Alma Mater Studiorum · Università di Bologna

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ELASTIC GRAVITY

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Abstract

We will describe a theory of emergent gravity, that allows us to identify the gravitational effects of dark matter as the deformation of the dark energy medium via linear elasticity under suitable conditions. To do so, we will introduce the necessary quantities and constants in three different chapters. In the first part we will review Newtons theory of gravitation as a classical view of gravity based on the universal gravitational constant and the gravitational potential. In the second part we will briefly see some of the main results in modern cosmology without being too specific. We will describe some experimental facts such as the rotation curves of spiral galaxies, which suggest the existence of dark matter. We will also explain why we need an acceleration scale as an alternative to dark matter theories and how the two can come together in emergent gravity. In the third part we will give a rigorous and yet elementary description of linear elasticity. We will introduce the strain and stress tensors, how they relate to each other and also see some important parameters that will help us to describe the dark energy medium in details. We will show the existence of a preferred direction in the dark energy medium that identifies an interface surface where all the identifications can be made. In the final chapter we will finally merge all notions given in the previous three chapters to formulate a simple, yet complete description of the theory, describing its weak and strong points. Once we have shown that gravitational and elastic quantities are somehow related under certain conditions in the dark energy medium, it will become easy to see the duality between certain laws, especially from the energetic point of view.

Abstract

Descriveremo una teoria di gravitá emergente, che ci permette sotto certe condizioni di identificare gli effetti gravitazionali della materia oscura come deformazioni del mezzo di energia oscura tramite elasticitá lineare. Per fare ció, descriveremo quantitá necessarie e costanti in tre differenti capitoli. Nella prima parte rivisiteremo la teoria della gravitazione di Newton da un punto di vista classico della gravitá basato sulla costante di gravitazione universale e sul potenziale gravitazionale. Nella seconda parte vedremo brevemente alcuni dei risultati principali in cosmologia moderna senza essere troppo specifici. Descriveremo alcuni fatti sperimentali come la velocitá di rotazione delle galassie a spirale, che suggerisce l'esistenza della materia oscura. Spiegheremo in oltre il motivo per cui abbiamo bisogno di una scala di accelerazione come alternativa alle teorie sulla materia oscura e come le due possono congiungersi in gravitá emergente. Nella terza parte daremo una rigorosa e al contempo elementare descrizione di elasticitá lineare. Introdurremo i tensori di sforzo e stress, come sono legati fra di loro e vedremo anche alcuni importanti parametri che ci aiuteranno a descrivere il mezzo di energia oscura nei dettagli. Mostreremo l'esistenza di una direzione preferenziale nel mezzo di energia oscura che identifica una superficie di interfaccia dove tutte le identificazioni possono essere fatte. Nel capitolo finale uniremo finalmente tutte le nozioni date nei tre capitoli precedenti per formulare una semplice, ma allo stesso tempo completa descrizione della teoria, descrivendo i suoi punti forti e deboli. Una volta che abbiamo mostrato che le quantitá gravitazionali ed elastiche sono relazionate sotto alcune condizioni nel mezzo di energia oscura, diventerá facile vedere la dualitá fra certe leggi, specialmente dal punto di vista energetico.

Contents

Chapter 1

Introduction

This paper is mainly based on Verlinde's work on emergent gravity. What we propose here is a formal approach to the laws that bind together gravity and elasticity. The main idea is that local density of entropy given by mass remains in the dark energy medium leaving an imprint. The result is that dark energy medium is deformed. We can apply classical linear theory of elasticity results to describe its behaviour. The deformation causes a stress, the medium relaxes very slowly and it generates a reaction elastic force on mass. It's this force, that dark energy medium exerts on mass, that causes an excess of gravity which we observe as dark matter. These hypothesis are highly speculative and emergent gravity is nowadays a really discussed and criticized theory but also a really interesting approach to dark matter problem and a good alternative to MOND[1] theories.

1.1 Summary

The correspondence equations shown in chapter 5, allow us to translate the response of then dark energy medium described by the elastic quantities of chapter 4, in the form of apparent gravitational quantities of chapter 2 thanks to the experimental results given in chapter 3. This identification can only be made once proper conditions are met and we are in the Newtonian regime.

Our main purpose is to describe the correspondence maps and explain the entities involved. In revisiting Newton's theory of gravitation we focus on the universal gravitational constant and gravitational acceleration through the basic relations of the classical theory in its most simplified version. Some of its applications will be discussed and some of its main results shown, such as the orbit of the earth around the sun. Even if a more complete covariant theory is available for describing such phenomena, we will deal with

[1]Modified Newtonian Dynamic (MOND) theories are about an hypothetical acceleration scale that is the boundary below which the usual Newton's law is no longer valid and needs to be modified.

a simple yet satisfactory version of it.

The transition of the Newtonian regime to the elastic one can only be understood with quantum theory in a de Sitter space which we shortly describe in the first part of Appendix A. Nonetheless we will give an insight at the reasons that led us on this path in the cosmology chapter. Recent experimental results indicated us that not Newton's theory of gravitation and not even Einstein's general relativity are enough to describe all the gravity related phenomena in the universe. Thereby we start thinking that may there be some intrinsic space-time properties that define a boundary where these laws are no more valid. Here's where the acceleration scale come into play, defining a limit where standard Newton's law, which can be derived by general relativity with the appropriate conditions, are no more valid. Although these ideas are not well accepted, they seems to lead to interesting theoretical results such as the one we are describing.

We can in fact identify the acceleration scale as a fundamental parameter that allow us to describe the precise laws that binds gravity and elasticity. The latter, is better described as linear elasticity since we will treat only the theory with a linear relation between the strain and stress tensor, known as linear Hooke's law. The phenomena described by theory on a generic medium are then translated into the entanglement entropy medium ones as we briefly see in Appendix A and thereby into the dark energy medium ones. As we will see, being able to describe this relation between dark energy medium deformations and apparent gravity gives us the chance to bind more deep laws such as Gauss law and Poisson equation in an elegant and concise way.

Albeit we are looking forward to discuss a covariant theory, our treaty will merely stop at the relation that gravitational self-energy and elastic energy have for the sake of simplicity and of course, because of the limits of the most recent theoretical results.

1.2 Theoretical framework

We will mainly focus on gravity and elasticity or "elastic gravity" as we call for short, leaving the really complex theoretical framework aside, which we will describe shortly in this section just to be clear on our main goal.

In this paper we will work on usual four dimensional space-time even if, to be consistent the theory requires an introduction of de Sitter space which is a direct generalization of a four dimensional space-time hypersphere by adding extra spatial dimensions. In this context entropy has a different behaviour then the one in usual space-time and justify our assumptions on local density of entropy which is given by mass that remains in dark energy medium.

In our work we will consider only classical fields that cover a patch or the whole manifold of space or space-time. Sometimes we will refer to fields as mediums where all the necessary approximations are supposed to be given.

As we all know, at small scales what really describes the universe behaviour is quantum mechanics and since we will deal with low acceleration scales it is mandatory to do so. We will anyway ignore the real structure of space-time and consider its entropy like an intrinsic property i.e. a given field. It is none less worthwhile to say that gravity is emergent from an entanglement structure of a microscopic theory, which is why we refer to this theory as emergent gravity.

When we speak of dark energy medium we mean a form of energy that permeates the entire universe and that can only be detected indirectly by observations of our universe accelerating expansion.

1.3 Conventions and notation

Since we will treat several topics, each with its own usual notation, we will stick to standard conventions used in general relativity even in classical theories, for an homogeneous notation between different chapters of this paper.

We recall that Greek lower-case indices $\alpha, \beta, \ldots, \omega \in \{0, 1, 2, 3\}$ and Latin lower-case indices $a, b, \ldots, z \in \{1, 2, 3\}$ unless specified otherwise, where 0 is reserved for temporal component and $1, 2, 3$ for spatial components. Indices between parenthesis will have different meaning to be specified from the context and they do not follow summation rule. We always chose to use upper and lower indices version of Einstein summation convention where the metric tensor will be $g_{ij} = \delta_{ij} = diag(1, 1, 1)$ unless specified differently. When we will call vectors and higher order tensors with their components, we will always implicitly assume to have fixed an orthonormal basis, unless specified otherwise. For example, if we consider two generic vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}^3$ we can write their inner product using our conventions as:

$$\mathbf{v} \cdot \mathbf{u} = v^i \delta_{ij} u^j = v^i u_i \equiv \sum_{i=1}^{3} v^i u_i = v^1 u_1 + v^2 u_2 + v^3 u_3$$

while we will reserve the notation $|u^i| = \sqrt{u^i u_i}$ for the modulus of a vector. The Minkowski metric will have signature $(-, +, +, +)$ where we use positive sign for spatial components and negative sign for temporal component.

We will mark *definitions* with blue italic font the first time they appear as relevant in our discussion while black italic font will be used for *important statements* and laws.

Shorthand notation LHS (left hand side) and RHS (right hand side) will be used to specify which side of an equation we are referring to.

We will usually refer to fields omitting their nature without any misconception since we can always define them as a quantity given for each point of our manifold.

We will use the dot on a quantity to indicate its total derivative with respect to time, which is meant to be proper time if not specified otherwise.

