the Minimax Theorem ([AH], Theorem 2.4.1) are satisfied, so we can say that

$$\min_{\mu \in \mathcal{X}} \sup_{f \in \mathcal{Y}} \mathcal{E}(\mu, f) = \sup_{f \in \mathcal{Y}} \min_{\mu \in \mathcal{X}} \mathcal{E}(\mu, f). \tag{1.7}$$

Now by the correspondence $(L^p)^* \cong L^{p'}$ we have

$$\sup_{f \in \mathcal{Y}} \mathcal{E}(\mu, f) = \sup_{\|f\|_{L^p} \leq 1} \int_M \check{\mathcal{G}}\mu(y) f(y) d\nu(y) = \|\check{\mathcal{G}}\mu\|_{L^{p'}}$$

and by homogeneity

$$\min_{\mu \in \mathcal{X}} \sup_{f \in \mathcal{Y}} \mathcal{E}(\mu, f) = \min_{\mu \in \mathcal{M}_+(K), \, \mu \neq 0} \frac{\|\check{\mathcal{G}}\mu\|_{L^{p'}}}{\mu(K)}.$$

On the other side

$$\min_{\mu \in \mathcal{X}} \mathcal{E}(\mu, f) = \min_{\mu \in \mathcal{X}} \int_K \mathcal{G}f(x) d\mu(x) \geq \min_{x \in K} \mathcal{G}f(x) = \mathcal{G}f(x_0).^5$$

By considering the Dirac measure $\delta_{x_0} \in \mathcal{X}$ we get that

$$\min_{\mu \in \mathcal{X}} \mathcal{E}(\mu, f) = \min_{x \in K} \mathcal{G}f(x)$$

and by homogeneity

$$\sup_{f \in \mathcal{Y}} \min_{\mu \in \mathcal{X}} \mathcal{E}(\mu, f) = \sup_{f \in L^p_+} \frac{\min_K \mathcal{G}f}{\|f\|_{L^p}} = \sup_{f \in \Omega_K} \frac{1}{\|f\|_{L^p}} = Cap_p(K)^{-1/p}.$$

---

[4] It follows by Banach-Alaoglu Theorem observing that $\mathcal{M}(K) \cong \mathcal{C}(K)^*$. See [Br], Theorem 3.16 for the proof.

[5] A LSC map over a compact set has a minimum point.

The equality (1.7) holds, hence

$$Cap_p(K)^{1/p} = \sup_{\mu \in \mathcal{M}_+(K),\, \mu \neq 0} \frac{\mu(K)}{\|\check{\mathcal{G}}\mu\|_{L^{p'}}},$$

which leads to (1.5) by means of another homogeneity argument. □

Theorem 1.13 ([AH], Theorem 2.5.3)**.** *Let $K \subset X$ be compact and $1 < p < +\infty$. Suppose that $Cap_p(K) < +\infty$. There exists $\mu^K \in \mathcal{M}_+(K)$ such that*

$$f^K = (\check{\mathcal{G}}\mu^K)^{p'-1} \qquad \nu\text{-almost everywhere.} \tag{1.8}$$

where f^K is the equilibrium function associated with K, and

$$\mu^K(K) = \|\check{\mathcal{G}}\mu^K\|^{p'}_{L^{p'}(\nu)} = \int_K \mathcal{G}f^K(x)d\mu^K(x) = Cap_p(K). \tag{1.9}$$

μ^K is an equilibrium measure *(or* capacitary measure*) associated with K.*

Proof. If $Cap_p(K) = 0$, there is nothing to prove.

By the preceding Theorem, we consider a maximizing sequence $\{\mu_j\}_{j=1}^{+\infty}$ such that $\|\check{\mathcal{G}}\mu_j\|_{L^{p'}} = 1$ for all j and $\lim_{j \longrightarrow +\infty} \mu_j(K) = Cap_p(K)^{1/p}$. Then we suppose that the sequence $\{\mu_j\}$ converges to $\mu_0 \in \mathcal{M}_+(K)$ with the weak* topology [6], so $\mu_0(K) = Cap_p(K)^{1/p}$. Now, by Proposition 1.4(2) and Fatou's Lemma, $\|\check{\mathcal{G}}\mu_0\|^{p'}_{L^{p'}} \leq \liminf_{j \longrightarrow +\infty} \|\check{\mathcal{G}}\mu_j\|^{p'}_{L^{p'}} = 1$. If $\|\check{\mathcal{G}}\mu_0\|_{L^{p'}} < 1$ held, then by picking a proper scalar multiple of μ_0 we would have an 'admissible' measure for the dual definition (1.5) whose relative value on K would be strictly greater that $Cap_p(K)^{1/p}$, leading to a contradiction: so $\|\check{\mathcal{G}}\mu_0\|_{L^{p'}} = 1$. By setting $\mu^K := \rho\mu_0$, where $\rho = Cap_p(K)^{1/p'}$, we get that

$$\mu^K(K) = \rho\mu_0(K) = \rho Cap_p(K)^{1/p} = Cap_p(K),$$

$$\|\check{\mathcal{G}}\mu^K\|^{p'}_{L^{p'}} = \rho^{p'} = Cap_p(K).$$

It remains to show that (1.8) holds: once proven, the third term of (1.9) would follow from the second one by using Fubini-Tonelli Theorem. Let

[6] The sequence is bounded, so there is a subsequence which converges weak* (Banach-Alaoglu).

$F := \{x \in K | \, \mathcal{G}f^K(x) < 1\}$. Then $Cap_p(F) = 0$ and from (1.6) - with F replacing A - we deduce that $\mu^K(F) = 0$. Thus

$$Cap_p(K) = \mu^K(K) \leq \int_K \mathcal{G}f^K(x) d\mu^K(x) = \int_M \check{\mathcal{G}}\mu^K(y) f^k(y) d\nu(y) \leq$$

$$\overset{\text{(Hölder)}}{\leq} \|\check{\mathcal{G}}\mu^K\|_{L^{p'}} \|f^K\|_{L^p} = Cap_p(K)^{1/p'} Cap_p(K)^{1/p} = Cap_p(K).$$

But then the equality holds when using Hölder, so we have that

$$(\check{\mathcal{G}}\mu^K)^{p'} = \alpha (f^K)^p \qquad \nu\text{-almost everywhere,} \tag{1.10}$$

with $\alpha > 0$ constant, unless one of the two terms is 0 ν-a.e.(then we are in the trivial case $Cap_p(K) = 0$). Otherwise we obtain $\alpha = 1$ by simply integrate (1.10), since we already know that $\int_M (f^K)^p \, d\nu$ and $\int_M (\check{\mathcal{G}}\mu^K)^{p'} \, d\nu$ are equal to $Cap_p(K)$. $\square$

Definition 1.7. Let $1 < p < +\infty$. It is defined a potential on X associated with the measure $\mu \in \mathcal{M}_+(X)$:

$$V_p^\mu(x) := (\mathcal{G}(\check{\mathcal{G}}\mu)^{p'-1})(x).$$

If $p \neq 2$, it is a *nonlinear* potential. If $\mu = \mu^K$ is the equilibrium measure of the compact set K, then $V_p^{\mu^K} = \mathcal{G}f^K$ and so

$$Cap_p(K) = \int_K V_p^{\mu^K}(x) d\mu^K(x).$$

Furthermore $V_p^{\mu^K}(x) \geq 1$ Cap_p-a.e.

Theorem 1.14 ([AH], Theorem 2.5.5)**.** *Let $K \subset X$ be a compact set, $1 < p < +\infty$ and $Cap_p(K) < +\infty$. Then*

- $V_p^{\mu^K}(x) \leq 1$ *for all* $x \in \operatorname{supp} \mu^K$,
- $Cap_p(K) = \max\{\mu(K) | \, \mu \in \mathcal{M}_+(K), V_p^\mu \leq 1$ *on* $\operatorname{supp} \mu\} = \mu^K(K)$.

Proof. Let $x_0 \in X$ be such that $V_p^{\mu^K}(x_0) = \mathcal{G}f^K(x_0) > 1$. Then, by considering the lower semi-continuity of $x \longmapsto \mathcal{G}f^K(x)$ [Proposition 1.4(1)], there exists a neighborhood $U \ni x_0$ where $\mathcal{G}f^K \geq 1 + \delta > 1$ for some $\delta > 0$. We know that $\mathcal{G}f^K \geq 1$ Cap_p-a.e. on K, so $\mathcal{G}f^K \geq 1$ μ^K-a.e. by means of (1.6). Thus

$$Cap_p(K) = \int_K \mathcal{G}f^K(x)d\mu^K(x) = \int_U \mathcal{G}f^K(x)d\mu^K(x) + \int_{K\setminus U} \mathcal{G}f^K(x)d\mu^K(x) \geq$$

$$\geq (1+\delta)\mu^K(U) + \mu^K(K\setminus U) = \mu^K(K) + \delta\mu^K(U) = Cap_p(K) + \delta\mu^K(U).$$

Since $\delta > 0$, $\mu^K(U) = 0$ and then $x_0 \notin \operatorname{supp}\mu^K$.

Now let $\mu \in \mathcal{M}_+(K)$ be a generic measure for which $V_p^\mu \leq 1$ on $\operatorname{supp}\mu$. Then

$$\|\check{\mathcal{G}}\mu\|_{L^{p'}}^{p'} = \int_M \check{\mathcal{G}}\mu(y)(\check{\mathcal{G}}\mu(y))^{p'-1}d\nu(y) \overset{\text{(Fubini)}}{=} \int_{\operatorname{supp}\mu} V_p^\mu(x)d\mu(x) \leq \mu(K),$$

so

$$\mu(K) \overset{(1.6)}{\leq} \|\check{\mathcal{G}}\mu\|_{L^{p'}} Cap_p(K)^{1/p} \leq \mu(K)^{1/p'} Cap_p(K)^{1/p}$$

$$\Longrightarrow \mu(K)^{1/p} \leq Cap_p(K)^{1/p} \Longrightarrow \mu(K) \leq Cap_p(K).$$

Hence the maximum is achieved for $\mu = \mu^K$ as $\mu^K(K) = Cap_p(K)$. □

1.4 Capacitability

We are interested in extending the dual definition to sets which are not necessarily compact.

Definition 1.8. A set $A \subset X$ is said to be Cap_p-*capacitable* if

$$Cap_p(A) = \sup\{Cap_p(K) |\, K \text{ compact}, K \subset A\}.$$

Lemma 1.15. *If* $\{K_j\}_{j=1}^{+\infty}$ *is a sequence of decreasing compact subsets of* X *and* $K := \bigcap_{j=1}^{+\infty} K_j$, *then*

$$\lim_{j\longrightarrow+\infty} Cap_p(K_j) = Cap_p(K).$$

Proof. If O is an open set containing K, then there exists an index j_0 such that $K_j \subset O$ for all $j \geq j_0$, so we deduce that

$$Cap_p(K) \leq \lim_{j\longrightarrow+\infty} Cap_p(K_j) \leq \inf_{O\supset K, O \text{ open}} Cap_p(O) = Cap_p(K)$$

by using outer regularity of Cap_p (Proposition 1.6). □

Lemma 1.16 ([AH] Proposition 2.3.12). *Let $1 < p < +\infty$. If $\{A_j\}_{j=1}^{+\infty} \subset X$ is an increasing sequence of sets, and $A := \bigcup_{j=1}^{+\infty} A_j$, then*

$$\lim_{j \longrightarrow +\infty} Cap_p(A_j) = Cap_p(A).$$

Furthermore, if $Cap_p(A)$ is finite, the sequence of equilibrium functions f^{A_j} converges to f^A strongly in $L^p(\nu)$.

Proof. Since $A_j \subset A$ for each j, $\lim_{j \longrightarrow +\infty} Cap_p(A_j) \leq Cap_p(A)$. So we suppose that $\ell := \lim_{j \longrightarrow +\infty} Cap_p(A_j) < +\infty$, otherwise the proof is trivial. Let $\{f^{A_j}\}$ be the sequence of equilibrium functions of A_j. As the sets $\overline{\Omega}_{A_j}$ are decreasing,

$$f^{A_j}, f^{A_k} \in \overline{\Omega}_{A_{j \wedge k}} \Longrightarrow \frac{f^{A_j} + f^{A_k}}{2} \in \overline{\Omega}_{A_{j \wedge k}},$$

for every couple of indices $j, k \in \mathbb{N}$, with $j \wedge k = \min\{j, k\}$. Then

$$\left\| \frac{f^{A_j} + f^{A_k}}{2} \right\|_{L^p}^p \geq Cap_p(A_{j \wedge k}) \Longrightarrow \liminf_{j,k \longrightarrow +\infty} \left\| \frac{f^{A_j} + f^{A_k}}{2} \right\|_{L^p}^p \geq \ell.$$

By Lemma 1.2 we deduce that the functions f^{A_j} converge (strongly) to $\widetilde{f}$ in $L^p(\nu)$, with $\widetilde{f} \geq 0$ and $\|\widetilde{f}\|_{L^p}^p = \ell$. Moreover, $\mathcal{G}\widetilde{f} \geq 1$ on each A_j, except for a set of null capacity, so it follows that $\mathcal{G}f \geq 1$ Cap_p-a.e. on A. Hence $\widetilde{f} \in \overline{\Omega}_A$ and

$$\ell \leq Cap_p(A) \leq \|\widetilde{f}\|_{L^p}^p = \ell :$$

thus $\ell = Cap_p(A)$ and $\widetilde{f} = f^A$ by uniqueness of the equilibrium function. $\square$

We showed that Cap_p is a nonnegative function on subsets of X such that

- $Cap_p(\emptyset) = 0$;
- $A \subset \widetilde{A} \Longrightarrow Cap_p(A) \leq Cap_p(\widetilde{A})$;
- $\lim_{j \longrightarrow +\infty} Cap_p(K_j) = Cap_p\left(\bigcap_{j=1}^{+\infty} K_j\right)$ for every sequence $\{K_j\}$ of decreasing compact sets.
- $\lim_{j \longrightarrow +\infty} Cap_p(A_j) = Cap_p\left(\bigcup_{j=1}^{+\infty} A_j\right)$ for every sequence $\{A_j\}$ of increasing sets.

These are all the hypothesis for the Capacitability Theorem by G. Choquet (see [Cho]):

Theorem 1.17. *All Borel sets are Cap_p-capacitable.*

Corollary 1.18 ([AH], Corollary 2.5.2)**.** *Whenever $A \subset X$ is a Borel set, we have that*

$$Cap_p(A)^{1/p} = \sup\{\mu(A) \,|\, \mu \in \mathcal{M}_+(X),\, \operatorname{supp}\mu \subset A,\, \|\check{\mathcal{G}}\mu\|_{L^{p'}(\nu)} \le 1\}. \quad (1.11)$$

Proof. If there exists $\mu \in \mathcal{M}_+(X)$ supported is in A, such that $\|\check{\mathcal{G}}\mu\|_{L^{p'}} \le 1$ and $\mu(A) = +\infty$, then $Cap_p(A) = +\infty$ by (1.6). Otherwise all the measures we consider are inner regular with respect to A:

$$\mu(A) = \sup\{\mu(K) \,|\, K \text{ compact, } K \subset A\}.$$

It follows that the right hand side of (1.11) is equal to

$$\sup_{K \subset A,\, K \text{ compact}} \Big\{ \sup\{\mu(K) \,|\, \mu \in \mathcal{M}_+(K),\, \|\check{\mathcal{G}}\mu\|_{L^{p'}(\nu)} \le 1\} \Big\} =$$

$$\text{(by Theorem 1.12)} = \sup_{K \subset A,\, K \text{ compact}} Cap_p(K)^{1/p} = Cap_p(A)^{1/p},$$

since A is Cap_p-capacitable. □

Now we are ready to extend the notion of equilibrium measure to sets which are not compact. Some further hypothesis on the measure space (M, ν) and the kernel g are required.

Theorem 1.19 ([AH], Theorem 2.5.6)**.** *Let $A \subset X$ be such that $Cap_p(A) < +\infty$, where $1 < p < +\infty$. Suppose that M is a locally compact topological space and the kernel g is chosen so that for every $\psi \in \mathcal{C}_c(M)$:*

i) $x \longmapsto \mathcal{G}\psi(x)$ *is continuous on X;*

ii) $\lim_{|x| \longrightarrow +\infty} \mathcal{G}\psi(x) = 0.$

Then there exists a measure $\mu^A \in \mathcal{M}_+(\overline{A})$, corresponding to A, which is related to the equilibrium function by

$$f^A = (\check{\mathcal{G}}\mu^A)^{p'-1} \qquad \nu\text{-a.e.}$$

and such that the following properties are satisfied:

$$V_p^{\mu^A}(x) = \mathcal{G}f^A(x) \geq 1 \quad Cap_p\text{-a.e. on } A;$$

$$V_p^{\mu^A}(x) = \mathcal{G}f^A(x) \leq 1 \quad \text{on } \operatorname{supp}\mu^A;$$

$$\mu^A(\overline{A}) = \|\check{\mathcal{G}}\mu^A\|_{L^{p'}(\nu)}^{p'} = \int_X V_p^{\mu^A}(x)d\mu^A(x) = Cap_p(A).$$

Proof. Let $\{O_i\}_{i=1}^{+\infty}$ be a decreasing sequence of open sets containing A such that $Cap_p(O_i) \downarrow Cap_p(A)$ as $i \longrightarrow +\infty$. This can be done since Cap_p is outer regular (Proposition 1.6). We define $G := \overline{A} \cap \left(\bigcap_{i=1}^{+\infty} O_i\right)$. Then $A \subset G \subset \overline{A}$ and

$$Cap_p(A) \leq Cap_p(G) \leq \inf\{Cap_p(\overline{A}),\, Cap_p(O_i),\, i \geq 1\} = Cap_p(A),$$

implying that $Cap_p(G) = Cap_p(A)$. G is a Borel set, so it is Cap_p-capacitable (Theorem 1.17) and there exists an increasing sequence of compact sets $K_j \subset G$ such that $Cap_p(G) = Cap_p(Q)$, where $Q := \bigcup_{j=1}^{+\infty} K_j \subset G$. The equilibrium function f^G belongs to both $\overline{\Omega}_A$ and $\overline{\Omega}_Q$, so $\|f^G\|_{L^p}^p = Cap_p(G) = Cap_p(A) = Cap_p(Q)$. This implies that $f^G = f^A = f^Q$ in $L^p(\nu)$, as they have the same norm. By Theorem 1.13 we can pick the equilibrium measures μ^{K_j}:

$$Cap_p(K_j) = \mu^{K_j}(K_j), \qquad f^{K_j} = (\check{\mathcal{G}}\mu^{K_j})^{p'-1}\ \nu\text{-a.e.} \quad \text{for all } j \geq 1.$$

All the quantities $\mu^{K_j}(\overline{A}) = \mu^{K_j}(K_j)$ are bounded by $Cap_p(Q)$. By possibly extracting a subsequence, the sequence μ^{K_j} has a weak* limit $\mu_* \in \mathcal{M}_+(\overline{A})$, so that $\mu_*(\overline{A}) \leq Cap_p(Q) = Cap_p(A)$. Let $\psi \in \mathcal{C}_c(M)$; the restriction $\mathcal{G}\psi|_{\overline{A}}$ can be approximated by functions in $\mathcal{C}_c(\overline{A})$, so

$$\lim_{j \longrightarrow +\infty} \mathcal{E}(\mu^{K_j}, \psi) = \lim_{j \longrightarrow +\infty} \int_{\overline{A}} \mathcal{G}\psi\, d\mu^{K_j} = \int_{\overline{A}} \mathcal{G}\psi\, d\mu_* = \int_M \check{\mathcal{G}}\mu_*\psi\, d\nu.$$

