Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
Dipartimento di Fisica e Astronomia
Corso di Laurea Magistrale in Fisica

Functional Methods in Quantum Field Theory

Relatore:
Prof. Roberto Zucchini

Presentata da:
Andrea Rondelli

Anno Accademico 2017/2018

FUNCTIONAL METHODS IN QUANTUM FIELD THEORY

Andrea Rondelli

March 1, 2018

(...)Who out of many, tell me, is the Skambha. (...)
Skambha set fast these two, the earth and heaven, Skambha maintained the ample air between them.
Skambha established the six spacious regions: this whole universe Skambha entered and pervaded.

Hymn VII, Atharva Veda.

(...) e quindi uscimmo a riveder le stelle.

Divina Commedia, Dante Alighieri

To my family, especially to Leonardo and Sofia

Contents

I FUNCTIONAL DETERMINANTS AND REGULARIZATIONS 5

1 Funcional Integrals 5

2 ζ -function Regularization 8

3 Heat Kernel 10

4 Cutoff in Proper Time 13

II APPLICATIONS TO QUANTUM THEORY 18

5 qp-symbols Quantization 18

6 Evolution Operator and Heat Kernel 22

7 The Continuum Limit 25

8 Path Integral of Probability Amplitude 26

9 Euclidean Formulation 27

10 An Example of Hamiltonian 27

III GAUGE THEORIES 30

11 Abelian Case - Electromagnetism 30

12 The Volume Form 33

13 Connections 35

14 Curvature 36

15 Change of Basis of Sections 37

16 Yang-Mills Action 38

17 Path-Ordered Exponential 39

18 Holonomy and Wilson Loop 43

19 Isometric Group Actions 44

20 Measures **45**

21 Faddeev-Popov Determinant **46**

IV MATHEMATICAL APPENDIX 48

A Gamma Function **48**

B Euler-Riemann Zeta Function **49**

C The Generalized Zeta Function **50**

D Analytic Continuation of Zeta Function **52**

E Euler's Reflection Formula **53**

F Exterior Derivative **53**

G Poincaré Lemma **56**

H Parallel Transport and Holonomy Group **56**

I Left and Right Cosets, Stabilizer and Orbits **57**

INTRODUCTION

The goal of the first part of this thesis is to study the main properties of functional integration, i.e. integrals over infinite dimensional Hilbert manifolds. In particular we will focus on "Gaussian-like" functional integral and the Jacobian determinants which arise when one performs a change of variable, namely a linear invertible mapping on Hilbert space (we approximate locally a Hilbert manifold with its tangent vector space). Since such determinants are as a rule divergent we must find a way to regularize them. In the language of perturbation theory the determinant of an operator is expressed as a single closed loop graph. We shall study two regularization techniques: zeta function regularization and proper time cutoff regularization. Both techniques "remove the infinity" allowing one to obtain a finite value for the determinant, but the values achieved differ by a finite quantity. We will compare the results and try to relate them.

The aim of the second part is to demonstrate the fundamental path-integral formula for probability amplitudes in QM, first obtained by Dirac and subsequently formalized by Feynman.[1][2] In order to do this, we introduce the *qp*-symbol formalism which allows us to write matrix elements of operators or products of operators on Hilbert space in terms of multiple phase space integrals. By using this approach, we shall express the kernel of time evolution operator in a suitable way and, letting the number of phase space integrations grow to infinity, we will obtain a functional integral formula for the probability amplitude.

As is well known, in QM the amplitude has the following physical meaning: its square absolute value is the probability of transition from an initial to a final state. Specifically, we will calculate the probability amplitude for a particle to propagate from point a to a point b in configuration space. The main feature of the path integral formula is that this probability is the resultant of the contributions of continuously infinitely many phase-space trajectories. One may interpret this result by saying that the particle can follow any path joining a to b with a weight measured by the value of the corresponding classical action, not only the classical path that makes the action stationary.

In the third and last part we begin to extend functional integration to gauge theory. In particular we will focus on some geometric important features of gauge fields, viewed in fiber bundles framework.

Part I

FUNCTIONAL DETERMINANTS AND REGULARIZATIONS

We try to describe a technique for regularizing quadratic path integrals in Minkowski or Euclidean spacetime background. This approach can be generalize to curved spacetime.[3]

1 Funcional Integrals

The functional integrals are integrals over some infinite-dimensional space (a space which can be or not be a linear one), in general somethings infinite-dimensional manifold. For example we can choose to integrate over an Hilbert space (which is of course a vector and Banach space) or a space of functions like $S(\mathbb{R}^n)$, the Schwartz's space. In Quantum Field Theory (QFT) one often work in $S(\mathbb{R}^n)$, the vector space of functions all of whose derivatives are rapidly decreasing to zero at infinity. Recall that $S(\mathbb{R}^n)$ is constituted by functions $f \in C^\infty(\mathbb{R}^n)$ such that for all multi-indexes α, β we have $\underset{x\in\mathbb{R}^n}{Sup} \left\| x^\alpha \partial^\beta f(x) \right\| < +\infty$, which implies that $\left| D^\beta f(x) \right| \leq C_{N,\beta}(1+|x|^2)^{-N}$ for every multiindex β and $N \in \mathbb{N}$, and for some appropriate constant $C_{N,\beta}$. We must recall that $S(\mathbb{R}^n)^- = L^2(\mathbb{R}^n, \mathbb{C})$, i.e. the closure of the Schwartz's space give the standard Quantum Mechanics Hilbert space formed by the square integrable functions. We will call J our functional integral.

Let us consider a functional $F(\varphi)$ with $\varphi : [a,b] \subset \mathbb{R} \to \mathbb{R}$. The $\varphi(t)$ are continuous functions of single real variable defined on a compact (closed and bounded) interval:

$$\varphi : [a,b] \to \mathbb{R} \qquad F[\varphi] : C([a,b]) \to \mathbb{C}, \qquad \varphi(t) \text{ continuous on } [a,b] \tag{1}$$

First of all we aim to "finitized" the situation, namely we want to make finite the number of dimensions of the functions φ' space, by operating some appropriate restriction on it. We choose to be in the special case in which φ are piecewise linear, segmented functions (whose graphs will be polygonal curves) on N small segments from the initial point a to the end point b :

$$\text{Let } \varphi \text{ be linear in } \quad [t_0,t_1],\ [t_1,t_2],\ \ldots,[t_{N-1},t_N]$$

with

$$t_i = a + i\frac{(b-a)}{N}\,, \qquad \forall i = 0,1,2,...,N$$

so that the two extremes are $t_0 = a$, $\quad t_N = b$.

We have divided the finite interval $[a, b]$ in N subintervals of length $\frac{b-a}{N}$ and we assume that the functions are linear in each of these sub-segments, so that in $[t_k, t_{k+1}]$ with $k = 0, 1, ..., N-1$; the function φ is completely determined by its values at the edges of the sub-segment or subinterval.
$\varphi(t_k)$ and $\varphi(t_{k+1})$ are sufficient to tell us all of the values assumed by φ in the subinterval $[t_k, t_{k+1}] \quad \forall\, k = 0, 1, ..., N-1$.
Therefore now the functional space is finite-dimensional, its dimension being $N+1$: we can in fact parameterize our space by the vector $(\varphi_0, \varphi_1, ..., \varphi_N)$.

$$\varphi(t_0) \equiv \varphi_0 \; ; \; \varphi(t_1) \equiv \varphi_1 \; ; \; .... \; ; \; \varphi(t_N) \equiv \varphi_N$$

The functional space is thus parameterized by these numbers, in the limit $N \to +\infty$. We now define

$$J_N := \int F[\varphi] \prod_{j=0}^{N} d\varphi(t_j) \tag{2}$$

So if we perform the limit of this integral:

$$J = \lim_{N\to+\infty} J_N = \int F[\varphi] \prod_{j=0}^{+\infty} d\varphi(t_j) = \int F\,[\varphi] \prod_{a \le t \le b} d\varphi(t)$$

Where the last equality follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, and we can formally write

$$J = \int \prod_{\forall t \in [a,b]} d\varphi(t)\, F\,[\varphi] \tag{3}$$

We ask ourselves if this limit always exists, the answer is: generally not. May not exist or exist infinite. A first treatment that can be done to make converge J is to identify the divergent part of it and redefine J_N so that this part cancels out.

Now we examine an example to illustrate this: the Gaussian functionals.
We want to consider a functional integral like
$J \equiv \int_{\mathcal{H}} dx \exp(-\frac{1}{2}\langle x, Ax\rangle)$ with $\bar{A} = A^{\dagger}$, $A > 0$
A is an essentially self-adjoint, positive definite operator on a Hilbert space $\mathcal{H}$ over the field of the real numbers $\mathbb{R}$. The finite approximation of J is J_N :
$J_N \equiv \int_{\mathcal{H}_N} dx \exp\left(-\frac{1}{2}\langle A_N x, x\rangle\right)$ with x finite-dimensional
and A_N is the restriction (which is unique) of A at a N-dimensional subspace of $\mathcal{H}$; A_N is a square symmetric matrix, which is also positive definite, and $\langle -, - \rangle$ is the real-valued scalar product.
Now, if $\lim_{N\to+\infty} \det A_N < +\infty$ we can easily define the limit of the determinants of the succession of matrices $\{A_N\}$ as the determinant of an operator A, but usually this does not occurs (the limit diverges). Nevertheless we formally write:
$\det A \equiv \lim_{N\to+\infty} \det A_N$
In our case the functional integral is

$$J_N = \int dx\; e^{-\frac{1}{2}\langle x, A_N x\rangle} = \frac{(2\pi)^{\frac{N}{2}}}{(\det A_N)^{\frac{1}{2}}} \tag{4}$$

Hence $\lim_{N\to+\infty} J_N = +\infty$. Since the factor $(2\pi)^{\frac{N}{2}}$ diverges, it is natural to redefine $dx_i^{'} \mapsto \frac{dx_i}{\sqrt{2\pi}}$. Upon doing this, we therefore have

$$J = \int dx \exp\left(-\frac{1}{2}\langle x, Ax\rangle\right) = (\det A)^{-\frac{1}{2}} \tag{5}$$

also when the limit of the sequence of the determinants $\{\det A_N\}_{N\in\mathbb{N}}$ does not exist. When this happens we have to regularize the determinant in other ways. The first way we want to examine is the so-called Zeta Function Regularization.

Now we generalize these basic facts about functional integration stating more precisely our conventions and definitions. If $\mathcal{F}$ is a real Hilbert manifold, then, for any $f \in \mathcal{F}$, the tangent space $T_f\mathcal{F}$ is a Hilbert space. A functional measure Df on $\mathcal{F}$ is defined by assigning a smoothly varying functional measure $D\delta f_f$ on the tangent space $T_f\mathcal{F}$ for each $f \in \mathcal{F}$ according to the following rules.
Assuming that $\mathcal{H}$ is a real Hilbert space with a symmetric sesquilinear form $\langle\cdot,\cdot\rangle$ we define the associated functional measure $D\phi$ on $\mathcal{H}$ as the translation invariant measure normalized so that

$$\int_{\mathcal{H}} D\phi \exp(-\|\phi\|^2/2) = 1. \tag{6}$$

Following equation (5) the functional determinant $\det(\Delta)$ of a positive self-adjoint linear operator $\Delta : \mathcal{H} \to \mathcal{H}$ is

$$(\det(\Delta))^{-1/2} = \int_{\mathcal{H}} D\phi \exp(-\langle\phi, \Delta\phi\rangle/2). \tag{7}$$

Furthermore the functional Dirac delta function $\delta(\phi)$ on $\mathcal{H}$ is normalized so that

$$\int_{\mathcal{H}} D\phi\, F(\phi)\delta(\phi) = F(0), \tag{8}$$

for any function $F : \mathcal{H} \to \mathbb{R}$. So the delta function acts in the usual way.
A linear invertible mapping $T : \mathcal{H}' \to \mathcal{H}$ of Hilbert spaces induces a change of functional integration variables $\phi = T\phi'$. Its Jacobian J_T satisfies

$$\int_{\mathcal{H}} D\phi\, F(\phi) = J_T \int_{\mathcal{H}'} D\phi'\, F(T\phi') \tag{9}$$

for any function $F : \mathcal{H} \to \mathbb{R}$. J_T is given by

$$J_T = (\det(T^\dagger T))^{1/2} \tag{10}$$

with the determinant defined according to (7).

Proof:

$$1=\int_{\mathcal{H}} D\phi e^{-\|\phi\|^2/2}=J_T\int_{\mathcal{H}'} D\phi' e^{-\left\|T\phi'\right\|^2/2}=J_T\int_{\mathcal{H}'} D\phi' e^{-\left\langle \phi',T^\dagger T\phi'\right\rangle/2}=$$
$$J_T\left(\det\left(T^\dagger T\right)\right)^{-1/2}\quad\Longrightarrow\quad J_T=\left(\det\left(T^\dagger T\right)\right)^{1/2}.$$

When a Hilbert space $\mathcal{H}$ is decomposable as an orthogonal direct sum of a collection of Hilbert spaces $\mathcal{H}_\alpha$, $\mathcal{H}=\bigoplus_\alpha \mathcal{H}_\alpha$, the functional measure $D\phi$ of $\mathcal{H}$ factorizes accordingly in the product of the functional measures $D\phi_\alpha$ of $\mathcal{H}_\alpha$,

$$D\phi=\prod_\alpha D\phi_\alpha. \tag{11}$$

Starting from

$$D\phi=C\prod_\alpha D\phi_\alpha$$

we can demonstrate that the constant C is equal to the unit:

$$1=\int_{\mathcal{H}} D\phi e^{-\|\phi\|^2/2}=C\int_{\oplus_\alpha\mathcal{H}_\alpha}\prod_\alpha D\phi_\alpha e^{-\sum_\alpha\|\phi_\alpha\|_\alpha^2/2}=$$
$$C\int_{\oplus_\alpha\mathcal{H}_\alpha}\prod_\alpha D\phi_\alpha\prod_\beta e^{-\|\phi_\beta\|_\beta^2/2}=C\int_{\oplus_\alpha\mathcal{H}_\alpha}\prod_\alpha\left(D\phi_\alpha e^{-\|\phi_\alpha\|_\alpha^2/2}\right)=$$
$$C\prod_\alpha\left\{\int_{\mathcal{H}_\alpha} D\phi_\alpha e^{-\|\phi_\alpha\|_\alpha^2/2}\right\}=C\prod_\alpha 1=C.$$

The other properties of functional integration are formal consequences of the above ones.

2 ζ -function Regularization

Let A be an essentially self-adjoint, positive definite operator on $\mathcal{H}$ with totally discrete spectrum. Denote by λ_k the non zero eigenvalues of A. We have thus $\lambda_k\in\mathbb{R}$, $\lambda_k>0$ for $k\in\mathbb{N}$. The ζ-function of A is then defined by the expression

$$\zeta_A(s):=\sum_{k=0}^{+\infty}\lambda_k^{-s} \tag{12}$$

For the sake of accuracy about the eventual degeneracy of the operator A, we call $\{\lambda_k\}_{k\in\mathbb{N}}$ the sequence of the eigenvalues counting multiplicity i.e. if $\deg\lambda_k=p$ then λ_k appears p times in the sequence; instead we call $\{\lambda_\nu\}_{\nu\in\mathbb{N}}$ the sequence of the eigenvalues without counting multiplicity so if $\deg\lambda_\nu=d_\nu$ in the series it appears with the "weight" d_ν. Then we have

$$\zeta_A\left(s\right)=\sum_{\nu=0}^{+\infty}d_\nu\lambda_\nu^{-s} \tag{13}$$

$$\zeta_A(s) = \mathrm{Tr}\left(\mathrm{A}^{-\mathrm{s}}\right) \tag{14}$$

Now we want to demonstrate this last equations. Let the operator A be a Hilbert-Schmidt operator, i.e. an essentially self-adjoint, strictly positive definite, compact and trace class operator. From the compactness of A it follows that the set of its eigenvalues $\{\lambda_k\}$ is at most countably infinite (enumerable) and $\lambda_j \underset{j\to\infty}{\to} 0$. Let also P_ν be the orthogonal projector over the ν-th eigenspace of $\mathcal{H}$, so every operator P_ν has the properties:

$i)P_\nu = P_\nu^\dagger \qquad ii)P_\nu P_\mu = \delta_{\nu\mu}P_\mu \qquad iii)\mathrm{Tr}(P_\nu) = d_\nu < +\infty$

All projectors have finite dimensional eigensubspaces. Then from the spectral theorem we have

$$A = \sum_{\nu=0}^{+\infty} \lambda_\nu P_\nu$$

Hence, under the assumptions made, we have a beautiful integral expression for the complex power of the operator A

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt\, t^{s-1} \sum_{k=0}^{+\infty} e^{-t\lambda_k} P_k \tag{15}$$

Proof:

$$A^{-s} = \sum_{\nu=0}^{+\infty} \lambda_\nu^{-s} P_\nu = \frac{1}{\Gamma(s)} \sum_{\nu=0}^{+\infty} \int_0^{+\infty} dt\, t^{s-1} e^{-t\lambda_\nu} P_\nu =$$
$$\Gamma(s)^{-1} \int_0^{+\infty} dt\, t^{s-1} \sum_{\nu=0}^{+\infty} e^{-t\lambda_\nu} P_\nu$$

Whence taking the trace of the last equation we get

$$\mathrm{Tr}\left(\mathrm{A}^{-\mathrm{s}}\right) = \Gamma(s)^{-1} \int\limits_{\mathbb{R}^+} dt\, t^{s-1} \sum_{\nu=0}^{+\infty} d_\nu e^{-\lambda_\nu t} = \sum_{\nu=0}^{+\infty} d_\nu \lambda_\nu^{-s} \equiv \sum_{k=0}^{+\infty} \lambda_k^{-s} = \zeta_A(s) \tag{16}$$

For some mathematical details of last passages eventually look in appendix the generalized Riemann zeta function. Now, using the derivative $\frac{d}{dx}a^{-x} = -\ln(a)\, a^{-x}$ it follows that, forgetting temporarily that we are manipulating a divergent series, formally we have

$$-\frac{d}{ds}\zeta_A(s)\bigg|_{s=0} = \sum_{k\geq 0} \ln \lambda_k = \ln\left(\prod_{k\geq 0} \lambda_k\right) \equiv \ln \det A$$

which suggests the following definition of $\det A$

$$\det A = \exp\left\{ -\frac{d}{ds}\zeta_A(s)\Big|_{s=0} \right\} \tag{17}$$

This last equation defines the regularized determinant of the operator A.
The "usual" function $\zeta(s)$ is convergent in the complex half-plane $\mathrm{Re}(s) > 1$ but by analytic continuation methods it is possible to extend its domain to all the complex plane $\mathbb{C}$ except the point 1. The reader who wishes to delve into this beautiful topic will find in the Mathematical Appendix some detailed calculations about how to make analytic continuation of generalized Riemann Zeta function and how it is possible to modify the original Euler's series to obtain the trace of an elliptic operator. In fact we need $\zeta_A(s)$ to be regular at the origin $s = 0$ in order to formula (17) to furnish a sensible definition of the regularized determinant of A. Indeed one has to derive the $\zeta_A(s)$ at $s = 0$ and only the analytic continued zeta function is regular at the origin. In fact it can be shown that the zeta-function regularization is well and uniquely defined, working with Hurwitz zeta function. [4]

3 Heat Kernel

We now take a scalar field φ defined in $\mathbb{R}^n$, satisfying the Klein-Gordon equation. Let us consider the following functional integral, i.e. the generating functional for a free scalar field φ

$$J = \int \exp\left\{ -\frac{1}{2}\int \left[(\nabla\varphi)^2 + m^2\varphi^2 \right] dx \right\} \prod d\varphi(x) \tag{18}$$

where the argument of the exponential above is the Klein-Gordon action. If the scalar field goes rapidly to zero at infinity the integration of the kinetic term of the Lagrangian gives

$\int d^n x (\nabla\varphi).(\nabla\varphi) = \oint_{\partial\mathbb{R}^n} d^{n-1}x\varphi\nabla\varphi - \int d^n x\varphi\nabla^2\varphi = -\int d^n x\varphi\triangle\varphi$

where the first surface integral goes to zero because $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and the surface over which one integrates is at infinity. Whence using (5) we have

$J = \int \exp\left\{ -\frac{1}{2}\int \varphi\left(-\triangle + m^2\right)\varphi dx \right\} \prod d\varphi(x) = \left(\det\left(-\triangle + m^2\right)\right)^{-\frac{1}{2}}$.

