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$\mathbb{Z}$-graded Lie superalgebras

Tesi di Laurea in Algebra

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‘A volte capita la vita che va
in mezzo ad un traffico algebrico’

Contents

Introduction

This thesis investigates the role of filtrations and gradings in the study of Lie (super)algebras.

In his paper [6] Kac indicates filtrations as the key ingredient used to solve the problem of classifying simple finite-dimensional primitive Lie superalgebras. In [9] he relates the problem of classifying simple infinite-dimensional linearly compact Lie superalgebras to the study and the classification of even transitive irreducible $\mathbb{Z}$-graded Lie superalgebras.

A $\mathbb{Z}$-graded Lie superalgebra is a Lie superalgebra $L = \oplus_{j \in \mathbb{Z}} L_j$ where the L_j's are $\mathbb{Z}_2$-graded subspaces such that $[L_i, L_j] \subset L_{i+j}$. Consequently, L_0 is a subalgebra of L and the L_j's are L_0-modules with respect to the adjoint action.

A (decreasing) filtration of a Lie (super)algebra L is a sequence of subspaces of L:

$$L = L_{-d} \supset L_{-d+1} \cdots \supset \cdots \supset L_0 \supset L_1 \supset \dots$$

such that $[L_i, L_j] \subset L_{i+j}$. The positive integer d is called the depth of the filtration. If L_0 is a maximal subalgebra of L of finite codimension and the filtration is transitive, i.e., for any non-zero $x \in \mathfrak{g}_k$ for $k \geq 0$, where $\mathfrak{g}_k = L_k / L_{k+1}$, there is $y \in \mathfrak{g}_{-1}$ such that $[x, y] \neq 0$, the filtration is called, after [12], a Weisfeiler filtration.

The associated $\mathbb{Z}$-graded Lie (super)algebra is of the form $\mathfrak{g} = GrL = \oplus_{k \geq -d} \mathfrak{g}_k$, and has the following properties:

(1) $\dim \mathfrak{g}_k < \infty$;

(2) $\mathfrak{g}_j = \mathfrak{g}_1^j$ for $j > 1$;

(3) if $a \in \mathfrak{g}_j$ with $j \geq 0$ and $[a, \mathfrak{g}_{-1}] = 0$, then $a = 0$;

(4) the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is irreducible.

A $\mathbb{Z}$-grading satisfying property (3) is called *transitive*, if it satisfies property (4) it is called *irreducible*. Besides, it is said of *finite growth* if $\dim \mathfrak{g}_n \leq P(n)$ for some polynomial P.

Weisfeiler's classification of such $\mathbb{Z}$-graded Lie algebras remained unpublished, but it is through these filtrations that Weisfeiler solved in a completely algebraic way the problem of classifying primitive linearly compact infinite-dimensional Lie algebras [12], a problem which had been first faced by Cartan [2] and then solved in [4] by the use of rather complicated methods from analysis.

Weisfeiler's idea leads Kac to the following classification theorem of infinite-dimensional Lie algebras, later generalized by Mathieu:

Theorem 0.1. *[5] Let L be a simple graded Lie algebra of finite growth. Assume that L is generated by its local part and that the grading is irreducible. Then L is isomorphic to one of the following:*

(i) a finite dimensional Lie algebra;

(ii) an affine Kac-Moody Lie algebra;

(iii) a Lie algebra of Cartan type.

Theorem 0.2. *[10] Let L be a simple graded Lie algebra of finite growth. Then L is isomorphic to one of the following Lie algebras:*

(i) a finite dimensional Lie algebra;

(ii) an affine Kac-Moody Lie algebra;

(iii) a Lie algebra of Cartan type;

(iv) a Virasoro-Witt Lie algebra.

The classification of simple finite-dimensional Lie superalgebras [6] is divided into two main parts, namely, that of classical and non-classical Lie superalgebras. A Lie superalgebra $L = L_{\bar{0}} + L_{\bar{1}}$ is called classical if it is simple and the representation of the Lie algebra $L_{\bar{0}}$ on $L_{\bar{1}}$ is completely reducible. In the case of such Lie superalgebras almost standard Lie algebras methods and techniques can be applied.

For the classification of the nonclassical simple Lie superalgebras L a Weisfeiler filtration is constructed and the classification of finite-dimensional $\mathbb{Z}$-graded Lie superalgebras with properties (1)–(4) is used. In the proof the methods developed in Kac's paper [5] for the classification of infinite-dimensional Lie algebras are applied and the Lie superalgebra L with filtration is reconstructed from the $\mathbb{Z}$-graded Lie superalgebra GrL. These methods rely on the connection between the properties of the gradings and the structure of the Lie (super)algebra. This thesis is focused on these properties to which Chapters 1, 2 and 3 are dedicated.

Chapter 4 is dedicated to the Lie superalgebras of vector fields $W(m,n)$ and $S(m,n)$. Here $W(m,n) = der\Lambda(m,n)$ where $\Lambda(m,n) = \mathbb{C}[x_1,\dots,x_m] \otimes \Lambda(n)$ is the Grassmann superalgebra and $S(m,n)$ is the derived algebra of $S'(m,n) = \{X \in W(m,n) \mid div(X) = 0\}$. If $n = 0$ these are infinite-dimensional Lie algebras, if $m = 0$ they are finite-dimensional Lie superalgebras.

If we set $\deg(x_i) = -\deg \frac{\partial}{\partial x_i} = 1$ for every even variable x_i, and $\deg(\xi_j) = -\deg \frac{\partial}{\partial \xi_j} = 1$ for every odd variable ξ_j, then we get a grading of $W(m,n)$ and $S(m,n)$, called the *principal grading*, satisfying properties $(1)-(4)$. The properties of this grading can be used to prove the simplicity of the Lie superalgebras $W(m,n)$ and $S(m,n)$ (see Sections 4.1.2 and 4.2.2).

We then classify, up to isomorphims, the strongly symmetric gradings of length 3 and 5 of $W(m,n)$ and $S(m,n)$, and give a detailed description of them. A $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$ is said symmetric if $\mathfrak{g} = \oplus_{i=-k}^{k}\mathfrak{g}_i$ for some $k < \infty$. If, in addition, the grading is transitive, generated by its local part and $\mathfrak{g}_{-i}$ and $\mathfrak{g}_i$ are isomorphic vector spaces, then the grading is called *strongly symmetric*. We say that a strongly symmetric grading has length three (resp. five) if $k = 1$ (resp. $k = 2$).

The study of such gradings is motivated by [11], where a correspondence between strongly-symmetric graded Lie superalgebras of length three and five and triple systems appearing in three-dimensional supersymmetric conformal field theories is established.

We prove the following results:

Theorem 0.3.

1. *If* $(m, n) \neq (0, 2), (1, 1)$ *the Lie superalgebra* $W(m, n)$ *has no strongly symmetric* $\mathbb{Z}-$*gradings of length three.*

2. *A complete list, up to isomorphisms, of strongly symmetric* $\mathbb{Z}-$*gradings of length three of the Lie superalgebras* $W(0, 2)$ *and* $W(1, 1)$ *is the following:*

   (a) $(|1, 1)$

   (b) $(|0, 1)$

   (c) $(0|1)$

Theorem 0.4. *A complete list, up to isomorphisms, of strongly symmetric* $\mathbb{Z}-$*gradings of length five of the Lie superalgebra* $W(m, n)$ *is the following:*

1. $(|1, 2)$ *for* $m = 0$ *and* $n = 2$

2. $(0, ..., 0|1, -1, 0, ..., 0)$

Theorem 0.5.

1. *If* $(m, n) \neq (1, 2)$ *then the Lie superalgebra* $S(m, n)$ *has no strongly symmetric* $\mathbb{Z}$*-grading of length three.*

2. *A complete list, up to isomorphisms, of strongly symmetric* $\mathbb{Z}-$*gradings of length three of the Lie superalgebra* $S(1, 2)$ *is the following:*

   (a) $(0|1, 1)$

   (b) $(0|1, 0)$

Theorem 0.6. *A complete list, up to isomorphisms, of strongly symmetric $\mathbb{Z}-$gradings of length five of the Lie superalgebra of $S(m,n)$ is the following:*

1. $(0,...,0|1,-1,0,...,0)$

2. $(0|2,1)$ *for* $m=1$ *and* $n=2$

Throughout this thesis the ground field is $\mathbb{C}$.

Introduzione

In questa tesi viene analizzato il ruolo di filtrazioni e graduazioni nello studio di (super)algebre di Lie.

Nel suo articolo [6] Kac indica le filtrazioni come ingrediente chiave utilizzato per risolvere il problema di classificare le superalgebre di Lie semplici, di dimensione finita, primitive. In [9] mette in relazione il problema di classificare le superalgebre di Lie semplici, di dimensione infinita, linearmente compatte, allo studio e classificazione delle superalgebre di Lie $\mathbb{Z}$-graduate even, transitive, irriducibili.

Una superalgebra di Lie $\mathbb{Z}$-graduata è una superalgebra di Lie $L = \oplus_{j\in\mathbb{Z}} L_j$ dove gli L_j sono sottospazi $\mathbb{Z}_2$-graduati tali che $[L_i, L_j] \subset L_{i+j}$. Ne segue che L_0 è una sottoalgebra di L e che gli L_j sono L_0-moduli rispetto all'azione aggiunta.

Una filtrazione (decrescente) di una (super)algebra di Lie L è una sequenza di sottospazi di L:

$$L = L_{-d} \supset L_{-d+1} \cdots \supset \cdots \supset L_0 \supset L_1 \supset \ldots$$

tale che $[L_i, L_j] \subset L_{i+j}$. L'intero positivo d è chiamato profondità della filtrazione. Se L_0 è una sottoalgebra massimale di L di codimensione finita e la filtrazione è transitiva, i.e., per ogni $x \in \mathfrak{g}_k$ non nullo, per $k \geq 0$, dove $\mathfrak{g}_k = L_k/L_{k+1}$, esiste $y \in \mathfrak{g}_{-1}$ tale che $[x, y] \neq 0$, la filtrazione è chiamata, seguendo [12], una filtrazione di Weisfeiler .

La (super)algebra di Lie $\mathbb{Z}$-graduata associata è della forma $\mathfrak{g} = GrL = \oplus_{k\geq -d}\mathfrak{g}_k$, e ha le seguenti proprietà:

(1) $\dim \mathfrak{g}_k < \infty$;

(2) $\mathfrak{g}_j = \mathfrak{g}_1^j$ per $j > 1$;

(3) se $a \in \mathfrak{g}_j$ con $j \geq 0$ e $[a, \mathfrak{g}_{-1}] = 0$, allora $a = 0$;

(4) la rappresentazione di $\mathfrak{g}_0$ su $\mathfrak{g}_{-1}$ è irriducibile.

Una $\mathbb{Z}$-graduazione che soddisfa la proprietà (3) è chiamata *transitiva*, se soddisfa la proprietà (4) è chiamata *irriducibile*. Inoltre, si dice che ha *crescita finita* se $\dim \mathfrak{g}_n \leq P(n)$ per qualche polinomio P.

La classificazione di Weisfeiler di tali algebre di Lie $\mathbb{Z}$-graduate rimase non pubblicata, ma fu grazie a queste filtrazioni che Weisfeiler risolse in un modo completamente algebrico il problema di classificare le algebre di Lie primitive, linearmente compatte, di dimensione infinita [12], un problema che venne prima affrontato da Cartan [2] e poi risolto in [4] con l'utilizzo di complicati metodi dell'analisi.

L'idea di Weisfeiler portò Kac al seguente teorema di classificazione di algebre di Lie infinito-dimensionali , in seguito generalizzato da Mathieu:

Teorema 0.1. *[5] Sia L un'algebra di Lie semplice graduata di crescita finita. Assumiamo che L sia generata dalla sua parte locale e che la graduazione sia irriducibile. Allora L è isomorfa a una delle seguenti:*

(i) un'algebra di Lie finito-dimensionale;

(ii) un'algebra di Lie di Kac-Moody di tipo affine;

(iii) un'algebra di Lie di tipo Cartan.

Teorema 0.2. *[10] Sia L un'algebra di Lie semplice graduata di crescita finita. Allora L è isomorfa a una delle seguenti algebre di Lie:*

(i) un'algebra di Lie finito-dimensionale;

(ii) un'algebra di Lie di Kac-Moody di tipo affine;

(iii) un'algebra di Lie di tipo Cartan;

(iv) un'algebra di Lie Virasoro-Witt.

La classificazione delle superalgebre di Lie semplici finito-dimensionali [6] è divisa in due parti principali, ossia, quella delle superalgebre di Lie classiche e non classiche. Una superalgebra di Lie $L = L_{\bar{0}} + L_{\bar{1}}$ è chiamata classica se è semplice e la rappresentazione dell'algebra di Lie $L_{\bar{0}}$ su $L_{\bar{1}}$ è completamente riducibile. Nel caso di tali superalgebre di Lie vengono applicati metodi e tecniche simili alle algebre di Lie.

Per la classificazione delle superalgebre di Lie non classiche, semplici L, si costruisce una filtrazione di Weisfeiler e si utilizza la classificazione di superalgebre di Lie $\mathbb{Z}$-graduate finito-dimensionali con le proprietà (1)–(4). Nella dimostrazione, vengono applicate le tecniche utilizzate nell'articolo di Kac [5] per la classificazione di algebre di Lie infinito-dimensionali e la superalgebra di Lie L con filtrazione è ricostruita dalla superalgebra di Lie $\mathbb{Z}$-graduata GrL. Queste tecniche si basano sul legame tra le proprietà delle graduazioni e la struttura della (super)algebra di Lie. Questa tesi studia queste proprietà, a cui sono dedicati i capitoli 1, 2 e 3.

Il capitolo 4 è dedicato alle superalgebre di Lie di campi vettoriali $W(m,n)$ e $S(m,n)$. $W(m,n) = der\Lambda(m,n)$ dove $\Lambda(m,n) = \mathbb{C}[x_1,\ldots,x_m] \otimes \Lambda(n)$ è la superalgebra di Grassmann e $S(m,n)$ è l'algebra derivata di $S'(m,n) = \{X \in W(m,n) \mid div(X) = 0\}$. Se $n = 0$ queste sono algebre di Lie infinito-dimensionali, se $m = 0$ sono superalgebre di Lie finito-dimensionali.

Se poniamo $\deg(x_i) = -\deg\frac{\partial}{\partial x_i} = 1$ per ogni variabile pari x_i, e $\deg(\xi_j) = -\deg\frac{\partial}{\partial \xi_j} = 1$ per ogni variabile dispari ξ_j, allora otteniamo una graduazione di $W(m,n)$ e $S(m,n)$, chiamata *graduazione principale*, che soddisfa le proprietà $(1)-(4)$. Le proprietà di questa graduazione possono essere usate per dimostrare la semplicità delle superalgebre di Lie $W(m,n)$ e $S(m,n)$ (Sezioni 4.1.2 e 4.2.2).

In seguito classifichiamo, a meno di isomorfismo, le graduazioni fortemente simmetriche di lunghezza 3 e 5 di $W(m,n)$ e $S(m,n)$, e diamo una loro descrizione. Una $\mathbb{Z}$-graduazione di una superalgebra di Lie $\mathfrak{g}$ è detta simmetrica se $\mathfrak{g} = \oplus_{i=-k}^{k}\mathfrak{g}_i$ per qualche $k < \infty$. Se, inoltre, la graduazione è transitiva, generata dalla parte locale e $\mathfrak{g}_{-i}$ and $\mathfrak{g}_i$ sono spazi vettoriali isomorfi, allora la graduazione è chiamata *fortemente simmetrica.* Diciamo che una graduazione fortemente simmetrica ha lunghezza tre (risp. cinque) se $k = 1$ (risp. $k = 2$).

Lo studio di tali graduazioni è motivato da [11], dove è stabilita una corrispondenza tra superalgebre di Lie con graduazione fortemente-simmetrica di lunghezza tre e cinque e sistemi tripli che intervengono nelle teorie di campo conforme supersimmetrico tridimensionali.

Otteniamo i seguenti risultati:

Teorema 0.3. 1. *Se* $(m, n) \neq (0, 2), (1, 1)$ *allora la superalgebra di Lie* $W(m, n)$ *non ha* $\mathbb{Z}$*-graduazioni fortemente simmetriche di lunghezza tre.*

2. *Una lista completa, a meno di isomorfismi, di* $\mathbb{Z}$*-graduazioni fortemente simmetriche di lunghezza tre delle superalgebre di Lie* $W(0, 2)$ *e* $W(1, 1)$ *è la seguente:*

   (a) $(|1, 1)$

   (b) $(|0, 1)$

   (c) $(0|1)$

Teorema 0.4. *Una lista completa, a meno di isomorfismi, di* $\mathbb{Z}-$*graduazioni fortemente simmetriche di lunghezza cinque della superalgebra di Lie* $W(m, n)$ *è la seguente:*

1. $(|1, 2)$ *per* $m = 0$ *e* $n = 2$

2. $(0, ..., 0|1, -1, 0, ..., 0)$

Teorema 0.5. 1. *Se* $(m, n) \neq (1, 2)$ *allora la superalgebra di Lie* $S(m, n)$ *non ha* $\mathbb{Z}$*-graduazioni fortemente simmetriche di lunghezza tre.*

2. *Una lista completa, a meno di isomorfismi, di* $\mathbb{Z}-$*graduazioni fortemente simmetriche di lunghezza tre della superalgebra di Lie* $S(1, 2)$ *è la seguente:*

(a) $(0|1,1)$

(b) $(0|1,0)$

Teorema 0.6. *Una lista completa, a meno di isomorfismi, di* $\mathbb{Z}-$*graduazioni fortemente simmetriche di lunghezza cinque della superalgebra di Lie* $S(m,n)$ *è la seguente:*

1. $(0,...,0|1,-1,0,...,0)$

2. $(0|2,1)$ *per* $m=1$ *e* $n=2$

In questa tesi il campo utilizzato è $\mathbb{C}$.

Chapter 0

Preliminaries on representations of semisimple Lie algebras

In this chapter we recall some basic facts about the irreducible representations of a semisimple Lie algebra.

0.1 Highest and lowest weights

We consider a semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Moreover, we consider a finite-dimensional representation ρ of $\mathfrak{g}$ on V or, equivalently a $\mathfrak{g}-$module V. If λ is an element of $\mathfrak{h}^*$ we set:

$$V_\lambda = \{v \in V \mid \rho(h)(v) = \lambda(h)v \ \ \forall h \in \mathfrak{h}\}$$

Definition 0.1. If $V_\lambda \neq 0$ we call λ a weight of ρ.

Definition 0.2. An element $v_\lambda \in V_\lambda$ is called a weight vector if $v_\lambda \neq 0$.

We denote by $\mathcal{L}_\rho$ the set of all the weights of ρ. It follows that $V = \oplus_{\lambda \in \mathcal{L}_\rho} V_\lambda$.
If the representation is the adjoint representation of $\mathfrak{g}$, then a weight α is

called a root of $\mathfrak{g}$. It follows that $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha$, with $\mathfrak{g}_0 = \mathfrak{h}$, and if $\alpha \neq 0$ we have $dim(\mathfrak{g}_\alpha) = 1$.

Definition 0.3. A nonzero vector $e_\alpha \in \mathfrak{g}_\alpha$ is said root vector.

Moreover if $[\mathfrak{g}_\alpha, V_\lambda] \neq 0$ and $\lambda + \alpha \in \mathcal{L}_\rho$ then $[\mathfrak{g}_\alpha, V_\lambda] \subset V_{\lambda+\alpha}$, on the other hand $[\mathfrak{g}_\alpha, V_\lambda] = 0$ if $\lambda + \alpha \notin \mathcal{L}_\rho$.
We know that the Killing form $(a, b) = tr(ad(a)ad(b))$ and its restriction to $\mathfrak{h}$ are nondegenerate. Therefore if $\alpha \neq 0$ we have $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha \neq 0$, where $h_\alpha \in \mathfrak{h}$ is the unique vector such that $\alpha(h) = (h_\alpha, h)$.
We denote by Δ the set of nonzero roots of $\mathfrak{g}$.

Definition 0.4. A subset Σ of Δ is called base if:

1. Σ is a basis of $\mathfrak{h}^*$
2. every root can be written as $\sum k_i\alpha_i$, with k_i all nonnegative or all nonpositive integers

We call positive (resp. negative) the roots for which all k_i are nonnegative (resp. nonpositive) and we denote the set of positive (resp. negative) roots by Δ^+ (resp. Δ^-). Moreover we call simple roots the elements of $\Sigma = \{\alpha_1, ..., \alpha_s\}$. We have that $\Delta = \Delta^+ \cup -\Delta^+$.
We denote by $\mathfrak{h}_0^*$ the linear span of Δ over $\mathbb{Z}$, it follows that the Killing form is positively definite on $\mathfrak{h}_0^*$ and $\mathcal{L}_\rho \subset \mathfrak{h}_0^*$.
If $\alpha \in \Delta$ and $\lambda \in \mathcal{L}_\rho$, the elements $\lambda + s\alpha$ are weights if $-p \leq s \leq q$ where p and q are non negative integers and $p - q = 2(\lambda, \alpha)/(\alpha, \alpha)$. We call numerical marks of $\lambda \in \mathfrak{h}^*$ the numbers $2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$. If $\lambda \in \mathcal{L}_\rho$, its numerical marks are integers.

Definition 0.5. We call $\lambda \in \mathfrak{h}_0^*$ dominant if its numerical marks are nonnegative.

Definition 0.6. A weight $\Lambda \in \mathcal{L}_\rho$ is said highest weight of ρ if $\Lambda + \alpha \notin \mathcal{L}_\rho$ for every $\alpha \in \Delta^+$.

Definition 0.7. A weight $M \in \mathcal{L}_\rho$ is said lowest weight of ρ if $M - \alpha \notin \mathcal{L}_\rho$ for every $\alpha \in \Delta^+$.

Definition 0.8. A nonzero vector $v \in V_\Lambda$, where Λ is the highest weight, is said highest weight vector of ρ.

Definition 0.9. A nonzero vector $v \in V_M$, where M is the lowest weight, is said lowest vector of ρ.

The highest and lowest vectors are unique up to scalars.
It is known that if ρ is an irreducible finite-dimensional representation of a semisimple algebra $\mathfrak{g}$ then Λ is dominant and for any dominant linear function Λ there is a unique finite-dimensional representation with highest weight Λ, up to isomorphisms.
Every representation ρ of $\mathfrak{g}$ in V induces a representation ρ^* of $\mathfrak{g}$ in V^*, ρ and ρ^* said contragredient, moreover $\lambda \in \mathcal{L}_\rho$ if and only if $-\lambda \in \mathcal{L}_{\rho^*}$. It follows that if Λ is the highest weight of ρ then $-\Lambda$ is the lowest weight of ρ^*.
If $\mathfrak{g}$ is simple, then its adjoint representation is irreducible and its highest weight is the highest root.

Definition 0.10. A Lie algebra $\mathfrak{g} \neq 0$ is said reductive if $Rad(\mathfrak{g}) = Z(\mathfrak{g})$.

Theorem 0.1.
1. *Let $\mathfrak{g}$ be a reductive Lie algebra, it follows that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$ and $[\mathfrak{g}, \mathfrak{g}]$ is either semisimple or* 0.
2. *If a nonzero Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, where V is finite-dimensional, acts irreducibly on V, then $\mathfrak{g}$ is reductive and $dim(Z(\mathfrak{g})) \leq 1$*

Proof. 1) If $\mathfrak{g}$ is abelian, the thesis is obvious. Let us consider $\mathfrak{g}$ a non abelian reductive Lie algebra. Then $\mathfrak{g}' = \mathfrak{g}/Z(\mathfrak{g})$ is semisimple, it follows, by Weyl's theorem, that $ad\mathfrak{g} \cong \mathfrak{g}'$ acts completely reducibly on $\mathfrak{g}$. From the semisimplicity of $\mathfrak{g}/Z(\mathfrak{g})$, it follows that $[\mathfrak{g}, \mathfrak{g}]/Z(\mathfrak{g}) \cong [\mathfrak{g}/Z(\mathfrak{g}), \mathfrak{g}/Z(\mathfrak{g})] = \mathfrak{g}/Z(\mathfrak{g})$, i.e. for all $x \in \mathfrak{g}$ there exist $y, z \in \mathfrak{g}$ such that $x = [y, z] + c$ for some $c \in Z(\mathfrak{g})$, that is $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + Z(\mathfrak{g})$. Since $Z(\mathfrak{g})$ is an $ad\mathfrak{g}-$submodule of $\mathfrak{g}$, then

$\mathfrak{g} = M \oplus Z(\mathfrak{g})$, where M is an ideal of $\mathfrak{g}$. We have that $[\mathfrak{g}, \mathfrak{g}] \subset [M, M] \subset M$, so $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$.
2) We denote by S the radical of $\mathfrak{g}$. By Lie's theorem the elements of S have a common eigenvector in V, hence $s.v = \lambda(s)v \ \forall s \in S$. Let $x \in \mathfrak{g}$, then $[s, x] \in S$, so $s.(x.v) = \lambda(s)x.v + \lambda([s, x])v$. But $\mathfrak{g}$ acts irreducibly on V, so every element of V is obtained acting by elements of $\mathfrak{g}$ on V and taking linear combinations. It follows that every matrix of $s \in S$ is triangular, with respect to a suitable basis, with only $\lambda(s)$ on the diagonal. But the trace of the elements of $[S, \mathfrak{g}]$ is zero, so λ is null on $[S, \mathfrak{g}]$. Then $s \in S$ acts diagonally as $\lambda(s)$ on V. We have that $S = Z(\mathfrak{g})$ and $dim(S) \leq 1$. □

0.2 Dynkin Diagrams

We know that a semisimple Lie algebra can be described by a Dynkin diagram. If we fix a set of simple roots $\Sigma = \{\alpha_1, ..., \alpha_s\}$, the numbers $< \alpha_i, \alpha_j >= 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ are non positive integers. The Dynkin diagram is composed by s nodes, which represent the simple roots, where the ith and the jth nodes are linked by $< \alpha_i, \alpha_j >< \alpha_j, \alpha_i >$ edges with an arrow pointing to α_i if $| < \alpha_i, \alpha_i > | < | < \alpha_j, \alpha_j > |$, i.e. if α_i is shorter than α_j. If ρ is an irreducible representation of $\mathfrak{g}$, it can be represented by a Dynkin diagram endowed with the numerical marks $2(\Lambda, \alpha_i)/(\alpha_i, \alpha_i)$ of the highest weight Λ of ρ, written by the corresponding nodes.

Chapter 1

Lie superalgebras

In this chapter we introduce some basic notions about superalgebras, Lie superalgebras and some examples.

1.1 Superalgebras

Definition 1.1. (M-grading) Let A be an algebra and M an abelian group, we define an M-grading on A a decomposition of A as $A = \oplus_{\alpha \in M} A_\alpha$, where the A_α's are subspaces of A such that $A_\alpha A_\beta \subset A_{\alpha+\beta}$.

We call an algebra A endowed with a grading as in Definition 1.1 *M-graded* and an element $a \in A_\alpha$ homogeneous of degree α. Moreover a subspace of A is called M-graded if $B = \oplus_{\alpha \in M}(B \cap A_\alpha)$. All subalgebras and ideals of an $M-$graded algebra are meant to be $M-$graded.

Definition 1.2. (Homomorphism) A homomorphism ϕ of two M-graded algebras A and A' is a homomorphism which preserves the grading, i.e. $\phi(A_\alpha) \subset A'_{\varphi(\alpha)}$, with φ an automorphism of M.

We are interested in the case $M = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$.

Definition 1.3. (Superalgebra) A superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$.

We call the elements of $A_{\bar{0}}$ even and the elements of $A_{\bar{1}}$ odd. If $a \in A_\alpha$ we will say that α is the parity of a and we will denote it by $p(a)$.

Definition 1.4. (Tensor product) Let A and B be superalgebras. We define $A \otimes B$ as the superalgebra with underlying space the tensor product of the spaces A and B and induced $\mathbb{Z}_2$-grading. The product is defined as follows:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{p(a_2)p(b_1)} a_1 a_2 \otimes b_1 b_2 \quad a_i \in A, b_i \in B$$

Definition 1.5. (Bracket) On a superalgebra we define the following bracket:

$$[a, b] = ab - (-1)^{p(a)p(b)} ba \tag{1.1}$$

We call abelian a superalgebra A in which $[a, b] = 0$ for all $a, b \in A$.