Chapter 2

Newtonian gravity

Newton's gravitational theory states that every point mass attracts each other point mass in the universe by a force inversely proportional to square distance and pointing along the line that intersect them. We will introduce the definition of conservative forces of which gravitational force is one. We will speak of Newton's gravitational law and give a brief introduction of gravitational potential theory.

2.1 Conservative forces

In nature, a force field F^i can only be of two types, determined by the condition that the path integral:

$$\int_{r_{(1)}}^{r_{(2)}} F^i dx_i \tag{2.1.1}$$

depends or less by the path that joins points $r_{(1)}$ and $r_{(2)}$. If path integral (2.1.1) do not depends on path between the two points but only on the actual points $r_{(1)}$ and $r_{(2)}$ then the force field is said to be a *conservative force field*.

If we fix point $r_{(1)}$ to $\bar{r}_{(1)}$ path integral (2.1.1) for a conservative force field will be a well defined function $U(r_{(2)})$:

$$\int_{\bar{r}_{(1)}}^{r_{(2)}} F^i dx_i = U(r_{(2)}) \tag{2.1.2}$$

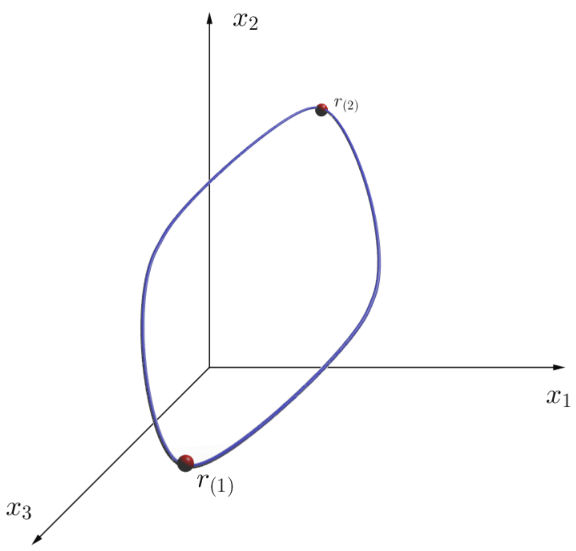


Figure 2.1: Two different generic paths (blue) joining the points $r_{(1)}$ and $r_{(2)}$ (red).

If we move now from $r_{(2)}$ to $r_{(2)} + dx^1$ along the direction x^1 we have:

$$U(r_{(2)} + dx^1) = \int_{\bar{r}_{(1)}}^{r_{(2)}+dx^1} F^i dx_i = U(r_{(2)}) + \int_{r_{(2)}}^{r_{(2)}+dx^1} F^i dx_i = U(r_{(2)}) + F^1 dx_1$$

so that:

$$U(r_{(2)} + dx^1) - U(r_{(2)}) = F^1 dx_1$$

and we can write the force as:

$$F^1 = \partial^1 U(r_{(2)}) \tag{2.1.3}$$

If we repeat this reasoning, moving now along x^2 and x^3 we obtain three equations in the form of (2.1.3) which can be written as:

$$F^i = \partial^i U(r_{(2)}) \tag{2.1.4}$$

We can now fix $r_{(2)}$ to $\bar{r}_{(2)}$ letting $r_{(1)}$ as variable and repeat the process above that led to (2.1.4) concluding that must exist a function $U(r_{(1)}, r_{(2)})$ such that:

$$F^i = \partial^i U(r_{(1)}, r_{(2)}) \tag{2.1.5}$$

where $\partial^i = \nabla^i$ is the usual gradient according to our notation.

2.2 Gravitational force

Classically, the force that keeps planets in orbit around the sun is called force of gravity and it was firstly studied by Isaac Newton.

The gravitational force that the sun of mass M exerts on earth of mass m is given by Newton's law of universal gravitation:

$$F^i = -G\frac{mM}{|r^i|^2}\hat{r}^i \tag{2.2.1}$$

where the sun has been placed in the origin of our system and where $|r^i|^2 = r^i r_i$ is the modulus square of position vector r^i which defines position of earth, of direction given by $\hat{r}^i = r^i/|r^i|$.

The constant G is the gravitational constant and it has a value of:

$$G \simeq 6,67 \times 10^{-11} m^3 s^{-2} kg^{-1} \tag{2.2.2}$$

According to Newton's law of universal gravitation (2.2.1), the gravitational force that acts on earth is proportional to its inertial mass m. This can only be explained by equivalence principle in Einstein's general relativity which states that inertial and gravitational masses are the same quantities i.e. inertial mass is gravitational charge [10].

Newton's gravitational force is a conservative force and as such, it is possible to write it as a gradient of a scalar function $\mathcal{U}(|r^i|)$ according to eq. (2.1.5):

$$F^i = -\partial^i \mathcal{U}(|r^i|) \tag{2.2.3}$$

where the sign choice is arbitrary but made such that the potential $\mathcal{U}(|r^i|)$ of our planet in the gravitational field of the sun is in the form:

$$\mathcal{U}(|r^i|) = -G\frac{mM}{|r^i|} \tag{2.2.4}$$

Since gravitational force is a conservative the total energy of the system must be conserved. In fact we can write energy for unit mass as:

$$\mathcal{E} = \frac{|\dot{r}^i|^2}{2} - \frac{GM}{|r^i|} \tag{2.2.5}$$

which is time independent, meaning that it does not depend on time explicitly.

Since gravitational potential is a central potential on the other hand we have that total angular momentum of the system, given by the vector product:

$$\ell^i = r^k p^j \varepsilon^i{}_{jk} \tag{2.2.6}$$

where p^j is the linear momentum, must be conserved. In fact, it can be shown from properties of Levi-Civita tensor[1] $\varepsilon^i{}_{jk}$ that defines the vector product, that the following relation must hold:

$$r_i \ell^i = r_i r^k p^j \varepsilon^i{}_{jk} = 0 \tag{2.2.7}$$

The above expression (2.2.7) is just saying that, since the three vectors involved are coplanar, they must have a null scalar triple product. This is actually an equation for a plane passing from the origin and with normal direction $\hat{k}^i$ parallel to the third component ℓ^3 of the angular momentum ℓ^i and since it is constant over time, the motion of earth around the sun will be in the plane [10]. The problem is thus reduced to a bidimensional problem and it is possible to determine the orbit of earth around the sun.

[1]The Levi-Civita tensor is defined by the property that it takes the value 1 for $\{i, j, k\} \in \{\{1, 2, 3\}, \{2, 3, 1\}, \{3, 1, 2\}\}$, it takes the value -1 for $\{i, j, k\} \in \{\{3, 2, 1\}, \{1, 3, 2\}, \{2, 1, 3\}\}$ and it takes the value 0 if two or more indices are the same.

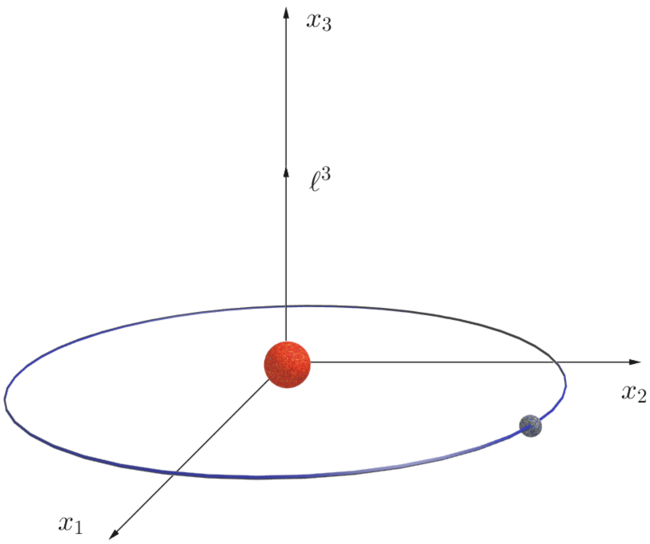


Figure 2.2: A tridimensional representation of earth orbit (blue line) around the sun (red) in the plane (x_1, x_2) while the third component of angular momentum is directed along x_3 axis.

2.3 Gravitational potential

Let's consider a direct generalization of Newton's gravitational law (2.2.1) by considering two generic bodies of mass m, m' and respectively of position vectors r^i, r'^i which reads:

$$F^i = mg^i \tag{2.3.1}$$

where the gravitational acceleration is given by:

$$g^i = G\frac{m'}{|\Delta r^i|^2}\hat{\Delta r^i} \tag{2.3.2}$$

as we define $\Delta r^i = r'^i - r^i$ and $\hat{\Delta r^i} = \dfrac{\Delta r^i}{|\Delta r^i|}$.

