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Dipartimento di Fisica e Astronomia
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Spectral properties of stochastic matrices: an application to random walks

Relatore:
Prof. Armando Bazzani

Presentata da:
Alessio Catanzaro

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Abstract

In questa tesi si affronta il problema di determinare la scala di tempo di rilassamento alla distribuzione stazionaria di un random walk sui nodi di un random network. A tale scopo si utilizza il teorema di Wigner sulla distribuzione degli autovalori di una matrice random per stimare il secondo autovalore della matrice stocastica che definisce il random walk nel limite termodinamico. In particolare nel primo capitolo si affrontano alcuni teoremi di teoria delle matrici random per determinare come si distribuiscano gli autovalori di matrici stocastiche i cui elementi opposti abbiano una correlazione o meno. Nel secondo capitolo si ripercorre la teoria generale delle catene di Markov e si collegano i risultati teorici alla determinazione del tempo di rilassamento di un random walk su un network. Infine si presentano alcuni risultati numerici a supporto delle tesi espresse in precedenza.

Introduction

The aim of this thesis is to provide a framework in which Random Matrix Theory can deliver appropriate results to the study of random walk on a network.
From the theory of Markov chains applied to random walks on graphs [3], [9], [12] we glean that main dynamical features may be derived from the spectral properties of the stochastic matrix defined by the network connectivity. Namely, the relaxation time scale to stationary state is defined by the second largest eigenvalue of the transition matrix, derived from the adjacencies of the network and the node degrees. This result is relevant in applications since a great variety of biological, economic and social systems can be represented with a random walk, and thus the study about the relaxation process towards the stationary state can be relevant to understand the behaviour of the considered system out of equilibrium. The Wigner theorem about spectral distribution of random symmetric matrices turns out to be a powerful tool to cope with this problem [16] [17]. This theorem is the act of birth of Random Matrix Theory, stating the limiting empirical spectral distribution of a symmetric matrices whose entries are randomly chosen with unitary variance converges weakly to a semicircle, and it is the analogue of the Central Limit theorem in the space of random matrices. A generalization of such theorem may be performed focusing on two main paths. The first

one tries to introduce non-unitary variance, which means as a physical consequence that we are relating the distribution support with the network features. We find that, reproducing Wigner arguments in the original proof and introducing a new parameter to measure the variance of the matrix elements, the relation between relaxation time and eigenvalues is brought to be a relation between the matrix elements' variance, or alternatively the connectivity of the graph. A numerical computation of the eigenvalues of a random graph has been done, and the results support the thesis of an inverse correlation between spectral gap and mean connectivity. The second path involves breaking the symmetry of the matrix, which physically means violating the Detailed Balance Condition (DBC) which ensures stationary equilibrium of the system to exist. In this case the result is to be obtained from a theorem, the Circle law [1], [14], about totally non-symmetric random matrices. This result is further generalized to obtain a theorem describing the measure of the eigenvalues of matrices whose elements show only partial correlation, called the elliptic law [10], [11], [13]. This theoretical study predicts that Non-Equilibrium Stationary States may show shorter relaxation time than equilibrium ones, according to previous results [7]. Running an analogue numerical simulation on matrices randomly perturbed out of DBC we find support for our hypothesis, as the distribution of eigenvalues stretches into an ellipsis. However, further analysis about the pertinence of such law may be needed to assert such results hold in general.

Contents

Chapter 1

Spectral Distribution of Random Matrices

Random Matrix Theory (RMT) dates back to the early development of nuclear physics in the 1950s. In an attempt to explain the energetic structure of nuclei of heavy elements, E. P. Wigner suggested that the energy levels should not be computed from Schrödingers' equation. Instead, the nucleus should be described by $n \times n$ Hamiltonian matrices with elements drawn from a probability distribution with only weak constraints dictated by symmetry considerations. Under these assumptions and a mild condition imposed on the probability measure in the space of matrices, one finds the joint probability density of the n eigenvalues. Based on this consideration, Wigner established the well-known semicircular law. Since then, RMT has been developed into a big research area in mathematical physics and probability. We will review some of the historic results and some of the most recent developments in a field that, since its birth, have found a extremely wide range of applications, spanning from nuclear physics to telecommunications engineering.

1.1 The Wigner Law

The cornerstone and historic origin of RMT is the Semicircular Law theorem, proved by Eugene P. Wigner in 1955 while trying to establish an approximated solution for large atomic nuclei. The model, although being somewhat accurate, was soon outdated. Nonetheless its core, *i.e.* the fact that the limiting distribution of large symmetric random matrices' eigenvalues tends to a semicircle function, soon laid the ground to a new field of research on random matrices which has been producing suitable results in many other fields of knowledge.

1.1.1 Preliminary definitions

In order to study the probability distribution of eigenvalues we need to introduce the Empirical Spectral Distribution of a random matrix.

Definition 1.1. Let A_n be a n x n symmetrical square matrix and λ_j its eigenvalues. We define the Empirical Spectral Distribution (ESD) of A, μ_A as

$$\mu_A(x) = \frac{1}{n}\#\{j < n : \lambda_j \leq x\} \tag{1.1}$$

If the matrix is not symmetrical the eigenvalues may be not real, so the above definition should be changed as follows:

$$\mu_A(x, y) = \frac{1}{n}\#\{j < n : \Re(\lambda_j) \leq x, \Im(\lambda_j) \leq y\} \tag{1.2}$$

where x, y are respectively the real and imaginary parts of the eigenvalues.

In order to prove the Wigner Law the first formula shall suffice, as the statement involves symmetric matrices. Nonetheless we will make use of the latter definition in the following theorems.

A Wigner matrix is a Hermitian matrix whose entries:

- on the diagonal are independent and identically distributed (iid) real random variables with finite variance.
- above the diagonal are iid real random variables having mean zero and unitary variance.

1.1.2 The Semicircle Law

We state the result that gave birth to random matrix theory, due to Wigner [16], [17]. The proof can be shown by using two different approaches: one involving Stieltjes transform and another, the one used historically by Wigner, that is more 'physical and is based on explicitly counting the moments of the distributions. We shall use the latter, but Stieltjes transform will be used later on, as it is a really powerful tool in random matrix theory.

Theorem 1.1.1 (Semicircle Law)**.** *Let X_n be a n x n Wigner matrix, and let $W_n = \frac{1}{\sqrt{n}}X_n$, then, as $n \to \infty$, $\mu_{W_n}(x)$ converges weakly to $\mu_{sc}(x)$, where*

$$\mu_{sc}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2}, & \text{if } x \in [-2,2] \\ 0, & \text{otherwise} \end{cases} \tag{1.3}$$

Before the proof, it is worth discussing that the theorem derives the weak convergence of the measure, that is the convergence under the law of the Large Numbers. This classic result in probability theory will be implicitly applied for all the theorems we will discuss.

Proof. The key concept of the proof is to show that the expected moments of W_n and those of the semicircular distribution coincide in probability. Therefore the first task to comply with is to calculate the moments of the semicircular distribution, that is $\int_{-\infty}^{\infty} x^k \mu_{sc}(x)dx$.

As we notice that $\mu_{sc}(x)$ is an even function on a symmetrical interval, all its odd moments are zero, therefore only the computation of even ones is required. Let m_{2k} be those moments, then

$$m_{2k} = \frac{1}{2\pi}\int_{-2}^{2} x^{2k}\sqrt{4-x^2}dx = \frac{1}{\pi}\int_{0}^{2} x^{2k}\sqrt{4-x^2}dx.$$

By substituting $x = 2\sqrt{y}$

$$m_{2k} = \frac{1}{\pi}\int_0^1 2^{2k+1}y^{k-\frac{1}{2}}\sqrt{1-y}dy =$$
$$= \frac{2^{2k+1}}{\pi}\frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(k+2)} = \frac{1}{k+1}\binom{2k}{k} = C_k$$

These moments are then equivalent to the so-called Catalan numbers C_k.

The second step of the proof is the calculation of the expected moments of the ESD. At first we get a nicer expression:

$$m_k(W_n) = m_k(\mu_{W_n}) = \int x^k \mu_{W_n} dx = \frac{1}{n}\sum_{i=1}^{n}\lambda_i^k =$$
$$= \frac{1}{n}tr(W_n^k) = \frac{1}{n^{1+\frac{k}{2}}}tr(X_n^k) = \frac{1}{n^{1+\frac{k}{2}}}\sum_{\mathbf{i}} X(\mathbf{i}).$$

where λ_i are the eigenvalues of W_n and $X(\mathbf{i}) = x_{i_1 i_2}x_{i_2 i_3}\cdots x_{i_k i_1}$, $\mathbf{i}$ representing the multi-index $(i_1, i_2, i_3, \cdots i_k)$.