The definition of associativity is the same as for algebras.

Remark 1. If A is an associative superalgebra, then the following identity holds:

$$[a, bc] = [a, b]c + (-1)^{p(a)p(b)} b[a, c] \tag{1.2}$$

Indeed, the left hand side is:

$$[a, bc] = a(bc) - (-1)^{p(a)p(bc)}(bc)a =$$
$$a(bc) - (-1)^{p(a)((p(b)+p(c))}(bc)a$$

The right hand side is:

$$(ab - (-1)^{p(a)p(b)} ba)c + (-1)^{p(a)p(b)} b(ac - (-1)^{p(a)p(c)} ca) =$$
$$(ab)c - (-1)^{p(a)p(b)}(ba)c + (-1)^{p(a)p(b)} b(ac) - (-1)^{p(a)((p(b)+(p(c))} b(ca) =$$
$$a(bc) - (-1)^{p(a)((p(b)+p(c))}(bc)a$$

Example 1. If M is an abelian group and $V = \oplus_{\alpha \in M} V_\alpha$ an M-graded space, we can consider $End(V)$ with the induced M-grading, i.e., $End(V) = \oplus_{\alpha \in M} End_\alpha V$ where:

$$End_\alpha V = \{a \in End V \quad | \quad a(V_s) \subseteq V_{s+\alpha}\}$$

If $M = \mathbb{Z}_2$ we have $End V = End_{\bar{0}} V \oplus End_{\bar{1}} V$

Example 2. If $\Lambda(n)$ is the Grassmann algebra in the variables $\xi_1, \dots, \xi_n$, we define $p(\xi_i) = \bar{1}$, for every $i \in \{1, ..., n\}$. $\Lambda(n)$ with this grading is said the Grassmann superalgebra.

1.2 Lie superalgebras

Definition 1.6. (Lie superalgebra) A superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with bracket $[,]$ is called a Lie superalgebra if the following conditions hold:

$$[a, b] = -(-1)^{p(a)p(b)}[b, a] \quad \textit{anticommutativity}$$
$$[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]] \quad \textit{Jacobi} \quad \textit{identity}$$

Remark 2. $\mathfrak{g}_{\bar{0}}$ is a Lie algebra. Besides, $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module with the action given by the bracket and the following map is a homomorphism of $\mathfrak{g}_{\bar{0}}$-modules:

$$\varphi : S^2\mathfrak{g}_{\bar{1}} \longrightarrow \mathfrak{g}_{\bar{0}}$$
$$(g_1, g_2) \longmapsto [g_1, g_2]$$

On the other side, a Lie superalgebra is completely determined by the Lie algebra $\mathfrak{g}_{\bar{0}}$, the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ and a map φ such that for $a, b, c \in \mathfrak{g}_{\bar{1}}$:

$$\varphi(a, b)c + \varphi(b, c)a + \varphi(c, a)b = 0$$

Example 3. Bracket (1.1) defines on an associative superalgebra A a Lie superalgebra structure that we will indicate by A_L. Indeed anticommutativity and the Jacobi identity follow from the definition of the bracket and associativity: for $a, b \in A$,

$$\begin{aligned}[a, b] =& ab - (-1)^{p(a)p(b)}ba = -(-1)^{p(a)p(b)}(-(-1)^{p(a)p(b)}ab + ba) = \\ & -(-1)^{p(a)p(b)}[b, a]\end{aligned}$$

Moreover:

$$[a,[b,c]] = [a, bc - (-1)^{p(b)p(c)}cb] =$$
$$a(bc) - (-1)^{p(b)p(c)}a(cb) - (-1)^{(p(b)+p(c))p(a)}(bc)a+$$
$$(-1)^{(p(b)+p(c))p(a)}(-1)^{p(b)p(c)}(cb)a =$$
$$a(bc) - (-1)^{p(b)p(c)}a(cb) - (-1)^{(p(b)+p(c))p(a)}(bc)a+$$
$$(-1)^{p(b)p(a)+p(c)(p(a)+p(b))}(cb)a$$

and:

$$[[a,b],c] + (-1)^{p(a)p(b)}[b,[a,c]] =$$
$$[ab - (-1)^{p(a)p(b)}ba, c] + (-1)^{p(a)p(b)}[b, ac - (-1)^{p(a)p(c)}ca] =$$
$$(ab)c - (-1)^{p(a)p(b)}(ba)c - (-1)^{p(c)(p(a)+p(b))}c(ab)+$$
$$(-1)^{p(a)p(b)}(-1)^{p(c)(p(a)+p(b))}c(ba)+$$
$$(-1)^{p(a)p(b)}b(ac) - (-1)^{p(a)p(b)}(-1)^{p(a)p(c)}b(ca)+$$
$$-(-1)^{p(a)p(b)}(-1)^{p(b)(p(a)+p(c))}(ac)b+$$
$$(-1)^{p(a)p(b)}(-1)^{p(a)p(c)}(-1)^{p(b)(p(a)+p(c))}(ca)b =$$
$$a(bc) - (-1)^{p(b)p(c)}a(cb) - (-1)^{(p(b)+p(c))p(a)}(bc)a+$$
$$(-1)^{p(b)p(a)+p(c)(p(a)+p(b))}(cb)a$$

1.3 Derivations

Definition 1.7. Let A be a superalgebra. We call $D \in End_s A$ a derivation of A of degree s, where $s \in \mathbb{Z}_2$, if:

$$D(ab) = D(a)b + (-1)^{sp(a)}aD(b) \quad (Leibniz\ \ rule)$$

We call $Der_s A \subset End_s A$ the space of derivations of degree s on A and $Der A = Der_{\bar{0}} A \oplus Der_{\bar{1}} A$. Notice that $Der A$ is not an associative subalgebra of $End A$, but it is a Lie subalgebra of $(End A)_L$.

Remark 3. Let us consider a Lie superalgebra $\mathfrak{g}$. Then the map:

$$ad_a(b) = [a,b] \quad for \quad a,b \in \mathfrak{g}$$

is a derivation, by the Jacobi identity. Derivations of this form are said inner derivations.

Remark 4. Inner derivations are an ideal of $\mathfrak{g}$. Indeed $[D, ad_a] = ad_{D(a)} \quad \forall D \in Der\mathfrak{g}, \forall a \in \mathfrak{g}$.

Example 4. Let us consider the Grassmann superalgebra $\Lambda(n) = \Lambda_{\bar{0}}(n) \oplus \Lambda_{\bar{1}}(n)$. Our purpose is to describe $Der\Lambda(n)$. We see $\Lambda(n)$ as the quotient $\tilde{\Lambda}(n)/I$ where $\tilde{\Lambda}(n)$ is the free associative superalgebra generated by $\xi_1, ..., \xi_n$ and I is the ideal generated by the relations $\xi_i\xi_j + \xi_j\xi_i$. The grading is given by setting $p(\xi_i) = \bar{1}$, $\forall i = 1, ..., n$. If $P, Q \in \tilde{\Lambda}(n)$ are homogeneous elements, then $[P, Q] = PQ - (-1)^{p(P)p(Q)}QP \in I$. Therefore let D be a derivation of $\tilde{\Lambda}(n)$ of degree s. We have:

$$D(\xi_i\xi_j + \xi_j\xi_i) = D(\xi_i)\xi_j + (-1)^s\xi_i D(\xi_j) + D(\xi_j)\xi_i + (-1)^s\xi_j D(\xi_i) =$$
$$(D(\xi_i)\xi_j + (-1)^s\xi_j D(\xi_i)) + (D(\xi_j)\xi_i + (-1)^s\xi_i D(\xi_j)) \in I$$

So $D(I) \subset I$. Notice that, by the Leibniz rule, $D(1) = 0$. Besides, by the Leibniz rule, a derivation D of $\tilde{\Lambda}(n)$ is completely determined by the values $D(\xi_i)$, therefore, if we choose $P_1, ..., P_n \in \Lambda(n)$, there is only one derivation D of $\Lambda(n)$ such that $D(\xi_i) = P_i$. Let us consider the relation $\frac{\partial}{\partial\xi_i}(\xi_j) = \delta_{ij}$, this defines a derivation on $\Lambda(n)$. So we can now write a derivation $D \in \Lambda(n)$ such that $D(\xi_i) = P_i$ in the following way:

$$D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial\xi_i}$$

1.4 The superalgebra $l(V)$, supertrace and bilinear forms

We consider a $\mathbb{Z}_2$-graded space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We already noticed that $EndV$, with the induced $\mathbb{Z}_2$-grading, is an associative superalgebra and $(EndV)_L$ is a Lie superalgebra. We shall denote $(EndV)_L$ by $l(V) = l(V)_{\bar{0}} \oplus l(V)_{\bar{1}}$ or $l(m, n)$, if $m = dim(V_{\bar{0}})$ and $n = dim(V_{\bar{1}})$.

Let us consider a basis $\{e_1, ..., e_m, e_{m+1}, ..., e_{m+n}\}$ of V where $\{e_1, ..., e_m\}$ is a basis of $V_{\bar{0}}$ and $\{e_{m+1}, ..., e_{m+n}\}$ a basis of $V_{\bar{1}}$. We call such a basis homogeneous. With respect to this basis every element of $l(V)$ has matrix of the form:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with α a $m \times m$ matrix, β a $m \times n$ matrix, γ a $n \times m$ matrix, δ a $n \times n$ matrix. An element of $l(V)_{\bar{0}}$ has a matrix of the form $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$, and an element of $l(V)_{\bar{1}}$ has a matrix of the form $\left(\begin{smallmatrix} 0 & \beta \\ \gamma & 0 \end{smallmatrix}\right)$.

Definition 1.8. (Supertrace) Let us consider an element $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ of $l(m, n)$. The supertrace of A is:

$$str(A) = tr\alpha - tr\delta$$

Since the supertrace does not depend on the choice of the homogeneous basis, we can consider the supertrace of A in any homogeneous basis.
Let us now introduce some definitions about bilinear forms. In the following $V = V_{\bar{0}} \oplus V_{\bar{1}}$ will be a $\mathbb{Z}_2$-graded space and f a bilinear form on V.

Definition 1.9. A bilinear form f on V is said *consistent* if $f(a, b) = 0 \quad \forall a \in V_{\bar{0}}, \forall b \in V_{\bar{1}}$.

Definition 1.10. A bilinear form f on V is said *supersymmetric* if $f(a, b) = (-1)^{p(a)p(b)} f(b, a)$.

Definition 1.11. A bilinear form f on a Lie superalgebra $\mathfrak{g}$ is said *invariant* if $f([a, b], c) = f(a, [b, c])$.

Proposition 1.1. *The bilinear form $str(ab)$ is consistent, supersymmetric and invariant on $l(V)$. Moreover:*

$$str([a, b]) = 0 \quad \forall a, b \in l(V).$$

Proof. Let us set $str(ab) = (a, b)$. We fix a homogeneous basis of $l(V)$. We start showing consistency: we consider $a = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in l(V)_{\bar{0}}$ and $b = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \in l(V)_{\bar{1}}$. Then $ab = \begin{pmatrix} 0 & \alpha\beta \\ \delta\gamma & 0 \end{pmatrix}$ so $(a, b) = 0$.
We now prove supersymmetry. Let us consider $a = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and $b = \begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & \tilde{\delta} \end{pmatrix}$, $ab = \begin{pmatrix} \alpha\tilde{\alpha} & 0 \\ 0 & \delta\tilde{\delta} \end{pmatrix}$, then $(a, b) = tr(\alpha\tilde{\alpha}) - tr(\delta\tilde{\delta}) = tr(\tilde{\alpha}\alpha) - tr(\tilde{\delta}\delta)$. If $a \in l(V)_{\bar{0}}$ and $b \in l(V)_{\bar{1}}$ supersymmetry follows from consistency. Finally we analyze the case $a, b \in l(V)_{\bar{1}}$, i.e. $a = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix}$ with respect to a homogeneous basis. We have $(a, b) = tr(\alpha\delta) - tr(\beta\gamma)$ and $(b, a) = tr(\gamma\beta) - tr(\delta\alpha)$, then $(a, b) = -(b, a)$. The property $str([a, b]) = 0 \quad \forall a, b \in l(V)$ is equivalent to supersymmetry.
It remains to show invariance, using (1.2) we get:

$$0 = str([b, ac]) = ([b, a], c) + (-1)^{p(a)p(b)}(a, [b, c])$$

therefore

$$([b, a], c) = -(-1)^{p(a)p(b)}(a, [b, c])$$

We conclude $-(-1)^{p(a)p(b)}([b, a], c) = ([a, b], c) = (a, [b, c])$ □

1.5 Classical Lie superalgebras

Definition 1.12. (Classical Lie superalgebra) Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite dimensional Lie superalgebra, $\mathfrak{g}$ is said classical if it is simple and $\mathfrak{g}_{\bar{1}}$ is a completely reducible $\mathfrak{g}_{\bar{0}}$-module.

1.5.1 A(m,n)

We define:

$$sl(m, n) = \{a \in l(m, n) \quad | \quad str(a) = 0\}$$

This is an ideal of $l(m, n)$ of codimension 1, since $\forall a, b \in l(m, n)$, $str[a, b] = 0$. If $m = n$ the set of elements of the form λI_{2n} is an ideal of $sl(m, n)$.

We set:

$$\mathbf{A(m,n)} = sl(m+1,n+1) \quad if \quad m \neq n \quad m,n \geq 0$$
$$\mathbf{A(n,n)} = sl(n+1,n+1)/ < I_{2n+2} >$$

1.5.2 B(m,n), D(m,n), C(n)

Let us consider a non degenerate, consistent, supersymmetric bilinear form F on V, such that $V_{\bar{0}}$ and $V_{\bar{1}}$ are orthogonal, $F_{V_{\bar{0}} \times V_{\bar{0}}}$ is symmetric and $F_{V_{\bar{1}} \times V_{\bar{1}}}$ is skew-symmetric. Then n must be even, say $n = 2r$.
We define the orthogonal-symplectic superalgebra $osp(m,n) = osp(m,n)_{\bar{0}} \oplus osp(m,n)_{\bar{1}}$ in the following way:

$$osp(m,n)_s = \left\{ a \in l(m,n)_s \quad | \quad F(a(x),y) = -(-1)^{sp(x)} F(x,a(y)) \right\}, \quad s \in \mathbb{Z}_2$$

Let us consider the case $m = 2l+1$. With respect to a conveniently chosen basis, the matrix of F becomes:

$$\left[\begin{array}{ccc|cc} 0 & I_l & 0 & & \\ I_l & 0 & 0 & & \\ 0 & 0 & 1 & & \\ \hline & & & 0 & I_r \\ & & & -I_r & 0 \end{array}\right]$$

hence an element of $osp(m,n)$ becomes of the form:

$$\left[\begin{array}{ccc|cc} a & b & u & x & x_1 \\ c & -a^T & v & y & y_1 \\ -v^T & -u^T & 0 & z & z_1 \\ \hline y_1^T & x_1^T & z_1^T & d & e \\ -y^T & -x^T & -z^T & f & -d^T \end{array}\right] \qquad (*)$$

where a is a matrix of size $l \times l$, b and c are skew-symmetric of size $l \times l$, d is $r \times r$, the matrices e and f are symmetric of size $r \times r$, u and v are column vectors of length l, x and y are of size $l \times r$, finally z is of a column vector of

length r.
Similarly, in the case $m = 2l$, if we choose a basis conveniently, the matrix of F becomes:

$$\left[\begin{array}{cc|cc} 0 & I_l & & \\ I_l & 0 & & \\ \hline & & 0 & I_r \\ & & -I_r & 0 \end{array}\right]$$

then a matrix of $osp(m,n)$ is of the same form as (*) up to deleting the central column and row.
We define:

$$\mathbf{B(m,n)} = osp(2m+1, 2n) \quad m \geq 0 \quad n > 0$$
$$\mathbf{D(m,n)} = osp(2m, 2n) \quad m \geq 2 \quad n > 0$$
$$\mathbf{C(n)} = osp(2, 2n-2) \quad n \geq 2$$

1.5.3 The superalgebras P(n), $n \geq 2$ and Q(n), $n \geq 2$

P(n) is a subalgebra of $sl(n+1, n+1)$, whose elements are of the form:

$$\left[\begin{array}{c|c} a & b \\ \hline c & -a^T \end{array}\right]$$

with $tr(a) = 0$, b symmetric and c skew-symmetric.
Before defining the elements of **Q(n)**, we consider the subalgebra $\tilde{Q}(n)$ of $sl(n+1, n+1)$ consisting of matrices of the form:

$$\left[\begin{array}{c|c} a & b \\ \hline b & a \end{array}\right]$$

with $tr(b) = 0$. The center of $\tilde{Q}(n)$ is $C =< I_{2n+2} >$ and we set $\mathbf{Q(n)} = \tilde{Q}(n)/C$.

Chapter 2

$\mathbb{Z}$-gradings

Let us introduce some definitions about $\mathbb{Z}$-gradings.

Definition 2.1. ($\mathbb{Z}$-graded Lie superalgebra) A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is said $\mathbb{Z}$-graded if:

$$\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$
$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad \forall i, j \in \mathbb{Z}$$

Definition 2.2. If $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is a $\mathbb{Z}$-graded Lie superalgebra s.t. $\mathfrak{g}_i = 0$ $\forall i < -d$ for some $d \in \mathbb{N}$, i.e, $\mathfrak{g} = \oplus_{i \geq -d} \mathfrak{g}_i$, we will say that d is the depth of the grading.

Definition 2.3. (Consistent $\mathbb{Z}$-grading) A $\mathbb{Z}$-grading is said consistent if:

$$\mathfrak{g}_{\bar{0}} = \oplus \mathfrak{g}_{2i} \quad \mathfrak{g}_{\bar{1}} = \oplus \mathfrak{g}_{2i+1}$$

From Definition 2.1 it follows that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ and $[\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_\mathrm{i}, \forall i \in \mathbb{Z}$, so the $\mathfrak{g}_i$'s are $\mathfrak{g}_0$-modules with respect to the adjoint representation restricted to $\mathfrak{g}_0$.

Example 5. Let us consider a $\mathbb{Z}_2$-graded space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ as $\mathbb{Z}-$graded, i.e $V = V_0 \oplus V_1$, then $l(V)$ is endowed with a $\mathbb{Z}-$grading, compatible with the $\mathbb{Z}_2-$grading, and $l(V) = l_{-1} \oplus l(V)_{\bar{0}} \oplus l_1$, where the elements of l_{-1} have matrix of the form $\left(\begin{smallmatrix} 0 & 0 \\ \gamma & 0 \end{smallmatrix}\right)$ and the elements of l_1 have matrix of the form $\left(\begin{smallmatrix} 0 & \beta \\ 0 & 0 \end{smallmatrix}\right)$.

Example 6. If we choose a homogeneus basis of $l(m, n)$, then the elements of $sl(m, n) = sl(m, n)_{-1} \oplus sl(m, n)_{\bar{0}} \oplus sl(m, n)_1$, seen as $\mathbb{Z}$-graded, are of the following form: the elements of $sl(m, n)_{\bar{0}}$ are matrices $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$ with $tr(\alpha) = tr(\delta)$, the elements of $sl(m, n)_1$ are matrices $\left(\begin{smallmatrix} 0 & \beta \\ 0 & 0 \end{smallmatrix}\right)$ and the elements of $sl(m, n)_{-1}$ are $\left(\begin{smallmatrix} 0 & 0 \\ \gamma & 0 \end{smallmatrix}\right)$, where α is a $m \times m$ matrix, β is a $m \times n$ matrix, γ is a $n \times m$ matrix, δ is a $n \times n$ matrix.

Example 7. The Lie superalgebra $osp(m, n)$ can be realized also in a different way. We consider a space $V_{\bar{0}}$ of dimension m endowed with a nondegenerate symmetric bilinear form $(,)_0$ and $V_{\bar{1}}$ a space of dimension $n = 2r$ endowed with a nondegenerate skew-symmetric bilinear form $(,)_1$. Therefore we define:

$$osp(m, n)_{\bar{0}} = \Lambda^2 V_{\bar{0}} \oplus S^2 V_{\bar{1}} \quad and \quad osp(m, n)_{\bar{1}} = V_{\bar{0}} \otimes V_{\bar{1}}$$

Moreover we set:

$$[a \wedge b, c] = (a, c)_0 b - (b, c)_0 a \quad with \quad a \wedge b \in \Lambda^2 V_{\bar{0}}, c \in V_{\bar{0}}$$
$$[a \circ b, c] = (a, c)_1 b + (b, c)_1 a \quad with \quad a \circ b \in S^2 V_{\bar{1}}, c \in V_{\bar{1}}$$

From these definitions, we obtain that the brackets on $\Lambda^2 V_{\bar{0}}$ and $S^2 V_{\bar{1}}$ are defined by:

$$[a \wedge b, c \wedge d] = [a \wedge b, c]d + c[a \wedge b, d]$$
$$[a \circ b, c \circ d] = [a \circ b, c]d + c[a \circ b, d]$$

Moreover, if we consider $a \otimes b, c \otimes d \in V_{\bar{0}} \otimes V_{\bar{1}}$ we define:

$$[a \otimes c, b \otimes d] = (a, b)_0 c \circ d + (c, d)_1 a \wedge b$$

We can now consider the following $\mathbb{Z}$-grading on $osp(m, n)$:

$$osp(m, n) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

In order to do this, we consider $V_{\bar{1}}$ as direct sum of isotropic subspaces $V'_{\bar{1}} \oplus V''_{\bar{1}}$, hence:

$$osp(m, n) = S^2 V'_{\bar{1}} \oplus (V_{\bar{0}} \otimes V'_{\bar{1}}) \oplus (V'_{\bar{1}} \otimes V''_{\bar{1}} \oplus \Lambda^2 V_{\bar{0}}) \oplus (V_{\bar{0}} \otimes V''_{\bar{1}}) \oplus S^2 V''_{\bar{1}}$$

Definition 2.4. (Irreducible Lie superalgebra) A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ is said irreducible if the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is irreducible.

Definition 2.5. (Transitive Lie superalgebra) A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ is said transitive if, given $a \in \mathfrak{g}_i, i \geq 0$, $[a, \mathfrak{g}_{-1}] = 0$ implies $a = 0$.

Definition 2.6. (Bitransitive Lie superalgebra) A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ is said bitransitive if it is transitive and in addition, given $a \in \mathfrak{g}_i, i \leq 0$, $[a, \mathfrak{g}_1] = 0$ implies $a = 0$.

Theorem 2.1. *Let $\mathfrak{g}$ be a simple $\mathbb{Z}$-graded Lie superalgebra which is generated by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then $\mathfrak{g}$ is bitransitive.*

Proof. Let $x \in \mathfrak{g}_i, i \geq 0$ such that $[x, \mathfrak{g}_{-1}] = 0$, we show that $x = 0$. Indeed let us consider:

$$I = \oplus_{k,l=0}^{+\infty}(ad\mathfrak{g}_1)^k(ad\mathfrak{g}_0)^l x$$

I is an ideal of $\mathfrak{g}$, indeed let $g \in \mathfrak{g}$ and $h \in I$; since $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ generates $\mathfrak{g}$, then $g = \sum[g_i, g_j]$ with $g_i, g_j \in \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, but every term of this sum is such that $[[g_i, g_j], h] \in I$ by definition of I. If $x \neq 0$ then I is nontrivial. But also $I \neq \mathfrak{g}$ since no elements of $\mathfrak{g}_k, k < i$ lie in I, due to its definition. This leads to a contradiction, so $x = 0$. Similarly if we choose $J = \oplus_{k,l=0}^{+\infty}(ad\mathfrak{g}_{-1})^k(ad\mathfrak{g}_0)^l x$ with $x \in \mathfrak{g}_i$, $i \leq 0$ we can show that bitransitivity holds.

□

2.1 Local Lie superalgebras

Definition 2.7. (Local Lie superalgebra) Let $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$-graded space which is the direct sum of the $\mathbb{Z}_2$-graded spaces $\mathfrak{g}_i, i = -1, 0, 1$. If $\forall i, j$ such that $|i + j| \leq 1$ there is a bilinear operation:

$$\mathfrak{g}_i \times \mathfrak{g}_j \longrightarrow \mathfrak{g}_{i+j}$$
$$(x, y) \longmapsto [x, y]$$

that is anticommutative ad satisfies the Jacobi identity, provided that the commutators in the identity are defined, then $\hat{\mathfrak{g}}$ is said a local Lie superalgebra.

Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie superalgebra, then $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a local Lie superalgebra which is called the local part of $\mathfrak{g}$. Transitivity and bitransitivity for local parts can be defined as for usual Lie superalgebras. In the following we consider $\mathbb{Z}$-graded Lie superalgebras generated by their local parts.

Definition 2.8. (Maximal Lie superalgebra) Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie superalgebra and let $\hat{\mathfrak{g}}$ be its local part, $\mathfrak{g}$ is called maximal if, given any other $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}'$, an isomorphism of the local parts $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ can be extended to a surjective omomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$.

Definition 2.9. (Minimal Lie superalgebra) Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie superalgebra and let $\hat{\mathfrak{g}}$ be its local part. Then $\mathfrak{g}$ is called minimal if, given any other $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}'$, an isomorphism of the local parts $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ can be extended to a surjective omomorphism of $\mathfrak{g}'$ onto $\mathfrak{g}$.