We can write the gravitational potential in a more general form using (2.1.5) and (2.3.1).

$$\Phi = -G\frac{m'}{|\Delta r^i|} \tag{2.3.3}$$

we can show that gravitational acceleration (2.3.2) is given by the gradient of the gravitational potential (2.3.3).

$$\partial^i\left(\frac{1}{|\Delta r^i|}\right) = \frac{\partial}{\partial r^i}\left[\sum_{i=1}^{3}(r'^i - r^i)^2\right]^{-\frac{1}{2}} =$$

$$= \left(-\frac{1}{2}\right)\left[-2(r'^i - r^i)\right]\left[\sum_{i=1}^{3}(r'^i - r^i)^2\right]^{-\frac{3}{2}} = \frac{\Delta r^i}{|\Delta r^i|^3}$$

So that we get exactly (2.3.2):

$$g^i = -\partial^i\Phi = Gm'\partial^i\left(\frac{1}{|\Delta r^i|}\right) = G\frac{m'}{|\Delta r^i|^2}\hat{\Delta r^i}$$

We can further generalize, by considering N mass points each of mass m_s and position vector $r^i_{(s)}$ with $s \in \{1, 2, \dots, N\}$. Gravitational potential at the generic point r^k will take the form [10]:

$$\Phi = -G\sum_{s=1}^{N}\frac{m_{(s)}}{|r^k_{(s)} - r^k|} \tag{2.3.4}$$

If we consider a continuous distribution of mass we will just substitute, with proper considerations, the sum with an integral over all the distribution volume V and the mass m with a mass density ρ, to get the following expression [10]:

$$\Phi = -G\int_V \frac{\rho(r'^i)}{|r'^i - r^i|}d^3{r'}_i \tag{2.3.5}$$

where $dm' = \rho(r'^i)dV'$ is the mass of the unit volume $dV' = d^3r'_i$. We can also write the gravitational acceleration by taking the gradient of (2.3.5):

$$g^i = -\partial^i\Phi \tag{2.3.6}$$

which yields by straightforward calculation:

$$g^i = G\int_V \frac{r'^i - r^i}{|r'^i - r^i|^3}\rho(r'^i)d^3{r'}_i \tag{2.3.7}$$

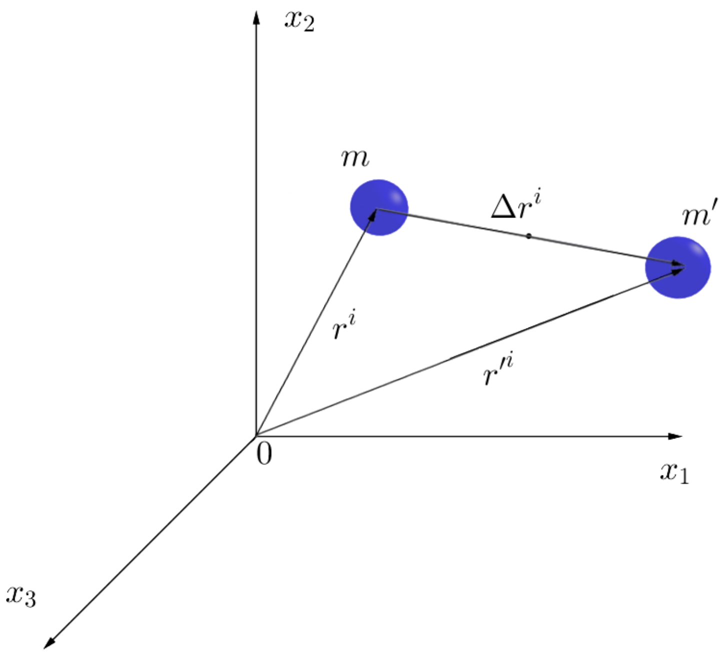


Figure 2.3: A tridimensional representation of two masses m and m' (blue) placed at distance $|\Delta r^i|$.

Chapter 3

Cosmology

Cosmology is the study of the universe at large spatial scales and long times, it study the origin and future of the universe trying to give a physical explanation of observative phenomena. We will speak of cosmological principle and scale factor to be able to introduce Hubble parameter. We will then explain why we need an acceleration scale and how it shows up.

3.1 Cosmological principle

Modern cosmology is based on principles that allow us to give a satisfying description from a physical point of view and mathematically simple at the same time. The most important principle is the *cosmological principle*:

The universe is homogeneous and isotropic.

Isotropy and homogeneity are observative facts that result to be evident on large spatial scales, such as the length scale of superclusters of galaxies[1]. We recall that homogeneity means that physical properties taken in exam do not depended on the point, while isotropy means that physical properties taken in exam do not depend on direction. If we assume as principle that does not exists any preferred observer in the universe, homogeneity is a direct consequence of isotropy [5]. We can in fact consider a universe isotropic for all possible observers and since none of them is preferential the universe will be homogeneous.

[1] Superclusters of galaxies are sets of gravitationally bound galaxies which exibit properties as a single.

Let's assume cosmological principle to be true and through this assumption we will build a universe model that will later confirm it. In the following construction we will consider the universe as filled with a homogeneous and isotropic fluid called *cosmic fluid* formed by galaxies [5].

3.2 FRW metric and scale factor

The cosmological principle implies existence of a space-time metric uniquely defined, also known as Friedman-Robertson-Walker metric shortly called FRW metric.

The general form of a space-time metric is:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{3.2.1}$$

where $g_{\mu\nu}$ is the *metric tensor*. Given the cosmological principle we have that components g_{0i} of metric tensor are null since, for a fixed time t, physical parameters cannot depend on point and are the same in all the space [11]. The space-time metric (3.2.1) can thus be written as:

$$ds^2 = -c^2dt^2 + d\ell^2 \tag{3.2.2}$$

where we have set $-c^2dt^2 = g_{00}dx^0dx^0$ and $d\ell^2 = g_{ij}dx^idx^j$ to be the *spatial metric*.

Let's now consider the surface of an hypersphere in $\mathbb{R}^4$. Using polar coordinates (ρ, θ, ϕ) and $w \in \mathbb{R}$ as fourth coordinate we can write the metric on this surface as:

$$dl^2 = d\rho^2 + \rho^2 d\Omega^2 + dw^2 \tag{3.2.3}$$

Where $d\Omega^2 = \sin^2\theta d\theta^2 + d\phi^2$ is the metric of a unit sphere in $\mathbb{R}^3$.

If we set $a \in \mathbb{R}^+ \setminus \{0\}$ to be the radius, the equation of the hypersphere will be:

$$\rho^2 + w^2 = a^2 \tag{3.2.4}$$

Differentiating (3.2.4) we obtain $\rho d\rho + wdw = 0$ which give us:

$$dw^2 = \frac{\rho^2 d\rho^2}{a^2 - \rho^2} \tag{3.2.5}$$

where we used again (3.2.4) to set $w^2 = a^2 - \rho^2$ on the denominator. By substituting (3.2.5) in (3.2.3) and using some simple algebraic manipulations we get:

$$dl^2 = \frac{d\rho^2}{1 - \left(\frac{\rho}{a}\right)^2} + \rho^2 d\Omega^2 \tag{3.2.6}$$

which is the metric of a three-dimensional isotropic and homogeneous space with positive Gaussian curvature $K = a^{-2} > 0$.

We can consider a direct generalization of (3.2.6) by formally considering $a \in \mathbb{C}$ so that $K \in \mathbb{R}$ and we can write the Gaussian curvature as [6]:

$$K = |K| sign(K) \tag{3.2.7}$$

where $k = sign(K)$ is called *curvature constant*:

$$k = \begin{cases} 1 & \text{if } K > 0 \\ 0 & \text{if } K = 0 \\ -1 & \text{if } K < 0 \end{cases} \tag{3.2.8}$$

Imposing a rescaling of radial coordinate $\rho = ar$ and considering as further generalization $a = a(t)$, which doesn't break homogeneity [11], we get the generalized version of (3.2.6):

$$dl^2 = a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \tag{3.2.9}$$

So that the *FRW metric* can be written as:

$$ds^2 = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \tag{3.2.10}$$

by substituting (3.2.9) into (3.2.2).

The new coordinates (t, r, θ, ϕ) are called *comoving coordinates* [5]. In this frame the coordinate t is the proper time of an observer which moves with the cosmic fluid at constant r, θ, ϕ.

The new function $a(t)$, which is a real valued function for an arbitrary real Gaussian curvature K, is the so called *cosmic scale factor*. It depends only on proper time t and it has dimensions of a length. We will define present scale factor as $a_o = a(t_o) = 1$ where t_o is present time.

This allow us to define:

$$\mathcal{H}(t) = \frac{\dot{a}(t)}{a(t)} \tag{3.2.11}$$

as the *Hubble parameter* and consequently:

$$\mathcal{H}_o = \frac{\dot{a}(t_o)}{a(t_o)} \tag{3.2.12}$$

as the *Hubble constant* which is Hubble parameter at present time.

FRW metric determines a three-dimensional surface Σ_t which evolves in time and has geometry uniquely determined by the value of k. From (3.2.8) we have only 3 possible cases:

(i) $k = 1$.

Spatial metric (3.2.4) takes the form:

$$dl^2 = a^2(t)\left(\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right) \tag{3.2.13}$$

So that Σ_t is a three-dimensional sphere and the metric takes the usual form with a coordinate transformation $r = \sin(X)$.

$$dl^2 = a^2(t)[dX^2 + \sin^2(X)d\Omega^2] \tag{3.2.14}$$

FRW metric (3.2.10) is:

$$ds^2 = -c^2dt^2 + a^2(t)[dX^2 + \sin^2(X)d\Omega^2] \tag{3.2.15}$$

which defines a closed universe.