We want to consider the matrix element of the exponential of the operator $A_x = -\triangle_x + m^2$:

$$\langle\, x, e^{-At}x_0 \,\rangle := K(x, x_0, t) \tag{19}$$

which satisfies the partial differential equation

$$\partial_t K(x, x_0, t) = -A_x K(x, x_0, t) \tag{20}$$

with the initial condition $K(x, x_0, t = 0) = \langle x, x_0\rangle = \delta(x - x_0)$
as it is straightforwardly verified:

$$\partial_t K(x, x_0, t) = \partial_t \left\langle x, e^{-A_x t}x_0 \,\right\rangle = -\, A_x K(x, x_0, t)$$

This equation is formally identical to the heat equation, so we will call $K(x, x_0, t)$ the heat kernel.

From equation (16) it follows that the zeta function of the operator A (in this case the operator has continuous spectrum) is

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt\, t^{s-1} \mathrm{Tr}\left(e^{-At}\right) \tag{21}$$

where $\mathrm{Tr}\left(e^{-At}\right) \equiv \int dx \left\langle x, e^{-At} x \right\rangle$ is the trace inside the integral which we want to be able to calculate. To this aim we have to evaluate the diagonal matrix element of the heat kernel $K(x, x, t)$. But let's first calculate the general matrix element $K(x, x_0, t)$. We can express the exponential operator in the momentum space, i.e. in Fourier transform as: $\exp\left[-t\left(-\triangle + m^2\right)\right] = \int d^n k\, |k\rangle\, e^{-t\left(k^2+m^2\right)} \langle k|$

Via Fourier decomposition the heat kernel, which is also the two-points Green function $G_t(x, x_0) \equiv K(x, x_0, t) = \left\langle x, \exp\left[-t\left(-\triangle + m^2\right)\right] x_0 \right\rangle$ becomes

$G_t(x, x_0) = \int d^n k \left\langle x, k \right\rangle e^{-t\left(k^2+m^2\right)} \langle k,\ x_0\rangle = \int d^n k \frac{e^{ikx}}{(2\pi)^{n/2}} e^{-t\left(k^2+m^2\right)} \frac{e^{-ikx_0}}{(2\pi)^{n/2}}$

where changing from one basis in the position representation to one basis in the momentum representation brings in the factors $\langle\, x, k \,\rangle = \frac{e^{ikx}}{(2\pi)^{n/2}}$, namely the usual plane waves. So the two-points Green function or propagator (of the field between two space points x and x_0 in the time interval t) can be written as

$$G_t(x, x_0) = (2\pi)^{-n} \int d^n k\, e^{ik(x-x_0)} e^{-t\left(k^2+m^2\right)}$$

Now to compute this integral we have to perform the inverse Fourier transform of a Gaussian function. For this purpose we set $x - x_0 \equiv \triangle x$, so the argument of the exponential of the integrand can be rewritten as $ik\triangle x - tk^2 - tm^2 = -\left(\sqrt{t}k - \frac{i}{2\sqrt{t}}\triangle x\right)^2 - \frac{(\triangle x)^2}{4t} - tm^2$, and putting it in the last expression of the heat kernel we get

$$G_t(x, x_0) = (2\pi)^{-n}\, e^{-\frac{(\triangle x)^2}{4t} - m^2 t} \int d^n k\, e^{-\left(k\sqrt{t} - \frac{i}{2\sqrt{t}}\triangle x\right)^2} =$$

$$(2\pi)^{-n}\, e^{-\frac{(\triangle x)^2}{4t} - m^2 t} \int d^n k\ \exp\left[\sum_{j=1}^{n} \left(\sqrt{t}k_j - \frac{i}{2\sqrt{t}}\triangle x_j\right)^2\right]$$

Using the fact that the function $g(z) = e^{-z^2}$, with $z \in \mathbb{C}$, is holomorphic and taking a rectangle in the complex plane as integration circuit, from Cauchy's integral theorem it follows that $I = \int_{\mathbb{R}^n} d^n y\ e^{-(y+ib)^2} = (\pi)^{n/2}$ with $y, b \in \mathbb{R}^n$ and making a suitable change of variable, namely $k_j\sqrt{t} \equiv y_j$ (with $j = 1, ..., n$), so $dk = t^{-n/2} dy$

we finally obtain

$$K(x,x_0,t) \equiv G_t(x,x_0) = \frac{2^{-n}}{(\pi t)^{n/2}} \exp\left\{-m^2 t - \frac{(x-x_0)^2}{4t}\right\} \tag{22}$$

which is valid for any dimension n. We remind that we are working in $\mathbb{R}^n$, $n \in \mathbb{N}$ being the spacetime dimension with $n-1$ space dimensions.

This result fits with the given initial condition because the right-hand side of (22) is known to provide a regularization of the Dirac δ-function for finite t

$$\lim_{t\to 0} K(x,x_0,t) = \delta(x-x_0)$$

where this limit is to be understood in a distributional sense. If we put ourselves in the four dimensional speciale case ($n=4$) it turns out the more familiar result

$$K(x,x_0,t) = \frac{1}{16\pi^2 t^2} \exp\left\{-m^2 t - \frac{(x-x_0)^2}{4t}\right\} \tag{23}$$

Which is the heat kernel for a real scalar field propagating in accordance with Klein-Gordon equation. Setting $x = x_0$ we obtain its diagonal element

$$K(x,x,t) = \frac{1}{16\pi^2 t^2} e^{-m^2 t} \tag{24}$$

Now, starting from this it is easy to calculate the zeta function of the operator $A = -\triangle + m^2$ using equation (21), but one finds that equation (24) implies that

$$\mathrm{Tr}\left(e^{-At}\right) \equiv \int dx \left\langle x, e^{-At} x\right\rangle = \frac{e^{-m^2 t}}{16\pi^2 t^2} \int dx \longmapsto +\infty$$

This integral diverges because the Euclidean space is unbounded, and it is a typical infrared divergence. Roughly speaking an infrared divergence is about big distance and low momentum or frequency, just like this case. The simplest way to overcome this drawback is by confining the field in a spatial "box" of finite volume.
If the space has finite volume, i.e. the field φ has bounded domain Ω ($\Omega \subset \mathbb{R}^4$) with measure$(\Omega) = V$ then we have

$$\mathrm{Tr}\left(\exp(-tA)\right) = \int_\Omega dx \frac{e^{-tm^2}}{16\pi^2 t^2} = \frac{V}{16\pi^2 t^2} e^{-tm^2} \tag{25}$$

and putting it inside equation (21) we easily obtain

$$\begin{aligned} \zeta_A(s) &= \frac{1}{\Gamma(s)} \int_0^\infty dt \, \frac{V}{16\pi^2 t^2} e^{-m^2 t} t^{s-1} = \frac{V}{\Gamma(s)\, 16\pi^2} \int_0^\infty dt \, e^{-tm^2} t^{s-3} = \\ &\frac{V}{16\pi^2 \Gamma(s)} \int_0^\infty dt' m^{-2} e^{-t'} t'^{s-3} m^{-2s+6} = \frac{V m^{-2s+4}}{16\pi^2 \Gamma(s)} \int_0^\infty dt \, t^{(s-2)-1} e^{-t} = \\ &\frac{V}{16\pi^2} \left(m^2\right)^{2-s} \frac{\Gamma(s-2)}{\Gamma(s)} \end{aligned}$$

Where we put $tm^2 \equiv t'$. Thus

$$\zeta_A(s) = \frac{V}{16\pi^2}\left(m^2\right)^{2-s}\frac{\Gamma(s-2)}{\Gamma(s)}$$

and using recursively the property of the gamma function $\Gamma(s) = (s-1)\,\Gamma(s-1)$ one gets

$$\zeta_A(s) = \frac{V}{16\pi^2}\left(m^2\right)^{2-s}\left[(s-1)(s-2)\right]^{-1}. \tag{26}$$

The last equation tells us that $\zeta_A(s)$ is analytically extended to all the complex plane except in the two points $s=1$ ans $s=2$ in which it exhibits two simple poles. Thus the generalized $\zeta-$function of the operator $A = -\triangle + m^2$ is analytic at the origin $s=0$ and we can do the derivative $\zeta_A'(s)\,|_{s=0}$. We saw that $-\frac{d}{ds}\zeta_A(s)\,|_{s=0} = \ln(\det A)$, and applying it to the present case we have

$$\frac{d}{ds}\zeta_A(s) = \frac{d}{ds}\left\{\frac{V}{16\pi^2}m^4 m^{-2s}\left(s^2-3s+2\right)^{-1}\right\} =$$

$$\frac{Vm^4}{16\pi^2}\left\{-\frac{m^{-2s}2\ln m}{(s^2-3s+2)} - \frac{m^{-2s}(2s-3)}{(s^2-3s+2)^2}\right\}$$

from which it follows that the derivative at the origin is

$$\zeta_A'(0) = \frac{Vm^4}{16\pi^2}\left\{-\ln m + \frac{3}{4}\right\}$$

and taking the exponential

$$\det\left(-\triangle + m^2\right) = \exp\left(-\zeta'(0)\right) = \exp\left\{\frac{Vm^4}{32\pi^2}\left(\ln m^2 - \frac{3}{2}\right)\right\} \tag{27}$$

The last equation defines the regularized determinant by zeta function of the Klein-Gordon operator in the Euclidean formulation.

Now to conclude this section we briefly review what we have shown. We have considered a Gaussian functional integral, i.e. an integral over an Hilbert space $\mathcal{H}$ of the exponential of a negative quadratic form such that the inner product $-\frac{1}{2}\langle x, Ax\rangle$ with A being the Klein-Gordon (strongly elliptic) operator $-\nabla^2 + m^2$ acting on real scalar fields ϕ . Then through Zeta function regularization we have calculated explicitly its determinant which is given by equation (27).

4 Cutoff in Proper Time

Let $A : \mathcal{H} \to \mathcal{H}$ be a self-adjoint positive operator, $\epsilon > 0$ an arbitrary "small" real number. The proper time regularized determinant of A is defined by

$$\ln\det{}_\epsilon A = -\sum_i \int_\epsilon^\infty \frac{e^{-\lambda_i t}}{t}dt. \tag{28}$$

Since from the spectral decomposition of the operator A we have $\sum_{i=0}^{\infty} e^{-t\lambda_i} = \mathrm{Tr}\left(e^{-At}\right)$ (remember that the italic indices also count multiplicity of eigenvalues) it follows that

$$\ln \det_{\varepsilon} A = -\int_{\varepsilon}^{\infty} t^{-1}\mathrm{Tr}\left(e^{-At}\right) dt.$$

Heuristically this definition can be understood as follows

$$-\sum_{i}\int_{\epsilon}^{\infty}\frac{e^{-\lambda_i t}}{t}dt = -\sum_{\substack{i\\ \lambda_i\le 1/\epsilon}}\int_{\varepsilon}^{\infty}\frac{e^{-\lambda_i t}}{t}dt - \sum_{\substack{i\\ \lambda_i> 1/\epsilon}}\int_{\epsilon}^{\infty}\frac{e^{-\lambda_i t}}{t}dt \simeq \sum_{\substack{i\\ \lambda_i\le 1/\epsilon}}\int_{\epsilon}^{\infty}\frac{e^{-\lambda_i t}}{t}dt$$

In the second sum the argument of the exponential must satisfy the inequality $\lambda_i t \geq \lambda_i \epsilon > 1$ thus the integrand is exponentially suppressed. Next we carry out the change of variable $\lambda_i t \equiv u$, obtaining

$$\begin{aligned}
-\sum_{\substack{i\\ \lambda_i\le 1/\epsilon}}\int_{\epsilon}^{\infty}\frac{e^{-\lambda_i t}}{t}dt &= -\sum_{\substack{i\\ \lambda_i\le 1/\epsilon}}\int_{\epsilon\lambda_i}^{\infty}\frac{e^{-u}}{u}du = \\
-\sum_{\substack{i\\ \lambda_i\le 1/\epsilon}}\left\{\int_{\epsilon\lambda_i}^{1}\frac{e^{-u}}{u}du + \int_{1}^{\infty}\frac{e^{-u}}{u}du\right\} &\simeq -\sum_{\substack{i\\ \lambda_i\le 1/\epsilon}}\int_{\lambda_i\epsilon}^{1}\frac{du}{u} = \\
+\sum_{\substack{i\\ \lambda_i\le 1/\epsilon}}\ln u\Big|_1^{\lambda_i\epsilon} &= \sum_{\lambda_i\le 1/\epsilon}\ln\lambda_i\epsilon = \\
\ln\left[\prod_{\lambda_i\le 1/\epsilon}(\lambda_i\epsilon)\right] &= \ln\det_{\lambda\le 1/\epsilon}(\epsilon A)\,.
\end{aligned}$$

From this calculation two important considerations arise. First, the parameter ϵ makes dimensionless the argument of the logarithm, as it must be. Second, in the limit for ϵ which tends to zero there are two asymptotic behaviors: on one hand ϵ "sends to zero the operator A" but on the other hand the cutoff upper-bound of its eigenvalues grows up to infinity. So we expect that these two features will, in some way, roughly compensate each other.

Taking $A = -\nabla + m^2$ equation (28) reads

$$\ln \det_{\epsilon}\left(-\triangle + m^2\right) = -\int_{\epsilon}^{\infty} dt\,\frac{1}{t}\mathrm{Tr}\left\{\exp\left[-t\left(-\triangle + m^2\right)\right]\right\} \qquad (29)$$

and using equation (25) we get

$$\ln \det_\epsilon \left(m^2 - \triangle\right) = -\frac{V}{16\pi^2} \int_\epsilon^\infty dt\, t^{-3} e^{-tm^2}. \tag{30}$$

One can calculate the last integral applying recursively integration by parts obtaining:

$$\int_\epsilon^\infty dt\, t^{-3} e^{-tm^2} = \frac{m^4}{2} \left\{ \frac{\epsilon^{-2}}{m^4} - \frac{2}{m^2}\epsilon^{-1} - \ln\epsilon - \ln m^2 + \frac{3}{2} + \Gamma'(1) \right\} + O(\epsilon \ln \epsilon) \tag{31}$$

PROOF:

$$\begin{aligned}
\int_\epsilon^\infty dt\, t^{-3} e^{-tm^2} &= \left[-\frac{t^{-2}}{2} e^{-m^2 t} \right|_\epsilon^\infty - \frac{m^2}{2} \int dt\, t^{-2} e^{-m^2 t} = \\
&\frac{1}{2}\epsilon^{-2} e^{-m^2\epsilon} - \frac{m^2}{2} \int_\epsilon^\infty dt\, t^{-2} e^{-m^2 t} = \\
\frac{1}{2}\epsilon^{-2} e^{-m^2\epsilon} + \frac{m^2}{2} &\left[e^{-m^2 t} t^{-1} \right]_\epsilon^\infty + \frac{m^4}{2} \int_\epsilon^\infty dt\, t^{-1} e^{-m^2 t} = \\
&\frac{\epsilon^{-2} e^{-m^2\varepsilon}}{2} - \frac{m^2}{2\varepsilon} e^{-m^2\varepsilon} + \frac{m^4}{2} \int_\epsilon^\infty dt\, \frac{e^{-m^2\varepsilon}}{t} = \\
e^{-m^2\epsilon} \left(\frac{1}{2\epsilon^2} - \frac{m^2}{2\epsilon} \right) + \frac{m^4}{2} &\left[(\ln t)\, e^{-m^2 t} \right|_\epsilon^\infty + \frac{m^6}{2} \int_\epsilon^\infty dt\, (\ln t)\, e^{-m^2 t} = \\
e^{-m^2\epsilon} \left(\frac{1}{2\epsilon^2} - \frac{m^2}{2\epsilon} \right) &- \frac{m^4 e^{-m^2\epsilon} \ln\epsilon}{2} + \frac{m^6}{2} \int_\epsilon^\infty dt\, e^{-m^2 t} \ln t
\end{aligned}$$

So we have

$$\int_\epsilon^\infty t^{-3} e^{-tm^2} dt = e^{-m^2\epsilon} \left(\frac{1}{2\epsilon^2} - \frac{m^2}{2\epsilon} - \frac{m^4}{2} \ln\epsilon \right) + \frac{m^6}{2} \int_\epsilon^\infty dt\, e^{-m^2 t} \ln t \tag{32}$$

We evaluate the remained integral by breaking it as follows

$$\begin{aligned}
\int_\epsilon^\infty dt\, e^{-m^2 t} \ln t = \int_0^\infty dt\, e^{-m^2 t} \ln t - \int_0^\epsilon dt\, e^{-m^2 t} \ln t = \\
\int_0^\infty dt\, e^{-m^2 t} \ln t + O(\epsilon \ln \epsilon)
\end{aligned}$$

and performing the change of variable $m^2 t \equiv t' \quad \Longrightarrow \quad dt = \frac{dt'}{m^2}$ we get

$$\begin{aligned}
\int_0^\infty dt\, e^{-m^2 t} \ln t = \int_0^\infty dt' m^{-2} e^{-t'} \left(\ln t' - \ln m^2 \right) = \\
\int_0^\infty dt\, \frac{e^{-t} \ln t}{m^2} - \int_0^\infty dt\, \frac{e^{-t} \ln m^2}{m^2} = m^{-2}\Gamma'(1) + \frac{\ln m^2}{m^2} \left[e^{-t} \right|_0^\infty = \\
m^{-2} \left[\Gamma'(1) - \ln m^2 \right].
\end{aligned}$$

Indeed

$$\Gamma'(s) = \int_0^\infty dt\, t^{s-1} e^{-t} \ln t \quad \Longrightarrow \quad \Gamma'(1) = \int_0^\infty dt\, e^{-t} \ln t \tag{33}$$

As is easily shown:

$$\Gamma'(s) = \frac{d}{ds}\left\{\int_0^\infty dt\, \frac{e^{-t}}{t} e^{s \ln t}\right\} = \int_0^\infty dt\, e^{-t} (\ln t)\, t^{s-1}.$$