Theorem 2.2. *Let $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a local Lie superalgebra. Then there exist a maximal $\mathbb{Z}$-graded Lie superalgebra and a minimal $\mathbb{Z}$-graded Lie superalgebra whose local parts are isomorphic to $\hat{\mathfrak{g}}$.*

Proof. Let us start from considering the free Lie superalgebra $F\hat{\mathfrak{g}}$ that is freely generated by $\hat{\mathfrak{g}}$ and let $\tilde{I}$ be the ideal of $F\hat{\mathfrak{g}}$ generated by the relations as $[x, y] = z$ in $\hat{\mathfrak{g}}$. We set $\tilde{\mathfrak{g}} = F\hat{\mathfrak{g}}/\tilde{I}$. Let us denote by π the natural projection of $F\hat{\mathfrak{g}}$ onto the quotient space $\tilde{\mathfrak{g}}$ and let $\tilde{\mathfrak{g}}_{-1} = \pi(\mathfrak{g}_{-1})$, $\tilde{\mathfrak{g}}_0 = \pi(\mathfrak{g}_0)$ and $\tilde{\mathfrak{g}}_1 = \pi(\mathfrak{g}_1)$. Let $\tilde{\mathfrak{g}}_-$ be the subalgebra generated by $\tilde{\mathfrak{g}}_{-1}$ and $\tilde{\mathfrak{g}}_+$ the subalgebra generated by $\tilde{\mathfrak{g}}_1$. It follows that $\tilde{\mathfrak{g}}_- \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_+ = \tilde{\mathfrak{g}}$, its local part is isomorphic to $\hat{\mathfrak{g}}$ and $\tilde{\mathfrak{g}} = \oplus_i \tilde{\mathfrak{g}}_i$, where $\tilde{\mathfrak{g}}_i = \tilde{\mathfrak{g}}_1^i$ and $\tilde{\mathfrak{g}}_{-i} = \tilde{\mathfrak{g}}_{-1}^i$, is a minimal Lie superalgebra.
In order to construct a maximal Lie superalgebra whose local part is isomorphic to $\hat{\mathfrak{g}}$, we consider the set:

$$L_2 = \{a \in F\hat{\mathfrak{g}} \quad | \quad [a, \mathfrak{g}_{-1}] \subset \mathfrak{g}_1\}$$

In the same way:

$$L_{-2} = \{a \in F\hat{\mathfrak{g}} \quad | \quad [a, \mathfrak{g}_1] \subset \mathfrak{g}_{-1}\}$$

Recursively, we define, if $i > 2$, $L_i = \{a \in F\hat{\mathfrak{g}} \quad | \quad [a, \mathfrak{g}_{-1}] \subset L_{i-1}\}$ for $i > 2$ and $L_i = \{a \in F\hat{\mathfrak{g}} \quad | \quad [a, \mathfrak{g}_1] \subset L_{i+1}\}$ for $i < -2$. It follows that $(\oplus_{i \leq -2} L_i) \oplus \hat{\mathfrak{g}} \oplus (\oplus_{i \geq 2} L_i)$ is a maximal Lie superalgebra. □

Theorem 2.3. i) *Let $\mathfrak{g}$ be a bitransitive $\mathbb{Z}$-graded Lie superalgebra, then $\mathfrak{g}$ is minimal.*

ii) *Let $\mathfrak{g}$ be a minimal $\mathbb{Z}$-graded Lie superalgebra. If its local part is bitransitive then $\mathfrak{g}$ is bitransitive.*

iii) *Two bitransitive $\mathbb{Z}$-graded Lie superalgebras are isomorphic if and only if their local parts are isomorphic.*

Proof. i)
Let $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a bitransitive $\mathbb{Z}$-graded Lie superalgebra. We suppose that $\mathfrak{g}$ is not minimal and that $\mathfrak{h}$ is a minimal superalgebra with local part isomorphic to the local part of $\mathfrak{g}$. Then there exits a surjective morphism:

$$\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$$

which is the extension of the isomoprhism between the local parts. Moreover $\mathfrak{h} \cong \mathfrak{g}/Ker(\varphi)$, where clearly $Ker(\varphi) \neq 0$, since $\mathfrak{g}$ is not minimal, is an ideal of $\mathfrak{g}$. We have $Ker(\varphi) \cap \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = 0$ because $\varphi_{|\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1}$ is an isomorphism. Then if $\mathfrak{g}$ is not minimal there exists an ideal $J \neq 0$ of $\mathfrak{g}$ such that $J \cap (\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1) = 0$. Let $k \in \mathbb{Z}$ be the smallest integer in module for which $(J \cap \mathfrak{g}_k) \neq 0$. Let us suppose for the sake of simplicity $k > 0$, then:

$$[J \cap \mathfrak{g}_k, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{k-1} \cap J = 0$$

From the minimality of k it follows $[J \cap \mathfrak{g}_k, \mathfrak{g}_{-1}] = 0$, and from transitivity $J \cap \mathfrak{g}_k = 0$ which leads to a contradiction.
ii) First we prove transitivity. Let us consider $z \in \mathfrak{g}_k$, $k \geq 2$ such that

$[z, \mathfrak{g}_{-1}] = 0$. Since $\mathfrak{g}$ is minimal, we know that it is generated by its local part. As in the proof of Theorem 2.1, $I = \oplus_{j,l=0}^{+\infty}(ad\mathfrak{g}_1)^j(ad\mathfrak{g}_0)^l z$ is an ideal of $\mathfrak{g}$ which is contained in $\oplus_{i\geq 2}\mathfrak{g}_i$ because $k \geq 2$. If we suppose $z \neq 0$, then $I \neq 0$, and this leads to a contradiction because $\mathfrak{g}/I$ has the same local part of $\mathfrak{g}$ and an isomorphism of their local parts can be extended, using the projection to the quotient, to an epimorphism from $\mathfrak{g}$ onto $\mathfrak{g}/I$, but not the viceversa. This contradicts the minimality of $\mathfrak{g}$. The same argument proves that if $z \in \mathfrak{g}_k$, $k \leq -2$ is such that $[z, \mathfrak{g}_1] = 0$ and we consider $I = \oplus_{j,l=0}^{+\infty}(ad\mathfrak{g}_{-1})^j(ad\mathfrak{g}_0)^l z$, the we obtain $z = 0$.
iii) follows from i).

□

Theorem 2.4. *Let $\mathfrak{g} = \oplus_i\mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie superalgebra generated by its local part. Suppose that a consistent supersymmetric invariant bilinear form $(,)$ is defined on $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ such that $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i+j \neq 0$. Then $(,)$ can be uniquely extended to a consistent supersymmetric invariant bilinear form on $\mathfrak{g}$.*

Proof. We start from setting $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i + j \neq 0$. We extend $(,)$ by induction when $x \in \mathfrak{g}_k$ and $y \in \mathfrak{g}_{-k}$ in order to keep the property of invariance. Since $\mathfrak{g}$ is generated by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, we can assume, up to linear combinations, that $x = [x_{k-s}, x_s]$, with $x_i \in \mathfrak{g}_i$ and $y = [y_{s-k}, y_{-s}]$ with $y_{-i} \in \mathfrak{g}_{-i}$. In order to maintain invariance, we define:

$$(x, y) = ([[x_{k-s}, x_s], y_{s-k}], y_{-s})$$

or

$$(x, y) = -(-1)^{p(x_{k-s})p(x_s)}(x_s, [x_{k-s}, [y_{s-k}, y_{-s}]])$$

let us show that this is a good definition. From hypothesis of induction, the extension is well defined if $0 < s < k$, so:

$$\begin{aligned}
&([[x_{k-s}, x_s], y_{s-k}], y_{-s}) = \\
&-(-1)^{p(x_{k-s})p(x_s)}([x_s, [x_{k-s}, y_{s-k}]], y_{-s}) + ([x_{k-s}, [x_s, y_{s-k}]], y_{-s}) = \\
&(-1)^{p(x_{k-s})p(x_s)}(-1)^{p(x_s)(p(x_{k-s})+p(y_{s-k}))}([x_{k-s}, y_{s-k}], [x_s, y_{-s}]) + \\
&+(-1)^{p(x_{k-s})(p(x_s)+p(y_{s-k}))+p(y_{-s})p(x_{k-s})}([x_s, y_{s-k}], [y_{-s}, x_{k-s}]) = \\
&(-1)^{p(x_s)p(y_{s-k})}([x_{k-s}, y_{s-k}], [x_s, y_{-s}]) + \\
&+(-1)^{p(x_{k-s})(p(x_s)+p(y_{s-k}))+p(y_{-s})p(x_{k-s})}([x_s, y_{s-k}], [y_{-s}, x_{k-s}]) = \\
&-(-1)^{p(x_s)p(y_{s-k})}(-1)^{p(x_s)p(y_{-s})}([[x_{k-s}, y_{s-k}], y_{-s}], x_s) + \\
&-(-1)^{p(x_{k-s})(p(x_s)+p(y_{s-k}))+p(y_{-s})p(x_{k-s})+p(y_{s-k})(p(y_{-s})+p(x_{k-s}))}(x_s, [[y_{-s}, x_{k-s}], y_{s-k}]) = \\
&-(-1)^{p(x_s)p(y_{s-k})+p(x_s)p(y_{-s})}([[x_{k-s}, y_{s-k}], y_{-s}], x_s) + \\
&-(-1)^{p(x_{k-s})p(x_s)+p(y_{-s})p(x_{k-s})+p(y_{s-k})p(y_{-s})}(x_s, [[y_{-s}, x_{k-s}], y_{s-k}]) = \\
&-(-1)^{p(x_s)p(y_{s-k})+p(x_s)p(y_{-s})+p(x_s)(p(y_{-s})+p(x_{k-s})+p(y_{s-k}))}(x_s, [[x_{k-s}, y_{s-k}], y_{-s}]) + \\
&-(-1)^{p(x_{k-s})p(x_s)+p(y_{-s})p(x_{k-s})+p(y_{s-k})p(y_{-s})}(x_s, [[y_{-s}, x_{k-s}], y_{s-k}]) = \\
&-(-1)^{p(x_s)p(x_{k-s})}(x_s, [[x_{k-s}, y_{s-k}], y_{-s}]) + \\
&-(-1)^{p(x_{k-s})p(x_s)+p(y_{-s})p(x_{k-s})+p(y_{s-k})p(y_{-s})}(x_s, [[y_{-s}, x_{k-s}], y_{s-k}]) = \\
&-(-1)^{p(x_{k-s})p(x_s)}[(x_s, [[x_{k-s}, y_{s-k}], y_{-s}]) + \\
&+(-1)^{p(y_{-s})p(x_{k-s})+p(y_{s-k})p(y_{-s})}(x_s, [[y_{-s}, x_{k-s}], y_{s-k}])] = \\
&-(-1)^{p(x_{k-s})p(x_s)}[(x_s, [[x_{k-s}, y_{s-k}], y_{-s}]) + \\
&+(-1)^{p(y_{-s})p(x_{k-s})+p(y_{s-k})p(y_{-s})+p(y_{s-k})(p(y_{-s})+p(x_{k-s}))+p(y_{-s})p(x_{k-s})} \\
&\qquad \cdot (x_s, [y_{s-k}, [x_{k-s}, y_{-s}]])] = \\
&-(-1)^{p(x_{k-s})p(x_s)}[(x_s, [[x_{k-s}, y_{s-k}], y_{-s}]) + \\
&+(-1)^{p(x_{k-s})p(y_{s-k})}(x_s, [y_{s-k}, [x_{k-s}, y_{-s}]])] = \\
&-(-1)^{p(x_{k-s})p(x_s)}(x_s, [x_{k-s}, [y_{s-k}, y_{-s}]])
\end{aligned}$$

□

Theorem 2.5. *Let $\mathfrak{g}$ be a simple superalgebra, then an invariant form on $\mathfrak{g}$ is either non degenerate or identically zero, and any two invariant forms on $\mathfrak{g}$ are proportional.*

Proof. If $(,)$ is an invariant form on $\mathfrak{g}$, then its radical is an ideal of $\mathfrak{g}$, so, due to the simplicity of $\mathfrak{g}$, the radical is the whole $\mathfrak{g}$ or 0. In the first case $(,)$ is identically zero, in the second it is non degenerate. We consider now two invariant forms α and β on $\mathfrak{g}$. We define, $\forall x \in \mathfrak{g}$, $\phi_x, \psi_x \in \mathfrak{g}^*$ such that $\forall y \in \mathfrak{g}$:

$$\phi_x(y) = \alpha(x, y) \quad and \quad \psi_x(y) = \beta(x, y).$$

Let us suppose that α is non degenerate, then there exists a unique morphism F of $\mathfrak{g}$-modules such that:

$$\begin{aligned} F : \mathfrak{g}^* &\longrightarrow \mathfrak{g}^* \\ \phi_x &\longrightarrow \psi_x \end{aligned}$$

Indeed, since α is non degenerate, F is uniquely determined because there exists a unique isomorphism

$$\begin{aligned} \gamma : \mathfrak{g} &\longrightarrow \mathfrak{g}^* \\ x &\longrightarrow \phi_x. \end{aligned}$$

It remains to show that F is indeed a morphism of $\mathfrak{g}$-modules. We first show that $z.\phi_x = \phi_{[z,x]}$, $\forall z \in \mathfrak{g}$. Indeed, we have:

$$\begin{aligned} z.\phi_x(y) = -(-1)^{p(z)p(x)}\phi_x(z.y) &= \\ -(-1)^{p(z)p(x)}\phi_x([z, y]) &= \\ -(-1)^{p(z)p(x)}\alpha(x, [z, y]) &= \quad (invariance) \\ -(-1)^{p(z)p(x)}\alpha([x, z], y) &= \\ \alpha([z, x], y) &= \\ \phi_{[z,x]}(y) & \end{aligned}$$

Similarly, we have: $z.\psi_x = \psi_{[z,x]}$, $\forall z \in \mathfrak{g}$. Then F is a a morphism of

$\mathfrak{g}$-modules, indeed:

$$\begin{aligned} F(z.\phi_x) = F(\phi_{[z,x]}) &= \\ \psi_{[z,x]} &= \\ z.\psi_x &= \\ z.F(\phi_x) & \end{aligned}$$

Finally $\mathfrak{g}$ is simple, so $\mathfrak{g}$ is an irreducible $\mathfrak{g}$-module, therefore $\mathfrak{g}^*$ is irreducible and, by Schur's Lemma, $F = \lambda I$. Then $\psi_x = \lambda\phi_x$, i.e. for every $y \in \mathfrak{g}$ $\beta(x, y) = \lambda\alpha(x, y)$. □

2.2 $\mathbb{Z}$-graded Lie superalgebras of depth 1

Theorem 2.6. *Let $\mathfrak{g} = \oplus_{i\geq -1}\mathfrak{g}_i$ be a $\mathbb{Z}$-graded transitive irreducible Lie superalgebra. If $(Z(\mathfrak{g}_0))_{\bar{0}}$ is nontrivial, then it is one dimensional, $(Z(\mathfrak{g}_0))_{\bar{0}} =< z >$, and $[z, g] = sg, \forall g \in \mathfrak{g}_s$.*

Proof. Let $0 \neq z \in (Z(\mathfrak{g}_o))_{\bar{0}}$. We define:

$$\begin{aligned} F : \mathfrak{g}_{-1} &\to \mathfrak{g}_{-1} \\ g &\mapsto [z, g] \end{aligned}$$

Then F is $\mathfrak{g}_0$-invariant, indeed if $g_0 \in \mathfrak{g}_0$:

$$\begin{aligned} &[g_0, F(g)] = [g_0, [z, g]] = \\ &[\underbrace{[g_0, z]}_{=0, z\in C}, g] + [z, [g_0, g]] = F([g_0, g]) \end{aligned}$$

By Schur's Lemma, since $\mathfrak{g}_{-1}$ is an irreducible $\mathfrak{g}_0$-module, $F = \lambda Id$. Since $z \neq 0$, we can choose it such that $\lambda = -1$. It follows that $(Z(\mathfrak{g}_0))_{\bar{0}} = < z >$, because if $y \in (Z(\mathfrak{g}_0))_{\bar{0}}$ then $\forall g \in \mathfrak{g}_{-1}$:

$$[z, g] = -g \quad and \quad [y, g] = \alpha g$$
$$\Downarrow$$
$$[\alpha z + y, g] = 0.$$

From transitivity $y = -\alpha z$. Let us use induction on $k > -1$. Suppose $[z, g_k] = kg_k, \forall g_k \in \mathfrak{g}_k$, $x \in \mathfrak{g}_{-1}$ and let $g_{k+1} \in \mathfrak{g}_{k+1}$ then:

$$\begin{aligned} [z,[x,g_{k+1}]] &= \\ [[z,x],g_{k+1}] + [x,[z,g_{k+1}]] &= \\ -[x,g_{k+1}] + [x,[z,g_{k+1}]] \end{aligned}$$

By the inductive hypothesis $[z, [x, g_{k+1}]] = k[x, g_{k+1}]$, so:

$$[x, [z, g_{k+1}] - (k+1)g_{k+1}] = 0$$

We conclude using transitivity. □

Theorem 2.7. *Let $\mathfrak{g} = \oplus_{i \geq -1}\mathfrak{g}_i$ be a $\mathbb{Z}$-graded transitive irreducible Lie superalgebra. If the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful, then $\mathfrak{g}$ is bitransitive.*

Proof. We set $V = \{a \in \mathfrak{g}_{-1} \mid [a, \mathfrak{g}_1] = 0\}$. V is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$, indeed if $g_0 \in \mathfrak{g}_0$, $a \in V$:

$$\begin{aligned} [[g_0,a],\mathfrak{g}_1] &= \quad (\textit{Jacoby identity}) \\ [g_0,[a,\mathfrak{g}_1]] - (-1)^{p(g_0)p(a)} \underbrace{[a,[g_0,\mathfrak{g}_1]]}_{=0, a\in V} &= \\ [g_0,[a,\mathfrak{g}_1]] = 0 \end{aligned}$$

We know that the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful, then $\mathfrak{g}_1 \neq 0$ and, by transitivity, $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \neq 0$. It follows $V \neq \mathfrak{g}_{-1}$, then by irreducibility $V = 0$. For the elements of $\mathfrak{g}_0$ the thesis is obvious from hypothesis. □

Theorem 2.8. *If a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is simple and $\mathfrak{g}_{\bar{1}} \neq 0$, then these conditions are necessary: the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is faithful and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$. Moreover if these two conditions hold and, in addition, the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is irreducible, then $\mathfrak{g}$ is simple.*

Proof. Let us consider $V = \{g \in \mathfrak{g}_{\bar{0}} \mid [g, \mathfrak{g}_{\bar{1}}] = 0\}$. V is the kernel of the adjoint representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$, so V is an ideal of $\mathfrak{g}_{\bar{0}}$, moreover, it is clear

from its definition that it is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple and $\mathfrak{g}_{\bar{1}} \neq 0$ then $V = 0$.
Let us now show that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$. Indeed, let us set $I = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \oplus \mathfrak{g}_{\bar{1}}$. Then I is an ideal of $\mathfrak{g}$, indeed if $g_{\bar{1}} \in \mathfrak{g}_{\bar{1}}$:

$$[g_{\bar{1}}, I] = \underbrace{[g_{\bar{1}}, [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]]}_{\in \mathfrak{g}_{\bar{1}} \subset I} + \underbrace{[g_{\bar{1}}, \mathfrak{g}_{\bar{1}}]}_{\in I}$$

and $g_{\bar{0}} \in \mathfrak{g}_{\bar{0}}$:

$$[g_{\bar{0}}, I] = [g_{\bar{0}}, [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]] + \underbrace{[g_{\bar{0}}, \mathfrak{g}_{\bar{1}}]}_{\in I} \subseteq$$
$$[[g_{\bar{0}}, \mathfrak{g}_{\bar{1}}], \mathfrak{g}_{\bar{1}}] + [\mathfrak{g}_{\bar{1}}, [g_{\bar{0}}, \mathfrak{g}_{\bar{1}}]] + [g_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset I$$

By simplicity of $\mathfrak{g}$, $I = \mathfrak{g}$, i.e. $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$.
Let us now suppose that the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is faithful and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$ and that, in addition, the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is irreducible, we then shall prove that $\mathfrak{g}$ is simple.
Let $0 \neq J = J_{\bar{0}} \oplus J_{\bar{1}}$ be an ideal of $\mathfrak{g}$. Then $[J_{\bar{1}}, \mathfrak{g}_{\bar{0}}] \subset J_{\bar{1}}$. It follows that $J_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-submodule of $\mathfrak{g}_{\bar{1}}$, hence, by irreducibility, we have either $J_{\bar{1}} = 0$ or $J_{\bar{1}} = \mathfrak{g}_{\bar{1}}$.
The first case cannot hold, since it would follow $[J_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset J_{\bar{1}} = 0$, but $\mathfrak{g}_{\bar{1}}$ is a faithful $\mathfrak{g}_{\bar{0}}$-module. Then $J_{\bar{1}} = \mathfrak{g}_{\bar{1}}$, hence $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subset J$. $J = \mathfrak{g}$.

□

Theorem 2.9. *If a $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$ is simple and $\mathfrak{g}_{-1} \neq 0$, then these conditions are necessary: $\mathfrak{g}$ is transitive and irreducible, $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$. Moreover if these conditions hold and in addition $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$ and $\mathfrak{g}_i = \mathfrak{g}_1^i, \forall i > 0$, then $\mathfrak{g}$ is simple.*

Proof. Let us prove the necessary conditions. Suppose that V is a $\mathbb{Z}_2$-graded subspace of $\mathfrak{g}$ such that: $[\mathfrak{g}_{-1}, V] \subset V$ and $[\mathfrak{g}_0, V] \subset V$. We set $\mathfrak{g}^+ = \oplus_{i \geq 1} \mathfrak{g}_i$ and $V^n = [\mathfrak{g}^+, [\mathfrak{g}^+, ...[\mathfrak{g}^+, V]...]]$, $\forall n \geq 0$, where n is the number of the $\mathfrak{g}^+$ factors. Then, clearly by its definition, $\tilde{V} = \sum_{n\geq 0} V^n$ is an ideal of $\mathfrak{g}$ containing V. Now take:

$$V = \{a \in \oplus_{i\geq 0}\mathfrak{g}_i \quad | \quad [a, \mathfrak{g}_{-1}] = 0\}.$$

From the previous observation, in this case $\tilde{V}$ is an ideal of $\mathfrak{g}$ contained in $\oplus_{i\geq 0}\mathfrak{g}_i$, so by simplicity of $\mathfrak{g}$, $\tilde{V} = 0$, i.e., $\mathfrak{g}$ is transitive. Moreover, if we choose V as a non zero $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$, it follows that $\tilde{V} \neq 0$ and $\tilde{V} \subset V \oplus (\oplus_{i\geq 0}\mathfrak{g}_i)$. By simplicity of $\mathfrak{g}$ it follows that $V = \mathfrak{g}_{-1}$ and irreducibility is proved. It remains to show that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$. Let us consider:

$$I = \mathfrak{g}_{-1} \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus \mathfrak{g}^+$$

We prove that I is an ideal of $\mathfrak{g}$, from which it follows that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$. Indeed, if $g_i \in \mathfrak{g}_i$ and $x = a + b + c \in I, \quad with \quad a \in \mathfrak{g}_{-1}, b \in [\mathfrak{g}_{-1}, \mathfrak{g}_1], c \in \mathfrak{g}^+$:

$$[g_i, x] = \underbrace{[g_i, a]}_{\in \mathfrak{g}_{i-1}} + \underbrace{[g_i, b]}_{\in \mathfrak{g}_i} + \underbrace{[g_i, c]}_{\in \oplus_{k\geq 1}\mathfrak{g}_{i+k}}$$

Note that if $i \geq 2$ $[g_i, x] \in I$; if $i = 1$, $[g_i, x] \in I$ because $[g_i, a] \in [\mathfrak{g}_{-1}, \mathfrak{g}_1]$; if $i = 0$ $[g_i, b] \in [\mathfrak{g}_{-1}, \mathfrak{g}_1]$ since $[\mathfrak{g}_{-1}, \mathfrak{g}_1]$ is an ideal; finally if $i = -1$, $[g_i, x] \in I$ since $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset I$.

Let us now show that if $\mathfrak{g}$ is transitive and irreducible, $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$, and if in addition $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$ and $\mathfrak{g}_i = \mathfrak{g}_1^i, \forall i > 0$, then $\mathfrak{g}$ is simple. Let $I \neq 0$ be a graded ideal of $\mathfrak{g}$, $I = \oplus_{i\geq -1} I_i$. It follows that $[\mathfrak{g}_0, I_{-1}] \subset I_{-1}$, hence, by irreducibility, either $I_{-1} = 0$ or $I_{-1} = \mathfrak{g}_{-1}$. If $I_{-1} = 0$ then $[\mathfrak{g}_{-1}, I_0] = 0$, hence, by transitivity, $I_0 = 0$. Similarly, it follows that $I_k = 0, k \geq 1$. But this is impossible since $I \neq 0$.
Therefore $\mathfrak{g}_{-1} \subset I$, then $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset I$. Since $\mathfrak{g}_0 \in I$, $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1 \subset I$. Finally $\mathfrak{g}_i = \mathfrak{g}_1^i, \forall i > 0$, then $I = \mathfrak{g}$. □

Theorem 2.10. *Let $\mathfrak{g} = \oplus_{i\geq -1}\mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie superalgebra such that $\mathfrak{g}_{-1} \neq 0$. Suppose that the grading is consistent. If $\mathfrak{g}$ is transitive and irreducible, $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$ and in addition the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful and $\mathfrak{g}_i = \mathfrak{g}_1^i, \forall i > 0$, then $\mathfrak{g}$ is simple.*

Proof. Let $I \neq 0$ be a graded ideal of $\mathfrak{g}$, $I = \oplus_{i\geq -1} I_i$. It follows that $[\mathfrak{g}_0, I_{-1}] \subset I_{-1}$, hence, by irreducibility, either $I_{-1} = 0$ or $I_{-1} = \mathfrak{g}_{-1}$. If $I_{-1} = 0$ then $[\mathfrak{g}_{-1}, I_0] = 0$, hence, by transitivity, $I_0 = 0$. Similarly, it follows that $I_k = 0, k \geq 1$. But this is impossible since $I \neq 0$. Therefore $\mathfrak{g}_{-1} \subset I$, then $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset I$. Since $\mathfrak{g}_0 \in I$, $[\mathfrak{g}_0, \mathfrak{g}_1] \subset I$. It remains to show $\mathfrak{g}_1 \subset I$, then $I = \mathfrak{g}$ and $\mathfrak{g}$ is simple since it does not contain non trivial ideals. Since $[\mathfrak{g}_0, \mathfrak{g}_1] \subset I$, it is sufficient to prove that $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$. Since the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is irreducible and faithful, it follows that $\mathfrak{g}_0$ is a reductive Lie algebra, in particular $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus Z(\mathfrak{g}_0)$ where $[\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple and $Z(\mathfrak{g}_0)$ is the center, with $dim(Z(\mathfrak{g}_0)) \leq 1$. From Theorem 2.6 it follows that if $Z(\mathfrak{g}_0) \neq 0$, then $Z(\mathfrak{g}_0) =< c >$ with $[c, x] = x \quad \forall x \in \mathfrak{g}_1$, but $c \in \mathfrak{g}_0 \subset I$, so $x \in I$.

If $Z(\mathfrak{g}_0) = 0$, we know that $\mathfrak{g}_0$ is semisimple, so $\mathfrak{g}_1$ is a completely reducible $\mathfrak{g}_0$-module, that is $\mathfrak{g}_1 = V_1 \oplus ... \oplus V_k$, with V_i irreducible $\mathfrak{g}_0$-modules.

It follows $[\mathfrak{g}_0, \mathfrak{g}_1] = \oplus_i [\mathfrak{g}_0, V_i] = \mathfrak{g}_1$. Indeed $[\mathfrak{g}_0, V_i] = V_i$, because V_i for every i is an irreducible $\mathfrak{g}_0$-module, and due to the fact that the representation on $\mathfrak{g}_1$ is faithful, it is non trivial. It is obvious that $[\mathfrak{g}_0, V_i] \subset V_i$, but in fact the equality holds due to irreducibility.

