ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

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Dipartimento di Fisica e Astronomia
Corso di Laurea Magistrale in Fisica

Diffusion in Hamiltonian Systems under Stochastic Perturbations and LHC Dynamic Aperture Issues

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To Rosa e Bubba for their love
and
To Manta e Costa for the journey

Sommario

In questo lavoro studiamo sistemi Hamiltoniani sottoposti a perturbazioni stocastiche e ne analiziamo applicazioni a problemi relativi all'apertura dinamica di LHC.

Il moto betatronico non lineare di particelle sottoposte all'azione degli elementi magnetici di un acceleratore é ben descritto da sistemi Hamiltoniani perturbati. Una approfondita comprensione della dinamica di singola particella é cruciale per il buon funzionamento della macchina. In particolare il problema della perdita di intensità del fascio é stato affrontato in passato e sono state proposte leggi di scala per l'apertura dinamica basate su stime di tipo Nekhoroshev.

Noi studiamo lo scenario in cui una Hamiltoniana integrabile é sottoposta a piccole perturbazioni stocastiche, rappresentanti l'interazione con l'ambiente, la cui ampiezza dipende solo dallo stato dinamico della particella.

Successivamente impostiamo un'equazione di Fokker-Planck per la densitá di probabilitá della variabile di azione per studiare la diffusione del sistema debolmente caotico introdotto sopra e proponiamo come coefficienti di diffusione stime di tipo Nekhoroshev e leggi di potenza, calibrate in certa misura fenomenologicamente.

Segue una nostra derivazione di quantitá semi-analitiche interessanti nello studio della diffusione di particelle a partire dall'equazione di Fokker-Placnk.

In fine relazioniamo il modello diffusivo cosí costruito con dati sperimentali, comparando le nostre stime con misure su perdite di intensitá nei fasci eseguite a LHC.

Abstract

In this work we study stochastically perturbed Hamiltonian systems and relate them to recent applications to LHC dynamic aperture issues.

Nonlinear betatronic motion of particles in magnetic lattice is well described by means of perturbed Hamiltonian systems. A proper understanding of the single-particle dynamics is crucial for the performances of the machine. In particular the problem of beam losses has been tackled in the past and scaling laws for the dynamic aperture have been proposed on the basis of Nekhoroshev-like estimates.

We study the scenario in which an integrable Hamiltonian undergoes small stochastic perturbations, which encapsulate the interactions with the environment, whose amplitude depends only on the particle dynamical state.

We then derive a Fokker-Planck equation for the probability density function of the action variable to study the diffusion of the weakly chaotic Hamiltonian system described above and propose to use either Nekhoroshev-like estimate or a power law, gauged phenomelogically to some extent, as diffusion coefficients.

Our derivation of semi-analitical quantities of relevance for particle diffusion starting from the Fokker-Planck equation is then carried out.

Finally we relate our diffusion model with experimental data comparing our estimates with measurements on beam losses at LHC.

Contents

Introduction

Particle accelerator projects for hign energy physics have reached a high degree of complexity and Accelerator Physics has to cope with new challenging problems. Due to the use of superconductig magnets the nonlinear effects are extremely important and study of the dynamic aperture problem that reduces the lifetime of the beam requires the applications of Nonlinear Dynamical Systems Theory.

In this thesis we consider some aspects of the problem of beam losses. Recently beam halo diffusion rate measurements in the the Large Hadron Collider (LHC) at CERN [1] has been performed [1] and data obtained from similiar experimental procedures, regarding the LHC dynamic aperture, has been supplied to us by the Hadrons Synchrotrons Single particle section of the Accelerators and Beam Physics Group at CERN's Beam Department to be studied.

The data represent decreases of the current, i.e. particle losses, over time in the storage ring for different set-ups of the multipolar magnetic elements. The beam is observed for long times (10^6s) and losses under 5% are recorded.

We want to describe in term of particle diffusion the beam quality degradation. Such description is valid if the macroscopic motion has a stochastic character [2, 3, 4, 5] as is the case in real machines. Indeed imperfections in the magnet, beam-gas scattering, ground motion, ripple in the power supplies and other uncontrolled effects are unavoidable. To tackle the problem we start from a stochastically perturbed Hamiltonian to describe the microscopic dynamics. The Hamiltonian approach to accelerator physics has been in use for decades [6, 7] and has been proved to be particularly suited to face non-linear promlems [8, 9]. Furthermore for hadron colliders is appropriate even to describe long times behaviour due to the possibility to neglect synchrotron radiation effects. The work is structured as follows.

In the first chapter some aspects of beam-dynamics are presented. We introduce the betatronic oscillations and write down the Hamiltonian for a charged particle in the ideal magnetic field in multipolar expansion, typical of synchrotron magnetic lattices. In the case of hadrons, a symplectic

[1] *Conseil Européen pour la Recherche Nucléaire*

description of the dynamics holds and this Hamiltonian serve as a starting point for the stochastic perurbation analysis in this work. We briefly tackle the equaiton of motion in the linear approximation and sieze the opportuinity to introduce some important quantities in accelerator physics. Finally we introduce the concept of dynamic aperture.

In chapter 2 we recollect some useful notions about Markov processes and stochastic differential equations. The Fokker-Planck equation is introduced and we describe the cumulant expansion approach to the study of stochastic dynamical systems.

Such procedure is used chapter 3 to derive a Fokker-Planck equation from a stochastic phase flow. In this chapter indeed we start form a weakly noisy Hamiltonian system to end up, after an averaging procedure, with a Fokker-Planck equation for the probability distribution function of the action variable (an invariant for the unperturbed motion).

In chapter 4 we propose to use a Nekhoroshev-like estimate and power laws to model diffusion in the beam dynamics. We start finding formal solution for the Fokker-Planck equation, then we analyse a simple case with a linear diffusion coefficient and derive closed formula for currents and particle flow which can be directly related to beam losses issues. At this point we devise an approximation procedure to obtain the correspective of these formula for the diffusion coefficients we are interested in. We test the goodness of the approach by means of numerical simulations in such a way we can safely use the formula to analyse beam losses.

In the last chapter we introduce the experimental data. These are measures of beam losses at LHC and throughout the chapter we compare the data with our diffusion model phenomenologically fitted on them.

All the numerical simulations in this thesis are performed using CraNiO; a numerical solver for general one dimensional Fokker-Planck equations, developed by the author, based on the Crank-Nicolson scheme.

Chapter 1

Betatronic motion and dynamic aperture

In this chapter basic notions on beam dynamics in the transverse plan will be given. Betatronic oscillations in linear approximation will be studied explicitly and central quantities of interest in accelerator physics introduced. Moreover it will be defined the concept of dynamic aperture and related to the problem of beam losses which is relevant in this thesis.

1.1 Betatronic oscillations and one-turn maps

In this section we introduce the transverse beam dynamics in a synchrotron, briefly describing the usual approach adopted in the study of single-particle motion and writing down the equations of betatronic oscillations.

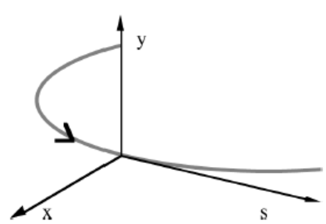


Figure 1.1: Frenet-Serret coordinate system [10]

In the study of the dynamics in the transverse plane (x,y) is usually adopted the Ferret-Serret system of coordinates shown in figure 1.1, where the evolution is followed with respect to the curvilinear coordinate s instead of time, a possibility due to the fact that there is a bijective correspondence between the longitudinal position of the particle and the time. This reference frame follows the path of an ideal particle always in $(0,0)$ on the transeverse plane, morover with $\rho(s)$ we denote the local radius of curvature which depends on the magnets and varies around the ring. Hence the coniugate momenta became the dimensionless quantities

$$p_x \equiv \frac{dx}{ds} \qquad p_y \equiv \frac{dy}{ds} \tag{1.1}$$

Therefore the dynamics take place on the four-dimensional phase space (x, p_x, y, p_y), neglecting the longitudinal motion.

The beam has to be controlled with magnetic fields. Thus the possibility to express the total field through a multipolar expansion in which every term can be linked to a different type of magnet with accordingly many poles is unvaluable [10]. This can be achieved if one consider a field with only transverse component B_x and B_y and the expansion reads in the Beth representation

$$B_y(x,y;s) + iB_x(x,y;s) = B_0\Big(\rho_0 \sum_{n=0}^{\infty}[k_n(s) + ij_n(s)]\frac{(x+iy)^n}{n!}\Big) \tag{1.2}$$

where $k_0(s) = -1/\rho_0$ and $j_0(s) = 0$. In equation (1.2) B_0 denotes the constant magnetic field provided by dipoles needed to keep on an orbit of radius ρ_0 a particle with nominal momentum p_0 (we assume all the particles at the same nominal energy and off-momentum effect are not considered) according to the relation

$$\frac{p_0}{e} = B_0\rho_0 \tag{1.3}$$

where the r.h.s. is often called the *magnetic rigidity*.

The coefficient $k_n(s)$ and $j_n(s)$ are respectivley the normal and skew gradients, defined in terms of the derivative of B_y and B_x

$$\begin{aligned} k_n(s) &\equiv \frac{1}{B_0\rho_0}\frac{\partial^n B_y}{\partial x^n}\Big|_(0,0;s) \\ j_n(s) &\equiv \frac{1}{B_0\rho_0}\frac{\partial^n B_x}{\partial x^n}\Big|_(0,0;s) \end{aligned} \qquad n = 1, 2, ... \tag{1.4}$$

Neglecting linear coupling, which amounts to set $j_1(s) = 0$, and performing other approximation it can be shown [11] that the equation of the betatron motion are

$$\begin{aligned} \frac{d^2y}{ds^2} + \Big(\frac{1}{\rho^2(s)} - k_1(s)\Big)x &= \mathrm{Re}\Big[\sum_{n=2}^{\infty}\frac{k_n(s) + ij_n(s)}{n!}(x+iy)^n\Big] \\ \frac{d^2y}{ds^2} + k_1(s)y &= -\mathrm{Im}\Big[\sum_{n=2}^{\infty}\frac{k_n(s) + ij_n(s)}{n!}(x+iy)^n\Big] \end{aligned} \tag{1.5}$$

In the following section we will look for solution to the linear case where only dipoles and quadrupoles are present, and this will allow us to gain more insights on the dynamics in an accelerator machine. The non-linear case with multipolar terms is a more realistic and far more challenging problem to tackle and will not be treated here because goes beyond the scope sof this thesis where a different kind of non-linearities in the beam dynamics are considered. For a tratment of non-linear betatronic motion, in which normal forms theory is fruitfully employed, we refer to [9].

We remark that the equation of motion (1.5) can be derived form the Hamiltonian

$$\begin{aligned} H(x,p_x,y,p_y) = \frac{p_x^2+p_y^2}{2} + \Big(\frac{1}{\rho^2(s)} - k_1(s)\Big)\frac{x^2}{2} + k_1(s)\frac{y^2}{2} + \\ - \operatorname{Re}\Big[\sum_{n=2}^{\infty} \frac{k_n(s)+ij_n(s)}{(n+1)!}(x+iy)^{n+1}\Big] \end{aligned} \tag{1.6}$$

This Hamiltonian it will be important in this work. Indeed it encodes all the effects of ideal multipolar magnets on the dynamic and through s-dependent canonical transformations can be cast in a form in which depend only on invariant of motion $H(I)$. In this form, considered in two dimensions in phase space, it will represent the integrable unperturbed part to which we will add stochastic perturbations and thus study the behaviour of the entire system.

We conclude this section introducing the single-element and one-turn maps. It is indeed possible to break the description of the particle dynamics for the whole circular accelerator magnetic lattice into the composition of the action of every single magnetic element.

To tackle this task, it is convenient to introduce the *transfer map* $M^{(n)}$ of the magnetic element $\mathcal{M}^{(n)}$. This is possible since on a single magnet the terms in the multipolar expansion are in good approximation constant functions. As in figure 1.2 we take S_{n-1} and S_n as the ends of $\mathcal{M}^{(n)}$; if the accelerator is cut in L pieces, we set $S_0 = 0$ and S_L will be the total lenght of the machine.

The map $M^{(n)}$ is then the Hamiltonian flows associated to (1.6) which evolves the phase space coordinates form $\mathbf{x}(S_{n-1})$ to $\mathbf{x}(S_n)$, where $\mathbf{x} = (x, p_x, y, p_y)$. More explicitly we have

$$\begin{aligned} x(S_n) &= M_1^{(n)}(\mathbf{x}(S_{n-1})) \\ p_x(S_n) &= M_2^{(n)}(\mathbf{x}(S_{n-1})) \\ y(S_n) &= M_3^{(n)}(\mathbf{x}(S_{n-1})) \\ p_y(S_n) &= M_4^{(n)}(\mathbf{x}(S_{n-1})) \end{aligned} \qquad M_j^{(n)} : \mathbb{R}^4 \to \mathbb{R} \quad j = 1,2,3,4 \tag{1.7}$$

Having defined the transfer map for the single element one can easily define the *one-turn map* M, also called the Poincaré map at section $S = S_0$,

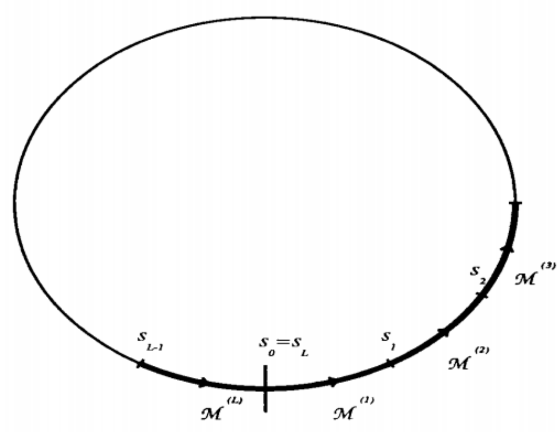


Figure 1.2: Sketch of a circular accelerator [9]

as their composition

$$M \equiv M^{(L)} \circ M^{(L-1)} \circ ... \circ M^{(1)} \tag{1.8}$$

Hence M is the operator which evolves the phase space coordinates $\mathbf{x}(S_0)$ after a complete turn

$$\mathbf{x}(S_L) = M(\mathbf{x}(S_0)) \tag{1.9}$$

For further contents about application of theory of maps in accelerator physics we refer to [9].

1.2 Linear betatron motion and relevant quantities in accelerator physics

In this section we briefly survey the linear betatron motion and seize the opportunity to introduce some of the most relevant quantities in the field of accelerator physics.

In the linear approximation, apart form dipoles and quadrupoles, every other term in the multipolar expansion of the magnetic field is neglected. The linear beam dynamics derived in this approximation describes how the beam is bent and focused in order to follow the designed trajectories.

The dipoles deal with bending the beam keeping particles in orbit at radius ρ_0 as defined in the previous section. Practically the particles do not follow a clean closed orbit but instead they undergo oscillations in the transvers plane, these oscillations are driven and controlled by quadrupoles. It is not possible to focus at the same time the beam along both axes, so in every magnetic element of the lattice there are quadrupoles which focus on the x-axis and defocus on the y-axis and viceversa. In figure 1.3 are reported idealized dipole and quadrupole magnets along with the respective magnetic fields and the currents that powers them.

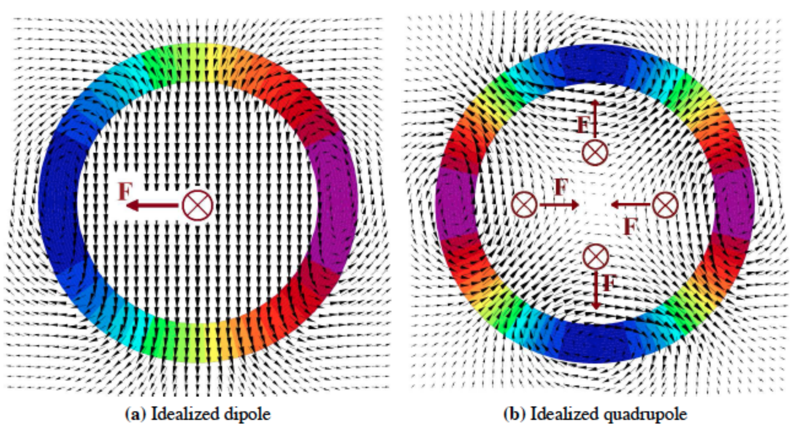


Figure 1.3: idealized dipole and quadrupole magnets. Arrows show the produced magnetic fields. The currents that power the magnets are represented with different color being the extreme blue for the current entering the page and violet for the current flowing outside the page. The forces acting on a test particle entering the page are also reported [12].

The linearized form of the equations of the betatronic motion (1.5), again neglecting the linear coupling, is

$$\begin{aligned} &\frac{d^2x}{ds^2} + \Big(\frac{1}{\rho^2(s)} - k_1(s)\Big)x = 0 \\ &\frac{d^2y}{ds^2} + k_1(s)y = 0 \end{aligned} \tag{1.10}$$

These are Hill's equations, i.e. oscillators with frquencies depending on s. A solution for these equations can be found by means of the ansatz [11]

$$\begin{aligned} x(s) &= \sqrt{\epsilon_x\beta_x(s)}\sin(\psi_x(s) + \delta_x) \\ y(s) &= \sqrt{\epsilon_y\beta_y(s)}\sin(\psi_y(s) + \delta_y) \end{aligned} \tag{1.11}$$

Thus we have the dynamics of two harmonic oscillators with s-dependent amplitude $\beta_{x,y}(s)$ and phase advance $\psi_{x,y}(s)$; $\delta_{x,y}$ and $\epsilon_{x,y}$ are constants.

The constants of motion $\epsilon_{x,y}$ are the *emittances* for single particle in the transverse plane. The amplitudes $\beta_{x,y}(s)$ are the *beta functions*, describing the envelope in which the betatronic oscillations in both directions undergo. Plugging the ansatz (1.11) in (1.10) one can find the relation between the phase advances and the beta functions which has to be fulfilled for the ansatz to be a consistent solution

$$\psi_{x,y}(s) = \int_0^s \frac{d\xi}{\beta_{x,y}(\xi)} \tag{1.12}$$

where considerations on the $\delta_{x,y}$ and other constant of integration have been done to end up with this result [9].

Other central quantities in accelerator physics are the machine *tunes*. The tunes are defined as the phase advances per turn divided by 2π

$$Q_{x,y} = \frac{\psi_{x,y}(S_L)}{2\pi} = \int_0^{S_L} \frac{d\xi}{\beta_{x,y}(\xi)} \tag{1.13}$$

This definition holds for every order in the multipole expansion and even when linear or non-linear perturbations are considered. In the present case, they are called the *linear tunes* and are often denoted as $\nu_{x,y}$.

The central role of tunes in the design and control of the beam emerges because of resonances [13, 14]. Resonance appears when the tunes of the machine have the same value of the frequency of periodic perturbations. Resonant condition thus lead to a deterioration of the beam and usually the operators want to mantain the system at a working point as far as possible from resonant ones, i.e. at tunes which are off the lines in typical working diagrams, as the one portrayed in figure 1.4. It should be noted that in some circumstances (and for some goals) resonant conditions are wanted, as for exemple in [15, 16].

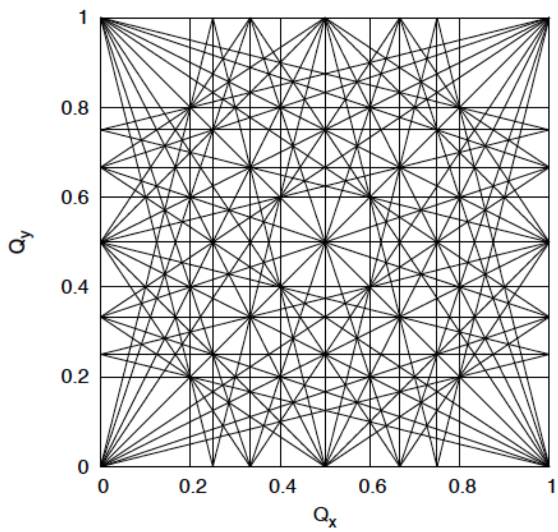


Figure 1.4: Resonance in tune space (Q_x,Q_y), where Q_x and Q_y stand for the fractional parts of the tunes, for $n \leq 5$ [12].

The general equation for resonant conditions reads

$$aQ_x + bQ_y = c \qquad a, b, c \in \mathbb{Z} \tag{1.14}$$

and the nth order resonances correspond to solutions for the tunes when a and b in the equation are such that $n = |a| + |b|$.

Consider now the conjugate momenta $p_{x,y}$. They behave as x and y and the so called *gamma function* $\gamma_{x,y}$ describe the envelope of their oscillations and are related to the beta functions through

$$\alpha_{x,y}(s) \equiv -\frac{1}{2}\frac{d\beta_{x,y}(s)}{ds} \tag{1.15}$$

These functions, togheter with the emittances, define ellipses in the phase space

$$\epsilon_x = \gamma_x(s)x^2 + 2\alpha_x(s)xp_x + \beta_x(s)p_x^2 \tag{1.16}$$

$$\epsilon_y = \gamma_y(s)y^2 + 2\alpha_y(s)yp_y + \beta_y(s)p_y^2 \tag{1.17}$$

These ellipses may vary their shape during the motion but their area, being $\pi\epsilon_{x,y}$, remain constant. This can also be seen as a consequence of Liouville's theorem which, in the case of hadron accelerators, holds being there synchrotron radiations almost neglectable. It can be noted that when the particles undergo acceleration their energy increase and the areas of the ellipses reduces, this reduction is called adiabatic dumping. Hence one can define normalized emittances which remain constant of motion regardless change in the particles energy

$$\epsilon^*_{x,y} \equiv \beta\gamma\epsilon_{x,y}$$

where here β and γ stand for the relativistic functions.

It is possible to deal with circles in the phase space instead of ellipses passing to the Courant-Snyder coordinates. Thus we pass to the new variables through the following transformation involving the α, β and γ functions

$$\begin{pmatrix} \hat{x} \\ \hat{p}_x \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta_x(s)}} & \frac{\alpha_x(s)}{\sqrt{\beta_x(s)}} \\ 0 & \sqrt{\beta_x(s)} \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

and the same goes for y and p_y

$$\begin{pmatrix} \hat{y} \\ \hat{p}_y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta_y(s)}} & \frac{\alpha_y(s)}{\sqrt{\beta_y(s)}} \\ 0 & \sqrt{\beta_y(s)} \end{pmatrix} \begin{pmatrix} y \\ p_y \end{pmatrix}$$

Figure 1.5 shows a sketch of the effects of the transformation on the geometry in the phase space.

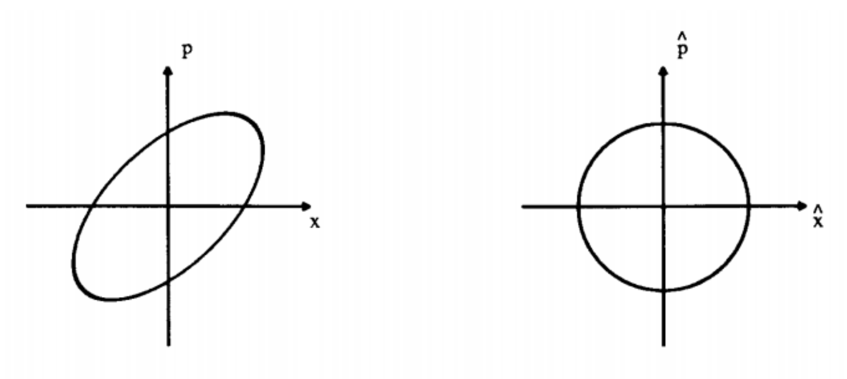


Figure 1.5: Effects of the Courant-Snyder coordinates transformation on the trajectories in the phase space [9].