(ii) $k = 0$.

Spatial metric (3.2.9) takes the form:

$$dl^2 = a^2(t)\left(dr^2 + r^2 d\Omega^2\right) \tag{3.2.16}$$

So that Σ_t is a flat three-dimensional hyper-surface and FRW metric (3.2.10) is:

$$ds^2 = -c^2dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2) \tag{3.2.17}$$

which defines a flat universe.

(iii) $k = -1$.

Spatial metric (3.2.9) takes the form:

$$dl^2 = a^2(t)\left(\frac{dr^2}{1+r^2} + r^2 d\Omega^2\right) \tag{3.2.18}$$

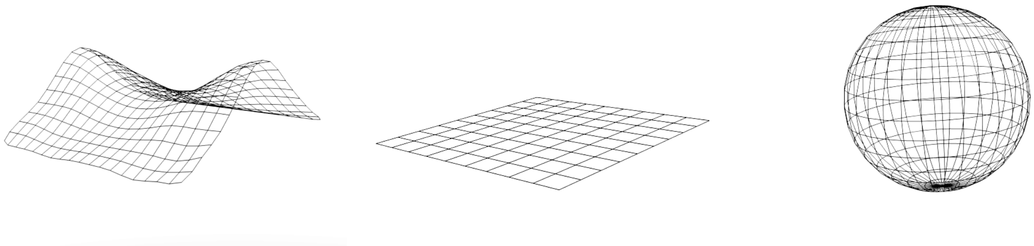
So that Σ_t is a three-dimensional hyperboloid and the metric takes the usual form with a coordinate transformation $r = \sinh(Y)$.

$$dl^2 = a^2(t)[dY^2 + \sinh^2(Y)d\Omega^2] \tag{3.2.19}$$

FRW metric (3.2.10) is:

$$ds^2 = -c^2dt^2 + a^2(t)[dY^2 + \sinh^2(Y)d\Omega^2] \tag{3.2.20}$$

which defines an open universe.



(a) Open Universe (b) Flat Universe (c) Closed Universe

Figure 3.1: Models of possible universes in two spatial dimensions.

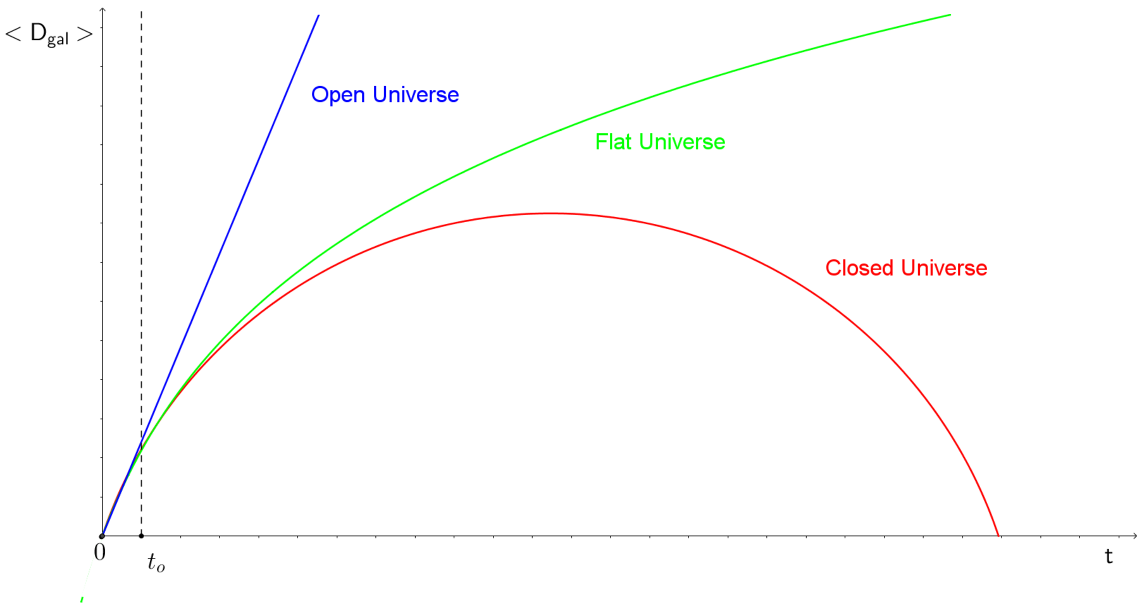


Figure 3.2: Average distance between glaxies $< D_{gal} >$ in function of cosmic time t in open universe (blue line), flat universe (green line) and open univse (red line).

3.3 Acceleration scale

It's well known from modern cosmology that spiral galaxies have a rotational speed, given by the dynamic of stars gravitationally bound, which rotate around the center of mass of the galaxy. It's possible to find out observationally rotational speed of stars in spiral galaxies in function of distance from center of mass [9]. It turns out to be approximatively constant even at great distances, fact in disagreement with Newton's theory.

If we consider that density of stars in a spiral galaxy decrease with radius, we expect a speed that decrease with increasing distance from center of mass.

These observative facts led to develop a theory that suppose the existence of an "hidden" mass called *dark matter*. Dark matter would be responsible of the intense gravitational field in periferic zones of spiral galaxies which leads to a greater rotational speed values than the expected ones [9].

An other approach to the problem is to consider that at small acceleration scales Newtonian gravity would not be valid. These assertion were firstly proposed in Modified Newtonian Dynamic (MOND) by Milgrom, who calculated that the *acceleration scale* α_0 which determines the transition between Newtonian dynamic and MOND must be:

$$\alpha_0 \simeq 1,2 \times 10^{-10} ms^{-2} \tag{3.3.1}$$

for his law to fit rotation curves data. It is interesting that, from theoretical point of view the acceleration scale α_0 can be defined as [3]:

$$\alpha_0 = \frac{c\mathcal{H}_0}{2\pi} \tag{3.3.2}$$

where c is the speed of light in vacuum, $\mathcal{H}_0$ is Hubble's constant (3.2.12) and 2π it's a normalization factor, showing that α_0 is in fact a fundamental quantity.

Even if Acceleration scale (3.3.2) was firstly determined in MOND theories, it will play an important role in elastic gravity as a fundamental constant and not as direct consequence of MOND. Its meaning will remain the same while the applications will be totally different. We will not treat it as a scale for which Newton's gravitational laws stop working, but as a scale where gravitation and elastic theory meets.

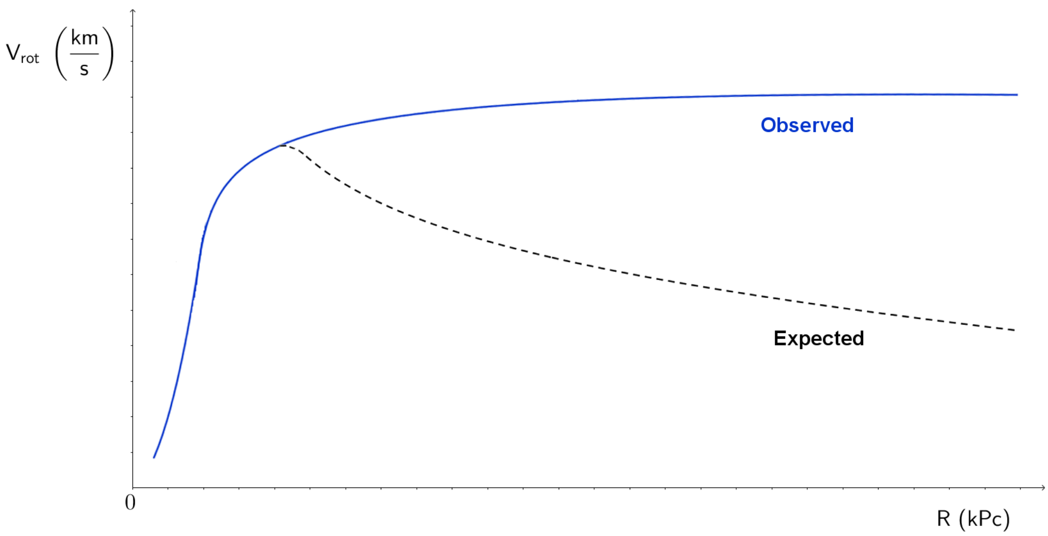


Figure 3.3: A graphic showing rotation velocity v_{rot} in function of distance from center of galaxy R. Observed rotation curve is the blue line while expected newtonian rotation curve is black dotted line.

Chapter 4

Linear elasticity

Linear elasticity is a branch of continuum mechanics that deals with infinitesimal strains. We will derive strain and stress tensors and show that they are bound by a linear relation also known as Hooke's law. We will also speak very shortly of elastic waves and finally introduce a preferential direction in the medium of extreme importance for our concern.

4.1 Strain tensor

When we apply a force to solid bodies they get deformed and as result they change in shape and volume [14]. The position of each point that makes the body is described by the radius vector x^i.