On the other side, the equilibrium function f^Q ($=f^A$ ν-a.e.) is the strong limit in $L^p(\nu)$ of the functions f^{K_j} by Lemma 1.16, and thus $\{(f^{K_j})^{p-1}\}$ is a bounded sequence in $L^{p'}(\nu)$. Up to subsequences, we can say that $\{(f^{K_j})^{p-1}\}$ converges weakly to $(f^A)^{p-1}$ in $L^{p'}(\nu)$, hence

$$\lim_{j \longrightarrow +\infty} \mathcal{E}(\mu^{K_j}, \psi) = \lim_{j \longrightarrow +\infty} \int_M \check{\mathcal{G}}\mu^{K_j}\psi\, d\nu = \lim_{j \longrightarrow +\infty} \int_M (f^{K_j})^{p-1}\psi\, d\nu =$$

$$= \int_M (f^A)^{p-1} \psi \, d\nu.$$

Then

$$\int_M \check{\mathcal{G}}\mu_* \psi \, d\nu = \int_M (f^A)^{p-1} \psi \, d\nu \quad \text{for all } \psi \in \mathcal{C}_c(M)$$

$$\Longrightarrow \check{\mathcal{G}}\mu_* = (f^A)^{p-1} \ \nu\text{-a.e.} \Longrightarrow (\check{\mathcal{G}}\mu_*)^{p'-1} = f^A \ \nu\text{-a.e.}$$

It follows that

$$Cap_p(A) = \|f^A\|_{L^p}^p = \|\check{\mathcal{G}}\mu_*\|_{L^{p'}}^{p'} = \int_X V_p^{\mu_*} \, d\mu_*$$

and

$$V_p^{\mu_*} = \mathcal{G}f^A \geq 1 \ Cap_p\text{-a.e. on } A.$$

Let $x_* \in \operatorname{supp} \mu_*$. Every open neighborhood of x_* has nonzero μ_*-measure, and so it has nonzero μ^{K_j}-measure definitively. By possibly passing to a subsequence of $\{K_j\}$, we can choose $x_j \in \operatorname{supp} \mu^{K_j}$ such that $x_j \longrightarrow x_*$, and we may also suppose that $f^{K_j}(y) \longrightarrow f^A(y)$ (ν-a.e.). Since the kernel $g(x, y)$ is LSC with respect to x,

$$V_p^{\mu_*}(x_*) = \mathcal{G}f^A(x_*) \leq \int_M \liminf_{j \longrightarrow +\infty} \left[g(x_j, y) f^{K_j}(y) \right] d\nu(y) \leq$$

$$\overset{\text{(Fatou)}}{\leq} \liminf_{j \longrightarrow +\infty} \mathcal{G}f^{K_j}(x_j) = \liminf_{j \longrightarrow +\infty} V_p^{\mu^{K_j}}(x_j) \leq 1.$$

The last inequality is a consequence of Theorem 1.14. So $V_p^{\mu_*} \leq 1$ on the support of μ_*. Eventually we obtain

$$\mu_*(\overline{A}) \leq Cap_p(A) = \int_X V_p^{\mu_*} \, d\mu_* = \int_{\operatorname{supp} \mu_*} V_p^{\mu_*} \, d\mu_* \leq \mu_*(\overline{A}),$$

hence $Cap_p(A) = \mu_*(\overline{A})$ and $\mu^A := \mu_*$ is a measure with all the properties we were looking for. $\square$

Chapter 2

ℓ^p-Capacities on the boundary of a tree

2.1 Trees with a root: main definitions

Generally, a *tree* is a connected graph without cycles[1]. The trees we consider in this chapter are more specific. Let $T = (V(T), E(T))$ be a tree, with a fixed root vertex o, from which starts a unique root edge ω. We suppose that the degree of the vertices is uniformly bounded: for every $x \in V(T)$ we have exactly one edge that ends at x, while the edges (sons) starting from x are more than one and their number is smaller than a uniform constant N.

Definition 2.1. Here are some notations and basic definitions.

- If x and y are two adjacent points of the tree, xy denotes the edge in $E(T)$ linking x and y. It is not oriented: $yx = xy$.

- A *path* between two vertices x and y is a finite sequence of edges $\alpha_1, \ldots, \alpha_n$ with $\alpha_j = x_{j-1}x_j$, $x_0 = x$, $x_n = y$. If we consider a path between x and y with no repeated edges, it is unique since we have no cycles on the tree. It is called the *geodesic* from x to y and it is denoted

[1] That is the usual definition given in graph theory. A graph with no cycles but more than one connected component is called a *forest*. For basic definitions of graph theory see for example [Di].

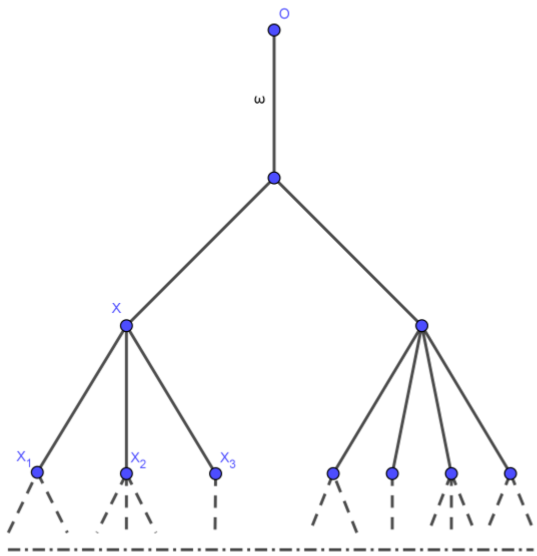

Figure 2.1: A tree with root vertex o and a root edge ω

by $[x, y]$. For adjacent vertices the geodesic $[x, y]$ reduces to the linking edge xy.

- We say that $x \leq y$ if x is the extreme of an edge along $[o, y]$: so we have a partial order on $V(T)$. From this order we can distinguish the extremes of each $\alpha \in E(T)$, the *beginning* $\mathrm{b}(\alpha)$ and the *end* $\mathrm{e}(\alpha)$, by imposing $\mathrm{b}(\alpha) < \mathrm{e}(\alpha)$.

- The *predecessors* of a vertex x are all the edges crossed by the geodesic between x and the root:

$$P(x) = [o, x].$$

  If $x \leq y$, then we have the inclusion $P(x) \subset P(y)$.

- Also the set of edges $E(T)$ has a natural partial order: we say that $\alpha \leq \beta$ if $\alpha \in P(\mathrm{e}(\beta))$.

- Let x, $y \in V(T)$; there exists a unique $z \in V(T)$ such that

$$P(z) = P(x) \cap P(y).$$

  We use the notation $x \wedge y$ for z. It is called the *confluent* of x and y.

2.2 The boundary of a tree as a metric space

Definition 2.2 (Natural distance)**.** A natural distance d on vertices of T is defined by counting the edges along the geodesic from a vertex to another one:

$$d : V(T) \times V(T) \longrightarrow \mathbb{R}_+ \qquad d(x, y) = \#[x, y].^2$$

We use the notation $d(x)$ for $d(o, x)$. It is easy to prove that d is a distance: for example, triangular inequality is simply proven by observing that $[x, y] \subset [x, z] \cup [y, z]$ for all x, y, $z \in V(T)$.

Definition 2.3 (Half-infinite geodesic)**.** Let $x \in V(T)$. A *half-infinite geodesic* starting from x is the union of geodesics

$$\bigcup_{n=1}^{+\infty} [x, y_n], \qquad \text{with } d(x, y_n) \overset{n \longrightarrow +\infty}{\longrightarrow} +\infty.$$

Let $\{x_n\}_{n=0}^{+\infty}$ be a sequence of vertices such that $x_0 = x$, $x_{n+1} > x_n$ and $x_n x_{n+1} \in E(T)$. The vertices x_n are exactly the extremes of a unique half-infinite geodesic (from x), so we will identify it with the sequence of vertices. The set of half-infinite geodesic starting from the root o is denoted by $\mathcal{H}_o$.

Remark 2.1. The $\{x_n\}_{n=0}^{+\infty}$ above are not Cauchy sequences on the metric space $(V(T), d)$, because $d(x_n, x_{n+1}) = 1$ for all $n \in \mathbb{N}$. We are looking for another distance, in order to define limits of half-infinite geodesics.

Definition 2.4. Let $0 < \delta < 1$ be a fixed constant. Then we define

$$\rho : V(T) \times V(T) \longrightarrow \mathbb{R}_+ \qquad \rho(x, y) = \sum_{\alpha \in [x,y]} \delta^{d(\mathrm{e}(\alpha))}.$$

Since d is a distance, it is easy to check that also ρ is a distance.

Remark 2.2. The metric space $(V(T), \rho)$ is bounded. Indeed for all x, $y \in V(T)$

$$\rho(x, y) \leq \rho(o, x) + \rho(o, y) < 2 \cdot \sum_{j=1}^{+\infty} \delta^j = \frac{2\delta}{1 - \delta} < +\infty,$$

so we have an upper bound which is independent from x and y.

[2] We consider $[x, x] = \varnothing$, so $d(x, x) = 0$.

Now we consider the completion $\overline{T}$ of the metric space $(V(T), \rho)$. The *boundary* of T is

$$\partial T := \overline{T} \setminus V(T).$$

The metric on $\overline{T}$ (and its subset ∂T) is just obtained from ρ by simply passing to the limit. We still use the same symbol ρ. By Remark 2.2, $\overline{T}$ and ∂T are bounded metric spaces.

Proposition 2.1. *There exists a bijective map*

$$J : \mathcal{H}_o \longrightarrow \partial T \qquad J(\{x_n\}) = \lim_{n \longrightarrow +\infty} x_n.$$

Proof. We split the proof into three parts.

1. Let $\{x_n\}_{n=0}^{+\infty} \in \mathcal{H}_o$. We first prove that it is a Cauchy sequence. Indeed, if $n < m$, then

$$\rho(x_n, x_m) = \sum_{j=n+1}^{m} \delta^j < \sum_{j=n+1}^{+\infty} \delta^j \longrightarrow 0 \quad \text{as } n, m \longrightarrow +\infty \text{ [3]}$$

since $d(x_j) = j$ for each j. Now let $\zeta \in \overline{T}$, $\zeta := \lim_{n \longrightarrow +\infty} x_n$. In order to show that the map J is well defined it remains to prove that $\zeta \in \partial T$. By contradiction, if ζ were in $V(T)$, then it would be definitively $x_n \neq \zeta$ and $d(x_n) > d(\zeta)$, because $\{x_n\}$ is strictly increasing; thus we would get

$$0 < \delta^{d(\zeta)+1} \leq \rho(x_n, \zeta) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty,$$

which is absurd.

2. Now we prove that J is a one-to-one map. Suppose that $\{x_n\}, \{y_n\} \in \mathcal{H}_o$ converge to the same limit point $\zeta \in \partial T$. Then $\rho(x_n, y_n) \geq \rho(x_{n-1}, y_{n-1})$ for all $n \geq 1$. The equality is obvious if $x_n = y_n$, because then $x_{n-1} = y_{n-1}$ since there is only one edge ending in x_n; otherwise $[x_n, y_n]$ must cross the vertices x_{n-1} and y_{n-1}, so we get that $[x_n, y_n] \supset [x_{n-1}, y_{n-1}]$ and the inequality holds.

[3] Because it is the tail of a convergent series.

The distance $\rho(x_n, y_n)$ grows as n increases, but on the other side $\rho(x_n, y_n)$ tends to 0 because the sequences $\{x_n\}$ and $\{y_n\}$ converge to the same limit ζ. Thus

$$\rho(x_n, y_n) \equiv 0 \Longrightarrow x_n = y_n \text{ for all } n.$$

Briefly, two half-infinite geodesics with the same limit coincide.

3. It remains to show that J maps $\mathcal{H}_o$ onto ∂T. Let $\zeta \in \partial T$: then it is the limit of a Cauchy sequence $\{y_n\} \subset V(T)$. By possibly extracting a subsequence we can suppose that for all $n \geq 0$

$$m > n \Longrightarrow \rho(y_n, y_m) < \delta^{d(y_n)+1}. \tag{2.1}$$

We define $\widetilde{x}_n := y_n \wedge y_{n+1}$ for each $n \geq 0$. First of all, $\rho(\widetilde{x}_n, y_n) = \rho(y_n, y_n \wedge y_{n+1}) \leq \rho(y_n, y_{n+1}) \longrightarrow 0$ and then

$$\lim_{n \longrightarrow +\infty} \widetilde{x}_n = \lim_{n \longrightarrow +\infty} y_n = \zeta.$$

Furthermore the inequality $\widetilde{x}_n \leq \widetilde{x}_{n+1}$ holds. Otherwise, since $\widetilde{x}_n$, $\widetilde{x}_{n+1}$ are both in $P(y_{n+1})$, it would be $\widetilde{x}_n > \widetilde{x}_{n+1}$, so we could deduce that $[y_{n+1}, y_{n+2}]$ contains all the edges of $[\widetilde{x}_{n+1}, \widetilde{x}_n]$ and

$$\delta^{d(y_{n+1})} \leq \delta^{d(\widetilde{x}_n)} \leq \rho(y_{n+1}, y_{n+2}) \overset{(2.1)}{<} \delta^{d(y_{n+1})+1},$$

but $\delta < 1$ and we would get a contradiction.

Moreover

$$\delta^{d(\widetilde{x}_n)+1} \leq \rho(y_n, y_{n+1}) \longrightarrow 0,$$

implying that $d(\widetilde{x}_n) \overset{n \longrightarrow +\infty}{\longrightarrow} +\infty$. Then $\Gamma := \bigcup_n [0, \widetilde{x}_n] \in \mathcal{H}_o$; by possibly removing repetitions and adding all the other vertices along Γ we get the sequence $\{x_n\}_{n=0}^{+\infty}$ of adjacent vertices descending from the root which uniquely represents Γ. We already know that it is a Cauchy sequence containing a subsequence whose limit is $\zeta \in \partial T$, so

$$J(\{x_n\}) = \lim_{n \longrightarrow +\infty} x_n = \zeta.$$

$\{x_n\}$ represents a geodesic in $\mathcal{H}_o$ and ζ is a generic element of the boundary of T, so we proved that J is onto ∂T.

In conclusion, the map J is well defined, injective and surjective. $\square$

Definition 2.5. In the context of the 'extended' tree $\overline{T}$ we can give some further definitions.

- The notion of predecessors can be extended to points of the boundary: if $\zeta \in \partial T$ we call $P(\zeta) := J^{-1}(\zeta)$, i.e. the predecessors of ζ are the edges along the half-infinite geodesics associated with ζ. The notation $[o, \zeta]$ is another way to indicate this geodesic.

- In a dual sens, the set of *successors* of an edge $\alpha \in E(T)$ is

$$S(\alpha) := \{\zeta \in \overline{T} |\, \alpha \in P(\zeta)\}.$$

- If $\alpha \in E(T)$, we define $\partial T_\alpha := \partial T \cap S(\alpha)$: it is the boundary of the subtree T_α given by the vertices of $S(\alpha) \cup \{\mathrm{b}(\alpha)\}$ and the edges of $E(T)$ between them; α is the root edge.

- The notion of confluent can be easily extended to the boundary. If $\zeta, \xi \in \overline{T}$, $\zeta \wedge \xi$ denotes the vertex $z \in V(T)$ such that $P(z) = P(\zeta) \cap P(\xi)$. The confluent of two distinct points of the boundary is just the last vertex on which the two half-infinite geodesics coincide.

Remark 2.3. It is not difficult to calculate the expression for the distance ρ between two point ζ, $\xi \in \partial T$: the edges on which the sum is performed are the ones along $[\zeta, \xi] = [\zeta \wedge \xi, \zeta] \cup [\zeta \wedge \xi, \xi]$, so

$$\rho(\zeta, \xi) = 2 \cdot \sum_{j=d(\zeta\wedge\xi)+1}^{+\infty} \delta^j = \frac{2\delta^{d(\zeta\wedge\xi)+1}}{1-\delta} = \left(\frac{2\delta}{1-\delta}\right) \delta^{d(\zeta\wedge\xi)}.$$

Remark 2.4. If ζ, ξ, $\eta \in \partial T$ is a generic third of points, $\zeta \wedge \eta$ and $\xi \wedge \eta$ are comparable since they both lie on $P(\eta)$:

- If $\zeta \wedge \eta > \xi \wedge \eta$, then the geodesic ξ separates from ζ at the very moment it separates from η. Thus

$$\zeta \wedge \xi = \xi \wedge \eta.$$

- If $\zeta \wedge \eta < \xi \wedge \eta$, we proceed in the same way, obtaining

$$\zeta \wedge \xi = \zeta \wedge \eta.$$

- If $\zeta \wedge \eta = \xi \wedge \eta$, ζ and ξ separates at the same vertex where they both separate from η, or at a successive point, so

$$\zeta \wedge \xi \geq \zeta \wedge \eta = \xi \wedge \eta.$$

Anyway

$$d(\zeta \wedge \xi) \geq \min\{d(\zeta \wedge \eta), d(\xi \wedge \eta)\}$$

and thus, since $\delta < 1$,

$$\rho(\zeta, \xi) \leq \left(\frac{2\delta}{1-\delta}\right) \delta^{\min\{d(\zeta\wedge\eta), d(\xi\wedge\eta)\}} = \left(\frac{2\delta}{1-\delta}\right) \max\{\delta^{d(\zeta\wedge\eta)}, \delta^{d(\xi\wedge\eta)}\}$$

$$\Longrightarrow \rho(\zeta, \xi) \leq \max\{\rho(\zeta, \eta), \rho(\xi, \eta)\}. \tag{2.2}$$

So we get an 'upgrade' of the triangular inequality for ρ: a metric satisfying (2.2) is usually called an *ultra-metric*.

2.2.1 Topological properties

Now we show some topological properties of the metric space $(\partial T, \rho)$.

Proposition 2.2. *For each* $\alpha \in E(T)$

$$\partial T_\alpha = \overline{B(\zeta, r_\alpha)} = B(\zeta, r_\alpha + \varepsilon_\alpha), \tag{2.3}$$

where ζ *is a generic point of* ∂T_α, $r_\alpha = \dfrac{2\delta^{d(\mathrm{e}(\alpha))+1}}{1-\delta}$ *and* $0 < \varepsilon_\alpha \leq 2\delta^{d(\mathrm{e}(\alpha))}$.

Proof. If $\xi \in \partial T_\alpha$, then

$$\rho(\zeta, \xi) \leq \rho(\mathrm{e}(\alpha), \zeta) + \rho(\mathrm{e}(\alpha), \xi) = 2 \cdot \sum_{j=d(\mathrm{e}(\alpha))+1}^{+\infty} \delta^j = \frac{2\delta^{d(\mathrm{e}(\alpha))+1}}{1-\delta} =$$

$$= r_\alpha < r_\alpha + \varepsilon_\alpha,$$

hence

$$\partial T_\alpha \subset \overline{B(\zeta, r_\alpha)} \subset B(\zeta, r_\alpha + \varepsilon_\alpha).$$

If $\xi \in \partial T \setminus \partial T_\alpha$, we can find $\beta \in E(T)$ such that $d(\mathrm{e}(\alpha)) = d(\mathrm{e}(\beta))$ and $\xi \in \partial T_\beta$, so

$$\rho(\zeta, \xi) \geq \sum_{j=d(\mathrm{e}(\alpha))}^{+\infty} \delta^j + \sum_{j=d(\mathrm{e}(\beta))}^{+\infty} \delta^j = r_\alpha + 2\delta^{d(\mathrm{e}(\alpha))} \geq r_\alpha + \varepsilon_\alpha.$$

As a consequence

$$B(\zeta, r_\alpha + \varepsilon_\alpha) \subset \partial T_\alpha$$

and so equalities in (2.3) are proven. □

Remark 2.5. It is easy to see that if two balls of ∂T have nonempty intersection, then each point belonging to both sets is a common center, so we conclude that one ball is included in the other ball.