□

Theorem 2.11. *Let $\mathfrak{g} = \oplus_{i\geq -1}\mathfrak{g}_i$ be a transitive irreducible Lie superalgebra with a consistent $\mathbb{Z}$-grading. If $\mathfrak{g}_1 \neq 0$ then $[\mathfrak{g}_0, \mathfrak{g}_0] \subset [\mathfrak{g}_{-1}, \mathfrak{g}_1]$.*

Proof. We notice that $V = [\mathfrak{g}_{-1}, [\mathfrak{g}_{-1}, \mathfrak{g}_1]] \neq 0$, indeed there exists $\mathfrak{g}_1 \ni a \neq 0$, so, since $\mathfrak{g}$ is transitive, we have $[a, \mathfrak{g}_{-1}] \neq 0$ and, again by transitivity, $[\mathfrak{g}_{-1}, [\mathfrak{g}_{-1}, a]] \neq 0$. Moreover V is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$. Indeed, if $g_0 \in \mathfrak{g}_0$, $\tilde{g}_{-1}$, $g_{-1} \in \mathfrak{g}_{-1}$, $g_1 \in \mathfrak{g}_1$, we have:

$$[g_0, [\tilde{g}_{-1}, [g_{-1}, g_1]]] =$$

$$\underbrace{[[g_0,\tilde{g}_{-1}],[g_{-1},g_1]]}_{\in V}+[\tilde{g}_{-1},[g_0,[g_{-1},g_1]]]=$$
$$[[g_0,\tilde{g}_{-1}],[g_{-1},g_1]]+\underbrace{[\tilde{g}_{-1},[[g_0,g_{-1}],g_1]]}_{\in V}+\underbrace{[\tilde{g}_{-1},[g_{-1},[g_0,g_1]]]}_{\in V}.$$

Let C be the centralizer of $[\mathfrak{g}_{-1},\mathfrak{g}_1]$ in $\mathfrak{g}_0$. By irreducibility of the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ and transitivity, it follows that $\mathfrak{g}_0$ is a reductive Lie algebra, in particular $\mathfrak{g}_0=[\mathfrak{g}_0,\mathfrak{g}_0]\oplus Z(\mathfrak{g}_0)$ where $[\mathfrak{g}_0,\mathfrak{g}_0]$ is semisimple and $Z(\mathfrak{g}_0)$ is the center of $\mathfrak{g}_0$, with $dim(Z(\mathfrak{g}_0))\leq 1$.
In order to prove the thesis it is enough to show that C is abelian: indeed $[\mathfrak{g}_{-1},\mathfrak{g}_1]$ is an ideal of $\mathfrak{g}_0$ and $\mathfrak{g}_0=L_1\oplus\ldots\oplus L_t\oplus Z(\mathfrak{g}_0)$, where we denote by L_i, with $1\leq i\leq t$, the simple ideals of $\mathfrak{g}_0$. In particular $[L_i,L_j]=0 \quad \forall i\neq j$. Let J be $[\mathfrak{g}_{-1},\mathfrak{g}_1]$, an ideal of $\mathfrak{g}_0$, the, up to reordering the indexes, we may assume $J=L_1\oplus\ldots\oplus L_k$ and $C_{\mathfrak{g}_0}(J)=L_{k+1}\ldots\oplus L_t\oplus Z(\mathfrak{g}_o)$. If C is abelian, then $C\subset Z(\mathfrak{g}_0)$ and $[\mathfrak{g}_0,\mathfrak{g}_0]\subset[\mathfrak{g}_{-1},\mathfrak{g}_1]$, otherwise $L_i\subset C$ for some i, but L_i is not abelian since it is simple.
It remains to show that C is abelian, i.e. $[a,b]=0,\forall a,b\in C$. Since $\mathfrak{g}$ is transitive, it is sufficient to prove that $[\mathfrak{g}_{-1},[a,b]]=[[[\mathfrak{g}_1,\mathfrak{g}_{-1}],\mathfrak{g}_{-1}],[a,b]]=0$ i.e. to show that for $t\in\mathfrak{g}_1$, $x,y\in\mathfrak{g}_{-1}$ and $a,b\in C$:

$$[[t,x],y],[a,b]]=0$$

Indeed:

$$[[t,x],y],[a,b]]=$$
$$[[[[t,x],y],a],b]+[a,[[[t,x],y],b]]=$$
$$\underbrace{[[[[t,x],y],a],b]}_{:=d}-\underbrace{[[[[t,x],y],b],a]}_{:=e}$$

where we used the fact that the $\mathbb{Z}$-grading is consistent. We have:

$$e=[[[[t,x],y],b],a]=$$

$$
\begin{aligned}
[[[t,x],[y,b]],a] - [[y,\underbrace{[[t,x],b]}_{=0, b\in C}],a] &= \\
[[[t,x],[y,b]],a] &= \quad (Jacobi\ \ identity) \\
[[t,\underbrace{[x,[y,b]]}_{\in[\mathfrak{g}_{-1},\mathfrak{g}_{-1}]=0}],a] + [[x,[t,[y,b]]],a] &= \\
[[x,[t,[y,b]]],a] &= \\
-[[[t,[y,b]],x],a] &= \\
-[[t,[y,b]],[x,a]] + [x,\underbrace{[[t,[y,b]],a]}_{=0, a\in C}] &= \\
-[[t,[y,b]],[x,a]].
\end{aligned}
$$

Moreover we observe that:

$$
\begin{aligned}
[[t,x],y] &= \\
-[y,[t,x]] &= \\
-[[y,t],x] + [t,\underbrace{[y,x]}_{\in[\mathfrak{g}_{-1},\mathfrak{g}_{-1}]=0}] &= \\
-[[t,y],x]
\end{aligned}
$$

Therefore we have:

$$
\begin{aligned}
d = [[[[t,x],y],a],b] &= \\
-[[[[t,y],x],a],b] &= \\
-[[[t,y],[x,a]],b] + [[x,\underbrace{[[t,y],a]}_{=0, a\in C}],b] &= \\
[[[x,a],[t,y]],b] &= \\
[[[[x,a],t],y],b] - [[t,\underbrace{[[x,a],y]}_{\in[\mathfrak{g}_{-1},\mathfrak{g}_{-1}]=0}],b] &= \\
[[[[x,a],t],y],b] &= \\
[[[t,[x,a]],y],b] &=
\end{aligned}
$$

$$[[t,[x,a]],[y,b]] - [y, \underbrace{[[t,[x,a]],b]}_{=0, b\in C}] =$$

$$[[t,[x,a]],[y,b]] =$$

$$[t, \underbrace{[[x,a],[y,b]]}_{\in[\mathfrak{g}_{-1},\mathfrak{g}_{-1}]=0}] + [[x,a],[t,[y,b]]] =$$

$$-[[t,[y,b]],[x,a]] = e.$$

□

Theorem 2.12. *Let $\mathfrak{g} = \oplus_{i\geq -1}\mathfrak{g}_i$ be a transitive irreducible Lie superalgebra with a consistent $\mathbb{Z}$-grading and $\mathfrak{g}_1 \neq 0$. Let the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ be irreducible and faithful, and denote by H a Cartan subalgebra of $\mathfrak{g}_0$, by F_Λ the highest weight vector of the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ and by E_M the lowest weight vector of the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$.*
Then:

a) *If $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are contragredient $\mathfrak{g}_0$-modules:*

1) $M = -\Lambda$

2) $[F_\Lambda, E_M] = h \neq 0$, $h \in H$

3) $[\mathfrak{g}_1, \mathfrak{g}_1]=0$

4) $\mathfrak{g}_{-1} \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus \mathfrak{g}_1$ *is simple*

b) *If $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are not contragredient:*

1) $[F_\Lambda, E_M] = e_\alpha$, *with $\alpha = \Lambda + M$ a nonzero root of $[\mathfrak{g}_0, \mathfrak{g}_0]$*

2) $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = [\mathfrak{g}_0, \mathfrak{g}_0]$

3) $[\mathfrak{g}_0, \mathfrak{g}_0]$ *is simple*

Proof. Let $\alpha_1, \dots, \alpha_m$ be a system of simple roots of $[\mathfrak{g}_0, \mathfrak{g}_0]$ with respect to H. It follows that: $\mathfrak{g}_{-1} =< [\cdots[F_\Lambda, e_{-\gamma_1}], \dots, e_{-\gamma_k}] >$, where $\gamma_1, \dots, \gamma_k \in \{\alpha_1, \dots, \alpha_m\}$ and $e_{-\gamma_i}$ is a root vector associated with $-\gamma_i$. Likewise, $\mathfrak{g}_1 =< [\cdots[E_M, e_{\delta_1}], \dots, e_{\delta_k}] >$, where $\delta_1, \dots, \delta_s \in \{\alpha_1, \dots, \alpha_m\}$ and e_{δ_i} is a root

vector associated with δ_i.
So we have: $[\mathfrak{g}_{-1}, \mathfrak{g}_1] =< [\dots[F_\Lambda, E_M], e_{\beta_1}, \dots, e_{\beta_t}] >$, with $\beta_1, \dots, \beta_t \in \{\alpha_1, \dots, \alpha_m, -\alpha_1, \dots, -\alpha_m\}$.
From the hypothesis $\mathfrak{g}_1 \neq 0$ so it follows from transitivity that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \neq 0$, then $[F_\Lambda, E_M] \neq 0$.
We have $[t, [F_\Lambda, E_M]] = (\Lambda + M)(t)[F_\Lambda, E_M], \forall t \in H$. If $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are contragredient $\mathfrak{g}_0$-modules then $\Lambda + M = 0$, 1a), hence $[F_\Lambda, E_M]$ lies in the centralizer of H in $\mathfrak{g}$ which coincides with H itself, 2a). If $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are not contragredient then $\Lambda + M \neq 0$ and $[F_\Lambda, E_M]$ is a root vector corresponding to the root $\Lambda + M$, 1b). We now prove $3a$). Le $\tilde{\mathfrak{g}}$ be the subalgebra generated by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, then $\tilde{\mathfrak{g}}$ is bitransitive. Indeed by Theorem 2.11 $[\mathfrak{g}_{-1}, \mathfrak{g}_1]$ is either $\mathfrak{g}_0$ or $[\mathfrak{g}_0, \mathfrak{g}_0]$. In the first case $\tilde{\mathfrak{g}}$ is simple because it satisfies the hypothesis of Theorem 2.10. Indeed, $\tilde{\mathfrak{g}}$ is transitive, $[\tilde{\mathfrak{g}}_{-1}, \tilde{\mathfrak{g}}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_1] = \tilde{\mathfrak{g}}_0$ by construction and the representation of $\tilde{\mathfrak{g}}_0$ on $\tilde{\mathfrak{g}}_{-1} = \mathfrak{g}_{-1}$ is irreducible: otherwise there would exist a non trivial $\tilde{\mathfrak{g}}_0$-submodule V of $\mathfrak{g}_{-1}$, but, from 2.6, the elements of $Z(\mathfrak{g}_0)$ act as scalars on V, then V would be a nontrivial $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$, which is impossible since $\mathfrak{g}_{-1}$ is irreducible (this argument proves also $4a$)). Then $\tilde{\mathfrak{g}}$ is simple and by Theorem 2.1 it is bitransitive. Now suppose that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = [\mathfrak{g}_0, \mathfrak{g}_0]$, $\tilde{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus Z(\mathfrak{g}_0) \oplus \mathfrak{g}_1$. Then by Theorem 2.6, $Z(\mathfrak{g}_0) =< z >$, where $[z, x] = x, \forall x \in \mathfrak{g}_1$, hence $\tilde{\mathfrak{g}}$ is bitransitive.
There exists an automorphism:

$$\varphi : \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

carrying the positive roots of $\mathfrak{g}_0$ in the negative ones and interchanges $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$. By Theorem 2.3 i), $\tilde{\mathfrak{g}}$ is minimal, so φ can be extended to an automorphism of $\tilde{\mathfrak{g}}$ that interchanges $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, hence $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$.
It remains to prove $2b$) and $3b$).
We know that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] =< [\dots[e_\alpha, e_{\beta_1}], \dots, e_{\beta_t}] >\subset \tilde{H}$, where $\tilde{H}$ is the simple ideal of $[\mathfrak{g}_0, \mathfrak{g}_0]$ which contains the root space of α. But by Theorem 2.11:

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset [\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset \tilde{H} \subset [\mathfrak{g}_0, \mathfrak{g}_0]$$

We conclude that $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = [\mathfrak{g}_0, \mathfrak{g}_0]$ and it is simple. $\square$

Theorem 2.13. *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a transitive Lie superalgebra which satisfies the hypotheses of Theorem 2.11. Then either $\mathfrak{g}_1$ is a faithful and irreducible $\mathfrak{g}_0$-module, or $dim(\mathfrak{g}_1) = 1$.*

Proof. Suppose $dim(\mathfrak{g}_1) > 1$. Let us suppose $\mathfrak{g}_1$ is not irreducible, then by Weyl's Theorem, $\mathfrak{g}_1 = \mathfrak{g}'_1 \oplus \mathfrak{g}''_1$, where $\mathfrak{g}'_1$ and $\mathfrak{g}''_1$ are $\mathfrak{g}_0$-submodules. If we apply Theorem 2.11 to $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}'_1$, it follows $[\mathfrak{g}_0, \mathfrak{g}_0] \subset [\mathfrak{g}_{-1}, \mathfrak{g}'_1]$. Then:

$$[[\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{g}''_1] \subset [[\mathfrak{g}_{-1}, \mathfrak{g}'_1], \mathfrak{g}''_1] \subset [\mathfrak{g}'_1, [\mathfrak{g}_{-1}, \mathfrak{g}''_1]] \subset \mathfrak{g}'_1$$

where we used the Jacoby identity and the fact that $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$. Since the sum of $\mathfrak{g}'_1$ and $\mathfrak{g}''_1$ is direct it follows that: $[[\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{g}''_1] = 0$. In the same way it follows $[[\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{g}'_1] = 0$. Hence $[[\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{g}_1] = 0$. Now let us prove that if $[[\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{g}_1] = 0$ then $dim(\mathfrak{g}_1) = 1$. Let $a \in \mathfrak{g}_1$, $a \neq 0$, and define:

$$\begin{aligned} F_a : \mathfrak{g}_{-1} &\to \mathfrak{g}_0 \\ y &\mapsto [a, y] \end{aligned}$$

Then F_a is a morphism of $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules: for $x \in [\mathfrak{g}_0, \mathfrak{g}_0]$, we have:

$$F_a([x, y]) = [a, [x, y]] = [\underbrace{[a, x]}_{\in [[\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{g}_1] = 0}, y] + [x, [a, y]] = [x, F_a(y)]$$

By Theorem 2.12 $F_a(\mathfrak{g}_{-1}) = [\mathfrak{g}_0, \mathfrak{g}_0]$, since $\mathfrak{g}_{-1}$ and $\mathbb{C}a$ are not contragredient. But $\mathfrak{g}_{-1}$ is irreducible and $[\mathfrak{g}_0, \mathfrak{g}_0]$ is an irreducible $[\mathfrak{g}_0, \mathfrak{g}_0]$-module since $[\mathfrak{g}_0, \mathfrak{g}_0]$ is simple by Theorem 2.12 3b), so $Ker(F_a) = 0$ then F_a is an isomorphism onto its image. Therefore $[\mathfrak{g}_0, \mathfrak{g}_0]$ and $\mathfrak{g}_{-1}$ are isomorphic, irreducible $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules, so they are isomorphic highest weight modules, hence $F_a = \lambda Id$. For $a_1, a_2 \in \mathfrak{g}_1$, $x \in \mathfrak{g}_{-1}$:

$$[a_1, x] = \lambda[a_2, x] \Rightarrow \; [a_1 - \lambda a_2, x] = 0 \Rightarrow \; a_1 = \lambda a_2$$

Therefore we proved that $dim(\mathfrak{g}_1) = 1$.
We conclude by showing that the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful. If it is

not faithful, then there exists a simple ideal J of $[\mathfrak{g}_0, \mathfrak{g}_0]$ such that: $[J, \mathfrak{g}_1] = 0$. Indeed $Ker(ad_{\mathfrak{g}_0})$ is an ideal and it cannot be contained in $Z(\mathfrak{g}_0)$, since the elements of $Z(\mathfrak{g}_0)$ act as scalars on $\mathfrak{g}_1$ by Theorem 2.6. So $Ker(ad_{\mathfrak{g}_0})$ is a simple ideal of the semisimple Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$.
Set $\mathfrak{g}_{-1} = V_1 \oplus \ldots \oplus V_k$, where V_i are J-submodules of $\mathfrak{g}_{-1}$ and let F_a be as defined above.
We observe:

$$\begin{aligned}(F_a)_{|V_i} : V_i &\to [\mathfrak{g}_0, \mathfrak{g}_0] = J \oplus C_{[\mathfrak{g}_0,\mathfrak{g}_0]}(J)\\ x &\mapsto [a, x]\end{aligned}$$

The V_i's are irreducible and faithful J-modules, then they are nontrivial. From the fact that F_a is an isomorphism onto its image and J acts trivially on $C_{[\mathfrak{g}_0,\mathfrak{g}_0]}(J)$, it follows $F_a(V_i) \subset J$, $\forall i$. Finally:

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset [\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset J \subset [\mathfrak{g}_0, \mathfrak{g}_0]$$

Then $J = [\mathfrak{g}_0, \mathfrak{g}_0]$, so $dim(\mathfrak{g}_1) = 1$ and this is a contraditcion. □

Chapter 3

Filtrations

In this chapter we explain some results on Lie superagebras with filtrations.

Definition 3.1. (Filtration) A filtration of a Lie superalgebra L is a sequence of $\mathbb{Z}_2$-graded subspaces $L_i, i \in \mathbb{Z}$, such that:

$$L = L_{-1} \supset L_0 \supset L_1 ...$$
$$L_i = L \quad \forall i \leq -1$$
$$[L_i, L_j] \subset L_{i+j} \quad \forall i, j \in \mathbb{Z}$$
$$\bigcap_{i \in \mathbb{Z}} L_i = 0$$

Definition 3.2. A Lie superalgebra with filtration is said transitive if:

$$L_i = \{a \in L_{i-1} \quad | \quad [a, L] \subset L_{i-1}\} \quad i > 0 \tag{3.1}$$

Remark 5. If we consider a Lie superalgebra L and a subalgebra L_0 of L which does not contain any nonzero ideal of L, then condition (3.1) together with $L_{-1} = L$ defines a filtration on L. Indeed $[L_i, L_j] \subset L_{i+j} \quad \forall i, j \in \mathbb{Z}$. This is obvious for $i \leq -1$ or $j \leq -1$. If $i, j \geq 0$ we proceed by induction on $i + j$:
if $i + j = 0$, that is $i = j = 0$, $[L_i, L_j] \subseteq L_{i+j}$ since L_0 is a subalgebra.

if $i+j>0$:

$$\begin{aligned}[L,[L_i,L_j]] &= \\ [\underbrace{[L,L_i]}_{\subset L_{i-1}},L_j] \pm [L_i,\underbrace{[L,L_j]}_{L_{j-1}}] &\subset \\ L_{i+j-1}\end{aligned}$$

by induction. Moreover it is obvious that $\bigcap_{i\in\mathbb{Z}} L_i$ is an ideal of L contained in L_0, so $\bigcap_{i\in\mathbb{Z}} L_i = 0$. This filtration is called the transitive filtration of the pair (L, L_0).

Let L be a filtered Lie superalgebra. Then we can consider the $\mathbb{Z}$-graded Lie superalgebra GrL, associated to L, defined as follows:

$$GrL = \oplus_{i\geq -1} Gr_i L, \quad Gr_i L = L_i/L_{i+1}$$

GrL is a $\mathbb{Z}_2$-graded Lie superalgebra, due to the $\mathbb{Z}_2$-grading of L_i, but the $\mathbb{Z}$-grading is not consistent in general.
If $\mathfrak{g} = \oplus_{i\geq -1}\mathfrak{g}_i$ is a $\mathbb{Z}$-graded Lie superalgebra, we can canonically consider the filtration given by $L_i = \oplus_{s\geq i}\mathfrak{g}_s$, the properties of Definition 3.1 are obviously verified by this L_i.

3.1 Proprierties of L and GrL

Proposition 3.1. *A Lie superalgebra L with filtration is transitive if and only if GrL is transitive.*

Proof. Let us suppose that GrL is transitive, i.e., for every $\bar{a} = a + L_{i+1} \in Gr_i L, i \geq 0$:

$$\begin{aligned} if \quad [\bar{a}, Gr_{-1}L] = 0 \quad then \quad \bar{a} = 0 \quad or, \quad equivantly, \\ if \quad [a, L] \subset L_i \quad then \quad a \in L_{i+1}, \quad i.e. \\ \{a \in L_i \quad | \quad [a, L] \subset L_i\} \subset L_{i+1} \quad \forall i \geq 0. \end{aligned}$$

Since the reverse inclusion is obvious, equality holds, i.e., L is transitive.

Now suppose that the filtration on L is transitive. Let $\bar{a} = a + L_{i+1} \in Gr_iL, i \geq 0$ such that $[\bar{a}, Gr_{-1}L] = 0$ or,equivalently, $[a, L] \subset L_i$. By the transitivity of the filtration on L, $a \in L_{i+1}$ that is $\bar{a} = 0$. □

Proposition 3.2. *Let L be a Lie superalgebra with filtration. If GrL is simple then L is simple.*

Proof. Let $I \neq 0$ be an ideal of L and set $\tilde{I} = \{\bar{a} \in GrL \quad | \quad a \in I\}$. $\tilde{I}$ is an ideal of GrL, indeed: let us consider $\bar{a} \in \tilde{I}$ and $\bar{u} \in GrL$. For the sake of simplicity we suppose $\bar{u} \in Gr_iL$ for some i. Then:

$$[\bar{u}, \bar{a}] = [u + L_{i+1}, a + L_{j+1}] = \overline{[u, a]} \in Gr_{i+j}L$$

Let $x \in I$ be a non zero element, then $x \in L_i$ where i is the minimal index such that $x \notin L_{i+1}$. It follows that $0 \neq \bar{x} \in Gr_iL \cap \tilde{I}$ and $\tilde{I} \neq 0$. Since GrL is simple, $\tilde{I} = GrL$, then $I = L$.

□

Remark 6. Let L be a Lie superalgebra with filtration. If there exist subspaces G_i such that $L_i = G_i \oplus L_{i+1}$ and $[G_i, G_j] \subset G_{i+j}, \forall i, j$, then we say that a $\mathbb{Z}$-grading consistent with the filtration is defined on L and if $dim(L) < \infty$ then $L \cong GrL$.

Theorem 3.3. *Let us consider a transitive finite-dimensional Lie superalgebra L with filtration. If GrL is consistently $\mathbb{Z}$-graded, the representation of Gr_0L on $Gr_{-1}L$ is irreducible and $Z(Gr_0L) \neq 0$, then $L \cong GrL$.*

Proof. We can apply Theorem 2.6 to GrL, so $Z(Gr_0L) =< z >$, with $[z, g] = sg, \forall g \in Gr_sL$. We consider the map:

$$\pi : L_0 \longrightarrow L_0/L_1 = Gr_0L$$
$$x \longmapsto x + L_1$$

We denote by $\tilde{z}$ an element of $\pi^{-1}(z)$. It follows that for $\bar{g} \in L_s/L_{s+1}$:

$$[\tilde{z}, \bar{g}] =$$
$$[z + L_1, g + L_{s+1}] = sg + L_{s+1}$$

So $\tilde{z}$ is diagonalizable in L, $L = \oplus_{i\geq -1} G_i$ and $L_s = G_s \oplus L_{s+1}$ where G_i is the eigenspace relative to the eigenvalue i. Then we obtained a $\mathbb{Z}$-grading consistent with the filtration on L. □

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra with a maximal proper subalgebra L_0 of L such that $L_{\bar{0}} \subset L_0$ and let us suppose that $L_{\bar{0}}$ does not contain nonzero ideals of L. Let us consider, as defined before, the filtration of the pair (L, L_0):

$$L_i = \{a \in L_{i-1} \quad | \quad [a, L] \subset L_{i-1}\} \quad i > 0.$$

Theorem 3.4. *Let $GrL = \oplus_{i\geq -1} Gr_i L$ be $\mathbb{Z}$-graded Lie superalgebra associated to the filtration of the pair (L, L_0), then:*

1. *GrL is transitive;*
2. *the $\mathbb{Z}$-grading of GrL is consistent;*
3. *GrL is irreducible;*
4. *if the representation of $L_{\bar{0}}$ on $L_{\bar{1}}$ is reducible, then $Gr_1 L \neq 0$.*

Proof. 1) From the definition of its filtration, L is transitive, then, by Proposition 3.1, GrL is transitive.
2) Since $L_{\bar{0}} \subset L_0$, $L/L_0 = Gr_{-1}L \subset (GrL)_{\bar{1}}$. We show, using induction and transitivity, that $Gr_i L \subset (GrL)_{\bar{0}}$ if i is even and $Gr_i L \subset (GrL)_{\bar{1}}$ if i is odd. If i is even, $[Gr_i L \cap (GrL)_{\bar{1}}, Gr_{-1}L] \subset Gr_{i-1}L \subset (GrL)_{\bar{1}}$ by the inductive hypothesis. Besides:

$$[Gr_i L \cap (GrL)_{\bar{1}}, Gr_{-1}L] \subset [(GrL)_{\bar{1}}, (GrL)_{\bar{1}}] \subset (GrL)_{\bar{0}}$$

It follows that $[Gr_i L \cap (GrL)_{\bar{1}}, Gr_{-1}L] = 0$, by transitivity $Gr_i L \cap (GrL)_{\bar{1}} = 0$ and $Gr_i L \subset (GrL)_{\bar{0}}$. The case i odd is similar.
3) Assume that GrL is reducible, then there exists $\tilde{L} \subsetneq L$ such that $\tilde{L} \supset L_0$ and $[L_0, \tilde{L}] \subset \tilde{L}$.

$\tilde{L} = L_0 \oplus V$ where $V \subset L_{\bar{1}}$, because $L_{\bar{0}} \subset L_0$, and $[V,V] \subset L_{\bar{0}} \subset L_0$. It follows:

$$[\tilde{L},\tilde{L}] = [L_0 \oplus V, L_0 \oplus V] = \underbrace{[L_0,L_0]}_{\subset L_0 \subset \tilde{L}} + \underbrace{[L_0,V]}_{\subset \tilde{L}} + \underbrace{[V,V]}_{\subset L_0 \subset \tilde{L}} \subset \tilde{L}$$

But this leads to a contradiction by the maximality of L_0.
4) We suppose $Gr_1 L = 0$, then $L_1/L_2 = 0$ that is $L_1 = L_2$. Note that:

$$L_2 = \{a \in L_1 \quad | \quad [a,L] \subset L_1\}$$

It follows that L_1 is an ideal of L, then $L_1 \cap L_{\bar{0}}$ is an ideal of $L_{\bar{0}}$. From the hypothesis $L_1 \cap L_{\bar{0}} = \{0\}$, i.e., $L_1 \subset L_{\bar{1}}$, so $Gr_0 L = L_0/L_1 = L_{\bar{0}}$. Since $Gr_1 L = 0$, then $GrL = Gr_{-1}L \oplus Gr_0 L$, but $Gr_0 L = L_{\bar{0}}$, then $Gr_{-1}L = L_{\bar{1}}$. Using 3) we conclude that the representation of $L_{\bar{0}}$ on $L_{\bar{1}}$ is irreducible. $\square$

Chapter 4

Superalgebras of vector fields

In this chapter we study the $\mathbb{Z}$-gradings of some Lie superalgebras of vector fields, focusing on the cases in which the grading is symmetric (see Definition 4.1) and of depth two.

4.1 The Lie superalgebra $W(m,n)$

We recall that $\Lambda(n)$ is the Grassmann algebra in the n odd indeterminates $\xi_1, ..., \xi_n$. Let $x_1, ..., x_m$ be even coordinates, we denote $\Lambda(m,n) = \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n)$ and $W(m,n)$ the space of its derivations:

$$W(m,n) = \left\{ \sum_{i=1}^{m} f_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} g_i \frac{\partial}{\partial \xi_i} \quad where \quad f_i, g_i \in \Lambda(m,n) \right\}.$$

The derivations $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial \xi_i}$ are determined by:

$$\frac{\partial}{\partial x_i}(x_j) = \delta_{ij} \quad \frac{\partial}{\partial x_i}(\xi_j) = 0$$
$$\frac{\partial}{\partial \xi_i}(x_j) = 0 \quad \frac{\partial}{\partial \xi_i}(\xi_j) = \delta_{ij}$$

We can define a $\mathbb{Z}$-grading on $W(m,n)$ by letting $deg(x_i) = a_i = -deg(\frac{\partial}{\partial x_i})$ and $deg(\xi_i) = b_i = -deg(\frac{\partial}{\partial \xi_i})$, where $a_i \in \mathbb{N}$ and $b_i \in \mathbb{Z}$ and we call it grading of type $(a_1, ..., a_m|b_1, ..., b_n)$. The grading of type $(1, ..., 1|1, ..., 1)$ is called *principal*, instead the grading of type $(1, ..., 1|0, ..., 0)$ is called *subprincipal*.

4.1.1 The principal grading

We study the principal grading of $W(m,n)$. First we observe that with this grading $W(m,n) = \oplus_{j=-1}^{\infty} W(m,n)_j$, i.e. it has depth one. We have:

$$W(m,n)_0 =< x_i \frac{\partial}{\partial x_j}, \xi_i \frac{\partial}{\partial x_j}, x_i \frac{\partial}{\partial \xi_j}, \xi_i \frac{\partial}{\partial \xi_j} >\cong \mathfrak{gl}(m,n)$$

The isomorphism is given by the map:

$$\begin{aligned} \Phi : W(m,n)_0 &\longrightarrow \mathfrak{gl}(m,n) \\ x_i \frac{\partial}{\partial x_j} &\longmapsto e_{i,j} \\ \xi_i \frac{\partial}{\partial x_j} &\longmapsto e_{i+m,j} \\ x_i \frac{\partial}{\partial \xi_j} &\longmapsto e_{i,j+m} \\ \xi_i \frac{\partial}{\partial \xi_j} &\longmapsto e_{i+m,j+m} \end{aligned}$$

where by $e_{l,k}$ we denoted the elementary matrix with 1 in position l, k.
Notice that

$$W(m,n)_{-1} =< \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial \xi_1}, ..., \frac{\partial}{\partial \xi_n} >\cong \mathbb{C}^{m|n}$$

and $W(m,n)_0$ acts on $W(m,n)_{-1}$ via the standard action, therefore the principal grading of $W(m,n)$ is irreducible.