So, continuing the initial calculation, one obtains

$$\begin{aligned}
\int_\epsilon^\infty t^{-3} e^{-tm^2} dt = e^{-m^2\epsilon}\left(\frac{1}{2\epsilon^2} - \frac{m^2}{2\epsilon} - \frac{m^4}{2}\ln\epsilon\right) + \\
\frac{m^4}{2}\left(\Gamma'(1) - \ln m^2\right) + O(\epsilon \ln \epsilon) = \\
\left(1 - m^2\epsilon + \frac{m^4\epsilon^2}{2} + O\left(\epsilon^3\right)\right)\left(\frac{1}{2\epsilon^2} - \frac{m^2}{2\epsilon} - \frac{m^4}{2}\ln\epsilon\right) + \\
\frac{m^4}{2}\Gamma'(1) - \frac{m^4}{2}\ln m^2 + O(\epsilon\ln\epsilon) = \\
\frac{1}{2\epsilon^2} - \frac{m^2}{2\epsilon} - \frac{m^4 \ln\epsilon}{2} - \frac{m^2}{2\epsilon} + \frac{m^4}{2} - \frac{m^6\epsilon}{4} + \\
\frac{m^4}{4} - \frac{m^6\epsilon}{4} - \frac{m^8\epsilon\ln\epsilon}{4} + \frac{m^4\Gamma'(1)}{2} - \frac{m^4\ln m^2}{2} + O\left(\epsilon^2\right) = \\
\frac{1}{2}\epsilon^{-2} - m^2\epsilon^{-1} - \frac{m^4}{2}\ln\epsilon + \frac{3m^4}{4} + \frac{m^4}{2}\Gamma'(1) - \frac{m^4\ln m^2}{2} + O(\epsilon\ln\epsilon).
\end{aligned}$$

QED

Now, inserting (31) in (30) we finally obtain

$$\ln \det{}_\epsilon\left(m^2 - \triangle\right) = \frac{Vm^4}{32\pi^2}\left\{-\frac{1}{m^4\epsilon^2} + \frac{2}{m^2\epsilon} + \ln\epsilon + \ln m^2 - \frac{3}{2} - \Gamma'(1)\right\} + O(\epsilon\ln\epsilon). \tag{34}$$

Evidently the first three terms diverge in the limit $\epsilon \to 0^+$, and the natural question arises about how one should treat them. We simply subtract them from the original definition and define the regularized determinant $\det' A$ of the Klein-Gordon operator A in the following way

$$\ln \det{}'\left(-\triangle + m^2\right) := \left\{\ln\det{}_\epsilon\left(-\triangle + m^2\right) + \frac{V}{32\pi^2\epsilon^2} - \frac{Vm^2}{16\pi^2\epsilon} - \frac{Vm^4}{32\pi^2}\ln\epsilon\right\}\Bigg|_{\epsilon\to 0^+}$$

So the regularized determinant becomes convergent and finite:

$$\ln\det{}' A = \frac{Vm^4}{32\pi^2}\left(\ln m^2 - \frac{3}{2} - \Gamma'(1)\right) \quad , \quad \text{with } A = -\triangle + m^2 \tag{35}$$

Comparing this result with one obtained through zeta function regularization, namely equation (27) we see immediately that they are different of a finite quantity:

$$\ln \det{}' A = \ln \det A - \frac{V m^4}{32\pi^2} \Gamma'(1) \tag{36}$$

where $\ln \det(A)$ is the one calculated via zeta function regularization. The first observation is that in zeta function regularization we have to add, compared to the cutoff in proper time determinant, a finite term directly proportional to $\Gamma'(1)$ and this fact is not fortuitous. In fact when we do the analytic continuation of zeta function we remove the point $s = 1$ from the complex plane where it remains a single pole of zeta function. Thus in that procedure we lost a term proportional to the first derivative of gamma function (which is related to zeta). The second observation is that the two regularized determinant differ of a finite quantity, once we have removed the remaining infinite part. [5]

Part II
APPLICATIONS TO QUANTUM THEORY

In this part, we shall use functional integral techniques to compute relevant physical quantities. The expressions we obtain, as in the case of the heat kernel approach, are often useful in the analysis of important issues in quantum mechanics and quantum field theory.

5 qp-symbols Quantization

Regarding the notation, since in this section we must distinguish between operators (acting on Hilbert spaces) and their respective *qp*-symbols (which are functions of position and momentum) we will use an "hat" over the operators whereas the *qp*-symbols are denoted without it. This notation will be used only in this section. We also set the Planck constant, as all the others physical constants, egual to one ($\hbar = 1$). Let us consider the operator $\hat{A} : \mathcal{H} \to \mathcal{H}$ defined by

$$\hat{A} := \sum_n a_n(q)\left(-i\frac{\partial}{\partial q}\right)^n \equiv \sum_n a_n(\hat{q})\,\hat{p}^n \tag{37}$$

where $\hat{A}$ acts on functions $f \in L^2(\mathbb{R}^n)$. $\hat{q}$ and $\hat{p}$ are the familiar quantum mechanical position and momentum operators. $a_n(\hat{q})$ is an arbitrary function of the position operator. We define the qp-symbol (or symbol) of $\hat{A}$ as

$$A = \sum_k a_k(q)\,p^k$$

In this formalism the quantization of the physical quantities, i.e. passing from the *qp*-symbol to the operator can be easily obtained by replacing $q \to \hat{q}$, $p \to \hat{p}$ imposing the condition of normal ordering: all the $\hat{q}'s$ must be to the left of all the $\hat{p}'s$. Now let define the function

$$\tilde{f}(q) := \hat{A}f(q) \tag{38}$$

with $\tilde{f}$ square-integrable whenever f is. The Fourier transform and its inverse will be useful to us:

$$\mathscr{F}\{f(q)\}(p) := \frac{1}{(2\pi)^{1/2}}\int dq\, e^{-iq.p} f(q) \equiv \bar{f}(p)$$

$$\check{\mathscr{F}}\left\{\bar{f}(p)\right\}(q) := \frac{1}{(2\pi)^{1/2}}\int dp\, e^{ip.q} \bar{f}(p) \equiv f(q)$$

So that we can express $\tilde{f}(q)$ as follows

$$\tilde{f}(q) = \frac{1}{2\pi} \int A(q,p) f(q_1) e^{ip.(q-q_1)} dp dq_1. \tag{39}$$

PROOF:

$$\tilde{f}(q) = \hat{A} f(q) = \sum_n \left\{ a_n(q) \left(\frac{1}{i} \frac{\partial}{\partial q} \right)^n \right\} \frac{1}{(2\pi)^{1/2}} \int dp\, e^{ip.q} \bar{f}(p) =$$
$$\frac{1}{(2\pi)^{1/2}} \sum_n \left[a_n(q) \left(\frac{1}{i} \frac{\partial}{\partial q} \right)^n \right] \int dp\, e^{ipq} \int \frac{dq_1}{(2\pi)^{1/2}} e^{-iq_1 p} f(q_1) =$$
$$\frac{1}{2\pi} \int dp dq_1 f(q_1) \sum_n \left\{ a_n(q) \frac{1}{i^n} \frac{\partial^n}{\partial q^n} e^{ipq} \right\} e^{-iq_1 p} =$$
$$\frac{1}{2\pi} \int dp dq_1 f(q_1) \sum_n \left\{ a_n(q) \left(\frac{1}{i} \right)^n (ip)^n \right\} e^{ip(q-q_1)} =$$
$$\frac{1}{2\pi} \int dp dq_1 f(q_1) e^{ip.(q-q_1)} \sum_n a_n(q) p^n \equiv \frac{1}{2\pi} \int dp dq_1 A(q,p) f(q_1) e^{ip.(q-q_1)}.$$

QED

Equation (39) define an integral representation of $\tilde{f}(q) = \hat{A} f(q)$ which allows us to express the operator $\hat{A}$ through its *qp*-symbol also when $A(q,p)$ in not polynomial, provided that we can exchange the series with the integral in the calculation above.

Using (39) we get an expression for the matrix entries $\langle q_2 \, , \hat{A} \, q_1 \rangle$ in terms of the *qp*-symbol $A(q,p)$:

$$\langle q_2 \, , \hat{A} \, q_1 \rangle = \frac{1}{2\pi} \int dp A(q_2, p) e^{-ip.(q_1 - q_2)} \tag{40}$$

PROOF:

$$\tilde{f}(q) = \hat{A} f(q) = \int dq_1 \langle q \, , \hat{A} \, q_1 \rangle \langle q_1, f \rangle = \int dq_1 f(q_1) \langle q \, , \hat{A} \, q_1 \rangle$$

Thus

$$\tilde{f}(q) = \int dq_1 f(q_1) \langle q \, , \hat{A} \, q_1 \rangle \tag{41}$$

Putting together (39) and (41) one gets

$$\tilde{f}(q) = \int dq_1 f(q_1) \langle q \, , \hat{A} \, q_1 \rangle = \frac{1}{2\pi} \int dq_1 f(q_1) \int dp A(q,p) e^{ip(q-q_1)} \quad \Longrightarrow$$
$$\langle q \, , \hat{A} \, q_1 \rangle = \frac{1}{2\pi} \int A(q,p) e^{-ip(q_1 - q)} dp$$

The last equation is the same of (40) with q instead of q_2, **QED**

Now let $\hat{A}, \hat{B}, \hat{C}$ be operators such that $\hat{C} = \hat{A}\hat{B}$, then we can express the qp-symbol of $\hat{C}$ in a integral form in terms of the other two operators as follows

$$C(q,p) = \frac{1}{2\pi} \int A(q,p_1)\, B(q_1,p)\, e^{-i(p_1-p)(q_1-q)} dq_1 dp_1 \tag{42}$$

Proof:

$$\begin{aligned}\langle q\,, \hat{C}\, q_2\rangle \equiv \langle q\,, \hat{A}\hat{B}\, q_2\rangle = \langle q\,, \hat{A} \int dq_1\; q_1\rangle \langle q_1\,, \hat{B}\, q_2\rangle = \\ \int dq_1\, \langle q\,, \hat{A}\, q_1\rangle \langle q_1\,, \hat{B}\, q_2\rangle = \\ \frac{1}{(2\pi)^2} \int dq_1 \int dp_1\; A(q,p_1)\, e^{-ip_1(q_1-q)} \int dp\; B(q_1,p)\, e^{-ip(q_2-q_1)}\end{aligned}$$

Where we have used equation (40) to rephrase the two scalar products. Comparing it with

$$\langle q\,, \hat{C}\, q_2\rangle = \frac{1}{2\pi} \int dp\; C(q,p)\, e^{-ip(q_2-q)}$$

and since one has

$$\begin{aligned}e^{-ip(q_2-q_1)} e^{-ip_1(q_1-q)} = e^{-ip(q_2-q)} e^{-ipq} e^{-ip_1(q_1-q)} e^{ipq_1} = \\ e^{-ip(q_2-q)} e^{-i(p_1-p)(q_1-q)}\end{aligned}$$

it follows that

$$\begin{aligned}\int dp\; e^{-ip(q_2-q)} C(q,p) = \\ \frac{1}{2\pi} \int dp \int dq_1 B(q_1,p)\, e^{-ip(q_2-q_1)} \int dp_1 A(q,p_1)\, e^{-ip_1(q_1-q)} = \\ \int dp\; e^{-ip(q_2-q)} \left\{ \frac{1}{2\pi} \int dq_1 \int dp_1 A(q,p_1)\, B(q_1,p)\, e^{-i(p_1-p)(q_1-q)} \right\}\end{aligned}$$

from which it is immediately deduced equation (42). **QED**

Now we want to extend formula (42) to the product of N operators $\hat{A}_1$, $\hat{A}_2$, ... , $\hat{A}_N$ and calculate its qp-symbol $(A_1 A_2 ... A_N)(q,p)$. So we have $n \equiv N-1$ products between N operators and we shall proceed in an heuristic way by calculating first the case $n=1$, second the case $n=2$ and so forth. Finally we might guess the general case for arbitrary $n \in \mathbb{N}$. In a second time we will give a rigorous demonstration of it.

- $n = 1$

$$(A_1 A_2)(q,p) = \frac{1}{2\pi} \int dq_1 dp_1\; A_1(q,p_1)\, A_2(q_1,p)\, e^{-i(p_1-p)(q_1-q)}$$

- $n = 2$

$$\left(A_1 A_2 A_3\right)(q,p) = \frac{1}{2\pi}\int dq_2 dp_2 \left(A_1 A_2\right)(q,p_2)\, A_3(q_2,p)\,.$$
$$\exp\left[-i\left(p_2 - p\right)\left(q_2 - q\right)\right] =$$
$$\frac{1}{(2\pi)^2}\int dq_1 dp_1 dq_2 dp_2 A_1(q,p_1)\, A_2(q_1,p_2)\, A_3(q_2,p)\,.$$
$$\exp\left[-i\left(p_1 - p_2\right)\left(q_1 - q\right) - i\left(p_2 - p\right)\left(q_2 - q\right)\right] =$$
$$\frac{1}{(2\pi)^2}\int dp_1 dq_1 dp_2 dq_2 A_1(q,p_1)\, A_2(q_1,p_2)\, A_3(q_2,p)\,.$$
$$\exp\left\{-i\left[p_1\left(q_1 - q\right) + p_2\left(q_2 - q_1\right) + p\left(q - q_2\right)\right]\right\}$$

- ... etc etc ...

- $n + 1 = N$, we guess the "chain structure" form:

$$\left(A_1 A_2 A_3 ... A_N\right)(q,p) = (2\pi)^{-N+1}\int dp_1 dq_1 dp_2 dq_2 ... dp_{N-1} dq_{N-1}.$$
$$A_1(q,p_1)\, A_2(q_1,p_2)\, A_3(q_2,p_3) ... A_{N-1}(q_{N-2},p_{N-1})\, A_N(q_{N-1},p)\,.$$
$$\exp\left\{-i\left[p_1\left(q_1 - q\right) + p_2\left(q_2 - q_1\right) + ..p_{N-1}\left(q_{N-1} - q_{N-2}\right) + p\left(q - q_{N-1}\right)\right]\right\} \tag{43}$$

Currently we give the strict proof of (43) by mathematical induction.

Proof:

We know that equation (43) is true for $n = 1$ $(N = 2)$, and assuming it is correct also for $n \in \mathbb{N}$ let us show that is valid for $n+1$ too. Namely we assume that (43) is true and we have to show that it involves the following equation

$$\begin{gathered}\left(A_1 A_2 ... A_N A_{N+1}\right)(q,p) = (2\pi)^{-N}\int \prod_{j=1}^{N} dp_j dq_j. \\ A_1(q,p_1)\, A_2(q_1,p_2) ... A_N(q_{N-1},p_N)\, A_{N+1}(q_N,p)\,. \\ \exp\left\{-i\left[p_1\left(q_1 - q\right) + p_2\left(q_2 - q_1\right) + ... + p_N\left(q_N - q_{N-1}\right) + p\left(q - q_N\right)\right]\right\}\end{gathered} \tag{44}$$

Starting from equation (42) with $A \equiv A_1 ... A_N$ and $B \equiv A_{N+1}$ and using (43) we obtain

$$\left(A_1A_2...A_NA_{N+1}\right)(q,p)=\frac{1}{2\pi}\int dq_Ndp_N\left(A_1A_2...A_N\right)(q,p_N)\,A_{N+1}(q_N,p)\,.$$

$$\exp\left\{-i\left[\left(p_N-p\right)\left(q_N-q\right)\right]\right\}=\frac{1}{(2\pi)^N}\int\prod_{j=1}^{N}dp_jdq_jA_1(q,p_1)\,A_2(q_1,p_2)\,...$$

$$...A_N(q_{N-1},p_N)\,A_{N+1}(q_N,p)\,.$$

$$\exp -i[p_1(q_1-q)+p_2(q_2-q_1)+...$$

$$...+p_N(q-q_{N-1})+p_N(q_N-q)+p(q-q_N)]=\frac{1}{(2\pi)^N}\int\prod_{j=1}^{N}dp_jdq_j.$$

$$A_1(q,p_1)\,A_2(q_1,p_2)\,A_3(q_2,p_3)\,...A_{N-1}(q_{N-2},p_{N-1})\,A_N(q_{N-1},p_N)\,A_{N+1}(q_N,p)\,.$$

$$\exp -i[p_1(q_1-q)+p_2(q_2-q_1)+p_3(q_3-q_2)+....$$

$$+p_{N-1}(q_{N-1}-q_{N-2})+p_N(q_N-q_{N-1})+p(q-q_N)].$$

So equation (43) is rigorously proved via mathematical induction. **QED**

6 Evolution Operator and Heat Kernel

Now we shall write the evolution operator $\hat{U}(t)$ through a time interval of duration t as the product of N evolution operators on a time interval of duration t/N and then take the limit $N\to\infty$:

$$e^{-it\hat{H}}=\lim_{N\to\infty}\left(e^{-i\frac{t}{N}\hat{H}}\right)^N \tag{45}$$

An approximation will be sufficient for our purpose: by Taylor expanding the exponential $e^{-i(t/N)\hat{H}}$ in the right hand side of (45) we get

$$e^{-it\hat{H}}\equiv\lim_{N\to\infty}\left(e^{-i\frac{t}{N}\hat{H}}\right)^N=\lim_{N\to\infty}\left(\hat{1}-i\frac{t}{N}\hat{H}+O\left(N^{-2}\right)\right)^N=$$

$$\lim_{N\to\infty}\left(\hat{1}-i\frac{t}{N}\hat{H}\right)^N$$

Indeed from the Taylor series of the exponential $e^x=\sum_{k=0}^{\infty}\frac{x^n}{n!}$ one gets $e^x=1+x+O\left(x^2\right)$ thus for N "very large" we have, with good approximation

$$\left(e^{-i\frac{t}{N}H}\right)(q,p)\simeq\left(1-i\frac{t}{N}H\right)(q,p)=1-i\frac{t}{N}H(q,p)\simeq e^{-i\frac{t}{N}H(q,p)}$$

So we have

$$\left(e^{-i\frac{t}{N}H}\right)(q,p)\simeq e^{-i\frac{t}{N}H(q,p)}\qquad\text{, for } N\gg 1 \tag{46}$$

Combining together equations (43), (45) and (46) we get

$$
\begin{aligned}
&[\exp(-itH)](q,p) \equiv \left[\exp\left(-i\frac{t}{N}H\right)\exp\left(-i\frac{t}{N}H\right)...\exp\left(-i\frac{t}{N}H\right)\right](q,p) = \\
&\frac{1}{(2\pi)^{N-1}}\int\prod_{\alpha=1}^{N-1} dq_\alpha dp_\alpha \left(e^{-i\frac{t}{N}H}\right)(q,p_1)\left(e^{-i\frac{t}{N}H}\right)(q_1,p_2)\left(e^{-i\frac{t}{N}H}\right)(q_2,p_3)... \\
&\qquad \left(e^{-i\frac{t}{N}H}\right)(q_{N-2},p_{N-1})\left(e^{-i\frac{t}{N}H}\right)(q_{N-1},p) \\
&\exp\left\{-i\left[p_1(q_1-q)+p_2(q_2-q_1)+...+p_{N-1}(q_{N-1}-q_{N-2})+p(q-q_{N-1})\right]\right\} = \\
&(2\pi)^{-N+1}\int\prod_{\alpha=1}^{N-1} dq_\alpha dp_\alpha \exp\left\{-\frac{it}{N}\left[\sum_{\alpha=1}^{N} H(q_{\alpha-1},p_\alpha)\right]-i\left[\sum_{\alpha=1}^{N} p_\alpha(q_\alpha-q_{\alpha-1})\right]\right\}
\end{aligned}
$$

with the initial conditions $q_N = q_0 = q$, and $p_N = p$.
Thus the symbol (or qp-symbol) of the unitary time evolution operator $e^{-it\hat{H}}$ is

$$
\begin{aligned}
&[\exp(-itH)](q,p) \equiv G(q,p,t) \simeq (2\pi)^{-N+1}\int\prod_{\alpha=1}^{N-1} dq_\alpha dp_\alpha \cdot \\
&\exp\left\{+i\frac{t}{N}\left[-\sum_{\alpha=1}^{N}\frac{p_\alpha(q_\alpha-q_{\alpha-1})}{t/N}-\sum_{\alpha=1}^{N} H(p_\alpha,q_{\alpha-1})\right]\right\}
\end{aligned}
\tag{47}
$$

with boundary conditions $q_N = q_0 = q$, and $p_N = p$ and when the approximation is good for large N.
We notice the cyclic structure of the integral above, so we can visualize the generalized coordinates $q_0, ..., q_N$ standing on a circle, starting from $q_0 = q$ and ending at $q_N = q$.
To cast expression (47) in a more natural form we perform a relabelling of indices and then a change of variable, as we show now. We want to invert the sign of the sum $-\sum_{\alpha=1}^{N} p_\alpha(q_\alpha - q_{\alpha-1})$ and in order to do this we exchange the extremes $1 \longleftrightarrow N$ of the set $\{\alpha\}_{\alpha=1}^{N}$ by performing the next permutation of indices:

- $\alpha = 1 \longrightarrow \beta = N$
- ...
- $\alpha = N \longrightarrow \beta = 1$

Thus we "guess" the relabelling $\beta = -\alpha + N + 1$. Since it is a permutation, i.e. a bijection from the set of indices $\{\alpha\}_{\alpha=1}^{N}$ to itself, the sum $\sum_{\alpha=1}^{N}$ does not change. Thus the argument between square brackets of the exponential of equation (47) becomes

$$
-\sum_{\alpha=1}^{N} \frac{p_\alpha (q_\alpha - q_{\alpha-1})}{t/N} - \sum_{\alpha=1}^{N} H(p_\alpha, q_{\alpha-1}) =
$$

$$
-\sum_{\beta=1}^{N} p_{-\beta+1+N} \frac{(q_{-\beta+1+N} - q_{-\beta+N})}{t/N} - \sum_{\beta=1}^{N} H(p_{-\beta+1+N}, q_{-\beta+N}) =
$$

$$
-\sum_{\beta=1}^{N} \tilde{p}_\beta \frac{(\tilde{q}_\beta - \tilde{q}_{\beta+1})}{t/N} - \sum_{\beta=1}^{N} H(\tilde{p}_\beta, \tilde{q}_{\beta+1}) = +\sum_{\beta=1}^{N} \tilde{p}_\beta \frac{(\tilde{q}_{\beta+1} - \tilde{q}_\beta)}{t/N} - \sum_{\beta=1}^{N} H(\tilde{p}_\beta, \tilde{q}_{\beta+1}).
$$

Where, in the step before the last, we have performed the change of variables $q_{-(\gamma-1)+N} := \tilde{q}_\gamma$, $p_{-(\gamma-1)+N} := \tilde{p}_\gamma$ so we have $q_{-\beta+1+N} = \tilde{q}_\beta$, $q_{-\beta+N} = \tilde{q}_{\beta+1}$ and $p_{-\beta+1+N} = \tilde{p}_\beta$. Then equation (47) can be rewritten, for $N \gg 1$ as

$$
\begin{aligned}
[\exp(-itH)](q,p) \equiv G(q,p,t) \simeq (2\pi)^{-N+1} \int \prod_{\alpha=2}^{N} dq_\alpha dp_\alpha . \\
\exp\left\{ +i \sum_{\alpha=1}^{N} \frac{t}{N} \left[p_\alpha \frac{(q_{\alpha+1} - q_\alpha)}{t/N} - H(p_\alpha, q_{\alpha+1}) \right] \right\}
\end{aligned} \tag{48}
$$

with the new edge conditions $q_1 = q_{N+1} = q$, $p_1 = p$. Note that the shift between the two arguments $q_{\alpha+1}$ and p_α of the Hamiltonian is very slight for $N \gg 1$. Now we can finally give an approximation to the heat kernel $\left\langle y, e^{-it\hat{H}} x \right\rangle$ substituting (48) into equation (40) (setting $\hat{A} \equiv e^{-it\hat{H}}$):

$$
\left\langle y, e^{-it\hat{H}} x \right\rangle = (2\pi)^{-1} \int dp \left(e^{-itH} \right)(y,p)\, e^{-ip.(x-y)} \equiv
$$

$$
(2\pi)^{-1} \int dp_1 G(y, p_1; t)\, e^{-i(x-y).p_1} \simeq (2\pi)^{-N} \int \prod_{\alpha=1}^{N} dp_\alpha \prod_{\alpha=2}^{N} dq_\alpha
$$

$$
\exp\left\{ i \sum_{\alpha=1}^{N} \left[p_\alpha (q_{\alpha+1} - q_\alpha) - p_1 (x-y) - \frac{t}{N} H(q_{\alpha+1}, p_\alpha) \right] \right\}
$$

with the boundary conditions $y = q = q_1 = q_{N+1}$, whereas for the moment x remains free. Note also that now al the momentums from p_1 to p_N are integrated. Now let us "manipulate" the first sum at the exponent of the last expression in the following way:

$$\sum_{\alpha=1}^{N} p_\alpha \left(q_{\alpha+1} - q_\alpha\right) - p_1 \left(x - y\right) =$$

$$p_1 \left(q_2 - q_1\right) - p_1 \left(x - y\right) + \sum_{\alpha=2}^{N} p_\alpha \left(q_{\alpha+1} - q_\alpha\right) =$$

$$p_1 \left(q_2 - y\right) + p_1 y - p_1 x + \sum_{\alpha=2}^{N-1} p_\alpha \left(q_{\alpha+1} - q_\alpha\right) + p_N \left(y - q_N\right) =$$

$$p_1 \left(q_2 - x\right) + \sum_{\alpha=2}^{N-1} p_\alpha \left(q_{\alpha+1} - q_\alpha\right) + p_N \left(y - q_N\right) = \sum_{\alpha=1}^{N} p_\alpha \left(q_{\alpha+1} - q_\alpha\right)$$

where is fixed that $q_1 = x$, $q_{N+1} = y$.
Thus finally one has a integral (approximate) expression for the heat kernel:

$$\begin{aligned} \left\langle y, e^{-it\hat{H}} x \right\rangle \simeq (2\pi)^{-N} \int \prod_{\alpha=1}^{N} dp_\alpha \prod_{\alpha=2}^{N} dq_\alpha . \\ \exp \left\{ i \sum_{\alpha=1}^{N} \frac{t}{N} \left[p_\alpha \frac{\left(q_{\alpha+1} - q_\alpha\right)}{t/N} - H\left(q_{\alpha+1}, p_\alpha\right) \right] \right\} \end{aligned} \tag{49}$$

with boundary conditions $q_1 = x$ and $q_{N+1} = y$ and for $N \gg 1$. The approximation gets better as N increases.

7 The Continuum Limit

We think of q_α , p_α as the values of continuous functions $q(\tau)$, $p(\tau)$ at $\tau = \alpha \frac{t}{N}$:

$$q_\alpha \equiv q\left(\alpha \frac{t}{N}\right) \qquad , \qquad p_\alpha \equiv p\left(\alpha \frac{t}{N}\right).$$

Thus we have

$$\sum_{\alpha=1}^{N} p_\alpha \frac{q_{\alpha+1} - q_\alpha}{t/N} - \sum_{\alpha=1}^{N} H\left(q_{\alpha+1}, p_\alpha\right) =$$

$$\sum_{\alpha=1}^{N} \frac{q\left((\alpha+1)\frac{t}{N}\right) - q\left(\alpha \frac{t}{N}\right)}{t/N} p\left(\alpha \frac{t}{N}\right) - \sum_{\alpha=1}^{N} H\left[q\left((\alpha+1)\frac{t}{N}\right), p\left(\alpha \frac{t}{N}\right)\right]$$

from which it is clear that when one takes the limit $N \to \infty$ the difference, but only in the arguments of the Hamiltonian, between $\frac{\alpha+1}{N}$ and $\frac{\alpha}{N}$ is very slight and thus negligible.
Now we take the limit $N \to +\infty$ of equation (49). While N grows up to infinity the time interval $\frac{t}{N} = \Delta t$ becomes infinitesimal and one can formally write:

$$\begin{cases} \frac{t}{N} \overset{N\to+\infty}{\longrightarrow} dt \\ \sum_{\alpha=1}^{N} \overset{N\to+\infty}{\longrightarrow} \int_0^t \end{cases}$$

We can call this procedure a limit of the continuum because it makes us pass from discrete quantities to continuous ones, and through it we pass from multiple integrals to functional ones, as we shall see now. According to the definition of the derivative as limit of the difference quotient we have

$$\lim_{N\to+\infty} \frac{q_{\alpha+1} - q_\alpha}{t/N} \equiv \lim_{\Delta t\to 0} \frac{q_{\alpha+1} - q_\alpha}{\Delta t} = \frac{dq(\tau)}{d\tau}.$$

Now let us implement the passage as follows

$$\left\langle y, e^{-it\hat{H}} x \right\rangle \overset{N\gg 1}{\simeq} \frac{1}{(2\pi)^N} \int \prod_{\alpha=2}^{N} dq_\alpha \prod_{\alpha=1}^{N} dp_\alpha .$$
$$\exp\left\{ i \sum_{\alpha=1}^{N} \frac{t}{N} \left[p_\alpha \frac{q_{\alpha+1} - q_\alpha}{t/N} - H(q_{\alpha+1}, p_\alpha) \right] \right\} \overset{N\to\infty}{\longrightarrow} \tag{50}$$
$$\int \prod_{\tau\in[0,t]} dp(\tau)\, dq(\tau) \exp\left\{ i \int_0^t d\tau \left[p(\tau)\, \dot{q}(\tau) - H(p(\tau), \dot{q}(\tau)) \right] \right\}$$

with $q(0) = x$, $q(t) = y$ and where the exponent can be recognized as the action , i.e. the functional defined by

$$S = \int_0^t d\tau \left[p(\tau) \frac{dq(\tau)}{d\tau} - H(q(\tau), p(\tau)) \right] = \int_0^t \mathcal{L}(\tau)\, d\tau \tag{51}$$

where $\mathcal{L}$ is the Lagrangian of the system.

8 Path Integral of Probability Amplitude

Equation (50), which we have obtained with considerable effort, is the celebrated Feynman's path-integral formula of probability amplitude in Quantum Mechanics, namely:

$$\left\langle y, \exp\left(-it\hat{H}\right) x \right\rangle = \int \prod_{0\le\tau\le t} dq(\tau)\, dq(\tau) \exp\left\{ iS\left[q(\tau), p(\tau) \right] \right\}. \tag{52}$$

It gives us a functional integral expression for the probability amplitude that a single quantum particle, whose dynamics is governed by the Hamiltonian $\hat{H}$, goes from a point x to a point y in space. This is called path integral because one has to integrate over *all paths* that go from the initial point to the ending one , i.e. over all possible phase-space trajectories such that $q(0) = x$ and

$q(t) = y$. It is remarkable to note that, compared with the classical case, here we do not integrate only over the trajectories that makes stationary the action, as the Hamilton principle tells us. With the appropriate modifications this formula is valid in Quantum Field Theory too. [6] One of the difference is that in modern QFT it is usually preferred to use Lagrangian formalism rather then Hamiltonian one, because the former is more appropriate to deal with relativistic theories. Indeed Lagrangian formulation ensures manifest relativistic spacetime invariants, whereas Hamiltonian operator involves a specific time coordinate choice.

9 Euclidean Formulation

If we continue analytically the time parameter t to purely imaginary values by the substitution $t \to (-it)$ then equation (49) becomes

$$\left\langle y, \exp\left(-t\hat{H}\right) x\right\rangle \overset{N \gg 1}{\simeq} (2\pi)^{-N} \int \prod_{\alpha=1}^{N} dp_\alpha \prod_{\alpha=2}^{N} dq_\alpha \cdot$$

$$\exp\left\{\frac{t}{N}\sum_{\alpha=1}^{N}\left[ip_\alpha \frac{(q_{\alpha+1}-q_\alpha)}{t/N} - H(q_{\alpha+1}, p_\alpha)\right]\right\} = (2\pi)^{-1} \cdot$$

$$\int \prod_{\alpha=1}^{N} \frac{dp_\alpha}{\sqrt{2\pi}} \prod_{\alpha=2}^{N} \frac{dq_\alpha}{\sqrt{2\pi}} \exp\left\{\sum_{\alpha=1}^{N} (t/N)\left[ip_\alpha \frac{q_{\alpha+1}-q_\alpha}{t/N} - H(q_{\alpha+1}, p_\alpha)\right]\right\}$$

This analytical continuation is called "Wick rotation" and it can be performed directly on the path-integral. Then taking the limit $N \to +\infty$ we get

$$\left\langle y, \exp\left(-t\hat{H}\right) x\right\rangle = \frac{1}{2\pi}\int \prod_{0 \le \tau \le t} dq(\tau)\, dp(\tau) \cdot$$
$$\exp\left\{i\int_0^t p(\tau)\frac{dq(\tau)}{d\tau}d\tau - \int_0^t H(q(\tau), p(\tau))\, d\tau\right\} \quad (53)$$

10 An Example of Hamiltonian

Le us consider now a quantum system governed by the following Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(q)\, p_i p_j + V(q) \quad (54)$$

where $\left(a^{ij}\right) \equiv a$ is a positive definite matrix and d is the dimension of the configuration space. We introduce the following notation

$q \equiv \left(q^1, q^2, ..., q^d\right) \in \mathbb{R}^d$ belonging to the configuration space whose square norm is given by $\qquad q^2 \equiv q \cdot q = a_{ij}(q)\, q^i q^j$

$$p \equiv (p_1, p_2, ..., p_d) \in \mathbb{R}^d \qquad \text{belonging to the cotangent space}$$

whose square norm is given by $\qquad p^2 \equiv p \cdot p = a^{ij}(q)\, p_i p_j$
And $(a_{ij}(q))$ is the inverse matrix of $\left(a^{ij}(q)\right)$. We view (q, p) as a fiber bundle, of which q is the basis coordinate, belonging to the configuration space and p is the fiber coordinate "standing on" q. For q fixed p is the coordinate of the cotangent space at q, i.e. q is a coordinate of a point of the configuration space and p is a coordinate for the cotangent space at that point. The collection of all tangent spaces to all points of configuration space is the cotangent bundle. Regarding sums over indices we use Einstein's notation.
Now using equation (49) we evaluate the approximate value of the heat kernel with the Hamiltonian operator given by (54):

$$\left\langle y, \exp\left(-it\hat{H}\right) x\right\rangle \overset{N \gtrsim 1}{\simeq} (2\pi)^{-N} \int \prod_{\alpha=1}^{N} dp_\alpha \prod_{\alpha=2}^{N} dq_\alpha \cdot$$

$$\exp\left\{ i \sum_{\alpha=1}^{N} \left[p_\alpha \cdot (q_{\alpha+1} - q_\alpha) - \frac{t}{2N} p_\alpha^\intercal a(q_{\alpha+1})\, p_\alpha - \frac{t}{N} V(q_{\alpha+1}) \right] \right\} = (2\pi)^{-N} \cdot$$

$$\int \prod_{\alpha=1}^{N} dp_\alpha \prod_{\alpha=2}^{N} dq_\alpha \exp\left\{ -i \frac{t}{N} \sum_{\alpha=1}^{N} \frac{1}{2} \left(p_\alpha - a^{-1}(q_{\alpha+1}) \frac{(q_{\alpha+1} - q_\alpha)}{t/N} \right)^\intercal a(q_{\alpha+1}) \cdot\right.$$

$$\left(p_\alpha - a^{-1}(q_{\alpha+1}) \frac{(q_{\alpha+1} - q_\alpha)}{t/N} \right) + \frac{it}{2N} \sum_{\alpha=1}^{N} \frac{(q_{\alpha+1} - q_\alpha)^\intercal}{t/N} a^{-1}(q_{\alpha+1}) \frac{(q_{\alpha+1} - q_\alpha)}{t/N} -$$

$$\left. -\frac{it}{N} \sum_{\alpha=1}^{N} V(q_{\alpha+1}) \right\} = \int \prod_{\alpha=2}^{N} dq_\alpha \cdot$$

$$\exp\left\{ i \sum_{\alpha=1}^{N} (t/N) \left[\frac{1}{2} \left(\frac{q_{\alpha+1} - q_\alpha}{t/N} \right)^\intercal a^{-1}(q_{\alpha+1}) \left(\frac{q_{\alpha+1} - q_\alpha}{t/N} \right) - V(q_{\alpha+1}) \right] \right\} \cdot$$

$$\int \prod_{\alpha=1}^{N} dp_\alpha \exp\left\{ \frac{-it}{2N} \sum_{\alpha=1}^{N} \left[p_\alpha - a^{-1}(q_{\alpha+1}) \frac{(q_{\alpha+1} - q_\alpha)}{t/N} \right]^\intercal a(q_{\alpha+1}) \cdot\right.$$

$$\left.\left[p_\alpha - a^{-1}(q_{\alpha+1}) \frac{(q_{\alpha+1} - q_\alpha)}{t/N} \right] \right\} = \left(\frac{2\pi}{it/N} \right)^{\frac{Nd}{2}} \int \prod_{\alpha=2}^{N} dq_\alpha \prod_{\alpha=2}^{N+1} (\det a(q_\alpha))^{-\frac{1}{2}} \cdot$$

$$\exp\left\{ i \sum_{\alpha=1}^{N} \frac{t}{N} \left[\frac{1}{2} \left(\frac{q_{\alpha+1} - q_\alpha}{t/N} \right)^\intercal a^{-1}(q_{\alpha+1}) \left(\frac{q_{\alpha+1} - q_\alpha}{t/N} \right) - V(q_{\alpha+1}) \right] \right\}.$$

Hence

$$
\begin{aligned}
&\left\langle y, \exp\left(-it\hat{H}\right)x\right\rangle \overset{N\gg 1}{\simeq} \left(\frac{2\pi}{it/N}\right)^{\frac{Nd}{2}} \int \prod_{\alpha=2}^{N} dq_\alpha \prod_{\alpha=2}^{N+1} \left(\det a^{-1}\left(q_\alpha\right)\right)^{\frac{1}{2}} \cdot \\
&\exp\left\{ i \sum_{\alpha=1}^{N} \frac{t}{N} \left[\frac{1}{2} \left(\frac{q_{\alpha+1}-q_\alpha}{t/N}\right)^{\intercal} a^{-1}\left(q_{\alpha+1}\right) \left(\frac{q_{\alpha+1}-q_\alpha}{t/N}\right) - V\left(q_{\alpha+1}\right) \right] \right\}
\end{aligned} \tag{55}
$$

where we have done the change of variables $p_\alpha^{'} = p_\alpha - \frac{N}{t} a^{-1}\left(q_{\alpha+1}\right)\left(q_{\alpha+1} - q_\alpha\right)$, which is a translation over momentums. Thus in virtue of properties of functional measure (invariance under translation) our functional integral is invariant under this change of variables: $dp_\alpha^{'} = dp_\alpha$. We also have used equation (7) to evaluate the Gaussian integral.