A deformed body has in general, points moved in $x^{i'}$, such that we can define a *displacement vector* as:

$$u^i = x^{i'} - x^i \tag{4.1.1}$$

With deformation distance between points changes. If we consider two points infinitesimally close before the deformation takes place, they will be joined by the radius vector dx^i and their distance will be $d\ell = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$ which can be cast as:

$$d\ell = \sqrt{dx^i dx_i} \tag{4.1.2}$$

where $dx^i dx_i = g_{ij} dx^i dx^j = \delta_{ij} dx^i dx^j$. After the deformation points will be joined by a radius vector given by $dx^{i'} = dx^i + du^i$ so that new distance between the two points will be $d\ell' = \sqrt{dx_{i'} dx^{i'}}$.

We can write the square of the new distance as:

$$\begin{aligned} d\ell'^2 = dx_{i'}dx^{i'} = (dx^i + du^i)(dx_i + du_i) = \\ = dx^i dx_i + du^i du_i + dx^i du_i + dx_i du^i \end{aligned}$$

Taking in account that $u^i = u^i(x^k)$ we have $du^i = \partial_k u^i dx^k$ and using (4.1.2) we find:

$$d\ell'^2 = d\ell^2 + \partial_k u^i \partial_l u_i dx^k dx^l + 2\partial_k u_i dx^k dx^i$$

Since $\partial_k u_i$ is symmetrical we can write $\partial_k u_i = \frac{1}{2}(\partial_k u_i + \partial_i u_k)$ which gives us:

$$\begin{aligned} d\ell'^2 = d\ell^2 + \partial_k u^i \partial_l u_i dx^k dx^l + (\partial_k u_i + \partial_i u_k) dx^i dx^k = \\ = d\ell^2 + (\partial_k u_i + \partial_i u_k + \partial_k u^l \partial_i u_l) dx^i dx^k \end{aligned}$$

Or alternatively:

$$d\ell'^2 = d\ell^2 + 2u_{ik} dx^i dx^k \tag{4.1.3}$$

Where:

$$u_{ik} = \frac{1}{2}(\partial_k u_i + \partial_i u_k + \partial_k u^l \partial_i u_l) \tag{4.1.4}$$

is the *strain tensor*.

The strain tensor u_{ik} given by eq. (4.1.4) is symmetrical:

$$u_{ik} = u_{ki} \tag{4.1.5}$$

we can in fact use the symmetry of $\partial_k u_i$ and write:

$$u_{ik} = \frac{1}{2}(\partial_k u_i + \partial_i u_k + \partial_k u^l \partial_i u_l) = \frac{1}{2}(\partial_i u_k + \partial_k u_i + \partial_i u^l \partial_k u_l) = u_{ki}$$

Since u_{ik} is symmetrical as seen in eq. (4.1.5), we can diagonalize it at any given point, meaning that it is possible to find a frame of axis called *principal axis of the strain tensor* in which the associated matrix is diagonal [14].

Non null components of diagonalized strain tensor matrix u_{11}, u_{22}, u_{33} are called *principal values of strain tensor*, commonly denoted as $u^{(1)}, u^{(2)}, u^{(3)}$.

If the strain tensor is diagonal $u_{ik}dx^i dx^k = u_{ii}dx^i dx^i$ and square distance (4.1.3) takes a very simple form:

$$d\ell'^2 = d\ell^2 + 2u_{ik}dx^i dx^k = \delta_{ik}dx^i dx^k + 2u_{ii}dx^i dx^i$$

and remembering the definition $u_{ii} = u^{(i)}$ of the principal strains, can also be written as:

$$d\ell'^2 = (\delta_{ii} + 2u^{(i)})dx^i dx^i \tag{4.1.6}$$

Where we intend as extraordinary case a sum on the index between pharentesis. Eq. (4.1.6) implies that a strain in any elemental volume, can be seen as composed of three independent strains in the directions of principal axis of the strain tensor [14].

4.2 Stress tensor

After a deformation, position of molecules that compose a body changes and the molecules go out of equilibrium position resulting in formation of internal forces called *internal stresses*. These forces are actually generated by intermolecular short range forces that bring the body back to its equilibrium position [14].

Let's consider a body after a deformation, which we will call *medium* from now on, of volume V and surface S. If F^i is force per volume unit and $F^i dV$ is force on volume element dV, the total force acting on the body will be:

$$\mathcal{F}^i = \int_V F^i dV \tag{4.2.1}$$

Since forces that exercise between adjacent volume elements balance to each other, the resulting force will be a surface force. For Gauss theorem we can write a volume integral of a vector F^i as surface integral if the vector can be written as divergence of rank 2 tensor σ^{ik}, namely $F^i = \partial_k \sigma^{ik}$. In this case, we write RHS of eq. (4.2.1) as:

$$\int_V F^i dV = \int_V \partial_k \sigma^{ik} dV \tag{4.2.2}$$

If we consider df_k the direction normal to surface element dS, using Gauss theorem we have:

$$\mathcal{F}^i = \oint_S \sigma^{ik} df_k \tag{4.2.3}$$

Where σ^{ik} is the *stress tensor*. The total force acting on body will be other than the integral of forces $\varphi^i = \sigma^{ik} df_k$ that act on surface elements dS, so that we can rewrite (4.2.3) as:

$$\mathcal{F}^i = \oint_S \varphi^i \tag{4.2.4}$$

4.3 Hooke's Law

Let's consider a deformed medium, in the case where the deformation changes in a way that u^i changes only by a small amount δu^i. If δR is work done by internal infinitesimal stresses per unit volume and according to eq. (4.2.2) force per unit volume is $F^i = \partial_k \sigma^{ik}$ then we can write:

$$\delta R = \partial_k \sigma^{ik} \delta u_i \tag{4.3.1}$$

Integrating (4.3.1) over all the medium volume :

$$\int_V \delta R dV = \int_V \partial_k \sigma^{ik} \delta u_i dV \tag{4.3.2}$$

which is total infinitesimal work done by internal stresses. Using integration by parts on RHS of (4.3.2) we obtain:

$$\int_V \partial_k \sigma^{ik} \delta u_i dV = \oint_S \sigma^{ik} \delta u_i df_k - \int_V \sigma_{ik} \partial_k (\delta u_i) dV$$

Considering the case of an infinite medium undeformed at infinity, integration surface S_∞ is at infinity and the surface integral will be zero, since we can formally put $\sigma^{ik} = 0$ on S_∞. There will only be left a volume term and using the fact that $\partial_k(\delta u_i) = \partial_i(\delta u_k)$

we get:

$$\int_V \partial_k \sigma^{ik} \delta u_i dV = -\int_V \sigma_{ik} \partial_k (\delta u_i) dV =$$
$$= -\frac{1}{2}\int_V \sigma_{ik} \partial_k (\delta u_i) dV - \frac{1}{2}\int_V \sigma_{ik} \partial_k (\delta u_i) dV =$$
$$= -\frac{1}{2}\int_V \sigma^{ik} [\partial_k (\delta u_i) + \partial_i (\delta u_k)] dV = -\frac{1}{2}\int_V \sigma^{ik} \delta[\partial_k u_i + \partial_i u_k] dV$$

In our approximation u^k can be considered small enough that strain tensor (4.1.4) takes the form:

$$u_{ik} = \frac{1}{2}(\partial_k u_i + \partial_i u_k) \tag{4.3.3}$$

where small second order term $\partial_k u^l \partial_i u_l$ is neglected, so that:

$$-\frac{1}{2}\int_V \sigma^{ik} \delta[\partial_k u_i + \partial_i u_k] dV = -\int_V \sigma^{ik} \delta u_{ik} dV$$

With this approximation, eq. (4.3.2) becomes:

$$\int_V \delta R dV = -\int_V \sigma^{ik} \delta u_{ik} dV \tag{4.3.4}$$

which leads to the relation between strain tensor, stress tensor and work done per unit volume:

$$\delta R = -\sigma^{ik} \delta u_{ik} \tag{4.3.5}$$

For the first law of classical thermodynamics, for a reversible process an infinitesimal variation of internal energy is given by [13]:

$$d\mathcal{E} = T d\mathcal{S} - dR \tag{4.3.6}$$

where $\mathcal{S}$ is total entropy and T is temperature of the system. Reversible means that we are considering only elastic deformations of our medium, process for which there is no residual strain. In fact, if we deal only with small deformations the medium will go back to its equilibrium once external forces stop acting.

An other requirement is that deformation process is slow enough to allow us to define thermodynamic equilibrium at each given instant of time. keeping this in mind, we can write free energy of the system as [14]:

$$F = \mathcal{E} - T\mathcal{S} \tag{4.3.7}$$

Differentiating (4.3.7) and using first law of thermodynamics (4.3.6) along with (4.3.5):

$$dF = d\mathcal{E} - T\mathcal{S} - \mathcal{S}dT = -dR - SdT = \sigma^{ik} du_{ik} - \mathcal{S}dT$$

Let's further suppose that deformation happens at fixed temperature so that:

$$dF = \sigma^{ik} du_{ik} \tag{4.3.8}$$

which leads to:

$$\sigma^{ik} = \frac{\partial F}{\partial u_{ik}} \tag{4.3.9}$$

Eq. (4.3.8) imply that we can expand F in powers of u_{ik} and stopping to second order we have:

$$F = F_0 + \frac{1}{2}\lambda u^2 + \mu u_{ik} u^{ik} \qquad ; \; i \neq k \tag{4.3.10}$$

where F_0 is a constant, $u = u^i{}_i$ is the trace of strain tensor and λ, μ are the so called Lamé parameters. If we differentiate eq. (4.3.10) we obtain:

$$dF = \lambda u du + 2\mu u_{ik} du^{ik} = \lambda u \delta_{ik} du^{ik} + 2\mu u_{ik} du^{ik} = (\lambda u \delta_{ik} + 2\mu u_{ik}) du^{ik}$$

and using eq. (4.3.8) it brings us to Hooke's law:

$$\sigma_{ik} = \lambda u \delta_{ik} + 2\mu u_{ik} \tag{4.3.11}$$

which describe the linear relation between stress tensor and strain tensor (4.3.3).