1.3 Dynamic aperture and beam losses

The stability in the particle orbits is an unavoidable condition for the proper functioning of an accelerator. This task is accomplished using dipoles, quadrupoles and multipolar magnets to control the beam and avoiding resonances. The goal of mantaining stable orbits is not merely to keep the particles inside the mechanical aperture of the machine, because, loosely speaking, non-linearities can reduce in a considerable way the region of phase space where it is possible to work. Thus defining and be able to measure and control the so called *dynamic aperture* (DA) is crucial.

The DA [9, 17, 18] is the region of the phase space (for the transverse dynamics) where the stable motion occurs. The topology of the phase space depends on the magnetic multipolar terms involved and it gets complicated if one considers all the four phase space dimensions of the transverse plan (x, p_x, y, p_y).
Nonetheless in simplified models such as the 2D (in phase space) Hénon map [9] one can visualize the DA and have some more intuition about its meaning. This map, which will not be analysed in details here, can be seen as the Poincaré map of a periodic Hamiltonian which describes the horizontal betatronic oscillations of a ring composed of L magnetic elements each long l (see fig. 1.2) with a sextupole treated in the thin lens approxiamtion.

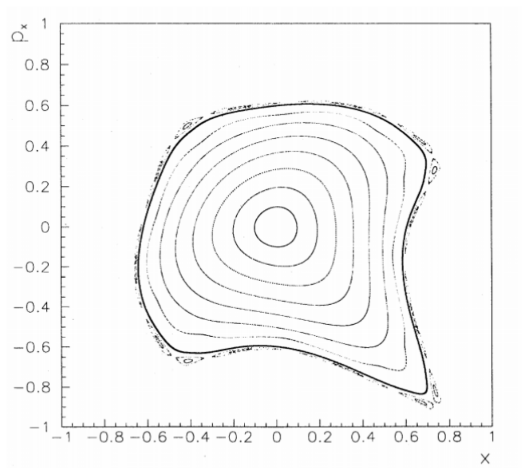


Figure 1.6: Phase portrait of the 2D Hénon map with linear tune $\nu_x = 0.28$. The last invariant torus is marked in boldface [17].

We assume (x, p_x) to be the Courant-Snyder coordinate in such a way that the action of the linear part of the map in one turn amounts to a rotation of angle $\omega = 2\pi\nu_x$, where ν_x is the linear tune. The phase space assume a well known structure reported in figure 1.6.

It is possible to appreciate that the origin is an elliptic fixed point surrounded of one dimensional Kolmogorov-Arnold-Moser tori. A result which is true only in 2D in phase space is that each of these connected invariant curves separate different domains of the phase space: if a particle has its initial conditions inside a torus, it cannot escape it. Thus there is a last torus before a sea of chaotic motion whose interior represent a set of initial conditions.
Therefore the DA can be defined in this case as the radius of the circle whose area equals the area of the set of initial conditions which lie in the interior of the last torus, i.e. the area of the stability domain.

In more realistic situations however is impossible to interpret the dynamic aperture in this way. In 4D indeed the existence of KAM tori does not imply stability of the motion because they are no longer topological barriers and Arnold's diffusion occurs.
Thus in such situations the dynamic aperture change (i.e. reduce) with the number of turns in the machine. Stability results can be obtained for large (finite) times for the typical models encountered in beam dynamics (Hénon-like mappings in any dimensions). We report here, strictly following [9], a theorem [19, 20] which is the generalizasion for symplectic maps of the Nekhoroshev theorem [21] for Hamiltonian flows.

THEOREM (Nekhoroshev). Let $\vec{F}(\vec{x})$ be a symplectic map in a $2D$ ($D \geq 2$) phase space, analytic in a polydisc of unit radius, having the origin as an elliptic fixed point. Let the frequencies vector $\vec{\omega} \in \mathbb{R}^D$ satisfy the following estimate

$$|\mathrm{e}^{i\vec{q}\vec{\omega}} - 1|^{-1} \leq \gamma|\vec{q}|^{\eta}$$

where $\vec{q}$ is an integer vector while γ and η positive constants.

Then any orbit in a polydisc of radius $\rho/2$ will remain in polydisc of radius ρ for a time $t \leq T$, where

$$T = T * \exp\Big[\Big(\frac{\rho^*}{\rho}\Big)^A\Big]$$

provided that $\rho \leq \rho^*$; the constants A,T^* and ρ^* depend on η and γ.

We can ask what happens if effects due to unpredictable perturbations are taken into account for the evaluation of DA and in general for beam stability analysis. These are the kind of effects we tackle in this thesis.
Differently from the case where only non-linearities due to ideal multipolare magnets and resonances are guilty for the reduction of the DA, when stochastic perturbations are considered, diffusion in the action occurs and the DA shrink even in two dimensional phase space.

Works to estimate time evolution of the DA in this case have been carried out, e.g. [18], but the task remain challenging.

Throughout the present work we develope some tools which could be useful. Indeed we construct a diffusion model for a stochastically perturbed Hamiltonian system ending up with a diffusion equation for the probability distribution of the action variable which could be related to the problem. Moreover as a guess for the diffusion coefficient in this equation we will also use Nekhoroshev-like estimates.

In the next chapters we will be more specific and in the last one we will make comparisons with data regarding beam losses.

Chapter 2

Markov processes and stochastic differential equations

In this chapter we will describe Planck's derivation of the Fokker-Planck equation, which describes the evolution of a probability density function, strictly following Van Kampen's treatise [22]. Moreover links between Stochastic differential equations (SDE) and the Fokker-Planck equation will be explored and the cumulant expansion will be portrayed. Throughout all the chapter only one dimension will be considered, both to keep the discussion simpler and due to the fact that a generalization to more dimensions is quite straightforward. Other pillars on the subject in literature are the books from Gardiner [23] and Risken [24].

2.1 Markov processes and the Chapman-Kolmogorov equation

A Markov process is a stochastic process which satisfy

$$P_{1|n-1}(y_n, t_n|y_1, t_1; ...; y_{n-1}, t_{n_1}) = P_{1|1}(y_n, t_n|y_{n-1}, t_{n_1}) \qquad \forall n \tag{2.1}$$

where $t_1 < t_2 < ... < t_n$. This amounts to say that the conditional probability density at t_n, given the value y_{n-1} at t_{n-1}, is completely determined and the knowledge of the values at previous times has no role. $P_{1|1}$ is the *transition probability.*

This property allows to find results which are not valid for general stochastic processes and makes Markov processes very tractable and fit for a large variety of applications. In fact, to fully characterize a stochastic process, the knowledge of the whole hierarchy of joint probability densities is needed

$$P_1(y_1, t_1), \quad P_2(y_1, t_1; y_2, t_2), \quad ..., \quad P_n(y_1, t_1; ...; y_n, t_n), \quad ...$$

Markov processes instead are fully determined by two functions. The hierarchy can be constructed entirely by knowing $P_1(y_1, t_1)$ and $P_2(y_2, t_2; y_1, t_1)$ and using the property (2.1). As an example, taking t_1, t_2, t_3, it is possible to find

$$\begin{aligned} P_3(y_1, t_1; y_2, t_2; y_3, t_3) &= P_2(y_1, t_1; y_2, t_2) P_{1|2}(y_3, t_3 | y_2, t_2; y_1, t_1) \\ &= P_1(y_1, t_1) P_{1|1}(y_2, t_2 | y_1, t_1) P_{1|1}(y_3, t_3 | y_2, t_2) \end{aligned} \tag{2.2}$$

It is possible to introduce the *Chapman-Kolmogorov equation* integrating the previous identity (2.2) over y_2 and dividing both sides by $P_1(y_1, t_1)$

$$P_{1|1}(y_3, t_3 | y_1, t_1) = \int P_{1|1}(y_3, t_3 | y_2, t_2) P_{1|1}(y_2, t_2 | y_1, t_1) dy_2 \tag{2.3}$$

where $t_1 < t_2 < t_3$, under the assumption that $P_1(y_1, t_1) \neq 0$.
Along with (2.3) there is an obvious necessary relation that can be stated

$$P_1(y_2, t_2) = \int P_{1|1}(y_2, t_2 | y_1, t_1) P_1(y_1, t_1) dy_1 \tag{2.4}$$

Every Markov process has to obey (2.3) and (2.4). Also, going the other way, any two functions P_1 and $P_{1|1}$ that obey these consistency conditions uniquely define a Markov process.

While through these equations it is often easy to check if a process one is dealing with and of which P_1 and $P_{1|1}$ are known is a Markov process, other equations that can be found starting from the Chapman-Kolmogorov are far more easy to handle when one is looking for a solution rather checking if one she already has is correct. This will be elaborated in the next sections.

Let us close this section defining stationary and homogeneous processes. Consider a closed, isolated physical system and one (or more) quantity $Y(t)$ which describes it and can be modeled as Markov process. This is for instance the case when in analysing physical phenomena one choses to look at a coarse-grained level such that there are perturbations which can be well approximated as random and uncorrelated between them with respect to the main dynamics.

When the system is in equilibrium, $Y(t)$ is stationary. In particular P_1 does not depend on time and amounts to the equilibrium distribution for Y one can calcluate through equilibrium statistical mechanics. The best known example of a stationary Markov process is the *Ornstein-Uhlenbeck* process, defined by

$$P_1(y_1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} y_1^2\right] \tag{2.5}$$

$$T_\tau(y_2 | y_1) = \frac{1}{\sqrt{2\pi(1 - e^{-2\tau})}} \exp\left[-\frac{(y_2 - y_1 e^{-\tau})^2}{2(1 - e^{-2\tau})}\right] \tag{2.6}$$

where, due to the fact that the transition probability of a stationary process only depends on one time $\tau = t_2 - t_1$, the notation

$$P_{1|1}(y_2, t_2; y_1, t_1) = T_\tau(y_2|y_1)$$

has been used which will be employed in the next section. This process is even more interesting for our goal because in the next chapters we will use it to model the stochastic perturbation to which the Hamiltonian system in analysis is subject. As a matter of fact it will be used to represent an actual (physical) noise, then the white noise (which is ideal) limit will be taken.

Homogeneous process are non-stationary Markov processes with transition probability depending on time differences alone. They usually occurs as subensembles of stationary Markov processes. As an exemple let a stationary Markov process $Y(t)$ be given by $P_1(y_1)$ and $T_\tau(y_2|y_1)$. Take a fixed time t_0 and a fixed value y_0. A non-stationary Markov process $Y^*(t)$ for $t \geq t_0$ can be defined

$$\begin{aligned} P_1^*(y_1, t_1) &= T_{t_1 - t_0}(y_1|y_0) \\ P_{1|1}^*(y_2, t_2|y_1, t_1) &= T_{t_2 - t_1}(y_2|y_1) \end{aligned}$$

Or more generally one may extract a subensemble in which at a given time t_0 the values of $Y(t_0)$ are distributed according to a given probability distribution $p(y_0)$

$$\begin{aligned} P_1^*(y_1, t_1) &= \int T_{t_1 - t_0}(y_1|y_0) p(y_0) dy_0 \\ P_{1|1}^*(y_2, t_2|y_1, t_1) &= T_{t_2 - t_1}(y_2|y_1) \end{aligned}$$

2.2 The Master equation

In this section the master equation it is derived from the Chapman-Kolmogorov equation.
It is a differential equation easier to handle than the latter, which is equivalent but is a functional equation, and susceptible to a different interpretation. Moreover it is possible to derive the Fokker-Planck equation as an approximation of the master equation in a sound way, this procedure will be carried on in the next section.

Take a Markov process, which we assume homogeneous for simplicity, thus we can write T_τ for the transition probability. Taking the limit of vanishing time difference τ' will give us the master equation, it is necessary so to establish how T_τ behaves as τ' tends to zero. it can be proved that under proper regularity assumptions the following statements holds for small τ'

$$T_{\tau'}(y_2|y_1) = (1 - a_0\tau')\delta(y_2 - y_1) + \tau' W(y_2|y_1) + \mathcal{O}(\tau'^2) \tag{2.7}$$

Here $W(y_2|y_1)$ is the *transition rate* from y_1 to y_2, thus

$$W(y_2|y_1) \geq 0 \tag{2.8}$$

The coefficient $1 - a_0\tau'$ in front of the delta function represent the chance that no transition takes place during τ', hence

$$a_0(y_1) = \int W(y_2|y_1)dy_2 \tag{2.9}$$

Which can also be drived from

$$\int T_{\tau'}(y_2|y_1)dy_2 = 1 \tag{2.10}$$

Inserting now this expression for $T_{\tau'}$ in the Chapman-Kolmogorov equation (2.3)

$$T_{\tau+\tau'}(y_3|y_1) = [1 - a_0(y_3)\tau']T_\tau(y_3|y_1) + \tau' \int W(y_3|y_2)T_\tau(y_2|y_1)$$

Dividing by τ', going to the limit $\tau' \to 0$ and using (2.9) one gets

$$\frac{\partial}{\partial \tau} T_\tau(y_3|y_1) = \int \Big[W(y_3|y_2)T_\tau(y_2|y_1) - W(y_2|y_3)T_\tau(y_3|y_1) \Big] dy_2 \tag{2.11}$$

This differential form of the Chapman-Kolmogorov equation is called the master equation.

A more intuitive and used form for the equation is

$$\frac{\partial P(y,t)}{\partial t} = \int \Big[W(y|y')P(y',t) - W(y'|y)P(y,t) \Big] dy' \tag{2.12}$$

This equation must be interpreted in the following way. Take a time t_1 and a value y_1 and consider the solution of (2.11) that is determined for $t > t_1$ by the initial condition $P(y, t_1) = \delta(y - y_1)$. This solution is the transition probability $T_{t-t_1}(y|y_1)$ of the Markov process - for any choice of t_1 and y_1. The master equation is *not* meant as an equation for the single time distribution $P_1(y, t)$.

An meaningful physical interpretation can be given to (2.12). $W(y|y')\Delta t$ represents the probabilities for the transition during a short time Δt, therefore for a particular system under analysis can be estimated by means of any approximation method suitable for short times. Using master equation then one determines the evolution of the system over long time periods. In this way the two scales can be treated separately, at the expense of *assuming* the Markov property.

2.3 The Fokker-Planck equation

The Fokker-Planck equation can be seen as a special type of master equation, used as an approximation to the actual equation, or as a model for more general Markov processes. Its elegant mathematical properties should not obscure the fact that its application in physical situations requires physical justification, which is not always obvious, in particular not in nonlinear systems, which happens to be the case in this work. In the next section we will see how the Fokker-Planck equation can be derived from a stochastic differential equation.

Planck derivation of the Fokker-Planck equation consists of an approximation to the master equation. One can start expressing the transition probability W as a function of the size r of the jump and of the starting point

$$W(y|y') = W(y';r), \quad r = y - y' \tag{2.13}$$

hence the master equation (2.11) reads

$$\frac{\partial P(y,t)}{\partial t} = \int W(y-r;r)P(y-r,t)dr - P(y,t)\int W(y;-r)dr \tag{2.14}$$

The *basic assumption* is that only small jumps are allowed, $W(y';r)$ is thus a sharply peaked function of r but varies very slowly with y'. Otherwise stated, $\exists\delta > 0$ such that

$$W(y';r) \approx 0 \qquad \text{for} \quad |r| > \delta \tag{2.15}$$

$$W(y'+\Delta y;r) \approx W(y';r) \qquad \text{for} \quad |\Delta y| < \delta \tag{2.16}$$

The *second assumption* is that the solution $P(y,t)$ varies slowly with y as well, in the same sense as (2.16). Thus one can deal with the shift from y to $y-r$ in the first integral in (2.14) through a Taylor expansion up to the second order

$$\frac{\partial P(y,t)}{\partial t} = \int W(y;r)P(y,t)dr - \int r\frac{\partial}{\partial y}\{W(y;r)P(y,t)\}dr + \frac{1}{2}\int r^2\frac{\partial^2}{\partial y^2}\{W(y;r)P(y,t)\}dr - P(y,t)\int W(y;-r)dr$$

The dependence of $W(y;r)$ on r is remain; an expansion with respect to this argument is not allowed as W varies rapidly with r. The first and the fourth terms cancel. One can write the other terms using the jump moments defined as

$$a_\nu(y) = \int_{-\infty}^{\infty} r^\nu W(y;r)dr \qquad (\nu = 0,1,2,...) \tag{2.17}$$

The result is the *Fokker-Planck equation*

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y}\{a_1(y)P(y,t)\} + \frac{1}{2}\frac{\partial^2}{\partial y^2}\{a_2(y)P(y,t)\} \tag{2.18}$$

Which, written in the usual notation where $\mu(y) \equiv a_1(y)$ and $D(y) \equiv a_2(y)/2$ stand respectively for the *drift* and *diffusion* coefficient, reads

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y}\{\mu(y)P(y,t)\} + \frac{\partial^2}{\partial y^2}\{D(y)P(y,t)\} \tag{2.19}$$

Including all terms of the Taylor expansion of (2.14) one obtains the *Kramers-Moyal expansion*

$$\frac{\partial P(y,t)}{\partial t} = \sum_{\nu=1} \frac{(-1)^\nu}{\nu!}\left(\frac{\partial}{\partial y}\right)^\nu \{a_\nu(y)P(y,t)\} \tag{2.20}$$

Formally (2.20) is equivalent to the master equation and it is therefore not easier to deal with, but it suggests that one may truncate after a suitable number of terms. The Fokker-Planck approximation assumes that all term after $\nu = 2$ are neglegible.

In the case of Gaussian processes though the Fokker-Plack equation is no longer an approximation, indeed being all the cumulant higher than the second null the equation is exact.

One feature worth to be remarked about the Fokker-Planck equation is the fact that it does not require the knowledge of the entire kernel $W(y|y')$ but merely of the functions $A(y)$ and $B(y)$. For any actual stochastic process these can be determined with a minimum of detailed knowledge about the underlying mechanism.

2.4 Cumulant expansion for stochastic differential equations

In this section we derive, following Van Kampen's approach in [22], the cumulant expansion for a nonlinear stochastic differential equation (SDE). First results are found for a linear case and then it is shown how to extend them for nonlinear one. We will end up with a Fokker-Planck equation that in the right conditions describes the evolution of a nonlinear stochastic dynamical system. We will use this result in the next chapter being the subject of this work an Hamiltonian system under nonlinear stochastic perturbation.

We begin considering the n-dimensional (with n finite) dynamical system

$$\dot{u}^{\nu}(t) = A^{\nu\mu}(t)u_{\mu}(t) = [A_0^{\nu\mu} + \alpha A_1^{\nu\mu}(t)]u_{\mu}(t) \qquad \nu, \mu = 1, 2, ..., n \tag{2.21}$$

which written in matrix notation reads

$$\dot{u}(t) = A(t)u(t) = [A_0 + \alpha A_1(t)]u(t)$$

A_0 is a constant matrix, $A_1(t)$ is a stochastic matrix, i.e. a matrix with one or more stochastic processes as elements, and α is a constant parameter defining the magnitude of the fluctuations.

We remark that here and throughout the work we adopt for stochastic differential equations such as (2.21) the Stratonovich interpretation. It can be shown indeed that within this interpretation if the system studied is a stochastically perdurbed Hamiltonian one then the stochastic phase flow mantain a symplectic character.

We will assume that the process $A_1(t)$ has a finte correlation time τ_c, which amounts to require that for t_1 and t_2 with $|t_1 - t_2| > \tau_c$, $\Big\langle A(t_1)A(t_2)\Big\rangle = 0$, i.e. they are uncorrelated.

We recall that an average for a generic function $f(\Xi, t)$ of a given stochastic process $\Xi(t)$ can be defined as

$$\Big\langle f(\Xi, t)\Big\rangle \equiv \int_{\xi\in\Omega} f(\xi, t)d\mu(\xi)$$

where Ω represent the functional space of all the possible realizations $\xi(t)$ of $\Xi(t)$ and accordingly $d\mu(\xi)$ is the probability mesure conforming to the path integral approach. Sometimes one can use directly a realization $\xi(t)$ to indicate the stochastic process.

Hence in the case of a matrix this is done for every element.

What we do now is to find an approximated formal solution to (2.21) in the form of a series expansion in powers of $\alpha\tau_c$. This last quantity is called the Kubo number, first defined in [25], and we demand $\alpha\tau_c \ll 1$.

First we pass to the interaction picture defining

$$v(t) \equiv \mathrm{e}^{-tA_0} u(t) \tag{2.22}$$

hence eq. (2.21) reads

$$\begin{aligned} \frac{d(\mathrm{e}^{tA_0} u)}{dt} &= [A_0 + \alpha A_1(t)]\mathrm{e}^{tA_0} u \\ \Longrightarrow \; \dot{v}(t) &= \alpha \mathrm{e}^{-tA_0} A_1(t) \mathrm{e}^{tA_0} v(t) \equiv \alpha V(t) v(t) \end{aligned}$$

Thus the formal solution for the last equation is

$$v(t) = \mathcal{T}\Big(\exp\Big\{\alpha \int_0^t V(t')dt'\Big\}\Big)a \tag{2.23}$$

where $\mathcal{T}$ stands for the time ordering operator and a is an integration constant which represent the initial condition.
We average now for fixed a

$$\Big\langle v(t) \Big\rangle = \Big\langle \mathcal{T}\Big(\exp\Big\{\alpha \int_0^t V(t')dt'\Big\}\Big) \Big\rangle a$$

by explicit calculation it is straightforward to see that the time ordering product commutes with the averaging process, hence

$$\Big\langle v(t) \Big\rangle = \mathcal{T}\Big(\Big\langle \exp\Big\{\alpha \int_0^t V(t')dt'\Big\} \Big\rangle\Big)a \tag{2.24}$$

the averaged exponential has the same form of the generating functional $G[k(t)]$ for the moments of V(t)

$$G[k] = \Big\langle exp\Big[i \int_{-\infty}^{\infty} k(t)V(t)dt\Big] \Big\rangle$$

By means of a formula for the cumulant generating functional which reads

$$\ln G[k] = \sum_{m=1}^{\infty} \frac{i^m}{m!} \int k(t_1)...k(t_m) \Big\langle\Big\langle V(t_1)...V(t_m) \Big\rangle\Big\rangle dt_1...dt_m$$

where the double-bracket notation denotes indeed the cumulants.
It is possible, *mutatis mutandis*, to express (2.24) as

$$\begin{aligned} \Big\langle v(t) \Big\rangle = \mathcal{T}\Big(\exp\Big\{\alpha \int_0^t dt_1 \Big\langle V(t_1) \Big\rangle + \frac{\alpha^2}{2} \int_0^t dt_1 dt_2 \Big\langle\Big\langle V(t_1)V(t_2) \Big\rangle\Big\rangle + ... \\ + \frac{\alpha^m}{m!} \int_0^t dt_1 dt_2 ... dt_m \Big\langle\Big\langle V(t_1)V(t_2)...V(t_m) \Big\rangle\Big\rangle + ... \Big\}\Big)a \end{aligned} \tag{2.25}$$

Note that it is possible to do this because, being under the time ordering operator the elements inside can be trated as if they commute. α now appears at the exponent. We remark that it is not an actual exponent because

in order to explicitate the action of the time ordering operator one has to expand it. Nonetheless we consider reliable the estimates and approximations in what follows.
Being $A_1(t)$ and hence $V(t)$ stochastic matrices with finite correlation time τ_c the cumulants in the previous expression do not vanish only in a domain of order τ_c. Thus the contribution from the m-fold integral in (2.25) come from a domain of order $t\tau_c^{m-1}$. As a consequance the m-th term in the exponent is of order

$$(\alpha t)(\alpha\tau_c)^{m-1}$$

At this point one can decide to truncate the expansion at a certain order and then the time ordering procedure should be performed. Such procedure in subtle and highly non trivial in as the order of the expansion grows. As a matter of fact it can be carried out only for the first two orders.