Part III

GAUGE THEORIES

We now try to lay the basic concepts of gauge theories in the geometric framework of fiber bundles. We will see that to every gauge field it is associated a connection, i.e. a rule for transporting along curves in a manifold a vector quantity that transforms according to a representation of a Lie group G and that the geometric meaning of the strength field is what is called curvature.

11 Abelian Case - Electromagnetism

To introduce classical gauge theories we consider the simplest not trivial example: Maxwell electromagnetic theory. As to notation we shall write vectors with uppercase latin characters. Maxwell equations, which describe the time evolution of electric and magnetic fields, read:

$$\begin{cases} \boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B} = \boldsymbol{0} & \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \\ \boldsymbol{\nabla} \times \boldsymbol{H} - \partial_t \boldsymbol{D} = \boldsymbol{J} & \boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho \end{cases} \tag{56}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{D}$ is the electric displacement and $\boldsymbol{H}$ is the magnetic field. Using Minkowski metric:

$$g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

setting $\epsilon^{0123} = 1$ and introducing the antisymmetric tensors

$$\begin{aligned} F_{\mu\nu} &= \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\mu\nu} \\ G_{\mu\nu} &= \begin{pmatrix} 0 & H_1 & H_2 & H_3 \\ -H_1 & 0 & D_3 & -D_2 \\ -H_2 & -D_3 & 0 & D_1 \\ -H_3 & D_2 & -D_1 & 0 \end{pmatrix} = -G_{\mu\nu} \end{aligned} \tag{57}$$

we can form the following 2-forms

$$\begin{aligned} F &:= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} \sum_{ijk=1}^{3} \epsilon_{ijk} B_i dx^j \wedge dx^k - \sum_{i=1}^{3} E_i dx^0 \wedge dx^i \\ G &:= \frac{1}{2} G_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} \sum_{ijk=1}^{3} \epsilon_{ijk} D_i dx^j \wedge dx^k + \sum_{i=1}^{3} H_i dx^0 \wedge dx^i \end{aligned} \tag{58}$$

called field strength and dual field strength respectively. Introducing also the 4-current $J_\mu = (\rho, J_1, J_2, J_3)$ we define the 3-form

$$j := \frac{1}{3!}\epsilon_{\mu\nu\lambda\rho}J^\mu \, dx^\nu \wedge dx^\lambda \wedge dx^\rho \tag{59}$$

which is the 4-current density.
Equations (56) can be re-expressed compactly in terms of the 2-forms (58) as

$$\begin{cases} dF = 0 \\ dG = j \end{cases} \tag{60}$$

Proof: Indeed, we have

$$F \equiv \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \Rightarrow dF = \frac{1}{2}\partial_\rho \, F_{\mu\nu}dx^\rho \wedge dx^\mu \wedge dx^\nu =$$

$$\frac{1}{2\cdot 3}\Big\{\partial_\rho F_{\mu\nu}dx^\rho \wedge dx^\mu \wedge dx^\nu + \partial_\mu F_{\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho +$$

$$\partial_\nu F_{\rho\mu}dx^\nu \wedge dx^\rho \wedge dx^\mu\Big\} = \frac{1}{6}\Big\{\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu}\Big\}dx^\rho \wedge dx^\mu \wedge dx^\nu.$$

It is clear that the 3 indices must all be different, otherwise (i.e. if at least 2 of them are equal) the wedge product gives zero. Let us consider the various components of this 3-form, with $\rho \neq \mu \neq \nu$:

- for $(\rho, \mu, \nu) = (i, j, k)$ (spatial components) one has

  $$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = \epsilon_{ijk}\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \pm\left(\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3\right) \equiv \pm \nabla \cdot B = 0$$

  which shows that

  $$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \text{ is equivalent to } \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0.$$

- for $(\rho, \mu, \nu) = (0, i, j)$ ("mixed" spacetime components) one has

  $$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} =$$

  $$\begin{cases} (i, j = 1, 2) = \partial_t B_3 + \partial_1 E_2 - \partial_2 E_1 = (\nabla \times E)_3 + \partial_t B_3 = 0 \\ (i, j = 2, 3) = \partial_t B_1 + \partial_2 E_3 - \partial_3 E_2 = (\nabla \times E)_1 + \partial_t B_1 = 0 \\ (i, j = 3, 1) = \partial_t B_2 + \partial_3 E_1 - \partial_1 E_3 = (\nabla \times E)_2 + \partial_t B_2 = 0. \end{cases}$$

  which shows that

  $$\boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B} = \boldsymbol{0} \text{ is equivalent to } \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0$$

Hence it follows that $dF = 0$, i.e. the 2-form F is closed (namely its exterior derivative is zero). Thus from the Poincaré Lemma we know that in a star-shaped (or simply connected) open set $\mathscr{U} \subset \mathbb{R}^n$ F is an exact form, that is there exists a 1-form A such that

$$F = dA \tag{61}$$

i.e. F is an exact form.
Now let us prove the second of equations (60):

$$dG := \frac{1}{2}\partial_\eta G_{\mu\nu} dx^\eta \wedge dx^\mu \wedge dx^\nu =$$

$$\frac{1}{6}\Big\{\partial_\eta G_{\mu\nu} dx^\eta \wedge dx^\mu \wedge dx^\nu + \partial_\mu G_{\nu\eta} dx^\mu \wedge dx^\nu \wedge dx^\eta + \partial_\nu G_{\eta\mu} dx^\nu \wedge dx^\eta \wedge dx^\mu\Big\} =$$

$$\frac{1}{6}\Big\{\partial_\eta G_{\mu\nu} + \partial_\mu G_{\nu\eta} + \partial_\nu G_{\eta\mu}\Big\} dx^\eta \wedge dx^\mu \wedge dx^\nu = \frac{1}{6}\Big\{\Big(\partial_i G_{jk} + \partial_j G_{ki} + \partial_k G_{ij}\Big)\cdot$$

$$dx^i \wedge dx^j \wedge dx^k + \Big(\partial_0 G_{ij} + \partial_i G_{j0} + \partial_j G_{0i}\Big)\cdot dx^0 \wedge dx^i \wedge dx^j\Big\} =$$

$$\frac{1}{3!}\Big\{\Big(\partial_1 D_1 + \partial_2 D_2 + \partial_3 D_3\Big) dx^1 \wedge dx^2 \wedge dx^3 + \text{cyclic permutations}+$$

$$\Big(\partial_0 D_3 - \partial_1 H_2 + \partial_2 H_1\Big) dx^0 \wedge dx^1 \wedge dx^2 + \Big(\partial_0 D_1 - \partial_2 H_3 + \partial_3 H_2\Big)\cdot$$

$$dx^0 \wedge dx^2 \wedge dx^3 + \Big(\partial_0 D_2 - \partial_3 H_1 + \partial_1 H_3\Big) dx^0 \wedge dx^3 \wedge dx^1 +$$

$$\text{cyclic permutations}\Big\} =$$

$$\frac{1}{3!}\Big\{\boldsymbol{\nabla}\cdot\boldsymbol{D}\ \ dx^i \wedge dx^j \wedge dx^k + \Big(\partial_t \boldsymbol{D} - \boldsymbol{\nabla}\times\boldsymbol{H}\Big) dx^0 \wedge dx^l \wedge dx^m\Big\} =$$

$$\frac{1}{3!}\epsilon_{\mu\nu\lambda\rho} J^\mu\ dx^\nu \wedge dx^\lambda \wedge dx^\rho \equiv j.$$

Thus $dG = j$. **QED**
One great advantage of rewriting Maxwell's fields equations in the more compact form (60)is that they are manifestly coordinate-free and thus relativistically invariant. Indeed equations (60) are totally independent from any choice of a particular coordinate system. Furthermore this equations do not require necessarily a given explicit metric. So this formalism can be generalized to curved spacetime too. It is important to remark that the field strength and its Hodge dual field are viewed as 2-forms of spacetime. Furthermore F admits, in a star-shaped region, a potential A give by equation (61) and understood as a 1-form of spacetime (not only the Minkowski one). It is possible to choose a Riemann manifold as spacetime and fields equations (60) do not change.
We can also rewrite equations (60) eliminating G from them:

$$\begin{cases} dF = 0 \\ \delta F = *j \end{cases} \tag{62}$$

Proof: It is easy to check component per component that $*F = G$, i.e. G is the Hodge star dual of the field strength F. For example:

$$(*F)_{01} = \frac{1}{2!}\epsilon_{01\rho\eta}F_{\mu\nu}g^{\mu\rho}g^{\nu\eta} = \frac{1}{2}\epsilon_{01\rho\eta}F^{\rho\eta} \equiv \frac{1}{2}\epsilon_{01}{}^{\rho\eta}F_{\rho\eta} =$$

$$\frac{1}{2}\left(\epsilon_{01}{}^{23}F_{23} + \epsilon_{01}{}^{32}F_{32}\right) = \epsilon_{01}{}^{23}F_{23} = \epsilon^{0123}F_{23} = F_{23} = B_1 = H_1 = G_{01}.$$

Proceeding in a similar way for all the others components of the tensor one finds $*F = G$, and one has

$$d * F = j \quad \Longleftrightarrow \quad *d * F = *j \quad \Longleftrightarrow \delta F = *j$$

where we use the equality $*d* = \delta$, which is demonstrated in Mathematical Appendix. **QED**

Furthermore, since $d^2 = 0$ (the exterior derivative is nilpotent) from the second of equations (60) it follows that $dj = 0$, i.e. we have 4-current density local conservation. Introducing the electromagnetic coupling constant g and defining

$$\tilde{A} = -igA, \quad \tilde{F} = -igF, \quad \tilde{j} = -\frac{i}{g}j$$

we can write the electromagnetic Lagrangian as

$$\mathcal{L}_{EM} = \frac{1}{2g^2}\tilde{F} \wedge *\tilde{F} + \tilde{j} \wedge \tilde{A}. \tag{63}$$

This Lagrangian is invariant under $U(1)$ local transformations group:

$$A'_\mu(x) = A_\mu(x) + i\partial_\mu\lambda(x) \tag{64}$$

12 The Volume Form

Let us consider a n-dimensional oriented manifold M equipped with a metric g. Then there exists a canonic volume form on M which can be constructed as follows. First we cover the manifold M with oriented charts $\varphi_\alpha : \mathscr{U}_\alpha \to \mathbb{R}^n$, in each of which we set a metric $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ and we define

$$\text{vol} := \left(|\det g_{\mu\nu}|\right)^{1/2} dx^1 \wedge dx^2 \wedge ... \wedge dx^n \tag{65}$$

called a volume n-form. We remember that the standard volume form in $\mathbb{R}^n$ is $\omega = dx^1 \wedge ... \wedge dx^n$. Let us take two different oriented charts φ, φ' ($\varphi : \mathscr{U} \to \mathbb{R}^n$, $\varphi' : \mathscr{U}' \to \mathbb{R}^n$), the latter one being equipped with a metric $g'_{\mu\nu} = g\left(\partial'_\mu, \partial'_\nu\right)$. In addiction to the volume form 65 associated to the chart φ we have then the volume form associated to the chart φ':

$$\text{vol}' = \left(|\det g'_{\mu\nu}|\right)^{1/2} dx'^1 \wedge dx'^2 \wedge ... \wedge dx'^n.$$

We want to find a volume form well defined over all the manifold M, that is we require that

$$\mathrm{vol} = \mathrm{vol}' \quad , \quad \text{on } \mathscr{U} \cap \mathscr{U}' \tag{66}$$

This equation holds true if we assume that the metric transforms as a covariant tensor of rank two.

PROOF: On the overlap of the two charts φ and φ' we have

$$dx'^{\nu} = T^{\nu}{}_{\mu} dx^{\mu} \quad , \quad \text{with} \quad T^{\nu}{}_{\mu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}$$

hence

$$dx'^1 \wedge \dots \wedge dx'^n = \left(\det T\right) dx^1 \wedge \dots \wedge dx^n. \tag{67}$$

Since the metric tensor $g_{\mu\nu}$ transforms with the inverse matrix T^{-1}:

$$g'_{\mu\nu} \equiv g\left(\partial'_{\mu}, \partial'_{\nu}\right) = g\left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\partial_{\alpha}, \frac{\partial x^{\beta}}{\partial x'^{\nu}}\partial_{\beta}\right) \equiv$$

$$\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} = \left(T^{-1}\right)^{\alpha}{}_{\mu} \left(T^{-1}\right)^{\beta}{}_{\nu} g_{\alpha\beta}$$

and taking the determinant of this equation it follows that

$$\det g'_{\mu\nu} = \left(\det T\right)^{-2} \det g_{\mu\nu}. \tag{68}$$

But since both charts φ φ' are oriented we have $\det T > 0$ and on the overlap $\mathscr{U} \cap \mathscr{U}'$ we can extract the square root and find

$$\sqrt{\left|\det g'_{\mu\nu}\right|} = \left(\det T\right)^{-1} \sqrt{\left|\det g_{\mu\nu}\right|}. \tag{69}$$

Or, defining $g := |\det g_{\mu\nu}|$ equivalently

$$\left(g'\right)^{1/2} = \left(\det T\right)^{-1} (g)^{1/2} \tag{70}$$

from which (on $\mathscr{U} \cap \mathscr{U}'$) it follows that

$$\left(g'\right)^{1/2} = \left(\det T\right)^{-1} (g)^{1/2} \Longleftrightarrow$$

$$\left(g'\right)^{1/2} dx'^1 \wedge \dots \wedge dx'^n = \left(\det T\right)^{-1} \left(\det T\right) (g)^{1/2} dx^1 \wedge \dots \wedge dx^n =$$

$$(g)^{1/2} dx^1 \wedge \dots \wedge dx^n$$

namely $\mathrm{vol} = \mathrm{vol}'$. **QED**

Thus we have found an invariant volume n-form defined by

$$\mathrm{vol} = g^{1/2}\, dx^1 \wedge \dots \wedge dx^n. \tag{71}$$

In general relativity it is used to write it simply as $\mathrm{vol} = \sqrt{g}\ d^n x$. Since this volume form is well defined over all an oriented manifold M (curved spacetime) and it is totally independent from any choice of the chart we can use it as volume measure in the action integral for gauge fields on curved spacetime:

$$dV = \sqrt{g}\ d^4 x \tag{72}$$

13 Connections

How can we differentiate a vector field in a (vector) fiber bundle in general? We need to extend the concept of differentiation to the vector bundle in such a way that we get infinitesimal variations of sections of the fiber bundle. This problem is not trivial. A section of a fiber bundle is a map that associates smoothly to each point x of the base manifold (physically the spacetime manifold) a vector $s(x)$ belonging to the fiber E_x lying above that point. To differentiate a vector field in a direction v tangent at a point x one would naively take the limit of the appropriate incremental ratio:

$$D_v s(x) := \lim_{\epsilon \to 0} \frac{s(x+\epsilon v) - s(x)}{\epsilon}, \qquad v \in T_x M.$$

This implies that one has to compute the difference of vectors belonging to distinct vector spaces. However this difference is not defined in any natural way because distinct fibers are different spaces. Let see it in greater detail.

Let E be a vector bundle over the manifold M: we view E as a smooth family of vector spaces parametrized by the corresponding point x of M. We denote as $\Gamma(E)$ the space of sections of E. A section s is a continuous map which associates a point of a fiber over $x \in M$ with each point x. A connection D of E associates with each vector field v over M a function $D_v : \Gamma(E) \to \Gamma(E)$ such that, $\forall v, w \in Vect(M)$, $\forall s, t \in \Gamma(s)$:

- i) $D_v(\alpha s) = \alpha D_v(s) \quad \forall \ \alpha \in \mathbb{R}$
- ii) $D_v(t+s) = D_v(t) + D_v(s)$
- iii) $D_v(fs) = v(f)\, s + f D_v s \quad \forall \ f \in C^\infty(M)$
- iv) $D_{v+w} s = D_v s + D_w s$
- v) $D_{fv} s = f D_v s \quad \forall \ f \in C^\infty(M)$

We note that the definition of connection extends the notion of vector field from functions to sections, and it depends on the vector field v defined over M: $D_v(s)$ is the covariant derivative of the section s along the direction of v. We must think D as a collection of functions D_v with all possible vector fields v over M. The third property above is nothing less that Leibniz rule and it makes D_v a derivative operator.

Now we try to make this definition less abstract and to do this let us express the connection in terms of local coordinates of M and a basis of sections of the bundle E. Let $\{x^\mu\}$ be a set of coordinates of a open subset $\mathscr{U}$ of the manifold M, $\{\partial_\mu\}$ the corresponding basis of the tangent bundle at $\mathscr{U}$ and let $\{e_i\}$ be a basis of sections of E over $\mathscr{U}$ (so $\{e_i\}$ spans all $\Gamma(E)$). We also set $D_\mu \equiv D_{\partial_\mu}$. Since for every section s of E $D_v(s)$ is also a section of E and the e_i form a basis in $\Gamma(E)$ then we can express in a unique way $D_\mu e_j$ (with arbitrary μ and j) as a linear combination of e_i with appropriate coefficients which will be functions on $\mathscr{U}$:

$$D_\mu e_j = A^k_{\mu j} e_k. \tag{73}$$

The functions $A^k_{\mu j}$ defined on $\mathscr{U}$ are the components of the connection (i.e. physically the gauge field). Let now see that the connection allows us to express explicitly the covariant derivative of any section of the bundle E in the direction of any vector field v. Writing s $(\forall s \in \Gamma(E))$ as $s = s^i e_i$ we have

$$D_v(s) \equiv D_{v^\mu \partial_\mu}(s) = v^\mu D_\mu\left(s^i e_i\right) = v^\mu\left[\left(\partial_\mu s^i\right) e_i + s^i D_\mu e_i\right] = \\ v^\mu\left\{\left(\partial_\mu s^i\right) e_i + A^i_{\mu k} e_i s^k\right\} = v^\mu\left(\partial_\mu s^i + A^i_{\mu k} s^k\right) e_i.$$

We obtain therefore the expression

$$D_v(s) = v^\mu\left(\partial_\mu s^i + A^i_{\mu k} s^k\right) e_i \tag{74}$$

From which it is manifest that the covariant derivative $D_v(s)$ is indeed a form of differentiation of the section s in the direction of v. Note that, physically speaking, the n-dimensional vector v is a spacetime quantity whereas the values of the section s belong to a "inner" space, the fiber, which is connected to spacetime but is "above" it. Equation (74) also tells us that the connection D associates a covariant derivative D_v to each vector field v defined on the manifold M once the connection components $A_\mu = \left(A^k_{\mu j}\right) \in C^\infty(\mathscr{U}, GL(d, \mathbb{C}))$ are given.

14 Curvature

Let E be a vector bundle over the manifold M equipped with a connection D. The "curvature" of a connection is somehow a measure of the failure of covariant derivative to commute. Given two vector fields v and w on M, we define the curvature $F(v, w)$ to be the operator acting on sections of E given by

$$F(v, w) := D_v D_w s - D_w D_v s - D_{[v,w]} s \equiv [D_v, D_w] - D_{[v,w]}. \tag{75}$$

In the simplest case of a trivial bundle with vector fiber V with a standard flat connection and where a section is a function $f : M \to V$ one has

$$F(v, w)s = vwf - wvf - [v, w]f = 0$$

A connection such that $F(v, w)s = 0$ for all vector fields v, w and sections s has vanishing curvature and is said to be flat. Curvature is manifestly antisymmetric:

$$F(v, w) = -F(w, v). \tag{76}$$

It is also a linear operation over $C^\infty(M)$ in each argument and in this sense it is often said to be a 'tensor':

$$F(fv, w)s = F(v, fw)s = F(v, w)(fs) = fF(v, w)s \tag{77}$$

for any $f \in C^\infty(M)$.