Corollary 2.3. *Every open ball B in the metric space $(\partial T, \rho)$ is closed and there exists an edge $\alpha \in E(T)$ such that $B = \partial T_\alpha$.*

Proof. Let $B = B(\zeta, r)$.

- If $r > \dfrac{2\delta}{1-\delta}$, then $B = \partial T = \partial T_\omega$ because the radius r is bigger than an upper bound for the diameter of the space (see Remark 2.2).
- Otherwise there exists $m \in \mathbb{N}$ such that

$$\frac{2\delta^{m+1}}{1-\delta} < r \leq \frac{2\delta^{m+1}}{1-\delta} + 2\delta^m = \frac{2\delta^m}{1-\delta}$$

and we can find an edge α so that $m = d(\mathrm{e}(\alpha))$. By Proposition 2.2 we infer that

$$B = \overline{B\left(\zeta, \frac{2\delta^{m+1}}{1-\delta}\right)} = \partial T_\alpha.$$ □

Remark 2.6. $\{\partial T_\alpha |\, \alpha \in E(T)\}$ is a base for the topology of ∂T, since it is the collection of all the balls of the metric space.

Corollary 2.4. *The metric space $(\partial T, \rho)$ is compact.*

Proof. We already know that the metric space ∂T is complete. It is also totally bounded: indeed the number of edges at a fixed distance d from the root is finite and so the space is covered by a finite number of balls of arbitrarily small radius. □

Proposition 2.5. *The metric space $(\partial T, \rho)$ is totally disconnected.*

Proof. Let be ζ and ξ a generic pair of distinct points of ∂T. Then, since the corresponding half-infinite geodesics from the root are different, there exists an edge $\alpha \in P(\zeta) \setminus P(\xi)$. But then $\zeta \in \partial T_\alpha$, $\xi \in \partial T \setminus \partial T_\alpha$. ∂T_α is an open and closed ball, so $\partial T \setminus \partial T_\alpha$ is an open set. Thus any couple of points of the boundary lie on two distinct connected components of the space. It follows that all the connected components cannot have more than a single point. □

Proposition 2.6 (Isolated points)**.** *The following are equivalent:*

i) $\zeta \in \partial T$ is an isolated point.

ii) If ζ corresponds to the sequence $\{x_n\}_{n=0}^{+\infty}$, there exists $n_0 \geq 0$ such that x_n has only one son for all $n \geq n_0$.

Proof. *i)* $\Longrightarrow$ *ii)*
If ζ is isolated, then we can find a ball $B(\zeta, r) \in \partial T$ such that $B(\zeta, r) = \{\zeta\}$. By Corollary 2.3 there exists $\alpha \in P(\zeta)$ for which $\partial T_\alpha = \{\zeta\}$, so the vertices of T_α cannot have more than one son (otherwise ∂T_α would contain another geodesic $\xi \neq \zeta$). We can choose n_0 such that $x_{n_0} = \mathrm{b}(\alpha)$ and then *ii)* follows.

ii) $\Longrightarrow$ *i)*
If *ii)* holds, let $\alpha := x_{n_0} x_{n_0+1}$. Then ζ is the unique point of ∂T_α, which is an open ball centered in ζ: hence ζ is an isolated point. □

Remark 2.7. As a consequence, supposing that there are no points in ∂T satisfying the condition *ii)* of the last Proposition, ∂T is a *perfect metric space*: it is totally disconnected and all its points are not isolated.

2.3 ℓ^p-Capacities on trees

We consider the metric space $(\overline{T}, \rho)$, the space $E(T)$ equipped with the counting measure ν and the following kernel:

$$g(\zeta, \alpha) = \chi(\alpha \in P(\zeta)) = \chi(\zeta \in S(\alpha)).$$

Let $f : E(T) \longrightarrow \mathbb{R}_+$ and $\mu \in \mathcal{M}_+(\overline{T})$. According to the given kernel, we define the potentials

$$If : \overline{T} \longrightarrow [0, +\infty], \quad If(\zeta) := \int_{E(T)} g(\zeta, \alpha) f(\alpha) d\nu(\alpha) = \sum_{\alpha \in P(\zeta)} f(\alpha), \quad (2.4)$$

$$I^*\mu : E(T) \longrightarrow [0, +\infty], \quad I^*\mu(\alpha) := \int_{\overline{T}} g(\zeta, \alpha) d\mu(\zeta) = \int_{S(\alpha)} d\mu(\zeta), \quad (2.5)$$

$$\mathcal{E}(\mu, f) := \int_{\overline{T}} If(\zeta) d\mu(\zeta) = \sum_{\alpha \in E(T)} I^*\mu(\alpha) f(\alpha). \quad (2.6)$$

The notations I and I^* correspond to $\mathcal{G}$ and $\check{\mathcal{G}}$ of Chapter 1, Definition 1.4. For our purposes, we will consider measures defined on the closed subspace $\partial T \subset \overline{T}$. If $\mu \in \mathcal{M}_+(\partial T)$, then $I^*\mu(\alpha) = \mu(\partial T \cap S(\alpha)) = \mu(\partial T_\alpha)$.

Definition 2.6 (ℓ^p-capacity of a subset of ∂T)**.** Let $1 \leq p < +\infty$ and $A \subset \partial T$. As in Definition 1.5 we call $\Omega_A := \{f \in \ell^p_+(E(T)) | \, If \geq 1 \text{ on } A\}$ and

$$Cap_p(A) := \inf_{f \in \Omega_A} \|f\|^p_{\ell^p(E(T))}.$$

This definition is a particular case of the one given in Chapter 1. All the properties we showed in the general setting still hold.

Proposition 2.7. *Let $1 < p < +\infty$. If $A \subset \partial T$ is a finite or countable set, then $Cap_p(A) = 0$.*

Proof. Obviously $Cap_p(\emptyset) = 0$. If $A = \{\zeta\}$, we define

$$h : E(T) \longrightarrow [0, +\infty] \quad h(\alpha) = \frac{1}{d(\mathrm{e}(\alpha))} \chi_{P(\zeta)}(\alpha).$$

Then $Ih(\zeta) = \sum_{n=1}^{+\infty} n^{-1} = +\infty$ and $\|h\|^p_{\ell^p(E(T))} = \sum_{n=1}^{+\infty} n^{-p} < +\infty$. $Cap_p(\{\zeta\}) = 0$ by Proposition 1.8. Finally, if A is finite or countable, σ-sub-additivity can be used:

$$Cap_p(A) \leq \sum_{\zeta \in A} Cap_p(\{\zeta\}) = 0. \qquad \square$$

2.3.1 The equilibrium function

Theorem 2.8. *Let $A \subset \partial T$. If $1 < p < +\infty$, then there exists a unique function $f^A \in \ell^p_+(E(T))$ such that*

$$If \geq 1 \quad Cap_p\text{-}a.e.\ on\ A; \tag{2.7}$$

and

$$Cap_p(A) = \|f^A\|^p_{\ell^p(E(T))}.$$

f^A is the function of least ℓ^p-norm satisfying (2.7)*.*

Proof. It is easy to show that $\Omega_A \neq \emptyset$ and so $Cap_p(A) < +\infty$. So we can conclude by Theorem 1.11. □

The function f^A is the *equilibrium function* (or *capacitary function*) of A on T.

Remark 2.8. Some reduction can be made on f^A.

1. The support of f^A is included in $P(A) := \bigcup_{\zeta \in A} P(\zeta)$.

   Indeed, starting with a function f satisfying (2.7), we can consider $f_1 = f\chi_{P(A)}$. Then $If_1 \geq 1$ Cap_p-a.e. on A, because the sum defining the potential I is made on all the edges of $P(A)$. The ℓ^p-norm of f_1 is less than or equal to the ℓ^p-norm of f and $\operatorname{supp} f_1 \subset P(A)$.

2. $If^A = 1$ Cap_p-a.e. on A and $If^A \leq 1$ on the whole boundary.

   Let f be a function satisfying (2.7). Defining $\Phi(\zeta) := \min\{1, If(\zeta)\}$, it is easy to check that for each $\alpha \in E(T)$

   $$0 \leq \Phi(\mathrm{e}(\alpha)) - \Phi(\mathrm{b}(\alpha)) \leq If(\mathrm{e}(\alpha)) - If(\mathrm{b}(\alpha)).$$

   Indeed $If(\mathrm{b}(\alpha)) \leq If(\mathrm{e}(\alpha))$ and we distinguish three cases:

   - If $1 < If(\mathrm{b}(\alpha))$, then $\Phi(\mathrm{e}(\alpha)) - \Phi(\mathrm{b}(\alpha)) = 1 - 1 = 0$.
   - If $If(\mathrm{b}(\alpha)) \leq 1 < If(\mathrm{e}(\alpha))$, then

     $$\Phi(\mathrm{e}(\alpha)) - \Phi(\mathrm{b}(\alpha)) = 1 - If(\mathrm{b}(\alpha))$$

     which is ≥ 0 and $< If(\mathrm{e}(\alpha)) - If(\mathrm{b}(\alpha))$.

- Otherwise $\Phi(\mathrm{e}(\alpha)) - \Phi(\mathrm{b}(\alpha)) = If(\mathrm{e}(\alpha)) - If(\mathrm{b}(\alpha))$.

So we are allowed to write $\Phi = I\varphi$ where $0 \leq \varphi \leq f$ is a function on edges such that $I\varphi$ is equal to 1 Cap_p-a.e. on A. Since $\|\varphi\|_{\ell^p(E(T))} \leq \|f\|_{\ell^p(E(T))}$, we get the reduction.

3. We consider a generic edge α and its sons $\alpha_1, \ldots, \alpha_m$. Without loss of generality we suppose $m = 2$. Given f so that (2.7) holds, let a, b, $c \geq 0$ be the values f takes on α, α_1 and α_2 respectively. We try to change them in order to minimize the ℓ^p-norm of the function. Since we do not want to change the values of the potential I on the boundary we consider the constraints

$$\begin{cases} a + b = k_1 := f(\alpha) + f(\alpha_1) \\ a + c = k_2 := f(\alpha) + f(\alpha_2) \end{cases} .$$

So we are just looking for an element of least p-norm in the intersection of the straight line in $\mathbb{R}^3$ $\{(a,b,c) \in \mathbb{R}^3 | a + b = k_1, a + c = k_2\}$ with the positive octant $\{(a,b,c) \in \mathbb{R}^3 | a, b, c \geq 0\}$: it obviously exists. Invoking the method of Lagrange multipliers we easily get a necessary condition for the minimum.

$$\mathcal{L}(a,b,c,\lambda_1,\lambda_2) = a^p + b^p + c^p + \lambda_1(k_1 - a - b) + \lambda_2(k_2 - a - c)$$

$$\partial_a \mathcal{L} = 0 \quad \Longrightarrow \quad pa^{p-1} = \lambda_1 + \lambda_2$$

$$\partial_b \mathcal{L} = 0 \quad \Longrightarrow \quad pb^{p-1} = \lambda_1$$

$$\partial_c \mathcal{L} = 0 \quad \Longrightarrow \quad pc^{p-1} = \lambda_2.$$

$$\Longrightarrow \boldsymbol{a^{p-1} = b^{p-1} + c^{p-1}}.$$

This argument shows that the equilibrium function of a set A in the boundary has the following property:

$$\text{for all } \alpha \in E(T), \quad f(\alpha)^{p-1} = \sum_{\beta,\, \mathrm{e}(\alpha) = \mathrm{b}(\beta)} f(\beta)^{p-1} \tag{2.8}$$

(the sum is made on all the sons of α).

2.3.2 The equilibrium measure

The information about an equilibrium function can be fully recovered by its behavior 'near the boundary'. To be more precise, given a function f satisfying (2.8) we can define an associated measure μ on the boundary of the tree in this way:

$$\mu \in \mathcal{M}_+(\partial T) \qquad \mu(\partial T_\alpha) = I^*\mu(\alpha) = f(\alpha)^{p-1}. \tag{2.9}$$

The sets ∂T_α are a base for the topology of ∂T (see Remark 2.6). It is known from measure theory that two finite measures coinciding on a base of the Borel σ-algebra and on the whole space are the same (on Borel sets). But then μ defines a unique measure: from (2.8) we get all the compatibility relations we need in order to have monotonicity and σ-additivity of μ. Vice versa, given a positive measure μ on the boundary we can get a nonnegative function on edges f for which (2.8) holds:

$$f(\alpha) = (I^*\mu(\alpha))^{p'-1}$$

where p' is the Hölder-conjugate exponent of p.

Suppose that $A \subset \partial T$ denotes a closed (hence compact) set. Let μ^A be the measure associated with the equilibrium function f^A: since the support of f^A lies on $P(A)$, then it easily follows that $\operatorname{supp} \mu^A \subset \overline{A} = A$. From Theorem 1.13, since μ^A is the unique measure for which $f^A = (I^*\mu^A)^{p'-1}$, it follows that

$$Cap_p(A) = \mu^A(A) = I^*\mu^A(\omega) = \|I^*\mu^A\|^{p'}_{\ell^{p'}(E(T))}, \tag{2.10}$$

As in Chapter 1, μ^A is called the *equilibrium measure* (or *capacitary measure*) of A on T. Observe that we have the *uniqueness* of μ^A on Borel sets. The notion can be extended to arbitrary sets by Theorem 1.19 [4].

Remark 2.9. Observe that $Cap_p(A)^{p'-1} = (I^*\mu^A(\omega))^{p'-1} = f^A(\omega) \leq 1$ because $If^A \leq 1$ on A, so we get the upper bound (for $p > 1$)

$$Cap_p(A) \leq 1.$$

[4] We do not need to verify the hypothesis made on the kernel, since in this setting characteristic functions on balls are continuous and the equality $I^*\mu^A = (f^A)^{p-1}$ can be proven pointwise.

As in Definition 1.7, for each $\mu \in \mathcal{M}_+(\partial T)$ and $\zeta \in \overline{T}$ we define the potential

$$V_p^\mu(\zeta) := I(I^*\mu)^{p'-1}(\zeta).$$

If we pick the equilibrium measure μ^A associated with $A \subset \partial T$, then

$$V_p^{\mu^A}(\zeta) = If^A(\zeta) = 1 \qquad Cap_p\text{-a.e. on } A$$

and $V_p^{\mu^A}(\zeta) \leq 1$ on supp μ^A. This follows from the second reduction we made on f^A (so we do not need to invoke Theorem 1.14 and Theorem 1.19 from the general theory).

2.4 Rescaling on subtrees

Equilibrium functions and measures on subtrees of the form T_α can be obtained by making a proper rescaling of the ones relative to the whole tree.

Proposition 2.9. *Let $A \subset \partial T$, $\alpha \in E(T)$ and $A_\alpha := A \cap S(\alpha) = A \cap \partial T_\alpha$. Suppose that $A_\alpha \neq \emptyset$ and $If^A(\mathrm{b}(\alpha)) < 1$. Then the equilibrium function for the set A_α on the subtree T_α is*

$$f_\alpha^{A_\alpha} \in \ell_+^p(E(T_\alpha)), \quad f_\alpha^{A_\alpha} = \frac{f^A|_{E(T_\alpha)}}{c_\alpha}, \quad \textit{where } c_\alpha := 1 - If^A(\mathrm{b}(\alpha)) \in]0,1].$$

Proof. The function $c_\alpha^{-1} f^A$ is admissible as an equilibrium function for A_α on the subtree T_α. Indeed, if we denote I_α the potential referred to T_α (i.e. the sum defining the potential is performed only on the edges $\beta \geq \alpha$) then for each $\zeta \in A_\alpha$

$$If^A(\mathrm{b}(\alpha)) + I_\alpha f^A(\zeta) = If^A(\zeta) = 1 \qquad Cap_p\text{-a.e. on } A_\alpha,$$

$$I_\alpha(c_\alpha^{-1} f^A)(\zeta) = c_\alpha^{-1}(I_\alpha f^A(\zeta)) \overset{Cap_p-a.e.}{=} c_\alpha^{-1} \underbrace{(1 - If^A(\mathrm{b}(\alpha)))}_{c_\alpha} = 1.$$

It remains to prove that $c_\alpha^{-1} f^A$ is the admissible function with least ℓ^p-norm on T_α. By contradiction, let ψ be a function in $\ell_+^p(E(T_\alpha))$ so that $I_\alpha \psi \overset{Cap_p-a.e}{=} 1$ on A_α and

$$\sum_{\beta \geq \alpha} \psi(\beta)^p < \sum_{\beta \geq \alpha} (c_\alpha^{-1} f^A(\beta))^p. \tag{2.11}$$

We consider the map

$$h(\beta) := \begin{cases} c_\alpha \psi & \text{if } \beta \geq \alpha \\ f^A & \text{if } \beta \ngeq \alpha \end{cases}.$$

We calculate its potential Ih:

$$Ih(\zeta) = \begin{cases} If^A(\zeta) \overset{Cap_p-a.e.}{=} 1 & \text{if } \zeta \in A \setminus A_\alpha \\ If^A(\mathrm{b}(\alpha)) + c_\alpha(I_\alpha \psi(\zeta)) \overset{Cap_p-a.e.}{=} 1 - c_\alpha + c_\alpha = 1 & \text{if } \zeta \in A_\alpha \end{cases}.$$

$$\Longrightarrow Ih(\zeta) = 1 \qquad Cap_p\text{-a.e. on } A.$$

Furthermore

$$\sum_{\beta \in E(T)} h(\beta)^p = \sum_{\beta \ngeq \alpha} f^A(\beta)^p + c_\alpha^p \sum_{\beta \geq \alpha} \psi(\beta)^p \overset{(2.11)}{<} \sum_{\beta \in E(T)} f^A(\beta)^p,$$

but f^A is the equilibrium function for A on T, so we have a contradiction (see Theorem 2.8). We proved that $c_\alpha^{-1} f^A$, restricted to $E(T_\alpha)$, is the requested equilibrium function. □

Proposition 2.10 (Equilibrium measure on a subtree)**.** *Let be $A \subset \partial T$, $\alpha \in E(T)$ and $A_\alpha = A \cap \partial T_\alpha$. Suppose that $I^* \mu^A(\alpha) > 0$. Then the equilibrium measure for the set A_α on the subtree T_α is*

$$\mu_\alpha^{A_\alpha} \in \mathcal{M}_+(\partial T_\alpha), \quad \mu_\alpha^{A_\alpha} = \frac{\mu^A|_{\partial T_\alpha}}{c_\alpha^{p-1}} = \frac{\mu^A|_{\partial T_\alpha}}{\left[1 - V_p^{\mu^A}(\mathrm{b}(\alpha))\right]^{p-1}}.$$

Proof. We know that $f^A(\alpha) = I^* \mu^A(\alpha)^{p'-1} > 0$. If ζ is a half-infinite geodesic crossing α, then $V_p^{\mu^A}(\mathrm{b}(\alpha)) = If^A(\mathrm{b}(\alpha)) < If^A(\zeta) \leq 1$ and so

$$c_\alpha = 1 - If^A(\mathrm{b}(\alpha)) = 1 - V_p^{\mu^A}(\mathrm{b}(\alpha)) > 0.$$

From Proposition 2.9 we know that $f_\alpha^{A_\alpha} = c_\alpha^{-1} f^A$ (restricted to the subtree T_α). Then

$$\mu_\alpha^{A_\alpha}(\partial T_\beta) \overset{(2.9)}{=} f_\alpha^{A_\alpha}(\beta)^{p-1} = c_\alpha^{1-p} f^A(\beta)^{p-1} \overset{(2.9)}{=} c_\alpha^{1-p} \mu^A(\partial T_\beta)$$

for all $\beta \geq \alpha$, hence $\mu_\alpha^{A_\alpha}$ coincides with $c_\alpha^{1-p} \mu^A$ as a Borel measure in ∂T_α. □