We remark that in the particular case where we are going to employ this approach the process is Gaussian, thus there will be no truncation at all. Indeed in orders higher then $m = 2$ cumulants of the same order appear and if the process happens to be Gaussian they are all null.

If we break it after $m = 1$

$$\Big\langle v(t)\Big\rangle = \mathcal{T}\Big(\exp\Big\{\alpha\int_0^t \Big\langle V(t_1)\Big\rangle dt_1\Big\}\Big)a$$

In this approximation the exponent is of order αt and the last expression is nothing but the solution to the ordinary differential equaiton

$$\frac{\partial}{\partial t}\Big\langle v(t)\Big\rangle = \alpha\Big\langle V(t)\Big\rangle\Big\langle v(t)\Big\rangle$$

At this level of approximation the stochasticity is simply averaged and the description of the averaged dynamics amounts to a system subject to a force consisting of a linear dumping term with a time-dependent coefficient. Back to the original picture, by means of (2.22), the equation reads

$$\frac{\partial}{\partial t}\Big\langle u(t)\Big\rangle = [A_0 + \alpha\Big\langle A_1(t)\Big\rangle]\Big\langle u(t)\Big\rangle \qquad (2.26)$$

Now we go further in the expansion and perform the truncation after $m = 2$ admitting an error of order $(\alpha t)(\alpha\tau_c)^2$

$$\begin{aligned} < v(t) > &= \mathcal{T}\Big(\exp\Big\{\alpha\int_0^t dt_1\Big\langle V(t_1)\Big\rangle + \frac{\alpha^2}{2}\int_0^t dt_1 dt_2\Big\langle\Big\langle V(t_1)V(t_2)\Big\rangle\Big\rangle\Big\}\Big)a \\ &= \mathcal{T}\Big(\exp\Big\{\alpha\int_0^t dt_1\Big\langle V(t_1)\Big\rangle + \alpha^2\int_0^t dt_1\int_0^{t_1} dt_2\Big\langle\Big\langle V(t_1)V(t_2)\Big\rangle\Big\rangle\Big\}\Big)a \end{aligned}$$

We define then

$$K(t_1) \equiv \int_0^{t_1}\Big\langle\Big\langle V(t_1)V(t_2)\Big\rangle\Big\rangle$$

Note that combining the cumulant in this one symbol only dependent from t_1 is not strictly correct, because when one considers the action of the time ordering operator it makes a difference to consider it in this way. Nonetheless it is possible to show that after the procedure is finished the error committed in doing so is higher in order $(\alpha\tau_c)$ than the cut off of the expansion, thus the result we are going to get is acceptable.
Hence the equation reads

$$\Big\langle v(t)\Big\rangle = \mathcal{T}\Big(\exp\Big\{\int_0^t dt'\Big[\alpha\Big\langle V(t')\Big\rangle + \alpha^2 K(t')\Big]\Big\}\Big)$$

which represent the solution of the ordinary differential equation

$$\frac{\partial}{\partial t}\Big\langle v(t)\Big\rangle = [\alpha\Big\langle v(t)\Big\rangle + \alpha^2 K(t)]\Big\langle v(t)\Big\rangle$$

which is valid, we recall, if we throw away terms higher than $(\alpha t)(\alpha\tau_c)^2$.
Back to the original picture we have

$$\frac{\partial}{\partial t}\Big\langle u(t)\Big\rangle = \Big[A_0 + \alpha\Big\langle A_1(t)\Big\rangle + \\ + \alpha^2\int_0^t d\tau\Big\langle\Big\langle A_1(t)\mathrm{e}^{\tau A_0}A_1(t-\tau)\mathrm{e}^{-\tau A_0}\Big\rangle\Big\rangle\Big]\Big\langle u(t)\Big\rangle \qquad (2.27)$$

which is an approximation for the averaged dynamics richer in informations than (2.26). We remark that the superior limit of the integral in the last expression can be sent to ∞ formally as soon as $t > \tau_c$. Hence we can write

$$\frac{\partial}{\partial t}\Big\langle u(t)\Big\rangle = \Big[A_0 + \alpha\Big\langle A_1(t)\Big\rangle + \\ + \alpha^2\int_0^\infty d\tau\Big\langle\Big\langle A_1(t)\mathrm{e}^{\tau A_0}A_1(t-\tau)\mathrm{e}^{-\tau A_0}\Big\rangle\Big\rangle\Big]\Big\langle u(t)\Big\rangle \qquad (2.28)$$

Now we consider a nonlinear stochastic differential equation. As for ordinary nonlinear differential equations one can pass to the associated Liouville equation to deal with a linear equation. Let $\Xi(t)$ be a continuous stochastic process, we introduce the nonlinear SDE

$$\dot{u}_\nu(t) = F_\nu(u, t; \Xi(t)) \qquad \nu = 1, 2, ..., n \qquad (2.29)$$

We get the hydrodinamic continuity relation

$$\frac{\partial\rho(u,t)}{\partial t} = -\frac{\partial}{\partial u_\nu}\{F_\nu(u,t;\Xi(t))\rho(u,t)\} \qquad (2.30)$$

where Einstein notation holds. We remark that usually one speak of Liouville equation when there is incompressibility, and such will be our case in dealing with Hamiltonian system, so the equation can be directly written as

$$\frac{\partial\rho(u,t)}{\partial t} = -F_\nu(u,t;\Xi(t))\frac{\partial\rho(u,t)}{\partial u_\nu}$$

but we keep the discussion general.
It is possible to see this case as an infinite dimensional analog of the linear SDE studied before. If one perform the following association with the quantities present in (2.21)

$$\rho \longleftrightarrow u$$
$$u \longleftrightarrow \nu$$
$$\frac{\partial}{\partial u_\nu} F_\nu \longleftrightarrow A$$

then she can exploit the already analyzed method to obtain an approximate equation for the average $\Big\langle \rho(u,t) \Big\rangle$ given an initial condition $\rho(u,0)$.
At this point we want to understand what this average tells us about the dynamics of the system.

The following lemma state the identification. Let $u_\nu(0) = a_\nu$ be an initial condition for which (2.29) has been solved, and let $p(u,t)$ the related probability distribution for u. Now let

$$\rho(u,0) = \delta(u-a) = \prod_\nu \delta(u_\nu - a_\nu)$$

the initial condition for a solution of (2.30). Then

$$\Big\langle \rho(u,t) \Big\rangle = p(u,t) \tag{2.31}$$

holds. To prove this consider that, for a given realization $\xi(t)$ of $\Xi(t)$ the phase flow $\Phi^t_\xi(a)$ associated to the nonlinear dynamical system (2.29) for initial condition a transform a to a certain u

$$\Phi^t_\xi(a) = a^t = u(t)$$

If all realizations $\xi(t)$ are taken into account we have

$$p(u,t) = \int_{\xi\in\Omega} \delta(u - \Phi^t_\xi(a)) d\mu(\xi) = \Big\langle \delta(u - a^t) \Big\rangle \tag{2.32}$$

For each $\xi(t)$ though, the flow density ρ in u-space satisfy

$$\rho(a^t, t) da^t = \rho(a,0) da$$

hence

$$\rho(u,t) = \rho(u^{-t}, 0) \frac{du^{-t}}{du} = \delta(u^{-t} - a) \frac{du^{-t}}{du}$$

For the proprieties of the delta function

$$\delta(u^{-t} - a) \frac{du^{-t}}{du} = \delta(u - a^t)$$

Thus, using (2.32), we proved the lemma.

Taking advantage of this lemma we can study the system approximating (2.30) for small fluctuation ending up with a Fokker-Planck equation for $p(u,t)$.
So we assume

$$F_\nu(u,t;\xi(t)) = F_\nu^{(0)}(u) + \alpha F_\nu^{(1)}(u;t) \tag{2.33}$$

There is no stochasticity in the first term and without loss of generality we assume $\left\langle F_\nu^{(1)}(u;t)\right\rangle = 0\ \forall u, t$. For our purpose, see the next chapter, we consider $F_\nu^{(0)}(u)$ and $F_\nu^{(1)}(u;t)$ not dependent explicitely on time but only through, respectively, $[u(t)]$ and $[u(t),\, \xi(t)]$. Furthermore we assume $\xi(t)$ to be a stationary process, as defined in the previous sections.
Using (2.33), eq. (2.30) reads

$$\dot{\rho}(u,t) = \{A_0(u) + \alpha A_1(u;t)\}\rho(u,t) \tag{2.34}$$

where

$$A_0\rho \equiv -\frac{\partial}{\partial u_\nu} F_\nu^{(0)}(u)\rho \qquad\qquad A_1\rho \equiv -\frac{\partial}{\partial u_\nu} F_\nu^{(1)}(u;t)\rho$$

Being this equation in the same form of (2.21) we may procede with the expansion in $(\alpha\tau_c)$, which gives now, breaking after the second order

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle \rho(u,t)\right\rangle = &-\frac{\partial}{\partial u_\nu} F_\nu^{(0)}(u)\left\langle \rho(u,t)\right\rangle + \\ &+ \alpha^2 \frac{\partial}{\partial u_\nu}\int_0^\infty d\tau \left\langle F_\nu^{(1)}(u;\tau)\mathrm{e}^{\tau A_0}\frac{\partial}{\partial u_\mu} F_\mu^{(1)}(u;t-\tau)\right\rangle \mathrm{e}^{-\tau A_0}\left\langle \rho(u,t)\right\rangle\end{aligned} \tag{2.35}$$

It is now worthwhile to express e^{tA_0} explicitly. By definition the action of e^{tA_0} on a function $f(u)$ is the solution of the unperturbed part of Liouville equation (2.34). Thus $\mathrm{e}^{tA_0} f(u) \equiv \psi(u,t)$, where $\psi(u,t)$ is the solution of

$$\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial u_\nu} F_\nu^{(0)}(u)\psi(u,t) \qquad \psi(u,0) = f(u)$$

which can be written as

$$\psi(u,t) = \psi(u^{-t},0)\frac{u^{-t}}{du}$$

where the deterministic unperturbed phase flow $\Phi^t(u)$ which gives us u^t is to be found from

$$\dot{u}^t = F_\nu^{(0)}(u^t) \qquad\qquad u^0 = u$$

Hence

$$\mathrm{e}^{tA_0} = f(u^{-t})\frac{du^{-t}}{du}$$

and using the lemma proved before we can substitute in (2.35)

$$\mathrm{e}^{-\tau A_0}\Big\langle \rho(u,t)\Big\rangle = p(u^\tau,t)\frac{du^\tau}{du}$$

In such a way that one can write

$$\mathrm{e}^{\tau A_0}\frac{\partial}{\partial u_\mu}F_\mu^{(1)}(u;t-\tau)p(u^\tau,t)\frac{du^\tau}{du} = \frac{du^{-\tau}}{du}\frac{\partial}{\partial u_\mu^{-\tau}}F_\mu^{(1)}(u^{-\tau};t-\tau)p(u,t)\frac{du}{du^{-\tau}}$$

Finally the second order approximation for our equation reads

$$\begin{aligned}\dot{p}(u,t) &= -\frac{\partial}{\partial u_\nu}F_\nu^{(0)}(u)p(u,t)+ \\ &+\alpha^2\frac{\partial}{\partial u_\nu}\int_0^\infty d\tau\Big\langle F_\nu^{(1)}(u;t)\frac{du^{-\tau}}{du}\frac{\partial}{\partial u_\mu^{-\tau}}F_\mu^{(1)}(u^{-\tau};t-\tau)\Big\rangle\frac{du}{du^{-\tau}}p(u,t)\end{aligned} \tag{2.36}$$

which is a second order differential equation for p, first order in time, and thus formally a Fokker-Planck equation. We will use this result to tackle our stochastically perturbed Hamiltonian system in the next chapter.

Chapter 3

Fokker-Planck equation for Hamiltonian systems under weak noise

In this chapter one dimensional Hamiltonian systems subject to small stochastic perturbations with the characteristics we are interested in will be introduced. Then, by means of the cumulant expansion previously described, a Fokker-Planck equaiton associated to the system will be derived. In the last section relying on rigorous theorem in stochastic processes an approximation to this equation, consisting in an averaging on the angle variable, will be deduced. The averaged equation will be the starting point to the analysis carried out in the rest of this work.

3.1 Hamiltonian systems under stochastic perturbations

We want to investigate diffusive behaviour in perurbed Hamiltonian systems. It is worth mentioning that transport in almost integrable Hamiltonian systems can occur even in absence of perturbations, i.e. Arnold's diffusion, (we refer to [26] and [27] for insights) but its relevance is limited by the critical dependence on initial conditions and it happens only in multidimensional systems. In real experiment such as the ones we are interested in (see chapter 5) it is reasonable to assume the presence of a stochastic noise that perturbs the nonlinear Hamiltonian dynamics.

Thus we analyze the effects of small stochastic noise on symplectic maps related to Hamiltonian systems. Such models have been investigated, e.g. in [28] and [29], and are susceptible to be described by means a Fokker-Planck equation. Through this chapter we will derive this description to end up with an equation which will serve as a starting point for the purpose of this work.

We start from the stochastically perturbed one dimensional Hamiltonian system in action(I)-angle(θ) variables described by

$$H(\theta, I; t) = H_0(I) + \epsilon\xi(t)H_1(\theta, I) \tag{3.1}$$

$\xi(t)$ is a continuous stochastic process and ϵ gives the magnitude scale of the perturbation and plays here the role of what in the previous chapter we called α, it will turn to be related to the diffusion time scale when we will pass to the Fokker-Planck equation.

To get the connection with the beam dynamics which is the physical problem we are tackling, it is possible to think at the system as follows. The unperturbed Hamiltonian $H_0(I)$ encode the integrable part of the single particle dynamic due to the ideal dipoles, quadrupoles and multipolar magnets and represent the energy of the particle, we consider the case in which $H_0 = E$ are compact invariant surfaces. Whereas H_1 is the interaction Hamiltonian which encodes the environmental noise such as not designed multipolar effects, particle interaction with residual molecule in the vacuum chamber (beam-gas interaction), interactions between the particles in the beam, to name some.

The stochastic process $\xi(t)$ is an Ornstein-Ulhenbeck process, thus stationary and Gaussian, with correlation function

$$\Big\langle \xi(t)\xi(t+\tau) \Big\rangle = \sigma^2\phi(\gamma\tau) \qquad \phi(\gamma\tau) \approx \mathrm{e}^{-\gamma|\tau|}$$

where γ^{-1} is the characteristic time scale that defines the noise evolution with rescpect to the unperturbed dynamics provided by H_0. The choice of this process is such that we can safely say we are representing a physical noise and we can perform the formal cumulant expansion described in the previous chapter. Nonetheless, due to the various nature of the phenomena involved and their time scales in relation to that of the unperturbed motion, we will assume the wite noise limit $\gamma, \sigma \to \infty$, $\sigma^2/\gamma \to 1/2$ so that $\sigma^2\phi(\gamma\tau) \to \delta(\tau)$ and in the following we will consider

$$\Big\langle \xi(t)\xi(t+\tau) \Big\rangle = \delta(\tau)$$

This limit is singular and for Hamiltonian systems it should be performed keeping the symplectic structure of the solution, thus using a Stratonovich interpretation for the symplectic phase flow.

Without loss of generality we also assume $\Big\langle \xi(t) \Big\rangle = 0$.

The evolution follows a symplectic dynamics

$$\dot{u} = J\frac{\partial H}{\partial u} \tag{3.2}$$

where $u = (\theta, I)$ and J is the symplectic matrix, in matrices notation we have

$$\begin{pmatrix} \dot{\theta} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial \theta} \\ \frac{\partial H}{\partial I} \end{pmatrix}$$

Thus we have a nonlinear stochastic differential equation for u which we are going to use as a starting point to perform the procedure portrayed in the previous chapter

$$\dot{\theta} = \nu(I) + \epsilon \xi(t) \frac{\partial H_1(\theta, I)}{\partial I} \tag{3.3}$$

$$\dot{I} = -\epsilon \xi(t) \frac{\partial H_1(\theta, I)}{\partial \theta} \tag{3.4}$$

being

$$\nu(I) \equiv \frac{dH_0(I)}{dI}$$

the characteristic frequency for the unperturbed motion.

3.2 Fokker-Planck description through cumulant expansion

To adapt the general approach for the cumulant expansion described in the previous chapter to our case we can start writing the Liouville equation associated to the stochastic phase flow Φ^t_ξ defined by 3.2

$$\frac{\partial}{\partial t}\rho(\theta,I,t) = -\dot{I}\frac{\partial}{\partial I}\rho(\theta,I,t) - \dot{\theta}\frac{\partial}{\partial\theta}\rho(\theta,I,t)$$

hence

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= -\frac{\partial H_0}{\partial I}\frac{\partial\rho}{\partial\theta} + \epsilon\xi(t)\Big\{\frac{\partial H_1}{\partial\theta}\frac{\partial\rho}{\partial I} - \frac{\partial H_1}{\partial I}\frac{\partial\rho}{\partial\theta}\Big\} = \\ &= -\nu(I)\frac{\partial\rho}{\partial\theta} + \epsilon\xi(t)D_{H_1}\rho \end{aligned} \tag{3.5}$$

where in the last line we introduced the Lie derivative operator for the interaction Hamiltonian

$$D_{H_1} \equiv \{H_1,\cdot\} = \frac{\partial H_1}{\partial u}J\frac{\partial}{\partial u}$$

where $\{\cdot,\cdot\}$ are the usual Poisson bracket.

Now we recall the general equation we found for the cumulant expansion which in this case, being the stochastic process Gaussian, is exact

$$\begin{aligned}\dot{p}(u,t) = &-\frac{\partial}{\partial u_\nu}F^{(0)}_\nu(u)p(u,t) + \\ &+\alpha^2\frac{\partial}{\partial u_\nu}\int_0^\infty d\tau\Big\langle F^{(1)}_\nu(u;t)\frac{du^{-\tau}}{du}\frac{\partial}{\partial u^{-\tau}_\mu}F^{(1)}_\mu(u^{-\tau};t-\tau)\Big\rangle\frac{du}{du^{-\tau}}p(u,t)\end{aligned} \tag{3.6}$$

One can now, using the notation $u = (\theta, I) \equiv (u_1, u_2)$, identify the generic terms in the equation with quantities from our specific problem

$$F^{(0)}_1 = \nu(I) \qquad F^{(0)}_2 = 0 \tag{3.7}$$

$$F^{(1)}_1 = \xi(t)\frac{\partial H_1}{\partial I} \qquad F^{(1)}_2 = -\xi(t)\frac{\partial H_1}{\partial\theta} \tag{3.8}$$

The symplectic phase flow $\Phi^t(u)$ for the unperturbed motion is defined by

$$\dot{u}^t = F^{(0)}$$

hence

$$\dot{I}^t = 0 \qquad \dot{\theta}^t = \nu(I)$$

which gives us the unperturbed map

$$I^t = I \qquad \theta^t = \theta + \nu(I)t \tag{3.9}$$

Finally for scope we evaluate the Jacobian

$$\frac{d(\theta, I)}{d(\theta^{-\tau}, I^{-\tau})} = 1 \tag{3.10}$$

and the expressions for the derivatives

$$\frac{\partial}{\partial \theta^{-\tau}} = \frac{\partial \theta}{\partial \theta^{-\tau}} \frac{\partial}{\partial \theta} + \frac{\partial I}{\partial \theta^{-\tau}} \frac{\partial}{\partial I} = \frac{\partial}{\partial \theta} \tag{3.11}$$

$$\frac{\partial}{\partial I^{-\tau}} = \frac{\partial \theta}{\partial I^{-\tau}} \frac{\partial}{\partial \theta} + \frac{\partial I}{\partial I^{-\tau}} \frac{\partial}{\partial I} = -\tau \nu'(I) \frac{\partial}{\partial \theta} + \frac{\partial}{\partial I} \tag{3.12}$$

where $\nu'(I)$ stands for $d\nu(I)/dI$.

We are ready now to perform the calculation

$$\begin{aligned}
\dot{p} = & -F_1^{(0)}(u,t)\frac{\partial p}{\partial \theta} + \\
& + \epsilon^2 \frac{\partial}{\partial \theta}\Big\{ \int_0^\infty d\tau \Big\langle F_1^{(1)}(u,t) \frac{\partial}{\partial \theta}\Big(F_1^{(1)}(u^{-\tau}, t-\tau)\Big)\Big\rangle p + \\
& + \int_0^\infty d\tau \Big\langle F_1^{(1)}(u,t) F_1^{(1)}(u^{-\tau}, t-\tau)\Big\rangle \frac{\partial p}{\partial \theta} + \\
& + \int_0^\infty d\tau \Big\langle F_1^{(1)}(u,t) \frac{\partial}{\partial I}\Big(F_2^{(1)}(u^{-\tau}, t-\tau)\Big)\Big\rangle p + \\
& + \int_0^\infty d\tau \Big\langle F_1^{(1)}(u,t) F_2^{(1)}(u^{-\tau}, t-\tau)\Big\rangle \frac{\partial p}{\partial I} + \\
& - \int_0^\infty d\tau \Big\langle F_1^{(1)}(u,t) \tau \nu'(I) \frac{\partial}{\partial \theta}\Big(F_2^{(1)}(u^{-\tau}, t-\tau)\Big)\Big\rangle p + \\
& - \int_0^\infty d\tau \Big\langle F_1^{(1)}(u,t) F_2^{(1)}(u^{-\tau}, t-\tau)\Big\rangle \tau \nu'(I) \frac{\partial p}{\partial \theta}\Big\} + \\
& + \epsilon^2 \frac{\partial}{\partial I}\Big\{ \int_0^\infty d\tau \Big\langle F_2^{(1)}(u,t) \frac{\partial}{\partial \theta}\Big(F_1^{(1)}(u^{-\tau}, t-\tau)\Big)\Big\rangle p + \\
& + \int_0^\infty d\tau \Big\langle F_2^{(1)}(u,t) F_1^{(1)}(u^{-\tau}, t-\tau)\Big\rangle \frac{\partial p}{\partial \theta} + \\
& + \int_0^\infty d\tau \Big\langle F_2^{(1)}(u,t) \frac{\partial}{\partial I}\Big(F_2^{(1)}(u^{-\tau}, t-\tau)\Big)\Big\rangle p + \\
& + \int_0^\infty d\tau \Big\langle F_2^{(1)}(u,t) F_2^{(1)}(u^{-\tau}, t-\tau)\Big\rangle \frac{\partial p}{\partial I} + \\
& - \int_0^\infty d\tau \Big\langle F_2^{(1)}(u,t) \tau \nu'(I) \frac{\partial}{\partial \theta}\Big(F_2^{(1)}(u^{-\tau}, t-\tau)\Big)\Big\rangle p + \\
& - \int_0^\infty d\tau \Big\langle F_2^{(1)}(u,t) F_2^{(1)}(u^{-\tau}, t-\tau)\Big\rangle \tau \nu'(I) \frac{\partial p}{\partial \theta}\Big\}
\end{aligned}$$

hence

$$
\begin{aligned}
\dot{p} = & -\nu(I)\frac{\partial p}{\partial \theta}+\\
& + \epsilon^2 \frac{\partial}{\partial \theta}\Big\{\int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \frac{\partial H_1(\theta,I)}{\partial I}\frac{\partial^2 H_1(\theta^{-\tau},I)}{\partial \theta \partial I}p+\\
& + \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \frac{\partial H_1(\theta,I)}{\partial I}\frac{\partial H_1(\theta^{-\tau},I)}{\partial I}\frac{\partial p}{\partial \theta}+\\
& - \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \frac{\partial H_1(\theta,I)}{\partial I}\frac{\partial^2 H_1(\theta^{-\tau},I)}{\partial I \partial \theta}p+\\
& - \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \frac{\partial H_1(\theta,I)}{\partial I}\frac{\partial H_1(\theta^{-\tau},I)}{\partial \theta}\frac{\partial p}{\partial I}+\\
& + \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \tau\nu'(I)\frac{\partial H_1(\theta,I)}{\partial I}\frac{\partial^2 H_1(\theta^{-\tau},I)}{\partial \theta^2}p+\\
& + \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \tau\nu'(I)\frac{\partial H_1(\theta,I)}{\partial I}\frac{\partial H_1(\theta^{-\tau},I)}{\partial \theta}\frac{\partial p}{\partial \theta}\Big\}+\\
& - \epsilon^2 \frac{\partial}{\partial I}\Big\{\int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \frac{\partial H_1(\theta,I)}{\partial \theta}\frac{\partial^2 H_1(\theta^{-\tau},I)}{\partial \theta \partial I}p+\\
& - \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \frac{\partial H_1(\theta,I)}{\partial \theta}\frac{\partial H_1(\theta^{-\tau},I)}{\partial I}\frac{\partial p}{\partial \theta}+\\
& + \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \frac{\partial H_1(\theta,I)}{\partial \theta}\frac{\partial^2 H_1(\theta^{-\tau},I)}{\partial I \partial \theta}p+\\
& + \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \frac{\partial H_1(\theta,I)}{\partial \theta}\frac{\partial H_1(\theta^{-\tau},I)}{\partial \theta}\frac{\partial p}{\partial I}+\\
& - \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \tau\nu'(I)\frac{\partial H_1(\theta,I)}{\partial \theta}\frac{\partial^2 H_1(\theta^{-\tau},I)}{\partial \theta^2}p+\\
& - \int_0^\infty d\tau \Big\langle \xi(t)\xi(t-\tau)\Big\rangle \tau\nu'(I)\frac{\partial H_1(\theta,I)}{\partial \theta}\frac{\partial H_1(\theta^{-\tau},I)}{\partial \theta}\frac{\partial p}{\partial \theta}\Big\}
\end{aligned}
$$