Proposition 4.1. *The principal grading of $W(m,n)$ is transitive.*

Proof. Let $a = \sum_{deg(P_i)\geq 1} P_i \frac{\partial}{\partial x_i} + \sum_{deg(Q_j)\geq 1} Q_j \frac{\partial}{\partial \xi_j}$ be an element of $W(m,n)_{\geq 0}$, where $P_i, Q_j \in \Lambda(m,n)$. We show that if $[a, W(m,n)_{-1}] = 0$ it follows $a = 0$. In fact if $[a, \frac{\partial}{\partial x_k}] = 0, \forall k = 1, ..., m$ we have:

$$-\sum_{deg(P_i)\geq 1} \frac{\partial P_i}{\partial x_k} \frac{\partial}{\partial x_i} - \sum_{deg(Q_j)\geq 1} \frac{\partial Q_j}{\partial x_k} \frac{\partial}{\partial \xi_j} = 0 \quad \forall k = 1, ..., m$$

From this we deduce that $\frac{\partial P_i}{\partial x_k} = \frac{\partial Q_j}{\partial x_k} = 0\ \forall i,j,k$. Analogously if $[a, \frac{\partial}{\partial \xi_r}] = 0, \forall r = 1,...,n$ we obtain $\frac{\partial P_i}{\partial \xi_r} = \frac{\partial Q_j}{\partial \xi_r} = 0\ \forall i,j,r$. So $P_i, Q_j \in \mathbb{C}$ and this leads to a contradiction. □

4.1.2 Simplicity

Theorem 4.2. *$W(m,n)$ is simple if $(m,n) \neq (0,1)$.*

Proof. We consider $W(m,n)$ with the principal grading. We have: $[W(m,n)_{-1}, W(m,n)_1] = W(m,n)_0$, in fact it is obvious that $W(m,n)_0 \supset [W(m,n)_{-1}, W(m,n)_1]$, on the other hand $[W(m,n)_{-1}, W(m,n)_1] \supset W(m,n)_0$ because:

$$
\begin{aligned}
x_i\frac{\partial}{\partial x_j} &= [\frac{\partial}{\partial x_i}, \frac{x_i^2}{2}\frac{\partial}{\partial x_j}] \\
\xi_i\frac{\partial}{\partial x_j} &= [\frac{\partial}{\partial x_j}, x_j\xi_i\frac{\partial}{\partial x_j}] \\
x_i\frac{\partial}{\partial \xi_j} &= [\frac{\partial}{\partial \xi_j}, x_i\xi_j\frac{\partial}{\partial \xi_j}] \\
\xi_i\frac{\partial}{\partial \xi_j} &= [\frac{\partial}{\partial \xi_j}, \xi_j\xi_i\frac{\partial}{\partial \xi_j}] \quad if\ i \neq j
\end{aligned}
$$

$$
\xi_i\frac{\partial}{\partial \xi_i} = \begin{cases} [\frac{\partial}{\partial x_1}, x_1\xi_i\frac{\partial}{\partial \xi_i}] & if\ m \geq 1 \\ [\frac{\partial}{\partial \xi_k}, \xi_k\xi_i\frac{\partial}{\partial \xi_i}] & if\ n \geq 2 \end{cases}
$$

Now let I be a nonzero ideal of $W(m,n)$ and let us show that $I = W(m,n)$. Indeed, due to the irreducibility of $W(m,n)_{-1}$ and the fact that $[I_{-1}, W(m,n)_0] \subset I_{-1}$, it follows $I_{-1} = 0$ or $I_{-1} = W(m,n)_{-1}$. In the first case we have that $[W(m,n)_{-1}, I_0] \subset I_{-1} = 0$, hence, by transitivity, $I_0 = 0$ and, proceeding in the same way, $I_i = 0\ \forall i$ which is impossible because $I \neq 0$.
So $I_{-1} = W(m,n)_{-1}$ and $W(m,n)_0 = [W(m,n)_{-1}, W(m,n)_1] \subset I$, hence it remains to show that a generic element of the type $PQ\frac{\partial}{\partial x_i}$ or $PQ\frac{\partial}{\partial \xi_j}$ lies in I, where $P \in \mathbb{C}[x_1,...,x_m]$ and $Q \in \Lambda(n)$. Suppose $m \geq 1$: we denote by $\tilde{P}$

an element of $\mathbb{C}[x_1, ..., x_m]$ such that $\frac{\partial \tilde{P}}{\partial x_i} = P$. It follows:

$$PQ\frac{\partial}{\partial x_i} = [\frac{\partial}{\partial x_i}, \tilde{P}Q\frac{\partial}{\partial x_i}]$$
$$PQ\frac{\partial}{\partial \xi_j} = [\frac{\partial}{\partial x_i}, \tilde{P}Q\frac{\partial}{\partial \xi_j}]$$

Now suppose $m = 0$ and $n \geq 2$: we show that a generic element of the type $Q\frac{\partial}{\partial \xi_i}$, with $deg(Q) \geq 2$ lies in I. Indeed, since $deg(Q) \geq 2$, there exists some $k \neq i$, such that:

$$Q\frac{\partial}{\partial \xi_i} = [\xi_k\frac{\partial}{\partial \xi_k}, Q\frac{\partial}{\partial \xi_i}]$$

We conclude $I = W(m, n)$. □

Remark 7. We now analyze the case $(m, n) = (0, 1)$ and notice that $W(0, 1)$ is not simple.

$$W(0, 1) = < \frac{\partial}{\partial \xi}, \xi\frac{\partial}{\partial \xi} >$$

and

$$[W(0, 1), W(0, 1)] = < \frac{\partial}{\partial \xi} > \subsetneq W(0, 1)$$

4.1.3 Subprincipal grading

Let us consider the subrincipal grading of $W(m, n)$, i.e., the grading of type $(1, ..., 1|0, ...0)$. We have:

$$\begin{aligned} W(m, n)_0 &= < P_i\frac{\partial}{\partial \xi_i}, P_i \in \Lambda(n) > + < x_i P_l\frac{\partial}{\partial x_j}, P_l \in \Lambda(n) > \\ &\cong \mathfrak{gl}(m) \otimes \Lambda(n) + W(0, n) \end{aligned}$$

The isomorphism is:

$$\begin{aligned} \Phi : W(m, n)_0 &\longrightarrow \mathfrak{gl}(m) \otimes \Lambda(n) + W(0, n) \\ x_i \otimes P_l\frac{\partial}{\partial x_j} + P_k\frac{\partial}{\partial \xi_r} &\longmapsto e_{i,j} \otimes P_l + P_k\frac{\partial}{\partial \xi_r} \end{aligned}$$

On the other hand we have that:

$$W(m, n)_{-1} = < P_i\frac{\partial}{\partial x_j},\ P_i \in \Lambda(n) > \cong \mathbb{C}^m \otimes \Lambda(n)$$

We observe that $W(m, n)$ with the subprincipal grading has depth 1.

Proposition 4.3. *$W(m,n)$ with the subprincipal grading is irreducible.*

Proof. Let $S \neq 0$ be a submodule of $W(m,n)_{-1} \cong \mathbb{C}^m \otimes \Lambda(n)$ and $z \in S$ a nonzero element. Then z is of the form:

$$z = \sum_k \alpha_k P_k \frac{\partial}{\partial x_k} \quad where \quad P_k \in \Lambda(n), \alpha_k \in \mathbb{C}$$

Let us suppose $\alpha_i \neq 0$ for an index i, we have:

$$[x_i \frac{\partial}{\partial x_1}, z] = -\alpha_i P_i \frac{\partial}{\partial x_1} \in S$$

We recall that $W(m,n)_0 \cong \mathfrak{gl}(m) \otimes \Lambda(n) \oplus W(0,n)$. By the action of $\mathfrak{gl}(m)$ on $\frac{\partial}{\partial x_1}$ we generate $P_i \otimes \mathbb{C}^m$. Moreover by the action of $W(0,n)$ on P_i we generate $1 \otimes \mathbb{C}^m$, finally by the action of $\mathfrak{gl}(m) \otimes \Lambda(n)$ on $1 \otimes \mathbb{C}^m$ we generate $\mathbb{C}^m \otimes \Lambda(n)$. □

Proposition 4.4. *$W(m,n)$ with the subprincipal grading is transitive.*

Proof. Let a be an element of $W(m,n)_{\geq 0}$ such that $[a, W(m,n)_{-1}] = 0$. The element a is of the form:

$$a = \sum_{deg(P_i)\geq 1} P_i Q_i \frac{\partial}{\partial x_i} + \sum_{deg(\tilde{P}_j)\geq 0} \tilde{P}_j \tilde{Q}_j \frac{\partial}{\partial \xi_j}, \quad P_i, \tilde{P}_j \in \mathbb{C}[x_1, ..., x_m],\ Q_i, \tilde{Q}_j \in \Lambda(n)$$

We have:

$$0 = [a, \frac{\partial}{\partial x_k}] = -\sum_{deg(P_i)\geq 1} \frac{\partial}{\partial x_k}(P_i Q_i) \frac{\partial}{\partial x_i} - \sum_{deg(\tilde{P}_j)\geq 0} \frac{\partial}{\partial x_k}(\tilde{P}_j \tilde{Q}_j) \frac{\partial}{\partial \xi_j}$$

We obtain that:

$$\frac{\partial}{\partial x_k}(P_i Q_i) = 0 \quad \forall i, k$$
$$\frac{\partial}{\partial x_k}(\tilde{P}_j \tilde{Q}_j) = 0 \quad \forall j, k$$

So we get that $P_i, \tilde{P}_j \in \mathbb{C}$, but $deg(P_i) \geq 1\ \forall i$, so $P_i = 0\ \forall i$. Therefore, including now the constants $\tilde{P}_j$ in the elements $\tilde{Q}_j$, $a = \sum \tilde{Q}_j \frac{\partial}{\partial \xi_j}$. Moreover we also know:

$$0 = [a, T(\xi_1, ..., \xi_n) \frac{\partial}{\partial x_k}]$$

This means:

$$\sum \tilde{Q}_j \frac{\partial T(\xi_1, ..., \xi_n)}{\partial \xi_j} \frac{\partial}{\partial x_k} = 0$$

Finally $\tilde{Q}_j \frac{\partial T(\xi_1,...,\xi_n)}{\partial \xi_j} = 0\ \forall j,\ \forall T \in \Lambda(n)$. In particular we choose $T = \xi_h$ $\forall h = 1, ..., n$ and get:

$$0 = \tilde{Q}_h \frac{\partial}{\partial \xi_h} \xi_h = \tilde{Q}_h$$

□

4.1.4 Symmetric gradings

Definition 4.1. (Symmetric grading) A $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$ is said symmetric if $\mathfrak{g} = \oplus_{i=-h}^{k} \mathfrak{g}_i$ with $h = k < \infty$.

Definition 4.2. (Strongly symmetric grading) A $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$ is said strongly symmetric if it is symmetric, transitive, generated by its local part and $\mathfrak{g}_{-i} \cong \mathfrak{g}_i$ as vector spaces $\forall i$.

Definition 4.3. (Strongly symmetric grading of length five (resp. three)) A $\mathbb{Z}$-grading on a Lie superalgebra $\mathfrak{g}$ is said strongly symmetric of length five (resp. three) if it is strongly symmetric and $h = k = 2$ (resp. $h = k = 1$).

Our aim is to obtain a complete list, up to isomorphisms, of strongly symmetric gradings of length five of the Lie superalgebra $W(m, n)$.

Remark 8. Notice that if $\deg(x_i) \neq 0$ for some i, then the length of the grading is not finite. Therefore a grading of $W(m, n)$ has finite length if an only if it is of type $(0, ..., 0|b_1, ..., b_n)$.

Remark 9.

1. If there exists an index $i \in \{1, ..., m\}$ such that a_i is odd, then the $\mathbb{Z}$-grading is not consistent. In fact $\frac{\partial}{\partial x_i} \in W_{\bar{0}}$ would lie in W_{-a_i}.

2. If there exists an index $j \in \{1, ..., n\}$ such that $|b_j|$ is even, then the $\mathbb{Z}$-grading is not consistent. In fact $\frac{\partial}{\partial \xi_j} \in W_{\bar{1}}$ would lie in W_{-b_j}.

3. A $\mathbb{Z}$-grading of type $(a_1,...,a_m|b_1,...,b_n)$, where all a_i's are even and all b_j's are odd, is consistent. Indeed, let $P \in \mathbb{C}[x_1,...,x_m]$ and $Q \in \Lambda(n)$: $deg(PQ\frac{\partial}{\partial x_i}) = deg(P)+deg(Q)+deg(\frac{\partial}{\partial x_i})$ and $deg(PQ\frac{\partial}{\partial \xi_i}) = deg(P)+deg(Q)+deg(\frac{\partial}{\partial \xi_i})$.

Now we start our analysis from $W(0,n)$ and then generalize it to $W(m,n)$.

4.1.5 $W(0,n)$, $n \geq 2$

First we consider a grading of type $(|b_1,...,b_n)$ where $b_i > 0\ \forall i$. We denote by k the maximal degree and by $-h$ the minimal degree of elements of $W(0,n)$ in this grading. It follows:

$$k = b_1 + b_2 + ... + b_n - min\,\{b_i\} \quad h = max\,\{b_i\}$$

So:

$$h = k \Leftrightarrow\ b_1 + b_2 + ... + b_n - min\,\{b_i\} = max\,\{b_i\} \Leftrightarrow\ n = 2$$

We first study the case $n = 2$.

A) $W(0,2)$

i) $\mathbb{Z}$-grading of type $(|b,B)$ where $0 < b \leq B$.

In this case $h = k = B$ and the degree that we can obtain are: $-b, -B, 0, b-B, B-b, B, b$

Remark 10. We are interested in $\mathbb{Z}$-grading such that $\mathfrak{g}_{-1} \neq 0$, so if a $\mathbb{Z}$-grading is such that $\mathfrak{g}_{-l} \neq 0$ for some $l > 0$ and $\mathfrak{g}_{-i} = 0$ for every $0 < i < l$, then we assume, up to isomorphisms, $l = 1$.

- If $B = b$ the grading becomes of type $(|b,b)$, we suppose $b = 1$. We have:

$$W(0,2)_{-1} =< \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} >\cong \mathbb{C}^2$$
$$W(0,2)_0 =< \xi_i\frac{\partial}{\partial \xi_j} >\cong \mathfrak{gl}(2)$$
$$W(0,2)_1 =< \xi_1\xi_2\frac{\partial}{\partial \xi_1}, \xi_1\xi_2\frac{\partial}{\partial \xi_2} >\cong \mathbb{C}^2$$

It is consistent and generated by its local part.

- If $B > b$ and $B = 2b$, with $b = 1$, so that $b - B = -b$, we have:

$$W(0,2)_{-2} =< \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{-1} =< \frac{\partial}{\partial \xi_1}, \xi_1 \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_0 =< \xi_i \frac{\partial}{\partial \xi_i} >$$
$$W(0,2)_1 =< \xi_1\xi_2 \frac{\partial}{\partial \xi_2}, \xi_2 \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_2 =< \xi_1\xi_2 \frac{\partial}{\partial \xi_1} >$$

It is generated by its local part and it is not consistent.

- If $B > b$ and $B > 2b$ so that $-b > b-B$, we have, assuming $b = 1$:

$$W(0,2)_{-B} =< \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{1-B} =< \xi_1 \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{-1} =< \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_0 =< \xi_i \frac{\partial}{\partial \xi_i} >$$
$$W(0,2)_1 =< \xi_1\xi_2 \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_{B-1} =< \xi_2 \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_B =< \xi_1\xi_2 \frac{\partial}{\partial \xi_2} >$$

It is not generated by its local part since $[W(0,2)_{-1}, W(0,2)_{-1}] = 0$.

- If $B > b$ and $B < 2b$ so that $-b < b - B$, we have, choosing

$b - B = 1$:

$$W(0,2)_{-b-1} =< \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{-b} =< \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_{-1} =< \xi_1 \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{0} =< \xi_i \frac{\partial}{\partial \xi_i} >$$
$$W(0,2)_{1} =< \xi_2 \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_{b} =< \xi_1\xi_2 \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{b+1} =< \xi_1\xi_2 \frac{\partial}{\partial \xi_1} >$$

It is not generated by its local part, since $[W(0,2)_{-1}, W(0,2)_{-1}] = 0$.

ii) $\mathbb{Z}$-grading of type $(|0,a)$ where $a > 0$.

We observe that $h = k = a$, so we choose $a = 1$. We have:

$$W(0,2)_{-1} =< \frac{\partial}{\partial \xi_2}, \xi_1 \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{0} =< \xi_i \frac{\partial}{\partial \xi_i}, \xi_1\xi_2 \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_{1} =< \xi_1\xi_2 \frac{\partial}{\partial \xi_1}, \xi_2 \frac{\partial}{\partial \xi_1} >$$

It is not consistent.

iii) $\mathbb{Z}$-grading of type $(|a,-b)$ where $a, b > 0$.

We observe that $h = k = a + b$, we can obtain the degrees $-b - a, -a, -b, 0, b, a, a + b$.

- If $a = b = 1$, then $h = k = 2$, we have:

$$W(0,2)_{-2} =< \xi_2 \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_{-1} =< \frac{\partial}{\partial \xi_1}, \xi_1\xi_2 \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_{0} =< \xi_i \frac{\partial}{\partial \xi_i} >$$
$$W(0,2)_{1} =< \xi_1\xi_2 \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{2} =< \xi_1 \frac{\partial}{\partial \xi_2} >$$

It is consistent and generated by its local part.

- If $a > b$, we have, choosing $b = 1$:

$$W(0,2)_{-a-1} =< \xi_2 \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_{-a} =< \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_{-1} =< \xi_1\xi_2 \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_{0} =< \xi_i \frac{\partial}{\partial \xi_i} >$$
$$W(0,2)_{1} =< \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{a} =< \xi_1\xi_2 \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{a+1} =< \xi_1 \frac{\partial}{\partial \xi_2} >$$

It is not generated by its local part, since $[W(0,2)_{-1}, W(0,2)_{-1}] = 0$.

- If $b > a$, we have a situation analogous to the previous one, obtaining a grading which is not generated by its local part.

Now we study $W(0,n)$, with $n \geq 3$.

B) $W(0,n)$**,** $n \geq 3$

We saw in the previous section that in this case there is no symmetric $\mathbb{Z}$-grading of type $(|b_1,...,b_n)$ where $b_i > 0$ $\forall i$. So the following cases are left: $b_i \geq 0$ for every i and $b_j = 0$ for some j, or $b_i > 0$ and $b_j < 0$ for some $i \neq j$. We observe that in both these options it follows that $W(m,n) = \oplus_{i=-h}^{k} W(m,n)_i$ with $h, k < \infty$ and:

$$h = \sum_{b_i \leq 0} |b_i| + max\{b_i \geq 0\}$$
$$k = \sum_{b_i \geq 0} b_i + |min\{b_i \leq 0\}|$$

Then, if we set $b_1 = max\{b_i\}$, $b_2 = min\{b_i\}$:

$$\begin{aligned} h = k &\Leftrightarrow \\ b_1 + ... + b_n = max\{b_i \geq 0\} - |min\{b_i \leq 0\}| &\Leftrightarrow \\ b_1 + ... + b_n = max\{b_i \geq 0\} + min\{b_i \leq 0\} &\Leftrightarrow \\ b_1 + ... + b_n = max\{b_i\} + min\{b_i\} &\Leftrightarrow \\ b_3 + ... + b_n = 0 & \end{aligned}$$

Remark 11. If a $\mathbb{Z}$-grading is of type $(|b_1,...,b_n)$ such that $b_i > 0$ and $b_j < 0$ for some $i \neq j$, it is sufficient, in order to study symmetric gradings of length five, to analyze a grading of type $(|B, b, 0, ..., 0)$, with $B > 0$, $b < 0$. Indeed if $b_i, b_j, b_k \neq 0$ for some distinct i, j, k, since $b_i > 0$ and $b_j < 0$, then $deg(\xi_i \xi_k \frac{\partial}{\partial \xi_j}) = b_i + b_k - b_j \geq 3$ if $b_k > 0$ or $deg(\xi_j \xi_k \frac{\partial}{\partial \xi_i}) = b_j + b_k - b_i \leq -3$ if $b_k < 0$.

i) $\mathbb{Z}$-grading of type $(|b_1,...,b_n)$ where $b_i \geq 0$ for every i and $b_j = 0$ for some j, it follows that $h = k$ if and only if the grading is of type $(|a, 0, ..., 0)$

where $a > 0$. We have, choosing $a = 1$:

$$W(0,n)_{-1} = < \frac{\partial}{\partial \xi_1} > \otimes \Lambda(\xi_2, ..., \xi_n)$$
$$W(0,n)_{1} = < \xi_1 \frac{\partial}{\partial \xi_2}, \xi_1 \frac{\partial}{\partial \xi_3}, ..., \xi_1 \frac{\partial}{\partial \xi_n} > \otimes \Lambda(\xi_2, ..., \xi_n)$$

Since $n \geq 3$, $dim(< \xi_1 \frac{\partial}{\partial \xi_2}, \xi_1 \frac{\partial}{\partial \xi_3}, ..., \xi_1 \frac{\partial}{\partial \xi_n} >) \geq 2$ and $W(0,n)_{-1} \ncong W(0,n)_1$.

ii) $\mathbb{Z}$-grading of type $(|b_1, ..., b_n)$such that $b_i > 0$ and $b_j < 0$ for some $i \neq j$. In particular we analyze a grading of type $(|B, b, 0, ..., 0)$, with $B > 0$, $b < 0$, in fact this is sufficient in order to study symmetric gradings of length five by Remark 11. We have $h = k = -b + B$, the possible degrees are $-B, b, b - B, B, -b, B - b, 0, B + b$. We notice that surely $-b - B \neq -B, b - B, -b, B - b$. So only the following possibilities remain: $-b - B = b$, $-b - B = B$ and $-b - B = 0$, which can be rewritten as $B = 2|b|$, $|b| = 2B$ and $|b| = B$. If no one of these hold, then $dim(W(0,n)_{-b-B}) = 0$ and $dim(W(0,n)_{b+B}) > 0$.

- $B = 2|b|$, we choose $|b| = 1$, so that the grading is $(|2, -1, 0, ..., 0)$. We have:

$$W(0,n)_{-2} = < \frac{\partial}{\partial \xi_1} > \otimes \Lambda(\xi_3, ..., \xi_n)$$
$$W(0,n)_{2} = < \xi_1 \xi_2 \frac{\partial}{\partial \xi_2}, \xi_1 \frac{\partial}{\partial \xi_3}, ..., \xi_1 \frac{\partial}{\partial \xi_n} > \otimes \Lambda(\xi_3, ..., \xi_n)$$

  It is not symmetric since $W(0,n)_{-2} \ncong W(0,n)_2$.

- $|b| = 2B$, we choose $B = 1$, so that the grading is $(|1, -2, 0, ..., 0)$. This is analogous to the previous one, $W(0,n)_{-2} \ncong W(0,n)_2$.

- $|b| = B$, we choose $B = 1$, so that the grading is $(|1,-1,0,...,0)$.

$$W(0,n)_{-2} =< \xi_2\frac{\partial}{\partial\xi_1} > \otimes\Lambda(\xi_3,...,\xi_n)$$
$$W(0,n)_{-1} =< \frac{\partial}{\partial\xi_1}, \xi_1\xi_2\frac{\partial}{\partial\xi_1}, \xi_2\frac{\partial}{\partial\xi_3}, ..., \xi_2\frac{\partial}{\partial\xi_n} > \otimes\Lambda(\xi_3,...,\xi_n)$$
$$W(0,n)_0 =< \xi_i\frac{\partial}{\partial\xi_i}, \xi_1\xi_2\frac{\partial}{\partial\xi_3}, ..., \xi_1\xi_2\frac{\partial}{\partial\xi_n}, \frac{\partial}{\partial\xi_3}, ..., \frac{\partial}{\partial\xi_n} > \otimes\Lambda(\xi_3,...,\xi_n)$$
$$W(0,n)_1 =< \frac{\partial}{\partial\xi_2}, \xi_1\xi_2\frac{\partial}{\partial\xi_2}, \xi_1\frac{\partial}{\partial\xi_3}, ..., \xi_1\frac{\partial}{\partial\xi_n} > \otimes\Lambda(\xi_3,...,\xi_n)$$
$$W(0,n)_2 =< \xi_1\frac{\partial}{\partial\xi_2} > \otimes\Lambda(\xi_3,...,\xi_n)$$

This grading is symmetric, not consistent and generated by its local part.

4.1.6 $W(m,n)$, $m \geq 1$, $n \geq 1$

The analysis of the $\mathbb{Z}$-grading of type $(0,...,0|b_1,...,b_n)$ of the Lie superalgebra $W(m,n)$ is similar to that of the grading of type $(|b_1,...,b_n)$ of the Lie superalgebra $W(0,n)$. Indeed, the following relations still hold:

$$h = \sum_{b_i \leq 0} |b_i| + max\{b_i \geq 0\}$$
$$k = \sum_{b_i \geq 0} b_i + |min\{b_i \leq 0\}|$$

Then:

$$h = k \Leftrightarrow$$
$$b_1 + ... + b_n = max\{b_i \geq 0\} - |min\{b_i \leq 0\}| \Leftrightarrow$$
$$b_1 + ... + b_n = max\{b_i \geq 0\} + min\{b_i \leq 0\}$$

Remark 12. In these formulas it is tacit that if either $\{b_i \geq 0\} = \emptyset$ or $\{b_i \leq 0\} = \emptyset$ we mean that $max\{b_i \geq 0\}$ or respectively $min\{b_i \leq 0\}$ are substituted by a 0.

The possibilities become:

i) $\mathbb{Z}$-grading of type $(0,...,0|b_1,...,b_n)$ where $b_i \geq 0$ $\forall i$, it follows that $h = k$ if and only if the grading is of type $(0,...,0|a,0,...,0)$ where $a > 0$. We have, choosing $a = 1$:

$$W(m,n)_{-1} =< \frac{\partial}{\partial \xi_1} > \otimes \mathbb{C}[x_1,...,x_m] \otimes \Lambda(\xi_2,...,\xi_n)$$
$$W(m,n)_1 =< \xi_1\frac{\partial}{\partial \xi_2}, \xi_1\frac{\partial}{\partial \xi_3},...,\xi_1\frac{\partial}{\partial \xi_n}, \xi_1\frac{\partial}{\partial x_1},...,\xi_1\frac{\partial}{\partial x_m} > \otimes \mathbb{C}[x_1,...,x_m]$$
$$\otimes \Lambda(\xi_2,...,\xi_n)$$

$W(m,n)_{-1} \cong W(m,n)_1$ if and only if $m = 1$ and $n = 1$, the grading becomes $(0|1)$. Indeed:

$$W(1,1)_{-1} =< \frac{\partial}{\partial \xi} > \otimes \mathbb{C}[x]$$
$$W(1,1)_0 =< P(x)\frac{\partial}{\partial x}, Q(x)\xi\frac{\partial}{\partial \xi} >$$
$$W(1,1)_1 =< \xi\frac{\partial}{\partial x} > \otimes \mathbb{C}[x]$$

This strongly symmetric grading of $W(m,n)$ of length three is not present in the list given in [1] because $W(1,1) \cong K(1,2)$ (for the definition of the Lie superalgebra $K(1,2)$ see [9]). We give a description of it.

$$W(1,1)_0 =< \xi\frac{\partial}{\partial \xi} > \otimes \mathbb{C}[x] + < \frac{\partial}{\partial x} > \otimes \mathbb{C}[x] \cong$$
$$I \rtimes W(1,0)$$

where I is an abelian ideal isomorphic, as a $W(1,0)-$module, to $\mathbb{C}[x]$. Indeed:

$$[P(x)\frac{\partial}{\partial x}, Q(x)\xi\frac{\partial}{\partial \xi}] = P\frac{\partial Q}{\partial x}\xi\frac{\partial}{\partial \xi}$$

$W(1,1)_{-1}$ is isomorphic, as a module, to $\mathbb{C}[x]$, $W(1,0)$ acts naturally on it, meanwhile I acts by multiplication on it. Indeed:

$$[P(x)\frac{\partial}{\partial x}, Q(x)\frac{\partial}{\partial \xi}] = P\frac{\partial Q}{\partial x}\frac{\partial}{\partial \xi}$$
$$[P(x)\xi\frac{\partial}{\partial \xi}, Q(x)\frac{\partial}{\partial \xi}] = -PQ\frac{\partial}{\partial \xi}$$

ii) $\mathbb{Z}$-grading of type $(0,...,0|b_1,...,b_n)$ where there exist a $b_i > 0$ and a $b_j < 0$ for some $i \neq j$. We focus on $(0,...,0|B,b,0,...,0)$, with $B > 0$, $b < 0$. Slightly adjusting the case ii) of $W(0,n), n \geq 3$, we obtain that the only symmetric grading, in which $W(m,n)$ is generated by its local part, is $(0,...,0|1,-1,0,...,0)$

Therefore we have proved the following results:

Theorem 4.5. *1. If $(m,n) \neq (0,2),(1,1)$ the Lie superalgebra $W(m,n)$ has no strongly symmetric $\mathbb{Z}-$gradings of length three.*

2. A complete list, up to isomorphisms, of strongly symmetric $\mathbb{Z}-$gradings of length three of the Lie superalgebras $W(0,2)$ and $W(1,1)$ is the following:

(a) $(|1,1)$

(b) $(|0,1)$

(c) $(0|1)$

Theorem 4.6. *A complete list, up to isomorphism, of strongly symmetric $\mathbb{Z}-$gradings of length five of the Lie superalgebra $W(m,n)$ is the following:*

1. $(|1,2)$ *for* $m=0$, $n=2$

2. $(0,...,0|1,-1,0,...,0)$

Remark 13. Neither $(|1,2)$ nor $(0,...,0|1,-1,0,...,0)$ is consistent.