Proof: By definition of curvature and using the Lie Brackets of vector fields we have

$$\begin{aligned} F(v, fw) = D_v D_{fw} - D_{fw} D_v - D_{f[v,w]+v(f)w} = \\ D_v f D_w - f D_w D_v - f D_{[v,w]} - v(f) D_w = \\ f D_v D_w + v(f) D_w - f D_w D_v - f D_{[v,w]} - v(f) D_w = \\ f[D_v, D_w] - f D_{[v,w]} \equiv f F(v,w). \end{aligned}$$

F is also linear in the first argument:

$$F(fv, w) = -F(w, fv) = -fF(w,v) = fF(v,w).$$

And finally we have $F(v,w)fs = fF(v,w)s$, indeed:

$$\begin{aligned} F(v,w)(fs) = D_v D_w(fs) - D_w D_v(fs) - D_{[v,w]}(fs) = \\ f D_v D_w s + v(f) D_w s + w(f) D_v s + v(w(f))s - f D_{[v,w]} s - ([v,w]f)s = \\ f[D_v, D_w]s - f D_{[v,w]} s = fF(v,w)s. \end{aligned}$$

Thus $F(v,w)$ defines a smooth linear map from $\Gamma(E)$ to itself. **QED**

By virtue of linearity property of the curvature $F(v,w)$ must correspond to a section of $\mathrm{End}(V)$ and we can write

$$F(v,w) = v^\mu w^\nu F_{\mu\nu}. \tag{78}$$

with

$$F_{\mu\nu} = F(\partial_\mu, \partial_\nu) = [D_\mu, D_\nu]. \tag{79}$$

Using a local basis of sections e_i we have

$$\begin{aligned} F_{\mu\nu} e_i = D_\mu D_\nu e_i - D_\nu D_\mu e_i = D_\mu(A^j_{\nu i} e_j) - D_\nu(A^j_{\mu i} e_j) = \\ \big((\partial_\mu A^j_{\nu i}) - (\partial_\nu A^j_{\mu i}) + A^j_{\mu k} A^k_{\nu i} - A^j_{\nu k} A^k_{\mu i}\big) e_j. \end{aligned}$$

And suppressing the internal indices i, j, k associated with the local basis of sections of E we can finally write the curvature in the more familiar way:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{80}$$

15 Change of Basis of Sections

Let $\mathscr{U}, \mathscr{V}$ be neighborhoods of the manifold M with $\mathscr{U} \cap \mathscr{V} \neq \emptyset$, $\{e_i\}_{i=1,\ldots,d}$ a basis of sections on $\mathscr{U}$ and $\{f_i\}_{i=1,\ldots,d}$ a basis of sections on $\mathscr{V}$. On the overlap $\mathscr{U} \cap \mathscr{V}$ we have $e_i = T^j{}_k f_j$ and it follows that

$$\overset{(e)}{A_\mu} = T^{-1} \overset{(f)}{A} \;\, T + T^{-1} \partial_\mu T \tag{81}$$

where

$$\begin{cases} D_\mu e_j = \overset{(e)}{A}{}^k_{\mu j} e_k \\ D_\mu f_l = \overset{(f)}{A}{}^m_{\mu l} f_m \end{cases} \tag{82}$$

Proof:

$$\overset{(e)}{A}{}^k_{\mu j} e_k = \overset{(e)}{A}{}^k_{\mu j} T^l{}_k f_l = D_\mu e_j = D_\mu \left(T^l{}_j f_l \right) = T^l{}_j D_\mu f_l + \partial_\mu \left(T^l{}_j \right) f_l =$$

$$T^l{}_j \overset{(f)}{A}{}^m_{\mu l} f_m + f_l \partial_\mu T^l{}_j \Longleftrightarrow \overset{(e)}{A}{}^m_{\mu j} T^k{}_m f_k = \left\{ T^m{}_j \overset{(f)}{A}{}^k_{\mu m} + \partial_\mu \left(T^k{}_j \right) \right\} f_k$$

from which it follows that

$$\overset{(e)}{A}{}^m_{\mu j} T^k{}_m = T^m{}_j \overset{(f)}{A}{}^k_{\mu m} + \partial_\mu T^k{}_j$$

or without indices in the form

$$T \overset{(e)}{A}_\mu = \overset{(f)}{A}_\mu T + \partial_\mu T.$$

Now we multiply both sides of last equation for T^{-1} from the left and we finally obtain the famous gauge transformation law

$$\overset{(e)}{A}_\mu = T^{-1} \overset{(f)}{A}_\mu T + T^{-1} \partial_\mu T. \tag{83}$$

QED

Thus we see clearly that a gauge transformation arises naturally when one performs a change of basis in the space of sections $\Gamma(E)$ of the fiber bundle. It is easy to show that equation (83) reduces to (64) if the gauge group of the bundle is U(1).

16 Yang-Mills Action

The basic data of a gauge theory are a Euclidean Riemannian manifold M and a gauge compact Lie group G. The fundamental field of the gauge theory is the gauge field $\mathcal{A}_\mu$, which is a 1-form over M with values in the Lie algebra $\mathfrak{g}$ of G. Introducing the gauge strength field (i.e. the curvature)

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + \left[\mathcal{A}_\mu, \mathcal{A}_\nu \right] \tag{84}$$

the action functional reads

$$S_{\text{eucl.}} = \frac{1}{4g^2_{YM}} \int dV \;\; g^{\mu\alpha} g^{\nu\beta} \langle \mathcal{F}_{\alpha\beta}, \mathcal{F}_{\mu\nu} \rangle = \frac{1}{4g^2_{YM}} \int dV \; \langle \mathcal{F}^{\mu\nu}, \mathcal{F}_{\mu\nu} \rangle \tag{85}$$

where g_{YM} is the coupling constant and $\langle \cdot\, , \, \cdot \rangle$ is a quadratic gauge-invariant form on the Lie algebra. The basic property that characterizes all gauge theories (Abelian and not) is the invariance under local G-transformations, where G is the compact Lie group. A gauge transformation is given by the laws

$$\begin{cases} {}^{\gamma}A_{\mu} = \gamma A_{\mu}\gamma^{-1} - \partial_{\mu}\gamma\gamma^{-1} \\ {}^{\gamma}\mathcal{F}_{\mu\nu} = \gamma\mathcal{F}_{\mu\nu}\gamma^{-1} \end{cases} \tag{86}$$

where $\gamma \in \mathrm{Map}(M, G)$ is the G-valued function that represent the local gauge transformation.
Let consider the Yang-Mills Lagrangian with fermionic term too

$$\mathcal{L}_{YM}(x) = -\frac{1}{4}F(x)^{a}_{\mu\nu}F^{\mu\nu a}(x) + \overline{\Psi}(x)\big(i\gamma^{\mu}D_{\mu} - M\big)\Psi(x). \tag{87}$$

This Lagrangian is invariant under the gauge transformation (86) if the quadratic form $\langle \cdot, \cdot \rangle$ is gauge-invariant, namely if for $X, Y \in \mathfrak{g}$ and $h \in G$ one has

$$\langle hXh^{-1}, hYh^{-1} \rangle = \langle X, Y \rangle. \tag{88}$$

So it follows that the quadratic form in the Yang-Mills action in invariant under gauge transformations:

$$\langle {}^{\gamma}\mathcal{F}_{\mu\nu}, {}^{\gamma}\mathcal{F}^{\mu\nu} \rangle = \langle \gamma\mathcal{F}_{\mu\nu}\gamma^{-1}, \gamma\mathcal{F}^{\mu\nu}\gamma^{-1} \rangle = \langle \mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu} \rangle. \tag{89}$$

Thus the action (85) is gauge invariant. The transposition of equation (86) in terms of operators (i.e. in second quantization) is expressed by

$$A'_{\mu}(x) = U_h(x)A_{\mu}(x)U_h^{-1}(x) + \partial_{\mu}U_h(x)U_h^{-1}(x) \tag{90}$$

where h is an element of the gauge group G and U_h is a unitary representation of G on Fock space.

17 Path-Ordered Exponential

Then, let G be a Lie group, $\mathfrak{g}$ its Lie algebra and V a r-dimensional vector space which is a representation space of $\mathfrak{g}$. Let $\rho : G \to GL(V)$ be a representation of $\mathfrak{g}$ on the linear space V and $\rho' : \mathfrak{g} \to \mathfrak{gl}(V)$ the associated representation of $\mathfrak{g}$. We consider a scalar field ϕ, which can be viewed as a V-valued 0-form: $\phi : \mathscr{U} \to V$ ($\mathscr{U} \subset \mathbb{R}^4$), and a local gauge transformation γ given by the G-valued function $\gamma : \mathscr{U} \to G$. The field ϕ transforms according to the law:

$$\phi'(x) = \rho(\gamma(x))\,\phi(x) \tag{91}$$

We shall understand that γ acts through the representation ρ and we shall write equation (91) in a more simple way as

$$\phi'(x) = \gamma(x)\,\phi(x) \tag{92}$$

Now let C be a curve with initial point x and end point y and let consider the parallel transport $\Gamma[C]$: the operator that associates with each vector of the space V_x a corresponding vector in the fiber V_y. Our construction will be gauge-invariant if

$$\Gamma'[C] = \gamma(y)\,\Gamma[C]\,\gamma^{-1}(x)\,. \tag{93}$$

Since the parallel transport of a vector should be a continuous operation $\Gamma[C]$ must be infinitesimally closed to the identity for an infinitesimal curve C. Now we consider a curve C parameterized as $Q(\tau) : [0,1] \rightarrow \mathscr{U}$ $(0 \leq \tau \leq 1)$. The tangent vector at C in $Q(\tau)$ is given by $\dot{Q}(\tau) \equiv \frac{d}{d\tau}Q(\tau)$. Let $\phi(\tau) \in V$ be the result of the parallel transport of $\phi(0)$ along the segment of the curve C that goes from $Q(0)$ to $Q(\tau)$. Similarly for $\epsilon \rightarrow 0^+$ $\phi(\tau+\epsilon)$ results from the parallel transport of $\phi(0)$ along the partial piece of C from $Q(\tau)$ to $Q(\tau+\epsilon)$. The infinitesimal generator of parallel transport at $Q(\tau)$ is the component of the gauge field $A_{Q(\tau)}$, along $\dot{Q}(\tau)$:

$$\phi(\tau+\epsilon) = \Gamma\left(C_{Q(\tau),Q(\tau+\epsilon)}\right)\phi(t) = \phi(\tau) - \epsilon A_{Q(\tau)}\left(\dot{Q}(\tau)\right)\phi(\tau) + O\left(\epsilon^2\right) = \left[1 - \epsilon A_\mu\Big(Q(\tau)\Big)\dot{Q}(\tau)^\mu\right]\phi(\tau) + O\left(\epsilon^2\right)$$

from which it follows the differential equation for the parallel transport

$$\frac{d}{d\tau}\phi(\tau) = -A_\mu\big(Q(\tau)\big)\dot{Q}(\tau)^\mu\,\phi(\tau) \tag{94}$$

Let see this from a more geometric perspective. Let E be a vector bundle over the manifold M equipped with a connection D, $\gamma : [0,T] \rightarrow M$ a smooth path that goes from a point p to a point q of M and let $u(t)$ be a section of the bundle E over the curve $\gamma(t)$. We want to write an equation which describes the 'parallel transport' of $u(t)$ along $\gamma(t)$. Calling $\gamma'(t)$ the tangent vector at $\gamma(t)$ we must require that

$$D_{\gamma'(t)}u(t) \equiv 0. \tag{95}$$

This equation states that the vector $u(t)$ is transported "parallel to itself" along $\gamma(t)$, since its variation along the direction in which the curve "runs" is zero. Equation (95) explicitly reads

$$D_{\gamma'(t)}u(t) = \frac{d}{dt}u(t) + A(\gamma'(t))\,u(t) = 0 \tag{96}$$

Thus one gets the equation of parallel transport:

$$\frac{d}{dt}u(t) = -A(\gamma'(t))\,u(t) \tag{97}$$

The formal solution of this differential equation is

$$u\left(t\right)=\sum_{n=0}^{+\infty}\Big\{\left(-1\right)^{n}\int_{0\le t_{n}\le t_{n-1}\le\ldots\le t_{1}\le t}dt_{n}...dt_{1}A\left(\gamma'\left(t_{1}\right)\right)...A\left(\gamma'\left(t_{n}\right)\right)\Big\}u_{0}. \tag{98}$$

PROOF: Let us solve equation (97). We first note that one particular solution is given by

$$u(t)=u_{0}-\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)u\left(t_{1}\right)\quad,\qquad u_{0}\equiv u(0) \tag{99}$$

Indeed one has

$$\frac{d}{dt}u(t)=-\frac{d}{dt}\Big\{\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)u\left(t_{1}\right)\Big\}=-A\left(\gamma'(t)\right)u\left(t\right) \tag{100}$$

Thus the expression (99) solves the equation of parallel transport, nevertheless it does not provide a solution but only an integral equation for the solution . So let us proceed by iteration, namely we substitute (99) into $u(t)$ in the integrand n times as follows:

$$\begin{aligned}
u(t)=u_{0}-\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)u\left(t_{1}\right)=\\
u_{0}-u_{0}\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)+\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)\int_{0}^{t_{1}}dt_{2}A\left(\gamma'(t_{2})\right)u\left(t_{2}\right)=\\
u_{0}-u_{0}\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)+u_{0}\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)\int_{0}^{t_{1}}dt_{2}A\left(\gamma'(t_{2})\right)-\\
u_{0}\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)\int_{0}^{t_{1}}dt_{2}A\left(\gamma'(t_{2})\right)\int_{0}^{t_{2}}dt_{3}A\left(\gamma'(t_{3})\right)+...+\\
(-1)^{n}\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)\int_{0}^{t_{1}}dt_{2}A\left(\gamma'(t_{2})\right)\cdot\\
\int_{0}^{t_{2}}dt_{3}A\left(\gamma'(t_{3})\right)....\int_{0}^{t_{n-1}}dt_{n}A\left(\gamma'(t_{n})\right)u(t_{n})=\\
\Big\{1-\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)+\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)\int_{0}^{t_{1}}dt_{2}A\left(\gamma'(t_{2})\right)-\\
\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)\int_{0}^{t_{1}}dt_{2}A\left(\gamma'(t_{2})\right)\int_{0}^{t_{2}}dt_{3}A\left(\gamma'(t_{3})\right)+...+\\
(-1)^{n}\int_{0}^{t}dt_{1}A\left(\gamma'(t_{1})\right)\int_{0}^{t_{1}}dt_{2}A\left(\gamma'(t_{2})\right)\cdot\\
\int_{0}^{t_{2}}dt_{3}A\left(\gamma'(t_{3})\right)....\int_{0}^{t_{n-1}}dt_{n}A\left(\gamma'(t_{n})\right)\frac{u(t_{n})}{u_{0}}\Big\}u_{0}
\end{aligned}$$

and taking the limit when n tends to infinity, since $u(t_n) \underset{n\to\infty}{\longrightarrow} u_0$, we finally get

$$u(t) = \sum_{n=0}^{+\infty}\Big\{(-1)^n \int_{0\le t_n\le t_{n-1}\le \ldots\le t_1\le t} A\left(\gamma'(t_1)\right).....A\left(\gamma'(t_n)\right)dt_n.....dt_1\Big\}u_0. \tag{101}$$

QED
It is apparent that the parameter $t \in [0, T]$ of the path γ is decreasingly ordered in the series of the above integrals and if t were a time parameter we should say that equation (101) is time ordered. Now we ask whether the infinite sum (101) converges. Surprisingly that sum converges!

Proof: Let give a norm in the vector space V and let define a norm in $\mathrm{End}(V)$ by $\|T\| := \sup_{\|u_0\|=1} \|Tu_0\|$. Furthermore defining $K := \sup_{t\in[0,T]} \|A\left(\gamma'(t)\right)\|$ we consider the $n-$th term of the series (101):

$$u_n \equiv (-1)^n \int_{0\le t_n\le t_{n-1}\le \ldots\le t_1\le t} dt_1....dt_n\ A\left(\gamma'(t_1)\right)....A\left(\gamma'(t_n)\right)u_0.$$

We have

$$\|u_n\| = |(-1)^n| \sup_{\|u_0\|=1}\left\|\int_{0\le t_n\le t_{n-1}\le \ldots\le t_1\le t} dt_1....dt_n\ A\left(\gamma'(t_1)\right)....A\left(\gamma'(t_n)\right)u_0\right\| \le$$
$$\sup_{\|u_0\|=1}\int_{0\le t_n\le t_{n-1}\le \ldots\le t_1\le t} dt_1....dt_n\ \|A\left(\gamma'(t_1)\right)\|....\|A\left(\gamma'(t_n)\right)\|\,\|u_0\| \le$$
$$\frac{K^n t^n \|T\|}{n!} \overset{n\to\infty}{\longrightarrow} 0\ .$$

Thus the series (101) is convergent and it defines a good solution $u(t)$ of equation (97). **QED**
Equation (101) can be re-expressed as

$$u(t) = \mathcal{P}\exp\left\{-\int_0^t A(\gamma'(s))ds\right\}u_0. \tag{102}$$

Proof:
Let us define the path-ordered product as

$$\mathcal{P}\Big\{A(\gamma'(t_1))...A(\gamma'(t_n))\Big\} := A(\gamma'(t_{\sigma(1)}))...A(\gamma'(t_{\sigma(n)})), \qquad t_{\sigma(1)} \ge ... \ge t_{\sigma(n)}$$

where $\sigma \in S_n$ (i.e. an element of the finite symmetric group of n objects) is a permutation such that larger values of t_i appear first in the product (of the gauge fields). Then we have

$$\int_{0\leq t_n \leq t_{n-1} \leq \ldots \leq t_1 \leq t} dt_1 \ldots . dt_n \, A\left(\gamma'(t_1)\right) \ldots . A\left(\gamma'(t_n)\right) =$$
$$\frac{1}{n!}\int_{t_i\in[0,t]} dt_1 \ldots dt_n \mathcal{P}\Big\{A(\gamma'(t_1))\ldots A(\gamma'(t_n))\Big\} \equiv \frac{1}{n!}\mathcal{P}\Big(\int_0^t A(\gamma'(s))ds\Big)^n. \tag{103}$$