Considering now that in the white noise limit

$$
\Big\langle \xi(t)\xi(t-\tau)\Big\rangle = \delta(\tau)
$$

the espression simplify to

$$
\begin{aligned}
\dot{p} = & -\nu(I)\frac{\partial p}{\partial \theta} + \frac{\epsilon^2}{2}\frac{\partial}{\partial I}H_1(\theta,I)\frac{\partial^2}{\partial \theta \partial I}\frac{\partial p}{\partial \theta}+\\
& + \frac{\epsilon^2}{2}\Big(\frac{\partial}{\partial I}H_1(\theta,I)\Big)^2\frac{\partial^2 p}{\partial \theta^2} - \frac{\epsilon^2}{2}\frac{\partial}{\partial I}H_1(\theta,I)\frac{\partial^2}{\partial \theta^2}H_1(\theta,I)\frac{\partial p}{\partial I}+\\
& - \epsilon^2\frac{\partial}{\partial I}H_1(\theta,I)\frac{\partial}{\partial \theta}H_1(\theta,I)\frac{\partial^2 p}{\partial \theta \partial I} - \frac{\epsilon^2}{2}\frac{\partial}{\partial \theta}H_1(\theta,I)\frac{\partial^2}{\partial I^2}H_1(\theta,I)\frac{\partial p}{\partial \theta}+\\
& + \frac{\epsilon^2}{2}\frac{\partial}{\partial \theta}H_1(\theta,I)\frac{\partial^2}{\partial \theta \partial I}H_1(\theta,I)\frac{\partial p}{\partial I} + \frac{\epsilon^2}{2}\Big(\frac{\partial}{\partial \theta}H_1(\theta,I)\Big)^2\frac{\partial^2 p}{\partial I^2}
\end{aligned}
$$

Now one can note that the square of the Lie derivative for the interaction Hamiltonian D_{H_1} reads

$$
\begin{aligned}
D_{H_1}^2 &= \Big(\frac{\partial H_1}{\partial \theta}\frac{\partial}{\partial I} - \frac{\partial H_1}{\partial I}\frac{\partial}{\partial \theta}\Big)\Big(\frac{\partial H_1}{\partial \theta}\frac{\partial}{\partial I} - \frac{\partial H_1}{\partial I}\frac{\partial}{\partial \theta}\Big) \\
&= \frac{\partial H_1}{\partial \theta}\frac{\partial^2 H_1}{\partial I \partial \theta}\frac{\partial}{\partial I} + \Big(\frac{\partial H_1}{\partial \theta}\Big)^2 \frac{\partial^2}{\partial I^2} + \\
&+ \frac{\partial H_1}{\partial I}\frac{\partial^2 H_1}{\partial I \partial \theta}\frac{\partial}{\partial \theta} + \Big(\frac{\partial H_1}{\partial I}\Big)^2 \frac{\partial^2}{\partial \theta^2} + \\
&- \frac{\partial H_1}{\partial \theta}\frac{\partial^2 H_1}{\partial I^2}\frac{\partial}{\partial \theta} - \frac{\partial H_1}{\partial \theta}\frac{\partial H_1}{\partial I}\frac{\partial^2}{\partial \theta \partial I} + \\
&- \frac{\partial H_1}{\partial I}\frac{\partial^2 H_1}{\partial \theta^2}\frac{\partial}{\partial I} - \frac{\partial H_1}{\partial I}\frac{\partial H_1}{\partial \theta}\frac{\partial^2}{\partial \theta \partial I}
\end{aligned}
$$

Therefore the expression for the Fokker-Planck equation we are looking for to describe the Hamiltonian system (3.1) is

$$
\dot{p}(\theta, I, t) = -\nu(I)\frac{\partial}{\partial \theta}p(\theta, I, t) + \frac{\epsilon^2}{2}D_{H_1}^2 p(\theta, I, t) \tag{3.13}
$$

This equation describes the evolution of the probability distribution function for both the angle and the action variable in time. It can be noted that the diffusion time scale results of order ϵ^{-2}. As we will argue in the next section in our situation in the limit of small amplitude for the noise it is possible to end up with a Fokker–Planck equation for the distribution function in the unperturbed action only.

3.3 Averaging procedure

In this section we perform an averaging procedure on equation (3.13) in order to derive a Fokker-Planck equation for the probability distribution of I, which is an invariant for the unperturbed dynamics, which turns out to be a good approximation of the first one.

The procedure lies on rigorous results which can be found in [30, 31, 32] and it is possible in the limit of small perturbation. In this scenario indeed, as pointed out in [29], if the small perturbation is of order ϵ it can be shown that the angle variable θ completely relaxes to a uniform distribution with a relaxation time proportional to $\epsilon^{-2/3}$ much shorter then the diffusion time scale ϵ^{-2}. Therefore the evolution of $p(\theta, I, t)$ through (3.13) is well approximated at the diffusion time scale by the same equaiton averaged, though, on the angle variable.

Let us define the average on the fast angle variable θ for a generic function f

$$\Big\langle f \Big\rangle_\theta \equiv \frac{1}{2\pi} \int_0^{2\pi} f d\theta \tag{3.14}$$

We separate the probability distribution in a mean and a fluctuating part, with respect of course to the average for the angle variable, thus

$$p(\theta, I, t) = p_0(I, t) + \epsilon p_1(\theta, I, t) \tag{3.15}$$

where

$$p_0 \equiv \Big\langle p \Big\rangle_\theta$$

and

$$\Big\langle p_1 \Big\rangle_\theta = 0$$

We stress the importance for the fluctuating part to be of order ϵ.
Now we look for the averaged equation keeping in mind that terms of order $\mathcal{O}(\epsilon^3)$ in the limit $\epsilon \to 0$ will be considered sufficiently small to be neglected. First we recall the Fokker-Planck equation (3.13) for $p = p_0 + \epsilon p_1$

$$\frac{\partial}{\partial t}(p_0 + \epsilon p_1) = -\nu(I)\frac{\partial}{\partial \theta}(p_0 + \epsilon p_1) + \frac{\epsilon^2}{2} D^2_{H_1}(p_0 + \epsilon p_1) \tag{3.16}$$

Then we average over the fast variable

$$\Big\langle \frac{\partial}{\partial t}(p_0 + \epsilon p_1) \Big\rangle_\theta = -\nu(I)\Big\langle \frac{\partial}{\partial \theta}(p_0 + \epsilon p_1) \Big\rangle_\theta + \frac{\epsilon^2}{2} \Big\langle D^2_{H_1}(p_0 + \epsilon p_1) \Big\rangle_\theta \tag{3.17}$$

which expressing explicitly the Lie derivative in terms of the Poisson bracket reads

$$\Big\langle \frac{\partial}{\partial t}(p_0 + \epsilon p_1) \Big\rangle_\theta = -\nu(I)\Big\langle \frac{\partial}{\partial \theta}(p_0 + \epsilon p_1) \Big\rangle_\theta + \frac{\epsilon^2}{2} \Big\langle \{H_1, \{H_1, (p_0 + \epsilon p_1)\}\} \Big\rangle_\theta$$

Now using the properties of the Poisson bracket and the commutativity of the average in the angle variable with the derivatives in the previous equation one gets

$$\frac{\partial}{\partial t}p_0 + \epsilon\frac{\partial}{\partial t}\Big\langle p_1\Big\rangle_\theta = -\nu(I)\frac{\partial}{\partial \theta}\Big\langle (p_0+\epsilon p_1)\Big\rangle_\theta + \frac{1}{2}\Big\langle \{H_1, \{H_1, (\epsilon^2 p_0 + \epsilon^3 p_1)\}\}\Big\rangle_\theta$$

The second term on the l.h.s. and the first term on the r.h.s. obviously vanish. Furthermore we neglect in the limit of vanishing ϵ the term in which appears ϵ^3 and in doing so we are already at the point in which the equation concerns uniquely the evolution of the distribution function in the unperturbed action

$$\frac{\partial}{\partial t}p_0 = \frac{\epsilon^2}{2}\Big\langle \{H_1, \{H_1, p_0\}\}\Big\rangle_\theta \tag{3.18}$$

It is possible to develope the Poisson bracket to end up with a more explicit form

$$\frac{\partial p_0}{\partial t} = \frac{\epsilon^2}{2}\Big[\Big\langle \frac{\partial H_1}{\partial \theta}\frac{\partial^2 H_1}{\partial \theta \partial I}\Big\rangle_\theta \frac{\partial p_0}{\partial I} + \Big\langle \Big(\frac{\partial H_1}{\partial \theta}\Big)^2\Big\rangle_\theta \frac{\partial^2 p_0}{\partial I^2} - \Big\langle \frac{\partial H_1}{\partial I}\frac{\partial^2 H_1}{\partial \theta^2}\Big\rangle_\theta \frac{\partial p_0}{\partial I}\Big]$$

thus

$$\begin{aligned}\frac{\partial p_0}{\partial t} = \frac{\epsilon^2}{2}\Big[&\frac{\partial p_0}{\partial I}\frac{\partial}{\partial I}\Big\langle \Big(\frac{\partial H_1}{\partial \theta}\Big)^2\Big\rangle_\theta - \frac{\partial p_0}{\partial I}\frac{\partial}{\partial \theta}\Big\langle \frac{\partial H_1}{\partial I}\frac{\partial H_1}{\partial \theta}\Big\rangle_\theta + \\ &+ \Big\langle \frac{\partial H_1}{\partial I}\frac{\partial^2 H_1}{\partial \theta^2}\Big\rangle_\theta \frac{\partial p_0}{\partial I} + \Big\langle \Big(\frac{\partial H_1}{\partial \theta}\Big)^2\Big\rangle_\theta \frac{\partial^2 p_0}{\partial I^2} - \Big\langle \frac{\partial H_1}{\partial I}\frac{\partial^2 H_1}{\partial \theta^2}\Big\rangle_\theta \frac{\partial p_0}{\partial I}\Big]\end{aligned}$$

The second term on the r.h.s. vanish and while the third and fifth cancel out. Therefore, rearranging the remaining terms and calling p_0 now simply p the equation finally reads

$$\frac{\partial}{\partial t}p(I,t) = \frac{1}{2}\frac{\partial}{\partial I}\Big[h^2(I)\frac{\partial}{\partial I}p(I,t)\Big] \tag{3.19}$$

where we defined

$$h(I) \equiv \epsilon\sqrt{\Big\langle \Big(\frac{\partial H_1(\theta, I)}{\partial \theta}\Big)^2\Big\rangle_\theta} \tag{3.20}$$

This is simply (3.13) averaged on the angle variable and where the limit $\epsilon \to 0$ has been considered, but it well describes the entire dynamics because of the fast relaxation of θ as pointed out at the beginning of the section.

This Fokker-Planck equation is self-adjoint thus its spectrum is real. We will analyze it in the next chapter, in particular for diffusion coefficients $h(I)$ which are relevant to our goals.

Chapter 4

Fokker-Planck equation approximation for beam losses estimate

In this chapter it will be described an approach to tackle the Fokker-Planck equation derived in the previous one. The specific goal of this effort is to evaluate the current associated to such euqation when considered in a finite domain having an absorbing barrier at one end and with diffusion coefficients of forms interesting to our enquiry on particle losses in beam dynamics.

First a formal expansion in eigenfunction for the probability distribution of the unperturbed action, with bounday conditions as just described, will be constructed casting the initial equation into an equivalent one in Smoluchowsky form. Then it will be found an explicit solution for a simple case which will serve as starting point to our approximation, also the evaluation of the current associated to the solution and the connection with losses in the probablility distribution will be carried out. Finally formulas for the losses in probability distribution will be found for more complicated diffusion coefficients after performing a suitable approximation. The accuracy of the approximation is analysed quantitatively in a dedicated section.

4.1 Smoluchowsky form and eigenfunctions expansion

In this section a formal solution to the equation obtained in the previous chapter, given specific boundary conditions, in terms of its eigenfunctions is derived. To achieve this we show that it is possible to cast the equation into a Smoluchowsky form by means of a nonlinear change of coordinates.

We start with Fokker-Planck equation (3.19)

$$\frac{\partial \rho(I,t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial I}\Big[h^2(I)\frac{\partial \rho(I,t)}{\partial I}\Big] \tag{4.1}$$

with $I \in [0, I_a]$, $I_a \in \mathbb{R}^+$. It is possible to introduce a new variable $x(I)$ such that

$$x = -\int_I^{I_a} dI' h^{-1}(I') \tag{4.2}$$

$$\frac{dI(x)}{dx} = h(I(x))$$

and

$$\rho(I,t)dI = \rho(I(x),t)h(I(x))dx = \rho'(x,t)dx \tag{4.3}$$

to preserve measure. The new Fokker-Planck equation reads

$$\begin{aligned}\frac{\partial \rho'(x,t)}{\partial t} &= \frac{h(I(x))}{2}\frac{dx}{dI}\frac{\partial}{\partial x}\Big[h(I(x))^2\frac{dx}{dI}\frac{\partial}{\partial x}\Big(\frac{\rho'(x,t)}{h(I(x))}\Big)\Big] \\ &= -\frac{1}{2}\frac{\partial}{\partial x}\Big[\frac{1}{h(I(x))}\frac{dh(I(x))}{dx}\rho'(x,t)\Big]+\frac{1}{2}\frac{\partial^2 \rho'(x,t)}{\partial x^2}\end{aligned}$$

Defining finally the effective potential

$$V(x) \equiv -\ln\left(h(I(x))\right)$$

equation (4.1) assumes the Smoluchowsky form

$$\frac{\partial \rho'}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\Big[\Big(\frac{dV(x)}{dx}\Big)\rho'\Big]+\frac{1}{2}\frac{\partial^2 \rho'}{\partial x^2} \tag{4.4}$$

where now the domain is $x \in (-\infty, 0]$ as one can appreciate from (4.2).

Now we will recollect few general facts. Consider the following one dimensional Fokker-Planck equation

$$\frac{\partial \rho(x,t)}{\partial t} = \mathcal{L}_{FP}\rho(x,t) \tag{4.5}$$

where, being $\mu(x)$ and $D(x)$ generic time independent drift and diffusion coefficients, the Fokker-Planck operator is defined as

$$\mathcal{L}_{FP} \equiv -\frac{\partial \mu(x)}{\partial x} + \frac{\partial^2 D(x)}{\partial x^2} \tag{4.6}$$

Then one can define (see [24]) a potential

$$\phi(x) \equiv \ln D(x) - \int^x \frac{\mu(x')}{D(x')}dx' \tag{4.7}$$

so that (4.6) can be cast as

$$\mathcal{L}_{FP} = \frac{\partial}{\partial x} D(x) \mathrm{e}^{-\phi(x)} \frac{\partial}{\partial x} \mathrm{e}^{\phi(x)}$$

From this operator one can construct an Hermitian one

$$\mathcal{L} \equiv \mathrm{e}^{-\phi(x)/2} \mathcal{L}_{FP} \mathrm{e}^{\phi(x)/2} \tag{4.8}$$

In fact for two distribution functions ρ_1 and ρ_2 both satisfying same boundary conditions it is straightforward to show that

$$\int_{x_{min}}^{x_{max}} \rho_1 \mathcal{L} \rho_2 dx = \int_{x_{min}}^{x_{max}} \rho_2 \mathcal{L} \rho_1 dx$$

Now, if ψ_ξ are eigenfunctions of $\mathcal{L}$ of eigenvalues λ_ξ, $\Psi_\xi = \mathrm{e}^{-\phi/2}\psi$ are eigenfunctions of $\mathcal{L}_{FP}$ associated to the same eigenvalues.

Let us start form the Smoluchowski equation which recalls (4.4)

$$\frac{\partial \rho}{\partial t} = \mathcal{L}_{FP}\rho = \frac{1}{2}\frac{\partial}{\partial x}\Big[\Big(\frac{dV(x)}{dx}\Big)\rho\Big] + D\frac{\partial^2 \rho}{\partial x^2} \tag{4.9}$$

so that the drift and diffusion coefficients are $\mu(x) \equiv -dV(x)/dx$ and $D(x) \equiv D = const.$ and we have

$$\phi(x) = \frac{V(x)}{2D}$$

where we avoided to add $\ln D$ because it can be readly canceled out anyway when one writes $\mathcal{L}_{FP}$ and $\mathcal{L}$. We want to look for the eigenfunctions ψ_ξ of the operator $\mathcal{L}$ related to (4.9) so we start setting up the equation for a new distribution $p(x,t)$

$$\frac{\partial p}{\partial t} = \mathcal{L}p = D\mathrm{e}^{V/4D}\frac{\partial}{\partial x}\Big[\mathrm{e}^{-V/2D}\frac{\partial}{\partial x}\Big(\mathrm{e}^{V/4D}p\Big)\Big]$$

which can be cast as

$$\frac{\partial p}{\partial t} = \frac{1}{4}\Big[\frac{d^2V}{dx^2} - \frac{1}{4D}\Big(\frac{dV}{dx}\Big)^2\Big]p + D\frac{\partial^2 p}{\partial x^2} \tag{4.10}$$

Thus the eigenfunctions ψ_ξ are such that they satisfy

$$\frac{d^2\psi_\xi}{dx^2} = -\frac{1}{D}[\lambda_\xi - a(x)]\psi_\xi \tag{4.11}$$

with

$$a(x) = \frac{1}{4}\Big[\frac{1}{4D}\Big(\frac{dV}{dx}\Big)^2 - \frac{d^2V}{dx^2}\Big]$$

so that, recalling $\Psi_\xi = \mathrm{e}^{-\phi/2}\psi_\xi$, the solution to (4.9) can be written as

$$\rho(x,t) = \int_0^\infty d\xi c_\xi(0)\mathrm{e}^{-\lambda_\xi t}\Psi_\xi(x) \tag{4.12}$$

We have the orthogonality

$$\delta(\xi - \xi') = \int_{x_{min}}^{x_{max}} dx \psi_\xi(x)\psi_{\xi'}(x) = \int_{x_{min}}^{x_{max}} dx \Psi_\xi(x)\Psi_{\xi'}(x)\mathrm{e}^{\phi(x)}$$

and completeness relations

$$\begin{aligned}
\delta(x - x') &= \int_0^\infty d\xi \psi_\xi(x)\psi_\xi(x') \\
&= \mathrm{e}^{\phi(x)} \int_0^\infty d\xi \Psi_\xi(x)\Psi_\xi(x') \\
&= \mathrm{e}^{\phi(x')} \int_0^\infty d\xi \Psi_\xi(x)\Psi_\xi(x') \\
&= \mathrm{e}^{\phi(x)/2+\phi(x')/2} \int_0^\infty d\xi \Psi_\xi(x)\Psi_\xi(x')
\end{aligned}$$

which allow to find

$$c_{\xi'}(0) = \int_{x_{min}}^{x_{max}} dx \mathrm{e}^{\phi(x)}\rho(x,0)\Psi_{\xi'}(x) \qquad\qquad \forall \xi' \in \mathbb{R}^+$$

If $\rho(0,t) = \delta(x - x_0)$ it follows

$$c_\xi(0) = \mathrm{e}^{\phi(x_0)}\Psi_\xi(x_0) \qquad\qquad \forall \xi \in \mathbb{R}^+$$

4.2 Linear potential and expressions for the current

We want to study now an useful case, i.e. equation (4.9) with $V(x) = -\nu x$ ($\nu \in \mathbb{R}^+$) and an absorbing boundary at $x = 0$ (thus $\rho(0,t) = 0$) and recalling that our domain is $x \in (-\infty, 0]$.

$$\frac{\partial \rho}{\partial t} = \frac{-\nu}{2}\frac{\partial \rho}{\partial x} + D\frac{\partial^2 \rho}{\partial x^2}$$

Equation (4.11) become

$$\frac{d^2\psi_\xi}{dx^2} = -\frac{1}{D}k_\xi^2\psi_\xi$$

where

$$k_\xi \equiv \left[\lambda_\xi - \frac{\nu^2}{16D}\right]^{1/2}$$

Thus

$$\psi_\xi(x) = A_\xi \sin\left(\frac{k_\xi}{\sqrt{D}}x\right)$$

The eigenvalues read

$$\lambda_\xi = \xi^2 D + \frac{\nu^2}{16D} \qquad\qquad \xi \in \mathbb{R}^+$$

and we can find the normalization constants A_ξ through

$$\begin{aligned}\delta(x-x') &= \int_0^\infty d\xi \psi_\xi(x)\psi_\xi(x') \\ &= \frac{1}{4}\int_{-\infty}^{\infty} d\xi A_\xi^2[\mathrm{e}^{i\xi(x-x')} - \mathrm{e}^{i\xi(x+x')}] \\ &= \frac{A_\xi^2\pi}{2}\Big[\delta(x-x') - \delta(x+x')\Big] \\ &= \frac{A_\xi^2\pi}{2}\delta(x-x')\end{aligned}$$

where in the third line $\delta(x+x') = 0$ because both $x, x' \in (-\infty, 0]$. Therefore

$$A_\xi = \sqrt{2/\pi} \qquad\qquad \forall\xi$$

Thus solution (4.12) using again as initial condition $\rho(x,0) = \delta(x-x_0)$ reads

$$\rho(x,t) = \frac{2}{\pi}\exp\left(-\frac{\nu(x_0-x)}{4D} - \frac{\nu^2 t}{16D}\right)\int_0^\infty d\xi \mathrm{e}^{\xi^2 Dt}\sin(\xi x_o)\sin(\xi x) \quad (4.13)$$