Theorem 2.11 (Recursive formula for capacity on a tree)**.** *Let ω be the root edge of T and $A \subset \partial T$. Let $\omega_1, \omega_2, \ldots, \omega_m$ be all the sons of ω. Let define $T_j = T_{\omega_j}$ and $A_j = A \cap \partial T_j$ for each $j = 1, \ldots, m$. If $Cap_{p,j}$ denotes the ℓ^p-capacity calculated with respect to the subtree T_j, the capacity of A satisfies the following formula:*

$$Cap_p(A) = \frac{\sum_{j=1}^m Cap_{p,j}(A_j)}{\left[1 + \left(\sum_{j=1}^m Cap_{p,j}(A_j)\right)^{p'-1}\right]^{p-1}}. \tag{2.12}$$

Proof. Let $\mu = \mu^A \in \mathcal{M}_+(\partial T)$ be the equilibrium measure for A on the whole tree T and $\mu_j = \mu^{A_j} \in \mathcal{M}_+(\partial T_j)$ the capacitary measures of A_j on the subtrees T_j. Since $\mathrm{e}(\omega) = \mathrm{b}(\omega_j)$ for all j, then the scaling constant $c := c_{\omega_1} = \ldots = c_{\omega_m}$ is

$$c = 1 - I(I^*\mu)^{p'-1}(\mathrm{e}(\omega)) = 1 - I^*\mu(\omega)^{p'-1} \overset{(2.10)}{=} 1 - Cap_p(A)^{p'-1}.$$

Using (2.10) and Proposition 2.10,

$$Cap_p(A) = I^*\mu(\omega) = \sum_{j=1}^m I^*\mu(\omega_j) = c^{p-1}\sum_{j=1}^m I^*\mu_j(\omega_j) \overset{(2.10)}{=} c^{p-1}\sum_{j=1}^m Cap_{p,j}(A_j).$$

It follows that

$$Cap_p(A)^{p'-1} = (1 - Cap_p(A)^{p'-1})\left(\sum_{j=1}^m Cap_{p,j}(A_j)\right)^{p'-1}$$

$$\Longrightarrow Cap_p(A)^{p'-1}\left[1 + \left(\sum_{j=1}^m Cap_{p,j}(A_j)\right)^{p'-1}\right] = \left(\sum_{j=1}^m Cap_{p,j}(A_j)\right)^{p'-1}.$$

$$\Longrightarrow Cap_p(A)^{p'-1} = \frac{\left(\sum_{j=1}^m Cap_{p,j}(A_j)\right)^{p'-1}}{\left[1 + \left(\sum_{j=1}^m Cap_{p,j}(A_j)\right)^{p'-1}\right]}.$$

By raising both members to the $(p-1)$-th power, we obtain (2.12). $\square$

Example 2.1 (The N-ary tree)**.** Let $N \geq 2$. T_N is the tree with root vertex o and root edge ω, such that each vertex in $V(T_N) \setminus \{o\}$ has exactly N sons. We have a kind of fractal structure: each subtree of successors $T_{N,\alpha}$, descending from the root $\alpha \in E(T_N)$, is isomorphic to the tree T_N, and so are their boundaries. Hence

$$k_N := Cap_p(\partial T_N) = Cap_{p,\alpha}(\partial T_{N,\alpha}) \qquad \text{for all } \alpha \in E(T_N).$$

By Theorem 2.11 we get that

$$k_N = \frac{N k_N}{[1 + (N k_N)^{p'-1}]^{p-1}}$$

and since $k_N > 0$

$$1 + N^{p'-1} k_N^{p'-1} = N^{p'-1}$$

$$\Longrightarrow k_N = Cap_p(\partial T_N) = \left(\frac{N^{p'-1} - 1}{N^{p'-1}}\right)^{p-1} = \left(1 - N^{-p'+1}\right)^{p-1}.$$

Observe that $\lim_{N \longrightarrow +\infty} Cap_p(\partial T_N) = 1$. If we choose $p = 2$, then the ℓ^2-capacity of the boundary of a N-ary tree is

$$Cap(\partial T_N) = 1 - \frac{1}{N} = \frac{N-1}{N}.$$

(When the index p is omitted, we mean $p = 2$). The equilibrium function relative to the boundary of T_N is

$$f^{\partial T_N}(\alpha) = \frac{N-1}{N^{d(\mathrm{e}(\alpha))}};$$

note that in this case $I f^{\partial T_N} = 1$ on the whole boundary.

These are the analogous formulas for capacities if $p = 3$ or $p = 3/2$:

$$Cap_3(\partial T_N) = \left(1 - \frac{1}{\sqrt{N}}\right)^2, \quad Cap_{3/2}(\partial T_N) = \sqrt{1 - \frac{1}{N^2}}.$$

We conclude this chapter with a Proposition which shows that there exist compact subsets in ∂T_2 with arbitrary ℓ^2-capacity $\leq 1/2$.

Proposition 2.12. *Let $p = 2$. For each real number $\tau \in [0, 1/2]$ there exists a compact subset G_τ on the boundary ∂T_2 of the binary tree, such that $Cap(G_\tau) = \tau$.*

	$p=2$	$p=3$	$p=\frac{3}{2}$
$N=2$	$\frac{1}{2}$	$\frac{3}{2}-\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
$N=3$	$\frac{2}{3}$	$\frac{2}{3}(2-\sqrt{3})$	$\frac{2}{3}\sqrt{2}$
$N=4$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}\sqrt{15}$
$N=5$	$\frac{4}{5}$	$\frac{2}{5}(3-\sqrt{5})$	$\frac{2}{5}\sqrt{6}$

Table 2.1: Some values of $Cap_p(\partial T_N)$

Proof. We define a map Λ from ∂T_2 to $[0,1]$. In T_2 we distinguish the two sons of a generic edge α: we call them α^- and α^+. Let $\{\alpha_j\}_{j\geq 1}$ be the increasing sequence of all the edges crossed by the geodesic $\zeta \in \partial T_2$, We identify ζ with the subset of natural numbers $\mathcal{J}(\zeta) := \{j \in \mathbb{N}|, \alpha_{j+1} = \alpha_j^+\}$ and we define

$$\Lambda(\zeta) := \sum_{j\in\mathcal{J}(\zeta)} 2^{-j}.$$

It is easy to check that Λ is a map *onto* $[0,1]$ (for each real $0 \leq r \leq 1$ there exists a binary expansion made up with negative powers of 2). It is not *one-to-one*, since for example

$$\frac{5}{8} = \begin{cases} \frac{1}{2}+\frac{1}{8} = \sum_{j=1,3} 2^{-j} \\ \frac{1}{2} + \sum_{j=4}^{+\infty} 2^{-j} = \sum_{j=1 \text{ or } j\geq 4} 2^{-j} \end{cases},$$

but the inverse images of points have exactly two elements. The map Λ is continuous: for every $\varepsilon > 0$ we choose $n_\varepsilon \in \mathbb{N}$ such that $2^{-n_\varepsilon} < \varepsilon$, and the inequality $d(\zeta,\xi) \leq 2\frac{\delta^{n_\varepsilon+1}}{1-\delta}$ implies that the first n_ε edges from the root coincide along the two geodesics ζ and ξ, so

$$|\Lambda(\zeta)-\Lambda(\xi)| \leq \sum_{j=n_\varepsilon+1}^{+\infty} 2^{-j} = 2^{-n_\varepsilon} < \varepsilon.$$

Let $\phi : [0,1] \longrightarrow \mathbb{R}$ be the function

$$\phi(x) = Cap(\Lambda^{-1}[0,x])$$

It is an increasing map and we know that $\phi(0) = 0$ and $\phi(1) = 1/2$. Since the ℓ^2-capacity Cap is sub-additive and monotonic, then

$$\phi(x+t) - \phi(x) \le Cap(\underbrace{\Lambda^{-1}[x, x+t]}_{\text{compact}}) \tag{2.13}$$

for all $x \in [0,1]$ and $0 \le t \le 1-x$. By Lemma 1.15, as $t \longrightarrow 0^+$, the right hand side of (2.13) goes to $Cap(\Lambda^{-1}\{x\})$, which is 0 because $\Lambda^{-1}\{x\}$ is a finite set (see Proposition 2.7). Hence ϕ is continuous and $\phi([0,1]) = [0, 1/2]$. The Lemma is proven by picking $G_\tau = \Lambda^{-1}[0, x_0]$ where $x_0 \in [0,1]$ is such that $\phi(x_0) = \tau$. $\square$

Chapter 3

Characterization of equilibrium measures

3.1 Characterization Theorem

Let T be a tree with a root, as in the previous Chapter. Our aim is to show that equilibrium measures on the boundary of T are characterized by a discrete nonlinear equation. For all $\alpha \in E(T)$, $\mu \in \mathcal{M}_+(\partial T)$ and $1 < p < +\infty$ we use the following notations:

$$\mathsf{M}(\alpha) := I^*\mu(\alpha), \qquad \mathcal{E}_{p,\alpha}(\mathsf{M}) := \|\mathsf{M}\|^{p'}_{L^{p'}(E(T_\alpha))} = \sum_{\beta \geq \alpha} \mathsf{M}(\beta)^{p'}.$$

The last one is usually called the *energy* of the function M (or the *energy* of the relative measure μ) on the subtree T_α.

Theorem 3.1 (Characterization, [AL])**.** *Let $\mu \in \mathcal{M}_+(\partial T)$. There exists a F_σ-set*[1] *$A \subset \partial T$ such that $\mu = \mu^A$ (μ is the equilibrium measure of A) if and only if $\mathsf{M} := I^*\mu$ satisfies*

$$\mathcal{E}_{p,\alpha}(\mathsf{M}) = \mathsf{M}(\alpha) \underbrace{\left[1 - I(\mathsf{M}^{p'-1})(\mathrm{b}(\alpha))\right]}_{c_\alpha} \tag{3.1}$$

for each $\alpha \in E(T)$ [c_α is the scaling constant of Proposition 2.9].

Furthermore, each function $f : E(T) \longrightarrow \mathbb{R}_+$ which is a solution to (3.1) *is such that $f = I^*\mu$, where μ is the equilibrium measure of a F_σ-set in ∂T.*

[1] In a topological space, a F_σ*-set* is a countable union of closed sets.

One implication of the Theorem is quite easy.

$(*)$ *Let* $\mu = \mu^A$ *be an equilibrium function for* $A \subset \partial T$*. Then the function* $\mathsf{M} = I^*\mu$ *satisfies the relations* (3.1).

Proof of $(*)$. Supposing that $\mathsf{M}(\alpha) = 0$, for all $\beta \geq \alpha$ we get that $0 \leq \mathsf{M}(\beta) = \mu(\partial T_\beta) \leq \mu(\partial T_\alpha) = \mathsf{M}(\alpha) = 0$; as a direct consequence

$$\mathcal{E}_{p,\alpha}(\mathsf{M}) = \sum_{\beta \geq \alpha} \mathsf{M}(\beta)^{p'} = 0$$

and (3.1) reduces to $0 = 0$.

Otherwise $\mathsf{M}(\alpha) > 0 \Longrightarrow c_\alpha > 0$; by the properties of equilibrium measures and Proposition 2.10 we get that

$$\begin{cases} c_\alpha^p Cap_{p,\alpha}(A_\alpha) = c_\alpha^p \|I^*\mu_\alpha\|^{p'}_{L^{p'}(E(T_\alpha))} = \underbrace{c_\alpha^{p+p'-pp'}}_{=c_\alpha^0=1} \|I^*\mu\|^{p'}_{L^{p'}(E(T_\alpha))} = \mathcal{E}_{p,\alpha}(\mathsf{M}) \\ c_\alpha^p Cap_{p,\alpha}(A_\alpha) = c_\alpha^p (I^*\mu_\alpha(\alpha)) = c_\alpha (I^*\mu(\alpha)) = c_\alpha \mathsf{M}(\alpha) \end{cases}$$

where μ_α denotes $\mu_\alpha^{A_\alpha}$. So $\mathcal{E}_{p,\alpha}(\mathsf{M}) = c_\alpha \mathsf{M}(\alpha)$, which is exactly (3.1). □

Next we show that a function satisfying (3.1) is the co-potential of a measure.

$(\triangle)$ *Suppose that* $f : E(T) \longrightarrow \mathbb{R}_+$ *is a function satisfying* (3.1) *for each edge: namely* $\mathcal{E}_{p,\alpha}(f) = f(\alpha)\left[1 - (If^{p'-1})(\mathrm{b}(\alpha))\right]$ *for all* $\alpha \in E(T)$*. Then* $f = I^*\mu$ *for some* $\mu \in \mathcal{M}_+(\partial T)$*.*

Proof of $(\triangle)$. Let $\alpha_1, \dots, \alpha_m$ be the sons of $\alpha \in E(T)$. From (3.1) we get that $If^{p'-1} \leq 1$ on $V(T)$. If $(If^{p'-1})(\mathrm{e}(\alpha)) = 1$, then $f(\beta) = 0$ for each $\beta \geq \alpha$ [otherwise we would find a vertex such that $(If^{p'-1})(x) > 1$]. By using (3.1) it is $f(\alpha)^{p'} = f(\alpha) \cdot 0 = 0$, so $0 = f(\alpha) = \sum_{j=1}^m f(\alpha_j)$.

Otherwise, let $(If^{p'-1})(\mathrm{e}(\alpha)) < 1$. We have that

$$\sum_{j=1}^m f(\alpha_j) \overset{(3.1)}{=} \sum_{j=1}^m \frac{\mathcal{E}_{p,\alpha_j}(f)}{1 - (If^{p'-1})(\mathrm{e}(\alpha))} = \frac{\mathcal{E}_\alpha(f) - f(\alpha)^{p'}}{1 - (If^{p'-1})(\mathrm{b}(\alpha)) - f(\alpha)^{p'-1}} \overset{(3.1)}{=}$$

$$= \frac{f(\alpha)\left[1 - (If^{p'-1})(\mathrm{b}(\alpha)) - f(\alpha)^{p'-1}\right]}{1 - If(\mathrm{b}(\alpha)) - f(\alpha)^{p'-1}} = f(\alpha),$$

so we conclude that $f(\alpha) = \sum_{j=1}^m f(\alpha_j)$. Then $\mu(\partial T_\alpha) := f(\alpha)$ extends (uniquely) to a measure on Borel sets $\mu \in \mathcal{M}_+(\partial T)$ such that $f = I^*\mu$. □

It remains to prove the other implication of the Theorem 3.1. We restrict our attention to the case $p = p' = 2$, in order to avoid some troubles with exponents. Moreover, we will see that this particular choice of p leads to an interesting geometric interpretation (see Section 3.3).

($\Diamond$) *Let* $p = 2$*,* $\mu \in \mathcal{M}_+(\partial T)$ *and* $\mathsf{M} := I^*\mu$*. Suppose that*

$$\mathcal{E}_\alpha(\mathsf{M}) := \sum_{\beta\geq\alpha} \mathsf{M}(\beta)^2 = \mathsf{M}(\alpha)\underbrace{[1 - I\mathsf{M}(\mathrm{b}(\alpha))]}_{c_\alpha} \quad \textit{for all } \alpha \in E(T). \tag{3.2}$$

Then there exists a F_σ*-set* $A \subset \partial T$ *such that* $\mu = \mu^A$*.*

Equation (3.2) implies that $1 - I\mathsf{M}(\mathrm{b}(\alpha)) \geq 0$ for each α; if we choose $\alpha = \omega$, it follows that the measure μ has *finite energy*, in the sense that

$$\mathcal{E}(\mathsf{M}) = \mathcal{E}_\omega(\mathsf{M}) = \sum_{\beta\in E(T)} \mathsf{M}(\beta)^2 < +\infty.$$

To prove ($\Diamond$) two simple lemmas are required.

Lemma 3.2. *If* $\mu \in \mathcal{M}_+(\partial T)$ *is so that* $\mathsf{M} = I^*\mu$ *satisfies* (3.2)*, then*

$$I\mathsf{M} = 1 \qquad \mu\text{-a.e. on } \partial T$$

Proof. For each $n \in \mathbb{N}$ we define the finite set $E_n := \{\alpha \in E(T) |\, d(\mathrm{e}(\alpha)) = n\}$. All the points $\zeta \in \partial T$ have a unique predecessor $\alpha_\zeta^{(n)}$ which is in E_n. We define the maps

$$\Psi_n(\zeta) := 1 - I\mathsf{M}(\mathrm{b}(\alpha_\zeta^{(n)})) \geq 0.$$

By taking the limit for $n \longrightarrow +\infty$ we get that $\Psi_n(\zeta) \downarrow 1 - I\mathsf{M}(\zeta) \geq 0$ and using Monotone Convergence Theorem we deduce that

$$\lim_{n\longrightarrow+\infty} \int_{\partial T} \Psi_n \, d\mu = \int_{\partial T} (1 - I\mathsf{M}) \, d\mu. \tag{3.3}$$

On the other side, the function Ψ_n can be written in the form

$$\Psi_n = \sum_{\alpha\in E_n} [1 - I\mathsf{M}(\mathrm{b}(\alpha))]\chi_{\partial T_\alpha}$$

and then, by Fubini-Tonelli Theorem,

$$0 \leq \int_{\partial T} \Psi_n \, d\mu = \sum_{\alpha\in E_n} [1 - I\mathsf{M}(\mathrm{b}(\alpha))] \int_{\partial T_\alpha} d\mu =$$

$$= \sum_{\alpha\in E_n} [1 - I\mathsf{M}(\mathrm{b}(\alpha))]\mathsf{M}(\alpha) \overset{(3.2)}{=} \sum_{\alpha\in E_n}\sum_{\beta\geq\alpha} \mathsf{M}(\beta)^2 \longrightarrow 0 \text{ as } n \longrightarrow +\infty,$$

as the series $\sum_{\beta\in E(T)} \mathsf{M}(\beta)^2$ converges. Combining with (3.3),

$$\int_{\partial T} \underbrace{(1 - I\mathsf{M})}_{\geq 0}\, d\mu = 0,$$

hence $I\mathsf{M} = 1$ μ-a.e. on ∂T. □

In this proof we showed that (3.2) implies $I\mathsf{M} \leq 1$. A point $\zeta \in \partial T$ is *irregular* for the measure μ if $I\mathsf{M}(\zeta) < 1$. The thesis of Lemma 3.2 can be reformulate by saying that the set of irregular points for μ has null μ-measure.

The second Lemma is a simple result of measure theory.

Lemma 3.3. *Fix $\varepsilon > 0$ and $\mu \in \mathcal{M}_+(\partial T)$. For every compact set $K \subset \partial T$ such that $\mu(K) = 0$ there exists a finite set of edges $\{\alpha_j\}_{j=1}^m$ such that*

i) ∂T_{α_j} are pairwise disjoint;

ii) $K \subset \bigcup_{j=1}^m \partial T_{\alpha_j}$;

iii) $\sum_{j=1}^m \mu(\partial T_{\alpha_j}) < \varepsilon$.