Now we need to calculate the expected value of these $m_k(W_n)$, that is

$$E(m_k(W_n)) = \frac{1}{n^{1+\frac{k}{2}}}\sum_{\mathbf{i}} E(X(\mathbf{i}))$$

where $E(x)$ means the expected value in probability of x. We shall make use of some results in combinatorics to do so.

For each $\mathbf{i} = (i_1, i_2, i_3, \cdots i_k)$, each i_j taking values from $(1, \cdots, n)$, build a graph G as follows: Draw a horizontal line and plot the $i_1, i_2, \cdots i_k$ of $\mathbf{i}$ on it. The distinct numbers are the vertices, then draw k directed edges e_j from i_j to i_{j+1} , $j = 1, \cdots, k$, where $i_{k+1} = i_1$ to close the graph. We will mark the number of each distinct i_j by t. Such a graph is called a Γ(k,t)-graph, and is defined by a triple (V, E, F), where V is the set of vertices, E is the set of edges and $F(e) = (v_1, v_2)$, that is the function which, given an edge, gives its terminals.

We say two Γ-graphs are isomorphic if they can be turned one into another by a permutation of $(1, \cdots, n)$, this defines classes of isomorphism, each containing $n(n-1)\cdots(n-t+1)$ graphs. We define a Γ-graph as canonical if the following properties hold:

- The vertex set is $V = \{1, \cdots, t\}$ and the edge set is $E = \{e_1, \cdots e_k\}$
- There is a function g from $(1, 2, \cdots, k)$ $(1, 2, \cdots, t)$ satysfing $g(1) = 1$ and $g(i) \leq max\{g(1), \cdots, g(i-1)\} + 1$ when $1 < i \leq k$
- Its function $F(e) = (g(i), g(i+1))$, with the condition $g(k+1) = g(1) = 1$ to keep it consistent with $i_{k+1} = i_1$.

These canonical graphs can be regrouped in three main categories:

- $\Gamma_1(k)$ graphs: every edge coincides with another edge of opposite direction, and the graph of non-coincident edges forms a tree.

- $\Gamma_2(k,t)$ graphs: there is at least one non coincident edge.
- $\Gamma_3(k.t)$ graphs: all the remaining canonical graphs.

Now it is time to count them. The number of $\Gamma_1(k)$ graphs when $k = 2p$ is in fact $\frac{1}{p+1}\binom{2p}{p}$. To show that let us proceed by defining $H : E \to \{-1, 1\}$, which takes the value 1 if the edge (called an innovation) links $(i_\alpha, i_{\alpha+1})$ and $i_{\alpha+1} \notin \{i_1, \cdots, i_\alpha\}$, and -1 otherwise. We call "characteristic sequence" of the graph the sequence $(H(e_1), H(e_2), \cdots H(e_k) = -1) = (a_1 = 1, a_2, \cdots, a_{2p-1}, a_{2p} = -1)$. There is a one-to-one relation between characteristic sequences and $\Gamma_1(2p)$ graphs, so it is equivalent to count the number of graphs or their characteristic sequences.

The number of possibilities to arrange arbitrarily p ones and p minus ones is of course $\binom{2p}{p}$. The number of non-characteristic sequences must be subtracted. In order to count them we shall notice at first that the characteristic sequences partial sums are always non-negative, though ending always to be zero. That is

$$S_l = \sum_{i=1}^{l} a_i = a_1 + a_2 + \cdots a_l \geq 0$$

If an S is negative (non-characteristic), that means there is an index h for which $S_h = -1$. We define a new sequence of b_j, taking values $b_j = a_j$ if $j \geq i$, and $b_j = -a_j$ if $j > i$. This sequence contains a -1 more than the previous one, and its number is $\binom{2p}{p-1}$.

Hence the number of characteristic sequences is

$$\binom{2p}{p} - \binom{2p}{p-1} = \frac{1}{p+1}\binom{2p}{p}.$$

For what concerns the number of Γ_3 graphs, we notice that those graph either contains a cycle of non-coincident edges or it contains coincidence class of at least three edges. That is $t \leq \frac{k+1}{2}$.

With this results, we can proceed to the evaluation of the sum.

We regroup the summation $\sum_{\mathbf{i}} E(X(\mathbf{i}))$ by the categories $\Gamma_1, \Gamma_2, \Gamma_3$, so that $E(m_k(W_n)) = \frac{S_1+S_2+S_3}{n^{1+\frac{k}{2}}}$.

Each S_j is the sum of all the canonical graph in the category j of the sum of all the isomorphic graph, that is

$$S_j = \sum_{\Gamma(k,t)\in\Gamma_j} \sum_{G(\mathbf{i})\in\Gamma(k,t)} E(X(G(\mathbf{i}))).$$

For the assumptions on the randomness of the matrix, $S_2 = 0$.

We know that the number of Γ_3 canonical graphs is $t \leq \frac{k+1}{2}$, therefore

$$|S_3| = \frac{O(n^t)}{n^{1+\frac{k}{2}}} = o(1)$$

.

The only contribution left is S_1, we already know that the number of canonical graphs is $\frac{1}{k+1}\binom{2k}{k}$. As we can notice, there are no such graphs if k is odd, so we set $k = 2m$. The formula to compute S_1 is then

$$\begin{aligned} S_1 = \frac{1}{n^{1+m}} \sum_{\Gamma(k,t)\in\Gamma_j} n(n-1)\cdots(n-m) = \\ = \frac{1}{k+1}\binom{2k}{k}(1-\frac{1}{m})\cdots(1-\frac{m}{n}) \to \frac{1}{k+1}\binom{2k}{k} \end{aligned}$$

So as the size n of the matrix approaches infinity, we have

$$E(m_k(W_n)) = \frac{1}{k+1}\binom{2k}{k} = C_k$$

As the expected moments of the ESD and those of the semi-circle law coincide, the two distributions also do.

$\square$

1.1.3 Generalizing the semicircular law

The previous result is an evergreen in Random Matrix Theory, and it has been used in a multitude of fields ranging from nuclear physics to telecommunications and finance. Thus we will try to generalize it. Indeed we may wonder what happens when the variance in the hypotheses is not unitary. While the physical motivation for this case shall be more clear in the second chapter, nonetheless it is a question worth asking *per se*. The answer to this question at its core relies on the fact that if we re-scale the variance we simply deform the support on which the distribution lies, then as the area must not vary, the circle is stretched into an ellipsis. However the following approach has some interest as we see that the Wigner law is recovered as we consider fluctuations.

Theorem 1.1.2. *Let X_n be a symmetric matrix whose elements follow a PDF with $E(x_{ij}) = 0$ and finite variance V^2 then, as $n \to \infty$, $\mu_{X_n}(x)$ converges weakly to $\sigma(x)$, where*

$$\sigma(x) = \begin{cases} \frac{1}{2\pi V^2}\sqrt{4V^2 - x^2}, & \textit{if } x \in [-2V, 2V] \\ 0, & \textit{otherwise} \end{cases}$$

Proof. Let us imagine to add small symmetric matrix δX_n to X_n. δx_{ij} are the matrix element, being the mean value $E(\delta x_{ij}) = 0$ and the variance $E(\delta x_{ij}^2) = v^2$. We make use of perturbation theory to calculate the variation of the eigenvalues in an interval $d\lambda$ around λ_i

$$Z(\lambda_i) = \delta x_{ij} + \sum_{i \neq j} \frac{|\delta x_{ij}|^2}{\lambda_i - \lambda_j} + \cdots$$

We can make considerations about those two terms. On the account of the former, the average contribution of the term δx_{ij} is to be neglected, as its mean is zero. While the latter does contribute, and we can estimate the term $|\delta x_{ij}|^2$ with the variance of the perturbation v^2. Then the continuum limit for the variation becomes

$$Z(\lambda) = \int v^2 \frac{\sigma(\lambda')d\lambda'}{\lambda - \lambda'}$$

To be precise we mean the principal part of this integral, as we are not interested in null eigenvalues to appear. Then we call the total variance $E(x_{ij}^2) = V^2$ and we calculate the variation of the number of eigenvalues in the interval $(\lambda, \lambda + d\lambda)$, that is

$$\begin{aligned} \sigma(\lambda + d\lambda, V^2)Z(\lambda + d\lambda, V^2) - \sigma(\lambda, V^2)Z(\lambda, V^2) = \\ = \frac{\partial(\sigma Z)}{\partial\lambda} d\lambda = -v^2 \frac{\partial\sigma}{\partial V^2} dV^2 \end{aligned}$$

If we re-scale all the matrix by a factor c, the eigenvalues scales as well by the same factor, while the variance scales as c^2, so

$$\sigma(c\lambda, c^2V^2)cd\lambda = \sigma(\lambda, V^2)d\lambda.$$

Putting $cV = 1$, we have

$$\sigma(\lambda, V^2) = \frac{1}{V}\sigma\left(\frac{\lambda}{V}, 1\right) = \frac{1}{V}\sigma_1\left(\phi = \frac{\lambda}{V}\right)$$

this yield

$$Z(\lambda, V^2) = \frac{v^2}{V} Z_1(\phi) \qquad \sigma(\lambda, V^2) = \frac{v^2}{V}\sigma_1(\phi)$$