Using (4.13) we are able to evaluate the current at the boundary $x = 0$ which we will simply call $J(t)$

$$
\begin{aligned}
J(t) &= -D\frac{\partial \rho}{\partial x}\Big|_{x=0} \\
&= -\frac{2D}{\pi}\exp\Big(-\frac{\nu x_0}{4D}-\frac{\nu^2 t}{16D}\Big)\int_0^{\infty} d\xi \xi e^{\xi^2 Dt}\sin(\xi x_o) \\
&= -\frac{D}{i\pi}\exp\Big(-\frac{\nu x_0}{4D}-\frac{\nu^2 t}{16D}\Big)\int_0^{\infty} d\xi \xi e^{\xi^2 Dt}(e^{i\xi x_0}-e^{-i\xi x_0}) \\
&= -\frac{D}{i\pi}\exp\Big(-\frac{\nu x_0}{4D}-\frac{\nu^2 t}{16D}\Big)\int_{-\infty}^{\infty} d\xi \xi e^{\xi^2 Dt+i\xi x_0} \\
&= \frac{D}{\pi}\exp\Big(-\frac{\nu x_0}{4D}-\frac{\nu^2 t}{16D}\Big)\frac{\partial}{\partial x_0}\int_{-\infty}^{\infty} d\xi e^{\xi^2 Dt+i\xi x_0} \\
&= \frac{D}{\pi}\exp\Big(-\frac{\nu x_0}{4D}-\frac{\nu^2 t}{16D}\Big)\frac{\partial}{\partial x_0}\sqrt{\frac{\pi}{Dt}}\exp-\Big(-\frac{x_0^2}{4Dt}\Big)
\end{aligned}
$$

i.e.

$$
J(x_0, t) = \frac{-x_0}{t\sqrt{4\pi Dt}}\exp\Big(-\frac{(x_0+\frac{\nu}{2}t)^2}{4Dt}\Big) \tag{4.14}
$$

where we specified the dependens on the peak of the initial delta condition. For a generic initial distribution $\rho(x, 0)$ the current reads

$$
J(t) = \int_{x_{min}}^{x_{max}} J(x,t)\rho(x,0)dx \tag{4.15}
$$

One can asks how $h(I)$ in our initial problem (4.1) should looks like to end up with a Smoluchowsky equation with a potential $V(x)$ which is linear. We study this case being the base of the enquiry of more complicated potentials. If in our Fokker-Planck equation (4.1) we chose

$$
h(I) = \nu I \qquad\qquad \nu \in \mathbb{R}^+
$$

in the finite domain $I \in [0, I_a]$ being an absorbing barrier in I_a and we want to cast it in the form (4.9) the change of variable reads

$$
x = -\int_I^{I_a} dI' h^{-1}(I') = -\frac{1}{\nu}\ln\Big(\frac{I_a}{I}\Big) = \frac{1}{\nu}\ln\Big(\frac{I}{I_a}\Big) \tag{4.16}
$$

$$
I = I_a \exp(\nu x)
$$

$$
V(x) = -\ln(h(I(x))) = -\nu x - \ln(I_a)
$$

$$
\Longrightarrow \frac{dV(x)}{dx} = -\nu \tag{4.17}
$$

$$\rho(I,t)dI = \rho(I(x),t)\nu I(x)dx = \rho'(x,t)dx \tag{4.18}$$

and we have, according to (4.4),

$$D = \frac{1}{2}$$

If one now recast (4.14) in terms of I_0, being the initial condition $\rho(I,0) = \delta(I - I_0)$, using relation (4.16) obtains the current through the absorbing barrier I_a for the initial problem

$$J_{I_a}(I_0,t) \equiv J(x(I_0),t) = \frac{-x(I_0)}{t\sqrt{4\pi Dt}} \exp\Big(-\frac{(x(I_0) + \frac{\nu}{2}t)^2}{4Dt}\Big) \tag{4.19}$$

In the same fashion the current for a generic distribution $\rho(I,0)$ reads

$$J_{I_a}(t) = \int_{-\infty}^{0} J(x,t)\rho'(x,0)dx = \int_{0}^{I_a} J(x(I),t)\rho(I,0)dI \tag{4.20}$$

where $\rho(I,0)$ and $\rho'(x,0)$ are related through (4.3).

Let's see now how to use the current to evaluate particle losses from the domain due to diffusion causing particles flow in presence of an absorbing barrier. Starting from

$$\frac{\partial \rho(I,t)}{\partial t} = -\frac{\partial J(I,t)}{\partial I}$$

one gets

$$\frac{\partial}{\partial t}\int_0^{I_a} \rho(I,t)dI = -J_{I_a}(t) \qquad \Longrightarrow \qquad 1 - g(t) = \int_0^t J_{I_a}(t')dt'$$

where $g(t)$ stands for the area under the distribution function at time t in the domain. Thus losses as functions of time are given by the integral

$$\int_0^t J_{I_a}(t')dt' \tag{4.21}$$

In figure (4.1) we show using sample parameters through independent numerical evaluation and through (4.21) that (4.19) and (4.20) are correct.

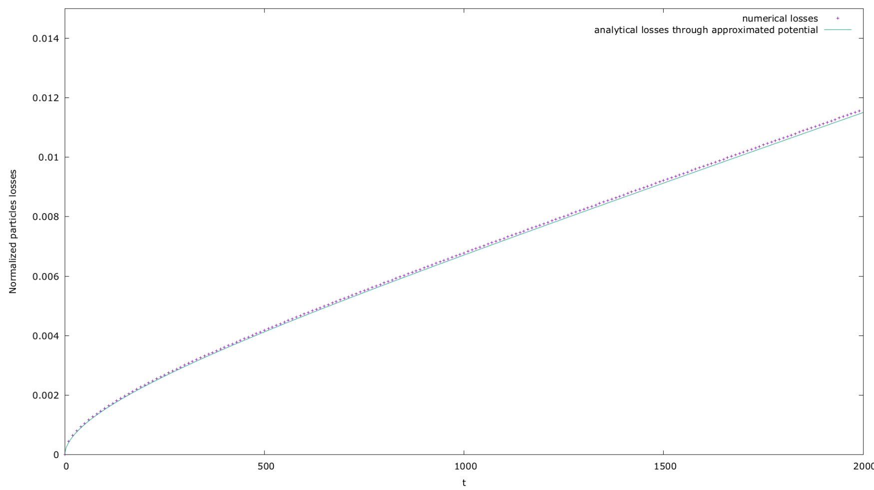


Figure 4.1: Normalized particle losses evaluated through numerical solution of (4.1) with $h(I) = \nu I$ and through formula (4.21) starting from the initial distribution $\rho(I, 0) = 1/2 \exp(-I/2)$ and using $\nu = 0.005$. On the orizontal axis there is not-rescaled time

4.3 Power law diffusion coefficient

Now we consider equation (4.1) whith

$$h(I) = \epsilon\Big(\frac{I}{I_*}\Big)^\beta \qquad\qquad \beta \in \mathbb{R}^+ \tag{4.22}$$

where $I_* > 0$ is a constant and $\epsilon > 0$ determines the time scale of the diffusion process. Again we assume $I \in [0, I_a]$ and an absorbing barrier in I_a.

First we rescale the time of our problem defining the new time $\tau = \epsilon^2 t$, so from equaiton (4.1) we end up with

$$\frac{\partial\rho(I,\tau)}{\partial\tau} = \frac{1}{2}\frac{\partial}{\partial I}\Big[\Big(\frac{I}{I_*}\Big)^{2\beta}\frac{\partial\rho(I,\tau)}{\partial I}\Big] \tag{4.23}$$

Following the previous sections, in this case the change of variables amounts to (now we will call $h = (I/I_*)^\beta$ for simplicity)

$$x = -\int_I^{I_a} dI' h^{-1}(I') = -\frac{I_a}{1-\beta}\Big(\frac{I_*}{I_a}\Big)^\beta + \frac{I_*^\beta}{1-\beta} I^{1-\beta} \tag{4.24}$$

$$V(x) = -\ln\left(h(I(x))\right) = -\beta \ln\Big(\frac{I(x)}{I_*}\Big) \tag{4.25}$$

Now we want a linear approximation for (4.25) to use current (4.14) as an approximation for the particle flow through barrier I_a associated to (4.23). Our interest is focused on scale of time and intensity of the diffusion that are related to experimental set up we are dealing with. We reproduce the experimental set up through simulations which involve equation (4.23).

We want to find in this case what should be identified with the ν coefficient (4.17). In fact the strategy is to approximate the potential to a linear one in every point of the domain expanding around that point, for every point evaluate the current for an initial peak centered there using ν and then use (4.20) to evaluate the current for a generic initial distribution $\rho(I,0)$. Taylor expansion in around $x = x_0$, $\forall x_0$, of the potential up to the first order leads to

$$\begin{aligned} V(x) &= -\beta \ln\Big(\frac{I(x_0)}{I_*}\Big) - \beta\Big[\frac{1}{I(x)}\frac{dI(x)}{dx}\Big]\Big|_{x=x_0} x + \mathcal{O}(x^2) \\ &\approx -\beta \ln\Big(\frac{I(x_0)}{I_*}\Big) - \Big[\frac{\beta}{I_*^{\beta}} I^{\beta-1}(x_0)\Big] x \\ &= -\beta \ln\Big(\frac{\Big(\frac{x_0(1-\beta)}{I_*^{\beta}} + I_a^{1-\beta}\Big)^{\frac{1}{1-\beta}}}{I_*}\Big) - \Big[\frac{\beta}{I_*^{\beta}}\Big(\frac{x_0(1-\beta)}{I_*^{\beta}} + I_a^{1-\beta}\Big)^{-1}\Big] x \end{aligned}$$

where the last line follow from

$$I(x) = \Big(\frac{x(1-\beta)}{I_*^{\beta}} + I_a^{1-\beta}\Big)^{\frac{1}{1-\beta}}$$

Thus we can identify $\nu = \nu(I_0)$

$$\nu(I_0) = \frac{\beta}{I_*^{\beta}} I^{\beta-1}(x_0) = \frac{\beta}{I_*^{\beta}} I_0^{\beta-1} \tag{4.26}$$

and treat analytically this scenario as an approximation using (4.14) for the current where ν is taken accordingly with (4.26), $D = 1/2$, $x(I_0)$ needs to be evaluated through (4.24) and $\rho(I,0)$ is the initial condition.

In figure (4.2) are portrayed results of a test of this approximation, for parameters and scale interesting for our purpose as we will see in the next chapter. The losses refers to an initial distribution $\rho(I,0) = \frac{1}{2}\exp -(I/2)$ which can be well representative of our experimental scenario. The agreement is quite good and it is possible to notice that after a while the losses evaluated through the analytical current underestimate the numerical prediction. This is expected, in fact linearizing the potential in each point amounts to using a slightly less intense drift causing an underestimate of particles flow. Later in this chapter is reported a quantitative and more insightfull analysis of the error commited and we will say more about it.

Further tests of this approximation of a power law potential with other parameters and confront with actual experimental data will be carried out in the next chapter.

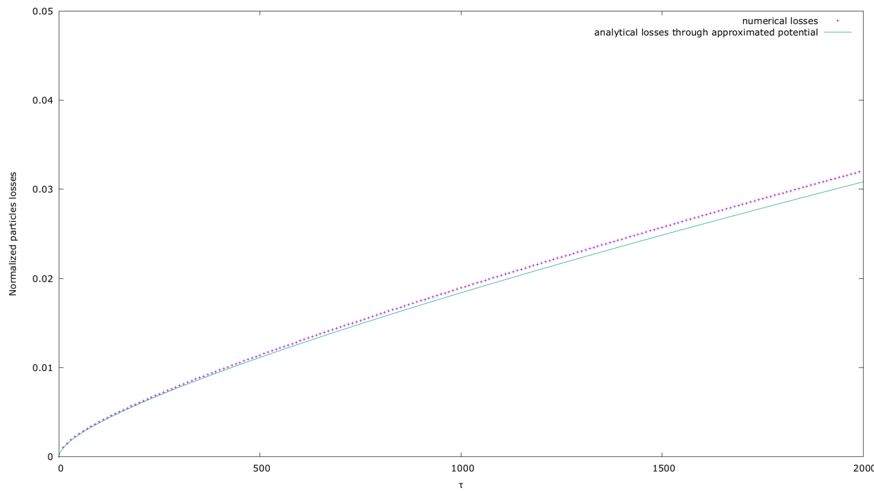


Figure 4.2: On the vertical axis we have the normalized particle losses and on the orizontal axis the rescaled time $\tau = \epsilon^2 t$. Parameters are set as $\beta = 2$, $I_* = 30$ and $I_a = 10$. The inital distribution reads $\rho(I, 0) = \frac{1}{2} \exp -(I/2)$. Losses are evaluated numerically starting form (4.23) and analytically using (4.21) with (4.20) in which the current is evaluated following the steps outlined this section.

4.4 Nekhoroshev-like diffusion coefficient

We will reproduce the same picture presented in the previous section, now with a Nekhoroshev-like diffusion coefficient

$$h(I) = \epsilon \exp\left[-\left(\frac{I_*}{I}\right)^{\alpha}\right] \tag{4.27}$$

where ϵ is again representative of the diffusion time scale, I_* a constant and α is usually related to the dimensionality of the system. Rescaling time ($\tau = \epsilon^2 t$) as before in equation (4.1), leads to

$$\frac{\partial \rho(I,\tau)}{\partial \tau} = \frac{1}{2} \frac{\partial}{\partial I}\left[\exp\left[-\left(\frac{I_*}{I}\right)^{\alpha}\right] \frac{\partial \rho(I,\tau)}{\partial I}\right] \tag{4.28}$$

For simplicity we will now call $h(I) = \exp\left[-\left(\frac{I_*}{I}\right)^{\alpha}\right]$. Changing variable to end up with the equation in the Smoluchowsky form now brigs us

$$V(x) = -\ln h(I(x)) = \left(\frac{I_*}{I(x)}\right)^{\alpha} \tag{4.29}$$

$$x = -\int_I^{I_a} dI' h^{-1}(I') = -\int_I^{I_a} dI' \exp\left(\frac{I_*}{I'}\right)^{\alpha} \tag{4.30}$$

Expanding potential (4.29) around $x = x_0$

$$V(x) = \left(\frac{I_*}{I(x_0)}\right)^{\alpha} - \alpha\left[\left(\frac{I_*}{I(x)}\right)^{\alpha}\frac{1}{I(x)}\frac{dI(x)}{dx}\right]\Big|_{x=x_0} x + \mathcal{O}(x^2)$$
$$\approx \left(\frac{I_*}{I(x_0)}\right)^{\alpha} - \frac{\alpha}{I(x_0)}\left(\frac{I_*}{I(x_0)}\right)^{\alpha}\exp\left[-\left(\frac{I_*}{I(x_0)}\right)^{\alpha}\right]x$$

$$\implies \frac{dV(x)}{dx} = -\frac{\alpha}{I(x_0)}\left(\frac{I_*}{I(x_0)}\right)^{\alpha}\exp\left[-\left(\frac{I_*}{I(x_0)}\right)^{\alpha}\right]$$

thus we can identify $\nu = \nu(I_0)$ in current (4.19) with

$$\nu(I_0) = \frac{\alpha}{I(x_0)}\left(\frac{I_*}{I(x_0)}\right)^{\alpha}\exp\left[-\left(\frac{I_*}{I(x_0)}\right)^{\alpha}\right]$$
$$= \frac{\alpha}{I_0}\left(\frac{I_*}{I_0}\right)^{\alpha}\exp\left[-\left(\frac{I_*}{I_0}\right)^{\alpha}\right]$$

and evaluating x in terms of I through (4.30) we can use (4.19) and (4.15) for an initial distribution $\rho(I, 0)$ to approximate particle flow through the barrier in I_a for the problem described by eq. (4.28).

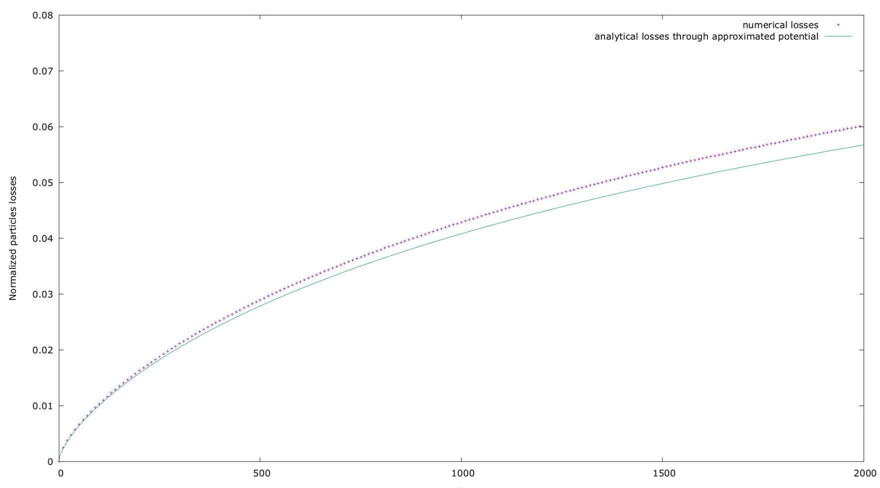


Figure 4.3: On the vertical axis we have the normalized particle losses and on the orizontal axis the rescaled time $\tau = \epsilon^2 t$. Parameters are set as $\alpha = 3/2$, $I_* = 12.6$ and $I_a = 10$. The inital distribution reads $\rho(I, 0) = \frac{1}{2}\exp -(I/2)$. Losses are evaluated numerically starting form (4.28) and analytically using (4.21) with (4.20) in which the current is evaluated following the steps outlined this section.

In figure (4.3) is reported an estimate of losses both numerically evaluated and through our approximation approach. Same considerations as in the previous section on power law potentials hold here.

4.5 Errors estimate

In this section we will be more quantitative about the agreement between the quantity we calculated with our approximation approach and the respective quantity obtained numerically using the actual diffusion coefficients. It is crucial to have a good correspondence between the two because if it happens to be the case one could use the analytic approximated forumlas in the attempt to say something about wheter or not the physics underlying the phenomena studied is related to the diffusion coefficients. In this work we lay the basis and gives some preliminar results, but for more extended analysis which could be carried out in the future the possibility to relay on analytical formulas (that can maybe be extended in more dimensions) could represent a huge advantage in terms of time and cost with respect to numerical solutions of PDEs.

Let us now procede with the error estimates. We consider the losses in particles obtained with numerical simulations and with the aim of the approximation portrayed in this chapter both for the Nekhoroshev-like and power law case. The parameters are set in such a way that the analysis is valid in regimes relevant to the experiment we are confronting our theoretical speculations with. In the next chapter we are going to justify this choices and be more clear about the dimensions involved, but for now we set the initial condition as

$$\rho(I,0) = \exp\{-I\}$$

and the absorbing barrier at

$$I_a = 6.5$$

morover for all the analysis in this section

$$\alpha = 3/2$$

and we will consider a power law with

$$\beta = 4$$

We thus explore a range, interesting to our purpose, of possible values for I_* both for the power law and the Nekhoroshev coefficient. Then we procede evaluating the normalized difference in particles losses between the numerical and the analytic approach, i.e.

$$[(\text{numeircal} \quad \text{losses}) - (\text{analytical} \quad \text{losses})]/(\text{numerical} \quad \text{losses})$$

In figures form 4.4 to 4.15 are reported some cases for both the coefficients.

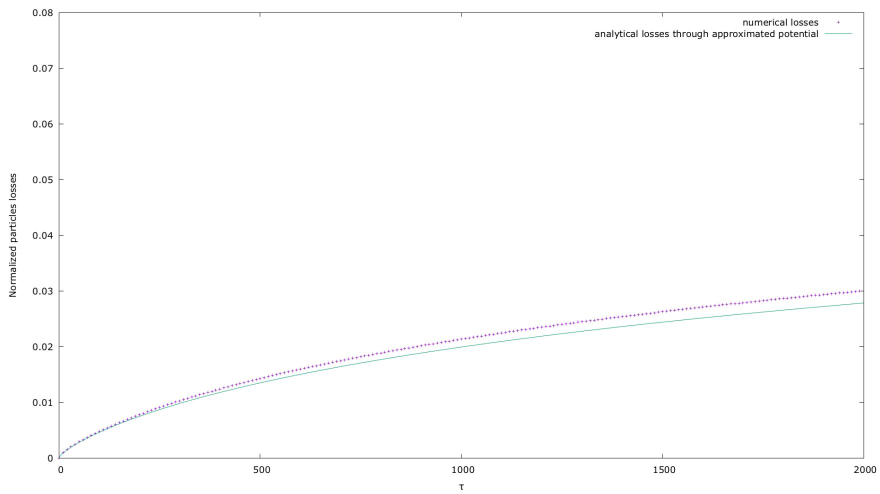


Figure 4.4: Normailzed particle losses for a Nekhoroshev diffusion coefficient with $I_* = 7$.

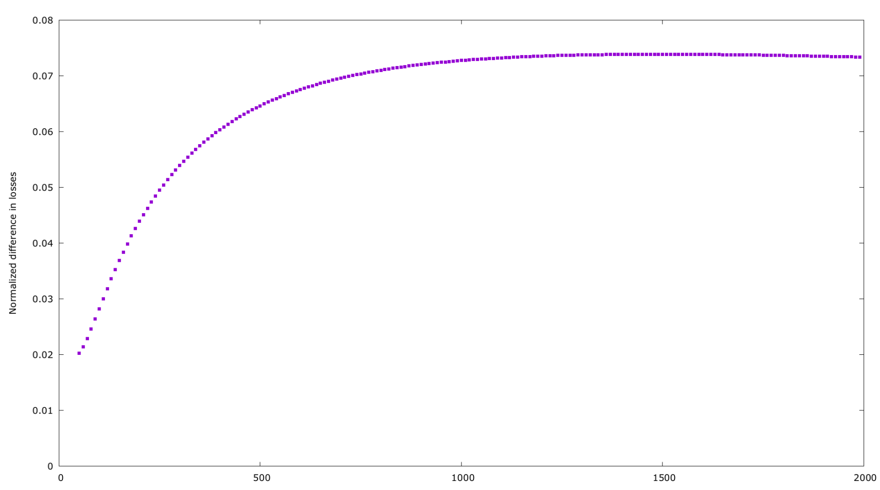


Figure 4.5: Normalized difference for the losses in figure 4.4

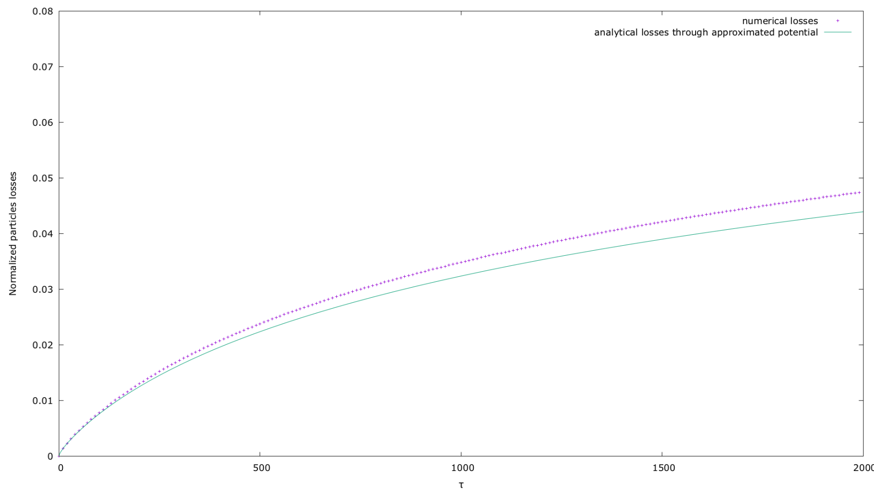


Figure 4.6: Normailzed particle losses for a Nekhoroshev diffusion coefficient with $I_* = 8$.