Proof. μ is an outer regular measure and $\mu(K) = 0$, so there exists an open set $U \supset K$ such that $\mu(U) < \varepsilon$. Then U is the union of balls which are centered in its points, so we get a open cover of K. Since K is compact, there is a finite number of balls $B_1, \ldots, B_m$ so that $K \subset (B_1 \cup B_2 \cup \ldots \cup B_m) \subset U$. By Remark 2.5 and Corollary 2.3 we may suppose that the balls are pairwise disjoint and $B_j = \partial T_{\alpha_j}$ for some $\alpha_j \in E(T)$. It remains to prove *iii*), but μ is additive and then

$$\sum_{j=1}^m \mu(\partial T_{\alpha_j}) = \mu\left(\bigcup_{j=1}^m \partial T_{\alpha_j}\right) \leq \mu(U) < \varepsilon.$$ □

Proof of ($\Diamond$). If $\mu \equiv 0$ there is nothing to prove. Since (3.2) holds for $\mathsf{M} = I^*\mu$, by Lemma 3.2 the set of irregular points

$$H := \{\zeta \in \partial T |\, I\mathsf{M}(\zeta) < 1\}$$

has null μ-measure. H is a countable union of the compact sets

$$H_s = \{\zeta \in \partial T \,|\, I\mathsf{M}(\zeta) \leq 1 - 2^{-s}\}, \qquad s \in \mathbb{N}, \quad H_s \uparrow H, \quad \mu(H_s) = 0.$$

Let $n \in \mathbb{N}$; by Lemma 3.3, for each s we can choose $\{\alpha_{j,s}^{(n)}\}_{j=1}^{m_s} \subset E(T)$ so that

$$H_s \subset \bigcup_{j=1}^{m_s} B_{j,s}^{(n)} \quad \text{where } B_{j,s}^{(n)} = \partial T_{\alpha_{j,s}^{(n)}}, \qquad \sum_{j=1}^{m_s} \mu(B_{j,s}^{(n)}) < \frac{1}{n} \cdot \frac{1}{2^s}.$$

We define the open sets $W_n = \bigcup_{s\geq 0} \bigcup_{j=1}^{m_s} B_{j,s}^{(n)} \supset H$. Then

$$\mu(W_n) \leq \sum_{s=1}^{+\infty} \sum_{j=1}^{m_s} \mu(B_{j,s}^{(n)}) < \frac{1}{n}.$$

Next we consider the compact sets $A_n := \partial T \setminus W_n$. Define $A := \bigcup_{n\geq 0} A_n$: it is a F_σ-set and by additivity of μ

$$\begin{aligned} \mu(\partial T) - \frac{1}{n} < \mu(\partial T) - \mu(W_n) = \mu(A_n) \leq \mu(\partial T) \\ \Longrightarrow \mu(A) = \lim_{n \longrightarrow +\infty} \mu(A_n) = \mu(\partial T). \end{aligned} \tag{3.4}$$

By construction, the set A does not contain any irregular point:

$$I(I^*\mu) = I\mathsf{M} = 1 \qquad \text{on A}.$$

Then we could guess that $\mu = \mu^A$. It is sufficient to show that

$$Cap(A) = \mathcal{E}(\mathsf{M}). \tag{3.5}$$

Since $\mathsf{M} = I^*\mu \in \Omega_A$, by definition of capacity

$$Cap(A) \leq \|\mathsf{M}\|^2_{\ell^2(E(T))} = \mathcal{E}(\mathsf{M}).$$

On the other side we may use the dual definition of capacity (Theorem 1.12) for the compact sets A_n:

$$Cap(A_n)^{1/2} = \sup\{\widehat{\mu}(A_n) \,|\, \widehat{\mu} \in \mathcal{M}_+(A_n),\, \|I^*\widehat{\mu}\|_{\ell^2} \leq 1\} =$$

$$= \sup \left\{ \frac{\widehat{\mu}(A_n)}{\|I^*\widehat{\mu}\|_{\ell^2}} \mid \widehat{\mu} \in \mathcal{M}_+(A_n),\, \widehat{\mu} \neq 0 \right\}.$$

If μ_n denotes the restriction $\mu|_{A_n}$ and $\mathsf{M}_n = I^*\mu_n$, then

$$Cap(A_n) \geq \frac{\mu_n(A_n)^2}{\mathcal{E}(\mathsf{M}_n)} = \frac{\mu(A_n)^2}{\mathcal{E}(\mathsf{M}_n)} \geq \frac{\mu(A_n)^2}{\mathcal{E}(\mathsf{M})}. \tag{3.6}$$

From (3.2) it follows that $\mathcal{E}(\mathsf{M}) = \mathsf{M}(\omega) = \mu(\partial T)$, and by Lemma 1.16 $Cap(A) = \lim_{n \longrightarrow +\infty} Cap(A_n)$; hence by taking the limit for $n \longrightarrow +\infty$ in (3.6):

$$Cap(A) \geq \frac{\mu(A)^2}{\mathcal{E}(\mathsf{M})} \overset{(3.4)}{=} \frac{\mu(\partial T)^2}{\mathcal{E}(\mathsf{M})} = \mathcal{E}(\mathsf{M}).$$

So (3.5) holds, and then $\mu = \mu^A$. □

Proposition 3.4. *Let $\mu \in \mathcal{M}_+(\partial T)$ and $\mathsf{M} = I^*\mu$. If M satisfies* (3.1) *for all the sons of $\alpha \in E(T)$, then M satisfies* (3.1) *for α.*

Proof. We prove the Proposition in the case $p = 2$. Let $\alpha_1, \ldots, \alpha_m$ be the sons of $\alpha \in E(T)$. Then $\mathsf{M}(\alpha) = \sum_{j=1}^{m} \mathsf{M}(\alpha_j)$, and for each $j = 1, \ldots, m$

$$\mathrm{b}(\alpha_j) = \mathrm{e}(\alpha), \quad I\mathsf{M}(\mathrm{b}(\alpha_j)) = I\mathsf{M}(\mathrm{e}(\alpha)) = I\mathsf{M}(\mathrm{b}(\alpha)) + \mathsf{M}(\alpha).$$

Since (3.2) holds for all α_j, by summing up the members, we obtain

$$\sum_{j=1}^{m} \mathcal{E}_{\alpha_j}(\mathsf{M}) = \underbrace{\left(\sum_{j=1}^{m} \mathsf{M}(\alpha_j) \right)}_{=\mathsf{M}(\alpha)} (1 - I\mathsf{M}(\mathrm{b}(\alpha)) - \mathsf{M}(\alpha))$$

$$\Longrightarrow \mathcal{E}_{\alpha}(\mathsf{M}) = \left(\sum_{j=1}^{m} \mathcal{E}_{\alpha_j}(\mathsf{M}) \right) + \mathsf{M}(\alpha)^2 = \mathsf{M}(\alpha) \left[1 - I\mathsf{M}(\mathrm{b}(\alpha)) \right].$$

The proof in the case $p \neq 2$ is the same: just replace $\mathcal{E}_{\alpha_j}(\mathsf{M})$ with $\mathcal{E}_{p,\alpha_j}(\mathsf{M})$ and $I\mathsf{M}$ with $I\mathsf{M}^{p'-1}$. □

Remark 3.1. Let $p = 2$: observe that if μ is an equilibrium measure, then $\mathsf{M} = I^*\mu$ is the corresponding equilibrium function (this fact is not true if $p \neq 2$). Thus we can also say that each function $f \in \ell^2_+(E(T))$ satisfying (3.2), i.e.

$$\mathcal{E}_{\alpha}(f) = f(\alpha) \left[1 - If(\mathrm{b}(\alpha)) \right] \quad \text{for all } \alpha \in E(T)$$

is such that $f = f^A$, where $A \subset \partial T$ is a F_σ-set.

3.2 Irregular points

Fix $p = 2$ and let $\mu = \mu^A$ be an equilibrium measure on ∂T, relative to a F_σ-set A in the boundary of the tree T. Without losing generality, we may suppose that $\operatorname{supp}\mu = \partial T$ (otherwise we reduce T to the tree T_A whose vertices are all the predecessors of A). In order to keep things simple, we consider $p = 2$ and $\mathsf{M} = I^*\mu$, as above. Most of the work we made to prove Characterization Theorem (Theorem 3.1) was dealing with irregular points. Let $H_\mu \subset \partial T$ be the set of all irregular points relative to μ; by Lemma 3.2 the set H_μ has null μ-measure. But the irregular points are not negligible: they play an important role while talking about equilibrium measures. In the next Proposition it is shown that if the set of irregular points for a measure with finite energy $\mu \in \mathcal{M}_+(\overline{A})$ has null capacity, then μ coincides with the equilibrium measure of A.

Proposition 3.5. *Let $A \subset \partial T$, $\mu \in \mathcal{M}_+(\overline{A})$, $\mathsf{M} = I^*\mu$ so that $I\mathsf{M} = 1$ Cap-a.e. on A and $\mathcal{E}(\mathsf{M}) < +\infty$. Then $\mu = \mu^A$.*

Proof. Let $\mathsf{M}^A := I^*\mu^A$, where $\mu^A \in \mathcal{M}_+(\overline{A})$ is the equilibrium measure. Then $I\mathsf{M}^A = 1$ Cap-a.e. on A and $\mathcal{E}(\mathsf{M}^A) = Cap(A) < +\infty$. We consider the difference $\widehat{\mathsf{M}} = \mathsf{M} - \mathsf{M}^A$. The set $\widehat{A} := \{\zeta \in A |\, I\widehat{\mathsf{M}}(\zeta) \neq 0\}$ is such that $Cap(\widehat{A}) = 0$; by (1.6)

$$\mu(\widehat{A}) \leq \sqrt{\mathcal{E}(\mathsf{M})}\sqrt{Cap(\widehat{A})}, \quad \mu^A(\widehat{A}) \leq \sqrt{\mathcal{E}(\mathsf{M}^A)}\sqrt{Cap(\widehat{A})}.$$

As $\mathcal{E}(\mathsf{M})$ and $\mathcal{E}(\mathsf{M}^A)$ are finite, we infer that $\mu(\widehat{A}) = \mu^A(\widehat{A}) = 0$ and

$$\underbrace{\int_{\partial T} I\widehat{\mathsf{M}}\, d\mu}_{\Gamma_1} - \underbrace{\int_{\partial T} I\widehat{\mathsf{M}}\, d\mu^A}_{\Gamma_2} = 0. \tag{3.7}$$

The integral Γ_1 can be written more explicitly:

$$\Gamma_1 = \int_{\partial T} \sum_{\alpha \in E(T)} \left[\widehat{\mathsf{M}}(\alpha)\chi_{P(\zeta)}(\alpha)\right] d\mu(\zeta) = \int_{\partial T} \sum_{\alpha \in E(T)} \left[\widehat{\mathsf{M}}(\alpha)\chi_{\partial T_\alpha}(\zeta)\right] d\mu(\zeta).$$

The integrand function could be negative somewhere: so we need to check whether we are allowed to switch the summation over $E(T)$ with the integration over ∂T. If we could do that, then we would obtain

$$\Gamma_1 = \sum_{\alpha \in E(T)} \left[\widehat{\mathsf{M}}(\alpha) \int_{\partial T} \chi_{\partial T_\alpha}(\zeta) d\mu(\zeta)\right] = \sum_{\alpha \in E(T)} \widehat{\mathsf{M}}(\alpha)\mathsf{M}(\alpha).$$

Nothing bad happens if we make the exchange, because

$$\sum_{\alpha\in E(T)}\int_{\partial T}|\widehat{\mathsf{M}}(\alpha)|\chi_{\partial T_\alpha}(\zeta)d\mu(\zeta)=\sum_{\alpha\in E(T)}|\widehat{\mathsf{M}}(\alpha)|\mathsf{M}(\alpha).$$

The last term is finite as $|\widehat{\mathsf{M}}|\mathsf{M}=\left|\mathsf{M}-\mathsf{M}^A\right|\mathsf{M}\le\mathsf{M}^2+(\mathsf{M}^A)^2$ and $\mathcal{E}(\mathsf{M})<+\infty$, $\mathcal{E}(\mathsf{M}^A)<+\infty$.

Analogously $\Gamma_2=\sum_{\alpha\in E(T)}\widehat{\mathsf{M}}(\alpha)\mathsf{M}^A(\alpha)$ and

$$0\overset{(3.7)}{=}\Gamma_1-\Gamma_2=\sum_{\alpha\in E(T)}\widehat{\mathsf{M}}(\alpha)^2=\|\widehat{\mathsf{M}}\|^2_{\ell^2(E(T))}.$$

So we conclude that $\mathsf{M}=\mathsf{M}^A$: this means that $\mu=\mu^A$ since they coincide on each ball of ∂T. □

Remark 3.2. Let $A\subset A'\subset\partial T$. From Proposition 3.5 we deduce that $Cap(A'\setminus A)=0\Longrightarrow\mu^{A'}=\mu^A$: the equilibrium measure does not change if we add, or remove, a set of null capacity (a countable set, for example).

In the next Example we build an equilibrium measure with 'a certain amount' of irregular points, in the sense of capacity.

Example 3.1. There exists a dense subset A on the boundary of the binary tree $T=T_2$ with arbitrarily small - but positive - capacity. More, A is dense in a stronger sense: for each ball $B\subset\partial T$ the intersection $A\cap B$ has positive capacity.

Let $0<\varepsilon<1/2$. If we take an edge α such that $d(\mathrm{e}(\alpha))=n$, then $Cap(\partial T_\alpha)=1/(n+1)$ [2]: this can be proven by using iteratively the recursive formula (2.12). We pick n_1 so that $(n_1+1)^{-1}<\varepsilon/2$ and define the compact set $K_1=\partial T_{\alpha_1}$ with $d(\mathrm{e}(\alpha_1))=n_1$, so $Cap(K_1)<\varepsilon/2$. Then we proceed iteratively: at the j-th step ($j\ge2$).

- if $E_n=\{\alpha\in E(T)|\,d(\mathrm{e}(\alpha))=n\}$, then $\#E_{n_{j-1}}=2^{n_{j-1}-1}<+\infty$;
- we pick $n_j>n_{j-1}$ so that for each edge $\beta_{j,k}$ in $E_{n_{j-1}}\setminus P(K_{j-1})$ ($k=1,\dots,q_j<\#E_{n_{j-1}}$) we can choose $\alpha_{j,k}\in E_{n_j}\cap S(\beta_{j,k})$ in order to have $0<Cap(\partial T_{\alpha_{j,k}})<\dfrac{1}{q_j}\cdot\dfrac{\varepsilon}{2^j}$ for each k;

[2] $Cap=Cap_\omega$ denotes the capacity with respect to the whole tree T.

- we define the compact set $K_j = K_{j-1} \cup \left(\bigcup_{k=1}^{q} \partial T_{\alpha_{j,k}} \right)$.

Then we get an increasing sequence of compact sets $\{K_j\}_{j \geq 1}$, which are finite unions of balls. Their capacity is smaller that ε:

$$Cap(K_j) < Cap(K_{j-1}) + \frac{\varepsilon}{2^j} < \ldots < \varepsilon \sum_{t=1}^{j} 2^{-t} \leq \varepsilon.$$

Defining $A := \bigcup_{j \geq 1} K_n$, it is $Cap(A) = \lim_{j \longrightarrow +\infty} Cap(K_j) \leq \varepsilon$. On the other side $\overline{A} = \partial T$, since any edge of the tree is a predecessor of K_j, for some j sufficiently large. Observe that

$$Cap(\partial T \setminus A) \geq Cap(\partial T) - Cap(A) \geq \frac{1}{2} - \varepsilon > 0.$$

Let $\mu^A \in \mathcal{M}_+(\partial T)$ be the equilibrium measure of A. Then $\operatorname{supp} \mu^A = \partial T$. Indeed if $\zeta \in \partial T$ and B is an open ball centered in ζ, there exists $\xi \in A \cap B$ by density. Thus, since A is a countable union of balls, there exists an edge α so that $\xi \in \partial T_\alpha \subset A \cap B$ and from the fact that ∂T_α has nontrivial capacity it follows that $0 < \mu^A(\partial T_\alpha) \leq \mu^A(A \cap B) \leq \mu^A(B)$ [3]: hence $\zeta \in \operatorname{supp} \mu^A$. It also follows that $Cap(A \cap B) > 0$ for each ball B, so we have density in the sense of ℓ^2-capacity.

The equilibrium measures μ^A and $\mu^{\partial T}$ have the same support, but they are completely different. In fact $\mu^A(\partial T) = \varepsilon < 1/2 = \mu^{\partial T}(\partial T)$. Furthermore, it is easy to show that $\mu^{\partial T}$ has no irregular points, since

$$\mu^{\partial T}(\partial T_\alpha) = \mathsf{M}^{\partial T}(\alpha) = 2^{-d(\mathrm{e}(\alpha))} \quad \text{for all } \alpha \in E(T)$$

$$\Longrightarrow I\mathsf{M}^{\partial T}(\zeta) = \sum_{j=1}^{+\infty} 2^{-j} = 1 \quad \text{for each } \zeta \in \partial T;$$

on the contrary, by Proposition 3.5, $\mu^A \neq \mu^{\partial T} \Longrightarrow Cap(H_{\mu^A}) > 0$: the set of irregular points of μ^A has a strictly positive capacity!

[3]Let $\mathsf{M}^A := I^*\mu^A$. If $\mathsf{M}^A(\alpha) = \mu^A(\partial T_\alpha)$ were 0, then $Cap(\partial T_\alpha) > 0 \Longrightarrow I\mathsf{M}^A(\mathrm{b}(\alpha)) = 1$, so we would get contradictions, unless M^A is zero along all the edges in $[o, \mathrm{b}(\alpha)]$, but this is absurd.

3.3 Equilibrium measures and square tilings

Let T be a tree and $p = 2$. By Theorem 3.1, μ is an equilibrium measure of a F_σ-subset $A \subset \partial T$ if and only if the corresponding function $I^*\mu = \mathsf{M}$ satisfies the relations (3.2):

$$\sum_{\beta \geq \alpha} \mathsf{M}(\beta)^2 = [1 - I\mathsf{M}(\mathrm{b}(\alpha))]\, \mathsf{M}(\alpha) \quad \text{for all } \alpha \in E(T).$$

Moreover we recall that the value of M on each edge is equal to the sum of all the values the function M assumes on the sons [4]:

$$\mathsf{M}(\alpha) = \sum_{\beta,\, \mathrm{e}(\alpha)=\mathrm{b}(\beta)} \mathsf{M}(\beta).$$

All these properties can be displayed in a geometrical fashion.

Definition 3.1. Let R be a rectangle in $\mathbb{R}^2$. By a change of (orthogonal) coordinates we may always suppose that $R = [0, a] \times [0, b]$. A *square tiling* of R is a collection $\mathcal{T} = \{Q_i\}_{i \in \mathcal{I}} \subset \mathbb{R}^2$ of squares such that

- $\bigcup_{Q \in \mathcal{T}} Q = \bigcup_{i \in \mathcal{I}} Q_i$ is dense in R;
- $(Q \cap Q') \subset \partial Q \cap \partial Q'$ for all $Q, Q' \in \mathcal{T}$.

Now every measure $\mu \in \mathcal{M}_+(\partial T)$ satisfying (3.2) can be associated with a square tiling $\mathcal{T}_\mu = \{Q_\alpha\}_{\alpha \in E(T)}$ of the rectangle $R = [0, \mathsf{M}(\omega)] \times [0, 1]$ such that $\mathrm{area}(Q_\alpha) = \mathsf{M}(\alpha)^2$. We call $Q_\omega = [0, \mathsf{M}(\omega)] \times [1 - \mathsf{M}(\omega), 1]$, where ω is the root edge. By induction, if $\alpha_1, \ldots, \alpha_N \in E(T)$ are the sons of α, then we split the bottom side of Q_α into N adjacent segments of respective length $\mathsf{M}(\alpha_1), \ldots \mathsf{M}(\alpha_N)$, and these segments are the upper sides of the squares $Q_{\alpha_1}, \ldots, Q_{\alpha_N}$. The edges β that are zeros of M correspond to squares that degenerate to points, and since all the edges descending from β are zeros of M, we may cut the whole subtree T_β without changing the representation. Relation (3.2) with $\alpha = \omega$ says that the squares of $\mathcal{T}_\mu$ - whose mutual intersections are along their boundaries by construction - cover the same area of a rectangle with base $\mathsf{M}(\omega)$ and height 1: as a consequence $\mathcal{T}_\mu$ fills the whole interior of R and so it is a square tiling as in Definition 3.1.