Thus defining the path-ordered exponential by

$$\mathcal{P}\exp\Big\{-\int_0^t A(\gamma'(s))ds\Big\} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\mathcal{P}\Big(\int_0^t A(\gamma'(s))ds\Big)^n \tag{104}$$

one has

$$u(t) = \mathcal{P}e^{-\int_0^t A(\gamma'(s))ds}u_0. \tag{105}$$

QED

The physical meaning of path-ordering may be that gauge fields are ordered in decreasing way following the path from the initial to the end point. If $A(\gamma'(s))$ is independent of the parameter s the path-ordered exponential reduces to the ordinary one. Instead when G is Abelian $A(\gamma'(s))$ commute for different values of s and the path-ordering operation has no effect:

$$\mathcal{P}\Big\{A(\gamma'(t_1))\ldots A(\gamma'(t_n))\Big\} := A(\gamma'(t_1))\ldots A(\gamma'(t_n))$$

and (105) reduces to

$$u(t) = e^{-\int_\gamma A}u_0 \tag{106}$$

This expression remembers very closely the phase acquired by a charged particle moving along a path through a magnetic field:

$$e^{-\frac{i}{\hbar}q\int_\gamma A}$$

18 Holonomy and Wilson Loop

Let us consider a piecewise smooth path γ and let break it up into maximal smooth pieces $\gamma_i : [t_i, t_{i+1}] \to M$ $(1 \leq n \leq n)$. We define the holonomy by

$$\mathscr{H}(\gamma, D) = \mathscr{H}(\gamma_n, D)\ldots\mathscr{H}(\gamma_1, D) \tag{107}$$

It is easy to show that holonomy $\mathscr{H}(\gamma, D)$ is affected in a simple way when we perform a gauge transformation g to the connection D:

$$\mathscr{H}(\gamma, D') = g(\gamma(t_f))\mathscr{H}(\gamma, D)g(\gamma(0)) \tag{108}$$

where applying the gauge transformation g to the connection the holonomy is the linear map that sends $u(0)$ to $u(t_f)$. If we consider the holonomy around a loop (i.e. a closed path) last equation reduces to

$$\mathscr{H}(\gamma, D') = g(q)\mathscr{H}(\gamma, D)g(q)^{-1} \tag{109}$$

for $q \in M$ a point of the loop. Thus if we take the trace of $\mathscr{H}(\gamma, D)$ we obtain a number that does not change under gauge transformations:

$$\mathrm{tr}(\mathscr{H}(\gamma, D')) = \mathrm{tr}\left[g(q)\mathscr{H}(\gamma, D)g(q)^{-1}\right] = \mathrm{tr}\left[\mathscr{H}(\gamma, D)\right]. \tag{110}$$

Thus $\mathrm{tr}(\mathscr{H}(\gamma, D))$ is gauge invariant and it is called Wilson loop:

$$W(\gamma, D) := \mathrm{tr}(\mathscr{H}(\gamma, D)). \tag{111}$$

19 Isometric Group Actions

Let us consider a left action $G \times M \to M$ of the group G over the manifold M. Supposing that G is equipped with a bi-invariant metric then for $g \in G$ the tangent space T_gG is endowed with a metric $(\cdot, \cdot)_g$ such that, for $\xi, \eta \in T_gG$:

$$i) \quad \left(L_{h*g}\xi, L_{h*g}\eta\right)_{hg} = \left(\xi, \eta\right)_g \ , \qquad \left(R_{h*g}\xi, R_{h*g}\eta\right)_{gh} = \left(\xi, \eta\right)_g \tag{112}$$

Then $(\cdot, \cdot)$ is the bi-invariant adjoint metric of the Lie algebra $\mathfrak{g} = T_eG$. From (112) follows that

$$\begin{aligned}(\mathrm{Ad}h\ \xi, \mathrm{Ad}h\ \eta)_e := (L_{h*h^{-1}} \circ R_{h^{-1}*e}\xi, L_{h*h^{-1}} \circ R_{h^{-1}*e}\eta)_{hh^{-1}} =& \\ (R_{h^{-1}*e}\xi, R_{h^{-1}*e}\eta)_{h^{-1}} = (\xi, \eta)_e&\end{aligned} \tag{113}$$

Thus we have a bi-invariant metric, i.e. a metric invariant under right and left translations. Suppose that also the manifold M is endowed with a metric. The left action $G \times M \to M$ is said to be isometric if the following property holds true. For any $g \in G$ let us define a map $A_g : M \to M$ given by $A_g(x) = gx$, with $x \in M$. Then we require that

$$\left(A_{g*x}u, A_{g*x}v\right)_{gx} = (u, v)_x \qquad u, v \in T_xM. \tag{114}$$

Then the tangent space T_xM is equipped with an invariant metric, i.e. the action of all group elements through the map A_g does not change the metric properties of the manifold. The action $G \times M \to M$ is also associated with other maps. Let $x \in M$, $B_x : G \to M$ the map defined as $B_x(g) = gx$ (with x fixed while g varies on all G). Then $B_{x*g} : T_gG \to T_{gx}M$ and $B_{x*e} : \mathfrak{g} \to T_xM$. Given $\xi \in \mathfrak{g}$ we can also consider the map $C_\xi(x) = B_{x*e}\xi$ which defines the so-called fundamental vector field of ξ. Finally if we consider an element h of the stabilizer of x we have $B_x(h) = hx = x$.

20 Measures

It is possible to define a measure $(d\mu)$ over G by assigning to each element g of the group a measure $(d\xi)_g$ in the tangent space T_gG. The measure $(d\xi)_g$ is a standard Lebesgue measure over T_gG with the normalization condition:

$$\int_{T_gG} (d\xi)_g \exp\left\{-\frac{1}{2}(\xi.\xi)_g\right\} = 1. \tag{115}$$

For $h \in G$ the map $L_h : G \to G$ is invertible and then also the map $L_{h*g} : T_gG \to T_{hg}G$ is invertible. So we can define the Jacobian $J(L_{h*g})$ of the left translation by the property:

$$J(L_{h*g}) \int_{T_gG} (d\xi)_g \exp\left\{-\frac{1}{2}(L_{h*g}\xi, L_{h*g}\xi)_{hg}\right\} = \int_{T_{hg}G} (d\xi)_{hg} \exp\left\{-\frac{1}{2}(\xi,\xi)_{hg}\right\} \tag{116}$$

which expresses the invariance of the measure under (left) translations. Since the scalar product is preserved by left translations: $(L_{h*g}\xi, L_{h*g}\xi)_{hg} = (\xi,\xi)_g$ it follows that

$$J(L_{h*g}) = 1. \tag{117}$$

Proceeding in a very similar way in the case of right translations, given by the invertible maps $R_h : G \to G$, one gets

$$J(R_{h*g}) = 1. \tag{118}$$

It must be clear that the measure $(d\mu)_g$ is defined over the manifold representing the locally compact Lie group G whereas the measure $(d\xi)_g$ is defined over the tangent space T_gG at that manifold in the point g. In any neighborhood of the group manifold we have

$$(d\mu)_g = (d\xi)_g. \tag{119}$$

From equations (116), (117) it follows that

$$\int_G J(L_{h*g})(d\mu)_g f(hg) = \int_G (d\mu)_{hg} f(hg) \equiv \int_G (d\mu)_g f(g) = \int_G (d\mu)_g f(hg) \tag{120}$$

and similarly

$$\int_G (d\mu)_g f(gh) = \int_G (d\mu)_g f(g). \tag{121}$$

Thus $(d\mu)_g$ is a bi-invariant Haar measure over G and leads to the definition of an integral for functions on G. An Haar measure assigns an "invariant volume" to subsets of a locally compact topological group G.

Now let us define a measure (dm) over the manifold M assigning to every $g \in G$ a measure $(du)_x$ on the tangent space T_xM. $(du)_x$ is defined as a standard Lebesgue measure normalized as usual:

$$\int_{T_xM} (du)_x \exp\left(-\frac{1}{2}(u,u)_x\right) = 1. \tag{122}$$

Since the maps $A_g : M \to M$ and $A_{g*h} : T_xM \to T_{gx}M$ are invertible we can define the associated Jacobian $J(A_{g*x})$ by

$$\int_{T_{gx}M} (d\mu)_{gx} \exp(-\frac{1}{2}(u,u)_{gx}) = J(A_{g*x}) \int_{T_xM} (du)_x \exp(-\frac{1}{2}(A_{g*x}u, A_{g*x}u)_{gx}). \tag{123}$$

Since $(A_{g*x}u, A_{g*x}u)_{gx} = (u,u)_x$ also this Jacobian must be equal to unit:

$$J(A_{g*x}) = 1. \tag{124}$$

When we pass to the tangent space TM to the manifold M we can use the fact that locally $(dm)_x = (du)_x$ in any neighborhood of x. Thus we have

$$\int_M J(A_{g*x})(dm)_x f(A_g x) = \int_M (dm)_x f(gx) = \int_M (dm)_{gx} f(gx) \equiv \int_M (dm)_x f(x) \tag{125}$$

which tells us that the left action of an element of G on the function $f : G \to G$ preserves the invariant measure on M (the same holds true for the right action).

21 Faddeev-Popov Determinant

Let us introduce the notion of the Faddeev-Popov determinant. Recall the map $B_x : G \to M$ that we defined as the left action of g on x. $B_x(g)$ is the orbit of the point x and it is not an invertible map since it is not injective. We define the Faddev-Popov function $W_{FP} : M \to \mathbb{R}$ by

$$W_{FP}(x)^{-1} = \int_G (d\mu)_g \, \delta(F \circ B_x(g)) \tag{126}$$

where F is a gauge-fixing vector-valued function. F must be chosen such that $\forall x \in M$, there exists a $g_x \in G$ that satisfies:

$$F \circ B_x(g_x) = 0. \tag{127}$$

W_{FP} is gauge-invariant, indeed for $h \in G$ one has:

$$W_{FP}^{-1}(hx) = \int_G (d\mu)_g \, \delta(F \circ B_x(gh)) = \int_G (d\mu)_g \, \delta(F \circ B_x(g)) = W_{FP}^{-1}(x) \tag{128}$$

where we have assumed that there exists a unique $h_g \in H_x$ (with H_x stabilizer of x) such that $g = g_x h_g$. From these properties it follows that

$$\frac{1}{Vol\,(G)} \int_M (dm)_x f(x) = \int_M (dm)_x f(x) W_{FP}\,(x)\,\delta\,(F(x)) \tag{129}$$

PROOF:

$$\begin{aligned}\int_M (dm)_x\ f(x) \cdot 1 = \int_M (dm)_x\ f(x) \int_G W_{FP}(x)\delta(F \circ B_x(g)) = \\ \int_G (d\mu)_g \int_M (dm)_x f(gx) W_{FP}(gx)\delta(F(gx)) = \\ \int_G (d\mu)_g \int_M (dm)_x f(\tilde{x}) W_{FP}(\tilde{x})\delta(F(\tilde{x})) = \\ \mathrm{Vol}(G) \int_M (dm)_x f(x) W_{FP}(x)\delta(F(x)).\end{aligned}$$

QED

An important special case of equation (129) is the following

$$\frac{1}{\mathrm{Vol(Gauge)}} \int_{\text{gauge fields}} (dA) e^{iS[A]} = \int_{\text{gauge fields}} (dA) e^{iS[A]} W_{FP}[A] \delta(\partial_\mu A^\mu). \tag{130}$$

Part IV
MATHEMATICAL APPENDIX

A Gamma Function

Let $z \in \mathbb{C}$, for $Re\,(z) > 0$ we can define tha Euler's Γ function by means of the integral

$$\Gamma\,(z) := \int_0^{+\infty} dt\, t^{z-1} e^{-t} \tag{131}$$

First of all let us see that for $Re\,(z) > 0$ this integral in well defined and convergent, indeed:
$\left|\int_0^{+\infty} dt\, t^{z-1} e^{-t}\right| \le \int_0^{+\infty} dt\, \left|t^{z-1} e^{-t}\right| = \int_0^{+\infty} dt\, e^{-t} t^{Re(z)-1} < +\infty$
iff $Re\,(z) > 0$, because the power $x^{-\alpha}$, with $\alpha > 0$, is integrable at the origin when $\alpha < 1$.
Furthermore the recurrence formula

$$\begin{cases} \Gamma\,(z+1) = z\Gamma\,(z) \\ \Gamma\,(1) = 1 \end{cases} \tag{132}$$

holds.
Indeed $\Gamma\,(1) = \int_0^{+\infty} dt\, e^{-t} = e^{-t}\big|_{+\infty}^{0} = +1$, and integrating by parts one finds
$\Gamma\,(z+1) = \int_0^{+\infty} dt\, t^z e^{-t} = -t^z e^{-t}\big|_0^{+\infty} + \int_0^{+\infty} dt\, e^{-t} z t^{z-1} = z \int_0^{+\infty} dt\, e^{-t} t^{z-1} = z\Gamma\,(z)$.

If we restrict our consideration to the special case $z = n \in \mathbb{Z}^+$ a very interesting property arises:

$$\Gamma\,(n+1) = n\Gamma\,(n) = n\,(n-1)\,\Gamma\,(n-1) = n\,(n-1)\,(n-2)\,...3 \cdot 2 \cdot 1 = n!$$

So we see that the Euler's Gamma function is a sort of generalization at the complex plane of the factorial:

$$\Gamma\,(n+1) = n! \quad , n \in \mathbb{N}$$

From these formulas it follows that $\Gamma\,(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)...(z+n)}$, for an arbitrary natural number n. As a matter of fact, by iteration we have

- $\Gamma\,(z+1) = z\Gamma\,(z)$
- $\Gamma\,(z+2) = (z+1)\,\Gamma\,(z+1) = (z+1)\,z\Gamma\,(z)$

- $\Gamma(z+3) = (z+2)(z+1)z\Gamma(z)$
- ...
- $\Gamma(z+n+1) = (z+n)(z+n-1)(z+n-2)...(z+2)(z+1)z\Gamma(z) \Rightarrow$

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)...(z+n)} \quad , \; n \in \mathbb{N} \tag{133}$$

The very last equation is useful for making the analytic continuation of Γ function over the complex left half-plane, i.e. to extend the domain of the gamma function to the left half-plane of the complex plane $\mathbb{C}$, apart from a countable infinite set of points. Indeed by analytic continuation tool our special function Γ is well defined everywhere in the complex plane apart the countable set of simple poles

$$z = 0, -1, -2, -3, ..., -n, ... \qquad , \qquad \forall n \in \mathbb{N}$$

An alternative definition of the gamma function is

$$\Gamma(z) = \lim_{k\to+\infty} \frac{k!k^{z-1}}{z(z+1)(z+2)...(z+k)}$$

which manifests too the numerable infinity of simple poles in $z \in \mathbb{Z}^-$.

B Euler-Riemann Zeta Function

The Euler-Riemann ζ-function is defined as

$$\zeta(z) := \sum_{n=1}^{+\infty} \frac{1}{n^z} \qquad , \; z \in \mathbb{C} \text{ with } Re(z) > 1 \tag{134}$$

A relation exists between Γ and ζ as we shall see now. We try to make the substitution $u = \frac{t}{n}$ in the integral which define Gamma function and we get

$$\Gamma(z) = \int_0^{+\infty} dt\, e^{-t} t^{z-1} = \int_0^{+\infty} du\, (un)^{z-1} e^{-nu} n = n^z \int_0^{+\infty} du\, e^{-nu} u^{z-1}$$

$n \in \mathbb{N}, \quad u \equiv t/n \Rightarrow dt = ndu$

which implies

$$\zeta(z)\Gamma(z) = \left\{\sum_{n=1}^{+\infty} n^{-z}\right\} n^z \int_0^{+\infty} du\, u^{z-1} e^{-nu} = \sum_{n=1}^{+\infty} \int_0^{+\infty} du \;\, u^{z-1} e^{-nu}$$

The last equality is allowed by the fact that $\forall u \in \mathbb{R}^+$ *and* $\forall n \in \mathbb{N} \quad \exists t \in \mathbb{R}^+$ such that $nu = t$.

The integrals above don't make troubles for $Re(z) > 1$ because the integrand function stays limitated. Applying the Lebesgue's dominated convergence theorem we get

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{+\infty} \frac{t^{z-1}}{e^t - 1} dt \tag{135}$$

because of the geometric series $\sum_{n=1}^{+\infty} e^{-nt} = \left(e^{t}-1\right)^{-1}$, since for $t \geq 0$ the absolute value of the ratio of the series is less than one and we get
$\sum_{n=1}^{+\infty} \left(e^{-t}\right)^{n} = \sum_{n=0}^{+\infty} \left(e^{-t}\right)^{n} - 1 = \frac{1}{1-e^{-t}} - 1 = \frac{1}{e^{t}-1}$

C The Generalized Zeta Function

It's clear that the Euler's Riemann $\sum_{k=1}^{+\infty} k^{-s} = \zeta_{N+1}(s)$, where N is the number operator, is different from the trace of the complex power of an operator $\sum_{k=1}^{+\infty} \lambda_k^{-s}$, so we want to extend the original definition of zeta function. During the proceedings we will find a Mellin transform. Let a_k be a real number such that $0 < a_k < 1$ and consider the following steps (where we set $t = u(k + a_k)$): Il

$$\Gamma(s) = \int_{\mathbb{R}^+} dt\, t^{s-1} e^{-t} = (k+a_k)^s \int_{\mathbb{R}^+} du\, u^{s-1} e^{-u(k+a_k)}$$

$$\Rightarrow (k+a_k)^{-s} = \Gamma(s)^{-1} \int_0^{+\infty} dt e^{-t(k+a_k)} t^{s-1}$$

Now we set $k + a_k \equiv \lambda_k$.

Then it follows that $\sum_{\nu=1}^{N} n_\nu \lambda_\nu^{-s} = \frac{1}{\Gamma(s)} \sum_{\nu=1}^{N} \int_0^{+\infty} dt\, t^{s-1} e^{-t\lambda_\nu} d_\nu$, and taking the limit $N \to +\infty$

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \sum_{\nu=1}^{+\infty} \int_{\mathbb{R}^+} dt\, t^{s-1} e^{-t\lambda_\nu} d_\nu$$

This last expression define a modified version of the generalized or Hurwitz zeta function:

$\zeta_{N+a}(s,a) := \sum_{n=0}^{+\infty} (a+n)^{-s} \qquad n \in \mathbb{N},\ s \in \mathbb{C}\ with\ Re(s) \geq 1+\delta,\ \delta > 0$

$a \in \mathbb{R}$ with e $0 < a \leq 1$

Proposition: Let $s \in \mathbb{C}$ such that $Re(s) > 1$, and $a \in \mathbb{R}$ such that $0 < a \leq 1$, then

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dx \frac{x^{s-1} e^{-ax}}{1-e^{-x}} \qquad , Re(s) > 1 \tag{136}$$

Proof:

$$\sum_{n=0}^{+\infty} (a+n)^{-s} = \lim_{n \to +\infty} \frac{1}{\Gamma(s)} \sum_{k=0}^{n} \int_0^{+\infty} dx\, x^{s-1} e^{-(a+k)x} \Longleftrightarrow$$

$$\Gamma(s)\zeta(s,a) = \lim_{n\to\infty}\sum_{k=0}^{n}\int_{\mathbb{R}^+} dx\, x^{s-1}e^{-(k+a)x} =$$

$$= \lim_{n\to\infty}\int_0^{+\infty} dx\, x^{s-1}e^{-ax}\sum_{k=0}^{n}\left(e^{-x}\right)^k = \lim_{n\to\infty}\int_{\mathbb{R}^+} dx\, x^{s-1}e^{-ax}\left(\frac{1-e^{-x(n+1)}}{1-e^{-x}}\right) =$$

$$= \int_0^{+\infty} dx\,\frac{x^{s-1}e^{-ax}}{1-e^{-x}} - \lim_{n\to\infty}\int_0^{+\infty} dx\,\frac{x^{s-1}e^{-(a+n+1)x}}{1-e^{-x}}$$

But the limit tend to zero as we shall see now. In fact we have $e^x > 1 + x \Longrightarrow e^x - 1 > x \Longrightarrow \frac{1}{e^x-1} = \frac{e^{-x}}{1-e^{-x}} < \frac{1}{x}$. Thus:

$$\left|\int_0^{+\infty} dx\,\frac{x^{s-1}e^{-x(n+a+1)}}{1-e^{-x}}\right| \le \int_0^{+\infty} dx\left|x^{s-1}\frac{e^{-x}}{1-e^{-x}}e^{-(n+a)x}\right| <$$

$$< \int_0^{+\infty} dx\,\frac{x^{Re(s-1)}e^{-x(n+a)}}{x} = \int_0^{+\infty} dx\, x^{Re(s)-2}e^{-x(n+a)} = \#$$

But $\Gamma(Re(s)-1) \equiv \int_0^{+\infty} dt\, e^{-t}t^{Re(s)-2}$ so acting a change of variable we find

$$\# = \int_0^{+\infty} dt\ (n+a)^{-1}\ \frac{e^{-t}t^{Re(s)-2}}{(n+a)^{Re(s)-2}} =$$

$$= (n+a)^{1-Re(s)}\int_0^{+\infty} dt\, e^{-t}t^{Re(s)-2} = (n+a)^{1-Re(s)}\,\Gamma(Re(s)-1)$$

$$\Longleftrightarrow \left|\int_0^{+\infty} dx\,\frac{x^{s-1}e^{-x(n+a+1)}}{1-e^{-x}}\right| < \Gamma(Re(s)-1)\,(n+a)^{1-Re(s)}$$

And the right hand side of the very last inequality tend to zero when $n \to +\infty$, under the assumption that $Re(s) > 1$. Hence

$$\lim_{n\to+\infty}\int_0^{+\infty} dx\,\frac{x^{s-1}e^{-(a+n+1)x}}{1-e^{-x}} = 0 \Longrightarrow \Gamma(s)\zeta(s,a) = \int_0^{+\infty} dx\,\frac{x^{s-1}e^{-ax}}{1-e^{-x}}$$

Thus we have proved that the equation (136) holds true. **QED**
One can get the same result applying the dominated convergence theorem of Lebesgue, but it is very quite similar.