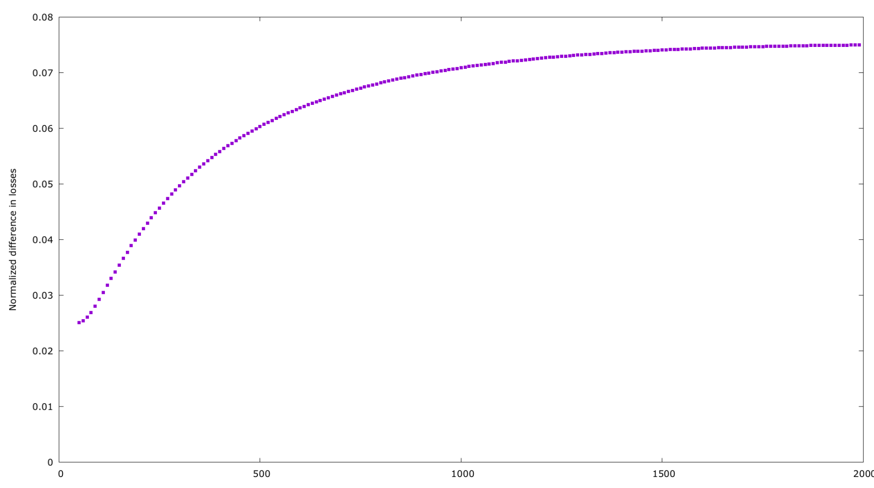


Figure 4.7: Normalized difference for the losses in figure 4.6

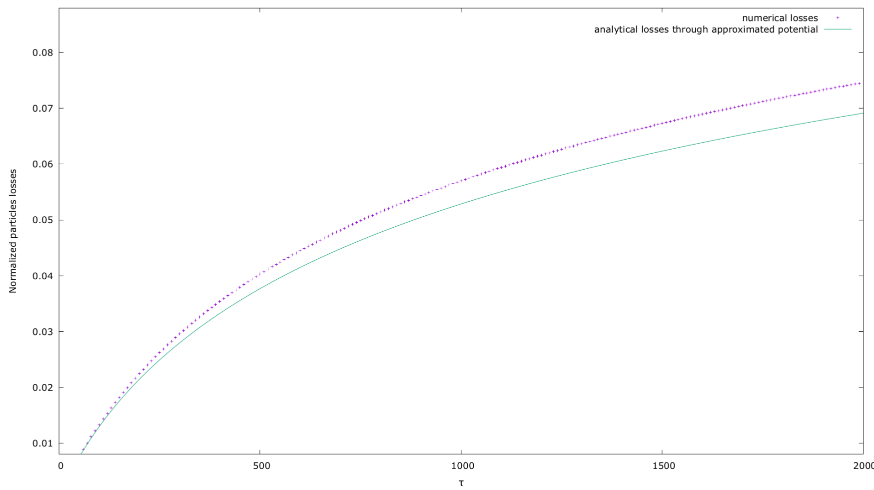


Figure 4.8: Normailzed particle losses for a Nekhoroshev diffusion coefficient with $I_* = 9$.

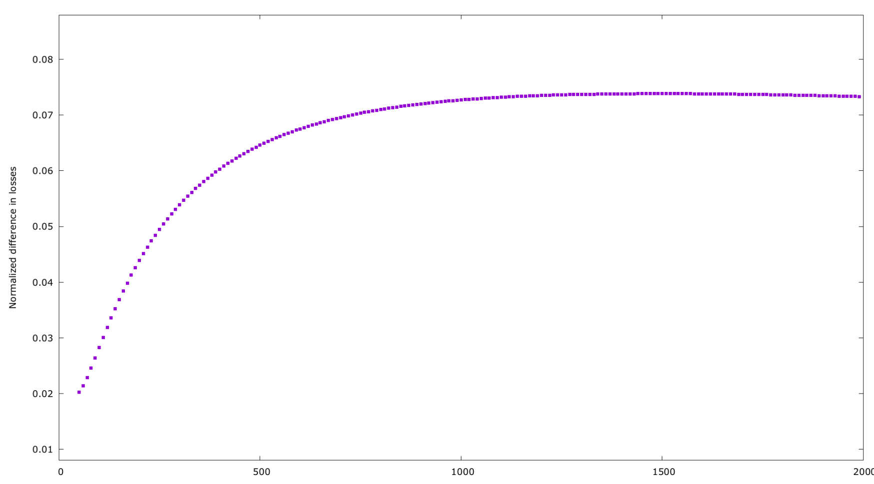


Figure 4.9: Normalized difference for the losses in figure 4.8

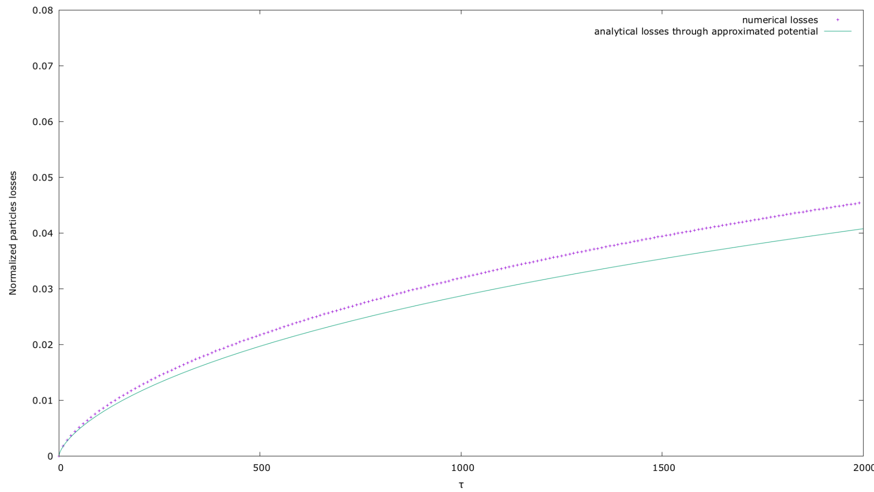


Figure 4.10: Normailzed particle losses for a Power law diffusion coefficient with $I_* = 7.5$.

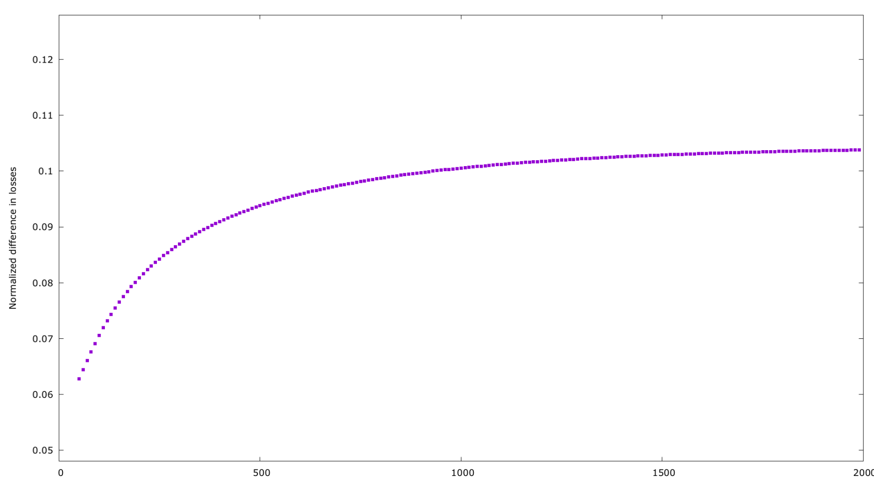


Figure 4.11: Normalized difference for the losses in figure 4.10

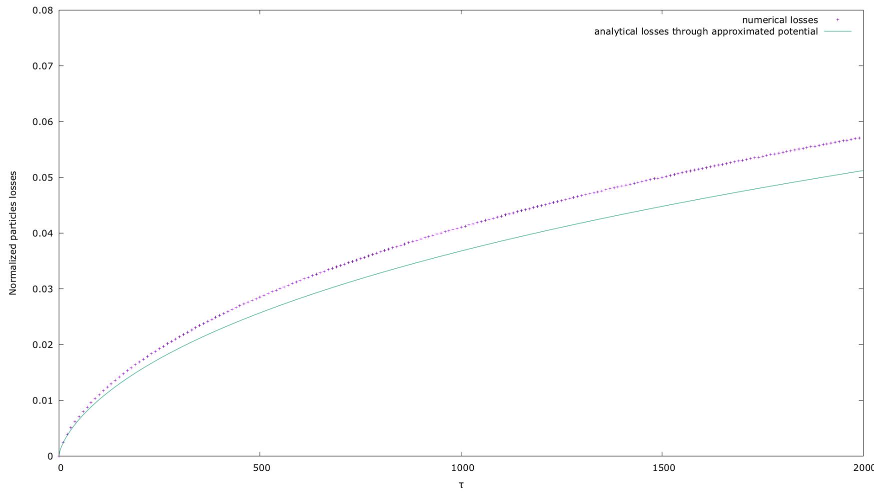


Figure 4.12: Normailzed particle losses for a Power law diffusion coefficient with $I_* = 8$.

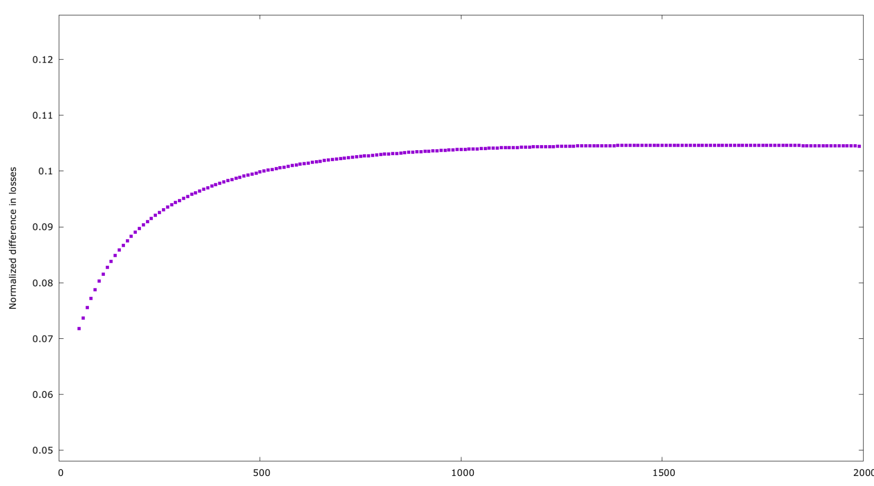


Figure 4.13: Normalized difference for the losses in figure 4.12

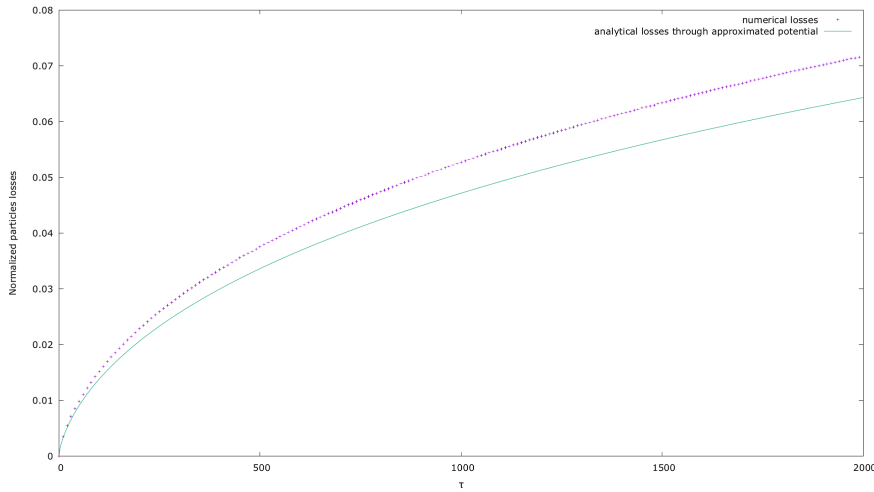


Figure 4.14: Normailzed particle losses for a Power law diffusion coefficient with $I_* = 8.5$.

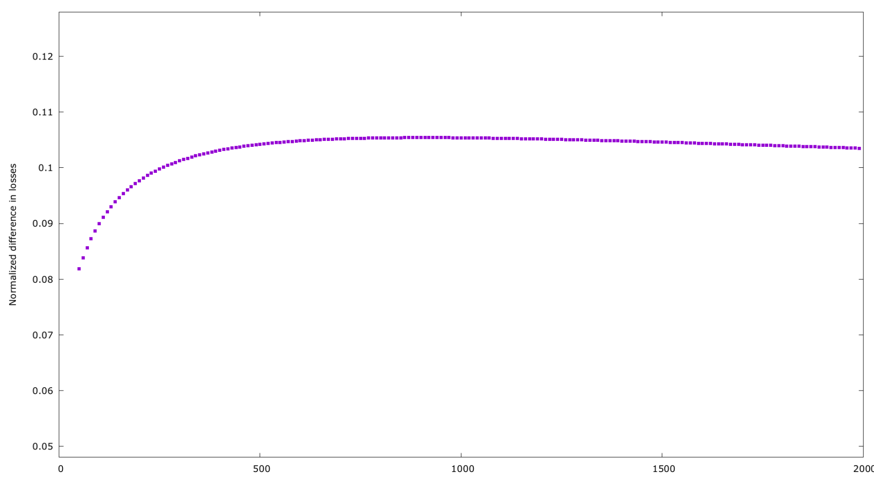


Figure 4.15: Normalized difference for the losses in figure 4.14

The remarkable result is that without exceptions after a transient phase the normalized difference stabilize around a precise value and remain constant. For the Nekhoroshev case the error committed approximateing the current stabilize between $7 - 7.5\%$. For the Power law the error is between $10 - 10.5\%$. Hence this result tells us that after a while the error on the approximated current before the time integration that gives the losses does not accumulate. This is understandable because both the analytical and the numerical losses, as can be seen in the figures in this section, at a certain point in time stop to grow and so their difference remain freezed at the value assumed at the beginning of this sort of stationary phase.

Given this results one can use the approximations of this chapter to scaled times τ, and thus times $t = \tau/\epsilon^2$ once the scale is defined, bigger then the one used up to now knowing the error commited in approximating actual Nekhoroshev and power law (with $\beta = 4$) diffusion coefficients.

4.6 Summary of the approach

Here we report schematically again the approach and recollect the important formulas. One start form F.P. equation

$$\frac{\partial\rho(I,t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial I}\Big(h^2(I)\frac{\partial\rho(I,t)}{\partial I}\Big) \tag{4.31}$$

with $I \in [0, I_a]$ being an absorbing barrier at I_a and cast it into a Smoluchowsky form

$$\frac{\partial\rho'}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\Big[\Big(\frac{dV(x)}{dx}\Big)\rho'\Big]+\frac{1}{2}\frac{\partial^2\rho'}{\partial x^2} \tag{4.32}$$

by means of

$$x = -\int_I^{I_a} dI' h^{-1}(I') \tag{4.33}$$

$$\frac{dI(x)}{dx} = h(I(x))$$

$$\rho(I,t)dI = \rho(I(x),t)\nu I(x)dx = \rho'(x,t)dx \tag{4.34}$$

$$V(x) \equiv -\ln\left(h(I(x))\right) \tag{4.35}$$

For $dV(x)/dx = -\nu$, with $\nu \in \mathbb{R}^+$ one is able to find the current at I_a for (4.31) when $\rho(I,0) = \delta(I_0 - I)$

$$J_{I_a}(I_0,t) \equiv J(x(I_0),t) = \frac{-x(I_0)}{t\sqrt{4\pi Dt}}\exp\Big(-\frac{(x(I_0)+\frac{\nu}{2}t)^2}{4Dt}\Big) \tag{4.36}$$

and the current for a generic distribution $\rho(I,0)$ reads

$$J_{I_a}(t) = \int_{-\infty}^0 J(x,t)\rho'(x,0)dx = \int_0^{I_a} J(x(I),t)\rho(I,0)dI \tag{4.37}$$

where (4.33) should be used to evaluate $x(I)$. Then for a generic $h(I)$ one can approximate the potential

$$V(x) = V(x_0) + \frac{dV(x)}{dx}\Big|_{x=x(I_0)} x + \mathcal{O}(x^2)$$

and obtain an estimate for what would be ν if she were treating a linear case

$$\nu(x(I_0)) = -\frac{dV(x)}{dx}\Big|_{x=x(I_0)} \tag{4.38}$$

Thus using (4.38) and (4.33) one can evaluate the currents (4.36) and (4.37) as approximations for the currents due to the generic diffusion coefficient $h(I)$.

In the specific we looked for, given the initial distribution $\rho(I,0) = 1/2\exp(-I/2)$, the current for a power low diffusion coefficient

$$h(I) = \left(\frac{I}{I_*}\right)^{\beta}$$

and for a Nekhoroshev-like diffusion coefficient

$$h(I) = \exp\left[-\left(\frac{I_*}{I}\right)^{\alpha}\right]$$

Chapter 5

Beam diffusion measurements in the LHC

This chaper will open with the presentation of the mesurements of beam losses which are the testing playground of the theoretical work of the previous chapters. Then a comparison between experimental data and predictions will be carried out, along with comments about the approach used for the enquiry and about what is possible to conclude given the experiments conducted so far.

5.1 The measurements

The data which are the reference for our theoretical work are the outcome of experiments conducted by the Hadrons Synchrotrons Single particle section (HSS) of the Accelerators and Beam Physics Group at CERN's Beam Department. Among other issues this section studies the mechanical aperture and magnetic field imperfections of the machines as installed and their impact on beam dynamics. Thus carries out experimental and theoretical beam physics and design studies aimed at improving the performance of CERN's circular hadron colliders and accelerators, in particular in terms of beam losses management.

Before introducing the experiment we recall that LHC consists of eight 2.45-km-long arcs, and eight 545-m-long straight sections called insertion regions (IRs). Each arc, with a regular lattice structure, contains 23 arc cells, and each arc cell has a FODO structure, 106.9 m long (see fig. 5.1), for the multipole magnets.

Dipole magnets are employed to bend the paths of the particles. If normal magnets were used in the 27 km-long LHC instead of superconducting magnets, the accelerator would have to be 120 kilometres long to reach the same energy. Powerful magnetic fields generated by the dipole magnets al-

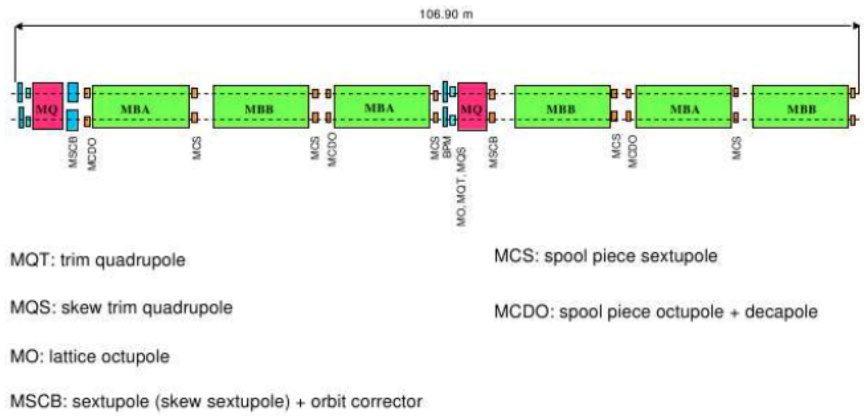


Figure 5.1: Basic multipolar magnetic cell (FODO) in LHC [33].

low the beam to handle tighter turns. When particles are bunched together, they are more likely to collide in greater numbers when they reach the LHC detectors. Quadrupoles help to keep the particles in a tight beam. They have four magnetic poles arranged symmetrically around the beam pipe to squeeze the beam either vertically or horizontally. Dipoles are also equipped with sextupole, octupole and decapole magnets, which correct for small imperfections in the magnetic field at the extremities of the dipoles.

The experiments are collocated at the insertion points of the IRs, where the beam collide. LHC has 100 collimators (see fig. 5.2) whose main purpose is to prevent damages to the superconducting magnets [34] and other sensitive elements and electronics absorbing beam halo particles and screening from radiation. Another use for the collimators though could be, as in our case, to serve as diagnostic tools.

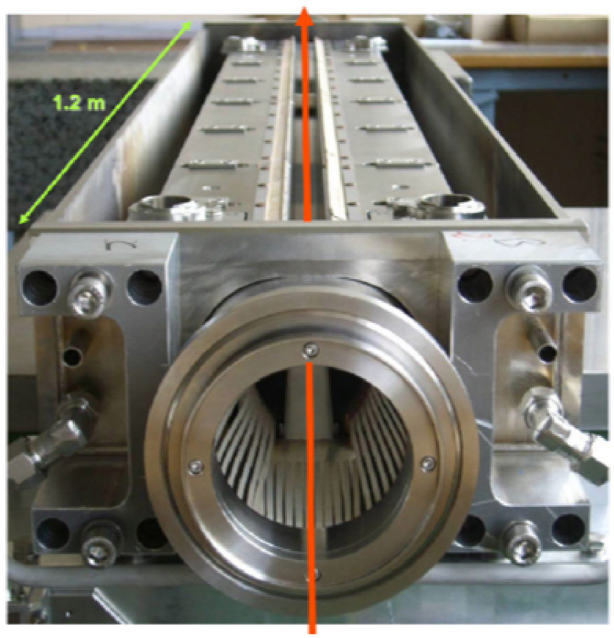


Figure 5.2: Photograph of a LHC collimator as seen from one end. The two blocks (jaws) of carbon, tungsten, or copper material need to be placed symmetrically on either side of and parallel to the beam to clean halo particles at maximum efficiency.[1]

The mesurements in the present analysis are mesurements of currents or, to be more precise, of the ratios between currents at a certain time and the currents at initial time. To have an idea of the experiment, it is by no means our aim to describe it accurately, we can think about the following scenario. The collimators are placed on both the axis of the tansverse plan of the motion at a certain distance from the center of the vacuum chamber. For the beam, with an essentially gaussian profile on both the axis, this collimators has the effects of absorbing barriers, when a particle reach a collimator it is lost. The beam during the experiment has of course constant longitudinal velocity in such a way that mesuring the ratios of a current at a certain time on the current at initial time amounts to mesure the normalized losses in particles in the beam. Mesurements are taken for several variation of intensity of octupole magnets (MCO) and decapole magnets (MCD). It is also kept track of the value of the sigma (hence the width) of the gaussian profiles, which are of order of 10^{-3}m, through time.
For similiar experiments on Beam diffusion using collimators we refer to [1].

In figures from 5.3 to 5.5 are reported the graphs with the outcome of the mesurements. In all the configurations losses remain under 5% in times of order 10^6 s.