If we insert those relations in the estimate for the series we get

$$Z_1(\phi) = P \int \frac{\sigma_1(\zeta)d\zeta}{\phi - \zeta}$$

while if we insert them in the variation relation we have

$$\frac{\partial(Z_1\sigma_1)}{\partial\phi} = \frac{1}{2}\frac{\partial(\phi\sigma_1)}{\partial\phi}$$

then, since for symmetry requirements when $\phi = 0$ also $Z_1 = 0$, we get to the equation

$$Z_1(\phi) = \frac{1}{2}\phi = P\int_{-\infty}^{+\infty}\frac{\sigma_1(\zeta)d\zeta}{\phi-\zeta}.$$

The generalized semicircular distribution $\sigma(\phi) = C\sqrt{A^2-\phi^2}$ where $|\phi| < A$ and 0 if $|\phi| > A$ satisfy this condition, as we integrate

$$\begin{aligned}
&\int\frac{\sqrt{A^2-\zeta^2}}{\phi-\zeta}d\zeta = \\
&\quad= \frac{1}{\sqrt{A^2-\phi^2}}(-\sqrt{A^2-\zeta^2}\sqrt{A^2-\phi^2}+ \\
&\quad+ A^2 ln\left(\sqrt{A^2-\zeta^2}\sqrt{A^2-\phi^2}+A-\phi\zeta\right)- \\
&\quad- \phi^2\ln\left(\sqrt{A^2-\zeta^2}\sqrt{A^2-\phi^2}+A^2-\phi\zeta\right)+ \\
&\quad+ \phi\sqrt{A^2-\phi^2}arctan\left(\frac{\zeta}{\sqrt{A^2-\zeta^2}}\right)+\phi^2- \\
&\quad- A^2\ln(\phi-\zeta))
\end{aligned}$$

Taking the principal part under the limit of ζ going from $-A$ to A the previous equation yields π as a totally unexpected result.

Therefore we have the conditions

$$C = \frac{1}{2\pi}$$

and

$$\int\sigma(\phi)d\phi = 1$$

Thanks to these constraints on the constants we can write the solution that is the generalized circular law for the variance V^2

$$\sigma(x, V^2) = \frac{1}{2\pi V^2}\sqrt{4V^2-x^2}$$

$\square$

1.2 The Circle Law

Wigner Semicircular law gave birth to modern random matrix theory, and its generalization, also known as the circular law conjecture, challenged scholars since the Fifties. A seminal result was given by Girko [5] that, despite he provided rather rough proof, introduced several tools and concepts that Tao and Vu and Bai used in their most recent proof [14], [1]. This result generalizes the semicircle law to the case of non symmetric random matrices. We shall review the concepts behind the proof, and later see the further generalization called the elliptic law.

1.2.1 The Stieltjes transform

A fundamental tool in Random matrix theory is the Stieltjes transform, whose definition and properties shall be stated in this section.

Definition 1.2. Let μ be a measure, then its Stieltjes transform is defined as:

$$s_\mu(z) = \int \frac{1}{x-z}\mu(x)dx \tag{1.4}$$

The Stieltjes transform has the following properties:

Theorem 1.2.1 (Inversion Formula)**.** *Let a, b be two continuity points for F, with $a < b$, then*

$$F([a,b]) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_a^b \Im s_F(x + i\epsilon)dx$$

Theorem 1.2.2. *Let $\{F_n\}$ be a sequence of Lipshitz functions, and* $\lim_{x\to\infty} F_n(x) = 0 \quad \forall\, n$*, and $F_n(x) \to F(x)$ as $n \to \infty$, then*

$$\lim_{n\to\infty} s_{F_n}(z) = s_F(z)$$

Hence, as the criteria for uniqueness of the limiting function are met only by the boundedness of $\{F_n\}$ and to our purposes we will deal with bounded ESD, we shall assume this limiting property to always hold true.

Being μ the ESD of the matrix A, then its Stieltjes transform, which will now on be written as s_μ, is

$$s_\mu(z) = \int \frac{1}{x+iy-z}\mu(x,y)dxdy = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{\lambda_k - z} \tag{1.5}$$

Moreover, if we have that μ is analytic everywhere (except in its poles) then its real part already determines the eigenvalues, and it is:

$$\begin{aligned}\Re(s_\mu) &= \frac{1}{n}\sum_{k=1}^{n}\frac{\Re(\lambda_k)-x}{|\lambda_k - z|^2} = \\ &= -\frac{1}{2n}\sum_{k=1}^{n}\frac{\partial}{\partial x}\ln\left(|\lambda_k - z|^2\right) = -\frac{1}{2}\frac{\partial}{\partial x}\int_0^\infty \ln(w)\ \nu(z,w)dw\end{aligned}$$

where by ν we intend the ESD of the matrix $H = (A-zI)^*(A-zI)$

The previous property is one of the key concept behind the circular law. As a matter of facts it gives a relation between the ESD of a non-Hermitian matrix A and its Hermitian counterpart H. In the following section we will use this result in order to compute the characteristic function of the ESD of a non-Hermitian matrix, thus establishing the circular law.

1.2.2 The Circle Law

Since the 1950s, right after Wigner's proof of the semicircular law, it has been conjectured the universality of this result, that is, in some sense, the analogue of central limit theorem with respect to random matrix eigenvalues. The central limit approach to the problem is in fact quite troubled, therefore the proof will be carried on in a different fashion.

Theorem 1.2.3. *Let X_n be a random matrix whose elements follow a bounded PDF with finite $2+\eta$ moments, ($\eta > 0$) mean zero and unitary variance, then its ESD tends to the circular law, i.e. its eigenvalues are uniformly distributed over the complex disc of radius one.*

Proof. For the sake of synthesis the proof we provide will not be complete as it would require and only the main features will be fully shown. The reader can anyway find more details about this result in [14] and [1].

We need to show that in the limit of large n the measure μ_n associated with the ESD of X_n converges to μ that determines the circular law. This shall be done by calculating the characteristic function c_n of the ESD μ_n (*i.e.* its Fourier transform) and then showing its limit converges to c that determines μ.

It is useful to recall that the characteristic polynomial can be expressed in terms of ESD as

$$\exp\left\{ n \int_0^\infty \ln(x)\ \mu(x) dx \right\} = det(A)$$

therefore we can exploit this formula to change the matrix whose ESD we will actually analyze, in this way:

$$\ln\left(det\left(\sqrt{(X - zI)(X - zI)^*} \right)\right) = n \int_0^\infty \ln(x)\ \nu(x) dx$$

where ν will be the ESD of the Hermitian matrix $(X - zI)(x - zI)^*$.

Then we compute the characteristic function of the real part of the Stieltjes transform of μ, that, as seen before, already determines the eigenvalues:

$$\iint_{\mathbb{R}^2} \Re(s_\mu) e^{ius+ivt} dsdt =$$
$$= -\frac{1}{n}\sum_{k=1}^{n} \iint \frac{s-\Re(\lambda_k)}{(\Re(\lambda_k)-s)^2+(\Im(\lambda_k)-t)^2} e^{ius+ivt} dtds =$$
$$= -\frac{1}{n}\sum_{k=1}^{n} \iint \frac{s}{s^2+t^2} e^{ius+ivt+iu\Re(\lambda_k)+iv\Im(\lambda_k)} dtds$$

It is to be remarked that the previous integration is not interchangeable, as Fubini theorem cannot be applied since $\frac{s}{s^2+t^2}$ is not integrable over the whole $\mathbb{R}^2$. We have then to iterate the integration:

$$\begin{aligned}\iint \frac{x}{x^2+y^2} e^{iux+ivy} dydx &= \pi \int sgn(x) e^{iux-|vx|} dx \\ &= 2\pi i \int_0^\infty sin(xu) e^{|v|x} dx \\ &= \frac{2\pi iu}{u^2+v^2}\end{aligned}$$

Inserting this result in the previous one, we get the fundamental Girko's identity:

$$c(u,v) = \iint_{\mathbb{R}^2} e^{iux+ivy} \mu_n(x,y) dxdy =$$
$$= \frac{u^2+v^2}{4\pi iu} \iint \left(\frac{\partial}{\partial x} \int_0^\infty \ln(w)\ \nu_n(z,w)dw \right) e^{iux+ivy} dxdy$$

The core problem at this point is that the logarithm is not bounded neither at 0 nor at ∞, so the convergence of ν_n to a limit does not imply the convergence of the whole expression. This could lead to some serious problems while exchanging limits or derivatives. For this reason, we shall use the result that under the conditions of the finiteness of $2+\eta$ moments, the logarithm is uniformly integrable, as stated in [10], Theorem 5.2.