We will define:

$$\mathcal{K} = \lambda + \frac{2}{3}\mu \tag{4.3.12}$$

as the bulk modulus where μ is the lamé parameter called rigidity modulus. The bulk modulus describes the medium response to uniform pressure while the rigidity modulus μ describes response to shear stress, for this reason the latter is also called shear modulus.

Further, we will also define:

$$M = \lambda + 2\mu \tag{4.3.13}$$

as the pressure waves modulus or P-waves modulus for short. M and μ determines pressure and shear waves' velocity in homogeneous isotropic media of density ρ, respectively via:

$$v_p = \sqrt{\frac{M}{\rho}} \tag{4.3.14}$$

$$v_s = \sqrt{\frac{\mu}{\rho}} \tag{4.3.15}$$

so that P-waves modulus and shear modulus must be positive:

$$M \geq 0 \quad , \quad \mu \geq 0 \tag{4.3.16}$$

4.4 Largest strain

For isotropic media properties do no depend on spatial direction and the strain tensor (4.1.4) can be written as sum of a symmetric traceless tensor and a diagonal tensor [17]:

$$u_{ij} = \frac{1}{3}u\delta_{ij} + \left(u_{ij} - \frac{1}{3}u\delta_{ij}\right) \tag{4.4.1}$$

where we will define:

$$v_{ij} = \frac{1}{3}u\delta_{ij} \tag{4.4.2}$$

as volumetric strain tensor and:

$$d_{ij} = u_{ij} - \frac{1}{3}u\delta_{ij} \tag{4.4.3}$$

as deviatoric strain tensor.

We can rewrite the strain tensor (4.4.1) as sum of a volumetric and a deviatoric part:

$$u_{ij} = v_{ij} + d_{ij} \tag{4.4.4}$$

Since strain and stress tensors are symmetrical and linearly related by Hooke's law (4.3.11) , they can be simultaneously diagonalized. In fact, we have that the condition of applicability of simultaneous diagonalization theorem $[u_{ij}, \sigma^{ki}] = 0$ is satisfied [1]:

$$\begin{aligned} [u_{ij}, \sigma^{ki}] &= u_{ij}\sigma^{ki} - \sigma^{ki}u_{ij} = u_{ij}(\lambda u\delta^{ik} + 2\mu u^{ki}) - (\lambda u\delta^{ik} + 2\mu u^{ki})u_{ij} = \\ &= \lambda u u_j{}^k + 2\mu u_{ij}u^{ki} - u_j{}^k\lambda u - 2\mu u^{ki}u_{ij} = 2\mu(u_{ij}u^{ki} - u^{ki}u_{ij}) = 2\mu[u_{ij}, u^{ki}] = 0 \end{aligned}$$

where we used (4.3.11) and (4.1.5). One can immediately see that $[u_{ij}, u^{ki}] = 0$ using strain tensor definition (4.1.4) and its symmetry property (4.1.5).

We have then a base of common eigenvectors $\{a^i_{(k)}\}$ of tensors u_{ij} and σ_{ij}. Let's consider the set $\{u^{(1)}, u^{(2)}, u^{(3)}, \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}\} = \{u^{(i)}, \sigma^{(i)}\}$ formed by eigenvalues of u_{ij} and σ_{ij} respectively, called principal values of strain and stress. It allow us to define the largest principle of strain and stress as [17]:

$$s = \max\{u^{(i)}, \sigma^{(i)}\} \tag{4.4.5}$$

The largest principle of strain and stress is an eigenvalue of u_{ij} or σ_{ij} and it has associated a common eigenvector $s^i \in \{a^i_{(k)}\}$ which can be normalized:

$$n^i = \frac{s^i}{|s^i|} \tag{4.4.6}$$

Deviatoric strain tensor (4.4.3) is symmetric since strain tensor is (4.1.5), so it must be diagonalizable. If we call $\{d^i_{(k)}\}$ its eigenvectors of eigenvalues $\{d^{(k)}\}$ then n^i will be parallel to a direction detected by $q^i \in \{d^i_{(k)}\}$ of eigenvalue $q \in \{d^{(k)}\}$ called maximal strain.

The eigenvalue equation then becomes:

$$d^i{}_j n^j = qn^i \tag{4.4.7}$$

Where n^i is the direction of maximal strain.

Eq. (4.4.7) defines the direction of maximal strain as the normalized eigenvector $n^i = q^i/|q^i|$ of the deviatoric strain tensor associated with the eigenvalue q such that $q^i/|q^i| = s^i/|s^i|$ where s^i is the eigenvector defined by the largest principle of strain and stress.

Chapter 5

Gravity-Elasticity

Elastic Gravity theory deals with the identification between dark energy medium deformation and observed gravity. We will give via assumption some "correspondence maps" that will allow us to translate the deformation of the medium in gravitational terms. Then, we will show that given these maps, it is possible to determine elastic counterparts of gravitational main results such as gauss' law and poisson equation.

5.1 Correspondence maps

We can make a correspondence between elastic and gravitational quantities once it is given the direction of maximal strain which determines a perpendicular interface surface S. This surface is the boundary between two spatial regions occupied by the dark energy medium in two different states. The only constants that will play a role in the identification are the acceleration scale and the universal gravitational constant [17].

Correspondence maps are relations which allow us to to translate the elastic response of the dark energy medium in terms of a gravitational field. These maps will be given as guessed since we have not the necessary tools to derive them from more deep laws. A general idea on how to prove the first correspondence relation is shown in Appendix A. Anyway, from these maps we will be able to show that they yield the exact form of many familiar relations, both in gravitational and elastic version.

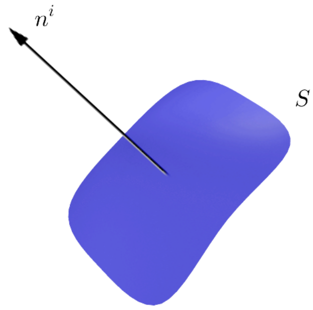


Figure 5.1: The interface surface S in blue and the direction of maximal strain n^i.

The displacement field (4.1.1) is related to the gravitational potential field (2.3.5) by:

$$u^i = \frac{\Phi}{\alpha_0} n^i \tag{5.1.1}$$

The strain tensor (4.3.3) is related to the gravitational acceleration (2.3.7) by:

$$u^{ij} n_j = -\frac{g^i}{\alpha_0} \tag{5.1.2}$$

while the stress tensor given by Hooke's law (4.3.11) is related to surface mass density:

$$\Sigma^i = -\frac{g^i}{4\pi G} \tag{5.1.3}$$

by the following relation:

$$\sigma^{ij} n_j = \alpha_0 \Sigma^i \tag{5.1.4}$$

where we recall that n_i is the direction of maximal strain given by eq. (4.4.7).

It is also possible to determine shear modulus of the dark energy medium and P-waves modulus (4.3.13) respectively as:

$$\mu = \frac{\alpha_0^2}{16\pi G} \tag{5.1.5}$$

$$M = 0 \tag{5.1.6}$$

Where the proof, which is quite complex, will be omitted and a general idea on how to derive them is shown in Appendix B.

5.2 Gauss law and Poisson equation

Gauss' law states that the total flux through a closed surface S is proportional to the charge inside that region [15]. Gravitationally, it can be derived if we consider a mass M enclosed in a sphere surface $\mathcal{A}$ of radius $|r^i|$. We recall that inertial mass M is actually a gravitational charge.

Since the net gravitational flux through the area element dA of the sphere is:

$$d\phi = g^i dA_i = gdA = \left(-\frac{GM}{|r^i|^2}\right)(|r^i|^2 \sin\theta d\theta d\varphi) = -MG\sin\theta d\theta d\varphi$$

the total flux through the entire sphere surface can be written as:

$$\oint_{\mathcal{A}} g^i dA_i = -MG\int_o^{\pi}\sin\theta d\theta\int_0^{2\pi} d\varphi = -4\pi MG$$

It can be shown that we can choose an arbitrarily shaped close surface S called Gaussian surface through which the same result apply:

$$\oint_S g^i dS_i = -4\pi MG \tag{5.2.1}$$

It takes a simpler form if it's expressed in terms of surface mass density (5.1.3):

$$\oint_S \Sigma^i dS_i = M \tag{5.2.2}$$

also known as gravitational Gauss law.