D Analytic Continuation of Zeta Function

Now our aim is to extend analytically the hyperfunction ζ , in such a way that its domain coincides with all the complex plane $\mathbb{C}$ apart from the single point $s = 1$

$$\mathfrak{D}(\zeta) = \mathbb{C} - \{1\} \tag{137}$$

Proof:

To this end let $z \in \mathbb{C}$ and let study the next function, very similar to the integrand of equation (136) extended to complex domain

$$z \in \mathbb{C}\ ,\ a \in \mathbb{R} \quad \text{with} \quad 0 < a \leq 1\ , \qquad f(z) = \frac{(-z)^{s-1}\, e^{-az}}{1 - e^{-z}}$$

The function $f(z)$ has simple poles when its denominator vanishes, i.e. in the points $z = 2\pi k i$, $k \in \mathbb{Z}$. Furthermore when $s \notin \mathbb{Z}$, $f(z)$ is a multivalued function with branch point $z = 0$.
We assume the cut of the complex plane along the positive semiaxis of abscissa coordinate x: $0 < \arg(z) < 2\pi \iff -\pi < \arg(-z) < \pi$. We set $z = e^{i\vartheta}$, with $0 \leq \vartheta \leq \pi$ and $0 < \epsilon < 1$ and taking a piecewise regular (i.e. smooth) curve $\mathcal{C}$ in the complex plane, as shown in the figure we integrate $f(z)$ along $\mathcal{C}$:

$$\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz = \int_{+\infty}^{+\epsilon} \frac{x^{s-1}e^{-\pi i(s-1)}e^{-ax}}{1-e^{-x}}dx +$$
$$\int_0^{2\pi} \frac{(-\epsilon e^{i\vartheta})^{s-1}e^{-a\epsilon(\cos\vartheta + i\sin\vartheta)}}{1-e^{-\epsilon(\cos\vartheta+i\sin\vartheta)}}(i\epsilon e^{i\vartheta})d\vartheta + \int_{+\epsilon}^{+\infty} \frac{e^{\pi i(s-1)}x^{s-1}e^{-ax}}{1-e^{-x}}dx$$

And taking the limit $\epsilon \to 0^+$ we find that the second integral makes zero:

$$\lim_{\epsilon\to 0^+} i\epsilon \int_0^{2\pi} \frac{(-\epsilon e^{i\vartheta})^{s-1}e^{-a\epsilon(\cos\vartheta+i\sin\vartheta)}}{1-e^{-\epsilon(\cos\vartheta+i\sin\vartheta)}}e^{i\vartheta}d\vartheta$$
$$\simeq \lim_{\epsilon\to 0^+} i \int_0^{\epsilon\pi} \frac{e^{i\vartheta s}(-\epsilon)^s}{\epsilon(\cos\vartheta + i\sin\vartheta) + O(\epsilon^2)}d\vartheta = \lim_{\epsilon\to 0^+} \epsilon^{s-1} \int_0^{2\pi} I(\vartheta)d\vartheta = 0$$

if $Re(s-1) > 1$. So, required that $Re(s-1) \geq 1 + \delta$ ($\delta > 0$) we get

$$\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz = \left[e^{\pi i(s-1)} - e^{\pi i(s-1)}\right] \int_{\epsilon\to 0^+}^{+\infty} dx \frac{x^{s-1}e^{-ax}}{1-e^{-x}} =$$
$$-2i\sin[\pi(1-s)] \int_{\epsilon\to 0^+}^{+\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx$$

and using the reflection formula of Euler

$$-2i\sin[\pi(1-s)] = \frac{-2i\pi}{\Gamma(1-s)\Gamma(s)} \tag{138}$$

we finally obtain

$$\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz = \frac{-2\pi i}{\Gamma(1-s)}\Gamma(s)^{-1}\int_0^{+\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx \equiv \frac{-2\pi i}{\Gamma(s-1)}\zeta(s,a).$$

Thus we have reached the following important formula:

$$\zeta(s,a) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz \tag{139}$$

from which it is manifest that the extended domain of zeta function must be all complex plane except for the unique point 1. Indeed we know that initially the zeta domain was the complex half-plane $Re(z) > 1$, and now ζ must have at least all the domain of $\Gamma(1-s)$ because the integral above defines an analytic function. And since we have

$$\mathfrak{D}(\Gamma(1-s)) = \mathbb{C} - \{1, 2, 3, ..., n, ...\} \tag{140}$$

and at the same time ζ is well defined in all points $s = 2, 3, 4, ..., n, ...$ we must infer that

$$\mathfrak{D}(\zeta) = \mathbb{C} - 1. \tag{141}$$

QED

E Euler's Reflection Formula

The reflection relation that we used previously reads

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)} \tag{142}$$

Proof:

F Exterior Derivative

The exterior derivative d maps p-forms into (p+1)-forms. Let $\mathscr{U} \subset \mathbb{R}^n$ be an open subset and let $\Lambda^p\mathscr{U}$ denote the set of all p-forms defined on it. We recall that we always use Einstein's notation about indices' contractions. For $p = 0$ the exterior derivative acts on functions $f \in \Lambda^0\mathscr{U}$ (0-forms are functions) as follows

$$df = \frac{\partial f}{\partial x^i}dx^i \quad , \quad x \in \mathscr{U}, \quad f : \mathscr{U} \to \mathbb{R}$$

so $df \in \Lambda^1\mathscr{U}$ and it is equivalent to the standard differential of a function. Instead if we evaluate the exterior derivative along a tangent vector $v = v^i\frac{\partial}{\partial x^i} \in T_x\mathscr{U}$ we find

$$(df)_x(v) = \langle df, v\rangle = \left\langle \frac{\partial f}{\partial x^i} dx^i, v^j \frac{\partial}{\partial x^j} \right\rangle = \frac{\partial f}{\partial x^i} \delta^i{}_j v^j = \frac{\partial f}{\partial x^i} v^i$$

from which it is evident that df gives the derivative of f along the direction of v. Let us now consider the exterior derivative of an arbitrary p-form.

Proposition: Let $\mathscr{U} \subseteq \mathbb{R}^n$, $\Lambda\mathscr{U} := \bigoplus_{p=0}^{n} \Lambda\mathscr{U}$. There exists a unique map $d : \Lambda\mathscr{U} \to \Lambda\mathscr{U}$ such that

- i) d is a linear map
- ii) for $p = 0$ it reduces to the differential just seen above
- iii) $d(\Lambda^p\mathscr{U}) \subset \Lambda^{p+1}\mathscr{U}$
- iv) Leibniz rule: $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^p \varphi \wedge d\psi$, $\varphi \in \Lambda^p\mathscr{U}$
- v) nilpotent: $d^2 = 0$

Proof: Let assume that there exists an operator d acting on p-forms in $\mathscr{U}$ having all the properties above and let us show that it is unique. Let $\varphi \in \Lambda^p\mathscr{U}$ be an arbitrary p-form, then we can express it as:

$$\varphi = (p!)^{-1} \varphi_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}. \tag{143}$$

From properties i), ii), iv) and v) it follows that

$$\begin{aligned} d\varphi &= \frac{1}{p!}\left\{d\left(\varphi_{i_1\dots i_p}\right) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} + (-1)^0 \varphi_{i_1\dots i_p} d\left(dx^{i_1} \wedge \dots \wedge dx^{i_p}\right)\right\} \\ &= \frac{1}{p!}\left(d\varphi_{i_1\dots i_p}\right) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} + \varphi_{i_1\dots i_p}\Big[\left(ddx^{i_1}\right) \wedge \dots \wedge dx^{i_{p-1}} \wedge dx^{i_p} + \\ &\qquad (-1)\, dx^{i_1} \wedge \left(ddx^{i_2}\right) \wedge \dots \wedge dx^{i_p} + \dots + \\ &\qquad (-1)^{p-1} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{p-1}} \wedge \left(ddx^{i_p}\right)\Big] = \frac{1}{p!}\left(d\varphi_{i_1\dots i_p}\right) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \end{aligned}$$

So we have just shown that the exterior derivative of a p-form generally expressed by equation (143) is given by:

$$d\varphi = \frac{1}{p!}\left(d\varphi_{i_1\dots i_p}\right) \wedge dx^{i_1} \wedge \dots\dots\dots \wedge dx^{i_p} \tag{144}$$

where $d\varphi_{i_1\dots i_p}$ is the exterior derivative of the functions $\varphi_{i_1\dots i_p}$. Since it is uniquely defined it follows that the map $d : \Lambda\mathscr{U} \to \Lambda\mathscr{U}$ exists unique. Now, by inverse, we have to demonstrate that if we define the operator d through equation (144) then all five properties i) - v) are satisfied.

- i) d is clearly a linear map.

- ii) for $p = 0$, i.e. when φ is a function, equation (144) reduces to

$$d\varphi = \frac{\partial \varphi}{\partial x^i} dx^i$$

which is the well known differential of a 0-form.

- iii) $d\left(\Lambda^p \mathscr{U}\right) \subset \Lambda^{p+1}\mathscr{U}$ is evidently true because given $\varphi \in \Lambda^p \mathscr{U}$ then $d\varphi$ expressed via (144) is an element of $\Lambda^{p+1}\mathscr{U}$.

- iv) Leibniz rule is fulfilled since, given arbitrary forms $\varphi \in \Lambda^p \mathscr{U}$ and $\psi \in \Lambda^q \mathscr{U}$:

$$\varphi = (p!)^{-1} \varphi_{i_1 \dots i_p} \; dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}$$
$$\psi = (p!)^{-1} \psi_{j_1 \dots j_q} \; dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_q}$$

Hence

$$\begin{aligned} d\left(\varphi \wedge \psi\right) = \frac{1}{p!q!} d\Big\{ \varphi_{i_1 \dots i_p} \psi_{j_1 \dots j_q} \; dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q} \Big\} = \\ \frac{1}{p!q!} \left(d\varphi_{i_1 \dots i_p}\right) \psi_{j_1 \dots j_q} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q} + \\ \frac{1}{p!q!} \varphi_{i_1 \dots i_p} \left(d\psi_{j_1 \dots j_q}\right) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q} = \\ \frac{1}{p!} \left\{ d\varphi_{i_1 \dots i_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \right\} \wedge \frac{1}{q!} \left\{ \psi_{j_1 \dots j_q} dx^{j_1} \wedge \dots \wedge dx^{j_q} \right\} + \\ \frac{1}{p!} \left\{ \varphi_{i_1 \dots i_p} \left(-1\right)^p dx^{i_1} \wedge \dots \wedge dx^{i_p} \right\} \wedge \frac{1}{q!} \left\{ d\psi_{j_1 \dots j_q} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q} \right\} = \\ d\varphi \wedge \psi + \left(-1\right)^p \varphi \wedge d\psi. \end{aligned}$$

- v) The operator d is nilpotent. Indeed we can write the exterior derivative of a p-form as

$$d\varphi = \frac{1}{p!} \frac{\partial}{\partial x^i} \varphi_{i_1} \dots i_p \; dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}.$$

Thus

$$\begin{aligned} d^2\varphi = (p!)^{-1} \frac{1}{2} \Bigg(\frac{\partial}{\partial x^j \partial x^i} \varphi_{i_1 \dots i_p} \; dx^j \wedge dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} + \\ \frac{\partial}{\partial x^i \partial x^i} \varphi_{i_1 \dots i_p} \; dx^i \wedge dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \Bigg) = \frac{1}{2} \left(p!\right)^{-1} \frac{\partial}{\partial x^j \partial x^i} \varphi_{i_1 \dots i_p} \\ \Bigg(dx^j \wedge dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} - dx^j \wedge dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \Bigg) \equiv 0. \end{aligned}$$

where we have used $dx^i \wedge dx^j = -dx^j \wedge dx^i$ and $\frac{\partial}{\partial x^j \partial x^i} = \frac{\partial}{\partial x^i \partial x^j}$. **QED**

G Poincaré Lemma

Proposition: If $\mathscr{U} \subset \mathbb{R}^n$ is an open and star-shaped subset, a form $\varphi \in \Lambda\mathscr{U}$ is closed if and only if it is exact. An open subset of $\mathbb{R}^n$ is called star-shaped if $\exists x_0 \in \mathscr{U}$ such that $\forall x \in \mathscr{U}$ the straight line connecting x and x_0 is entirely contained in $\mathscr{U}$.

Proof: Let $\varphi \in \Lambda^1\mathscr{U}$. The demonstration can then be generalized to $p > 1$. A 1-form φ can always be expressed as $\varphi = \varphi_i dx^i$ and we already know that an exact form is closed. So we have to show that, in a star-shaped domain, if φ is closed (i.e. $d\varphi = 0$) then is exact:

$$d\varphi \equiv \partial_k\varphi_i dx^k dx^i = \frac{1}{2}\left(\partial_k\varphi_i - \partial_i\varphi_k\right)dx^k dx^i = 0 \iff \partial_k\varphi_i = \partial_i\varphi_k.$$

So we can define a 0-form (i.e .a function) $f(x) := \int_0^1 dt\varphi_i(tx)\, x^i$ such that its exterior derivative is

$$df = d\int_0^1 dt\;\varphi_i(tx)\, x^i = \int_0^1 \left(\partial_k\varphi_i(tx)\;\; tx^i dx^k + \varphi_i(tx)\,\delta^i{}_k dx^k\right) =$$
$$\int_0^1 \left(\partial_i\varphi_k(tx)\;\; tx^i dx^k + \varphi_i(tx)\, dx^i\right) = \int_0^1 dt\;\frac{d}{dt}\left(\varphi_i(tx)\, t dx^i\right) =$$
$$\varphi_i(x)\, dx^i \equiv \varphi.$$

Since the straight line going from $t = 0$ to $t = 1$ is all contained in $\mathscr{U}$. Thus there exists a function f such that

$$df = \varphi \tag{145}$$

and this equation tells us that φ is an exact form. **QED**

H Parallel Transport and Holonomy Group

Given two curves $C : x \to y$ and $C' : y \to z$ over the manifold M, let consider their concatenation $C' \circ C : x \to z$, the inverse curve $C^{-1} : y \to x$ and the constant curve $C_0 : x \to x$. The operator of parallel transport $\Gamma(C) : V_x \longrightarrow V_y$ must satisfy the following properties:

- i) $\Gamma(C' \circ C) = \Gamma(C')\Gamma(C)$
- ii) $\Gamma(C^{-1}) = \Gamma(C)^{-1}$
- iii) $\Gamma(C_0) = id_{V_x}$

where id_{V_x} is the identity operator on the vector space V_x, which is a copy of V at the point x of M. It should be clear that

$$\Gamma(C) : v \in V_x \longrightarrow \Gamma(C)v \in V_y \tag{146}$$

that is the parallel transport associates (linearly) vectors belonging to different fibers over distinct points of the base manifold. Indeed since the differential equation that defines parallel transport is linear the map $\Gamma(C)$ is also linear. Now let us see what is holonomy along a path C. It is clear that the parallel transport depends on the particular path and this dependence leads to the notion of holonomy group. Let E be a principal bundle endowed with a connection A. If we parallel transport a $x \in E$ along a closed curve C we return to the fibre in which x lies. Thus there must exists a unique $g \in G$ such that $\Gamma_C(x) = xg$. As C varies over all closed paths the corresponding elements of G form a subgroup called the holonomy group $\mathscr{H}(x)$ of A at x. The main property of the holonomy group is

$$\mathscr{H}(xg) = g^{-1}\mathscr{H}g \qquad \forall g \in G \tag{147}$$

and furthermore for any curve C in M one has

$$\mathscr{H}(x) = \mathscr{H}(\Gamma_C(x)). \tag{148}$$

I Left and Right Cosets, Stabilizer and Orbits

Given a group G and a subgroup $H \subset G$, H can act on G from the left (right) through the left (right) translation $L_h : H \times G \to G$ ($R_h : G \times H \to G$), with $h \in H$. L_h and R_h are defined by

$$L_h(g) = hg \quad , \qquad R_h(g) = gh. \tag{149}$$

We define the orbit of $g \in G$ as

$$\mathrm{Orb}_g = \{R_h(g) \ \mid h \in H\} = \{\tilde{g} \in G \mid gh = \tilde{g}, h \in H, g \in G\} \tag{150}$$

Orbits are equivalence classes and we call the quotient or coset space G/H the partition of G induced by orbits. The stabilizer H_g of $g \in G$ is defined as

$$H_g := \{h \in G \mid gh = g \ \ \forall g \in G\}. \tag{151}$$

References

[1] Richard P. Feynman, Albert R. Hibbs. Quantum Mechanics and Path Integrals

[2] Paul A. M. Dirac. Lectures on Quantum Mechanics and Relativistic Field Theory

[3] S.W. Hawking. Zeta Function Regularization of Path Integrals in Curved Spacetime - Communications in Mathematical Physics

[4] Emilio Elizalde. Zeta-Function Regularization is Uniquely Defined and Well - http://arxiv.org/abs/hep-th/9308028

[5] Albert S.Schwarz. Quantum Field Theory and Topology

[6] Steven Weinberg. The Quantum Theory of Fields (Volume I - Foundations)

[7] Albert S. Schwarz. Topology for Physicists

[8] M. Gockeler, T.S. Differential Geometry, Gauge Theories, and Gravity

[9] Baez, Muniain. Gauge Fields, Knots and Gravity

[10] Samuel L.Goldberg. Curvature and Homology