We now give a description of these gradings:

- $(|1, 2)$:

$$W(0,2)_{-2} =< \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_{-1} =< \frac{\partial}{\partial \xi_1}, \xi_1 \frac{\partial}{\partial \xi_2} >$$
$$W(0,2)_0 =< \xi_i \frac{\partial}{\partial \xi_i} >$$
$$W(0,2)_1 =< \xi_1\xi_2 \frac{\partial}{\partial \xi_2}, \xi_2 \frac{\partial}{\partial \xi_1} >$$
$$W(0,2)_2 =< \xi_1\xi_2 \frac{\partial}{\partial \xi_1} >$$

We have that $W(0,2)_0$ is an abelian Lie algebra of dimension two. $W(0,2)_{-2}$ and $W(0,2)_2$ are $W(0,2)_0$-modules of dimension 1 isomorphic to $\mathbb{C}$. $W(0,2)_{-1} =< \frac{\partial}{\partial \xi_1} > \oplus < \xi_1 \frac{\partial}{\partial \xi_2} >$, where $< \frac{\partial}{\partial \xi_1} >$ and $< \xi_1 \frac{\partial}{\partial \xi_2} >$ are $W(0,2)_0$-modules of dimension 1. Finally $W(0,2)_1 =< \xi_1\xi_2 \frac{\partial}{\partial \xi_2} > \oplus < \xi_2 \frac{\partial}{\partial \xi_1} >$ which are $W(0,2)_0$-modules of dimension 1.

- $(0, ..., 0|1, -1, 0, ..., 0)$:

$$W(m,n)_{-2} =< \xi_2 \frac{\partial}{\partial \xi_1} > \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2)$$
$$W(m,n)_{-1} =< \xi_2 > \otimes W(m, n-2) \oplus$$
$$(< \frac{\partial}{\partial \xi_1} > \oplus < \xi_1\xi_2 \frac{\partial}{\partial \xi_1} >) \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2)$$
$$W(m,n)_0 =< \xi_1\xi_2 > \otimes W(m, n-2) \rtimes (< \xi_1 \frac{\partial}{\partial \xi_1}, \xi_2 \frac{\partial}{\partial \xi_2} > \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2) \oplus$$
$$W(m, n-2))$$
$$W(m,n)_1 =< \xi_1 > \otimes W(m, n-2) \oplus$$
$$(< \frac{\partial}{\partial \xi_2} > \oplus < \xi_1\xi_2 \frac{\partial}{\partial \xi_2} >) \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2)$$
$$W(m,n)_2 =< \xi_1 \frac{\partial}{\partial \xi_2} > \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2)$$

where by $\Lambda(n-2)$ we mean $\Lambda(\xi_3, ..., \xi_n)$. $W(m,n)_0$ is not simple since it contains a non trivial abelian ideal, i.e. $I := <\xi_1\xi_2> \otimes W(m, n-2)$. $<\xi_1\frac{\partial}{\partial\xi_1}, \xi_2\frac{\partial}{\partial\xi_2}> \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2)$ acts on I by multiplication, indeed let $P, \tilde{P} \in \mathbb{C}[x_1, ..., x_m]$ and $Q, \tilde{Q} \in \Lambda(n-2)$:

$$[PQ\xi_1\frac{\partial}{\partial\xi_1}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial x_j}] = PQ\tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial x_j}$$
$$[PQ\xi_1\frac{\partial}{\partial\xi_1}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial\xi_j}] = PQ\tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial\xi_j}$$
$$[PQ\xi_2\frac{\partial}{\partial\xi_2}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial x_j}] = PQ\tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial x_j}$$
$$[PQ\xi_2\frac{\partial}{\partial\xi_2}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial\xi_j}] = PQ\tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial\xi_j}$$

$W(m, n-2)$ acts on I by adjoint action, let $X, Y \in W(m, n-2)$:

$$[X, \xi_1\xi_2 Y] = \xi_1\xi_2[X, Y]$$

The grading is not irreducible, indeed $<\xi_1\xi_2\frac{\partial}{\partial\xi_1}> \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2)$ is a proper submodule of $W(m,n)_{-1}$.
I and $<\xi_1\frac{\partial}{\partial\xi_1}> \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2)$ act trivially on this submodule, $<\xi_2\frac{\partial}{\partial\xi_2}> \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2)$ acts by multiplication:

$$[P\xi_1\xi_2 Q\frac{\partial}{\partial x_j}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}] = 0$$
$$[P\xi_1\xi_2 Q\frac{\partial}{\partial\xi_j}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}] = 0$$
$$[P\xi_1 Q\frac{\partial}{\partial\xi_1}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}] =$$
$$P\tilde{P}\xi_1 Q\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1} - (-1)^{p(Q)(p(\tilde{Q})+1)}P\tilde{P}\xi_1\xi_2\tilde{Q}Q\frac{\partial}{\partial\xi_1} =$$
$$P\tilde{P}\xi_1 Q\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1} - P\tilde{P}\xi_1 Q\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1} = 0$$
$$[P\xi_2 Q\frac{\partial}{\partial\xi_2}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}] = -P\tilde{P}\xi_2 Q\xi_1\tilde{Q}\frac{\partial}{\partial\xi_1}$$

$W(m, n-2)$ acts on it by derivation:

$$[PQ\frac{\partial}{\partial x_j}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial \xi_1}] = P\frac{\partial \tilde{P}}{\partial x_j}Q\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial \xi_1}$$
$$[PQ\frac{\partial}{\partial \xi_j}, \tilde{P}\xi_1\xi_2\tilde{Q}\frac{\partial}{\partial \xi_1}] = P\tilde{P}Q\xi_1\xi_2\frac{\partial \tilde{Q}}{\partial \xi_j}\frac{\partial}{\partial \xi_1}$$

I acts trivially on $W(m, n)_{-2}$:

$$[P\xi_1\xi_2Q\frac{\partial}{\partial \xi_j}, \tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial \xi_1}] = 0$$
$$[P\xi_1\xi_2Q\frac{\partial}{\partial x_j}, \tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial \xi_1}] = 0$$

$< \xi_1\frac{\partial}{\partial \xi_1}, \xi_2\frac{\partial}{\partial \xi_2} > \otimes \mathbb{C}[x_1, ..., x_m] \otimes \Lambda(n-2)$ acts on $W(m, n)_{-2}$ by multiplication. $W(m, n-2)$ acts on $W(m, n)_{-2}$ by derivation:

$$[PQ\frac{\partial}{\partial x_j}, \tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial \xi_1}] =$$
$$P\frac{\partial \tilde{P}}{\partial x_j}Q\xi_2\tilde{Q}\frac{\partial}{\partial \xi_1}$$
$$[PQ\frac{\partial}{\partial \xi_j}, \tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial \xi_1}] =$$
$$-PQ\xi_2\frac{\partial \tilde{Q}}{\partial \xi_j}\frac{\partial}{\partial \xi_1}$$

4.2 The Lie superalgebra $S'(m, n)$

We call divergence of a vector field $D = \sum_{i=1}^m f_i\frac{\partial}{\partial x_i} + \sum_{i=1}^n g_i\frac{\partial}{\partial \xi_i} \in W(m, n)$ the expression:

$$divD = \sum_{i=1}^m \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n (-1)^{p(g_i)}\frac{\partial g_i}{\partial \xi_i}$$

We denote by $S'(m, n)$ the subspace of $W(m, n)$ consisting of vector fields with zero divergence, $S'(m, n)$ is a subalgebra of $W(m, n)$.

Moreover we call $S(m, n)$ the derived algebra of $S'(m, n)$. A $\mathbb{Z}$-grading on $W(m, n)$ induces gradings on $S'(m, n)$ and $S(m, n)$.

4.2.1 The principal grading

The principal grading of $W(m,n)$ induces a grading on $S'(m,n)$ that we still call principal. With respect to this grading:

$$S'(m,n) = \oplus_{i=-1}^{\infty} S'(m,n)_i$$

where:

$$S'(m,n)_0 =< x_i\frac{\partial}{\partial x_i} + \xi_j\frac{\partial}{\partial \xi_j}, x_i\frac{\partial}{\partial x_j}, \xi_i\frac{\partial}{\partial \xi_j}, x_i\frac{\partial}{\partial \xi_j}, \xi_i\frac{\partial}{\partial x_j},\ i \neq j >\cong \mathfrak{sl}(m,n)$$

The isomorphism is given by the map:

$$\begin{aligned} \Phi : S'(m,n)_0 &\longrightarrow \mathfrak{sl}(m,n) \\ x_i\frac{\partial}{\partial x_i} + \xi_j\frac{\partial}{\partial \xi_j} &\longmapsto e_{i,i} + e_{j+m,j+m} \\ x_j\frac{\partial}{\partial x_j} \quad i \neq j &\longmapsto e_{i,j} \\ \xi_i\frac{\partial}{\partial \xi_j} \quad i \neq j &\longmapsto e_{i+m,j+m} \\ x_i\frac{\partial}{\partial \xi_j} &\longmapsto e_{i,j+m} \end{aligned}$$

Moreover:

$$S'(m,n)_{-1} =< \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial \xi_1}, ..., \frac{\partial}{\partial \xi_n} >\cong \mathbb{C}^{m|n}$$

So with this grading $W(m,n)$ is irreducible because $S'(m,n)_{-1}$ acts via the standard action on $S'(m,n)_0$.

Proposition 4.7. *$S'(m,n)$ with the principal grading is transitive.*

Proof. It follows from the transitivity of the principal grading of $W(m,n)$. □

4.2.2 Simplicity

Theorem 4.8. *$S'(m,n)$ is simple if $m > 1$ or $m = 0$ and $n \geq 3$.*

Proof. We prove the simplicity of $S'(m,n)$ using the principal grading. First we observe that $[S'(m,n)_{-1}, S'(m,n)_1] = S'(m,n)_0$, in fact it is sufficient to show that $[S'(m,n)_{-1}, S'(m,n)_1] \supset S'(m,n)_0$:

$$x_i \frac{\partial}{\partial x_j} = [\frac{\partial}{\partial x_i}, \frac{x_i^2}{2}\frac{\partial}{\partial x_j}] \quad i \neq j$$

$$\xi_i \frac{\partial}{\partial \xi_j} = \begin{cases} [\frac{\partial}{\partial \xi_k}, \xi_k \xi_i \frac{\partial}{\partial \xi_j}] & if \quad n \geq 3 \quad and \quad m = 0 \\ [\frac{\partial}{\partial x}, x\xi_i \frac{\partial}{\partial \xi_j}] & if \quad m > 1 \end{cases} \quad i \neq j$$

$$x_i \frac{\partial}{\partial x_i} + \xi_j \frac{\partial}{\partial \xi_j} = [\frac{\partial}{\partial x_k}, x_k x_i \frac{\partial}{\partial x_i} + \xi_j x_k \frac{\partial}{\partial \xi_j}] \quad \exists x_k \neq x_i \quad because \quad m > 1$$

$$x_i \frac{\partial}{\partial \xi_j} = [\frac{\partial}{\partial x_k}, x_k x_i \frac{\partial}{\partial \xi_j}] \quad \exists x_k \neq x_i \quad because \quad m > 1$$

$$\xi_i \frac{\partial}{\partial x_j} = [\frac{\partial}{\partial x_k}, x_k \xi_i \frac{\partial}{\partial x_j}] \quad \exists x_k \neq x_j \quad because \quad m > 1$$

$$x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} = \frac{1}{4}[\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}, (x_i^2 + 2x_i x_j + x_j^2)\frac{\partial}{\partial x_i} + \\ - (x_i^2 + 2x_i x_j + x_j^2)\frac{\partial}{\partial x_j}] + \\ - x_j \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_j} \quad i \neq j$$

$$\xi_i \frac{\partial}{\partial \xi_i} - \xi_j \frac{\partial}{\partial \xi_j} = \begin{cases} [\frac{\partial}{\partial x}, x\xi_i \frac{\partial}{\partial \xi_i} - x\xi_j \frac{\partial}{\partial \xi_j}] & m > 1 \\ [\frac{\partial}{\partial \xi_k}, \xi_k \xi_i \frac{\partial}{\partial \xi_i} - \xi_k \xi_j \frac{\partial}{\partial \xi_j}] & n \geq 3 \end{cases}$$

Now let I be a nonzero ideal. We will show that $I = S'(m,n)$. In fact from the irreducibility of $S'(m,n)_{-1}$ and the fact that $[I_{-1}, S'(m,n)_0] \subset I_{-1}$, it follows $I_{-1} = 0$ or $I_{-1} = S'(m,n)_{-1}$. In the first case we have that $[S'(m,n)_{-1}, I_0] \subset I_{-1} = 0$, by transitivity we have $I_0 = 0$ and, proceeding in the same way, $I_i = 0$ $\forall i$ which is impossible because $I \neq 0$.
It follows that $I_{-1} = S'(m,n)_{-1}$ and $S'(m,n)_0 = [S'(m,n)_{-1}, S'(m,n)_1] \subset I$. It remains to show that an element of degree higher than 0 lies in I. Let us first analyze the case $m = 0$ and $n \geq 3$.

It is enough to prove that a system of generators of $S'(m,n)_k, k < n-1$, lies in I. Indeed we have:

$$\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{k+1}}\frac{\partial}{\partial\xi_i} = \frac{1}{2}[\xi_{i_1}\frac{\partial}{\partial\xi_{i_1}} - \xi_i\frac{\partial}{\partial\xi_i}, \xi_{i_1}\xi_{i_2}\cdots\xi_{i_{k+1}}\frac{\partial}{\partial\xi_i}] \quad i \neq i_1,...,i_{k+1}$$

$$\xi_i\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}\frac{\partial}{\partial\xi_i} - \xi_j\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}\frac{\partial}{\partial\xi_j} =$$
$$[\xi_{i_1}\frac{\partial}{\partial\xi_{i_1}} - \xi_i\frac{\partial}{\partial\xi_i}, \xi_i\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}\frac{\partial}{\partial\xi_i} - \xi_j\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}\frac{\partial}{\partial\xi_j}]$$
$$i,j \neq i_1,...,i_{k+1},\ i \neq j$$

If $m > 1$ we have:

$$x_{j_1}^{\lambda_{j_1}}\cdots x_{j_l}^{\lambda_{j_l}}\xi_{i_1}\cdots\xi_{i_t}\frac{\partial}{\partial\xi_j} =$$
$$[\frac{\partial}{\partial x_{j_1}}, \frac{x_{j_1}^{\lambda_{j_1}+1}}{\lambda_{j_1}+1}\cdots x_{j_l}^{\lambda_{j_l}}\xi_{i_1}\cdots\xi_{i_t}\frac{\partial}{\partial\xi_j}] \quad j \neq i_1\cdots i_t, \lambda_{j_1}+...+\lambda_{j_l}+t = k+1$$
$$x_{j_1}^{\lambda_{j_1}}\cdots x_{j_l}^{\lambda_{j_l}}\xi_{i_1},...,\xi_{i_{t-1}}(-\xi_j\frac{\partial}{\partial\xi_j} - \frac{x_{j_s}}{\lambda_{j_s}+1}\frac{\partial}{\partial x_{j_s}}) =$$
$$[\frac{\partial}{\partial x_{j_s}}, x_{j_1}^{\lambda_{j_1}}\cdots\frac{x_{j_s}^{\lambda_{j_s}+1}}{\lambda_{j_s}+1}\cdots x_{j_l}^{\lambda_{j_l}}\xi_{i_1}\cdots\xi_{i_{t-1}}(-\xi_j\frac{\partial}{\partial\xi_j} - \frac{x_{j_s}}{(\lambda_{j_s}+2)}\frac{\partial}{\partial x_s})]$$
$$x_{j_1}^{\lambda_{j_1}}\cdots x_{j_l}^{\lambda_{j_l}}\frac{\partial}{\partial x_s} =$$
$$[\frac{\partial}{\partial x_{j_1}}, \frac{x_{j_1}^{\lambda_{j_1}+1}}{\lambda_{j_1}+1}\cdots x_{j_l}^{\lambda_{j_l}}\frac{\partial}{\partial x_s}] \quad s \neq \lambda_{j_1},...,\lambda_{j_l}, \lambda_{j_1}+...+\lambda_{j_l} = k+1$$
$$x_s^{\lambda_s}x_{j_1}^{\lambda_{j_1}}\cdots x_{j_l}^{\lambda_{j_l}}\frac{\partial}{\partial x_s} + \lambda_s x_s^{\lambda_s-1}x_{j_1}^{\lambda_{j_1}}\cdots x_{j_l}^{\lambda_{j_l}}\xi_i\frac{\partial}{\partial\xi_i} =$$
$$[\frac{\partial}{\partial x_s}, \frac{x_s^{\lambda_s+1}}{\lambda_s+1}x_{j_1}^{\lambda_{j_1}}\cdots x_{j_l}^{\lambda_{j_l}}\frac{\partial}{\partial x_s} + x_s^{\lambda_s}x_{j_1}^{\lambda_{j_1}}\cdots x_{j_l}^{\lambda_{j_l}}\xi_i\frac{\partial}{\partial\xi_i}]$$
$$x_s^{\lambda_s}x_{j_1}^{\lambda_{j_1}}\cdots x_{j_l}^{\lambda_{j_l}}\frac{\partial}{\partial x_s} - \lambda_s x_s^{\lambda_s-1}x_{j_1}^{\lambda_{j_1}}\cdots\frac{x_t^{\lambda_t+1}}{\lambda_t+1}\cdots x_{j_l}^{\lambda_{j_l}}\frac{\partial}{\partial x_t} =$$
$$[\frac{\partial}{\partial x_s}, \frac{x_s^{\lambda_s+1}}{\lambda_s+1}x_{j_1}^{\lambda_{j_1}}\cdots x_{j_l}^{\lambda_{j_l}}\frac{\partial}{\partial x_s} - x_s^{\lambda_s}x_{j_1}^{\lambda_{j_1}}\cdots\frac{x_t^{\lambda_t+1}}{\lambda_t+1}\cdots x_{j_l}^{\lambda_{j_l}}\frac{\partial}{\partial x_t}]$$

□

Remark 14. Let us analyze the cases in which $S'(m,n)$ is not simple.

i) $m=0$ and $n=1$, then $P(\xi)\frac{\partial}{\partial\xi}\in S'(0,1)$ if and only if $P(\xi)=a\in\mathbb{C}$, so $S'(0,1)=<\frac{\partial}{\partial\xi}>$ which is abelian.

ii) $m=0$ and $n=2$, we have:

$$S'(0,2)=<\xi_1\frac{\partial}{\partial\xi_2},\xi_2\frac{\partial}{\partial\xi_1},\xi_1\frac{\partial}{\partial\xi_1}-\xi_2\frac{\partial}{\partial\xi_2},\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_2}>$$

We notice that $<\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_2}>$ is a non trivial ideal.

iii) $m=1$ $\forall n$: $S'(1,n)=S(1,n)+\mathbb{C}\xi_1\xi_2\cdots\xi_n\frac{\partial}{\partial x}$.
First we show that $\xi_1\xi_2\cdots\xi_n\frac{\partial}{\partial x}\notin S(1,n)$, in fact:

$$\begin{aligned}&[P(x,\xi)\frac{\partial}{\partial x}+\sum_{l=1}^{n}Q_l(x,\xi)\frac{\partial}{\partial\xi_l},R(x,\xi)\frac{\partial}{\partial x}+\sum_{j=1}^{n}T_j(x,\xi)\frac{\partial}{\partial\xi_j}]= \qquad (4.1)\\ &P(x,\xi)\frac{\partial R(x,\xi)}{\partial x}\frac{\partial}{\partial x}-(-1)^{p(P)p(R)}R(x,\xi)\frac{\partial P(x,\xi)}{\partial x}\frac{\partial}{\partial x}+\\ &\sum_{j=1}^{n}(P(x,\xi)\frac{\partial T_j(x,\xi)}{\partial x}\frac{\partial}{\partial\xi_j}-(-1)^{p(P)(p(T_j)+1)}T_j\frac{\partial P}{\partial\xi_j}\frac{\partial}{\partial x})+\\ &\sum_{l=1}^{n}(Q_l(x,\xi)\frac{\partial R}{\partial\xi_l}\frac{\partial}{\partial x}-(-1)^{p(R)(p(Q_l)+1)}R(x,\xi)\frac{\partial Q_l(x,\xi)}{\partial x}\frac{\partial}{\partial\xi_l})+\\ &\sum_{l,j}[Q_l\frac{\partial}{\partial\xi_l},T_j\frac{\partial}{\partial\xi_j}]\end{aligned}$$

Therefore the term $\xi_1\xi_2\cdots\xi_n\frac{\partial}{\partial x}$ can come out from this bracket only if one of these holds:

- If $P(x,\xi)\frac{\partial R(x,\xi)}{\partial x}\frac{\partial}{\partial x}=\xi_1\cdots\xi_n\frac{\partial}{\partial x}$, then it should be $P=\xi_{i_1}\cdots\xi_{i_t}$, $R=x\xi_{i_{t+1}}\cdots\xi_{i_n}$, such that $\xi_{i_1}\cdots\xi_{i_t}\xi_{i_{t+1}}\cdots\xi_{i_n}=\xi_1\cdots\xi_n$. Then:

$$\begin{aligned}&P\frac{\partial}{\partial x}+\sum_{l=1}^{n}Q_l\frac{\partial}{\partial\xi_l}=\\ &\xi_{i_1}\cdots\xi_{i_t}\frac{\partial}{\partial x}+\sum_{l=1}^{n}Q_l\frac{\partial}{\partial\xi_l}\end{aligned}$$

The condition of null divergence gives:

$$\sum_{l=1}^{n}(-1)^{P(Q_l)}\frac{\partial Q_l}{\partial \xi_l}=0$$

And:

$$R(x,\xi)\frac{\partial}{\partial x}+\sum_{j=1}^{n}T_j(x,\xi)\frac{\partial}{\partial \xi_j}=$$

$$x\xi_{i_{t+1}}\cdots\xi_{i_n}\frac{\partial}{\partial x}+\sum_{j=1}^{n}T_j(x,\xi)\frac{\partial}{\partial \xi_j}$$

The condition of null divergence gives:

$$\xi_{i_{t+1}}\cdots\xi_{i_n}+\sum_{j=1}^{n}(-1)^{P(T_j)}\frac{\partial T_j}{\partial \xi_j}=0$$

In this case the terms of (4.1) that involve $\frac{\partial}{\partial x}$ become:

$$\xi_1\cdots\xi_n\frac{\partial}{\partial x}-\sum_j(-1)^{p(P)(p(T_j)+1)}T_j\frac{\partial P}{\partial \xi_j}\frac{\partial}{\partial x}+\sum_l Q_l\frac{\partial R}{\partial \xi_l}\frac{\partial}{\partial x}$$

We now observe that $\xi_1\cdots\xi_n\frac{\partial}{\partial x}$ cannot be canceled by neither the terms $\sum_l Q_l\frac{\partial R}{\partial \xi_l}\frac{\partial}{\partial x}$ because they contain x nor the terms $T_j\frac{\partial P}{\partial \xi_j}\frac{\partial}{\partial x}$ if $T_j\neq\xi_{i_{t+1}}\cdots\xi_{i_n}\xi_j$ where $j\neq i_{t+1},...,i_n$.