Then, assuming that the limit of the ν_n exists and is $\lim_{n\to\infty}\nu_n = \nu$, we calculate the limiting expression

$$
\begin{aligned}
\frac{\partial}{\partial x}\int_0^\infty \ln(w)\ \nu(z,w)dw &= \frac{1}{\pi}\iint_{x^2+y^2\le 1}\frac{2(s-x)}{(s-x)^2+(t-y)^2}dxdy \\
&= \Re\left(\frac{2}{\pi}\iint_{x^2+y^2\le 1}\frac{1}{(s-x)+i(t-y)}dxdy\right) \\
&= \Re\left(\frac{2}{\pi}\int_0^1\left(\int_0^{2\pi}\frac{\rho d\theta}{z-\rho e^{i\theta}}\right)d\rho\right) \\
&= \Re\left(\frac{2}{\pi i}\int_0^1\left(\int_{|\xi|=\rho}\frac{d\xi}{\xi(z-\xi)}\right)d\rho\right) \\
&= \Re\left(\frac{2}{\pi i}\int_0^{|z|\wedge 1}\frac{2\pi i}{z}\rho d\rho\right) \\
&= 2\Re\left(\frac{|z|\wedge 1}{z}\right) \\
&= \begin{cases}\frac{2s}{s^2+t^2} & \text{if } s^2+t^2>1\\ 2s & \text{otherwise}\end{cases}
\end{aligned}
$$

For the sake of notation, we used the wedge symbol $\wedge$ to indicate the lesser of the two integration extremes.

As we insert this result into Girko's identity, we get an expression of the characteristic function that coincides with the Circle law one, therefore also the measures coincide.

□

1.3 The Elliptic Law

The long-debated Circular law has, in its hypotheses, the condition that the matrix should be random. We need to refine this statement by making clear what we really mean by random entries. Next theorem provides a clearer framework in which such statements can be asserted. Indeed the previous theorems have to deal respectively with symmetric and totally non-symmetric matrices, and we will try to infer such results by introducing a way to measure the symmetry of a random matrix. Let $E(x_{ij}x_{ji})$ be the expected value of the product of the elements x_{ij}, x_{ji} of the matrix X, that is its weighted average value. We immediately notice that, in order for a matrix to be asymmetric, such expected value must be zero, and, if we want it to be symmetric, its value must be unitary. These results appears to be some kind of limiting cases to a wider distribution law, which we will try to assert by making use of this new parameter of symmetry of random matrices.

A useful approach to this problem is one that recalls electrostatics. Let $s_\mu(z)$ be the Stieltjes transform of the ESD μ of the matrix $Iz - A$, let us write it as

$$s_\mu(z) = \int \frac{\mu(w)}{z-w} dw$$

the analogy begins to appear as we integrate $s_\mu(z)$ along the border ∂R of an arbitrary region R

$$\begin{aligned}\int_{\partial R} s_\mu(z) \frac{dz}{2\pi i} &= \frac{1}{n}\sum_{k=1}^{n} \int_{\partial R} \frac{1}{z-w}\frac{dz}{2\pi i} \\ &= \frac{1}{n}\sum_{k=1}^{n} 1 = \int_R \mu(w) d^2 w\end{aligned}$$

Then we recall the Divergence theorem:

$$\int_R \left(\frac{\partial}{\partial x} s_\mu(z) + i \frac{\partial}{\partial y} s_\mu(z) \right) \frac{d^2 z}{2\pi} = \int_{\partial R} s_\mu(z) \frac{dz}{2\pi i} = \int_R \mu(z) d^2 z$$

The analogy is then settled, as s_μ is the fundamental solution of a potential U_μ defined by $2\Re(s_\mu) = -\frac{\partial U_\mu}{\partial x}, 2\Im(s_\mu) = -\frac{\partial U_\mu}{\partial y}$ and which obeys to the analogue of Poisson equation

$$\nabla^2 U_\mu = -4\pi\mu. \tag{1.6}$$

An useful reference to expand this analogy can be found in [8]. Keeping this in mind, it is time to state the main result of this section.

Theorem 1.3.1. *Let X_n be a matrix whose elements x_{ij} are independent and follow a random PDF, with $E(x_{ij}) = 0$, $E(x_{ij}^2) = 1$ and $E(x_{ij}x_{ji}) = \rho$. Then, as n approaches ∞, the ESD μ_n of X_n tends to the limit*

$$\mu(x, y) = \begin{cases} \frac{1}{\pi(1-\rho^2)} & \textit{if } \frac{x^2}{(1+\rho)^2} + \frac{y^2}{(1-\rho)^2} \leq 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. This proof will be similar to that of [13], however some of the steps taken there were not completely legitimate (namely the commutativity and integrability of ln).

As we have done before, we need to calculate the potential

$$U_\mu(x, y) = -\frac{1}{n} \ln \left(det \left((X - zI)(x - zI)^* \right) \right)$$

and as previously, and is stated in [10], we assume the logarithm is uniformly integrable and commutes with the expected value operator.

Then the integral becomes

$$U_\mu(z) = \frac{1}{n}\ln\left(E\left(\int\prod_i \frac{d^2w_i}{\pi}\times\exp\left\{-\epsilon\sum_i |w_i|^2 - \sum_{i,j,k} w^*(z^*\delta_{ik} - A^T_{ik})(z\delta_{kj} - A_{kj})\right\}\right)\right)$$

where we have added a small parameter ϵ to avoid zero eigenvalues to appear, as in [15]. Then we introduce $r = \frac{1}{n}\sum_i z_i z_i^*$

$$e^{nU_\mu} = \int\prod_i \frac{d^2w_i}{\pi}\exp\left\{-n\left(\epsilon r + \ln(1+r) + \frac{rx^2}{1+r(1+\rho)} + \frac{ry^2}{1+r(1-\rho)}\right)\right\}$$

and by substituting $\sigma = \frac{1}{r}$ we get

$$e^{nU_\mu} = \frac{n^n}{\Gamma(n)}\int_0^\infty \frac{d\sigma}{\sigma}\exp\left\{-n\left(\frac{\epsilon}{\sigma} + \ln\left(1+\frac{1}{\sigma}\right) + \frac{x^2}{\sigma+1+\rho} + \frac{y^2}{\sigma+1-\rho}\right)\right\}$$

.

As we are interested in the limit of large n, this integral can be estimated by the saddle point method, that is we approximate the integral to be peaked over the point of maximum of σ. The equation for its maximum point is

$$\frac{1}{1+\sigma^2} - \frac{\epsilon}{\sigma^2} - \frac{x^2}{(\sigma+1+\rho)^2} - \frac{y^2}{(\sigma+1-\rho)^2} = 0$$

and the Green function/Stieltjes transform at that point is

$$s_\mu(x,y) = \frac{x}{(\sigma+1+\rho)} - i\frac{y}{(\sigma+1-\rho)}.$$

Now it is time to resolve the limit of ϵ, if we expand in powers of σ we get

$$\epsilon = \sigma^2\left(1 - \frac{x^2}{(1+\rho)^2} - \frac{y^2}{(1-\rho)^2}\right) + o(\sigma^3)$$

then, we have that inside the ellipse whose semiaxes are $1+\rho$ and $1-\rho$, $\sigma \sim \sqrt{\epsilon} \to 0$ the Stieltjes transform is

$$s_\mu(x,y) = \frac{x}{(1+\rho)} - i\frac{y}{(1-\rho)}$$

On the other hand we have to evaluate s_μ outside the ellipse. This can be done by solving the saddle point equation and substituting the result for σ in the expression for s_μ, yielding the following result:

$$s_\mu(z) = \begin{cases} \frac{x}{(1+\rho)} - i\frac{y}{(1-\rho)} & \text{if } \frac{x^2}{(1+\rho)^2} + \frac{y^2}{(1-\rho)^2} \leq 1 \\ \frac{z}{2\rho}(1 - \sqrt{1-\frac{4\rho}{z^2}}) & \text{otherwise} \end{cases}$$

As these expression for s_μ are inserted in the previous relations

$$2\Re(s_\mu) = -\frac{\partial U_\mu}{\partial x} = E_x \qquad -2\Im(s_\mu) = -\frac{\partial U_\mu}{\partial y} = E_y$$

using the Poisson-like equation

$$\nabla^2 U_\mu = -\nabla \cdot E = -4\pi\mu$$

we get the thesis.

□

1.3.1 Consistency of the Elliptic Law

The Elliptic Law looks like the most powerful tool to tackle into the analysis of spectrum of a random matrix, as it is a statement about a really wide class of matrices. Nevertheless we must retrieve the classical results as the semicircular law to verify its consistency in the theory of random matrices. We immediately notice that the Circle law is won back just by having $\rho = 0$. At a first glance the limit as $\rho \to \pm 1$ may look singular, as one of the terms, either the real or the imaginary one diverges. Besides the fact that this divergence can be naively explained noticing that either the real or the imaginary part of the eigenvalues have zero value, we proceed in a more formal way.