[4] Because of the additivity of μ.

Remark 3.3. The tiling $\mathcal{T}_\mu$ reflects the combinatorial structure of the tree T in this sense: if two edges $\alpha, \beta \in E(T)$ are consecutive in T (i.e. they have a common extreme) and $\alpha < \beta$, then the corresponding tiles Q_α and Q_β have a nonempty intersection, which lies on the upper side of Q_β. We say that Q_α and Q_β have a *vertical contact.*

Observe that there is a bijection

$$E(T) \longrightarrow V(T) \setminus \{o\}$$
$$\alpha \longmapsto \mathrm{e}(\alpha),$$

so we may also label the squares with vertices $\neq o$: for all $x \neq o$, $Q_x = Q_\alpha$ where α is the unique edge such that $x = \mathrm{e}(\alpha)$. In this case the combinatorial structure becomes the following: if two vertices are connected by an edge, the corresponding tiles have a vertical contact.

Remark 3.4. By the Characterization Theorem we know that $\mu = \mu^A$, where $A \subset \partial T$ is a F_σ-set. So

$$\mathsf{M}(\omega) = I^*\mu(\omega) = Cap(A).$$

This shows that the associated tiling $\mathcal{T}_\mu$ covers a rectangle of height 1 and base $Cap(A)$: we obtain a geometric representation of the capacity $Cap(A)$. Analogously, the squares relative to the edges $\geq \alpha$ give a tiling of a 'sub-rectangle' of height $1 - I\mathsf{M}(\mathrm{b}(\alpha)) = c_\alpha$ and base $\mathsf{M}(\alpha)$. By considering this 'sub-tiling' and making a rescaling (i.e. a homogeneous dilation) in order to normalize the height to 1, we get the already known result:

$$Cap_\alpha(A_\alpha) = \mathsf{M}_\alpha(\alpha) = \frac{\mathsf{M}(\alpha)}{1 - I\mathsf{M}(\mathrm{b}(\alpha))}.$$

Example 3.2. We consider a tree T^0 built in this way: each edge α has three sons if $d(\mathrm{b}(\alpha))$ is even, otherwise it has two sons. We may think T^0 as a tree which is contained in T_3 and that includes T_2, so we have the estimates $1/2 \leq Cap(\partial T^0) \leq 2/3$. If ω_1, ω_2 and ω_3 are the sons of ω (the root edge), then the subtrees $T^0_{\omega_j}$ are identical to a tree T', which is obtained by attaching two copies of T^0 to the end of the root edge. So it follows by (2.12) that

$$x := Cap(\partial T^0) \neq 0, \quad y := Cap(\partial T') = Cap(\partial T^0_{\omega_j}) \quad \begin{cases} x = \dfrac{3y}{1+3y} \\ y = \dfrac{2x}{1+2x} \end{cases} \quad \Longrightarrow$$

$$\Longrightarrow x = \frac{\frac{6x}{1+2x}}{1+\frac{6x}{1+2x}} \Longrightarrow x = \frac{6x}{1+8x} \Longrightarrow 1 = \frac{6}{1+8x} \Longrightarrow 1+8x = 6 \Longrightarrow$$

$$\Longrightarrow \boldsymbol{Cap(\partial T^0) = \frac{5}{8}}.$$

The corresponding equilibrium measure $\mu = \mu^{\partial T^0}$ corresponds to the function

$$\mathsf{M}(\alpha) = \frac{5}{8} \cdot \left(\frac{1}{2}\right)^{\lfloor m_\alpha/2 \rfloor} \cdot \left(\frac{1}{3}\right)^{m_\alpha - \lfloor m_\alpha/2 \rfloor} \qquad \text{where } m_\alpha = d(\mathrm{b}(\alpha))$$

and the associated tiling is displayed in Figure 3.1.

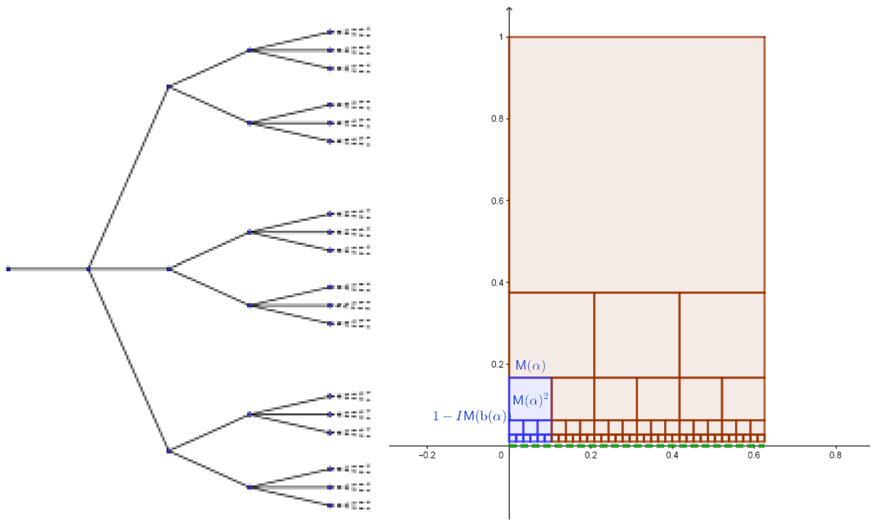

Figure 3.1: The tree T^0 and the tiling associated with $\mu = \mu^{\partial T^0}$. The height is 1 and the base is $Cap(\partial T^0) = 5/8$.

Remark 3.5 (Irregular points). In the square tiling of a rectangle $\mathcal{T}_\mu$ associated with an equilibrium measure μ, irregular points for μ correspond to 'vertical' sequences of adjacent tiles starting from the upper side of the rectangle and stopping (at the limit) without reaching the bottom edge. The equilibrium measure of Example 3.2 has no irregular points, as for each $\zeta \in \partial T^0$

$$I\mathsf{M}(\zeta) = \frac{5}{8}\left(1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{3}\cdot\frac{1}{6} + \frac{1}{6^2} + \frac{1}{3}\cdot\frac{1}{6^2} + \dots\right) =$$

$$= \frac{5}{8}\left(1+\frac{1}{3}\right)\left(\sum_{j=0}^{+\infty} 6^{-j}\right) = \frac{5}{8}\cdot\frac{4}{3}\cdot\frac{6}{5} = 1.$$

This fact is displayed in Figure 3.1, where all the descending sequences of consecutive squares end at the base of the rectangle. Analogously, if we take a N-ary tree T_N, the equilibrium measure of its boundary is such that all the points in ∂T_N are regular (see Example 2.1).

On the contrary, the measure μ^A of Example 3.1, defined on the boundary of the binary tree T_2, has a lot of irregular points. Its corresponding tiling $\mathcal{T}_{\mu^A}$ has the same combinatorial pattern as the (regular) tiling $\mathcal{T}_{\mu^{\partial T_2}}$, but it has a narrower base ($\varepsilon < 1/2$) and there are uncountable vertical sequences of tiles stopping sharply before reaching the bottom side. See Figure 3.2.

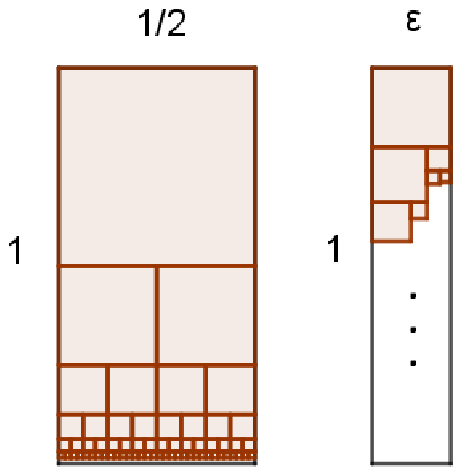

Figure 3.2

Remark 3.6. It is quite easy to extend the notions of capacity and equilibrium functions/measures in the case there are some vertices in the tree T with no sons. They are called the *leaves* of T. In this case each leaf is both a vertex of the tree and a point of the boundary ∂T. It is easy to show that the p-capacity of a leaf x is

$$Cap_p(\{x\}) = \frac{1}{d(x)^{p-1}} > 0.$$

Indeed $d(x) = \#P(x)$ and the capacitary function is $f^{\{x\}} = \dfrac{1}{d(x)}\chi_{P(x)}$. Many results of the previous Chapter still hold: for example we are allowed to use the recursive formula (2.12). The main difference we have to consider is the fact that there are points of the boundary with nontrivial capacity (the leaves).

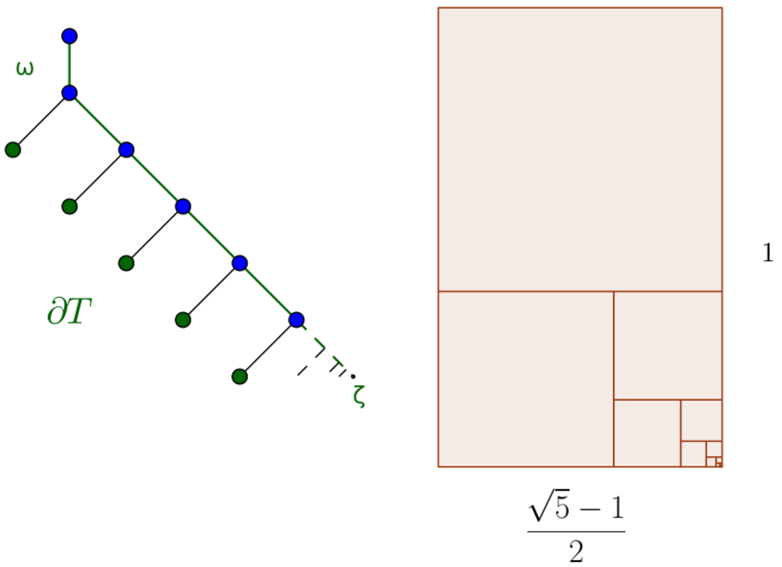

Figure 3.3: The tree T_φ and its associated tiling.

Example 3.3. We consider the tree T_φ as in Figure 3.3: each 'left' son ends at a leaf, and every 'right' son has two sons. The boundary ∂T_φ is a countable discrete set of leaves, plus a unique half-infinite geodesic ζ, which is an accumulation point (so ∂T_φ is homeomorphic to the closure of the sets of natural numbers $\mathbb{N}\cup\{+\infty\}$). We may use the formula (2.12) with $p = 2$ in order to evaluate $k := Cap(\partial T_\varphi)$: all the subtrees descending from the 'right' sons coincide with T_φ, whereas all the 'left' subtrees consist of a unique edge. By (2.12)

$$k = \frac{1+k}{1+1+k} \tag{3.8}$$

$$\Longrightarrow 2k + k^2 = 1 + k \Longrightarrow k^2 + k - 1 = 0 \Longrightarrow k = \frac{\sqrt{5}-1}{2} = \boldsymbol{\varphi - 1},$$

where φ is the *golden ratio*. This result becomes clear if we consider the square tiling associated with the tree (see Figure 3.3): indeed the covered rectangle is a *golden rectangle* and $\varphi = \dfrac{1}{\varphi - 1}$ is the ratio between the height and the base. Observe that the formula (3.8) may be rewritten in order to obtain the continued fraction expression for φ:

$$k = \cfrac{1}{1 + \cfrac{1}{1+k}} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}$$

$$\Longrightarrow \varphi = 1 + k = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}} = [1; 1, 1, 1, 1, \ldots].$$

Chapter 4

Probabilistic approach on graphs

4.1 Simple random walk on a graph

In the paper by K. L. Chung [Chu] it is shown how, in the general theory, a problem about equilibrium measures can be studied by considering exit times of Brownian motions. Our aim is to show that we can build a kind of square tiling with reference to a graph that is not necessarily a tree, and the sizes of tiles are in terms of probabilities.

Definition 4.1. Let $G = (V, E)$ be a locally finite connected graph. For every $v \in V$ the *neighborhood* of v is the set

$$\mathcal{N}_v := \{u \in V \mid uv \in E\}.$$

A *simple random walk* (s.r.w.) on G is a discrete-time stationary Markov chain $\{X_n\}_{n\geq 0}$ with values in V, such that the transition probabilities have a uniform distribution on the neighborhood:

$$\mathbb{P}(X_{n+1} = v_2 \mid X_n = v_1) = \ldots = \mathbb{P}(X_1 = v_2 \mid X_0 = v_1) = \begin{cases} \dfrac{1}{\#\mathcal{N}_{v_1}} & \text{if } v_1 v_2 \in E \\ 0 & \text{otherwise} \end{cases}.$$

We will use the notation $\mathbb{P}^v(\cdot) := \mathbb{P}(\cdot \mid X_0 = v)$. See [Wo] for definitions and results about Markov chains. We just recall the significant notions of *transience* and *recurrence*.

Definition 4.2. We say that a locally finite connected graph $G = (V, E)$ is *transient* (with respect to a simple random walk) if for each $v \in V$

$$\mathbb{P}^v(\forall\, n > 0,\ X_n \neq v) > 0, \tag{4.1}$$

where $\{X_n\}_{n\geq 0}$ denotes a simple random walk on G. Since the graph G is connected, the fact that (4.1) holds for a single $v \in V$ implies that (4.1) holds for each vertex, so the graph G is transient. By Markov property it follows that $\mathbb{P}^v(X_n = v \text{ for infinite } n) = 0$.

A *recurrent* graph is a graph which is not transient: for each $v \in V$

$$\mathbb{P}^v(\exists\, n > 0,\ X_n = v) = 1$$

and by Markov property $\mathbb{P}^v(X_n = v \text{ for infinite } n) = 1$.

4.2 The tiling of a cylinder

We now assert a result by I. Benjamini and O. Schramm (from the paper [BS]): a square tiling is associated with a graph G which has an embedding in $\mathbb{R}^2$. The lengths are in terms of probability and the covered figure is a cylinder (not a rectangle).

Definition 4.3. Let $G = (V, E)$ be a planar, locally finite graph, embedded in $\mathbb{R}^2$. We say that a subset $W \subset V$ is *absorbing* if there is a nonzero probability that a random walk reaches $V \setminus W$ only for a finite number of times:

$$\mathbb{P}^v(X_n \notin W \text{ for finite } n) > 0 \quad \text{for all } v \in V\ ^1.$$

We say that G is *uniquely absorbing* if, considering the embedding G_0 of an arbitrary finite subgraph of G, there exists a unique connected component D_0 of $\mathbb{R}^2 \setminus G_0$ such that $V \cap D_0$ is absorbing. Note that this definition strongly depends on the way the graph is embedded in the real plane.

Definition 4.4. An *oriented* edge $\vec{\alpha}$ of G is an edge of the graph together with an orientation: one extreme is the *beginning* of the edge α [b($\vec{\alpha}$)], and the

[1] This property is independent from the starting point of the s.r.w., so we could write 'for some $v \in V$' (equivalently).

other one is the *end* of the edge α [$\mathrm{e}(\vec{\alpha})$]. We will use the vector notation $\vec{\alpha}$. The opposite orientation is denoted by $-\vec{\alpha}$: $\mathrm{b}(-\vec{\alpha}) = \mathrm{e}(\vec{\alpha})$ and $\mathrm{e}(-\vec{\alpha}) = \mathrm{b}(\vec{\alpha})$. The notation $|\vec{\alpha}| = |-\vec{\alpha}|$ denotes the 'not oriented' edge corresponding to $\vec{\alpha}$ (and its opposite).

Theorem 4.1 ([BS] Theorem 4.1)**.** *Let $G = (V, E)$ be a planar connected (locally finite) graph. Suppose that G is transient and there exists an embedding in $\mathbb{R}^2$ so that G is uniquely absorbing. We fix a 'root' vertex $o \in V$ and we define, for each $v \in V$, $h(v)$ as the probability that a simple random walk starting from v will never reach the root o at a finite time:*

$$h(v) := \mathbb{P}^v(\forall\, n \geq 0,\, X_n \neq o).$$

Let $\vec{E}(v)$ be the set of oriented edges $\vec{\alpha}$ in G such that $\mathrm{b}(\vec{\alpha}) = v$, let $\vec{E}_+(v) := \{\vec{\alpha} \in \vec{E}(v) |\, h(v) \leq h(\mathrm{e}(\vec{\alpha}))\}$ and $\vec{E}_-(v) := \{\vec{\alpha} \in \vec{E}(v) |\, h(v) \geq h(\mathrm{e}(\vec{\alpha}))\}$. We call $\eta := \sum_{\vec{\alpha} \in \vec{E}(o)} h(\mathrm{e}(\vec{\alpha}))$ and we consider the cylinder $S \times [0, 1[$ where $S = \mathbb{R}/\eta\mathbb{Z} \cong \mathbb{S}^1$. Then there exists a 'square' tiling [2] $\mathcal{T}$ such that

1. *$\mathcal{T} = \{Q_\alpha\}_{\alpha \in E}$.*

2. *If u, v are the extremes of $\alpha \in E$ and $h(u) \leq h(v)$, then $Q_\alpha = \theta_\alpha \times [h(u), h(v)]$, where θ_α is an arc of S whose length is $h(v) - h(u)$.*

3. *If we call $\theta(v) = \bigcup_{\vec{\alpha} \in \vec{E}(v)} \theta_{|\vec{\alpha}|}$, then $\theta(v)$ is connected and whenever $v \neq o$ we have*
$$\theta(v) = \bigcup_{\vec{\alpha} \in \vec{E}_+(v)} \theta_{|\vec{\alpha}|} = \bigcup_{\vec{\alpha} \in \vec{E}_-(v)} \theta_{|\vec{\alpha}|}.$$

4. *For almost every $x \in S$ and for every $0 \leq T < 1$, $\{x\} \times [0, T]$ is contained in the union of finitely many squares of $\mathcal{T}$.*

We do not prove the Theorem, but we refer to [BS]. The article by A. Georgakopoulos [Ge] shows more details on this construction.

Remark 4.1. The squares Q_α are at a height which becomes smaller (closer to 0) as the extremes of α get closer to o. The third statement of Theorem

[2] The tiles are in fact *curved* squares.

4.1 may be interpreted in this way: each vertex v corresponds to a connected 'horizontal' arc at a height of $h(v)$, and the squares intersecting this arc are the ones associated with the edges connecting v; these squares lie just above [under] $\theta(v)$ if the relative edges belongs in $\vec{E}_+(v)$ [$\vec{E}_-(v)$] and the intersections with $\theta(v)$ correspond the whole bottom [upper] side of the tiles. The fourth point can be explained by saying that there might be some 'irregular' vertical sequences of tiles, but their projection over S has null measure.

Remark 4.2. We can think Theorem 4.1 as a generalization on graphs of the easiest implication in Theorem 3.1 [(∗), case $p = 2$], which assures the existence of a tiling with the combinatorial structure given by a tree T, as shown in Section 3.3 (an equilibrium measure gives the dimension of each tile). In the case of graphs, the map on edges

$$\alpha \longmapsto h(\mathrm{e}(\alpha)) - h(\mathrm{b}(\alpha)) = \mathbb{P}^{\mathrm{e}(\alpha)}(\forall\, n \geq 0,\, X_n \neq o) - \mathbb{P}^{\mathrm{b}(\alpha)}(\forall\, n \geq 0,\, X_n \neq o)$$

can be thought as the 'equilibrium measure' on the infinite boundary of the graph.

It may be interesting to show if the 'inverse implication' holds, i.e. we would like to know if every tiling of the cylinder $S \times [0, 1[$, whose structure comes from the graph G as in Theorem 4.1, is associated to a kind of 'equilibrium measure' like the one above (possibly referred to a proper subset in the boundary of G).