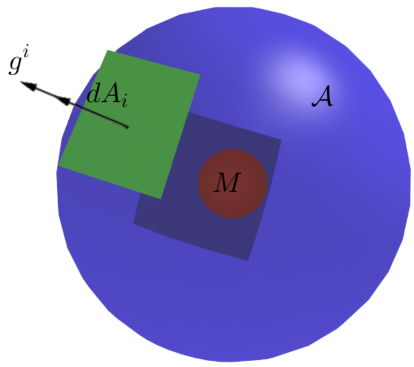


Figure 5.2: A tridimensional representation of the spherical gaussian surface $\mathcal{A}$ in blue, the mass M in red and the area element dA in green. The direction of the vector dA_i is the same of the gravitational field g^i.

We can now apply the correspondence map (5.1.4) to (5.2.2) and get:

$$\oint_S \Sigma^i dS_i = \oint_S \frac{\sigma^{ij} n_j}{\alpha_0} dS_i = M$$

multiplying both sides by n^j and α_0 we have:

$$\oint_S \sigma^{ij} n_j n^j dS_i = M \alpha_0 n^j$$

and recalling that $n^j n_j = 1$ we finally end up with:

$$\oint_S \sigma^{ij} dS_i = -f^j \tag{5.2.3}$$

which is the elastic counterpart for gravitational Gauss law (5.2.2) where we have defined:

$$f^j = -M \alpha_0 n^j \tag{5.2.4}$$

as the elastic charge vector that is the point force acting on the points of the medium. This point force acts as a point source for the elastic displacement field u^k.

Using Gauss theorem we can write LHS of eq. (5.2.1) as:

$$\oint_S g^i dS_i = \int_V \partial_i g^i dV$$

while its RHS can be written as:

$$-4\pi GM = -4\pi G \int_V \rho dV = \int_V (-4\pi G\rho) dV$$

where V is the volume of the Gaussian surface S and ρ is the mass density. Bringing LHS and RHS results together, the relation:

$$\int_V \partial_i g^i dV = \int_V (-4\pi G\rho) dV \tag{5.2.5}$$

give us gravitational Gauss law in differential form:

$$\partial_i g^i = -4\pi G\rho \tag{5.2.6}$$

It takes a simpler form using surface mass density (5.1.3):

$$\partial_i \Sigma^i = \rho \tag{5.2.7}$$

Using correspondence map (5.1.4) we find:

$$\partial_i \Sigma^i = \partial_i \left(\frac{\sigma^{ij} n_j}{\alpha_0} \right) = \rho$$

Again, multiplying both sides by n^j and α_0 we have:

$$\partial_i \sigma^{ij} n_j n^j = \rho \alpha_0 n^j$$

and again recalling that $n^j n_j = 1$ we obtain:

$$\partial_i \sigma^{ij} = -b^j \tag{5.2.8}$$

which is the elastic counterpart for gravitational Gauss law in differential form where we have defined:

$$b^j = -\rho \alpha_0 n^j \tag{5.2.9}$$

as the density of elastic charge vector that is force per unit volume acting on the medium.

We can see that gravitational Gauss law (5.2.7) is a Poisson equation once we express the gravitational acceleration (2.3.7) in terms of the gravitational potential (2.3.5) with relation (2.3.6):

$$\partial_i \Sigma^i = \partial_i \left(-\frac{g^i}{4\pi G} \right) = \frac{\partial_i \partial^i \Phi}{4\pi G} = \rho$$

so that Gravitational Poisson equation is:

$$\partial_i \partial^i \Phi = 4\pi G \rho \tag{5.2.10}$$

where, according to our notation $\partial_i \partial^i = \nabla_i \nabla^i$ is the laplacian operator.

We can apply the correspondence maps (5.1.1) and (5.2.9) to Gravitational Poisson equation (5.2.10) LHS and RHS respectively:

$$\partial_i \partial^i \Phi = \partial_i \partial^i (\alpha_0 u^k n_k) = 4\pi G \rho = 4\pi G \left(-\frac{b^j n_j}{\alpha_0} \right)$$

to get the corresponding elastic Poisson equation:

$$\partial_i \partial^i \Psi = -\frac{\chi}{4\mu} \tag{5.2.11}$$

which is elastic counterpart for gravitational Poisson equation (5.2.10), where we have defined:

$$\Psi = -u^k n_k \tag{5.2.12}$$

as the medium elastic potential and:

$$\chi = \rho \alpha_0 \tag{5.2.13}$$

as the elastic charge density which is the modulus of the elastic charge density vector (5.2.9):

$$|b^j| = \sqrt{b^j b_j} = \sqrt{\rho^2 \alpha_0^2 n^j n_j} = \rho \alpha_0 = -b^j n_j = \chi$$

Taking the modulus of the point force (5.2.4) we have that the elastic charge is:

$$\mathcal{E} = M\alpha_0 \tag{5.2.14}$$

with the dimensions of a force.

5.3 Energy correspondence

Gravitational self-energy of a mass configuration is energy of self-interaction, in other words, it's energy with negative sign needed to dissociate the mass into point masses and bring them to infinity. Gravitational self-energy can be expressed in terms of the acceleration field (2.3.7) and surface mass density (5.1.3) as [17]:

$$U = \frac{1}{2}\int_V g^i \Sigma_i dV \tag{5.3.1}$$

If we apply correspondence maps (5.1.2) and (5.1.4) we get:

$$U = \frac{1}{2}\int_V g^i \Sigma_i dV = \frac{1}{2}\int_V (-u^{ij} n_j \alpha_0) \left(\frac{\sigma_{ij} n^j}{\alpha_0}\right) dV$$

and since it's an energy, we can take it with a positive sign by formally considering $U' = -U$ as:

$$U' = \int_V u^{ij} \sigma_{ij} dV \tag{5.3.2}$$

which is the elastic counterpart for gravitational self energy, also called elastic energy.

Chapter 6

Conclusion

We have not yet found any evidence that at these scales Newtonian mechanics and usual physics laws should stop working but the great discrepancy between luminous and dark matter gravitational effects point us indirectly in this direction. Anyway, our physical knowledge on these phenomena is still based on "non local" observations and this may be an important factor which we should take care of.

Hereby we conclude that we tried to describe a way to measure this apparent gravity effects we attribute to dark matter from a thermodynamic point of view as an entropy related phenomenon. More precisely, we mean entropy related to dark energy, where the latter can be thought as a positive energy lifted from negative ground state energy by excitations of the microscopic degrees of freedom.

The main result we want to emphasize is that, once we introduced the correspondence maps, we were able to derive the elastic version of the Gauss law and Poisson equation just by applying them to their gravitational counterparts. Moreover the elastic energy is directly derived from the gravitational self-energy, underlining that there can be a more deep connection between the two theories. More precisely it is interesting to notice that to a rank 1 tensor in gravitation corresponds a rank 2 tensor in linear elasticity.

It is natural then, to raise the question whether or not the non-Newtonian regime can be somehow related to non-linear elasticity and to continuum mechanics in general and if so, where it could lead.

The results we obtained are still non relativistic in the sense that they should be written in covariant form to properly agree with Einstein's general relativity. After all, Emergent gravity is still a young theory and many researches are still going on nowadays.

Chapter 7

Acknowledgements

I want to thank my family, who supported me during the writing of the thesis and all the way through my studies. I want to thank my supervisor Prof. Roberto Casadio for offering me such an interesting and stimulating topics to work on. I also want to give a special thanks to my colleagues for sharing their ideas and their thoughts on the subject.

Appendix A

Correspondence maps derivation

As we already pointed out, we are working in a de Sitter space[1] which can be denoted as $\mathrm{dS}_{d-1} \subset \mathbb{R}^{1,d-1}$ where d is the number of space-time dimensions, with the metric induced by the standard Minkowski metric. Globally it can be written as a closed FRW universe (3.2.15) and it can be easy visualized as the hyperboloid with one sheet [4]:

$$-x_0^2 + \sum_{j=1}^{d-1} x_j^2 = L^2 \tag{A.1}$$

where L is a positive real number. A remarkable property of de Sitter space is that entanglement entropy has, besides the usual area law that appears in Anti de Sitter space, a volume law which comes into play when we add matter. This volume law enters in competition with the area law and causes microscopic de Sitter states to show glassy behaviours. This means that at long time scales these states exhibit properties due to their ergodic dynamics, impossible to observe at human time scales. When we add matter to de Sitter space its entanglement entropy content decreases and the resulting distribution of entanglement entropy density with respect to its equilibrium position is described by a displacement vector u^i, where now $i \in \{1, 2, \ldots, d-1\}$.

An other fundamental property of de Sitter space is that it has a cosmological horizon[2] L and no asymptotic spatial infinity. The entanglement entropy on the horizon can be interpreted as quantifying the amount of entanglement between two opposite sides of the horizon. These facts allow us to treat a de Sitter space much like we would treat a black hole and use some of the known laws with appropriate precautions.

As we know from general relativity the metric outside the mass for a static patch will

[1] de Sitter space is the simplest solution of Einstein equations with a positive cosmological costant.

[2] Cosmological horizon is a measure of the distance at which two particles can no longer exchange information in a classical way.

take the form [16]:

$$ds^2 = -\left(1 - \frac{r^2}{L^2} + 2\Phi(r)\right) dt^2 + \left(1 - \frac{r^2}{L^2} + 2\Phi(r)\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{A.2}$$

also known as de Sitter-Schwarzschild solution, where $\Phi(r)$ is the Newton potential due to mass M:

$$\Phi(r) = -\frac{8\pi GM}{(d-2)\Omega_{d-2} r^{d-3}} \tag{A.3}$$

and Ω_{d-2} is the surface area of a unit $(d-2)$-sphere.