We first integrate the Elliptic Law with respect to y and to x, thus yielding respectively the distribution of the real and imaginary parts.

$$\mu_R(x) = \int_{-(1-\rho)\sqrt{1-\frac{x^2}{(1+\rho)^2}}}^{+(1-\rho)\sqrt{1-\frac{x^2}{(1+\rho)^2}}} \frac{1}{\pi(1-\rho^2)}dy = \frac{2\sqrt{(1+\rho)^2 - x^2}}{\pi(1+\rho)^2}$$

$$\mu_I(y) = \int_{-(1+\rho)\sqrt{1-\frac{y^2}{(1-\rho)^2}}}^{+(1+\rho)\sqrt{1-\frac{y^2}{(1-\rho)^2}}} \frac{1}{\pi(1-\rho^2)}dx = \frac{2\sqrt{(1-\rho)^2 - y^2}}{\pi(1-\rho)^2}$$

Then we have two limiting cases. One is the symmetric one, *i.e.* $\rho = E(x_{ij}x_{ji}) = E(x_{ij}^2) = 1$. In this circumstance we know that there are only real eigenvalues, and as a matter of fact the imaginary part distribution tends to a Dirac Delta centered in zero. We immediately recognize the Semicircular Law

$$\lim_{\rho\to 1} \mu_R(x) = \lim_{\rho\to 1} \frac{2\sqrt{(1+\rho)^2 - x^2}}{\pi(1+\rho)^2} = \frac{\sqrt{4-x^2}}{2\pi} = \mu_{sc}(x)$$

The same procedure can be applied to the case in which $\rho = E(x_{ij}x_{ji}) = -1$, that is total anti-symmetry. As before

$$\lim_{\rho\to -1} \mu_I(y) = \lim_{\rho\to -1} \frac{2\sqrt{(1-\rho)^2 - y^2}}{\pi(1-\rho)^2} = \frac{\sqrt{4-y^2}}{2\pi} = \mu_{sc}(y)$$

we recognize the Semicircular Law for the imaginary part, while the Delta distribution of the real part tells us it has null value. This fact is consistent with the theorem of linear algebra that states skew-symmetric matrices have only pure imaginary eigenvalues, and it is to be remarked that we found an analogue to the semicircular law that is not explicitly present in the literature.

Chapter 2

Random Walks on Networks

2.1 Markov Processes on Networks

Random walks on graphs are ubiquitous in applied physics; from chemical reaction pathways to transportation systems, from finance to gene networks, we always model the system as a random walk on a network. It is indeed a quite simple idea: we formalize a random walk by assuming that, having a graph, each node exchanges particles with its neighbours, the proportion according to a certain transition matrix, called stochastic matrix. Let us consider a process that consists in a particle moving from a state j to a state i, being the probability of doing so dependent only on such states. This kind of processes, *i.e.* those processes in which transitions depend only on the present system state, are called Markov processes (chains), from Russian physicist Andrey Markov who first studied them. As a matter of fact a Markov process can be regarded as a random walk on a certain graph. The passage from state j at time t to state i at time $t + 1$ will be performed on the network by a jump from node j to node i. The probability of this event to occur is described by the Kolmogorov continuity equation

$$p_i(t+1) = \sum_j \pi_{ij} p_j(t) \tag{2.1}$$

where we need to introduce π_{ij}, that is the probability of passing from node j to node i. Of course for π_{ij} there are the boundedness constraints:

$$\sum_j \pi_{ij} = 1 \quad \text{and} \quad 0 \leq \pi_{ij} \leq 1 \tag{2.2}$$

We arrange those coefficients in a matrix Π and we call it the transition probability matrix, or *stochastic matrix*. Indeed, the stochastic matrix wholly represents the process, as it describes every possible shift from one state to another.

Then it is worthwhile introducing the connectivity (or adjacency) matrix C, whose elements c_{ij} may assume only the values 0 if the nodes i j are disconnected and 1 if there exists a link between the two. This matrix gives us the possible transitions between states, but not their weights. To recollect the stochastic matrix from the connectivity matrix we need to introduce a graph metrics, that is weights for each node. We shall call this metric matrix G, whose entries are defined by

$$g_{ij} = \frac{\delta_{ij}}{d(j)}$$

and the stochastic matrix is given the product $CG = \Pi$. Being the product of two matrices which in general do not commute, Π is not symmetric. Indeed it is symmetric with respect to the metrics introduced by G, this fact being shown by taking into account the scalar product $< \mathbf{u}, \mathbf{v} >= \mathbf{u} \cdot G\mathbf{v}$

$$< \mathbf{u}, \Pi\mathbf{v} >= \mathbf{v} \cdot GCG\mathbf{v} = CG\mathbf{u} \cdot G\mathbf{v} =< \Pi\mathbf{u}, \mathbf{v} >$$

Such condition of symmetry is in fact more profound than just a mere calculation. Indeed it is a consequence of the equilibrium condition, as it will be shown in the followings that the nodes degrees define the stable condition of the walk. As the matrix's symmetry is recovered, so does the orthogonality of the eigenvectors, with respect to the new metrics. Moreover, we can claim all the eigenvalues are real, indeed by making a similarity transformation

$$G^{\frac{1}{2}}\Pi G^{-\frac{1}{2}} = G^{\frac{1}{2}}CG^{\frac{1}{2}} = \Sigma$$

Since the second matrix is symmetric its eigenvalues are all real, and as similarity transformations do not alter the spectrum of a matrix, the eigenvalues of Π are real. The entries of Σ will have the form $\sigma_{ij} = \frac{c_{ij}}{\sqrt{d(i)d(j)}}$, and will have to satisfy the constraints:

$$\sum_i \sqrt{d(i)d(j)}\sigma_{ij} = 1$$

$$E(\sigma_{ij}) = \frac{d(i)d(j)}{M\sqrt{d(i)d(j)}} = \frac{\sqrt{d(i)d(j)}}{M}$$

where $M = \sum_i d(i)$. Moreover these entries shall fulfill the condition

$$\sum_i \sqrt{\frac{d(i)}{d(j)}}\sigma_{ij} = \sum_i \frac{c_{ij}}{d(j)} = 1$$

We now compute the variance of the elements of Σ. As we know that c_{ij} is a variable taking only values of 0 and 1, the variance reads

$$Var(\sigma_{ij}) = \frac{1}{M}\left(1 - \frac{d(i)d(j)}{M}\right)$$

And by taking the thermodynamic limit of large N we can estimate $M \simeq N\bar{d}$, thus yielding the variance to be

$$Var(\sigma_{ij}) \simeq \frac{1}{N\bar{d}} + O(N^{-2}). \tag{2.3}$$

2.1.1 Stationary Distribution

We can derive some useful properties from the stochastic matrix to analyze the process. For instance, we might get interested in finding the stable solution of the random walk. A *stationary distribution* can be defined as follows:

Definition 2.1. Let $\mathbf{p}^*$ be the vector of probabilities, then $\mathbf{p}^*$ is a stationary distribution if

$$\mathbf{p}^* = \Pi \mathbf{p}^* \tag{2.4}$$

That is $\mathbf{p}^*$ is the eigenvector of Π with eigenvalue $\lambda^* = 1$

Indeed we can recast the Kolmogorov equation as

$$\mathbf{p}(t+1) = \Pi \mathbf{p}(t) \tag{2.5}$$

where we simply put the p_is in a vector p. This recursive equation yields

$$\mathbf{p}(t) = \Pi^t \mathbf{p}(0) \tag{2.6}$$

then, substituting $\mathbf{p}(t) = \mathbf{p}(t+1) = \mathbf{p}^*$ we get the previous definition.