4.3 Trees, equilibrium measures and probability

Proposition 4.2. *Let $p = 2$. Suppose that T is a tree with a root and no leaves, as in Chapter 2. Then T is a transient graph if and only if $Cap(\partial T) > 0$.*

Proof. This result is a particular case of [LP], Theorem 2.11: it asserts that T is transient if and only if there exists a function $\Theta \in \ell^2_+(E(T))$ such that

$$\Theta(\omega) = 1, \quad \Theta(\alpha) = \sum_{\beta,\, \mathrm{e}(\alpha)=\mathrm{b}(\beta)} \Theta(\beta) \quad \text{for each } \alpha \in E(T).$$

If $Cap(\partial T) > 0$, then we choose $\Theta = Cap(\partial T)^{-1} f^{\partial T}$ (a rescaling of the equilibrium function). The function Θ has the requested properties since $f^{\partial T}(\omega) = I^* \mu^{\partial T}(\omega) = Cap(\partial T)$ and $f^{\partial T}$ satisfies (2.8) with $p = p' = 2$.

On the contrary, supposing that there exists Θ as above, we can build a measure $\mu_\Theta \in \mathcal{M}_+(\partial T)$ by setting for each $\alpha \in E(T)$

$$\mu_\Theta(\partial T_\alpha) = \Theta(\alpha) \Longrightarrow \Theta = I^* \mu_\Theta.$$

In particular $\mu_\Theta(\partial T) = I^* \mu_\Theta(\omega) = \Theta(\omega) = 1$. Since Θ has finite ℓ^2-norm, by dual definition of capacity (Theorem 1.12)

$$Cap(\partial T) = \sup_{\mu \neq 0} \frac{\mu(\partial T)^2}{\|I^* \mu\|_{\ell^2}^2} \geq \frac{\mu_\Theta(\partial T)^2}{\|I^* \mu_\Theta\|_{\ell^2}^2} = \frac{1}{\|\Theta\|_{\ell^2}^2} > 0. \qquad \square$$

Proposition 4.3. *Let $Cap(\partial T) > 0$. For each $x \in V(T)$*

$$\mathbb{P}^x(\exists \lim_{n \longrightarrow +\infty} X_n \in \partial T) = 1.$$

Proof. From Proposition 4.2 we know that T is transient, so a s.r.w. on T crosses a generic vertex x for infinite times with zero probability. Now $\{X_n\}$ is a random sequence on $(\overline{T}, \rho)$, which is a complete metric space [See Section 2.2]. If it does not converge, it is not a Cauchy sequence; but then we deduce that the natural distance d from the root is bounded, and this implies that X_n passes through a finite number of vertices. So there exists a vertex crossed by X_n for infinite times and $\mathbb{P}^x(\nexists \lim_{n \longrightarrow +\infty} X_n) = 0$ by transience. If X_n converges to $y \in V(T)$, then $X_n = y$ definitively, as y is an isolated point of $\overline{T}$: the probability is obviously null. Hence X_n converges to the boundary of T almost surely. $\square$

Corollary 4.4. *Let $Cap(\partial T) > 0$. Then T has a uniquely absorbing embedding in the real plane.*

Sketch of the proof. The tree T has an embedding in $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus T_0$ is connected for each finite subgraph T_0 of T. By Proposition 4.3 it is clear that the set of vertices in $\mathbb{R}^2 \setminus T_0$ is absorbing. $\square$

Let $Cap(\partial T) > 0$ and let $\mu = \mu^{\partial T}$ be the equilibrium measure of the boundary: following Section 3.3 we can build a square tiling $\mathcal{T}_\mu = \{Q_\alpha\}_{\alpha \in E(T)}$

of $R = [0, Cap(\partial T)] \times [0,1]$; the side of Q_α has a length of $\mathsf{M}(\alpha) = I^*\mu(\alpha)$. If we do a 180-degrees rotation to R and we paste the left side on the right one, we get the tiling of a cylinder which satisfies each property of the construction in Theorem 4.1. This suggests the idea that M is somehow related to the 'probability' function h in this sense: for all $\alpha \in E(T)$

$$\mathsf{M}(\alpha) = h(\mathrm{e}(\alpha)) - h(\mathrm{b}(\alpha)) =: dh(\alpha).$$

We explicit the probabilistic meaning of dh:

$$dh(\alpha) = \mathbb{P}^{\mathrm{e}(\alpha)}(\forall\, n \geq 0,\ X_n \neq o) - \mathbb{P}^{\mathrm{b}(\alpha)}(\forall\, n \geq 0,\ X_n \neq o) =$$

$$= \mathbb{P}^{\mathrm{b}(\alpha)}(\exists\, n \geq 0,\ X_n = o) - \mathbb{P}^{\mathrm{e}(\alpha)}(\exists\, n \geq 0,\ X_n = o) = {}^{3}$$

$$= \mathbb{P}^{\mathrm{b}(\alpha)}(\exists\, n \geq 0,\ X_n = o) - \mathbb{P}^{\mathrm{b}(\alpha)}(\exists\, n \geq 0,\ X_n = o)\mathbb{P}^{\mathrm{e}(\alpha)}(\exists\, n \geq 0,\ X_n = \mathrm{b}(\alpha)) =$$

$$= \mathbb{P}^{\mathrm{b}(\alpha)}(\exists\, n \geq 0,\ X_n = o)[1 - \mathbb{P}^{\mathrm{e}(\alpha)}(\exists\, n \geq 0,\ X_n = \mathrm{b}(\alpha))] =$$

$$= \boldsymbol{\mathbb{P}^{\mathrm{b}(\alpha)}(\exists\, n \geq 0,\ X_n = o)\mathbb{P}^{\mathrm{e}(\alpha)}(\forall\, n \geq 0,\ X_n \neq \mathrm{b}(\alpha))}.$$

In particular $Cap(\partial T) = \mathsf{M}(\omega) = \mathbb{P}^{\mathrm{e}(\omega)}(\forall\, n \geq 0,\ X_n \neq o)$: the capacity of ∂T corresponds to the *probability that a simple random walk from* $\mathrm{e}(\omega)$ *never visits the root vertex* o. This fact is also true in the recurrent case $Cap(\partial T) = 0$, as every s.r.w. reaches o with probability 1.

Furthermore, for each $x \in V(T)$

$$I\mathsf{M}(x) = \sum_{\alpha \in P(x)} \mathsf{M}(\alpha) = \ldots = h(x) - \underbrace{h(o)}_{=0} = h(x) = \mathbb{P}^x(\forall\, n \geq 0,\ X_n \neq o)$$

$$\Longrightarrow 1 - I\mathsf{M}(x) = \mathbb{P}^x(\exists\, n \geq 0,\ X_n = o).$$

Observe that this probabilistic interpretation of capacity is consistent with rescaling on the subtree T_α:

$$Cap_\alpha(\partial T_\alpha) = \frac{\mathsf{M}(\alpha)}{1 - I\mathsf{M}(\mathrm{b}(\alpha))} = \mathbb{P}^{\mathrm{e}(\alpha)}(\forall\, n \geq 0,\ X_n \neq \mathrm{b}(\alpha))$$

for all $\alpha \in E(T)$.

[3] By the Markov property, since every walk from $\mathrm{e}(\alpha)$ to o crosses $\mathrm{b}(\alpha)$.

Remark 4.3. We recall that the function M is *additive*, in the sense that

$$\mathsf{M}(\alpha) = \sum_{\beta,\, \mathrm{e}(\alpha)=\mathrm{b}(\beta)} \mathsf{M}(\beta) \quad \text{for each } \alpha \in E(T).$$

The fact that also dh has the additivity property, which is stated in the third point of Theorem 4.1,is a simple exercise on Markov chains. Let $\alpha \in E(T)$ and $\alpha_1, \dots \alpha_m$ its sons (i.e. $\mathrm{e}(\alpha) = \mathrm{b}(\alpha_j)$ for each $j = 1, \dots, m$). By Markov property we deduce that h has the *mean value property*:

$$h(\mathrm{e}(\alpha)) = \mathbb{P}^{\mathrm{e}(\alpha)}(\forall\, n \geq 0,\, X_n \neq o) =$$

$$= \frac{1}{m+1}\left(\mathbb{P}^{\mathrm{b}(\alpha)}(\forall\, n \geq 0,\, X_n \neq o) + \sum_{j=1}^{m} \mathbb{P}^{\mathrm{e}(\alpha_j)}(\forall\, n \geq 0,\, X_n \neq o)\right) =$$

$$\frac{1}{m+1}\left(h(\mathrm{b}(\alpha)) + \sum_{j=1}^{m} h(\mathrm{e}(\alpha_j))\right).$$

It follows that

$$\sum_{j=1}^{m} dh(\alpha_j) = \left(\sum_{j=1}^{m} h(\mathrm{e}(\alpha_j))\right) - (m+1)h(\mathrm{e}(\alpha)) + h(\mathrm{e}(\alpha)) =$$

$$= h(\mathrm{e}(\alpha)) - h(\mathrm{b}(\alpha)) = dh(\alpha).$$

Remark 4.4. If we consider a closed subset $A \subset \partial T$, by reasoning as above on the subtree T_A made up with the edges of $P(A)$, we could get the following probabilistic interpretation of capacity and equilibrium measure:

$$Cap(A) = \mathbb{P}^{\mathrm{e}(\omega)}(\forall\, n \geq 0,\, X_n^A \neq o),$$

$$\mu^A(\partial T_\alpha) = \mathsf{M}^A(\alpha) = \mathbb{P}^{\mathrm{b}(\alpha)}(\exists\, n \geq 0,\, X_n^A = o)\mathbb{P}^{\mathrm{e}(\alpha)}(\forall\, n \geq 0,\, X_n^A \in S(\alpha)),$$

where $\alpha \in E(T_A)$ and X_n^A is a simple random walk on T_A. Moreover, since $\partial T_A = A$ and

$$\mathbb{P}^{\mathrm{e}(\omega)}(\exists \lim X_n^A \in A) = \begin{cases} 1 & \text{if } Cap(A) > 0, \text{ by Proposition 4.3} \\ 0 & \text{if } Cap(A) = 0, \text{ by recurrence of } T_A \end{cases},$$

then we may say that $Cap(A)$ is the *probability that a simple random walk on T_A starting from* $\mathrm{e}(\omega)$ *reaches A (at the limit), without visiting the root vertex o.*

If T is a tree with some leaves, we need to make some adjustments, but eventually we obtain an analogous result: the capacity of a set $A \subset \partial T$ is the *probability that simple random walk on* T_A *starting from* $\mathrm{e}(\omega)$ *reaches* A *(at a finite time or at the limit) before visiting the root vertex* o *for the first time.*

Chapter 5

Finite tilings with prescribed combinatorics

5.1 Triangulations of a quadrilateral

Now we explain a construction from the paper [Sc] written by O. Schramm. A particular kind of graphs is considered. Let $\Delta = (V, E, F)$ be a *finite* triangulation of a closed topological 2-dimensional disk: we mean that Δ is a *(geometric) simplicial complex* which is also a planar disk as a topological space (up to homeomorphisms); V is the set of *vertices* (0-simplexes), E is the set of *edges* (1-simplexes) and F is the set of triangular *faces* (2-simplexes). The juxtaposition uvw denotes the face delimited by the vertices u, v and w. Suppose that the (triangulated) boundary of the disk can be split into 4 nontrivial arcs:

$$\partial\Delta = B_1 \cup B_2 \cup B_3 \cup B_4.$$

The B_j are unions of adjacent edges in E such that $B_j \cap B_k = \emptyset$ if $k - j \equiv 2 \pmod 4$ and it reduces to a vertex when $k - j \equiv \pm 1 \pmod 4$. The collection

$$\Delta = (V, E, F; B_1, B_2, B_3, B_4)$$

is called a *triangulation of a quadrilateral*. An example is shown in Figure 5.1. We intend to associate Δ with a square tiling of a rectangle which has some good combinatorial properties with respect to Δ.

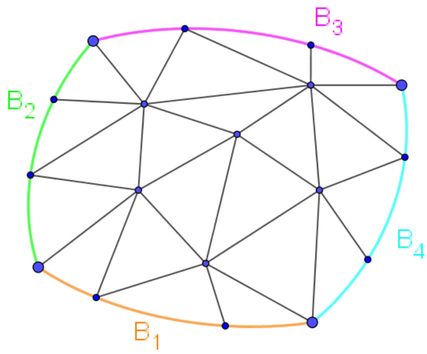

Figure 5.1: A triangulation of a quadrilateral

Theorem 5.1 ([Sc], Theorem 1.3)**.** *Let* $\Delta = (V, E, F; B_1, B_2, B_3, B_4)$ *be a triangulation as above. Then there exists a tiling* $\mathcal{T}_\Delta = \{Q_v\}_{v \in V}$ *(indexed by the vertices) which covers a rectangle* $R = [0, h^{-1}] \times [0, h]$ *of unitary area so that*

$$uv \in E \Longrightarrow Q_u \cap Q_v \neq \emptyset. \tag{5.1}$$

Furthermore, if we distinguish the four edges of R

$$R_1 = [0, h^{-1}] \times \{0\}, \quad R_2 = \{0\} \times [0, h],$$

$$R_3 = [0, h^{-1}] \times \{h\}, \quad R_4 = \{h^{-1}\} \times [0, h],$$

we require that

$$v \in B_j \Longleftrightarrow Q_v \cap R_j \neq \emptyset. \tag{5.2}$$

Under these restrictions, the tiling $\mathcal{T}_\Delta$ *and the height* h *of the rectangle* R *are uniquely determined.*

Remark 5.1. An example of what we get is displayed in Figure 5.2. It may happen that two squares intersect at a common corner point although the corresponding vertices are not connected by an edge. This phenomenon is due to the fact that boundaries of squares are not smooth. Moreover some tiles could degenerate into points (they may be thought as squares whose side has null length).

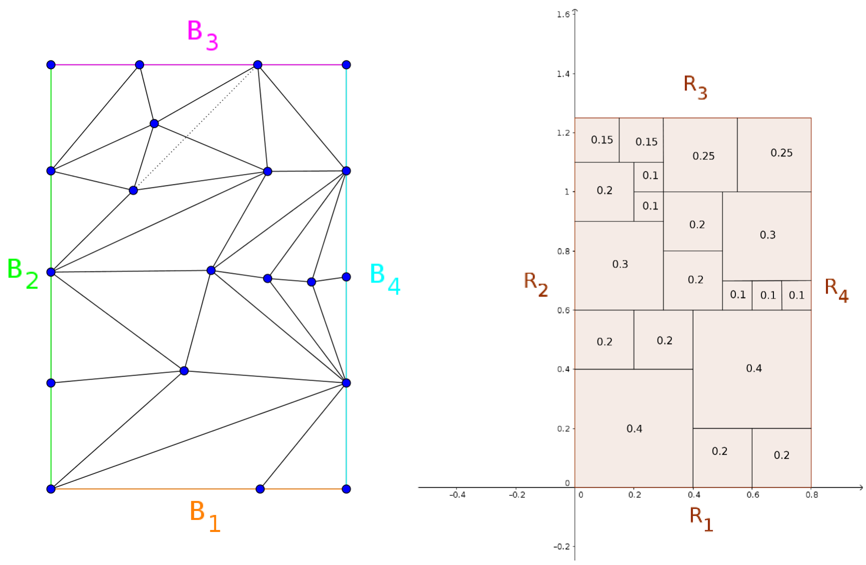

Figure 5.2: A triangulation of a quadrilateral and its associated tiling with the relative dimensions. The dotted line in the left figure is not an edge of the triangulation, but the corresponding squares have a common corner. In this example $h = 5/4$.

5.2 Extremal length on graphs

In order to prove the preceding Theorem, a significant tool is the definition of *(discrete) extremal length* on a graph.

Definition 5.1. Let $G = (V, E)$ be a finite connected graph, and let U_1, U_2 be two disjoint nontrivial subsets of V. We call a *metric* on G a generic nonnegative function $m : V \longrightarrow [0, +\infty[$ [1]. For each path (of vertices)

$$\gamma = \{v_0, \ldots, v_n\} \qquad \text{such that } v_{j-1}v_j \in E \text{ for all } j = 1, \ldots, n$$

we define the *m-length* of γ as

$$\lambda_m(\gamma) := \sum_{v \in \gamma} m(v) = \sum_{j=0}^{n} m(v_j).$$

Note that a path made by a single vertex v is of m-length $m(v)$, which could be strictly positive. We denote by $\Gamma(U_1, U_2)$ the set of all paths starting from

[1] As in [Sc], the metric is defined on vertices, instead of edges.

a point in U_1 and finishing at some point in U_2. The (U_1, U_2)-*length* of the metric m is

$$l(m) := \inf_{\gamma \in \Gamma(U_1, U_2)} \lambda_m(\gamma).$$

Let $1 < p < +\infty$. The *p-extremal length* of $(G; U_1, U_2)$ is

$$L_p(G; U_1, U_2) := \sup_{m \neq 0} \underbrace{\frac{l(m)^p}{\|m\|_p^p}}_{\widehat{l}(m)}. \tag{5.3}$$

where m is a generic metric on the vertices of G.

Remark 5.2. Observe that for every constant factor $\kappa > 0$ we have that

$$\lambda_{\kappa m}(\gamma) = \kappa \lambda_m(\gamma) \Longrightarrow l(\kappa m) = \kappa l(m), \quad \widehat{l}(\kappa m) = \widehat{l}(m).$$

Thus (5.3) can be rewritten in these ways by homogeneity:

$$L_p(G; U_1, U_2) := \sup_{m \neq 0} \widehat{l}(m) = \sup_{\|m\|_p = 1} l(m)^p = \frac{1}{\inf\limits_{l(m)=1} \|m\|_p^p}. \tag{5.4}$$

Observe that the last term looks like the reciprocal of a capacity: we will explain this fact in Section 5.4.

Lemma 5.2 ([Sc], Lemma 3.1)**.** *Let $G = (V, E)$ be as above and fix the disjoint sets of vertices U_1, U_2. There exists an* extremal metric m_0*, i.e. a metric for which the supremum in* (5.3) *is reached:*

$$\widehat{l}(m_0) = \max_{m \neq 0} \widehat{l}(m).$$

Up to a multiplication by a positive constant factor, it is unique.

Proof. We consider the subset $\mathfrak{M}_1 = \{m |\, l(m) \geq 1\}$. It is clearly nonempty and $0 \notin \mathfrak{M}_1$. $\mathfrak{M}_1$ is convex: l acts on metrics as a super-additive operator and then it follows that

$$l(\kappa m + (1-\kappa)\widetilde{m}) \geq \kappa l(m) + (1-\kappa) l(\widetilde{m}) \geq \kappa + 1 - \kappa = 1$$

for every $\kappa \in [0,1]$ and $m, \widetilde{m} \in \mathfrak{M}_1$. Furthermore, $\mathfrak{M}_1$ is closed: since the number of vertices is finite, then if we pick $m_j \longrightarrow m_*$ with $m_j \in \mathfrak{M}_1$ we get that

$$1 \leq \lambda_{m_j}(\gamma) = \sum_{v \in \gamma} m_j(v) \overset{j \longrightarrow +\infty}{\longrightarrow} \sum_{v \in \gamma} m_*(v) = \lambda_{m_*}(\gamma)$$

for every path $\gamma \in \Gamma(U_1, U_2)$, so $l_{m_*} \geq 1$ and $m_* \in \mathfrak{M}_1$. Using Proposition 1.3, there exists a unique element $m_0 \in \mathfrak{M}_1$ so that $\|m_0\|_p = \min_{l(m)\geq 1} \|m\|_p$. It is trivial to show that $l(m_0) = 1$, so the statement of the Lemma follows from (5.4). □

5.3 Proof of Theorem 5.1

Let $\Delta = (V, E, F; B_1, B_2, B_3, B_4)$ be a triangulation as in the statement of Theorem 5.1. $G = (V, E)$ is the 1-skeleton of the simplicial complex Δ. We first show that if a square tiling with the requested properties exists, then the dimension of all the tiles necessarily gives an extremal metric, relative to the opposite sides B_1 and B_3 of the quadrilateral. (In the following, we fix $U_1 = B_1$, $U_2 = B_3$).