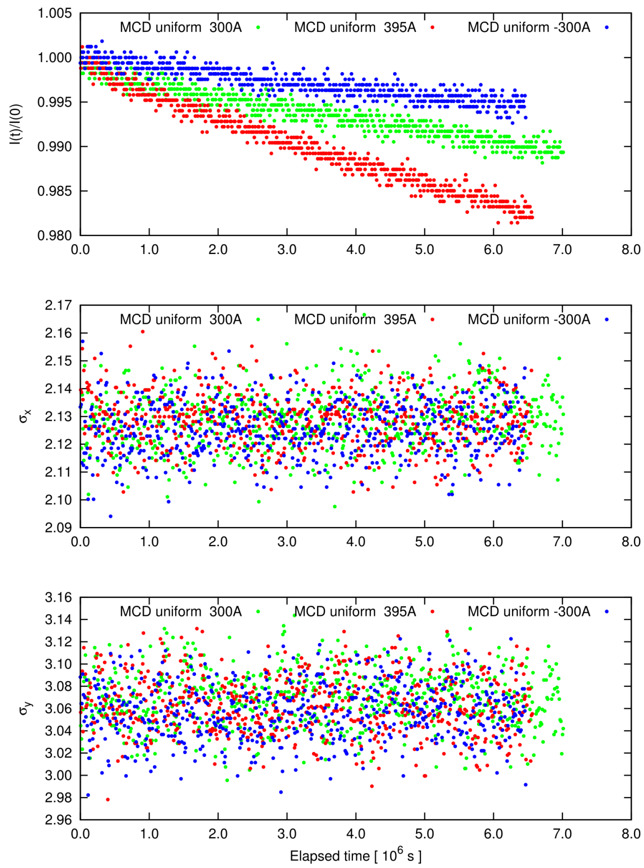


Figure 5.3: The upper panel shows mesurements of currents over time normalized by their initial value. In these set up of the machine the decapole magnets (MCD) are powered by currents of -300A, 300A and 395A. Below it is reported for each set up the evolution of the width of the gaussian beam in terms of its sigma for both the coordinates of the transverse plan.

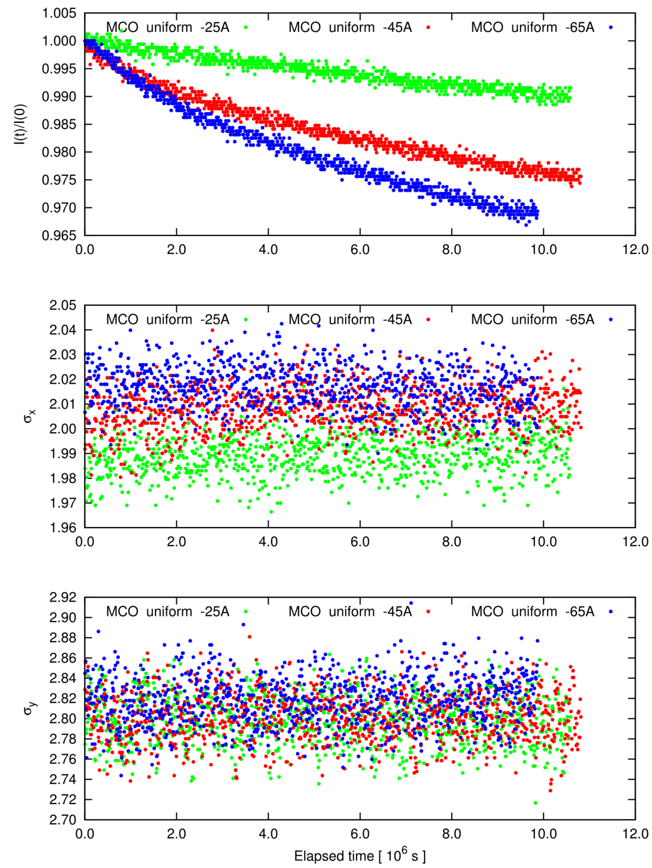


Figure 5.4: The upper panel shows mesurements of currents over time normalized by their initial value. In these set up of the machine the octupole magnets (MCO) are powered by currents of -25A, -45A and -65A. Below it is reported for each set up the evolution of the width of the gaussian beam in terms of its sigma [mm] for both the coordinates of the transverse plan.

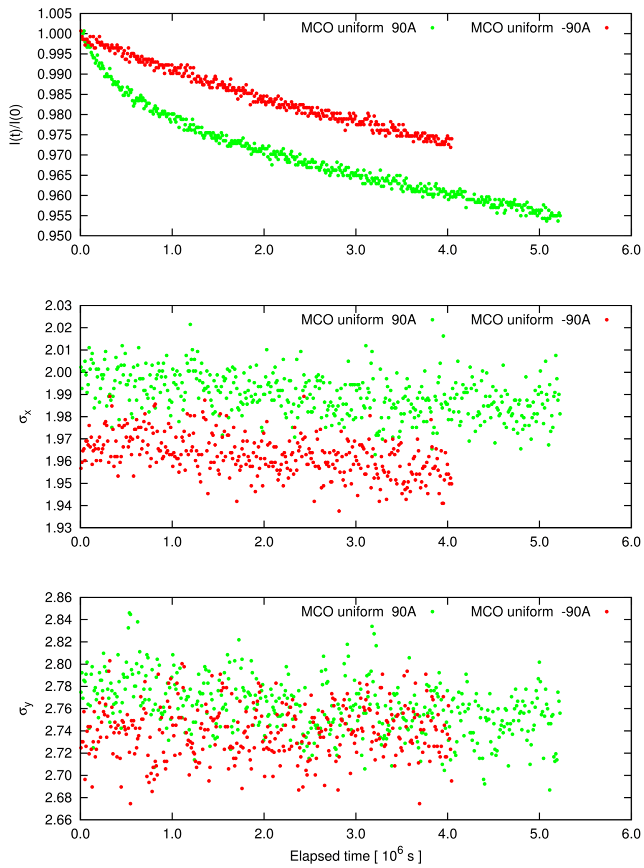


Figure 5.5: The upper panel shows mesurements of currents over time normalized by their initial value. In these set up of the machine the octupole magnets (MCO) are powered by currents of -90A and 90A. Below it is reported for each set up the evolution of the width of the gaussian beam in terms of its sigma [mm] for both the coordinates of the transverse plan.

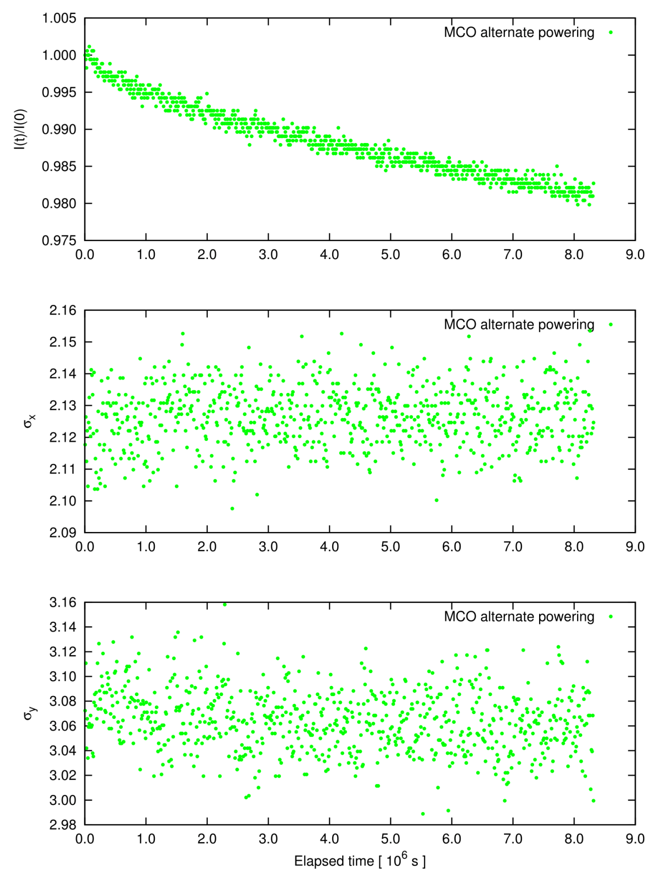


Figure 5.6: The upper panel shows mesurements of current over time normalized by its initial value. In these set up of the machine the octupole magnets (MCO) are powered with alternate currents around the ring. Below it is reported the evolution of the width of the gaussian beam in terms of its sigma [mm] for both the coordinates of the transverse plan.

5.2 Comparisons

In the Hamiltonian we described in the first chapter and hence the unperturbed part in our analysis on stochastic perturbations are encoded all the informations about the ideal dipoles, quadrupoles and multipole magnets. Then our approach in the previous chapters was to represent every other influence in the dynamics by stochastic perturbations, e.g. not designed and unwanted nonlinearities due to the magnets, particles interacting with each other, beam-gas scattering and so on. Differences in such influences, encoded in an accordingly different perturbative part of the Hamiltonian, which is impossible to determine by means of the microscopic dynamics, produce different effects. The diffusion coefficient in the Fokker-Planck which we end up with in this enquiry depend only on this last part of the Hamiltonian and we made some ansatzs on its form based on outcomes of the experimetns.

Thus variations in the current that powers the multipole magnets, besides being represented in the unperturbed Hamiltonian, bring with them an unpredictable a priori difference in the noise term. A different intensity in a multipole magnet could e.g. bring more unwanted nonlinearities due to unavoidable imperfections in the magnet itself, or could correct unwanted nonlinearities already present in the dipoles or quadrupoles, or other effects not related to undesigned nonlinearities.

Whatever the reasons they produce a mesurable different diffusive behaviour so we want to relate them to few parameters present in the estimate of the diffusion coefficient in the Fokker-Planck equation (3.19) at the center of this work

$$\frac{\partial}{\partial t}p(I,t) = \frac{1}{2}\frac{\partial}{\partial I}\left[h^2(I)\frac{\partial}{\partial I}p(I,t)\right]$$

Hence we fit phenomenologically this parameters. Even if related to an approximatoin for the diffusion coefficients *à la* Nekhoroshev and power law, the derivation of a current in a (semi)analytical form gives us a little more theoretical insight on the physics and allows to further studies, e.g. parametric scans, at a lower cost with respect to current obtained numerically solving the Fokker-Planck equation.

Despite the fact that mesurements have been take for the entire transverse plane, we study the situation for one dimension only, the x-axis. Considering a flat beam is not bad as an assumption, after previous mesurements [35] indeed was concluded that the horizontal tail was more populated then the vertical one. Hence in the analysis the data about the losses are treated as if they are actually for one dimension.

We define now the parameters for the cases under study. We remark that, given the informations, in this enquiry we chose to left only two parameter to be found fitting the data, the others being fixed by such informations.
We recall that troughout the work we called time the evolution parameter t but it has dimensiosn of lenght [mm] since we are in Frenet-Serret coordinates. Thus when we portray the results of the fit in seconds we use the

speed of light approximated as $c = 3 \cdot 10^{11}$mm/s to convert the quantities.

Consider the diffusion coefficients analysed in the previous chapter (4.27)

$$h(I) = \epsilon \exp\left[-\left(\frac{I_*}{I}\right)^{\alpha}\right]$$

and (4.22)

$$h(I) = \epsilon\left(\frac{I}{I_*}\right)^{\beta}$$

The exponent α is chosen to be

$$\alpha = 3/2$$

while, for the power law, we choose to study only

$$\beta = 4$$

Knowing the sigmas for the various configurations we can use as initial condition, in the calculations of current (4.37)

$$J_{I_a}(t) = \int_0^{I_a} J(x(I), t)\rho(I, 0)dI$$

the expression

$$\rho(I, 0) = \frac{1}{\sigma}\exp\{-(I/\sigma)\}$$

Indeed in the position variable the initial distribution is a gaussian and in our case the action is proportional to the squared positon, thus the previous expression is a good estimate for the initial distribution in the action variable. Furthermore we set for the abosrbing barrier

$$I_a = 6.5\sigma \quad [\text{mm}]$$

At this point it is possible to rescale the action $I \to I/\sigma$ and, calling with an abuse of notiation I and I_a the rescaled variables, use for the comparison with every data set

$$\rho(I, 0) = \exp\{-I\}$$
$$I_a = 6.5$$

where the action is now adimensional and expressed in unity of σ.

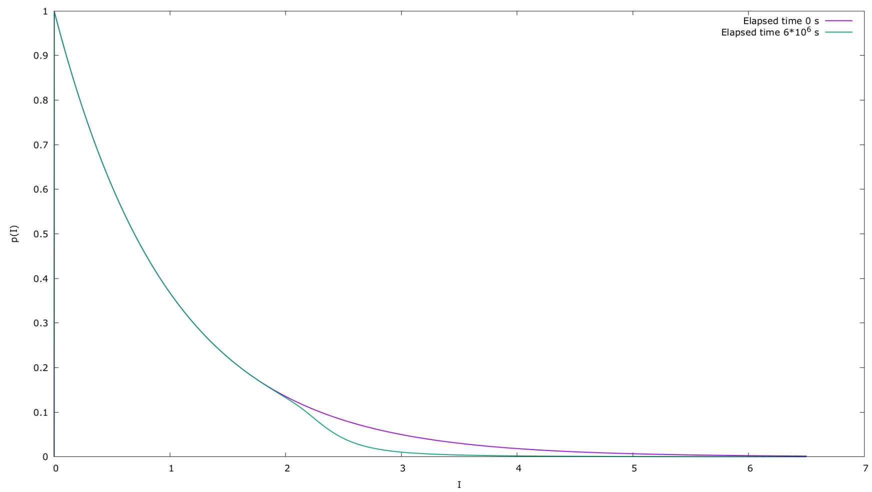
Therefore the only parameters left to determine are ϵ and I_*.

In this preliminary study we will use a rigrous best fit procedure only for I_*, while we device a criterion to set ϵ in a more qualitative but still objective fashion. In the first instance from the data can be deduced that the factor ϵ^2 indicative of the diffusion time scale is of order approximatively 10^{-15}-10^{-17} mm.

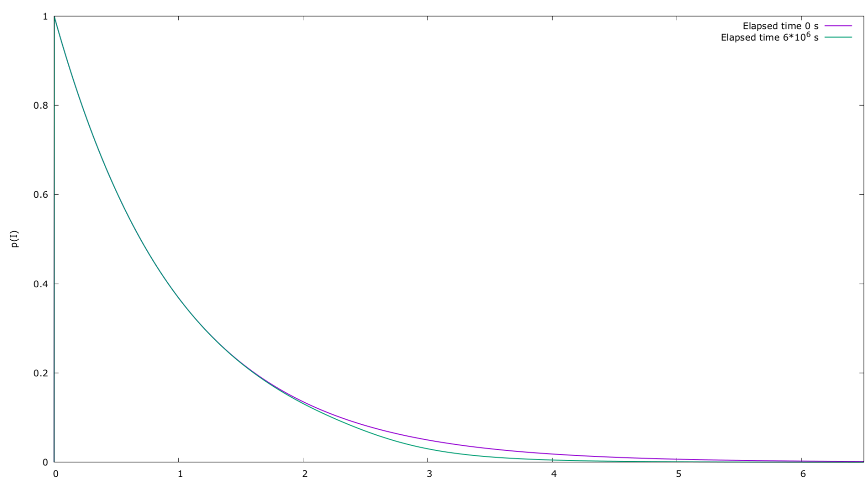
But it is possible to be more precise and use an ϵ^2 between 10^{-17} and 10^{-16} mm if after $\approx 1\%$ of losses the profile of the data resemble a straight line (see red dots in fig. 5.5); while if the profile assume the curved shape as in the green dots in fig. 5.5 one should set $\epsilon^2 \approx 10^{-15}$ mm. In measurements for which losses remain under 1% it is harder to appreciate the difference.

The enquiry procedes as follows. For every set of data which represent the outcomes of the mesurements we use criterion above to set ϵ, then with more rigor we look for which I_* for the Nekhoroshev coefficient and which I_* for the power law coefficient best fit the data. Our aim is to discern if there is a difference in the possibility to well reproduce the data, and well reproduce a larger variety of them, using a Nekhoroshev or a power law coefficient.

To have an intuition of the diffusive behaviour, we report in figure 5.7 exemples of the evolution of the beam profile in the action variable respectively for the Nekhoroshev-like and the power law coefficient. This profiles are obtained numerically solving the Fokker-Planck equation (3.19)



(a)



(b)

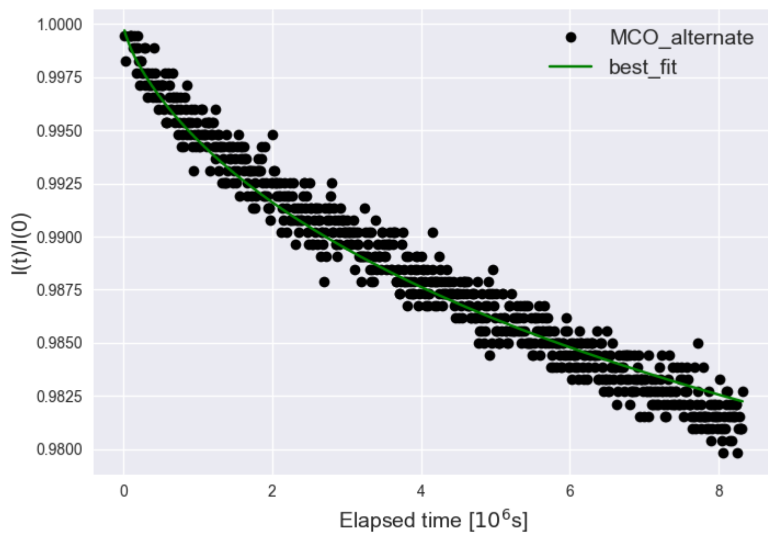
Figure 5.7: Beam profile evolution in the action variable for the Nekhoroshev (a) and power law (b) diffusion coefficient. Parameters are set as specified in this section and $I_* = 7$ for (a) and $I_* = 8.5$ for (b)

In figures from [?] to 5.16 we portray best fit for the data in the parameter I_* and with ϵ^2 set according to the criterion previously introduced.

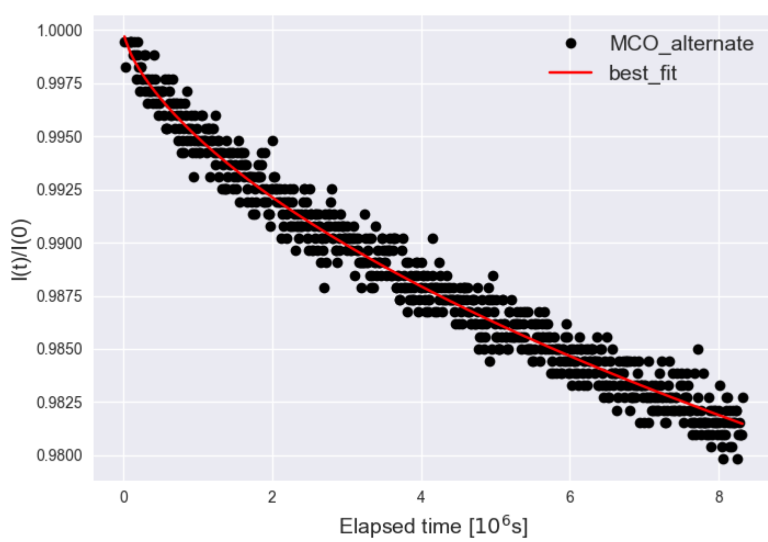
We remark the good agreement for both, the model based on the Nekhoroshev-like diffusion coefficient and the one based on power law; the fits are inside the spread of the experimental data. In the regime where $\epsilon^2 \approx 10^{-16} - 10^{-17}$ mm is however possible to appreciate a qualitatively slightly better correspondence in the case of Nekhoroshev with respect to the power law one.

We stress that this is not due to the fact that we used the same ϵ^2 for both, indeed a small in this parameter does not modify considerably the outcome and a change of scale result in a even worst agreement.

This is a preliminary study, and without in-depth quantitative analysis definitive conclusion can't be stated. Nonetheless there are indications that in the slower diffusive regime it could be possible to discern between Nekhoroshev-like coefficient and power law in measurements of this kind.

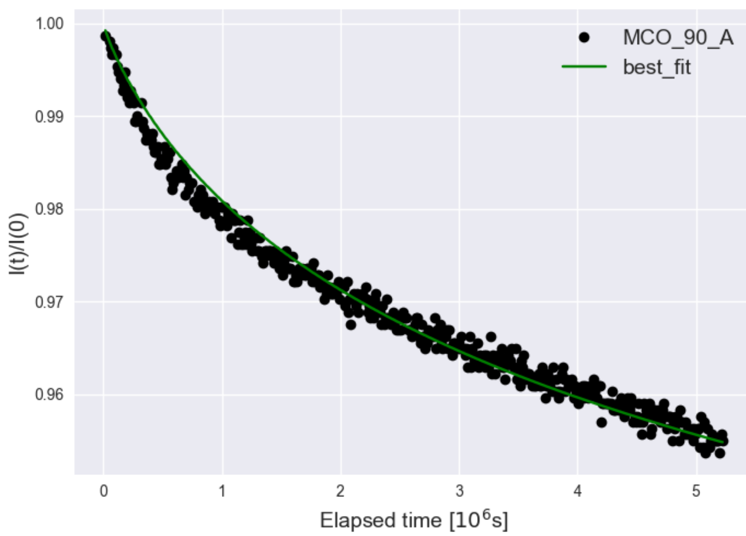


(a)



(b)

Figure 5.8: Dots represent mesurements of current over time normalized by its initial value for a machine set up with alternate power for the octupole magnets (MCO). Solid line stands for the losses evaluated through our approximation in the case of a Nekhoroshev diffusion coefficient with $I_* = 10.33$ (a) and power law diffusion coefficient with $I_* = 10.67$ (b). In this case we set $\epsilon^2 = \frac{10}{9} \cdot 10^{-15} \quad [\text{mm}]$.



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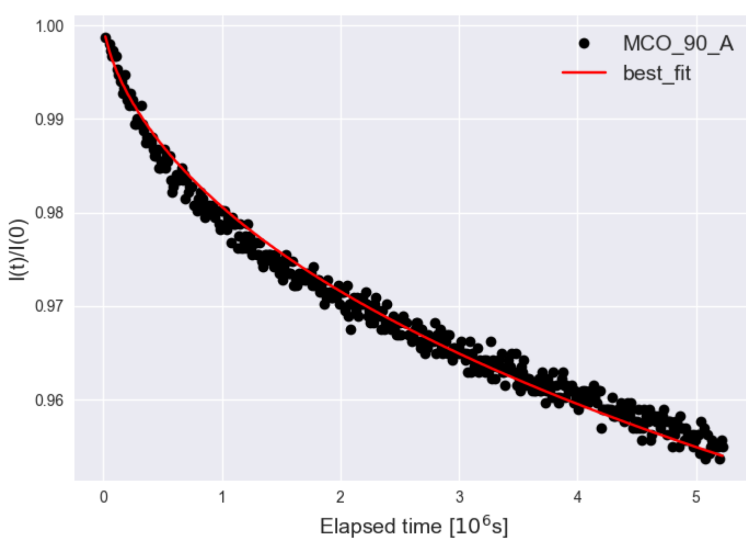
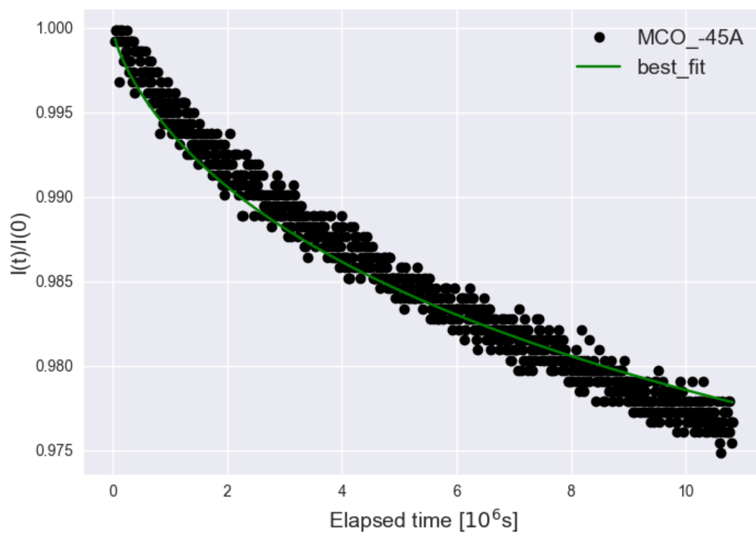


(b)

Figure 5.9: Dots represent mesurements of current over time normalized by its initial value for a machine set up with the octupole magnets (MCO) powered by 90A. Solid line stands for the losses evaluated through our approximation in the case of a Nekhoroshev diffusion coefficient with $I_* = 7.82$ (a) and power law diffusion coefficient with $I_* = 8.08$ (b). In this case we set $\epsilon^2 = \frac{10}{9} \cdot 10^{-15} \quad [\mathrm{mm}]$.