Obviously the stationary distribution is the eigenvector of the stochastic matrix with unitary eigenvalue. If we recover the symmetrized matrix Σ we have by construction that an eigenvector with eigenvalue one can be made by node degrees $(\sqrt{d(1), d(2), \cdots, d(N)})$. This sets the condition for an invariant linear space orthogonal to the stationary state

$$\sum_i \sqrt{d(i)} x_i = 0 \tag{2.7}$$

Then, re-normalizing the vector

$$\widehat{\mathbf{e}}_1 = \frac{1}{\sqrt{M}}(\sqrt{d(1), d(2), \cdots, d(N)}) \tag{2.8}$$

we complete to an orthonormal basis with an orthogonal basis of the invariant subspace $\widehat{\mathbf{e}}_i$ $i = 2, \cdots, N$. Thanks to these basis vectors we can define an orthogonal matrix

$$O = \begin{pmatrix} \widehat{\mathbf{e}}_1 \\ \widehat{\mathbf{e}}_2 \\ \cdots \\ \widehat{\mathbf{e}}_N \end{pmatrix} \tag{2.9}$$

which allows us to decompose the stochastic matrix in the form

$$\Sigma' = O^T W O \tag{2.10}$$

where W is the matrix

$$W = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & w_{22} & w_{23} & \cdots & w_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & w_{N2} & w_{N3} & \cdots & w_{NN} \end{pmatrix} \tag{2.11}$$

and W is a symmetric matrix with mean zero and finite variance, as in the statement of the Wigner law. Let us return to the calculation of the variance of the elements of the stochastic matrix. We may decompose the entries as

$$\sigma'_{ij} = e_{1_i} e_{1_j} + \sum_{h,k \geq 2} e_{ki} w_{kh} e_{hj} = \sqrt{d(i)d(j)}M + \sum_{h,k \geq 2} e_{ki} w_{kh} e_{hj}$$

and compute the variance

$$Var(\sigma'_{ij}) = E\left(\sum_{h,k\geq 2} e_{ki}w_{kh}e_{hj}\sum_{h',k'\geq 2} e_{k'i}w_{k'h'}e_{h'j}\right) =$$
$$= \sum_{h,k\geq 2} e_{ki}E(w_{hk}^2)e_{hj}e_{ki}e_{hj}$$

Then, setting $E(w_{hk}^2) = \rho^2$ we recast

$$Var(\sigma'_{ij}) = \rho^2 - \rho^2\left(\frac{d(i)}{M} + \frac{d(j)}{M}\right) + \rho^2\frac{d(i)d(j)}{M^2} =$$
$$= \rho^2\left(1 - \frac{d(i)}{M}\right)\left(1 - \frac{d(j)}{M}\right) \simeq \frac{1}{N\bar{d}}$$

where the last estimate comes from the fact we took the limit of large N, that is we approximate $M \simeq N\bar{d}$ and we use the previous result $\rho^2 = \frac{1}{N\bar{d}} + O(N^{-2})$.

2.1.2 Relaxation Time

As we have just seen, the matrix Σ is symmetric, thus we can decompose it with orthogonal matrices O in the form

$$\Sigma = O^T\Lambda O = \sum_i \lambda_i \mathbf{u}_i\mathbf{u}_i^T \qquad (2.12)$$

where the $\mathbf{u}_i$s are the eigenvectors of eigenvalue λ_i. In order to evolve our process we now calculate the power of the stochastic matrix.

$$\Pi^t = G^{\frac{1}{2}}\Sigma^t G^{-\frac{1}{2}} = G^{-\frac{1}{2}}\sum_l \lambda_i^t\mathbf{u}_i\mathbf{u}_i^T G^{\frac{1}{2}} = \sum_l \lambda_i^t\mathbf{u}_i^R\mathbf{u}_i^L \qquad (2.13)$$

where for the sake of readability we introduced $\mathbf{u}_i^R = \mathbf{u}_i^T G^{\frac{1}{2}}$ and $\mathbf{u}_i^L = G^{-\frac{1}{2}}\mathbf{u}_i$, respectively the right and left re-normalized eigenvectors of Π.

The evolved probability is then

$$p_i(t) = \sum_j \lambda_j^t < \mathbf{p}(0), \mathbf{u}_i^R > (u_j^L)_i = \sum_j \lambda_j^t c_j(0)(u_j^L)_i \qquad (2.14)$$

This formula is called the spectral representation. We may wonder then about the long run expression. In order to deal with it we need to make the continuous limit of such Markov process. The continuous version is made by substituting the discrete time steps by a distribution of steps, and it is natural to think of it as Poissonian. Then, the spectral representation formula yields

$$p_i(t) = \sum_j c_j(0)(u_j^L)_i \sum_n \lambda_j^n \frac{e^{-t} t^n}{n!} = \sum_j c_j(0)(u_j^L)_i e^{-(1-\lambda_j)t} \qquad (2.15)$$

The definition of relaxation time derives from this formula.

Definition 2.2. Let λ_* be the second greatest eigenvalue of Π, let $\gamma_* = 1 - \lambda_*$ be the spectral gap of the matrix. Then the relaxation time of the chain is

$$\tau_* = \frac{1}{\gamma_*} = \frac{1}{1 - \lambda_*} \qquad (2.16)$$

This relation between a dynamical property such as the relaxation time and a structural feature of the underlying graph would provide an intriguing insight when applying RMT theorems to the transition matrix.

2.2 Master Equation

We now take a wider look to the theory of random walk. The need to introduce a equation for the whole dynamics arises as we increase the number of particles moving among the nodes. We introduce N particles moving in the K nodes, and a function χ^α to describe the network state, that is

$$\chi_i^\alpha(t) = \begin{cases} 1 \text{ if the particle } \alpha \text{ is in the node } i \text{ at the time } t \\ 0 \text{ otherwise} \end{cases}$$

The state of each node will be described as

$$n_i(t) = \sum_\alpha \chi_i^\alpha(t)$$

and the probability of finding the particle α in the node i will be the mean value of χ over all the possible realization of the random walk of α

$$p_i^\alpha(t) = E(\chi_i^\alpha(t))$$

The generalized Kolmogorov equation shall read

$$\chi_i^\alpha(t + \Delta t) = \sum_j \xi_{ij}^\alpha(\Delta t, t)\chi_{j^\alpha}(t)$$

where Ξ^α is a matrix that describes the transition probabilities for each particle. As the average value is taken, we must recover the stochastic matrix $\pi_{ij} = E(\xi_{ij}^\alpha)$ for each α.

The total number of particles in the node i will be then

$$n_i(t) = \sum_\alpha \chi_i^\alpha(t + \Delta t) = \sum_j \sum_\alpha \xi_{ij}^\alpha \chi_{ij}^\alpha(t) = \sum_j \xi_{ij} n_j(t)$$

whereas the variation of the particles shall be

$$n_i(t + \Delta t) - n_i(t) = \sum_j \xi_{ij}(t)n_j(t) - \xi_{ij}(t)n_i(t)$$

that is simply the quantification of the particles that are approaching the node minus those who leave it.

The mean dynamics for each particle will read

$$\bar{n}_i(t+\Delta t) - \bar{n}_i(t) = \sum_j \pi_{ij}(\Delta t) n_j(\Delta t) - \pi_{ji}(\Delta t) n_i(\Delta t)$$

This, however is the most general case, that assumes the transition rates may varies over time. It is a reasonable condition to impose that the transition rate should be constant or quasi-constant

$$\pi_{ij} = \widehat{\pi}_{ij}\Delta t + o(\Delta t)$$
$$\pi_{ii} = 1 - \widehat{\pi}_{ii}\Delta t = 1 - \Delta t \sum_{i \neq j} \widehat{\pi}_{ij}$$

That allows the continuum limit to be taken yielding

$$\frac{d\bar{n}_i}{dt} = \sum_i \widehat{\pi}_{ij}\bar{n}_j - \sum_j \widehat{\pi}_{ji}\bar{n}_i$$

Then, we try to recast this expression in terms of probability flow, that is we are more interested in studying the condition for which the states are equivalent in probability. The Kolmogorov equation is recast

$$p(\mathbf{n}, t+\Delta t) = \sum_{\mathbf{\Delta n}} \Pi(\mathbf{n} - \mathbf{\Delta n} \rightarrow \mathbf{\Delta n}) p(\mathbf{n} - \mathbf{\Delta n}, t)$$

and, as before, for each node we get

$$\Delta n_i = \sum_j \xi_{ij}(t) n_j(t) - \xi_{ji}(t) n_i(t)$$

and the condition of regularity we impose becomes

$$\pi_{ij}(\mathbf{n} - \mathbf{\Delta n} \rightarrow \mathbf{n}, \Delta t) = \frac{1}{N}\widehat{\pi}_{ij}(n_j + 1)\Delta t + o(\Delta t)$$

leading to a new form of the Kolmogorov equation

$$p(\mathbf{n}, t + \Delta t) = \frac{1}{N} \sum_{ij} E_j^+ E_i^- \pi_{ij} n_j p_j(\mathbf{n}, t) \Delta t + o(\Delta t)$$

where we introduced a kind of creation (E^+) and destruction (E^-) operator for the particle in the nodes (a jump from node j to i is equivalent to a creation of a particle in the node i and a destruction in the node j, weighted accordingly to the transition matrix). We re-normalize the regularized π_{ij} as

$$\frac{1}{N} \sum_{i,j} \widehat{\pi}_{ij} n_j \Delta t = 1 + o(\Delta t)$$

which yields

$$p(\mathbf{n}, t + \Delta t) - p(\mathbf{n}, t) = \frac{1}{N} \sum_{i,j} E_j^+ E_i^- \widehat{\pi}_{ij} n_j p(\mathbf{n}, t) \Delta t - \\ - \frac{1}{N} \sum_{i,j} \widehat{\pi}_{ij} n_i p(\mathbf{n}, t) \Delta t + o(\Delta t)$$

which, in the continuum limit for t gives the Master Equation

$$\frac{\partial p}{\partial t} = \frac{1}{N} \sum_{i,j} E_j^+ E_i^- \widehat{\pi}_{ij} n_j p(\mathbf{n}, t) - \frac{1}{N} \sum_{i,j} \widehat{\pi}_{ji} n_i p(\mathbf{n}, t)$$

2.2.1 Stationary Solution of Master Equation

The master equation has a general solution in the form

$$p(\mathbf{n}, t) = N! \sum_{\lambda} e^{-(1-\lambda)t} c_\lambda \prod_{k}^{K} \frac{(v_k^\lambda)^{n_k}}{n_k!}$$

where the λs are the eigenvalues of the matrix $\widehat{\Pi}$ and v^λ their respective eigenvectors. If the eigenvalues are real, this solution converges to a stationary probability state, that is

$$p_s(\mathbf{n}) = N! \prod_k^K \frac{v_k^{n_k}}{n_k!}$$

A stationary solution means that the probability currents from node j to node i are null. The condition that ensures the reality of the eigenvalues is, as we know from algebra, the symmetry of the stochastic matrix. However, it is not the symmetry with respect to canonical metrics, but we have to introduce a more refined condition.