So we focus on:

$$R(x,\xi)\frac{\partial}{\partial x}+\sum_{j=1}^{n}T_j(x,\xi)\frac{\partial}{\partial \xi_j}=$$

$$x\xi_{i_{t+1}}\cdots\xi_{i_n}\frac{\partial}{\partial x}+\sum_{j\neq i_{t+1},\ldots,i_n}\alpha_j\xi_{i_{t+1}}\cdots\xi_{i_n}\xi_j\frac{\partial}{\partial \xi_j}$$

The divergence condition becomes:

$$\xi_{i_{t+1}}\cdots\xi_{i_n}(1-\sum_{j\neq i_{t+1},\ldots,i_n}\alpha_j)=0$$

Therefore:

$$\xi_1 \cdots \xi_n \frac{\partial}{\partial x} - \sum_j (-1)^{p(P)(p(T_j)+1)} T_j \frac{\partial P}{\partial \xi_j} \frac{\partial}{\partial x} =$$

$$\xi_1 \cdots \xi_n \frac{\partial}{\partial x} - \sum_{j \neq i_{t+1},\ldots,i_n} (-1)^{t(n-t+2)} \alpha_j \xi_{i_{t+1}} \cdots \xi_{i_n} \xi_j \frac{\partial \xi_{i_1} \cdots \xi_{i_t}}{\partial \xi_j} \frac{\partial}{\partial x} =$$

$$\xi_1 \cdots \xi_n \frac{\partial}{\partial x} - \sum_{j \neq i_{t+1},\ldots,i_n} (-1)^{t(n-t+2)} \alpha_j \xi_{i_{t+1}} \cdots \xi_{i_n} \xi_{i_1} \cdots \xi_{i_t} \frac{\partial}{\partial x} =$$

$$\xi_{i_{t+1}} \cdots \xi_{i_n} ((-1)^{t(n-t)} - \sum_{j \neq i_{t+1},\ldots,i_n} (-1)^{t(n-t+2)}) \alpha_j \xi_{i_1} \cdots \xi_{i_t} \frac{\partial}{\partial x} =$$

$$(-1)^{t(n-t)} \xi_{i_{t+1}} \cdots \xi_{i_n} (1 - \sum_{j \neq i_{t+1},\ldots,i_n} \alpha_j) \xi_{i_1} \cdots \xi_{i_t} \frac{\partial}{\partial x} = 0$$

- There exists a $\bar{j}$ such that $T_{\bar{j}} \frac{\partial P}{\partial \xi_{\bar{j}}} = \xi_1 \xi_2 \cdots \xi_n$, then $T_{\bar{j}} = \xi_{i_1} \cdots \xi_{i_t} \xi_{\bar{j}}$, $P = \xi_{\bar{j}} \xi_{i_{t+2}} \cdots \xi_{i_n}$, such that $\xi_{i_1} \cdots \xi_{i_t} \xi_{\bar{j}} \xi_{i_{t+2}} \cdots \xi_{i_n} = \xi_1 \xi_2 \cdots \xi_n$. So:

$$P \frac{\partial}{\partial x} + \sum_{j=1}^{n} Q_j \frac{\partial}{\partial \xi_j} =$$

$$\xi_{\bar{j}} \xi_{i_{t+2}} \cdots \xi_{i_n} \frac{\partial}{\partial x} + \sum_{j=1}^{n} Q_j \frac{\partial}{\partial \xi_j}$$

The condition of zero divergence becomes:

$$\sum_{j=1}^{n} (-1)^{P(Q_j)} \frac{\partial Q_j}{\partial \xi_j} = 0$$

Then:

$$R \frac{\partial}{\partial x} + \sum_{j=1}^{n} T_j \frac{\partial}{\partial \xi_j} =$$

$$R \frac{\partial}{\partial x} + \sum_{j \neq \bar{j}} T_j \frac{\partial}{\partial \xi_j} + \xi_{i_1} \cdots \xi_{i_t} \xi_{\bar{j}} \frac{\partial}{\partial \xi_{\bar{j}}}$$

The condition of zero divergence becomes:

$$\frac{\partial R}{\partial x} + \sum_{j \neq \bar{j}} (-1)^{P(T_j)} \frac{\partial T_j}{\partial \xi_j} - \xi_{i_1} \cdots \xi_{i_t} = 0$$

In this case the terms of (4.1) that involve $\frac{\partial}{\partial x}$ become:

$$\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\frac{\partial R}{\partial x}\frac{\partial}{\partial x} - \sum_{j\neq\bar{j},i_1,\dots,i_t}(-1)^{(n-t)(p(T_j)+1)}T_j\frac{\partial\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}}{\partial\xi_j}\frac{\partial}{\partial x}+$$
$$-(-1)^{(n-t)(t+2)}\xi_{i_1}\cdots\xi_{i_t}\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\frac{\partial}{\partial x}+\sum_l Q_l\frac{\partial R}{\partial\xi_j}\frac{\partial}{\partial x}=$$
$$\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\frac{\partial R}{\partial x}\frac{\partial}{\partial x} - \sum_{j\neq\bar{j},i_1,\dots,i_t}(-1)^{(n-t)(p(T_j)+1)}T_j\frac{\partial\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}}{\partial\xi_j}\frac{\partial}{\partial x}+$$
$$-(-1)^{(n-t)(t+2)}\xi_1\cdots\xi_n\frac{\partial}{\partial x}+\sum_l Q_l\frac{\partial R}{\partial\xi_l}\frac{\partial}{\partial x}$$

Now we analyze these subcases:

1. If $\frac{\partial R}{\partial x}\neq 0$ we have that the terms $\sum_l Q_l\frac{\partial R}{\partial\xi_j}\frac{\partial}{\partial x}$ cannot cancel $\xi_1\cdots\xi_n\frac{\partial}{\partial x}$, we focus on $R=\beta x\xi_{i_1}\cdots\xi_{i_t}$ and $T_j=\alpha_j\xi_{i_1}\cdots\xi_{i_t}\xi_j$ $j\neq\bar{j},i_1,...,i_t$ that can cancel $\xi_1\cdots\xi_n\frac{\partial}{\partial x}$. In this case the terms of (4.1) that involve $\frac{\partial}{\partial x}$ become:

$$\beta\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\xi_{i_1}\cdots\xi_{i_t}\frac{\partial}{\partial x}+$$
$$-\sum_{j\neq\bar{j},i_1,\dots,i_t}(-1)^{(n-t)(t+2)}\alpha_j\xi_{i_1}\cdots\xi_{i_t}\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\frac{\partial}{\partial x}+$$
$$-(-1)^{(n-t)(t+2)}\xi_{i_1}\cdots\xi_{i_t}\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\frac{\partial}{\partial x}+\sum_l Q_l\frac{\partial R}{\partial\xi_j}\frac{\partial}{\partial x}$$

So:

$$\beta\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\xi_{i_1}\cdots\xi_{i_t}\frac{\partial}{\partial x}+$$
$$-\sum_{j\neq\bar{j},i_1,\dots,i_t}(-1)^{(n-t)(t+2)}\alpha_j\xi_{i_1}\cdots\xi_{i_t}\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\frac{\partial}{\partial x}+$$
$$-(-1)^{(n-t)(t+2)}\xi_{i_1}\cdots\xi_{i_t}\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\frac{\partial}{\partial x}=$$
$$((-1)^{t(n-t)}\beta-\sum_{j\neq\bar{j},i_1,\dots,i_t}((-1)^{(n-t)(t+2)}\alpha_j)-(-1)^{(n-t)(t+2)})\xi_1\cdots\xi_n\frac{\partial}{\partial x} \tag{4.2}$$

But the condition of zero divergence of $R\frac{\partial}{\partial x}+\sum_{j=1}^{n}T_j\frac{\partial}{\partial \xi_j}$ becomes:

$$\frac{\partial R}{\partial x}+\sum_{j\neq\bar{j}}(-1)^{P(T_j)}\frac{\partial T_j}{\partial \xi_j}-\xi_{i_1}\cdots\xi_{i_t}=$$
$$\beta\xi_{i_1}\cdots\xi_{i_t}+\sum_{j\neq\bar{j},i_1,\dots,i_t}(-1)^{t+1}(-1)^{t}\alpha_j\xi_{i_1}\cdots\xi_{i_t}-\xi_{i_1}\cdots\xi_{i_t}=$$
$$(\beta-\sum_{j\neq\bar{j},i_1,\dots,i_t}\alpha_j-1)\xi_{i_1}\cdots\xi_{i_t}=0$$

Therefore (4.2) becomes:

$$((-1)^{t(n-t)}\beta-\sum_{j\neq\bar{j},i_1,\dots,i_t}((-1)^{(n-t)(t+2)}\alpha_j)-(-1)^{(n-t)(t+2)})\xi_1\cdots\xi_n\frac{\partial}{\partial x}=$$
$$(-1)^{t(n-t)}(\beta-\sum_{j\neq\bar{j},i_1,\dots,i_t}(\alpha_j)-1)\xi_1\cdots\xi_n\frac{\partial}{\partial x}=0$$

2. If $\frac{\partial R}{\partial x}=0$, and $R=\beta\xi_{i_1}\cdots\xi_{i_t}$, then the terms of (4.1) that involve $\frac{\partial}{\partial x}$ become:

$$\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\frac{\partial R}{\partial x}\frac{\partial}{\partial x}-\sum_{j\neq\bar{j},i_1,\dots,i_t}(-1)^{(n-t)(p(T_j)+1)}T_j\frac{\partial\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}}{\partial\xi_j}\frac{\partial}{\partial x}+$$
$$-(-1)^{(n-t)(t+2)}\xi_1\cdots\xi_n\frac{\partial}{\partial x}+\sum_{l}Q_l\frac{\partial R}{\partial \xi_l}\frac{\partial}{\partial x}=$$
$$-\sum_{j\neq\bar{j},i_1,\dots,i_t}(-1)^{(n-t)(p(T_j)+1)}T_j\frac{\partial\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}}{\partial\xi_j}\frac{\partial}{\partial x}+$$
$$-(-1)^{(n-t)(t+2)}\xi_1\cdots\xi_n\frac{\partial}{\partial x}+\sum_{l=i_1,\dots,i_t}\beta Q_l\frac{\partial\xi_{i_1}\cdots\xi_{i_t}}{\partial\xi_l}\frac{\partial}{\partial x}$$

Focusing on $T_j=\alpha_j\xi_{i_1}\cdots\xi_{i_t}\xi_j$ $j\neq\bar{j},i_1,...,i_t$ and $Q_l=\gamma_l\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\xi_l$, that can cancel $\xi_1\cdots\xi_n$ the last expression becomes:

$$
\begin{aligned}
&-\sum_{j\neq\bar{j},i_1,\ldots,i_t}(-1)^{(n-t)(t+2)}\alpha_j\xi_{i_1}\cdots\xi_{i_t}\xi_j\frac{\partial\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}}{\partial\xi_j}\frac{\partial}{\partial x}+\\
&-(-1)^{(n-t)(t+2)}\xi_1\cdots\xi_n\frac{\partial}{\partial x}+\\
&+\sum_{l=i_1,\ldots,i_t}\beta\gamma_l\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\xi_l\frac{\partial\xi_{i_1}\cdots\xi_{i_t}}{\partial\xi_l}\frac{\partial}{\partial x}=\\
&-\sum_{j\neq\bar{j},i_1,\ldots,i_t}(-1)^{(n-t)(t+2)}\alpha_j\xi_1\cdots\xi_n\frac{\partial}{\partial x}+\\
&-(-1)^{(n-t)(t+2)}\xi_1\cdots\xi_n\frac{\partial}{\partial x}+\\
&+\sum_{l=i_1,\ldots,i_t}\beta\gamma_l\xi_{\bar{j}}\xi_{i_{t+2}}\cdots\xi_{i_n}\xi_{i_1}\cdots\xi_{i_t}\frac{\partial}{\partial x}=\\
&-\sum_{j\neq\bar{j},i_1,\ldots,i_t}(-1)^{(n-t)(t+2)}\alpha_j\xi_1\cdots\xi_n\frac{\partial}{\partial x}+\\
&-(-1)^{(n-t)(t+2)}\xi_1\cdots\xi_n\frac{\partial}{\partial x}+\sum_{l=i_1,\ldots,i_t}\beta\gamma_l(-1)^{(n-t)t}\xi_1\cdots\xi_n\frac{\partial}{\partial x}=\\
&(-1)^{(n-t)t}(-\sum_{j\neq\bar{j},i_1,\ldots,i_t}\alpha_j-1+\sum_{l=i_1,\ldots,i_t}\beta\gamma_l)\xi_1\cdots\xi_n\frac{\partial}{\partial x} \qquad (4.3)
\end{aligned}
$$

But the condition of zero divergence of $R\frac{\partial}{\partial x}+\sum_{j=1}^{n}T_j\frac{\partial}{\partial\xi_j}$ becomes:

$$
\begin{aligned}
&\frac{\partial R}{\partial x}+\sum_{j\neq\bar{j}}(-1)^{p(T_j)}\frac{\partial T_j}{\partial\xi_j}-\xi_{i_1}\cdots\xi_{i_t}=\\
&\sum_{j\neq\bar{j},i_1,\ldots,i_t}(-1)^{t+1}(-1)^t\alpha_j\xi_{i_1}\cdots\xi_{i_t}-\xi_{i_1}\cdots\xi_{i_t}=\\
&(-\sum_{j\neq\bar{j},i_1,\ldots,i_t}\alpha_j-1)\xi_{i_1}\cdots\xi_{i_t}=0
\end{aligned}
$$

On the other hand the condition of zero divergence of

$P\frac{\partial}{\partial x} + \sum_{l=1}^{n} Q_l \frac{\partial}{\partial \xi_l}$ becomes:

$$\frac{\partial P}{\partial x} + \sum_l (-1)^{p(Q_l)} \frac{\partial Q_l}{\partial \xi_l} =$$
$$\sum_l \gamma_l (-1)^{n-t+1} (-1)^{n-t} \xi_{\bar{j}} \xi_{i_{t+2}} \cdots \xi_{i_n} = 0$$

Therefore (4.3) becomes 0.

This proves that $\xi_1 \cdots \xi_n \frac{\partial}{\partial x} \notin S(1,n)$.

Now we show that every element different from $\xi_1\xi_2\cdots\xi_n\frac{\partial}{\partial x}$ lies in $S(1,n)$. In order to do this, we consider the principal grading and prove that a basis of $[S'(1,n), S'(1,n)]_k$ lies in $S(1,n)$. Indeed we have:

1. If $k < n-1$:

$$x^{k+1}\frac{\partial}{\partial \xi_i} = [\frac{\partial}{\partial x}, \frac{x^{k+2}}{k+2}\frac{\partial}{\partial \xi_i}]$$
$$x^{k+1}\frac{\partial}{\partial x} + (k+1)x^k \xi_i \frac{\partial}{\partial \xi_i} = \frac{1}{k+2}[\frac{\partial}{\partial x}, x^{k+2}\frac{\partial}{\partial x} + (k+2)x^{k+1}\xi_i\frac{\partial}{\partial \xi_i}]$$
$$x^h \xi_{i_1}\cdots\xi_{i_{k+1-h}}\frac{\partial}{\partial x} - (-1)^{k+2-h} h x^{h-1}\xi_i\xi_{i_1}\cdots\xi_{i_{k+1-h}}\frac{\partial}{\partial \xi_i} =$$
$$\frac{1}{h+1}[\frac{\partial}{\partial x}, x^{h+1}\xi_{i_1}\cdots\xi_{i_{k+1-h}}\frac{\partial}{\partial x} - (-1)^{k+2-h}(h+1)x^h\xi_i\xi_{i_1}\cdots\xi_{i_{k+1-h}}\frac{\partial}{\partial \xi_i}]$$
$$\xi_{i_1}\cdots\xi_{i_{k+1}}\frac{\partial}{\partial x} = [\frac{\partial}{\partial \xi_i}, \xi_i\xi_{i_1}\cdots\xi_{i_{k+1}}\frac{\partial}{\partial x}] \quad i \neq i_1, ..., i_{k+1}$$
$$\xi_{i_1}\cdots\xi_{i_{k+1}}\frac{\partial}{\partial \xi_i} = [\frac{\partial}{\partial x}, x\xi_{i_1}\cdots\xi_{i_{k+1}}\frac{\partial}{\partial \xi_i}] \quad i \neq i_1, ..., i_{k+1}$$

2. If $k = n-1$ can be treated in the same way, except for the element $\xi_1\xi_2\cdots\xi_n\frac{\partial}{\partial x}$

3. If $k > n-1$: $x^{k+1}\frac{\partial}{\partial \xi_i}$, $x^{k+1}\frac{\partial}{\partial x} + (k+1)x^k\xi_i\frac{\partial}{\partial \xi_i}$, $x^h\xi_{i_1}\cdots\xi_{i_{k+1-h}}\frac{\partial}{\partial x} - (-1)^{k+2-h}hx^{h-1}\xi_i\xi_{i_1}\cdots\xi_{i_{k+1-h}}\frac{\partial}{\partial \xi_i}$ where $k+1-h \leq n-1$ can be obtained as seen in the first case.

Proposition 4.9. *$S(1,n)$ is simple if $n \geq 2$.*

Proof. We shall prove the statement using principal grading. We show that $[S(1,n)_{-1}, S(1,n)_1] = S(1,n)_0$, indeed:

$$\xi_i \frac{\partial}{\partial \xi_j} = [\frac{\partial}{\partial x}, x\xi_i \frac{\partial}{\partial \xi_j}], i \neq j$$

$$x\frac{\partial}{\partial x} + \xi_j \frac{\partial}{\partial \xi_j} = [\frac{\partial}{\partial x}, \frac{1}{2}x^2 \frac{\partial}{\partial x_i} + \xi_j x \frac{\partial}{\partial \xi_j}]$$

$$x\frac{\partial}{\partial \xi_j} = [\frac{\partial}{\partial x}, \frac{1}{2}x^2 \frac{\partial}{\partial \xi_j}]$$

$$\xi_i \frac{\partial}{\partial x} = [\frac{\partial}{\partial \xi_i}, \xi_j \xi_i \frac{\partial}{\partial x}], i \neq j$$

$$\xi_i \frac{\partial}{\partial \xi_i} - \xi_j \frac{\partial}{\partial \xi_j} = [\frac{\partial}{\partial x}, x\xi_i \frac{\partial}{\partial \xi_i} - x\xi_j \frac{\partial}{\partial \xi_j}]$$

Now let I be a nonzero ideal of $S(1,n)$. We will show that $I = S(1,n)$. By the irreducibility of $S(1,n)_{-1} = S'(1,n)_{-1}$ and the fact that $[I_{-1}, S(1,n)_0] \subset I_{-1}$, it follows $I_{-1} = 0$ or $I_{-1} = S(1,n)_{-1}$. In the first case we have that $[S(1,n)_{-1}, I_0] \subset I_{-1} = 0$, by transitivity we have $I_0 = 0$ and, proceeding in the same way, $I_i = 0 \ \forall i$ which is impossible because $I \neq 0$.
So $I_{-1} = S(1,n)_{-1}$ and $S(1,n)_0 = [S(1,n)_{-1}, S(1,n)_1] \subset I$,
It remains to show that an element of degree $k > 0$ lies in I.

$$x^{k+1-t}\xi_{i_1}\cdots\xi_{i_t}\frac{\partial}{\partial \xi_j} =$$

$$[\frac{\partial}{\partial x}, \frac{x^{k+2-t}}{k+2-t}\xi_{i_1}\cdots\xi_{i_t}\frac{\partial}{\partial \xi_j}] \quad j \neq i_1 \cdots i_t$$

$$x^{k+1-t}\xi_{i_1}, ..., \xi_{i_{t-1}}(-\xi_j \frac{\partial}{\partial \xi_j} - \frac{x}{k+2-t}\frac{\partial}{\partial x}) =$$

$$[\frac{\partial}{\partial x}, \frac{x^{k+2-t}}{k+2-t}\xi_{i_1}\cdots\xi_{i_{t-1}}(-\xi_j \frac{\partial}{\partial \xi_j} - \frac{x}{(k+3-t)}\frac{\partial}{\partial x})]$$

$$x^{k+1}\frac{\partial}{\partial x} + (k+1)x^k \xi_i \frac{\partial}{\partial \xi_i} =$$

$$[\frac{\partial}{\partial x}, \frac{x^{k+2}}{k+2}\frac{\partial}{\partial x} + \frac{x^{k+1}}{k+1}\xi_i \frac{\partial}{\partial \xi_i}]$$

□

Remark 15. $S(1,1)$ is not simple. Indeed $< \frac{\partial}{\partial \xi} > \otimes \mathbb{C}[x]$ is a non zero ideal of $S(1,1)$.

4.2.3 Subprincipal grading

The subprincipal grading is that of type $(1,...,1|0,...0)$. We have:

$$\begin{aligned} S'(m,n)_0 &=< x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j} \quad i\neq j, x_i\frac{\partial}{\partial x_j} \quad i\neq j > \otimes\Lambda(\xi_1,...,\xi_n) + S'(0,n) \\ &\cong \mathfrak{sl}(m)\otimes\Lambda(\xi_1,...,\xi_n) + S'(0,n) \end{aligned}$$

The isomorphism is:

$$\begin{aligned} \Phi : S'(m,n)_0 &\longrightarrow \mathfrak{sl}(m)\otimes\Lambda(\xi_1,...,\xi_n) + S'(0,n) \\ (x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j})\otimes P(\xi_1,...,\xi_n) \quad i\neq j &\longmapsto (e_{i,i}-e_{j,j})\otimes P(\xi_1,...,\xi_n) \\ x_i\frac{\partial}{\partial x_j}\otimes P(\xi_1,...,\xi_n) \quad i\neq j &\longmapsto e_{i,j}\otimes P(\xi_1,...,\xi_n) \\ S'(0,n)\ni P &\longmapsto P \end{aligned}$$

On the other hand we have that:

$$S'(m,n)_{-1} =< \frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_m} > \otimes\Lambda(n) \cong \mathbb{C}^m\otimes\Lambda(n)$$

We observe that $S'(m,n)$ with the subprincipal grading has depth 1.

Proposition 4.10. *$S'(m,n)$ with the subprincipal grading is irreducible.*

Proof. Let $S\neq 0$ be a submodule of $S(m,n)_{-1}\cong\mathbb{C}^m\otimes\Lambda(n)$ and $z\in S$ a nonzero element. Then z is of the form:

$$z=\sum_k \alpha_k P_k\frac{\partial}{\partial x_k} \quad where \quad P_k\in\Lambda(n),\alpha_k\in\mathbb{C}$$

Let us suppose $\alpha_i\neq 0$ for an index i. Then we have:

$$[x_i\frac{\partial}{\partial x_1}, z] = -\alpha_i P_i\frac{\partial}{\partial x_1}\in S$$

We recall that $S'(m,n)_0\cong\mathfrak{sl}(m)\otimes\Lambda(n)\oplus S'(0,n)$. By the action of $\mathfrak{sl}(m)$ on $\frac{\partial}{\partial x_1}$ we generate $P_i\otimes\mathbb{C}^m$. Moreover by the action of $S'(0,n)$ on P_i we generate $1\otimes\mathbb{C}^m$, finally by the action of $\mathfrak{sl}(m)\otimes\Lambda(n)$ on $1\otimes\mathbb{C}^m$ we generate $\mathbb{C}^m\otimes\Lambda(n)$. □

Proposition 4.11. *$S'(m,n)$ with subprincipal grading is transitive.*

Proof. Let $a \neq 0$ be an element of $S'_{i\geq 0}(m,n)$ and suppose $[a, S'_{-1}(m,n)] = 0$. Since $S'_{-1}(m,n) = W_{-1}(m,n)$, we have, by the transitivity of $W(m,n)$ with the subprincipal grading, that $a = 0$.

□

4.2.4 Symmetric gradings

Our aim is to obtain a complete list, up to isomorphisms, of strongly symmetric gradings of length five of the Lie superalgebra $S'(m,n)$.

Remark 16. We notice that we are interested only in $\mathbb{Z}$-gradings of type $(0,...,0|b_1,...,b_n)$ or $(a|)$, in fact if there exists an $a_i \neq 0$ and $m \geq 2$, the maximal degree k would not be finite, because for example an element of the form $x_1^l \frac{\partial}{\partial x_2}$ would lie in S' for every l. On the other hand if $m = 1$ and $n \geq 1$ the maximal degree k would not be finite, because, similarly, an element of the form $x^l \frac{\partial}{\partial \xi_1}$ would lie in S' for every l. Moreover the gradings of type $(0,...,0|b_1,...,b_n)$ and $(a|)$ are of finite depth, because the squares of the ξ_i's are zero.
The grading of type $(a|)$ is very elementary, indeed, if we suppose $a = 1$ $S'(1,0) =< \frac{\partial}{\partial x} >= S'(1,0)_{-1}$.

We will start our analysis from $S'(0,n)$ and then generalize it to $S'(m,n)$.

4.2.5 $S'(0,n)$

We first consider a grading of type $(|b_1,...,b_n)$ where $b_i > 0\ \forall i$. We denote by k the maximal degree and $-h$ the minimal degree of elements of $S'(0,n)$ with such a grading. We set $max(b_i) = b_2$ and $min(b_i) = b_1$ It follows:

$$k = b_2 + ... + b_n - min\{b_i\} = b_2 + ... + b_n - b_1$$
$$h = max\{b_i\} = b_2$$

So:

$$h = k \Leftrightarrow$$
$$-b_1 + b_2 + ... + b_n = b_2 \Leftrightarrow$$
$$b_3 + ... + b_n = b_1 \Leftrightarrow$$
$$n = 3 \quad and \quad b_3 = b_1$$

Therefore we first study the case $n = 3$ and grading $(|b, B, b), B \geq b$. We have $h = k = B$ and the following two possibilities:

1. If $B = b$, supposing $b = 1$:

   $S'(0,3)_{-1} =< \frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}, \frac{\partial}{\partial\xi_3} >$
   $S'(0,3)_1 \supset< \xi_1\xi_2\frac{\partial}{\partial\xi_3}, \xi_1\xi_3\frac{\partial}{\partial\xi_2}, \xi_2\xi_3\frac{\partial}{\partial\xi_1}, \xi_1\xi_3\frac{\partial}{\partial\xi_1} - \xi_2\xi_3\frac{\partial}{\partial\xi_2}, \xi_1\xi_2\frac{\partial}{\partial\xi_2} - \xi_1\xi_3\frac{\partial}{\partial\xi_3} >$

   Then $dim(S'(0,3)_{-1}) < dim(S'(0,3)_1)$.

2. If $B > b$ we have:

   $$S'(0,3)_{-B} =< \frac{\partial}{\partial\xi_2} >$$
   $$S'(0,3)_B \supset< \xi_1\xi_2\frac{\partial}{\partial\xi_3}, \xi_2\xi_3\frac{\partial}{\partial\xi_1} >$$

   Then $dim(S'(0,3)_{-B}) < dim(S'(0,3)_B)$.

Therefore now we study gradings of type $(|b_1, ..., b_n)$ with $b_i \geq 0\ \forall i$ such that $b_j = 0$ for some j, or such that $b_i > 0$ and $b_j < 0$ for some $i \neq j$. First we analyze what happens for $n = 2$ and then $n \geq 3$.

A) $S'(0,2)$

The possibilities are:

i) $(|0,a)$ with $a>0$. We suppose $a=1$. Then:

$$S'(0,2)_{-1} =< \frac{\partial}{\partial\xi_2}, \xi_1\frac{\partial}{\partial\xi_2} >$$
$$S'(0,2)_0 =< \xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2}, \frac{\partial}{\partial\xi_1} >$$
$$S'(0,2)_1 =< \xi_2\frac{\partial}{\partial\xi_1} >$$

Therefore $dim(S'(0,2)_{-1}) > dim(S'(0,2)_1)$.

ii) $(|a,-b)$ with $a,b>0$ and $a>b$ then $h=k=a+b$ and:

$$S'(0,2)_{-b} = 0$$
$$S'(0,2)_b =< \frac{\partial}{\partial\xi_2} >$$

Therefore $dim(S'(0,2)_{-b}) < dim(S'(0,2)_b)$.

iii) $(|a,-b)$ with $a,b>0$ and $a<b$ then $h=k=a+b$, It is analogous to the previous one.

iv) $(|a,-a)$ with $a>0$, we suppose $a=1$, we have $h=k=2$ and:

$$S'(0,2)_{-2} =< \xi_2\frac{\partial}{\partial\xi_1} >$$
$$S'(0,2)_{-1} =< \frac{\partial}{\partial\xi_1} >$$
$$S'(0,2)_0 =< \xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2} >$$
$$S'(0,2)_1 =< \frac{\partial}{\partial\xi_2} >$$
$$S'(0,2)_2 =< \xi_1\frac{\partial}{\partial\xi_2} >$$

This grading not generated by its local part, since $[S'(0,2)_{-1}, S'(0,2)_{-1}] = 0$.

B) $S'(0,n)$**,** $n \geq 3$

Let $(|b_1,...,b_n)$ be a $\mathbb{Z}-$grading such that $b_i \geq 0$ $\forall i$ and $b_j = 0$ for some j, or such that $b_i > 0$ and a $b_j < 0$ for some $i \neq j$.

We observe that in both these cases $S'(0,n) = \oplus_{i=-h}^{k} S'(0,n)_i$ with $h, k < \infty$ and:

$$h = \sum_{b_i \leq 0} |b_i| + max\,\{b_i \geq 0\}$$
$$k = \sum_{b_i \geq 0} b_i + |min\,\{b_i \leq 0\}\,|$$

Then, if we set $b_1 = max\,\{b_i\}$, $b_2 = min\,\{b_i\}$:

$$h = k \Leftrightarrow \;\; b_3 + ... + b_n = 0$$

i) $\mathbb{Z}$-grading of type $(|b_1,...,b_n)$ where $b_i \geq 0$ $\forall i$ and $b_j = 0$ for some j. Notice that $h = k$ if and only if the grading is of type $(|a, 0, ..., 0)$ where $a > 0$. We have, choosing $a = 1$:

$$S'(0,n)_{-1} =< \frac{\partial}{\partial \xi_1} > \otimes \Lambda(\xi_2, ..., \xi_n)$$
$$S'(0,n)_1 =< \xi_1 \frac{\partial}{\partial \xi_2} > \otimes \Lambda(\xi_3, ..., \xi_n) + < \xi_1 \frac{\partial}{\partial \xi_3} > \otimes \Lambda(\xi_2, \xi_4, ..., \xi_n) + ...$$
$$+ < \xi_1 \frac{\partial}{\partial \xi_n} > \otimes \Lambda(\xi_2, \xi_3, ..., \xi_{n-1}) + < \xi_1 \xi_i \frac{\partial}{\partial \xi_i} - \xi_1 \xi_j \frac{\partial}{\partial \xi_j} > \otimes \Lambda(\xi_k, k \neq i, j, 1)$$
$$i \neq j, \quad i, j \neq 1$$

Therefore:

$$dim(S'(0,n)_{-1}) = 2^{n-1}$$
$$dim(S'(0,n)_1) = (n-1)2^{n-2} + (n-2)2^{n-3}$$
$$dim(S'(0,n)_{-1}) = dim(S'(0,n)_1) \Leftrightarrow$$
$$2^{n-1} = (n-1)2^{n-2} + (n-2)2^{n-3} \Leftrightarrow$$

$$2^{n-1} = 2^{n-3}(2n-2+n-2) \Leftrightarrow$$
$$4 = 3n-4 \Leftrightarrow$$
$$n = 8/3$$

So these two spaces have always different dimensions.

ii) $\mathbb{Z}$-grading of type $(|b_1,...,b_n)$ with $b_i > 0$ and $b_j < 0$ for some $i \neq j$. Let us analyze the grading of type $(|B,b,0,...,0)$, with $B > 0$, $b < 0$. In fact this is sufficient, by Remark 11, in order to study symmetric gradings of length five. We have $h = k = -b+B$, the possible degrees are $-B, b, b-B, B, -b, B-b, 0, B+b$. We notice that $-b-B \neq -B, b-B, -b, B-b$. Therefore we have the following possibilities:

iia) $-b-B = b$, i.e., $B = 2|b|$;

iib) $-b-B = B$, i.e., $|b| = 2B$;

iic) $-b-B = 0$, i.e., $|b| = B$.