Lemma 5.3 ([Sc], Lemma 4.1)**.** *Let $\mathcal{T}_\Delta = \{Q_v\}$ be a square tiling satisfying* (5.1)*,* (5.2) *and covering the rectangle $R = [0, h^{-1}] \times [0, h]$ of unitary area. Let $s(v)$ be the side length of Q_v and fix $p = 2$. Then s is an extremal metric, with respect to $(G; B_1, B_3)$:*

$$L_2(G; B_1, B_3) = \widehat{l}(s) = l(s)^2.$$

In particular it is the unique extremal metric of unitary 2*-norm, and the height of R is the (B_1, B_3)-length of s.*

Proof. Considering the metric s, is it easy to show that $\|s\|_2 = 1$ and $l(s) = h$.

Now let m be a generic metric on the vertices which is not identically zero. For each $t \in [0, h^{-1}]$ we consider the vertical line $r_t = \{t\} \times \mathbb{R}$ and the set $\widetilde{\gamma}_t = \{v \in V | \, r_t \cap Q_v \neq \emptyset\}$. By the properties of $\mathcal{T}_\Delta$, $\widetilde{\gamma}_t$ contains a path from B_1 to B_3 (see Figure 5.3) and thus

$$l(m) \leq \sum_{v \in \widetilde{\gamma}_t} m(v) = \sum_{v \in V} m(v) \chi_{\widetilde{\gamma}_t}(v).$$

If we integrate over $t \in [0, h^{-1}]$, then

$$h^{-1} l(m) \leq \int_0^{h^{-1}} \sum_{v \in V} m(v) \chi_{\widetilde{\gamma}_t}(v) dt \overset{\text{(Fubini)}}{=} \sum_{v \in V} m(v) \int_0^{h^{-1}} \chi_{\widetilde{\gamma}_t}(v) dt.$$

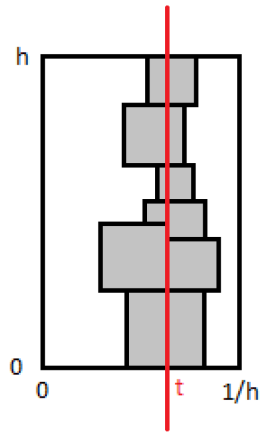

Figure 5.3: The vertical line r_t crosses a sequence of squares associated with a path (of vertices) from the bottom side to the upper side.

The last integral is equal to $s(v)$, hence

$$\frac{l(m)}{l(s)} \leq \sum_{v \in V} m(v)s(v) \leq \|m\|_2 \|s\|_2 = \|m\|_2$$

by using Cauchy-Schwarz inequality. We conclude that s is an extremal metric since for every metric $m \neq 0$

$$\widehat{l}(m) = \frac{l(m)^2}{\|m\|_2^2} \leq l(s)^2 = \widehat{l}(s). \qquad \square$$

Theorem 5.4 ([Sc], Theorem 5.1)**.** *Let $\Delta = (V, E, F; B_1, B_2, B_3, B_4)$ be a triangulation of a quadrilateral and let $G = (V, E)$ be its 1-skeleton. Fixed $p = 2$, we consider the extremal metric for $(G; B_1, B_3)$ s such that $\|s\|_2 = 1$ (it is unique by Lemma 5.2). Let define for each $v \in V$:*

$$x(v) = \inf_{\gamma \in \Gamma(B_2, \{v\})} \lambda_s(\gamma), \quad y(v) = \inf_{\gamma \in \Gamma(B_1, \{v\})} \lambda_s(\gamma),$$

$$Q_v = [x(v) - s(v), x(v)] \times [y(v) - s(v), y(v)].$$

Then $\{Q_v\}_{v \in V}$ is a tiling of the rectangle $R = [0, h^{-1}] \times [0, h]$ with $h = l(s)$, which satisfies the properties (5.1) *and* (5.2)*.*

Proof. We first prove that the tiling $\{Q_v\}$ satisfies (5.1). First we observe that $x(v) \geq s(v)$ and $y(v) \geq s(v)$ for each vertex $v \in V$. Let $uv \in E$ be an edge of G. Then a path in $\Gamma(B_2, \{v\})$ can be obtained by following a path from B_2 to u and then walking along uv. So we infer that

$$x(v) \leq x(u) + s(v) \Longrightarrow 0 \leq x(v) - s(v) \leq x(u).$$

The role of u and v can be interchanged and, if we consider B_1 instead of B_2, we get the same inequalities with y replacing x. So it follows that

$$[x(u) - s(u), x(u)] \cap [x(v) - s(v), x(v)] \neq \emptyset$$

$$[y(u) - s(u), y(u)] \cap [y(v) - s(v), y(v)] \neq \emptyset,$$

otherwise some of the inequalities we just got would not be true. We conclude that $Q_u \cap Q_v \neq \emptyset$.

Now we define

$$\widehat{R}_1 = [0, +\infty[\times\{0\}, \quad \widehat{R}_2 = \{0\} \times [0, +\infty[,$$

$$\widehat{R}_3 = [0, +\infty[\times[h, +\infty[, \quad \widehat{R}_4 = [h^{-1}, +\infty[\times[0, +\infty[.$$

If $v \in B_1$, then $y(v) = s(v)$ and so Q_v intersects $\widehat{R}_1$; if $v \in B_2$, then $x(v) = s(v)$ and thus $Q_v \cap \widehat{R}_2 \neq \emptyset$; if v belongs to B_3, then

$$h = l(s) = \inf_{\gamma \in \Gamma(B_1, B_3)} \lambda_s(\gamma) \leq \inf_{\gamma \in \Gamma(B_1, \{v\})} \lambda_s(\gamma) = y(v),$$

which means that Q_v has some points in common with the region $\widehat{R}_3$.

The proof of $v \in B_4 \Longrightarrow Q_v \cap \widehat{R}_4 \neq \emptyset$ is more difficult. Let γ be a path of least s-length connecting B_2 to $v \in B_4$. It is enough to prove that $\lambda_s(\gamma) = x(v)$ is greater than h^{-1}. For $t \geq 0$ we define the metrics s_t:

$$s_t = s + t\chi_\gamma.$$

For every path in $\Gamma(B_1, B_3)$ must cross γ at least once, we deduce that

$$l(s_t) \geq l(s) + t \Longrightarrow \left.\frac{d}{dt}\right|_{t=0^+} l(s_t) = \lim_{t \longrightarrow 0^+} \frac{l(s_t) - l(s)}{t} \geq 1. \tag{5.5}$$

Since $s_0 = s$ is the extremal metric of unitary norm, $\widehat{l}(s) \geq \widehat{l}(s_t)$ and

$$0 \geq \left.\frac{d}{dt}\right|_{t=0^+} \widehat{l}(s_t) = \left.\frac{d}{dt}\right|_{t=0^+} \frac{l(s_t)^2}{\|s_t\|_2^2} \overset{(5.5)}{\geq} 2l(s) - l(s)^2 \left.\frac{d}{dt}\right|_{t=0^+} \|s_t\|_2^2 =$$

$$= 2h - h^2 \sum_{v\in\gamma} \frac{d}{dt}\bigg|_{t=0^+} (s(v)+t)^2 = 2\left(h - h^2 \sum_{v\in\gamma} s(v)\right) = 2\left(h - h^2\lambda_s(\gamma)\right),$$

$$\Longrightarrow \lambda_s(\gamma) \geq h/h^2 = h^{-1}.$$

Now we prove that $R \subset \bigcup_{v\in V} Q_v$. We assign parameterizations by a standard 2-simplex to each face of Δ, with compatibility relations on edges. So we can consider the center of a face (or an edge) in Δ. We construct a piecewise linear map f from Δ to $\bigcup_{v\in V} Q_v$:

- For each $v \in V$, let $f(v)$ be a point in Q_v. If $v \in B_j$, then we choose $f(v) \in \widehat{R}_j$. There are no troubles: if $v \in B_j \cap B_k$, $k - j \equiv \pm 1 \pmod 4$, then it is easy to show that $Q_v \cap \widehat{R}_j \cap \widehat{R}_k$ is nonempty.

- Let $uv \in E$. Let $b(uv)$ be the center of uv. Then we pick $f(b(uv)) \in Q_u \cap Q_v$. If u and v lie on a side B_j, then Q_u, Q_v and $\widehat{R}_j$ intersect each other at pairs: so it is a geometric fact that $Q_u \cap Q_v \cap \widehat{R}_j \neq \emptyset$ and we choose $f(b(uv))$ in this intersection.

- Let $uvw \in F$ and let $b(uvw)$ be its center. Then Q_u Q_v and Q_w intersect each other at pairs: again we deduce that their intersection is not empty and we pick $f(b(uvw)) \in Q_u \cap Q_v \cap Q_w$.

- We extend the map f so that it becomes an affine map on each face of the barycentric subdivision Δ' [2].

$f(\Delta) = \bigcup_{v\in V} Q_v$ follows since f maps all the vertices of a generic face in Δ' to points in a certain square Q_{v_0}, observing that f is affine on the face and Q_{v_0} is convex. Let consider the restriction of f along $\partial\Delta$: its image is a polygonal path in $\mathbb{R}^2 \setminus \operatorname{int} R$. So we deduce that $f(\partial\Delta)$ is homotopic to $\partial R = R_1 \cup R_2 \cup R_3 \cup R_4$ in $\mathbb{R}^2 \setminus \operatorname{int} R$.

On the other side, Δ is a closed disk (as a topological space), hence it is simply connected and $\partial\Delta$ is homotopic to a constant in Δ. As f is a

[2] It is the simplicial complex whose faces are of the form $u\,b(uv)\,b(uvw)$, where uvw is a generic face of Δ.

continuous function,

$$\begin{cases} f(\partial\Delta) \sim \text{constant} & \text{in } \bigcup_{v\in V} Q_v \\ f(\partial\Delta) \sim \partial R & \text{in } \mathbb{R}^2 \setminus \operatorname{int} R \end{cases} \Longrightarrow$$

$$\Longrightarrow \partial R \sim \text{constant in } \mathcal{S} := \left(\bigcup_{v\in V} Q_v\right) \cup \left(\mathbb{R}^2 \setminus \operatorname{int} R\right). \tag{5.6}$$

So the tiles Q_v must include the whole rectangle R: otherwise ∂R would be a simple loop in $\mathcal{S}$ turning around some 'holes' in the space, thus it would be $\partial R \nsim$ constant in $\mathcal{S}$, contradicting (5.6).

Moreover $1 = \text{area}(R) \leq \text{area}\left(\bigcup_{v\in V} Q_v\right) \leq \sum_{v\in V} s(v)^2 = \|s\|_2^2 = 1$, so we deduce that

$$R = \bigcup_{v\in V} Q_v, \quad \text{area}\left(\bigcup_{v\in V} Q_v\right) = \sum_{v\in V} \text{area}(Q_v) = 1.$$

As a consequence we have a tiling of R (the second relation implies that the squares do not overlap).

To conclude, we shall verify (5.2). If $v \in B_j$, then we already know that Q_v intersects $\widehat{R}_j$, $Q_v \subset R$ and $\widehat{R}_j \cap R = R_j$, so we infer that $Q_v \cap R_j \neq \emptyset$. Let $v \notin \bigcup_j B_j$. If we consider the neighborhood $\mathcal{N}_v = \{u \in V | \, uv \in E\}$, there exist a circular path on the graph which turns around v reaching all the vertices of $\mathcal{N}_v$, since the graph is the 1-skeleton of a triangulation. Hence Q_v is surrounded by a finite sequence of tiles touching each other cyclically, so it cannot lie on any side of R. If v belongs only to one side B_k of the quadrilateral, we find a simple path connecting all the vertices in the neighborhood which starts and ends at B_k, so one side of Q_v lies on R_k, and the other three are surrounded by a finite number of consecutive tilings, hence Q_v does not intersect R_j as $j \neq k$. Analogously, if we take $v \in B_j \cap B_k$ we deduce that Q_v touches only the two sides R_j and R_k. Eventually, (5.2) is proven. □

Proof of Theorem 5.1. Theorem 5.4 assures the existence of $\mathcal{T}_\Delta$. Uniqueness follows from Lemma 5.3: indeed, if we could build two different tilings $\{Q_v\}$, $\{\widetilde{Q}_v\}$ with the requested combinatorial properties, then we would find $v_0 \in V$

such that Q_{v_0} and $\widetilde{Q}_{v_0}$ have different sizes, obtaining two different extremal metrics of unitary norm: this contradicts the uniqueness assertion of Lemma 5.2. □

5.4 What we can say about trees?

Let T be a *finite* tree with a root edge $\omega = [o, o']$. The boundary ∂T contains all the leaves. From Remark 3.3 we have a bijection between $E(T)$ and $V(T) \setminus \{o\}$ (each edge is identified with its end), so there is an 'identity'

$$\{m : V(T) \setminus \{o\} \to \mathbb{R}_+\} \longleftrightarrow \{f : E(T) \to \mathbb{R}_+\}$$

$$m \longmapsto f, \quad f(\alpha) = m(\mathrm{e}(\alpha)), \quad \|f\|_2 = \|m\|_2.$$

So a positive function f defined on $E(T)$ is identified with a metric m defined on the 'uprooted' tree T^* (i.e. the tree T without the root vertex o and the root edge ω). We choose $U_1 = \{o'\}$ and $U_2 = \partial T = \partial T^*$. $\Gamma = \Gamma(U_1, U_2)$ contains all the paths (of vertices) from o' to the leaves. If m denotes a metric on T^* (and f the corresponding function on $E(T)$), $x \in \partial T$ and $\gamma \in \Gamma$ is the geodesic from o' to x, then $\lambda_m(\gamma) = If(x)$ and

$$l(m) = \inf_{\gamma \in \Gamma} \lambda_m(\gamma) = \inf_{x \in \partial T} If(x).$$

By homogeneity

$$Cap(\partial T) = \inf\{\|f\|_2^2 |\, If \geq 1 \text{ on } \partial T\} = \inf\{\|m\|_2^2 |\, l(m) \geq 1\} =$$

$$= \inf_{m \neq 0} \frac{\|m\|_2^2}{l(m)^2} = \left(\sup_{m \neq 0} \widehat{l}(m)\right)^{-1} = L_2\left(T^*; \{o'\}, \partial T\right)^{-1}$$

We have just obtained that:

Capacity is the reciprocal of extremal length.

Furthermore, the (unique) extremal metric with unitary norm s is strictly related to the equilibrium measure on the boundary:

$$s(\mathrm{e}(\alpha)) = \frac{\mathsf{M}(\alpha)}{Cap(\partial T)^{1/2}} = \frac{\mathsf{M}(\alpha)}{\mathsf{M}(\omega)^{1/2}}. \tag{5.7}$$

We may think $\mu = \mu^{\partial T}$ as a positive function on leaves. The function M is simply

$$\mathsf{M}(\alpha) = \sum_{x \in S(\alpha) \cap \partial T} \mu(x).$$

(5.7) can be displayed in a geometric way. Let $\mathcal{T}_\mu = \{Q_\alpha\}$ be the square tiling associated with μ, as in Section 3.3. It covers the rectangle $R = [0, Cap(\partial T)] \times [0, 1]$ of area is $Cap(\partial T) \leq 1$. By making a homogeneous dilation we transform R into the rectangle of unitary area $R' = [0, Cap(\partial T)^{1/2}] \times [0, Cap(\partial T)^{-1/2}]$. The tile Q'_α is the enlarged counterpart of Q_α. By the combinatorial properties of the tiling and following the steps in the proof of Lemma 5.3, it may be deduced that the dimensions of dilated squares give the extremal metric s:

$$s(\mathrm{e}(\alpha)) = \text{side length of } Q'_\alpha = Cap(\partial T)^{-1/2} \mathsf{M}(\alpha) = \mathsf{M}(\omega)^{-1/2} \mathsf{M}(\alpha).$$

Remark 5.3. We have considered only finite trees since the definition of extremal length/metric was given for finite graphs. We could try to extend all this setting to an infinite tree T, but then the argument which comes from the proof of Lemma 5.3 cannot be used anymore, since there are irregular points in general and it may be $l(s) \neq h$. This should not be a surprise! If the Lemma were true, we would have a unique square tiling whose combinatorial pattern comes from T; but, as it is shown in Remark 3.5, there might be different square tilings of (different) rectangles with the structure of T: building a compatible tiling is not enough to conclude about extremal metric, equilibrium measure, capacity of the boundary, etc.

Bibliography

[AH] D. R. Adams, L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der mathematischen Wissenshaften **314**, Springer-Verlag, Berlin Heidelberg 1996

[AL] N. Arcozzi, M. Levi, in preparation

[ARSW] N. Arcozzi, R. Rochberg, E. T. Sawyer, B. D. Wick, *Potential Theory on Trees, Graphs and Ahlfors-regular Metric Spaces*, Potential Analysis **41** (2014), 317-366

[BS] I. Benjamini, O. Schramm, *Random walks and harmonic functions on infinite planar graphs using square tilings*, The Annals of Probability **24** (1996), 1219-1238

[Be] M. Bersani, *Shrinking condensers on trees and euclidean spaces*, thesis, 2012, `http://amslaurea.unibo.it/id/eprint/3765`

[Br] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011

[BSST] R. L. Brooks, C. A. B. Smith, A. H. Stone, W. T. Tutte, *The dissection of rectangles into squares*, Duke Math. J. **7**, 1 (1940), 312-340

[Cho] G. Choquet, *Theory of Capacities*, Annales de l'institut Fourier **5** (1954), 131-295

[Chu] K. L. Chung, *Probabilistic approach in potential theory to the equilibrium problem*, Annales de l'institut Fourier **23**, 3 (1973), 313-322

[Di] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics **173**, Springer, 2017

[Ge] A. Georgakopoulos, *The boundary of a square tiling coincides with the Poisson boundary*, Invent. math. **203** (2016), 773-821

[LP] R. Lyons, Y. Peres, *Probability on Trees and Networks*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2017

[Ru] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1987 (3rd Edition)

[Sc] O. Schramm, *Square tilings with prescribed combinatorics*, Israel Journal of Mathematics **84** (1993), 97-118

[So] P. M. Soardi, *Potential Theory on Infinite Networks*, Lecture Notes in Mathematics **1590**, Springer-Verlag, Berlin Heidelberg 1994

[Wo] W. Woess, *Catene di Markov e Teoria del Potenziale nel Discreto*, Quaderni dell'Unione Matematica Italiana **41**, Pitagora Editrice, Bologna 1996 (Italian)

Ringraziamenti

Innanzitutto vorrei ringraziare il professore Nicola Arcozzi per la supervisione della tesi e per i continui spunti e stimoli che mi ha fornito negli ultimi mesi, e il dottor Matteo Levi per i tanti chiarimenti e suggerimenti che si sono rivelati estremamente utili e preziosi nella stesura.

Più in generale vorrei ringraziare tutti i professori (di Matematica e non solo) che ho incontrato in questi diciotto anni di percorso scolastico e universitario: ognuno ha (a suo modo) contribuito alla mia formazione e mi ha consentito di arrivare fino a questo punto, di arrivo ma anche di partenza.

Ringrazio i miei compagni di università per tutti i piccoli momenti di confronto e i coinquilini che mi hanno sopportato in questi anni a Bologna.

Infine la dedica più ovvia, ma ugualmente sentita, alla mia famiglia, che ha sempre creduto in me anche nei momenti in cui avevo poca autostima: grazie di tutto!