Without the mass M the horizon would be at distance L and the entropy on the horizon, also know as Bekenstain-Hawking entropy [2], would simply be:

$$\mathcal{S}_{DE}(L) = \frac{A(L)}{4G\hbar} \tag{A.4}$$

where the subscript DE stands for Dark Energy and $A(L) = \Omega_{d-2} L^{d-2}$ is the area of the horizon. Also, note that in these reasonings we implicitly set $c = 1$, fact that we will bring back at the end of the discussion.

The addition of the mass M caused the horizon to move from L to $L + u(L)$ where $u(L) = \Phi(L)L$ with the approximation $\Phi(L) << 1$ is the displacement of the horizon:

$$u(L) = -\frac{8\pi GM}{(d-2)\Omega_{d-2} L^{d-4}} \tag{A.5}$$

which results to be negative, in agreement with the fact that the horizon has been restricted.

As result, the entropy change in the dark energy medium on the horizon due to the addition of mass $\mathcal{S}_M(L)$ is negative and can be calculated by formally considering $dL \sim u(L)$:

$$\mathcal{S}_M(L) = u(L)\frac{d\mathcal{S}_{DE}}{dL} = -\frac{2\pi}{\hbar} ML \tag{A.6}$$

The entropy reduction (A.6) correspond to a reduction of entanglement between the two sides of the horizon where the mass M was added.

Since we know from general relativity that mass reduces the growth rate of the area as a function of the geodesic distance [17], we can immediately write with proper considerations that for a generic radius $r < L$ the reduction entropy due the addition of a mass M as:

$$\mathcal{S}_M(r) = -\frac{2\pi}{\hbar} M r \tag{A.7}$$

which shows that the mass M reduces in general, reduces the entanglement entropy of the region with radius r by $\mathcal{S}_M(r)$. This result is true for de Sitter, flat space and Anti de Sitter. We can see that:

$$\mathcal{S}_M(r) = \frac{r}{L} \mathcal{S}_M(L) \tag{A.8}$$

showing that the ratio between r and L is the scale factor between entropy removed from a spherical region of radius r (A.7) and the total de Sitter entropy removed (A.6).

From a similar reasoning, we can deduce that when there is no mass added the de Sitter entropy of a spherical region of radius r must be similar to Bekenstein-Hawking entropy calculated in r but with r/L as scaling factor:

$$\mathcal{S}_{DE}(r) = \frac{r}{L} \frac{A(r)}{4G\hbar} \tag{A.9}$$

So that when we set $r = L$ we get exactly the expression of Bekenstein-Hawking entropy (A.4).

From the fact that entropy removed by mass $\mathcal{S}_M(r) \propto r$ we deduce that $u(r) \propto r^{-(d-3)}$ seen that the Newton potential $\Phi(r) \propto r^{-(d-3)}$. This lead us to the conclusion that the displacement of the spherical surface due to the addition of the mass must be [17]:

$$u(r) = \Phi(r) L \tag{A.10}$$

We can rewrite that expression now considering again c to fix the dimensions and bring back the normalization factor 2π we introduced in the acceleration scale (3.3.2) to end up with:

$$u(r) = \Phi(r) \frac{2\pi}{c^2} L \tag{A.11}$$

We can now write the horizon as the Hubble length scale defined by:

$$L = \frac{c}{\mathcal{H}_0} \tag{A.12}$$

so that the above expression becomes:

$$u(r) = \Phi(r)\frac{2\pi}{\mathcal{H}_0 c} = \frac{\Phi(r)}{\alpha_0} \tag{A.13}$$

which is the radial displacement of the spherical surface corresponding to the removal of entropy medium in a symmetrical way due to the addition of mass.

If we want to translate the entropy medium displacement in terms of dark energy medium displacement we need to move ourself in the direction of maximal strain as we already pointed out. This bring us to translate the radial direction for the entropy medium displacement as the direction of maximal strain for the dark energy medium displacement:

$$u^i = \frac{\Phi}{\alpha_0} n^i \tag{A.14}$$

Showing correspondence map (5.1.1) once we have set $d = 4$ so that $i \in \{1, 2, 3\}$ as usual.

Appendix B

Shear and P-waves modulus derivation

Let's consider the killing symmetry associated with the horizon. From Noether's theorem we have a conserved current and as such, a Noether charge $Q[\xi]$ which depends on the killing vector ξ^a and can be defined by the following expression:

$$\int_{hor} Q[\xi] = \frac{\hbar}{2\pi}\mathcal{S} \tag{B.1}$$

where the integration is performed on the horizon and $\mathcal{S}$ is the horizon entropy.

Using Wald's formalism [12], when there is no stress energy on the bulk[1] we can use the above expression (B.1) to write the first law of thermodynamics in de Sitter space as:

$$\frac{\hbar}{2\pi}\delta\mathcal{S} + \delta\mathcal{H}_\xi = 0 \tag{B.2}$$

Where $\mathcal{H}_\xi$ is the Hamiltonian associated with the killing symmetry.

In our approximation where the stress is concentrated in a small region with radius negligible with respect to the Hubble length L, we will consider the variation of the Hamiltonian as an integral extended from the horizon to a surface at infinity S_∞ of the variation of the Noether charge:

$$\delta\mathcal{H}_\xi = \int_{S_\infty} (\delta Q[\xi] - \xi \cdot \delta B) \tag{B.3}$$

[1]The term "Bulk" is used in theoretical physics when we speak of spaces with higher dimensions than the usual 11 dimensions used in M-theory.

where the second term $\xi \cdot \delta B$ vanishes on the horizon for a suitable choice of the (d-1)-form B.

So that the Hamiltonian can be written as:

$$\mathcal{H}_\xi = \int_{S_\infty} (Q[\xi] - \xi \cdot B) \tag{B.4}$$

If we now choose ξ^a to be an asymptotic time translation t^a we can define the canonical energy as:

$$\mathcal{E} = \int_{S_\infty} (Q[t] - t \cdot B) \tag{B.5}$$

and following Wald's calculations [12], this allow us to define ADM mass[2] [7]. We remark that we assumed that at spatial infinity the surface integral gets finite and the metric approaches a well defined asymptotic form. It can be shown that once we set $c = 1$ we can write ADM mass as [17]:

$$M_{ADM} = \int_{S_\infty} (Q[t] - t \cdot B) = \frac{1}{16\pi G} \int_{S_\infty} (\partial^j h^i{}_j - \partial^i h) dA_i \tag{B.6}$$

where $h^i{}_j$ is the spatial metric and $h = h^j{}_j$ its trace.

We will now make some approximations, that will allow us to treat ADM mass (B.6), which can be defined only at spatial infinity, as mass in the interior of de Sitter space.

The approximations are the following:

- The surface S_∞ is sufficiently large that the gravitational field of the mass M is negligible.
- The surface S_∞ is sufficiently small so that its radius r_∞ is negligible with respect to the Hubble length L.

Given the above approximations, if we pick a particular surface S_∞ with a spherical symmetry and the spatial metric with the following form:

$$h_{ij} = \delta_{ij} + 2\Phi(r) n_i n_j \tag{B.7}$$

[2]The Arnowitt-Deser-Misner formalism (ADM formalism) is an approach to general relativity which allow us to view the gravitational field as an Hamiltonian system [8].

with $n_i = r_i/|r_i|$ we can write the expression for the mass [17]:

$$M = -\frac{1}{8\pi G}\int_{r_\infty} \Phi(r)\partial^j n_j dA \tag{B.8}$$

where $\Phi(r)$ is the Newton potential defined as:

$$\Phi(r) = -\frac{8\pi GM}{(d-2)\Omega_{d-2}r^{d-3}} \tag{B.9}$$

and Ω_{d-2} is the surface area of a unit $(d-2)$-sphere.

It can be shown that, given our particular spherical choice of the surface S_∞ and the correspondence map (5.1.1), we can write the mass expression as [17]:

$$M = \frac{\alpha_0}{8\pi G}\int_{S_\infty}(u^{ij}n_j - un^i)dA_i \tag{B.10}$$

We can justify eq. (B.10) by saying that we chose a particular reference frame in de Sitter space with respect to which we define the mass M. This reference frame is chosen such that the gravitational regime makes the transition to the elastic one.

We can now multiply both sides by α_0 so that on LHS we obtain a force and on RHS we get:

$$M\alpha_0 = \frac{\alpha_0^2}{8\pi G}\int_{S_\infty}(u^{ij}n_j - un^i)dA_i = \int_{S_\infty}\sigma^{ij}n_j dA_i \tag{B.11}$$

where we used the familiar Hooke's Law (4.3.11) and stress tensor physical meaning as seen in eq. (4.2.3). The expression above (B.11) lead us to write the the lamé parameters as follow:

$$\mu = \frac{\alpha_0^2}{16\pi G} \tag{B.12}$$

$$\lambda = -\frac{\alpha_0^2}{8\pi G} \tag{B.13}$$

showing eq. (5.1.5) and eq. (5.1.6).

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