If none of these cases holds, then $\dim(S'(m,n)_{-b-B}) = 0$ and $\dim(S'(m,n)_{b+B}) > 0$, hence we rule this possibility out. In case *iia)* (resp. *iib)*) we can assume $|b| = 1$ (resp. $B = 1$) hence getting a grading of depth three. Now suppose $|b| = B$, and set $B = 1$, i.e., consider the grading of type $(|1,-1,0,...,0)$. We have:

$$S'(0,n)_{-2} =< \xi_2 \frac{\partial}{\partial \xi_1} > \otimes \Lambda(\xi_3,...,\xi_n)$$
$$S'(0,n)_{-1} =< \frac{\partial}{\partial \xi_1} > \otimes \Lambda(\xi_3,...,\xi_n) +$$
$$< \xi_1\xi_2 \frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_j} \frac{\partial}{\partial \xi_1} + (-1)^{p(Q)} \xi_2 Q(\xi_3,...,\xi_n) \frac{\partial}{\partial \xi_j} > \ j \geq 3$$
$$S'(0,n)_0 =< Q(\xi_3,...,\xi_n)\xi_1 \frac{\partial}{\partial \xi_1} - Q(\xi_3,...,\xi_n)\xi_2 \frac{\partial}{\partial \xi_2} > +$$
$$< Q(\xi_3,...,\xi_n) \frac{\partial}{\partial \xi_i} - \xi_1 \frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_i} \frac{\partial}{\partial \xi_1} > + \xi_1\xi_2 \otimes S(m,n-2)$$

$$S'(0,n)_1 =< \frac{\partial}{\partial \xi_2} > \otimes \Lambda(\xi_3, ..., \xi_n) +$$
$$< \xi_1\xi_2 \frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_j}\frac{\partial}{\partial \xi_2} + (-1)^{p(Q)}\xi_1 Q(\xi_3,...,\xi_n)\frac{\partial}{\partial \xi_j} > \ j \geq 3$$
$$S'(0,n)_2 =< \xi_1 \frac{\partial}{\partial \xi_2} > \otimes \Lambda(\xi_3, ..., \xi_n)$$

Note that this grading is symmetric, it is consistent if and only if $n = 2$ and it is generated by its local part.

4.2.6 $S'(m, n)$, $m > 1$ and $n \geq 2$

The analysis of the $\mathbb{Z}$-grading of type $(0, ..., 0|b_1, ..., b_n)$ of the Lie superalgebra $S'(m, n)$ is similar to that of the grading of type $(|b_1, ..., b_n)$ of the Lie superalgebra $S'(0, n)$. Indeed, the following relations still hold:

$$h = \sum_{b_i \leq 0} |b_i| + max\,\{b_i \geq 0\}$$
$$k = \sum_{b_i \geq 0} b_i + |min\,\{b_i \leq 0\}\,|$$

Then:

$$h = k \Leftrightarrow$$
$$b_1 + ... + b_n = max\,\{b_i \geq 0\} - |min\,\{b_i \leq 0\}\,| \Leftrightarrow$$
$$b_1 + ... + b_n = max\,\{b_i \geq 0\} + min\,\{b_i \leq 0\}$$

Remark 17. As in the general case of $W(m, n)$, in these formulas we mean that if either $\{b_i \geq 0\} = \emptyset$ or $\{b_i \leq 0\} = \emptyset$ then $\max\{b_i \geq 0\} = 0$ (resp. $\min\{b_i \leq 0\} = 0$).

The following possibilities may thus occur:

i) $b_i \geq 0\ \forall i$:

in this case $h = k$ if and only if the grading is, up to isomorphisms, of type $(0,...,0|a,0,...,0)$ with $a>0$. Let us set $a=1$. Then we have:

$$S'(m,n)_{-1} =< \frac{\partial}{\partial\xi_1} > \otimes\Lambda(\xi_2,...,\xi_n)\otimes\mathbb{C}[x_1,...,x_m]$$
$$S'(m,n)_1 =< P\frac{\partial Q(\xi_2,...,\xi_n)}{\partial\xi_j}\xi_1\frac{\partial}{\partial x_i} - (-1)^{p(Q)+1}\frac{\partial P}{\partial x_i}Q(\xi_2,...,\xi_n)\xi_1\frac{\partial}{\partial\xi_j} >$$
$$j\geq 2$$

where $P\in\mathbb{C}[x_1,...,x_m]$ and $Q\in\Lambda(\xi_3,...,\xi_n)$.

ii) $b_i>0$ and $b_j<0$ for some $i>j$.

In order to study symmetric gradings of length five it is sufficient to analyze gradings of type $(0,\dots,0|B,b,0,\dots,0)$ with $B>0$, $b<0$, by Remark 11. Then $h=k=-b+B$ and the degrees which appear are: $-B,b,b-B,B,-b,B-b,0,B+b$. Notice that $-b-B\neq -B,b-B,-b,B-b$. Therefore we have the following possibilities:

iia) $-b-B=b$, i.e., $B=2|b|$;

iib) $-b-B=B$, i.e., $|b|=2B$;

iic) $-b-B=0$, i.e., $|b|=B$.

If none of these cases holds, then $\dim(S'(m,n)_{-b-B})=0$ and $\dim(S'(m,n)_{b+B})>0$, hence we rule this possibility out. In case *iia*) (resp. *iib*)) we can assume $|b|=1$ (resp. $B=1$) hence getting a grading of depth three. Now suppose $|b|=B$, and set $B=1$, i.e., consider the grading of type $(0,...,0|1,-1,0,...,0)$, let $P\in\mathbb{C}[x_1,...,x_m]$ and $Q\in\Lambda(\xi_3,...,\xi_n)$:

$$S'(m,n)_{-2} =< \xi_2\frac{\partial}{\partial\xi_1} > \otimes\Lambda(\xi_3,...,\xi_n)\mathbb{C}[x_1,...,x_m]$$

$$S'(m,n)_{-1} =< \frac{\partial}{\partial \xi_1} > \otimes \Lambda(\xi_3, ..., \xi_n) \otimes \mathbb{C}[x_1, ..., x_m] +$$
$$< P\frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_j}\xi_2\frac{\partial}{\partial x_i} - (-1)^{p(Q)+1}\frac{\partial P}{\partial x_i}Q(\xi_3,...,\xi_n)\xi_2\frac{\partial}{\partial \xi_j} > +$$
$$< P\xi_1\xi_2\frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_j}\frac{\partial}{\partial \xi_1} + (-1)^{p(Q)}P\xi_2 Q(\xi_3,...,\xi_n)\frac{\partial}{\partial \xi_j} > \ j \geq 3$$
$$S'(m,n)_0 =< PQ(\xi_3,...,\xi_n)\frac{\partial}{\partial x_i} + (-1)^{p(Q)}\frac{\partial P}{\partial x_i}\xi_1 Q(\xi_3,...,\xi_n)\frac{\partial}{\partial \xi_1},$$
$$PQ(\xi_3,...,\xi_n)\xi_1\frac{\partial}{\partial \xi_1} - PQ(\xi_3,...,\xi_n)\xi_2\frac{\partial}{\partial \xi_2},$$
$$P\frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_j}\frac{\partial}{\partial x_i} - (-1)^{p(Q)}\frac{\partial P}{\partial x_i}Q(\xi_3,...,\xi_n)\frac{\partial}{\partial \xi_j},$$
$$P\xi_1\xi_2\frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_j}\frac{\partial}{\partial x_i} - (-1)^{p(Q)}\frac{\partial P}{\partial x_i}\xi_1\xi_2 Q(\xi_3,...,\xi_n)\frac{\partial}{\partial \xi_j} > \ j \geq 3$$
$$S'(m,n)_1 =< \frac{\partial}{\partial \xi_2} > \otimes \Lambda(\xi_3, ..., \xi_n) \otimes \mathbb{C}[x_1, ..., x_m] +$$
$$< P\frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_j}\xi_1\frac{\partial}{\partial x_i} - (-1)^{p(Q)+1}\frac{\partial P}{\partial x_i}Q(\xi_3,...,\xi_n)\xi_1\frac{\partial}{\partial \xi_j} > +$$
$$< P\xi_1\xi_2\frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_j}\frac{\partial}{\partial \xi_2} + (-1)^{p(Q)}P\xi_1 Q(\xi_3,...,\xi_n)\frac{\partial}{\partial \xi_j} > \ j \geq 3$$
$$S'(m,n)_2 =< \xi_1\frac{\partial}{\partial \xi_2} > \otimes \Lambda(\xi_3, ..., \xi_n)\mathbb{C}[x_1, ..., x_m]$$

This grading is symmetric, consistent if and only if $n = 2$ and generated by its local part.

4.2.7 $S(1,n)$, $n \geq 2$

We start by analyzing the grading of type $(0|b_1, ..., b_n)$ with $b_i > 0$ for every i. Recall that $S(1,n) = S'(1,n) \setminus \langle \xi_1 \cdots \xi_n \frac{\partial}{\partial x} \rangle$, hence $S(1,n) = \oplus_{i=-h}^{k} S(1,n)_i$ where, if we set $b_1 = min(b_i)$ and $b_2 = max(b_i)$:

$$h = b_2 \ k = b_2 + ... + b_n$$

Then:

$$k = h \Leftrightarrow b_2 + ... + b_n = b_2 \Leftrightarrow b_3 + ... + b_n = 0 \Leftrightarrow n = 2$$

The following possibilities may then occur:

i) $(0|b,b)$ where $b=1$, that is $(0|1,1)$. We have:

$$S(1,2)_{-1} = < \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} > \otimes \mathbb{C}[x]$$

$$S(1,2)_1 = < rx^{r-1}\xi_1\xi_2\frac{\partial}{\partial \xi_1} - x^r\xi_2\frac{\partial}{\partial x}, rx^{r-1}\xi_1\xi_2\frac{\partial}{\partial \xi_2} - x^r\xi_1\frac{\partial}{\partial x} > \quad r \geq 0$$

Note that this grading is symmetric and consistent.

ii) $(0|b,B)$, $B > b$. This grading is symmetric of length 5 if and only if $b=1$ and $B=2$. Then we have:

$$S(1,2)_{-2} = < \frac{\partial}{\partial \xi_2} > \otimes \mathbb{C}[x]$$

$$S(1,2)_{-1} = < \frac{\partial}{\partial \xi_1}, \xi_1\frac{\partial}{\partial \xi_2} > \otimes \mathbb{C}[x]$$

$$S(1,2)_0 = < \xi_1\frac{\partial}{\partial \xi_1} - \xi_2\frac{\partial}{\partial \xi_2} > \otimes \mathbb{C}[x] + < x^r\frac{\partial}{\partial x} + rx^{r-1}\xi_i\frac{\partial}{\partial \xi_i}, i=1,2 >$$

$$S(1,2)_1 = < \xi_2\frac{\partial}{\partial \xi_1} > \otimes \mathbb{C}[x] + < rx^{r-1}\xi_1\xi_2\frac{\partial}{\partial \xi_2} - x^r\xi_1\frac{\partial}{\partial x} > \quad r \geq 0$$

$$S(1,2)_2 = < x^r\xi_2\frac{\partial}{\partial x} - rx^{r-1}\xi_1\xi_2\frac{\partial}{\partial \xi_1} >$$

This grading is symmetric and generated by its local part, but not consistent.

Finally we consider the $\mathbb{Z}$-grading of type $(0|b_1,...,b_n)$, where either $b_i > 0$ for some i and $b_j < 0$ for some j or $b_i \geq 0$ for every i and $b_k = 0$ for at least one k. The analysis of these cases can be carried out as for $S'(m,n)$ with $m > 1$ and $n \geq 2$, keeping in mind that $\xi_1 \cdots \xi_n \frac{\partial}{\partial x} \notin S(1,n)$. Notice, though, that the grading of type $(0|1,0)$ of $S(1,2)$ is strongly symmetric of length three. Indeed, let us consider $S(1,n)$ with the grading of type $(0|1,0,...,0)$. Then we have:

$$S(1,n)_{-1} = < \frac{\partial}{\partial \xi_1} > \otimes \Lambda(\xi_2,...,\xi_n) \otimes \mathbb{C}[x]$$

$$S(1,n)_1 = < P(x)\frac{\partial Q(\xi_2,...,\xi_n)}{\partial \xi_i}\xi_1\frac{\partial}{\partial x} - (-1)^{p(Q)+1}\frac{\partial P(x)}{\partial x}Q(\xi_2,...,\xi_n)\xi_1\frac{\partial}{\partial \xi_i} >$$

$$for \quad i \geq 2$$

Therefore $S(1,n)_1$ is isomorphic to $n-1$ copies of $S(1,n)_{-1}$, that is $S(1,n)_1 \cong S(1,n)_{-1}$ if and only if $n=2$.

If $n=2$ we obtain:

$$S(1,2)_{-1} =< \frac{\partial}{\partial \xi_1} > \otimes \Lambda(\xi_2) \otimes \mathbb{C}[x]$$
$$S(1,2)_1 =< \xi_1 \frac{\partial}{\partial \xi_2} > \otimes \mathbb{C}[x] + < x^r \xi_1 \frac{\partial}{\partial x} - r x^{r-1} \xi_1 \xi_2 \frac{\partial}{\partial \xi_2} >$$

Therefore we have proved the following results:

Theorem 4.12. *1. If $(m,n) \neq (1,2)$ then the Lie superalgebra $S(m,n)$ has no strongly symmetric $\mathbb{Z}$-grading of length three.*

2. A complete list, up to isomorphisms, of strongly symmetric $\mathbb{Z}-$gradings of length three of the Lie superalgebra $S(1,2)$ is the following:

(a) $(0|1,1)$

(b) $(0|1,0)$

Theorem 4.13. *A complete list, up to isomorphisms, of strongly symmetric $\mathbb{Z}-$gradings of length five of the Lie superalgebra of $S(m,n)$ is the following:*

1. $(0,...,0|1,-1,0,...,0)$

2. $(0|2,1)$ *for $m=1$ and $n=2$*

We now give a description on the strongly symmetric $\mathbb{Z}-$gradings of length five of the Lie superalgebra of $S(m,n)$.

1. $S(1,2)$ with grading $(0|2,1)$. It follows that:

$$S(1,2)_0 =< \xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2} > \otimes \mathbb{C}[x] + < x^r \frac{\partial}{\partial x} + r x^{r-1} \xi_1 \frac{\partial}{\partial \xi_1} > \cong$$
$$\mathbb{C}[x] \rtimes W(1,0)$$

and $I :=< \xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2} > \otimes\mathbb{C}[x]$ is a non trivial abelian ideal. Indeed:

$$[x^r\frac{\partial}{\partial x} + rx^{r-1}\xi_1\frac{\partial}{\partial\xi_1}, x^t\frac{\partial}{\partial x} + tx^{t-1}\xi_1\frac{\partial}{\partial\xi_1}] =$$
$$tx^{r+t-1}\frac{\partial}{\partial x} + t(t-1)x^{r+t-2}\xi_1\frac{\partial}{\partial\xi_1}$$
$$- rx^{r+t-1}\frac{\partial}{\partial x} - r(r-1)x^{r+t-2}\xi_1\frac{\partial}{\partial\xi_1}$$
$$[P(x)(\xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2}), Q(x)(\xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2})] = 0$$

$W(0,1)$ acts naturally on I, indeed:

$$[x^r\frac{\partial}{\partial x} + rx^{r-1}\xi_1\frac{\partial}{\partial\xi_1}, P(x)\xi_1\frac{\partial}{\partial\xi_1} - P(x)\xi_2\frac{\partial}{\partial\xi_2}] =$$
$$\frac{\partial P}{\partial x}x^r\xi_1\frac{\partial}{\partial\xi_1} - \frac{\partial P}{\partial x}x^r\xi_2\frac{\partial}{\partial\xi_2}$$

Moreover:

$$S(1,2)_{-1} =< \frac{\partial}{\partial\xi_1} > \otimes\mathbb{C}[x]\oplus < \xi_1\frac{\partial}{\partial\xi_2} > \otimes\mathbb{C}[x] \cong$$
$$S_1 \oplus S_2$$

with S_1 and S_2 $S(1,2)_0$-modules. In particular: $S_1 \cong \mathbb{C}[x]^{(-1)}$ and $S_2 \cong \mathbb{C}[x]^{(1)}$, where by $\mathbb{C}[x]^{(\lambda)}$ we denote the twisted action of $W(1,0)$ on $\mathbb{C}[x]$ defined as follows, for $X \in W(1,0)$, $\lambda \in \mathbb{C}$ and $P \in \mathbb{C}[x]$:

$$X.P = X(P) + \lambda div(X)P$$

Indeed:

$$[x^r\frac{\partial}{\partial x} + rx^{r-1}\xi_1\frac{\partial}{\partial\xi_1}, Q(x)\frac{\partial}{\partial\xi_1}] =$$
$$x^r\frac{\partial Q}{\partial x}\frac{\partial}{\partial\xi_1} - rQx^{r-1}\frac{\partial}{\partial\xi_1}$$
$$[x^r\frac{\partial}{\partial x} + rx^{r-1}\xi_1\frac{\partial}{\partial\xi_1}, Q(x)\xi_1\frac{\partial}{\partial\xi_2}] =$$
$$x^r\frac{\partial Q}{\partial x}\xi_1\frac{\partial}{\partial\xi_2} + rQx^{r-1}\xi_1\frac{\partial}{\partial\xi_2}$$

Moreover $\mathbb{C}[x]$ acts on S_1 by multiplication for -1 and on S_2 by multiplication for 2. Indeed:

$$[P(x)(\xi_1\frac{\partial}{\partial\xi_1}-\xi_2\frac{\partial}{\partial\xi_2}),Q\frac{\partial}{\partial\xi_1}]=-PQ\frac{\partial}{\partial\xi_1}$$
$$[P(x)(\xi_1\frac{\partial}{\partial\xi_1}-\xi_2\frac{\partial}{\partial\xi_2}),Q\xi_1\frac{\partial}{\partial\xi_2}]=2PQ\xi_1\frac{\partial}{\partial\xi_2}$$

Finally $S(1,2)_{-2}=<\frac{\partial}{\partial\xi_2}>\otimes\mathbb{C}[x]$ is isomorphic, as a module, to $\mathbb{C}[x]$, it is a $W(1,0)-$module with respect to the natural action, meanwhile it is a $\mathbb{C}[x]-$module with respect to the product action.

2. $S(m,n)$ with grading $(0,...,0|1,-1,0,...,0)$:

$$S'(m,n)_0=<PQ\frac{\partial}{\partial x_i}+(-1)^{p(Q)}\frac{\partial P}{\partial x_i}\xi_1Q\frac{\partial}{\partial\xi_1}>+$$
$$<PQ\xi_1\frac{\partial}{\partial\xi_1}-PQ\xi_2\frac{\partial}{\partial\xi_2}>+$$
$$<P\frac{\partial Q}{\partial\xi_j}\frac{\partial}{\partial x_i}-(-1)^{p(Q)}\frac{\partial P}{\partial x_i}Q\frac{\partial}{\partial\xi_j}>+$$
$$<P\xi_1\xi_2\frac{\partial Q}{\partial\xi_j}\frac{\partial}{\partial x_i}-(-1)^{p(Q)}\xi_1\xi_2\frac{\partial P}{\partial x_i}Q\frac{\partial}{\partial\xi_j}>\ j\geq 3=$$
$$W(m,0)\otimes\Lambda(n-2)\oplus I_1\oplus S(m,n-2)\oplus I_2$$

where $P\in\mathbb{C}[x_1,...,x_m]$ and $Q\in\Lambda(n-2)$ and by $\Lambda(n-2)$ we mean $\Lambda(\xi_3,...,\xi_n)$.

$I_1\cong\mathbb{C}[x_1,...,x_m]\otimes\Lambda(n-2)$ and $I_2\cong<\xi_1\xi_2>\otimes S(m,n-2)$ are abelian ideals. The ideals I_1 and I_2 commute, indeed let $P,\tilde{P}\in\mathbb{C}[x_1,...,x_m]$ and $Q,\tilde{Q}\in\Lambda(n-2)$:

$$[PQ\xi_1\frac{\partial}{\partial\xi_1}-PQ\xi_2\frac{\partial}{\partial\xi_2},\tilde{P}\xi_1\xi_2\frac{\partial\tilde{Q}}{\partial\xi_j}\frac{\partial}{\partial x_i}-(-1)^{p(\tilde{Q})}\xi_1\xi_2\frac{\partial\tilde{P}(x)}{\partial x_i}\tilde{Q}\frac{\partial}{\partial\xi_j}]=$$
$$PQ\tilde{P}\xi_1\xi_2\frac{\partial\tilde{Q}}{\partial\xi_j}\frac{\partial}{\partial x_i}-(-1)^{p(\tilde{Q})}PQ\xi_1\xi_2\frac{\partial\tilde{P}(x)}{\partial x_i}Q\frac{\partial}{\partial\xi_j}+$$
$$PQ\tilde{P}\xi_2\xi_1\frac{\partial\tilde{Q}}{\partial\xi_j}\frac{\partial}{\partial x_i}-(-1)^{p(\tilde{Q})}PQ\xi_2\xi_1\frac{\partial\tilde{P}(x)}{\partial x_i}Q\frac{\partial}{\partial\xi_j}=0$$

$S(m,n-2)$ acts by derivation on I_1, $W(m,0)$ acts on I_1 by derivation, $\Lambda(n-2)$ by multiplication.

$S(m,n-2)$ and $W(m,0)\otimes\Lambda(n-2)$ act on I_2 via the adjoint action.

Moreover:

$$S'(m,n)_{-1} =< \frac{\partial}{\partial \xi_1} > \otimes \Lambda(n-2) \otimes \mathbb{C}[x_1,...,x_m]+$$
$$< P(x)\frac{\partial R(\xi_3,...,\xi_n)}{\partial \xi_j}\xi_2\frac{\partial}{\partial x_i} - (-1)^{p(R)+1}\frac{\partial P(x)}{\partial x_i}R(\xi_3,...,\xi_n)\xi_2\frac{\partial}{\partial \xi_j} > \ j \geq 3$$
$$< P(x)\xi_1\xi_2\frac{\partial Q(\xi_3,...,\xi_n)}{\partial \xi_j}\frac{\partial}{\partial \xi_1} + (-1)^{p(Q)}P(x)\xi_2 Q(\xi_3,...,\xi_n)\frac{\partial}{\partial \xi_j} > \ j \geq 3 =$$
$$S_1 + S_2 + S_3 \cong$$
$$\mathbb{C}[x_1,...,x_m] \otimes \Lambda(n-2) + S(m,n-2) + \mathbb{C}[x_1,...,x_m] \otimes W(0,n-2)$$

By direct and long computations one can see that the following inclusions hold:

$$[I_1 \otimes \Lambda(n-2), S_1] \subset S_1$$
$$[W(m,0) \otimes \mathbb{C}[x_1,...,x_m], S_1] \subset S_1$$
$$[S(m,n-2), S_1] \subset S_1$$
$$[I_2 \otimes \Lambda(n-2), S_1] \subset S_2 + S_3$$
$$[I_1 \otimes \Lambda(n-2), S_2] \subset S_2 + S_3$$
$$[W(m,0) \otimes \mathbb{C}[x_1,...,x_m], S_2] \subset S_2 + S_3$$
$$[S(m,n-2), S_2] \subset S_2$$
$$[I_2 \otimes \Lambda(n-2), S_2] = 0$$
$$[I_1 \otimes \Lambda(n-2), S_3] \subset S_3$$
$$[W(m,0) \otimes \mathbb{C}[x_1,...,x_m], S_3] \subset S_2 + S_3$$
$$[S(m,n-2), S_3] \subset S_2 + S_3$$
$$[I_2 \otimes \Lambda(n-2), S_3] = 0$$

Therefore this grading is not irreducible, since $S_2 + S_3$ is a submodule.

Finally:

$$S'(m,n)_{-2} =< \xi_2\frac{\partial}{\partial \xi_1} > \otimes \Lambda(\xi_3,...,\xi_n)\mathbb{C}[x_1,...,x_m]$$

Therefore $W(m,0)\otimes\Lambda(n-2)$ acts on $S'(m,n)_{-2}$ by $(-1)-$twisted action. Indeed, let $Q,\tilde{Q}\in\Lambda(n-2)$:

$$[P(x)Q\frac{\partial}{\partial x_i}+(-1)^{p(Q)}\frac{\partial P(x)}{\partial x_i}\xi_1 Q\frac{\partial}{\partial\xi_1},\tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}]=$$
$$P\frac{\partial\tilde{P}}{\partial x_i}Q\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}-(-1)^{p(Q)p(\tilde{Q})}(-1)^{p(Q)}\frac{\partial P(x)}{\partial x_i}\tilde{P}\xi_2\tilde{Q}Q\frac{\partial}{\partial\xi_1}=$$
$$P\frac{\partial\tilde{P}}{\partial x_i}Q\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}-\frac{\partial P(x)}{\partial x_i}Q\tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}$$

$\mathbb{C}[x]\otimes\Lambda(n-2)$ acts on $S'(m,n)_{-2}$ by multiplication:

$$[P(x)Q(\xi_3,...,\xi_n)\xi_1\frac{\partial}{\partial\xi_1}-P(x)Q(\xi_3,...,\xi_n)\xi_2\frac{\partial}{\partial\xi_2},\tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}]=$$
$$-PQ\tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}-(-1)^{p(Q)p(\tilde{Q})}(-1)^{p(Q)}\tilde{P}\xi_2\tilde{Q}PQ\frac{\partial}{\partial\xi_1}=$$
$$-PQ\tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}-PQ\tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}=$$
$$-2PQ\tilde{P}\xi_2\tilde{Q}\frac{\partial}{\partial\xi_1}$$

and $S(m,n-2)$ acts on $S'(m,n)_{-2}$ by derivation:

$$[P(x)\frac{\partial Q(\xi_3,...,\xi_n)}{\partial\xi_j}\frac{\partial}{\partial x_i}-(-1)^{p(Q)}\frac{\partial P(x)}{\partial x_i}Q(\xi_3,...,\xi_n)\frac{\partial}{\partial\xi_j},\tilde{P}\tilde{Q}\xi_2\frac{\partial}{\partial\xi_1}]=$$
$$P(x)\frac{\partial Q(\xi_3,...,\xi_n)}{\partial\xi_j}\frac{\partial\tilde{P}}{\partial x_i}\tilde{Q}\xi_2\frac{\partial}{\partial\xi_1}-(-1)^{p(Q)}\frac{\partial P(x)}{\partial x_i}Q(\xi_3,...,\xi_n)\tilde{P}\frac{\partial\tilde{Q}}{\partial\xi_j}\xi_2\frac{\partial}{\partial\xi_1}$$

$<\xi_1\xi_2>\otimes S(m,n-2)$ acts on $S'(m,n)_{-2}$ trivially:

$$[P(x)\xi_1\xi_2\frac{\partial Q}{\partial\xi_j}\frac{\partial}{\partial x_i}-(-1)^{p(Q)}\xi_1\xi_2\frac{\partial P(x)}{\partial x_i}Q\frac{\partial}{\partial\xi_j},\tilde{P}\tilde{Q}\xi_2\frac{\partial}{\partial\xi_1}]=0$$

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