Let us write the probability density current

$$J_{ij} = \sum_n \widehat{\pi}_{ij} \frac{n}{N} p_j(n,t) - \sum_n \widehat{\pi}_{ji} \frac{n}{N} p_i(n,t)$$

The stationary condition is equivalent to impose the flow of probability from one node to another to be zero when the equilibrium is reached. That is, from the previous equation

$$\pi_{ij} p_j = \pi_{ji} p_i$$

This condition, called the detailed balance condition, is crucial to achieve stationary equilibrium in a network. If the detailed balance is not to be held, and therefore the generalized symmetry of the transition matrix vanishes, the eigenvalues in the solution of the master equation may be in general belonging to the complex field. As such, stationary equilibrium is no more, but a new kind of balance appears: Non-Equilibrium Steady States (NESS).

2.2.2 Breaking the Symmetry

We previously discussed that the stochastic matrix Π is symmetric with respect to the metrics induced by the node degrees G, as we simply compute

$$(G\Pi)^T = (GCG)^T = GCG = G\Pi$$

as both the connectivity matrix C and the metrics G are symmetric. This property is equivalent to the generalized detailed balance condition

$$\pi_{ij} g_{jk}^{-1} = \pi_{kj} g_{ji}^{-1}$$

Which says, at its core, that node-weighted transition probabilities from node j to i are the same. We proceed as before by decomposing the stochastic matrix Π into

$$\Pi = G^{-\frac{1}{2}} \Sigma' G^{\frac{1}{2}} = G^{-\frac{1}{2}} (O^T W O) G^{\frac{1}{2}}$$

then a perturbation Ξ is added to W, yielding to a new non-symmetric but still stochastic matrix Π'

$$\Pi' = G^{-\frac{1}{2}} (\Sigma' + O^T \Xi O) G^{\frac{1}{2}} = G^{-\frac{1}{2}} (\Sigma'') G^{\frac{1}{2}}$$

The addition of such perturbation can be compared to a decorrelation of the opposite matrix elements. As we have done before, we compute the expected value

$$E(\sigma''_{ij}\sigma''_{ji}) = E\left(\left(\sigma'_{ij} + \sum_{h,k\geq 2} e_{ki}\xi_{kh}e_{hj}\right)\left(\sigma'_{ji} + \sum_{h',k'\geq 2} e_{k'j}\xi_{k'h'}e_{h'i}\right)\right)$$
$$= Var(\sigma'_{ij})$$

Then, the variance of σ''_{ij} will read

$$Var(\sigma''_{ij}) = Var(\sigma'_{ij}) + Var(\xi_{ij}) = Var(\sigma'_{ij}) + \frac{\epsilon^2}{N}$$

where if the perturbation is of order ϵ its variance shall be of order ϵ^2. Estimating as before

$$Var(\sigma''_{ij}) \simeq \frac{1}{N\bar{d}}$$

we have the relation

$$E(\sigma''_{ij}\sigma''_{ji}) = \frac{1}{N\bar{d}} - \frac{\epsilon^2}{N}$$

We notice that this result is similar to the one we previously obtained, but we can no longer apply neither the Wigner law nor its generalization for the variance, as the matrix is not symmetric. Nonetheless we also notice that if we perturb Σ with a matrix whose elements are of order $\epsilon = \frac{1}{\sqrt{\bar{d}}}$ the expected value of the product vanishes as the opposite elements show no correlation at all. In this case the spectrum shall be studied with a generalized circular law. Physically, that would mean that there would be no equilibrium at all, as imaginary eigenvalues of the same magnitude of the real ones would modulate the solution by oscillations. This violation of the Detailed Balance Condition needs further theoretical and experimental studies. Still, when the order of the perturbation is small, we recognize the condition under which an Elliptic law taking into account a non-unitary variance may pertain, giving information about NESS. As a matter of facts, it is shown by numerical simulations that as we decorrelate the opposite elements by a perturbation the spectral gap increases. Hence a theoretical explanation can be inferred to what may seem be emerging from simulations [7].

Chapter 3

Final Remarks

3.1 Random Networks Spectra

This last chapter is intended to introduce some results that support some of the theoretical predictions previously made. Namely, a hint about the validity of our proposition about the relation between spectral gap and connectivity is shown in the graphs we display below. Before that, however, we may discuss some technicalities about the creation of the networks.

3.1.1 The Erdös-Rényi Model

As we know by now, a graph $G(N, N_l)$ is a set of N vertices, or nodes, and N_l edges, or links, that connect the nodes. The number of links that start from a node is said to be the degree (d) of the node. Despite having a multitude of possibilities to make a graph to test our statements, we shall restrict our attention to a specific class of graphs, namely random graph. The most celebrated model for building a random graph is surely that due to Paul Erdös and Alfréd Rényi, in which two nodes (i, j) may be linked (or not) according to a given probability p (or $1 - p$). Indeed, having N nodes, we expect to have $p\frac{N(N-1)}{2}$ links in the whole graph. The average link number $\bar{N}_l$ will be $\bar{N}_l = p\binom{N}{2}$, and its variance can be computed according to a binomial distribution $Var(N_l) = p(1-p)\binom{N}{2}$. Moreover when studying properties of random graphs, the limit of large N is performed, as it provides a nice expression for the probability of having a certain average degree:

$$p(d) = \binom{N-1}{d} p^d (1-p)^{N-1-d} \simeq \frac{\bar{d}^d e^{-\bar{d}}}{N_l!}$$

where $\bar{d} = \frac{2}{N}\binom{N}{2} = p(N-1)$ is the average degree. According to various values p, graphs may look rather different one to another. As a matter of facts, if $p < \frac{1}{N}$ the whole graph will not be connected, but various detached components will be present, increasing p to be of order $\sim \frac{1}{N}$ $(\bar{d} = 1)$ a phase transition-like phenomenon occurs, and a maximal connected component of order $O(N)$ appears. In the following computations the graph will be set up not to be disconnected.

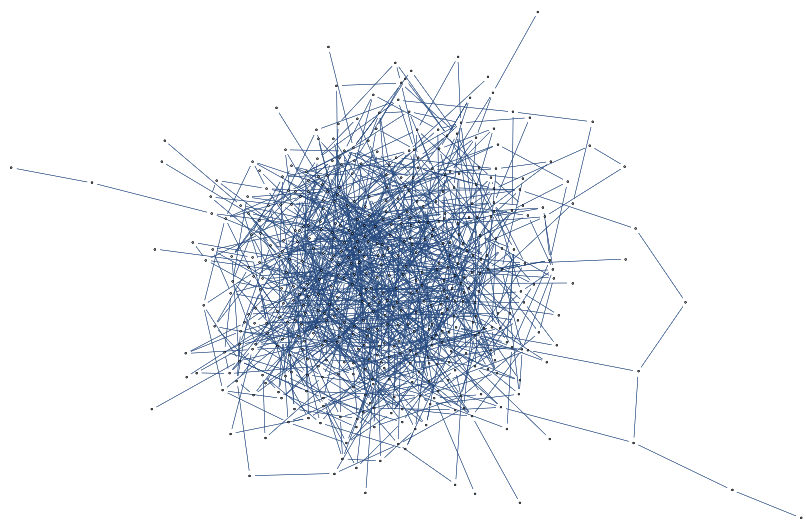
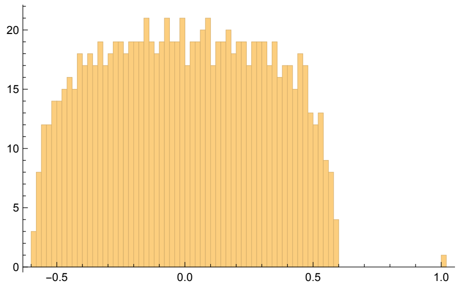


FIGURE 3.1 – A generated example of Erdös-Rényi Random graph, with $p = 0.01$ and $N = 500$.