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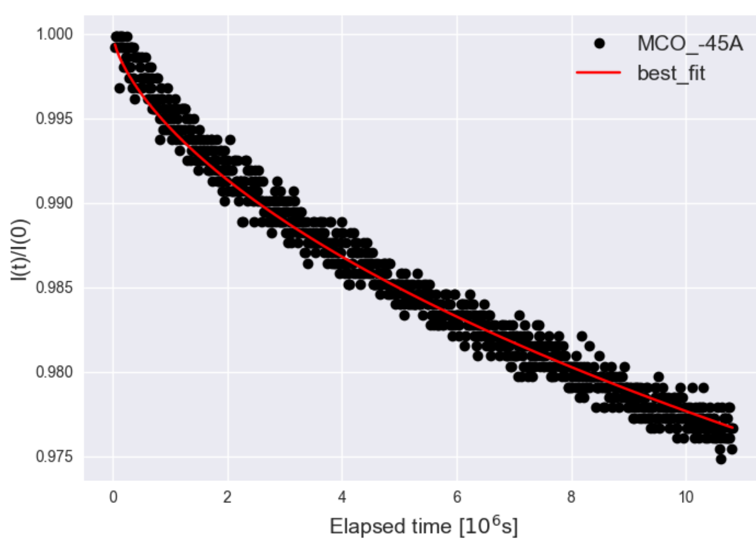
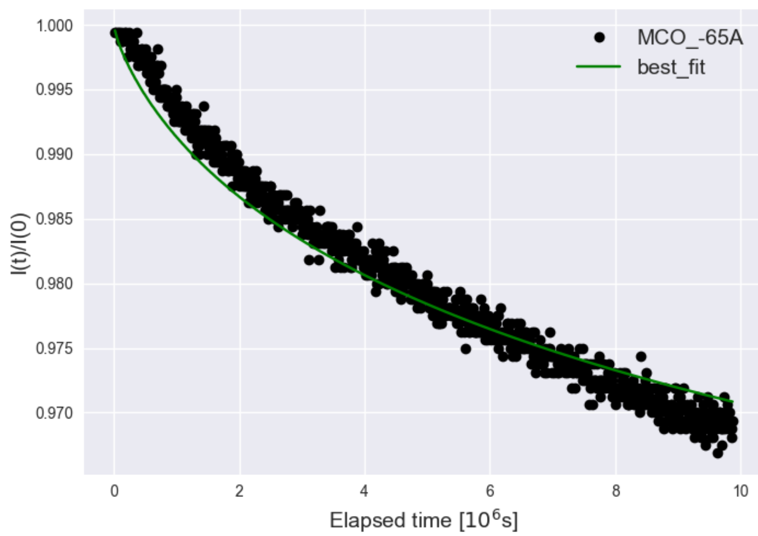
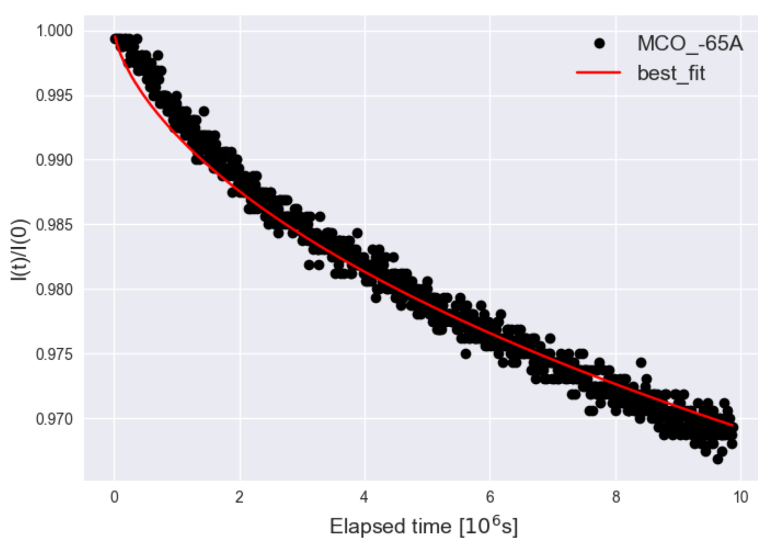


(b)

Figure 5.10: Dots represent mesurements of current over time normalized by its initial value for a machine set up with the octupole magnets (MCO) powered by -45A. Solid line stands for the losses evaluated through our approximation in the case of a Nekhoroshev diffusion coefficient with $I_* = 10.09$ (a) and power law diffusion coefficient with $I_* = 10.47$ (b). In this case we set $\epsilon^2 = \frac{10}{9} \cdot 10^{-15}$ [mm]

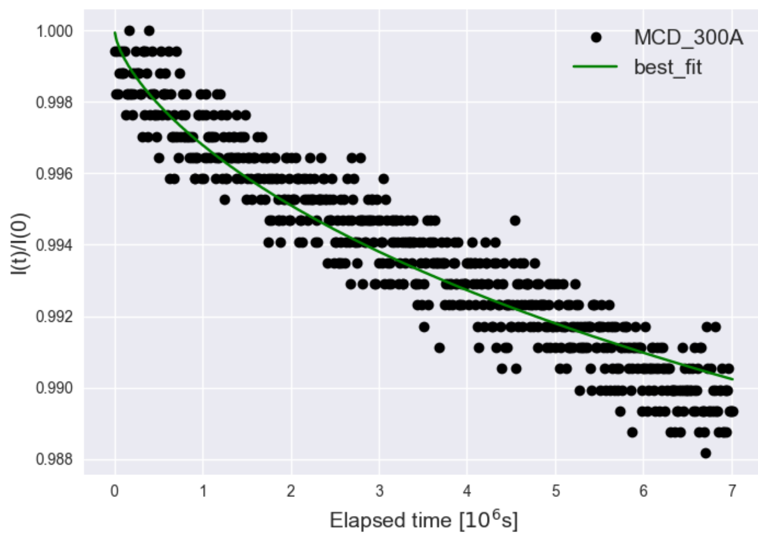


(a)

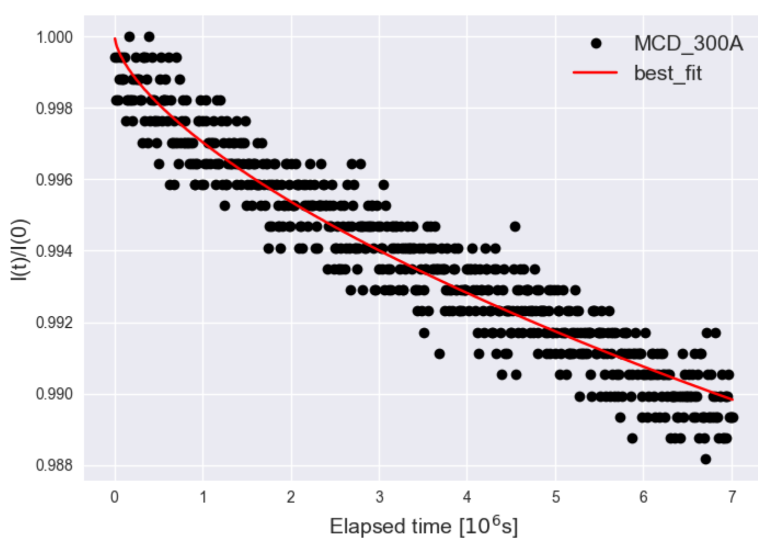


(b)

Figure 5.11: Dots represent mesurements of current over time normalized by its initial value for a machine set up with the octupole magnets (MCO) powered by -65A. Solid line stands for the losses evaluated through our approximation in the case of a Nekhoroshev diffusion coefficient with $I_* = 9.37$ (a) and power law diffusion coefficient with $I_* = 9.71$ (b). In this case we set $\epsilon^2 = \frac{10}{9} \cdot 10^{-15} \quad [\text{mm}]$

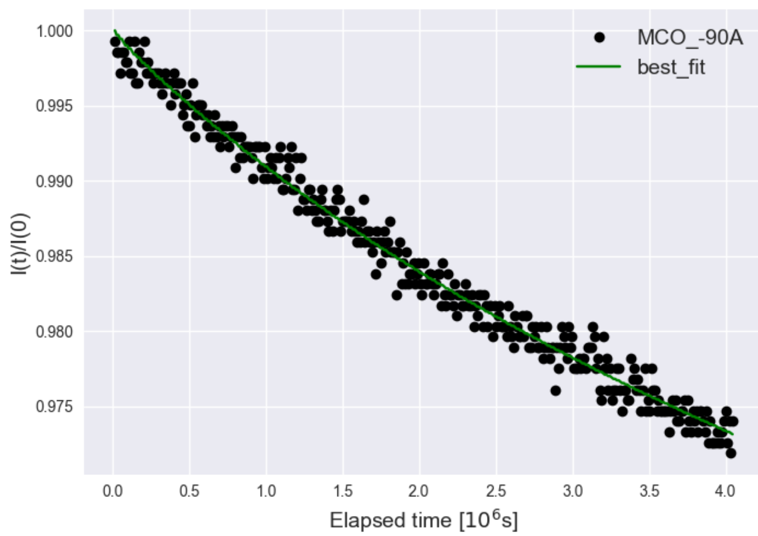


(a)

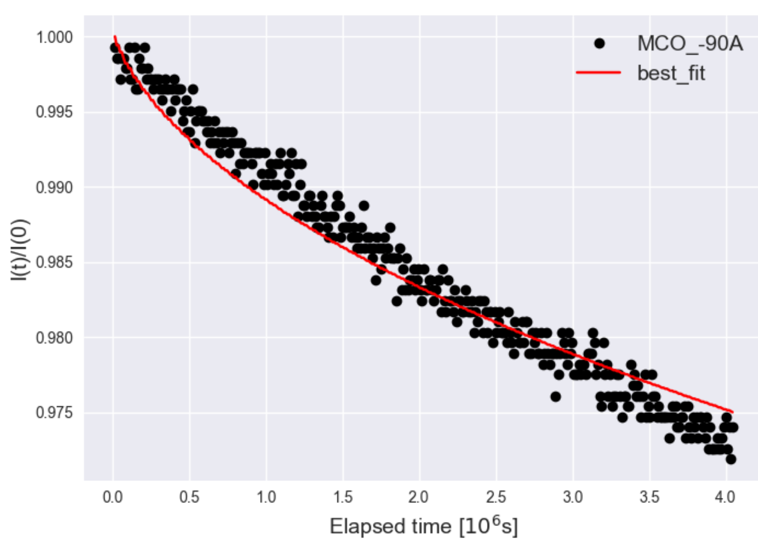


(b)

Figure 5.12: Dots represent mesurements of current over time normalized by its initial value for a machine set up with the octupole magnets (MCD) powered by 300A. Solid line stands for the losses evaluated through our approximation in the case of a Nekhoroshev diffusion coefficient with $I_* = 11.5$ (a) and power law diffusion coefficient with $I_* = 11.86$ (b). In this case we set $\epsilon^2 = \frac{10}{9} \cdot 10^{-15} \quad [\text{mm}]$

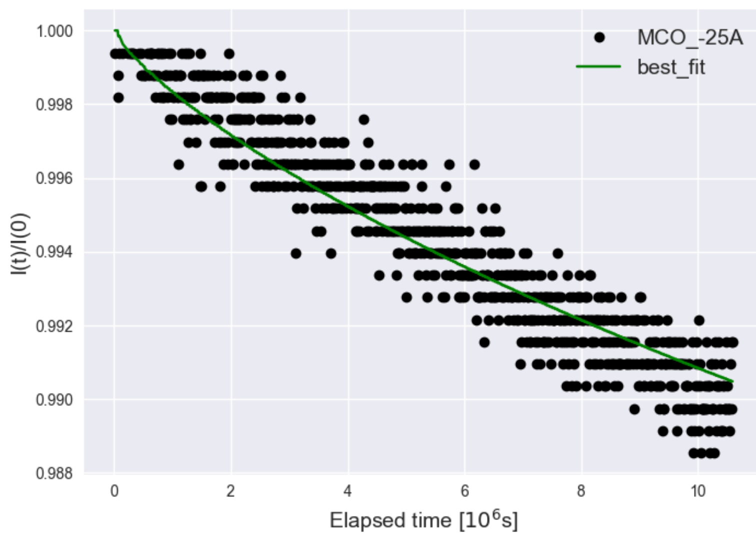


(a)

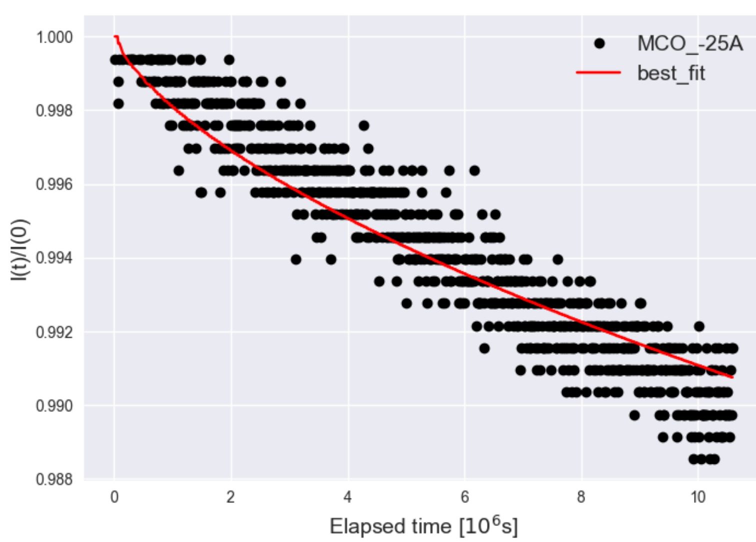


(b)

Figure 5.13: Dots represent mesurements of current over time normalized by its initial value for a machine set up with the octupole magnets (MCO) powered by -90A. Solid line stands for the losses evaluated through our approximation in the case of a Nekhoroshev diffusion coefficient with $I_* = 6.7$ (a) and power law diffusion coefficient with $I_* = 7.22$ (b). In this case we set $\epsilon^2 = \frac{10}{5.55} \cdot 10^{-16} \quad [\text{mm}]$

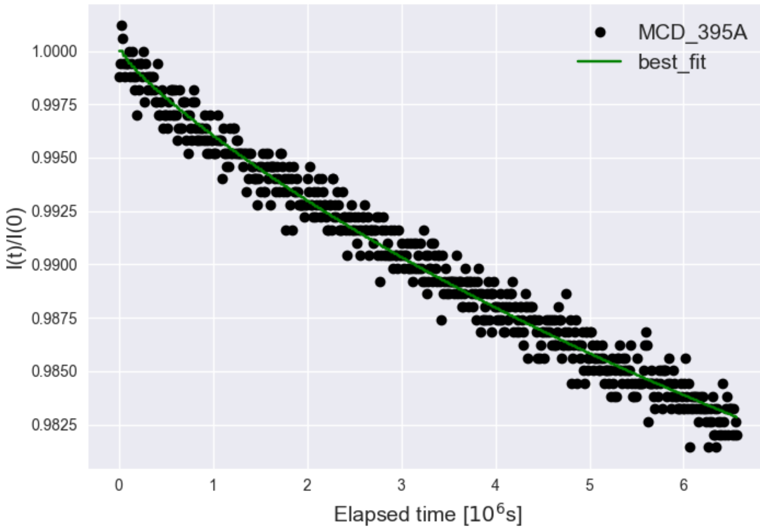


(a)



(b)

Figure 5.14: Dots represent mesurements of current over time normalized by its initial value for a machine set up with the octupole magnets (MCO) powered by -25A. Solid line stands for the losses evaluated through our approximation in the case of a Nekhoroshev diffusion coefficient with $I_* = 8.8$ (a) and power law diffusion coefficient with $I_* = 9.11$ (b). In this case we set $\epsilon^2 = \frac{10}{1.275} \cdot 10^{-17}$ [mm]



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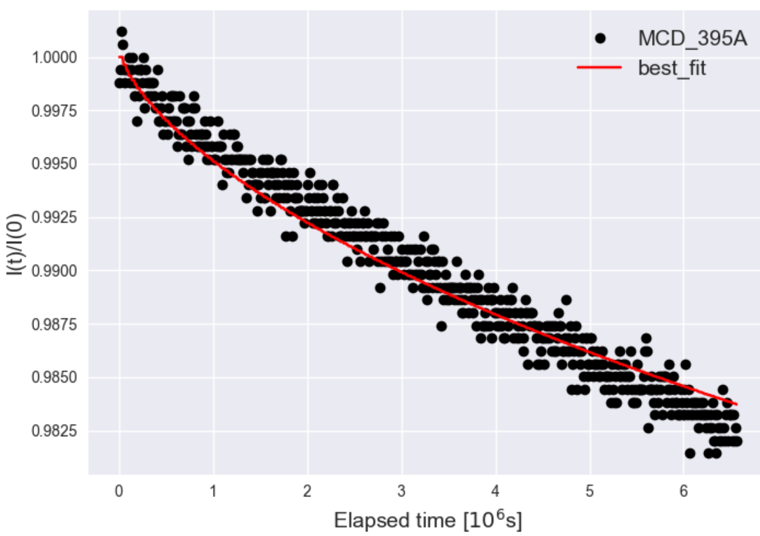
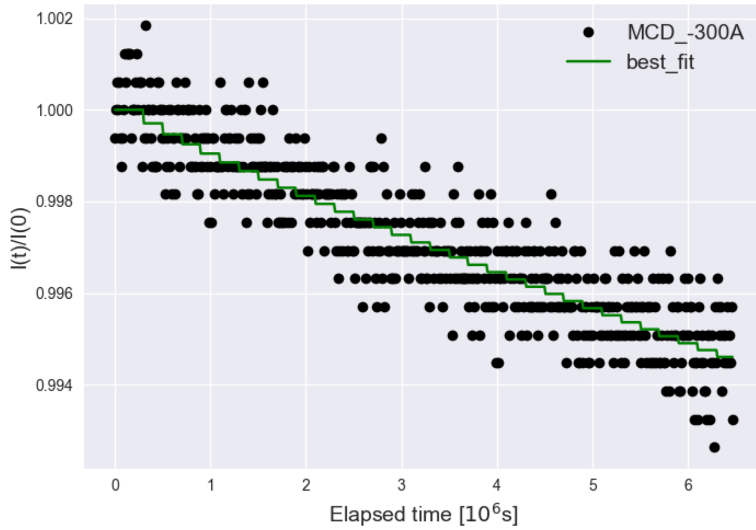
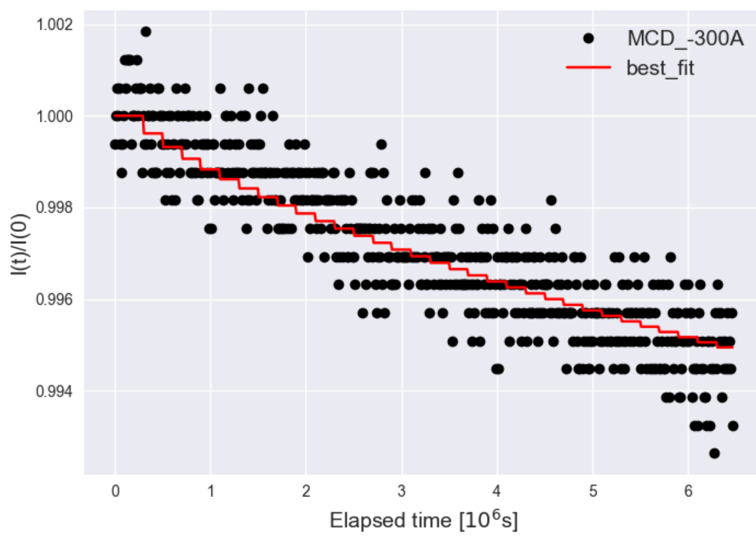


(b)

Figure 5.15: Dots represent mesurements of current over time normalized by its initial value for a machine set up with the octupole magnets (MCD) powered by 395A. Solid line stands for the losses evaluated through our approximation in the case of a Nekhoroshev diffusion coefficient with $I_* = 7.5$ (a) and power law diffusion coefficient with $I_* = 7.96$ (b). In this case we set $\epsilon^2 = \frac{10}{9} \cdot 10^{-16} \quad [\text{mm}]$



(a)



(b)

Figure 5.16: Dots represent mesurements of current over time normalized by its initial value for a machine set up with the octupole magnets (MCD) powered by -300A. Solid line stands for the losses evaluated through our approximation in the case of a Nekhoroshev diffusion coefficient with $I_* = 6.8$ (a) and power law diffusion coefficient with $I_* = 7.85$ (b). In this case we set $\epsilon^2 = \frac{10}{6} \cdot 10^{-17}$ [mm]

Conclusions

In this work we considered some aspects of beam losses in hadron synchrotrons and confronted our results with measurements regarding the Large Hadron Collider (LHC) at CERN [1].

We first recollected basic notions of transverse beam dynamics and relevant quantities for accelerator physics. Among these, the concept of dynamic aperture was introduced and we highlighted its close connection with the problem of particle losses.

After the description of useful tools in stochastic dynamics we described how we wanted to model the noise which affects the dynamics of a particle (hadron) in a synchrotron. Indeed we started from a noisy Hamiltonian system adding a small stochastic perturbation term to an integrable Hamiltonian. Through proper considerations we ended up with a Fokker-Planck equation for the probability distribution of the action variable (an invariant for the unperturbed motion), which is suited for the description of particle diffusion.

Our goal was to connect this model for the noisy system to quantities relevant for the comparisons with the measurements. Hence we derived formulas for the particle flow starting from a simpler linear case. Then we proposed two options for a diffusion coefficient useful for realistic models, Nekhoroshev-like estimates and power laws, in which few parameters were left to be gauged phenomenologically. We developed an approximation technique starting from the linear case in order to obtain semi-analytical formulas for the particle flow. Other than giving more insights on the problem such expressions are extremely useful if one is interested in analysing the relation of a particular choice in the form of the diffusion coefficient and the measure of particle losses. Indeed a great computational advantage is clear.

Finally, after introducing the experimental data supplied to us by the Hadrons Synchrotrons Single particle section of the Accelerators and Beam Physics Group at CERN's Beam Department, we tested our theoretical work with the measurements.

The data are the outcome of measurements of particle losses over time in the LHC for different set-ups of the machine. Thus we used our model to fit

[1] *Conseil Européen pour la Recherche Nucléaire*

the data setting properly the free parameters in the diffusion coefficients we proposed. The ultimate goal is to derive an effective form for the diffusion coefficient which allows to reproduce the experimental outcomes and enables to study the dependence from the parameters of the machine. In this work we provided a first step in this direction.

We found that it is possible to obtain a good agreement using both Nekhoroshev-like and the power law coefficient in the model to fit the data. Although, being this a preliminary study, more in-depth quantitative analysis should be performed.

One of our future goal is to check if with the available data it is possible to discern if the physics is better described, assuming either one or the other form for the diffusion coefficient. Besides, we want to extend our phenomenological fit in a more rigorous fashion to more of the free parameters and be more quantitative about the agreement.

We remark that we restricted this work to only one dimension, thus an effort towards extending the analysis to the whole transverse plane is in our plans.

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