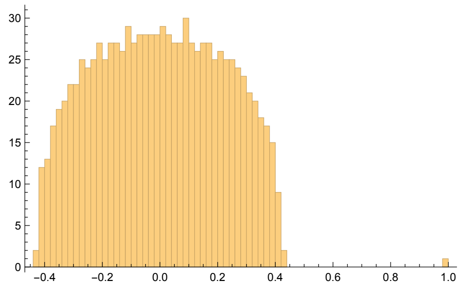
3.1.2 Some Results

We now display some of the consequences of the considerations we have made in the previous chapters. We established an approximate relation $V^2 \simeq \frac{1}{Nd}$ between the variance of the transition matrix elements and the basic structural properties (number of nodes N, average degree d) of the underlying graph. Moreover in the first chapter we discussed how the semicircular law evolves as the variance changes from the unit. The purpose of this section is to show that those results can be merged. A simulation has been run, building a random graph from Erdös-Rényi model (described in the appendix), fixing the total number of nodes and varying the probability of connection among them. Then we composed the transition from the connectivity one and the vertex degree one, and we analyzed its spectrum. What we expect from the theory is that as we increase the average connectivity the spectral gap increases. The results show indeed that as the probability of connection increase, the variance, thus the second-greatest eigenvalue, decrease.

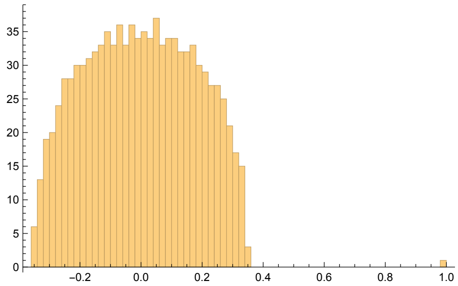
We must not be lead astray from the fact that the unitary eigenvalue may seem to void the semicircular law distribution. The existence of this eigenvalue is guaranteed by Perron-Frobenius theorem, indeed we have shown that Wigner law is the limiting distribution up to null set, as the singlet 1 is with respect to the interval upon which the semicircular law is defined.



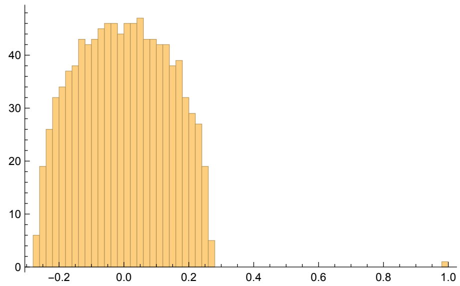
(A) $p = 0.01$



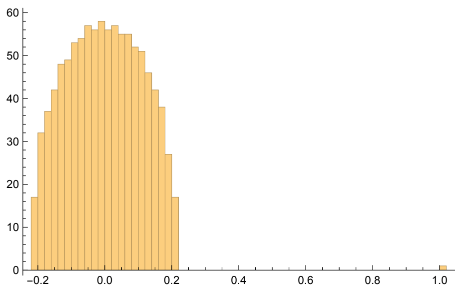
(B) $p = 0.02$



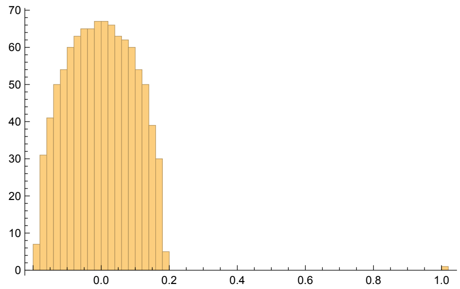
(C) $p = 0.03$



(D) $p = 0.05$



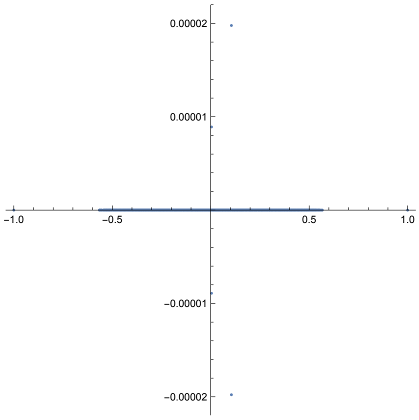
(E) $p = 0.075$



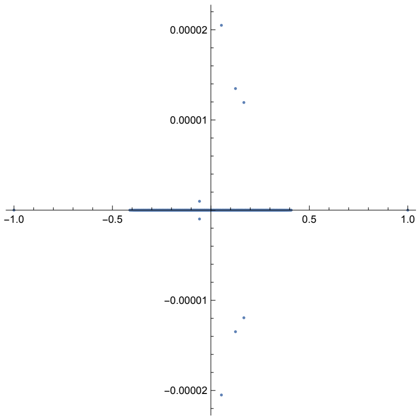
(F) $p = 0.1$

FIGURE 3.2 – Spectral occurrences of transition matrices from Erdös-Rényi graphs with 1000 nodes. Probability of connection p defines the connectivity by $\bar{d} = p(N-1)$.

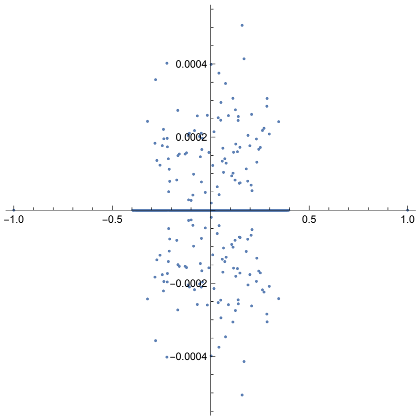
A similar procedure has been carried to analyze the validity of our proposition about the consistency of the elliptic law in case of perturbation. 1000 nodes graphs have been built according to p, and a small perturbation has been added to the stochastic matrix of the network. We show the cases in which random perturbations of order 10^{-2}, 10^{-1} and of the same magnitude have been applied to the matrix elements. The results seem to confirm what we expected, as we perturb the matrix, the distribution over the complex plan is deformed into an ellipsis from a dense line lying on the real axis (that generates semicircular distribution in the case the perturbation is removed). Moreover the spectral gap increases as we break symmetry in such a manner, this fact being a qualitatively explanation of what emerges from literature, *i.e.* that detailed balance violation reduces relaxation time [7].



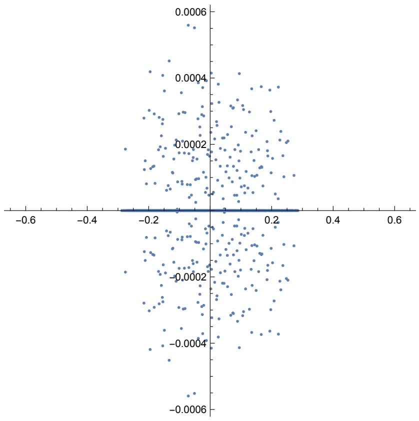
(A) $p = 0.01$, $W = 10^{-2}$



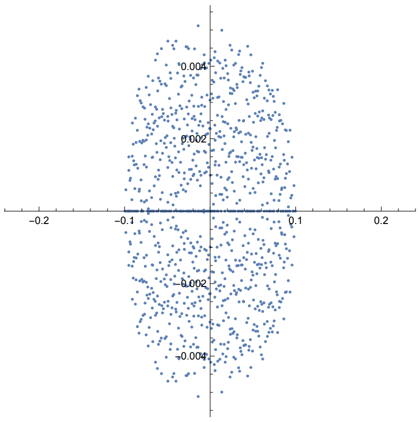
(B) $p = 0.02$, $W = 10^{-2}$



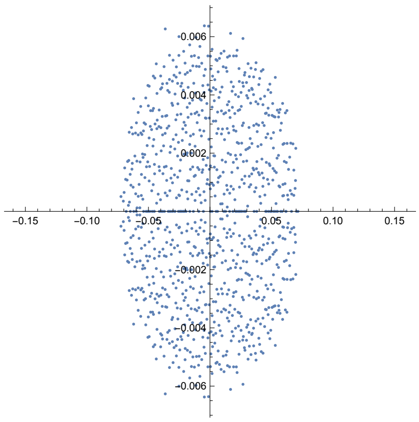
(C) $p = 0.01$, $W = 10^{-1}$



(D) $p = 0.02$, $W = 10^{-1}$



(E) $p = 0.01$, $W = 1$



(F) $p = 0.02$, $W = 1$

FIGURE 3.3 – Perturbations. On the horizontal axis the real part, whereas on the vertical axis the imaginary one. By we p define the connectivity of the graph, W stands for perturbation order with respect to the magnitude of the matrix elements.

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